Optimal Pollution Control

by

Daniel Milanes Perez

A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematical Finance

Department of Mathematical and Statistical Sciences
University of Alberta

Abstract

We consider the problem of a firm that wants to maximize its earnings. Production generates pollution as a by-product and has a negative impact on the environment. This negative impact causes disutility. The firm determines the optimal production rate and chooses between two types of technologies available. Each technology generates different pollution levels and benefits from production. We model this as a mixed classical-switching stochastic control problem. By analyzing the related differential equation, we are able to obtain an explicit solution that allows us to derive interesting managerial insights.

Preface

This research has been conducted under the supervision of Professor Abel Cadenillas. No part of this thesis has been previously published.

Acknowledgements

First and foremost, I would like to express my deepest gratitude to Professor Abel Cadenillas. His constant assistance, guidance, and support, even towards the end of his life, made the development of this thesis possible. I am profoundly grateful to him for introducing me to the fascinating area of stochastic control and optimization. Our numerous discussions and his insightful comments reflected his profound knowledge and experience, and were invaluable to me.

I would also like to thank Professor Christoph Frei for his support and guidance during the completion of this thesis. His willingness to take me on as a student, along with his insightful comments and suggestions, were instrumental in helping me finish this thesis. I am also deeply grateful for his efforts to help me transition into life after graduate school.

I would like to thank Professor Alexander Melnikov, Professor Thomas Hillen, and Professor Bin Han for agreeing to serve as members of my defence committee.

Special thanks go out to my family for their support and to my friends in Edmonton, who helped me enjoy graduate school.

Table of Contents

List of Figures

Chapter 1

Introduction

Economies can only grow sustainably if they simultaneously manage the growing urgency of environmental degradation and climate change. Failing to do so not only leads to catastrophic impacts on the environment, including the depletion of natural resources and more frequent and severe droughts and extreme weather events, but it also exacerbates health and social inequalities, forcing millions into extreme poverty (see, for example, Hardy [2003] and Islam and Winkel [2017]). Additionally, it weakens countries' ability to withstand future shocks.

According to Guo et al. [2021] the situation is already grave, with the planet on path to a 2.0–2.6$°C$ temperature rise by mid-century, surpassing the 1.5$°C$ Paris Agreement target. Continuation of this trajectory would cut total global economic value by 10% by 2050.

Addressing this issue requires a fundamental transformation of all sectors, including energy, manufacturing, transport, infrastructure, agriculture, forestry and land use. Market forces alone are insufficient and governments have the responsibility to take the lead. Many have set targets to achieve net-zero carbon emissions by specified dates, ranging from 2030 in Uruguay and 2035 in Finland to 2050 for most other countries.

Governments can choose from a wide range of policy interventions and financing measures to support the transformation of energy and industrial systems, improve energy efficiency, tackle environmental pollution, and protect and replenish natural capital. Many have already adopted different measures, including green taxes on harmful environmental activities, tighter regulations, and new environmental standards and certification for energy performance, emissions and pollutants.

In this thesis, we take the point of view of a firm facing these policies. We consider a firm that it is looking to maximize its earnings. During the production process, pollution occurs as a by-product and has a negative impact on the environment. This environmental damage comes along with some disutility in the form of policy interventions. We assume that there are two technologies available, a pollutive "brown" technology and a cleaner but less productive "green" alternative. At any point in time the firm decides the production rate and which type of technology to use.

A similar problem was studied in Moser et al. [2014], they focused on the circumstances under which the switch between different technologies occurs. In their paper, they modeled the dynamics of pollution as deterministic. However, considering various factors such as meteorological conditions, which can influence the absorption capabilities of the environment and the deposition sites of pollutants (see Nahorski and Ravn [2000]), we model pollution as a stochastic process.

The early papers on economic dynamic pollution problems emerged in the beginning of the 1970s (see, for example, Keeler et al. [1972]). Since then, the prevailing approach in the literature has been to model pollution as a deterministic process. Nevertheless, there have been instances where uncertainty has been incorporated into pollution dynamics. For example, Mosiño and Pommeret [2015] studied the tradeoff between environmental quality and economic performance within a two-stage framework, modelling pollution as a stochastic process. Similarly, Zemel [2012] studied the effect of uncertainty on precautionary behaviour.

In what follows, we apply the theory of classical stochastic control combined with optimal switching control to solve the problem described above. We are able to obtain an explicit solution that allows us to derive interesting managerial insights.

The solution is as follows. The production rate will be a decreasing function of pollution. Changes in technologies will take place optimally when the pollution reaches a certain level. Indeed, we shall prove that when the firm is using green technology, whenever the level of pollution is above a threshold, the firm will continue to use green technology; but when the level of pollution reaches that threshold it is optimal to change technology. The optimal policy when the firm is using brown technology is the following: whenever the level of pollution is below a threshold, the firm will continue to use brown technology; but when the level of pollution reaches that threshold it is optimal to change technology.

The remaining of this thesis is organized as follows. Chapter 2 describes the pollution model and present the environmental problem as a classical-switching control problem. In Chapter 3, we characterize the solution to the problem, and in Chapter 4, we obtain a solution. This is done in two steps, first we obtain a candidate solution, and then we prove rigorously that this candidate is indeed the optimal solution. Chapter 5 is devoted to a comparative statistics analysis. Chapter 6 concludes.

Chapter 2

The Pollution Model

Consider a one-dimensional Brownian motion W, which will represent the uncertainty in the evolution of the pollution stock, on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $\{\mathcal{F}_t;\ t \geq 0\}$ be the augmented filtration generated by W. We denote

$$\begin{aligned} X &:= \text{pollution stock}, \\ u &:= \text{production rate}. \end{aligned}$$

We consider two possible regimes, and denote the set of regimes by $\mathbb{I} = \{1, 2\}$. Regime 1 represents green technology, and regime 2 represents brown technology. We denote

$$I_t := \text{regime at time } t.$$

We assume that X is an adapted stochastic process given by

$$dX_t = (\rho_{I_t} u_t - \alpha X_t)\, dt + \sigma X_t\, dW_t, \quad X_0 = x > 0. \tag{2.1}$$

The stochastic process u_t represents the production rate of the firm at time t. The constant $0 < \rho_i \leq 1$ denotes the pollution intensity of technology i. Among the technologies, green technology causes less pollution, so $\rho_1 < \rho_2$. The constant $\alpha > 0$ represents the natural decay rate of pollution, while $\sigma > 0$ signifies the volatility in the evolution of the pollution stock. We assume that $\alpha > 2\sigma$. We fix a constant $\delta > 0$, which will be used to denote the discount rate.

Definition 2.1 (The controls)**.** *A mixed classical-switching stochastic control is a double*

$$(u, T) = (u; \tau_1, \tau_2, \dots, \tau_n, \dots).$$

Here, u is a classical control. This means that $u : [0, \infty)\times\Omega \mapsto \mathbb{R}$ is an $\{\mathcal{F}_t\}$-adapted stochastic process. Moreover, T is the switching control. This means that $0 \le \tau_1 < \tau_2 < \dots < \tau_n < \dots$ is a sequence of increasing stopping times. At each stopping time, the firm decides to switch to a different technology.

Definition 2.2 (Admissible controls)**.** *A mixed classical-switching stochastic control (u, T) is admissible if it satisfies the following conditions:*

$$P\{\lim_{n\to\infty} \tau_n \le t\} \;=\; 0 \quad \forall t \ge 0, \tag{2.2}$$

$$E\left[\int_0^\infty \mathrm{e}^{-\delta t}\, u_t^2 \, dt\right] \;<\; \infty, \tag{2.3}$$

$$E\left[\int_0^\infty \mathrm{e}^{-\delta t}\, X_t^4 \, dt\right] \;<\; \infty. \tag{2.4}$$

We will denote by $\mathcal{A}$ the class of admissible controls.

Problem 2.1. *The firm wants to select the control $(u, T) \in \mathcal{A}$ that maximizes the functional J defined by*

$$J(x, i; u, T) \;:=\; E\left[\int_0^\infty \mathrm{e}^{-\delta t}\, f(X_t, u_t, I_t)\, dt - \sum_{n=1}^{\infty} \mathrm{e}^{-\delta\tau_n}\, k_{\iota_{n-1},\iota_n} \mathbb{1}_{\{\tau_n<\infty\}}\right], \tag{2.5}$$

where ι_n represents the regime in the interval $[\tau_n, \tau_{n+1})$, and

$$f(X_t, u_t, I_t) = \beta_{I_t} u_t - u_t^2 - \lambda_{I_t} X_t^2,$$
$$\beta_i, \lambda_i, k_{ij} \in (0, \infty).$$

We assume that production yields a concave benefit $\beta_i u - u^2$, and incurs a damage $\lambda_i x^2$ through pollution. Here, β_i and λ_i represent the marginal benefit and damage associated with the use of technology i, respectively. The brown technology is both more productive and more damaging than the green technology, leading to the inequalities $\beta_1 \leq \beta_2$ and $\lambda_1 \leq \lambda_2$. Additionally, the constant k_{ij} reflects the cost of transitioning from regime i to regime j. We assume that this cost is higher when we are shifting from brown to green technology, i.e., $k_{12} < k_{21}$.

According to Section 2, Part F of Nordhaus [2019], increasing the cost of using pollutive technologies relative to less pollutive ones incentivizes firms to adopt low-carbon technologies, which is essential for effective climate-change policy. Following this, to avoid having a solution where it is optimal to continue using brown technology, the value of λ_2/λ_1 must be sufficiently large. In our model it is enough to have $\lambda_2/\lambda_1 \geq \rho_2^2/\rho_1^2$.

Furthermore, we assume that the marginal benefit and pollution intensities of both technologies are related by $\beta_2/\beta_1 \geq \rho_2/\rho_1$. This inequality indicates that the improvement in benefits from using a brown technology, relative to using a green technology, is greater than or equal to the increase in pollution intensity associated with the use of a brown technology, relative to the use a green technology.

To summarize, we assume that the parameters satisfy the following properties:

$$\rho_1 < \rho_2, \quad \alpha > 2\sigma^2, \quad \beta_1 \leq \beta_2, \quad \lambda_1 \leq \lambda_2, \quad k_{12} < k_{21}, \quad \frac{\beta_2}{\beta_1} \geq \frac{\rho_2}{\rho_1} \quad \text{and} \quad \frac{\lambda_2}{\lambda_1} \geq \frac{\rho_2^2}{\rho_1^2}. \tag{2.6}$$

Remark 2.1. *Note that conditions (2.3) and (2.4) imply that (2.5) is well defined:*

$$\begin{aligned}
\left| E\left[\int_0^\infty \mathrm{e}^{-\delta t} f(X_t, u_t, I_t)\, dt \right] \right| &\leq E\left[\left| \int_0^\infty \mathrm{e}^{-\delta t} f(X_t, u_t, I_t)\, dt \right| \right] \\
&\leq E\left[\int_0^\infty \left| \mathrm{e}^{-\delta t} f(X_t, u_t, I_t) \right| dt \right] \\
&= E\left[\int_0^\infty \mathrm{e}^{-\delta t} \left| (\beta_{I_t} u_t - u_t^2 - \lambda_{I_t} X_t^2) \right| dt \right] \\
&\leq E\left[\int_0^\infty \mathrm{e}^{-\delta t} \beta_{I_t} |u_t|\, dt \right] + E\left[\int_0^\infty \mathrm{e}^{-\delta t} u_t^2\, dt \right]
\end{aligned}$$

$$
\begin{aligned}
& +E\left[\int_0^\infty \mathrm{e}^{-\delta t}\,\lambda_{I_t} X_t^2 dt\right] \\
= \;& E\left[\int_0^\infty \mathrm{e}^{-\delta t}\,\beta_{I_t}|u_t|\mathbb{1}_{\{|u_t|\le 1\}}\,dt\right] \\
& +E\left[\int_0^\infty \mathrm{e}^{-\delta t}\,\beta_{I_t}|u_t|\mathbb{1}_{\{|u_t|>1\}}\,dt\right] \\
& +E\left[\int_0^\infty \mathrm{e}^{-\delta t}\,u_t^2\,dt\right] \\
& +E\left[\int_0^\infty \mathrm{e}^{-\delta t}\,\lambda_{I_t} X_t^2\mathbb{1}_{\{|X_t|\le 1\}}\,dt\right] \\
& +E\left[\int_0^\infty \mathrm{e}^{-\delta t}\,\lambda_{I_t} X_t^2\mathbb{1}_{\{|X_t|>1\}}\,dt\right] \\
\le \;& E\left[\int_0^\infty \mathrm{e}^{-\delta t}\,\beta_{I_t}\mathbb{1}_{\{|u_t|\le 1\}}\,dt\right] \\
& +E\left[\int_0^\infty \mathrm{e}^{-\delta t}\,\beta_{I_t} u_t^2\mathbb{1}_{\{|u_t|>1\}}\,dt\right] \\
& +E\left[\int_0^\infty \mathrm{e}^{-\delta t}\,u_t^2\,dt\right] \\
& +E\left[\int_0^\infty \mathrm{e}^{-\delta t}\,\lambda_{I_t}\mathbb{1}_{\{|X_t|\le 1\}}\,dt\right] \\
& +E\left[\int_0^\infty \mathrm{e}^{-\delta t}\,\lambda_{I_t} X_t^4\mathbb{1}_{\{|X_t|>1\}}\,dt\right] \\
< \;& \infty.
\end{aligned}
$$

Chapter 3

The Value Function

Let us denote the value function by V. That is, for every $x \in (0, \infty)$ and $i \in \mathbb{I}$,

$$V(x, i) := \sup \{J(x, i; u, T); (u, T) \in \mathcal{A}\}.$$

Let us consider the operator $\mathcal{L}^u$ defined by

$$\mathcal{L}^u \psi(x, i) = \frac{1}{2}\sigma^2 x^2 \psi_{xx}(x, i) + (\rho_i u - \alpha x)\psi_x(x, i).$$

Now we intend to characterize the value function and an associated optimal strategy.

Definition 3.1 (VI)**.** *We say that a function $v : (0, \infty) \times \mathbb{I} \mapsto \mathbb{R}$ satisfies the variational inequalities (VI) for Problem 2.1 if for every $x \in (0, \infty)$,*

$$\max \left\{ \max_u \left[\mathcal{L}^u v(x, 1) - \delta v(x, 1) + f(x, u, 1)\right], \right. \left. v(x, 2) - v(x, 1) - k_{12} \right\} = 0, \tag{3.1}$$

$$\max \left\{ \max_u \left[\mathcal{L}^u v(x, 2) - \delta v(x, 2) + f(x, u, 2)\right], \right. \left. v(x, 1) - v(x, 2) - k_{21} \right\} = 0. \tag{3.2}$$

For each regime $i \in \mathbb{I}$, a solution v of the VI separates the interval $(0, \infty)$ into two disjoint regions: a continuation region, where the controller stays in the current regime

$$\mathcal{C}_i := \{x \in (0,\infty) : v(x,i) - v(x,j) - k_{ij} < 0\},$$

and a switching region, where the controller chooses to switch regimes

$$\mathcal{S}_i := \{x \in (0,\infty) : v(x,i) - v(x,j) - k_{ij} = 0\}.$$

From a solution to the VI it is possible to construct the following mixed classical-switching control.

Definition 3.2. *Let v be a solution of the VI. The following mixed classical-switching control*

$$(u^v, T^v) = (u^v, \tau_1^v, \tau_2^v, \dots, \tau_n^v, \dots)$$

is called the VI-control associated with v if: for all $i \in \mathbb{I}$,

$$P\{\forall (t, X_t^v) \in [0,\infty) \times X^i : u_t^v \in \arg\max_u [\mathcal{L}^u v(x,i) - \delta v(x,i) + f(x,u,i)]\} = 1,$$

$$\tau_1^v := \inf\{t \geq 0 : v(X_t^v, \iota_0) = v(X_t^v, \iota_1) + k_{\iota_0,\iota_1}\}$$

and, for every $n \geq 2$

$$\tau_n^v := \inf\{t > \tau_{n-1}^u : v(X_t^v, \iota_{n-1}) = v(X_t^v, \iota_n) + k_{\iota_{n-1},\iota_n}\}.$$

Here, X^v is the trajectory determined by (u^v, T^v), and X^i is the set where X_t is in regime i.

Theorem 3.1 (Verification Theorem)**.** *Let $v : (0,\infty) \times \mathbb{I} \mapsto \mathbb{R}$ be a function that satisfies the VI. Suppose that v is either convex or concave, and has quadratic growth; that is, there exists $K \in (0,\infty)$ such that $|v(x,\cdot)| \leq K(1+x^2)$. Additionally, assume that $v(\cdot,i) \in C^2(\mathcal{C}_i)$. Then, for every $x \in (0,\infty)$:*

$$V(x) \leq v(x).$$

Furthermore, if the VI-control (u^v, T^v) corresponding to v is admissible, then it is an optimal mixed classical-switching control, and for every $x \in (0,\infty)$,

$i \in \mathbb{I}$:

$$V(x,i) = v(x,i) = J(x,i;u^v,T^v).$$

Proof. Let (u,T) be an admissible control, and denote by $X = X^{u,T}$ the trajectory determined by (u,T). For every $t > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathrm{e}^{-\delta(t\wedge\tau_n)}\, v(X_{t\wedge\tau_n}, I_{t\wedge\tau_n}) - v(X_0, I_0) \\
&= \sum_{j=1}^{n} \Big\{ \mathrm{e}^{-\delta(t\wedge\tau_j)}\, v(X_{t\wedge\tau_j}, I_{t\wedge\tau_j -}) - \mathrm{e}^{-\delta(t\wedge\tau_{j-1})}\, v(X_{t\wedge\tau_{j-1}}, I_{t\wedge\tau_{j-1}}) \Big\} \\
&\quad + \sum_{j=1}^{n} \mathbb{1}_{\{\tau_j \le t\}}\, \mathrm{e}^{-\delta\tau_j} \{ v(X_{\tau_j}, I_{\tau_j}) - v(X_{\tau_j}, I_{\tau_j -}) \}.
\end{aligned}$$

Note that by the definition of the switching control T, $X_s \in \mathcal{C}_{I_{t\wedge\tau_j}}$, $\forall s \in [t\wedge\tau_j, t\wedge\tau_{j+1})$. Since $v(\cdot,i) \in C^2(\mathcal{C}_i)$, we can apply Itô's formula in the interval $[t\wedge\tau_j, t\wedge\tau_{j+1})$.

$$\begin{aligned}
&\mathrm{e}^{-\delta(t\wedge\tau_n)}\, v(X_{t\wedge\tau_n}, I_{t\wedge\tau_n}) - v(X_0, I_0) \\
&= \sum_{j=1}^{n} \Big\{ \int_{t\wedge\tau_{j-1}}^{t\wedge\tau_j} \mathrm{e}^{-\delta s} (\mathcal{L}^u v(X_s, I_s) - \delta v(X_s, I_s))\, ds \\
&\qquad + \int_{t\wedge\tau_{j-1}}^{t\wedge\tau_j} \mathrm{e}^{-\delta s}\, \sigma v'(X_s, I_s) X_s\, dW_s \Big\} \\
&\quad + \sum_{j=1}^{n} \mathbb{1}_{\{\tau_j \le t\}}\, \mathrm{e}^{-\delta\tau_j} \{ v(X_{\tau_j}, I_{\tau_j}) - v(X_{\tau_j}, I_{\tau_j -}) \}.
\end{aligned}$$

Since v satisfies equations (3.1)–(3.2), we have

$$\begin{aligned}
&\mathrm{e}^{-\delta(t\wedge\tau_n)}\, v(X_{t\wedge\tau_n}, I_{t\wedge\tau_n}) - v(X_0, I_0) \\
&\le \sum_{j=1}^{n} \Big\{ \int_{t\wedge\tau_{j-1}}^{t\wedge\tau_j} -\mathrm{e}^{-\delta s}\, f(X_s, u_s, I_s)\, ds + \int_{t\wedge\tau_{j-1}}^{t\wedge\tau_j} \mathrm{e}^{-\delta s}\, \sigma v'(X_s, I_s) X_s\, dW_s \Big\} \\
&\quad + \sum_{j=1}^{n} \mathbb{1}_{\{\tau_j \le t\}}\, \mathrm{e}^{-\delta\tau_j}\, k_{\iota_{j-1},\iota_j}.
\end{aligned}$$

The inequality becomes an equality for the VI-control associated with v. Taking expectations, we have

$$\begin{aligned} & v(X_0, I_0) - E\big[\,\mathrm{e}^{-\delta(t\wedge\tau_n)}\, v(X_{t\wedge\tau_n}, I_{t\wedge\tau_n})\big] \\ & \geq E\Bigg[\sum_{j=1}^{n}\bigg\{\int_{t\wedge\tau_{j-1}}^{t\wedge\tau_j} \mathrm{e}^{-\delta s}\, f(X_s, u_s, I_s)\, ds - \int_{t\wedge\tau_{j-1}}^{t\wedge\tau_j} \mathrm{e}^{-\delta s}\, \sigma v'(X_s, I_s) X_s\, dW_s\bigg\} \\ & \qquad - \sum_{j=1}^{n} \mathbb{1}_{\{\tau_j \leq t\}}\, \mathrm{e}^{-\delta\tau_j}\, k_{\iota_{j-1},\iota_j}\Bigg], \end{aligned}$$

with equality for the VI-control associated with v. Taking the limit as $n \to \infty$ on both sides we get

$$\begin{aligned} & v(X_0, I_0) - \lim_{n\to\infty}\Big\{E\big[\,\mathrm{e}^{-\delta(t\wedge\tau_n)}\, v(X_{t\wedge\tau_n}, I_{t\wedge\tau_n})\big]\Big\} \\ & \geq \lim_{n\to\infty}\Bigg\{E\bigg[\int_0^{t\wedge\tau_n} \mathrm{e}^{-\delta s}\, f(X_s, u_s, I_s)\, ds - \int_0^{t\wedge\tau_n} \mathrm{e}^{-\delta s}\, \sigma v'(X_s, I_s) X_s\, dW_s \\ & \qquad\qquad - \sum_{j=1}^{n} \mathbb{1}_{\{\tau_j \leq t\}}\, \mathrm{e}^{-\delta\tau_j}\, k_{\iota_{j-1},\iota_j}\bigg]\Bigg\}. \end{aligned}$$

When v is convex or concave and of quadratic growth, its derivative is of linear growth (see Theorem 6.7(ii) of Evans and Gariepy [2015]), then it follows from (2.4) that

$$\begin{aligned} & E\bigg[\int_0^{\infty} \big\{\,\mathrm{e}^{-\delta s}\, \sigma v'(X_s, I_s) X_s\big\}^2\, ds\bigg] < \infty \\ & \Rightarrow \lim_{n\to\infty} E\bigg[\int_0^{t\wedge\tau_n} \mathrm{e}^{-\delta s}\, \sigma v'(X_s, I_s) X_s\, dW_s\bigg] = 0. \end{aligned}$$

Thus,

$$\begin{aligned} & v(X_0, I_0) - \lim_{n\to\infty}\Big\{E\big[\,\mathrm{e}^{-\delta(t\wedge\tau_n)}\, v(X_{t\wedge\tau_n}, I_{t\wedge\tau_n})\big]\Big\} \\ & \geq E\bigg[\int_0^{t} \mathrm{e}^{-\delta s}\, f(X_s, u_s, I_s)\, ds - \sum_{j=0}^{\infty} \mathbb{1}_{\{\tau_j \leq t\}}\, \mathrm{e}^{-\delta\tau_j}\, k_{\iota_{j-1},\iota_j}\bigg], \end{aligned}$$

with equality for the VI-control associated with v. Next, will prove that the limit of the left-hand side as $t \to \infty$ is $v(X_0, I_0)$. From the quadratic growth of v, (2.2), and (2.4), we have

$$-E\big[\,e^{-\delta t}\,K(1+X_t^2)\big] \le \lim_{n\to\infty} E\big[\,e^{-\delta(t\wedge\tau_n)}\,v(X_{t\wedge\tau_n}, I_{t\wedge\tau_n})\big] \le E\big[\,e^{-\delta t}\,K(1+X_t^2)\big].$$

It follows from (2.4) and Fubini's theorem that

$$\begin{aligned}&\lim_{t\to\infty} E\big[\,e^{-\delta t}\,K(1+X_t^2)\big]=0\\ &\Rightarrow \lim_{t\to\infty}\big\{v(X_0,I_0)-\lim_{n\to\infty}E\big[\,e^{-\delta(t\wedge\tau_n)}\,v(X_{t\wedge\tau_n},I_{t\wedge\tau_n})\big]\big\}=v(X_0,I_0).\end{aligned}$$

Furthermore,

$$\begin{aligned}&\lim_{t\to\infty}E\Bigg[\sum_{j=0}^{\infty}\bigg\{\int_{t\wedge\tau_{j-1}}^{t\wedge\tau_j} e^{-\delta s}\,f(X_s,u_s,I_s)\,ds-\mathbb{1}_{\{\tau_j\le t\}}\,e^{-\delta\tau_j}\,k_{\iota_{j-1},\iota_j}\bigg\}\Bigg]\\ &=\lim_{t\to\infty}E\bigg[\int_0^t e^{-\delta s}\,f(X_s,u_s,I_s)\,ds\bigg]-\lim_{t\to\infty}E\Bigg[\sum_{j=1}^{\infty}\mathbb{1}_{\{\tau_j\le t\}}\,e^{-\delta\tau_j}\,k_{\iota_{j-1},\iota_j}\Bigg].\end{aligned}$$

For the first expected value, we have

$$\begin{aligned}\bigg|\int_0^t e^{-\delta s}\,f(X_s,u_s,I_s)\,ds\bigg| &\le \int_0^t e^{-\delta s}\,|f(X_s,u_s,I_s)|\,ds\\ &\le \int_0^\infty e^{-\delta s}\,|f(X_s,u_s,I_s)|\,ds,\end{aligned}$$

which, by Remark 2.1, is integrable. Applying the dominated convergence theorem, we get

$$\lim_{t\to\infty}E\bigg[\int_0^t e^{-\delta s}\,f(X_s,u_s,I_s)\,ds\bigg]=E\bigg[\int_0^\infty e^{-\delta s}\,f(X_s,u_s,I_s)\,ds\bigg].$$

Applying the monotone convergence theorem for the second expected value, we get

$$\lim_{t\to\infty}E\Bigg[\sum_{j=1}^{\infty}\mathbb{1}_{\{\tau_j\le t\}}\,e^{-\delta\tau_j}\,k_{\iota_{j-1},\iota_j}\Bigg]=E\Bigg[\sum_{j=1}^{\infty}e^{-\delta\tau_j}\,k_{\iota_{j-1},\iota_j}\Bigg].$$

Hence, for every $(u, T) \in \mathcal{A}$:

$$v(X_0, I_0) \geq E\left[\int_0^\infty \mathrm{e}^{-\delta s} f(X_s, I_s)\, ds - \sum_{j=1}^{\infty} \mathrm{e}^{-\delta \tau_j} k_{\iota_{j-1},\iota_j}\right] = J(X_0, I_0; u, T),$$

with equality for the VI-control associated with v. Therefore, for every $x \in (0, \infty)$, $i \in \mathbb{I}$:

$$v(x, i) \geq \sup \{J(x, i; u, T); (u, T) \in \mathcal{A}\},$$

and the second part of the theorem follows. □

Chapter 4

Solution

4.1 Construction of the Solution

In this section, we construct a candidate solution for the VI (3.1)–(3.2).

We start with

$$\max_u \left[\frac{1}{2}\sigma^2 x^2 v_{xx}(x,1) + (\rho_1 u - \alpha x) v_x(x,1) - \delta v(x,1) + \beta_1 u - u^2 - \lambda_1 x^2 \right]. \tag{4.1}$$

We see that u is given by

$$u(x,1) = \frac{\beta_1}{2} + \frac{\rho_1}{2} v_x(x,1). \tag{4.2}$$

Replacing (4.2) in (4.1), we obtain the non-linear ordinary differential equation

$$\frac{1}{2}\sigma^2 x^2 v_{xx}(x,1) + \frac{\rho_1^2}{4} v_x^2(x,1) + \left(\frac{\rho_1 \beta_1}{2} - \alpha x \right) v_x(x,1) - \delta v(x,1) + \frac{1}{4}\beta_1^2 - \lambda_1 x^2 = 0.$$

A solution of this non-linear ordinary differential equation is

$$v(x,1) = a_1 + b_1 x + c_1 x^2, \tag{4.3}$$

where

$$
\begin{aligned}
c_1^{\pm} &= \frac{1}{2\rho_1^2}\left\{2\alpha+\delta-\sigma^2 \pm \sqrt{(2\alpha+\delta-\sigma^2)^2+4\lambda_1\rho_1^2}\right\}, \\
b_1^{\pm} &= \frac{-\beta_1\rho_1 c_1^{\pm}}{c_1^{\pm}\rho_1^2-\alpha-\delta}, \\
a_1^{\pm} &= \frac{1}{4\delta}\left(\beta_1+\rho_1 b_1^{\pm}\right)^2.
\end{aligned}
$$

We conjecture that $v(\cdot,1)$ is concave, hence we select

$$
c_1 = \frac{1}{2\rho_1^2}\left\{2\alpha+\delta-\sigma^2 - \sqrt{(2\alpha+\delta-\sigma^2)^2+4\lambda_1\rho_1^2}\right\},
$$

and have

$$
\begin{aligned}
b_1 &= \frac{-\beta_1\rho_1 c_1}{c_1\rho_1^2-\alpha-\delta}, \\
a_1 &= \frac{1}{4\delta}\left(\beta_1+\rho_1 b_1\right)^2.
\end{aligned}
$$

Then, $c_1 < 0$, $b_1 < 0$, and $a_1 > 0$. Equation (4.3) implies

$$
u(x,1) = \frac{\beta_1}{2} + \frac{\rho_1}{2}(b_1+2c_1x) = \left(\frac{\beta_1}{2}+\frac{b_1\rho_1}{2}\right) + \rho_1 c_1 x.
$$

Similarly, we consider

$$
\max_u \left[\frac{1}{2}\sigma^2 x^2 v_{xx}(x,2) + (\rho_2 u - \alpha x)v_x(x,2) - \delta v(x,2) + \beta_2 u - u^2 - \lambda_2 x^2\right]. \tag{4.4}
$$

We see that u is given by

$$
u(x,2) = \frac{\beta_2}{2} + \frac{\rho_2}{2} v_x(x,2). \tag{4.5}
$$

Replacing (4.5) in (4.4), we obtain the non-linear ordinary differential equation

$$
\frac{1}{2}\sigma^2 x^2 v_{xx}(x,2) + \frac{\rho_2^2}{4} v_x^2(x,2) + \left(\frac{\rho_2\beta_2}{2} - \alpha x\right) v_x(x,2) - \delta v(x,2) + \frac{1}{4}\beta_2^2 - \lambda_2 x^2 = 0.
$$

A solution of this non-linear ordinary differential equation is

$$v(x,2) = a_2 + b_2 x + c_2 x^2, \tag{4.6}$$

where

$$\begin{aligned} c_2^{\pm} &= \frac{1}{2\rho_2^2}\left\{2\alpha+\delta-\sigma^2 \pm \sqrt{(2\alpha+\delta-\sigma^2)^2+4\lambda_2\rho_2^2}\right\}, \\ b_2^{\pm} &= \frac{-\beta_2\rho_2 c_2^{\pm}}{c_2^{\pm}\rho_2^2-\alpha-\delta}, \\ a_2^{\pm} &= \frac{1}{4\delta}\left(\beta_2+\rho_2 b_2^{\pm}\right)^2. \end{aligned}$$

We conjecture that $v(\cdot,2)$ is concave, hence we select

$$c_2 = \frac{1}{2\rho_2^2}\left\{2\alpha+\delta-\sigma^2 - \sqrt{(2\alpha+\delta-\sigma^2)^2+4\lambda_2\rho_2^2}\right\},$$

and have

$$\begin{aligned} b_2 &= \frac{-\beta_2\rho_2 c_2}{c_2\rho_2^2-\alpha-\delta}, \\ a_2 &= \frac{1}{4\delta}\left(\beta_2+\rho_2 b_2\right)^2. \end{aligned}$$

Then, $c_2 < 0$, $b_2 < 0$, and $a_2 > 0$. Equation (4.6) implies

$$u(x,2) = \frac{\beta_2}{2} + \frac{\rho_2}{2}(b_2+2c_2x) = \left(\frac{\beta_2}{2}+\frac{\rho_2 b_2}{2}\right) + \rho_2 c_2 x.$$

We conjecture that there exists $M_1 < M_2$ with the following property. If the system starts in regime 1 (green technology), then (M_1,∞) is the continuation region and $(0,M_1]$ is the intervention region. If the system starts in regime 2 (brown technology), then $(0,M_2)$ is the continuation region and $[M_2,\infty)$ is the intervention region.

Therefore, our candidate for value function V is the function $v : (0,\infty)\times\mathbb{I}\mapsto\mathbb{R}$ defined by

$$v(x,1) := \begin{cases} a_1 + b_1 x + c_1 x^2 & \text{if } x \in (M_1, \infty) \\ a_2 + b_2 x + c_2 x^2 - k_{12} & \text{if } x \in (0, M_1] \end{cases} \tag{4.7}$$

and

$$v(x,2) := \begin{cases} a_2 + b_2 x + c_2 x^2 & \text{if } x \in (0, M_2) \\ a_1 + b_1 x + c_1 x^2 - k_{21} & \text{if } x \in [M_2, \infty). \end{cases} \tag{4.8}$$

Our candidate for optimal control satisfies the following property. If the system starts in regime 1, then the production should be given by

$$u(x,1) = \left(\frac{\beta_1}{2} + \frac{b_1\rho_1}{2}\right) + c_1\rho_1 x \tag{4.9}$$

when the pollution is above M_1. As soon as the pollution is below M_1, there should be a change from green technology to brown technology.

If the system starts in regime 2, then the production should be given by

$$u(x,2) = \left(\frac{\beta_2}{2} + \frac{b_2\rho_2}{2}\right) + c_2\rho_2 x \tag{4.10}$$

when the pollution is below M_2. As soon as the pollution is above M_2, there should be a change from brown technology to green technology.

To specify the values of M_1 and M_2, we conjecture that $v(\cdot,1)$ is continuous at M_1 and $v(\cdot,2)$ is continuous at M_2. From equation (4.7), we have

$$a_1 + b_1 M_1 + c_1 M_1^2 = a_2 + b_2 M_1 + c_2 M_1^2 - k_{12}.$$

Thus,

$$M_1^{\pm} = \frac{1}{2(c_1 - c_2)}\left\{b_2 - b_1 \pm \sqrt{(b_2 - b_1)^2 - 4(c_1 - c_2)(a_1 - a_2 + k_{12})}\right\}. \tag{4.11}$$

We select M_1^{+}. From equation (4.8), we have

$$a_2 + b_2 M_2 + c_2 M_2^2 = a_1 + b_1 M_2 + c_1 M_2^2 - k_{21}.$$

Thus,

$$M_2^{\mp} = \frac{1}{2(c_1 - c_2)} \left\{ b_2 - b_1 \mp \sqrt{(b_2 - b_1)^2 - 4(c_1 - c_2)(a_1 - a_2 - k_{21})} \right\}. \tag{4.12}$$

We select M_2^+.

We have the following lemma:

Lemma 4.1. *The following inequalities hold:*

1. $c_1 > c_2$,
2. $b_1 > b_2$,
3. $M_i^- < M_i^+,\ i \in \mathbb{I}$, *and*
4. $M_2^+ > M_1^+$.

Proof. Proof of 1.

$$\begin{aligned} c_1 - c_2 &= \frac{(\rho_2^2 - \rho_1^2)(2\alpha + \delta - \sigma^2) - \sqrt{\rho_2^4(2\alpha + \delta - \sigma^2)^2 + 4\lambda_1 \rho_1^2 \rho_2^4}}{2\rho_1^2 \rho_2^2} \\ &\quad + \frac{\sqrt{\rho_1^4(2\alpha + \delta - \sigma^2)^2 + 4\lambda_2 \rho_2^2 \rho_1^4}}{2\rho_1^2 \rho_2^2} \\ &\geq \frac{(\rho_2^2 - \rho_1^2)(2\alpha + \delta - \sigma^2) - \sqrt{\rho_2^4(2\alpha + \delta - \sigma^2)^2 + 4\lambda_1 \rho_1^2 \rho_2^4}}{2\rho_1^2 \rho_2^2} \\ &\quad + \frac{\sqrt{\rho_1^4(2\alpha + \delta - \sigma^2)^2 + 4\lambda_1 \rho_2^4 \rho_1^2}}{2\rho_1^2 \rho_2^2}. \end{aligned}$$

The last inequality follows from the assumption that $\lambda_2/\lambda_1 \geq \rho_2^2/\rho_1^2$. For a moment, assume that the last expression were negative. This means

$$\begin{aligned} &(\rho_2^2 - \rho_1^2)(2\alpha + \delta - \sigma^2) \\ &< \sqrt{\rho_2^4(2\alpha + \delta - \sigma^2)^2 + 4\lambda_1 \rho_1^2 \rho_2^4} - \sqrt{\rho_1^4(2\alpha + \delta - \sigma^2)^2 + 4\lambda_1 \rho_2^4 \rho_1^2}. \end{aligned}$$

Taking the square on both sides we get

$$\begin{aligned}&(\rho_2^2-\rho_1^2)^2(2\alpha+\delta-\sigma^2)^2\\ &<\left(\sqrt{\rho_2^4(2\alpha+\delta-\sigma^2)^2+4\lambda_1\rho_1^2\rho_2^4}-\sqrt{\rho_1^4(2\alpha+\delta-\sigma^2)^2+4\lambda_1\rho_2^4\rho_1^2}\right)^2.\end{aligned}$$

Then,

$$\begin{aligned}&(\rho_2^4+\rho_1^4-2\rho_2^2\rho_1^2)(2\alpha+\delta-\sigma^2)^2\\ &<(\rho_2^4+\rho_1^4)(2\alpha+\delta-\sigma^2)^2+8\lambda_1\rho_1^2\rho_2^4\\ &-2\left(\rho_2^4(2\alpha+\delta-\sigma^2)^2+4\lambda_1\rho_1^2\rho_2^4\right)^{\frac{1}{2}}\left(\rho_1^4(2\alpha+\delta-\sigma^2)^2+4\lambda_1\rho_1^2\rho_2^4\right)^{\frac{1}{2}}\end{aligned}$$

and thus

$$\begin{aligned}&\rho_2^2\rho_1^2(2\alpha+\delta-\sigma^2)^2+4\lambda_1\rho_1^2\rho_2^4\\ &>\left(\rho_2^4(2\alpha+\delta-\sigma^2)^2+4\lambda_1\rho_1^2\rho_2^4\right)^{\frac{1}{2}}\left(\rho_1^4(2\alpha+\delta-\sigma^2)^2+4\lambda_1\rho_1^2\rho_2^4\right)^{\frac{1}{2}}.\end{aligned}$$

Taking the square again,

$$\begin{aligned}&\rho_2^4\rho_1^4(2\alpha+\delta-\sigma^2)^4+8\rho_1^4\rho_2^6\lambda_1(2\alpha+\delta-\sigma^2)^2+16\lambda_1^2\rho_1^4\rho_2^8\\ &>\rho_2^4\rho_1^4(2\alpha+\delta-\sigma^2)^4+4\lambda_1\rho_1^2\rho_2^4(2\alpha+\delta-\sigma^2)^2(\rho_2^4+\rho_1^4)+16\lambda_1^2\rho_1^4\rho_2^8.\end{aligned}$$

This implies

$$\begin{aligned}8\lambda_1\rho_1^4\rho_2^6(2\alpha+\delta-\sigma^2)^2 &> 4\lambda_1\rho_1^2\rho_2^4(2\alpha+\delta-\sigma^2)^2(\rho_2^4+\rho_1^4)\\ 2\rho_1^2\rho_2^2 &> \rho_2^4+\rho_1^4\\ 0 &> (\rho_2^2-\rho_1^2)^2,\end{aligned}$$

which is not possible. Therefore, $c_1>c_2$.

Proof of 2.

$$\begin{aligned} b_1 - b_2 &= \frac{\beta_2\rho_2 c_2}{c_2\rho_2^2 - \alpha - \delta} - \frac{\beta_1\rho_1 c_1}{c_1\rho_1^2 - \alpha - \delta} \\ &= \frac{\beta_2\rho_2 c_2(c_1\rho_1^2 - \alpha - \delta) - \beta_1\rho_1 c_1(c_2\rho_2^2 - \alpha - \delta)}{(c_2\rho_2^2 - \alpha - \delta)(c_1\rho_1^2 - \alpha - \delta)} \\ &= \frac{\rho_2\rho_1 c_1 c_2(\beta_2\rho_1 - \beta_1\rho_2) - (\beta_2\rho_2 c_2 - \beta_1\rho_1 c_1)(\alpha + \delta)}{(c_2\rho_2^2 - \alpha - \delta)(c_1\rho_1^2 - \alpha - \delta)} \end{aligned}$$

By (2.6), $\beta_2\rho_1 \geq \beta_1\rho_2$, and by the first part of this lemma, $\beta_2\rho_2 c_2 - \beta_1\rho_1 c_1 < 0$. Since $c_i < 0$, the denominator is positive, and $\rho_2\rho_1 c_1 c_2 > 0$. Therefore, $b_1 > b_2$.

Proof of 3.

$$\begin{aligned} M_1^+ - M_1^- &= \frac{2\sqrt{(b_2 - b_1)^2 - 4(c_1 - c_2)(a_1 - a_2 + k_{12})}}{2(c_1 - c_2)}, \\ M_2^+ - M_2^- &= \frac{2\sqrt{(b_2 - b_1)^2 - 4(c_1 - c_2)(a_1 - a_2 - k_{21})}}{2(c_1 - c_2)}. \end{aligned}$$

Since $c_1 > c_2$, $M_i^+ > M_i^- \ \forall i \in \mathbb{I}$.

Proof of 4.

$$\begin{aligned} M_2^+ - M_1^+ = &\frac{\sqrt{(b_2 - b_1)^2 - 4(c_1 - c_2)(a_1 - a_2 - k_{21})}}{2(c_1 - c_2)} \\ &- \frac{\sqrt{(b_2 - b_1)^2 - 4(c_1 - c_2)(a_1 - a_2 + k_{12})}}{2(c_1 - c_2)}. \end{aligned}$$

Since $c_1 > c_2$, we only need to prove that the difference of the square roots is positive. Suppose it were negative. Then

$$\begin{aligned} &\sqrt{(b_2 - b_1)^2 - 4(c_1 - c_2)(a_1 - a_2 + k_{12})} \\ &> \sqrt{(b_2 - b_1)^2 - 4(c_1 - c_2)(a_1 - a_2 - k_{21})} \end{aligned}$$

which implies

$$\begin{aligned} (b_2 - b_1)^2 - 4(c_1 - c_2)(a_1 - a_2 + k_{12}) &> (b_2 - b_1)^2 - 4(c_1 - c_2)(a_1 - a_2 - k_{21}) \\ \Rightarrow k_{12} &< -k_{21}, \end{aligned}$$

which is not possible. Therefore, $M_2^+ > M_1^+$. $\square$

Equations (4.9) and (4.10) can be rewritten as

$$u \;=\; \left(\frac{\beta_{I(t)}}{2} + \frac{b_{I(t)}\rho_{I(t)}}{2}\right) + c_{I(t)}\rho_{I(t)}X(t). \tag{4.13}$$

This implies that (2.1) can be written as

$$dX_t = \left\{\frac{\beta_{I(t)}}{2} + \frac{b_{I(t)}\rho_{I(t)}}{2} + (c_{I(t)}\rho_{I(t)} - \alpha)X_t\right\} dt + \sigma X_t\, dW_t, \quad X_0 = x > 0. \tag{4.14}$$

According to Section 5.6 of Karatzas and Shreve [1998], we observe that the solution of the stochastic differential equation (4.14), when technology i is used, is given by

$$X_t \;=\; Z_t\left\{x + \frac{(\beta_i + b_i\rho_i)}{2}\int_0^t \mathrm{e}^{-((c_i\rho_i-\alpha-\frac{1}{2}\sigma^2)s+\sigma W_s)}\; ds\right\}, \tag{4.15}$$

where

$$Z_t \;=\; \mathrm{e}^{(c_i\rho_i-\alpha-\frac{1}{2}\sigma^2)t+\sigma W_t}\,.$$

Note that

$$\beta_i + b_i\rho_i = \beta_i - \frac{\beta_i\rho_i^2 c_i}{c_i\rho_i^2 - \alpha - \delta} = \frac{-\beta_i(\alpha+\delta)}{c_i\rho_i^2 - \alpha - \delta} > 0.$$

This implies that (4.15) is always positive. Additionally, since (4.13) is decreasing in the pollution level ($c_i < 0$), there exists a positive value of X given by

$$X(t) = -\frac{\beta_{I(t)} + b_{I(t)}\rho_{I(t)}}{2c_{I(t)}\rho_{I(t)}}, \tag{4.16}$$

that makes the production rate equal to zero. If the pollution surpasses the value specified by (4.16), the production rate turns negative, indicating that the firm incurs costs to mitigate and reduce the pollution level.

4.2 Verification of the Solution

The way the continuation and intervention region are defined in our candidate solution reveals that, depending on the values of M_1^+ and M_2^+, these regions might be empty. The next lemma establishes conditions on the parameters that determine the sign of M_1^+ and M_2^+.

Lemma 4.2.

1. *Assume that* $k_{12} < a_2 - a_1$. *Then,* M_1^+ *and* M_2^+ *are both positive.*

2. *Assume that* $k_{12} \geq a_2 - a_1$ *and* $k_{21} > a_1 - a_2$. *Then,* M_2^+ *is positive, and* M_1^+ *is less than or equal to zero, or it does not exist.*

Proof. From Lemma 4.1, $c_1 > c_2$ and $b_1 > b_2$, hence the sign of M_1^+ and M_2^+ is going to be determined by the sign of $-4(c_1 - c_2)(a_1 - a_2 + k_{12})$ and $-4(c_1 - c_2)(a_1 - a_2 - k_{21})$, respectively.

Proof of 1.

$$\begin{aligned} & k_{12} < a_2 - a_1 \Rightarrow 0 > a_1 - a_2 + k_{12} \\ \Rightarrow \quad & 0 < -4(c_1 - c_2)(a_1 - a_2 + k_{12}). \end{aligned}$$

Therefore, M_1^+ is positive. From Lemma 4.1, $M_2^+ > M_1^+$, hence $M_2^+ > 0$.

Proof of 2.

$$\begin{aligned} & k_{12} \geq a_2 - a_1 \Rightarrow 0 \leq a_1 - a_2 + k_{12} \\ \Rightarrow \quad & 0 \geq -4(c_1 - c_2)(a_1 - a_2 + k_{12}). \end{aligned}$$

Therefore, M_1^+ is less than or equal than zero, or it does not exist. Moreover,

$$\begin{aligned} & k_{21} > a_1 - a_2 \Rightarrow 0 > a_1 - a_2 - k_{21} \\ \Rightarrow \quad & 0 < -4(c_1 - c_2)(a_1 - a_2 - k_{21}). \end{aligned}$$

Therefore, M_2^+ is positive. □

To prove the optimality of the proposed candidate solution, we first need to establish some assumptions about the parameters of our model. We begin by defining the following constants:

$$R_1^{\pm} = \frac{-\left(c_2\left(\rho_1\beta_1-\rho_2\beta_2\left(\frac{\rho_1^2c_2-\alpha-\delta}{\rho_2^2c_2-\alpha-\delta}\right)\right)\right)\pm\left(\left(c_2\left(\rho_1\beta_1-\rho_2\beta_2\left(\frac{\rho_1^2c_2-\alpha-\delta}{\rho_2^2c_2-\alpha-\delta}\right)\right)\right)^2\right.}{2\left(c_2(\frac{\rho_1^2}{\rho_2^2}-1)(2\alpha+\delta-\sigma^2))+(\lambda_2\frac{\rho_1^2}{\rho_2^2}-\lambda_1)\right)}$$

$$-\frac{\left.4\left(c_2(\frac{\rho_1^2}{\rho_2^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)\left(\frac{1}{4}(\beta_1+\rho_1b_2)^2-\delta a_2+\delta k_{12}\right)\right)^{\frac{1}{2}}}{2\left(c_2(\frac{\rho_1^2}{\rho_2^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_2\frac{\rho_1^2}{\rho_2^2}-\lambda_1)\right)}$$

$$R_2^{\pm} := \frac{-\left(c_1\left(\rho_2\beta_2-\rho_1\beta_1\left(\frac{\rho_2^2c_1-\alpha-\delta}{\rho_1^2c_1-\alpha-\delta}\right)\right)\right)\pm\left(\left(c_1\left(\rho_2\beta_2-\rho_1\beta_1\left(\frac{\rho_2^2c_1-\alpha-\delta}{\rho_1^2c_1-\alpha-\delta}\right)\right)\right)^2\right.}{2\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)}$$

$$-\frac{\left.4\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)\left(\frac{1}{4}(\beta_2+\rho_2b_1)^2-\delta a_1+\delta k_{21}\right)\right)^{\frac{1}{2}}}{2\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)}$$

Assumption 4.1. *We consider the following:*

1. $M_2^+ > R_2^-$
2. $M_1^+ < R_1^+$
3. $k_{12} \le a_2 - \frac{1}{4\delta}(\beta_1+\rho_1b_2)^2$
4. $k_{21} \le a_1 - \frac{1}{4\delta}(\beta_2+\rho_2b_1)^2$

Theorem 4.1. *Suppose Assumption 4.1, parts 1–3 hold. Also, assume* $k_{12} < a_2 - a_1$. *Let* $v : (0,\infty)\times\mathbb{I}\mapsto\mathbb{R}$ *be defined by*

$$v(x,1) = \begin{cases} a_1+b_1x+c_1x^2 & \text{if } x\in(M_1^+,\infty) \\ a_2+b_2x+c_2x^2-k_{12} & \text{if } x\in(0,M_1^+] \end{cases} \tag{4.17}$$

and

$$v(x,2) = \begin{cases} a_2+b_2x+c_2x^2 & \text{if } x\in(0,M_2^+) \\ a_1+b_1x+c_1x^2-k_{21} & \text{if } x\in[M_2^+,\infty). \end{cases} \tag{4.18}$$

Then, v is the value function.

The optimal control satisfies the following property: If the system starts in regime 1, then the production should be given by (4.9) when the pollution is above M_1^+. As soon as the pollution is below M_1^+, there should be a change from green to brown technology. If the system starts in regime 2, then the production should be given by (4.10) when the pollution is below M_2^+. As soon as the pollution is above M_2^+, there should be a change from brown to green technology.

Proof. If the company is using technology i, and produces according to (4.13), then (2.1) can be written as

$$dX_t = \left\{ \frac{\beta_i + b_i\rho_i}{2} + (c_i\rho_i - \alpha)X_t \right\} dt + \sigma X_t \, dW_t.$$

Denoting $f(x) := \frac{\beta_i+b_i\rho_i}{2} + (c_i\rho_i - \alpha)X_t$ and $L(x) := \sigma X_t$, the moments of the process X_t are given by the following differential equation (Section 5.6, Särkkä and Solin [2019]):

$$\frac{dE[X_t^n]}{dt} = nE[X_t^{n-1}f(x)] + \frac{n(n-1)}{2}E[X_t^{n-2}L^2(x)].$$

Solving this differential equation for $n = 4$ we obtain:

$$\begin{aligned} E[X_t^4] = {} & C_1 \, \mathrm{e}^{(4(c_i\rho_i-\alpha)+6\sigma^2)t} + C_2 \, \mathrm{e}^{(c_i\rho_i-\alpha)t} + C_3 \, \mathrm{e}^{3(c_i\rho_i-\alpha+\sigma^2)t} \\ & + C_4 \, \mathrm{e}^{(2c_i\rho_i-2\alpha+\sigma^2)t} + C_5, \end{aligned}$$

for some constants C_i, $i = 1, \dots, 5$. We can see that the $E[X_t^4]$ is finite for each value of t, and

$$\int_0^\infty \mathrm{e}^{-\delta t} \, E[X_t^4] \, dt < \infty,$$

for $c_i < 0$ and $\alpha > 2\sigma^2$. Since $v(x, \cdot)$ is also finite for each value of x and the production rate is linear in x, the admissibility conditions in Definition 2.2 are satisfied.

The functions (4.17) and (4.18) are concave, have quadratic growth, and are of class C^2 in their continuation regions. To be a solution of the system (3.1)–(3.2), they should satisfy the following equations and inequalities:

$$\begin{aligned}
v(x,2) - v(x,1) - k_{12} &= 0 \quad \text{if} \quad x \in (0, M_1^+] && (4.19)\\
v(x,2) - v(x,1) - k_{12} &< 0 \quad \text{if} \quad x \in (M_1^+, \infty) && (4.20)\\
v(x,1) - v(x,2) - k_{21} &= 0 \quad \text{if} \quad x \in [M_2^+, \infty) && (4.21)\\
v(x,1) - v(x,2) - k_{21} &< 0 \quad \text{if} \quad x \in (0, M_2^+) && (4.22)
\end{aligned}$$

and

$$\begin{aligned}
\max_u \left[\mathcal{L}^u v(x,1) - \delta v(x,1) + f(x,1)\right] &= 0 \quad \text{if} \quad x \in (M_1^+, \infty) && (4.23)\\
\max_u \left[\mathcal{L}^u v(x,1) - \delta v(x,1) + f(x,1)\right] &< 0 \quad \text{if} \quad x \in (0, M_1^+] && (4.24)\\
\max_u \left[\mathcal{L}^u v(x,2) - \delta v(x,2) + f(x,2)\right] &= 0 \quad \text{if} \quad x \in (0, M_2^+) && (4.25)\\
\max_u \left[\mathcal{L}^u v(x,2) - \delta v(x,2) + f(x,2)\right] &< 0 \quad \text{if} \quad x \in [M_2^+, \infty) && (4.26)
\end{aligned}$$

Because of the way we selected the functions (4.17) and (4.18), the equalities (4.19), (4.21), (4.23) and (4.25) hold. From (4.20) and (4.22) we have the following:

$$\begin{aligned}
&v(x,2) - v(x,1) - k_{12}\\
&= \begin{cases} (a_2 - a_1) + (b_2 - b_1)x + (c_2 - c_1)x^2 - k_{12} & \text{if} \quad x \in (M_1^+, M_2^+), \\ -k_{12} - k_{21} & \text{if} \quad x \in [M_2^+, \infty), \end{cases}
\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}
&v(x,1) - v(x,2) - k_{21}\\
&= \begin{cases} -k_{12} - k_{21} & \text{if} \quad x \in (0, M_1^+], \\ (a_1 - a_2) + (b_1 - b_2)x + (c_1 - c_2)x^2 - k_{21} & \text{if} \quad x \in (M_1^+, M_2^+). \end{cases}
\end{aligned} \tag{4.28}$$

Equation (4.27) is negative in the interval $[M_2^+, \infty)$. By Lemma 4.1, Part 1, $c_1 > c_2$. Then

$$(a_2 - a_1 - k_{12}) + (b_2 - b_1)x + (c_2 - c_1)x^2$$

is concave and has zeros at M_1^- and M_1^+. By Lemma 4.1, part 3, we know that $M_1^- < M_1^+$. Hence, this function is negative in the interval (M_1^+, M_2^+) and (4.20) is satisfied.

Equation (4.28) is negative in the interval $(0, M_1^+]$. By Lemma 4.1 part 1, $c_1 > c_2$. Then

$$(a_1 - a_2 - k_{21}) + (b_1 - b_2)x + (c_1 - c_2)x^2$$

is convex and has zeros at M_2^- and M_2^+. We can see that

$$\begin{aligned} M_1^+ - M_2^- &= \frac{\sqrt{(b_2 - b_1)^2 - 4(c_1 - c_2)(a_1 - a_2 + k_{12})}}{2(c_1 - c_2)} \\ &+ \frac{\sqrt{(b_2 - b_1)^2 - 4(c_1 - c_2)(a_1 - a_2 - k_{21})}}{2(c_1 - c_2)} > 0. \end{aligned}$$

Hence, the function is negative in the interval (M_1^+, M_2^+) and (4.22) is satisfied.

From (4.24), $v(\cdot, 1)$ should satisfy

$$\frac{1}{2}\sigma^2 x^2 v_{xx}(x,1) + \frac{\rho_1^2}{4} v_x^2(x,1) + \left(\frac{\rho_1 \beta_1}{2} - \alpha x\right) v_x(x,1) - \delta v(x,1) + \frac{1}{4}\beta_1^2 - \lambda_1 x^2 < 0$$

in $(0, M_1^+]$. Simplifying the left-hand side of the previous equation, we obtain

$$\begin{aligned} & x^2 \left(c_2 (\frac{\rho_1^2}{\rho_2^2} - 1)(2\alpha + \delta - \sigma^2) + (\lambda_2 \frac{\rho_1^2}{\rho_2^2} - \lambda_1) \right) \\ & + x \left(c_2 \left(\rho_1 \beta_1 - \rho_2 \beta_2 \left(\frac{\rho_1^2 c_2 - \alpha - \delta}{\rho_2^2 c_2 - \alpha - \delta} \right) \right) \right) + \frac{1}{4}(\beta_1 + \rho_1 b_2)^2 - \delta a_2 + \delta k_{12}. \end{aligned} \tag{4.29}$$

Since $c_2 < 0$, according to (2.6), the coefficient of x^2 is positive. Thus, (4.29) is negative in $(0, M_1^+]$ if it has two roots, the smaller root being less or equal than zero and the larger root being greater than M_1^+. The roots are given by

$$R_1^\pm = \frac{-\left(c_2\left(\rho_1\beta_1 - \rho_2\beta_2\left(\frac{\rho_1^2 c_2 - \alpha - \delta}{\rho_2^2 c_2 - \alpha - \delta}\right)\right)\right) \pm \left(\left(c_2\left(\rho_1\beta_1 - \rho_2\beta_2\left(\frac{\rho_1^2 c_2 - \alpha - \delta}{\rho_2^2 c_2 - \alpha - \delta}\right)\right)\right)^2\right.}{2\left(c_2(\frac{\rho_1^2}{\rho_2^2} - 1)(2\alpha + \delta - \sigma^2)) + (\lambda_2 \frac{\rho_1^2}{\rho_2^2} - \lambda_1)\right)}$$

$$-\frac{4\left(c_2(\frac{\rho_1^2}{\rho_2^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)\left(\frac{1}{4}(\beta_1+\rho_1 b_2)^2-\delta a_2+\delta k_{12}\right)\Bigg)^{\frac{1}{2}}}{2\left(c_2(\frac{\rho_1^2}{\rho_2^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_2\frac{\rho_1^2}{\rho_2^2}-\lambda_1)\right)}.$$

Equation (4.29) has two roots when the square root in the previous equation is positive, which occurs when

$$k_{12}<\frac{\left(c_2\left(\rho_1\beta_1-\rho_2\beta_2\left(\frac{\rho_1^2c_2-\alpha-\delta}{\rho_2^2c_2-\alpha-\delta}\right)\right)\right)^2}{4\delta\left(c_2(\frac{\rho_1^2}{\rho_2^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_2\frac{\rho_1^2}{\rho_2^2}-\lambda_1)\right)}-\frac{1}{4\delta}(\beta_1+\rho_1 b_2)^2+a_2.$$

Since (4.29) has to be less or equal to zero at zero, we have that $k_{12}\leq a_2-\frac{1}{4\delta}(\beta_1+\rho_1 b_2)^2$. This implies that $R_1^-<0$. Then, (4.29) is negative in $(0,M_1^+]$ if

$$k_{12}\leq a_2-\frac{1}{4\delta}(\beta_1+\rho_1 b_2)^2 \quad\text{and}\quad M_1^+<R_1^+.$$

In other words, Assumption 4.1, parts 2–3, imply that (4.24) holds.

From (4.26), $v(\cdot,2)$ should satisfy:

$$\frac{1}{2}\sigma^2x^2v_{xx}(x,2)+\frac{\rho_2^2}{4}v_x^2(x,2)+\left(\frac{\rho_2\beta_2}{2}-\alpha x\right)v_x(x,2)-\delta v(x,2)+\frac{1}{4}\beta_2^2-\lambda_2x^2<0$$

in $[M_2^+,\infty)$. Simplifying the left-hand side of the previous equation, we obtain

$$\begin{aligned}&x^2\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)\\&+x\left(c_1\left(\rho_2\beta_2-\rho_1\beta_1\left(\frac{\rho_2^2c_1-\alpha-\delta}{\rho_1^2c_1-\alpha-\delta}\right)\right)\right)+\frac{1}{4}(\beta_2+\rho_2 b_1)^2-\delta a_1+\delta k_{21}.\end{aligned}\tag{4.30}$$

Since $c_1<0$, according to (2.6), the coefficient of x^2 is negative. Equation (4.30) is negative in $[M_2^+,\infty)$ if its largest root is smaller than M_2^+, or if it has no roots. The roots are given by

$$R_2^{\pm}=\frac{-\left(c_1\left(\rho_2\beta_2-\rho_1\beta_1\left(\frac{\rho_2^2c_1-\alpha-\delta}{\rho_1^2c_1-\alpha-\delta}\right)\right)\right)\pm\left(\left(c_1\left(\rho_2\beta_2-\rho_1\beta_1\left(\frac{\rho_2^2c_1-\alpha-\delta}{\rho_1^2c_1-\alpha-\delta}\right)\right)\right)^2\right.}{2\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)}$$

$$-\frac{4\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)\left(\frac{1}{4}(\beta_2+\rho_2 b_1)^2-\delta a_1+\delta k_{21}\right)\Bigg)^{\frac{1}{2}}}{2\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)}.$$

Equation (4.30) has roots when the square root in the previous equation is greater or equal to zero, which occurs when

$$k_{21}\geq\frac{\left(c_1\left(\rho_2\beta_2-\rho_1\beta_1\left(\frac{\rho_2^2 c_1-\alpha-\delta}{\rho_1^2 c_1-\alpha-\delta}\right)\right)\right)^2}{4\delta\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)}-\frac{1}{4\delta}(\beta_2+\rho_2 b_1)^2+a_1. \quad (4.31)$$

Note that since $k_{12}<a_2-a_1$, $a_1-a_2<0$. Then, by Lemma 4.1, part 2, $a_1-\frac{1}{4\delta}(\beta_2+\rho_2 b_1)^2<0$. This implies that (4.31) always hold, with strict inequality. Then, (4.30) is negative in $[M_2^+,\infty)$ if

$$M_2^+>R_2^-,$$

In other words, Assumption 4.1, part 1, implies that (4.26) is satisfied. Hence, (4.19)–(4.26) are satisfied and v is a solution of the VI. □

Theorem 4.2. *Suppose Assumption 4.1, part 1 holds. Also, assume that $k_{12}>a_2-a_1$ and $k_{21}>a_1-a_2$. Let $v:(0,\infty)\times\mathbb{I}\mapsto\mathbb{R}$ be defined by*

$$v(x,1) \;=\; a_1+b_1x+c_1x^2 \quad (4.32)$$

and

$$v(x,2) \;=\; \begin{cases} a_2+b_2x+c_2x^2 & \text{if } x\in(0,M_2^+) \\ a_1+b_1x+c_1x^2-k_{21} & \text{if } x\in[M_2^+,\infty). \end{cases} \quad (4.33)$$

Then, v is the value function.

The optimal control satisfies the following property: If the system starts in regime 1, then the production should be given by (4.9). If the system starts in regime 2, then the production should be given by (4.10) when the pollution is below M_2^+. As soon as the pollution is above M_2^+, there should be a change from brown to green technology.

Proof. The proof of the admissibility of the controls associated with v is the same as the one in Theorem 4.1.

The functions (4.32) and (4.33) are concave, have quadratic growth, and are of class C^2 in their continuation regions. To be a solution of the system (3.1)–(3.2), they should satisfy the following equations and inequalities:

$$\begin{aligned} v(x,2) - v(x,1) - k_{12} &< 0 \quad \text{if} \quad x \in (0,\infty) && (4.34)\\ v(x,1) - v(x,2) - k_{21} &= 0 \quad \text{if} \quad x \in [M_2^+,\infty) && (4.35)\\ v(x,1) - v(x,2) - k_{21} &< 0 \quad \text{if} \quad x \in (0,M_2^+) && (4.36) \end{aligned}$$

and

$$\begin{aligned} \max_u \left[\mathcal{L}^u v(x,1) - \delta v(x,1) + f(x,1)\right] &= 0 \quad \text{if} \quad x \in (0,\infty) && (4.37)\\ \max_u \left[\mathcal{L}^u v(x,2) - \delta v(x,2) + f(x,2)\right] &= 0 \quad \text{if} \quad x \in (0,M_2^+) && (4.38)\\ \max_u \left[\mathcal{L}^u v(x,2) - \delta v(x,2) + f(x,2)\right] &< 0 \quad \text{if} \quad x \in [M_2^+,\infty]. && (4.39) \end{aligned}$$

Because of the way we selected the functions (4.32) and (4.33), the equalities (4.35), (4.37), and (4.38) hold. For (4.34) and (4.36) we have

$$\begin{aligned} &v(x,2) - v(x,1) - k_{12}\\ &= \begin{cases} (a_2 - a_1) + (b_2 - b_1)x + (c_2 - c_1)x^2 - k_{12} & \text{if} \quad x \in (0,M_2^+)\\ -k_{12} - k_{21} & \text{if} \quad x \in [M_2^+,\infty) \end{cases} \qquad (4.40) \end{aligned}$$

and

$$\begin{aligned} &v(x,1) - v(x,2) - k_{21}\\ &= (a_1 - a_2) + (b_1 - b_2)x + (c_1 - c_2)x^2 - k_{21} \quad \text{if} \quad x \in (0,M_2^+). \qquad (4.41) \end{aligned}$$

Equation (4.40) is negative in the interval $[M_2^+,\infty)$. By Lemma 4.1, part 1, $c_1 > c_2$. Then

$$(a_2 - a_1 - k_{12}) + (b_2 - b_1)x + (c_2 - c_1)x^2$$

is concave. If M_1^+ does not exists, the function does not have roots, and hence it is negative in $(0, M_2^+)$. If M_1^+ exists, the function has roots at M_1^- and M_1^+. By Lemma 4.1, part 3 and Lemma 4.2 part 2, we have $M_1^- < M_1^+ \leq 0$. Hence, the function is negative in the interval $(0, M_2^+)$, therefore (4.34) is satisfied.

By Lemma 4.1, part 1, $c_1 < c_2$. Thus (4.41) is convex and has zeros at M_2^- and M_2^+. Since $b_1 > b_2$, $M_2^- < 0$. Hence, (4.41) is negative in the interval $(0, M_2^+)$, therefore (4.36) is satisfied.

From equation (4.39), $v(\cdot, 2)$ should satisfy:

$$\frac{1}{2}\sigma^2 x^2 v_{xx}(x,2) + \frac{\rho_2^2}{4} v_x^2(x,2) + \left(\frac{\rho_2\beta_2}{2} - \alpha x\right) v_x(x,2) - \delta v(x,2) + \frac{1}{4}\beta_2^2 - \lambda_2 x^2 < 0$$

in $[M_2^+, \infty)$. Simplifying the left-hand side of the previous equation we obtain

$$\begin{aligned} & x^2\left(c_1(\frac{\rho_2^2}{\rho_1^2} - 1)(2\alpha + \delta - \sigma^2) + (\lambda_1\frac{\rho_2^2}{\rho_1^2} - \lambda_2)\right) \\ & + x\left(c_1\left(\rho_2\beta_2 - \rho_1\beta_1\left(\frac{\rho_2^2 c_1 - \alpha - \delta}{\rho_1^2 c_1 - \alpha - \delta}\right)\right)\right) + \frac{1}{4}(\beta_2 + \rho_2 b_1)^2 - \delta a_1 + \delta k_{21}. \end{aligned} \tag{4.42}$$

Since $c_1 < 0$, according to (2.6), the coefficient of x^2 is negative. Equation (4.42) is negative in $[M_2^+, \infty)$ if its largest root is smaller than M_2^+, or if it has no roots. The roots are given by

$$\begin{aligned} R_2^\pm = & \frac{-\left(c_1\left(\rho_2\beta_2 - \rho_1\beta_1\left(\frac{\rho_2^2 c_1 - \alpha - \delta}{\rho_1^2 c_1 - \alpha - \delta}\right)\right)\right) \pm \left(\left(c_1\left(\rho_2\beta_2 - \rho_1\beta_1\left(\frac{\rho_2^2 c_1 - \alpha - \delta}{\rho_1^2 c_1 - \alpha - \delta}\right)\right)\right)^2\right.}{2\left(c_1(\frac{\rho_2^2}{\rho_1^2} - 1)(2\alpha + \delta - \sigma^2) + (\lambda_1\frac{\rho_2^2}{\rho_1^2} - \lambda_2)\right)} \\ & \frac{\left.- 4\left(c_1(\frac{\rho_2^2}{\rho_1^2} - 1)(2\alpha + \delta - \sigma^2) + (\lambda_1\frac{\rho_2^2}{\rho_1^2} - \lambda_2)\right)\left(\frac{1}{4}(\beta_2 + \rho_2 b_1)^2 - \delta a_1 + \delta k_{21}\right)\right)^{\frac{1}{2}}}{2\left(c_1(\frac{\rho_2^2}{\rho_1^2} - 1)(2\alpha + \delta - \sigma^2) + (\lambda_1\frac{\rho_2^2}{\rho_1^2} - \lambda_2)\right)}. \end{aligned}$$

Equation (4.42) has roots when the square root in the previous equation is greater than or equal to zero, which occurs when

$$k_{21} \geq \frac{\left(c_1\left(\rho_2\beta_2 - \rho_1\beta_1\left(\frac{\rho_2^2 c_1 - \alpha - \delta}{\rho_1^2 c_1 - \alpha - \delta}\right)\right)\right)^2}{4\delta\left(c_1(\frac{\rho_2^2}{\rho_1^2} - 1)(2\alpha + \delta - \sigma^2) + (\lambda_1\frac{\rho_2^2}{\rho_1^2} - \lambda_2)\right)} - \frac{1}{4\delta}(\beta_2 + \rho_2 b_1)^2 + a_1. \tag{4.43}$$

Since $k_{21} > a_1 - a_2$, by Lemma 4.1, part 2, $a_1 - \frac{1}{4\delta}(\beta_2 + \rho_2 b_1)^2 < a_1 - a_2$. This implies that (4.43) holds, with a strict inequality. Then, (4.42) is negative in $[M_2^+, \infty)$ if

$$M_2^+ > R_2^-.$$

In other words, Assumption 4.1, part 1, implies that (4.39) is satisfied. Hence, (4.34)–(4.39) are satisfied and v is a solution of the VI. □

Theorem 4.3. *Suppose Assumption 4.1, part 4 holds. Let $v : (0, \infty) \times \mathbb{I} \mapsto \mathbb{R}$ be defined by*

$$v(x,1) = a_1 + b_1 x + c_1 x^2 \qquad (0,\infty) \tag{4.44}$$

$$v(x,2) = a_1 + b_1 x + c_1 x^2 - k_{21} \qquad (0,\infty) \tag{4.45}$$

Then, v is the value function, and the production function is given by (4.9).

Proof. The proof of the admissibility of the controls associated with v is the same as the one in Theorem 4.1.

The functions (4.44) and (4.45) are concave, have quadratic growth and are of class C^2 in their continuation regions. To be a solution of the system (3.1)–(3.2), they should satisfy the following equations and inequalities:

$$v(x,2) - v(x,1) - k_{12} < 0 \quad \text{if} \quad x \in (0,\infty) \tag{4.46}$$

$$v(x,1) - v(x,2) - k_{21} = 0 \quad \text{if} \quad x \in (0,\infty) \tag{4.47}$$

and

$$\max_u [\mathcal{L}^u v(x,1) - \delta v(x,1) + f(x,1)] = 0 \quad \text{if} \quad x \in (0,\infty) \tag{4.48}$$

$$\max_u [\mathcal{L}^u v(x,2) - \delta v(x,2) + f(x,2)] < 0 \quad \text{if} \quad x \in (0,\infty). \tag{4.49}$$

Because of the way we selected the functions (4.44) and (4.45), the equalities (4.47) and (4.48) hold. From (4.46) we have:

$$v(x,2) - v(x,1) - k_{12} = -k_{12} - k_{21}.$$

This is negative, thus (4.46) is satisfied. From equation (4.49), $v(\cdot, 2)$ should satisfy:

$$\frac{1}{2}\sigma^2x^2v_{xx}(x,2)+\frac{\rho_2^2}{4}v_x^2(x,2)+\left(\frac{\rho_2\beta_2}{2}-\alpha x\right)v_x(x,2)-\delta v(x,2)+\frac{1}{4}\beta_2^2-\lambda_2x^2<0$$

in $(0,\infty)$. Simplifying the left-hand side of the previous equation, we obtain

$$x^2\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right) \qquad (4.50)$$
$$+x\left(c_1\left(\rho_2\beta_2-\rho_1\beta_1\left(\frac{\rho_2^2c_1-\alpha-\delta}{\rho_1^2c_1-\alpha-\delta}\right)\right)\right)+\frac{1}{4}(\beta_2+\rho_2b_1)^2-\delta a_1+\delta k_{21}.$$

Since $c_1<0$, according to (2.6), the coefficient of x^2 is negative. Equation (4.50) is negative in $(0,\infty)$ if it does not have a root or its biggest root is smaller or equal to zero. The roots are given by

$$R_2^{\pm}=\frac{-\left(c_1\left(\rho_2\beta_2-\rho_1\beta_1\left(\frac{\rho_2^2c_1-\alpha-\delta}{\rho_1^2c_1-\alpha-\delta}\right)\right)\right)\pm\left(\left(c_1\left(\rho_2\beta_2-\rho_1\beta_1\left(\frac{\rho_2^2c_1-\alpha-\delta}{\rho_1^2c_1-\alpha-\delta}\right)\right)\right)^2\right.}{2\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)}$$
$$\left.-\frac{4\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)\left(\frac{1}{4}(\beta_2+\rho_2b_1)^2-\delta a_1+\delta k_{21}\right)\right)^{\frac{1}{2}}}{2\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)}.$$

Equation (4.50) has no roots when the square root in the previous equation is negative, which happens when

$$k_{21}<\frac{\left(c_1\left(\rho_2\beta_2-\rho_1\beta_1\left(\frac{\rho_2^2c_1-\alpha-\delta}{\rho_1^2c_1-\alpha-\delta}\right)\right)\right)^2}{4\delta\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)}-\frac{1}{4\delta}(\beta_2+\rho_2b_1)^2+a_1.$$

For (4.50) to be less or equal than zero at zero, we need to have $k_{21}\le\frac{1}{4\delta}(\beta_2+\rho_2b_1)^2+a_1$. Then, (4.50) is negative in $(0,\infty)$ when:

$$k_{21}<\frac{\left(c_1\left(\rho_2\beta_2-\rho_1\beta_1\left(\frac{\rho_2^2c_1-\alpha-\delta}{\rho_1^2c_1-\alpha-\delta}\right)\right)\right)^2}{4\delta\left(c_1(\frac{\rho_2^2}{\rho_1^2}-1)(2\alpha+\delta-\sigma^2)+(\lambda_1\frac{\rho_2^2}{\rho_1^2}-\lambda_2)\right)}-\frac{1}{4\delta}(\beta_2+\rho_2b_1)^2+a_1.$$

or

$$
\begin{aligned}
k_{21} &\geq \frac{\left(c_1\left(\rho_2\beta_2 - \rho_1\beta_1\left(\frac{\rho_2^2 c_1 - \alpha - \delta}{\rho_1^2 c_1 - \alpha - \delta}\right)\right)\right)^2}{4\delta\left(c_1(\frac{\rho_2^2}{\rho_1^2} - 1)(2\alpha + \delta - \sigma^2) + (\lambda_1\frac{\rho_2^2}{\rho_1^2} - \lambda_2)\right)} - \frac{1}{4\delta}(\beta_2 + \rho_2 b_1)^2 + a_1, \\
k_{21} &\leq a_1 - \frac{1}{4\delta}(\beta_2 + \rho_2 b_1)^2, \quad \text{and} \\
0 &> R_2^-,
\end{aligned}
$$

When $k_{21} \leq a_1 - \frac{1}{4\delta}(\beta_2 + \rho_2 b_1)^2$, since $c_1 < 0$, the sign of R_2^- is determined by the sign of

$$
\left(\rho_2\beta_2 - \rho_1\beta_1\left(\frac{\rho_2^2 c_1 - \alpha - \delta}{\rho_1^2 c_1 - \alpha - \delta}\right)\right). \tag{4.51}
$$

Next, we prove that (4.51) is positive.

$$
\begin{aligned}
\rho_2\beta_2 - \rho_1\beta_1\left(\frac{\rho_2^2 c_1 - \alpha - \delta}{\rho_1^2 c_1 - \alpha - \delta}\right) &= \frac{\rho_2\beta_2(\rho_1^2 c_1 - \alpha - \delta) - \rho_1\beta_1(\rho_2^2 c_1 - \alpha - \delta)}{\rho_1^2 c_1 - \alpha - \delta} \\
&= \frac{\rho_2\rho_1 c_1(\beta_2\rho_1 - \beta_1\rho_2) + (\alpha + \delta)(\rho_1\beta_1 - \rho_2\beta_2)}{\rho_1^2 c_1 - \alpha - \delta}.
\end{aligned}
$$

Since $c_1 < 0$, and by (2.6), $\beta_2/\beta_1 > \rho_2/\rho_1$ and $\rho_1\beta_1 < \rho_2\beta_2$, (4.51) is positive and $R_2^- < 0$. Then, (4.50) is negative in $(0, \infty)$ when:

$$
k_{21} \leq a_1 - \frac{1}{4\delta}(\beta_2 + \rho_2 b_1)^2
$$

In other words, Assumption 4.1, part 4, implies that (4.49) is satisfied. Hence, (4.46)–(4.49) are satisfied and v is a solution to the VI. □

Chapter 5

Comparative Statistics

In this chapter, we study how changes in the parameter values affect the solution. Specifically, we study the effect of these changes in the values of M_1^+ and M_2^+.

Given that $c_1 > c_2$, equations (4.11) and (4.12) show that M_1^+ decreases with k_{12} while M_2^+ increases with k_{21}. This implies a direct correlation between the parameter k_{ij} and the duration for which technology i is utilized: an increase in k_{ij} extends the usage period of technology i, whereas a decrease shortens it.

To further explore the impact of other parameters on our model, we introduce a base case example. This example will serve as a reference point for analyzing how variations in different parameters affect the solution:

Example 5.1. *Consider the parameter values*

$$\beta_1 = 1,\ \beta_2 = 5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ k_{12} = 4.85,\ k_{21} = 6,\ \alpha = 2,\ \sigma = 0.3,\ \delta = 1,\ \rho_1 = 0.7,\ \textit{and}\ \rho_2 = 0.9.$$

In this case the conditions for Theorem 4.1 are satisfied and we have

$$\begin{aligned} a_1 &= 0.23446675137941369, & a_2 &= 5.133116648102141, \\ b_1 &= -0.04512187606806963, & b_2 &= -0.5208039044650348, \\ c_1 &= -0.1996866337412412, & c_2 &= -0.3831178742357036. \end{aligned}$$

The control u is given by

$$\begin{aligned} u(x,1) &= 0.483 - 0.14x, \\ u(x,2) &= 2.27 - 0.34x, \end{aligned}$$

and the value function is given by

$$v(x,1) = \begin{cases} 0.24 - 0.05x - 0.2x^2 & \text{if } x \in (0.0986, \infty), \\ 0.28 - 0.52x - 0.38x^2 & \text{if } x \in [0, 0.0986], \end{cases}$$

$$v(x,2) = \begin{cases} 5.13 - 0.52x - 0.38x^2 & \text{if } x \in [0, 6.5198), \\ -5.76 - 0.05x - 0.2x^2 & \text{if } x \in [6.5198, \infty). \end{cases}$$

In Figure 5.1, we study the changes in α, σ, δ and λ_2/λ_1. We see that an increase in the natural rate of decay α increases the values of M_1^+ and M_2^+. In other words, increasing the value of α decreases the time we spend using the green technology. Indeed, since the environment recovers faster from pollution, we can use the brown technology for a longer period of time. An increase in the uncertainty of the evolution of pollution stock σ leads to a decrease in the values of M_1^+ and M_2^+, meaning that we spend more time using the green technology. A similar effect is observed with an increase in δ. As we pointed out in Chapter 2, an increase in the cost of using pollutive technologies relative to less pollutive ones incentivizes firms to adopt low-carbon technologies. We can see this in Figure 5.1, as λ_2/λ_1 increases, the values of M_1^+ and M_2^+ decrease, indicating that the firm transitions faster to green technology and spends more time using it.

In Figure 5.2, we study the changes in β_1, β_2, ρ_1 and ρ_2. We see that an increase in the marginal benefit of the green technology β_1, makes the values of M_1^+ and M_2^+ decrease, indicating that we spend more time using green technology. On the other hand, an increase in the marginal benefit of the brown technology β_2 causes the values of M_1^+ and M_2^+ to increase, meaning that we spend more time using the brown technology. An increase in the pollution intensity of green technology ρ_1 leads to an increase in M_1^+ and a decrease in M_2^+. This means that we are going to switch faster between

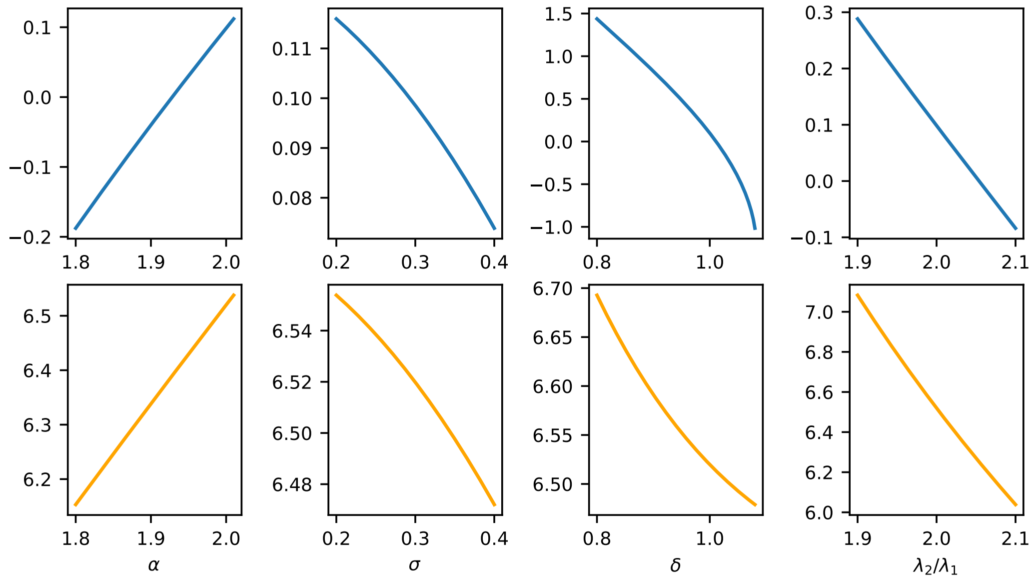

Figure 5.1: Effect of α, σ, δ and λ_2/λ_1. Effect in the value of M_1^+ (blue; upper panels) and M_2^+ (orange; lower panels) as α, σ, δ and λ_2/λ_1 change.

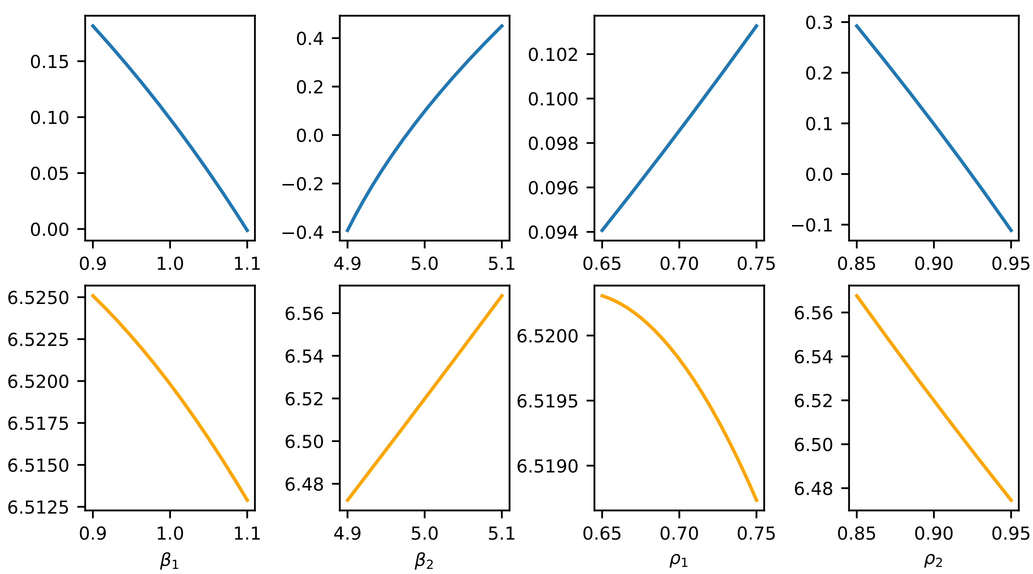

Figure 5.2: Effect of β_1, β_2, ρ_1, and ρ_2. Effect in the value of M_1^+ (blue; upper panels) and M_2^+ (orange; lower panels) as β_1, β_2, ρ_1, and ρ_2 change.

technologies. An increase in the pollution intensity of brown technology ρ_2 leads to a decrease in the values of M_1^+ and M_2^+, indicating that we spend more time using the green technology.

Chapter 6

Conclusion

The main mathematical contribution of this thesis is the solution of a mixed classical-switching stochastic control problem with different profit functions and diffusion regimes. Additionally, we believe that the pollution model we have presented allows for a more comprehensive economic analysis than previous models in the literature. Indeed, in an example, we analyzed the dependence of the solution on the various model parameters, and such a comparative statistics yielded meaningful results.

To enhance model tractability, we made certain specifications that could be adjusted in future research to increase realism and possibly gain further economic insights. For instance, representing the pollution decay rate as a non-linear function of pollution would account for reduced environmental recovery capacity beyond a critical pollution level. Additionally, we could incorporate public perception of the firm as a factor in the damage caused by pollution. Another interesting future research direction could be to extend this model to multiple firms, taking their interactions and cumulative pollution into account.

Bibliography

L.C. Evans and R.F. Gariepy. *Measure Theories and Fine Properties of Functions, Revised Edition*. CRC Press, 2015. URL `https://books.google.ca/books?id=e3R3CAAAQBAJ`.

J. Guo, D. Kubli, P. Saner, and P. Ronke. *The Economics of Climate Change: No Action Not an Option*. Swiss Re Institute, 2021. URL `https://books.google.ca/books?id=1lJxzgEACAAJ`.

J.T. Hardy. *Climate Change: Causes, Effects, and Solutions*. Wiley, 2003. URL `https://books.google.ca/books?id=TKOhTRyFVIkC`.

S.N. Islam and J. Winkel. Climate change and social inequality. *UN Department of Economic and Social Affairs (DESA) Working Papers*, (152):1–25, 2017.

I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer New York, NY, 1998.

E. Keeler, M. Spence, and R. Zeckhauser. The optimal control of pollution. *Journal of Economic Theory*, 4(1):19–34, 1972.

E. Moser, A. Seidl, and G. Feichtinger. History-dependence in production-pollution-trade-off models: a multi-stage approach. *Annals of Operations Research*, 222(1):457–481, 2014.

A. Mosiño and A. Pommeret. Switching to clean(er) technologies in a stochastic environment. MPRA Paper 83841, University Library of Munich, Germany, June 2015. URL `https://ideas.repec.org/p/pra/mprapa/83841.html`.

Z. Nahorski and H.F. Ravn. A review of mathematical models in economic environmental problems. *Annals of Operations Research*, 97:165–201, 2000.

W. Nordhaus. Climate change: The ultimate challenge for economics. *American Economic Review*, 109(6):1991–2014, 2019. doi: 10.1257/aer.109.6.1991. URL https://www.aeaweb.org/articles?id=10.1257/aer.109.6.1991.

S. Särkkä and A. Solin. *Applied Stochastic Differential Equations*. Institute of Mathematical Statistics Textbooks. Cambridge University Press, 2019.

A. Zemel. Precaution under mixed uncertainty: Implications for environmental management. *Resource and Energy Economics*, 34(2):188–197, 2012. ISSN 0928-7655. doi: https://doi.org/10.1016/j.reseneeco.2011.11.002. URL https://www.sciencedirect.com/science/article/pii/S0928765511000753.