University of Alberta

Epistemology, Normativity and Mathematics Education

by

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Dedication and Acknowledgement

This work is dedicated to my wife, Anne-Marie Pedersen. This thesis could not have been possible were it not for Anne-Marie's, encouragement, support and assistance. Our children, Michaela and Madelena, were patient and generous in sharing their father with philosophy. At times, I am sure that all three wished that the thesis would go away and that I would spend less time reading and writing, but their support and love stayed strong throughout.

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Abstract

Theories of mathematics education can aim for several goals: to account for the nature of mathematical content and knowledge, to describe and explain the teaching and learning of mathematics, and to make sense of performance and thinking within mathematics. Four prominent contemporary theories—von Glasersfeld's radical constructivism, Lakoff and Núñez's cognitive science, Ernest's social constructivism and Davis's complexity science—all claim to naturalize mathematics and mathematics education. Each in its turn fails in its ambitions, but in instructive ways.

These four theories are analyzed in terms of their programmatic content. Each theorist has a declared political intent, namely, to provide an alternative to perceived elitism in mathematics education. The political claims of each theory are compared against the theory's epistemological aspirations, and in each case an insuperable gap is shown.

The theories' epistemological claims are tested against the Kripke-Wittgenstein paradox. Each fails to account of the possibility of mathematical understanding because it cannot provide the normative content of rule following. Positive responses to the paradox are provided through appeals to the work of Charles Taylor and Thomas Nagel.

In addition, none of the theories gives a defensible account of mathematical intersubjectivity. Habermas's theory of communicative action is shown to provide sufficient resources to give a rich account of intersubjectivity of mathematics education.

Building on Taylor's, Nagel's and Habermas's theories, suggestions for mathematics education research are given. I argue that it is necessary for educational purposes to assume the content and practice of mathematics as part of the initially unquestioned background against which mathematical thinking and learning can take place and be understood by assessors and researchers. Further, I argue that in order to do mathematics,

one must treat mathematical objects as something that transcend the first-person perspective. Finally, I argue that a communicative model of understanding holds promise in educational assessment and research, because it may provide means of inferring student understanding either through direct discourse or via models that reconstruct internalized dialogue. One understands a mathematical concept, I claim, insofar as one can provide reasons that would be compelling to any relevantly situated interlocutor, at least in principle.

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Part One: Philosophy and Mathematics Education

One central point of mathematics education research is to understand how it is that mathematics is learned, so that improved teaching, materials and policies can be implemented. To that end, mathematics education research is seen to be sensitive to three main ideas. First, it must be aware of issues surrounding the nature of mathematical knowledge. It is important to differentiate the mathematical from the non-mathematical in order to assess mathematics education. Second, mathematics education theory must have something to say about the teaching and learning of mathematics. This is important both from the vantage point of practical teaching and learning, but also ethically and politically. Not only is it important to know how to teach mathematics; it is desirable to have some sense of *why* it is to be taught. Third, mathematics education research should have something to say about mathematical thoughts and performances and the relationship between the two.

1 Introduction

The student touched buttons on the calculator: 2...2...0...9...√. Numbers appeared on the screen: 47. She copied the numbers into her notebook. A question puzzles many of us: Did she understand the mathematics? More specifically, this thesis will address the question, 1) What must be true about a person in order to say that the person understands a mathematical concept, proposition or justification? Teaching is an attempt to bring a student to understanding, and educational assessment is an attempt to judge the content of a student's understanding. This thesis will also address the question, 2) What must be true about two persons to say that they have come to a point of mutual understanding of a mathematical concept, proposition or justification? Two subsidiary questions immediately follow: a) What is the justification for educational interest in mathematical understanding? and, b) Do current theories of mathematics education provide adequate answers for questions 1 and 2?

I argue that four currently prominent theories of mathematics education deprive us of an adequate answer to questions 1 and 2, because they attempt to account for mathematics as if it were a natural phenomenon, as if it could be explained like the phases of the moon, the existence of mountains, or the circulation of the blood. It is my contention that in order to provide a defensible account of teaching and learning mathematics, a theory of mathematics education must incorporate normativity into its epistemology. Further, this normativity cannot be reduced to other, non-normative elements of the theory. Normativity, I argue, is not successfully naturalized by contemporary theories of mathematics education. Indeed, it is unlikely that normativity can be naturalized (Nagel, 1997; Searle, 1998).

There is more to mathematical understanding than the ability to repeat mathematical truths. Mathematical understanding at least involves having accurate mathematical concepts, defensible beliefs about mathematical propositions, and the ability to provide mathematical justifications. Mathematical justification is a norm-governed activity that involves the use of speech acts to bring interlocutors into a position of mutual understanding and conviction regarding the truth of a mathematical proposition or the soundness of a mathematical procedure. In the preceding example, the judgment that the student understands the mathematics is contingent upon her having some mathematically relevant concepts that guide her in pushing the buttons on her calculator. She must also have good reasons for performing a particular sequence of button pushes. I claim that it is in the articulation of those reasons that she most directly demonstrates her mathematical understanding. If she can reach mutual understanding of her mathematical reasons with an interlocutor, and these reasons justify her actions, then the interlocutor has grounds for saying that the student understands the mathematics. Further, I argue that understanding is related to the process of coming to mutual understanding with an interlocutor. A theory of mathematics education must be able to account for the norm-governed intersubjective actions that bring about mutual understanding of mathematical concepts, propositions and reasons, including mutual knowledge of understanding.

If the above can be defended, then it is incumbent on any theory of mathematics education to articulate the conditions under which mathematical reasons are formulated, communicated and justified in educational contexts. Such a theory should account for differences in cognitively relevant circumstances, such as age and maturity. The acquisition of mathematics, I claim, involves individually comprehended concepts, propositions and justifications, and intersubjectively realized speech acts that bring interlocutors into mutual understanding of these concepts, propositions and justifications. One point of this thesis is to show that current prominent theories of mathematics education fail to theorize mathematics learning adequately because they do not adequately account for the normatively governed actions which bring about mutual mathematical understanding.

That the four theories I shall discuss fail to provide defensible accounts of mathematics education will become clear in chapters 5-7. The consequences of not challenging them are potentially disastrous because the influence of mathematics educational theory is felt in two significant ways. First, researchers' understanding of mathematics, mathematics teaching and mathematics learning has influence on public policy. Decisions regarding curriculum content and standards and the assessment of students and teachers are informed by beliefs about what it is that ought to be done in mathematics classrooms and how it is determined that these things are being done. Suppose a researcher held the belief that the knowledge of an individual student is unimportant, if the total knowledge possessed by the members of the student's class meets some minimum standard. Should such a view come to inform curriculum, then a likely outcome would be that some students would fail to learn much mathematics at all. If the researcher's view is not defensible—and I do not believe that such a view is defensible—then the foreseeable consequences of the policy are indefensible. A student has a legitimate expectation of benefiting from public education. If that benefit is denied as a consequence of a public policy, then that policy requires scrutiny.

More directly, faculties of education are usually, among other things, training centres for prospective teachers. The practices of pre-service teachers are influenced by the presented ideas of what mathematics, mathematics learning and mathematics teaching entail. For example, Davis, Sumara and Luce-Kapler (2000) have published an undergraduate textbook for pre-service teachers that promotes the very controversial view of mathematics education that I describe in section 3.4. I will explore some possible consequences of the adoption of this doctrine in a later chapter.

Suppose, as above, a mathematics education researcher believed that the knowledge of individual students is irrelevant, provided the total knowledge possessed by the student's class meets some minimum standard. Even if this researcher had no influence on public policy, it is possible that this view would be presented to prospective teachers. The possibility that it would influence the prospects of students in the near future is substantial. My point is this: even without the force of public policy, extant theoretical views of mathematics education are likely to have effects on the educational experiences of children. It is my goal to provide coherent and sound critique and guidance to mathematics education researchers and, consequently, to mathematics teachers.

I aim to show that four prominent theories of mathematics education do not sufficiently describe mathematical thinking, nor do they account for intersubjective mathematical understanding. Chapter 2 develops some basic ideas of what I argue to be essential features of theories of mathematics education: in short, they must be consistent with our best understandings of the content and practices of mathematics and of education. Central to both mathematics and education, I claim, is *thinking*. Mathematics education theory must account for mathematical thinking in individual agents, and it must provide the means of understanding how it is possible to establish that another person possesses meaningful mathematical concepts and is able to justify mathematical judgments. Further, such theory must account for the possibility of intersubjective understanding: how can one person come to mutual understanding with another person regarding mathematical concepts, propositions or justifications. Finally, I argue that the processes that make mathematical intersubjectivity possible are precisely those that make mathematical understanding possible.

Chapter 3 provides a brief exegesis of four influential contemporary theoretical approaches to mathematics education: radical constructivism, cognitive science, social constructivism, and an ecological epistemology informed by complexity science. All four of these theories are found in the literatures of many fields of social inquiry and all have found strong champions within mathematics education. From radical constructivism, I look at the work of its originator, Ernst von Glasersfeld (Glasersfeld, 1988; Glasersfeld, 1991; Glasersfeld, 1994; Glasersfeld, 1995; Glasersfeld, 2005; Steffe, Cobb, & Glasersfeld, 1988). In cognitive science, George Lakoff and Raphael Núñez have provided a comprehensive treatise, *Where Mathematics Comes From* (Lakoff & Núñez, 2000). For social constructivism in mathematics education, I turn to two authors: Paul Ernest (Ernest, 1991; Ernest, 1994a; Ernest, 1994b; Ernest, 1994c; Ernest, 1998) and Reuben Hersh (Davis & Hersh, 1980; Hersh, 1994; Hersh, 1997). For the application of complexity science to mathematics education, I focus on the work of Brent Davis (Davis, 2004; Davis & Simmt, 2003; Davis & Sumara, 2002; Davis & Sumara, 2003; Davis & Sumara, 1997; Davis, Sumara, & Luce-Kapler, 2000; Towers & Davis, 2002).

Chapter 4 recasts the four theoretical positions in light of the desiderata of chapter 2. To what extent is each account consistent with what is uncontroversially mathematical and educational? Where inconsistencies are identified, they will need to be examined individually to identify whether they represent theoretical failure on the part of the educational theory, or whether they identify places where a theoretical transition can be seen as an epistemic gain or loss. That is, it is possible that inconsistencies between currently accepted views of mathematics and education and proposed educational theory represent opportunities to expand or improve theoretical understanding in some way.

Chapter 5 articulates and criticizes the overt and covert political aspirations of the four theories. All are explicit in claiming that the adoption of their epistemologies will lead to social benefit. In each case, it is claimed that alternative epistemologies ossify a class structure in mathematics, with only a select few people able to attain mathematical understanding. I show to be dubious the connection between mathematics epistemology and mathematical hierarchy.

In chapter 6, the problem of understanding moves to the fore. Wittgenstein's *Philosophical Investigations* (Wittgenstein, 1968) occupies a central place here. Wittgenstein raises a profound skeptical argument that appears to make all rule-following, including mathematics and language, to be impossible. Wittgenstein takes the basic structures of meaning to be rule-governed. Following Kripke (Kripke, 1982), I accept that Wittgenstein's argument, taken on its own terms, does show meaning to be impossible. The four theories are placed in relation to the skeptical problem. Since each claims to provide a comprehensive account of mathematics, mathematical understanding and mathematical learning, it is reasonable to see if they can deal with Wittgenstein's problem. After showing how they each fail to adequately deal with Wittgenstein's skeptic, I turn to solutions provided by Kripke, Charles Taylor (Taylor, 1995) and Thomas Nagel (Nagel, 1997) to find possible ways for mathematics education theorists to proceed.

The problem of the intersubjectivity of mathematical knowledge dominates chapter 7. Following from chapter 6, I suggest that a theory of mathematics education would require some response to Wittgenstein's skeptical problem similar to the Taylor or Nagel solutions. The question now becomes whether such positions can account for the intersubjectivity of mathematical knowledge. Using a portion of Habermas's theory of language, I show that either approach leads to plausible theoretical descriptions of mathematical intersubjectivity in an educational setting.

Chapter 8 completes the thesis. Here, I take the conclusions of chapters 5-7 to show some features of a plausible theoretical approach to mathematics education. I conclude with a brief statement of a research program in mathematics education and philosophy of education that could follow from this thesis.

2 Desiderata for Theories of Mathematics Education

What conditions must mathematics education theories satisfy? A simple first answer is that such theory must, at the very least, be consistent with and be relevant to what is true about mathematics and education. A theory of mathematics education that cannot distinguish what is clearly and unambiguously mathematical—say, counting—from the clearly and unambiguously non-mathematical—say, eating—is not sufficient for the task. The theory need not agree with all possible suggestions of what might be mathematical—an activity such as knitting might lead to some legitimate disagreement as to whether it is mathematical in part or not—but the theory must not contradict the clear cases. In fact, it is possible that a good theory could be a source of some arguments one way or the other on difficult cases such as knitting.

Second, a theory of mathematics education must be consistent with what is clearly and unambiguously educational. That mathematics can be taught and learned is beyond question. Thus, a theory of mathematics education must acknowledge that it is possible that a teacher can know some mathematics that the student does not, and that through some meaningful interactions the student can come to know at least some of the mathematics that he or she did not know beforehand. A theory that claimed that mathematics is unteachable would violate common experience and common sense. In what follows, I raise some of the questions that this discussion helps bring to the fore.

Beyond consistency with other considered judgments, a theory of mathematics education must also inform research and practice. It must be possible to use the theory to interpret teaching and learning in a way that is comprehensible and helpful both for researchers and practitioners. A theory of mathematics education should help teachers to teach, and learners to learn. And it should help observers of teaching and learning to account for what is happening. In short, educational theories should be of some utility in explaining and improving educational practice.

Finally, a theory of mathematics education must account for both mathematical performance and mathematical thinking. It is not sufficient that students reproduce solutions to problems in mathematics: they must be able to have thoughts appropriate to the mathematics they are doing. Looking up solutions in books, or copying them from the board is not in itself doing mathematics; students need to understand reasons for what they are doing for their work to be considered mathematical. This is not an uncontroversial point, as will become clear as I articulate the four theories of mathematics education in chapter 3. Yet, the burden of justification on any theory of mathematics education that displaces thinking from central consideration is substantial.

2.1 An Account of the Nature of Mathematics

While it is possible that a theory of mathematics education could enlarge or otherwise shape our conceptions of what mathematical activity is, it is unlikely to do so in a radical way. This is principally because much of what is seen to be mathematical has been stably so for over two thousand years. All cultures with mathematics developed calculations. Many worked with concepts such as area and volume. In the West, proof has been a key part of mathematical activity since well before Euclid. A theory of mathematics education that challenged the mathematical status of these activities would need to make an overwhelmingly strong case, or risk appearing to be frivolous.

Nonetheless, acceptable modes of mathematical practice have changed historically. Arguments that were judged to be adequate at one time might be judged to be lacking in another one. Many of Euler's proofs, for example, were considered inadequate shortly after his death. Several 19^{th} century mathematicians put considerable effort into bringing Euler's results "into repute" by providing proofs meeting the increased standards of rigor that had been adopted. Nonetheless, Euler's work was not judged to be non-mathematical. Further, the activities normally called mathematical are not necessarily all closely related. It is possible that the account provided by a theory of mathematics education that captured the process of learning to multiply fractions might be quite different from a theory that dealt with the ability to make mathematical conjectures, or a theory that dealt with the proofs or refutations of conjectures. It is possible that a single over-arching theory can account for all of mathematical practice; but it is also possible that no theory can do all that. Regardless, whatever is offered as a theory of mathematics or of mathematics education must be consistent with clear cases of mathematical content and practice in all its unambiguous contexts.

A further salient feature of mathematics with which an educational theory must be consistent is the attachment of mathematics to experience. Not only is the physical world often meaningfully described in mathematical terms, but mathematical experience is often best described in physical terms. Consider, on the one hand, the clear and obvious relationship between Euclidean geometry and the building of physical structures. If one wishes to find the center of a rectangular floor, the theorem that the diagonals of a rectangle are mutually bisecting is very useful. If one wishes to construct the set of Real numbers, physical experience with linear ordering is very useful. There appears to be a two-way relationship between the mathematical and the physical that is significant in mathematical experience.

2.2 The Possibility of Education

One thing is clear about mathematics education: it is possible to teach and to learn mathematics. This possibility points to the mutual knowledge of mathematics: if one person has a mathematical thought, it is possible for that person to help bring another to understand it. One of the most remarkable features of mathematical intersubjectivity is that those mathematical ideas from the ancient world that somehow make it to modern attention are comprehensible to us. A mathematical thinker from thousands of years ago was able to make representations of those thoughts in such a way that modern thinkers with none of the originator's language or culture are able to understand. For example, ancient Babylonian arithmetic, using either cuneiform or circular notation, is read and understood by many modern historians. The arithmetic is comprehensible *as arithmetic* and can be easily translated into modern notation (Eves, 1969). This suggests that the objects of these ancient mathematical writings are meaningful in a very wide array of cultural settings. The possibility and actuality of mathematical communication must be a part of any theory of mathematics education.

In section 2.1, I referred to changing standards of mathematical rigor over time. Analogously, the expectations of rigor change from one situation to another. No one would expect an 8-year-old's justification of a formula for the computation of the area of a circle to be the same as the justification given by an undergraduate mathematics student.

In the preceding two examples, I have made a distinction between mathematical thinking and mathematical performance; and I have suggested that both have educational significance. Mathematical thoughts are fundamental to all mathematical activity, yet the expression of these thoughts is not only possible, but is an equally fundamental part of what it is to do mathematics. It is conceivable that simple mathematical performances such as computation could be seen as communicative acts. Certainly they are public in a sense that thoughts are not. And education, as it is normally conceived in the modern world, involves some sort of assessment based upon these communicative acts. Attempts to theorize mathematics education should recognize both the private-cognitive and public-expressive aspects of mathematical experience and take both into account.

The key idea is that mathematical thoughts and mathematical communications have a mathematical referent. When the ancient Babylonian mathematician made a sum, the sum was of some numbers. These numbers and this sum are understandable to modern thinkers. What is at stake, as I will develop in chapters 6 and 7, is the nature of this understanding. This is so because the teacher and student are in a position similar to that of the ancient mathematician and you and me. When the Babylonian mathematician wrote his sum, he could not know precisely what thoughts his future readers would have, but he nonetheless wrote his symbols expecting that they would be meaningful. Similarly, the teacher cannot know what thoughts the students will have, but nonetheless expects to teach the students some of the teacher's mathematical thoughts, and expects that there is a meaningful comparison to be made between the students' writings and the mathematical thoughts to which the lesson referred. In chapter 3, I will explicate four very different accounts of the relationships between mathematical thoughts and mathematical performances, including communication.

2.3 Mathematical Thought and Performance

An electronic calculator performs an arithmetic algorithm with greater speed and accuracy than I do in most cases. It does not appear to be the case, however, that the calculator is doing mathematics, because it does not have any concept of what it is doing. On the other hand, by entering the numbers into the calculator and pressing the buttons corresponding to the algorithm I wish followed, I am doing mathematics even though I am not performing the algorithm. What separates the human from the machine in this case is an interpretation of mathematical meaning. Let me clarify this with a more specific example. Imagine a right-angled plane triangle with legs of length 5cm and 12cm. How long is the hypotenuse of this triangle? You may simply recall the answer 13cm from previous experience, or you may be able to perform the relevant calculations mentally. Or you may use a calculator to come up with the solution. In each of these cases, it is clear that a person solving the problem is engaged in mathematics because that person's thoughts are not only mathematical, but they are relevant to the problem. Alternately, someone who randomly punched numbers into the calculator then wrote down whatever showed up on the display would not seem to be doing mathematics of any kind, even if that person were to claim 13cm as the solution to the problem. Someone who failed to arrive at the correct solution, or perhaps even to arrive at any solution at all might still be said to be doing mathematics. Again, relevant thinking is what separates the mathematical solutions from the non-mathematical. The person who simply misperformed the calculation can still be doing mathematics, while the person who copies another's solution without any understanding of the meaning of the symbols cannot.

The central arguments of this thesis are independent of views of mathematical ontology. Whether one holds a realist view that claims that mathematical objects are discovered as a mind-independent feature of reality, or one holds the view that mathematical objects are constructed through human practice, the arguments still hold. I defend a view that insists one must *deal* with mathematical objects as though they were independent of oneself in order to do mathematics. I remain silent on whether this pragmatic dealing corresponds to a true ontology. Further, I argue that once an appropriate pragmatic stance toward mathematical objects is taken, one must not only display certain mathematical performances, but one must also have mathematically relevant thoughts in order to do mathematics. To see that the display of mathematical performance is not necessary for the doing of mathematics, consider the possibility of a person reading the problem, mentally computing the length of the hypotenuse and going no further. Insofar as we can imagine such a situation, we can imagine mathematics being done without any trace of a publicly identifiable performance. In education, the teacher is obliged to assess the mathematical understanding of students, but has no direct access to their thoughts. It is not surprising, then, that student assessment is largely based upon performance, from which the teacher infers understanding. This has been complicated in recent years with the introduction of increasingly sophisticated calculators and computer programs. Student mathematics is often demonstrated through electronically mediated performance. The 5-12-13 triangle problem, for example, might be given to a student with access to computational technology. I will raise three possibilities, but will not discuss them deeply at this point. Rather, my purpose is to show the sort of thinking with which a fruitful theory of mathematics education must be able to deal.

Solution #1: The student has a pre-programmed right-angled triangle program. She runs the program and is prompted to enter the lengths of the two legs and the program types the output "13" to the screen. She then writes this number in her notebook.

Solution #2: The student uses a graphical program such as "Geometer's Sketchpad". The student uses the construction tools to create segments 5cm and 12cm long. With her mouse, she arranges the segments to be perpendicular at their endpoints. She constructs the third side of the triangle with the mouse. The program calculates the length of the third side to be 12.9999997, which she writes in her notebook.

Solution #3: The student enters the vector (5,12) in her calculator, and then pushes the button that calculates the modulus of the vector. She writes the output 13 in her notebook. I do not believe that the theories I will consider in chapter 3 adequately deal with how these simple situations should be interpreted.

2.4 Summary

I have presented two broad categories of consideration for theories of mathematics education. On the one hand, these theories must be consistent with that which is mathematical. I have not delved deeply into the nature of the mathematical, but I have asserted that at a minimum an educational theory must be consistent with thoughts and acts that are clearly mathematical. Further, such a theory must recognize that mathematical performance has an explicitly mathematical object.

To account for the possibility of teaching and learning, any credible theory of mathematics education must acknowledge that mathematics is at least partly acquired in a social process. The intersubjectivity of mathematics is a part of common experience, and it appears to be central to the social process of mathematics education.

Behind ideas of mathematics and education, I have placed the question of what it means to have mathematical thoughts. As I explore four prominent theories of mathematics education, I will first interrogate them on their own terms. But ultimately, I will raise the question of whether these theories give an adequate account of mathematical thinking. That they are unable to do so will be argued in chapters 5 and 6, and a constructive argument suggesting some necessary features of a more successful theory will be presented in chapters 7 and 8.

Part Two: Epistemology Becomes Entangled in Mathematics Education

Part two begins with an exegesis of the four theories. The main claims of Ernst von Glasersfeld's radical constructivism, Paul Ernest's and Reuben Hersh's social constructivism, Lakoff and Núñez's cognitive science approach, and Brent Davis's ecologically-informed complexity theory account are brought forward with illustrative examples.

Chapter 4 catalogues the main mathematical and philosophical commitments of the four theories. Their positions on the nature of mathematics and its relationship to the world of experience are explored, as well as their commitments to stability and change in mathematical results. Educational considerations include performance, thinking, intersubjectivity and justifications for mathematics education.

3 Four Theories of Mathematics Education

This chapter provides an exegesis of four contemporary theories of mathematics education: radical constructivism, a cognitive science approach, social constructivism and complexity science. The first three can profitably be seen as post-behavioral theories. That is, advocates for all three theories claim that the performance of mathematics is not only physical behavior, but is also a cognitive activity. All three recognize individual thinking as fundamental to mathematics. Where they differ strongly is in their claims about what warrants mathematical beliefs. The fourth, complexity science, claims to be a "meta-discourse" (Davis et al., 2003) that transcends the first three.

Radical constructivism is Ernst von Glasersfeld's Piagetian theory of individual learning that grounds mathematical thinking in sensory-motor experience. Knowledge, on this view, is not seen as correspondence to an external reality, but rather as a state of momentary cognitive stability. Thus, one comes to acquire and to maintain certain mathematical beliefs not because they are judged to be true, but because they are satisfying in some other way.

George Lakoff and Raphael Núñez have produced a provocative view of mathematical understanding that sees knowledge as a consequence of human neural development. On their cognitive science view, mathematics is inevitable in that it is a consequence of the way that human brains are structured, but it has no meaning outside of human cognitive and social experience. This view is similar to radical constructivism in that it sees mathematics in terms of individual consciousness. It is different in that it is a universal thesis, claiming that the development of mathematics is contingent on the evolutionary heritage of all normally developed human beings.

The third view I consider is Paul Ernest's mathematics education as social constructivism. Hacking (1999) takes exception to Ernest's terminology. 'Constructivism' is the term used for an active research program within mathematics. Ernest would have done better, says Hacking, to have used the word 'constructionism' in his title. I agree; but for the sake of clarity I will stick to Ernest's terminology. I simultaneously refer to the work of Reuben Hersh to help clarify some key concepts. Social constructivists do not see mathematical knowledge as a matter of individual judgment, but instead see it as a performance of individuals, receiving warrant from a discursive community. Regardless of the content of my mathematical ideas, social constructivists argue, they do not qualify as knowledge unless they meet the standards of approval of a community of knowers. Further, this view suggests that the intersubjective warranting process forms the standards for what can be claimed, and even for what can be considered to be an appropriate object of knowledge.

In complexity science, I focus on Brent Davis's contributions. He typically co-authors papers, so his work shows the influence of others in the field. Nonetheless, the works I have cited are consistent with one another in tone and content, and are representative of a unified view. The section on complexity science will be different from the first three. Because Davis does not put forward an explicit theory of mathematics education, it is difficult for me to state precisely what he has to say about the specific mechanisms by which mathematics is meaningful and is known. He does provide an over-arching scheme for talking about mathematics and provides a socialized view of mathematics knowledge. He cites von Glasersfeld, Lakoff and Núñez, and Ernest favorably in his work (e.g., Davis et al., 2003) so I will rely on my exposition of those three to help make Davis's position as comprehensible as I can.

My exegesis of von Glasersfeld's and of Lakoff and Núñez's work will be much more explicitly mathematical than will be my exegesis of Ernest and Davis. This is a reflection of the content of their presentations. Both von Glasersfeld and Lakoff and Núñez provide concrete examples of the mechanisms by which an individual can come to learn some elementary mathematics. Ernest and Davis are not interested in theorizing individual cognition, and instead rely on generalities across groups of thinkers to make their points. For me, the lack of mathematical concreteness in Ernest and Davis represents both an irritant and a shortcoming of their theories.

3.1 Ernst von Glasersfeld: Mathematics as Personal Construction

The psychologist Ernst von Glasersfeld champions *radical constructivism*, a theory of learning he developed as an extension and clarification of Piaget's theory of learning. Von Glasersfeld follows Piaget's work on evolutionary epistemology. For Piaget, knowledge amounts to no more than schemes of action and stable beliefs that allow an organism to survive in the world. Stable beliefs that arise from social interaction, says Piaget, are contingent upon stable schemes and beliefs that have their origins in sensory-motor experience (Piaget, 1978). Leslie Steffe, a specialist in mathematics education at the University of Georgia, has worked extensively with von Glasersfeld to focus radical constructivism on issues related to the learning of mathematics. The result of this collaboration is a theory of learning that claims to be an evolutionary model of individual adaptations. Further, von Glasersfeld claims that radical constructivism offers a naturalized epistemology of mathematics.

Von Glasersfeld sees radical constructivism as being opposed to two main competitors. Radical constructivism, he argues, is superior to behaviorism as a theory of learning and superior to mathematical realism as an epistemology. His objection to behaviorism is simple: learning involves thinking, which is precisely what behaviorism refuses to theorize. He extends this criticism to any view of learning that focuses primarily on performance rather than cognition. Von Glasersfeld's objection to realism is also simple to state. All experience is necessarily undergone from an individual perspective. Sensory data and feedback from physical action in the world provide the raw materials that cognition organizes. This organization is of experience itself, not of the world; each individual lacks direct access to either an unmediated world or to other people's thoughts. Therefore, von Glasersfeld claims, all knowledge can be only of personal experience and thoughts informed by experience or other thoughts. This leads von Glasersfeld to the conclusion that knowledge cannot come to a person "ready-made" but must necessarily be constructed by that person. "Radical", then, refers to the sharp break between the individual knower and both the world and other knowers; and "constructivism" refers to the internal processes by which individuals come to have knowledge.

3.1.1 Viability

Von Glasersfeld sees knowledge not as a true or reliable depiction of reality, but as a heuristic, something to help an individual achieve goals and minimize instability.

> Radical constructivism is uninhibitedly instrumentalist. It replaces the notion of "truth" (as true representation of an independent reality) with the notion of "viability" within the subjects' experiential world. Consequently, it refuses all metaphysical commitments and claims to be no more than one possible model of thinking about the only world we can come to

> know, the world we construct as living subjects. Because this is a difficult and shocking change of attitude when one first comes to it, I want to reiterate once more that it would be misguided to ask whether radical constructivism is true or false, for it is intended, not as a metaphysical conjecture, but as a conceptual tool whose value can be gauged only by using it. (Glasersfeld, 1995, p. 22)

Von Glasersfeld's conception of viability comes from evolutionary theory. An adaptation is said to be viable if and only if it is consistent with the organism's continuing survival. The shading of elk, from dark on the back to white on the belly, could be seen as a viable evolutionary response to predation because its success is seen in the continued survival of the elk as a species. On an individual level, the movement of a *particular* elk from one valley to another is seen as a viable response if and only if the movement resolved some perturbation in the elk's existence, such as lack of food.

Von Glasersfeld extends the notion of viable adaptation to cognitive life. The belief that 2+3=5 may be viable in that it appears to have become part of the evolutionary inheritance of *Homo sapiens*. A particular person's belief that there are an infinite number of prime numbers is viable only if it is a stabilizing response to a perturbation. In short, radical constructivism is a view that holds beliefs about the world as neither true nor false, but as viable. A belief is viable only when it produces some level of cognitive stability, even if only for a short time. “A thinking subject has no reason to change his or her way of thinking as long as there is no awareness of failure” (Glasersfeld, 1991, p. xviii). Thoughts are modified or displaced not because they are false, but because their modification or displacement contributes to decreased cognitive instability. I can believe that 2+3=4 only so long as that belief causes me no perturbation; I will modify my belief only if I become aware of some inadequacy with it. Von Glasersfeld is not explicit as to the range of inadequacies I might find, but they include beliefs about the truth or falsity of the proposition, even though, as von Glasersfeld maintains, I cannot obtain certainty or finality in such matters.

So, von Glasersfeld sees belief formation as purposive and adaptive. The proposed mechanisms by which beliefs are formed come directly from Piaget: accommodation and assimilation. These concepts from biology survive intact from their well-known use in Piaget's work (e.g., Piaget, 1952). When a new behavior or belief is grafted onto an existing network of behaviors and beliefs without disruption to them, the new behavior or belief is said to be *assimilated.* When an already-established behavior or belief is modified, the new behavior or belief is said to have been accommodated.

Cognitive accommodation and assimilation are integrated into feedback and control mechanisms in an overarching system of thinking and doing. Von Glasersfeld uses cybernetic imagery to describe human cognition as similar to control systems. The cognizing agent actively constructs concepts and attempts to act or reflect upon them. Actions in the world (including thinking) provide feedback that is either confirmatory or contradictory to the constructions. Confirmatory feedback maintains a state of cognitive homeostasis, while contradictory feedback creates cognitive perturbations that are dealt with through accommodation and/or assimilation until stability is reached once more. This new stability, argues von Glasersfeld, is knowledge. Thus knowledge, he claims, does not represent an external reality, but is merely a representation of thoughts that have provided a state of cognitive stability in an individual (Glasersfeld, 1995, ch. 3 and 8).

3.1.2 Knowledge, Experience and Reality

As indicated above, von Glasersfeld has produced a model of concept formation, largely informed by Piaget's developmental psychology. Von Glasersfeld argues that concepts and beliefs are formed through a number of naturalistically-describable processes of coping with experience. The individual, argues von Glasersfeld, has access to the world only mediately, though the senses, language, etc. From this observation, he adopts a perspectival view. He argues that since all experience is from a particular perspective, it follows that knowledge of an unmediated reality is not possible. This is not, von Glasersfeld takes pains to explain, a solipsistic view. He does not deny that there is a world outside of the individual human mind; rather he simply denies that anyone has access to it.

What makes von Glasersfeld's view *radical*? For von Glasersfeld, knowledge is a purely individualistic phenomenon. Knowing of any sort, he argues, takes place entirely and only in an individual's mind. Collective understanding, on this view, is reducible to individual knowing by multiple individuals. This view sees all knowing as "essentially subjective, and though I may find reasons to believe that my experience may not be unlike yours, I have no way of knowing that it is the same" (Glasersfeld, 1991, p. 1). I'm not certain if von Glasersfeld is being ironic here. If knowing is subjective in the sense he describes, then he does know whether his experience is alike to someone else's. It seems more likely to me that he is using 'knowing' in the quoted material to refer to the idea of 'knowing something true' that he is rejecting.

Von Glasersfeld champions radical constructivism as "a theory of *active knowing*, rather than a traditional theory of knowledge or epistemology. From this standpoint, as Piaget maintained fifty years ago, *knowledge serves to organize experience* not to depict an experiencer-independent reality" (Glasersfeld, 1991, p. xix). So, we see that von Glasersfeld claims that radical constructivism is not only a psychological theory, he insists that his theory do double duty as both an explanation of learning and as an epistemology, albeit one that redefines knowledge in an idiosyncratic way.

> From this perspective, cognitive structures—action schemes, concepts, rules, theories, and laws—are evaluated primarily by the criterion of success, and success must ultimately be understood in terms of the organism's efforts to gain, maintain, and extend its internal equilibrium in the face of perturbations. (Glasersfeld, 1995, p. 74)

So, von Glasersfeld has established a view that he claims accounts for belief formation. Beliefs are formed as an internalized response to experience. When individuals find themselves in a state of internal disequilibrium, cognitive structures are modified to re-establish equilibrium. Further—and this is the epistemological point—von Glasersfeld claims that there is no test for the truthfulness of beliefs. Truth is not a meaningful or helpful notion within radical constructivism; all that matters on this view, is stability. Further, there is no normative component to this view. Von Glasersfeld's point is not that we *ought* to seek cognitive equilibrium; he claims that it is just what we do do.

3.1.3 Language and Intersubjectivity

For von Glasersfeld, neither the notions of truth nor of objectivity have any importance. Nonetheless, he maintains that radical constructivism is not an "anything goes" theory. In order to avoid facile relativism, he invokes intersubjectivity as the "highest, most reliable level of experiential reality" (Glasersfeld, 1995, p. 119). The idea is that communication with others can provide some level of corroboration of our knowledge. Note, though, that under radical constructivism, our knowledge of others is restricted to the viability of our cognitive constructions, as we utilize them to assuage our internal dissonances. Von Glasersfeld is aware of this tension. First he claims, following Kant, that our concepts of others are constructions that make them comprehensible to us as the sort of creatures that we conceive ourselves to be. As a child begins interacting with (experiences of) others, the child begins to see that behaving as though those others had senses, experiences and thoughts of types similar to her or his own, the child becomes able to do, say and believe things in accordance with these new experiences. What is experienced as interaction and communication with other rational beings serves to permit a large array of accommodations and assimilations to areas of practical and cognitive perturbation.

> Once this level of sophistication is reached, a great deal of time is spent explaining, predicting, and attempting to control these "others". That is to say, one now has populated one's experiential field with models of others who move, perceive, plan, think, feel, and even philosophize, others to whom one imputes the kinds of concepts, schemes, and rules one has oneself abstracted from experience.
>
> At this point, these models are thought to have some of the knowledge we ourselves have found viable in our own dealings with experience. Thus, when we make a prediction about how one of these others will behave in a given situation, the prediction is based on a particular piece of knowledge which we have imputed to that other. If, then, the other does

> what we predicted, we may say that the piece of knowledge was found to be viable not only in our own sphere of actions, but also in that of the other. This bestows a second order of viability to the knowledge and the reasoning we assumed the other to have and act on. (Glasersfeld, 1995, p. 120)

What makes this possible, argues von Glasersfeld, is that our constructions of reality are not simply free-floating, but are constrained by our interactions in the world. Radical constructivism, it must be remembered, does not deny that others and the world exist. Rather it claims that our only access to what is outside of us is through what is inside of us. All we have, on this view, is our own experiences and our cognitive sorting and interpretations of them. Thus, our experience of intersubjectivity serves as simply one further source of experience with which we can construct concepts that help us to assimilate and accommodate beliefs that help establish homeostasis. The experience of intersubjective agreement, then, does not bring us closer to the truth; it provides viable cognition.

3.1.4 Mathematics

As I mentioned in 3.1, von Glasersfeld has focused a considerable amount of his work on how radical constructivism can be applied to the learning of mathematics. Not surprisingly, his work on the acquisition of basic mathematical concepts is informed by Piaget's case studies in the area (Piaget, 1997). I will follow von Glasersfeld's development of the child's basic conceptions of number (Glasersfeld, 1995, ch. 9).

The construction of a concept of number, according to von Glasersfeld, begins with the development of conceptions of unity and plurality. Both unity and plurality are a consequence, he argues, of periodic attention. As the child pays attention to some sensory experience, the attention is momentarily focused and unfocused. If a period of sensory attention is framed by periods of sensory inattention, the experience is said to be an abstraction of unity. Although he does not say so, von Glasersfeld's account relies on two key assumptions. First, the account is only comprehensible if one accepts that there is a distinction between attention and inattention to be noticed. Second, von Glasersfeld assumes that an isolated person can generalize from noticed distinctions.

> What constitutes the abstract concept of *number* is the attentional pattern abstracted from the counting procedure. In this pattern, it is irrelevant what the focused moments of attention are actually focused on. The salient features are: (1) the iteration of moments that are focused on *some* unitary items and attentional moments that are not; (2) that the iterated sequence itself is bounded by unfocused moments; and that the focused moments are coordinated with number words. (Glasersfeld, 1995, p. 172)

For example, my attention is currently alternating between this essay and the sounds of a Leonard Cohen CD. I focus on one, then the other, but not on both simultaneously. This sensory attention to one, then the other, von Glasersfeld would argue, allows me to reflect on them as separate. Because my attention to one is distinguishable from my attention to the other, I am able to perceive *one essay* and *one song*. The experience of differential attention allows me to abstract the concept of different objects. From here, I can abstract the concepts of unity and plurality. Further, if my attention is on the music and one song becomes distinguishable from another, I am able to distinguish my essay from either of the songs, or from a plurality of songs. Attention to such patterns allows me to notice similarities of experience from one pattern to the next. Noticing these similarities can lead to the asking of questions of what it is that is being felt as similar, conceptual accommodation or assimilation can occur to induce cognitive stability. Encounters with the practice of verbalizing a counting sequence (e.g. 1, 2, 3 . . .) permit the child to generalize the attention-inattention sequence to numerical conceptualizing. This introduces a social component to learning small cardinals. For me to learn to count songs, I need to develop a habit of repeating a sequence of sounds in one-to-one correspondence to the songs. I am able to do this, von Glasersfeld claims, because I have noticed a distinction between songs as tokens and between number sounds as tokens and can conceive of each song-number token pair as a unity. Number is not abstracted from collections of objects, nor is it abstracted from my experiencing of collections of objects; according to radical constructivism, number is abstracted from my *dealing* with experiences of collections of objects.

So, on this view, one learns the abstract concept of *number* through a conceptual iteration of attention and inattention, combined with the socially acquired routine of counting. This, argues, von Glasersfeld, accounts for the seemingly indubitable results of elementary arithmetic. There is intersubjective agreement on the number word sequence and on the procedure for counting. It is impossible, given this agreement, for 2+2 to be anything but 4. On the other hand, it is not clear how the rest of mathematics—the less obvious content—can be justified on this view. How does the abstraction of plurality and unity lead to the theorem that every positive integer greater than 1 can be uniquely expressed as a product of prime numbers? At a more fundamental level, it is not clear how objects can be distinguished, or attention differentiated from inattention without mathematical concepts being in place already. It seems that I must have some sense of unity and plurality before I can make the distinction von Glasersfeld takes as unproblematic. It appears that I must accept certain features of mathematics for von Glasersfeld's explanation to make any sense. I believe that this is a very strong rejoinder to the radical constructivist account, and explore the issue more fully in chapters 6 and 7.

3.1.5 A Research Example

In collaboration with Leslie Steffe and Paul Cobb, von Glasersfeld published a collection of observations of 6 first-grade children (Steffe et al., 1988). The point of the study was to look at how children internalize number concepts at an early age, and how they manage to abstract from reflections on physical experience to come to understand and perform first-grade mathematics. Steffe, Cobb and von Glasersfeld's interpretations of their observations give some idea of what radical constructivism is intended to theorize.

Steffe et al. established a series of discrete stages of development that the children were seen to pass through as they acquired concepts of number and arithmetic. Notably, they began with counting on their fingers, and matching their fingers with objects to be counted, even in the absence of the objects themselves. "In general, if a child reenacts the activity of counting a spatial configuration, the motor acts involved co-occur with the isolation of elements in a visualized pattern" (Steffe et al., 1988, p. 20). The central claim is that the motor operations are constructed by the child as an abstraction from reflections on experience. Such abstractions or reflections are, of course, not observed directly, but are inferred from activities, such as viewing a group of three objects and a group of four objects, then being asked how many objects there are in total when they are not all observable by the child. The common strategy is to hold up three fingers of one hand and four of the other, then count all raised fingers.

From perception and motor work, the children were then seen to construct verbal units. They were able to perform simple arithmetic verbally, without physical operations except the counting itself. After verbalizing, the children then moved to a stage where they could perform arithmetic without any externalizations. That is, they could abstractly go from 3+4 to 7 without outwardly counting. The longest period of study involved movement from the mental abstractions to lexical and syntactical representations of arithmetic: they were ultimately able to work with symbols.

Throughout their reporting, Steffe, Cobb and von Glasersfeld are able to couch their observations of the children in the language of radical constructivism, but they are unable to show just what is being constructed. Their descriptive language indicates the plausibility of the claim that concepts are constructed by children. Von Glasersfeld's stronger claim, that each mathematical object (the number 4, say) has no ontology beyond individual constructions is not put to any kind of test in this research.

To be sure, the subjects acquire ever-increasing skill with arithmetic and do, indeed, learn to perform at increasing levels of abstraction, and with increasing acuity with written symbols. A trivial sense of construction is easily seen: the children somehow manage to develop schemes for reporting what they are thinking, and these schemes must amount to some sort of cognitive construction. What is not clear, though, is whether von Glasersfeld's epistemology has any relevance to this sort of research. That is, radical constructivism's descriptive psychology appears to be detachable from its epistemological content: the observations of how children acquire number concepts are plausibly interpreted as internalized constructions regardless of from where the warrant of mathematical belief is derived.

3.2 George Lakoff and Raphael Núñez: Mathematics as Cognitive Mapping

Lakoff and Núñez attempt to account for mathematics and mathematical learning from the perspective of cognitive science. Like von Glasersfeld, they claim that their theory of learning underwrites an epistemology of mathematics. Mathematics is a peculiarly human activity that is a consequence of the physical structures of human beings, they argue. It is unlikely, on this view, that other sentient creatures (intelligent aliens say) would have a mathematics similar to human mathematics unless they had a physical structure (including neurological and sensory structures) similar to ours. Mathematics, for Lakoff and Núñez, is not an abstraction of truths that are independent of human experience; rather, it is an expression of the ways in which humans encounter the world.

One important consequence of this view is that it places the possibility of mathematics squarely within human physiology. Lakoff and Núñez do not deny that language and culture can have a part to play in the learning and practice of mathematics; what they do deny is that the *basic* conceptual structures of mathematics are to be found in culture or language. On this view, all humans with functioning brains and senses already possess the key building blocks of all human mathematics.

This view is educationally significant in that it makes positive claims with regard to the basic mechanisms by which mathematics is learned. It thus implies a research program into the development of the basic skills, dispositions, etc. that are required for *anyone* to learn *any* mathematics.

But Lakoff and Núñez claim to present more than a theory of learning. They claim that their understanding of the embodiment of mathematics underwrites a theory of why mathematics appears to be true to humans. In short, they claim that human physiology dictates which mathematical thoughts are thinkable, and that this in turn explains why we believe mathematics to be true. To do this, Lakoff and Núñez contrast their scientific view of mathematical epistemology with what they call the *romance of mathematics*. I will look at this purported romance in greater detail in section 5.3.

3.2.1 Getting Started: Subitizing and Embodied Cognition

Subitizing refers to the ability to recognize small numbers without counting. (The word derives from the Latin *subitus*, meaning 'sudden'.) Healthy adult humans are able to look at small groupings of objects (fewer than 7, roughly) and "instantly" tell how many are in the group. This is a skill that appears not to require any arithmetic as it is normally understood. Interestingly, the ability has been inferred from observations in infants as young as two or three days (Wynn, 1992), but the inference remains controversial (Feigenson, Carey, & Spelke, 2002). Other experimental results appear to indicate the ability of non-human mammals to subitize (Boysen & Capaldi, 1993). Lakoff and Núñez consider this basic ability to be foundational to human mathematics. The subitizing ability of small children appears to have a limit of about four distinct objects. After infancy, the ability to combine and to partition small, subitizable groups leads to the earliest arithmetical experiences.

"Very basic arithmetic uses at least the following capacities: subitizing, perception of simple arithmetic relationships, the ability to estimate numerosity with close approximations (for bigger arrays) and the ability to use symbols, calculate and memorize shorter tables" (Lakoff et al., 2000, p. 26). So, beginning with the perception of small numbers, on this view, humans are able to extend and generalize arithmetic far beyond immediate experience. In fact, anyone who can perform addition can perform an operation with infinite normative implications. That is, if you have the concept of addition, you know how to perform an infinite number of additions. I will consider in greater detail the infinite normative implications of arithmetic in chapter 6.

What is crucial to this view is that arithmetic begins with basic, innate abilities and quickly expands through experience. A child may be able to distinguish a group of three objects from a group of four objects at an early age, but it is through experience combining the groups that the concepts of number and arithmetic become possible. Thus, according to Lakoff and Núñez, the earliest arithmetic in any human involves sensory experience with groups of objects with small cardinalities. The mechanisms by which this experience extends to abstraction are to be found in metaphorical mapping within the brain. Basic experiences are metaphorically related to one another in the mind, then through processes of blending and conflation are extended into the more comprehensive concepts associated with higher mathematics.

3.2.2 Grounding Metaphors

Lakoff and Núñez claim that there are four basic metaphors upon which arithmetical experience is built. These grounding metaphors, the 4Gs as Lakoff and Núñez call them, are:

1. Arithmetic as object collection.
2. Arithmetic as object construction.
3. The measuring stick metaphor.
4. Arithmetic as motion along a path.

I will briefly describe the four metaphors to give some of the flavor of the ideas. In the previous section, I noted that Lakoff and Núñez claim that children first begin to extend innate arithmetic into arithmetic with larger cardinals. The metaphoric process is introduced through everyday experiences, such as a small child playing with blocks. Subitizing allows for the addition or removal of one or two or three blocks to be recognized by the child.

> Such regular correlations, we hypothesize, result in neural connections between sensory-motor physical operations like taking away objects from a collection and arithmetic operations like the subtraction of one number from another. Such neural connections, we believe, *constitute a conceptual metaphor* at the neural level—in this case the metaphor that Arithmetic is Object Collection (Lakoff et al., 2000, pp. 54-55).

Lakoff and Núñez argue that this basic metaphor is interesting it two ways. First, it explains how the child is able to perform simple pre-symbolic and pre-linguistic mathematics through thinking of one group of objects as having a property—cardinality—in common with another. Second, Lakoff and Núñez claim that the metaphor "is a precise mapping from the domain of physical objects to the domain of numbers" (2000, p. 55). It is not clear what Lakoff and Núñez mean by "precise mapping." In mathematical terms this can mean one-one, one-many, or many-one. The context seems to imply one-one. If so, Lakoff and Núñez are claiming that a mathematical operation—one-one mapping—perfectly captures a physical phenomenon—the human conception of cardinal number. As I explore in section 5.3, this is one of the things Lakoff and Núñez claim that mathematics cannot do. It is a simple move, say Lakoff and Núñez, from the object collection metaphor to arithmetic, as all the laws of arithmetic normally associated with positive integers are readily experienced in physical counting.

The "Arithmetic as Object Construction" metaphor comes from the recognition that finite sets can be partitioned into non-overlapping subsets. Without a formal concept of set, Lakoff and Núñez argue, children are able to form ideas of such statements as "five is made up of two and three." Thus, they are able to abstract the idea that if whole numbers can be partitioned into smaller numbers, then the operation is reversible and whole numbers can be constructed from the composition of smaller ones. This allows for extension into positive rational numbers; the unit fraction $1/n$ is easily seen as a partitioning of one object into n equal parts.

The first two metaphors may provide some insight into how cardinals are conceptualized, but they are not sufficient for the construction of the real numbers. The third metaphor, "The Measuring Stick" allows humans to imagine numbers along a continuum. By imagining a concatenation of uniform sticks in a linear array, it becomes possible to imagine numbers falling between the extremities of the sticks. Combining this with the insights into rational numbers provided through extension of the counting and constructing metaphors, rational numbers emerge quite naturally. The significant difference between the constructing and measuring stick metaphors is that the latter implies the notion of *continuity*. By blending the metaphors of numbering with the metaphor of measuring, argue Lakoff and Núñez, we get rational numbers. This leads to the possibility of wondering whether rational numbers fill every point along our measuring stick, and to the construction of the irrational numbers. Lakoff and Núñez argue that Dedekind's celebrated construction of the real numbers is a clear example of the blending of the metaphors of number and line, with the added concept of closure

(2000, ch. 13). Dedekind showed that all real numbers can be constructed by a process that lists all rational numbers in ascending order, and then partitions the set into two subsets, with all the elements of one subset less than all the elements of the other. Each place where such a partition, or cut, can be made corresponds to a real number. Dedekind showed that his method of cutting identifies each real number uniquely.

The fourth grounding metaphor involves conceptualizing number as realized through motion along a path. In addition to the obvious correspondence of the numbers associated with the measuring stick metaphor with the position of a point traveling along the stick, the notion of motion gives a natural association of zero as a location within an ordering, rather than as an arbitrarily selected endpoint of a measuring rod. Further, negatives are naturally placed within a path relative to zero, simply through the reversal of the direction of movement. Lakoff and Núñez assert that these inferences as "natural" but do not explain why that is so.

The point of the grounding metaphors is that all functioning humans have access to the basic materials necessary to comprehend arithmetic at this level. It is their simplicity that makes them accessible to us; they are teachable and learnable.

3.2.3 Conflation

Lakoff and Núñez begin, as does von Glasersfeld, with the idea that much of mathematical understanding begins at a sensory level. Further, Lakoff and Núñez claim that a certain level of hard-wiring of mathematical concepts is even more fundamental than our ability to learn and perform mathematics (Lakoff et al., 2000, ch. 4). To explain our ability to move beyond our very basic innate ability to compare and order small ordinals toward more elaborate mathematics, Lakoff and Núñez invoke the notion of *conflation.*

Lakoff and Núñez claim that humans have a number of basic ways of experiencing the physical world and that these ways are (imperfectly) generalizable to other domains of experience and thought, including, of course, mathematics. A key set of conceptual schemata relate to physical proximity: e.g. *above, contact, support, inside* and *outside*. These correspond to a number of physical experiences. These and other basic conceptual schemata and metaphors link to mathematical experience through conflation. The idea is that from early development, sensory-motor and subjective experiences frequently tend to occur together, leading to the concepts being formed in conjunction.

> Conflation is part of embodied cognition. It is the simultaneous activation of two distinct areas of our brains, each concerned with distinct aspects of our experience, like the physical experience of warmth and the emotional experience of affection. In a conflation, the two kinds of experience occur inseparably. The coactivation of two or more parts of the brain generates a single complex experience . . . . It is via such conflations that neural links across domains are developed—links that often result in conceptual metaphor, in which one domain is conceptualized in terms of the other. (Lakoff et al., 2000, p. 42)

An example of this conflation is given by Lakoff and Núñez in their description of basic logic. *Modus tollens*, for example, is reputed to be metaphorical cross-mapping of "container" schemata: "Given two *Container schemas* A and B and an object Y, if A is in B and Y is outside B, then Y is outside A" (Lakoff et al., 2000, p. 44). In short, the claim is that when I use *modus tollens*, I have conflated my experience of physical containment with the particulars of the case under question. Abstract logic is derived from spatial experience.

3.2.4 A Mathematical Example

The preceding does not exhaust the conceptual schemes provided by Lakoff and Núñez, but it does provide some of the basics of their view. As with radical constructivism, much of elementary mathematics is assumed. That is, the most basic arithmetic is seen as innate, and generalizations to simple arithmetic are seen to be natural unfoldings of basic cognitive capacities for generalizations from physical experiences to abstract mathematics. Lakoff and Núñez claim that their conceptual scheme of metaphorical mapping is isomorphic with arithmetic as it is understood by mathematicians.

> The mechanism is as follows: In each conflation of innate arithmetic with a source domain, the inferences of innate arithmetic fit those of the source domain (say, object collection). Just as 3-1=2 abstractly, if you take one object from a collection of three objects, you get a collection of two objects. In other words, the inferences of abstract innate arithmetic hold when it is conceptually blended with object collection. (Lakoff et al., 2000, p. 77)

Lakoff and Núñez suggest that this implies that the arithmetic we have is fundamentally linked to our cognitive structures, because all of arithmetic is claimed to be accounted for in this model. To me, the claim implies something rather different. If what Lakoff and Núñez say about arithmetic fitting the world and fitting our metaphorical structures is true, then it is reasonably inferred that arithmetical truths correspond to facts that are not contingent upon human cognitive structures. The example of 3-1=2 seems to imply just that: the arithmetic truth corresponds to a fact about groups of objects, not only to a fact about human neurology.

A strikingly original part of Lakoff and Núñez's work is the case studies that end *Where Mathematics Comes From.* They attempt to show that their metaphorical analysis can demonstrate Euler's famous identity $e^{i\pi}+1=0$. The example is not simple; it involves a string of about a dozen metaphors and metaphoric blends (it is difficult to say precisely where one metaphor ends and another begins). In brief, the meaning of π is derived from metaphorical blending of lines and circles, with π merging from physical experiences with rotations. The number *e* is claimed to emerge from a conceptual mapping of products onto sums. That is, the concept of logarithm provides the cognitive procedure for conceiving of products as sums (i.e. log (a*b) =log (a) +log (b) and $p^q * p^r = p^{q+r}$). Lakoff and Núñez claim that this property of logarithms is sufficient to ground the concepts; other rules of logarithms are a consequence of the "multiplication as addition" transformation. The base of the natural logarithm, *e*, provides the interesting case of logarithmic transformation "that maps sums onto products and whose rate of change is exactly the same as its size" (Lakoff et al., 2000, p. 415). The imaginary unit, *i*, is conceived, like π, in terms of rotations in the plane. It is well known that if a complex number, *z*, is multiplied by *i*, the product has a magnitude equal to that of *z*, and has an angle of rotation of 90° or $\frac{\pi}{2}$ radians.

As these three ideas are combined, the meaning of Euler's famous statement is supposed to become clear. Recall that logarithms are claimed to be conceptualized in terms of mappings of products onto sums. Using analysis similar to the development of *e* in terms of its rate of change, Lakoff and Núñez show that the function $f(x) = \cos(x) + i * \sin(x)$ has a rate of change proportional to its value for any real x. From this point, Lakoff and Núñez take a fairly standard approach to showing that $\cos(x) + i * \sin(x) = e^{ix}$ given the equal derivatives and equal values when x = 0. From here, it is simple substitution to get $e^{i\pi} = -1$ and $e^{i\pi} + 1 = 0$. The point of the case study is not to provide novel mathematics, but to show how simple metaphors and their blending can provide insight into how humans are able to conceptualize the identity, and to provide grounds for believing its truth on the basis of metaphorical extension of the everyday experiences of number as motion along a path (linear or circular), rates of change, addition and multiplication. The result is something rather like a school mathematics lesson, but with emphasis on the metaphors themselves rather than their mathematical referents.

3.3 Paul Ernest and Reuben Hersh: Mathematics as Social Construction

Paul Ernest is a mathematics educator and philosopher, specializing in mathematics education who has been writing on the relevance of social construction to mathematics education since the mid 1980s. His central concern is to articulate what he sees as the cognitive and epistemological background behind mathematics learning and communication. To this end, Ernest has explored the tacit dimensions of mathematical performance and the conversational and rhetorical dimensions of mathematical communication.

Reuben Hersh is a mathematician who has turned some of his attention to philosophy of mathematics and mathematics education in the past twenty years. Hersh believes that most mathematicians cling to a Platonic view of an indubitable mathematics that is stultifying for students of the subject. What is missing, Hersh argues, is a recognition of the humanistic traditions in mathematics that place cooperation among and competition between ideas represented by individual mathematicians as central to mathematical activity. And this is crucial for Hersh: the importance of mathematics is found in its activity, not in its content.

What both Ernest and Hersh emphasize is that any account of mathematics that focuses on a deductive reconstruction to justify claims to mathematical knowledge misses the most important features of mathematical activity. Mathematical activity, on this account, involves individuals participating in a communal practice and producing work for others to judge. These judgments become part of the work in a hermeneutical context. Thus, that which is justified is precisely that which is *judged* to be justified. And as judgments are the result of contingent human practices, so must mathematical justifications be contingent also. On the social constructivist view, then, what is epistemologically relevant is not the truth of mathematical claims, but their acceptability. And mathematics education, on this view, ought to be concerned with induction into a form of life, not with the acquisition of eternal truths.

My exegesis will mainly focus on Ernest, but reference to Hersh will help clarify some key concepts. Hersh supports Ernest's ideas on mathematical philosophy, especially as they relate to education. Both label themselves as social constructivists, and both recognize that mathematical ideas are partially constrained by the logic of mathematical practice, and their emergence is highly contingent upon social events. Neither Ernest nor Hersh attempt to account for this "logic", taking it as an irreducible element of practice. Further, both see mathematical proof as merely persuasive, rather than compelling.

3.3.1 Mathematics and Historicity

While von Glasersfeld and Lakoff and Núñez both argue that mathematics is a phenomenon best understood in terms of individual thinkers, Ernest sees mathematics as embedded in partially contingent social practices. Some of what counts as mathematics today possibly could not have done so in the past, and may not be so considered in the future. Not only are the results of mathematical practice socially embedded, according to Ernest, but so are the standards of warrant and the sets of questions considered to be mathematically relevant. For example, non-Euclidean geometries have been widely accepted as mathematically meaningful and interesting only in the last century and a half. While the techniques for working with non-Euclidean geometries have been available for much longer, it wasn't until the community of mathematicians more widely took interest that such work was seen to be important. It is conceivable, on this view, that non-Euclidean geometries could have remained undeveloped for all of human history. Or, they might have emerged much sooner, and in a different context than they did. These possibilities are a consequence of human interactions, not of the fundamental structure of mathematics.

Central to this conception of mathematics is communication, what Ernest calls *conversation*. Conversation, in this context, refers to a language activity that is recognizably mathematical. This ranges from a casual discussion of sums to the more elaborately played-out dialectical movements between mathematicians. It is this Hegelian turn that makes history significant in the social constructivist view. Ernest argues that mathematical ideas are the product of an ongoing dialectic between mathematical conversants. This is not only the case at the level of the professional mathematician, Ernest claims, it is the case in all expressions of mathematical ideas. If a student proposes a solution to a teacher, its acceptability is negotiated through the norms of the particular educational setting. The student's understanding of mathematics is shaped by these interchanges, with the student offering further ideas, and the teacher and the student's peers offering critique. Thus, the student learns mathematics not by moving toward a Platonic ideal, but through negotiating the norms and practices of the learning environment.

3.3.2 Wittgenstein and Lakatos

Ernest acknowledges a debt to the works of Wittgenstein and Lakatos as a basis for his social constructivist thesis. From Wittgenstein, Ernest takes the notion that mathematics is a practice comprehensible only from within a particular, contingent form of life. From Lakatos, he takes the notion that most mathematical exposition hides the dialectical process of mathematical discovery and proof negotiation. I will briefly consider both of these positions in turn.

Both Hersh and Ernest see Wittgenstein as significant in the development of a social view of mathematics. Wittgenstein famously argued that much of what appears to be human knowledge is irreducibly part of shared forms of life. One does not invent or discover, say, addition; rather one participates in a community in which it is done. Some philosophers have suggested that Wittgenstein is merely a conventionalist: mathematics is simply *what we do*; someone else might do it differently. Ernest denies this as an acceptable interpretation of Wittgenstein, while Hersh sees it as a correct interpretation of Wittgenstein's words, but as simply mathematically untenable. For Hersh, what is important in Wittgenstein's observations is that mathematics is a fundamentally human, social activity. It is *what we do*, but it is also grounded in shared experience (Hersh, 1997). Ernest claims that Wittgenstein saw the force of the sort of objection that Hersh brings forward, and that he recognized that although mathematics is embedded in language games and forms of life precisely because it is a shared activity, it is not arbitrary. For Ernest, forms of life do have internal logics, even if they are informal. In the end, Hersh and Ernest agree that mathematical activity demands a commitment to some organizing mathematical principles within a community of practice.

From Lakatos, Ernest and Hersh take the idea that mathematics is fallible, precisely because it is not an abstraction of a human-independent reality, but is a system of human convention. Or to put it differently, it is an abstraction of reality, but mathematical reality is a purely human invention: mathematics is seen to be created, not discovered. Lakatos's brilliant *Proofs and Refutations* is a rational reconstruction of the historical development of the proof of the Descartes-Euler conjecture that any polyhedron with F faces, E edges and V vertices satisfies the condition that $V-E+F=2$ (Lakatos, 1976). What Lakatos shows is that the progress of all the relevant mathematical ideas to the conjecture are developed through a historical dialectical process of (putative) proof followed by refutations and the subsequent reformulation of the definitions, scope and/or structure of the proof. The process iterates through several generations of putative proof. What this shows, Ernest claims, is that mathematical concepts, their definitions and results go through a process of major change, and that this change is a consequence not of discoveries more closely approximating a Platonic ideal, but of negotiated justifications, contingent upon human interactions. Ernest attempts to shift attention away from questions of what constitutes the truth in mathematics, toward the question of how mathematical knowledge is negotiated.

> In summary, the social constructivist thesis is that objective knowledge of mathematics exists in and through the social world of human action, interactions and rules, supported by individuals' subjective knowledge of mathematics (and language and social life), which need constant re-creation. Thus, subjective knowledge recreates objective knowledge, without the latter being reducible to the former. (Ernest, 1991, p. 83)

Ultimately, though, many mathematical results are stable. The Euler-Descartes conjecture now enjoys a topological reformulation that was not available to Euler. It would be absurd to suggest that Euler foresaw twentieth-century developments in topology. What is important, says Ernest, is that Euler's intuition was shared with a community of mathematicians, and that for well over a century, his intuition provided the raw materials—including motivation—for a number of significant mathematical developments. The contribution to modern mathematics Euler made with his conjecture is not explicable in terms of an ahistorical truth of the conjecture. But a historical understanding of the human processes of negotiation within a shared form of life does provide, on this view, a means for understanding the significance of the conjecture, and for appreciating its contribution to mathematical understanding within the community.

3.3.3 Conversation and Rhetoric

Ernest sees conversation as relevant to mathematical knowledge in three ways. First, conversation is intersubjective communication. Much of the language use of the mathematics classroom, from teacher-centered instruction to questions-and-answers to test-writing and grading is of this type. Second, we have cultural conversation. This is the movement of ideas across wide expanses of time through oral or written tradition. In mathematics, this is usually written. Third, Ernest sees the value of the internalized conversation of single individuals. These three linguistic contexts of mathematics form the basis of mathematical forms of life. And as outlined above, Ernest sees forms of life as ontologically and epistemologically primitive in the acquisition and justification of mathematical knowledge (Ernest, 1994b).

The written tradition of mathematics, at least in the West, is clearly monological, with Euclid providing the most emulated model of mathematical exposition. In Euclid and his successors in all branches of mathematics, definitions, axioms, and rules of inference are established to provide a rhetorical device by which the conclusions of mathematical argumentation appear to be natural and inevitable. But, as Lakatos points out, the definitions and axioms are often convoluted and "unnatural" to such an extent that the mathematical initiate can have no hope of seeing how they came about (Lakatos, 1976). They came about, argues Ernest, through the dialectic of proofs and refutations first articulated by Lakatos, and further generalized by Ernest. Ernest sees the project of mathematics education as improved through being informed by the dialectical form of a "Generalized Logic of Mathematical Discovery." This dialectic involves a number of individuals engaged in a cycle of thesis (proposal of a mathematical idea), antithesis (an evaluative response to the proposal), synthesis (reevaluation and modification of the proposal) until the proposal is either rejected outright, or takes a stable form which is seen to be acceptable, though fallible.

3.3.4 Objectivity and Subjectivity

Ernest acknowledges that mathematical understanding has both subjective and objective components. Subjectively, each individual thinker has concepts that are part of that person's understanding of the world. In addition, there is an intersubjective component to mathematical understanding. There is substantial agreement on much of the content of mathematics, and it appears to be the case that mathematical objects, regardless of their genesis, are available for inspection *as objects*. You and I have concepts of, say, the Pythagorean Theorem that are unlikely to be identical. Nonetheless, argues Ernest, the referent of our concepts—the theorem itself—is an object whose existence transcends your or my concepts of it. If I never came to know it, the Pythagorean Theorem would nonetheless exist.

What is crucial to Ernest's view is that he sees mathematical discourse as objective. He is not concerned with mathematical objects, but with human conversations about mathematical objects.

> The claim of social constructivism is that all explicit objective mathematical knowledge, that which was traditionally termed mathematical *knowledge*, depends on linguistic utterance and symbolic representation, which originates with and is rooted in human conversation. In particular, at the center of the social constructivist philosophy of mathematics is located a dialogical social process for warranting objective mathematical knowledge based on Lakatos's logic of mathematical discovery. . . . This utilizes a dialectical form or logic, alternating between a thesis and its criticism—that is, between contradictory voices—to arrive at a higher synthesis. (Ernest, 1998, p. 149)

The movement in mathematical dialectic is one that begins with subjective understanding of some mathematical phenomenon, whose existence, once declared, is objective. An articulation of this understanding becomes an object within the intersubjective realm of mathematical discourse. Through responses, queries, clarifications, demarcation of domains of applicability, etc. the warrant of the claim becomes clearer and more precise, and the claim (usually in some modified form) gains a level of social acceptability. This social acceptability is the warrant of the claim, and it is in this way that mathematical knowledge is seen to be socially constructed.

This illustrates the key idea of social constructivism within mathematics education. The claim is not that all mathematical concepts are socially constructed, nor is it that the results of mathematics are contingent upon social practices. What is central to Ernest's position is that the *warrant* of mathematical claims is socially constituted, and that the *acceptability* of mathematical propositions is socially contingent. Further, it is through the social processes of warranting that mathematical ideas become refined and improved. Euclidean rhetoric of axioms, rules of inference and definitions leading inexorably toward theorems is not simply misleading; it is blatantly false as an account of mathematical practice. Non-trivial theorems are not the natural consequence of naturally intuited principles: they are the result of considerable negotiation across space and time, by many subjects. Through the intersubjective dialectic of proposed proofs and refutations, natural-seeming principles become not-so-obvious, but workable ones. And theorems simply do not emerge from the field of discourse, but help to shape the sorts of considerations that can become possible. Mathematical practice is largely responsible for the shape of mathematical content.

3.3.5 Education

Hersh claims two fundamental connections between his philosophy of mathematical practice and education. First he claims that the fact that education in mathematics is possible tells us something about the nature of mathematics.

> The teaching of mathematics *should* affect the philosophy of mathematics, in the sense that mathematics can be taught. A philosophy that obscures the teachability of mathematics is unacceptable. Platonists and formalists ignore this question. If mathematical objects were an other-worldly, nonhuman reality (Platonism), or symbols and formulas whose meaning is irrelevant (formalism), it would be a mystery how we can teach or learn it. Its teachability is at the heart of the humanist conception of mathematics. (Hersh, 1997, pp. 237-238)

The claim seems to be that if mathematical objects are socially constructed, then they can plausibly be passed on through human practices, but if they are not, then the possibility of education seems remote. Since education clearly is possible, Hersh claims, the plausibility of social constructivism is supported.

Hersh's second key educational comment suggests that the attitude of the teacher towards mathematics is important to the teaching of the subject. Further, he claims that humanist mathematics, being historical, links teachers and students to the discipline:

> Platonism can justify a student's certainty that it's impossible for her/him to understand mathematics. Platonism can justify the belief that some people can't learn math. Elitism in education and Platonism in philosophy naturally fit together. Humanist philosophy, on the other hand, links mathematics with people, with society, with history. It can't do damage the

> way formalism and Platonism can. It could even do good. It could narrow the gap between pupil and subject matter. (Hersh, 1997, p. 238)

Hersh's claim is that humanistically conceived mathematics is less inherently elitist than its alternatives. Hersh is careful to note that the educationally desirable features of constructivism do not provide a warrant for the correctness of the doctrine, but he nonetheless champions the putative benefit of inclusiveness, suggesting that "it's not unexpected that a philosophy epistemologically superior is educationally superior" (Hersh, 1997, p. 238).

Ernest's views on teaching mathematics have been influenced by Hersh's. Ernest claims that all teaching and learning rest upon implicit philosophy. Ernest claims that what a teacher or student believes about the ultimate nature of mathematical objects informs methods of instruction and attempts to come to grips with new or difficult concepts. This gives his philosophy of mathematics an important social agenda. Ernest wishes to convince educators that mathematics is a humanistic, socially constructed endeavor. This he claims, will battle the elitism that Hersh joins him in condemning.

3.4 Brent Davis: Complexity as a Meta-Discourse in Mathematics Education

Brent Davis's research in mathematics education is informed by complexity science, and forms part of a larger project to theorize diversity in learning. Davis argues for a socialized, constructivist emergentism. Emergentism is the view that when systems such as brains attain a certain level of organization, new properties such as consciousness come into existence. These new properties, on this account, are not explainable in terms of the physical or biological phenomena from which they emerge. Traditional emergentism, a philosophy of mind popular in the first half of the twentieth century, consists of three doctrines: ontological physicalism, property emergence, and irreducibility of emergents (Kim, 1996). Davis rejects ontological physicalism in favor of ontological constructivism: he consistently eschews discussion of physical reality in favor of discourses. Property emergence is expanded to include discursive emergence. On this view, phenomena such as consciousness are not "things" with an independent existence; rather they are properties that emerge from complex systems such as brains and discursive communities. Further, no theoretical reduction is possible beneath the level of the emergents. The correlation of consciousness with brain activity, for example, could not be predicted based upon the constituent elements of brains; it is only through observations of brains and consciousness that we are able to see any relationship between the two.

Davis wishes to apply the same sort of analysis to learners, learning environments and epistemic communities that the traditional emergentist applies to brains. For Davis, structures such as mathematics classrooms are complex in the same way that "learning systems" such as brains are, continually producing phenomena that transcend the constituent elements understood singly. He relies upon fractal imagery to suggest that the complexity of groups of learners bear structural similarities to the complexity of individual learners. This nesting of self-similar structures, he claims, goes far beyond the traditional objects of educational theory.

> With regard to objects of study, contemporary nonrepresentational theories of knowing and knowledge tend to be focused on particular phenomena, as opposed to the broad category of phenomena that are addressed under the umbrella of complexity. . . . Complexity science, on

> the other hand, is concerned with a range of nested learning systems, which includes the co-implicated processes of individual sense-making and collective knowledge-generation (among other layers of activity that extend at least from the subcellular to the planetary). (Davis et al., 2003, p. 142)

Systems are nested, Davis argues, in that human discourse is but one level of complexity within a much larger structure: bodily systems, brains, and cultures are complex structures with similarities to one another.

Further, Davis claims that the emergents of complexity are irreducible: no analysis of individual children can account for the emergence of communal mathematical understanding and creation. Only through a holistic appraisal of the community, says Davis, can one understand the content and meaning of the emergent mathematics. Emergent properties can be meaningfully contrasted with resultant properties. Resultant properties are simply additive: my shoes are the predictable sum of the bits of leather and thread that make them up. Even the property of "shoeness" is not considered novel enough to be emergent, even though the constituents of my shoes lacked it until they were fully assembled.

Davis asserts that complexity science is not an epistemology or theory of learning *per se*, but it is a “meta-discourse" that assists in understanding the partial results other theories can provide.

> The main attraction of complexity science is that it provides a means of reading across the concerns and contributions of radical, social, and critical constructivist discourses. At the same time it speaks to the

> multileveled, deliberate, and practical concerns of formal education. In particular, it prompts us to suggest that, in terms of the range of complex forms, the teacher's main attentions should perhaps be focused on the establishment of a classroom collective—that is, one ensuring that conditions are met for the possibility of a mathematical community. (Davis et al., 2003, pp. 163-164)

Further, Davis maintains that the generality of complexity science is reflected only in its applicability to complex phenomena, not in terms of any broad-based definite conclusions or predictions. Thus, complexity science makes the general claim that each classroom is different, but that the details of the differences are unknowable in general. This is not to say that *anything* might happen; the world imposes constraints on the possibilities inherent in any situation. The range of possibilities is nonetheless very large and is fundamentally unpredictable. A reasonable analogy would be current understandings of the weather, a complex phenomenon. We can predict with a great deal of certainty that in Canada, the summer will be warmer than the winter. And we can usually predict what the weather will be like five minutes from now. But if we scale our attention to, say, the daytime high temperature 54 days from now, we are unlikely to be able to make predictions with any great accuracy. According to Davis, the mathematics teacher faces the same predicament: we know roughly where our class's mathematical understandings are going, but we neither know exactly how we will get there, nor do we know the precise shape of the future understandings of the students.

Ultimately, Davis claims that there are some general principles of complex systems that can guide the teacher to practices that might allow the classroom to become an enhanced environment that facilitates greater learning and understanding than it might otherwise do.

3.4.1 Physical versus Biological Metaphors

Central to Davis's work is a biological interpretation of constructivism. If knowledge is constructed, argues Davis, the resultant structures are more like organisms than architecture. A physical structure is built with a plan and purpose in mind; knowledge is not like that. Knowledge emerges from activities in subtly unpredictable ways.

> Used in such phrases as "the structure of an organism" and "the structure of an ecosystem," the word points to the complex history of organic forms. Structure in such cases is both caused and accidental, both familiar and unique, both dependent and autonomous. (Towers & Davis, 2002, p. 316)

Davis and his colleagues use the notion of biological structure to describe educational behaviors in the classroom, socialized learning processes of informal groups, and the structure of the discipline of mathematics. Like von Glasersfeld, Davis sees knowledge as a temporary stability in belief that fits with experience, and is seen as adequate in some way. Unlike von Glasersfeld, Davis claims that while an individual can have knowledge, it is also possibly an emergent phenomenon of a social group. Thus, when a student claims to have provided a proof of a theorem, on this view, it is conceivable that the knowledge of the proof is not to be found solely in the individual student's mind, but in the totality of the mathematical discourse of the group. In complex environments, objects such as proofs may reside only partially in any student's repertoire, and the partial contributions of several individuals may be required for the larger result to become possible. In such cases, the proof is only comprehensible in terms of the historical processes that facilitated its emergence.

3.4.2 Complicated and Complex

Much of Davis's analysis depends on the distinction he draws between the *complicated* and the *complex.* On this view, deterministic systems that are both theoretically and practically predictable are said to be complicated. Thus, clockwork is complicated, because its behavior is thoroughly predictable as the sum of its parts. Brains, weather systems, and classrooms, argues Davis, are not to be understood merely as the sums of their constituent parts.

> These are *complex* systems. They exceed their components. They are more spontaneous, unpredictable and volatile—that is, alive—than complicated systems. Unlike complicated (mechanical) systems, which are constructed with particular purposes in mind, complex systems are self-organizing, self-maintaining, dynamic, and adaptive. (Davis et al., 2000, p. 58)

Davis maintains that classrooms are adaptive, complex systems. It is common for teachers to note that no two classes are identical and what is meaningful and exciting in one group can be fruitless and dull in another. Further, the sorts of properties that emerge in learning environments, including norms and cognitive achievements, appear to be the unpredictable result of the interactions of the members of the group. Davis and Sumara describe occasions of emergence within mathematics classrooms.

> It was not unusual for the activity in the classroom to take completely unanticipated but (in terms of the subject matter) appropriate turns.

> Insights would "spread" through the room. Conversations would begin with a few and grow to include most or all. Understanding would emerge that appeared to be shared and constructed by many. (Davis et al., 1997)

On this view, classes of students, weather systems, ant colonies and other complex systems are literally alive: they spontaneously adapt to their surroundings and literally learn from experience.

3.4.3 Emergence

So what does it mean for properties to emerge from complex systems? Davis is hard to pin down on this point. He makes general statements such as "a complex phenomenon is emergent, meaning that it is composed of and arises in the co-implicated activities of individual agents" (Davis et al., 2003, p. 138). I can find no definition of the term 'co-implicated' but it seems to refer to an incomplete causal contribution that is bound to two or more interacting bodies or agents.

Davis sees mathematics classrooms as potentially but not necessarily complex. It is possible, on his view, for a dedicated teacher to remove most of the complexity-supporting elements from a classroom. A room that is kept silent, except for the teacher's talking, that encourages only individualized working on closed problems might succeed in being complicated without being complex. That is, the mathematical activity of the room as a whole would be no more than the sum of the mathematical activities of the individuals in the seats. One significant difference in the two possibilities is to be found in the establishment of teacher authority. The authoritative teacher (to risk caricature) is one who strives to eliminate the unpredictable elements of classroom learning, one who establishes what is to be learned, then controls the environment such that everyone gets precisely that. While broad and deep learning is possible in such an environment, Davis sees it as less likely than in a complex environment.

The complex classroom is one where the teacher does not control the outcomes of the students' explorations, where communication of ideas is a welcome and integral part of learning, and where the minimal constraints of rationality, mathematical practice and reciprocally respectful behavior circumscribe activity. In such environments, Davis claims, the mathematical experience of the complex collective is likely to exceed what would happen in the merely complicated classroom.

> Emergent events cannot be caused, but they might be occasioned. A shift in interpretive focus is implicit here, away from what *must* or *should* happen toward what *might* or *could* happen. Pragmatically speaking, decisions around planning are more about setting boundaries and conditions for activity than about predetermining outcomes and means—proscription rather than prescription. (Davis et al., 2003, p. 146)

This amounts to both a practical pedagogical as well as a moral and political prescription. Davis's work is strongly influenced by his vision of the political significance of diversity in educational environments; I explore this further in section 3.4.6 and in chapter 5. While his writings often claim disinterested neutrality on issues of superior pedagogy, his examples consistently support the facilitation of a complex learning environment, both on grounds of increased mathematical output of such groups, but also on grounds of equality and recognition of difference. Behind the discourses of schooling, and the meta-discourse of complexity, lies a Foucauldian recognition of the power that is complicit in every level of his analysis.

3.4.4 Interobjectivity

Davis claims that the notion of intersubjectivity is not helpful in understanding the world. He defines intersubjectivity is "the notion that all human knowledge is a matter of social accord—usually tacit and always enabled and constrained by the associations that are already established within language" (Davis, 2004, p. 203). Of course, in standard philosophical discourse, intersubjectivity is much less than this. As I argue in chapter 7, intersubjectivity can be considered only as the coming to a mutual understanding. This understanding is between persons and is about some state of affairs in the world, including abstract states of affairs. It is clear that Davis is not working with a standard conception of intersubjectivity. In contrast to his view of intersubjectivity, Davis claims that agents are complicit in shaping those things they attempt to describe, becoming themselves part of the object of their description.

> An important principle here is that descriptions of the universe are actually part of the universe—and, hence, the universe changes as descriptions of the universe change. It is for this reason that these attitudes toward the generation of knowledge are known as participatory epistemologies. Knowledge, in this frame, is understood to inhere in interactions—that is, to be embodied or enacted in the ever-unfolding choreography of action within the universe. In other words, knowledge isn't out there. What we know is acted out in what we do, and what we do contributes to the unfolding of the cosmos. (Davis, 2004, p. 101)

On the interobjectivist view, when two people come to mutual understanding of some state of affairs, they are complicit in that state of affairs in some way. That is, our coming to know something affects that thing. Davis might even reject the above formulation because he does not accept that people can understand states of affairs as things separate from themselves. Agents partially create the states of affairs they describe and in this creative act become part of the state of affairs that they describe.

Davis's thesis that knowledge and knowers are part of the world is trivially true. That the universe changes as a consequence of human knowledge seems also to be trivial and true. As I learn I interact with my environment, I (presumably) undergo physiological changes and so forth. What is not clear is how this could matter to mathematics education. Suppose that it is true that my thinking of the fundamental theorem of arithmetic somehow changes the universe. Now what? It remains the case that every natural number greater than 1 can be uniquely expressed as a product of primes. If I forget the theorem, or mistakenly believe that it is false, what significant change has been made to mathematics? For that matter, what significant change has been made to my or anyone else's attempt to learn or apply the theorem? I cannot see what evidence one could provide to show my complicity in the Fundamental Theorem of Arithmetic.

3.4.5 Knowledge

Davis writes of knowledge from the enactivist perspective that he more actively championed in the early 1990s. He rejects depictions of knowledge that place it as the possession of individual knowers.

> What happens if we reject the pervasive knowledge-as-object . . . metaphor and adopt, instead, an understanding of knowledge-as-action—or, better yet, knowledge-as-(inter)action? Or, to frame it differently, what if we were to reject the *self*-evident axiom that cognition is located *within*

> cognitive agents who are cast as isolated from one another and distinct from the world, and insist instead that all cognition exists in the *interstices* of a complex ecology of organismic relationality? (Davis et al., 1997, p. 4)

Davis does not distinguish knowledge from understanding. This is not surprising because he sees knowledge as an emergent phenomenon, contingent on shared activities within a community. Understanding, on this view, is an activity wherein agents participate in a verbal community bound by disciplinary constraints. Thus, when a mathematics classroom engages in a discussion of the possibility of, say, calculating probabilities of certain events, understanding may emerge through this discussion. Further, Davis, argues, understanding cannot be adequately construed as the possession of any individual in the room, but it must be seen as contextual and participatory. Knowledge of the relative probabilities of a straight flush and a full house in poker emerges from the work that is done; it does not simply reside in one person's thoughts independently of the activities and social constraints and freedoms that brought the understanding into existence. Or so Davis claims.

3.4.6 Authority

Davis's work takes on a political tone when the topic of legitimate authority arises. As we saw, for von Glasersfeld or Lakoff and Núñez, the primary authority in mathematical understanding is the sense of certainty that an agent has upon understanding a mathematical proposition. They would agree with Ernest that communication within a mathematical community provides further warrant to their beliefs, but nonetheless see the experience of the individual as primary. For Ernest, mathematics is governed by a community of experts bound by norms of evidence and argumentation. Davis is not prepared to accept either form of epistemic regulation, deferring instead to the norms that emerge in any particular mathematical community. "Within a complex system, appropriate action can only be conditioned by external authorities, not imposed. The system itself 'decides' what is and is not acceptable" (Davis et al., 2003, p. 152).

This is not to say that any mathematical community can make up any rules it wishes:

> It is evident that control or authority might be redistributed—without getting caught in the trap of relativism. That is, the distribution or sharing of authority across the classroom community need not be a matter of "anything goes." On the contrary, with the emergence of any complex collective, standards of acceptable response and acceptable activity—of rightness and wrongness—inevitably arise. (Davis et al., 2003, p. 154)

Like von Glasersfeld, Davis rejects relativism, but does not make clear how he avoids it. He seems to assume that mathematically complex systems will self-organize in ways that are recognizably mathematical and will not develop wild norms of mathematical performance or warrant. The two examples he and Simmt bring forth (Davis et al., 2003) suggest that some sort of regulation is necessary. One example is the mathematics classroom with a teacher. Davis and Simmt suggest that the teacher has a regulative role to play in constraining the normative possibilities of the classroom's complex development. Davis and Simmt's second example is of a group of teachers who came together to learn mathematics at the demand of their school board. Here the mathematical norms are at least partially provided by the professor of the class in which they were registered. In both cases, the authorization of the norms of the group may be somewhat distributed, but they do not seem to be exclusively so, despite Davis and Simmt's apparent claim to the contrary. Davis is prepared to accept external regulation of mathematical acceptability, but he gives no clear idea of which regulations are appropriate or why they are so.

In the end, Davis wishes to promote the fostering of complexity in classroom learning environments. This is motivated partially by his notion of diversity in agency, and in his socialized view of knowing and learning.

> In terms of schooling, the notion of decentralized control should be interpreted neither as a condemnation of the teacher-centered classroom nor an endorsement of the student-centered classroom. Rather, it compels us to question an assumption that underlies both teacher-centered and learner-centered arguments—namely, that the locus of learning is the individual. (Davis et al., 2003, p. 152)

It is easy to see that Davis's application of complexity science has a political agenda; it is a challenge to authority in mathematics. Davis places learners and teachers on equal footing, and challenges established mathematicians as arbiters of mathematical content and practice. Even were I to accept this political agenda as desirable, it is not clear that the agenda's attachment to Davis's theory is necessary. I will look further into Davis's commitments in section 5.5.

Davis's work raises two large questions for me. First, what is the evidence for believing that mathematics learning is reasonably described as complex? Davis, so far as I can tell, provides no evidence for so believing. Second, even if I accept that mathematics learning can be understood this way, why should I be compelled to question the location of learning?

3.4.7 Dubious Distinctions

Because Davis's work relies on results from each of the other three theories, much of my critique of them will implicitly criticize Davis's theory. Still, Davis makes use of a number of idiosyncratic concepts and formulations that warrant some discussion.

First, in his general discussions of complexity science, Davis does not distinguish the objects of his inquiry—complex systems—from the theoretical tools used to analyze them—complexity science. Consider Davis's belief that classrooms are fundamentally unpredictable. It is true that many mathematically nonlinear system models are highly sensitive to variations in initial conditions. If we compare numerical predictions in some models in which small changes are made to the input conditions, it is often the case that the predictions are wildly divergent. Weather models are famous for having this feature. Suppose a nonlinear weather model predicted light showers with one set of inputs, but a very small change in a single parameter led the model to predict a tornado. This difference in outputs is worthy of further investigation. One does not wish simply to assert that because the model might predict light showers or a tornado with equal probability, unless one is certain that that is the best prediction that can be made. What is needed is empirical investigation. It could be that the line between the occurrence of light showers and tornadoes is very thin, or it could be that the model fails to fit the phenomena over certain domains. One cannot simply assume that because a model gives unexpected results that the object it models will cooperate and behave the same way; the valid domain of the model must be established. Davis does make such assumptions. It remains an empirical question whether classrooms exhibit the phenomena that Davis claims they do. Making matters even fuzzier, Davis does not provide any explicit models that can be tested. He asserts that complexity science can model learning, but he never provides any testable models.

Suppose we grant that models are forthcoming, and that Davis is still engaged in preliminary exposition of his ideas. Even at the general level that he presents his work, there are still very difficult distinctions to be made. Consider the complicated vs. complex opposition. Complicated systems, according to Davis, do not exceed the sum of their parts. They are, as Davis repeatedly points out, like clockwork. What sense are we to make of this? A clock, when fully assembled does exceed the sum of its parts. A clock bears information about time, and a box of clock parts does not. Perhaps, Davis meant something more than this. In section 3.4.2, I quoted Davis as claiming that complicated systems are designed for a purpose, whereas complex systems are adaptive. It is hard to see what this distinction could mean. I own a watch that self-corrects at regular intervals to keep its time reasonably close to a satellite time signal. This watch appears to be designed for a particular purpose, while simultaneously being adaptive. It is easy to think of human-built machines that possess an even higher degree of adaptation than does my watch. I do not see how this distinction can be made more precise than Davis has made it, and I do not see that it has any educational significance.

Complexity science typically deals with numerical models of phenomena. These models are testable, at least in part. The usefulness of a particular complex model is contingent on its predictive and its explanatory power. Nothing in Davis's work has predictive power. I do not see how the adoption of any of his suggestions could lead a researcher or teacher to predict any significant educational outcome. Even if we take Davis's claim that outcomes have an inherent element of unpredictability, it should still be possible to see a range of likely outcomes emerge from a useful model. Davis puts all his emphasis on explanation, rather than prediction. For the explanations to have much credibility, they require greater warrant than assertion.

4 Mathematics Education: Some Commitments

The theories of mathematics education that I have presented differ not only in the details of what they claim, but also in the range of issues that they declare to be of interest. Below, I provide an account of what each of the four theorists places on the table as desirable in a theory of mathematics education, and contrast their views (I will count Ernest and Hersh as one.) In so doing, I will highlight features that I claim to be significant to theories of mathematics education.

4.1 An Account of the Nature of Mathematics

All four agree that mathematics is to be at least partially described as a human activity. One of the reasons that mathematics is significant to educators is that it is an activity in addition to being a collection of results. Nonetheless, mathematics has a content that requires explanation. Why do we hold some results to be true (or at least justified) and not others? Why is "if z is a root of a polynomial, then so is its complex conjugate" a mathematical statement? And why is "if Jane is coming to the party, then so is Joe" not a mathematical statement, regardless of its truth value? It would be desirable for a theory of mathematics education to explain this distinction; at the very least it must be consistent with it.

All four theorists emphasize the first, humanistic, concern of mathematics. That is, they try to account for what it is that humans do when they are doing mathematics. Von Glasersfeld sees mathematics not as "an empirical abstraction from sensory-motor experience, but a reflective abstraction from the experiencer's own mental operations" (Glasersfeld, 1994, p. 6). My concept of, say, the number four is seen by von Glasersfeld as an abstraction from my thinking, not from my experiencing. He sees mathematics as abstractions from and operations upon concepts, and nothing more. Each individual mathematical thinker is operating solely upon personal conceptions of mathematics, and nothing else. If I were to prove the Pythagorean Theorem, on this view, what I would be doing is constructing my own conception of the content of the theorem, and constructing my own concepts of the steps of the proof, and I would have no possibility of making reference to anything beyond my own concepts. On von Glasersfeld's view, all mathematics is personal; my concept of the Pythagorean Theorem is all I have to go on, for the theorem has no referent beyond my concepts. This leaves unanswerable questions of why mathematics ought to be done one way and not another. If I cannot reconcile my beliefs about the sum of the squares on the sides of planar right triangles with my experience, it is not clear that it is my mathematical belief that ought to be changed; I might just as well change my interpretations of my experiences. Perhaps I simply need to drop the expectation that I can generalize at all.

For Lakoff and Núñez, the nature of mathematical content is explained by the nature of mathematical activity. Humans perform mathematics because they are neurologically predisposed to do so and they have cultural forms that support its expansion. The basic elements of mathematics—subitizing, metaphoric blending, etc.—are always ready to be used by a human agent. Higher mathematics can be acquired through further applications of a few basic capacities. Those statements and practices that we call mathematical are those that are direct consequences of the cognitive processes that lead to mathematics. This only appears to be circular, but is not because the content and practice of mathematics are claimed to be inseparable and interdependent. Human mathematics is the only mathematics of which we can conceive. Once we observe practice, we observe content, and conversely.

Ernest sees mathematics as recognizable from within a mathematical form of life. Those who are doing (non-trivial) mathematics know that that is what they are doing. They see themselves and others as performing mathematics by force of their shared activities. On this view, the question, "who is doing mathematics?" is similar to the question, "who is speaking English?" To answer the second, we look for people who belong to the group that sees itself as speaking English and compare clear cases with less clear ones. So it is with mathematics. We look at people who are unambiguously doing mathematics (say, mathematicians) as our clear cases and compare others with this group to see how well they match up. As for identifying the subject matter of mathematics, we can see that, on this view, mathematics is whatever mathematicians are doing when they say they are doing mathematics, and other mathematicians agree with the assessment. The question "is this mathematics?" is answerable by appeal to the mathematical community; the judgment as to what is mathematics may change by consensus. The question of why one ought to follow the judgment of the community simply is not addressed.

Davis assumes that the mathematical can be distinguished from the non-mathematical in a way that is outside the domain of his theory. That is, he expects the teacher to circumscribe the activities of the classroom in such a way that the activities of the students remain related to mathematics as it is known by the teacher. He makes this point through the invocation of the notion of *liberating constraints*. The central idea of liberating constraints is that if a student's work is to be meaningfully mathematical, it cannot be focused on or by just anything. One principal task of the teacher is to focus student work in a way that on the one hand allows for freedom and flexibility, and on the other, prevents work from going horribly astray from the intended subject matter at hand

(Davis et al., 2000, ch. 2C). For example, an investigation into quadratic equations might become an investigation into the elliptical orbits of planets. The teacher must ensure that the movement from an investigation into elliptical orbits does not stray too far from the original object of study. It is easy to imagine students' interests moving into numerous non-mathematical areas from there. A reasonable analogy might be with a sporting contest. The referee's job is not to dictate the outcome of the game, but rather to ensure that the game is played properly. Similarly, the teacher's job is not (necessarily) to dictate the outcome of student learning, but to ensure that relevant learning takes place.

By this, Davis is implying that there is a correct, or perhaps an appropriate, range of possibilities that the teacher ought to circumscribe for a given lesson. The details of the learning are indeterminate, but the circumscription of mathematical boundaries by the teacher is clearly informed by some knowledge that is outside of the learning space itself.

Davis is not attempting to theorize the nature of this background mathematical knowledge. Presumably, it forms one of the discourses that his complexity science forms a meta-discourse over. If, however, it turns out that the worldwide community of mathematicians is a complex system, then the content of mathematics emerges from the activities of this group, and it is fundamentally unpredictable. Mathematics, for Davis, is not the object of mathematical enquiry; it is an emergent property of mathematical activity. That is, the consensus as to what counts or does not count as mathematical will come from the dynamics of the group interactions and can only be seen *ex post facto*. This is similar to the social constructivist position, but there is one significant difference. Namely, for Ernest and Hersh, consensus is negotiated within the global community of mathematicians; Davis claims that significant consensus occurs at many different scales of complex community. The consensus of a handful of mathematical friends has an epistemological significance for Davis that it does not for Ernest or Hersh.

4.1.1 Attachment to Experience

Humans frequently use mathematics to describe the world in ways that are meaningful. We can count our children, estimate time of travel, calculate trajectories and the like. While it is an open question whether we have explicitly mathematical experiences, it is clear that our experiences are often comprehensible in mathematical terms. All four theories under consideration not only acknowledge this, they see it as providing educational opportunity.

For von Glasersfeld, mathematics is an abstraction from reflections. Its development and expansion is a result of conscious attempts to achieve an abstractive abatement of cognitive perturbation. It appears, on this view, to be simply a matter of fact that our mathematical abstractions nicely relate to our abstractions of non-mathematical phenomena. I find no direct reference in von Glasersfeld that gives insight into why, for example, humans are able to apply their mathematics to such phenomena as the relationship between gravitational attraction and the square of the distance between two bodies.

Lakoff and Núñez see mathematics as naturally attached to the world because it is a consequence of human experience in the world.

- The effectiveness of mathematics in the world is a tribute to evolution and to culture. Evolution has shaped our bodies and brains so that we have inherited neural capacities for the basics of number and for primitive spatial relations. Culture has made it possible for millions of astute observers of nature, through millennia of trial and error, to develop and pass on more and more sophisticated mathematical tools—tools shaped to describe what they have observed. . . .

- In the minds of those millions who have developed and sustained mathematics, conceptions of mathematics have been devised to fit the world as perceived and conceptualized. . . . The mathematization of ordinary human ideas is an ordinary human enterprise. (Lakoff et al., 2000, p. 378)

Further to this, Lakoff and Núñez argue that arithmetic appears to have an existence independent of individual thinkers because of a metaphorical conflation of the ideas of number and the experience of objects. A clear example is found in the educational practice of identifying real numbers with points on a line. Children have experience with linear motion, and are able to easily understand the idea of ordering in linear space (e.g., "point A is to the left of point W"). Through the invocation of the metaphor of linear ordering, Lakoff and Núñez argue, children are able to conceptualize the ordering of real numbers (e.g. $-\pi$ is less than -3 because it corresponds to a point to the left of the point corresponding to -3). Because we are able to treat disparate ideas with similar verbal patterns to those with which we are able to deal with physical objects, the conceptualizations we have of numbers come to be seen in ways similar to those we have of physical objects. In short, Lakoff and Núñez claim that mathematics lines up with experience precisely because non-trivial mathematical thoughts are metaphorical thoughts, abstracted from human experience with the physical world.

Neither Ernest nor Hersh give an account of why we experience the world mathematically, but both acknowledge that we do.

Reasons for mathematics attaching to experience are outside of Davis's work as well. To be sure, he is prepared to utilize this connection as a pedagogical tool, but his "meta-discourse" of complexity science makes no pretense to theorize the relationship between mathematics and experience that is not directly mathematical.

4.1.2 Simple and Advanced Mathematics

It is not clear how closely basic arithmetic, such as "add two" is conceptually or pedagogically related to more advanced arithmetic, such as "find the sum of the negative integral powers of 2", or to highly specialized mathematics such as the proof of the four-colour map theorem. In everyday language, we refer to all these results and activities as mathematics, but how can we theorize a mathematics education that recognizes such a broad range of results and activities?

For both von Glasersfeld and Lakoff and Núñez the simplest mathematics is accounted for by basic human physiology. The ability to subitize is described neurally by Lakoff and Núñez, as is the ability to work with the fundamental metaphoric structures and processes of cognition. Von Glasersfeld sees the apprehension of number as a fundamental ability to abstract from moments of attention and inattention. In spite of the technical differences in these positions, for my purposes they are equivalent. Both claim that some very elementary mathematical abilities are innate. As outlined in section 3.2.4, Lakoff and Núñez claim that complex mathematics does follow clearly and comprehensibly from basic mathematics. Their presentation suggests a fairly simple foundationalism for mathematics, but does not claim it directly.

Neither Ernest nor Davis gives attention to the most elementary of mathematical operations. Ernest takes them as a given for any member of a mathematical community.

He assumes that any participant in a mathematical dialog already has elementary mathematics in her repertoire. Davis makes similar assumptions. In both cases, their theories are not in conflict with the possibility, or even the need, for the most basic of mathematical capacities: they simply do not theorize it.

4.1.3 Stability and Change

New mathematics is reasonably under scrutiny. Putative proofs are not immediately warranted, and require considerable time and energy for them to be considered so. No one would be surprised to hear that an alleged proof is found to be lacking in some way. Nevertheless, when results are accepted, they can have an effect on mathematical practice; and they are definitely considered to be additions to the body of knowledge generally accepted by the mathematical community. It is desirable that a theory of mathematics account for this. A theory of mathematics education must also be sensitive to this in that the rules of warrant are a part of a student's induction into the mathematical community.

A large number of mathematical results have not undergone revision in a very long time. No one seriously doubts the correctness of basic numerical computation, nor do they doubt that every positive integer greater than one can be uniquely expressed as a product of primes, etc. How is such mathematical stability to be explained?

For von Glasersfeld and Lakoff and Núñez, the stability of basic results is a consequence of their conceptual nearness to the most basic of mathematical understandings and abilities. That is, a result such as the fundamental theorem of arithmetic is not logically far removed from the most basic, innate arithmetic. The line of reasoning connecting the arithmetic that we cannot possibly doubt to innate arithmetic is short enough that a human can easily follow it without missing some important steps. It is not surprising that more complex results are harder to make stable. The further we get along in our chains of reasoning, the more likely it is that we have made errors of omission or of logic.

Social constructivists simply see mathematical stability as an interesting fact that likely cannot be explained.

> The reproducibility of a mathematical calculation is comparable only to the reproducibility of a physical measurement or experiment. . . . Just as there is lawfulness and stability in certain parts of the physical world, there is lawfulness and stability in certain parts of the social-conceptual world. Why should this be so? I do not know. I suspect it's as fruitless as the same question about the physical world. (Hersh, 1994, p. 19)

I doubt that Ernest is in complete agreement here. It seems to me that Hersh has simply given up the fight at a critical moment: the stability and consistency of mathematical reason is a very significant phenomenon, even if it is one difficult to theorize adequately.

Davis's theory refers to mathematical stability in two contexts. On the one hand, he clearly advocates that the teacher constrain the possibilities for mathematical interpretations that are permissible in the classroom. This is the idea of "liberating constraints" mentioned earlier. There is no reason for Davis to assume that such constraints enjoy long-term status as warranted mathematical results; they only need to be stable long enough to gain acceptance in the professional community. The possibility that things can change is established by the nature of any complex system: "The system itself 'decides' what is and is not acceptable" (Davis et al., 2003, p. 153) . Such a view seems to disregard the importance of individual thinking in mathematics. Could a "mathematical system" decide that there is only a finite number of prime numbers? What could Davis say about the individual who resists the system to assert that there is a compelling proof to the contrary? It appears to me that Davis must insist that the system gets the last word in such disputes, and that while the individual may have a morally legitimate claim to be heard, he or she has no mathematically legitimate claim that there are an infinite number of prime numbers.

4.2 Educational Considerations

It is not surprising that the categories I have listed as having educational significance—*Mathematical Performance and Thinking*; *Mathematical Intersubjectivity*; *and Reasons to Teach, Learn and Perform Mathematics*—could also fit under the nature of mathematics itself. Certainly, Hersh would simply assert as an empirically-testable claim that teachability and learnability are properties of mathematics. I suspect that he is right on that point. Nonetheless, I believe that the division I am making is worthwhile because it will allow me to structure my response to the four theories under consideration.

4.2.1 Mathematical Performance and Mathematical Thinking

Mathematical activity can be conceived both in terms of performance and in terms of cognition. That is, one can be seen to *do* mathematics in a public way. When I perform a calculation, solve a problem, or articulate a mathematical argument, I am making a mathematical performance. I am able to do these things because I am having mathematical thoughts. Whether mathematical thinking is separable from mathematical performance cannot be prejudged, but any theory of mathematics education must be able to account for both phenomena. On the surface, it appears plausible that one can think mathematically without engaging in a public mathematical performance; but it is considerably less plausible that one could make a public mathematical performance without thinking, fraud notwithstanding. Of course, it is the business of educators to make reasonable inferences about the thinking behind students' mathematical performances.

Both von Glasersfeld and Lakoff and Núñez see mathematics as fundamentally embodied. They claim that the ability to think mathematically is contingent on the ability to perceive. The basis of elementary mathematical thinking is causally connected, on these views, to sensory-motor experience. For von Glasersfeld, by the time reflections become mathematical they are removed from the original sensory-motor experiences and have become reflections on reflections. In spite of similar grounding in experience, in the mechanisms by which mathematics is extended beyond the sensory-motor, Lakoff and Núñez and von Glasersfeld differ considerably.

Von Glasersfeld suggests that the individual agent moves from basic arithmetic to higher mathematical reasoning through individual operations. That is, once the basic concepts of unity, plurality and number are internalized, all further arithmetic comes about from the cognitive activity of the agent. By reflecting on my concepts (not my experiences; concepts are somehow abstracted from experiences), I am able to construct the meanings that are called mathematical. I can have experiences that I interpret as confirmation or rejection of these concepts, but these experiences do not directly change my concepts. The experiences provide raw materials from which I abstract and on which I reflect. These abstractions and reflections can provoke me into further thinking and possible refinement of my mathematical concepts.

Lakoff and Núñez offer very specific mechanisms that allow individual agents to move from innate mathematics (subitizing, for example) to higher mathematics. These are the processes of metaphorical mapping, including the metaphorical blending and conflation depicted in my general exegesis. Mathematics is consistent, and calculation is regular simply because the "algorithmic structure of arithmetic has been carefully put together to mirror rational processes and to be usable when those processes are disengaged" (Lakoff et al., 2000, p. 98). The argument appears to be that since our framing of mathematics is contingent on our cognitive framing of physical experience, the regularity of physical experience is mapped onto our mathematical frames. These regularities are assumed throughout their work and the possibility of their requiring justification is not raised. Further, they claim that mathematics has been constructed so as to be stable, but they give no explanation of how this is possible.

Ernest does not deal directly with questions of the mathematical performance of individual agents. He does recognize the significance of the questions and raises the desirability of an account of performance being consistent with his social constructivist view of the large structure of mathematical understanding within a community of interlocutors. See section 4.2.2 below for Ernest's stance on the significance of intersubjective agreement within mathematical communities.

Davis acknowledges that individuals can have mathematical thoughts and perform mathematical activities, but he does not consider them to be important objects of analysis in his theory of complex learning environments. As I elaborate in the next section, Davis sees complex group formation as an expansion of the abilities and understandings of individuals, as the group itself possesses mathematical knowledge and ability that transcends that of the individuals in the group.

4.2.2 Mathematical Intersubjectivity

Why is there widespread agreement on the results of mathematics? It is clear that two people sharing a language and culture can have a mathematical conversation. At least the simpler results are accessible to anyone who can understand the basic operations and follow a simple argument. There is little difficulty, for example, in teaching people how to keep score at a sporting event, or how to balance a bankbook, or how to calculate the floor area of their homes, etc. More significantly, results can be shared across culture, language and time. Results from ancient manuscripts in Egypt, China and India are comprehensible to people all over the world. These results fit neatly into existing mathematical schemata. Somehow, humans can acquire mathematical ideas through verbal, textual, symbolic, or diagrammatic interchanges with others.

For von Glasersfeld, the experiences we label as intersubjective are simply a means of equilibration. The cognitively perturbed student says, “I can't find the equation of the plane,” and the teacher replies, “The scalar product of a vector in a plane and a vector normal to the plane is zero,” and the student makes a connection, resolving her perplexity. Her understanding was facilitated by the teacher's response. My account of the student-teacher exchange is not prescriptive; it is purely descriptive. Anything the teacher said that eased the student's perturbation (“don't bother; it won't be on the test anyway”) could count as a mathematically intersubjective experience.

Lakoff and Núñez acknowledge that mathematical ideas, once conceived, are legitimate objects of human activity, including communications. That we can speak of mathematical ideas is as unsurprising as the fact that we can speak of the weather. The reason that intersubjective agreement is possible is that the basic building blocks of mathematical concepts are built into the neural machinery of all normally functioning humans. All humans are built so that they *can* conceive of the same mathematical ideas, so it becomes possible for them to actually do it, at least largely. Similarly, all humans are built so that they can use language. The ability to communicate mathematically, then, is part of what it means to be a functioning human being.

Ernest sees intersubjectivity as central to his conception of mathematics and mathematics education. On this view, it is intersubjectivity that makes mathematics meaningful. If it were not for discussion, and socially negotiated norms of warrant, there would be no non-trivial mathematics. Nonetheless, he does acknowledge the difficulty that social constructivist theories face in accounting for the mathematical experience of the individual. He explicitly rejects the possibility of a reconciliation of his view with von Glasersfeld's, and suggests the possibility of an extension of Vygotsky's constructivist theory of mind to accord with Ernest's views (Ernest, 1994a).

Davis adopts the idiosyncratic notion of interobjectivity, in which agents are not seen as independent in terms of their mathematical activities or understandings. Mathematics is not to be shared or communicated from one agent to another, on this view; mathematics emerges from social activities. What is significant, he says, is that while individuals can have their own mathematical ideas, they cannot directly exchange them. What they can do, under the right conditions, is present one another with mathematical stimulations that lead to two things. First, the stimulations might provoke thought; second the combined mathematical activity produces mathematics that is somehow greater than the sum of the individual thoughts. When learning is construed as complex, human agents appear to fade into the background, and their ideas, performances and interactions become mathematical "neighbors" in a state of continual interplay.

> The "neighbors" in mathematical communities are not physical bodies or social groupings. That is, personal and group interactions may not be as vital or useful as is commonly assumed. Rather, in mathematics, these neighbors that must "bump" against one another are ideas, hunches, queries, and other manners of representation. (Davis et al., 2003, p. 156)

The point here is that the complex mathematical community cannot be meaningfully represented as a collection of individual knowers engaged in overlapping discourse. Since the mathematical understanding of the group is greater than the sum of the understandings of the individuals, it is not found in simple intersubjectivity, but is instead found irreducibly situated in the community. Recall Davis's insistence that when systems are merely complicated, they are reducible to the sum of their parts; only in complex systems does knowledge emerge that transcends the knowledge of the individuals that comprise the group. What is important, according to Davis, is not that individual people come together, but rather that the concepts that are part of the possession of the group do. This does not appear to deny that individuals do have some of their own knowledge; it denies that this knowledge and its sharing exhausts the knowledge of the community.

4.2.3 Reasons to Teach, Learn and Perform Mathematics

Why do we care to teach, learn or perform mathematics? None of the theorists deal with this question directly, although Lakoff and Núñez and Hersh provide partial responses. They view mathematics as culturally significant and, as such, worthy of preservation.

Radical constructivism is explicitly non-normative. The only reason one has for doing or thinking *anything* is the attainment of some sort of equilibrium. Equilibrium is not a state that one is normatively called to establish; it is simply something that an organism seeks. If one suffers a perturbation that one believes can be alleviated through mathematical teaching, learning, or performing, then one has a reason to do these things. Otherwise, there is no reason to act.

Lakoff and Núñez see mathematics as an individual and cultural means of dealing with the world. To them, mathematics represents not only a useful means of understanding the world and of operating in it, but also it is a triumph of human culture, and is worthy of teaching, learning and performing to the same degree that any cultural achievement is.

> - . . . Human intelligence is multifaceted and . . . many forms of intelligence are vital to human culture. Mathematical intelligence is one of them—not greater or lesser than musical intelligence, artistic intelligence, literary intelligence, emotional and interpersonal intelligence, and so on. . . .
> - Human beings have been responsible for the creation of mathematics, and we remain responsible for maintaining and extending it. (Lakoff et al., 2000, pp. 378-379)

It is hard to say just what Lakoff and Núñez mean by a responsibility for mathematics. The context of the quotation makes it appear to be little more than a rhetorical device. If it is in fact a normative position, there is still much to be argued. It is not clear from the rest of their theory why anyone ought to feel responsible for the maintenance and extension of mathematics.

Ernest recognizes that a normative element is required to justify mathematics and mathematics education, even though it is strictly outside of his theory. Hersh takes a line similar to that taken by Lakoff and Núñez, claiming that mathematics is a human art, worthy of preservation for the same reasons that any other human art may be.

I find it difficult to find reasons for participation in mathematics in Davis's view. Since he sees the mathematics of a complex organization as having significance that transcends the mathematics of the individuals within the group, it is difficult to see why individuals would want to participate. It appears that he has appealed to an ideal of the maximization of mathematical content within a group. If this is so, it is not clear why one would want this. I do not see what incentive an individual could have to see the mathematical content of a group increased unless there were some benefit to that individual.

As I noted in section 3.4.6, Davis appeals to an ideal of inclusion in his work on complex learning environments. Still, his reasons for inclusion seem to be that diversity and redundancy within groups are necessary for complex phenomena to emerge. This raises a number of normative issues. First, why do we wish groups to possess these emergent phenomena? Second, what is the point of education for the individual in such a situation? Third, is it defensible to "use" students in this way? I revisit these questions in section 5.5.

4.3 Summary

I began by wondering what must be true about a person in order to say that the person understands a mathematical concept, proposition or justification. This I take to be a question central to mathematics education. Further, I suggest that it is a question that theories of mathematics education must be able to address. The reason is that if mathematics is to be learned through an educational process—whatever it might be—it is worthwhile to know what has been learned. If theory is to have any hope of informing practice, then theory must provide some way of testing whether learning has taken place. Second, I raised the question of what must be true about two people to say that they have come to mutual understanding of a mathematical concept, proposition or justification. This question is also crucial to education in that the point of instruction is to bring students to an understanding of some sort, and the point of assessment is to bring the teacher to an understanding of the content of the student's understanding.

Of the four theories under consideration, Lakoff and Núñez's cognitive science approach offers by far the most comprehensive account of how a person could learn mathematics. It provides an account of basic and advanced mathematics, the attachment of mathematics to physical reality, stability of results and intersubjectivity. Von Glasersfeld offers a thinner account of these phenomena, but at least acknowledges that these issues are significant. I suspect that there is a reasonable amount of inter-theoretical translation possible between the two theories.

Ernest and Hersh do not provide much in the way of an account of individual understanding of mathematics. As social constructivists they appear to be mostly interested in the phenomenon of mathematics, writ large. That is, they are theorizing why groups of people agree on certain rules and norms of mathematics, and how these norms are somewhat flexible over time. Ernest does recognize that individual mathematics does need to be accounted for, and suggests that von Glasersfeld cannot do so in a way that is consistent with social constructivism.

Davis gives an account of complexity within learning communities, without theorizing other (non-complex) communities. Davis relegates to the margins of his theory many of the questions I have raised. Intersubjectivity is rejected in favor of the more obscure notion of interobjectivity; individual thinking and stability of results are part of the assumed background of Davis's work, and do not come up as possible areas of dispute.

None of the theories provides a compelling account of mathematical intersubjectivity.

Part Three: Philosophical Problems Remain

In Part Three, the theories are further examined, first in terms of their moral and political contexts, then in terms of their epistemological and intersubjective contents.

In chapter 5, the four theories are shown to be programmatic, and are reassessed in terms of their political commitments. All are shown to explicitly oppose elitism in mathematics and to make claims supporting the beneficial social consequences of their adoption. These claims are shown to be poorly supported. Each theorist both fails to show that the adoption of their theory is justified on ethical or political grounds and that the adoption of their points of view by the educational community would have the consequences they claim. These arguments do not challenge the epistemological claims of the four theories. While chapter 5 explores the normative context of the four theories, chapters 6 and 7 examine their explicitly epistemological content.

Chapter 6 tests the four theories against the Kripke-Wittgenstein paradox. The paradox provides a vehicle for showing whether or where each theory claims that knowledge of mathematics is possible. Radical constructivism allies itself with the skeptic who claims knowledge to be impossible. Social constructivism is aligned with Kripke's socialized "skeptical solution" to the paradox. Powerful anti-skeptical responses to the paradox are provided by Charles Taylor and Thomas Nagel. Taylor grounds knowledge in action and in intersubjectivity; Nagel grounds some knowledge in fundamental thoughts. Both provide resources that go far beyond those offered by the four theories. Lakoff and Núñez's theory is seen to be partially aligned with Taylor's arguments.

In chapter 7, the desiderata for theories of mathematics education explored in chapter

2 are reexamined. The positive arguments from chapters 5 and 6 address some but not all of the issues. Habermas's theory of intersubjectivity and communicative rationality is shown to provide sufficient resources to address the remaining issues.

5 The Politics of Educational Theory

In the preceding chapters, I have attempted to reproduce fairly the main features of four theories of mathematics education. My purpose here is to look at each of the theories in terms of its ethical and political commitments. In this chapter, I show how von Glasersfeld's radical constructivism, Ernest's and Hersh's social constructivism, Lakoff and Núñez's cognitive science and Davis's ecological complexity all take political stances towards mathematics education. Each theorist claims an ethical or political, as well as an epistemological, problem with mathematics and mathematics education as practiced today. Common to all four theories is a desire for education to have a democratizing influence; there is a sense that through reconceptualizing mathematical learning and knowledge, some laudable political end has been achieved. Each offers a position that explicitly addresses an epistemological problem while simultaneously supporting a desired political change.

5.1 Politics and Theories of Mathematics Education

In this section I take my inspiration from Ian Hacking's provocative book, *The Social Construction of What?* Hacking sees social constructionism as a family of theories, each family member being as much about politics as it is about epistemology. (As I noted in chapter 3, Hacking prefers the term *social constructionism* to Ernest's, Hersh's or von Glasersfeld's *constructivism.*)

> So what are social constructions and what is social constructionism? With so many inflamed passions going the rounds, you might think that we first want a definition to clear the air. On the contrary, we first need to confront the point of social construction analyses. Don't ask for the meaning, ask

what's the point. (Hacking, 1999, p. 5)

Hacking argues that the main point of social construction analysis has been its use for raising consciousness. One argues for the construction of some phenomenon in order to call its apparent naturalness or inevitability into question. To pick a relatively straightforward example, one might reasonably claim that *genius* is socially constructed. The *idea* of genius is obviously a construction: I cannot imagine anyone disagreeing with that. What about geniuses themselves? Do we discover genius as a part of the fabric of the universe, as tokens of a natural kind, or is genius a projection of our social practices? Here we have reasonable grounds for debate. It could be that it is an observer-independent fact about humans that some have exceptional abilities that are captured in a significant way by the concept of genius. It could also be that without certain social structuring (privilege, capital, etc.) geniuses could not exist. I do not aim to resolve this question, but only to see how it is intelligible. If one takes seriously the claim that by labeling certain people as geniuses, certain forms of social privilege are preserved, one can see the point of wondering whether the labels themselves are appropriate. One might reasonably wonder if the label has what Mannheim called an "extra-theoretical function", and if unmasking this function might serve to counteract it in some way (Mannheim & Kecskemeti, 1952). The constructionist wishes to show that categorizations are not inevitable, and that they can have extra-theoretical functions, such as the preservation of privilege. In effect, constructionists typically engage in an unmasking of ideology from within categories that are currently taken for granted. Hacking contends that:

> Social construction work is critical of the status quo. Social constructionists about *X* tend to hold that:

(1) *X* need not have existed, or need not be at all as it is. *X*, or *X* as it is at present, is not determined by the nature of things; it is not inevitable.

Very often they go further and urge that:

(2) *X* is quite bad as it is.

(3) We would be much better off if *X* were done away with, or at least radically transformed. (Hacking, 1999, p. 6)

I believe that Hacking's strategy of analyzing social construction claims within their political matrices reveals some very important features of many such claims. Hacking unmasks the unmaskers. If theories of mathematics education arise at least partially as attempts to challenge a social order—and I claim that these four do just that—then it is reasonable to strive to show the political content of the alternatives they offer. Hacking's strategy does not apply only to social construction claims; it will point to some crucial features of von Glasersfeld's radical constructivism, Ernest's and Hersh's social constructivism, Lakoff and Núñez's cognitive science and Davis's complexity science.

Clearly, in all four theories, one *X* that is claimed not to be inevitable is a conception of mathematics that claims mathematical truth to be independent of human experience. Further, each sees a need for reform in the way that mathematics is taught, and each claims that his theory can assist in setting things right. The common target of all four theories is the notion that mathematics is an elite activity. Each claims that mathematics is widely perceived as a deep, dark, esoteric domain, inaccessible to most people. In their analyses, these authors attempt to show that by reconceptualizing mathematics and mathematics education, popular beliefs about the nature of mathematics may be changed, and that change will be for the better. If mathematics is demystified, goes the claim, then it can be understood, used and enjoyed by a wider public.

5.2 The Politics of Radical Constructivism

Von Glasersfeld presents radical constructivism as a statement of human freedom and dignity. He consistently contrasts his views with behaviouristic accounts of education. He argues—correctly, I think—that there is a fundamental difference between an education that trains students to produce correct responses and one that brings children to an understanding of the reasons that one ought to bring forward to produce some responses and not others. Further, he claims—correctly again, to my mind—that the latter conception of education is superior to the former. While I would champion such a position on ethical and normative-epistemological grounds, von Glasersfeld relies upon an instrumental justification.

> I am convinced that, in general, students will be more motivated to learn something, if they can see why it would be useful to know it. Most of the goals that determine the instrumental value of a piece of knowledge are not so arcane that students will not be capable of sharing them. This goes from meeting the prosaic material needs of everyday life to the generation of peace of mind on the abstract level of the individual's organization of experiential reality. (Glasersfeld, 1995, p. 177)

Of course, one might wonder whether one can hold these ideals and achieve von Glasersfeld's goals while denying radical constructivism. If one could—and I think it is clear that many people who are not radical constructivists do hold these ideals—then von

Glasersfeld's appeal seems a bit weak. The ideals do not support his epistemology, and his epistemology does not imply these ideals. He seems to assume that others will share his desire to see students enjoy the mathematical experience.

Throughout his writing, von Glasersfeld presents himself as a good-natured humanist. His interest in mathematics education seems to come from a genuine desire to share pleasant cognitive experiences with others. The joy of learning, he maintains, comes from discovering for oneself what one knows. This is the real point of radical constructivism. Von Glasersfeld sees a tedious instrumentalism in American schools—students learn only enough correct responses to pass exams and move to the next level of a program—and he wishes to see instrumentalism replaced with a more authentic curiosity and passion for learning. Ironically, he sees an instrumental approach to learning as the way ahead. Von Glasersfeld believes that if students see knowledge as theirs and theirs alone, even if it is only a temporary abatement of a psychological perturbation, they will come to an autonomous appreciation and a deeper understanding of the subject matter that they are studying.

Radical constructivism is in most regards an astonishingly immodest theory. It claims simultaneously to undermine traditional conceptions of knowledge, experience and learning. Nonetheless, when it comes to details of how the theory may affect individuals, von Glasersfeld retreats to a charmingly modest stance.

> As radical constructivism holds that there is never only *one* right way, it could not produce a fixed teaching procedure. At best it may provide the negative half of a strategy....Constructivism cannot tell teachers new things to do, but it may suggest why certain attitudes and procedures are

> fruitless or counter-productive; and it may point out opportunities for teachers to use their own spontaneous imagination. (1995, p. 177)

In general pronouncements about teaching and learning, von Glasersfeld makes modest, apparently commonsensical claims. Few educators believe that performance is more important than understanding, or deny that motivation is important to learning. Yet few appeal to doctrines so austere that they claim that the ultimate aims of education amount to no more than the temporary abatement of perturbation. His hope that radical constructivism will point out appropriate times for teachers to use their "spontaneous imaginations" seems laudable enough. Von Glasersfeld claims that radical constructivism has utility in showing why some attempts to teach are "futile and counter-productive", but that simultaneously it cannot provide positive guidance in producing a pedagogy that is not so. Even the modest task of eliminating tempting but ineffectual strategies provides positive guidance. One wonders if he is being a little coy here, maintaining his professed lack of certain knowledge, while simultaneously promoting an educational agenda. Most teachers likely would agree that there is rarely only one right way to teach something, but surely even von Glasersfeld would agree that some of the strategies not rejected by radical constructivism will help more students to learn more mathematics than some others. If not, surely there must be some local conditions under which one of the non-rejected strategies will appear to be preferable to the others. *Preferable*, on this view, is defined in terms of the accomplishments of individual learners.

Ultimately, radical constructivism is an individualist theory, supporting liberal ideals of autonomy and self-regulation. In fact, these ideals are inescapable for the radical constructivist. Von Glasersfeld provides a thick notion of the thinking agent, acting in the world for no purpose other than the attenuation and elimination of perturbation. Anything that matters, on this view, is something that matters to a single person. The agent in a radical constructivist world is a sort of super-Benthamite, seeking only to maximize personal benefit, as measured by the abatement of perturbation. To what else could the radical constructivist appeal?

But radical constructivism provides no means for mobilizing action on these ideals. As von Glasersfeld takes great pain to argue, according to the radical constructivist position there are no grounds for mutual agreement on ideals, educational or otherwise. It could be that educational policy makers and practitioners agree to a radical constructivist point of view, but there is no reason for them to extend the benefits of perturbationary abatement to anyone other than themselves. In the happy coincidence that teachers happen to *want* students to learn—though I see no reason for this to be more than a random happenstance in the radical constructivist world—they can merrily teach students to enjoy all the joys of mathematics that von Glasersfeld claims. Unfortunately, radical constructivism provides no grounds for compelling teachers to teach, beyond fear of retribution. Even if teachers could be found who were willing to teach, it is not clear how a radical-constructivism-informed society could justify education of the sort that von Glasersfeld champions. If people do not want their perturbations abated through education, it is difficult to see how the radical constructivist can justify imposing a particular abatement on those people.

The difficulty the radical constructivist faces is in finding reasons for some actions and not others. For if the point of any action is neither more nor less than the alleviation of perturbation, then I can see no grounds for recommending one course of action that alleviates over another that is equally efficacious. Why educate when, say, drugging or brainwashing would just as effectively alleviate perturbations? Even if a society were developed that *just happened* to agree that such methods of perturbationary alleviations were not be tolerated, it is difficult to see on what grounds the agreement could survive a challenge. If everyone involved agreed to the position that each individual is radically separated from the universe, with no hope of knowledge that others even exist, then basic concepts such as *right, duty,* or *violation* cannot have any force. There is no hope, on this view, of even coming to an agreement as to what these terms mean.

Of course, no radical constructivist would grant me my argument. We do find ourselves in broad agreement about good ways to treat one another. Or, at least I have the sense that we do, and I have the sense that what I perceive as other people are in agreement with me, and so on. But the radical constructivists cannot escape so easily. By their own admission, they have no grounds for believing that others even exist, let alone that they agree with us. No amount of wiggling or special pleading within the framework of radical constructivism can have any hope of saving the basic ethical and political intuitions behind the theory.

As I will argue in greater detail in chapters 6 and 7, norm-laden activities such as education and mathematics require a theory that accounts for this normativity. Radical constructivism has narrowed itself to such an extent that it leaves no room for normativity. Further, it is not clear how normativity can be appended to radical constructivism without undermining the theory itself.

5.3 Lakoff and Núñez and the "Romance of Mathematics"

Lakoff and Núñez see their work as a scientific alternative to a Platonic folk mathematical epistemology they believe to be common in mathematicians, educators and the general public. As we are creatures whose existence is an evolutionary accident, there is no *prima facie* reason to assume that our capacities correspond to eternal truths rather than to contingent features of our engagement in the world as we find it. So far, so good. I doubt that any of the theorists I consider in this thesis—including myself—would disagree with the basic idea behind the sentiment. The problematic move Lakoff and Núñez make is to assume that there are only two possibilities: either mathematics provides access to incorrigible extra-human truth, or mathematics is a mere contingency of human physiological structure and historical engagement. For this chapter, I will only look at the politics of Lakoff and Núñez's championing of contingency; a deeper look at its epistemological consequences will appear in the chapter to follow.

Lakoff and Núñez give a lot of attention to their perceived Platonic enemy. They present their theory of "embodied mathematics" in opposition to the following sorts of beliefs (such opinions, Lakoff and Núñez dub "the Romance of Mathematics").

- Mathematics is an objective feature of the universe; mathematical objects are real; mathematical truth is universal, absolute and certain.
- What human beings believe about mathematics . . . has no effect on what mathematics really is. Mathematics would be the same even if there were no human beings, or beings of any sort. Though mathematics is abstract and disembodied, it is real.
- Mathematicians are the ultimate scientists, discovering absolute truths, not just about this physical universe but about any possible universe.
- Since logic itself can be formalized as mathematical logic, mathematics characterizes the very nature of rationality.

- Since rationality defines what is uniquely human, and since mathematics is the highest form of rationality, mathematical ability is the apex of human intellectual capacities. Mathematicians are therefore the ultimate experts of rationality itself.
- The mathematics of physics resides *in* physical phenomena themselves—there are ellipses in the elliptical orbits of the planets, fractals in the fractal shapes of leaves and branches, logarithms in the logarithmic spirals of snails. This means that "the book of nature is written in mathematics," which implies that the language of mathematics is the language of nature and that only those who know mathematics can truly understand nature.
- Mathematics is the queen of the sciences. It defines what precision is. The ability to make mathematical models and do mathematical calculations is what makes science what it is. As the highest science, mathematics applies to and takes precedence over all other science. Only mathematics itself can characterize the ultimate nature of mathematics. (Lakoff et al., 2000, pp. 339-340)

Even Lakoff and Núñez acknowledge that it is unlikely that anyone believes *everything* in the list; they nonetheless claim that each statement is widely held by many people. Further, they assert that adherence to even some of the principles of the above "romance" is possibly debilitating because it leads to elitism. They claim a connection exists between a belief in a Platonic mathematics and a belief that mathematical knowledge is somehow privileged, and is inaccessible to the majority of people.

Embodied mathematics, say Lakoff and Núñez, is the possession of all (2000, chapter 15).

Take a moment to look at the elements of Lakoff and Núñez's list. First, they claim that they are opposed to the view that mathematics is an objective feature of the universe. It is not clear what they mean by this. The claim to objectivity is conflated with the claim that mathematical objects are real. I think that there is little doubt that they are real: when I say, "Let C be a circle passing through point P," the circle is real regardless of whether its existence is contingent upon human practices, or whether its existence transcends those practices. Nonetheless it is hard to escape the suspicion that I am at cross-purposes with Lakoff and Núñez here. Mathematical objects are not "in the world" in the same way that mountains, courage and fools are. The main problem is that the deeper contest here is not over mathematical objects, *per se*, but over the meaning of "objective" and "real". As Hacking points out, these words are often used as though their meanings were transparent and stable. Hacking calls such words "elevator words" because they instantly shift the level of discourse. Discourse about "reality" takes place at a different level than does talk about the weather or differential equations. "The difficulties with these nouns and adjectives provide one reason for being wary of arguments in which they are used, especially when we are asked to glide from one to the other without noticing how thin is the ice over which we are skating" (Hacking, 1999, p. 23). As for *absolute truth*, I have some idea of how to make sense of the term, but I do not know who claims it to be a salient feature of mathematics or mathematics education.

The second claim, that humans have no effect on the content of mathematics, seems unlikely to be held by many people. Who could doubt that some mathematical constructions are the products of human mathematicians? The truth of propositions may be outside of the mathematician's control, but that is a different matter, which I will discuss in chapter 7.

The third claim is to the effect that mathematicians are "ultimate scientists." Their framing of the statement is troublesome. On the one hand, it seems a reasonable belief that mathematicians are able to make claims that are true of this and any possible universe, or at least any universe that they can imagine. As I will argue in chapter 6, I doubt that anyone could imagine a universe in which it is not true that "two plus two equals four", or where the law of contraposition does not hold. This belief does not seem particularly well described as the work of an "ultimate scientist" unless we are willing to expand our definitions of the proper domain of science. I suspect that the phrase is not intended to be taken too seriously, with Lakoff and Núñez really trying to call into question the purported universality of mathematics.

Next, they claim that "since logic itself can be formalized as mathematical logic, mathematics characterizes the very nature of rationality." Of course, no serious student of the issues could believe that mathematical logic exhausts rationality; it is perhaps true that some people do not know this. If so, Lakoff and Núñez are correct to challenge the view. The fifth bullet is a follow-up from the fourth. Again, if some people hold that mathematical logic exhausts rationality, then they might hold the view that mathematicians are the ultimate arbiters of rationality. The prevalence of such a belief would be best assessed empirically. Regardless, I suspect that few people would make such a strong claim about the rationality of mathematicians—including mathematicians themselves.

The sixth bullet claims that someone believes that the mathematics of the physical world resides in the phenomena under description. Lakoff and Núñez use the parabolic trajectory of a projectile as an example. It is absurd, they claim, to believe that the function $h(t) = h_0 + V_0 t + \frac{1}{2} a t^2$ is somehow embedded in the space in which the projectile travels, or that its path somehow contains the formula. I agree. It is hard to imagine any thoughtful person believing that the function is embedded in the physical world in such a crude fashion. The relation of the function to the projectile is that of a model to an object. If one looks at the problem with certain preconditions in mind (i.e. definitions of displacement, velocity, acceleration, time, etc.), then one might expect to find some correspondences between what is observed and what is calculated. If the mathematics and the data agree, the mathematics stands; if they do not, new models—not new data—are called for. I am not denying that the model has an expressive function in addition to the predictive function. The summands each represent a salient feature of the idealized physical system under description. The second half of the bullet deals with the claim that one cannot truly understand nature without understanding mathematics. The redundant expression "truly understand" apparently is rhetorical. I presume that they claim that many people believe that mathematically described science is somehow richer than other kinds. I don't know if this is widely believed, nor do I have a strong opinion on the claim. It seems to me that much of science is richly described mathematically. I believe that Lakoff and Núñez claim that it is not necessarily the case that mathematical descriptions are superior to others. I agree, but don't see that their epistemology leads to that conclusion.

This leads to the final bullet. It is not clear what they mean by mathematics taking "precedence over all other science". The point is little more than a string of non sequiturs. The belief that mathematics is precise does not imply the odd claim that mathematics defines precision. Nor does it lead to the belief that mathematics is the highest science, nor that it is the single most powerful defining feature of science.

The list of alleged highlights of the romance of mathematics is a caricature. As noted above, most of the points Lakoff and Núñez raise are overstated. Some, such as the claim that many people believe that mathematical logic contains all of rationality seem just wildly out of line. Questions of what people believe are empirical claims, and Lakoff and Núñez make no attempt to establish whether anyone actually believes anything in their list. Their point is that mathematics is held in high esteem in contemporary intellectual culture, and in this they are undoubtedly right. They are also correct, I believe, in claiming that many people have unreasonable beliefs about mathematics, and probably ought to be engaged in some meaningful discussions about those beliefs.

Perhaps I can charitably agree with Lakoff and Núñez that extravagant beliefs about the nature of mathematical truth are behind some people's misgivings about mathematics. Suppose I agree that a belief that mathematics is true regardless of human beliefs and that one can come to understand mathematics only through pure reason is common. The holding of such beliefs is independent of whether they are justifiable. Even if people do have strange and debilitating ideas about Platonic mathematics, these beliefs could be true. What, then, could be Lakoff and Núñez's purpose in claiming that people believe odd, vaguely Platonic things about mathematics?

Recall Hacking's note about the aspirations of the theoretical unmasker. Lakoff and Núñez claim that a belief in human-transcendent mathematical truth is not inevitable, and that it is harmful and that we would be better off without such beliefs. Lakoff and Núñez further claim that mathematical Platonism serves the extra-theoretical function within mathematical discourse of making mathematics appear mysterious to the uninitiated, preserving mathematics as the private domain of mathematicians.

> The Romance serves the purposes of the mathematical community. It helps to maintain an elite and then justify it. It is part of a culture that rewards incomprehensibility, in which it is the norm to write only for an audience of the initiated—to write in symbols rather than clear exposition and in maximally accessible language. The inaccessibility of most mathematical writing tends to perpetuate the Romance and, with it, its ill effects: the alienation of other educated people from mathematics, and the inaccessibility of mathematics to people who are interested in it and could benefit from it. (Lakoff et al., 2000, p. 341)

So here we have a political point to the theory: embodied mathematics is democratized mathematics. Hacking's formulation is particularly apt in this case. The X that is claimed not to be a necessary feature of the world, that is quite bad and that we would be better of without is the notion that mathematics is a difficult or mysterious domain, accessible only to a select cabal of specialists. But surely all this is much too quick.

Mathematics, like many other human activities, requires a great deal of work to master. This is empirically borne out by the fact that so few people become mathematicians. Perhaps there are ways to make it easier to become a mathematician. Even so, it seems unlikely that the simple observation that mathematics has a physiological basis is sufficient for the task. Nor is there any strong reason to believe that such an observation would be sufficiently liberating to the aspiring mathematician to make an appreciable difference in the desire to pursue mathematical learning. The view that mathematics is comprehensible only due to human physiology is not incompatible with the view that in spite of this it possesses mind-independent validity.

5.4 Social Constructivism: Humanist Mathematics and Left-Wing Politics

Rueben Hersh explicitly connects epistemology and political commitment. He runs through a list of eminent thinkers about the nature of mathematics and makes speculations about their political commitments. After (good-naturedly) enumerating the epistemologies of thinkers from Plato through Descartes and Spinoza, on to Leibniz, Kant and Frege, right up to Hilbert, Brouwer and Pólya, Hersh notes a correspondence between a thinker's epistemology and his politics.

> Conservative politics and Platonist philosophy of mathematics don't *imply* each other...[but] the numbers do suggest a *correlation*....Political conservatism opposes change. Mathematical Platonism says the world of mathematics never changes.
>
> Political conservatism favors an elite over the lower orders. In mathematics teaching, Platonism suggests that the student either can "see" mathematical reality or she/he can't.
>
> A humanist/social constructivist/social conceptualist/quasi-empiricist/naturalist/maverick philosophy of mathematics pulls mathematics out of the sky and sets it on earth. This fits with left-wing anti-elitism—its historic striving for universal literacy, universal higher education, universal access to knowledge, and culture. (Hersh, 1997, pp. 245-246)

Hersh claims that an adherence to the belief that mathematics is true regardless of human beliefs about it leads to social stratification. Even if we grant Hersh his (clearly not randomly sampled) list and agree that as a sample it fairly represents a correlation between Platonist (or realist) mathematics and political elitism in some population, the correlation has no bearing on what we *ought* to believe about mathematics, epistemology and education. Suppose we agree that mathematics is a social construction, as Hersh would have us do. It does not follow that people will be more able to "see" mathematics after such an agreement than without it. The Riemann hypothesis will be no more comprehensible to the average person who believes that it represents an instance of human convention than it would be to the person who believes that the zeta function is part of the fabric of human-independent reality. But I need not pull out the esoteric examples. Even more simple results such as the uniqueness of the prime factorization of natural numbers do not suddenly become easier to understand if the ontology of numbers is understood in a different way.

Recall from section 3.3.5, Hersh makes two related claims about mathematical ontology and education. First, he claims that it is mysterious how mathematics could be teachable if it had an existence external to human practices. Second, he claims that the belief in human-transcendent mathematics leads to defeatist beliefs of the inaccessibility of mathematics to many students. Both of these claims are fallacious. First, regardless of the ultimate existence (whatever that might mean) of mathematical objects, it is trivially true that humans deal not with the objects themselves, but with their concepts of the objects. Concepts *are* accessible to humans both individually and socially. We can have mathematical thoughts, and we can communicate the content of those thoughts. Whether the thoughts and concepts make reference to human practices or whether they refer to human-independent truths has no bearing on whether anyone can have them, and whether they can be socially shared. Hersh is simply missing the fundamental distinction between an object and a reference to that object.

The second assertion is an empirical claim. Perhaps it is the case that if students are taught that mathematics is a human creation they will find it easier to learn. Even if this is so (and I have my doubts), I cannot see how this has bearing on ontology or epistemology. If the assertion that beliefs about the ontology of mathematical objects have an effect on mathematical learning is true, then there are pedagogical implications, regardless of the defensibility of the ontological claim. One strategy might be to discover more about the psychological connections to the difficulty and deal with whatever secondary problems lead to the block. Another might be to emphasize the constructedness of concepts in learning. It certainly does not follow that the teacher must take *any* stand on a constructivist epistemology to work around whatever psychological blocks follow student difficulties.

5.5 Complexity, Inclusion and the Environment

Davis argues that individuals are fruitfully construed as components of complex learning systems. Two interesting political themes permeate Davis's work. First, he focuses on the importance of functional difference in the components of a complex system. Second, he repeatedly stresses that complex systems, like fractal drawings, possess self-similarity at different levels of focus. What is true of one part of the system is true for the larger system within which that part is nested, and so on. The first theme appears to be a statement of the importance of individuals who might otherwise be seen to be outside of the mainstream, however that is construed. The second theme he takes to imply an ecological ethic.

The significance of the individual is revealed through the recognition of the importance of diversity within a complex collective. In section 3.4.5, I quoted Davis as claiming both that in complex organizations, authority and control are distributed amongst the elements of the organization, and that the educator whose practice is informed by complexity science does not see the individual child as the "locus of learning". Both of these statements suggest that Davis's theory places the authority of understanding outside of the expert. Mathematical knowledge no longer becomes the domain of the expert community of mathematicians, or a part of the structure of cognitive reality, it becomes a distributed resource, equally under your and my control and authority as it is under Andrew Wiles's. This is a strong claim, requiring justification. Davis provides no reason to believe that distributed authority provides superior mathematics—by any criterion—than does appeal to authoritative individuals, or to persuasive arguments. Even if we do not take the question of epistemic authority to such a strong limit, what sense can be made of the notion that the locus of learning need not be the individual learner, but is rather the complex mathematical collective to which the learner belongs? Davis makes frequent comments to the effect that *if* one looks at learning in one way it appears to be the actions of isolated individuals. But, he argues, if one looks at learning as a collective activity, then it can be profitably construed as a fractal-like growth of learning within parts of the complex system and within the system as a whole. Further, he claims that the researcher as observer of complexity becomes part of that complex learning collective through the act of interpretation.

Towers and Davis (2002) give an example of two grade six girls, Kayleigh and Carrie, solving a problem involving counting the number of tiles required to make a path around a rectangular swimming pool. The students were assigned the task of finding the number of 1-unit square paving stones that would be required to pave the perimeter of an n-unit by n-unit pool. The task was built in stages, with concrete examples (i.e. 1x1, 2x2, etc.) preceding the request for generalization. The girls worked independently for the majority of the activity, but took time to discuss results with one another, with the class and with the teacher. Towers and Davis noted that the girls' understandings developed through a number of informal stages of reflection, discussion and expansion. Each was a part of the other's developing understanding: "their collective action simultaneously occasions differing understandings and converges on shared understandings, revealing a collective dynamic that is complex and co-evolving" (p. 331).

The authors do not deny that the girls do understand different things in different ways *as individuals*. Towers and Davis note for example that "Kayleigh...prefers to work without the use of diagrams...the diagrams may be a prompt for her but they are not a critical part of her thinking," whereas Carrie "uses the diagrams not only as prompts but as thinking tools...for her the diagrams *are* the mathematics" (pp. 325-326).

Towers and Davis hedge their bets by alternating such phrases as "it becomes clear that each student is complicit in the unfolding understanding of the other" with "while it is possible to construe their activities as two separate events...there may be something to be gained by analyzing the structure of the shared activity" (p. 326). The first remark strongly indicates that there is a way that things are and that way is described through the language of complexity, and the second suggests that their model is just one of a number of plausible strategies for describing student learning.

I am not sure what is supposed to be clear about Davis's assertion of complicity. If he means that interlocutors can help one another to learn, then his claim is uncontroversial.

Switching focus from the individual-as-learner to the individuals-as-complicit-in-the-collective suggests a switching of valuation in classroom instruction. In the latter conception, students function not merely to learn, but to participate in a development of learning that is asymmetrically shared within the room. *Shared* on this interpretation refers not to people possessing the same thing, but to people working together on a common project. When shared learning experience is valued above individual learning experience, roles of differing amounts of learning can become valued. In complex systems so construed, it is not necessary that everyone learn or perform all the mathematics that is learned, it is only necessary that each member play some important functional part in the overarching project. It may be the case that a complex system requires a certain amount of redundant material, and a small amount of novel, "rule-breaking" material in order to continue to change within its environment. It does not follow from this that there are good political or moral grounds to treat children in ways to foster such systems. It would be cold comfort to a parent to read a report card that says "Susie is functionally important to our classroom in continuing to provide redundant thoughts and work to our ongoing discoveries." Nor is it likely that a parent would want to read, "Helen continues to perform activities that are different from the rest of the class. We fully expect that one of them will eventually assist the class in developing a novel bit of knowledge." Yes, these comments are a parody, but they are not wildly removed from Davis's ideology. He does not deny that children should learn, he denies that their personal learning is a necessary part of a defensible education. What is not defensible, Davis argues, is an education that sees its purpose as helping individual children toward individual understanding of some subject matter. Davis and Sumara claim that an education aimed at the individual learner is:

> the ineffective and potentially damaging practice of regarding learners as isolated, detemporalized, and decontextualized subjects. Knowledge will continue to be separated from knowers, and educators will continue to be oblivious to the formation and transformation of our culture. In a phrase, school will continue to be a place of mindless action, seeking to fit the inevitable complexity of existence into a framework of logical order. (Davis et al., 1997, p. 11)

The aims of education, it appears, are not to be found in a consideration of what a child may get from the process, but rather from the role the child is invited to play within the larger cultural organism.

> To understand the phenomenon of personal cognition, one must simultaneously regard the agent as an autonomous agent working to fit in with her or his context, as a component of a larger social order, as a complex collective of dynamic bodily subsystems, and so on.
>
> These bodies are structurally similar—where once again, structure is used in the sense of dynamic organic emergence and not more stable physical organization. (Towers et al., 2002, p. 335)

The idea that individuals possess a structural similarity to the collectivities to which they belong underwrites a second ethical thread in Davis's work. He champions an ecological ethic inspired by Fritjof Capra's *deep ecology*. Capra's influence is felt in Davis's continual echoing of phrases to the effect that "the whole exceeds the sum of the parts." I dealt with logical problems with Davis's and Capra's rendering of this maxim in section 3.4.7. For here, I simply want to point out the ethical ideal it underwrites.

The ecological sensibility that permeates Davis's work is a constant reminder that the individual human is part of something much bigger than himself. The human societies within which the individual is embedded are embedded in yet larger networks of interaction. Humans form only a small part of the biomass of our planet, which is part of a larger universe, and so on. Turning the telescope in the other direction, we see that we are made up of smaller systems, right down to our DNA, and perhaps beyond. Davis claims, following Capra, that these nested systems possess a self-similar structure that is describable with the language of complexity science. This description, according to Davis, serves to maintain our educational sensibilities in alignment with our view of our place within the cosmos. He claims that his theory brings forward "the assumed interconnectedness of the cosmos, the need for close attendance to the emergence of possibilities, and the insistence on ethical action" (Davis et al., 2003, ¶ 38).

5.6 Summary and Reflection

In this chapter I have claimed that each of the four theories of mathematics education has a political agenda attached to it. Each sees problems of inclusion in mathematics: that is, each agrees that mathematics is largely perceived as an esoteric activity, and each would like to do something about that. Each, in his turn, claims that his theory brings mathematics out of the realm of the supernatural, and places it in the hands of ordinary people.

Von Glasersfeld's political aspiration of an inclusive liberalism seems to be in direct conflict with the isolation of atomic persons as conceived under radical constructivism. He explicitly wishes individuals to share in meaningful cognitive activities, but has placed them so theoretically distant from one another that it is difficult to see how one could justify the cooperation needed to bring sharing about.

The social constructivists Hersh and Ernest see their position as directly challenging elitism by making mathematics a product of regular human discourse. Hersh claims that if Platonism or Formalism were correct theories of mathematics, then mathematicians would have access to something beyond the reach of most humans; constructivism places mathematics within the reach of any person participating in the mathematical form of life (Ernest has also written approvingly of this). The claim, as I argued, is unsustainable at both ends. That it would be nice if mathematics were easily accessible tells us nothing about its nature, nor does it provide good ways to teach it or to learn it. I also raised the doubt as to whether beliefs about the source of mathematics would make it easier for anyone to learn it.

Lakoff and Núñez commit the same fallacy as the social constructivists. They wish to place mathematics in human lived experience to make it more accessible to all. Their arguments against opposing epistemologies proved to be straw men, and their positive positioning faces the same objections as do Hersh and Ernest.

Davis's complexity science attempts to bring mathematics to the masses through conceptualizing it as participatory, rather than as a discipline to be acquired. His position stumbles as well. Most notably, it is not clear why anyone would want to participate in the pageant of complexity, rather than learn for themselves. Nor is it clear how one could make a legitimate political case to make students responsible for a collective knowing from which they may or may not directly benefit. Finally, I raised some doubts about the legitimacy of Davis's commitment to a particular view of human learning as participation in an ecological movement.

6 A Skeptical Background

In the preceding chapter, I argued that the four theories are offered within political and ethical contexts, some of which are explicitly articulated, and some of which are implicit. These theories of learning and of knowing are unable to provide the political and ethical justifications that they claim. My arguments left the theories' epistemological claims untouched. This chapter provides a challenge to the epistemological aspirations of any theory of mathematics education that claims to give an account of mathematical understanding: Can such a theory respond to a persistent skeptic about the possibility of understanding?

One of the most puzzling aspects of human understanding is that it is possible. As organisms whose existence on this planet appears to be largely contingent upon random events, and as social beings whose personal and cultural histories appear also to be contingent upon random events, how is it that we can understand anything about the world? In this study, my interest is in mathematical understanding. To understand mathematical concepts, propositions, or justifications, one must be able to formulate them, to follow rules of inference and manipulation and to give some account of why these are reasonable things to do. Central to my argument is the view that understanding is related to the ability to provide reasons. When one understands, I argue, one can provide reasons for one's claims. It is not necessary that one have all reasons ready immediately—indeed, I argue that this is rarely the case—but one must be able to provide reasons upon need. I take this view to be very general. For example I can understand things that I believe to be false, such as the Ptolemaic theory of the solar system. In such a case, my understanding is of the content of the theory and of the reasons that at one time were offered in its support, not of my beliefs about the solar system.

In this chapter, I look at mathematical understanding in light of Wittgenstein's celebrated "rules and private language" arguments. Wittgenstein's arguments render unintelligible the idea that the content of rules and understanding can be extracted from naturalistic facts about a person or group of people. I argue that no noncircular account of mathematical understanding is possible. In order to claim that a person understands some mathematics, I must appeal both to facts about the individual and to the content of the mathematics that is claimed to be understood. As I understand Wittgenstein's arguments, they show that analysis of the psychological processes of understanding cannot simultaneously establish the normative character of that understanding.

Analyzing the simple arithmetical procedure of adding two to an arbitrary natural number, Wittgenstein shows that no finite accounting of naturalistically describable facts about a person are sufficient to show that the person understands the rule "add two." To put the point another way, there is no way to retrieve the full content of a rule solely through an analysis of a rule-follower. The conclusion of the Wittgensteinian paradox is general: if arithmetical rules cannot be described in naturalistic terms, neither can any other rules. Physical and phenomenological descriptions are insufficient to capture the richness of simple rule-following. No empirical data about a person's rule following in the past can indicate what that person *ought to do* to follow the rule in the future.

Theories of mathematics education would be in trouble if they were to conclude that there is no hope of children learning to add two to an arbitrary natural number. Clearly, people do learn to add two, and they appear to understand many important features of the rule, including its applicability to an infinite number of situations.

In this chapter, I apply each of the four theories of mathematics education to the simple problem of adding two. If they fail to provide an adequate account of what it means for a person to learn to add two, and if they fail to account for how a person could learn to add two, then the theories are inadequate for the purposes that they claim. Indeed, I will argue that none of the four provides a sufficient response to Wittgenstein's skeptical challenge: how is it possible to understand how to follow a simple mathematical rule? I can sketch the challenge schematically.

a) *I follow rules.*

It appears to be impossible that I can follow rules because:

b) *Rules have infinite implications.*

c) *I am finite.*

d) *There is not enough of me to account for infinite implications.*

e) *How can I possibly understand rules well enough to follow them?*

For von Glasersfeld, the skeptical challenge cannot be answered, and he argues that no account of knowledge—including his own—that claims more than temporary cognitive stability is tenable. Lakoff and Núñez claim that the search for knowledge takes us only to ourselves as humans and not to a world that is independent of us. Ernest and Hersh argue that human mathematical knowledge is awareness of a consensus within a contingent language community. Davis claims that knowledge is participatory, and that knowers are participants in a larger knowledge scheme of which they are only a small part. Each theorist claims to provide a response to the skeptical challenge, and each response leads to a thesis of epistemic contingency. In each case, I believe that a serious attempt is made to overcome a fundamental problem of knowledge. Each theorist, apart from von Glasersfeld, claims to give a modest, contingent response to the question, "How is understanding possible?"

To adequately frame the skeptical problem and provide responses to it, I will present the arguments as described by Saul Kripke in his reading of Wittgenstein's *Philosophical Investigations.* I do this for three reasons. First, by using Wittgenstein's provocative examples Kripke provides a deep and penetrating explanation of the problem. Second, each of the four theorists in focus in this thesis makes reference to Wittgenstein in the context of skepticism. Third, Kripke's treatment of Wittgenstein has prompted outstanding responses from Charles Taylor and Thomas Nagel, from which I draw anti-skeptical resources.

6.1 The Kripke-Wittgenstein Paradox

Wittgenstein's (1968) arguments about rules and private language provide a penetrating discussion of why the understanding of a rule—however simple—cannot be identified with marks on paper, behaviours, verbalizations, or even brain states. Such descriptions, Wittgenstein repeatedly shows, cannot provide sufficient information to extract the infinite implications of the rules they purport to describe. A sufficiently persistent interlocutor (which Wittgenstein provides) can always find ways to misunderstand these marks, behaviours, words or brain states. Some key facet of understanding eludes naturalistic accounts of meaning. Kripke (1982) shows how Wittgenstein demonstrates that whatever it is to follow a rule, no catalog of empirical data about the individual rule-follower can be sufficient to indicate how the individual should follow the rule in the future. That is, the rule itself cannot be retrieved from facts about the individual. Further, rule followers are not and cannot be aware of a wide range of issues that bear directly on the correct application of the rule. In spite of these facts, people do correctly follow rules. The paradox is this: if nothing about me is sufficient to show anyone—including me—that I understand a rule, how is it that I, in fact, do understand rules?

The relevance of the Kripke-Wittgenstein paradox to this study should be clear. I wish to articulate the conditions under which someone can be said to understand a mathematical concept, proposition or justification. I do not claim that the judgment that X understands P is a simple "yes/no" judgment. A judgment of understanding is relative to other factors within a person's experience: a five-year-old's understanding of addition is different from an algebraist's. The mathematics educator is rarely interested in saying “X understands P,” but rather wishes to say “X understands P within context C.”

Wittgenstein begins by showing that for a person to understand how to follow a rule, an infinite number of conditions must hold. To add two to an arbitrary whole number, for example, one must not only be able to follow a rule that has implications for an infinite number of cases, one must also not follow the rule incorrectly in any of the infinite number of ways that one could. By incorrectly following the rule, Wittgenstein is not referring to the making of mathematical errors; he is referring to operations that are simply not equivalent to addition, even when applied correctly.

Wittgenstein invites us to imagine that a teacher believes herself to be teaching a child how to perform the simple arithmetical operation “add two.” The teacher runs through a number of sample sequences, with such instructions as, “count up two, but don’t say the first one out loud.” The student seems to understand, and verbalizes along with the teacher, “2, 4, 6, 8.” The teacher believes that the child understands the rule. Every time the teacher prompts the student, the student provides the expected response. In this circumstance, a teacher would likely be prepared to grant that the student understands the rule "add two". The teacher cannot be sure, however, that if the next day she gave the student a number of "add two" problems, that the student would not answer "5" every time, but the day following that return to providing the expected responses. Perhaps the student follows a rule that means the same as the expected "add two" except on Tuesdays, when the rule he follows says to respond "five" to each arithmetical query. Of course, the teacher couldn't have known this on Monday, and she is not sure about what will happen on other days. In spite of all this, no teacher would hesitate to infer that the student understands "add two" after the initial set of responses. Such inferences are based partially upon the content of the student's responses and partially upon the teacher's expectations. The student has provided too little data for the teacher to be *certain* that the student is adding in the normal way so the teacher compares the expected responses with the given responses and makes a judgment. The judgment that the child understands "add two" in the expected way necessarily involves the *importation* of the mathematical content of the rule. By this I mean that the student's responses do not contain the content of the rule; they do not indicate what the student will—or ought to do—in response to future queries. In this case, the teacher's judgment must be informed by her own grasp of "add two"; she cannot simply abstract it from observations of her student.

On the one hand, the example seems far-fetched; on the other, it represents a true educational dilemma. Teachers consistently attempt to infer student understanding from isolated performance. Every teacher knows that students often are able to provide correct responses in spite of having inadequate or patently false concepts; and at other times, students whom the teacher believes to understand the relevant concepts provide incorrect responses. That student errors are not often as wildly off the mark as the example above does not change the basic point. Neither correct nor incorrect responses in themselves provide sufficient information to guarantee an inference to understanding; some of the content of the rule must be brought into the judgment. As Thomas Nagel (1997) argues, one cannot make judgments about the rule "add two" without already having the thought "add two" in place.

To put the same point another way, the set of correct responses to a number of addition problems is consistent with an infinite number of alternative mathematical functions. An explanation of the responses must be finite, and cannot account for all possible objections. The rule "add two" has infinite normative implications in that there are an infinite number of pairs of numbers n and $n+2$, and there are an infinite number of pairs of numbers that do not satisfy the relation. A human, on the other hand, is capable of encoding only a finite amount of information. No facts about a person are sufficient to exhaust the informational implications of "add two". The number of ways of misinterpreting "add two" is infinite, but somehow we believe that these are, indeed, misinterpretations. We are convinced that we mean "plus" and not "quus." (Quus is an alternative function provided by Kripke, such that x quus $y = x + y$ except when x and y are both greater than 57, in which case x quus $y = 5$.) And we believe that any of the alternative functions that could be offered are not the ones we mean. If we are able to recognize these deviant readings as deviant, how do we do so? It does not seem reasonable that our understanding of the rule could include thoughts that have already taken care of an infinite number of alternatives. Yet somehow we are not tempted by the infinite number of deviant readings, so we appear to possess some way of telling the right way to follow a rule from the wrong ones. Whatever this way is, it must consist of something other than natural facts about us.

In summary, Kripke shows that no collection of natural facts about the student who is learning to add two is sufficient to show her understanding of the arithmetical rule. Past responses cannot distinguish the correct interpretations from all the wrong ones, and neither can any other natural facts about the student during these performances, including records of mental or brain activities. More troubling is Kripke's conclusion that not only can others not infer my understanding from natural facts about my past, neither can I.

If we can accept the above, it is easy to see how it leads to a paradox. There is nothing special about "add two". The mathematical example is a convenient one, but the argument is independent of the example's mathematical content. If facts about me cannot account for "add two", then neither can they account for my other apparently rule-governed thoughts and actions. We appear to be unable to tell whether we meant "plus" or "quus" from our usage in the past, but neither can we fix the ways in which we meant words to refer to objects or states of affairs. Nor can we fix the grammar of our language, nor any of the rules apparently governing social discourse. If "add two" cannot be defended, neither can any other case of putative rule-following.

> If there was no such thing as my meaning plus rather than quus in the past, neither can there be any such thing in the present. When we initially presented the paradox, we perforce used language, taking present meanings for granted. Now we see, as we expected, that this provisional concession was indeed fictive. There can be no fact as to what I mean by "plus" or by any other word at any time. The ladder must finally be kicked away. (Kripke, 1982, p. 21)

The paradox suggests that there are insufficient grounds even to believe that word meanings are stable across different uses by the same agent. Even though the argument seems to indicate that we must "kick the ladder away", to do so would be to make the argument itself come tumbling down, since it relies on stable meanings for its own terms.

Note that it is insufficient to appeal to more basic definitions to save the meaning of "plus" since those too are subject to precisely the same questioning the skeptic gives at this level. Addition does not simply consist in a generalization from a finite number of performances. In order to add, I follow an algorithm. Even here, the skeptic can raise the same objection as before. How do I know that the algorithmic procedures I followed in the past are identical with the ones I follow today? If I provide further argument to support the algorithm, the skeptic simply questions the stability of *those* terms. Wittgenstein's skeptic appears to be able to cast doubt upon my ability to bring forward reasons of any sort. The paradox is general; there is nothing in the argument about "add two" that does not apply to any other meaningful rule following. If the paradox shatters arithmetical rules, it shatters all the others as well, including rules of reference and grammar. Meaning and understanding appear to reach beyond the sum of all natural facts that can be stated about the person who means or understands.

The Kripke-Wittgenstein paradox is striking for a number of reasons. First, it is simple to express: finite accounting of finite beings cannot account for the understanding of rules. Second, one cannot help but object throughout: we *can* follow rules; we *can* explain rules; we *can* teach and learn rules. The paradox provides a *reductio*. If the argument is correct, then understanding seems to be impossible, but the argument itself can be understood. Something has to give. The difficulty is in finding the faulty assumption.

Before looking at responses to the paradox, consider the problem in light of radical constructivism. Radical constructivism holds that understanding is a purely individualistic construction. One understands a rule only insofar as one is not perturbed to a point of finding it inadequate. Presumably, one must also actually use the rule or believe it to be useful for the radical constructivist to say that one understands it. If one accepts the radical constructivist view, then understanding is explicable in terms of a finite accounting of an agent. Under the radical constructivist description, understanding, if there is such a thing, is fully disclosed through facts about an individual. Recall that the radical constructivist claims that understanding is no more than stable belief. The stability of a belief is claimed to be no more than a consequence of the abatement of perturbation (see section 3.1.2). Consider Wittgenstein's "add two" example in light of radical constructivism. The child continues the sequence "2, 4, 6, 8," with the responses "10, 12, 14, 16." The radical constructivist observer concludes only that the child continues in a way that seems natural to the child, and that the child's response is identical to how the researcher would respond herself. The radical constructivist is in a position identical to Wittgenstein's imaginary interlocutor. The radical constructivist attempts to account for understanding by accounting only for responses and their stability over time. Like Wittgenstein's interlocutor, the radical constructivist then concludes that since the finite analysis is correct and complete, one can only conclude that understanding is a much weaker notion that I have argued so far. I have maintained that part of what it means to understand is to be able to make some justification of what one *ought* to do to follow a rule. Von Glasersfeld champions a purely descriptive view of understanding. Radical constructivism makes no claims about how things ought to be done now or in the future. It claims only to tell a plausible story of how individuals organize their experiences.

> It is the knower who segments the manifold of experience into raw elementary particles, combines these to form viable 'things', abstracts concepts from them, relates them by means of conceptual relations, and thus constructs a relatively stable experiential reality. The viability of these concepts and constructs has a hierarchy of levels that begins with simple repeatability in the sensory-motor domain and turns, on levels of higher abstraction, into operational coherence, and ultimately concerns the non-contradictoriness of the entire repertoire of conceptual structures. (Glasersfeld, 2005, ¶ 3)

Not only has the skeptic got it right, says von Glasersfeld, but that is the end the story. He holds that there is no adequate response to the skeptical problem because there is no reason to believe that understanding reaches beyond temporary abatement, nor that understanding has a normative component. Our commonsense notion of understanding is cast away. But von Glasersfeld cannot get away so easily. He provides no reason to believe that his words have meaning, no more than Wittgenstein's skeptical interlocutor does. Whatever it is that von Glasersfeld means, he gives no reason to agree or disagree with him because he denies the possibility that there is anything to agree or disagree with. One either finds that his story abates a perturbation or not; either way, von Glasersfeld does not claim to tell the truth, because he cannot do so (see sections 3.1.1 and 3.1.2). Even the evasion that one can accept radical constructivism if one finds one's beliefs about the theory to abate perturbation cannot be believed to have stable meaning. I can now correct the first sentence of this paragraph. Von Glasersfeld cannot say that the interlocutor is correct, but he does say that he agrees with the interlocutor.

There is an even more troubling aspect to von Glasersfeld's account. He claims that individual agents abstract mathematics from experience. As Wittgenstein's interlocutor continually reminds us, there are an infinite number of rules that we could abstract from the same experience. How is it that people consistently abstract "plus" rather than "quus" from their experiences? It appears that some form of regulation beyond what von Glasersfeld is willing to concede must be necessary for this abstraction to take place.

In the sections that follow, I outline three attempts to overcome the Kripke-Wittgenstein paradox. Kripke, Taylor and Nagel provide responses that acknowledge the skeptical argument in the way that von Glasersfeld might, but nonetheless claim that meaning and knowledge are possible. In all three arguments, the faulty premise is identified as the assumption that understanding is solely to be found in the agent. All three claim that at least some component of understanding is outside of the individual. Where they differ is in where they locate this component.

6.1.1 Kripke's Solution to the Paradox

Kripke acknowledges that Wittgenstein's paradox appears to make "*all* language, *all* concept formation, to be impossible, indeed unintelligible" (Kripke, 1982, p. 62). Suggesting that Wittgenstein's skeptical paradox is unanswerable on its own terms, Kripke proposes a *skeptical solution* to it. He does not attempt to show that Wittgenstein's skepticism is unwarranted. Rather, he admits the validity of Wittgenstein's skeptical argument, but argues that meaning is still possible. "Nevertheless, our ordinary practice or belief is justified because—contrary appearances notwithstanding—it need not require the justification the skeptic has shown to be untenable" (1982, p. 66). This move allows Kripke to shift his gaze from an analysis of how an agent can mean something with words to a description of how words are used. Let us acknowledge, says Kripke, that there is no fact as to whether I mean plus or quus. It does not follow, he claims, that I cannot mean "plus". The important question here is not "what does Joe mean?" but rather, "What is Joe doing?"

> Wittgenstein replaces the question, "What must be the case for this sentence to be true?" by two others: first, "Under what conditions may this form of words be appropriately asserted (or denied)?"; second, given an answer to the first question, "What is the role, and the utility, in our lives of our practice of asserting (or denying) the form of words under these conditions?" (1982, p. 73)

Kripke acknowledges that the above formulation is less than apt for sentences other than declaratives. His more general point is that we can judge the correctness or appropriateness of a sentence not in terms of its content, but in terms of its use. "Add two", then, is identified through a combination of considerations of what makes it acceptable and of how it is used in a form of life. This makes the judgment that a student has performed addition correctly not a question of the truth of the judgment, but of the conditions under which it can be appropriately asserted. "All that is needed to legitimize assertions that someone means something is that there be roughly specifiable circumstances under which they are legitimately assertable, and that the game of asserting them under such conditions has a role in our lives" (1982, pp. 77-78). There is no correspondence between a person's addition and any "facts of the matter" on this view. We cannot address the question of what rule, precisely, a putative adder is following. All we can do is make observations about how putative adding is used. From use, we can make tentative judgments, following the best practical judgments at our disposal.

It is this skeptical response to the skeptical challenge that Kripke claims to underwrite Wittgenstein's famous rejection of "private language". Kripke argues that if we accept the view that it is through use that actors are judged to be rule-followers, then we can make no sense of the claim that a person in isolation is a rule-follower. This is so, he claims, because there is no fact of the matter as to whether a person's actions accord with a rule; "Joe is adding" is a judgment made from outside of Joe, made in light of the normative binding of a community. "To think one is obeying a rule is not to obey a rule. Hence it is not possible to obey a rule 'privately'; otherwise thinking one was obeying a rule would be the same thing as obeying it" (Wittgenstein, 1968, §202). Kripke argues that Wittgenstein's point is not that an isolated person cannot follow rules, but that the judgment that the person follows rules requires others. The isolated individual cannot judge his or her own rule following, Kripke says, because there is no background against which to make this judgment. In order to make the judgment that Joe is following a rule, one must be able to recognize the rule; in the absence of outside scrutiny, all Joe could say is that he is doing what he is moved to do. The reliance on external assertability conditions makes it impossible for Joe to affirm his own activities as rule-following. This consequence of Kripke's reading of Wittgenstein makes the skeptical solution seem implausible. One is tempted to object that a hermit can follow rules, just as anyone else can. Kripke does not deny this. The skeptical solution does not tell us whether rules are followed; rather it provides an explanation of how we come to judge that rules are followed.

Kripke suggests that communally governed assertability does not lead to an unbridled relativism. "Others will then have justification conditions for attributing correct or incorrect rule following to the subject, and this will *not* be simply that the subject's own authority is unconditionally to be accepted" (Kripke, 1982, p. 89). This is in accord with Wittgenstein's blunt assertion: "Certainly, the propositions 'Human beings believe that twice two is four' and 'Twice two are four' do not mean the same" (Wittgenstein, 1968, p. 226). In a very important sense, Kripke's skeptical solution leaves unanswered the question of how meaning is possible. On Kripke's view, the teacher can infer understanding from student performance based upon observed behavior. The judgment that the child is correctly following the rule comes not from observed correspondence between behaviors and the rule, but from observed correspondence between the child's behaviors and the teacher's expectations, and between the teacher's expectations and those of the mathematical community.

> Now what do I mean when I say that the teacher judges that, for certain cases, the pupil must give the "right" answer? I mean that the teacher judges that the child has given the same answer that he himself would give. Similarly, when I said that the teacher, in order to judge that the child is adding, must judge that, for a problem with larger numbers, he is applying the "right" procedure even if he comes out with a mistaken result, I mean that he judges that the child is applying the procedure he himself is inclined to apply. (1982, p. 90)

To verify one's claim to the ability to add, then, one must be acknowledged by others as an adder, according to Kripke. One's putative rule following can be checked by others. The rule can be accepted in two ways: either the putative adder agrees with the community in announcing sums, or she gives sums that diverge from the community in ways in which the community recognizes acceptable methodology and understandable error. In both cases the notion of the truth of the judgment of correct rule following has been replaced with the notion of the assertability of the judgment. Further, the judgment that someone correctly follows a rule is conditional: all Kripke requires is that the conditions of rule following be "roughly specifiable."

Kripke's "skeptical solution" to the paradox likely is quite congenial for Ernest and Hersh. Like Kripke, they are prepared to replace talk of truth conditions with talk of assertability conditions. The rightness or wrongness of rule following is not found in judgments about truth, or correspondence to facts. Rightness or wrongness, they say, is a consequence of communal acceptability. Thus, they claim that when the student carries on "1000, 1002, 1004, ..." the teacher can make a judgment that the student understands the concept and the student may infer from the teacher's status as a knower that he performs mathematics correctly. Note how Kripke's description of teaching and learning aligns with the view that warrant is a matter of community consensus, not of individual beliefs, justified in isolation.

> Consider the example of a small child learning addition. It is obvious that his teacher will not accept just any response from the child. On the contrary, the child must fulfill various conditions if the teacher is to ascribe to him mastery of the concept of addition....From this we can discern rough assertability conditions for such a sentence as "Jones means addition by 'plus'." *Jones* is entitled, subject to correction by others, provisionally to say, "I mean addition by 'plus'," whenever he has the

> feeling of confidence—"now I can go on!"—that he can give 'correct' responses in new cases; and *he* is entitled, again provisionally and subject to correction by others, to judge a new response to be 'correct' simply because it is the response that he is inclined to give. (Kripke, 1982, p. 90)

Jones, like his interlocutors, makes provisional judgments of his ability in light of the available evidence. Kripke's move from truth to assertability is a move of practical reason, a move from certainty to best judgments. In section 6.1.3, I present Thomas Nagel's view that the adoption of assertability need not force the abandonment of the notion of truth. In short, the abandonment of certainty as a requirement for knowledge does not necessitate the abandonment of all claims to truth, nor must it place putative claims to certainty outside of epistemological discourse.

I find the constructivist move to assertability conditions unconvincing for two reasons. Both reasons are related to the constructivist failure to account for the content of rule following. I am not convinced that there is nothing to say beyond a description of how we come to agreements. Conformity to the community is assumed to be unproblematic, but to ascertain this conformity requires that a certain state of affairs obtains, i.e. that the community accepts certain uses of words, symbols, and so on. How can I establish *that*? Beyond the problem of how one can know that consensus has been reached, I still remain convinced that greater resources for making sense of rule following and understanding are available than Kripke or the constructivists appear to be ready to admit. First, as I show in the next section, Taylor argues that one can frequently bring forth novel reasons for believing a proposition. These novel reasons are convincing in spite of their complete lack of prior community support. Second, it is implausible that the feeling of necessity that accompanies understanding is a consequence of a dubbing of competency from an external assertability expert. The feeling of the logical "must" comes from the force of the concept, not from the force of acceptance. These arguments will be presented in the sections that follow. The first is based on comments from Charles Taylor and the second from Thomas Nagel.

6.1.2 Taylor's Solution

Taylor recognizes Kripke's contribution in articulating the paradox accurately and meaningfully, but disputes Kripke's skeptical solution. Taylor argues that understanding is partially a social practice, as Kripke argues, but that it is also a part of an agent's physical and social engagement in the world.

> Why can someone always misunderstand? And why do we not have to resolve all these potential questions before we can understand ourselves? The answer to these two questions is the same. Understanding is always against a background of what is taken for granted, just relied on. Someone can always come along who lacks this background, and so the plainest things can be misunderstood. (Taylor, 1995, p. 167)

Taylor describes two ways to understand the background against which thinking occurs. The first suggests that the background is simply given to us; it is not susceptible to any justification. Lakoff and Núñez imply that human neurology at least partially forms the background for thinking. Kripke places the community as the bedrock of human understanding. Taylor is suspicious of these lines of thinking for a simple reason: when pushed, we can often conjure justifications for our actions as we follow rules. If the background of our thinking were really outside of us, Taylor argues, such justifications or explanations would be beyond us. This is a simple, yet powerful point. Recall §217 of the *Philosophical Investigations*.

> "How am I to obey a rule?"—if this is not a question about causes, then it is about the justification for my following the rule the way I do.
> If I have exhausted the justifications I have reached bedrock, and my spade is turned. Then I am inclined to say: "This is simply what I do."

Taylor wonders where it is that justifications run out, and where our spades are turned. If I were asked how to add, say, 68+5, I could give an answer of some sort; maybe I would say "68" and then count up five orally and with my fingers. Suppose that my interlocutor were not satisfied and asked how I knew how to perform the addition in the way I described, I could come up with another presentation. I could give a wide range of justifications to my addition before giving up and saying, "This is simply what I do." This is what mathematics teachers do every day: they answer unanticipated questions by making connections between a concept and other lived experiences. This is the second way of looking at the background against which thinking occurs, and it is the one that Taylor endorses. The background is seen as incorporating understanding. Through our practices in everyday life, we make sense of the world, often in an inarticulate or unarticulated way. Yet, when called upon, we can reflect upon the background and formulate reasons, challenge ideas and give explanations. The background is never fully disclosed, but can be brought to light fragmentarily.

The Kripke-Wittgenstein paradox arises because it appears that a naturalistic accounting of the teacher's ability to bring forward such answers implies that the answers were already present in the teacher's mind before the question was asked. But this is absurd. After reflection, often over long periods of time, new ways of articulating simple ideas come forward. How often does one think of an improved response long after the opportunity to present it has passed? The point is that wherever it is that justifications run out, it is well beyond our everyday conscious application of rules. Taylor's insight here is that we can come up with examples because our rules make sense within aspects of our living in the world. Recall Lakoff and Núñez's claim that the metaphor of counting arises from human dealing with discreet objects (section 3.2.2). Now Taylor wants no part of the claim that the source of our concept of number can be identified, but he does argue that activities such as piling stones provide material for us to articulate our understanding of counting.

Taylor takes examples of this sort to suggest that the background against which thinking occurs is not fixed. If the background were "hard wired" in our brains, or were simply given by the community, then revisions, reexaminations, reexplications and so forth would not be possible. Rather, Taylor suggests that the background of thinking is grounded in unarticulated practices. Most of the time, we are not called upon to provide propositions or explanations to account for what we are doing. But when called upon, we can and do, albeit often imperfectly. The links between our practices and thoughts are "not simply de facto, but . . . make a kind of sense, which is precisely what we would try to spell out in the articulation" (1995, p. 168).

There is an obvious similarity between Taylor's view and large parts of the Lakoff and Núñez model. Like Taylor, Lakoff and Núñez claim that mathematical understanding is grounded in physical engagement in the world. They see the cardinal numbers as generalizations of experiences with discrete objects. But this is something more than

Taylor will allow. One may encounter number within a form of life, on Taylor's view, without the existence of number being contingent on any particular experience. The relationship to particular experiences comes about through attempts to come to grips with meanings of number concepts. The teacher might use any of Lakoff and Núñez's metaphors to introduce children to number concepts, but this gives no account of why the metaphors appear to be true or of how the number concepts come to have meaning for people. Lakoff and Núñez attempt to provide a comprehensive account of why humans have the mathematics they do. Taylor is attempting something much more modest. Taylor claims that human understanding, or meaning-making, is a sort of thinking within the experiences of everyday life. Ideas make sense against a background of engagement in the world, and from this background, people are able to express the sense they make of their ideas.

Taylor's position implies that the mathematics teacher has a concept of number that is partially explicit, but also partially inarticulate. In teaching, some part of the explicitly articulated concept of number is brought to the student, who will have some understanding of what the teacher is saying. Some of the student's understanding will be explicit, but much of it will not be. The student thinks against a background of belief and engagement with the world, and will attempt to make sense of the teacher's lessons in relation to that background. Both the teacher and student have access to explanatory resources of which they are not immediately aware. The student may say, "Oh, addition is just like building with blocks!" The teacher might then ask the student to explain why addition is like building with blocks, and through an exchange of some sort, a negotiation of a meaningful representation of addition with building blocks can come to a fuller explication for both.

A deep difference between Taylor and Lakoff and Núñez is that Taylor refuses to ontologize understanding. According to Taylor, our concepts do provide information about the world and facilitate interpretation of experience in the world, but it is a mistake to claim that our representations are identical to, or are constitutive of external reality. Lakoff and Núñez provide something of a "creation myth" account of both how humans come to acquire mathematics and why mathematics appears to be true. But what can be the point of such a myth? Lakoff and Núñez can plausibly explain modes of thinking about mathematics, but what evidence could there possibly be that indicates that their metaphors create mathematical truth?

Taylor has no interest in such myths. As agents embodied in the world, we always find ourselves in the midst of things; we live engaged in a world that is full of people, these people have ideas that they share, we engage in practices such as reading, listening, speaking and participating in social activities. We have mathematical thoughts in the midst of all this: we do not create them *ex nihilo*, nor do we rely only on a set of already-established metaphorical mappings. That we can give accounts of our beliefs in various forms, and from various perspectives, many of which we are not and cannot be aware in advance provides no evidence for believing that the truth of our beliefs is contingent upon these abilities.

Wittgenstein makes this point with regard to language use: "one forgets that a great deal of stage-setting in the language is presupposed if the mere act of naming is to make sense" (Wittgenstein, 1968, §257). Taylor's point is more general than this. A great deal of stage-setting is presupposed if any kind of understanding—including mathematical understanding—is to make sense.

6.1.3 Nagel's Solution

For Nagel, Kripke's formulation of Wittgenstein's comments on rules and private language form a *reductio*. Wittgenstein takes the ostensibly simple proposition that some fact or facts about a person can explain what he means when following a rule, and then shows that this cannot be the case. "The unacceptable conclusion . . . is that thought is impossible. The faulty assumption . . . is that to think or speak is simply to do something, in the right circumstances and against the right background, which can be described without specifying its intentional content" (Nagel, 1997, pp. 42-43). By this, Nagel suggests that any understanding of mathematical thinking rests upon mathematical thoughts that cannot be explained in naturalistic terms. To understand someone's thought "add two" one must also have the thought "add two". Further, such basic arithmetical thoughts, once one has them, necessarily aspire to universality. One cannot coherently believe that 2+2=4, but only for me, in my current circumstances. To have this qualification, one would have to be able to coherently believe that it is possible that $2+2\neq4$. What could such a thought be?

Nagel notes that the aspiration for universality in reason is separable from the aspiration for certainty. That is, one can believe that certain forms of reasoning are universally valid, while suspending judgment on the conclusions obtained from practical engagement in those forms, or being prepared to revise the belief in universal validity on the basis of convincing new arguments. Consider the argument form of contraposition. Here one argues that if P implies Q and that Q is not true, then P is not true. One can claim that this form of logical argumentation is valid, without making claims about the soundness of any particular arguments that are putatively so structured. Further, one can believe that a piece of reasoning has universal validity, while leaving open the possibility of further revision. A mathematical proof, for example, claims to hold for all relevant applications of its content—this is a claim of universal validity, even if we acknowledge the possibility that the proof may contain an error that we have missed. This is hardly a novel position, but it is one that Nagel insists merits frequent repetition.

> Suppose, to take an extreme example, we are asked to believe that our logical and mathematical and empirical reasoning manifest historically contingent and culturally local habits of thought and have no wider validity than that. This appears on the one hand to be a thought about how things really are, and on the other hand to deny that we are capable of such thoughts. Any claim as radical and controversial as that would have to be supported by a powerful argument, but the claim itself seems to leave us without the capacity for such arguments. (1997, p. 14)

The point is not merely that unqualified skepticism is self-refuting, but that for reasoning to occur at all, there is a wide range of basic thoughts that *must* be accepted. One cannot sincerely say, "My words have no meaning," and expect that the statement could be taken seriously by anyone, including the speaker. One cannot say that one can provide a valid proof of a mathematical theorem, but that the proof represents only a contingent feature of the prover's life. If the proof is valid, then it is valid for everyone and for always. Objections to the validity of the proof also aspire to universal validity. To accept or challenge a mathematical proof, according to Nagel, one must have mathematical thoughts of a certain type. That is, mathematical claims are understood from *within* mathematics. No account of sensory-motor experience or cognitive mapping can show the validity of a proof. Once the analysis of thought is seen to require certain types of thinking among its basic elements, then all attempts to explain thinking without reference to thoughts must be rejected.

Nagel's charge suggests that von Glasersfeld, Lakoff and Núñez, and Ernest and Hersh have put forward logically unsustainable theories. His point is that once a theorist says, "this is the way mathematics is" he has put forward a claim of universal validity. If von Glasersfeld says that mathematical understanding is the temporary abatement of cognitive perturbation, he has made a substantive claim about the way things are, his protests to the contrary notwithstanding (see section 3.1.1). Even if von Glasersfeld claims that radical constructivism is only one vision amongst other competitors, the truth of which cannot be known, he is still making a substantive claim about the way that things are—i.e. that it is true that the truth of radical constructivism is indeterminate. But he simultaneously claims that such substantive claims are impossible, undermining his own claims. This much of radical constructivism is trivially self-refuting.

Nagel is likely sympathetic to Taylor's suggestion that the background against which thought occurs is grounded in an agent's physical engagement in the world, but he doubts that such engagement gets "the last word" in justification. Nagel argues that eventually justifications must come to an end, that eventually no more reasons *can* be given. For Nagel, the last word in many, but not all, epistemological claims cannot coherently be given to contingent facts about the agent, or the agent's community, but must be given to particular thoughts. The last word in arithmetical justifications goes to arithmetical thoughts about what is the case.

> To say that nothing that happens when I hear the instruction "Add two" determines the correct way to carry it out for any arbitrary integer depends

> on restricting one's conception of "what happens" to what can be described in abstraction from its intentional content, and then asking for a retrieval of the intentional content from this denuded material—which is of course impossible. The fallacy is that of thinking one can get "outside" of the thought "Add two" and understand it as a naturalistically describable event. But that is impossible. The thought is more fundamental than any facts about mental pictures or how we find it natural to go on. It is a mistake to pose the question by stepping back from the thought "Add two" itself, looking at the words or accompanying mental images apart from their content, and then asking what their content consists in. (1997, p. 42)

By this, Nagel makes the startling claim that in order to provide a coherent account of whether someone understands an arithmetical rule, part of the account must involve an articulation of the rule in terms of some arithmetical thoughts. Such thoughts are not reducible to naturalistically describable phenomena; they are atomic to certain types of thinking. Do not give an account of what mathematics consists in, says Nagel, unless the account contains some of the mathematical thoughts it claims to explain. Recall section 3.2.2, where I refer to Lakoff and Núñez's attempt to explain the concept of number in terms of object collections, object constructions, measuring sticks, and motion along a path. It is not clear that these metaphors can explain *anything* about number without some presuppositions about basic arithmetic being in place beforehand; we need to know what *number* means before we can construct the study and make judgments about what the results mean. In "numbers as object collection," we are asked to believe that through experience with objects, humans somehow come to the realization that a collection of 5 objects can be partitioned into a group of 2 and a group of 3. But how does this explain anything about the proposition 2+3=5? We can see that the object collection metaphor is a valid model of the proposition only if we understand something about arithmetic already. It is not clear how anyone could move from the objects to the arithmetic without something else going on in their minds. That something else appears to be arithmetic. A naturalistic explanation of number that relies on object collection does not appear to explain why the proposition 2+3=5 is true or why it is acceptable to anyone who does not already possess some arithmetical beliefs. The reason the metaphor appears to explain something is that it agrees with concepts we already hold, not because those concepts consist of the metaphor only. There is a plausible pedagogical case to be made that sensible metaphors such as these have practical use in the classroom because they might help students to clarify certain concepts, but Lakoff and Núñez claim considerably more from their metaphors than that. Nagel's observations strongly suggest that Lakoff and Núñez are claiming much more than they can reasonably support with their theory. Mathematical thoughts may be expanded and clarified through metaphor, but their composition cannot be reduced to non-mathematical metaphors.

Lakoff and Núñez claim that mathematical knowledge and truth are contingent upon human physiology, and Ernest and Hersh claim that they are contingent upon human social convention. These claims are more modest than are von Glasersfeld's, in that they refer only to mathematical truth and knowledge and not to knowledge in general. I will apply Nagel's arguments to their arguments strictly within the domain of mathematics and mathematics education.

If Nagel is right, then the social constructivists, cognitive scientists and complexivists must surrender their claims to comprehensiveness. They may be able to account for some of mathematical understanding within the scope of their theories, but they must necessarily leave some unaccounted for. Whatever it may mean to understand a mathematical proposition, it must mean more than acceptance of the canons of a community, or possession of a number of metaphors, or coupling with other members of a group. Some of the bits and pieces will be fundamental mathematical thoughts that are presupposed in any possible articulation of the proposition.

Suppose that Taylor is correct in suggesting that embodied engagement provides the practical background against which mathematical thinking is performed. Suppose further that a teacher who claims to understand the addition algorithm is able to provide novel, valid responses to queries of his reasons for his faith in the algorithm. Eventually, he will run out of metaphors. As Wittgenstein shows, the number of potential questions is infinite. The teacher is bound finally to say, "Enough!" if his interlocutor gets down to "But how do you know that 2+2=4?" There is no adequate response to this question if it is sincerely asked, according to Nagel. Anyone who can have the concepts of "2", "+", "=" and "4" cannot coherently doubt that 2+2=4. Neither can any account of why I believe that 2+2=4 be made that does not implicitly contain those concepts. To understand a complex arithmetic concept one must have a sense of basic arithmetic, and this arithmetic is one of the places where Wittgenstein's spade is turned.

I recently asked my children, aged 8 and 11 at the time, if they thought it possible that I could give them an addition problem involving two positive integers such that they would not know how to proceed. The 11-year-old asked me if the numbers were infinitely large. I answered that they were not. She asked if I required that she actually perform the addition if the numbers were huge. I replied that I only wanted to know if she felt she would know how the operation was to be done. After a minute or two, she confidently told me that she would know how to proceed. The 8-year-old was less confident. She was prepared to allow that there might be types of addition problems such that she would not know how to proceed. Now think about these responses in light of the present discussion. How can I evaluate their responses in terms of mathematics? It is at best obscure to evaluate their answers in terms of community standards, or in terms of metaphorical mapping, or cognitive perturbations. The 11-year-old clearly understood the infinite normative consequences of her ability to add and the 8-year-old did not. Whatever judgment I might make about their understanding, I have to make it in terms of the mathematical thoughts that comprise integer addition. One child possesses the concept in a reasonably complete way and the other does not. If I were to further probe, "How do you know?" what meaningful responses might they be able to give me? At the bottom of this mathematical discussion lies a mathematical thought that is decisive. The thought is not decisive because the mathematical community agrees to it; the community agrees because the thought is decisive.

The significance of Nagel's argument, relative to this study, is that it puts mathematical *thoughts* front and center in mathematics education research. At one time, such a notion might have been considered obvious, but in light of the four theories discussed in previous chapters, it is not considered so today. For von Glasersfeld, the point of mathematics education is the abatement of cognitive perturbations. Lakoff and Núñez see mathematics education as a matter of forming the right pre-mathematical metaphors. Ernest and Hersh see mathematics education as alignment with accepted conventions. Davis, whom I have barely mentioned in this chapter, sees good mathematics education as the creation of learning communities. His theory informs such community-building by "reading across" other constructivist discourses: he is prepared to work with the epistemological views provided by the others. Many of the pedagogical activities that these views recommend likely have merit in mathematics education, but the theories nonetheless miss what Nagel sees as fundamental: mathematical thoughts are a basic, irreducible element of mathematics and mathematics education. Psychological theories can provide insight into how these thoughts are acquired, and provide recommendations on the appropriate development of these thoughts, but they can provide no insight into what mathematics is or why mathematical propositions are true or false.

6.2 What Has Been Resolved?

This brief exposition of the Kripke-Wittgenstein paradox is not intended to provide any new analysis of the problem of understanding. Rather, it provides a basis to look at a question fundamental to any comprehensive theory of mathematics education. If it truly is the case that Wittgenstein's skeptic cannot be given a "straight" answer, then we must conclude that no noncircular account of mathematical understanding can be given. That is, there is no way to derive the content of mathematics from an analysis of humans, either taken individually or collectively. The full content of mathematical understanding is simply not present in minds or collectivities. I have outlined two plausible suggestions as to where the missing content of mathematical understanding is to be found. Nagel proposes that certain mathematical thoughts reach beyond the individual thinker and point to features of mathematical concepts. He acknowledges that he does not have a positive theory of how this is possible, and suggests that such a theory might be impossible to establish. For Nagel, it is likely that certain basic thoughts form the bedrock where Wittgenstein's spade is turned. Taylor suggests that the bits of mathematical thoughts that are outside of the individual agent are encoded in human engagement with the world. The background against which we think is rich and varied and includes both social and physical structuring. The background is simply too large ever to be brought fully into relief by any agent, but bits of it can be called up for reference. I do not claim that either of these positions has decisively settled the Wittgensteinian problem, but both provide considerably more intelligible resources to the educational researcher than do any of the mathematical "grand theories" I have been criticizing.

Without an adequate response to the Wittgensteinian skeptic, von Glasersfeld, Lakoff and Núñez, Ernest and Hersh, and Davis must surrender part of their doctrines. None of these theorists provide sufficient normative resources to ground the content of mathematics in the ways that they attempt. This leaves open the question of whether their theories can describe mathematical activities, including learning and understanding if supplemented with a normative element. As Taylor and Nagel make clear, it is reasonable to believe that people understand mathematics and it is not plausible that this understanding can be explained merely in terms of physiology or social practices. Something of what it means to be an agent acting in the world (Taylor) and something of what it means to have a particular type of thought (Nagel) appear to be required to make sense of a person's mathematical understanding.

Part of Lakoff and Núñez's position is superficially similar to Taylor's. As I argued above, one key difference is that Lakoff and Núñez claim that the human activity of metaphorically mapping mathematics onto experience is what makes mathematics true, an issue on which Taylor is silent. Taylor rather sees human activity as part of what makes claims about the world comprehensible and articulable and subject to rational scrutiny. This opposition makes any reconciliation between the two views highly non-trivial. I believe that a program such as Lakoff and Núñez's could be created such that it reverses its order of explanation to align with Taylor's view. If so, it would have a very different explanatory structure than does Lakoff and Núñez's present program. Such a program would see the identified metaphorical structure of mathematical thinking as evidence for the reliability of mathematical beliefs, and as a means by which such beliefs can be checked for consistency, both internally and with other types of beliefs. If successful, this program would provide a very important portion of a theory that spells out some of the inarticulate background against which mathematical thinking occurs. It may also provide some insight into how mathematical concepts are formed and revised. It could not, however, form the court in which the truth of mathematical propositions is ultimately determined.

Nagel provides at least a part of that court. As Nagel argues, thoughts cannot be fully disclosed in terms of naturalistically describable phenomena. His view is entirely consistent with Taylor's in that Taylor does not attempt to describe mathematical thoughts in terms of other features of humans or human existence. What Nagel does do is provide some understanding of how individual people can bring their thoughts into discourse, and how they can provide reasons for their beliefs so that they can be evaluated and assessed. Reductions of the sorts offered by the educational theorists I have been assessing are seen not to be consistent with these reflections on thinking and understanding. Understanding is seen not as a property of me or of my community, but of my ability to relate my thoughts to other features of my experience, both independently and communally.

I have not delved into the nature of shared understanding and its relationship to my study. That is the main focus of chapter 7. I wish to extend Nagel's and Taylor's insights further to make sense of the experienced reality of shared understanding. Whatever its imperfections and limitations, human discourse renders meaningful mathematical communication possible. We are able to express our mathematical ideas to one another, and we are able to teach and learn mathematical concepts. To provide a sense of how this is possible, I will appeal to Jürgen Habermas's theory of language. Habermas provides a strong argument that claims that intersubjective sharing and rational understanding are mutually intertwined in such a way that a theory of one must incorporate a theory of the other.

7 On Language and Understanding

In chapter 6, I argued that it is impossible to give a non-circular account of mathematics in terms of human actions, beliefs and behaviours. My claim is that the intentional features of mathematical thoughts cannot be reduced to physical or phenomenal features of agents, taken either singly or collectively. If my arguments are correct, then the mathematics education researcher must find resources outside of psychological or sociological theories to account for the content of mathematics and for the content of mathematical understanding. This conclusion provides good grounds to discard large portions of the "grand theories" from von Glasersfeld, Lakoff and Núñez, Ernest and Davis. These theories attempt to explain too many phenomena with the set of resources at their disposal. Although each explanation is different from each of the others, each theorist takes his account of how people learn to do mathematics and expands it into an account of the justification of mathematical knowledge and into an account of mathematical ontology. While I have left open the question of how much of mathematical learning these theories can accurately depict, I have shown that they fail in their ambitions to account for mathematical epistemology and ontology, and that their theoretical resources are incapable of completing their projects. In short, naturalistic psychological or sociological theories necessarily fail to provide the normative basis of mathematics.

As an alternative to von Glasersfeld, Lakoff and Núñez, Ernest, Hersh, and Davis, I have proposed that mathematics education researchers follow the lead of philosophers that have engaged Wittgenstein's skeptical paradox and have provided resources that keep mathematical content analytically separable from mathematical learning. The two exemplars I have provided—Charles Taylor and Thomas Nagel—do not provide the only possible ways out of the Wittgensteinian paradox, but they do present plausible and interesting approaches. Theories informed by Taylor's or Nagel's positions are not necessarily going to provide successful or complete accounts of mathematics and mathematics education, but they do hold promise for the researcher by providing external accounts of normativity, while leaving available a wide space for theories of learning.

In this chapter, I further develop some aspects of a possibly successful theory in mathematics education by showing what must be accomplished to account for intersubjectivity. I begin by recalling chapter 6: successful theories must acknowledge that some of the content of mathematical thoughts must come from a source *outside* of the agent. When one makes a mathematical statement, the qualifier "for me" or "for us" is empty. When one makes a mathematical statement, one is making a statement that claims universal validity; it is not only true for the speaker, it aspires to be true, full stop. The Pythagorean Theorem is not a statement about my particular psychological state, nor is it as statement about my community; it is a personless and tenseless statement whose application is determined by its truth or its justifiable assertability, not by the desires or actions of the person or persons who encounter it. I mean nothing grand by *truth* in this context. A mathematical proposition is often a claim of provability within a system, or a claim of consistency or inconsistency with other results. Such claims have truth values. As Habermas says, a truth value is something that a proposition cannot lose.

It is my purpose to show how an educational researcher might develop the key ideas of the four grand theories without succumbing to their excesses and inconsistencies. My approach is to look to extant theory within philosophy and the social sciences to build piecewise a picture of mathematical intersubjectivity. I carry this project further in the concluding chapter. I do not claim that the approach that I put forward here is the only possible way to proceed; rather, I aim to show that there are sufficient resources within the mainstream philosophical and social scientific literature to meet the demands of contemporary educational theorists. To that end, I offer but one reasonable alternative.

7.1 Strange Bedfellows

In the sections to follow, I build an argument using insights from three very different philosophers. Following the arguments in chapter 6, I appeal to Thomas Nagel's realist epistemology. From Nagel, I take the view that there are some thoughts that are foundational to understanding other thoughts. For this study, I rely on the insight that in order to judge whether a person understands a mathematical concept, proposition or justification, some of that judgment must come from *within* mathematics; to understand mathematical thinking one must have mathematical thoughts. Charles Taylor provides the educational insight that to provide reasons for mathematical beliefs, one can often rely upon comparisons and evaluations that come from active engagement in the world; mathematical thoughts are not simply abstracted from experience; they can also be justified from experience. I do not take Taylor to believe that one can simply *exit* from mathematical thinking to provide these comparisons and evaluations, but that the mathematical thoughts are a part of a network of action and belief from which the agent cannot exit. I have already provided the basics of these two positions. As I suggested earlier, Taylor argues that part of what it means to engage in the world is to engage with others. To further articulate what this might mean, I appeal to Jürgen Habermas's transcendental pragmatism (Habermas, 1984; Habermas, 1987; Habermas, 1990; Habermas, 1998; Habermas, 2003). Habermas provides an account of communicative action that shows how two people coming to a mutual understanding about some matter necessarily involves the realistic belief that some matters exist outside of those people. Further, the positions two people take in communicative action rely upon the same normative justificatory practices that underwrite epistemological activity.

In spite of substantial theoretical differences between these three philosophers, they provide an overlapping account of a position of practical reason in which mathematics educators and education researchers find themselves. Educators and researchers are usually not concerned with where mathematics comes from; they want to know what is going on in successful or failed mathematical exchanges. How do teachers and students come to a mutual understanding of some mathematical issues? What are some of the conditions under which such mutual understanding fails to come about?

7.2 A Problem Revisited

In section 2.3 I briefly considered three student solutions to the problem of calculating the hypotenuse of a plane right-angled triangle with legs of length 5cm and 12cm.

Solution #1: The student has a pre-programmed right-angled triangle program. She runs the program and is prompted to enter the lengths of the two legs and the program types the output "13" to the screen. She then writes this number in her notebook.

Solution #2: The student uses a graphical program such as "Geometer's Sketchpad". The student uses the construction tools to create segments 5cm and 12cm long. With her mouse, she arranges the segments to be perpendicular at their endpoints. She constructs the third side of the triangle with the mouse. The program calculates the length of the third side to be 12.9999997, which she writes in her notebook.

Solution #3: The student enters the vector (5,12) in her calculator, and then pushes the button that calculates the modulus of the vector. She writes the output 13 in her notebook.

Each of the above scenarios provides too little detail for us to be able to make judgments about each student's understanding of the problem, or of the mathematics behind the solution. All three students demonstrate at least that they are able to apply some relevant skill to the solution of the problem. All three rely upon calculating technology of some kind to produce their solutions. The first student applies a straightforward calculator program, designed to solve problems of this type. What the student recognizes beyond the program's applicability is not evident. The second student makes an electronic model of the problem and measures to find the requested length. The student makes no use of methods that suggest generality, and again there is little evidence to support inferences regarding the student's understanding. The third student has transformed the problem from one about right-angled triangles to one about vector addition. Even so, it is too hasty to infer much about that student's understanding, as it is possible that the student has been shown how to use a particular tool without the student's having any clear ideas about why the tool works, or even what a modulus is.

I bring these examples forward to illustrate the need for contextual evidence in the assessment of mathematical understanding. It is important for teachers and researchers to have access to considerable information about students to be able to infer details of their understanding. This information, as I shall argue in this chapter, can come from the application of theories of how mutual understanding is possible.

As I argued in chapter 6, mathematical thinking takes place against a background of intelligibility. Some of this background is undoubtedly informed by the physiological reality of being a human being. We move about in the world in a certain way, we perceive physical phenomena using sensory apparatus peculiar to our species, and so on. Indeed, both von Glasersfeld and Lakoff and Núñez argue that such considerations form great parts of the background to mathematical thinking. The extent to which this is so is an empirical question that cannot be settled here, but von Glasersfeld's and Lakoff and Núñez's *prima facie* cases seem to me to be very strong. On the other hand, Ernest and Hersh both claim that a significant portion of the background against which we have mathematical thoughts is social, including community practices and language-rich interactions. This seems plausible to me also, but again there appear to be a number of empirical questions to be addressed beyond the initial plausibility arguments they provide. Additionally, both von Glasersfeld and Lakoff and Núñez acknowledge the intersubjective nature of mathematical knowledge, but they provide very little detail as to what this might involve.

Given that humans have strongly similar but not identical bodies, and that they inhabit social spaces that are somewhat similar, but that differ in a number of possibly important ways, it is plausible that research programs into both similarities and differences in the context of mathematical thinking will be fruitful. The nature of the similarities and differences of both the physical and the social structures behind thinking will require elaboration and testing. In this chapter, I will focus only on the linguistically mediated portion of the background of mathematical thinking. Following Habermas, I suggest that the background of communicative action is partially shared with the background of much of human rationality. Further, I argue that the background is best understood not in physical terms, but in terms of its symbolic structuring.

In spite of failing in their attempts to explain the content of mathematics and mathematical understanding in naturalistic terms, there is still some plausibly worthwhile material in the four theories. In particular, all four recognize the possibility of coming to mutual understanding with other people regarding mathematical concepts. Further, they acknowledge that this mutual understanding is a significant part of what it means to teach and to learn mathematics. Davis provides an idiosyncratic account that is not compatible with the other three; he claims that the concept of intersubjectivity can profitably be replaced with *interobjectivity*. I do not find his arguments compelling, as I outlined in chapter 3.

Unfortunately, none of the four mathematics education theories appear to provide a rich enough account of intersubjectivity to allow for a deep understanding of the socially constituted components of the background of mathematical thinking. That mutual understanding of mathematics is possible is evidenced in a number of ways. A mundane example can be seen in everyday discourse. People are able to exchange goods, pay taxes and get along numerically with obvious ease. We simply do not see disagreements over the correct way to charge GST that go beyond momentary misunderstanding. More exotically, ancient documents in long dead languages contain mathematical demonstrations and calculations that can be understood by modern people. Although it is not always easy to understand what is intended, it is nonetheless possible to bridge enormous gaps of language, culture and time to understand, for example, the formula for calculating the area of a circle provided in the Rhind Papyrus nearly 4000 years ago. It does not follow from these observations that all possible mathematical ideas can be communicated to all people in all circumstances; but they do provide good grounds for optimism.

In the following section, I outline a theoretical view of how it is possible for two people to come to a mutual understanding of mathematical ideas. Habermas does not deal specifically with mathematics; his theory of communicative action is very general. I trace through some of the main features of his theory and provide a very modest background to how it works. In section 7.4, I apply Habermas's, Nagel's and Taylor's insights to some general issues in mathematics education and mathematics educational research.

7.3 Habermas's Theory of Communicative Action

As Austin (1962) showed, to speak is to act. The significance of Austin's move is that, if correct, much of the analysis of speech acts can be undertaken in ways similar to those used by social theorists to analyze other types of meaningful behaviour. Austin, Searle and Habermas have all provided typologies of speech acts. There is much similarity between the three; in particular Searle's and Habermas's show refinement of Austin's pioneering work (Austin, 1962; Habermas, 1998; Searle, 1969; Searle, 1979). For this thesis, I focus on Habermas's work, but the analysis I provide is consistent with both Searle's and Austin's theories. Habermas acknowledges that there are several types of action that are possible through speech, but he claims that action aimed toward the attainment of mutual understanding is the dominant form. Habermas claims that other types of action (teleological or strategic action, for example) are parasitic upon communicative action. By 'parasitic' Habermas means that the intentional and purposive content of teleological or strategic action can be explicated in terms of the intentional and purposive content of communicative action. When one uses language strategically, one must rely upon the well understood structures and functions that language has at other times. This does not imply, however, that teleological or strategic actions are reducible to communicative action, either ontologically or analytically.

Habermas defines communicative action as utterances that form part of a discourse of mutual understanding without reservation. The caveat "without reservation" is intended to differentiate communicative from strategic action. Strategic action is action that is used solely for the purpose of doing something that does not require mutual understanding. The command, "Copy these notes," might elicit cooperation in a student, but it is unlikely to bring the teacher and student into a position of mutual understanding because the student understands the teacher's intentions, but the teacher has made no movement to understand the student on the matter of copying. The student's cooperation, on my simple assumptions for the example, is a matter of submission before power. Suppose that a teacher intends that a student come to understand a mathematical concept. Teaching in this case involves working toward a state of mutual understanding between the teacher and student regarding the content of the mathematical concept. For the teacher and student to come to a position of mutual understanding about the concept, one of two things must occur. The teacher may bring forward defensible and persuasive reasons for holding particular beliefs, and the student must accept them. Alternatively, the student may come to believe that defensible and persuasive reasons are available and that the teacher could provide them if asked. If the teacher tells the student, "A tangent to a circle is perpendicular to the radius at the point of tangency," the student may or may not come to share the teacher's belief. The student may repeat the statement for personal gain (e.g. test scores) or the student may adopt a position toward the statement that recognizes its justification, or its potential justification. Only by adopting a stance that acknowledges the justification or potential justification of the statement, Habermas argues, is mutual understanding between the student and teacher possible. What is crucial to this distinction is that the student at some time either recognizes defensible reasons for the belief that the tangent is perpendicular to the radius, or that the student at least recognizes the possibility of an interlocutor providing such reasons.

Central to Habermas's conception of communicative action is the idea that such action is a consequence of social reality while simultaneously being constitutive of it. Communicative action potentially brings two persons to a situation of mutual understanding about some state of affairs that they can both take a stance toward. Communicative action, then, involves working within a shared social space, but the social action that it makes possible is part of what defines that space where cooperation is possible.

> The "aims" that an actor pursues *in* language and can realize only in cooperation with another actor cannot be described as though they resembled conditions that we can bring about by intervening causally in the world. For the actor, the aims of action oriented respectively towards success and toward reaching understanding are situated on different levels: either in the objective world or, beyond all entities, in the linguistically constituted lifeworld. (Habermas, 1998, p. 204)

That communicative action is not merely causally effected in the world is a consequence of the need for two actors to contribute cooperatively to bring it about. A teacher cannot force a student to understand; the student must participate in the matter in some way. This is evidenced in the example of the tangent and radius, above. If the teacher brings the theorem forward and the student merely memorizes it without any understanding (assume for the moment that this is possible), then the teacher's actions are causally efficacious, but the student and teacher are not in a position of mutual understanding. They have not moved into a position where they have a sense of the other's access to the content of the lesson, neither have they accessed one another's rationally justifiable reasons for believing that content. If, however, the student appeals to reasons, then the beginnings of communicative action are in place. If this distinction is sustainable, then the place of assessment in the classroom comes into relief. If understanding is to be assessed, then it is incumbent on the assessor to either provide opportunities for the student to give rationally justifiable reasons for her responses or to find some other means of inferring the student's rational justifications.

Recall the 5-12-13 triangle problem. In order to distinguish rote from rationally motivated responses, the assessor must have some access to the student's reasons. That is, the assessment of understanding is contingent upon the reasonable belief that the student can provide justification for her responses. I will return to this in section 7.4.

Habermas argues that communicative action can be intelligibly reconstructed in terms of validity claims that are necessarily raised by interlocutors. The idea is that when A uses language to come to a mutual understanding of some matter with B, A simultaneously raises three important claims to which B must assent in order for the communicative action to be successful. The three implicit claims relate to the three facets of communicative action. First, A claims that A's words truthfully depict some matter in the world (this can include indirect use, such as irony). Second, A claims to be in a position to make such a claim. Third, A claims to be sincere. B must accept these three claims, even provisionally, in order for mutual understanding to occur. These claims—to truth, normative rightness and truthfulness—are universal claims, transcending all local standards of evaluation. That is, when I claim to be telling the truth, I expect that other people are in principle able to see my claim as true. Even if I modestly claim that I am reporting things as they seem to me, that is a universal claim to truth in that I expect that anyone who understood my position would see that I am truly reporting things as they seem to me. When I claim to be normatively authorized to make a claim, I expect that anyone who examined my position would agree. Again, I can make a very weak claim to authorization, and still expect that my reasons are binding on other rational agents. Even if I believe that my authority is debatable, *that* position is intended to be acceptable to anyone relevantly placed to look at the situation. Finally, the sincerity condition is such that I am claiming to not be deliberately deceiving my interlocutor. The interlocutor who doubts my sincerity can come to some understanding of the matter about which I speak, but cannot come to a mutual understanding with me, as we ultimately believe different things about our collective situation.

7.3.1 The Lifeworld

Communicative action takes place enmeshed in shared beliefs, experiences and norms; Habermas calls this matrix the *lifeworld*. In section 6.1.2, I brought forward Taylor's argument that thinking occurs against a background of intelligibility. Habermas expands this notion to include not only human physical engagement in the world, but also the social world that makes shared activity possible. "Communicative action can be understood as a circular process in which the actor is two things in one: an *initiator* who masters situations through actions for which he is accountable and a *product* of the traditions surrounding him, of groups whose cohesion is based on solidarity to which he belongs, and of processes of socialization in which he is reared" (Habermas, 1990, p. 135). By saying that an agent is accountable, Habermas means that the rational agent is able to provide a justification for action, belief or claim if asked to do so. This is complemented with the idea that a person is to some degree a product of the social structures he lives within. Habermas does not claim that these two facets of human life are isolated from one another; they are very thoroughly intertwined.

The social world, Habermas argues, is *symbolically structured.* This structuring is in part a consequence of the semantic structure of natural language. Within language, certain features of physical and social reality can be articulated at any given time. Further to this, norms, practices, economic systems, laws and traditions provide material for working within the world, for communicating about our world and our place in it, and for further changing the lifeworld itself.

Although Habermas does not write about mathematical communities, the idea of a symbolically structured background is especially salient here. Mathematical concepts, beyond the simple examples of chapter 6, are usually given an explicit symbolic formulation. Children learn to factor polynomials not by abstractly considering the polynomial as an object, but by considering it symbolically. Different symbolic representations are possible, and different representations are likely to be used in different contexts. Regardless, the symbolic structures of mathematical objects such as polynomials are related to other symbols within the community. Roman characters, area models or iconic representations come to mind. Each of these models reaches into the community, pointing toward other features of shared life. The language of the classroom is intended to connect the student to a specialized community, such as the mathematical community, mediated through the overlapping structures of teacher and student lifeworld interactions. Through a combination of specialized mathematical language and everyday language, teachers attempt to bring students to a position of shared understanding of the meaning of the objects under discussion. In an important way, the communicative actions that bring students to understand mathematical objects are also a process of expansion of the shared lifeworld of teacher and student. Not only does the student come to understand something that the teacher understands, but the student and teacher can and do make new symbolic connections between features of their own lives and the matters at hand.

I will stray from my argument to deal with a constructivist objection to the above. Both radical and social constructivists argue that symbols are the true objects of mathematical understanding. That is, they claim that when people believe that they are discussing mathematical objects, polynomials for example, they are talking about symbols, not about objects. On these views, talk about polynomials is always talk about concepts, not about polynomials as objects. My arguments in chapter 6 and the theory of language I have been explicating in this chapter suggest that this constructivist position is unsustainable. I do not need to enter ontological discussions about whether polynomials have a human-transcendent existence. What is central to the position that I have been developing is that in order to think or speak about polynomials at all, I must consider them as objects that have properties that are outside of me or my linguistic community. Anything I can say about polynomials, I can say without referring to me. To say that the polynomial $3n^2+n+2$ has no real roots is to say something about the polynomial, not to say something about me, my concepts, or my friends. Similarly, when students are taught representations of polynomials, there is nothing magical or special about the particular representations or symbols that are used.

The available symbolic resources become part of the shared experiences within overlapping lifeworlds. Through these symbols, mathematical intersubjectivity is mediated. Regardless, the symbols should not be confused with their referents. For example, some students learn to factor trinomials through the manipulation of characters on a page; others learn factoring through physical manipulatives such as algebra tiles. In both cases, the symbols that are chosen are chosen to facilitate communication, not to be the end of the task of factoring. The choice of symbolic representations, then, is not a matter of finding that which truly is equivalent to the mathematical concepts or procedures that are to be learned. The choice of symbols ought to be made with an eye to what best mediates the development of computational facility and mathematical understanding. That the factoring of trinomials over the real numbers can be modeled with algebra tiles is a consequence of an isomorphism between two fields; this in no way implies that trinomial factoring is identical with algebra-tile modeling.

7.3.2 Rationality

Mathematics is a rational activity and mathematics learning involves initiations into mathematical forms of life. A child learns to do mathematics by learning to recognize mathematics, by performing mathematical operations and procedures, and by producing and assessing mathematical justifications. All of these are clearly rational processes. It may be the case that other, non-rationally motivated processes are involved at some stage of these actions, but the larger structures are all publicly rational. Much of the remainder of this chapter is dedicated to developing a sense of the relationship between mathematical rationality and mathematical intersubjectivity. The key insight is that to understand is to be able to provide reasons, and that to be in a position of mutual understanding with another, one must be able, at least in principle, to provide publicly assessable reasons.

Habermas argues that rationality is rooted in three things: knowledge, action and speech. These are not to be understood as independent foundational structures, but as interrelated and entwined structures that integrate, rather than merely support, rationality in its various manifestations: epistemic, teleological and communicative. Further, Habermas claims that human rationality is expressed through but it is not reducible to discourse.

> The rationality of a person is proportionate to his expressing himself rationally and to his ability to give account for his expressions in a reflexive stance. A person expresses himself rationally insofar as he is oriented performatively toward validity claims: we can say that he not only behaves rationally but is himself rational if he can give account for his orientation toward validity claims. (Habermas, 1998, p. 310)

Note that on this account, rationality is embedded in discourse in two ways. First, Habermas claims that discursive practice is necessary for the expression of rationality, and second, it is through discourse that rational self-reflexivity is in evidence. This does not imply that one cannot be rational in isolation; rather it is through social activity that rationality is informed and expressed.

7.3.2.1 Communicative Rationality

Habermas's theory depends upon the centrality of communication to rationality. To understand a concept or proposition one must be able to provide reasons that are comprehensible to anyone relevantly situated to reconstruct your reasoning. That is, the universal features of understanding are related to the universal features of language that make mutual understanding possible. These features are not merely semantical.

> There is a peculiar rationality, inherent not in language as such but in the

> communicative use of linguistic expressions, that can be reduced neither to the epistemic rationality of knowledge . . . nor to the purposive-rationality of action . . . .This *communicative rationality* is expressed in the unifying force of speech oriented toward reaching understanding, which secures for the participating speakers an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world. (Habermas, 1998, p. 315)

Communicative rationality is the use of illocutionary acts to attempt to bring a speaker and a listener into mutual understanding. The illocutionary aim of communicative rationality is to both bring about understanding of intentions, and to bring about acceptance of reasons. It is in the acceptance, or acceptability, of speech acts that they may be judged as rational. One understands something only insofar as one can provide reasons that would be persuasive to oneself and to other people. Central to Habermas's position is the notion that acceptability is contingent upon acceptance from more than one point of view.

The reaching of mutual understanding cannot be seen as a mere perlocutionary effect. For A to bring B into a position of mutual understanding, B has to be actively involved in the process. A's implicit claims to truth, normative appropriateness and truthfulness must be accepted by B. At any time, B can break off the movement toward mutual understanding by simply refusing one of A's claims.

> First, the illocutionary aims cannot be defined independently of the linguistic means of reaching understanding; as Wittgenstein made clear,

> the telos of reaching understanding is inherent in the linguistic medium itself. Second, the speaker cannot intend her aim as something to be effected causally, because the "yes" or "no" of the hearer is a rationally motivated position. . . . Finally, speakers and hearers confront one another in a performative attitude as first and second persons, not as opponents or objects. (Habermas, 1998, p. 316)

This can be seen in the daily exchanges between teachers and students. If the student's words produce nothing more than "uh huh" from the teacher, the student cannot reasonably infer that understanding has been reached. A purposeful exchange of words and ideas that show that the student and teacher understand one another's meanings is essential to the achievement of communicative rationality. The requirement of assent in mutual understanding points to one of the dangers in multiple-choice tests. In response to a test item, the student darkens in the bubble for choice C. Some teleological and communicative rationality is in evidence, but it is of a very superficial nature. Even with a battery of such items, it is hard to see how one can infer a great deal of epistemic rationality—the point of most assessment in schools.

7.3.2.2 Epistemic Rationality

Epistemic rationality refers to claims to knowledge of what is the case. Habermas sees epistemic rationality in terms of propositions: statements that can be true or false. A statement such as the Goldbach conjecture—each even whole number greater than two is expressible as the sum of two primes—is a proposition because it is either true or false; currently no one knows which the case is. Beliefs about the truth or falsity of the Goldbach conjecture are expressions of epistemic rationality; the believer's justifications of such beliefs are evidence of the depth of epistemic rationality she possesses. A claim to knowledge, then, is a claim to the ability to provide reasonable justification of a proposition. It is not necessary that a knowledge claim be demonstrably true for it to be rational, but only that the claimant be able to make an intelligible and defensible case for the belief. Epistemic irrationality is the holding of beliefs regardless of their justification, or lack thereof.

For the purpose of this study, I assert that epistemic rationality is a legitimate end of mathematics education. An argument for epistemic rationality as a goal of education could be that it supports the goal of personal autonomy. It could be argued to be respectful to persons as persons that they are able to provide reasons for their actions. Appeal could be made to pragmatic ends in education. Whatever it is that students are to get out of instruction in mathematics, presumably at least some of it is expressible in propositions. I cannot imagine a mathematics education that did not expect that students take a position regarding the truth of propositions. What sense can we make of a view of mathematics education that refuses to take a stand on whether the sum of the angles of a plane triangle is equal to two right angles? The objection that plane geometry is not the only conceivable geometry gets us nowhere, as the objection is itself a mathematical proposition, and for it to even be considered we must consider its truth value. Accepting the objection is not a problem either; all that needs to be done is to recognize that the question is set against one particular background and not another.

7.3.2.3 Teleological Rationality

Teleological rationality is a consequence of the intentionality of purposive action. That is, teleological rationality is expressed through action oriented toward the attainment of some set goal, however modest.

> A successful actor has acted rationally only if he...knows why he was successful (or why he could have realized the set goal in normal circumstances) and if this knowledge motivates the actor (at least in part) in such a way that he carries out his action for reasons that can at the same time explain its possible success. (Habermas, 1998, pp. 313-314)

There is an obvious connection between teleological and epistemic rationality. If an agent is to carry out a rational plan to achieve some end, then the plan must be informed by beliefs about what is or is not the case, i.e. the plan is informed by propositions. In each of the strategies employed in the 5-12-13 triangle problem, the student is using a tool as a means to the end of computing the length of the hypotenuse. The student's understanding of the problem is partially expressed through action, but the full richness of understanding is not.

In mathematics education, we often see a distinction between what students are expected to know and what they are expected to be able to do. It is one thing to be able to find the length of the hypotenuse of a right-angled triangle; it is another to understand why a particular method of finding it is correct. The provocative element of the three suggested solutions to the 5-12-13 triangle problem is that it is difficult to know why each student chose the strategy that she did. Many assessments of student learning would simply take the correct side length as an indication that the student possesses sufficient understanding of the problem. That is, teleological rationality is taken as a proxy for epistemic rationality. My analysis suggests that this is a mistake. I do not claim that teleological and epistemic rationality are isolated from one another; the problem in education is that their interdependence is such that it is difficult to say what evidence of one type of rationality says about the nature of the other.

7.4 Rationality, Communicative Action and Mathematics Education

I have been piecing together a view of mathematical rationality that is grounded in communicative action. I am now in a position to deal with the two explicit questions that began this study.

1) What must be true about a person in order to say that the person understands a mathematical concept, proposition or justification?
2) What must be true about two persons to say that they have come to a point of mutual understanding of a mathematical concept, proposition or justification?

It turns out that the answer I give to the first question is closely related to the answer I give to the second. To understand a mathematical concept, proposition or justification, one must take a stance that recognizes the claims to universal validity upon which the concept, proposition or justification is contingent. That is, the concept, proposition or justification must be seen as understandable by other rational agents. I understand a concept, proposition or justification when I know the conditions under which another person can accept it as true. This is superficially similar to Kripke's suggestion, as explicated in chapter 6. The difference is that Kripke identifies the conditions under which A can judge that B understands; Habermas is identifying the conditions under which the understanding occurs. Kripke's position (or at least his reading of Wittgenstein) leads to the argument against private rule following: under the Kripkean interpretation, it is impossible to make the judgment that a person understands without entering into a common form of life with that person. Fair enough. But suppose that an isolated person wished to check whether he, himself, understands something. Surely we can make sense of that scenario. The Habermasian solution that I am promoting here suggests that the isolated person understands if he can take a position in which he imagines what it would be for someone other than himself to agree. Insofar as he can convince himself that an arbitrary (but appropriately placed) interlocutor could come to mutual understanding with him, then he is justified in believing that he understands. Of course, a genuine interlocutor under conditions of practical reason might disagree with him; this could cause him to re-evaluate his reasons.

The advantage of the Habermasian view over Kripke's (or, I contend, over Ernest's or Hersh's) is that it captures the social constructivist intuitions about how people make judgments about one another's understanding, while at the same time providing a means by which an individual can make sense of his own understanding. Let me recast this in terms of the 5-12-13 triangle problem.

Suppose that a student were faced with the problem of finding the length of the hypotenuse of a right-angled triangle with legs of length 5cm and 12cm. Whichever method of solution the student chooses, the choosing itself indicates some teleological rationality. Suppose further that the student wonders why this particular method is a good one for this particular problem. The very asking of that question implies its universality. The question makes no sense unless the student holds out the possibility that there are ego-transcendent reasons for using one method rather than another. Without holding out that possibility, the student could not even formulate the question, "Why ought I to do it this way?" I have deliberately used the word "ought" in this context to recall the fundamental normativity of mathematics. There are right ways and wrong ways to proceed, and as soon as one enters into consideration of right and wrong, one has gone beyond the first person, into a second-person stance.

For two people to come to a position of mutual understanding, they must take the stance with one another that the rational agent takes with herself. For a student to understand some mathematics, she must be able to bring forth reasons for someone else to agree about the propositional content of the mathematics. To reach mutual understanding with another person about the mathematics in question, she must either convince the other person that her reasons are good ones, or at least that she could provide such reasons if required to do so. Note the convergence with Taylor's notion of practical engagement in the world. We do not have our justifications always ready; often we must find resources within the lifeworld to support our beliefs. When a teacher assesses a student's understanding, the student's mathematics is presented to the teacher. The teacher then attempts to determine whether the student has provided rationally defensible justification for the content of the work. Alternately, the teacher may be satisfied that the student could provide such justifications if pressed on the matter. This is a serious difficulty in educational assessment.

For the teacher to accept a justification as it is given the teacher must make inferences about the teacher's and student's overlapping lifeworlds. Some of this is unproblematic. There are ways of living in the world that are taken for granted, including language and certain types of physical and mathematical experience. In the case of assessment by the classroom teacher, this background is likely to be well understood on the basis of shared experiences and of periodic communicative action bringing the student and teacher into shared understanding on relevantly similar matters. A blind assessment such as a standardized test poses a much more difficult theoretical and practical challenge.

Whatever the student response consists in, the blind assessor must make inferences about a student that he has never met, and of whose mathematical experiences the assessor has only a vague notion. One way out of this situation is to enter into discourse with the student. Why did you do this? Explain what you are attempting here? Through the giving of reasons, the student is able to demonstrate the relevant understanding required in a mathematics examination. Of course, there are institutional barriers to this approach. First, many tests are blindly assessed, making communication between the student and the assessor impossible. Teachers typically do not have sufficient resources to engage in comprehensive questioning with each student with regard to each desired educational outcome.

Testing theorists are aware of this difficulty. Given the institutional barriers to genuine discourse, it is desirable that a proxy for discourse be developed. It is conceivable that inferences of epistemic understanding from teleological performance can be made through the judicious application of cognitive models (Norris, Macnab, & Phillips, in press). Such models would be structured on principles similar to those developed in chapters 6 and 7. That is, epistemic understanding is seen as an orientation to some propositions whose truth is independent of the first-person perspective. Epistemic understanding is seen to be expressible through the same communicative processes that underwrite a movement between two people toward mutual understanding. These models, however they are structured, *may* be validated through detailed discursive practices between assessors and students relevantly similar to those who will be tested. However, relevant similarity will likely prove to be a difficult criterion to develop. The validation group that will serve as proxies for the tested group will need to be chosen such that their lifeworlds match the test group's lifeworlds in ways that provide overlap between relevant mathematical experiences. It seems to me that this overlapping is as much an empirical question as it is a philosophical one. My sketch suggests possibilities; I leave practical questions of workability to the testing community.

Two significant conclusions come from my arguments in this chapter. The first, rather modest conclusion is that discourse is a reliable way to assess student understanding. The second, more striking conclusion is that understanding itself can be modeled on discourse.

The modest conclusion that having discourse with students is a reliable way to assess their understanding might seem something of a let-down at this point. Surely everyone knows this already! A quick review of chapters 2 and 3 of this thesis contradicts such a supposition. Von Glasersfeld calls intersubjective experience a powerful source of confirmation of individual constructions, while simultaneously denying that intersubjectivity can occur. Ernest and Hersh claim that intersubjectivity is possible, but deny that intersubjective agreement is about any objects outside of the symbols of discourse. Davis claims that intersubjectivity is not possible, and that through shared discourse individuals become part of the object on which they are focusing. The sole theorists consistent with my line of reasoning are Lakoff and Núñez. They acknowledge that intersubjectivity is necessary for the development and maintenance of the discipline of mathematics. Further, their physicalistic foundationalism implies a universality to mathematics consistent with the Habermasian thesis. I suspect that a cognitive science approach, purged of its epistemological and ontological excesses, can serve as a workable and interesting theory of mathematics education.

The stronger conclusion is that understanding can be reconstructed on a discursive model. To say that Jane understands the proposition "angles inscribed in a circle that are subtended by the same arc are congruent" is to say that Jane could bring another person to a mutual understanding of the proposition. Jane understands the proposition only insofar as she can provide reasons that are compelling to another person, similarly situated in the world. If Jane is not engaged with an interlocutor, she can still construct her understanding internally by raising questions for herself, and by taking a stance that she believes would satisfy any relevantly placed interlocutor. That is, Jane's understanding can be reconstructed as an interior dialogue wherein she recognizes that publicly comprehensible reasons for her belief *can be* provided and, further, she is able to provide them.

These observations have the potential to guide education in two important ways. First, models of instruction might be profitably informed by the notion of communicative action. One of the purposes of instruction is to bring students and teachers into mutual understanding of some concept or proposition. Lifeworld considerations as well as considerations of teleological, epistemic and communicative rationality may help to frame the scope and content of instruction. Assessment involves the student bringing the teacher to a position of mutual understanding of what the student already believes to be the case. Assessment models may be profitably based upon communicative action as well.

Part Four: A Way Ahead

The thesis ends with a summary of the arguments already presented, and with some preliminary thoughts about future research projects that build on Taylor's, Nagel's and Habermas's theories. It is necessary for educational and research purposes to assume the content and practice of mathematics as part of the initially unquestioned background against which mathematical thinking and learning can take place. I suggest that a communicative model of understanding holds promise in educational assessment and research because it may provide means of inferring student understanding either through direct discourse or via models that reconstruct internalized dialogue. Mathematics education research has access to many powerful resources well beyond those provided by von Glasersfeld, Lakoff and Núñez, Ernest, Hersh and Davis.

8 Desiderata Revisited

In chapter 2 I laid out three general desires for theories of mathematics education. I suggested that useful theories of mathematics education must have something significant to say about:

1) The nature of mathematics,
2) Educational practice, and
3) Mathematical thought and performance.

I do not claim that theories of mathematics education must make substantial contributions to all of the above. Such theories may clarify some aspect of one of these areas, so long as it is consistent with defensible accounts of the others.

Of course, it is possible that none of the above can be completely articulated by a single simple theory. If we consider mathematics as a collection of practices and results, there is no obvious reason that a single theory can have very much worthwhile to say about it. As I have noted at several points in this thesis, standards of rigor and clarity in mathematics have been fluid across culture, time and circumstance. What appear to be well-formed expressions in a grade 4 classroom might not appear to be well-formed to a mathematician, at least not without some charity.

The four mathematics education theories I have considered in this study all contain claims to substantial contributions to scholarship in all three areas. My arguments have cast doubt on the possibility that these theories can have much of substance to offer in the way of explicating the nature of mathematics because they all attempt to give a reductive, non-normative account of a fundamentally normative activity. All four theories attempt to revolutionize the teaching of mathematics through revealing the subject matter to be a consequence of human contingency. In each case, the claim to contingency leads to a claim to making mathematics more accessible to more people. I do not show that these programs are incoherent, but my arguments in chapter 5 suggest that regardless of what one believes about the project of democratizing mathematics education, such a project is independent of the ontological and epistemological considerations provided in the four theories. In chapter 6, I invoked Wittgenstein's celebrated arguments about rule following to show that the four theories attempt to do what Wittgenstein shows to be impossible: they try to explain the content of mathematics and mathematics understanding in naturalistically descriptive terms. Useful theories of mathematics education, I argued, must import some of the content and practice of mathematics into their descriptions of what teachers and learners are doing. Finally, I argued in chapter 7 that mathematical thinking and performance is closely tied to notions of mathematical intersubjectivity. Regardless of the ontology of mathematical objects, mathematical intersubjectivity requires that the objects be treated as publicly accessible and whose properties are independent.

In this final chapter, I recount my arguments and set out some modest directions for further research in mathematical thinking, performance and understanding.

8.1 An Account of the Nature of Mathematics

It should be clear now that it is not necessary that a theory of mathematics education provide a new or compelling account of the nature of mathematics. Not only is such an account unnecessary to the project of understanding the human acquisition of mathematical concepts, it is very unlikely that anyone could account for mathematics and mathematics learning in a single, unified theory. At the most, one might expect complementary theories of mathematics and mathematics education. I have argued that it is necessary for educational purposes to assume the content and practice of mathematics as part of the (initially) unquestioned background against which mathematical thinking and learning are to take place and to be understood by assessors and researchers.

As I argued in chapters 5 and 6, von Glasersfeld's radical constructivism, Lakoff and Núñez's cognitive science, Ernest's and Hersh's social constructivism and Davis's complexity science all attempt to provide accounts of both the content of mathematics, and why we have confidence in its methods and results. All four fail in their ambitions, each for much the same reason as the others. Each attempts to ground the content of mathematics and mathematical understanding in mathematical practice.

In each case, the move to account for mathematics in terms of its acquisition appears to be partially motivated by a desire to make mathematics more accessible to more people. Each promotes his theory as a social or political treatment of elitism in mathematics. In each case, the claim is made that the adoption of the theory would have the positive social consequence of freeing people from the burden of elitism in mathematics, making the mathematical experience more readily accessible to all. In each case, this seems to me to be a dubious claim for two reasons. First, it is never established that beliefs about the nature of mathematics have any bearing on the learning of the subject. This is an empirical question for which none of the theorists provides any evidence, and one upon which this thesis has cast doubt. Second, even if the empirical question were established, one still has the burden of evidence for the coherence and applicability of the theory at hand. As I have argued in chapters 6 and 7, that evidence is not provided in any of the four cases. Note also that if a student's beliefs about the nature of mathematics were to pose an impediment to the learning of mathematics, it does not follow that the student must change those beliefs in order to learn mathematics. The student could be taught to overcome this conflict in a number of ways, including learning more about the gap between the content of mathematics and the learning of the subject matter. Again, this is an empirical, not a philosophical problem for education.

At a more fundamental level, the Kripke-Wittgenstein paradox provides strong arguments leading me to doubt that the content of mathematics can be derived solely from an analysis of the practice of agents, taken either singly or collectively. In order to determine what mathematics another person is performing, one must already be engaged in some related mathematical thinking. Von Glasersfeld's and Lakoff and Núñez's attempts to show that number is abstracted from experience beg more questions than they answer. The process of abstraction is at best a mysterious one; what could it mean to abstract number from experience? The proposed explanation appears to be nothing more than the observation that a person at one time does not have a concept of number, but at a later time does have some such concept. It is not clear how the invocation of the vague concept of abstraction from experience explains anything.

Regardless, I believe that attempts to analyze the mechanisms through which individuals learn to think and perform mathematically are potentially very important to mathematics education. These mechanisms are the appropriate objects of empirical theories of learning. They cannot be derived *a priori* from ideological statements of what mathematics *ought* to be.

8.2 The Possibility of Education

Reuben Hersh sees the possibility of learning its content and being initiated into its practices as one of the defining features of mathematics. Certainly, from the point of view of mathematics as a human activity, he is correct. Curiously, his and Ernest's social constructivist views offer very little in way of concrete educational description or prescription. That mathematics education involves initiation into forms of life has been well understood since well before anyone even considered mathematics education as an academic discipline. The social constructivist claim that the content of mathematics is a consequence of group acceptability gains nothing for the enterprise, so far as I can see. In order to deal with mathematics, as I argued in chapter 7, one must treat mathematical objects as something that transcend the first person perspective. No amount of argument over whether this is an ontologically sound move can change that basic conceptual truth. That a humanized mathematics might have certain propagandistic advantages, as suggested by von Glasersfeld, Ernest, Hersh and Lakoff and Núñez I will leave as an open empirical question; there is to date no compelling argument to begin such a campaign.

There is some promise in the sort of work initiated by Steffe and Cobb (1988), inspired by von Glasersfeld's theory (see section 3.1.5). They combine empirical and interpretive methods to try to find physical mechanisms by which small children make numerical sense of physical experience. No ontology or epistemology of number is required for this research. The researchers begin with a firm concept of the arithmetic that they believe that the children will learn, and their analysis and interpretations will be based on that immersion. I see no difficulty with and indeed find considerable promise in such an approach. When the interpretations drift into von Glasersfeld's ontology and epistemology (and they do) then the research loses its way.

I am more optimistic about research inspired by Lakoff and Núñez than I am about that inspired by von Glasersfeld. The notion that much of human reasoning can be partially analyzed in terms of metaphors and other generalizations is interesting and provocative. Further, the idea that such metaphors can come from experiences as well as currently held concepts is worthy of further empirical study. Again, I cannot see any utility in hanging onto Lakoff and Núñez's dubious ontology and epistemology. As I argued in chapters 6 and 7, the move from why people believe X to why people ought to believe X is neither simple nor direct and it cannot be made without supplementing empirical theory with normative considerations. Once normative considerations enter the description, then Lakoff and Núñez's reductive account loses much of its purpose.

I dealt briefly with Davis's notion of interobjectivity in chapter 3. While the concept is consistent with the view that mathematics can be taught and learned, it is not clear what practical gain is made in adopting the position that each act of understanding is an act of complicity. Suppose that it were true that every time a teacher teaches students that $\ln(a*b) = \ln(a) + \ln(b)$ the very nature of logarithms is somehow changed. Presumably, the change would be very small. To my knowledge, no one has ever observed such a change, nor even provided a theoretical estimate of its significance. Even so, what possible difference could such an effect have upon the teaching and learning of logarithms? I accept the general claim that whenever people do something the universe is slightly altered as true but trivial. Whatever the effects are, they seem to be utterly insignificant to the nature of the objects of mathematical inquiry.

In summary, none of the four theories offers the revolution in our standard conceptions of mathematics and mathematics education that it claims. Some of von Glasersfeld's and Lakoff and Núñez's claims about how mathematics is processed internally are worthy of further study. There is much to learn still about the ways that mathematical thinking takes place, and there is an important role for such research in education. Whatever it is that radical constructivism or cognitive science have to offer mathematics education, it will be in the area of the psychology of learning. None of the proffered epistemologies offer any substantial gain to educational theory.

8.3 Mathematical Thought and Performance

In chapter 2, I emphasized the difference between a person and a calculating machine. The machine bears information that can be interpreted in a mathematical way; a person does mathematics by making that interpretation. The calculator or computer does not do mathematics any more than a ruler and compass do. Tools lack the intentionality required for the understanding that underwrites mathematical performance. The appropriate role of individual tools in the learning of mathematics is an interesting question, but not one that any of the work described in this thesis pursues.

I take it as uncontroversial that mathematical thinking is causally efficacious in mathematical performance. Recall once again the 5-12-13 triangle problem. In each of the three student performances, it is reasonable to infer that the students had reasons for solving the problem in the ways that they did. Each student had some set of beliefs about the content of the problem and applied some level of understanding to those beliefs to choose and follow through with a strategy. Beliefs and understanding form part of the causal nexus of each students' performance.

Educators have long struggled with how to deal with belief and understanding in the explanation of performance. Performances are considerably easier to observe and assess than is understanding. As I argued in chapter 6, observation of performance alone is not sufficient to infer the content of the rules that a person is following because such observations are not sufficient on their own to support inferences to understanding. In chapter 7, I argue that such inferences become possible only with the importation of physical and lifeworld structures that allow for the rational reconstruction of communicative action. Through processes of coming to mutual understanding, two agents can negotiate mutual claims to truth, normative rightness and truthfulness. It is through communicative action that we are able to make judgments about another's understanding. This points to a possibly fruitful research program in educational assessment.

In some circumstances it is possible and desirable to engage in language-rich interactions with students. Oral examinations, for example, provide the opportunity for students to attempt to bring the examiner into a mutual understanding with the student with regard to some of the student's mathematical concepts. The examiner can reasonably suspend judgement until the student provides adequate reasons for holding certain beliefs, but reliable (though fallible) judgments of understanding are possible. In examinations in which extensive dialogue is not possible, examiners may be able to construct models of student understanding based upon the notion of an internalized dialogue. I have argued that one understands a mathematical concept insofar as one can provide reasons that would be compelling to any relevantly situated interlocutor, at least in principle.

8.4 Further Research

I close with some thoughts about possible future application and extension of my research. I believe that this thesis can contribute to fruitful empirical research in mathematics education by clarifying some of the reasonable applications of mathematics education theory. I also believe that my results can lead to some further philosophical explorations.

From my discussion of the Kripke-Wittgenstein paradox, I would like to see further work regarding the limitations of naturalistic theory in education. Researchers should be able to better clarify their theoretical constructions if they are mindful of the limitations of their naturalistic descriptions, and take care to articulate explicitly their supporting normative structures. It is likely that by taking features of mathematics as given, rather than observed, the researcher will be better able to focus on the mechanisms that underwrite concept formation and understanding, as well as those that related understanding to performance.

Also from chapter 6 comes an awareness of the pervasiveness of mathematical "creation myths." It seems somehow significant to some researchers that they get to the bottom of where mathematics comes from. My arguments strongly suggest that this is an issue of no moment to education. A detailed treatment of what such claims entail, how they are unlikely to be successful and how, even if successful, they are unlikely to have much to give to eduational research or practice will be a positive contribution to the philosophy of mathematics education. I plan to further explore the nature of mathematical reasoning as practical reason. That is, one performs mathematics not from a foundational bedrock up, but from within the horizon of a lifeworld.

From chapter 7, the need for a more detailed model of mathematical rationality emerges. I have provided an outline of how one could be created from Habermas's theory of language, but considerably greater detail will need to be added for it to be useful.

Finally, this thesis should lead to a further exploration of the relationship between the bringing of interlocutors into intersubjective agreement and the individual possession of understanding. If, as I have argued, a robust model of individual understanding can be developed around the idea of communicative action, then the potential for improved instructional strategies and assessment procedures could follow.

Much of this thesis has been dedicated to an analysis of what theories of mathematics education cannot do. The arguments all turn on the necessity of normative considerations in the interpretation of mathematically meaningful activity. It is my hope that whatever merits my negative arguments have, they are overshadowed by a positive research program that brings a greater understanding of mathematics education through a combination of naturalistic and normative theories. It is my sincere hope that one consequence of this work is that it places the subject matter of mathematics prominently in front of mathematics education research.

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