On The General Theory of Optional Stochastic Processes and Financial Markets Modeling

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Mathematical Finance

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Abstract

Optional processes including optional semimartingales are not necessarily right or left continuous. However, optional semimartingales have right and left limits. Moreover, optional processes may exist on "un-usual" stochastic basis where the increasing information-filtrations are not complete or right continuous.

Elements of the stochastic calculus of optional processes is reviewed. The linear stochastic differential equations with respect to optional semimartingales is solved. A solution of the non-homogeneous linear stochastic differential equation and a proof of Gronwall inequality are given in this framework. Existence and uniqueness of solutions of stochastic equations of optional semimartingales under monotonicity condition is derived. Comparison theorem of solutions of stochastic equations of optional semimartingale under Yamada conditions is presented with a useful application to mathematical finance.

Furthermore, a financial market model based on optional semimartingales is proposed and a method for finding local martingale deflators is given. Several examples of financial applications are given: a laglad jump diffusion model, Optimal debit repayment and a defaultable bond with a stock portfolio. Also, a pricing and hedging theory of a contingent claims for these markets is treated with optional semimartingale calculus.

Finally, a new theory of defaultable markets on "un-usual" probability spaces is presented. In this theory, default times are treated as stopping times in the broad sense where no enlargement of filtration and invariance principles are required. However, default process, in this context, become optional processes of finite variation and defaultable cash-flows become optional positive semimartingales.

Preface

This research has been motivated and conducted under the supervision of Professor Alexander Melnikov and would not have been possible without his support. The thesis was initially motivated by the solution of the filtering problem for special semimartingales on usual probability space in relation to the existence of a single probability measure on this space. The singleness of the probability measure is key for the existence and uniqueness of solution of the filtering problem. It turns out that the filtering problem and singleness of probability measure are strongly connected to the problem of semimartingale invariance under changes of filtrations either by restriction or enlargement. The main question we wanted to address in filtering was: what if in certain filtering problems there were many probability measures? If this is the case, we will loose the right continuity of filtration which is required for constructing the solutions of the filtering problem. Right-continuity of filtration and its completeness are key for solving the filtering problem.

This realization lead us to a new stochastic calculus; The stochastic calculus of processes on unusual probability spaces – probability spaces where the information sigma algebras are neither complete nor right or left continuous. These are bizarre spaces and processes with complicated jump structures. They were actually studied by famous mathematicians like Meyer, Dellacharie and most importantly by Galchuk. Unfortunately, this form of stochastic calculus remained largely ignored by the larger community of probabilist and financial mathematicians. So, the usual versions of stochastic calculus saw a huge applications into finance and economics.

However, it turns out that the calculus of process on unusual spaces is intimately connected with the filtering problem. Moreover, as we will show in this work that the calculus of optional processes will have profound influence in mathematical finance, especially in the areas of defaultable markets, theory of optimal consumption, stochastic control, and many more.

In chapter 1 and 2 we introduce the stochastic calculus of optional process and motivate its use in mathematical finance. Chapter 3 expands on the calculus of processes on unusual probability spaces to solve problems in the theory of stochastic equations such as comparison theorem which is important for approximation of solutions for certain financial problems. Chapter 4 and 5 are motivated by the need to apply the calculus of optional processes on unusual spaces to financial markets and markets with defaults. Throughout the thesis we try to give useful financial examples of the most important and fundamental problems in finance: ladlag jump diffusion market, defaultable zero coupon bond and portfolio, debit repayments etc.

Acknowledgments

First and foremost, I would like to express my sincere gratitude to my advisor Prof. Alexander Melnikov for the support of my Ph.D study and research, and for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in research and writing of this thesis. I could not have imagined having a better mentor for my studies. Also, I would like to thank my thesis committee: Prof. Byron Schmuland, Prof. Alexander Litvak, Prof. Christoph Frei, Prof. Vakhtang Putkaradze and Prof. Alexandru Badescu for their careful reading of my thesis and insightful comments.

I am grateful for the financial support I have received from the Department of Mathematical and Statistical Sciences. I thank all the staff in the Main Office, especially, Prof. Jochen Kuttler, Prof. Thomas Hilen, Tara Schuetz, Leona Guthrie, Rick Mickalonis and Patti Bobowski. This doctoral dissertation was supported by Natural Sciences and Engineering Research Council of Canada discovery grant #5901.

Finally, I would like to thank my Parents Nabeel and Soad for instilling in me the love of knowledge and for their love, support and motivation. Also I would like to thank my siblings Mohanned and Lulu for their support and love. Finally,

"**Gratias ago** ∞".

Contents

Chapter 1

Introduction

The stochastic basis $\left(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}\right)$ – a probability space $(\Omega, \mathbf{P})$ equipped with a non-decreasing family of σ-algebras also known as a filtration or information flow $\mathcal{F}_t \in \mathbf{F}$, $\mathcal{F}_s \subseteq \mathcal{F}_t$, for all $s \leq t$ is a key notion of the general theory of stochastic processes. The theory of stochastic processes is well-developed under the so-called "usual conditions": $\mathcal{F}_t$ is complete for all time t, that is $\mathcal{F}_0$ is augmented with $\mathbf{P}$ null sets from $\mathcal{F}$, and $\mathcal{F}_t$ is right-continuous, $\mathcal{F}_t = \mathcal{F}_{t+} = \cap_{u>t}\mathcal{F}_u$. Under these convenient conditions adapted processes from a very wide class, known as semimartingales, that can be seen as processes with right-continuous and left limits paths (RCLL).

The stochastic calculus of RCLL semimartingales on usual spaces is also a well developed part of the theory of stochastic processes. This has a number of excellent applications in different areas of modern probability theory, mathematical statistics and other fields specially mathematical finance. Many fundamental results and constructs of modern mathematical finance were also proved with the help of the general theory of stochastic processes under the usual conditions. It is difficult to imagine how to get these results using other techniques and approaches.

Moreover, these two areas of research and applications are interconnected with each other; Namely, not only the general theory of stochastic processes, often called stochastic analysis, is important for mathematical finance, but also, the needs of mathematical finance sometimes leads to fundamental results for stochastic analysis. An excellent example of such influence, is the so-called optional or uniform Doob-Meyer decomposition of positive supermartingale.

Nevertheless the pervasiveness of the "usual conditions" in stochastic analysis there are examples showing the existence of a stochastic basis without the "usual conditions", see for instance Fleming & Harrington (2011), p.24 [1]: If we suppose

$$X_t = \mathbf{1}_{t>t_0}\mathbf{1}_A,$$

where A is $\mathcal{F}$-measurable with $0 < \mathbf{P}(A) < 1$ then filtration $\mathcal{F}_t$ generated by the history of X, $\mathcal{F}_t = \sigma(X_s, s \leq t)$ is not right continuous at t_0, i.e. $A \notin \mathcal{F}_{t_0}$,but $A \in \mathcal{F}_{t_0+}$, and it is not possible to make it right continuous. Furthermore, is it really natural to assume the usual conditions are "true"? The completion and right continuity are arbitrary construct to make is easy to carry on with analysis and prove certain results that would have been rather difficult to prove otherwise. Let us consider the notion of completion; Completion requires that we know a-priori all the null sets of $\mathcal{F}$ and augment the initial σ-algebra $\mathcal{F}_0$ with these null sets. In other words it is an initial completion of the probability space by future null sets. Furthermore, assuming that the σ-algebra is right continuous $\mathcal{F}_t$ is also rather unnatural; it means that the immediate future is equivalent to the present which is different from the immediate past! Also, with right continuity of filtration, events like $(\tau(\omega) = t)$ for all t must have a null total probability. All this, leads us to believe that the usual conditions are too restrictive and rather unnatural to assume.

Moreover, as we have noted earlier that stochastic processes on usual probability spaces leads to a calculus of RCLL semimartingales only. However, as we will see later in this work that there are many processes that are not RCLL, for example, consider the sum or product of a left continuous and a right continuous semimartingales. Does a calculus of such processes exists? and on what types of probability spaces can it be defined? We know that such processes must be mostly excluded from the framework of stochastic processes on usual basis, and, if they were to be considered their use must be loaded with assumptions and restrictions.

Consequently, famous experts of stochastic analysis, Dellacherie and Meyer (1972) [2], initiated studies of stochastic processes without the assumptions of the usual conditions. Dellacherie called this case the "*unusual conditions*", and the stochastic basis became known as the unusual stochastic basis or the unusual probability space. We will follow this terminology in this work. Dellacherie and Meyer began their studies with the process,

$$\text{"}\mathbf{E}\left[X|\mathcal{F}_t\right]\text{"} \tag{1.1}$$

where X is some *bounded* random variable in $\mathcal{F}$ and $\mathcal{F}_t$ is not complete or right or left continuous. Their goal was to find out if there exist a reasonable adapted modification of the conditional expectation (1.1). They have proved the following projection theorem,

Theorem 1 *Let X be a bounded random variable then there is a version X_t of the martingale $\mathbf{E}\left[X|\mathcal{F}_t\right]$ possessing the following properties: X_t is an optional process and for every stopping time T, $X_T\mathbf{1}_{T<\infty} = \mathbf{E}\left[X\mathbf{1}_{T<\infty}|\mathcal{F}_T\right]$ a.s.*

It turns out that optional processes on unusual stochastic basis are neither left or right continuous. However, it was found that optional martingales have right and left limits (RLL) but are not necessarily right or left continuous. Actually, stochastic processes that are RLL appear to exist even when the usual conditions are satisfied. Mertens (1972) [3] showed that, under the usual conditions, for X, a positive optional strong supermartingale of class D with $X_{0-} = X_0$ and $X_\infty = 0$, there exists an integrable, i.e. $\mathbf{E}[A] < \infty$, increasing, predictable, RLL process A such that,

$$X_T = \mathbf{E}\left[A_\infty|\mathcal{F}_T\right] - A_T, \tag{1.2}$$

for every stopping time T. Moreover, A is unique and the following equalities between processes holds: $\Delta^+ A = A_+ - A = X - {}^pX$, $\Delta A = A - A_- = X - {}^o(X_+)$ where ${}^pX$ and ${}^oX$ are the predictable and optional projections of the process X, respectively. In particular, A is right continuous if and only if X is right continuous and left continuous if and only if $X_- = {}^pX$.

An optional process X is an optional strong martingale (resp. supermartingale) if for every bounded stopping time T, X_T is integrable and for every pair of bounded stopping times S, T such that $S < T$, $X_S = \mathbf{E}[X_T|F_S]$ (resp. $X_S > \mathbf{E}[X_T|F_S]$ a.s.). Mertens decomposition (1.2) was later proved [4] under the unusual conditions.

In general, RCLL supermartingales are optional strong supermartingale where the usual conditions are not needed here. Under the usual conditions there exist many optionally strong supermartingales that are not RCLL. For example, the optional projection of a not necessarily right continuous decreasing process is an optional strong supermartingale is not RCLL, which is also the case for left-potentials. Similarly, the limit of a decreasing sequence of RCLL positive supermartingales is an optional strong supermartingale that is in general not RCLL. However, under the usual conditions and on bounded intervals every optional strong martingale is the optional projection of a constant process and hence RCLL.

Many other mathematicians have contributed directly and indirectly to the theory of optional processes on unusual probability spaces these are Doob (1975) [5] and Lepingle (1977) [6], Horowitz (1978) [7], Lenglart (1980) [8]. However, much of the theoretical foundation and stochastic calculus of the theory of stochastic processes on unusual probability spaces was formulated mostly by Gal'chuk in several papers published in the period between 1975 and 1985 [9, 10, 11]. Further development was done by Gasparyan [12, 13, 14, 15, 16] and Khun and Stroh [17] and Abdelghani and Melnikov [18, 19, 20, 21, 22, 23]. In these publications, a parallel theory of stochastic analysis was constructed for optional processes on unusual probability spaces. The existence of such theory calls for a new initiative for its further developments as well as for further applications in a very well-developing area of mathematical finance as a natural and promising reserve for further studies.

It is necessary to mention that up to now there are no fundamental papers primarily devoted to interconnections between optional processes on unusual spaces and mathematical finance. However, we can mention here few research problems that are not treated with the methods of the calculus of optional processes on unusual spaces but show some glimpses of relevance to the problem we are going to study, optional processes on "unusual conditions", and perhaps should be studied in this framework.

The first one, is a recent development in mathematical finance specially in pricing of derivative contracts and hedging under transaction costs (see [24] for details) that hints to the importance and needed application of the calculus of optional processes under "unusual conditions" to price derivatives and hedge under transaction costs. Also, we mention the papers by Jacobson (2005) [25] who studied time to ruin of an insurance portfolio in which the claims process have two-sided jumps. In a similar topic, Asmussen et al (2004) [26] studied a class of Levy processes with phase type jumps in both directions. Furthermore, in models [27] with stochastic dividends paid at random times, there is an opportunity to treat these problems in the context of optional semimartingale theory more naturally. Duffie [28] presented a new approach to modeling term structures of bonds and other contingent claims that are subject to default risk. Perhaps Duffie's method could be studied with the methods of optional calculus. Kuhn [29, 30] also studied the problem of optimal portfolios of a small investor in a limit order market and modeling of capital gains taxes for trading strategies of infinite variation under the usual conditions but where optional ladlag processes seems to creep in. In the "usual" theory RCLL-semimartingales are automatically optional however in the "unusual" case they are not, and it is necessary to assume that the class of semimartingales under the unusual conditions are optional processes so as to provide the existence of regular modifications which admit right and left limits (RLL). We believe that the theory optional processes will offer a natural foundation and a versatile set of tools for modeling financial markets.

1.1 Objectives

The goal of this work is to revisit stochastic analysis of optional processes on unusual stochastic basis. To develop new results and bring the methods and techniques of optional processes to mathematical finance. With the hope to consider bigger classes of financial markets as well as provide new insights into mathematical finance problems.

1.2 Organization of the Thesis

The thesis is organized as follows: The current chapter introduced notions of the "unusual basis" and optional processes. Also, it gave a historical development of the theory and highlighted some of the areas in mathematical finance where it can be applied. The next chapter presents auxiliary materials on stochastic calculus of optional processes.

The following chapter introduces the important topic of stochastic equations: definition and properties of stochastic logarithms, the solution of the nonhomogeneous linear equation involving optional semimartingales, proof of Gronwall lemma involving optional processes, existence and uniqueness of solutions under monotonicity conditions and comparison of solutions of stochastic equations involving optional processes with a simple financial application.

In chapter four, we describe the optional semimartingale model of financial market as a portfolio of assets (x, X) which are optional semimartingales, where X is the principle security and x the reference one and study the ratio, $R = X/x$. We state a criterion for which R is a local martingale and compute a local martingale transform Z such that ZR is a local optional martingale. Furthermore, we study specific examples of general market models: those that are described by stochastic exponentials and the ones modeled by stochastic logarithms. Also, we describe pricing and hedging in these markets and give several financial examples: ladlag jumps diffusion model; a basket of stocks some are right continuous while other are left continuous such as a market index; a portfolio of a defaultable bond and a stock; a stock with the option to trade its dividends; finally the optimal debt repayment problem.

Chapter 5 is dedicated to defaultable markets; A summary of current approaches to defaultable markets is presented. A new unusual approach to the credit risk problem is developed. Defaultable claims are redefined in terms of optional processes and a pricing theory derive We present several financial examples: a zero coupon defaultable bond, credit risk swaps and a stock with defaultable dividend stream.

Finally, in chapter 6, we provide a summary of results and identify future research problems.

Chapter 2

Stochastic Processes on Unusual Spaces

In this chapter we review some elements of the stochastic analysis of stochastic processes on unusual probability spaces. We will simply list fundamental results without proof, but we will remark on essential consequences of definitions and theorems. This chapter is mostly based on Galchuk work [10, 11].

2.1 Foundation

Let there be given a complete but "unusual" probability space,

$$\left(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbf{P}\right), \quad t \in \mathbb{R}_+ = [0, \infty),$$

where $\mathcal{F}_t \subseteq \mathcal{F},\ \mathcal{F}_s \subseteq \mathcal{F}_t,\ s \leq t$. This space is complete because $\mathcal{F}$ contains all its $\mathbf{P}$ null sets. The space is unusual in the sense that the family $\mathbf{F}$ is not assumed to be complete, right or left continuous. We shall see that probability spaces of this nature are interesting, as they bring about new mathematical phenomena of stochastic processes and they offer a versatile set of tools for modeling financial markets.

We also introduce the σ-algebras $\mathbf{F}_+ = (\mathcal{F}_{t+})$, $\mathbf{F}^{\mathbf{P}}_+ = (\mathcal{F}^{\mathbf{P}}_{t+})$ and $\mathbf{F}^{\mathbf{P}} = (\mathcal{F}^{\mathbf{P}}_t)$. $\mathbf{F}^{\mathbf{P}}_+$ ($\mathbf{F}^{\mathbf{P}}$

respectively) is obtained from $\mathbf{F}_+$ by adding $\mathbf{P}$-null sets to $\mathcal{F}_{0+}$ (respectively from $\mathbf{F}$ by adjoining to $\mathcal{F}_0$). Also, let $\mathcal{O}(\mathbf{F})$ and $\mathcal{P}(\mathbf{F})$ be the optional and predictable σ-algebras on $(\Omega, \mathbb{R}_+)$, respectively. We say the σ-algebra $\mathcal{O}(\mathbf{F})$ on $\Omega \times [0,\ \infty)$ is optional if it is generated by all right-continuous $\mathcal{F}_t$-adapted processes having limits on the left. Optional σ-algebra can also be generated by the sets $\{(\omega,\ t) : S(\omega) \leq t < T(\omega)\}$, where $S,\ T$ run through the set of all Markov times. On the other hand, we say the σ-algebra $\mathcal{P}(\mathbf{F})$ on $\Omega \times [0,\ \infty)$ is predictable if it is generated by all left-continuous $\mathcal{F}_t$-adapted processes having limits on the right or by sets $\{(\omega,\ t) : S(\omega) < t \leq T(\omega)\}$, where $S,\ T$ run through the set of all Markov times where Markov time is defined below.

A random process $X = (X_t),\ t \in [0, \infty)$, is said to be *optional* if it is $\mathcal{O}(\mathbf{F})$-measurable. Optional processes are progressively measurable, and thus clearly measurable. In general, Optional processes have right and left limits but are not necessarily continuous. For an optional stochastic process we can define the following:

(a) $X_- = (X_{t-})_{t\geq 0}$ and $X_+ = (X_{t+})_{t\geq 0}$ with $X_{0-} = X_0$ and $X_{t-} = \lim_{s\uparrow t} X_s$,

(b) $\triangle X = (\triangle X_t)_{t\geq 0}$ where $\triangle X_t = X_t - X_{t-}$ and $\triangle^+ X = (\triangle^+ X_t)_{t\geq 0}$ where $\triangle^+ X_t = X_{t+} - X_t$.

A random process $X = (X_t),\ t \in \mathbb{R}_+$, is said to be predictable if $\mathcal{P}(\mathbf{F})$-measurable and *strongly predictable* if **(a)** $(X_t) \in \mathcal{P}(\mathbf{F})$, and **(b)** $(X_{t+}) \in \mathcal{O}(\mathbf{F})$. In particular this means that, for every stopping time T the random variables $X_{T+} I_{T<\infty}$ is $\mathcal{F}_T$-measurable, and $X_{T-} I_{T<\infty}$ is $\mathcal{F}_{T-}$-measurable. We denote by $\mathcal{P}_s(\mathbf{F})$ or $\mathcal{P}_s$ the set of strongly predictable processes.

A random variable $T : (\Omega, \mathcal{F}) \longrightarrow ([0, \infty] \times \mathcal{B}([0, \infty]))$ is called a random time. The random variable T with values in $[0, \infty]$ is a Markov time m.t. if the set $(T \leq t) \in \mathcal{F}_t$ for any t. The random variable T with values in $[0, \infty]$ is an $\mathbf{F}$-stopping time (s.t.) if it is a Markov time and the set $\mathbf{P}(T < \infty) = 1$ and we write T is $\mathbf{F}$-s.t. T is a stopping time in the wide (broad) sense if $\mathbf{F}_+$-stopping time and we write T is w.s.s.t. or s.t.b.. Let $\mathcal{T}$ be the class of all s.t.'s and $\mathcal{T}_+$ the class of all s.t.b. wide-sense stopping time. We say the $\mathbf{F}$-s.t. is predictable if there exists a sequence of (S_n), $n \in \mathbb{N}$, such that $\lim S_n = T$ a.s. and $S_n < T$ a.s. on the set $(T > 0)$ for all $n \in \mathbb{N}$. The $\mathbf{F}$-s.t. T is said to be totally inaccessible if $\mathbf{P}(S = T < \infty) = 0$ for every predictable $\mathbf{F}$-s.t. S. For every $\mathbf{F}^{\mathbf{P}}$-s.t. T there exists an $\mathbf{F}_+$-s.t. U such that $T = U$ a.s. Moreover, for every event $L \in \mathbf{F}_T^{\mathbf{P}}$ there exists an event $M \in \mathbf{F}_{U+}$ such that $L = M$ a.s.. In other words, $\mathbf{F}^{\mathbf{P}} \subseteq \mathbf{F}_+$. For every predictable $\mathbf{F}_+^{\mathbf{P}}$-s.t. T there exists a predictable $\mathbf{F}$-s.t. T' such that $T = T'$ a.s. For every $\mathbf{F}_+$-s.t. T there exist a sequence of $\mathbf{F}$-s.t. $(S_n), n \in \mathbb{N}$ with nonintersecting graphs and an $\mathbf{F}_+$-s.t. T such that $[T] \subseteq [T'] \cup [\cup_n S_n]$ and $\mathbf{P}(T' = U < \infty) = 0$ for every $\mathbf{F}$-s.t. U. It follows that T' is a totally inaccessible s.t.b.. Suppose T is a s.t. there exist a totally inaccessible s.t. T' and a sequence of predictable s.t. (T_n) with pairwise non-intersecting graphs such that $[T] \subseteq [T'] \cup [\cup_n T_n]$.

A process $A = (A_t),\ t \in \mathbb{R}_+$, belongs to the space J_{loc} if there exists a sequence $(R_n),\ R_n \in \mathcal{T}_+,\ n \in \mathbb{N},\ R_n \uparrow \infty$ a.s., such that $A\mathbf{1}_{[0,R_n]} \in J$ for any n where J is a space of processes. The measurable set D in the product space $(\Omega \times \mathbb{R}_+, \mathbf{F} \times \mathcal{B}(\mathbb{R}_+))$ is called *negligible* if $\mathbf{P}(\pi(D)) = 0$, where $\pi(D)$ is the projection of D on Ω. Note that according to this definition the set $\pi(D) \in \mathcal{F}$, since $\mathcal{F}$ is a complete σ-algebra. We say that the measurable processes $X = (X_t)$ and $Y = (Y_t)$ are indistinguishable if the set $((\omega, t) : X_t(\omega) \neq Y_t(\omega))$ is negligible. Moreover, for every $\mathcal{P}(\mathbf{F}_+^P)$-predictable process X there exists a $\mathcal{P}(\mathbf{F})$-predictable process X' indistinguishable from it.

Finally and before I get to the details of stochastic analysis on unusual probability spaces, I like to present the following theorem which was due to Galchuk [31] and is the extension of Dellacherie and Meyer (1972) [2] projection theorem for bounded random variables.

Theorem 2 *Let X be an integrable random variable. There exists a modification X_t of the martingale $(\mathbf{E}[X|\mathcal{F}_t])$ such that X_t is an optional process and for any Markov time T*

$$X_T \mathbf{1}_{(T<\infty)} = \mathbf{E}[X\mathbf{1}_{(T<\infty)}|\mathcal{F}_T] \qquad a.s., \tag{2.1}$$

If another optional modification $(\tilde{X}_t)$ exists satisfying (2.1), then X_t and $\tilde{X}_t$ are indistinguishable.

This theorem became the foundation of stochastic calculus on unusual probability spaces. In the following sections we review the different types of stochastic processes that evolve on unusual probability spaces.

2.2 Increasing and Finite Variation Processes

If there is any useful knowledge or singular facts to be gained from a random process is to know whether it is increasing or its variability finite. These notions of finding deterministic facts about stochastic processes is captured by the following definitions of increasing and finite variation processes.

2.2.1 Definitions

Definition 3 *We shall say that the process $A = (A_t)$, $t \in \mathbb{R}_+$, is increasing if it is non-negative, its trajectories do not decrease and, for any t, the random variable A_t is $\mathcal{F}_t$-measurable. The collection of such processes will be denoted by $\mathcal{V}^+(\mathbf{F}, \mathbf{P})$ or $\mathcal{V}^+$ for short. We shall call an increasing process A integrable if $\mathbf{E}A_\infty < \infty$, and locally integrable if there is a sequence $(R_n) \subset \mathcal{T}_+$, $R_n \uparrow \infty$ a.s., such that $\mathbf{E}A_{R_n+} < \infty$ for all $n \in \mathbb{N}$. The collection of such processes is denoted by $\mathcal{A}^+$ ($\mathcal{A}^+_{loc}$ respectively).*

Definition 4 *A process $A = (A_t)$, $t \in \mathbb{R}_+$, is a finite variation process if it has finite variation on every segment $[0, t]$, $t \in \mathbb{R}_+$ a.s., i.e. $\mathbf{Var}(A)_t < \infty$, for all $t \in \mathbb{R}_+$ a.s. where*

$$\mathbf{Var}(A)_t = \sum_{0 \leqq s < t} |\Delta^+ A_s| + \int_0^{t-} |dA^r_s|.$$

We shall denote by $\mathcal{V}(\mathbf{F}, \mathbf{P})$ the set of $\mathbf{F}$-adapted finite variation processes ($\mathcal{V}$ for short).

We shall say that the process of finite variation $A = (A_t)$ belongs to the space $\mathcal{A}$ of integrable finite variation processes if $\mathbf{E}[\mathbf{Var}(A)_\infty] < \infty$. The process $A = (A_t)$, $t \in \mathbb{R}_+$, belongs to $\mathcal{A}_{loc}$ if there is a sequence $(R_n) \subset \mathcal{T}_+$, $R_n \uparrow \infty$, such that $A\mathbf{1}_{[0,R_n]} \in \mathcal{A}$ for any $n \in \mathbb{N}$, i.e., for any

n, the quantity

$$\mathbf{Var}(A)_{R_n} = \sum_{0 \le s \le R_n < \infty} |\Delta^+ A_s| + \int_0^{R_n} |dA_s^r|$$

is integrable.

This brings us to an important result which is the decomposition of increasing and finite variation processes.

2.2.2 Decomposition Results

Theorem 5 *An increasing process A can be decomposed to $A = A^r + A^g = A^c + A^d + A^g$ where*

$$\begin{aligned} A_t^d &= \sum_{s \le t} \Delta A_s, \ \ \Delta A_s = A_s - A_{s-}, \\ A_t^g &= \sum_{s < t} \Delta^+ A_s, \ \ \Delta^+ A_s = A_{s+} - A_s, \end{aligned}$$

where the series converge absolutely. A^c is a continuous process, A^r is a right-continuous process.

Theorem 6 *A finite variation process $A = (A_t)$, $t \in \mathbb{R}_+$, can be written in the form $A_t = A_t^r + A_t^g = A_t^c + A_t^d + A_t^g$,*

$$A_t^d = \sum_{s \le t} \Delta A_s, \quad A_t^g = \sum_{s < t} \Delta^+ A_s,$$

where the series converge absolutely and A^c is a continuous process of bounded variation ($\sup_t \mathbf{Var}(A^c)_t < \infty$, $t \in \mathbb{R}_+$) and A^r is a right-continuous process of finite variation.

2.3 Martingales

A martingale is a process where knowledge of the past history of the process does not predict its future average. In other words, the expectations of the future is not different from the present given knowledge of all prior observations. Martingales and their variants super and submartingale and their local versions are well understood in the general theory of stochastic processes on the usual basis. Here we give definitions of optional martingale super and sub martingales and their localized versions and their properties on unusual basis.

2.3.1 Definitions

Definition 7 *We say that X is an optional martingale (supermartingale, submartingale) if*

*(**a**) X is an optional process, $X \in \mathcal{O}(\mathbf{F})$,*

*(**b**) The random variable $X_T \mathbf{1}_{T<\infty}$ is integrable for any stopping time T, $T \in \mathcal{T}$, and*

*(**c**) There exists an $\mathcal{F}$-measurable integrable random variable ξ (i.e. $\mathbf{E}|\xi| < \infty$) such that $X_T = \mathbf{E}[\xi|\mathcal{F}_T]$ (respectively $X_T \geq \mathbf{E}[\xi|\mathcal{F}_T]$, $X_T \leq \mathbf{E}[\xi|\mathcal{F}_T]$) a.s. on $(T < \infty)$ for any stopping time, $T \in \mathcal{T}$.*

We let $\mathcal{M}(\mathbf{F}, \mathbf{P})$, $\mathcal{M}$ for short, denote the set of integrable martingales, i.e. ξ is integrable. We say $X \in \mathcal{M}^2$ a square integrable martingale if $\mathbf{E}|\xi|^2 < \infty$.

Since, any optional process can be transformed by a continuous change of variable to a separable process then optional martingale, super and sub martingales are separable processes. Separable processes have left-hand and right-hand limits at each point $t \in \mathbb{R}_+$, see [32] (Chapter VI, Theorem 3) and [4]. To this separable process, we can apply the theorem on the existence of a limit at infinity and right and left limits at each point [10]. Consequently, as $t \to \infty$ the martingale X_t has a limit a.s. equal to ξ in L^1. This allows us to consider X_t on $\mathbb{R}_+ \cup \{\infty\}$ by letting $X_\infty = \xi$. Therefore, the equality $X_T = \mathbf{E}[\xi|\mathcal{F}_T]$ holds a.s. for all $T \in \mathcal{T}$. Also, the class of supermartingales given by the last definition is sufficiently broad. Indeed, let $X = (X_t)$, $t \in \mathbb{R}_+$, be an arbitrary supermartingale for which there is an integrable variable ξ such that $X_t \geq \mathbf{E}[\xi|\mathcal{F}_t]$ a.s. for any $t \in \mathbb{R}_+$. Then, there exists a process X^* that is a modification of X and satisfies the conditions of the given definition. Moreover, the optional sampling theorem holds for X^*; namely, $X^*_S \geq \mathbf{E}[X^*_T|\mathcal{F}_S]$ a.s.for any S, $T \in \mathcal{T}$ with $S \leq T$. We remark that for $T = \infty$ it is assumed that $X^*_\infty = \xi$. We remark also that a supermartingale

X^* has a limit in L^1 as $t \to \infty$ (see [6], Chapter VI, Theorem 6), and it can be assumed without loss of generality to coincide with ξ.

Definition 8 *We shall call the optional process $X = (X_t)$, $t \in \mathbb{R}_+$, an (optional) local martingale and write $X \in \mathcal{M}^1_{loc}(\mathbf{F}, \mathbf{P})$, $\mathcal{M}^1_{loc}$ for short (respectively $\mathcal{M}^2_{loc}$), if there exists a sequence $(R_n, X^{(n)})$, $n \in \mathbb{N}$, where $R_n \in \mathcal{T}_+$, $X^{(n)} \in \mathcal{M}^1$ (respectively, $X^{(n)} \in \mathcal{M}^2$), $R_n \uparrow \infty$ a.s. such that $X = X^{(n)}$ on the stochastic interval $[0, R_n]$ and the random variables X_{R_n+} is integrable for any $n \in \mathbb{N}$. Instead of $\mathcal{M}^1_{loc}$, we shall write $\mathcal{M}_{loc}$.*

2.3.2 Decomposition Results

Theorem 9 *If $X \in \mathcal{M}_{loc}$ then X can be decomposed into a sum,*

$$\begin{aligned} X &= X^r + X^g, \\ X^r &= X^c + X^d, \end{aligned}$$

where X^r, X^c, X^d, and $X^g \in \mathcal{M}_{loc}$, the trajectories of X^c are continuous, X^d are right-continuous, and X^g are left-continuous and X^r is right-continuous. The processes X^d and X^g are orthogonal to any continuous $Y \in \mathcal{M}_{loc}$.

Denote by $\mathcal{M}^c_{loc}, \mathcal{M}^d_{loc}, \mathcal{M}^g_{loc}$ the set of continuous right-continuous, left-continuous local martingales and $\mathcal{M}^r_{loc} \supseteq \mathcal{M}^c_{loc} \cup \mathcal{M}^d_{loc}$. the set of all right continuous martingales. We shall interpret similarly the notations $\mathcal{M}^{2,c}_{loc}$, $\mathcal{M}^{2,d}_{loc}$, $\mathcal{M}^{2,g}_{loc}$ and $\mathcal{M}^{2,r}_{loc}$ for subsets of $\mathcal{M}^2_{loc}$. locally square integrable optional martingales.

The following theorems and lemmas highlights some of the properties of martingales super and sub martingales.

Theorem 10 *Suppose that X is a positive optional process, such that $\mathbf{E}[X_t|\mathcal{F}_s] \leq X_s$ a.s. for any $s \leq t < \infty$ and let $U = \sum_{t \geq 0} \left[(\Delta X_t)^2 + (\Delta^+ X_t)^2\right]$ be the quadratic variation of jumps of X. Then, If $X \leq C$ then $\mathbf{E}[U|\mathcal{F}_0] \leq 2CX_0$ and $\mathbf{P}\left[U > a^2|\mathcal{F}_0\right] \leq 3X_0 a^{-1}$ for any $a \in \mathbb{R}_+$.*

Lemma 11 *Let the process $A \in \mathcal{A}_{loc}$, be $\mathbf{F}$-adapted and its trajectories be left-continuous. Then, (**a**) There exist a unique left-continuous strongly predictable process $A^o \in \mathcal{A}_{loc}$ such that for any non-negative optional processes X,*

$$\mathbf{E}\int_0^\infty X_s dA_{s+} = \mathbf{E}\int_0^\infty X_s dA^o_{s+}.$$

in particular, for any $S,\ T \in \mathcal{T},\ S \leq T$,

$$\mathbf{E}\left[\int_S^{T-} X_t dA_{t+} | \mathcal{F}_s\right] = \mathbf{E}\left[\int_S^{T-} X_t dA^o_{t+} | \mathcal{F}_s\right].$$

*This means that $A - A^o \in \mathcal{M}_{loc}$. (**b**) If the sequence $(R_n) \subset \mathcal{T}_+$ absorb the jumps of the process A and possesses the property $\mathbf{P}(R_n = T < \infty) = 0$ for all $n \in \mathbb{N}$, $T \in \mathcal{T}$, then the process A^o is continuous and is known as the dual optional projection of the process A.*

Theorem 12 *Let Y be an optional process then there exist a unique process $Z \in \mathcal{M}^d_{loc}$ with $\Delta Z = Y$ if and only if the predictable projection ${}^pY = 0$ and $\left(\sum_{s<\cdot} Y_s^2\right)^{1/2} \in \mathcal{A}_{loc}$.*

Theorem 13 *Let Y be an $\mathcal{O}(\mathbf{F}_+)$-measurable process. Then there exist a unique process $Z \in \mathcal{M}^g_{1oc}$ with the property $\Delta^+ Z = Y$ if and only if the optional projection ${}^oY = 0$ and $\left(\sum_{s<\cdot} Y_s^2\right)^{1/2} \in \mathcal{A}_{loc}$.*

Now we define what do we mean by an absorbing sequence.

Definition 14 *The sequence $(T_n) \subset \mathcal{T}_+$ absorbs the jumps of the process X if for any $T \in \mathcal{T}_+$ for which the set $[T] \cap (\bigcup_n [T_n]) = \varnothing$ we have and $\Delta X_T = \Delta^+ X_T = 0$ a.s. on $(T < \infty)$.*

This brings us, in my opinion, to the most fundamental theorem in optional calculus. That is, for any optional process all of its jumps are absorbed by three sequences of jumps: those that are predictable, those that are inaccessible but immediately visible, i.e. adapted to $\mathbf{F}$, and those that are inaccessible but not visible, i.e. adapted to $\mathbf{F}_+$. Many theorems and proofs rely on this theorem. It provides us, as we will see later, the foundation for defining canonical and components decomposition of optional semimartingales.

Theorem 15 *Suppose X is an optional process whose trajectories have limits from the left and right a.s.. Then there exist a sequence of predictable stopping times (S_n), totally inaccessible stopping times (T_n) and totally inaccessible stopping times in the broad-sense (U_n) absorbing all jumps of the process X and having the following properties: the graphs of these stopping times are mutually non-intersecting within each sequence.*

Before we proceed with additional results in decomposition of optional processes let us state some basic results about the space of square integrable optional martingales.

Suppose $X \in \mathcal{M}^2$ then

$$\mathbf{E}\left[\sum_{t<\infty}(\Delta X_t)^2 + (\Delta^+ X_t)^2\right] \leq \mathbf{E}X_\infty^2.$$

Let us introduce the norm $||X||^2 = \mathbf{E}X_\infty^2$ for $X \in \mathcal{M}^2$. With the norm $||X||$, the space $\mathcal{M}^2$ is turned to a complete normed and separable space. The separability follows from the separability of the space L^2 of variables X_∞ and completeness comes as a result of Doob's inequality

$$\mathbf{P}\left(\sup_t |X_t^n - X_t^m| > \epsilon\right) \leq \frac{1}{\epsilon}||X^n - X^m||^2$$

for optional martingales X^n and X^m. Indeed if the sequence (X_n) is Cauchy in $\mathcal{M}^2$, then from it we can choose a subsequence converging a.s. uniformly to an optional modification of the martingale $\mathbf{E}[X_\infty|\mathcal{F}_t] \in \mathcal{M}^2$, where $X_\infty = \lim_{n\to\infty} X_\infty^n$.

Lemma 16 *Suppose $X \in \mathcal{M}^2$. Then,*

$$X = X^c + X^g + X^d,$$

where $X^g,\ X^c,\ X^d \in \mathcal{M}^2$, X^c is a continuous process, $X_0^c = 0$, X^g and X^d have one-sided limits, X^g is continuous from the left, X^d from the right, and $X_0^g = 0$. X^g (X^d respectively) is orthogonal to every martingale $Y \in \mathcal{M}_{loc}$ whose trajectories are one-sided continuous and do not have common discontinuity times with X^g (X^d respectively). Moreover, X^g (X^d respectively) is orthogonal to every martingale $Y \in \mathcal{M}^2$ whose trajectories are continuous from the right (from the left respectively). The decomposition is unique.

To prove the above theorem one must use the fact that there exist a sequence of predictable and totally inaccessible stopping times and totally inaccessible stopping times in the wide sense that absorb all the jumps of the process X and define the processes X^g and X^d as

$$
\begin{aligned}
X_t^d &= \sum_n \left[\Delta X_{S_n}\mathbf{1}_{(S_n \le t)} + \left(\Delta X_{T_n}\mathbf{1}_{(T_n \le t)} - A_t^n\right)\right], \\
X_t^g &= \sum_n \left[\Delta^+ X_{S_n}\mathbf{1}_{(S_n < t)} + \Delta^+ X_{T_n}\mathbf{1}_{(T_n < t)} + \left(\Delta^+ X_{U_n}\mathbf{1}_{(U_n < t)} - B_t^n\right)\right],
\end{aligned}
$$

where A^n and B^n are strongly predictable processes and the sums converge in mean-square.

If $X \in \mathcal{M}_{loc}^2$, then $X = X^g + X^c + X^d$, with $X^g, X^c, X^d \in \mathcal{M}_{loc}^2$ with the same orthogonality conditions on $Y \in \mathcal{M}_{loc}^2$.

Lemma 17 *Every local martingale $X \in \mathcal{M}_{loc}$ can be decomposed to $X = \tilde{X} + \breve{X}$, where $\tilde{X} \in \mathcal{M}_{loc}^2$ and $\breve{X} \in \mathcal{M}_{loc} \cap \mathcal{A}_{loc}$.*

Lemma 18 *Every local optional martingale $X \in \mathcal{M}_{loc}$ has a unique representation $X = X^g + X^c + X^d$, where $X^g,\ X^c,\ X^d \in \mathcal{M}_{loc}$, X^c is continuous, $X_0^c = 0$, X^g and X^d have one-sided limits, X^g is continuous from the left, $X_0^g = 0$, X^d is continuous from the right, and $X^g(X^d)$ is orthogonal to every martingale $Y \in \mathcal{M}_{loc}$ whose trajectories are one-sided continuous and do not have common discontinuity times with $X^g(X^d)$.*

2.3.3 Quadratic Variation

For the stochastic integral with respect to RCLL martingale and semimartingales a fundamental role is played by the unique increasing predictable process $\langle X \rangle$, i.e. angle-bracket process, with $\mathbf{E}X_T^2 = \mathbf{E}\langle X \rangle_T$ for every stopping time T. In other words, the process $X^2 - \langle X \rangle \in \mathcal{M}$. For the case of integration with respect to optional martingales in order to determine the process $\langle X \rangle$ for $X \in \mathcal{M}^2$ it suffices to consider the case where the trajectories of X are continuous from the left. Indeed, by the decomposition of optional martingales we have $X = X^g + X^c + X^d$ where all three components are mutually orthogonal. Then, for every stopping time T.

$$\mathbf{E}X_T^2 = \mathbf{E}\left[(X_T^c)^2 + (X_T^d)^2 + (X_T^g)^2\right] = \mathbf{E}\left[\langle X^c \rangle_T + \langle X^d \rangle_T + (X_T^g)^2\right]$$

For two processes $X, Y \in \mathcal{M}^2$ there exists a unique strongly predictable process $\langle X, Y \rangle \in \mathcal{A}$ *such that* $XY - \langle X, Y \rangle \in \mathcal{M}$, *where*

$$\langle X, Y \rangle = \frac{1}{2}[\langle X + Y \rangle - \langle X \rangle - \langle Y \rangle].$$

Now we define the quadratic variation of the square integrable martingale.

Definition 19 *Suppose* $X \in \mathcal{M}^2$ *and* X^c *is a continuous component. Let*

$$[X, X]_t = \langle X^c \rangle_t + \sum_{s \le t} (\Delta X_s)^2 + \sum_{s<t} \left(\Delta^+ X_s\right)^2, \ t \in \mathbb{R}_+.$$

The process $[X, X]$ *is increasing,* $\mathbf{F}$*-consistent, and integrable.*

Since $X^c \perp X^g + X^d$ we have $X^2 - [X, X] \in \mathcal{M}$. By $[X, Y] = \frac{1}{2}([X+Y, X+Y] - [X, X] - [Y, Y])$ for X, $Y \in \mathcal{M}^2$ and using the above definition we find that,

$$[X, Y]_t = \langle X^c, Y^c \rangle_t + \sum_{s \le t} \Delta X_s \Delta Y_s + \sum_{s<t} \Delta^+ X_s \Delta^+ Y_s.$$

Furthermore, we find that $XY - [X, Y] \in \mathcal{M}$.

Definition 20 *Suppose X, $Y \in \mathcal{M}_{loc}$. We say that X is orthogonal to Y ($X \perp Y$) if $XY \in \mathcal{M}_{loc}$.*

2.3.4 Elementary Processes

Elementary processes are simple processes associated with jump times of an optional process. They play an important role in many theorems and in the definition of stochastic integral with respect to optional semimartingales. Furthermore, it turns out, as we will show later, that elementary processes are important in credit risk modeling in finance.

Theorem 21 *Suppose T is a totally inaccessible stopping time ξ be an $\mathcal{F}_T$-measurable integrable ($\mathbf{E}\xi^2 < \infty$ respectively) random variable and $Y_t = \xi \mathbf{1}_{T \le t}$, then*
a) *There exists a unique predictable process $A \in \mathcal{A}$ (a nondecreasing $B \in \mathcal{A}$ respectively) such that the process $Z = Y - A \in \mathcal{M}$ ($Z = Y - Z \in \mathcal{M}^2$ and $Z^2 - B \in \mathcal{M}$ respectively). Moreover the process A (B respectively) is continuous.*
b) *Z is orthogonal to every martingale $X \in \mathcal{M}_{loc}$ whose trajectories are right or left continuous and do not have common jump times with Z. Moreover, in the case $Z \in \mathcal{M}^2$ the process Z is orthogonal to every martingale $X \in \mathcal{M}^2$ whose trajectories are left continuous.*

Theorem 22 *Suppose T is a predictable or totally inaccessible s.t., and the random variable ξ is integrable ($\mathbf{E}\xi^2 < \infty$ respectively) and $\mathcal{F}_{T+}$-measurable. Let $Y_t = \xi \mathbf{1}_{T < t}$ for $t > 0$ and $Y_0 = 0$.*
a) *There exists a unique right continuous strongly predictable process $A = (A_t) \in \mathcal{A}$, $A_0 = 0$ (respectively increasing $B \in \mathcal{A}$, $B_0 = 0$) such that $Z = Y - Z \in \mathcal{M}$ ($Z = Y - A \in \mathcal{M}^2$ and $Z^2 - B \in \mathcal{M}$ respectively).*
b) *Z is orthogonal to every martingale $X \in \mathcal{M}_{loc}$ whose trajectories are continuous from the left or the right and which do not have common discontinuity times with Z. Moreover, in the case $Z = Y - A \in \mathcal{M}^2$, Z is orthogonal to every martingale $X \in \mathcal{M}^2$ whose trajectories are right continuous.*

Theorem 23 *Suppose T is a totally inaccessible s.t.b., where $\mathbf{P}(T = S < \infty) = 0$ for all s.t.b. S. Suppose the random variables ξ is $\mathbf{F}_{T+}$-measurable and integrable (respectively, $\mathbf{E}\xi^2 < \infty$). Let $Y_t = \xi \mathbf{1}_{T<t},\ t > 0,$ and $Y_0 = 0$.*

***a**) There exists a continuous process $A = (A_t) \in \mathcal{A}$ (respectively nondecreasing $B = (B_t) \in \mathcal{A}$) such that $Z = Y - A \in \mathcal{M}$ (respectively $Z = Y - A \in \mathcal{M}^2$ and $Z^2 - B \in \mathcal{M}$). The process A (respectively B) is unique in the class of strongly predictable processes.*

***b**) Z is orthogonal to every martingale $X \in \mathcal{M}_{loc}$ whose trajectories are continuous from the left or the right and which do not have common jump times with Z. If $Z \in \mathcal{M}^2$, then Z is orthogonal to every martingale $X \in \mathcal{M}^2$ whose trajectories are continuous from the right.*

Theorem 24 *Suppose T is a predictable stopping time and ξ is a random variable which is $\mathcal{F}_T$-measurable and integrable (respectively $\mathbf{E}\xi^2 < \infty$). Let $Y_t = \xi \mathbf{1}_{T \le t}$.*

1) There exists a unique right continuous predictable process $A \in \mathcal{T}$ (respectively $B \in \mathcal{T}$) such that $Z = Y - A \in \mathcal{M}$ (respectively $Z = Y - A \in \mathcal{M}^2$ and $Z^2 - B \in \mathcal{M}$).

2) Z is orthogonal to every martingale $X \in \mathcal{M}_{loc}$ whose trajectories are continuous from the left or the right and which do not have discontinuities in common with Z. If $Z \in \mathcal{M}^2$, then Z is orthogonal to every martingale $X \in \mathcal{M}^2$ whose trajectories are continuous from the left.

2.3.5 Decomposition of Super and Sub Martingales

Definition 25 *A nonnegative optional supermartingale X will be called a potential if $\lim_{t\to\infty} \mathbf{E}X_t = 0$.*

Definition 26 *Let X be an optional supermartingale,*

***a**) Class D: We say that X belongs to the class D if the family of random variables $X_T,\ T \in \mathcal{T}_+$, is uniformly integrable. Also,*

***b**) Class DL: We say that X belongs to the class DL if the family of variables $X_T,\ T \in \mathcal{T}_+,\ T \le a$, is uniformly integrable for any $a,\ 0 \le a < \infty$.*

Let A and B be two random processes. We say that B *majorizes* A (and write $A \leq B$) if outside some set of $\mathbf{P}$-measure zero we have $A_t \leq B_t$ for any $t \in \mathbb{R}_+$.

Theorem 27 *Riesz decomposition. Let X be an optional supermartingale. Then, the following two properties are equivalent:*
(a) X majorizes some optional submartingale.
(b) There exist an optional martingale Y and a potential Z such that $X = Y + Z$. This decomposition is unique (to within indistinguishability). The martingale Y majorizes each submartingale Y' satisfying the condition $Y' \leq X$.

Theorem 28 *Uniqueness. Let X be an optional supermartingale of class D that admits a decomposition $X = Y - A$, where Y is an optional martingale, A is an increasing strongly predictable integrable process and $A_0 = 0$. Then this decomposition is unique to within negligibility.*

Theorem 29 *Existence. Let X be an optional supermartingale of class D. There exist an increasing strongly predictable integrable process A, $A_0 = 0$, and an optional martingale Y such that $X = Y - A$.*

Lemma 30 *Let X be a potential of the class D. Then there exist an optional martingale m and a strongly predictable increasing integrable process A, $A_0 = 0$, such that $X = m - A$.*

Theorem 31 *An optional supermartingale X admits a decomposition $X = Y - A$ (where $Y \in \mathcal{M}_{loc}$ and A is an increasing strongly predictable locally integrable process with $A_0 = 0$) if and only if X belongs to the class DL. This decomposition is unique to within indistinguishability.*

Theorem 32 *An optional submartingale X admits a decomposition $X = Y + A$ (where $Y \in \mathcal{M}_{loc}$ and A is an increasing strongly predictable locally integrable process with $A_0 = 0$) This decomposition is unique to within indistinguishability.*

Lemma 33 *Let (Y_t) be an optional submartingale for which there exists an integrable random variable Y such that for any stopping time T, $Y_T \mathbf{1}_{(T<\infty)} \leq \mathbf{E}(Y \mathbf{1}_{T<\infty} | F_T)$ a.s.. Then, for any $c > 0$, $P(\sup_t Y_t > c) \leq c^{-1} EY^+$ where $Y^+ = \max(Y, 0)$.*

2.4 Semimartingales

Semimartingales are the most essential process in stochastic analysis and in mathematical finance. Essentially, semimartingales are a composite process, composed of martingales and finite variation or increasing processes. Optional semimartingales are a class of semimartingales that are optional processes. Furthermore, on unusual probability spaces optional semimartingales have left and right limits but may not be right or left continuous. As a result, they possess complicated jump structures. In this section we define optional semimartingales and list some of their properties.

Definition 34 *The random process X is called an optional semimartingale if it is representable in the form*

$$X = X_0 + M + A, \ M \in \mathcal{M}_{loc}, \ A \in \mathcal{V}, \ M_0 = 0. \tag{2.2}$$

A semimartingale X is called special semimartingale if $A \in \mathcal{A}_{loc} \cap \mathcal{P}_s$ a strongly predictable process.

Let us denote by $\mathcal{S}(\mathbf{F}, \mathbf{P})$ $\mathcal{S}$ for short (respectively, $\mathcal{S}p$) the set of all (respectively, special) semimartingales. For an n-dimensional process $X = (X^1, ..., X^n) \in \mathcal{S}$ if $X^i \in \mathcal{S}, \ i = 1,...,n$.

The decomposition of special optional semimartingale decomposition is unique. Consequently, one can decompose a semimartingale further to

$$\begin{aligned} X &= X_0 + X^r + X^g \\ &= X_0 + A^r + M^r + A^g + M^g \\ &= X_0 + A^r + A^g + M^c + M^d + M^g. \end{aligned}$$

This decomposition is essential for formulating stochastic integration with respect to optional semimartingale and for defining the canonical and components representations of an optional semimartingale.

2.5 Calculus of Optional Processes

2.5.1 Integration with respect to Finite Variation and Increasing Processes

For an increasing or finite variation process A with the decomposition $A = A^r + A^g$ and a $\mathcal{B}([0,\infty[)$-measurable function K we would like to define an integral $Y = \int KdA$ having the following properties: **(a)** the process Y is can be written in the form $Y = Y^r + Y^g$ where Y^g is a left-continuous process and Y^r is a right-continuous process; **(b)** the process Y has bounded variation,

$$\sum_{0 \leq t < \infty} |\Delta^+ Y_t| + \int_{0+}^{\infty} |dY_t^r| < \infty,$$

and **(c)**

$$\Delta Y_t = K_t \Delta A_t, \quad \Delta^+ Y_t = K_t \Delta^+ A_t, \quad Y_t^r = \int_{[0,t]} K_s dA_s^r,$$

It is easy to see that by virtue of properties (a), (b) and (c) the definition is proper and yields

a unique process Y. It is important to note that integrals with respect to increasing or finite variation processes are understood in the Lebesgue-Stieltjes sense.

Let us summarize our findings with a definition of the integral with respect to increasing and finite variation processes on unusual probability spaces.

Definition 35 *By the integral of $\mathcal{B}([0,\infty[)$-measurable function K with respect to the increasing process (finite variation process) A we mean the process $Y = K \circ A$, where*

$$K \circ A_t = \int_0^t K_s dA_s = \int_{0+}^t K_{s-} dA_s^r + \int_0^{t-} K_s dA_{s+}^g.$$

Here we present some precursor lemmas and corollaries that are essential for characterizing the properties of integrals with respect to increasing or finite variation processes.

Lemma 36 *Suppose H is a measurable nonnegative function and A is an increasing right (left) continuous process. By the integral*

$$H \circ A_\infty = \int_0^\infty H_{s-} dA_s \left(= \int_0^\infty H_s dA_{s+} \right),$$

we mean the Lebesgue-Stieltjes integral defined trajectory wise. In particular,

$$H \circ A_t = \int_0^t H_{s-} dA_s \left(= \int_0^{t-} H_s dA_{s+} \right)$$

Corollary 37 *If the function H is such that for all t,*

$$\int_0^t |H_{s-}| dA_s < \infty \quad \left(\int_0^{t-} |H_s| dA_{s+} < \infty \right) \ a.s.$$

then the process $H \circ A$ is continuous from the right (left), has a limit from the left (right) and

$$\Delta(H \circ A)_t = H_t \Delta A_t, \quad \Delta(H \circ A)_0 = H_0 \Delta A_0 = H_0 A_0,$$
$$\Delta^+(H \circ A)_t = H_t \Delta^+ A_t$$

Corollary 38 *Suppose A is a right continuous increasing process, and X is a nonnegative optional martingale. Then for every s.t.b. T*

$$\mathbf{E}\int_0^T X_{t+}dA_t = \mathbf{E}X_{T+}A_T,$$

where it is assumed that $X_{\infty+} = X_\infty$.

2.5.2 Projections

Theorem 39 *Suppose $Y \in \mathcal{A}_{loc}$ is a left continuous increasing process. And suppose the sequence (T_n) s.t.b. absorbs the jumps of the process Y and has the property that $P(T_n = T < \infty) = 0$ for every n and an arbitrarily s.t. T. Then there exists a unique continuous increasing process $A \in \mathcal{A}_{loc}$ such that for every nonnegative measurable process X*

$$\mathbf{E}\int_0^{\infty-} {}^oX_t dY_{t+} = \mathbf{E}\int_0^{\infty-} X_t dA_t,$$

where ${}^oX$ is the optional projection of the process X.

Recall that the *optional projection* of the process X is the $\mathcal{O}(\mathbf{F})$-measurable process ${}^oX$ *such that* $\mathbf{E}\left(X_T\mathbf{1}_{T<\infty}\right) = \mathbf{E}\left({}^oX_T\mathbf{1}_{T<\infty}\right)$ for every s.t. T.

Theorem 40 *Suppose $A \in \mathcal{A}_{loc}$ is a left continuous increasing process. There exists a unique right-continuous $\mathcal{P}(\mathbf{F})$-predictable increasing process $B \in \mathcal{A}_{loc}$ such that for every positive $\mathcal{P}(\mathbf{F})$-predictable process X*

$$\mathbf{E}\int_0^{\infty-} X_s dA_s = \mathbf{E}\int_0^{\infty-} X_s dB_s.$$

In particular if S and T are s.t.b., $S \le T$, then

$$\mathbf{E}\left[\int_{]S,T]} X_s dA_s|\mathcal{F}_S\right] = \mathbf{E}\left[\int_{]S.T]} X_s dB_s|\mathcal{F}s\right]$$

a.s. on the set $(T < \infty)$.

If in the proceeding theorem the process A has jumps only at totally inaccessible s.t. then B is a continuous process.

2.5.3 Integration with respect to Martingales

Stochastic integrals with respect to martingales play an important role in various applications of martingale theory. Under the usual conditions a martingale X has right continuous trajectories and if $X \in \mathcal{M}^2$, the process h is predictable and $\mathbf{E}(h^2 \circ \langle X \rangle_\infty) < \infty$, then there exists a unique process $Y \in \mathcal{M}^2$ with right continuous trajectories, such that, $\langle Y, Z \rangle_T = h \circ \langle X, Z \rangle_T$ for all right continuous martingales $Z \in \mathcal{M}^2$ and s.t. T. The process Y is called a stochastic integral and denoted as

$$Y = h \circ X = (h \circ X)_t = \left(\int_0^t h_s dX_s \right).$$

In the case without the usual conditions, a stochastic integral is defined with respect to an optional martingale X so that the result is also an optional martingale. Suppose $X \in \mathcal{M}^2$ an optional martingale and take into account the decomposition, $X = X^c + X^d + X^g$ and the fact that the integral with respect to the right continuous martingale $X^c + X^d$ is defined in the usual sense. Then, all that we need to define is the integral for the component X^g.

Let us give an analogous result for defining the stochastic integral $h \circ X^g \in \mathcal{M}^g_{loc}$ with the property $\Delta^+ h \circ X^g = h \Delta^+ X^g$, where h is an optional process.

Definition 41 *Suppose the simple function h has the form*

$$h = \sum_{i=0}^{n} h_{t_i} \mathbf{1}_{[t_i.t_{i+1}[}(t),$$

where h_{t_i} is a bounded $\mathcal{F}_{t_j}$-measurable random variable $0 \le i \le n$, and the s.t. $0 = t_0 < t_1 <$

$... < t_n < t_{n+1} = \infty$ *yield a partition of the line. Let*

$$h \circ X^g_\infty = \sum_{i=0}^{n} h_{t_i} \left(X^g_{t_{i+1}} - X^g_{t_i} \right) = \int_0^\infty h_t dX^g_{t+}$$

In particular, for every s.t. T,

$$h \circ X^g_T = \sum_{i=0}^{n} h_{t_i} (X^g_{t_{i+1} \wedge T} - X^g_{t_i \wedge T}) = \int_0^{T-} h_t dX^g_{t+}.$$

We call $h \circ X^g$ the stochastic integral of the function h with respect to X^g.

Theorem 42 *Properties of stochastic integrals:*

(a) The process $h \circ X^g$ is continuous from the left and has limits from the right.

(b) $h \circ X^g \in \mathcal{M}^2$ and

for every s.t.b. T the following assertions hold:

(c) $\Delta^+ h \circ X^g_T = h_T \Delta^+ X^g_T$ a.s. on $(T < \infty)$,

(d) $\mathbf{E} Z_T h \circ X^g_T = \mathbf{E} h \circ \langle Z^g, X^g \rangle_T = \mathbf{E} h \circ [Z^g, X^g]_T$ for $Z \in \mathcal{M}^2$,

(e) $\langle Z, h \circ X^g \rangle = h \circ \langle Z^g, X^g \rangle$ and $[Z, h \circ X^g] = h \circ [Z^g, X^g]$, $Z \in \mathcal{M}^2$.

We are going to use the symbol "$\odot$" to denote integral with respect to the left-continuous part of the integrals. The "$\cdot$" symbol will denote the integral with respect to the right continuous part of the integral. The symbol "$\circ$" will stand for the general integral with respect to any optional process. Sometimes we are going to use the different symbols interchangeably if it is clear from context which integral we intend to use.

Theorem 43 *For every function $h \in \mathsf{L}_2(\langle X^g \rangle)$ there exists a unique process $h \odot X^g$ having properties (a)-(e) of (42).*

Theorem 44 *Let $X = X^g \in \mathcal{M}^2$ then for every $Y \in \mathcal{M}^2$*

$$\mathbf{E}X_\infty Y_\infty = \mathbf{E}\left[\sum_{t<\infty} \Delta^+ X_t \Delta^+ Y_t.\right]$$

Theorem 45 *Suppose $h \in \mathsf{L}_2(\langle X^g \rangle)$ and $X = X^g \in \mathcal{M}^2$. Then the Lebesgue-Stieltjes integral $h^{(S)} \odot X^g$ coincides with the stochastic integral $h \odot X^g$.*

Recall that for $X \in \mathcal{M}_{loc}$ we associate the process $[X, X]$, setting

$$[X, X]_t = \langle X^c, X^c \rangle_t + \sum_{s \leq t} (\Delta X_s)^2 + \sum_{s<t} (\Delta^+ X_s)^2, \ t \in \mathbb{R}_+.$$

The sums on the right side are finite a.s. for all $t \in \mathbb{R}_+$.

Definition 46 *We use $\mathsf{H}^{1,g}$ to denote the collection of local martingales $X = X^g$ for which $\|X\|_{\mathsf{H}^{1,g}} = \mathbf{E}([X, X]_\infty)^{1/2} < \infty$.*

Theorem 47 *The space $\mathsf{H}^{1,g}$ with the seminorm $\|\cdot\|_{\mathsf{H}^{1,g}}$ is a complete normed space.*

Lemma 48 *If $X = X^g \in \mathcal{M}_{loc}$, then $X \in \mathsf{H}^{1,g}_{loc}$.*

We proceed to the definition of the stochastic integral with respect to a local martingale. As in the class of square integrable martingales, it is sufficient to define the integral with respect to the process $X = X^g$.

Theorem 49 *Suppose $X = X^g \in \mathcal{M}_{loc}$, $X_0 = 0$, and the function h is optional and has the property that the process $(h^2 \circ [X, X]_t)^{1/2} \in \mathcal{A}_{loc}$. Then the following assertions are true:*
***(a)** For every local martingale N the increasing process $\left(\int_{[0,t[} |h_s| |d[X, N^g]_{s+}|\right)$ is finite for*

$t \in \mathbb{R}_+$.

(b) *There exists a unique local martingale* $Y = h \circ X (= Y^g)$ *such that for every local martingale* N, $[Y, N] = h \circ [X, N^g]$.

(c) $\Delta^+ Y_T = h_T \Delta^+ X_t$ *a.s. on* $(T < \infty)$ *for all s.t.b.* T.

Definition 50 *h an optional function, and suppose they satisfy the conditions*

$$(f^2 \cdot [X^r, X^r])^{1/2} \in \mathcal{A}_{loc}, \quad (h^2 \odot [X^g, X^g])^{1/2} \in \mathcal{A}_{loc},$$

where $X^r = X^c + X^d$, $[X^r, X^r]_t = \langle X^c, X^c \rangle_t + \sum_{s \leq t} (\triangle X_s)^2$ *and* $[X^g, X^g]_t = \sum_{s<t} (\triangle^+ X_s)^2$. *Then, there exists a unique process* $Y \in \mathcal{M}_{loc}$, *which we can write as*

$$Y = (f, h) \odot X = f \cdot X^r + h \odot X^d, \tag{2.3}$$

possessing the properties $\triangle Y = f \triangle X$, $\triangle^+ Y = h \triangle^+ X$, $[Y, Z] = f \cdot [X^r, Z^r] + h \odot [X^g, Z^g]$ *for any* $Z \in \mathcal{M}_{loc}$.

Again, we would like to emphasis the definition of the notations we are using for integrals with respect to optional processes. So, here is a summary,

Notation 51 *The operator "$\circ$" to denote the stochastic optional integral in general. The operator "$\cdot$" for the stochastic integral with respect to RCLL processes and "$\odot$" for the Gal'chuk integral with respect to left continuous processes. Often, we will use "$\circ$" to mean either the regular stochastic integral "$\cdot$" or the Gal'chuk integral "$\odot$" if it is clear from context which one we are working with. The superscript "r" will denote RCLL processes, "d" will denote discrete RCLL processes, "c" denote continuous processes and "g" will denote RLL processes.*

2.5.4 Integration with respect to Semimartingales

A stochastic integral with respect to optional semimartingale was defined by Gal'chuk as

$$Y_t = h \circ X_t = \int_0^t h_s dX_s = h \cdot X_t + h \odot X_t,$$

where

$$h \cdot X_t = \int_{0+}^t h_{s-} dX_s^r = \int_{0+}^t h_{s-} dA_s^r + \int_{0+}^t h_{s-} dM_s^r,$$
$$h \odot X_t = \int_0^{t-} h dX_{s+}^g = \int_0^{t-} h_s dA_{s+}^g + \int_0^{t-} h_s dM_{s+}^g.$$

Recall that the integral with respect to the finite variation processes or strongly predictable process A^r and A^g are interpreted as Lebesgue integrals. $\int_{0+}^t h_{s-} dM_s^r$ is our usual stochastic integral with respect to RCLL local martingale whereas $\int_0^{t-} h_s dM_{s+}^g$ is Gal'chuk stochastic integral [10, 11] with respect to left continuous local martingale. A direct extension of the above integral to a larger class of integrands is given by the bilinear form $(f, g) \circ X_t$,

$$Y_t = (f, g) \circ X_t = f \cdot X_t^r + g \odot X_t^g,$$

where Y_t is again an optional semimartingale $f_- \in \mathcal{P}(\mathbf{F})$, and $g \in \mathcal{O}(\mathbf{F})$ [18]. Hence, for the stochastic integral with respect to optional semimartingales, the space of integrands is the product space of predictable and optional processes, $\mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$. From now on we are going to use the operator "$\circ$" to denote the stochastic optional integral "$\cdot$" the regular stochastic integral with respect to RCLL semimartingales and "$\odot$" for the Gal'chouk stochastic integral $\phi \odot X^g$ with respect to left continuous semimartingales.

2.5.5 Random Measures and Canonical Decomposition

The canonical representation of semimartingales is of fundamental importance in stochastic analysis. It is also essential to our development of stochastic equations driven by optional semimartingales. The canonical representation of an optional semimartingale can be seen as a natural consequence of the decomposition

$$X = X_0 + X^c + X^d + X^g.$$

Let us elaborate on this point; If $m^c \in \mathcal{M}^c_{loc}$ is the martingale part of X^c, $m^d \in \mathcal{M}^d_{loc}$ of X^d and $m^g \in \mathcal{M}^g_{loc}$ of X^g then one can write $\alpha = (X^c - m^c) + \left(X^d - m^d\right) + (X^g - m^g)$. α is strongly predictable with locally integrable variation. Therefore,

$$\begin{aligned} X &= X_0 + X^c + X^d + X^g \\ &= X_0 + m^c + (X^c - m^c) + m^d + \left(X^d - m^d\right) + m^g + (X^g - m^g) \\ &= X_0 + \alpha + m^c + m^d + m^g. \end{aligned}$$

m^d and m^g are discrete local martingales that are representable in terms of an underlying measures of right and left jumps, respectively. These measures of jumps are referred to as integer valued random measures. We describe the integer random measure representation of discrete martingales briefly and refer the reader to the paper by Gal'chuk [11] for details.

Consider the Lusin space $(\mathbb{E}, \mathscr{E})$ where $\mathbb{E} = \mathbb{R}\backslash\{0\}$ and $\mathscr{E} = \mathcal{B}(\mathbb{E})$ is the Borel σ-algebra in $\mathbb{E}$. Also, define the spaces

$$\begin{gathered} \widetilde{\Omega} = \Omega \times \mathbb{R}_+ \times \mathbb{E}, \quad \widetilde{\mathbb{E}} = \mathbb{R}_+ \times \mathbb{E}, \quad \widetilde{\mathscr{E}} = \mathcal{B}(\mathbb{R}_+) \times \mathscr{E} \\ \widetilde{\mathcal{O}}(\mathbf{F}) = \mathcal{O}(\mathbf{F}) \times \mathscr{E}, \quad \widetilde{\mathcal{O}}(\mathbf{F}_+) = \mathcal{O}(\mathbf{F}_+) \times \mathscr{E}, \quad \widetilde{\mathcal{P}}(\mathbf{F}) = \mathcal{P}(\mathbf{F}) \times \mathscr{E}. \end{gathered} \tag{2.4}$$

On $\widetilde{\Omega}$ let $\mu(\omega, \cdot, \cdot)$ and $\eta(\omega, \cdot, \cdot)$ be integer valued measures defined on the σ-algebra $\widetilde{\mathscr{E}}$. A non-negative random set function $\mu(\omega, \Gamma)$, $\omega \in \Omega$, $\Gamma \in \widetilde{\mathscr{E}}$, is called a random measure on $\widetilde{\mathscr{E}}$ if $\mu(\cdot, \Gamma) \in \mathcal{F}$ for any $\Gamma \in \widetilde{\mathscr{E}}$ and $\mu(\omega, \cdot)$ is a σ-finite measure on $\left(\widetilde{\mathbb{E}}, \widetilde{\mathscr{E}}\right)$ for each $\omega \in \Omega$. A random measure is called integer-valued if $\mu(\omega, \Gamma) \in \{0, 1, ..., +\infty\}$ and $0 \leq \mu(\omega, \{t\} \times E) \leq 1$ for all

(ω, Γ). For the optional semimartingale $X \in \mathcal{O}(\mathbf{F})$ we define μ and η as

$$
\begin{aligned}
\mu(\omega, dt, dx) &= \sum_{s>0} \mathbf{1}_{(\triangle X_s \neq 0)} \boldsymbol{\varepsilon}_{(s,\triangle X_s)} (dt, dx) , \\
\eta(\omega, dt, dx) &= \sum_{s\geq 0} \mathbf{1}_{(\Delta^+ X_s \neq 0)} \boldsymbol{\varepsilon}_{(s,\triangle^+ X_s)} (dt, dx) ,
\end{aligned}
$$

where $\boldsymbol{\varepsilon}_{(a,b)} (dx, dy)$ is the Dirac measure. Here μ is the measure of right jumps while η the measure of the left jumps. Note, η is $\mathcal{O}(\mathbf{F}_+)$-optional with its compensator λ being $\mathcal{O}(\mathbf{F})$-optional. On the other hand μ is $\mathcal{O}(\mathbf{F})$-optional with its compensator ν being $\mathcal{P}(\mathbf{F})$-predictable. The predictable and optional compensators of jump measures are defined by the following lemmas.

Compensator of Random Measure

On $(\tilde{\Omega}, \tilde{\mathcal{O}}(\mathbf{F}))$ define the measure M^P_μ by setting $M^P_\mu(f) = \mathbf{E}\left(f \cdot \mu_\infty\right)$ where $f \in \tilde{\mathcal{O}}(\mathbf{F})$, $f \geq 0$.

Lemma 52 *Let the optional measure μ be such that the measure M^P_μ is $\tilde{\mathcal{P}}$-σ-finite, i.e. the restriction of M^P_μ to $(\tilde{\Omega}, \tilde{\mathcal{P}})$ is σ-finite. Then, there exist a unique to within a set of $\mathcal{P}$-null measure a predictable measure $v = \nu(\omega, dt, dx)$ such that for any function $f \in \tilde{\mathcal{O}}(\mathbf{F})$, $f \geq 0$, one has $M^P_\mu(f) = M^P_\nu(f)$. The measure v can be written in the form*

$$
\nu(\omega, dt, dx) = dA_t(\omega) K(\omega, t, dx),
$$

where A is an increasing predictable right-continuous process, $K(\omega, t, dx)$ is the kernel of the space $(\Omega \times \mathbb{R}_+, \mathcal{P}(\mathbf{F}))$ into $(\mathbb{E}, \mathcal{E})$. If the measure μ does not load any predictable stopping times whatsoever then the same is true of v, and the process $f \cdot \nu$ is continuous for any function $f \in \tilde{\mathcal{P}}$, $f \geq 0$. Moreover, for any $S \in \mathcal{T}^p$ and any $f \in \tilde{\mathcal{P}}$, $f \geq 0$, on $(S < \infty)$,

$$
\mathbf{E}\left[\int_E f(S, x) \mu(\{S\}, dx) | \mathcal{F}_{S-}\right] = \int_{\mathbb{E}} f(S, x) \nu(\{S\}, dx).
$$

The process $f \cdot \nu$ is the dual predictable projection for the process $f \cdot \mu$, $f \in \tilde{\mathcal{P}}$, $f \geq 0$. If $f \cdot \mu \in \mathcal{A}_{loc}$, then the process $f \cdot \mu - f \cdot v \in \mathcal{M}_{loc}$. In the case of integer-valued μ, outside a set of $\mathbf{P}$-null measure, $0 \leq \nu(\omega, \{t\} \times E) \leq 1$ for all $t \in \mathbb{R}_+$. The measure v is called the compensator of the measure μ.

Let η be an $\mathcal{O}(\mathbf{F}_+)$-optional measure. On the space $(\tilde{\Omega}, \tilde{\mathcal{O}}(\mathbf{F}_+))$ define the measure M^P_η by setting $M^P_\eta(f) = \mathbf{E} f \odot \eta_\infty$ for any $f \in \tilde{\mathcal{O}}(\mathbf{F}_+)$, $f \geq 0$.

Lemma 53 *Let the $\mathcal{O}(\mathbf{F}_+)$-optional measure η be such that the measure M^P_η is $\tilde{\mathcal{O}}(F)$-σ-finite. Then: There exists a unique (to within a set of $\mathbf{P}$-null measure) optional measure $\lambda = \lambda(\omega, dt, dx)$ such that, for any $f \in \tilde{\mathcal{O}}(\mathbf{F})$, $f \geq 0$, one has $M^P_\eta(f) = M^P_\lambda(f)$; the measure λ can be written in the form*

$$\lambda(\omega, dt, dx) = dA_t(\omega) K(\omega, t, dx),$$

where A is an increasing right-continuous $\mathcal{O}(\mathbf{F})$-measurable process, $K(\omega, t, dx)$ is the kernel from the space $(\Omega \times \mathbb{R}_+, \mathcal{O}(\mathbf{F}))$ into $(\mathbb{E}, \mathcal{E})$. If the measure η does not load any stopping time $T \in \mathcal{T}$ whatever, then the same is true for the measure λ, and the process $f \odot \lambda$ is continuous for any $f \in \tilde{\mathcal{O}}(\mathbf{F})$, $f \geq 0$. The process $f \odot \lambda_+$ is the dual optional projection for the process $f \odot \eta_+$ for $f \in \tilde{\mathcal{O}}(\mathbf{F})$, $f \geq 0$. In particular, for $f \in \tilde{\mathcal{O}}(\mathbf{F})$, $f \geq 0$ and $T \in \mathcal{T}$, $\mathbf{E}\left[\int_{\mathbb{E}} f(T, x)\eta(\{T\}, dx) | \mathcal{F}_T\right] = \mathbf{E}\left[\int_{\mathbb{E}} f(T, x)\lambda(\{T\}, dx) | \mathcal{F}_T\right]$ a.s..on $(T < \infty)$; if $f \odot \eta_+ \in \mathcal{A}_{loc}$, then the process $f \odot \eta_t - f \odot \lambda_t \in \mathcal{M}_{loc}$, where

$$f \odot \eta_t - f \odot \lambda_t = \int_{[0,t[\times \mathbb{E}} f \eta(ds, dx) - \int_{[0,t[\times \mathbb{E}} f \lambda(ds, dx).$$

If the measure η is integer-valued, then there exists a modification of the measure λ such that outside a set of $\mathbf{P}$-null measure, $0 \leq \lambda(\omega, \{t\}, E) \leq 1$ for any $t \in \mathbb{R}_+$. We shall call the measure λ the compensator of the $(\mathcal{O}(\mathbf{F}_+)$-optional measure η.

Integral Representation of Discrete Martingales

Functions that are integrands over the space of integer valued measures μ and η (integrators) are either $\widetilde{\mathcal{O}}(\mathbf{F})$-optional for the measure η and $\widetilde{\mathcal{P}}(\mathbf{F})$-predictable for the measure μ. That is, for functions $h \in \widetilde{\mathcal{P}}(\mathbf{F})$ and $k \in \widetilde{\mathcal{O}}(\mathbf{F})$ the integrals,

$$\begin{aligned} h \cdot (\mu - \nu)_t &= \int_{0+}^{t} \int_{\mathbb{E}} h(\omega, s, u)(\mu - \nu)(ds, du), \\ k \odot (\eta - \lambda)_t &= \int_{0}^{t-} \int_{\mathbb{E}} k(\omega, s, u)(\eta - \lambda)(ds, du), \end{aligned}$$

where $\mu - \nu \in \mathcal{M}^d_{loc}(\mathbf{F})$ and $\eta - \lambda \in \mathcal{M}^g_{loc}(\mathbf{F}_+)$, are well defined with the following properties,

$$\begin{aligned} \Delta h \cdot (\mu - \nu)_t &= \int_{\mathbb{E}} h(\omega, s, u)(\mu - \nu)(ds, du), \\ \Delta^+ k \odot (\eta - \lambda)_t &= \int_{\mathbb{E}} k(\omega, s, u)(\eta - \lambda)(ds, du). \end{aligned}$$

Now, lets consider the following spaces of functions,

$$\begin{aligned} G^i(\mu) &= \left\{ h \in \widetilde{\mathcal{P}}(\mathbf{F}) | \left(\sum_{0<s\leq \cdot} \left| \int_{\mathbb{E}} h(\omega, s, u)(\mu - \nu)(ds, du) \right|^2 \right)^{\frac{i}{2}} \in \mathcal{A}_{loc}(\mathbf{F}) \right\}, \\ G^i(\eta) &= \left\{ k \in \widetilde{\mathcal{O}}(\mathbf{F}) | \left(\sum_{0\leq s< \cdot} \left| \int_{\mathbb{E}} k(\omega, s, u)(\eta - \lambda)(ds, du) \right|^2 \right)^{\frac{i}{2}} \in \mathcal{A}_{loc}(\mathbf{F}) \right\}, \end{aligned}$$

then it is necessary and sufficient conditions that $h \in G^i(\mu)$ and $k \in G^i(\eta)$ for the integrals $h \cdot (\mu - \nu) \in \mathcal{M}^{i,d}_{loc}(\mathbf{F})$ and $k \odot (\eta - \lambda) \in \mathcal{M}^{i,g}_{loc}(\mathbf{F})$ to exist with unique elements.

Having well defined integer valued measure and stochastic integrals with respect to these measures one can finally write a representation of discrete local martingales. So, the local martingales m^d and m^g are represented as

$$\begin{aligned} m_t^d &= \int_{0+}^{t} \int_{\mathbb{E}} \kappa(\omega, s, u)(\mu - \nu)(ds, du), \\ m_t^g &= \int_{0}^{t-} \int_{\mathbb{E}} \kappa(\omega, s, u)(\eta - \lambda)(ds, du). \end{aligned}$$

where $\kappa(\omega, t, u) = u\mathbf{1}_{|u|\le 1}$ and $\varsigma(\omega, t, u) = u\mathbf{1}_{|u|>1}$, ν is the compensator of the measure μ and λ is that of ν (i.e. $\nu = \left(X^d - m^d\right)$ and $\lambda = (X^g - m^g)$).

Canonical Decomposition

Therefore, if $X = (X_t)$, $t \in [0, \infty[$, is an optional semimartingale in $\mathbb{R}$ then X admits the canonical representation

$$\begin{aligned} X_t &= a_t + m_t + \int_{0+}^{t} \int_{\mathbb{E}} \kappa(\omega, s, u)(\mu - \nu)(ds, du) + \int_{0}^{t-} \int_{\mathbb{E}} \kappa(\omega, s, u)(\eta - \lambda)(ds, du) \\ &\quad + \int_{0+}^{t} \int_{\mathbb{E}} \varsigma(\omega, s, u)\mu(ds, du) + \int_{0}^{t-} \int_{\mathbb{E}} \varsigma(\omega, s, u)\eta(ds, du) \end{aligned}$$

where $\kappa(\omega, t, u) = u\mathbf{1}_{|u|\le 1}$ and $\varsigma(\omega, t, u) = u\mathbf{1}_{|u|>1}$, a is strongly predictable with locally integrable variation ($a \in \mathcal{P}_s \cap \mathcal{A}_{loc}$), and m a continuous local martingale ($m \in \mathcal{M}_{loc}^c$). μ and η are integer valued measures where μ is the measure of right jumps and η the measure of the left jumps. ν is the compensator for μ and λ is that of ν. Using optional stochastic integral notations one can write the canonical representation equation as

$$X = a + m + \kappa \cdot (\mu - \nu) + \varsigma \cdot \mu + \kappa \odot (\eta - \lambda) + \varsigma \odot \eta, \tag{2.5}$$

$$\begin{aligned} \varsigma \cdot \mu &= \int_{0+}^{t} \int_{\mathbb{E}} \varsigma(\omega, s, u)\mu(ds, du), \quad \kappa \cdot (\mu - \nu) = \int_{0+}^{t} \int_{\mathbb{E}} \kappa(\omega, s, u)(\mu - \nu)(ds, du) \\ \varsigma \odot \eta &= \int_{0}^{t-} \int_{\mathbb{E}} \varsigma(\omega, s, u)\eta(ds, du), \quad \kappa \odot (\eta - \lambda) = \int_{0}^{t-} \int_{\mathbb{E}} \kappa(\omega, s, u)(\eta - \lambda)(ds, du) \end{aligned}$$

We remark that the existence of this decomposition was shown in [11].

2.5.6 Change of Variable Formula

The stochastic integral and the change of variable formula are the corner stones of stochastic calculus. With these two we can construct many more optional semimartingales.

Theorem 54 *Suppose X is an n-dimensional optional semimartingale, i.e., $X = (X^1, \ldots, X^n)$, $X^k = X_0^k + A^k + M^k \in S$, $k = 1,\ldots,n$, and $F(x) = F(x^1, \ldots, x^n)$ is a twice continuously differentiable function on $\mathbb{R}^n$. Then the process $F(X) \in \mathcal{S}$, and for all $t \in \mathbb{R}_+$,*

$$\begin{aligned} F(X_t) &= F(X_0) + \sum_{k=1}^{n} \int_{0+}^{t} \partial^k F(X_{s-}) d(A^{kr} + M^{kr})_s \\ &+ \sum_{k=1}^{n} \int_{0}^{t-} \partial^k F(X_s) d(A^{kg} + M^{kg})_{s+} \\ &+ \frac{1}{2} \sum_{k.l=1}^{n} \int_{0+}^{t} \partial^k \partial^l F(X_{s-}) d\langle M^{kc}, M^{lc} \rangle_s \\ &+ \sum_{0<s\leq t} \left[F(X_s) - F(X_{s-}) - \sum_{k=1}^{n} \partial^k F(X_{s-}) \Delta X_s^k \right] \\ &+ \sum_{0\leq s<t} \left[F(X_{s+}) - F(X_s) - \sum_{k=1}^{n} \partial^k F(X_s) \Delta^+ X_s^k \right], \end{aligned}$$

where ∂^k is the differentiation operator with respect to the k^{th} coordinate.

The above formula can be written compactly in the following way,

$$F(X) = F(X_0) + \partial F(X) \circ X + \frac{1}{2} \partial^2 F(X) \circ [X, X].$$

To justify this, the parts the formula above take the following form,

$$\partial F(X) \circ X = \partial F(X) \circ X^c + \partial F(X) \circ X^d + \partial F(X) \circ X^g$$

and

$$\frac{1}{2} \partial^2 F(X) \circ [X, X] = \frac{1}{2} \partial^2 F(X) \circ \left(\langle X^c, X^c \rangle + \Delta X^d \Delta X^d + \Delta^+ X^g \Delta^+ X^g \right)$$

and $\frac{1}{2}\partial^2 F(X) \circ \left(\Delta X^d \Delta X^d\right) = \Delta F(X) - \partial F(X_-) \circ X^d$ and $\frac{1}{2}\partial^2 F(X) \circ (\Delta^+ X^g \Delta^+ X^g) = \Delta^+ F(X) - \partial F(X) \circ X^g$.

2.5.7 Stochastic Inequalities

Kunita Watanabe inequality is a generalization of the Cauchy Schwarz inequality to integrals of stochastic processes. It has many applications in stochastic analysis. For square integrable optional martingales we have the following generalization of Kunita Watanabe inequality;

Theorem 55 *Kunita-Watanabe Inequalities. Suppose X, $Y \in \mathcal{M}^2$, and H and K are measurable random processes. Then, a.s. for every s.t.b. T*

$$\int_0^T |H||K||d\left\langle X, Y\right\rangle| \leq \sqrt{H^2 \circ \langle X\rangle_T}\sqrt{K^2 \circ \langle Y\rangle}$$

also,

$$\int_0^T |H||K||d\left[X, Y\right]| \leq \sqrt{H^2 \circ [X, X]_T}\sqrt{K^2 \circ [Y, Y]_T}$$

Chapter 3

Stochastic Equations

Stochastic equations have numerous applications in Engineering, Finance, and Physics and are a central part of stochastic analysis. Moreover, and perhaps a fundamental aspect of stochastic equations is that they provide a way of manufacturing complex processes from ones that are simpler. For example, a geometric Brownian motion is constructed from the simpler Weiner process. The impetus of progress in the study of stochastic equations came as a result of the development of semimartingale integration theory which gave the study of stochastic equations a strong theoretical basis. As research in stochastic integration theory progresses to realm beyond the usual probability spaces and RCLL processes, stochastic equations will also be advanced in those directions too. This is the main goal of this chapter, to study stochastic equations driven by optional semimartingales.

Here we extend stochastic equations to optional semimartingales on "unusual" stochastic basis. We cover the important topics of stochastic linear equations, stochastic exponential and logarithms, solutions of the nonhomogenous stochastic linear equation, Gronwall lemma, existence and uniqueness of solution of stochastic equations under monotonicity conditions and comparison lemma under Yamada conditions.

3.1 Stochastic Linear Equations, Exponentials and Logarithms

Stochastic exponentials and logarithms are indispensable tools of financial mathematics. They describe relative returns, link hedging with the calculation of minimal entropy and utility indifference. Moreover, they determine the structure of the Girsanov transformation.

For optional semimartingales the stochastic exponential was defined by Gal'chuk [11]; If $X \in \mathcal{S}(\mathbf{F}, \mathbf{P})$ then there exists a unique semimartingale $Z \in \mathcal{S}(\mathbf{F}, \mathbf{P})$ such that

$$Z_t = Z_0 + Z \circ X_t = Z_0 + \int_{0+}^{t} Z_{s-} dX_s^r + \int_0^{t-} Z_s dX_{s+}^g, = Z_0 \mathcal{E}(X)_t, \tag{3.1}$$

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}\langle X^c, X^c \rangle \right) \prod_{0 \le s \le t} (1 + \Delta X_s) e^{-\Delta X_s} \prod_{0 \le s < t} (1 + \Delta^+ X_s) e^{-\Delta^+ X_s}.$$

Two useful properties of stochastic exponentials are the inverse and product formulas;

Lemma 56 *The product of stochastic exponentials is* $\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y])$.

Proof. Using the change variable formula and definition of stochastic exponential,

$$\begin{aligned}
\mathcal{E}(X)\mathcal{E}(Y) &= 1 + \mathcal{E}(X) \circ \mathcal{E}(Y) + \mathcal{E}(Y) \circ \mathcal{E}(X) + [\mathcal{E}(X), \mathcal{E}(Y)] \\
&= 1 + \mathcal{E}(X)\mathcal{E}(Y) \circ Y + \mathcal{E}(X)\mathcal{E}(Y) \circ X + \mathcal{E}(X)\mathcal{E}(Y) \circ [X, Y] \\
&= 1 + \mathcal{E}(X)\mathcal{E}(Y) \circ (X + Y + [X, Y]) \\
&= \mathcal{E}\left(X + Y + [X, Y]\right).
\end{aligned}$$

■

Lemma 57 *The inverse of stochastic exponential for the semimartingale* X *is* $\mathcal{E}^{-1}(X) = \mathcal{E}(-X^*)$, *where,*

$$X_t^* = X_t - \langle X^c, X^c \rangle_t - \sum_{0 < s \le t} \frac{(\triangle X_s)^2}{1 + \triangle X_s} - \sum_{0 \le s < t} \frac{(\triangle^+ X_s)^2}{1 + \triangle^+ X_s}.$$

Proof. Suppose there is a X^* such that $\mathcal{E}(-X^*)\,\mathcal{E}(X) = \mathcal{E}(X - X^* - [X, X^*]) = 1$, which implies that $X - X^* - [X, X^*] = 0$ hence $X^* = X - [X, X]$,

$$\begin{aligned} X^* &= X - \langle X^c \rangle_t + \sum_{s \leq t} (\Delta X_s)^2 + \sum_{s<t} \left(\Delta^+ X_s\right)^2 \\ &= X - \langle X^c \rangle_t + \sum_{0<s\leq t} \frac{(\triangle X_s)^2 + (\triangle X_s)^3}{1 + \triangle X_s} + \sum_{0\leq s<t} \frac{\left(\triangle^+ X_s\right)^2 + \left(\triangle^+ X_s\right)^3}{1 + \triangle^+ X_s} \end{aligned}$$

since the sums of $(\triangle X)^3 / (1 + \triangle X)$ is zero, to see this consider

$$\sum (\triangle X)^3 / (1 + \triangle X) = \left[\sum (\triangle X)^2 / (1 + \triangle X), \sum \triangle X\right] = 0$$

and as well as for $\left(\triangle^+ X\right)^3 / (1 + \triangle^+ X)$ then

$$X^* = X - \langle X^c \rangle_t + \sum_{0<s\leq t} \frac{(\triangle X_s)^2}{1 + \triangle X_s} + \sum_{0\leq s<t} \frac{\left(\triangle^+ X_s\right)^2}{1 + \triangle^+ X_s}.$$

■

The stochastic logarithm is defined by the following theorem.

Theorem 58 *Let Y be a real valued optional semimartingale such that the processes Y_- and Y do not vanish then the process*

$$X_t = \frac{1}{Y} \circ Y_t = \int_{0+}^{t} \frac{1}{Y_{s-}} dY_s^r + \int_0^{t-} \frac{1}{Y_s} dY_{s+}^g, \quad X_0 = 0, \tag{3.2}$$

also denoted by $X = \mathcal{L}\text{og}Y$ is called the stochastic logarithm of Y, is the unique semimartingale X such that $Y = Y_0 \mathcal{E}(X)$. Moreover, if $\Delta X \neq -1$ and $\Delta^+ X \neq -1$ we also have

$$\begin{aligned} \mathcal{L}\text{og}Y_t &= \log\left|\frac{Y_t}{Y_0}\right| + \frac{1}{2Y^2} \circ \langle Y^c, Y^c \rangle_t - \sum_{0<s\leq t} \left(\log\left|1 + \frac{\Delta Y_s}{Y_{s-}}\right| - \frac{\Delta Y_s}{Y_{s-}}\right) \\ &\quad - \sum_{0\leq s<t} \left(\log\left|1 + \frac{\Delta^+ Y_s}{Y_s}\right| - \frac{\Delta^+ Y_s}{Y_s}\right). \end{aligned} \tag{3.3}$$

It is important to note that the process Y need not be positive for $\mathcal{L}\text{og}(Y)$ to exist, in accordance with the fact that the stochastic exponential $\mathcal{E}(X)$ may take negative values.

Proof. The assumption that Y_- and Y don't vanish implies that $S_n = \inf\left(t : |Y_{t-}| \leq \frac{1}{n}\right) \uparrow \infty$, hence, $1/Y_-$ is locally bounded, likewise, $T_n = \inf\left(t : |Y_t| \leq \frac{1}{n}\right) \uparrow \infty$, hence, $1/Y$ is also locally bounded and the stochastic integral in (3.2) make sense. Let $\tilde{Y} = Y/Y_0$ then $\tilde{Y}_0 = 1$ and by (3.2), $X = (1/\tilde{Y}) \circ \tilde{Y}$. Thus,

$$1 + \tilde{Y} \circ X = 1 + \tilde{Y} \circ \left(\frac{1}{\tilde{Y}} \circ \tilde{Y}\right) = 1 + \left(\tilde{Y}\frac{1}{\tilde{Y}}\right) \circ \tilde{Y} = \tilde{Y},$$

or $\tilde{Y} = \mathcal{E}(X)$. Furthermore, $\Delta X = \Delta Y/Y_- \neq -1$ and $\Delta X = \Delta^+ Y/Y \neq -1$. Let $\tilde{X}$ be any other semimartingale satisfying $Y = \mathcal{E}(\tilde{X})$. With $\tilde{Y} = Y/Y_0$, yields $\tilde{Y} = \mathcal{E}(\tilde{X})$. So, $\tilde{Y} = 1 + \tilde{Y} \circ \tilde{X}$ or $Y = Y_0 + \tilde{Y} \circ \tilde{X}$. Since $X_0 = 0$ we have,

$$\tilde{X} = \frac{\tilde{Y}}{\tilde{Y}} \circ \tilde{X} = \frac{1}{\tilde{Y}} \circ (Y - Y_0) = \frac{1}{\tilde{Y}} \circ Y = X,$$

and we get uniqueness.

To prove representation (3.3) we have to apply Gal'chuk extension [11] of Ito's lemma to the optional semimartingale $\log|Y|$. Since the log explodes at 0 we consider: for each n, the C^2 function f_n on $\mathbb{R}$, with $f_n(x) = \log|x|$ when $|x| \geq 1/n$. Then for all n and $t < T_n$ and $t < S_n$ we have

$$\begin{aligned}\log|Y_t| &= \log|Y_0| + \frac{1}{Y} \circ Y_t - \frac{1}{2Y^2} \circ \langle Y^c, Y^c \rangle_t + \sum_{0<s\leq t}\left(\Delta \log|Y_s| - \frac{\Delta Y_s}{Y_{s-}}\right) \\ &\quad + \sum_{0\leq s<t}\left(\Delta^+ \log|Y_s| - \frac{\Delta^+ Y_s}{Y_s}\right).\end{aligned}$$

This together with (3.2) yields (3.3) for $t < T_n$ and and $t < S_n$. Since $T_n \uparrow \infty$ and $S_n \uparrow \infty$ then we obtain (3.3) everywhere. ■

Now, we present some of the properties of stochastic logarithms.

Lemma 59 *(a) If X is a semimartingale satisfying $\Delta X \neq -1$ and $\Delta^+ X \neq -1$ then $\mathcal{L}\text{og}(\mathcal{E}(X)) = X - X_0$. (b) If Y is a semimartingale such that Y and Y_- do not vanish, then $\mathcal{E}(\mathcal{L}\text{og}(Y)) = Y/Y_0$. (c) For any two optional semimartingales X and Z we get the following identities:*

$$\mathcal{L}\text{og}\,(XZ) = \mathcal{L}\text{og}X + \mathcal{L}\text{og}Z + [\mathcal{L}\text{og}X, \mathcal{L}\text{og}Z]\,;$$

and

$$\mathcal{L}\text{og}\left(\frac{1}{X}\right) = 1 - \mathcal{L}\text{og}\,(X) - \left[X, \frac{1}{X}\right].$$

Proof. (a)

$$\mathcal{L}\text{og}(\mathcal{E}(X)) = \mathcal{E}(X)^{-1} \circ \mathcal{E}(X) = \frac{\mathcal{E}(X)}{\mathcal{E}(X)} \circ X = X - X_0.$$

(b) Let $Z = \mathcal{E}(\mathcal{L}\text{og}(Y))$ then by (a) we find that

$$\mathcal{L}\text{og}\,(Z) = \mathcal{L}\text{og}\,(\mathcal{E}(\mathcal{L}\text{og}(Y))) = \mathcal{L}\text{og}(Y) - \mathcal{L}\text{og}(Y_0) = \mathcal{L}\text{og}(Y/Y_0).$$

Therefore, $Z = Y/Y_0$.

(c) By the integral representation of the stochastic logarithm we find

$$\mathcal{L}\text{og}\,(XZ) = \frac{1}{XZ} \circ (X \circ Z + Z \circ X + [X, Z]) = \mathcal{L}\text{og}X + \mathcal{L}\text{og}Z + [\mathcal{L}\text{og}X, \mathcal{L}\text{og}Z]\,.$$

Using integration by parts and the integral definition the stochastic logarithm,

$$\mathcal{L}\text{og}\left(\frac{1}{X}\right) = X \circ \left(\frac{1}{X}\right) = 1 - \frac{1}{X} \circ X - \left[X, \frac{1}{X}\right] = 1 - \mathcal{L}\text{og}\,(X) - \left[X, \frac{1}{X}\right].$$

■

3.2 Nonhomogeneous Linear Stochastic Equation

A generalization of the stochastic exponential integral equation (3.1) is the nonhomogeneous linear stochastic integral equation [33], $X = G + X \circ H$. This equation has a natural application in finance: G is a stochastic cash flow, H is the interest rate of the money market account and X is the time value of the cash flow accumulated in the money market account. Here we will give the solution of the nonhomogeneous linear stochastic integral equation to the case where G and H are optional semimartingales.

Theorem 60 *Consider the nonhomogeneous linear stochastic integral equation,*

$$\begin{aligned} X_t &= G_t + \int_0^t X_s dH_s \\ &= G_t + \int_{0+}^t X_{s-} dH_s^r + \int_0^{t-} X_s dH_{s+}^g \end{aligned} \tag{3.4}$$

$G = (G_t)_{t\geq 0} \in \mathcal{S}(\mathbf{F}, \mathbf{P})$ *is an optional semimartingale. The solution is*

$$\begin{aligned} X_t &= \mathcal{E}_t(H)\left[G_0 + \int_0^t \mathcal{E}_s(H)^{-1} d\tilde{G}_s\right], \\ d\tilde{G}_t &= dG_t - d\left[G, \tilde{H}\right]_t, \\ \tilde{H}_t &= H_t^c + \sum_{0<s\leq t} \frac{\triangle H_s}{1+\triangle H_s} + \sum_{0\leq s<t} \frac{\triangle^+ H_s}{1+\triangle^+ H_s}. \end{aligned} \tag{3.5}$$

Proof. Lets consider a solution of the following form $X_t = \mathcal{E}_t(H)Z_t$ where Z_t is related to G_t. The differential of X_t is

$$\begin{aligned} dX_t &= d\left(\mathcal{E}_t(H)Z_t\right) \\ &= X_t dH_t + \mathcal{E}_t(H)\left\{dZ_t + d[H,Z]_t\right\}. \end{aligned} \tag{3.6}$$

Comparing equation (3.4) to equation (3.6) we find that $dG_t = \mathcal{E}_t(H)\left\{dZ_t + d[H,Z]_t\right\}$. We would like to solve this equation for Z_t. To do so we choose

$$\tilde{H}_t = H_t^c + \sum_{0<s\leq t} \frac{\triangle H_s}{1+\triangle H_s} + \sum_{0\leq s<t} \frac{\triangle^+ H_s}{1+\triangle^+ H_s}$$

and compute the quadratic variation of G with $\tilde{H}$. Note that the quadratic variation for optional processes belongs to the space $\mathcal{A}_{loc}$ and is defined as

$$[G, \tilde{H}]_t = \langle G^c, \tilde{H}^c \rangle_t + \sum_{0<s\leq t} \Delta G_s \Delta \tilde{H}_s + \sum_{0\leq s<t} \Delta^+ G_s \Delta^+_s \tilde{H}.$$

Hence,

$$\begin{aligned}
d\left[G, \tilde{H}\right]_t &= \mathcal{E}_t(H)\left\{d\left[Z, \tilde{H}\right]_t + d\left[[H, Z], \tilde{H}\right]_t\right\} \\
&= \mathcal{E}_t(H)\left\{d\left[Z^c, H^c\right]_t + \frac{\triangle H_t \triangle Z_t}{1 + \triangle H_t} + \frac{\triangle^+ H_t \triangle^+ Z_t}{1 + \triangle^+ H_t}\right. \\
&\left. + \frac{(\triangle H_t)^2}{1 + \triangle H_t}\triangle Z_t + \frac{(\triangle^+ H_t)^2}{1 + \triangle^+ H_t}\triangle^+ Z_t\right\} \\
&= \mathcal{E}_t(H)\left\{d\left[Z^c, H^c\right]_t + \triangle H_t \triangle Z_t \left(\frac{1 + \triangle H_t}{1 + \triangle H_t}\right)\right. \\
&\left. + \triangle^+ H_t \triangle^+ Z_t \left(\frac{1 + \triangle^+ H_t}{1 + \triangle^+ H_t}\right)\right\} \\
&= \mathcal{E}_t(H)\left\{d\left[Z^c, H^c\right]_t + \triangle H \triangle Z + \triangle^+ H_t \triangle^+ Z_t\right\} \\
&= \mathcal{E}_t(H) d\left[Z, H\right]_t.
\end{aligned}$$

Then we calculate Z in the following way,

$$\begin{aligned}
dG_t &= \mathcal{E}_t(H)\left\{dZ_t + d[H, Z]_t\right\} \\
\mathcal{E}_t(H)^{-1} dG_t &= dZ_t + d[H, Z]_t \\
&= dZ_t + \mathcal{E}_t(H)^{-1} d\left[G, \tilde{H}\right]_t \\
dZ_t &= \mathcal{E}_t(H)^{-1}\left[dG_t - d\left[G, \tilde{H}\right]_t\right]
\end{aligned}$$

Note that,

$$\begin{aligned}
\left[[H, Z], \tilde{H}\right] &= \left[\langle H^c, Z^c \rangle + \sum_{s\leq \cdot} \triangle H \triangle Z + \sum_{s<\cdot} \triangle^+ H \triangle^+ Z, \tilde{H}\right] \\
&= \left[\langle H^c, Z^c \rangle, \tilde{H}\right] + \left[\left(\sum_{s\leq \cdot} \triangle H \triangle Z\right), \tilde{H}\right] + \left[\left(\sum_{s<\cdot} \triangle^+ H \triangle^+ Z\right), \tilde{H}\right]
\end{aligned}$$

$\left[\langle H^c, Z^c\rangle, \tilde{H}\right] = 0$ because $\langle H^c, Z^c\rangle$ is continuous locally bounded variation processes and $\tilde{H}$ a semimartingale. Then,

$$\begin{aligned}\left[[H,Z],\tilde{H}\right] &= \left[\left(\sum_{s\leq\cdot}\triangle H\triangle Z\right),\tilde{H}\right] + \left[\left(\sum_{s<\cdot}\triangle^{+}H\triangle^{+}Z\right),\tilde{H}\right] \\ &= \sum_{s\leq t}\left(\triangle H_s\right)^2\triangle Z_s + \sum_{s<t}\left(\triangle^{+}H_s\right)^2\triangle^{+}Z_s.\end{aligned}$$

■

Another important application of stochastic exponential is Gronwall lemma.

3.3 Gronwall Lemma

The Gronwall lemma is a fundamental inequality in analysis and has far reaching consequences. For example, a fundamental problem in the study of differential or integral equations or their stochastic generalization is that of existence and uniqueness of solutions for which many variants of Gronwall's lemma were extensively used. Basically, Gronwall's lemmas allows us to put bounds on functions that satisfies an integral or differential inequality by a solution of a supposed equality. In stochastic analysis the lemma of Gronwall is essential and many extensions have been proposed see for example Metivier [34], Melnikov [35] and others [36, 37, 38]. It is used to study the stability of solutions of stochastic equations of semimartingales. Here we will extend Gronwell lemma to optional semimartingale in unusual probability spaces.

Lemma 61 *Assume an unusual probability space and let X be an optional process and H be optional increasing process and C_t a constant such that*

$$\begin{aligned}X_t &\leq C_t + \int_0^t X_s dH_s \\ &= C_t + \int_{0+}^t X_{s-}dH_s + \int_0^{t-} X_s dH_{s+}\end{aligned}$$

for all $t \in [0,\infty)$*. Then,*

$$X_t \le C_t \mathcal{E}_t(H).$$

Proof. Let

$$N_t = C_t + \int_0^t X_s dH_s - X_t$$

then $N_t \ge 0$ for all t. Therefore,

$$X_t = C_t - N_t + \int_0^t X_s dH_s$$

is a nonhomogeneous stochastic integral equation whose solution is given by

$$\begin{aligned} X_t &= \mathcal{E}_t(H)\left[G_0 + \int_0^t \mathcal{E}_s(H)^{-1} d\tilde{G}_s\right], \qquad (*) \\ G_t &= C_t - N_t \\ \tilde{G}_t &= G_t - \left[G, \tilde{H}\right]_t, \\ \tilde{H}_t &= H_t^c + \sum_{0<s\le t} \frac{\triangle H_s}{1+\triangle H_s} + \sum_{0\le s<t} \frac{\triangle^+ H_s}{1+\triangle^+ H_s}. \end{aligned}$$

Since H is increasing then $\triangle H_s \ge 0$ and $\triangle^+ H_s \ge 0$ and therefore $\tilde{H}$ is *increasing*. Hence, $\left[G, \tilde{H}\right] = 0$, $\tilde{G}_t = G_t = C_t - N_t \le C_t$ for all t since $N_t \ge 0$. Knowing all this, we can write (*) as

$$\begin{aligned} X_t &= \mathcal{E}_t(H)\left[G_0 + \int_0^t \mathcal{E}_s(H)^{-1} d\tilde{G}_s\right] \\ &= \mathcal{E}_t(H)\left[G_0 + \int_0^t \mathcal{E}_s(H)^{-1} dG_s\right] \\ &\le C_t \mathcal{E}_t(H), \end{aligned}$$

where $\int_0^t \mathcal{E}_s(H)^{-1} dG_s \le 0$ and $G_0 \le C_0$. ■

Next we develop the theory of existence and uniqueness of solutions of stochastic equations of optional semimartingales given that the coefficients satisfy a version of the monotonicity conditions. Monotonicity conditions are less stringent assumptions than Lipschitz conditions.

3.4 Existence and Uniqueness of Solution under Monotonicity Conditions

A central problem in the theory of stochastic equations is the study of existence and uniqueness of solutions under certain conditions placed on the semimartingale driver and on coefficients of this semimartingale. A plethora of stochastic equations, models and proofs were proposed (see [39, 40] for a review). However, little was done in showing existence and uniqueness of solution of stochastic equations driven by optional semimartingales except for the work of Gasparyan in 1985 on existence and uniqueness under Lipschitz conditions [12].

In this chapter, following Gyong [41] exposition for the proof of existence and uniqueness of solution of stochastic equations of RCLL semimartingales under monotonicity conditions, we will consider the question of existence and uniqueness of the following equations for RLL semimartingales on unusual spaces under monotonicity conditions,

$$x = \xi + a(\cdot, x) \circ A + b(\cdot, x) \circ M, \tag{3.7}$$

where A is an increasing strongly predictable optional process, M is a locally square integrable optional martingale (i.e. $\mathcal{M}^2_{loc}(\mathbf{F}, \mathbf{P})$) taking values in a Hilbert space $\mathsf{H}(\mathbf{F}, \mathbf{P})$ and ξ is a random variable in the Euclidean space $\mathbb{R}^d$. Under some conditions on the random functions a and b the existence and uniqueness of a strong solution taking values in $\mathbb{R}^d$ will be demonstrated.

3.4.1 Preliminaries

Consider the the unusual probability space.$(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. Also consider the Banach space, G, with the Borelian σ-algebra $\mathcal{B}(\mathsf{G})$ and $x = (x_t)_{t\geq 0}$ is a G-valued mapping defined on, $\mathbb{R}_+ \times \Omega$, then we shall say that x is a G-valued process. A G-valued process is said to be predictable if it is measurable relative to $\mathcal{P}(\mathbf{F})$ and $\mathcal{B}(\mathsf{G})$. We shall say that x is a G-valued optional process if it is measurable relative to $\mathcal{O}(\mathbf{F})$ and $\mathcal{B}(\mathsf{G})$. We denote by $\mathcal{P}(\mathsf{G})$ the set of all predictable G-valued processes, by $\mathcal{O}(\mathsf{G})$ the set of all optional G-valued processes and by $\mathcal{M}^2_{loc}(\mathsf{G})$ the set of all G-valued locally square integrable optional martingales M with respect to the family $(\mathcal{F}_t)_{t\geq 0}$. We assume, as usual, that $M_0 = 0$. Let us denote the set of all increasing predictable RLL real-valued processes $A = (A_t)_{t\geq 0}$ with the convention $A_0 = 0$ by $\mathcal{V}^+$ and let $\mathcal{V} = \mathcal{V}^+ - \mathcal{V}^+$ be the set of finite variation processes.

Let H be a separable Hilbert space and $\mathsf{G} = \mathbb{R}^d$ a d-dimensional Euclidean space. Also, $\mathsf{L}_1(\mathsf{H},\mathsf{H})$ denotes the Banach space of the nuclear operators on H, $\mathsf{L}_2(\mathsf{H},\mathbb{R}^d)$ the Hilbert space of the Hilbert-Schmidt operators on H into $\mathbb{R}^d$, $\mathsf{H}\otimes_1\mathsf{H}$ the projective tensor product of H by itself and $\mathsf{H}\otimes_2\mathbb{R}^d$ the Hilbertian tensor product of H by $\mathbb{R}^d$.

Let us identify $\mathsf{L}_1(\mathsf{H},\mathsf{H})$ with $\mathsf{H}\otimes_1\mathsf{H}$ and $\mathsf{L}_2(\mathsf{H},\mathbb{R}^d)$ with $\mathsf{H}\otimes_2\mathbb{R}^d$. If $Q \in \mathsf{L}_1(\mathsf{H},\mathsf{H})$ is a non-negative operator, then $\mathsf{L}_Q(\mathsf{H},\mathbb{R}^d)$ denotes the set of all linear not necessarily bounded operators C mapping $Q^{1/2}(\mathsf{H})$ into $\mathbb{R}^d$, such that $CQ^{1/2} \in \mathsf{L}_2(\mathsf{H},\mathbb{R}^d)$.

Recall that if $M \in \mathcal{M}^2_{loc}(\mathsf{H})$ then there exist processes $\langle M\rangle \in \mathcal{P}(\mathbb{R}) \cap \mathcal{V}^+$ and $\langle\langle M\rangle\rangle \in \mathcal{P}(\mathsf{H}\otimes_1\mathsf{H})$, such that $M^2 - \langle M\rangle$ and $M\otimes_1 M - \langle\langle M\rangle\rangle$ are local optional martingales taking values in $\mathbb{R}$ and $\mathsf{H}\otimes_1\mathsf{H}$, respectively and $\langle M\rangle_0 = 0 \in \mathbb{R}$, $\langle\langle M\rangle\rangle_0 = 0 \in \mathsf{H}\otimes_1\mathsf{H}$. If $x, y \in \mathbb{R}^d$, then xy denotes the scalar product of x and y in $\mathbb{R}^d$, $|x| = (xx)^{1/2}$ and if $L \in \mathsf{L}_2(\mathsf{H},\mathbb{R}^d)$, then $|L|$ denotes the Hilbert-Schmidt norm of L. It is the same as the norm in $\mathsf{H}\otimes_2\mathbb{R}^d$.

$A \in \mathcal{V}^+$ and $M \in \mathcal{M}^2_{loc}(\mathsf{H})$ and let $V \in \mathcal{V}^+$, such that $dV_t \geq dA_t$, $dV_t \geq d\langle\langle M\rangle\rangle_t$, and $dV_t \geq d\langle M\rangle_t$. Let $\mathcal{L}$ denote the set of all real-valued non-negative predictable processes which are locally integrable with respect to dV_t. Then there exist processes,

$$R := \tfrac{dA}{dV} \in \mathcal{P}(\mathsf{G})$$

$$Q := \tfrac{d\langle\langle M\rangle\rangle}{dV} \in \mathcal{P}(\mathsf{H}\otimes_1\mathsf{H})$$

or,

$$\begin{aligned} A_t &= \int_{0+}^{t} \left(\frac{dA}{dV}\right)_{s-} dV_s^r + \int_0^{t-} \left(\frac{dA}{dV}\right)_s dV_{s+}^g \\ &= R \circ V_t \\ \langle\langle M \rangle\rangle_t &= \int_{0+}^{t} Q_{s-} dV_s^r + \int_0^{t-} Q_s dV_{s+}^g \\ &= Q \circ V_t \end{aligned}$$

for every $t \in \mathbb{R}_+$ and for almost all $\omega \in \Omega$.

If the process z is such that $z \in \mathsf{L}_Q(\mathsf{H}, \mathbb{R}^d)$, $zQ^{1/2} \in \mathcal{P}(\mathsf{H} \otimes_2 \mathbb{R}^d)$, and $\mathbf{E}\left[\left|z\sqrt{Q}\right|^2 \circ V_t\right] < \infty$, then the stochastic integral

$$y_t = z \circ M_t \tag{3.8}$$

is defined for the usual case (see [42]) and generalization to the unusual case is obvious. Also, one can show that a.s.

$$\langle y \rangle_t = \left|z\sqrt{Q}\right|^2 \circ V_t; \tag{3.9}$$

Therefore, by property (3.9), the integration (3.8) may be extended to a larger class of processes z satisfying the conditions above, and $\langle y \rangle_t < \infty$ a.s.

3.4.2 Existence and Uniqueness Theorem

Let $A \in \mathcal{V}^+(\mathbb{R})$, $M \in \mathcal{M}_{loc}^2(\mathsf{H})$ and let ξ be an $\mathcal{F}_0$-measurable random variable in $\mathbb{R}^d$. Let $V \in \mathcal{V}^+(\mathbb{R})$, such that $dV_t \geq dA_t$ and $dV_t \geq d\langle M \rangle_t$. Let us denote $d\langle\langle M \rangle\rangle_t / dV_t$ by Q_t. We assume that the random functions a and b satisfy the following conditions:

1. a is an $\mathbb{R}^d$-valued, $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable function on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d$, and for almost all (t, ω) (with respect to $\mathbf{P} \times dV_t$) it is continuous in x, and for every fixed $x \in \mathbb{R}^d$ it is locally integrable with respect to dA_t for almost all ω.

2. $\beta^1 := b_- \sqrt{Q_-}$ and $\beta^2 := b\sqrt{Q}$ are $\mathsf{L}_2(\mathsf{H}, \mathbb{R}^d)$-valued $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable functions on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d$

3. For each $R \geq 0$ there exists a process $K_t(R) \in \mathcal{L}$, such that, for every x, $y \in \mathbb{R}^d$ $|x| \leq$

R, $|y| \leq R$ the following inequalities hold for almost all (t, ω) (with respect to $\mathbf{P} \times dV_t$):

i. Monotonity Condition:

$$2(x_- - y_-)(\alpha^1(t,x) - \alpha^1(t,y)) + \triangle V_t \left|\alpha^1(t,x) - \alpha^1(t,y)\right|^2 \qquad (3.10)$$
$$+|\beta^1(t,x) - \beta^1(t,y)|^2 \leq K_t(R)|x_- - y_-|^2$$
$$2(x-y)(\alpha^2(t,x) - \alpha^2(t,y)) + \triangle^+ V_t \left|\alpha^2(t,x) - \alpha^2(t,y)\right|^2 \qquad (3.11)$$
$$+|\beta^2(t,x) - \beta^2(t,y)|^2 \leq K_t(R)|x-y|^2$$

ii. Restriction on Growth:

$$2x_-\alpha^1(t,x) + \triangle V_t|\alpha^1(t,x)|^2 + |\beta^1(t,x)|^2 \leq K_t(R)(1 + |x_-|^2), \qquad (3.12)$$
$$2x\alpha^2(t,x) + \triangle^+ V_t|\alpha^2(t,x)|^2 + |\beta^2(t,x)|^2 \leq K_t(R)(1 + |x|^2), \qquad (3.13)$$

where $\alpha^1 := a_- \left(\frac{dA}{dV}\right)_-$, $\alpha^2 := a\left(\frac{dA}{dV}\right)$and $\beta^1 := b_-\sqrt{Q_-}$ and $\beta^2 := b\sqrt{Q}$.

Therefore, the *existence uniqueness theorem* is,

Theorem 62 *If the* conditions 1-3 *hold, then there exists one and only one* $\mathbb{R}^d$*-valued adapted RLL process* x *satisfying the equation*

$$x_t = \xi + a(\cdot, x) \circ A_t + b(\cdot, x) \circ M_t.$$

The proof of the existence uniqueness theorem (62) will come as a result of a sequence of several lemmas and statements presented next. So we begin by the following simple observations: $\alpha^1(t,x)$ and $\alpha^2(t,x)$ satisfies the Lipschitz conditions in x on the sets $\{(t,\omega) : \triangle V_t > 0\}$ and $\{(t,\omega) : \triangle^+ V_t > 0\}$, respectively; And for $|x| \leq R$ and $|y| \leq R$ we have

$$2 \triangle V_t|\alpha^1(t,x_-) - \alpha^1(t,y_-)| \leq (4 + K_t(R) \triangle V_t)|x_- - y_-|, \qquad (3.14)$$

which follows immediately from (3.10) if we multiply both sides of the inequality by $\triangle V$ and use the inequality $p^2 \geq 4p - 4$. Similarly, from (3.11) we get

$$2 \triangle^+ V_t |\alpha^2(t,x) - \alpha^2(t,y)| \leq (4 + K_t(R) \triangle^+ V_t)|x - y|. \tag{3.15}$$

Notation 63 *Consider a pure jump process X and a process Y then we write*

$$\begin{aligned} [X,Y]_t &= \int_{0+}^{t} \triangle X_{s-} dY_s + \int_0^{t-} \triangle^+ X_s dY_{s+} \\ &= \mathbf{\Delta} X \circ Y_t \end{aligned}$$

where $\mathbf{\Delta} X = (\triangle X, \triangle^+ X)$.

The following lemma will be an important tool in the proof of theorem (62).

Lemma 64 *Let φ be a solution of the equation*

$$\varphi_t = 1 + \int_{0+}^{t} \gamma_s \varphi_{s-} dV_s^r + \int_0^{t-} \gamma_s \varphi_s dV_{s+}^g, \tag{3.16}$$

where $\gamma \in \mathcal{L}$. Let $y \in \mathcal{P}(\mathbb{R})$ and $y = B + N$, where $B \in \mathcal{P}(\mathbb{R}) \cap \mathcal{V}$ and $N \in \mathcal{M}^2_{loc}(\mathbb{R})$. Then, the following equalities also hold

$$\varphi_t^{-1} = 1 - \int_{0+}^{t} \gamma_t \varphi_{t-}^{-1} dV_t^r - \int_0^{t-} \gamma_t \varphi_t^{-1} dV_t^g \tag{3.17}$$

and

$$\varphi_t^{-1} |y_t|^2 = \varphi^{-1} (2y + \mathbf{\Delta} B) \circ B + \varphi^{-1} \circ \langle N, N \rangle - \varphi^{-1} |y|^2 \gamma \circ V + m'_t, \tag{3.18}$$

where

$$m'_t = \varphi^{-1} 2 (y + \mathbf{\Delta} B) \circ N + \varphi^{-1} \circ ([N,N] - \langle N, N \rangle)$$

consequently, m'_t is a local martingale.

Proof. Equation (3.17) follows from Galchuk-Ito formula:

$$\begin{aligned}\varphi_t^{-1} &= 1 - \frac{1}{\varphi^2} \circ \varphi_t + \frac{1}{2}\frac{1}{\varphi^3}\,[\varphi]_t = 1 - \frac{\gamma}{\varphi} \circ V_t + \frac{1}{2}\frac{\gamma^2}{\varphi}\,[V]_t \\ &= 1 - \gamma\varphi^{-1} \circ V_t\end{aligned}$$

since $[V]=0$. The equality (3.18) can be derived as follows,

$$\begin{aligned}
\varphi_t^{-1}|y|_t^2 &= \varphi^{-1}\circ|y|_t^2+|y|^2\circ\varphi_t^{-1}\\
&= \varphi^{-1}\circ(2y\circ y+[y])_t-|y|^2\gamma\varphi^{-1}\circ V_t\\
&= \varphi^{-1}2y\circ y+\varphi^{-1}\circ[y]_t-|y|^2\gamma\varphi^{-1}\circ V_t\\
&= \varphi^{-1}2y\circ y+\varphi^{-1}\boldsymbol{\Delta}y\circ y_t-|y|^2\gamma\varphi^{-1}\circ V_t\\
&= \varphi^{-1}\left(2y+\boldsymbol{\Delta}y\right)\circ y-\varphi^{-1}|y|^2\gamma\circ V\\
&= \varphi^{-1}\left(2y+\boldsymbol{\Delta}y\right)\circ(B+N)-\varphi^{-1}|y|^2\gamma\circ V\\
&= \varphi^{-1}2y\circ B+\varphi^{-1}\boldsymbol{\Delta}y\circ B+\varphi^{-1}2y\circ N\\
&\quad+\varphi^{-1}\boldsymbol{\Delta}y\circ N-\varphi^{-1}|y|^2\gamma\circ V\\
&= \varphi^{-1}2y\circ B+\varphi^{-1}\circ[y,B]+\varphi^{-1}2y\circ N\\
&\quad+\varphi^{-1}\circ[y,N]-\varphi^{-1}|y|^2\gamma\circ V\\
&= \varphi^{-1}2y\circ B+\varphi^{-1}\circ[B+N,B]+\varphi^{-1}2y\circ N\\
&\quad+\varphi^{-1}\circ[B+N,N]-\varphi^{-1}|y|^2\gamma\circ V\\
&= \varphi^{-1}2y\circ B+\varphi^{-1}\circ[B,B]+\varphi^{-1}\circ[N,B]+\varphi^{-1}2y\circ N\\
&\quad+\varphi^{-1}\circ[B,N]+\varphi^{-1}\circ[N,N]-\varphi^{-1}|y|^2\gamma\circ V\\
&= \varphi^{-1}2y\circ B+\varphi^{-1}\boldsymbol{\Delta}B\circ B+\varphi^{-1}2y\circ N+2\varphi^{-1}\circ[B,N]\\
&\quad+\varphi^{-1}\circ[N,N]-\varphi^{-1}|y|^2\gamma\circ V\\
&= \varphi^{-1}\left(2y+\boldsymbol{\Delta}B\right)\circ B+\varphi^{-1}2y\circ N+2\varphi^{-1}\circ[B,N]\\
&\quad+\varphi^{-1}\circ[N,N]-\varphi^{-1}|y|^2\gamma\circ V\\
&= \varphi^{-1}\left(2y+\boldsymbol{\Delta}B\right)\circ B+\varphi^{-1}\circ\langle N,N\rangle-\varphi^{-1}|y|^2\gamma\circ V\\
&\quad+\varphi^{-1}2\left(y+\boldsymbol{\Delta}B\right)\circ N+\varphi^{-1}\circ([N,N]-\langle N,N\rangle)
\end{aligned}$$

■

We shall often make use of the fact that $\varphi^{(1)}\varphi^{(2)}\Delta V\in\mathcal{L}$ if $\varphi^{(1)}$ and $\varphi^{(2)}$ belong to $\mathcal{L}$. This follows from the simple inequality

$$\left(\varphi^{(1)}\varphi^{(2)}\mathbf{\Delta}V\right)\circ V_t \leq \left(\varphi^{(1)}\circ V_t\right)\left(\varphi^{(2)}\circ V_t\right).$$

Now we are ready to prove the *uniqueness of solution* part of theorem (62);

Proof. of theorem (62), [$Uniqueness$]; Now we are going to prove the uniqueness of the solution. Let x and y be RLL processes satisfying equation (3.7). Let φ be the solution of the equation $\varphi = 1 + K(R)\varphi\circ V_t$. From lemma (64), (for sake of simple notation we omit the variable s in the functions a, α , b and β) then

$$\begin{aligned}
\varphi^{-1}\left|x-y\right|^2 &= \varphi^{-1}\left(2\left(x-y\right)\left(a(x)-a(y)\right)+\left(a(x)-a(y)\right)^2\mathbf{\Delta}A\right)\circ A \\
&\quad +\varphi^{-1}\left(b(x)-b(y)\right)^2\circ\langle M,M\rangle-\varphi^{-1}\gamma|x-y|^2\circ V \\
&\quad +\varphi^{-1}2\left(\left(x-y\right)+\left(a(x)-a(y)\right)\mathbf{\Delta}A\right)\left(b(x)-b(y)\right)\circ M \\
&\quad +\varphi^{-1}\left(b(x)-b(y)\right)^2\circ\left([M,M]-\langle M,M\rangle\right),
\end{aligned}$$

Since $\langle M\rangle = Q\circ V$ and $A = \Lambda\circ V$ then

$$\begin{aligned}
\varphi^{-1}\left|x-y\right|^2 &= \varphi^{-1}2\left(x-y\right)\left(a(x)-a(y)\right)\Lambda\circ V \\
&\quad +\varphi^{-1}\left(a(x)-a(y)\right)^2\Lambda^2\mathbf{\Delta}V\circ V \\
&\quad +\varphi^{-1}\left(b(x)-b(y)\right)^2 Q\circ V-\varphi^{-1}\gamma|x-y|^2\circ V \\
&\quad +\varphi^{-1}2\left(x-y\right)\left(b(x)-b(y)\right)\circ M \\
&\quad +\varphi^{-1}\left(b(x)-b(y)\right)^2\circ\left([M,M]-\langle M,M\rangle\right).
\end{aligned}$$

Let $m'' = \varphi^{-1}2\left(x-y\right)\left(b(x)-b(y)\right)\circ M+\varphi^{-1}\left(b(x)-b(y)\right)^2\circ\left([M,M]-\langle M,M\rangle\right)$, $\alpha^1 := a_-\Lambda_-$,

$\alpha^2 := aR$ and $\beta^1 := b_-\sqrt{Q_-}$ and $\beta^2 := b\sqrt{Q}$ then

$$\begin{aligned}
\varphi^{-1}|x-y|^2 &= \varphi^{-1}2(x-y)\left(\alpha^1(x)-\alpha^1(y)\right)\cdot V + \varphi^{-1}2(x-y)\left(\alpha^2(x)-\alpha^2(y)\right)\odot V \\
&\quad +\varphi^{-1}\left(\alpha^1(x)-\alpha^1(y)\right)^2 \triangle V\cdot V + \varphi^{-1}\left(\alpha^1(x)-\alpha^1(y)\right)^2 \triangle^+ V\odot V \\
&\quad +\varphi^{-1}\left(\beta^1(x)-\beta^1(y)\right)^2\cdot V + \varphi^{-1}\left(\beta^2(x)-\beta^2(y)\right)^2\odot V \\
&\quad -\varphi^{-1}\gamma|x-y|^2\cdot V - \varphi^{-1}\gamma|x-y|^2\odot V + m'' \\
&= \varphi^{-1}\Big[2(x-y)\left(\alpha^1(x)-\alpha^1(y)\right) + \left(\alpha^1(x)-\alpha^1(y)\right)^2\triangle V \\
&\quad +\left(\beta^1(x)-\beta^1(y)\right)^2 - \gamma_-|x_- - y_-|^2\Big]\cdot V \\
&\quad +\varphi^{-1}\Big[2(x-y)\left(\alpha^2(x)-\alpha^2(y)\right) + \left(\alpha^2(x)-\alpha^2(y)\right)^2\triangle^+ V \\
&\quad +\left(\beta^2(x)-\beta^2(y)\right)^2 - \gamma|x-y|^2\Big]\odot V + m''
\end{aligned}$$

Since the first two terms are negative in accordance to the monotonicity conditions and $\varphi^{-1}|x-y|^2 \geq 0$ we get,

$$0 \leq \varphi^{-1}C^1\cdot V + \varphi^{-1}C^1\odot V + m'' \leq m''.$$

So, for each $R \geq 0$ let us define the stopping time in the broad sense

$$\tau(R) = \inf\left(t : \max\left(|x_t|,|y_t|\right) \geq R\right).$$

We get

$$\begin{aligned}
\varphi^{-1}_{t\wedge\tau(R)}\left|x_{t\wedge\tau(R)} - y_{t\wedge\tau(R)}\right|^2 &= \int_{t\wedge 0+}^{\tau(R)\wedge t} \varphi^{-1}_{s-}\big[2(x_{s-}-y_{s-})(\alpha^1(x_{s-})-\alpha^1(y_{s-})) \\
&\quad +\Delta V_s\left|\alpha^1(x_{s-})-\alpha^1(y_{s-})\right|^2 + \left|\beta^1(x_{s-})-\beta^1(y_{s-})\right|^2 \\
&\quad -K_s(R)|x_{s-}-y_{s-}|^2\big]\,dV^r_s \\
&\quad +\int_0^{\tau(R)\wedge t-} \varphi^{-1}_s\big[2(x_s-y_s)(\alpha^2(x_s)-\alpha^2(y_s)) \\
&\quad +\Delta^+V_s\left|\alpha^2(x_s)-\alpha^2(y_s)\right|^2 + \left|\beta^2(x_s)-\beta^2(y_s)\right|^2 \\
&\quad -K_s(R)|x_s-y_s|^2\big]\,dV^g_s \\
&\quad +m''_{t\wedge\tau(R)},
\end{aligned}$$

where m'' is a local martingale ($m''_0 = 0$). Hence, by conditions (3.10) and (3.11) we have

$$0 \leq \varphi^{-1}_{t\wedge\tau(R)} \left|x_{t\wedge\tau(R)} - y_{t\wedge\tau(R)}\right|^2 \leq m''_{t\wedge\tau(R)}.$$

Therefore $m''_{\tau(R)}$ is a non-negative local martingale and by Fatou's lemma it implies that it is a non-negative supermartingale. As $m'_0 = 0$, it follows that $m''_t = 0$ and consequently $\varphi^{-1}_s \left|x_s - y_s\right|^2 = 0$ for $s \in [0, \tau(R)]$ for almost all $\omega \in \Omega$. Since, $\varphi^{-1}_s > 0$ as $\tau(R) \uparrow \infty$ a.s., it follows that $x_s = y_s$ for all $s \in \mathbb{R}_+$ almost surely. ■

The existence of the solution will be proved next, after additional lemmata.

Lemma 65 *For $T > 0$ and $R > 0$ we have*

$$\begin{aligned}\int_{0+}^{T} |x_-| \sup_{x\leq R} |a(t, x_-)| dA_t + \int_{0}^{T-} |x| \sup_{x\leq R} |a(t, x)| dA_{t+} &< \infty, \\ \int_{0+}^{T} \sup_{|x_-|\leq R} |\beta(t, x_-)|^2 dV_t + \int_{0}^{T-} \sup_{|x|\leq R} |\beta(t, x)|^2 dV_{t+} &< \infty.\end{aligned}$$

for almost all ω.

Proof. The right-continuous case was proved by [41]. The left-continuous case is a simple change to the proof of the right-continuous case. ■

Lemma 66 *If f is a real locally bounded function on $\mathbb{R}^d$, n is an integer and $N = \sup\{|f(x)| : |x| \leq n\}$, then there exists a real function $\tilde{f}$, such that $\tilde{f}(x) = f(x)$ for $|x| \leq n$, $f(x) = 0$ for $|x| \geq n + 1$ and*

$$\begin{gathered}|\tilde{f}(x)| \leq |f(x)|, \\ |f(x) - f(y)|^2 \leq |f(x) - f(y)|^2 + N^2|x - y|^2\end{gathered}$$

for every x, $y \in \mathbb{R}^d$. Moreover, if f is continuous in x and it is a measurable function of some parameters, then these are valid for f too.

Proof. The right-continuous case was proved by [41]. The argument for proving the left-continuous case similar to the right continuous case with the proper replacement of right with left continuous. ■

Lemma 67 *For every fixed integer n there exist functions $\tilde{a}$, $\tilde{b}$ defined on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d$, such that, $\tilde{a} = a$, $\tilde{b} = b$ for $|x| \leq n$ and $\tilde{a} = 0 \in \mathbb{R}^d$, $\tilde{b} = 0 \in \mathsf{L}_2(\mathsf{H}, \mathbb{R}^d)$ for $|x| \geq n+3$. Moreover $\tilde{\alpha}^1 := \tilde{a}_- \left(\frac{dA}{dV}\right)_-$ and $\tilde{\beta}^1 := \tilde{b}_- Q_-^{1/2}$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$ measurable and $\tilde{\alpha}^2 := \tilde{a}\left(\frac{dA}{dV}\right)$ and $\tilde{\beta}^2 := \tilde{b} Q^{1/2}$ are $\mathcal{O} \times \mathcal{B}(\mathbb{R}^d)$ measurable and all are continuous in x. Furthermore, there exists a process $L \in \mathcal{L}$, such that for all x, $y \in \mathbb{R}^d$ the following inequalities hold,*

$$L_t \geq \left|\tilde{\alpha}^1(t, x_-)\right| + \left|\tilde{\beta}^1(t, x_-)\right|^2 \tag{3.19}$$

$$L_t \geq \left|\tilde{\alpha}^2(t, x)\right| + \left|\tilde{\beta}^2(t, x)\right|^2 \tag{3.20}$$

$$\begin{aligned} L_t|x_- - y_-|^2 &\geq 2(x_- - y_-)(\tilde{\alpha}^1(t, x_-) - \tilde{\alpha}^1(t, y_-)) \\ &\quad + \Delta V_t |\tilde{\alpha}^1(t, x_-) - \tilde{\alpha}^1(t, y_-)|^2 + |\tilde{\beta}^1(t, x_-) - \tilde{\beta}^1(t, y_-)|^2 \end{aligned} \tag{3.21}$$

$$\begin{aligned} L_t|x - y|^2 &\geq 2(x - y)(\tilde{\alpha}^2(t, x) - \tilde{\alpha}^2(t, y)) \\ &\quad + \Delta^+ V_t |\tilde{\alpha}^2(t, x) - \tilde{\alpha}^2(t, y)|^2 + |\tilde{\beta}^2(t, x) - \tilde{\beta}^2(t, y)|^2 \end{aligned} \tag{3.22}$$

for almost all (t, ω) with respect to $\mathbf{P} \times dV_t$.

Proof. Since $Q_t(\omega)$ is a self adjoint nuclear operator for every (t, ω), for every (t, ω) there exists a countable set of vectors $\{e_i(t, \omega)\}_{i=1}^{\infty}$ that form an orthonormal basis in $\overline{Q^{1/2}(\mathsf{H})}$ and $e_i(t, \omega)$'s are eigenvectors of $Q_t^{1/2}(\omega)$. In addition, by virtue of the $\mathcal{P}$-measurability of Q, we may assume that $e_i \in \mathcal{P}(\mathsf{H})$ for every i.

Now, let the functions $b_j := b e_j$. Truncate each coordinate function j to a maximum n and apply lemma (66). Let us denote these new functions by $\tilde{b}_j$. For every fixed (t, ω, x) let $\tilde{b}(t, x)$ be a linear operator in $Q^{1/2}(\mathsf{H})$ defined by $\tilde{b}(t, x) e_j = \tilde{b}_j(t, x)$. Since $|Q|_1 \leq 1$, the norm of Q in $\mathsf{H} \otimes_1 \mathsf{H}$, it follows by (66) that there is $\bar{A}$ the closure of $A \subset \mathsf{H}$. Then there exists a process

$L^{(1)} \in \mathcal{L}$, such that

$$\begin{aligned}
\left|\left(\tilde{b}(t,x_-) - \tilde{b}(t,y_-)\right) Q_{t-}^{1/2}\right|^2 &\leq L_t^{(1)} |x_- - y_-|^2 + \left|(b(t,x_-) - b(t,y_-)) Q_{t-}^{1/2}\right|^2, \quad (3.23)\\
\left|\left(\tilde{b}(t,x) - \tilde{b}(t,y)\right) Q_t^{1/2}\right|^2 &\leq L_t^{(1)} |x - y|^2 + \left|(b(t,x) - b(t,y)) Q_t^{1/2}\right|^2. \quad (3.24)
\end{aligned}$$

Let $\eta \in \mathrm{C}_0^\infty(\mathbb{R}^d)$, such that $0 \leq \eta \leq 1$ and $\eta = 1$ for $|x| \leq n+2$, $\eta = 0$ for $|x| \geq n+3$. Let $\tilde{a}(t,x) = a(t,x)\eta(x)$, $\tilde{\alpha}^1 = \tilde{a}_- \,(dA/dV)_-$ and $\tilde{\alpha}^2 = \tilde{a}\,(dA/dV)$. By (65) there exists a process $L^{(2)}$ such that

$$\begin{aligned}
|\alpha^1(t,x)| + |\beta^1(t,x)|^2 &\leq L_t^{(2)}, \quad (3.25)\\
|\alpha^2(t,x)| + |\beta^2(t,x)|^2 &\leq L_t^{(2)}. \quad (3.26)
\end{aligned}$$

Now we prove that there exists a process $L^{(3)}$, such that,

$$\begin{aligned}
&2(x_- - y_-)(\alpha(t,x_-) - \alpha(t,y_-)) + \triangle V_t\,|\alpha(t,x_-) - \alpha(t,y_-)|^2 \leq L_t^{(3)}|x_- - y_-|^2 \quad (3.27)\\
&\quad + \eta(x_-)\eta(y_-)\left\{2(x_- - y_-)(\alpha(t,x_-) - \alpha(t,y_-)) + \triangle V_t|\alpha(t,x_-) - \alpha(t,y_-)|^2\right\}.\\
&2(x-y)(\alpha(t,x) - \alpha(t,y)) + \triangle^+ V_t\,|\alpha(t,x) - \alpha(t,y)|^2 \leq L_t^{(3)}|x-y|^2 \quad (3.28)\\
&\quad + \eta(x)\eta(y)\left\{2(x-y)(\alpha(t,x) - \alpha(t,y)) + \triangle^+ V_t|\alpha(t,x) - \alpha(t,y)|^2\right\}.
\end{aligned}$$

Since (3.27) and (3.28) is symmetric with respect to x, y and is obvious if $\eta(x) = \eta(y)$ $\forall$ $(\eta^2 \leq \eta)$, it suffices to consider the case $0 \leq \eta(x) < \eta(y)$. In the latter case $|y| \leq n+3$ and the left-hand side of (3.27) and (3.28) is not greater than

$$\begin{aligned}
&\eta(x_-)2(x_- - y_-)(\alpha^1(t,x_-) - \alpha^1(t,y_-)) + \triangle V_t|\alpha^1(t,x_-) - \alpha^1(t,y_-)|^2\\
&+2(x_- - y_-)\alpha^1(t,y_-)(\eta(x_-) - \eta(y_-)) + |\alpha(t,y_-)|^2(\eta(x_-) - \eta(y_-))^2 \triangle V_t\\
&+2 \triangle V_t \eta(x_-)|\alpha(t,x_-) - \alpha(r,y_-)||\alpha(t,y_-)||\eta(x_-) - \eta(y_-)|,
\end{aligned}$$

and similarly

$$\eta(x)2(x-y)(\alpha(t,x)-\alpha(t,y))+\triangle^{+}V_t|\alpha(t,x)-\alpha(t,y)|^2$$
$$+2(x-y)\alpha(t,y)(\eta(x)-\eta(y))+|\alpha(t,y)|^2(\eta(x)-\eta(y))^2\triangle^{+}V_t$$
$$+2\triangle^{+}V_t\eta(x)|\alpha(t,x)-\alpha(r,y)\|\alpha(t,y)\|\eta(x)-\eta(y)|.$$

By (3.14) and (3.15) the last term can be estimated by

$$\begin{aligned} L_t^{(4)}|x_- - y_-|^2 &= 2C\sup_{|y_-|\leqq n+3}|\alpha^1(t,y_-)|(4+K_t(n+3)\triangle V_t)|x_- - y_-|^2, \\ L_t^{(4)}|x-y|^2 &= 2C\sup_{|y|\leqq n+3}|\alpha^2(t,y)|(4+K_t(n+3)\triangle^{+}V_t)|x-y|^2, \end{aligned}$$

where C is Lipschitz constant of η. From lemma (65) and conditions (3.14) and (3.15) it follows that $L^{(4)}\in\mathcal{L}$. Moreover using the trivial estimations

$$\begin{aligned} |\alpha(t,y_-)|^2(\eta(x_-)-\eta(y_-))^2\triangle V_t &\leq C^2\sup_{|y_-|\leq n+3}|\alpha(t,y_-)|^2\triangle V_t|x_- - y_-|^2, \\ \alpha(t,y)|^2(\eta(x)-\eta(y))^2\triangle^{+}V_t &\leq C^2\sup_{|y|\leq n+3}|\alpha(t,y)|^2\triangle^{+}V_t|x-y|^2 \end{aligned}$$

and

$$\begin{aligned} 2(x_- - y_-)\alpha(t,y_-)(\eta(x_-)-\eta(y_-)) &\leq 2C\sup_{|y_-|\leqq n+3}|\alpha(t,y_-)\|x_- - y_-|^2, \\ 2(x-y)\alpha(t,y)(\eta(x)-\eta(y)) &\leq 2C\sup_{|y|\leqq n+3}|\alpha(t,y)\|x-y|^2, \end{aligned}$$

by virtue of lemma (65) and conditions (3.14) and (3.15) we get (3.27) and (3.28). If $|x_-|\leq n+2$ and $|y_-|\leq n+2$, then $\eta(x_-)=\eta(y_-)=1$ and, by virtue of (3.23, 3.24), (3.27, 3.28) and (3.10, 3.11) the inequality (3.21, 3.22) is valid with $L=L^{(1)}+K(n+3)$. If $|x_-|\geq n+1$, $|y_-|\geq n+1$, then $\beta(t,x_-)=\beta(t,y_-)=0$ and (3.21) and (3.22) holds with $L=L^{(3)}+K(n+3)$. If one of the values $|x_-|$, $|y_-|$ is smaller than $n+1$ and the other is greater than $n+2$, then $|x-y|\geq 1$ and one of the values of $\beta(t,x_-)$, $\beta(t,y_-)$ is zero. Therefore, in this case (3.25, 3.26) implies that

$$\left|\beta^1(t,x_-)-\beta^1(t,y_-)\right|^2 \le L_t^{(2)}\left|x_- - y_-\right|. \tag{3.29}$$

and If $|x| \le n+2$ and $|y| \le n+2$, then $\eta(x)=\eta(y)=1$ and, by virtue of (3.23, 3.24), (3.27, 3.28) and (3.10, 3.11) the inequality (3.21, 3.22) is valid with $L = L^{(1)} + K(n+3)$. If $|x| \ge n+1$, $|y| \ge n+1$, then $\beta(t,x)=\beta(t,y)=0$ and (3.8) holds with $L = L^{(3)} + K(n+3)$. If one of the values $|x|$, $|y|$ is smaller than $n+1$ and the other is greater than $n+2$, then $|x-y| \ge 1$ and one of the values of $\beta(t,x)$, $\beta(t,y)$ is zero. Therefore, in this case (3.25, 3.26) implies that

$$\left|\beta^2(t,x)-\beta^2(t,y)\right|^2 \le L_t^{(2)}|x-y|.$$

Consequently, from (3.23-3.29) we get that the inequalities (3.19, 3.24) and (3.21, 3.22) *hold* with $L = L^{(1)} + L^{(2)} + L^{(3)} + K(n+3)$. ■

Lemma 68 *Let $L \in \mathcal{L}$ such than for every x, $y \in \mathbb{R}^d$*

$$|\alpha^1(t,x_-)| + |\beta^1(t,x_-)|^2 \le L_t \tag{3.30}$$

$$|\alpha^2(t,x)| + |\beta^2(t,x)|^2 \le L_t \tag{3.31}$$

$$2|x_- - y_-||\alpha^1(t,x_-)-\alpha^1(t,y_-)| + |\beta^2(t,x_-)-\beta^2(t,y_-)|^2 \le L_t|x_- - y_-|^2 \tag{3.32}$$

$$2|x-y||\alpha^2(t,x)-\alpha^2(t,y)| + |\beta^2(t,x)-\beta^2(t,y)|^2 \le L_t|x-y|^2 \tag{3.33}$$

hold for $\mathbf{P} \times dV_t$ – almost all (t,ω). Then the Eq. (3.7) has one and only one solution.

Proof. Let $\gamma_t^1 = L_t(6 + 4 \triangle V_t)$ and $\gamma_t^2 = L_t(6 + 4 \triangle^+ V_t)$ let φ be the solution of the equation (3.16). Let us define an iteration procedure as follows: $x_t^0 = \xi$ and for $n \ge 0$

$$x_t^{n+1} = \xi + \int_{0+}^{t} a(s,x_{s-}^n)dA_s + \int_0^{t-} a(s,x_s^n)dA_{s+} + \int_{0+}^{t} b(s,x_{s-}^n)dM_s + \int_0^{t-} b(s,x_s^n)dM_{s+} \tag{3.34}$$

If we set $\psi_t = \exp(-|\xi|)\varphi_t^{-1}$ and apply lemma (64) to $\left|x_t^{n+1} - x_t^n\right|^2 \psi_t$, then we obtain

$$\begin{aligned}
\left|x_t^{n+1} - x_t^n\right|^2 \psi_t \quad = \quad & \int_{0+}^{t} \psi_s \left\{2(x_{s-}^{n+1} - x_{s-}^n)(\alpha(x_{s-}^n) - \alpha(x_{s-}^{n-1}))\right. \\
& + \triangle V_s|\alpha(x_{s-}^n) - \alpha(x_{s-}^{n-1})|^2 + |\beta(x_{s-}^n) - \beta(x_{s-}^{n-1})|^2 \\
& \left. -\gamma_s^1|x_{s-}^{n+1} - x_{s-}^n|^2\right\} dV_s \\
& + \int_{0}^{t-} \psi_s \left\{2(x_s^{n+1} - x_s^n)(\alpha(x_s^n) - \alpha(x_s^{n-1}))\right. \\
& + \triangle^+ V_s|\alpha(x_s^n) - \alpha(x_s^{n-1})|^2 + |\beta(x_s^n) - \beta(x_s^{n-1})|^2 \\
& \left. -\gamma_s^2|x_s^{n+1} - x_s^n|^2\right\} dV_{s+} \\
& +m_t^n,
\end{aligned}$$

where m_t^n is a local martingale ($m_0^n = 0$). Hence, using the assumption (3.32, 3.33) and the simple inequality $2|pr| \leq |p|^2 + |r|^2$ we get

$$\begin{aligned}
\left|x_\iota^{n+1} - x_t^n\right|^2 \psi_t \quad \leq \quad & \int_{0+}^{t} \psi_s \left\{\frac{1}{4}\gamma_s|x_{s-}^n - x_{s-}^{n-1}|^2 - \frac{1}{2}\gamma_s^1|x_{s-}^{n+1} - x_{s-}^n|^2\right\} dV_s \qquad (3.35) \\
& + \int_{0}^{t-} \psi_s \left\{\frac{1}{4}\gamma_s|x_s^n - x_s^{n-1}|^2 - \frac{1}{2}\gamma_s^2|x_s^{n+1} - x_s^n|^2\right\} dV_{s+} \\
& +m_t^n. \qquad (3.36)
\end{aligned}$$

Let τ be an arbitrary stopping time in the broad sense and $\{\tau^j\}$ an increasing sequence of stopping times in the broad sense, such that $\lim_{i\to\infty} \tau^i = \infty$ (a.s.) and each τ^i reduces the local martingale m_t^n (i.e. $(m_{t\wedge\tau}^n)_{t\geq 0}$ is a uniformly integrable martingale). After replacing t with $t \wedge \tau \wedge \tau^i$ in (3.35) and then taking the expectations and letting $t \to \infty$, we obtain

$$\begin{aligned}
& 4\mathbf{E}|x_\tau^{n+1} - x_\tau^n|^2\psi_\tau + 2\mathbf{E}\int_{0+}^{\tau} \gamma_s|x_{s-}^{n+1} - x_{s-}^n|^2\psi_s dV_s \qquad (3.37) \\
& +2\mathbf{E}\int_{0}^{\tau-} \gamma_s|x_s^{n+1} - x_s^n|^2\psi_s dV_{s+} \qquad (3.38) \\
\leq\ & \mathbf{E}\int_{0+}^{\tau} \psi_s\gamma_s|x_{s-}^n - x_{s-}^{n-1}|^2 dV_s + \mathbf{E}\int_{0}^{\tau-} \psi_s\gamma_s|x_s^n - x_s^{n-1}|^2 dV_{s+}
\end{aligned}$$

(Per definition we set $\left|x_\tau^{n+1}-x_\tau^n\right|^2\psi_\tau=0$ on $\{\tau=\infty\}$. Similarly we can show that

$$\begin{aligned}
4\mathbf{E}|x_\tau^1|^2\psi_\tau+2\mathbf{E}\int_{0+}^{\tau}|x_{s-}^1|^2\gamma_s\psi_s dV_s+2\mathbf{E}\int_{0+}^{\tau}|x_s^1|^2\gamma_s\psi_s dV_{s+}\\
\leq \mathbf{E}\int_{0+}^{\tau}|\xi|^2\psi_s\gamma_s dV_s+\mathbf{E}\int_{0}^{\tau-}|\xi|^2\psi_s\gamma_s dV_s\\
=\mathbf{E}\left(|\xi|^2\left[\int_{0+}^{\tau}\psi_s\gamma_s dV_s+\int_{0}^{\tau-}\psi_s\gamma_s dV_s\right]\right)\\
=\mathbf{E}\left\{|\xi|^2\exp(-|\xi|)(1-\varphi_\tau^{-1})\right\}<\infty.
\end{aligned}$$

Consequently, carrying out an iteration on the inequality (3.37) we obtain

$$\mathbf{E}\left(\int_{0+}^{\tau}\gamma_s\left|x_{s-}^{n+1}-x_{s-}^n\right|^2\psi_s dV_s+\int_{0}^{\tau-}\gamma_s\left|x_s^{n+1}-x_s^n\right|^2\psi_s dV_{s+}\right)\leq C2^{-n}$$

and

$$\mathbf{E}\left|x^{n+1}-x^n\right|^2\psi_\tau\leq C2^{-n}$$

Hence, for $\tau=\tau(n)=\inf\left\{t:|x_t^{n+1}-x_t^n|^2\psi_t\geq n^{-4}\right\}$ we have $n^{-4}\mathbf{P}[\tau(n)<\infty]\leq C2^{-n}$. Consequently, $\mathbf{P}[\sup_{t\geq 0}\left|x_t^{n+1}-x_t^n\right|\psi_t^{1/2}\geq n^{-2}]\leq Cn^4 2^{-n}$.

By Borel-Cantelli lemma, convergence of $\sum_{n=0}^{\infty}n^4 2^{-n}$ implies that $\sum_{n=0}^{\infty}\left|x_t^{n+1}-x_t^n\right|\psi_t^{1/2}$ converges uniformly with probability 1. Consequently, with probability 1 the sequence of the adapted RLL processes $x_t^n=\xi+\sum_{i=0}^{n-1}(x_t^{i+1}-x_t^i)$ converges uniformly in on every bounded interval to an adapted RLL process, say x_t. It remains to show that x_t satisfies equation (3.7). For fixed $R>0$ let us define the stopping time $\tau(R)=\inf(t:|x_t|\geq R)$. Since with probability 1, $x_t^n\to x_t$ uniformly in t on every bounded interval, in accordance with the Lebesgue theorem for $n\to\infty$.

$$\begin{aligned}
\int_{0+}^{t\wedge\tau(R)}a(s,x_{s-}^n)dA_s &\to \int_{0+}^{t\wedge\tau(R)}a(s,x_{s-})dA_s \quad (a.s.),\\
\int_{0}^{t\wedge\tau(R)-}a(s,x_s^n)dA_{s+} &\to \int_{0}^{t\wedge\tau(R)-}a(s,x_s)dA_{s+} \quad (a.s.).
\end{aligned}$$

and

$$\left\langle \int_0 (b(s,x^n_{s-}) - b(s,x_{s-}))dM_s \right\rangle_{t\wedge\tau(R)} = \int_{t\wedge]0,\tau(R)]} |\beta(s,x^n_{s-}) - \beta(s,x_{s-})|^2 dV_s$$
$$+ \int_{t\wedge[0,\tau(R)[} |\beta(s,x^n_s) - \beta(s,x_s)|^2 dV_{s+}$$
$$\to 0$$

Similar definition will be made later too a.s., because of the continuity of a and b in x and lemma (65). Hence, letting $n\to\infty$ in (3.34), it follows that x_t satisfies the equation (3.7). ■

Lemma 69 *Let $\tilde{a}$ and $\tilde{b}$ the random functions defined in lemma (67). If we replace a with $\tilde{a}$ and b with $\tilde{b}$ in the Eq. (3.7), then (3.7) admits one and only one solution.*

Proof. First we approximate the functions a and b by smooth functions with respect to x. Let

$$\mathrm{J}(z) = \begin{cases} C\exp\left(\frac{-1}{1-|z|^2}\right) & if \quad |z|<1 \\ 0 & if \quad |z|\geq 1 \end{cases},$$

in $\mathbb{R}^d$ where $C\in\mathbb{R}$ such that $\int_{\mathbb{R}^d}\mathrm{J}(z)dz = 1$. For every integer k and for every fixed $(t,\omega)\in\mathbb{R}_+\times\Omega$ we define

$$\tilde{a}^k(t,x) = \int_{\mathbb{R}^d} \tilde{a}\left(t, x-k^{-1}z\right)\mathrm{J}(z)dz \tag{3.39}$$

and

$$\tilde{b}^k(t,x) = \int_{\mathbb{R}^d} \tilde{b}(t,x-k^{-1}z)\mathrm{J}(z)dz. \tag{3.40}$$

Further let $\tilde{\alpha}^{1,k} := \tilde{a}^k_-\,(dA/dV)_-$, $\tilde{\alpha}^{2,k} := \tilde{a}^k\,(dA/dV)$ and $\tilde{\beta}^{1,k} := \tilde{b}^k_- Q^{1/2}_-$ and $\tilde{\beta}^{2,k} := \tilde{b}^k Q^{1/2}$. Since for every k the functions $|\tilde{a}^k|$, $|\tilde{b}^k|$, $|(\partial/\partial x)\tilde{a}^k|$, $|(\partial/\partial x)b^\kappa|$ can be estimated for fixed (t,ω) by the maximums of $|\tilde{a}|$ and $|\tilde{b}|$ in x, and since $\left|\tilde{\alpha}^i(t,x)\right| + \left|\tilde{\beta}^i(t,x)\right|^2 \leq L_t$, $i=1,2$, it follows that for every k the functions $\tilde{a}^k$ and $\tilde{b}^k$ satisfy the inequalities (3.30, 3.31) and (3.32, 3.33) in lemma (68).

Consequently for every k the equation

$$x_t^k = \xi + \int_{0+}^{t} \tilde{a}^k(s, x_{s-}^k) dA_s + \int_0^{t-} \tilde{a}^k(s, x_s^k) dA_{s+} + \int_{0+}^{t} b^k(s, x_{s-}^k) dM_s + \int_0^{t-} b^k(s, x_s^k) dM_{s+} \tag{3.41}$$

admits one and only one solution $(x_t^k)_{t\geq 0}$. We shall show that x_t^k converges as $k \to \infty$, and this convergence is uniform in t on every bounded interval. Let φ be the solution of the equation $d\varphi_t = L_t \varphi_{t-} dV_t^r + L_t \varphi_t dV_{t+}^g$, $\varphi_0 = 1$. Application of formula (3.18) yields

$$\begin{aligned}
\left| x_t^k - x_t^l \right|^2 \varphi_r^{-1} &= \int_{0+}^{t} \varphi_{s-}^{-1} \Big\{ 2(x_{s-}^k - x_{s-}^l)(\tilde{\alpha}^{1,k}(x_{s-}^k) - \tilde{\alpha}^{1,l}(x_{s-}^l)) \\
&\quad + \Delta V_s |\tilde{\alpha}^{1,k}(x_{s-}^k) - \tilde{\alpha}^{1,l}(x_{s-}^l)|^2 + |\tilde{\beta}^{1,k}(x_{s-}^k) - \tilde{\beta}^{1,l}(x_{s-}^l)|^2 \\
&\quad - L_s \left| x_{s-}^k - x_{s-}^l \right|^2 \Big\} dV_s^r \\
&\quad + \int_0^{t-} \varphi_s^{-1} \Big\{ 2(x_s^k - x_s^l)(\tilde{\alpha}^{2,k}(x_s^k) - \tilde{\alpha}^{2,l}(x_s^l)) \\
&\quad + \Delta V_s |\tilde{\alpha}^{2,k}(x_s^k) - \tilde{\alpha}^{2,l}(x_s^l)|^2 + |\tilde{\beta}^{2,k}(x_s^k) - \tilde{\beta}^{2,l}(x_s^l)|^2 \\
&\quad - L_s \left| x_s^k - x_s^l \right|^2 \Big\} dV_s^g + m_t^k.
\end{aligned}$$

Using (3.39) and the Schwarz inequality, after simple calculations we get the integrand is not greater than $\tilde{b}^k(t,x)e_j = \int_{\mathbb{R}^d} \tilde{b}_j(t, x - k^{-1}z) \mathrm{J}(z) dz$, where e_j and $\tilde{b}_j$ have been defined in the proof of Lemma (67).

$$\begin{aligned}
I_s^1 &= \varphi_s^{-1} \int_{\mathbb{R}^d} \mathrm{J}(z) \Big\{ 2 \left((x_{s-}^k - k^{-1}z) - \left(x_{s-}^l - l^{-1}z \right) \right) \left(\tilde{\alpha}^1(x_{s-}^k - k^{-1}z) - \tilde{\alpha}^1(x_{s-}^l - l^{-1}z) \right) \\
&\quad + \Delta V_s \left| \tilde{\alpha}^1(x_{s-}^k - k^{-1}z) - \tilde{\alpha}^1(x_{s-}^l - l^{-1}z) \right|^2 + \left| \tilde{\beta}^1(x_{s-}^k - k^{-1}z) - \tilde{\beta}^1(x_{s-}^l - l^{-1}z) \right|^2 \\
&\quad - L_s \left| (x_{s-}^k - k^{-1}z) - (x_{s-}^k - l^{-1}z) \right|^2 \Big\} dz \\
&\quad + 2(k^{-1} - l^{-1}) \varphi_s^{-1} \int_{\mathbb{R}^d} \mathrm{J}(z) z \left| \tilde{\alpha}^1(x_{s-}^k - k^{-1}z) - \tilde{\alpha}^1(x_{s-}^l - l^{-1}z) \right| dz \\
&\quad + L_s (k^{-1} - l^{-1})^2 \varphi_s^{-1} \int_{\mathbb{R}^d} z^2 \mathrm{J}(z) dz.
\end{aligned}$$

and

$$
\begin{aligned}
I_s^2 &= \varphi_s^{-1}\int_{\mathbb{R}^d}\mathrm{J}(z)\left\{2\left((x_s^k-k^{-1}z)-\left(x_s^l-l^{-1}z\right)\right)\left(\tilde{\alpha}^2(x_s^k-k^{-1}z)-\tilde{\alpha}^2(x_s^l-l^{-1}z)\right)\right.\\
&+\triangle^+ V_s\left|\tilde{\alpha}^2(x_s^k-k^{-1}z)-\tilde{\alpha}^2(x_s^l-l^{-1}z)\right|^2+\left|\tilde{\beta}^2(x_s^k-k^{-1}z)-\tilde{\beta}^2(x_s^l-l^{-1}z)\right|^2\\
&\left.-L_s\left|(x_s^k-k^{-1}z)-(x_s^k-l^{-1}z)\right|^2\right\}dz\\
&+2(k^{-1}-l^{-1})\varphi_s^{-1}\int_{\mathbb{R}^d}\mathrm{J}(z)z\left|\tilde{\alpha}^2(x_s^k-k^{-1}z)-\tilde{\alpha}^2(x_s^l-l^{-1}z)\right|dz\\
&+L_s(k^{-1}-l^{-1})^2\varphi_s^{-1}\int_{\mathbb{R}^d}z^2\mathrm{J}(z)dz.
\end{aligned}
$$

Hence, by virtue of the inequality (3.21, 3.22) we get

$$
\begin{aligned}
I_s^1 &\leq C_1(k^{-1}-l^{-1})L_s\varphi_s^{-1}+C_2(k^{-1}-l^{-1})^2\varphi_s^{-1}L_s,\\
I_s^2 &\leq C_1(k^{-1}-l^{-1})L_s\varphi_s^{-1}+C_2(k^{-1}-l^{-1})^2\varphi_s^{-1}L_s,
\end{aligned}
$$

where $C_1=4\int|z|\mathrm{J}(z)dz$, $C_2=\int z^2\mathrm{J}(z)dz$. This implies that

$$
\begin{aligned}
\left|x_t^k-x_t^l\right|^2\varphi_t^{-1} &\leq C_1\left(k^{-1}-l^{-1}\right)\int_{0+}^t L_s\varphi_s^{-1}dV_s^r+C_2(k^{-1}-l^{-1})\int_{0+}^t\varphi_s^{-1}L_sdV_s^r\\
&+C_1\left(k^{-1}-l^{-1}\right)\int_0^{t-}L_s\varphi_s^{-1}dV_{s+}^g+C_2(k^{-1}-l^{-1})\int_0^{t-}\varphi_s^{-1}L_sdV_{s+}^g\\
&+m_t^k.
\end{aligned}
$$

From lemma (64) we have

$$
\begin{aligned}
\int_{0+}^t L_s\varphi_s^{-1}dV_s^r &= l-\varphi_t^{-1}<1,\\
\int_0^{t-}L_s\varphi_s^{-1}dV_{s+}^g &= l-\varphi_t^{-1}<1,
\end{aligned}
$$

consequently,

$$
\left|x_t^k-x_t^l\right|^2\varphi_t^{-1}\leq 2C_1(k^{-1}-l^{-1})+2C_2\left(k^{-1}-l^{-1}\right)^2+m_t^k.
$$

Hence, similarly as before, we obtain that for any stopping time τ

$$\mathbf{E}\left|x_\tau^k - x_\tau^l\right|^2 \varphi_\tau^{-1} \leq 2C_1(k^{-1} - l^{-1}) + 2C_2\left(k^{-1} - l^{-1}\right)^2 \tag{3.42}$$

We define next the stopping times $\tau = \tau(k,l) = \inf\left(t : |x_t^k - x_t^l|^2\varphi_t^{-1} \geq \epsilon\right)$. Since

$$\mathbf{P}\left(\sup_{t\geq 0}|x_t^k - x|^2\varphi_t^{-1} \geq \epsilon\right) \leq \mathbf{P}\left(\left|x_\tau^k - x\right|^2 \varphi_\tau^{-1}| \geq \epsilon\right) \leq \frac{1}{\epsilon}\mathbf{E}\left|x_\tau^k - x_\tau^l\right|^2 \varphi_\tau^{-1},$$

from (3.42) we get $\mathbf{P}\left(\sup_{t\geq 0}\left|x_t^k - x_t^l\right|^2 \varphi_t^{-1} \geq \epsilon\right) \to 0$ as $k, l \to \infty$. Consequently, x_t^k converges in probability, moreover this convergence is uniform in t on every bounded interval. If we select a subsequence $x_t^{k_i}$ converging with probability 1 uniformly in t on every bounded interval, then as before, by taking limit in (3.41) we get the assertion of lemma (69). ■

Now we are going to complete the proof of theorem (62).

Proof. of theorem (62), [*Existeness*]; For every integer n let a^n and b^n denote the functions $\tilde{a}$ and $\tilde{b}$, respectively, defined in lemma (67). For every n by virtue of lemma (69) there exists an adapted ladlag process x_t^n, such that

$$\begin{aligned} x_t^n &= \xi + \int_{0+}^t a^n(s, x_{s-}^n)dA_s^r + \int_0^{t-} a^n(s, x_s^n)dA_{s+}^g \\ &\quad + \int_{0+}^t b^n(s, x_{s-}^n)dM_s^r + \int_0^{t-} b^n(s, x_s^n)dM_{s+}^g. \end{aligned}$$

Let us define the stopping times $\tau^n = \inf(t : |x_t^n| \geq n)$ and $\tau^{nm} = \tau^n \wedge \tau^m$. Since $a^n(t,x) = a(t,x)$, $b^n(t,x) = b(t,x)$ for $|x| \leq n$, it follows that the process $x_{t\wedge\tau}^n$, $x_{t\wedge\tau}^m$ satisfy the same equation

$$\begin{aligned} dz_t &= \mathbf{1}_{(t<\tau^{nm})}a(t, z_{t-})dA_t^r + \mathbf{1}_{(t<\tau^{nm})}a(t, z_t)dA_{t+}^g \\ &\quad + \mathbf{1}_{(t<\tau^{nm})}b(t, z_{t-})dM_t^r + \mathbf{1}_{(t<\tau^{nm})}b(t, z_t)dM_{t+}^g \\ z_0 &= \xi. \end{aligned} \tag{3.43}$$

The uniqueness of the solution of (3.43) implies that $x_t^n = x_t^m$ on $[0, \tau^{nm}]$ almost surely. Consequently, if $n \leq m$, then $\tau^n \leq \tau^m$ (a.s.). Hence, it follows that there exists a stopping time τ such that $\tau = \lim_{n\to\infty} \tau^n$ (a.s.), and we can define an adapted cadlag process x_t so that $x_t = \lim_{n\to\infty} x_t^n$ almost surely on $[0, \tau]$. From (3.43) it follows that for every n and for every $t \in \mathbb{R}_+$,

$$\begin{aligned} x_{t\wedge\tau^n} &= x^n_{t\wedge\tau^n} = \xi + \int_{0+}^{t\wedge\tau^n} a(s, x_{s-}) dA_s^r + \int_0^{t\wedge\tau^n -} a(s, x_s) dA_{s+}^g \\ &+ \int_{0+}^{t\wedge\tau^n} b(s, x_{s-}) dM_s^r + + \int_0^{t\wedge\tau^n -} b(s, x_s) dM_{s+}^g \end{aligned} \tag{3.44}$$

holds with probability 1. It remains to show that $\tau = \infty$ (a.s.). Let φ_t be the solution of the equation $d\varphi_t = K_t(1)\varphi_{t-} dV_t^r + K_t(1)\varphi_t dV_t^g$, $\varphi_0 = 1$, and let $\psi_t = \varphi_t^{-1} \exp(-|\xi|)$. Using lemma (64), (3.10, 3.11) and (3.42) we get

$$\mathbf{E}\left(|x^n_{\tau^n}|^2 \psi_{\tau^n} \mathbf{1}_{(\tau^n<\infty)} \right) \leq \mathbf{E}\left(|\xi|^2 \exp\left(-|\xi|\right) \right) = const.$$

Hence $n^2 \mathbf{E}\psi_{\tau^n}\mathbf{1}_{(\tau^n<\infty)} \leq$ *const.*, consequently $\mathbf{E}\psi_{\tau^n}\mathbf{1}_{(\tau^n<\infty)} \to 0$ $(n \to \infty)$, therefore $\tau^n \to \infty$.
∎

We are done with existence and uniqueness of solutions. Now we present our work [19] on comparison theorem.

3.5 Comparison of Solutions

Another equally important results in the study stochastic equations are comparison theorems. Comparison theorems allows us to compare solutions of related stochastic equations. With a comparison theorem one finds that knowing the structure of the stochastic equation and the set of all possible initial conditions, a stochastic ordering of some sort can be established between processes that are solutions of these stochastic equations. Many have studied comparison of solutions of stochastic equations: Skorokhod [43] proved a comparison theorem for diffusion equations discovering that the solution of these equations must be a nondecreasing function of their drift coefficient. This result was established in another way in [44] under weaker conditions on the diffusion coefficient. In [39] and [45], the comparison theorem was carried over to the case of equations with integrals with respect to continuous martingales. In [10] Gal'chuk considered equations of a more general form, namely equations containing integrals with respect to continuous martingales and integer-valued random measures where the coefficients of the semimartingale are not lipschitz but satisfy weaker conditions similar to those of Yamada [44]. Again, the solution is a nondecreasing function of the drift coefficient and in some sense of the jump functions.

Our goal for this chapter is to study comparison of solutions of stochastic equations driven by optional semimartingales in unusual probability spaces under more general conditions placed on the coefficient of the stochastic equation. To do so, first, we define stochastic equations with respect to *components of optional semimartingales* then, we prove the comparison theorem. Finally, we give an illustrative example of a possible application of comparison theorem to finance.

Suppose that we are given a complete but unusual probability space.$(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. Let us also introduce $\mathcal{G}(\mathbf{F})$ progressive in addition to $\mathcal{O}(\mathbf{F})$ and $\mathcal{P}(\mathbf{F})$ on $\Omega \times \mathbb{R}_+$. Recall that $\mathcal{G}$ is generated by all progressively measurable processes. We assume that, unless otherwise specified, that, all processes considered here are $\mathbf{F}$-consistent, and their trajectories have right and left limits but are not necessarily right or left continuous.

3.5.1 Component Representation

The *canonical* and *component* representation of semimartingales is of fundamental importance in stochastic analysis. It is also essential to our development of stochastic integral equations driven by optional semimartingales. The canonical and component representation of optional semimartingale can be seen as a natural consequence of the decomposition

$$X = X_0 + X^c + X^d + X^g.$$

where X^c is a continuous optional semimartingale with decomposition, $X^c = a + m$, where a is continuous strongly predictable with locally integrable variation ($a \in \mathcal{P}_s \cap \mathcal{A}_{0,loc}$), and m a continuous local martingale ($m \in \mathcal{M}^c_{0,loc}$). The discrete optional semimartingale parts, $X^d = a^d + m^d$ and $X^g = a^g + m^g$, (a^d, $a^g \in \mathcal{A}_{loc}$, $m^d \in \mathcal{M}^d_{loc}$, $m^g \in \mathcal{M}^g_{loc}$) are representable in terms of an some underlying measures of right and left jumps, respectively. These measures' of jumps, are referred to as integer valued random measures. We describe the integer random measure representation of discrete martingales briefly and refer the reader to the paper by Gal'chuk [11] for details.

Here we are going to consider the Lusin space $(\mathbb{E}, \mathscr{E})$ where $\mathbb{E} = (\mathbb{R}^d \backslash \{0\}) \cup \{\delta^d\} \cup \{\delta^g\}$; δ^d and δ^g are some supplementary points or is the set of processes with finite variation on any segment $[0,t]$, $\mathbf{P}$-a.s.; $\mathscr{E} = \mathcal{B}(\mathbb{E})$ is the Borel σ-algebra in $\mathbb{E}$. Also, define the spaces

$$\widetilde{\Omega} = \Omega \times \mathbb{R}_+ \times \mathbb{E}, \quad \widetilde{\mathbb{E}} = \mathbb{R}_+ \times \mathbb{E}, \quad \widetilde{\mathscr{E}} = \mathcal{B}(\mathbb{R}_+) \times \mathscr{E}, \quad \widetilde{\mathcal{G}} = \mathcal{G} \times \mathcal{B}(\mathbb{E}), \tag{3.45}$$
$$\widetilde{\mathcal{O}}(\mathbf{F}) = \mathcal{O}(\mathbf{F}) \times \mathscr{E}, \quad \widetilde{\mathcal{O}}(\mathbf{F}_+) = \mathcal{O}(\mathbf{F}_+) \times \mathscr{E}, \textit{ and } \widetilde{\mathcal{P}}(\mathbf{F}) = \mathcal{P}(\mathbf{F}) \times \mathscr{E}.$$

It was shown in [11] that there exist sequences $\{S_n\}$, $\{T_n\}$, and $\{U_n\}$ for $n \in \mathbb{N}$ of predictable stopping time (s.t.), totally inaccessible stopping time and totally inaccessible stopping time in the broad sense (s.t.b.) respectively, absorbing all jumps of the process X such that the graphs of these stopping times do not intersect within each sequence. On $\widetilde{\Omega}$ let $\mu^i(\omega,\cdot,\cdot)$, $p^i(\omega,\cdot,\cdot)$ and $\eta^g(\omega,\cdot,\cdot)$ where $i \in (d,g)$ be integer valued measures defined on the σ-algebra $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{E})$ that are associated with the sequences of stopping times that are associated with X. On the σ-algebra $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{E})$ we define the random integer-valued measures by the relations,

$$p^d(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(S_n, \beta^d_{S_n}), \quad p^g(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(S_n, \beta^g_{S_n}),$$
$$\mu^d(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(T_n, \beta^d_{T_n}), \quad \mu^g(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(T_n, \beta^g_{U_n}),$$
$$\eta^g(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(T_n, \beta^g_{T_n}),$$

where $B \in \mathcal{B}(\mathbb{R}_+)$, $\Gamma \in \mathcal{B}(\mathbb{E})$, $\beta_t^d = \Delta X_t$ if $\Delta X_t \neq 0$ and $\beta_t^d = \delta^d$ if $\Delta X_t = 0$, $\beta_t^g = \Delta^+ X_t$ if $\Delta^+ X_t \neq 0$, $\beta_t^g = \delta^g$ if $\Delta^+ X_t = 0$, $t > 0$, $\mathbf{1}_A(x)$ is the indicator of the set A. For the measures $\mu^d(\omega, \cdot)$ and $\mu^g(\omega, \cdot)$ there exists unique random measures $\nu^d(\omega, \cdot)$ and $\nu^g(\omega, \cdot)$, respectively, on $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{E})$ such that, for any non-negative functions $\varphi_d \in \widetilde{\mathcal{P}}(\mathbf{F})$ and $\varphi_g \in \widetilde{\mathcal{O}}(\mathbf{F})$

(i) The process

$$\int_{0+}^{t} \int_{\mathbb{E}} \varphi_d(s,u) v^d(ds,du), \quad \int_{0}^{t-} \int_{\mathbb{E}} \varphi_g(s,u) v^g(ds,du)$$

is $\mathcal{P}(\mathbf{F})$-measurable and $\mathcal{O}(\mathbf{F})$-measurable respectively.

(ii) The equalities

$$\begin{aligned} \mathbf{E} \int_{0+}^{\infty} \int_{\mathbb{E}} \varphi_d(s,u) \mu^d(ds,du) &= \mathbf{E} \int_{0+}^{\infty} \int_{\mathbb{E}} \varphi_d(s,u) \nu^d(ds,du) \\ \mathbf{E} \int_{0}^{\infty} \int_{\mathbb{E}} \varphi_g(s,u) \mu^g(ds,du) &= \mathbf{E} \int_{0}^{\infty} \int_{\mathbb{E}} \varphi_g(s,u) \nu^g(ds,du) \end{aligned}$$

are valid.

The measures ν^i, $i \in (d,g)$ possesses the property $0 \leq \nu^i(\omega, \{t\} \times \mathbb{E}) \leq 1$ for all ω and t except for some set of $\mathbf{P}$ measure zero. We denote by $\lambda^i(\omega, \cdot)$, $i \in (d,g)$ the analogous measures for $p^i(\omega, \cdot)$. and $\theta^g(\omega, \cdot)$ that of $\eta^g(\omega, \cdot)$. The measures ν^i, λ^i and θ^g are called the (dual) predictable projections (compensator) for the measures μ^i, p^i and η^g, respectively. Note, μ^g, p^g, and η^g are $\mathcal{O}(\mathbf{F}_+)$-optional with their compensator' being $\mathcal{O}(\mathbf{F})$-optional. On the other hand μ^d and p^d is $\mathcal{O}(\mathbf{F})$-optional with their compensator' being $\mathcal{P}(\mathbf{F})$-predictable.

Having defined integer valued measures and stochastic integrals with respect them one can write a representation of discrete optional semimartingales,

$$\begin{aligned} X_t^d &= \int_{0+}^{t} \int_{\mathbb{E}} u \mathbf{1}_{|u| \leq 1} (\mu^d - \nu^d)(ds,du) + \int_{0+}^{t} \int_{\mathbb{E}} u \mathbf{1}_{|u|>1} \mu^d(ds,du) + \int_{0+}^{t} \int_{\mathbb{E}} u p^d(ds,du), \\ X_t^g &= \int_{0}^{t-} \int_{\mathbb{E}} u \mathbf{1}_{|u| \leq 1} (\mu^g - \nu^g)(ds,du) + \int_{0}^{t-} \int_{\mathbb{E}} u \mathbf{1}_{|u|>1} \mu^g(ds,du) + \int_{0}^{t-} \int_{\mathbb{E}} u p^g(ds,du) \\ &\quad + \int_{0}^{t-} \int_{\mathbb{E}} u \eta^g(ds,du). \end{aligned}$$

With $X^c = a + m$, we can write the component decomposition of X as

$$\begin{aligned} X &= X_0 + a + m \\ &+ \int_{0+}^{t} \int_{\mathbb{E}} u \mathbf{1}_{|u|\leq 1} (\mu^d - \nu^d)(ds, du) + \int_{0+}^{t} \int_{\mathbb{E}} u \mathbf{1}_{|u|>1} \mu^d(ds, du) + \int_{0+}^{t} \int_{\mathbb{E}} u p^d(ds, du), \\ &+ \int_{0}^{t-} \int_{\mathbb{E}} u \mathbf{1}_{|u|\leq 1} (\mu^g - \nu^g)(ds, du) + \int_{0}^{t-} \int_{\mathbb{E}} u \mathbf{1}_{|u|>1} \mu^g(ds, du) + \int_{0}^{t-} \int_{\mathbb{E}} u p^g(ds, du) \\ &+ \int_{0}^{t-} \int_{\mathbb{E}} u \eta^g(ds, du). \end{aligned}$$

Note that the local martingales of the process X are,

$$\begin{aligned} m^d &= \int_{0+}^{t} \int_{\mathbb{E}} u \mathbf{1}_{|u|\leq 1} (\mu^d - \nu^d)(ds, du), \\ m^g &= \int_{0}^{t-} \int_{\mathbb{E}} u \mathbf{1}_{|u|\leq 1} (\mu^g - \nu^g)(ds, du). \end{aligned}$$

and the characteristics of the process X is $\left(a, \langle m, m \rangle, \nu^d, \lambda^d, \nu^g, \lambda^g, \theta^g\right)$. For further details on the construction of the component decomposition of optional semimartingales, see [11].

Now, lets consider integrals with respect to the components of X. The process, a, is continuous locally finite variation process that is strongly predictable, ($a \in \mathcal{P}_s \cap \mathcal{A}_{0,loc}$). An integral of a function, f, with respect, a, is well defined in the Lebesgue-Stieltjes sense, $f \cdot a_t \in \mathcal{A}_{loc}$, where the integral is over the interval $[0, t]$ and f is $\mathcal{P}(\mathbf{F})$-measurable. For the continuous local martingale $m \in \mathcal{M}^c_{0,1oc}$, If the function g is $\mathcal{P}(\mathbf{F})$-measurable and $\left[|g|^2 \cdot \langle m, m \rangle\right]^{j/2} \in \mathcal{A}_{1oc}$ then the stochastic integral, $g \cdot m \in \mathcal{M}^{j,c}_{1oc}$, $j = 1, 2$, is well defined; again, the integral is over the interval $[0, t]$.

Integral with respect to random measures are

$$k_d * \mu_t^d = \int_{0+}^{t} \int_{\mathbb{E}} k_d(s,u)\mu^d(ds,du), \quad k_g * \mu_t^g = \int_{0}^{t-} \int_{\mathbb{E}} k_g(s,u)\mu^g(ds,du),$$
$$h_d * (\mu^d - \nu^d)_t = \int_{0+}^{t} \int_{\mathbb{E}} h_d(\omega,s,u)(\mu^d - \nu^d)(ds,du),$$
$$h_g * (\mu^g - \nu^g)_t = \int_{0}^{t-} \int_{\mathbb{E}} h_g(\omega,s,u)(\mu^g - \nu^g)(ds,du),$$
$$r_d * p^d = \int_{0+}^{t} \int_{\mathbb{E}} r_d(\omega,s,u)p^d(ds,du), \quad r_g * p^g = \int_{0}^{t-} \int_{\mathbb{E}} r_g(\omega,s,u)p^g(ds,du),$$
$$w_g * \eta^g = \int_{0}^{t-} \int_{\mathbb{E}} w_g(\omega,s,u)\eta^g(ds,du),$$

where "$*$" means integral with respect to random measures for any type of jump and differences are recognized by the symbols d and g for right and left jumps, respectively.

If the function h_d is $\widetilde{\mathcal{P}}(\mathbf{F})$-measurable and $\left[|h_d|^2 * \nu^d\right]^{j/2} \in \mathcal{A}_{1oc}$, then the stochastic integral $h_d * (\mu^d - \nu^d) \in \mathcal{M}_{1oc}^{j,d}$, $j = 1,2$, is well defined. If the function h_g is $\widetilde{\mathcal{O}}(\mathbf{F})$-measurable and $\left[|h_g|^2 * \nu^g\right]^{j/2} \in \mathcal{A}_{1oc}$, then the stochastic integral $h_g * (\mu^g - \nu^g) \in \mathcal{M}_{1oc}^{j,g}$, $j = 1,2$, is also well defined. If k_d is $\widetilde{\mathcal{G}}(\mathbf{F})$-measurable and $|k_d| * \mu^d \in \mathcal{V}$, then the integral $f * \mu^d \in \mathcal{V}$ is defined (see [46]). And, If k_g is $\widetilde{\mathcal{G}}(\mathbf{F})$-measurable and $|k_g| * \mu^g \in \mathcal{V}$, then the integral $k_g * \mu^g \in \mathcal{V}$ is defined.

If r_d is $\widetilde{\mathcal{P}}(\mathbf{F})$-measurable, $\left[|r_d|^2 * p^d\right]^{j/2} \in \mathcal{A}_{1oc}$ and for any predictable stopping time S, we have that $\mathbf{E}\left[r_d(S,\beta_S^d)|\mathcal{F}_{S_-}\right] = 0$ a.s., then the stochastic integral $r_d * p^d \in \mathcal{M}_{1oc}^{j,d}$, $j = 1,2$, is defined. And, If $r_d \in \widetilde{\mathcal{G}}(\mathbf{F})$ and $|r_d| * p^d \in \mathcal{V}$ then the integral $r_d * p^d \in \mathcal{V}$ is defined (see [47]). Note that the facts used below in the theory of martingales can be found in [48, 49, 46]. If r_g is $\widetilde{\mathcal{O}}(\mathbf{F})$-measurable, $\left[|r_g|^2 * p^g\right]^{j/2} \in \mathcal{A}_{1oc}$ and for any totally inaccessible stopping time T, $\mathbf{E}\left[r_g(T,\beta_T^g)|\mathcal{F}_T\right] = 0$ a.s., then the stochastic integral $r_g * p^g \in \mathcal{M}_{1oc}^{j,g}$, $j = 1,2$, is defined. And, If $r_g \in \widetilde{\mathcal{G}}(\mathbf{F})$ and $|r_g| * p^g \in \mathcal{V}$ then the integral $r_g * p^g \in \mathcal{V}$ is defined. If w_g is $\widetilde{\mathcal{O}}(\mathbf{F})$-measurable, $\left[|w_g|^2 * \eta^g\right]^{j/2} \in \mathcal{A}_{1oc}$ and for any totally inaccessible stopping time in the broad sense U, $\mathbf{E}\left[w_g(U,\beta_U^g)|\mathcal{F}_U\right] = 0$ a.s., then the stochastic integral $w_g * \eta^g \in \mathcal{M}_{1oc}^{j,g}$, $j = 1,2$, is defined. And, If $w_g \in \widetilde{\mathcal{G}}(\mathbf{F})$ and $|w_g| * \eta^g \in \mathcal{V}$ then the integral $w_g * \eta^g \in \mathcal{V}$ is defined.

Notation 70 *We have used $i \in (d,g)$ to clearly identify the different types of optional semi-*

martingales. However, from now on, for convenience, we are going to identify "d" by 1 the right-continuous discrete component of the semimartingale and "g" by 2 the left-continuous discrete part of the semimartingale to give a concise description (i.e. $i \in (1,2)$).

With new notations, the optional semimartingale X has the following components representation,

$$\begin{aligned} X &= X_0 + a + m \\ &+ \int_{0+}^{t}\int_{\mathbb{E}} U(\mu^1 - \nu^1)(ds,du) + \int_{0+}^{t}\int_{\mathbb{E}} V\mu^1(ds,du) + \int_{0+}^{t}\int_{\mathbb{E}} up^1(ds,du) \\ &+ \int_{0}^{t-}\int_{\mathbb{E}} U(\mu^2 - \nu^2)(ds,du) + \int_{0}^{t-}\int_{\mathbb{E}} V\mu^2(ds,du) + \int_{0}^{t-}\int_{\mathbb{E}} up^2(ds,du) \\ &+ \int_{0}^{t-}\int_{\mathbb{E}} u\eta(ds,du). \end{aligned}$$

where $U = u\mathbf{1}_{|u|\leq 1}$, $V = u\mathbf{1}_{|u|>1}$ and $\eta = \eta^g$. The component representation of optional semimartingale will be the representation form that we will use to construct the comparison lemma.

Before we get to the main theorem we need to extend the change of variables formula of the component representation of semimartingales in the usual conditions (cf. [47]) to optional semimartingales in the unusual case.

Lemma 71 *Suppose an optional semimartingale $Y = \left(Y^1, Y^2, ..., Y^k\right)$ is defined by the relation*

$$\begin{aligned} Y_t &= Y_0 + f \cdot a_t + g \cdot m_t + (r + w) * \eta_t, \\ &+ \sum_j UH_j * (\mu^j - \nu^j)_t + Vh_j * \mu^j_t + (k_j + l_j) * p^j_t, \end{aligned}$$

where all the integrals are well defined. Consider the function $F(y) = F(y^1, y^2, ..., y^k)$ to be twice continuously differentiable on $\mathbb{R}^k$.

Then the process $F(Y) = (F(Y_t))_{t\geq 0}$ is an optional semimartingale and has the representation

$$\begin{aligned} F(Y_t) &= F(Y_0) + F'(Y)f \cdot a_t + F'(Y)g \cdot m_t + \frac{1}{2}F''(Y)g^2 \cdot \langle m, m\rangle_t \\ &+ \sum_j U\left[F\left(Y + H_j\right) - F(Y)\right] * (\mu^j - \nu^j)_t \\ &+ \sum_j V\left[F\left(Y + h_j\right) - F(Y)\right] * \mu^j_t \\ &+ \sum_j U\left[F\left(Y + H_j\right) - F(Y) + F'(Y)H_j\right] * \nu^j_t \\ &+ \sum_j \left[F\left(Y + (k_j + l_j)\right) - F(Y)\right] * p^j_t \\ &+ \left[F\left(Y + (r + w)\right) - F(Y)\right] * \eta_t. \end{aligned}$$

Proof. Gal'chuk [47] proved the change of variable formula for semimartingales under the usual conditions. Extending the proof to optional semimartingale is straight forward. ■

3.5.2 Comparison Theorem

Let there be given an optional semimartingale Z with components: a continuous locally integrable process $a \in \mathcal{A}_{1oc}$ with $a_0 = 0$, a continuous martingale $m \in \mathcal{M}^c_{1oc}$ with $m_0 = 0$ and integer-valued measures μ^j, p^j for$j = 1,\ 2$ and η with predictable and optional projections ν^j, λ^j, and θ respectively.

We shall consider the equations

$$\begin{aligned} X^i_t &= X^i_0 + f^i(X^i) \cdot a_t + g(X^i) \cdot m_t \\ &+ \sum_j Uh_j(X^i) * (\mu^j - \nu^j)_t + Vh^i_j(X^i) * \mu^j_t + \left(k^i_j(X^i) + l^i_j(X^i)\right) * p^j_t \\ &+ \left(r^i(X^i) + w^i(X^i)\right) * \eta_t, \end{aligned} \tag{3.46}$$

where $U = \mathbf{1}_{|u|\leq 1}$ and $V = \mathbf{1}_{|u|>1}$ and the dependence on the arguments is as follows:

$$
\begin{aligned}
f^i(X^i) &= f^i(\omega, s, X^i_{s-}), & g(X^i) &= g^i(\omega, s, X^i_{s-}) \\
h_1(X^i) &= h_1(\omega, s, u, X^i_{s-}), & h_2(X^i) &= h_2(\omega, s, u, X^i_s) \\
h^i_1(X^i) &= h^i_1(\omega, s, u, X^i_{s-}), & h^i_2(X^i) &= h^i_2(\omega, s, u, X^i_s) \\
k^i_1(X^i) &= k^i_1(\omega, s, u, X^i_{s-}), & k^i_2(X^i) &= k^i_2(\omega, s, u, X^i_s) \\
l^i_1(X^i) &= l^i_1(\omega, s, u, X^i_{s-}), & l^i_2(X^i) &= l^i_2(\omega, s, u, X^i_s) \\
r^i(X^i) &= r^i(\omega, s, u, X^i_s), & w^i(X^i) &= w^i(\omega, s, u, X^i_s)
\end{aligned}
$$

for $i = 1, 2$; In another way to describe the processes X^i for $i = 1, 2$,

$$
\begin{aligned}
X^i_t &= X^i_0 + A^i_t(X^i) + M_t(X^i), \\
A^i_t(X^i) &= f^i(X^i) \cdot a_t + \sum_j V h^i_j(X^i) * \mu^j_t + \big(k^i_j(X^i) + l^i_j(X^i)\big) * p^j_t \\
&\quad + \big(r^i(X^i) + w^i(X^i)\big) * \eta_t, \\
M_t(X^i) &= g(X^i) \cdot m_t + \sum_j U h_j(X^i) * (\mu^j - \nu^j)_t
\end{aligned}
$$

the martingale part $M(X^i) \in \mathcal{M}_{1oc}$ and finite variation process $A^i(X^i) \in \mathcal{V}$ form the process X^i. It is also assumed that for, $i = 1, 2$, the functions above satisfy these conditions, (D1) $f^i(\omega, s, x)$ and $g(\omega, s, x)$ are defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$

and $\mathcal{P}(\mathbf{F}) \times \mathcal{B}(\mathbb{R})$-measurable,

(D2) $Uh_1(\omega, s, u, x)$ is defined on $\Omega \times \mathbb{R}_+ \times \mathbb{E} \cap (|u| \leq 1) \times \mathbb{R}$

and $\mathcal{P}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E} \cap (|u| \leq 1)\right) \times \mathcal{B}\left(\mathbb{R}\right)$-measurable,

(D3) $Uh_2(\omega, s, u, x)$ is defined on $\Omega \times \mathbb{R}_+ \times \mathbb{E} \cap (|u| \leq 1) \times \mathbb{R}$

and $\mathcal{O}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E} \cap (|u| \leq 1)\right) \times \mathcal{B}\left(\mathbb{R}\right)$-measurable,

(D4) $Vh_1^i(\omega, s, u, x)$ is defined on$\Omega \times \mathbb{R}_+ \times \mathbb{E} \cap (|u| > 1) \times \mathbb{R}$

and $\mathcal{G}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E} \cap (|u| > 1)\right) \times \mathcal{B}\left(\mathbb{R}\right)$-measurable,

(D5) $Vh_2^i(\omega, s, u, x)$ is defined on $\Omega \times \mathbb{R}_+ \times \mathbb{E} \cap (|u| > 1) \times \mathbb{R}$

and $\mathcal{G}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E} \cap (|u| > 1)\right) \times \mathcal{B}\left(\mathbb{R}\right)$-measurable,

(D6) $k_1^i(\omega, s, u, x)$ is defined on $\Omega \times \mathbb{R}_+ \times \mathbb{E} \times \mathbb{R}$

and $\mathcal{P}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E}\right) \times \mathcal{B}(\mathbb{R})$-measurable such that $k_1^i(X^i) * p^1 \in \mathcal{M}_{1oc}^{1,r}(\mathbf{F})$,

(D7) $k_2^i(\omega, s, u, x)$ is defined on$\Omega \times \mathbb{R}_+ \times \mathbb{E} \times \mathbb{R}$

and $\mathcal{O}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E}\right) \times \mathcal{B}(\mathbb{R})$-measurable such that $k_2^i(X^i) * p^2 \in \mathcal{M}_{1oc}^{1,g}(\mathbf{F})$,

(D8) $l_1^i(\omega, s, u, x)$ is defined on $\Omega \times \mathbb{R}_+ \times \mathbb{E} \times \mathbb{R}$

and $\mathcal{G}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E}\right) \times \mathcal{B}(\mathbb{R})$-measurable such that $l_1^i(X^i) * p^1 \in \mathcal{V}$,

(D9) $l_2^i(\omega, s, u, x)$ is defined on $\Omega \times \mathbb{R}_+ \times \mathbb{E} \times \mathbb{R}$

and $\mathcal{G}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E}\right) \times \mathcal{B}(\mathbb{R})$-measurable such that $l_2^i(X^i) * p^2 \in \mathcal{V}$,

(D10) $r^i(\omega, s, u, x)$ is defined on$\Omega \times \mathbb{R}_+ \times \mathbb{E} \times \mathbb{R}$

and $\mathcal{O}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E}\right) \times \mathcal{B}(\mathbb{R})$-measurable such that $r^i(X^i) * \eta \in \mathcal{M}_{1oc}^{1,g}(\mathbf{F})$,

(D11) $w^i(\omega, s, u, x)$ is defined on$\Omega \times \mathbb{R}_+ \times \mathbb{E} \times \mathbb{R}$

and $\mathcal{G}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E}\right) \times \mathcal{B}(\mathbb{R})$-measurable such that $w^i(X^i) * \eta \in \mathcal{V}$,

Note that the two in k^2 is an index whereas in $|k|^2$ it is an exponent.

Let us formulate conditions under which the comparison theorem will be proved:

(A1) $X_0^2 \geq X_0^1$;

(A2) $f^2(s, x) > f^1(s, x)$ for any $s \in \mathbb{R}_+,\ x \in \mathbb{R},\ f^i(s, x)$ are continuous in (s, x), $i = 1, 2$;

(A3) There exists a non-negative nondecreasing function $\rho(x)$ on $\mathbb{R}_+$ and a $\mathcal{P}(\mathbf{F})$-measurable non-negative function G such that

$$|g(s,x)-g(s,y)| \leq \rho(|x-y|)G(s),$$
$$|G|^2\cdot\langle m,m\rangle_s<\infty \ \ a.s., \ \ \int_0^{\epsilon}\rho^{-2}(x)dx=\infty \ \ for\ any \ \ s\in\mathbb{R}_+,\ \epsilon>0,\ x,y\in\mathbb{R};$$

(A4) There exists a non-negative $\widetilde{\mathcal{P}}(\mathbf{F})$-measurable function H_1 and $\widetilde{\mathcal{O}}(\mathbf{F})$-measurable function H_2 such that

$$\begin{aligned}|h_1(s,u,x)-h_1(s,u,y)| &\leq \rho(|x-y|)H_1(s,u), \ \ |H_1|^2 * \nu_s^1<\infty,\\ |h_2(s,u,x)-h_2(s,u,y)| &\leq \rho(|x-y|)H_2(s,u), \ \ |H_2|^2 * \nu_s^2<\infty,\end{aligned}$$

a.s. for any $s\in\mathbb{R}_+,\ u\in\mathbb{E},\ x,y\in\mathbb{R}$;

(A5) For any $s\in\mathbb{R}_+,\ u\in\mathbb{E},\ x,y\in\mathbb{R},\ y\geq x$,

$$h_1(s,u,y)\geq h_1(s,u,x), \quad h_2(s,u,y)\geq h_2(s,u,x)$$
$$y+h_1^2(s,u,y)\mathbf{1}_{|u|>1}\geq x+h_1^1(s,u,x)\mathbf{1}_{|u|>1},$$
$$y-h_2^2(s,u,y)\mathbf{1}_{|u|>1}\geq x-h_2^1(s,u,x)\mathbf{1}_{|u|>1},$$
$$y+h_1(s,u,y)\mathbf{1}_{|u|\leq 1}+(k_1^2+l_1^2)(s,u,y)\geq x+h_1(s,u,x)\mathbf{1}_{|u|\leq 1}+(k_1^1+l_1^1)(s,u,x),$$
$$y-h_2(s,u,y)\mathbf{1}_{|u|\leq 1}-(k_2^2+l_2^2)(s,u,y)-\left(r^2+w^2\right)(s,u,x)\geq$$
$$x-h_2(s,u,x)\mathbf{1}_{|u|\leq 1}-(k_2^1+l_2^1)(s,u,x)-\left(r^1+w^1\right)(s,u,x);$$

(A6) The functions $\left(r^i+w^i\right)(s,u,x)$ and $(k_j^i+l_j^i)(s,u,x)$ are continuous in $(s,u,x),\ i=1,2$ and $j=1,2$,

$$\begin{aligned}(k_j^2+l_j^2)(s,u,x) &> (k_j^1+l_j^1)(s,u,x)\\ \left(r^2+w^2\right)(s,u,x) &> \left(r^1+w^1\right)(s,u,x)\end{aligned}$$

for any $s\in\mathbb{R}_+,\ u\in\mathbb{E},\ x\in\mathbb{R}$;

(A7) For $i = 1, 2$ and $j = 1, 2$,

$$
\begin{gathered}
\left|f^i(X^i)\right| \cdot a \in \mathcal{A}_{1oc}, \\
\left|g(X^i)\right|^2 \cdot \langle m, m\rangle \in \mathcal{A}_{1oc}, \quad \left|h_j(X^i)\right|^2 * \nu^j \in \mathcal{A}_{1oc}, \\
\left|l_j^i(X^i)\right| * p^j \in \mathcal{A}_{1oc}, \quad \left[\left|k_j^i(X^i)\right|^2 * p^j\right]^{1/2} \in \mathcal{A}_{1oc}, \\
\left|r^i(X^i)\right| * \eta \in \mathcal{A}_{1oc}, \quad \left[\left|w^i(X^i)\right|^2 * \eta\right]^{1/2} \in \mathcal{A}_{1oc},
\end{gathered}
$$

and

$$\mathbf{E}[k_1^i(S, \beta_S^d, X_{S-}^i)|\mathcal{F}_{S-}] = 0$$

a.s. for any predictable stopping time S and

$$\mathbf{E}[k_2^i(T, \beta_T^g, X_T^i)|\mathcal{F}_T] = 0$$

a.s., for any totally inaccessible stopping time T, and

$$\mathbf{E}\left[w(U, \beta_U^g)|\mathcal{F}_U\right] = 0$$

a.s., for any totally inaccessible stopping time in the broad sense $U,\ i = 1, 2$.

To formulate the next assumption we need to introduce the sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers $a_0 = 1 > a_1 > \cdots,\ \lim_{n\to\infty} a_n = 0$, by the relations

$$\int_{a_{n+1}}^{a_n} \rho^{-2}(x)dx = n + 1, \quad n = 0, 1, \cdots.$$

Now let us write the last assumption

(A8) We assume that there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive numbers such that $\varepsilon_n \leq a_{n-1} - a_n$ for all $n \in \mathbb{N}$ and

$$\frac{1}{n}\left[\frac{\rho(a_{n-1})}{\rho(a_{n-1} - \varepsilon_n)}\right]^2 \to 0, \quad n \to \infty.$$

It is easy to verify that condition A8 is satisfied by Holder class functions ρ with index $\alpha = 1/2+\epsilon,\ \epsilon > 0$, and is not satisfied by functions of this class with index $\alpha = 1/2$.

Theorem 72 *Let there exist strong solutions X^i, $i = 1, 2$, of equations (1) and let conditions A1-A8 hold. Then off some set of $\mathbf{P}$-measure zero $X_t^2 \geq X_t^1$ for any $t \in \mathbb{R}_+$.*

Before proving the theorem, let us perform a useful reduction of the problem.

Lemma 73 *If the comparison theorem is valid for the equations in (3.47),*

$$\begin{aligned} Y_t^i &= X_0^i + f^i(Y^i) \cdot a_t + g(Y^i) \cdot m_t \\ &\quad + \sum_j Uh_j(Y^i) * (\mu^j - v^j)_t + \left(k_j^i(Y^i) + l_j^i(Y^i)\right) * p_t^j \\ &\quad + \left(r^i(Y^i) + w^i(Y^i)\right) * \eta_t \end{aligned} \tag{3.47}$$

with functions X_0^i, f^i, g, h_j, k_j^i, l_j^i, r^i, and w^i satisfying conditions A1-A8, then, it is also valid for equations (3.46).

Proof. Let $\{\tau_n\}_{n\in\mathbb{N}}$, $\tau_0 = 0$, be a nondecreasing sequence of totally inaccessible stopping times and stopping times in the broad sense, absorbing the jumps of the processes $h_j^i(X^i) * \mu^j$, $i = 1, 2$ and $j = 1, 2$, from equations (3.46). On $]\tau_0, \tau_1[$ equations (3.46) and (3.47) coincide. Since $Y^2 \geq Y^1$, on this interval then $X^2 \geq X^1$ on this interval. On the boundary of this interval, from equations (3.46) we find that at time τ_1,

$$\begin{aligned} X_{\tau_1}^2 &= X_{\tau_1-}^2 + \Delta X_{\tau_1}^2 = X_{\tau_1-}^2 + h_1^2\left(\tau_1, \beta_{\tau_1}^1, X_{\tau_1-}^2\right) \mathbf{1}_{|\beta_{\tau_1}^1|>1} \\ &\geq X_{\tau_1-}^1 + h_1^1(\tau_1, \beta_{\tau_1}^1, X_{\tau_1-}^1) \mathbf{1}_{|\beta_{\tau_1}^1|>1} = X_{\tau_1-}^1 + \Delta X_{\tau_1}^1 = X_{\tau_1}^1, \end{aligned}$$

and

$$\begin{aligned} X^2_{\tau_0} &= X^2_{\tau_0+} - \Delta^+ X^2_{\tau_0} \\ &= X^2_{\tau_0+} - h^2_2\left(\tau_0, \beta^2_{\tau_0}, X^2_{\tau_0}\right) \mathbf{1}_{|\beta^2_{\tau_0}|>1} \\ &\quad - \left(r^2(\tau_0, \beta^2_{\tau_0}, X^2_{\tau_0}) + w^2(\tau_0, \beta^2_{\tau_0}, X^2_{\tau_0})\right) \mathbf{1}_{|\beta^2_{\tau_0}|>1} \\ &\geq X^1_{\tau_0+} - h^1_2(\tau_0, \beta^2_{\tau_0}, X^1_{\tau_0}) \mathbf{1}_{|\beta^2_{\tau_0}|>1} \\ &\quad - \left(r^1(\tau_0, \beta^2_{\tau_0}, X^1_{\tau_0}) + w^1(\tau_0, \beta^2_{\tau_0}, X^1_{\tau_0})\right) \mathbf{1}_{|\beta^2_{\tau_0}|>1} \\ &= X^1_{\tau_0+} - \Delta^+ X^1_{\tau_0} = X^1_{\tau_0}; \end{aligned}$$

Therefore, by condition A5 the comparison theorem holds for (3.46) on $[0, \tau_1]$.

Now let us suppose that the comparison theorem for (3.46) holds on $[0, \tau_n]$, $n \geq 1$ and prove it holds on $]\tau_n, \tau_{n+1}]$. On $]\tau_n, \infty[$ consider the equations $(i = 1, 2)$

$$\begin{aligned} Y^i_t &= X^i_{\tau_n} + \int_{\tau_n+}^t f^i(Y^i_s) da_s + \int_{\tau_n+}^t g(Y^i_s) dm_s \\ &\quad + \int_{\tau_n+}^t \int_{\mathbb{E}} U h_1(Y^i) d(\mu^1 - \nu^1)_s \\ &\quad + \int_{\tau_n+}^{t-} \int_{\mathbb{E}} U h_2(Y^i_s) d(\mu^2 - \nu^2)_s + \int_{\tau_n+}^t \int_{\mathbb{E}} (k^i_1(Y^i_s) + l^i_1(Y^i_s)) dp^1 \qquad (3.48) \\ &\quad + \int_{\tau_n+}^{t-} \int_{\mathbb{E}} (k^i_2(Y^i_s) + l^i_2(Y^i_s)) dp^2_s + \int_{\tau_n+}^{t-} \int_{\mathbb{E}} \left(r^i(Y^i_s) + w^i(Y^i_s)\right) d\eta_s \end{aligned}$$

Let us transform (3.48) to the form (3.47). For this we make the substitution $t - \tau_n = s$ and set

$$\mathcal{F}_s^{(n)} = \mathcal{F}_{s+\tau_n}, \quad s \in [0,\infty[, \quad a_s^{(n)} = a_{s+\tau_n}, \quad m_s^{(n)} = m_{s+\tau_n},$$
$$\left(\mu^{1(n)} - \nu^{1(n)}\right)(]\rho,\varsigma],\Gamma) = \left(\mu^1 - \nu^1\right)(]\rho+\tau_n,\varsigma+\tau_n],\Gamma),$$
$$\left(\mu^{2(n)} - \nu^{2(n)}\right)([\rho,\varsigma[,\Gamma) = \left(\mu^2 - \nu^2\right)([\rho+\tau_n,\varsigma+\tau_n[,\Gamma),$$
$$p^{1(n)}(]\rho,\varsigma],\Gamma) = p^1(]\rho+\tau_n,\varsigma+\tau_n],\Gamma),$$
$$p^{2(n)}([\rho,\varsigma[,\Gamma) = p^2([\rho+\tau_n,\varsigma+\tau_n[,\Gamma),$$
$$\eta^{(n)}([\rho,\varsigma[,\Gamma) = \eta([\rho+\tau_n,\varsigma+\tau_n[,\Gamma),$$
$$Y_s^{i(n)} = Y^i_{s+\tau_n}, \quad X_s^{i(n)} = X^i_{s+\tau_n}.$$

Introduce further the functions $f^{i(n)}$, $g^{(n)}$, $h_j^{(n)}$, $k_j^{i(n)}$, $l_j^{i(n)}$, $r^{i(n)}$, and $w^{i(n)}$ setting

$$f^{i(n)}(s,x) = f^i(s+\tau_n,x), \quad h_j^{(n)}(s,u,x) = h_j(s+\tau_n,u,x),$$

and proceed analogously for the remaining functions. Equations (3.48) take on the form

$$\begin{aligned} Y_s^i \;&=\; X_0^{i(n)} + f^{i(n)}\left(Y^{i(n)}\right)\cdot a_s^{(n)} + g^{(n)}\left(Y^{i(n)}\right)\cdot m_s^{(n)} \\ &\quad + \sum_j U h_j^{(n)}(Y^{i(n)}) * (\mu^j - \nu^j)_s^{(n)} + (k_j^{i(n)} + l_j^{i(n)})(Y^{i(n)}) * p_s^{j(n)} \\ &\quad + \left(r^{i(n)}(Y^{i(n)}) + w^{i(n)}(Y^{i(n)})\right) * \eta_t^{(n)} \end{aligned}$$

for $(i = 1,2)$ and $s \in [0,\infty[$.

These are equations of the form (3.47) with integrands satisfying conditions A1-A8. By the assumption of the lemma, the comparison theorem holds for these equations. Their solutions for $s \in]0,\tau_{n+1}-\tau_n[$ coincide with the solutions of (3.46) on $]\tau_n,\tau_{n+1}[$. Hence the comparison theorem for (3.46) can be extended to the interval $]0,\tau_{n+1}[$. Arguing just as in the case of $[0,\tau_1]$ we extend the comparison theorem for (3.46) to the points τ_n and τ_{n+1}. Then repeating these arguments, we prove the result for equation (3.46) for all $t \in [0,\infty[$. ■

Proof. of $[Comparison]$, (72); By lemma (73), it suffices to establish comparison result for (3.47). So it begins; By A7, all the integrals in (3.47) are defined. Not to resort to an additional localization arguments, we shall assume that for $(i = 1, 2)$,

$$\mathbf{E}\left[|G|^2 \cdot \langle m, m\rangle_\infty\right] < \infty, \quad \mathbf{E}\left[|H_j|^2 * \nu^j_\infty\right] < \infty,$$

$$\begin{aligned}&\mathbf{E}\Big[|f^i(X^i)| \cdot a_\infty + |g(X^i)|^2 \cdot \langle m, m\rangle_\infty \\ &\quad + \textstyle\sum_j U|h_j(X^i)|^2 * \nu^j_\infty + |l^i_j(X^i)| * p^j_\infty + \left(|k^i_j(X^i)|^2 * p^j_\infty\right)^{1/2} \\ &\quad + \left|r^i(Y^i)\right| * \eta_\infty + w^i(Y^i) * \eta_\infty\Big] < \infty.\end{aligned}$$

Let us introduce the sets

$$A = \left\{\omega : X^2_0(\omega) > X^1_0(\omega)\right\}, \quad B = \left\{\omega : X^2_0(\omega) = X^1_0(\omega)\right\}.$$

I. First we prove the theorem on the set B. Let $Y^i_0 = X^i_0 \mathbf{1}_B$, $\tilde{f}^i = f^i \mathbf{1}_B$, and similarly define $\tilde{g}$, $\tilde{h}_j$, $\tilde{k}^i_j$, $\tilde{l}^i_j$, $\tilde{r}^i$ and $\tilde{w}^i$. Consider the equations $(i = 1, 2)$,

$$\begin{aligned} Y^i_t &= Y^i_0 + \tilde{f}^i(Y^i) \cdot a_t + \tilde{g}(Y^i) \cdot m_t \\ &\quad + \sum_j U\tilde{h}_j(Y^i) * (\mu^j - \nu^j)_t + (\tilde{k}^i_j + \tilde{l}^i_j)(Y^i) * p^j_t \\ &\quad + \left(\tilde{r}^i(Y^i) + \tilde{w}^i(Y^i)\right) * \eta_t. \end{aligned} \tag{3.49}$$

It is clear that $Y^i = X^i$ on B. Define the quantity T as follows:

$$\begin{aligned} T &= \inf\Big\{t > 0 : \tilde{f}^1(t, Y^1_{t-}) > \tilde{f}^2(t, Y^2_{t-}) \\ &\quad\ or \ \ (\tilde{k}^1_1 + \tilde{l}^1_1)(t, \beta^1_t, Y^1_{t-}) > (\tilde{k}^2_1 + \tilde{l}^2_1)(t, \beta^1_t, Y^2_{t-}) \\ &\quad\ or \ \ (\tilde{k}^1_2 + \tilde{l}^1_2)(t, \beta^2_t, Y^1_t) > (\tilde{k}^2_2 + \tilde{l}^2_2)(t, \beta^2_t, Y^2_t) \\ &\quad\ or \ \ (\tilde{r}^1 + \tilde{w}^1)(t, \beta^2_t, Y^1_t) > (\tilde{r}^2 + \tilde{w}^2)(t, \beta^2_t, Y^2_t)\Big\} \end{aligned}$$

We have $Y_0^2 = Y_0^1$, and, by A2 and A6,

$$\tilde{f}^2(0, Y_0^2) > \tilde{f}^1(0, Y_0^1),$$
$$(\tilde{k}_1^2 + \tilde{l}_1^2)(0, \beta_0^1, Y_0^2) > (\tilde{k}_1^1 + \tilde{l}_1^1)(0, \beta_0^1, Y_0^1),$$
$$(\tilde{k}_2^2 + \tilde{l}_2^2)(0, \beta_0^2, Y_0^2) > (\tilde{k}_2^1 + \tilde{l}_2^1)(0, \beta_0^2, Y_0^1),$$
$$(\tilde{r}^2 + \tilde{w}^2)(0, \beta_0^2, Y_0^2) > (\tilde{r}^1 + \tilde{w}^1)(0, \beta_0^2, Y_0^1)$$

Since $\beta_t^1 \to \beta_0^1$ and $\beta_t^2 \to \beta_0^2$, $Y_t^i \to Y_0^i$, $t \downarrow 0$, and the functions $\tilde{f}^i(t,x)$, $(\tilde{k}_j^i + \tilde{l}_j^i)(t,u,x)$ and $(\tilde{r}^i + \tilde{w}^i)(t,u,x)$ are continuous in (t,u,x), it follows from what has been said that $T > 0$ a.s. on the set B.

Let $v = t \wedge T$. Set $R = Y^2 - Y^1$,

$$\begin{aligned} R_v &= Y_v^2 - Y_v^1 = R_0 + \left(\tilde{f}^2(Y^2) - \tilde{f}^1(Y^1)\right) \cdot a_t + \left(\tilde{g}(Y^2) - \tilde{g}(Y^1)\right) \cdot m_t \\ &\quad + \sum_j U\left(\tilde{h}_j(Y^2) - \tilde{h}_j(Y^1)\right) * (\mu^j - \nu^j)_t + \left[(\tilde{k}_j^2 + \tilde{l}_j^2)(Y^2) - (\tilde{k}_j^1 + \tilde{l}_j^1)(Y^1)\right] * p_t^j \\ &\quad + \left(\left(\tilde{r}^2 + \tilde{w}^2\right)(Y^2) - \left(\tilde{r}^1 + \tilde{w}^1\right)(Y^1)\right) * \eta_t. \end{aligned}$$

and considering the properties of stochastic integrals we have

$$\begin{aligned} \mathbf{E}R_v &= \mathbf{E}\Bigg[\left(\tilde{f}^2(Y^2) - \tilde{f}^1(Y^1)\right) \cdot a_v \\ &\quad + \sum_j \left(\tilde{l}^2(Y^2) - \tilde{l}^1(Y^1)\right) * \lambda_v^j + \left(\tilde{w}^2(Y^2) - \tilde{w}^1(Y^1)\right) * \theta_v\Bigg]. \end{aligned} \tag{3.50}$$

Now let $\{\psi_n(x)\}_{n\in\mathbb{N}}$ be a sequence of non-negative continuous functions such that $\text{supp}\psi_n \subseteq (a_n, a_{n-1})$,

$$\int_{a_n}^{a_{n-1}} \psi_n(x)dx = 1, \quad \psi_n(x) \le \frac{2}{n}\rho^{-2}(|x|), \quad x \in \mathbb{R},$$

and the maximum of ψ_n is attained at $a_{n-1} - \epsilon_n$, where the sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ satisfies A8. Set

$$\varphi_n(x) = \int_0^{|x|} dy \int_0^y \psi_n(u)du, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Clearly,

$$\varphi_n \in C^2(R^1), \quad \varphi_n(x) \uparrow |x|, \quad n \to \infty, \quad |\varphi_n'| \le 1,$$
$$\varphi_n''(x) = \psi_n(x) \le \frac{2}{n}\rho^{-2}(|x|), \quad x \in \mathbb{R}. \tag{3.51}$$

Then, by lemma on change of variable formula for the component representation of optional semimartignales,

$$\begin{aligned}
\varphi_n(R_v) &= \varphi_n(R_0) + \varphi_n'(R)\left(\tilde{f}^2(Y^2) - \tilde{f}^1(Y^1)\right)\cdot a_v + \varphi_n'(R)\left(\tilde{g}(Y^2) - \tilde{g}(Y^1)\right)\cdot m_v \\
&+ \frac{1}{2}\varphi_n''(R)\left(\left|\tilde{g}(Y^2) - \tilde{g}(Y^1)\right|^2\right)\cdot \langle m, m\rangle_v \\
&+ \sum_j U\left[\varphi_n\left(R + \tilde{h}_j(Y^2) - \tilde{h}_j(Y^1)\right) - \varphi_n(R)\right] * (\mu^j - \nu^j)_v \\
&+ \sum_j U\left[\varphi_n\left(R + \tilde{h}_j(Y^2) - \tilde{h}_j(Y^1)\right) - \varphi_n(R) + \varphi_n'(R)\left(\tilde{h}_j(Y^2) - \tilde{h}_j(Y^1)\right)\right] * \nu_v^j \\
&+ \sum_j \left[\varphi_n\left(R + (\tilde{k}_j^2 + \tilde{l}_j^2)(Y^2) - (\tilde{k}_j^1 + \tilde{l}_j^1)(Y^1)\right) - \varphi_n(R)\right] * p_v^j \\
&+ \left[\varphi_n\left(R + \left(\tilde{r}^2 + \tilde{w}^2\right)(Y^2) - \left(\tilde{r}^1 + \tilde{w}^1\right)(Y^1)\right) - \varphi_n(R)\right] * \eta_v
\end{aligned}$$

Let

$$\begin{aligned}
I_1(v) &= \left[\varphi_n'(R)(\tilde{f}^2(Y^2) - \tilde{f}^1(Y^1))\right]\cdot a_v, \\
I_2(v) &= \frac{1}{2}\left[\varphi_n''(R)|\tilde{g}(Y^2) - \tilde{g}(Y^1)|^2\right]\cdot \langle m, m\rangle_v, \\
I_3(v) &= \sum_j U\left[\varphi_n\left(R + \tilde{h}_j(Y^2) - \tilde{h}_j(Y^1)\right) - \varphi_n(R) - \varphi_n'(R)\left(\tilde{h}_j(Y^2) - \tilde{h}_j(Y^1)\right)\right] * \nu_v^j \\
I_4(v) &= \sum_j \left[\varphi_n\left(R + (\tilde{k}_j^2 + \tilde{l}_j^2)(Y^2) - (\tilde{k}_j^1 + \tilde{l}_j^1)(Y^1)\right) - \varphi_n(R)\right] * p_v^j \\
I_5(v) &= \left[\varphi_n\left(R + \left(\tilde{r}^2 + \tilde{w}^2\right)(Y^2) - \left(\tilde{r}^1 + \tilde{w}^1\right)(Y^1)\right) - \varphi_n(R)\right] * \eta_v.
\end{aligned}$$

and write

$$
\begin{aligned}
\varphi_n(R_v) &= [\varphi_n'(R)(\tilde{g}(Y^2)-\tilde{g}(Y^1))]\cdot m_v \\
&\quad + \sum_j U\left[\varphi_n\left(R+\tilde{h}_j(Y^2)-\tilde{h}_j(Y^1)\right)-\varphi_n(R)\right]*(\mu^j-\nu^j)_v \\
&\quad + \sum_{\kappa=1}^{5} I_\kappa(v).
\end{aligned}
$$

Taking the expectation we get

$$
\mathbf{E}\varphi_n(R_v) = \mathbf{E}\sum_{\kappa=1}^{5} I_\kappa(v). \tag{3.52}
$$

Since $\tilde{f}^2(v, Y^2_{v-}) > \tilde{f}^1(v, Y^1_{v-})$ for $v < T$ and $|\varphi_n'| \le 1$, we have

$$
\mathbf{E}I_1(v) \le \mathbf{E}\left[\left(f^2(Y^2)-\tilde{f}^1(Y^1)\right)\cdot a_v\right].
$$

Further, by A3 and property (D7), the relations

$$
\begin{aligned}
\mathbf{E}|I_2(v)| &\le \frac{1}{2}\left(\max_{a_n\le x\le a_{n-1}}[\varphi_n''^2(|x|)\rho^2(|x|)]\right)\mathbf{E}\left[|G|^2\cdot\langle m,m\rangle_v\right] \\
&\le \frac{1}{n}\mathbf{E}|G|^2\cdot\langle m,m\rangle_\infty \to 0,\ \ n\to\infty,
\end{aligned}
$$

are valid.

Applying Taylor's formula, noting A4 and A8 and property (D7) we get

$$
\begin{aligned}
|I_3(v)| &\le \sum_j \frac{1}{2}\left[\left|\varphi_n''\left(R+\alpha(\tilde{h}_j(Y^2)-\tilde{h}_j(Y^1))\right)\right|\left|\tilde{h}_j(Y^2)-\tilde{h}_j(Y^1)\right|^2\right]*\nu^j_v \\
&\le \sum_j \frac{1}{2}\left[|H_j|^2\rho^2(|R|)|\varphi_n''(R+\alpha(\tilde{h}(Y^2)-\tilde{h}(Y^1)))|\right]*\nu^j_v,
\end{aligned}
$$

where $0\le\alpha\le 1$.

Consider here three cases:

(i) if $|R| \in [a_n, a_{n-1}]$, then by (D7) and A8,

$$|I_3| \leq \sum_j \frac{1}{2}|H_j|^2 * \nu^j_\infty \left(\frac{1}{n}\rho^2(a_{n-1})\rho^{-2}(a_{n-1} - \epsilon_n) \right)$$

(ii) if $|R| < a_n$, then, by A5 and A8,

$$\begin{aligned} |I_3| &\leq \sum_j \frac{1}{2}\rho^2(a_n) \left[|H_j|^2 \left| \varphi''_n \left(R + \alpha \left(\tilde{h}(Y^2) - \tilde{h}(Y^1) \right) \right) \right| \right] * \nu^j_\infty \\ &\leq \sum_j \frac{1}{2}|H|^2 * v^j_\infty \frac{1}{n}\rho^2(a_{n-1})\rho^{-2}(a_{n-1} - \epsilon_n); \end{aligned}$$

(iii) if $|R| > a_{n-1}$, then by A5 and property (D7) we find that $\varphi''_n(R+\alpha(\tilde{h}(Y^2)-\tilde{h}(Y^1))) = 0$ and $I_3 = 0$.

From what has been said it follows that

$$\mathbf{E}|I_3(v)| \leq \sum_j \frac{1}{2}\mathbf{E}|H_j|^2 * \nu^j_\infty \left(\frac{1}{n}\rho^2(a_{n-1})\rho^{-2}(a_{n-1} - \epsilon_n) \right) \to 0, \quad n \to \infty.$$

Again using Taylor's formula and noting that

$$(\tilde{k}^2_j + \tilde{l}^2_j)(v, \beta^1_v, Y^2_{v-}) > (\tilde{k}^1_j + \tilde{l}^1_j)(v, \beta^1_v, Y^1_{v-})$$

and

$$(\tilde{k}^2_2 + \tilde{l}^2_2)(v, \beta^2_v, Y^2_v) > (\tilde{k}^1_2 + \tilde{l}^1_2)(v, \beta^2_v, Y^1_v)$$

for $v < T$, we have

$$\begin{aligned} \mathbf{E}I_4(v) &= \mathbf{E}\sum_j \left[\varphi'_n \left(R + \alpha \left(\left(\tilde{k}^2_j + \tilde{l}^2_j \right)(Y^2) - \left(\tilde{k}^1_j + \tilde{l}^1_j \right)(Y^1) \right) \right) \left(\left(\tilde{k}^2_j + \tilde{l}^2_j \right)(Y^2) - \left(\tilde{k}^1_j + \tilde{l}^1_j \right)(Y^1) \right) \right] * p^j_v \\ &\leq \mathbf{E}\sum_j \left[\left(\tilde{k}^2_j + \tilde{l}^2_j \right)(Y^2) - \left(\tilde{k}^1_j + \tilde{l}^1_j \right)(Y^1) \right] * p^j_v. \end{aligned}$$

Now noting that $\mathbf{E}\left[\tilde{k}_j^2(Y^2)-\tilde{k}_j^1(Y^1)\right]*p_v^j=0$, we obtain

$$\begin{aligned}\mathbf{E}I_4(v) &\leq \mathbf{E}\sum_j\left[\tilde{l}_j^2(Y^2)-\tilde{l}_j^1(Y^1)\right]*p_v^j \\ &= \mathbf{E}\sum_j\left[\tilde{l}_j^2(Y^2)-\tilde{l}_j^1(Y^1)\right]*\lambda_v^j.\end{aligned}$$

Applying Taylor's formula one more time and noting that

$$\left(\tilde{r}^2+\tilde{w}^2\right)(v,\beta_v^2,Y_v^2)>\left(\tilde{r}^1+\tilde{w}^1\right)(v,\beta_v^2,Y_v^2)$$

for $v<T$, we have

$$\begin{aligned}\mathbf{E}I_5(v) &= \mathbf{E}\left[\varphi_n'\left(R+\alpha\left(\left(\tilde{r}^2+\tilde{w}^2\right)(Y^2)-\left(\tilde{r}^1+\tilde{w}^1\right)(Y^1)\right)\right)\left(\left(\tilde{r}^2+\tilde{w}^2\right)(Y^2)-\left(\tilde{r}^1+\tilde{w}^1\right)(Y^1)\right)\right]*\eta_v \\ &\leq \mathbf{E}\left[\left(\tilde{r}^2+\tilde{w}^2\right)(Y^2)-\left(\tilde{r}^1+\tilde{w}^1\right)(Y^1)\right]*\eta_v.\end{aligned}$$

Now noting that $\mathbf{E}\left[\tilde{r}^2(Y^2)-\tilde{r}^1(Y^1)\right]*\eta_v=0$, we obtain

$$\mathbf{E}I_5(v)\leq\mathbf{E}\left[\tilde{w}^2(Y^2)-\tilde{w}^1(Y^1)\right]*\eta_v=\mathbf{E}\left[\tilde{w}^2(Y^2)-\tilde{w}^1(Y^1)\right]*\theta_v.$$

by the estimates of $\mathbf{E}I_i,\ f=1,...,5$, and the fact that

$$\mathbf{E}\varphi_n(R)=\mathbf{E}\varphi_n(Y^2-Y^1)\uparrow\mathbf{E}\left|Y^2-Y^1\right|$$

as $n\to\infty$, we have from (8)

$$\begin{aligned}\mathbf{E}|Y_v^2-Y_v^1| &\leq \mathbf{E}\left[(\tilde{f}^2(Y^2)-\tilde{f}^1(Y^1))\cdot a_v+\sum_j(\tilde{l}_j^2(Y^2)-\tilde{l}_j^1(Y^1))*\lambda_v^j+\left(\tilde{w}^2(Y^2)-\tilde{w}^1(Y^1)\right)*\theta_v\right] \\ &= \mathbf{E}\left(Y_v^2-Y_v^1\right),\end{aligned}$$

where the last equality follows as a result of equation (3.50). Since the processes $Y^i,\ i=1,2$, are RLL, it follows from the derived inequality that off some set of $\mathbf{P}$-measure zero, $Y_v^2\geq Y_v^1$

(hence, also $X_v^2 \geq X_v^1$) for $v \leq T$ a.s. on the set B.

Now, how about we consider the quantity

$$\varrho = \inf\left(t > T : Y_t^2 < Y_t^1\right).$$

Let us show that $\varrho = \infty$ a.s. Naturally for $t < \varrho$ or on $[0, \varrho[$ the inequality $Y_t^2 \geq Y_t^1$ is valid a.s. whereas $]\varrho, \infty[$ $Y_t^2 < Y_t^1$ by definition of ϱ. Using (3.49) we obtain, that at time ϱ,

$$Y_\varrho^i = Y_{\varrho-}^i + \tilde{h}_1(\varrho, \beta_\varrho^1, Y_{\varrho-}^i)\mathbf{1}_{|\beta_\varrho^1| \leq 1} + \left(\tilde{k}_1^i + \tilde{l}_1^i\right)(\varrho, \beta_\varrho^1, Y_{\varrho-}^i).$$

And by A5 we obtain that $Y_\varrho^2 \geq Y_\varrho^1$. Hence, $Y_\varrho^2 \geq Y_\varrho^1$ is true a.s. on $[0, \varrho]$.

Now, let us introduce the sets

$$C = \left\{Y_\varrho^2 > Y_\varrho^1, \varrho < \infty\right\}, \quad D = \{Y_\varrho^2 = Y_\varrho^1, \varrho < \infty\}.$$

From the definition of ϱ and the processes Y^i, $i = 1, 2$, It follows that $\mathbf{P}(C) = 0$.

For the set D, we repeat the same arguments that was carried out, above, for B. Moreover, if $\varrho < \infty$, then there is a stopping time S, $\mathbf{P}(S > \varrho, \varrho < \infty) > 0$ such that $Y_t^2 \geq Y_t^1$ for $\varrho < t \leq S$. The latter will contradict the definition of ϱ. Hence, $\varrho = \infty$ a.s.

Now let us prove the theorem for the set A. Consider the quantity $\hat{\varrho} = \inf(t > 0 : X_t^2 < X_t^1)$, where the X^i, $i = 1, 2$, are the solutions of equations (3.47). For $t < \hat{\varrho}$ we have: $X_t^2 \geq X_t^1$ a.s. on A. Just as above, computing the jumps of the X^i at time ϱ and noting condition A5 we get: $X_{\hat{\varrho}}^2 \geq X_{\hat{\varrho}}^1$ a.s. on A.

Introduce the sets

$$\hat{C} = \left\{X_{\hat{\varrho}}^2 > X_{\hat{\varrho}}^1, \hat{\varrho} < \infty\right\}, \quad \hat{D} = \left\{X_{\hat{\varrho}}^2 = X_{\hat{\varrho}}^1, \hat{\varrho} < \infty\right\}.$$

From the definition of the stopping time $\hat{\varrho}$ and the right continuity of the processes X^i it follows that $\mathbf{P}(\hat{C}) = 0$. For the set $\hat{D}$ we repeat the arguments made in Section 1 for the set B, first carrying out a time shift by the quantity $\hat{\varrho}$, as was done in Lemma 2.2. We obtain: $X^2 \geq X^1$ a.s. on A. ■

The theorem is proved.

3.5.3 Financial Example

In this section we give an example showing how the stochastic domination/comparison theorems can be used in mathematical finance. For the purpose of demonstration we shall restrict our attention to the simplest cases only.

Example 74 *The constant elasticity of variance (CEV) model was proposed by Cox and Ross [50]. It is often used in mathematical finance to capture leverage effects and stochasticity of volatility. It is also widely used by practitioners in the financial industry for modeling equities and commodities. Consider a modified version of the CEV model where the stock price is said to satisfy the following integral equation,*

$$\begin{aligned} S_t &= \rho S \cdot A_t + \sigma S^\alpha \cdot M_t, \quad S_0 = s, \\ A_t &= t + V * \mu^1 + V * \mu^2 \\ M_t &= W_t + U * \left(\mu^1 - \nu^1\right)_t + U * \left(\mu^2 - \nu^2\right)_t \end{aligned} \tag{3.53}$$

where ρ and σ are constants and the martingale M is a jump-diffusion process with left and right jumps. W_t is the Wiener process, $\mu^1 - \nu^1$ is the measure of right jumps and $\mu^2 - \nu^2$ is the measure of left jumps. For $B \in \mathcal{B}(\mathbb{R}_+)$ and $\Gamma \in \mathcal{B}(\mathbb{E})$ the jump measures are defined

$$\begin{aligned} \mu^1 (B \times \Gamma) &: = \# \left\{ \left(t, \Delta L_t^1\right) \in B \times \Gamma | t > 0 \text{ such that } \Delta L_t^1 \neq 0 \right\} \\ \mu^2 (B \times \Gamma) &: = \# \left\{ \left(t, \Delta^+ L_t^2\right) \in B \times \Gamma | t > 0 \text{ such that } \Delta^+ L_t^2 \neq 0 \right\} \end{aligned}$$

where L_t^1 and L_t^2 are independent Poisson with constant intensities γ^1 and γ^2 respectively and compensators $\nu^1 = \gamma^1 t$ and $\nu^2 = \gamma^2 t$.

Let

$$\begin{aligned} F(x) &= \frac{1}{\sigma}\int_s^x u^{-\alpha}du = \frac{x^{1-\alpha}-s^{1-\alpha}}{\sigma(1-\alpha)}, \\ F'(x) &= \frac{x^{-\alpha}}{\sigma}, \quad F''(x) = \frac{-\alpha x^{-\alpha-1}}{\sigma}. \end{aligned}$$

where $0 < \alpha < 1$*. Denote* $X_t = F(S_t)$ *and applying Itô's formula we get*

$$\begin{aligned} X_t &= \rho S F'(S)\circ A_t + \sigma S^{\alpha}F'(S)\circ M_t + \frac{\sigma^2}{2}F''(Y)S^{2\alpha}\circ [M,M]_t \\ &= \frac{\rho}{\sigma}S_t^{1-\alpha}\circ A_t - \frac{\alpha\sigma}{2}S^{\alpha-1}\circ [M,M]_t + M_t \\ &= \frac{\rho}{\sigma}\left(\sigma(1-\alpha)X + s^{1-\alpha}\right)\circ A_t - \frac{\alpha\sigma}{2}\left(\sigma(1-\alpha)X + s^{1-\alpha}\right)^{-1}\circ [M,M]_t + M_t \end{aligned}$$

where $[M,M]_t = \left(1+\gamma^1+\gamma^2\right)t$*.*

With the comparison theorem proved above, we can give an estimate of the process X_t *from above by a new process* Y_t*, satisfying the equation,*

$$Y_t = \frac{\rho}{\sigma}\left(\sigma(1-\alpha)Y + s^{1-\alpha}\right)\cdot A_t + M_t, \quad Y_0 = 0$$

which is essentially an Ornstein-Uhlenbeck process with left and right jumps. Applying the comparison theorem to X_t *and* Y_t *yields that,* $Y_t \geq X_t = F(S_t)$ *a.s. therefore*

$$S_t \leq F^{-1}(Y_t) \qquad a.s. \tag{3.54}$$

Now lets consider an increasing function f *with an option payoff* $f(S_T)$*. Assuming zero interest rates, the price of such option is given by* $\tilde{\mathbf{E}}f(S_T)$ *for an appropriate martingale measure* $\tilde{P}$ *(see [18] where existence of* $\tilde{P}$ *is discussed). Using inequality (3.54) we have that* $\tilde{\mathbf{E}}f(S_T) \leq \tilde{\mathbf{E}}f(F^{-1}(Y_T))$ *and thus we obtain an estimate for the option price for which* $\tilde{\mathbf{E}}f(F^{-1}(Y_T))$ *is easier to compute.*

Chapter 4

Financial Markets on Unusual Spaces

In current theories of mathematical finance, financial markets are modeled by probability spaces that satisfy the usual conditions and market processes that are right-continuous semimartingales. In this framework, two of the most fundamental problems of finance were considered. These are the problem of portfolio optimization and hedging and pricing of contingent claims. In both cases, the existence of a risk-neutral martingale deflators (measures) is a way to thwart arbitrage (see [51, 52, 53, 54, 55, 56, 57]). Hence, finding martingale deflators became the central goal of arbitrage pricing theory and at large mathematical finance.

Here we develop a general model of financial markets based on optional semimartingales on *unusual probability spaces*. Our motivation is to expand our understanding of financial markets and in turn introduce new problems in the mathematic of stochastic processes. We propose, a rational framework for these markets and develop methods and tools for this purpose. The chapter begins by defining a general optional semimartingale market and portfolios. Then, several methods for finding martingale transforms (defalors) are presented. Hereinafter, we study pricing and hedging in these markets and give several examples including: Black-Scholes with left and right jumps and pricing of European call option, a protfolio of defaultable bond and a stock, an instrument with the option to trade its dividends and debit repayment problem.

The results we present here are largely based on the work of Abdelghani and Melnikov [18].

4.1 The Market Model

Our market consists of two types of securities x and X and a portfolio $\pi = (\eta, \xi)$ which is composed of the optional processes η and ξ. η is the volume of the reference asset x while ξ is the volume of the security X. Suppose $x_t > 0$ and $X_t \geq 0$ for all $t \geq 0$ and write the ratio process $R_t = X_t/x_t$. Then, the value of the portfolio is

$$Y_t = \eta_t + \xi_t R_t. \tag{4.1}$$

Furthermore, we restrict the portfolio, π, to be self-financing; that is we must have,

$$Y_t = Y_0 + \xi \circ R_t. \tag{4.2}$$

Reconciling equations (4.1) and (4.2) we get

$$C_t = \eta_t + R \circ \xi_t + [\xi, R]_t = C_0.$$

where C_t is the consumption process with its initial value C_0. Since the ratio process R is optional semimartingale then ξ, evolves in the space $\mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$ with the predictable part determining the volume of R^r and the optional part determining the volume of R^g. R^r is the right-continuous part of X/x and R^g is its left-continuous part. Also, η belongs to the space $\mathcal{O}(\mathbf{F})$. Furthermore, for the integral in equation (4.2) to be well defined ξ must be R-integrable,

$$\int_0^\infty \xi_s^2 d[R, R]_s \in \mathcal{A}_{loc}.$$

Remark 75 *It is also possible to write the portfolio as $Z_t = \eta_t x_t + \xi_t X_t$ where $Y = Z/x$. However, in this case the volume of the reference asset, η is a process that evolves in $\mathcal{P}(\mathbf{F}) \times$*

$\mathcal{O}(\mathbf{F})$. From this we consider there a variety of portfolios that are possible in unusual financial markets.

4.2 Martingale Transforms

Here we present methods for finding local martingale transforms (deflators) for these markets. A local martingale deflator is a strictly positive supermartingale multiplier used to transform the value process of a portfolio to a supermartingale (i.e. a local martingale). It is important for finding a fiar-price and hedging strategy for a claim.

4.2.1 The Stochastic Exponentials Approach

We suppose that the dynamics of securities in our market follow the stochastic exponentials,

$$\begin{aligned} X_t &= X_0 \mathcal{E}_t(H), \\ x_t &= x_0 \mathcal{E}_t(h) \end{aligned}$$

where x_0 and X_0 are $\mathcal{F}_0$-measurable random variables. $h = (h_t)_{t \geq 0}$ and $H = (H_t)_{t \leq 0}$ are optional semimartingales admitting the representations,

$$\begin{aligned} h_t &= h_0 + a_t + m_t, \\ H_t &= H_0 + A_t + M_t \end{aligned}$$

with respect to (w.r.t) $\mathbf{P}$. $a = (a_t)_{t \geq 0}$ and $A = (A_t)_{t \geq 0}$ are locally bounded variation processes and predictable. $m = (m_t)_{t \geq 0}$ and $M = (M_t)_{t \geq 0}$ are optional local martingales.

First, we shall study, when the ratio process, R, is a local optional martingale w.r.t. the initial measure $\mathbf{P}$, i.e. When is $R \in \mathcal{M}_{loc}(\mathbf{F}, \mathbf{P})$ for $t \geq 0$? R in exponential form can be written as

$$R_t = \frac{X_t}{x_t} = R_0 \mathcal{E}(H)_t \mathcal{E}^{-1}(h)_t,$$

and using the properties of stochastic exponentials [21] we find

$$\begin{aligned} R_t &= R_0\mathcal{E}(H_t)\mathcal{E}^{-1}(h_t) = R_0\mathcal{E}(H_t)\mathcal{E}(-h_t^*) \\ &= R_0\mathcal{E}(H_t - h_t^* - [H, h^*]_t) \end{aligned}$$

where

$$\begin{aligned} H_t - h_t^* - [H, h^*]_t &= H_t - h_t + \langle h^c, h^c\rangle_t + \sum_{0<s\le t} \frac{(\triangle h_s)^2}{1+\triangle h_s} \\ &+ \sum_{0\le s<t} \frac{(\triangle^+ h_s)^2}{1+\triangle^+ h_s} - \langle H^c, h^c\rangle_t - \sum_{0<s\le t} \triangle H_s \triangle h_s^* \\ &- \sum_{0\le s<t} \triangle^+ H_s \triangle_s^+ h^* \\ &= H_t - h_t + \langle h^c, h^c\rangle_t + \sum_{0<s\le t} \frac{(\triangle h_s)^2}{1+\triangle h_s} \\ &+ \sum_{0\le s<t} \frac{(\triangle^+ h_s)^2}{1+\triangle^+ h_s} - \langle H^c, h^c\rangle_t - \sum_{0<s\le t} \triangle H_s \frac{\triangle h_s}{1+\triangle h_s} \\ &- \sum_{0\le s<t} \triangle^+ H_s \frac{\triangle^+ h_s}{1+\triangle^+ h_s} \\ &= H_t - h_t + \langle h^c, h^c - H^c\rangle_t + \sum_{0<s\le t} \frac{\triangle h_s(\triangle h_s - \triangle H_s)}{1+\triangle h_s} \\ &+ \sum_{0\le s<t} \frac{\triangle^+ h_s\,(\triangle^+ h_s - \triangle^+ H_s)}{1+\triangle^+ h_s}. \end{aligned}$$

Let $J^g = \sum_{0\le s<t} \frac{\triangle^+ h_s(\triangle^+ h_s - \triangle^+ H_s)}{1+\triangle^+ h_s}$, $J^d = \sum_{0<s\le t} \frac{\triangle h_s(\triangle h_s - \triangle H_s)}{1+\triangle h_s}$ and write

$$\Psi(h_t, H_t) = H_t - h_t + \langle h^c, h^c - H^c\rangle_t + J_t^d + J_t^g.$$

Then, the ratio R satisfies

$$R_t = R_0 + \int_{0+}^{t} R_{s-} d\Psi_s + \int_0^{t-} R_s d\Psi_{s+}.$$

Considering the properties of stochastic integrals we get $\Psi(h, H) \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F}) \Rightarrow R \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$

and if $\triangle^{+}\Psi \neq -1$ and $\triangle\Psi \neq -1$ then $\Psi(h, H) \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F}) \Leftrightarrow R \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$. Given the decomposition of h and H one can write,

$$\begin{aligned}\Psi(h, H) &= (A-a) + \langle m^c, m^c - M^c \rangle + (M-m) + J^d + J^g \\ &= (A-a) + \langle m^c, m^c - M^c \rangle + \tilde{J}^d + \tilde{J}^g + (M-m) \\ &+ \left(J^g - \tilde{J}^g\right) + \left(J^d - \tilde{J}^d\right),\end{aligned} \tag{4.3}$$

where $(M-m) + \left(J^g - \tilde{J}^g\right) + \left(J^d - \tilde{J}^d\right) \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$. Thus, $\Psi(h, H) \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$ if and only if

$$(A-a) + \langle m^c, m^c - M^c \rangle + \tilde{J}^d + \tilde{J}^g = 0.$$

If Ψ is a local optional martingale then R is a local optional martingale and we are done. Otherwise, we have to find a strictly positive transformation $Z \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$ that will render $ZR \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$. Z is known as the local martingale transform or deflator. Since our market ratio process R is positive by definition and it *would not make any financial sense* to search for a local martingale transform Z that leads to $Z_t R_t \leq 0$ for some t when $R_t > 0$ for all t. Therefore, we will restrict ourselves to a set of possible local martingale transforms that are strictly positive, i.e. $Z > 0$ a.s. $\mathbf{P}$. For a strictly positive Z, we can define $N \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$ with $N = \mathcal{L}\text{og}(Z) = Z^{-1} \circ Z$ or $Z = \mathcal{E}(N)$. To find N we have the following theorem;

Theorem 76 *Given $R = R_0 \mathcal{E}(\Psi(h, H))$ where $\Psi(h, H)$ as in equation (4.3) and $Z = \mathcal{E}(N)$ where Z, $N \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$ and $Z > 0$ then $ZR \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$ is a local optional martingale if and only if*

$$(A-a) + \langle m^c - N^c, m^c - M^c \rangle + \tilde{K}^d + \tilde{K}^g = 0,$$

where $\tilde{K}^d$ and $\tilde{K}^g$ are the compensators of the processes

$$\begin{aligned} K^d &= \sum_{0<s\leq t} \frac{(\triangle h_s - \triangle N_s)(\triangle h_s - \triangle H_s)}{1 + \triangle h_s}, \\ K^g &= \sum_{0\leq s<t} \frac{(\triangle^{+} h_s - \triangle^{+} N_s)(\triangle^{+} h_s - \triangle^{+} H_s)}{1 + \triangle^{+} h_s}. \end{aligned}$$

Proof. Suppose $Z_t = \mathcal{E}(N)_t \in \mathcal{M}_{loc}(\mathbf{P},\mathbf{F})$, $Z_t > 0$ for all t such that $ZR \in \mathcal{M}_{loc}(\mathbf{P},\mathbf{F})$ then

$$\begin{aligned} ZR &= R_0\mathcal{E}(N)\mathcal{E}(\Psi(h,H)) = \mathcal{E}(N + \Psi(h,H) + [N,\Psi(h,H)]) \\ &= R_0\mathcal{E}(\Psi(h,H,N)), \end{aligned}$$

where

$$\begin{aligned} \Psi(h,H,N) &= N_t + H_t - h_t + \langle h^c, h^c - H^c\rangle_t + J^d_t + J^g_t \\ &+ [N,H] - [N,h] + [N,J^d] + [N,J^g] \\ &= N_t + H_t - h_t + \langle h^c, h^c - H^c\rangle_t + J^d_t + J^g_t \\ &+ \langle N^c, H^c\rangle_t + \sum_{0<s\le t} \triangle N_s \triangle H_s + \sum_{0\le s<t} \triangle^+ N_s \triangle^+ H_s \\ &- \langle N^c, h^c\rangle_t - \sum_{0<s\le t} \triangle N_s \triangle h_s - \sum_{0\le s<t} \triangle^+ N_s \triangle^+ h_s \\ &+ \sum_{0<s\le t} \triangle N_s \frac{\triangle h_s(\triangle h_s - \triangle H_s)}{1+\triangle h_s} \\ &+ \sum_{0\le s<t} \triangle^+ N_s \frac{\triangle^+ h_s\,(\triangle^+ h_s - \triangle^+ H_s)}{1+\triangle^+ h_s} \end{aligned}$$

hence,

$$\begin{aligned} \Psi(h,H,N) &= N_t + H_t - h_t + \langle h^c - N^c, h^c - H^c\rangle_t \\ &+ \sum_{0<s\le t} \frac{\triangle h_s(\triangle h_s - \triangle H_s)}{1+\triangle h_s} + \triangle N_s\,(\triangle H_s - \triangle h_s) \\ &+ \triangle N_s \frac{\triangle h_s(\triangle h_s - \triangle H_s)}{1+\triangle h_s} \\ &+ \sum_{0<s\le t} \frac{\triangle^+ h_s\,(\triangle^+ h_s - \triangle^+ H_s)}{1+\triangle^+ h_s} + \triangle^+ N_s\left(\triangle^+ H_s - \triangle^+ h_s\right) \\ &+ \triangle^+ N_s \frac{\triangle^+ h_s\,(\triangle^+ h_s - \triangle^+ H_s)}{1+\triangle^+ h_s}. \end{aligned}$$

therefore,

$$\begin{aligned}\Psi(h,H,N) &= N_t + H_t - h_t + \langle h^c - N^c, h^c - H^c\rangle_t \\ &+ \sum_{0<s\le t} \frac{(\triangle h_s - \triangle N_s)(\triangle h_s - \triangle H_s)}{1+\triangle h_s} \\ &+ \sum_{0<s\le t} \frac{(\triangle^+ h_s - \triangle^+ N_s)(\triangle^+ h_s - \triangle^+ H_s)}{1+\triangle^+ h_s}.\end{aligned}$$

Let

$$\begin{aligned}K^d &= \sum_{0<s\le t} \frac{(\triangle h_s - \triangle N_s)(\triangle h_s - \triangle H_s)}{1+\triangle h_s}, \\ K^g &= \sum_{0\le s\le t} \frac{(\triangle^+ h_s - \triangle^+ N_s)(\triangle^+ h_s - \triangle^+ H_s)}{1+\triangle^+ h_s}\end{aligned}$$

and write

$$\Psi(h,H,N) = N_t + H_t - h_t + \langle h^c - N^c, h^c - H^c\rangle_t + K^d + K^g.$$

So, if $\Psi(h,H,N) \in \mathcal{M}_{loc}(\mathbf{P},\mathbf{F})$ then $ZR \in \mathcal{M}_{loc}(\mathbf{P},\mathbf{F})$. And if $\triangle^+\Psi(h,H,N) \neq -1$ and $\triangle\Psi(h,H,N) \neq -1$ then $\Psi(h,H,N) \in \mathcal{M}_{loc}(\mathbf{P},\mathbf{F}) \Leftrightarrow ZR \in \mathcal{M}_{loc}(\mathbf{P},\mathbf{F})$. Now, let us take into consideration the decomposition of H and h and write

$$\Psi(h,H,N) = (A-a) + (M-m+N) + \langle (m-N)^c, (m-M)^c\rangle + K^d + K^g.$$

So, $\Psi(h,H,N)$ is a local optional martingale under $\mathbf{P}$ if

$$(A-a) + \langle m^c - N^c, m^c - M^c\rangle + \tilde{K}^d + \tilde{K}^g = 0 \tag{4.4}$$

where $\tilde{K}^d$ and $\tilde{K}^g$ are the compensators of K^d and K^g, respectively. ■

By finding all $N \in \mathcal{M}_{loc}(\mathbf{P},\mathbf{F})$ such that the above equation (4.4) is valid and $\mathcal{E}(N) > 0$ we find the set of all appropriate local optional martingale transforms Z such that ZR is a local optional martingale. Note that if Z is a local martingale transform such that ZR is a local martingale then it is true for all self financing strategies π.

Theorem 77 *If Z is a local martingale transform of R, that is ZR is a local optional martingale, and π is a self financing portfolio which is R-integrable then ZY_t^π is a local optional martingale.*

Proof. Z is a local martingale transform of R therefore $Z > 0$. $\pi = (\eta, \xi)$ is self financing and R-integrable, then $Y_t^\pi = Y_0 + \xi \circ R_t$ and $Z_t Y_t^\pi$ can be written as,

$$\begin{aligned} d\left(Z_t Y_t^\pi\right) &= Z_t dY_t^\pi + dZ_t Y_t^\pi + d\left[Z_t, Y_t^\pi\right] \\ &= Z_t \xi_t dR_t + dZ_t\left(\eta_t + \xi_t R_t\right) + d\left[Z_t, Y_t^\pi\right] \\ &= Z_t \xi_t dR_t + dZ_t \xi_t R_t + dZ_t \eta_t + \xi_t d\left[Z_t, R_t\right] \\ &= \xi_t\left(Z_t dR_t + dZ_t R_t + d\left[Z_t, R_t\right]\right) + dZ_t \eta_t \\ &= \xi_t d\left(Z_t R_t\right) + dZ_t \eta_t. \end{aligned}$$

This leads us to the following result

$$Z_t Y_t^\pi = \xi \circ Z_t R_t + \eta \circ Z_t.$$

$\eta \circ Z_t$ and $\xi \circ Z_t R_t$ are local optional martingales therefore their sum $Z_t Y_t^\pi$ is a well defined local optional martingale. Note, that we have implicitly used the fact that η is bounded, i.e. comes from the fact that π is a self financing and also that,

$$\int_0^\infty \xi_t^2 d\left[ZR\right]_t \in \mathcal{A}_{loc}.$$

■

Remark 78 *On the other hand, if we know that there exist a Z such that ZY^π is a local optional martingale then what can we say about the portfolio π and the product ZR? It is* *reasonable to suppose that $Z = \mathcal{E}(N) > 0$, π-self-financing, ξ is R-integrable and η is bounded. In this case, $\xi \circ Z_t R_t = Z_t Y_t^\pi - \eta \circ Z_t$ is a sum of two local optional martingales and therefore a local optional martingale itself, for any optional process ξ, in particular for $\xi = 1$; therefore ZR is a local optional martingale.*

Remark 79 *If $Z \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$ and $Z > 0$ then one can define $\tilde{\mathbf{P}}_t = \int_\Omega Z_t d\mathbf{P}_t$ is a new measure equivalent to $\mathbf{P}$, i.e. $\tilde{\mathbf{P}} \overset{loc}{\sim} \mathbf{P}$, and $Z_t = \frac{d\tilde{P}_t}{dP_t}$.*

Now, is there a way to construct N knowing equation (4.4)? We begin by making an educated guess, choosing N as,

$$N = [\alpha_c \;\; \alpha_d \;\; \alpha_g] \circ \left[m^c \;\; m^d \;\; m^g\right]^\mathsf{T} + \left[\beta_c \;\; \beta_d \;\; \beta_g\right] \circ \left[M^c \;\; M^d \;\; M^g\right]^\mathsf{T}, \tag{4.5}$$

where $\alpha_c, \beta_c, \alpha_d, \beta_d \in \mathcal{P}(\mathbf{F})$ and $\alpha_g, \beta_g \in \mathcal{O}(\mathbf{F})$. Note, with α's and β's the local optional martingales (m, M) span a subspace of the local optional martingales space $\mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$. So if N takes the representation above (4.5) then what is the solution of equation (4.4)? Substitute for N in (4.4) by equation (4.5) we get

$$\begin{aligned}
\langle m^c - N^c, m^c - N^c \rangle &= \left(1 - 2\alpha_c + \alpha_c^2\right) \langle m^c, m^c \rangle + \beta_c^2 \langle M^c, M^c \rangle, \\
K^d &= \sum_{0<s\le t} \frac{\left((1-\alpha_r)\triangle m_s^r - \beta_r \triangle M_s^r\right)\left(\triangle m_s^r - \triangle M_s^r\right)}{1 + \triangle m_s^r}, \\
K^g &= \sum_{0<s\le t} \frac{\left((1-\alpha_g)\triangle^+ m_s^g - \beta_g \triangle^+ M_s^g\right)\left(\triangle^+ m_s^g - \triangle^+ M_s^g\right)}{1 + \triangle^+ m_s^g}.
\end{aligned}$$

The compensator $\left(\tilde{K}^d, \tilde{K}^g\right)$ of K^d, K^g are hard to compute in general and will have to be evaluated on a problem by problem basis.

4.2.2 The Stochastic Logarithm Approach

Here we consider an alternative approach for finding local martingale deflators, using the methods of stochastic logarithms. What is interesting about this approach is that we don't have to define the process R as a stochastic exponential of an underlying process Ψ. All that is required is that the ratio process R and its predictable version R_{-} not to vanish, except on sets of measures zero.

Consider the following lemmas;

Lemma 80 *Suppose $X = X + A + M$, $x = x + a + m$, $R = X/x$ and $R_{-} \neq 0$ and $R \neq 0$ a.s. $\mathbf{P}$, then $\mathcal{L}\mathrm{og}(R)$ is a local optional martingale if and only if*

$$\frac{1}{X} \circ A - \frac{1}{x} \circ a + \frac{1}{x^2}[m, m] - \frac{1}{xX} \circ [M, m] = -1.$$

Proof. Consider $\mathcal{L}\mathrm{og}\,(R_t)$; using Gal'chuk lemma and properties of stochastic logarithm,

$$\begin{aligned}
\mathcal{L}\mathrm{og}\,(R_t) &= \mathcal{L}\mathrm{og}\,(X_t) + \mathcal{L}\mathrm{og}\left(\frac{1}{x_t}\right) + \left[\mathcal{L}\mathrm{og}\,(X)\,, \mathcal{L}\mathrm{og}\left(\frac{1}{x}\right)\right]_t \\
&= \mathcal{L}\mathrm{og}\,(X_t) + 1 - \mathcal{L}\mathrm{og}\,(x_t) - \left[x, \frac{1}{x}\right]_t \\
&\quad + \left[\mathcal{L}\mathrm{og}\,(X)\,, 1 - \mathcal{L}\mathrm{og}\,(x) - \left[x, \frac{1}{x}\right]\right]_t \\
&= 1 + \mathcal{L}\mathrm{og}\,(X_t) - \mathcal{L}\mathrm{og}\,(x_t) - \left[x, \frac{1}{x}\right]_t - \left[\mathcal{L}\mathrm{og}\,(X)\,, \mathcal{L}\mathrm{og}\,(x)\right]_t \\
&= 1 + \frac{1}{X} \circ X_t - \frac{1}{x} \circ x_t - \left[x, \frac{1}{x}\right]_t - \frac{1}{xX} \circ [X, x]_t\,.
\end{aligned}$$

knowing that $x^{-1} = -x^{-2} \circ x + 2/3x^{-3} \circ [x, x]$ then $\mathcal{L}\mathrm{og}\,(R_t) \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$ if

$$\frac{1}{X} \circ A - \frac{1}{x} \circ a + \frac{1}{x^2}[m, m] - \frac{1}{xX} \circ [M, m] = -1.$$

■

Lemma 81 *Suppose that R_{-} and R don't vanish then $\mathcal{L}\mathrm{og}(R) \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F}) \Leftrightarrow R \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$.*

Proof. Suppose $R \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$; since $\mathcal{L}\mathrm{og}\,(R) = R^{-1} \circ R$ and $R^{-1} \circ R$ is a local martingale then $\mathcal{L}\mathrm{og}(R) \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$. Now suppose that $\mathcal{L}\mathrm{og}(R) \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$ then $R^{-1} \circ R$ is a local martingale and since R_{-} and R don't vanish then it must be that $R \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$. ■

If R is not a local martingale then one can find a local martingale deflator $Z > 0$ such that ZR is a local martingale. The following lemma helps with finding local martingale deflators.

Lemma 82 *Let $X = X + A + M$, $x = x + a + m$, $R = X/x$ and suppose that R_- and R don't vanish then $\mathcal{L}\text{og}(ZR)$ is a local optional martingale if and only if*

$$1 + \frac{1}{X} \circ A - \frac{1}{x} \circ a + \frac{1}{x^2} \circ [m, m] - \frac{1}{xX} \circ [M, m] + \frac{1}{ZX} \circ [Z, M] - \frac{1}{xZ} \circ [Z, m] = -1$$

furthermore if $Z = \mathcal{E}(N) > 0$ then

$$\frac{1}{X} \circ A - \frac{1}{x} \circ a + \frac{1}{x^2} \circ [m, m] - \frac{1}{xX} \circ [M, m] + \frac{1}{X} \circ [N, M] - \frac{1}{x} \circ [N, m] = -1.$$

Proof. Consider $\mathcal{L}\text{og}\,(ZR)$, using Galchuk lemma and properties of stochastic logarithms,

$$\begin{aligned}
\mathcal{L}\text{og}\,(ZR) &= \mathcal{L}\text{og}\,(Z) + \mathcal{L}\text{og}\,(R) + [\mathcal{L}\text{og}\,(Z)\,, \mathcal{L}\text{og}\,(R)] \\
&= \mathcal{L}\text{og}\,(Z) + 1 + \mathcal{L}\text{og}\,(X) - \mathcal{L}\text{og}\,(x) - \left[x, \frac{1}{x}\right] \\
&\quad - [\mathcal{L}\text{og}\,(X)\,, \mathcal{L}\text{og}\,(x)] \\
&\quad + \left[\mathcal{L}\text{og}\,(Z)\,, 1 + \mathcal{L}\text{og}\,(X) - \mathcal{L}\text{og}\,(x) - \left[x, \frac{1}{x}\right]\right] \\
&\quad - [\mathcal{L}\text{og}\,(Z)\,, [\mathcal{L}\text{og}\,(X)\,, \mathcal{L}\text{og}\,(x)]] \\
&= 1 + \mathcal{L}\text{og}\,(Z) + \mathcal{L}\text{og}\,(X) - \mathcal{L}\text{og}\,(x) - \left[x, \frac{1}{x}\right] \\
&\quad - [\mathcal{L}\text{og}\,(X)\,, \mathcal{L}\text{og}\,(x)] \\
&\quad + [\mathcal{L}\text{og}\,(Z)\,, \mathcal{L}\text{og}\,(X)] - [\mathcal{L}\text{og}\,(Z)\,, \mathcal{L}\text{og}\,(x)] \\
&= 1 + \frac{1}{X} \circ X - \frac{1}{x} \circ x - \left[x, \frac{1}{x}\right] - \frac{1}{xX} \circ [X, x] \\
&\quad + \frac{1}{ZX} \circ [Z, X] - \frac{1}{xZ} \circ [Z, x]
\end{aligned}$$

then

$$\frac{1}{X} \circ A - \frac{1}{x} \circ a + \frac{1}{x^2} \circ [m, m] - \frac{1}{xX} \circ [M, m] + \frac{1}{ZX} \circ [Z, M] - \frac{1}{xZ} \circ [Z, m] = -1$$

and when $Z = \mathcal{E}(N)$ we get

$$\begin{aligned}
\mathcal{L}\text{og}\,(ZR) &= 1 + \mathcal{L}\text{og}\,(Z) + \mathcal{L}\text{og}\,(X) - \mathcal{L}\text{og}\,(x) - \left[x, \frac{1}{x}\right] \\
&\quad - \left[\mathcal{L}\text{og}\,(X), \mathcal{L}\text{og}\,(x)\right] \\
&\quad + \left[\mathcal{L}\text{og}\,(Z), \mathcal{L}\text{og}\,(X)\right] - \left[\mathcal{L}\text{og}\,(Z), \mathcal{L}\text{og}\,(x)\right] \\
&= 1 + N + \frac{1}{X} \circ X - \frac{1}{x} \circ x + \frac{1}{x^2} \circ [m, m] \\
&\quad - \frac{1}{xX} \circ [X, x] + \frac{1}{X} \circ [N, X] - \frac{1}{x} \circ [N, x]
\end{aligned}$$

then $\mathcal{L}\text{og}\,(ZR)$ is a local martingale if

$$N + \frac{1}{X} \circ A - \frac{1}{x} \circ a + \frac{1}{x^2} \circ [m, m] - \frac{1}{xX} \circ [M, m] + \frac{1}{X} \circ [N, M] - \frac{1}{x} \circ [N, m] = -1$$

■

Lemma 83 *Suppose that R_- and R don't vanish then $\mathcal{L}\text{og}(ZR) \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F}) \Leftrightarrow ZR \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$.*

Proof. Similar to lemma (82) ■

4.3 Pricing and Hedging

A contingent claim is an integrable or square-integrable random variable, $\Lambda \in \mathcal{F}$. Λ generates the optional local martingale process $\Lambda_t = \mathbf{E}[\Lambda|\mathcal{F}_t]$ for $t \in [0, T]$, for some final time T. In the market (x, X), for Λ to be priced and for there to exist a hedge portfolio, Λ_t, must admit an integral representation in terms of the ratio process R. If R is a local martingale then Gal'chuk [11] theorem 2.3 gives that for any $(\alpha, \beta) \in \mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$, $(\alpha, \beta) \circ R$ is again a local optional martingale in $(\mathbf{F}, \mathbf{P})$.

So, let $\pi = (\eta, \xi)$ be a general portfolio where $\xi = (\alpha, \beta) \in \mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$; Hence $\pi \in$

$\mathcal{O}(\mathbf{F}) \times \mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$. Let

$$\begin{aligned} \mathfrak{R} &= \{(\alpha,\beta)\circ R : \ \xi = (\alpha,\beta) \in \mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F}) \\ &\quad , (\eta,\xi) \text{ is self-financing } \eta \in \mathcal{O}(\mathbf{F}) \text{ and } \xi \text{ is } R\text{-integrable}\}, \end{aligned}$$

be the space of local optional martingales *generated by* the ratio process R. Furthermore, let the space $\mathfrak{A} \subseteq \mathcal{O}(\mathbf{F}) \times \mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$ be the set of all admissible portfolios such that $(\alpha,\beta)\circ R \in \mathfrak{R}$.

If $\Lambda \in \mathfrak{R}$ then there exist a portfolio $\pi^\Lambda = \left(\eta^\Lambda, \alpha^\Lambda, \beta^\Lambda\right)$ such that

$$\begin{gathered} \Lambda_t = \Lambda_0 + \int_0^t \xi_t^\Lambda dR_t = \Lambda_0 + \int_{0+}^t \alpha_{t-}^\Lambda dR_t^r + \int_0^{t-} \beta_t^\Lambda dR_{t+}^g, \qquad (4.6) \\ \Lambda_t = \eta_t^\Lambda + \xi_t^\Lambda R_t = \eta_t^\Lambda + \alpha_t^\Lambda R_t^r + \beta_t^\Lambda R_t^g, \ \textit{and} \\ C_t = \eta_t + R \circ \xi_t^\Lambda + [\xi^\Lambda, R]_t = C_0. \end{gathered}$$

The processes ξ^Λ together with η^Λ form the hedge portfolio for Λ in the market (x, X) and Λ_t is the value of the claim over time $t \in [0,T]$ such that $\Lambda_T = \Lambda$ and the claim price is Λ_0.

The portfolio $\left(\eta^\Lambda, \alpha^\Lambda, \beta^\Lambda\right)$ is not unique. In a sense, there exist many portfolios such that $\Lambda_T = \Lambda$. This implies that that there are many possible initial fare prices for the contingent claim Λ. Therefore, unusual stochastic markets are *fundamentally incomplete*. Furthermore, $\left(\eta^\Lambda, \alpha^\Lambda, \beta^\Lambda\right)$ is not like traditional hedging portfolios of predictable integrands; η^Λ and β^Λ are optional processes. Traditional portfolios can be approximated by a *single-simple-trading strategy* over the underlying asset, see [17] and references therein. However, ξ^Λ can not be approximated by a single-simple-trading strategy since α^Λ is predictable and β^Λ is optional. Also, it cannot be traded by an agent in the current structure of financial markets. The problem is that components of the price process R, the right-continuous part R^r and the left-continuous part R^g have to be traded differently. Moreover, even if we let $\phi \in \mathcal{O}(\mathbf{F})$ and consider a restricted version of ξ^Λ in which $\xi^\Lambda = (\alpha^\Lambda, \beta^\Lambda) = \left(\phi_-, \phi\right)$ where $\phi_- \in \mathcal{P}(\mathbf{F})$, even in this case, the integrals in equation (4.10) cannot be approximated by a single simple trading strategy. Indeed, Galchuk [11] optional semimartingales integration theory showed that α^Λ and β^Λ integrands can only be approximated by two simple functions in $\mathsf{L}^2(\langle R^r\rangle, \langle R^g\rangle)$ or $\mathsf{H}^2([R^r], [R^g])$. The real problem for these portfolios is not α^Λ, because α^Λ is predictable, therefore a possible trading strategy in the classical view of financial markets. But β^Λ is optional and is the source of the "problem" in our classical view of financial systems. Since β^Λ may not be possible! then we are lead to the fact that only a subset of contingent claims in optional semimartingale markets can be hedged by predictable portfolios α which is less than total set of contingent claims possible in this market. Therefore, once more, we arrive at the conclusion that optional semimartingale markets are *inherently incomplete*.

However, there is an alternative viewpoint. For a portfolio $\pi = (\eta, \xi)$ with consumption, C. The consumption process C is understood to be the dynamic addition, either of, funds, spending, dividends, charity payouts, commission payments, tax payments or debit repayments: that is of-course under the usual assumptions, R is RCLL local martingale and C an increasing/decreasing predictable process or in some cases optional finite variation process, the value process Y is either a supermartingale or submartingale admitting the representation $Y = Y_0 + \xi \cdot R - C$.

In the context of optional semimartingale market (x, X) under the risk neural measure where the ratio process R is a local optional martingale we can describe the value process by,

$$Y = Y_0 + \alpha \cdot R^r + \beta \odot R^g - C. \tag{4.7}$$

If C is an RLL increasing/decreasing strongly predictable process then Y is an optional super/submartingale by decomposition of optional super/submartingale [10]. If C is finite variation then Y is a semimartingale with its decomposition having the same form as equation (4.7). We can in this case consider $\beta \in \mathcal{O}(\mathbf{F})$ as part of a general consumption plan,

$$D = C - \beta \odot R^g,$$

and consider the following optimization-hedging problem,

$$
\begin{aligned}
u &= \min_{(\alpha,D)\in\mathfrak{A}} \mathbf{E}\left[(\Lambda - \alpha \cdot R^r_T + D_T)^2\right], \\
D_T &= \beta \odot R^g_T - C_T \in \mathcal{O}(\mathbf{F}), \\
\alpha &\in \mathcal{P}(\mathbf{F}),
\end{aligned}
$$

where Λ is the contingent claim we like to hedge, u is its optimal price, (α, D) is the admissible hedge and $\mathfrak{A}$ the set of admissible trade-consumption plans. This optimization problem is interesting but its solution is out of the scope of this work.

Next we consider financial examples.

4.4 Financial Examples

Here we present examples of a value process Y^π that is a jump diffusion with a combination of left and right jumps, corresponding to a portfolio π. and evolving on unusual probability space. A value process such as Y^π can be the result either one of the following market structures that we present below.

4.4.1 Ladlag Jumps Diffusion Model

Let us consider the augmented Black-Scholes model with left and right jumps,

$$
\begin{aligned}
x_t &= x_0 + \int_{0+}^{t} r x_s ds \qquad (4.8)\\
X_t &= X_0 + \int_{0+}^{t} X_{s-}\left(\mu ds + \sigma dW_s + a dL^r_s\right) + \int_0^{t-} b X_s dL^g_{s+},
\end{aligned}
$$

where $L^r_t = L_t - \lambda t$, $L^g_t = -\bar{L}_{t-} + \gamma t$, and r, μ, σ, a, and b are constants. W is diffusion term and L and $\bar{L}$ are independent Poisson with constant intensity λ and γ respectively. Let $\mathcal{F}_t$ be the natural filtration that is neither right or left contiguous. Let the initial money market account be x_0 and the initial price X_0. We can write X as $X_t = X_0\mathcal{E}(H)$, where

$H_t = \mu t + \sigma W_t + a\left(L_t - \lambda t\right) + b\left(\gamma t - \bar{L}_{t-}\right)$, with $H_0 = 0$, and $x_t = x_0 \exp(rt)$ so that $h_t = rt$. In some sense this model is simpler than Merton jump diffusion model [54] in which Merton assumed that the coefficient a "jump-amplitudes" of the Poisson process is a random variable having a normal distribution but complicated by the fact that we are adding a left jump Poisson process. In this case the ratio process is

$$
\begin{aligned}
R_t &= \frac{X_0}{x_0} \exp\left\{H_t - \frac{1}{2}\langle H^c, H^c\rangle - rt\right\} \prod_{0 \leq s < t} \left[\left(1 + \triangle^+ H_s\right) e^{-\triangle^+ H_s}\right] \\
&\times \prod_{0 < s \leq t} \left[\left(1 + \triangle H_s\right) e^{-\triangle H_s}\right] \\
&= X_0 \exp\left\{\left(\mu - r - \frac{1}{2}\left(\sigma^2 - \lambda a^2 - \gamma b^2\right)\right) t + \sigma W_t\right\} \\
&\times \prod_{0 < s \leq t} \left[\left(1 + a \triangle L_t\right) e^{-a \triangle L_t}\right] \prod_{0 \leq s < t} \left[\left(1 - b \triangle^+ \bar{L}_{s-}\right) e^{-b \triangle^+ \bar{L}_{s-}}\right],
\end{aligned}
$$

and is not a local optional martingale. So we want $Z = \mathcal{E}(N)$ for which we have to find a local martingale N such that $\Psi(h, H, N)$,

$$
\begin{aligned}
\Psi(h, H, N) = N_t + H_t - h_t + \langle h^c - N^c, h^c - H^c\rangle_t \\
+ \sum_{0 < s \leq t} \frac{\left(\triangle h_s - \triangle N_s\right)\left(\triangle h_s - \triangle H_s\right)}{1 + \triangle h_s} \\
+ \sum_{0 < s \leq t} \frac{\left(\triangle^+ h_s - \triangle^+ N_s\right)\left(\triangle^+ h_s - \triangle^+ H_s\right)}{1 + \triangle^+ h_s}.
\end{aligned}
$$

Computing a local martingale deflator

It makes sense to start with the guess $N_t = \varsigma W_t + c\left(L_t - \lambda t\right) + d\left(\gamma t - \bar{L}_{t-}\right)$ an optional local martingale that will render Z an optional scaling factor. If N is as we chose above then

$$\begin{aligned}
\Psi(h,H,N) &= \varsigma W_t + c\left(L_t - \lambda t\right) + d\left(\gamma t - \bar{L}_{t-}\right) + \mu t \\
&+ \sigma W_t + a\left(L_t - \lambda t\right) + b\left(\gamma t - \bar{L}_{t-}\right) - rt \\
&+ \left\langle \left[\varsigma W_t + c\left(L_t - \lambda t\right) + d\left(\gamma t - \bar{L}_{t-}\right)\right]^c \right. \\
&\left. , \left[\sigma W_t + a\left(L_t - \lambda t\right) + b\left(\gamma t - \bar{L}_{t-}\right)\right]^c \right\rangle \\
&+ \sum_{0<s\leq t} ac\triangle L_s \triangle L_s + \sum_{0\leq s\leq t} bd\triangle_s^+ \bar{L}_{s-}\triangle_s^+ \bar{L}_{s-} \\
&= \left(\varsigma + \sigma\right) W_t + \left(a + c\right)\left(L_t - \lambda t\right) \\
&+ \left(b + d\right)\left(\gamma t - \bar{L}_{t-}\right) \\
&+ \left(\mu - r\right) t + \varsigma\sigma t + ac\lambda t + bd \\
&+ \gamma t + acL_t + bd\bar{L}_{t-} + ac\lambda t - ac\lambda t \\
&+ bd\gamma t - bd\gamma t \\
&= \left(\varsigma + \sigma\right) W_t + \left(a + c\right)\left(L_t - \lambda t\right) \\
&+ \left(b + d\right)\left(\gamma t - \bar{L}_{t-}\right) + ac(L_t - \lambda t) \\
&- bd\left(\gamma t - \bar{L}_{t-}\right) + ac\lambda t + bd\gamma t \\
&+ \left(\mu - r\right) t + \varsigma\sigma t + ac\lambda t + bd\gamma t
\end{aligned}$$

therefore,

$$\begin{aligned}
\Psi(h,H,N) &= \left(\varsigma + \sigma\right) W_t + \left(a + c + ac\right)\left(L_t - \lambda t\right) \\
&+ \left(b + d - bd\right)\left(\gamma t - \bar{L}_{t-}\right) \\
&+ \left(\mu - r + \varsigma\sigma + 2ac\lambda + 2bd\gamma\right) t
\end{aligned} \tag{4.9}$$

is local martingale if

$$\mu - r + \varsigma\sigma + 2ac\lambda + 2bd\gamma = 0. \quad (*)$$

So we have to find (ς, c, d) such that the last statement (*) is true, or in other words

$$[\sigma, \quad 2a\lambda, \quad 2b\gamma]\,[\varsigma, \quad c, \quad d]^{\mathsf{T}} = r - \mu.$$

Trying to solve the equation above leads to infinitely many solutions which means that our market, the market of Black-Scholes with left and right jumps is incomplete. Many of these solutions are interesting; for example, one possible solution is $(\varsigma, c, d) = (\sigma, a, b)/|(\sigma, a, b)|^2$. Another interesting solution is to let $d = 0$ which leads to right continuous local martingale measure. Yet another solution is a one which will eliminate the effects of jumps on drift that is by letting $d = -1/b\gamma$ and $c = 1/a\lambda$, in this case $\varsigma = (r - \mu)/\sigma$. Now that we have found local martingale measure we are going to use this knowledge to price a European call option in this market.

Pricing and hedging of a European call option

A European call option is a contingent claim. Generally, a contingent claim is a random variable $Y \in \mathcal{F}_T$ at some time T. Y generates the optional martingale process $Y_t = \mathbf{E}[Y|\mathcal{F}_t]$ for $t \in [0, T]$. In the optional Black-Scholes market (x, X) equation (4.8), we are going to *assume* that Y_t is a solution of the integral

$$Y_t = Y_0 + \int_{0+}^{t} \alpha_{t-} dR_t^r + \int_0^{t-} \beta_t dR_{t+}^g, \tag{4.10}$$

where $R = X/x$. Recall that Y also satisfies the portfolio equation $Y_t = \eta_t + \xi_t R_t$ and $\xi_t R_t = \alpha_t R_t^r + \xi_t R_t^g$ where $\xi = (\alpha_-, \beta) \in \mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$. Under the risk neutral measure $\mathbf{Q} \sim \mathbf{P}$ where $\mathbf{Q}_t = \int_\Omega Z_t d\mathbf{P}_t$ where Z is the martingale transform of R is strictly larger than 0. $\mathbf{Q}$ exists if $Z > 0$. In the context of our example of the Black-Scholes market this fact is warranted as we have established above. Furthermore, when pricing a contingent claim we are only concerned with those contingent claims that can be written as in equation (4.10).

Now lets turn our attention to the problem at hand, pricing and hedging a European contingent claim. Lets us consider pricing of a European call option whose value at maturity is $C_T = (X_T - \mathsf{K})^+$ where K is the strike price and T is the maturity date. In a way, we can think of C_T as the value of a portfolio at time T. The normalized value of this portfolio or the option is

$$\tilde{C}_T = \frac{C_T}{x_T} = \left(\frac{X_T}{x_T} - \frac{\mathsf{K}}{x_T}\right)^+.$$

where $x_T = e^{rT}$ is the discounting factor. At time t and under the risk neutral measure $\tilde{C}_T$ is a local optional martingale whose value is given by

$$\tilde{C}_t = \mathbf{E_Q}\left[\frac{C_T}{x_T}|\mathcal{F}_t\right] = \mathbf{E_Q}\left[\left(R_T - e^{-rT}\mathsf{K}\right)^+|\mathcal{F}_t\right]. \ (*)$$

where R_T is the ratio process. Since $\tilde{C}_t = C_t/e^{rt}$ then we can write the above equation (*) as

$$\begin{aligned} C_t &= e^{rt}\mathbf{E_Q}\left[\left(R_T - e^{-rT}\mathsf{K}\right)^+|\mathcal{F}_t\right] \\ &= e^{rt}\mathbf{E_Q}\left[\left(R_T - e^{-rT}\mathsf{K}\right)\mathbf{1}_{(R_T \geq e^{-rT}\mathsf{K})}|\mathcal{F}_t\right]. \end{aligned}$$

The value of the portfolio at time $t = 0$ which is the option price at the time of its offering is

$$C_0 = \mathbf{E_Q}\left[\frac{C_T}{x_T}\right] = \mathbf{E_Q}\left[\left(R_T - e^{-rT}\mathsf{K}\right)^+\right].$$

To compute the value C_t of the option we must first choose $\mathbf{Q}$. an appropriate local martingale measure. Previously, we have shown that there exist infinitely many choices. Here we are going to choose a $\mathbf{Q}$ in a way which makes our calculations of the price simpler. However, in practice the option seller would want to choose $\mathbf{Q}$ which maximize the price of the option while the buyer would want to choose one to minimize the price of the option. To choose $\mathbf{Q}$ we must choose the parameters that makes $\Psi(h, H, N)$ a martingale. According to equation (4.9) we are going to choose

$$\left\{c = \frac{-a}{1+a}, \quad d = \frac{-b}{1-b}, \quad \varsigma = \frac{r - \mu + 2\lambda\frac{a^2}{1+a} + 2\gamma\frac{b^2}{1+b}}{\sigma}.\right\}$$

This choice leads to the normalized price R under $\mathbf{Q}$ to be a function of just the Wiener process W,

$$\begin{aligned}
R_t &= R_0 \mathcal{E}_t\left(\Psi(h, H, N)\right) = R_0 \mathcal{E}\left(\left(\varsigma + \sigma\right) W_t\right) \\
&= X_0 \exp\left(\left(\varsigma + \sigma\right) W_t - \frac{1}{2}\left(\varsigma + \sigma\right)^2 t\right), \quad x_0 = 1.
\end{aligned}$$

Hence, the option value at time t is given by

$$\begin{aligned}
C_t &= e^{rt}\mathbf{E}_{\mathbf{Q}}\left[\left(R_T - e^{-rT}\mathsf{K}\right)\mathbf{1}_{(R_T \geq e^{-rT}\mathsf{K})}|\mathcal{F}_t\right] \\
&= e^{rt}\mathbf{E}_{\mathbf{Q}}\left[\left(R_T - e^{-rT}\mathsf{K}\right)\mathbf{1}_{(R_T \geq e^{-rT}\mathsf{K})}|\mathcal{F}_t\right] \\
&= e^{rt}\mathbf{E}_{\mathbf{Q}}\left[R_T\mathbf{1}_{(R_T \geq e^{-rT}\mathsf{K})}|\mathcal{F}_t\right] - e^{-r(T-t)}\mathsf{K}\mathbf{Q}\left(R_T \geq e^{-rT}\mathsf{K}|\mathcal{F}_t\right)
\end{aligned}$$

and its price at the time of its initial offering, $t = 0$, is

$$C_0 = \mathbf{E}_{\mathbf{Q}}\left[R_T\mathbf{1}_{(R_T \geq e^{-rT}\mathsf{K})}\right] - e^{-rT}\mathsf{K}\mathbf{Q}\left(R_T \geq e^{-rT}\mathsf{K}\right).$$

Now lets compute the price C_0. We will start by computing

$$\begin{aligned}
\mathbf{Q}\left(R_T > e^{-rT}\mathsf{K}\right) &= \mathbf{Q}\left(\exp\left(\left(\varsigma + \sigma\right) W_T - \frac{1}{2}\left(\varsigma + \sigma\right)^2 T\right) > \frac{e^{-rT}\mathsf{K}}{X_0}\right) \\
&= \mathbf{Q}\left(W_T \geq \frac{1}{\left(\varsigma + \sigma\right)}\left[-rT - \ln\left[\frac{X_0}{\mathsf{K}}\right] + \frac{1}{2}\left(\varsigma + \sigma\right)^2 T\right]\right) \\
&= \mathbf{Q}\left(\mathsf{Z} < \frac{1}{\left(\varsigma + \sigma\right)\sqrt{T}}\left[\ln\left[\frac{X_0}{\mathsf{K}}\right] + \left(r - \frac{1}{2}\left(\varsigma + \sigma\right)^2\right) T\right]\right) \\
&= \Phi\left(\tilde{\mathsf{K}}\right).
\end{aligned}$$

where Z here is a standard normal random variable and

$$\tilde{\mathsf{K}} = \frac{1}{(\varsigma+\sigma)\sqrt{T}}\left[\ln\left[\frac{X_0}{\mathsf{K}}\right] + (r - \frac{1}{2}(\varsigma+\sigma)^2)T\right].$$

The other part of the price is the expectation

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}}\left[R_T \mathbf{1}_{(R_T \geq e^{-rT}\mathsf{K})}\right] &= \frac{X_0}{\sqrt{2\pi}}\int_{-\infty}^{\tilde{\mathsf{K}}} \exp -\frac{1}{2}\left(z^2 - 2(\varsigma+\sigma)\sqrt{T}z + (\varsigma+\sigma)^2 T\right) dz \\
&= \frac{X_0}{\sqrt{2\pi}}\int_{-\infty}^{\tilde{\mathsf{K}}} \exp -\frac{1}{2}\left(z + (\varsigma+\sigma)\sqrt{T}\right)^2 dz \\
&= X_0 \Phi\left(\tilde{\mathsf{K}} + (\varsigma+\sigma)\sqrt{T}\right).
\end{aligned}$$

Therefore, the price of the option is going to be given by

$$C_0 = X_0 \Phi\left(\tilde{\mathsf{K}} + (\varsigma+\sigma)\sqrt{T}\right) - e^{-rT}\mathsf{K}\Phi\left(\tilde{\mathsf{K}}\right).$$

Note that this is formula has the same form as the regular Black-Scholes pricing formula for a European call option. However, the volatility σ has been changed by ς "the effective volatility" that is a result of the left and right jumps. Now that we have computed the price of the option how do we go about finding the evolution of the price through time.

The evolution of the price of European call option can also be derived in the following way

$$\begin{aligned}
C_t &= e^{rt}\mathbf{E}_{\mathbf{Q}}\left[\left(R_T - e^{-rT}\mathsf{K}\right)\mathbf{1}_{(R_T \geq e^{-rT}\mathsf{K})} | \mathcal{F}_t\right] \\
&= e^{rt}\mathbf{E}_{\mathbf{Q}}\left[R_T \mathbf{1}_{(R_T \geq e^{-rT}\mathsf{K})} | \mathcal{F}_t\right] - e^{-r(T-t)}\mathsf{K}\mathbf{Q}\left(R_T \geq e^{-rT}\mathsf{K} | \mathcal{F}_t\right)
\end{aligned}$$

Note that $R_T \geq e^{-rT}\mathsf{K}$ if and only if

$$W_T \geq \frac{1}{(\varsigma+\sigma)}\left[-\ln\left[\frac{X_0}{\mathsf{K}}\right] - \left[r - \frac{1}{2}(\varsigma+\sigma)^2\right]T\right]$$

which is also true if and only if

$$W_T - W_t \geq \frac{1}{(\varsigma + \sigma)} \left[-\ln \left[\frac{X_0}{\mathsf{K}} \right] - \left[r - \frac{1}{2} (\varsigma + \sigma)^2 \right] (T - t) \right]$$

by using a simple time change from T by $T-t$ and knowing that W_{T-t} has the same distribution as $W_T - W_t$. Replacing the inequality in $\mathbf{Q}\left(R_T \geq e^{-rT}\mathsf{K}|\mathcal{F}_t\right)$ by the inequality involving $W_T - W_t$ therefore

$$\begin{aligned}
&\mathbf{Q}\left(R_T \geq e^{-rT}\mathsf{K}|\mathcal{F}_t\right) \\
&\Longleftrightarrow \mathbf{Q}\left(W_T \geq \frac{1}{(\varsigma + \sigma)} \left[-rT - \ln \left[\frac{X_0}{\mathsf{K}} \right] + \frac{1}{2} (\varsigma + \sigma)^2 T \right] |\mathcal{F}_t \right) \\
&\Longleftrightarrow \mathbf{Q}\left(W_T - W_t \geq \frac{-1}{(\varsigma + \sigma)} \left[\ln \left[\frac{X_0}{\mathsf{K}} \right] - \left[r - \frac{1}{2} (\varsigma + \sigma)^2 \right] (T - t) \right] |\mathcal{F}_t \right) \\
&= \mathbf{Q}\left(W_T - W_t \geq \frac{-1}{(\varsigma + \sigma)} \left[\ln \left[\frac{X_0}{\mathsf{K}} \right] - \left[r - \frac{1}{2} (\varsigma + \sigma)^2 \right] (T - t) \right] \right) \\
&= \Phi\left(\tilde{\mathsf{K}}_t\right)
\end{aligned}$$

where

$$\tilde{\mathsf{K}}_t = \frac{1}{(\varsigma + \sigma)\sqrt{T - t}} \left[\ln \left[\frac{X_0}{\mathsf{K}} \right] + (r - \frac{1}{2} (\varsigma + \sigma)^2) (T - t) \right].$$

For the other term

$$\begin{aligned}
&\mathbf{E}_{\mathbf{Q}}\left[R_T \mathbf{1}_{(R_T \geq e^{-rT}\mathsf{K})}|\mathcal{F}_t\right] \\
&= \mathbf{E}_{\mathbf{Q}}\left[R_t \exp\left((\varsigma+\sigma)(W_T - W_t) - \frac{1}{2}(\varsigma+\sigma)^2 (T-t)\right)\mathbf{1}_{(R_T \geq e^{-rT}\mathsf{K})}|\mathcal{F}_t\right] \\
&= R_t \mathbf{E}_{\mathbf{Q}}\left[\exp\left((\varsigma+\sigma)(W_T - W_t) - \frac{1}{2}(\varsigma+\sigma)^2 (T-t)\right)\right. \\
&\left.\mathbf{1}_{\left(W_T - W_t \geq \frac{-1}{(\varsigma+\sigma)}\left[\ln\left[\frac{X_0}{\mathsf{K}}\right] - \left[r - \frac{1}{2}(\varsigma+\sigma)^2\right](T-t)\right]\right)}|\mathcal{F}_t\right] \\
&= R_t \mathbf{E}_{\mathbf{Q}}\left[\exp\left((\varsigma+\sigma)(W_T - W_t) - \frac{1}{2}(\varsigma+\sigma)^2 (T-t)\right)\right. \\
&\left.\mathbf{1}_{\left(W_T - W_t \geq \frac{-1}{(\varsigma+\sigma)}\left[\ln\left[\frac{X_0}{\mathsf{K}}\right] - \left[r - \frac{1}{2}(\varsigma+\sigma)^2\right](T-t)\right]\right)}\right] \\
&= R_t \mathbf{E}_{\mathbf{Q}}\left[\exp\left((\varsigma+\sigma)\sqrt{T-t}\mathsf{Z} - \frac{1}{2}(\varsigma+\sigma)^2 (T-t)\right)\right. \\
&\left.\mathbf{1}_{\left(\sqrt{T-t}\mathsf{Z} \geq \frac{-1}{(\varsigma+\sigma)}\left[\ln\left[\frac{X_0}{\mathsf{K}}\right] - \left[r - \frac{1}{2}(\varsigma+\sigma)^2\right](T-t)\right]\right)}\right] \\
&= R_t \Phi\left(\tilde{\mathsf{K}}_t + (\varsigma+\sigma)\sqrt{T-t}\right).
\end{aligned}$$

So the price of the option evolves according to

$$C_t = e^{rt} R_t \Phi\left(\tilde{\mathsf{K}}_t + (\varsigma+\sigma)\sqrt{T-t}\right) + e^{-r(T-t)}\mathsf{K}\Phi\left(\tilde{\mathsf{K}}_t\right).$$

Hedging of a European Call option

Since $\tilde{C}_t$ is a local martingale under $\mathbf{Q}$ then by martingale representation theorem we can write

$$C_t = e^{-r(T-t)}\mathbf{E}_{\mathbf{Q}}\left[C_T\right] + e^{rt}\int_0^t \beta_s dR_s \text{ a.s. } \mathbf{Q}.$$

This representation of C_t will help us determine the replicating (hedging) portfolio by computing

$$\beta_t = \frac{d\left[\tilde{C}, R\right]_t}{d\left[R, R\right]_t}.$$

Note that

$$\begin{aligned}\left[\tilde{C}, R\right]_t &= \left[R_t \Phi\left(\tilde{\mathsf{K}}_t + (\varsigma + \sigma)\sqrt{T-t}\right) + e^{-r(T-t)}\mathsf{K}\Phi\left(\tilde{\mathsf{K}}_t\right), R_t\right] \\ &= \left[R_t \Phi\left(\tilde{\mathsf{K}}_t + (\varsigma + \sigma)\sqrt{T-t}\right), R_t\right] + e^{-r(T-t)}\mathsf{K}\left[\Phi\left(\tilde{\mathsf{K}}_t\right), R_t\right] \\ &= \Phi\left(\tilde{\mathsf{K}}_t + (\varsigma + \sigma)\sqrt{T-t}\right)[R, R]_t .\end{aligned}$$

where $\left[\Phi\left(\tilde{\mathsf{K}}_t\right), R_t\right] = 0$. Therefore the hedging strategy is

$$\begin{aligned}\beta_t &= \frac{d\left[\tilde{C}, R\right]_t}{d\,[R, R]_t} = \frac{\Phi\left(\tilde{\mathsf{K}}_t + (\varsigma + \sigma)\sqrt{T-t}\right) d\,[R, R]_t}{d\,[R, R]_t} \\ &= \Phi\left(\tilde{\mathsf{K}}_t + (\varsigma + \sigma)\sqrt{T-t}\right).\end{aligned}$$

and $\eta_t = e^{-rT}\mathsf{K}\Phi\left(\tilde{\mathsf{K}}_t\right)$.

Remark 84 *Note that in general hedging of contingent claims in optional semimartingale markets leads to portfolios that are not predictable. However, in this example we have found predictable hedging portfolios. So how is this possible! It turns out that, in some special cases one can choose a risk neutral measure that absorbs, at least all the left jumps in the market and renders the market processes right continuous or even in some cases continuous like what we have shown in the example above.*

4.4.2 A Basket of Stocks

Here we present an example of a market of optional semimartingales where it is possible to trade in the "usual sense". Furthermore, we discuss portfolio structure in this market. Consider the market: a money market account x and two assets X^1 and X^2 evolving according to

$$\begin{aligned} x_t &= 1 \\ X^1_t &= X^1_0 + \int_{0+}^{t} X^1_{s-}\left(\mu ds + \sigma dW_s + a dL^r_s\right), \\ X^2_t &= X^2_0 + \int_{0}^{t-} b X^2_s dL^g_{s+}, \end{aligned}$$

where $L^r_t = L_t - \lambda t$, $L^g_t = -\bar{L}_{t-} + \gamma t$, and μ, σ, a, and b are constants. W is diffusion term and L and $\bar{L}$ are independent Poisson processes with constant intensities λ and γ respectively. The initial prices are X^1_0 and X^2_0. In this case, one can write a portfolio of this market as

$$\begin{aligned} Y_t &= Y_0 + \alpha \cdot X^1_t + \beta \odot X^2_t, \\ &= \eta_t + \alpha_t X^1_t + \beta_t X^2_t. \end{aligned}$$

In this case, (η, α, β) can be traded independently since X^1_t and X^2_t are two different assets that are available in this market. Also, each trading strategy can be approximated by a simple trading strategy.

The portfolio value process Y is an optional semimartingale defined on the filtration generated by the pair X^1 and X^2 which is not necessarily right continuous. One notices here that even if the individual processes comprising the market are either right or left continuous the market information as a whole is not necessarily right or left continuous. Optional semimartingale in unusual probability spaces provide a way for dealing with complicated market structure such as the examples we have presented above.

4.4.3 A Defaultable Bond and a Stock

Consider a market composed of a money market account x and assets X evolving according to $x_t = x_0\mathcal{E}(h)_t$ and $X_t = X_0\mathcal{E}(H)_t$ where

$$\begin{aligned} h_t &= rt + bL^g_t, \ h_0 = 0, \\ H_t &= \mu t + \sigma W_t + aL^d_t, \ H_0 = 0. \end{aligned}$$

$L^d_t = L_t - \lambda t$, $L^g_t = -\bar{L}_{t-} + \gamma t$, and r, μ, σ, a, and b are constants. W is diffusion term and L and $\bar{L}$ are Poisson with constant intensity λ and γ respectively. Let $\mathcal{F}_t$ be the natural filtration that is neither right or left continuous. Let the initial money market account be x_0 and the initial price X_0.

In this example we have modeled the money market account value by a left continuous process. A similar model was given by Duffie [28] for bonds that can experience defaults. We believe that the model we present above is a better description of a portfolio of stocks and bonds than RCLL processes on usual probability space.

In this case the ratio process is $R_t = \frac{X_0}{x_0}\mathcal{E}(H_t - h^*_t - [H, h^*]_t)$. We want to find $Z = \mathcal{E}(N)$ such that ZR is a local martingale. In section 4.1 we have shown that associated with the product $ZR = \frac{X_0}{x_0}\mathcal{E}(\Psi(h, H, N))$ is the process $\Psi(h, H, N)$. To compute a reasonable form for $\Psi(h, H, N)$,

$$\begin{aligned} \Psi(h, H, N) = N_t + H_t - h_t + \langle h^c - N^c, h^c - H^c \rangle_t \\ + \sum_{0<s\leq t} \frac{(\triangle h_s - \triangle N_s)(\triangle h_s - \triangle H_s)}{1 + \triangle h_s} \\ + \sum_{0<s\leq t} \frac{(\triangle^+ h_s - \triangle^+ N_s)(\triangle^+ h_s - \triangle^+ H_s)}{1 + \triangle^+ h_s}, \end{aligned}$$

it makes sense to suppose that $N_t = \varsigma W_t + cL^d_t + \theta L^g_t$ is an optional local martingale for which

Z an optional local martingale deflator. In this case $\Psi(h, H, N)$ is

$$\begin{aligned}
\Psi(h, H, N) &= \left[\varsigma W_t + cL_t^d + \theta L_t^g\right] + \left[\mu t + \sigma W_t + aL_t^d\right] - \left[rt + bL_t^g\right] \\
&+ \left\langle \left[rt + bL_t^g\right]^c - \left[\varsigma W_t + cL_t^d + \theta L_t^g\right]^c, \left[rt + bL_t^g\right]^c - \left[\mu t + \sigma W_t + aL_t^d\right]^c \right\rangle \\
&+ \sum_{0<s\leq t} ac\left(\triangle L_s^d\right)^2 + \sum_{0\leq s<t} \frac{b\,(b-\theta)\,(\triangle^+ L_s^g)^2}{1 + b\triangle^+ L_s^g} \\
&= (\mu - r + \varsigma\sigma)\,t + (\varsigma + \sigma)\,W_t + (c + a)\,L_t^d + (\theta - b)\,L_t^g + acL_t \\
&+ (b - \theta)\,L_t^g + b\,(\theta - b)\,[L_t^g, L_t^g] \\
&= (\mu - r + \varsigma\sigma)\,t + (\varsigma + \sigma)\,W_t + (c + a)\,L_t^d + acL_t + b\,(\theta - b)\,\bar{L}_{t-}.
\end{aligned}$$

W_t and L_t^d are martingales. $\Psi(h, H, N)$ is a martingale if and only if $\mu - r + \varsigma\sigma + ca\lambda + \theta b - b^2\gamma = 0$. The solution of this equation leads to infinitely many solutions which means the market is incomplete. Many of these solutions are interesting; for example, one possible solution is to let $\theta = 0$ which leads to right continuous local martingale deflator. Yet another solution is a one which will eliminate the effects of jumps on drift that is by letting $\theta = -1/b$ and $c = 1/a\lambda$, in this case $\varsigma = \left(r - \mu + b^2\gamma\right)/\sigma$.

4.4.4 Optimal Debt Repayment

The gross public debt of a government is the total of all its borrowings minus repayments [58]. The debt ratio X of a government at time t is defined as

$$X_t := \frac{\textit{Gross public debit at time } t}{\textit{Gross domestic product (GDP) at time } t}.$$

A reasonable dynamics of public debit was described by Blanchard 2009 [59].

Here we will generalize the optimal debit repayment problem in the context of optional semimartingale markets. As we describe the problem it will become apparent that an optional semimartingale description of the problem is a better more natural setting for the problem, especially in the case where left and right continuous processes mix. We also propose a method of solution but we don't solve the problem in its entirety here. A complete solution requires a treatment of optimal control in the context of optional semimartingale market which is a much larger endeavor.

Suppose we are on an unusual probability space, $\left(\Omega,\, \mathcal{F},\, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}\,,\, \mathbf{P}\right)$, $t \in [0,\infty)$. X and C are adapted processes in this space. The model we propose for debit repayment is,

$$X_t = x + X \circ H_t - C_t, \tag{4.11}$$

where, x is the initial debit, H is an optional semimartingale and C is a positive left continuous process. H commonly has the following form,

$$H_t = \mu t + \sigma W_t + \rho L^d_t, \quad L^d_t = L_t - \lambda t, \tag{4.12}$$

where $\mu = (r - g) \in (-\infty, \infty)$ and $\sigma \in (0, \infty)$ are constants, and W is a Brownian motion and L is Poisson with constant intensity λ. $x \in (0, \infty)$ is the initial debt ratio. $r \in [0, \infty)$ is the real interest rate on debt and $g \in (-\infty, \infty)$ is the rate of economic growth. σ and ρ are the volatility of debt which can be the result of many factors. For example, if part of the debt was issued in foreign currencies, then, in this case σ captures the volatility of the exchange rates or as a result of changes in the GDP. The process C is the cumulative repayments, given by the government, to control the debt ratio X, with $C_0 = c > 0$. Since the intention of a government is to reduce its debt then C aught to be non-negative and non-decreasing and $\mathbf{F}$-adapted *left-continuous* with right limits process (RLLC). But, note that the process H is RCLL. Therefore, X is with right and left limits (RLL) but is not necessarily left or right-continuous.

The conceived value of debit after repayment is given by,

$$Y^C_t = y + h(X) \circ A_t + C_t.$$

where A is the government's discount-factor, e.g. $A_t = \frac{e^{-\lambda t}}{-\lambda}$, $\lambda \in (0, \infty)$ is the discount rate. The function $h : (0, \infty) \to [0, \infty)$, is a cost function which we assume to be convex, $h \geq 0$ and have the following form $h(\varsigma) = \alpha\varsigma^{2n} + \beta$. $\alpha > 0$ and $\beta \geq 0$ are constants and $n \geq 1$ an integer. β is a scale parameter and α represents the importance of public debt to the government. n captures the risk aversion attitude of policy makers towards debt. A lower n means that a country have never had a default or suffered a severe debt crisis (e.g. Canada) while a higher n implies that the country has experienced a serious debt problems (e.g. Greece) [60]. The goal is to chose C in a way to minimize the total cost function,

$$\mathbf{J}(x; C) := \mathbf{E}_x \left[Y_T^C - y \right]$$

for some final time $T \in]0, \infty]$. In other words, what a government wants to do is,

$$V(x) := \inf_{C \in \mathfrak{C}} \mathbf{J}(x; C),$$

where $V : (0, \infty) \to \mathbb{R}$ represents the smallest cost that can be achieved when the initial debt ratio is x considering all the admissible controls $C \in \mathfrak{C}$. V is non-negative, increasing and convex. Furthermore, we require C to be admissible if $\mathbf{J}(x; C) < \infty$.

The most common approaches for solving optimization problems in stochastic analysis are stochastic control method and the so called martingale method. Stochastic control goes back to Robert Merton [61]. Merton's main idea consists of interpreting portfolio problems as a stochastic control problem to which he then applied standard methods of stochastic control theory, which requires the formulation of an Hamilton-Jacobi-Bellman equation. The martingale method represents the second main approach for solving optimal portfolio problems. It was introduced by Cox & Huang (see [62]). The martingale method decomposes the optimal portfolio problem into a sequence of optimization problems; First one determines the optimal payments (consumption, final wealth, debit payments) by convex analysis then one determines of the corresponding optimal portfolio process. Implicit feature of this method is that it requires us to transform the portfolio process to a martingale hence its name the martingale method.

In this particular example if we were to use the martingale approach one needs to transform the process $\mathcal{E}(H)$ to a martingale by some martingale deflator Z. For the stochastic control approach one has to device an HJB equation in the context of optional semimartingale markets. Both these methods will be considered in future work. However, here we will present in a limited scope the martingale method.

First, we must find the local martingale transform Z such that $Z\mathcal{E}(H)$ is a local martingale. For our specific example, where H is given by equation (4.12). Hence we can consider $Z = \mathcal{E}(aW + b(L_t - \lambda t))$. Therefore,

$$\begin{aligned} Z\mathcal{E}(H) &= \mathcal{E}(aW + b(L_t - \lambda t))\mathcal{E}(H) \\ &= \mathcal{E}((\mu + a\sigma + b\rho\lambda)\, t + (\sigma + a)W_t + (\rho + b)(L_t - \lambda t)) \end{aligned}$$

implies that we must have $\mu + a\sigma + b\rho\lambda = 0$.

Equation (4.11) is a nonhomogeneous stochastic equation whose solution is give by,

$$\begin{aligned} X_t &= (x - C_t) + X \circ H_t, \\ &= \mathcal{E}_t(H)\left[(x - c) - \mathcal{E}(-H^*) \circ C_t\right]. \end{aligned}$$

Given a choice of a and b such that $Z\mathcal{E}(H)$ is a local martingale then the expected value of X

$$\begin{aligned} \mathbf{E}_x[ZX] &= x + \mathbf{E}_x[Z \circ X + X \circ Z + [Z, X]] \\ &= x + \mathbf{E}_x[Z \circ ((x - C) + X \circ H) + [Z, (x - C) + X \circ H]] \\ &= x + \mathbf{E}_x[-Z \circ C + ZX \circ (H + [\xi, H])] \\ &= x + \mathbf{E}_x[-ZC + C \circ Z] = x - \mathbf{E}_x[ZC], \end{aligned}$$

where $Z = \mathcal{E}(\xi)$ and note that $H + [\xi, H]$ is a local martingale hence $\mathbf{E}_x[ZX] = x - \mathbf{E}_x[ZC]$. Also $ZX > 0$ hence a supermartingale. Therefore, the optimization problem with constraint is given by

$$\inf_{C \in \mathfrak{C}} \mathbf{E}_x \left(h(X) \circ A_T + C_T \right),$$

$$x - \mathbf{E}_x Z_T C_T > 0.$$

The solution of this equation is going to be the subject of future work.

4.5 Concluding Remarks

Left continuous processes are a natural occurrence in financial markets, for example, defaultable bonds [28], stochastic dividends payments [63, 64], transaction costs [24, 65] and natural consumption processes. Therefore, the calculus of optional processes is an indispensable tool in the future study of mathematical finance.

In this chapter, we have described the optional semimartingale model of financial markets and a procedure of finding local martingale deflators for this market. Also, we have presented several illustrative examples.

Chapter 5

Defaultable Markets

A default risk is the possibility that any *counterparty* in a financial agreement will not fulfill their contractual obligations. It is a form of credit risk; Credit risk encompasses any risk associated with credit linked events. Examples of credit risk events are changes in credit quality, i.e. upgrades or downgrades in credit ratings, variations of credit spreads, and default events. Our focus here is risk of counterparty default.

But we like to make clear the distinction between a counterparty credit risk and reference credit risk. Reference credit risk is the situation when all parties of a contract are default free, but because of some specific features of the contract the credit risk of a reference entity (e.g. labor market) becomes an essential component of the contract final settlement. In other words, reference risk is the contract risk that is associated with a 3^{rd} party whose not a signatory to the contract. Credit derivatives are financial instruments that allow market participants to isolate and trade the reference credit risk. The goal of credit derivatives is to transfer the reference risk completely or partially to one of counterparties of the contract or to a third party.

On the other hand, counterparty risk is the risk that a counterparty will default. It is an important feature of all of over-the-counter (OTC) derivatives. Unlike exchange-traded contracts OTCs are not backed by the clearinghouse or an exchange. So each counterparty is exposed to the default risk of the other party. In practice, parties to an OTC are sometimes required to post collateral or mark to market periodically. Counterparty risk emerged in contracts such as vulnerable claims and defaultable swaps.

In order to assess the value of a contract correctly one needs to quantify the credit risk of all counterparties. Quantitative models of credit risk formally defines a *credit risk event* as a *random event* whose occurrence affects the ability of the counterparty in a financial contract to fulfill a contractual commitment. Credit events may not be directly observed by the parties in a financial contract. The vast majority of research in credit risk is concerned with modeling of the random time at which default event occurs which what became known as *default time*. Some approaches to defaultable term structure allowed for the possibility of intermediate credit events that are associated with changes in the credit quality of a corporate bond, which migrates between various rating classes. In these approaches the modeling of multiple random times as a result of credit migration became an important issue to consider.

Equally important in credit risk modeling is to formulate hedging strategies to insure against the risk of default by any of the counterparties. Another important problem arising in modeling of the credit risk is modeling of recovery rules. Recovery rules specifies a payment to the contract holder in case of default. The recovery payments together with the notional amount of the contract determine the potential cash flows associated with the contract.

Therefore, the main objective of the quantitative models of the credit risk is to provide ways to price and to hedge financial contracts that are sensitive to credit risk events. Needless to say that any approach to pricing credit risk should aim at producing intuitive, practical and *internally consistent* (e.g. arbitrage-free) financial model. Towards this end, two competing methodologies have emerged in order to model default or credit migration times and recovery rates. These are the structural approach and the reduced-form approach. We will give a short overview of these approaches in the next section.

However rich the literature of structural and reduced form modeling approaches to credit risk, our aim is to make it richer. Therefore, in this chapter we present a new approach for the valuation of contingent claims that are subject to default risk based on the stochastic calculus of optional processes on unusual probability spaces. In our view the calculus of optional processes allows for more of a natural and realistic modeling of default events. Moreover, we present applications to the term structure of interest rates for corporate or sovereign bonds and the valuation of credit-spread option. Our work here is largely based on our paper [20].

5.1 Current Approaches to Credit Risk Modeling

There are two main types of default risk models: structural and reduced-form models. In structural models the value of the firm determines if a default event occur. In this case, default events are predictable. This approach was founded by Merton [66, 67]. In his formulation, the value of the firm is shared by shareholders and debt-holders. Share-holders receive a positive payoff whenever the face value owed to creditors can be reimbursed, otherwise they receive nothing. The shareholders' claim is just a call on the value of assets of the firm. Thus, a bond is simply a right to reimburse the face amount with the sale of a put to shareholders on the assets of the firm. In 1976, Black and Cox provided an extension of Merton's model, where default takes place whenever the value of the assets of the firm drops below a boundary. Further contributions to structural models were provided by [68, 69, 70, 71, 72, 71, 73]. An advantage of structural models is that one can see how the corporate conditions affect the default rate. However, the value of the firm is not a tradable asset hence the parameters of the structural model are difficult to estimate. Reduced form approaches appeared as a result of this limitation in structural models.

In reduced form models: the firm value is not modeled and plays only an auxiliary role; Default time is modeled as a stopping time that is *not predictable* thus default arrives as a total surprise to all counterparties. Formally, random times of default are a *totally inaccessible stopping times* on an enlarged filtration that encompasses the default free market information and information that is a result of default processes. The main computational tool in this approach is the specification of the conditional probability of default given market information. This probability follows a process with a jump at default time. In most cases the probability depends on an intensity parameter called the hazard rate. The hazard rate maybe a constant value over time or it maybe stochastic implying a term structure for the probabilities of default.

In practice hazard rate is estimated either by a fit to historical probability or to current market data by calibration. Since probability of default is often associated with hazard rate or and intensity of default reduced form models were also commonly known as hazard rate models or intensity based models. Reduced form models are well studied by many mathematicians [74, 75, 76, 77, 28, 78, 79, 80, 81, 82].

A combined approach was proposed in [83, 84, 85]. The basic idea is to postulate that the hazard rate is directly linked to the current value of the firm. Reduced-form models with this specific feature are referred to as hybrid models. In this set-up, the default time is still a totally inaccessible stopping time but the likelihood of default may grow rapidly when the total value of the firm's assets approaches some barrier. Finally, for additional study of the mathematics of credit risk we recommend the papers [86, 87] and the book [88].

Now let us describe the basic mathematical setup of structural and reduced form models to clarify some of the concepts we have discussed above. Consider the usual probability space $\left(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}\right)$ where the market evolves. Structural form models relies on the assumption that a default time τ is a predictable stopping time with respect to $\mathbf{F}$. Thereby, the associated hazard process $H_t = \mathbf{1}_{(\tau \leq t)}$ is $\mathcal{F}_{t-}$-measurable for all t. $\mathbf{F}$ is also where the underlying market assets are measurable. It is common in structural models to specify default time in terms of a barrier process B by

$$\tau := \inf\{t > 0 : t \geq 0, X_t \leq B_t\}$$

with the usual convention that the infimum over the empty set equals $+\infty$ and X is the value of a firm.

In reduced form models the default comes as a surprising event which is not measurable in the filtration that supports the underlying assets. That is default time is a random time that occurs outside the filtration $\mathbf{F}$, i.e. $(\tau \leq t) \notin \mathcal{F}_t$ for all time t. To circumvent this problem and a way to analyze reduced form models is to change the random default times to inaccessible stopping times. To do this, a new filtration is construct that incorporates the filtration, $\mathbf{H} = (\mathcal{H}_t)_{t\geq 0}$, generated by the hazard process $H_t = \mathbf{1}_{(\tau \leq t)}$ and the filtration $\mathcal{F}_t$, that is to define $\mathcal{G}_t = \sigma(\mathcal{F}_t \vee \mathcal{H}_t)$. This enlargement of the filtration $\mathbf{F}$ by the filtration of hazard process $\mathbf{H}$ leads to several problems in trying to establish a rational pricing theory. Usually, enlargement of filtration changes the properties of martingales and semimartingale in the same way a measure change does. To deal with these effects and to establish existence of local martingale measures for rational pricing one has to invoke two invariance principles known as the **H** and **H'** hypothesis also known as the immersion properties. **H** hypothesis states that every local martingale in the smaller filtration $\mathcal{F}$ is a local martingale in the larger filtration $\mathcal{G}$ meaning that default does not affect the martingale properties of these processes. While the **H'** hypothesis states that a semimartingale under the smaller filtration $\mathcal{F}$ remains a semimartingale under the larger filtration $\mathcal{G}$. Much of this theory has been elaborated by many outstanding mathematicians (see [86, 87] for a good review).

Understanding default is essential for pricing and hedging of contingent claims affected by it. To describe defaultable claims let us define default models on a more convenient footing; Suppose that we are given the underlying probability space $(\Omega, \mathcal{G}, \mathbf{G} = (\mathcal{G}_t)_{t\geq 0}\, \mathbf{Q})$ which is also endowed with the filtration $\mathbf{F} = (\mathcal{F}_t)_t \geq 0$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ for any t where $\mathcal{G}$ is the enlarged filtration encompassing information about assets and defaults. The probability measure $\mathbf{Q}$ is the spot martingale measure for the market. The real world probability measure will be denoted by $\mathbf{P}$. All processes: firms value, assets and default, are defined on $(\Omega, \mathcal{G}, \mathbf{Q})$. In this space a *defaultable claim* is defined as a quintuple $DCT = (X, A, \tilde{X}, Y, \tau)$ where τ is default time a non-negative random variable defined on the underlying probability space $(\Omega, \mathcal{G}, \mathbf{Q})$ such that $\mathbf{Q}(\tau < +\infty) = 1$.For convenience it is usually assumed that $\mathbf{Q}(\tau = 0) = 0$ and $\mathbf{Q}(\tau > t) > 0$ for all t. $H_t = \mathbf{1}_{(\tau \leq t)}$ is a the default process associated with τ which is a right continuous jump process. $\mathbf{H}$ is the filtration generated by the process H. The enlarged filtration $\mathbf{G} = (\mathcal{G}_t)_{t\geq 0}$, $\mathbf{G} = \mathbf{H} \vee \mathbf{F} = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ contains all required information to price and hedge the defaultable claim DCT. It should be emphasized that the default time τ is not necessarily a stopping time with respect to the filtration $\mathbf{F}$. On the other hand, τ is, of course, a stopping time with respect to the filtration $\mathbf{G}$. However, if $\mathcal{H}_t \subseteq \mathcal{F}_t$ for any t the reduced form models become structure model as τ is measurable in $\mathbf{F}$ and $\mathbf{F} = \mathbf{G}$.

For pricing of defaultable contingent claims with reduced form approaches the majority of work is focused on computing of the conditional expectation of payoffs given that the default has not occurred yet and the immersion property satisfied. Two approaches arose here, the intensity based and density based approaches. However, only the density approach turned out to be useful for the case of after default where the intensity approaches were inadequate. Intensity of default can be deduced from the density of default. But the reverse does not hold except when the $\mathbf{H}$-hypothesis hold (for details of this theory see [86]).

While there are still open problems to consider in structural and reduce form approaches to credit risk modeling we choose to take a different approach to the problem. Our approach is based on the calculus of processes on unusual spaces. In the following sections we present many results including the definition of defaultable markets on unusual probability spaces, pricing and hedging of defaultable cash-flows, facts about the conditional probability of default on unusual spaces and several examples.

5.2 Defaultable Markets on Unusual Spaces

Now lets take a closer look at the mechanics of default and how it affects the value of a firm, X. In this case suppose the firm remains in existence after default but its value changes. Let us fix an instance of time, t, If default is predictable or a stopping time in $\mathcal{F}_t$, i.e. $(\tau \leq t) \in \mathcal{F}_t$, then default is a result of internal factors. All information about default process is incorporated in $\mathcal{F}_t$. On the other hand, if $(\tau \leq t) \notin \mathcal{F}_t$ (i.e. $H_t = \mathbf{1}_{(\tau \leq t)}$ is not $\mathcal{F}_t$-measurable) then the default time τ is a random time that is result of external factors. However, after default takes place at time, t, all surprising information about default gets incorporated in future values of the firm X. Lets look at this in hind site; suppose that at time t a random default event τ occurs. If X_t is RCLL and $\mathcal{F}_t = \mathcal{F}_{t+}$ then, obviously, $(\tau \leq t) \notin \mathcal{F}_{t+}$, however, loosely speaking, we may write

$$\text{"}\,(\tau \leq t) \in \bigcap_{s>t+} \mathcal{F}_s\text{"},$$

to mean that information about default time τ at t is part of the future market filtration. In reduced form modeling, to be precise an enlarged filtration $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$ is constructed and the information about default is incorporated in future values of the firm is subsumed in the definition of the enlarged filtration $\mathbf{G}$.

To avoid complexities of other approaches and without artificial construction of new enlarged filtrations and requiring immersion properties to be satisfied as a way to get decomposition results, we propose a different approach based on the stochastic calculus of optional processes;

Let $\left(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}\right)$, $t \in \mathbb{R}_+$, be the stochastic basis and that the financial market stays on this space. A defaultable market will consist of at least the following two objects: Y is the asset or firm's value for the duration of its life. Y_t is $\mathcal{F}_t$-measurable; And τ the time of default such that

$$\tau \textit{ is a totally inaccessible stopping time in the broad sense, } \tau \in \mathcal{T}(\mathbf{F}_+).$$

In this case we define the default (hazard) process H as,

$$H_t = \mathbf{1}_{(\tau < t)}.$$

The process H is optional and left-continuous and $\mathbf{F}$-measurable.however note that $\mathbf{1}_{(\tau \leq t)}$ is $\mathcal{F}_{t+}$-measurable. To illustrates the mechanics of our "*optional approach*" to default we will present an example of a simple market structure however point out that the ideas we propose here are general and can be carried over to other more delicate market structures.

Consider a firm whose value is given by the process Y and assumes its final value η realized at the end of time, i.e. ∞. Let $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ represent the history of the firm, the good, the bad and defaults. However bad, we assume that the firm never vanquishes, in other words $Y > 0$ for all time and η is strictly positive random variable measurable with respect to $\mathcal{F}_\infty$. All is well with the firm except for a single default event $\tau \in \mathcal{T}(\mathbf{F}_+)$ that happens at some time $t \geq 0$. Furthermore, we assume that the value of the firm before default τ is evolving according to X and immediately after default according to x. Both x and X are optional semimartingales adapted to $\mathbf{F}$. Also, the default process $H = \mathbf{1}_{(\tau<\cdot)}$ is a left continuous optional semimartingale with respect to $\mathbf{F}$. Hence, the value process Y of the firm can be written as

$$Y_t = (1 - H_t)X_t + H_t x_t = X_t - H_t (X_t - x_t).$$

In the integral representation we have,

$$\begin{aligned} Y_t &= X_t - H \circ (X - x)_t + (X - x) \circ H_t + [X - x, H] \qquad (5.1) \\ &= X_t - \int_{0+}^{t} H_{s-} d(X_s - x_s) - \int_{0}^{t-} H_s d(X_{s+} - x_{s+}) \\ &\quad + \int_{0+}^{t} (X_{s-} - x_{s-})\, dH_s + \int_{0}^{t-} (X_s - x_s)\, dH_{s+} \\ &\quad + \int_{0+}^{t} \triangle H_s d(X_s - x_s) + \int_{0}^{t-} \triangle^+ H_s d(X_{s+} - x_{s+}). \end{aligned}$$

But since H has a single left jump and otherwise continuous, $\triangle H = 0$ then

$$\int_{0+}^{t} \triangle H_s d(X_s - x_s) = 0, \quad \int_{0}^{t-} \triangle^+ H_s d(X_{s+} - x_{s+}) = (X_{\tau+} - x_{\tau+}) H_{t-},$$
$$\int_{0+}^{t} (X_{s-} - x_{s-})\, dH_s = \int_{\tau+}^{t} (X_{s-} - x_{s-})\, ds,$$
$$\int_{0}^{t-} (X_s - x_s)\, dH_{s+} = \int_{\tau}^{t-} (X_s - x_s)\, ds$$

Substituting in equation (5.1),

$$\begin{aligned} Y_t &= X_t - \int_{0+}^{t} H_{s-} d(X_s - x_s) - \int_{0}^{t-} H_s d(X_{s+} - x_{s+}) \\ &\quad + \int_{\tau}^{t} (X_s - x_s)\, ds + (X_{\tau+} - x_{\tau+}) H_{t-}. \end{aligned}$$

To highlight the effects of default τ on the value of the firm Y, which will manifest itself as a left jump on the value of Y at time of default, we are going to assume that both x and X are continuous processes. In this special case equation (5.1) simplifies to,

$$Y_t = X_t - \int_0^t H_s d\left(X_s - x_s\right) + \int_\tau^t \left(X_s - x_s\right) ds + \left(X_{\tau+} - x_{\tau+}\right) H_{t-}.$$

Next we consider the evaluation of the standard defaultable claims.

5.3 Defaultable Claims and Cash-Flows

Here we consider pricing and hedging of defaultable claims and cash flows. Let the market evolve on the "unusual" probability space $\left(\Omega, \mathcal{F}, \mathbf{F} = \left(\mathcal{F}_t\right)_{0 \le t \le \tilde{T}}, \mathbf{P}\right)$ with time horizon date is $\tilde{T} > 0$. The probability measure $\mathbf{P}$ is the real-world or statistical probability, as opposed to, the spot local optional martingale measure or the risk-neutral probability $\mathbf{Q}$. We also have $\mathbf{F}_+$ the smallest right-continuous enlargement of $\mathbf{F}$. The filtration $\mathbf{F}$ supports the following objects:

1. The value process, Y, of the financial entity of defaultable instrument(s) or at a larger scale the total value of the firm's assets; Y is an optional semimartingale under $\mathbf{P}$ and $\mathbf{Q}$;

2. The promised contingent claim, Λ, representing the firm's liabilities to be redeemed at time $T \le \tilde{T}$. Λ is the payoff received by the owner of the claim at time T if there was no default prior to time T;

3. The process A, with $A_0 = 0$ models the promised dividends if there was no default prior to time T, i.e., the firm's liabilities stream that is redeemed continuously or discretely over time to the holder of a defaultable claim. We assume that A is predictable with respect to the reference filtration $\mathbf{F}$;

4. The recovery claim ρ is the payoff received at time T if default occurs prior to the claims maturity date T;

5. The recovery process R specifies the recovery payoff at time of default if it occurs prior to the maturity date T;

6. Finally and most importantly is the default time τ which is a random time that is $\mathbf{F}_+$-measurable. In this case, we are going to define the default process by

$$H_t = \mathbf{1}_{(\tau<t)}$$

which is optional and left continuous with respect to $\mathcal{F}_t$.

Furthermore, we assume that the processes X, R, and A are progressively measurable with respect to the filtration $\mathbf{F}$, and that the random variables, Λ is $\mathcal{F}_T$-measurable and ρ is at least $\mathcal{F}_{T+}$-measurable. Also, we assume without mentioning that all random objects introduced above are at least RLL processes and satisfy suitable integrability conditions that are needed for evaluating integrals.

This brings us to the recovery rules. If default occurs after time T, the promised claim Λ is paid in full at time T. Otherwise, default time $\tau \leq T$ and depending on the agreed upon recovery rules either the amount R_τ is paid at the time of default τ, or the amount ρ is paid at the maturity date T. Therefore, in its more general setting we consider simultaneously both kinds of recovery payoff, and thus define a defaultable claim formally as a quintuple,

$$DCT = (A, \Lambda, \rho, R, H).$$

Notice that the information structures $\mathbf{F}$ and $\mathbf{F}_+$ and the real-world probability $\mathbf{P}$ are intrinsic components of the definition of a defaultable claim.

In most practical situations either $R = 0$ or $\rho = 0$ type of recovery payoffs. There are few more simpler cases. A claim that does not pay any dividends $A = 0$ with the assumption that no payment received after default $\rho = 0$ or the defaultable claim $DCT = (\Lambda, \rho, H)$.

Now, let us define the dividend pay process of a defaultable claim as,

Definition 85 *The* dividend *process D of a defaultable claim $DCT = (A, \Lambda, \rho, R, H)$ equals*

$$D_t = \tilde{X}\mathbf{1}_{(t\geq T)} + (1 - H) \circ A_t + R \circ H_t, \tag{5.2}$$

where $\tilde{X} = \Lambda(1 - H_T) + \rho H_T$. *The process* D *is optional and* $\mathbf{F}$*-measurable.*

The process D is finite variation over finite time segments, including $[0, T]$. Since,

$$\begin{aligned} (1-H)\circ A_t &= \int_0^t \mathbf{1}_{(\tau \geq u)} dA_u = \left(A_t^r + A_{t-}^g\right)\mathbf{1}_{(\tau \geq t)} + \int_0^\tau dA_u \\ &= \left(A_t^r + A_{t-}^g\right)\mathbf{1}_{(\tau \geq t)} + \int_{0+}^\tau dA_u^r + \int_0^{\tau-} dA_{u+}^g \\ &= \left(A_t^r + A_{t-}^g\right)\mathbf{1}_{(\tau \geq t)} + \left(A_\tau^r + A_{\tau-}^g\right)\mathbf{1}_{(\tau < t)} \\ &= (1-H_t)\left(A_t^r + A_{t-}^g\right) + \left(A_\tau^r + A_{\tau-}^g\right)H_t, \end{aligned} \tag{5.3}$$

and

$$R\circ H_t = \int_0^t R_u dH_u = \left(R_{(\tau\wedge t)}^r + R_{(\tau\wedge t)-}^g\right)\mathbf{1}_{(\tau<t)} = \left(R_\tau^r + R_{\tau-}^g\right)\mathbf{1}_{(\tau<t)} = \left(R_\tau^r + R_{\tau-}^g\right)H_t \tag{5.4}$$

and the promised payoff $\tilde{X}$ is finite.

Furthermore, using equations (5.3) and (5.4) one can simplify the definition of the dividend process,

$$\begin{aligned} D_t &= \tilde{X}\mathbf{1}_{(t\geq T)} + (1-H_t)\left(A_t^r + A_{t-}^g\right) + \left(A_\tau^r + A_{\tau-}^g\right)H_t + \left(R_\tau^r + R_{\tau-}^g\right)H_t \\ &= \tilde{X}\mathbf{1}_{(t\geq T)} + (1-H_t)A_t^r + \left(A_\tau^r + R_\tau^r\right)H_t + (1-H_t)A_t^g + \left(A_{\tau-}^g + R_{\tau-}^g\right)H_t. \end{aligned} \tag{5.5}$$

Now, we place ourselves in the framework of a financial market model were there exist a martingale deflator measure $\mathbf{Q}$ equivalent to $\mathbf{P}$. We define the realized value of a defaultable claim in this market as the discounted value of the dividend process D.

Definition 86 *The* ex-dividend *price process* $X(\cdot, T)$ *of a defaultable claim* $DCT = (A, \Lambda, \rho, R, H)$ *which settles at time* T *is given by*

$$\begin{aligned} X(t,T) &= B_t \mathbf{E}_{\mathbf{Q}}\left(B^{-1}\circ D_T - B^{-1}\circ D_t | \mathcal{F}_t\right) \\ &= B_t \mathbf{E}_{\mathbf{Q}}\left(\int_{t+}^T B_{u-}^{-1} dD_u + \int_t^{T-} B_u^{-1} dD_{u+} | \mathcal{F}_t\right), \quad \forall t\in[0,T]. \end{aligned} \tag{5.6}$$

Under $\mathbf{Q}$, the ex-dividend process, $B_t^{-1}X(t,T)$, is a local optional martingale – $B_s^{-1}X(s,T) = \mathbf{E}_{\mathbf{Q}}\left[B_t^{-1}X(t,T)|\mathcal{F}_s\right]$ for all $s \leq t$. Expression (5.6) is referred to as the risk-neutral valuation formula of a defaultable claim. See [53, 76, 89, 33] for its definition in usual probability spaces.

For brevity, write $X_t = X(t,T)$ and combine (5.2) with (5.6) and knowing that H is left continuous we obtain

$$X_t = B_t\mathbf{E}_{\mathbf{Q}}\left(B_T^{-1}\tilde{X} + \int_{t+}^{T} B_{u-}^{-1}(1-H_{u-})dA_u + \int_t^{T-} B_u^{-1}(1-H_u)dA_{u+} + \int_t^{T-} B_u^{-1}R_u dH_{u+}|\mathcal{F}_t\right).$$

Next we are going to provide a formal justification of Definition (5.6) based on the existence of local martingale deflator arguments. We will consider a portfolio of assets one of which is a defaultable cash-flow.

5.4 Portfolios with Defaultable Cash-Flows

Consider a portfolio of 3 primary securities: $S^2 = B$ the value process of a money market account, S^1 a default free non-dividend-paying assets, and S^0 a dividend paying asset, $S^0 = D$ with $D_0 = 0$ with the possibility of default. Introduce the discounted price processes $\tilde{S}^i$ by setting $\tilde{S}_t^i = S_t^i/S_t^2$. The market lifespan is the time interval $\left[0,\tilde{T}\right]$ and $\phi = (\phi^0, \phi^1, \phi^2)$ is an $\mathbf{F}$-optional self-financing trading strategy on $\left(S^0, S^1, S^2\right)$.

To begin with let us examine a simple trading strategy: suppose that at time 0 we purchase one unit of the 0^{th} asset at the initial price S_0^0 and hold it until time T, then invest all the proceeds from dividends in the money market account. More specifically, we consider a buy-and-hold strategy $\phi = (1, 0, \phi^2)$. Then, the associated wealth process U equals

$$U_t = S_t^0 + \phi_t^2 B_t, \quad \forall t \in [0,\tilde{T}], \tag{5.7}$$

with initial wealth $U_0 = S_0^0 + \phi_0^2 B_0$. Since ϕ is self-financing we obtain

$$U_t - U_0 = S_t^0 - S_0^0 + \phi^2 \circ B_t + D_t \tag{5.8}$$

and $B \circ \phi_t^2 + [\phi^2, B] = 0$, where D is the dividend payed by S^0. Now, lets us divide $(U_t - S_t^0)$ by B, and use the product rule,

$$\begin{aligned} d\left(B_t^{-1}\left(U_t - S_t^0\right)\right) &= B_t^{-1} d\left(U_t - S_t^0\right) + \left(U_t - S_t^0\right) dB_t^{-1} + d\left[B^{-1}, \left(U - S^0\right)\right]_t \\ &= \phi^2 B_t^{-1} dB_t + B_t^{-1} dD_t + \phi_t^2 B_t dB_t^{-1} + d\left[B^{-1}, \phi^2 B\right]_t, \end{aligned}$$

but since $B^{-1} \circ B + B \circ B^{-1} = B^{-2} \circ [B, B]$ and

$$\begin{aligned} \left[B^{-1}, \phi^2 B\right] &= \left[B^{-1}, \phi^2 \circ B + B \circ \phi^2 + \left[\phi^2, B\right]\right] = \phi^2 \circ \left[B^{-1}, B\right] \\ &= \phi^2 \circ \left[-B^{-2} \circ B + B^{-3} \circ [B, B], B\right] \\ &= -B^{-2} \phi^2 \circ [B, B], \end{aligned}$$

then we find the simple relation

$$\begin{aligned} B_t^{-1}\left(U_t - S_t^0\right) &= \phi^2 B_t^{-1} \circ B_t + \phi^2 B \circ B_t^{-1} - \phi^2 B^{-2} \circ [B, B]_t + B^{-1} \circ D_t \\ &= B^{-1} \circ D_t. \end{aligned}$$

And, on the interval $[t, T]$ we have

$$\begin{gathered} B_T^{-1}\left(U_T - S_T^0\right) - B_t^{-1}\left(U_t - S_t^0\right) = B^{-1} \circ D_T - B^{-1} \circ D_t, \\ B_T^{-1} U_T - B_t^{-1} U_t = B_T^{-1} S_T^0 - B_t^{-1} S_t^0 + B^{-1} \circ D_T - B^{-1} \circ D_t. \end{gathered} \tag{5.9}$$

Now we are ready to derive the risk-neutral valuation formula for the ex-dividend price S_t^0. To this end, we assume that our model admits a spot optional martingale measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ such that the discounted wealth process $B^{-1} U^{\phi}$ of any admissible self-financing trading strategy ϕ follow local optional martingales under $\mathbf{Q}$ with respect to the filtration $\mathbf{F}$. Moreover, we make an assumption that the market value at time t of the 0^{th} security comes exclusively from the future dividends stream; this amounts to postulate that $S_T^0 = 0$.

We shall refer to S^0 as the ex-dividend price of the 0^{th} asset – the defaultable claim. Given that,

$$\mathbf{E_Q}\left(B_T^{-1}U_T - B_t^{-1}U_t|\mathcal{F}_t\right) = 0,$$
$$\mathbf{E_Q}\left(B_t^{-1}S_t^0|\mathcal{F}_t\right) = B_t^{-1}S_t^0,$$
$$S_T^0 = 0,$$

and in conjunction with equation (5.9) we arrive at the definition of the value of defaultable claim,

$$\begin{aligned}\mathbf{E_Q}\left(B_T^{-1}U_T - B_t^{-1}U_t|\mathcal{F}_t\right) &= \mathbf{E_Q}\left(B_T^{-1}S_T^0 - B_t^{-1}S_t^0 + B^{-1}\circ D_T - B^{-1}\circ D_t|\mathcal{F}_t\right), \\ B_t^{-1}S_t^0 &= \mathbf{E_Q}\left(B^{-1}\circ D_T - B^{-1}\circ D_t|\mathcal{F}_t\right).\end{aligned}$$

Hence, $B_t^{-1}S_t^0$ is an $\mathbf{F}$ local optional martingale under $\mathbf{Q}$ and

$$B_t^{-1}S_t^0 = \mathbf{E_Q}\left(B^{-1}\circ D_T - B^{-1}\circ D_t|\mathcal{F}_t\right). \tag{5.10}$$

Let us now examine trading with a general self-financing trading strategy $\phi = (\phi^0, \phi^1, \phi^2)$. The associated wealth process is, $U_t(\phi) = \sum_{i=0}^{2}\phi_t^i S_t^i$. Since ϕ is self-financing then it must be that, $U_t(\phi) = U_0(\phi) + G_t(\phi)$ for every $t \in [0, \tilde{T}]$, where the gains process $G(\phi)$ is,

$$\begin{aligned}G_t(\phi) \quad : &= \phi^0\circ D_t + \sum_{i=0}^{2}\phi^i\circ S_t^i \\ &= \phi^0\circ\left(S_t^0 + D_t\right) + \phi^1\circ S_t^1 + \phi^2\circ S_t^2.\end{aligned}$$

As before, $S^2 = B$ the value of our money market account, $S_t^0 = X(t,T)$ is the ex-dividend paying asset and $S^1 = x$ is the default free, non-dividend paying instrument. The term "$\phi^0\circ\left(S_t^0 + D_t\right)$" of the gain process is the gain acquired as a result of trading ϕ^0 of the 0^{th} asset having current value S_t^0 and payed dividend D_t.

Next we show that for a general trading strategy that the discounted wealth process of a portfolio where one of its assets is a defaultable cash flow is a local optional martingale under

$\mathbf{Q}$. This theorem is particularly important if we want to find a hedging strategy for default or plan to price contingent claims in this market.

Theorem 87 *For any self-financing trading strategy, ϕ, the discounted wealth process $B_t^{-1}U_t(\phi)$ follows a local optional martingale under $(\mathbf{F}, \mathbf{Q})$.*

Proof. Given $\tilde{S}^i = B^{-1}S^i = S^i \circ B^{-1} + B^{-1} \circ S^i + \left[B^{-1}, S^i\right]$ and $\left[B^{-1}, D\right] = 0$ since D is finite variation optional process, then, by product rule we get,

$$
\begin{aligned}
B_t^{-1}U_t(\phi) &= U_t(\phi) \circ B_t^{-1} + B_t^{-1} \circ U_t(\phi) + \left[B^{-1}, U(\phi)\right]_t \\
&= B^{-1}\phi^0 \circ D_t + \sum_{i=0}^{2} B^{-1}\phi^i \circ S_t^i + \sum_{i=0}^{2} \phi^i S^i \circ B_t^{-1} \\
&\quad + \left[B^{-1}, U_0(\phi) + \phi^0 \circ D_t + \sum_{i=0}^{2} \phi^i \circ S_t^i\right] \\
&= B^{-1}\phi^0 \circ D_t + \sum_{i=0}^{2} B^{-1}\phi^i \circ S_t^i + \sum_{i=0}^{2} \phi^i S^i \circ B_t^{-1} + \phi^0 \circ \left[B^{-1}, D\right]_t \\
&\quad + \sum_{i=0}^{2} \phi^i \circ \left[B^{-1}, S^i\right]_t \\
&= \phi^0 B^{-1} \circ D_t + \phi^0 \circ \left[B^{-1}, D\right]_t + \sum_{i=0}^{2} \phi^i B^{-1} \circ S_t^i + \phi^i S^i \circ B_t^{-1} + \phi^i \circ \left[B^{-1}, S^i\right]_t \\
&= \phi^0 B^{-1} \circ D_t + \sum_{i=0}^{2} \phi^i \circ B_t^{-1} S_t^i = \phi^0 \circ \hat{S}_t^0 + \sum_{i=1}^{2} \phi^i \circ \tilde{S}_t^i
\end{aligned}
$$

where the process $\hat{S}^0$ is given by the formula $\hat{S}_t^0 := \tilde{S}_t^0 + B^{-1} \circ D_t$. $\tilde{S}^i$ are local optional martingales in $(\mathbf{F}, \mathbf{Q})$. So, to finalize the proof, it suffices to observe that in view of equation (5.10) the process $\hat{S}^0$ satisfies $\hat{S}_t^0 = \mathbf{E}_{\mathbf{Q}}\left(B^{-1} \circ D_T | \mathcal{F}_t\right)$, and thus it follows a local martingale under $\mathbf{Q}$. ■

It is important to notice that $\hat{S}_t^0$ is the discounted cumulative dividend price at time t of the 0^{th} asset. Furthermore, if we assume that the local martingale measure $\mathbf{Q}$ is unique then every integrable contingent claim is attainable and the valuation formula can be justified by means of replication. Otherwise, when a martingale probability measure is not unique the right-hand side of formula (5.10) may depend on the choice of a particular martingale measure.

Notation 88 *From now we are going to work in the deflated probability space* $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{Q})$. *Therefore, we will use the expectation operator* $\mathbf{E}$ *to mean* $\mathbf{E_Q}$ *the expected values under the measure* $\mathbf{Q}$.

5.5 The Probability of Default

An essential component of the defaultable cash-flow D equation (5.2) is the default process $H_t = \mathbf{1}_{(\tau<t)}$. The ex-dividend price process equation (5.6) relies implicitly on of conditional expected value of $\mathbf{1}_{(\tau<t)}$ given all known up to time t market information $\mathcal{F}_t$. Our goal here is to study the following process,

$$F_t = \mathbf{Q}\left(\tau < t|\mathcal{F}_t\right),$$

known as the conditional probability of default or the *hazard process* and its properties. Associate with the hazard process is the *survival process*, G,

$$G_t := 1 - F_t = \mathbf{Q}\left(\tau \geq t|\mathcal{F}_t\right).$$

We begin by showing that F is a submartingale.

Lemma 89 *The process* F (G, *resp.*) *follow a bounded, non-negative* $\mathbf{F}$-*sub*(*super, resp.*) *martingale under* $\mathbf{Q}$.

Proof. Consider $(\tau < t) \subseteq (\tau < s)$ then for any $0 \leq t \leq s$ we have

$$\begin{aligned} \mathbf{E}(F_s|\mathcal{F}_t) &= \mathbf{E}(\mathbf{Q}\left(\tau < s|\mathcal{F}_s\right)|\mathcal{F}_t) \\ &= \mathbf{Q}\left(\tau < s|\mathcal{F}_t\right) \geq \mathbf{Q}\left(\tau < t|\mathcal{F}_t\right) = F_t, \end{aligned}$$

and so the process F (the survival process G, resp) follows a bounded, non-negative $\mathbf{F}$-submartingale ($\mathbf{F}$-supermartingale, resp.) under $\mathbf{Q}$. ■

Furthermore, since $F \geq 0$ then let Γ be such that $1 - F_t = \exp(-\Gamma_t)$. Γ is the *hazard-rate process.* Equivalently, it can be defined as

Definition 90 *Suppose $F \geq 0$ then the hazard-rate process is given by*

$$\Gamma_t := -\ln G_t = -\ln(1 - F_t), \quad \forall t \in \mathbb{R}_+. \tag{5.11}$$

and since $G_0 = 1$, it is clear that $\Gamma_0 = 0$. In view of the equality $\mathbf{Q}(\tau < +\infty) = 1$ then it must be that $\lim_{t\to\infty} \Gamma_t = \infty$.

Now that we have defined default process and associated hazard-rate process in the context of *unusual probability spaces* there are few points we must consider before moving forward with pricing and hedging of defaultable claims on unusual spaces. In the classical theory of default processes the default process is defined as $\mathbf{1}_{(\tau \leq t)}$ and its hazard process is defined as $\mathbf{E}\left(\mathbf{1}_{(\tau \leq t)} | \mathcal{G}_t\right)$, where τ is an inaccessible stopping time with respect to $\mathcal{G}_t$. In our formalism, $\mathcal{F}_{t+}$ takes the place of $\mathcal{G}_t$. So, why is that we didn't consider defining the hazard process by,

$$\mathbf{E}(\mathbf{1}_{(\tau \leq t)} | \mathcal{F}_{t+}) = \mathbf{Q}(\tau \leq t | \mathcal{F}_{t+})?$$

We didn't define the hazard process by $\mathbf{Q}(\tau \leq t | \mathcal{F}_{t+})$ for several reasons; First, if we suppose $(\tau = t)$ then $\mathcal{F}_{t+}$ contain information about the market after default has occurred. This information is not available to market participants before default. Therefore, it is sound to consider that the information known to market participants is $\mathcal{F}_t$ which is the information upon which pricing and hedging must be carried on; Second, in the classical theory the default-free market filtration is assume to be right-continuous, $\mathcal{F}_{t+} = \mathcal{F}_t$, hence τ must be considered as an event outside the underlying filtration associated with default-free market processes. Whereas in the new theory $(\tau = t)$ information is incorporated in the market immediate future information, $\mathcal{F}_{t+}$.

However, it is still interesting to consider the relation between the *"ex-hazard" process*, $\mathbf{Q}(\tau \leq t|\mathcal{F}_{t+})$ and the hazard process $\mathbf{Q}(\tau < t|\mathcal{F}_t)$. So, let $\bar{F}_t = \mathbf{Q}(\tau \leq t|\mathcal{F}_{t+})$ and consider some of its properties. We begin by the simple result;

Lemma 91 *The ex-hazard process $\bar{F}$ can be decomposed to the hazard process F and the jump hazard process δ. Both $\bar{F}$ and δ are $\mathbf{F}_+$ measurable while$F$ is $\mathbf{F}$ measurable.*

Proof. Consider $\mathbf{E}(\bar{F}_t|\mathcal{F}_t)$,

$$\begin{aligned}
\mathbf{E}(\bar{F}_t|\mathcal{F}_t) &= \mathbf{E}(\mathbf{E}(\mathbf{1}_{(\tau\leq t)}|\mathcal{F}_{t+})|\mathcal{F}_t) = \mathbf{E}(\mathbf{1}_{(\tau\leq t)}|\mathcal{F}_t) \\
&= \mathbf{E}(\mathbf{1}_{(\tau<t)} + \mathbf{1}_{(\tau=t)}|\mathcal{F}_t) = F_t + \mathbf{E}(\mathbf{1}_{(\tau=t)}|\mathcal{F}_t) \\
&= F_t + \delta_t
\end{aligned}$$

where we have used $\mathbf{1}_{(\tau\leq t)} = \mathbf{1}_{(\tau<t)} + \mathbf{1}_{(\tau=t)} = H_t + \mathbf{1}_{(\tau=t)}$. ■

We can find δ_t by formula

$$\delta_t = \lim_{h\to 0} \frac{1}{h}\mathbf{Q}(t \leq \tau < t+h|\mathcal{F}_t).$$

Moreover, we can take the relation between events in $\mathbf{F}_+$ to events in $\mathbf{F}$ further. To this end, we would like to compute the conditional expectation, $\mathbf{E}(\mathbf{1}_{(\tau\geq t)}Y|\mathcal{F}_{t+})$ or $\mathbf{E}(\mathbf{1}_{(\tau>t)}Y|\mathcal{F}_{t+})$ such that Y is $\mathbf{F}_+$-measurable in terms of conditional expectation with respect to $\mathbf{F}$. The following lemma helps us do just that.

Lemma 92 *For any $\mathbf{F}_+$-measurable random variable Y and any $t \geq 0$ we have*

$$\mathbf{E}(\mathbf{1}_{(\tau\geq t)}Y|\mathcal{F}_{t+}) = \mathbf{Q}(\tau \geq t|\mathcal{F}_{t+}) \frac{\mathbf{E}(\mathbf{1}_{(\tau\geq t)}Y|\mathcal{F}_t)}{\mathbf{Q}(\tau \geq t|\mathcal{F}_t)}. \tag{5.12}$$

But, since H_t is $\mathcal{F}_{t+}$ measurable then

$$\mathbf{E}(\mathbf{1}_{(\tau\geq t)}Y|\mathcal{F}_{t+}) = \mathbf{1}_{(\tau\geq t)}\mathbf{E}(Y|\mathcal{F}_{t+}) = \mathbf{1}_{(\tau\geq t)}\frac{\mathbf{E}(\mathbf{1}_{(\tau\geq t)}Y|\mathcal{F}_t)}{\mathbf{Q}\left(\tau\geq t|\mathcal{F}_t\right)}. \tag{5.13}$$

In particular, for any $t<s$,

$$\mathbf{E}(\mathbf{1}_{(\tau\geq t)}\mathbf{1}_{(\tau<s)}|\mathcal{F}_{t+}) = \mathbf{Q}\left(t\leq\tau<s|\mathcal{F}_{t+}\right) = \mathbf{1}_{(\tau\geq t)}\frac{\mathbf{Q}\left(t\leq\tau<s|\mathcal{F}_t\right)}{\mathbf{Q}\left(\tau\geq t|\mathcal{F}_t\right)}. \quad (*) \tag{5.14}$$

Proof. It is enough to establish the first statement. Let us denote $C=(\tau\geq t)$. To prove, we need to verify that

$$\mathbf{E}\left(\mathbf{1}_C Y\mathbf{Q}(C|\mathcal{F}_t)|\mathcal{F}_{t+}\right) = \mathbf{E}(\mathbf{1}_C\mathbf{E}(\mathbf{1}_C Y|\mathcal{F}_t)|\mathcal{F}_{t+}).$$

Put another way, we need to show that for any $A\in\mathcal{F}_{t+}$ we have

$$\int_A \mathbf{1}_C Y\mathbf{Q}(C|\mathcal{F}_t)d\mathbf{Q} = \int_A \mathbf{1}_C\mathbf{E}(\mathbf{1}_C Y|\mathcal{F}_t)d\mathbf{Q}.$$

Since that, for any $A\in\mathcal{F}_{t+}$ there exist a $B\in\mathcal{F}_t$, such that $A\cap C=B\cap C$ for all t, so

$$\begin{aligned}
\int_A \mathbf{1}_C Y\mathbf{Q}(C|\mathcal{F}_t)d\mathbf{Q} &= \int_{A\cap C} Y\mathbf{Q}(C|\mathcal{F}_t)d\mathbf{Q} = \int_{B\cap C} Y\mathbf{Q}(C|\mathcal{F}_t)d\mathbf{Q} \\
&= \int_B \mathbf{1}_C Y\mathbf{Q}(C|\mathcal{F}_t)d\mathbf{Q} = \int_B \mathbf{E}(\mathbf{1}_C Y|\mathcal{F}_t)\mathbf{Q}(C|\mathcal{F}_t)d\mathbf{Q} \\
&= \int_B \mathbf{E}(\mathbf{1}_C\mathbf{E}(\mathbf{1}_C Y|\mathcal{F}_t)|\mathcal{F}_t)d\mathbf{Q} = \int_{B\cap C}\mathbf{E}(\mathbf{1}_C Y|F_t)d\mathbf{Q} \\
&= \int_{A\cap C}\mathbf{E}(\mathbf{1}_C Y|\mathcal{F}_t)d\mathbf{Q} = \int_A \mathbf{1}_C\mathbf{E}(\mathbf{1}_C Y|\mathcal{F}_t)d\mathbf{Q}.
\end{aligned}$$

■

If the hazard rate process Γ exists then one can deduce the following lemma,

Lemma 93 $\mathbf{Q}\left(t\leq\tau<T|\mathcal{F}_{t+}\right) = \mathbf{1}_{(\tau\geq t)}\mathbf{E}\left(1-e^{\Gamma_t-\Gamma_T}|\mathcal{F}_t\right).$

Proof. Combining formula (5.14) with the definition of the hazard process in terms of the Γ process we obtain,

$$\begin{aligned}\mathbf{Q}\left(t \leq \tau < T|\mathcal{F}_{t+}\right) &= \mathbf{1}_{(\tau \geq t)} \frac{\mathbf{Q}\left(t \leq \tau < T|\mathcal{F}_t\right)}{\mathbf{Q}\left(\tau \geq t|\mathcal{F}_t\right)} = \mathbf{1}_{(\tau \geq t)} \frac{\mathbf{E}(\mathbf{1}_{(\tau \geq t)}\mathbf{1}_{(\tau < T)}|\mathcal{F}_t)}{\exp\left(-\Gamma_t\right)} \\ &= \mathbf{1}_{(\tau \geq t)} e^{\Gamma_t} \mathbf{E}\left(e^{-\Gamma_t} - e^{-\Gamma_T}|\mathcal{F}_t\right) \\ &= \mathbf{1}_{(\tau \geq t)} \mathbf{E}\left(1 - e^{\Gamma_t - \Gamma_T}|\mathcal{F}_t\right).\end{aligned}$$

■

Moreover, it is convenient to postulated that Γ_t is absolutely continuous with respect to a measure m_t. Specifically, we assume that the hazard process Γ of τ admits the following integral representation,

$$\Gamma_t = \gamma \circ m_t, \quad \forall t \geq 0,$$

for some non-negative, $\mathbf{F}$-progressively measurable stochastic process γ, with integrable sample paths. In addition, we assume that $\gamma \circ m_\infty = \infty$, $\mathbf{Q}$-a.s. The process γ is called the $\mathbf{F}$-intensity of τ. It is also customary to refer to γ as the stochastic intensity of τ, especially when the choice of the reference filtration $\mathbf{F}$ is clear from the context. In terms of the stochastic intensity of a default time, the conditional probability of the default event $(t \leq \tau < T)$, given the information $\mathcal{F}_t$ available at time t, equals,

Corollary 94

$$\mathbf{Q}\left(t \leq \tau < T|\mathcal{F}_{t+}\right) = \mathbf{1}_{(\tau \geq t)} \mathbf{E}\left(1 - e^{-(\gamma \circ m_T - \gamma \circ m_t)}|\mathcal{F}_t\right), \tag{5.15}$$

Since the event $(\tau \leq t)$ manifestly belongs to the σ-field $\mathcal{F}_{t+}$, we have

$$\mathbf{Q}\left(\tau < T|\mathcal{F}_{t+}\right) = H_t + \mathbf{1}_{(\tau \geq t)} \mathbf{E}\left(1 - e^{-(\gamma \circ m_T - \gamma \circ m_t)}|\mathcal{F}_t\right).$$

Since the event $(\tau > t)$ *also belongs to* $\mathcal{F}_{t+}$, *we obtain*

$$\mathbf{Q}\left(t \leq \tau < T|\mathcal{F}_{t+}\right) + \mathbf{Q}\left(\tau > T|\mathcal{F}_{t+}\right) = \mathbf{Q}\left(\tau \geq t|\mathcal{F}_{t+}\right) = \mathbf{1}_{(\tau \geq t)},$$

so that the conditional probability of the non-default event $(\tau > T)$ *equals*

$$\mathbf{Q}\left(\tau \geq t|\mathcal{F}_{t+}\right) = \mathbf{1}_{(\tau \geq t)}\mathbf{E}\left(e^{-(\gamma \circ m_T - \gamma \circ m_t)}|\mathcal{F}_t\right).$$

Proof. Obvious. ■

Finally, the default time τ is a stopping time in the broad sense meaning that $\tau \in \mathcal{F}_{\tau+}$ or the set $(\tau \leq t) \in \mathcal{F}_{t+}$. Therefore, since $(\tau \leq t - 1/n) \in \mathcal{F}_{(t-1/n)+} \subseteq \mathcal{F}_t$ for all n then it must be that $(\tau < t) \in \mathcal{F}_t$. Galchuk [10] showed that for the process $\xi \mathbf{1}_{(\tau < t)}$ where ξ is $\mathcal{F}_{\tau+}$-measurable and integrable random variable then it can be decomposed to an optional martingale that is $\mathcal{F}_t$ measurable and an $\mathcal{F}_t$-measurable continuous finite variation process. Hence for the hazard-process H_t we will have the following decomposition,

$$H_t = \mathbf{1}_{(\tau < t)} = M_t + \mu_t$$

where $M \in \mathcal{M}(\mathbf{F}, \mathbf{Q})$ a *optional martingale* that is $\mathbf{F}$-measurable and μ is $\mathbf{F}$-measurable continuous finite variation process.

Now let us consider, evaluating $\mathbf{E}\left[H_T|\mathcal{F}_t\right]$, for $T > t$

$$\begin{aligned} F_t &= \mathbf{E}\left[\mathbf{1}_{(\tau < T)}|\mathcal{F}_t\right] = \mathbf{Q}\left(\mathbf{1}_{(\tau < T)}|\mathcal{F}_t\right) = \mathbf{E}\left[M_T + \mu_T|\mathcal{F}_t\right] = M_t + \mathbf{E}\left[\mu_T|\mathcal{F}_t\right] \\ &= H_t + \mathbf{E}\left[\mu_T - \mu_t|\mathcal{F}_t\right]. \end{aligned}$$

Consequently one can write,

$$\begin{aligned}\mathbf{Q}\left(t \leq \tau < T|\mathcal{F}_t\right) &= \left[\left(1-H_t\right)H_T|\mathcal{F}_t\right] = \mathbf{E}\left[H_T - H_t|\mathcal{F}_t\right] = \mathbf{E}\left[H_T|\mathcal{F}_t\right] - H_t \\ &= \mathbf{E}\left[\mu_T - \mu_t|\mathcal{F}_t\right] = \mathbf{E}\left[\mu_T|\mathcal{F}_t\right] - \mu_t.\end{aligned}$$

This ends our exposition about probability of default. Now we go back to defaultable claim and evaluation of defaultable dividends process.

5.6 Valuation of a Defaultable Claims

Our next goal is to establish a convenient representation of the value of a defaultable claim in terms of the probability of default. The *ex-dividend* value of defaultable claim is,

$$B_t^{-1}X(t,T) = \mathbf{1}_{(\tau \geq t)}\mathbf{E}\left(B^{-1}\circ D_T - B^{-1}\circ D_t|\mathcal{F}_t\right)$$

and in terms of $D_t = \left(\Lambda\left(1-H_T\right)+\rho H_T\right)\mathbf{1}_{(t\geq T)} + (1-H)\circ A_t + R\circ H_t$,

$$\begin{aligned}B_t^{-1}X_t &= \mathbf{E}\left[B_T^{-1}\left(\Lambda\left(1-H_T\right)+\rho H_T\right)|\mathcal{F}_t\right] \\ &\quad +\mathbf{E}\left[\int_{t+}^{T} B_{u-}^{-1}(1-H_{u-})dA_u + \int_t^{T-} B_u^{-1}(1-H_u)dA_{u+}|\mathcal{F}_t\right] \\ &\quad +\mathbf{E}\left[\int_t^{T-} B_u^{-1}R_u dH_{u+}|\mathcal{F}_t\right].\end{aligned}$$

First we will start with the easy case $\mathbf{E}\left[B_T^{-1}\Lambda\left(1-H_T\right)|\mathcal{F}_t\right]$. Let $\lambda_T = B_T^{-1}\Lambda$ therefore $\lambda_t = \mathbf{E}\left(\lambda_T|\mathcal{F}_t\right)$ is a martingale. λ_t can be thought of as the *default-free* contingent claim price at time t. Also, let $g_t = \mathbf{E}\left(G_T|\mathcal{F}_t\right)$ where $G_T = \mathbf{E}\left(\left(1-H_T\right)|\mathcal{F}_T\right)$. Hence, $g_t = \mathbf{E}\left(1-H_T|\mathcal{F}_t\right)$. g is also a martingale since $g_s = \mathbf{E}\left(g_t|\mathcal{F}_s\right)$ for any $s<t$ and $g_T = \mathbf{E}\left(G_T|\mathcal{F}_T\right) = G_T = \mathbf{E}\left(1-H_T|\mathcal{F}_T\right)$.

Lemma 95 *The value of* $\tilde{\Lambda}_t = \mathbf{E}\left[B_T^{-1}\Lambda\left(1-H_T\right)|\mathcal{F}_t\right]$ *at time* t *is given by*

$$\tilde{\Lambda}_t = \mathbf{E}\left(B_T^{-1}\Lambda|\mathcal{F}_t\right)G_t + \mathbf{E}\left[\int_t^{T-}\left[\mathbf{E}\left(B_T^{-1}\Lambda|\mathcal{F}_u\right) + \triangle^{+}\mathbf{E}\left(B_T^{-1}\Lambda|\mathcal{F}_u\right)\right]dG_{u+}|\mathcal{F}_t\right]. \quad (5.16)$$

Proof. Using the product rule for $\lambda_T g_T$ we find that,

$$\lambda_T (1 - H_T) = \lambda_t (1 - H_t) + \int_t^T \lambda_u d(1 - H_u) + \int_t^T (1 - H_u) d\lambda_u + \int_t^{T-} \triangle^+ \lambda_u d(1 - H_u).$$

Since H is a left-continuous finite variation optional process with a left jump then the quadratic variation,

$$[\lambda, H]_t = \sum_{0 \leq u < t} \triangle^+ \lambda_u \triangle^+ H_{u+} = \int_0^{t-} \triangle^+ H_u d\lambda_{u+} = \int_0^{t-} \triangle^+ \lambda_u dH_{u+}.$$

We choose the definition $[\lambda, H] = \triangle^+ \lambda \odot H$. The conditional expected value of $\lambda_T (1 - H_T)$ is

$$\mathbf{E}(\lambda_T (1 - H_T) | \mathcal{F}_t) = \mathbf{E}\left(B_T^{-1} \Lambda (1 - H_T) | \mathcal{F}_t\right)$$

and

$$\begin{aligned} \mathbf{E}(\lambda_T (1 - H_T) | \mathcal{F}_t) &= \mathbf{E}\left[\lambda_t (1 - H_t) + \int_t^T (1 - H_u) d\lambda_u + \int_t^{T-} \lambda_u d(1 - H_{u+}) | \mathcal{F}_t\right] \\ &\quad + \mathbf{E}\left[\int_t^{T-} \triangle^+ \lambda_u d(1 - H_{u+}) | \mathcal{F}_t\right] \\ &= \lambda_t \mathbf{E}(1 - H_t | \mathcal{F}_t) + \mathbf{E}\left[\int_t^{T-} \left(\lambda_u + \triangle^+ \lambda_u\right) dG_{u+} | \mathcal{F}_t\right]. \end{aligned}$$

where we have used the fact that $\mathbf{E}\left[\int_t^T g_u d\lambda_u | \mathcal{F}_t\right] = 0$ since λ is a local martingale. And,

$$\mathbf{E}(\lambda_t (1 - H_t) | \mathcal{F}_t) = \lambda_t \mathbf{E}(1 - H_t | \mathcal{F}_t).$$

Thus, we arrive at the result,

$$\mathbf{E}\left[B_T^{-1} \Lambda (1 - H_T) | \mathcal{F}_t\right] = \mathbf{E}\left(B_T^{-1} \Lambda | \mathcal{F}_t\right) G_t + \mathbf{E}\left[\int_t^{T-} \left[\mathbf{E}\left(B_T^{-1} \Lambda | \mathcal{F}_u\right) + \triangle^+ \mathbf{E}\left(B_T^{-1} \Lambda | \mathcal{F}_u\right)\right] dG_{u+} | \mathcal{F}_t\right].$$

■

Remark 96 *Note also that we have replaced $1 - H$ with its survival process G. This is the* *result of the following statement,*

$$\begin{aligned}\mathbf{E}\left[\int_t^{T-} \alpha_u dH_{u+}|\mathcal{F}_t\right] &\leftarrow \sum_{i=0}^{n} \mathbf{E}\left[\alpha_{t_i} \mathbf{E}\left(H_{t_{i+1}\wedge T} - H_{t_i\wedge T}|\mathcal{F}_{t_i}\right)|\mathcal{F}_t\right] \\ &= \sum_{i=0}^{n} \mathbf{E}\left[\alpha_{t_i} G_{t_{i+1}\wedge T} - G_{t_i\wedge T}|\mathcal{F}_t\right] \\ &= \mathbf{E}\left[\left(\sum_{i=0}^{n} \alpha_{t_i} G_{t_{i+1}\wedge T} - G_{t_i\wedge T}\right)|\mathcal{F}_t\right].\end{aligned}$$

For the second term, $\mathbf{E}\left(B_T^{-1}\rho H_T|\mathcal{F}_t\right)$: let $\tilde{\rho}_T = B_T^{-1}\rho$ then $\tilde{\rho}_t = \mathbf{E}\left(\tilde{\rho}_T|\mathcal{F}_t\right)$ is a martingale.

Lemma 97 *The value of* $\tilde{\rho}_t = \mathbf{E}\left(\tilde{\rho}_T|\mathcal{F}_t\right)$ *is given by*

$$\tilde{\rho}_t = \mathbf{E}\left(\tilde{\rho}_T|\mathcal{F}_t\right) = \tilde{\rho}_t\left(1 - G_t\right) - \mathbf{E}\left[\int_t^{T-} \left(\tilde{\rho}_u + \triangle^+\tilde{\rho}_u\right) dG_{u+}|\mathcal{F}_t\right]$$

Proof. Applying similar algebraic methods as the ones we have used in the above lemma we arrive at

$$\begin{aligned}\mathbf{E}\left(\tilde{\rho}_T H_T|\mathcal{F}_t\right) &= \mathbf{E}\left[\tilde{\rho}_t H_t + \int_t^T H_u d\tilde{\rho}_u + \int_t^{T-} \tilde{\rho}_u dH_u + \int_t^{T-} \triangle^+\tilde{\rho}_u dH_u|\mathcal{F}_t\right] \\ &= \tilde{\rho}_t\left(1 - G_t\right) - \mathbf{E}\left[\int_t^{T-} \left(\tilde{\rho}_u + \triangle^+\tilde{\rho}_u\right) dG_{u+}|\mathcal{F}_t\right].\end{aligned}$$

■

Lemma 98 *The value of defaultable claim recovery stream is*

$$\mathbf{E}\left[\int_t^{T-} B_u^{-1} R_u dH_{u+}|\mathcal{F}_t\right] = -\mathbf{1}_{(t\leq\tau<T)}\mathbf{E}\left[\int_t^{T-} B_u^{-1} R_u dG_{u+}|\mathcal{F}_t\right]$$

Proof. As for the third term of the ex-dividend value of defaultable claim we simply find that

$$\begin{aligned}\mathbf{E}\left[\int_t^{T-} B_u^{-1} R_u dH_{u+}|\mathcal{F}_t\right] &= \mathbf{1}_{(t\leq\tau<T)}\mathbf{E}\left[\int_t^{T-} B_u^{-1} R_u dF_{u+}|\mathcal{F}_t\right] \\ &= -\mathbf{1}_{(t\leq\tau<T)}\mathbf{E}\left[\int_t^{T-} B_u^{-1} R_u dG_{u+}|\mathcal{F}_t\right]\end{aligned}$$

■

Finally we look at the process $\mathbf{E}\left[B^{-1}(1-H)\circ A_T - B^{-1}(1-H)\circ A_t|\mathcal{F}_t\right]$.

Lemma 99 *The value of dividend payout stream is given by*

$$\begin{aligned}&\mathbf{E}\left[B^{-1}(1-H)\circ A_T - B^{-1}(1-H)\circ A_t|\mathcal{F}_t\right] \\ &\quad= \mathbf{E}\left[\int_{t+}^{T} B_{u-}^{-1}(1-F_{u-})dA_u + \int_t^{T-} B_u^{-1}(1-F_u)dA_{u+}|\mathcal{F}_t\right] \\ &\quad= \mathbf{E}\left[\int_{t+}^{T} B_{u-}^{-1}G_{u-}dA_u + \int_t^{T-} B_u^{-1}G_u dA_{u+}|\mathcal{F}_t\right].\end{aligned}$$

Proof. We find that

$$\begin{aligned}&\mathbf{E}\left[B^{-1}(1-H)\circ A_T - B^{-1}(1-H)\circ A_t|\mathcal{F}_t\right] \\ &\quad= \mathbf{E}\left[\int_{t+}^{T} B_{u-}^{-1}(1-H_{u-})dA_u + \int_t^{T-} B_u^{-1}(1-H_u)dA_{u+}|\mathcal{F}_t\right] \\ &\quad= \mathbf{E}\left[\int_{t+}^{T} B_{u-}^{-1}(1-F_{u-})dA_u + \int_t^{T-} B_u^{-1}(1-F_u)dA_{u+}|\mathcal{F}_t\right].\end{aligned}$$

This result is simply a consequence of the definition of the stochastic integral involving optional processes. ■

This essentially ends the valuation of defaultable cash-flow. We have been able to establish a convenient representation of the value of a defaultable claim in terms of the probability of default. Now we provide some examples.

5.7 Illustrative Examples

5.7.1 Zero-Coupon Defaultable Bond

The price of a zero-coupon bond with value \$1 at maturity is $B_t\mathbf{E}\left(B_T^{-1}|\mathcal{F}_t\right)$, however, for a bond that experience default we must compute $B_t\mathbf{E}\left(B_T^{-1}\mathbf{1}_{(\tau\geq T)}|\mathcal{F}_t\right)$. Lemma (95) tells how to compute the current price of $B_t\mathbf{E}\left(B_T^{-1}\mathbf{1}_{(\tau\geq T)}|\mathcal{F}_t\right)$. Let $\lambda_T = B_T^{-1}$ and $\lambda_t = \mathbf{E}\left(\lambda_T|\mathcal{F}_t\right)$ then

$$\begin{aligned}
\mathbf{E}\left(\lambda_T\left(1-H_T\right)|\mathcal{F}_t\right) &= \mathbf{E}\left[\lambda_t\left(1-H_t\right)+\int_t^T\left(1-H_u\right)d\lambda_u+\int_t^{T-}\lambda_u d\left(1-H_{u+}\right)|\mathcal{F}_t\right] \\
&\quad +\mathbf{E}\left[\int_t^{T-}\triangle^+\lambda_u d\left(1-H_{u+}\right)+\int_t^{T-}\triangle^+\lambda_u d\left(1-H_{u+}\right)|\mathcal{F}_t\right] \\
&= \lambda_t\mathbf{E}\left(1-H_t|\mathcal{F}_t\right)+\mathbf{E}\left[\int_t^{T-}\left(\lambda_u+\triangle^+\lambda_u\right)dG_{u+}|\mathcal{F}_t\right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
B_t^{-1}X(t,T) &= \mathbf{E}\left(B_T^{-1}\mathbf{1}_{(\tau\geq T)}|\mathcal{F}_t\right)=\mathbf{E}\left(B_T^{-1}\mathbf{1}_{(\tau\geq T)}|\mathcal{F}_t\right) \\
&= \mathbf{1}_{(t>\tau)}\lambda_t G_t+\mathbf{1}_{(t\leq\tau<T)}\mathbf{E}\left[\int_t^{T-}\left(\lambda_u+\triangle^+\lambda_u\right)dG_{u+}|\mathcal{F}_t\right].
\end{aligned}$$

Suppose that, $B_t = e^{rt}$, with a constant interest rate r and the survival process admits a constant intensity λ, $G_t = e^{-\lambda t}$. Then,

$$\begin{aligned}
X(t,T) &= \mathbf{1}_{(t>\tau)}e^{-r(T-t)}e^{-\lambda t}+\mathbf{1}_{(t\leq\tau<T)}e^{rt}\mathbf{E}\left[\int_t^T-\lambda e^{-rT}e^{-\lambda u}du|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\left[\mathbf{1}_{(t>\tau)}e^{-\lambda t}+\mathbf{1}_{(t\leq\tau<T)}\left(e^{-\lambda T}-e^{-\lambda t}\right)\right] \\
&= e^{-r(T-t)}e^{-\lambda t}\left(\mathbf{1}_{(t>\tau)}+\mathbf{1}_{(t\leq\tau<T)}\left(e^{-\lambda(T-t)}-1\right)\right).
\end{aligned}$$

5.7.2 Credit Default Swap

A credit default swap is a contract in which the holder of a defaultable asset buys an insurance against the default of an asset. If the maturity time is T, the fee rate function is $\kappa(t)$ and the recovery function is $\delta(t)$ are given then a CDS of characteristics (T, κ, δ) is the contract in which the buyer of default protection pays a fee at a rate κ up to default time or to maturity and receives the amount $\delta(\tau)$ at default from the protection seller. The price of the CDS at time t is given by the difference of the value of the protective leg and premium leg:

$$CDS(t,T) = \mathrm{Prot}_t - \mathrm{Prem}_t \equiv \mathbf{E}\left(\delta(\tau)\mathbf{1}_{t\leq\tau<T}|\mathcal{F}_t\right) - \mathbf{E}\left(\mathbf{1}_{t\leq\tau}\int_t^{\tau\wedge T}\kappa(s)ds|\mathcal{F}_t\right)$$

Since the survival process can be decompose to $G = M - \mu$ then every leg writes:

$$\mathrm{Prot}_t = \mathbf{1}_{(t\leq\tau<T)}\mathbf{E}\left(\int_t^T \delta(s)dH_s|\mathcal{F}_t\right) = \mathbf{1}_{(t\leq\tau<T)}e^{\Gamma_t}\mathbf{E}\left(\int_t^T \delta(s)dA_s|\mathcal{F}_t\right)$$

and

$$\mathrm{Prem}_t = \mathbf{1}_{t\leq\tau}\mathbf{E}\left(\int_t^T (1-H_s)\kappa(s)ds|\mathcal{F}_t\right) = \mathbf{1}_{t\leq\tau}e^{\Gamma_t}\mathbf{E}\left(\int_t^T \kappa(s)e^{-\Gamma_s}ds|\mathcal{F}_t\right)$$

Putting the two legs together we get the price of the CDS:

$$CDS(t,T) = \mathbf{1}_{t\leq\tau}e^{\Gamma_t}\mathbf{E}\left(\int_t^T \delta(s)dA_s - \kappa_s e^{-\Gamma_t}ds|\mathcal{F}_t\right).$$

This essentially ends the valuation of defaultable cash-flow. Now we construct the general default process.

5.8 General Default Process

In structural modeling on usual probability space the default time turns out to be predictable. However, on unusual probability spaces where the market filtration is not right continuous and market processes are not necessary right or left continuous a default time given by,

$$\tau := \inf\{t > 0 : t \geq 0, X_t \leq B_t\}$$

where the a barrier process, B, and value process X are optional process is not necessarily predictable. In fact, τ could either be predictable, accessible or inaccessible stopping time or an inaccessible stopping time in the broad sense depending on the interaction between X and B. For example, suppose X is continuous but B has jump at an inaccessible stopping time in the broad sense σ, such that $X_\sigma > B_\sigma$, then $\tau = \sigma$ a default time that is not predictable. This simple result lead us to realize a form of a generalization of the default process upon which one need not distinguish between structural and reduced form models. This generalization is based on two theorems: the decomposition of optional processes to continuous discrete right continuous and discrete left continuous parts and that there existence a sequence of stopping times that are predictable and inaccessible and inaccessible stopping time in the broad sense that absorbed all the jumps of an optional process.

To construct the general default process we need to make use of the component representation of optional processes. Therefore, we are going to consider the Lusin space $(\mathbb{E}, \mathscr{E})$ where $\mathbb{E} = \{1\} \cup \{\delta^d\} \cup \{\delta^g\}$; δ^d and δ^g are some supplementary points or is the set of processes with finite variation on any segment $[0, t]$, $\mathbf{P}$-a.s.; $\mathscr{E} = \mathcal{B}(\mathbb{E})$ is the σ-algebra in $\mathbb{E}$. Also, define the spaces

$$\widetilde{\Omega} = \Omega \times \mathbb{R}_+ \times \mathbb{E}, \quad \widetilde{\mathbb{E}} = \mathbb{R}_+ \times \mathbb{E}, \quad \widetilde{\mathscr{E}} = \mathcal{B}(\mathbb{R}_+) \times \mathscr{E}, \quad \widetilde{\mathcal{G}} = \mathcal{G} \times \mathcal{B}(\mathbb{E}),$$
$$\widetilde{\mathcal{O}}(\mathbf{F}) = \mathcal{O}(\mathbf{F}) \times \mathscr{E}, \quad \widetilde{\mathcal{O}}(\mathbf{F}_+) = \mathcal{O}(\mathbf{F}_+) \times \mathscr{E}, \ \textit{and} \ \widetilde{\mathcal{P}}(\mathbf{F}) = \mathcal{P}(\mathbf{F}) \times \mathscr{E}.$$

It was shown in [11] that there exist sequences $\{S_n\}$, $\{T_n\}$, and $\{U_n\}$ for $n \in \mathbb{N}$ of predictable stopping time (s.t.), totally inaccessible stopping time and totally inaccessible stopping time in the broad sense (s.t.b.) respectively, absorbing all jumps of an optional process, H, such that the graphs of these stopping times do not intersect within each sequence. On $\widetilde{\Omega}$ let $\mu^i(\omega, \cdot, \cdot)$, $p^i(\omega, \cdot, \cdot)$ and $\eta^g(\omega, \cdot, \cdot)$ where $i \in (d, g)$ be integer valued measures defined on the σ-algebra

$\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{E})$ that are associated with the sequences of stopping times associated with H. On $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{E})$ we define the random integer-valued measures by the relations,

$$p^d(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(S_n, \beta^d_{S_n}), \quad p^g(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(S_n, \beta^g_{S_n}),$$
$$\mu^d(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(T_n, \beta^d_{T_n}), \quad \mu^g(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(T_n, \beta^g_{T_n}),$$
$$\eta^g(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(U_n, \beta^g_{U_n}),$$

where $B \in \mathcal{B}(\mathbb{R}_+)$, $\Gamma \in \mathcal{B}(\mathbb{E})$, $\beta^d_t = \Delta H_t$ if $\Delta H_t \neq 0$ and $\beta^d_t = \delta^d$ if $\Delta H_t = 0$, $\beta^g_t = \Delta^+ H_t$ if $\Delta^+ H_t \neq 0$, $\beta^g_t = \delta^g$ if $\Delta^+ H_t = 0$, $t > 0$. Let the measures ν^i, λ^i and θ^g are called the (dual) predictable projections (compensator) for the measures μ^i, p^i and η^g, respectively.

Now we define the default process;

Definition 100 *A process of default, H, is a finite variation optional process that is bounded, $H \in \{0, 1\}$. Furthermore, we write H as*

$$H_t = \int_{0+}^{t} \int_{|u|\leq 1} u \left(\mu^d(ds, du) + p^d(ds, du) \right) + \int_0^{t-} \int_{|u|\leq 1} u \left(\mu^g(ds, du) + p^g(ds, du) + \eta^g(ds, du) \right) \tag{5.17}$$

with the characteristics $\left(\langle H^c, H^c \rangle, \nu^d, \lambda^d, \nu^g, \lambda^g, \theta^g\right)$.

The general default process, H, equation (5.17) describes default as a result of internal and external factors.

5.9 Concluding Remarks

This chapter presents a new approach to modeling market processes and contingent claims that are subject to default risk. As in previous reduced-form models, we treat default as an unpredictable event governed by a default conditional probability process. Our approach is new, general and unique. It will open the road for further development of modeling of defaultable markets.

We like to point out some essential aspects of our approach to modeling default. First, we have avoided the requirement that the market information be initially complete. Also, we have avoided all sort of invariance principles and enlargement and restriction of filtrations. Furthermore, we have generalized the concept of a default process and gave a general framework upon which one does not need to distinguish between structural and reduced form models.

Chapter 6

Conclusions

Unusual stochastic spaces are probability spaces where the information algebras are not right continuous or are complete. On these spaces lives certain types of processes that are optional and ladlag. In this research we have advanced the calculus of optional processes in different ways and developed a mathematical framework for financial markets on unusual probability spaces.

In advancing the calculus of optional processes, we presented a solution to the nonhomogeneous linear stochastic equation, proved Gronwall lemma, existence and uniqueness of solutions of stochastic equations of optional semimartingales under monotonicity conditions and proved a version of the comparison theorem for stochastic equations of optional semimartingales.

For financial markets on unusual spaces, we have defined a new market model and established a rational pricing and hedging methodology using martingale deflator methods. We have also developed examples of markets using the stochastic exponential and logarithms approaches. Moreover, we have given several examples where it is natural to treat a financial market with the calculus of optional processes. These examples are: a ladlag jumps diffusion model, a basket of stocks some of which are right continuous while others are left continuous, a portfolio of a defaultable bond and a stock, and optimal debt repayment problem.

Furthermore, we have developed a new theory for defaultable markets on unusual spaces. We have defined defaultable claims and cash-flows in this setting and produced a rational pricing and hedging theory. Finally we solved two fundamental examples of defaultable claims: the zero-coupon defaultable bond and credit default swap.

This thesis is not the end of our work on stochastic calculus on unusual spaces; we have several problems that we are currently working on. The first problem we are working on, which is a fundamental problem to financial mathematics, is uniform Doob-Meyer decomposition. The second problem, which is what have kick-started the whole project and is a fundamental problem in stochastic analysis, is the filtering problem where the underlying measure is not unique.

Finally, It is best, perhaps a fortune, that in this possible world stochasticity manifests. For that is otherwise, we loose free-choice and sciences will not be a worthwhile endeavor.

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