Optimal Insurance Contracts in Two Dynamic Models

by

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Abstract

As one type of principal-agent problem, the insurance contract models are closely related to the extent of information disclosure. We construct two new insurance contract models with full information and adverse selection respectively. The full information model is a continuous-time model in which an insurer proposes an insurance contract to a potential insured. Motivated by climate change and catastrophic events, we assume that the number of claims process is a shot-noise Cox process. The insurer selects the premium to be paid by the potential insured and the amount to be paid for each claim. In addition, the insurer can request some actions from the potential insured to reduce the number of claims. The insurer wants to maximize his expected total utility, while the potential insured signs the contract if his expected total utility for signing the contract is greater than or equal to his expected total utility when he does not sign the contract. We obtain an analytical solution for the optimal premium, the optimal amount to be paid for each claim, and the optimal actions of the insured. This leads to interesting managerial insights. The adverse selection model deals with multi-period insurance contracts between an insurer and the insureds of two risk types. The insurer offers a menu of contracts from which the insured can choose one that fits his type. We allow more than two outcome states. The loss amount is a positive random variable that can take two or more values. Accordingly, the traditional self-selection principal-agent model with pure adverse selection is not appropriate. To the traditional model, we add a new constraint that puts a boundary on

the premium and the compensation. We obtain the optimal contracts that maximize the insurer's utility and distinguish the types of insureds. We also explain why the traditional model is inappropriate and why the constraint of boundaries is necessary.

Preface

The research in this thesis results from a collaboration with Dr. Abel Cadenillas at the University of Alberta. I was the key investigator of all the research projects in Chapters 2 and 3.

The results of Chapter 2 of this thesis are published as W. Liu and A. Cadenillas, "Optimal insurance contracts for a shot-noise Cox claim process and persistent insured's action", Insurance: Mathematics and Economics, vol 109, 69-93.

I was responsible for both theoretical proofs and economic analyses as well as the manuscript composition of the published paper above. Professor Cadenillas was the supervisory author. He provided feedback and suggestions to improve the manuscripts and contributed to the final editing and corrections of the paper.

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Chapter 1

Introduction

Insurance mathematics studies different risks and ways to manage these risks. The definition of a risk is the possibility of the occurrence of a hazard. For individuals, the risk can be illness, disability, and death. For properties, the risk can be the properties are stolen or damaged by catastrophes. For businesses, the risk can be the loss from lawsuits. This dissertation uses stochastic processes to simulate and understand risks. Insurance mathematics applies mathematical model and solve problems in insurance. This dissertation aims to find the optimal insurance contracts applying stochastic control methods. Insurance is a policy in which the insurer provides the insured financial protection against losses in exchange for a fee. The fee is called premium and the financial protection is called compensation. The insurance contract details the conditions and circumstances under which the insurer will compensate the insured, the amount of premium, and the amount of compensation. For example, the model in Chapter 2 of this thesis deals with an insurance contract that contains the premium, the compensation, and the insured's action. The insured's action is the condition under which the insurer will compensate the insured. The insurer requires the insured to do some actions to reduce the probability of accidents. If the insured does not meet this condition, the insurer will not make the compensation.

Due to the amount of information revealed between two parties to the contract, a contract model is categorized into one of the three following categories:

full information, moral hazard, and adverse selection. In this thesis, we create a new insurance contract model with full information in Chapter 2 and a new insurance contract model with adverse selection in Chapter 3.

1.1 The Full Information Model

In a finite time horizon, we consider continuous-time model in which an insurer proposes an insurance contract to a potential insured. The insurance contracts are supposed to be effective in a finite time period, so we only consider the risk, premium, compensation, and other related elements within this time horizon. The setting of full information means that all the information is observable to both the insurer and the insured.

It has been standard in the actuarial sciences literature to assume that the total claim amount process is a compound Poisson process with deterministic intensity, or equivalently that the number of claims process is a Poisson process with deterministic intensity. The intensity is a deterministic process, thus the evolvement of the intensity is revealed at the beginning of the period. See, for example, Bühlmann [1970], Medhi [1982], Lindskog and McNeil [2003], and Moore and Young [2006]. However, there are many important cases in which a Poisson process with deterministic intensity does not represent well the total number of claims. For instance, Beard et al. [1984] shows that the standard Poisson process is not an appropriate model for the number of claims in catastrophe, fire, and some other types of insurance. Instead, Beard et al. [1984] suggests considering stochastic intensity. This means the intensity is a stochastic process, thus the intensity will be a random variable at every moment.

The Cox process, also called the doubly stochastic Poisson process, is a generalized Poisson process with stochastic intensity. We consider a Cox process where the intensity is a shot noise process. The shot noise process can be used to model the stochastic nature of catastrophic events. Due to climate change, natural disasters occur more frequently. The losses caused by catastrophes are usually enormous, so it is important to insure against losses caused by this type of events. Dassios and Jang [2003] explains that claims arising from catastrophic events depend on the intensity of natural disasters, and that one of the processes that can be used to measure the impact of catastrophic events is the shot noise process. Further, Dassios and Jang [2003] and Schmidt [2014] explain in detail the application of shot-noise Cox process in catastrophe insurance, although they do not study optimal insurance contracts. Following Dassios and Jang [2003] and Schmidt [2014], we adopt a shot-noise Cox process to count the number of claims. Besides catastrophe insurance, our model is also appropriate for other types of insurance. For example, Dassios et al. [2015] points out that the shot-noise Cox process models very well the number of traffic accidents if the rate of the event arrival is large.

We consider two cases: the insured does not intervene through his actions to reduce the number of claims, and the insured intervenes through his actions to reduce the number of claims. In the first case, we assume that the number of claims process is a shot-noise Cox process. In the second case, we assume that the number of claims process is a Cox process but the actions of the insured can affect the shot noise intensity. Equation (2.1.2) shows how the insured's actions $a = \{a_t; t \in [0, T]\}$ affect the intensity. The first case is the special case of the second case in which the actions of the insured are null.

We allow the actions of the insured to be persistent. That is, the actions of the insured at any point in time are effective until maturity. For instance, in flood insurance, the insurer may require the insured to bring the property up to some standards. See the national flood insurance program of the Federal Emergency Management Agency (2022). This action of the property owner will reduce the probability of having a loss caused by floods, and its protection against flood will last from the time of action. However, along with aging and wear, the flood-resistance equipment becomes less protective over time. Thus, we further assume that the action is discounted by time. We will discuss further details of persistent actions in Section 2.1. Hoffmann et al. [2021], Hopenhayn and Jarque [2010], Jarque [2010], and Mukoyama and Şahin [2005] have also considered persistent actions. We present a model in which persistent actions affect a Cox process.

The insurer selects the premium to be paid by the potential insured, the amount to be paid for each claim, and also requests some actions from the potential insured. The potential insured has a cost associated with his actions. Section 2.2 presents details on the utility and cost functions of the insurer and the potential insured. The insurer wants to maximize his expected total utility, while the potential insured signs the contract if his expected total utility for signing the contract is greater than or equal to his expected total utility when he does not sign the contract. Thus, the problem studied in our thesis is different from other papers (such as Zou and Cadenillas [2014], and Zou and Cadenillas [2017]) in which an insurer has already designed an insurance contract (which might not be the optimal insurance contract) and decides its optimal liability. To the best of our knowledge, we obtain, for the first time in the literature, an analytical solution for the optimal premium, the optimal amount to be paid for each claim, and the optimal actions of the insured when the number of claims process is a Cox process. The analytical solution leads to interesting managerial insights. For instance, we show that the optimal expected action decreases over time. Furthermore, the insured will perform less expected action over time to reach the reservation utility when he does not enter the insurance market. Jarque [2010] presents the same trend of the optimal action only through a numerical example while we prove it with an analytical solution in a general setting. Our result challenges the assumption of Mukoyama and Şahin [2005] that the principal prefers the agent to insert the highest action all the time. The decreasing trend of the optimal actions results from action persistence, where the earlier action reduces the loss further because it is effective for a relatively long period. We also present an example.

1.2 The Adverse Selection Model

Adverse selection is a situation in which buyers or sellers have more or better information than the other party in a transaction. Typically, only incomplete information is presented in the insurance market. The insurer cannot have full access to the risk level of the insured while the insured has a better understanding of the risk level. The insurance contracts implemented based on incomplete information usually cannot provide full insurance. Collecting information is important for the insurer to offer a contract that provides better insurance. Naturally, multi-period models are proposed (Cooper and Hayes [1987], Dionne [1983], Dionne and Lasserre [1985], Dionne and Lasserre [1987]). Compared to one-period models (Stiglitz [1977], Chade and Schlee [2012]), the multi-period models allow us to collect information from previous periods and to design contracts for later periods according to the information collected. Adverse selection is an allocation problem that exists in many markets, including insurance markets. For example, automobile insurance and home insurance contracts consider deductibles. The existence of adverse selection is also confirmed by many empirical tests (Shi et al. [2012], Browne [1992]). Cohen and Siegelman [2010] reviewed the empirical tests on adverse selection in insurance markets.

Discussions on adverse selection have been done not only in insurance markets but also in general principal-agent problems. In the literature dealing with adverse selection, different mechanisms are presented to sort out the types, even though the basic essence of all the mechanisms is experience rating. Self-selection, one of the mechanisms, is widely practiced (Stiglitz [1977], Cooper and Hayes [1987], Cvitanić and Zhang [2007], Halac et al. [2016], Chade and Schlee [2012]). With self-selection, the insurer reveals the unobserved information through the insured's selection from a menu of contracts. The second mechanism is discussed in Dionne [1983] and Dionne and Lasserre [1985]. The risk revelation results from a Stackelberg game (Dionne et al. [2000]). The insured reports his risk level in the first period and receives punishment if his outcome does not reflects his reported type. Another mechanism is shown in Hoffmann et al. [2022]. It reveals the information by extending the time of observation. We will consider an experience rating model with multi-period and self-selection.

Among the models with pure adverse selection and self-selection, this is the first time that considers more than two outcome states. The outcome in each period is an accident or no accident in Stiglitz [1977], Cooper and

Hayes [1987], Chade and Schlee [2012] and the loss amount from an accident is fixed. These models possess objective, participation constraints, and incentive compatibility constraints. Differently, the outcome in a period in our model is a nonnegative random variable representing the loss amount. The loss amount can be a discrete or continuous random variable. However, the solutions will be infeasible if the traditional self-selection model is applied when there are more than two states of loss. So, to adjust to the new situation, we modify the traditional model and add a new constraint that provides boundaries on the premium and compensation.

Suppose there are two types of potential insureds in the insurance market: low risk and high risk. Each insured's type is private information not observable to the insurer. The insurer can observe the insured's loss at each period which provides a clue of the insured's type. More periods can reveal more information about the type. Other tools the insurer can use to sort the types are the premium and compensation. How to formulate the premium and compensation for an insured of low risk? Here is a straightforward idea. The insurer could charge a high premium at every period, to pay a high compensation when the loss is low, and to punish when the loss is high. This contract is acceptable to the low-risk insured because the possibility for the low-risk insured to have a high loss is low. However, the high-risk insured will not choose this contract because it is highly likely for him to get punished while paying an expensive premium.

The proportion of each type is observable to both the insured and the insurer. We suppose the contract is effective in a finite number of periods. At the beginning of each period, the insured pays the premium. At the end of each period, the insurer compensates the insured according to the loss amount and the information obtained from the previous periods. We will set up a pure adverse selection model and solve for the optimal insurance contracts. The objective is to maximize the insurer's expected utility. We find the optimal premium and compensation for each type in each period and sort the types. The optimal contracts also reflect the idea discussed in the last paragraph. We obtain for the first time in the literature, solutions to a pure adverse selection problem with self-selection. Chade and Schlee [2012] obtain a solution only for a specific utility function. We also explain why the traditional model is inappropriate and why the constraints are necessary.

Other literature discusses principal-agent problems with adverse selection from different aspects. In Ramsay et al. [2013], the insured possesses two possible outcome states but the utility functions depend on states. Ma et al. [2020] discusses a two-period model. The tools of low compensation and the increase and decrease in the bonus are introduced to distinguish the risk types. Hellwig [2010] develops a technique for incentive problems with any type distribution. Jeleva and Villeneuve [2004] introduces the imprecise probabilities to adverse selection and agents differ in the perception of risks. Without purchasing the contracts, the insured's utility is a common value despite the types in Baron and Myerson [1982] and Maskin and Riley [1984]. Jullien [2000] allows type-dependent reservation utility and challenges the property obtained in Baron and Myerson [1982] and Maskin and Riley [1984].

Chapter 2

Full Information and Persistent Effort

This chapter consists of five sections. Section 2.1 presents the total claim amount model and Section 2.2 presents the problem that we study in this chapter. The solution is presented in Section 2.3. Section 2.4 discusses the value of the reservation utility. An example is presented in Section 2.5.

2.1 The Total Claim Amount Process

We consider a finite time horizon $[0, T]$. There are two possibilities: the insured does not affect the risky external environment and the insured affects the risky external environment.

If the insured does not affect the risky external environment, then the total claim amount process $\Psi = \{\Psi(t); t \in [0, T]\}$ is given by

$$\Psi(t) = \sum_{i=1}^{\mathcal{N}(t)} l_i = l_1 + l_2 + \cdots + l_{\mathcal{N}(t)},$$

where $\mathcal{N}(t)$ is the number of claims up to time $t \in [0, T]$ and $\{l_1, l_2, \cdots, l_{\mathcal{N}(t)}\}$ are the amounts claimed until time t. We make the following assumptions.
a) The random variables $\{l_1, l_2, l_3, \cdots\}$ are independent and identically distributed. Furthermore, their range is $\mathcal{R}_l$ and $\inf \mathcal{R}_l > 0$.

b) The sequence of random variables $\{l_1, l_2, l_3, \cdots\}$ are independent of the stochastic process $\mathcal{N} = \{\mathcal{N}(t); t \in [0, T]\}$.
c) The stochastic process $\mathcal{N} = \{\mathcal{N}(t); t \in [0, T]\}$ is a shot-noise Cox process with stochastic intensity rate $I = \{I(t); t \in [0, T]\}$ given by

$$I(t) = \theta \sum_{i=0}^{\mathcal{M}(t)} Y_i e^{\delta(\tau_i - t)} = \theta \sum_{i=0}^{\mathcal{M}(t)} Y_i e^{-\delta(t-\tau_i)}. \tag{2.1.1}$$

In the above equation, θ represents the risk level of the insured, $\mathcal{M}(t)$ counts the number of risky events exposed to the insured from time 0 to time t, Y_i is the jump size caused by the i-th random risky event, τ_i is the time when the i-th risky event occurs, and δ is the rate of decay. The effect of a risk event happening at time τ lasts in the time period $[\tau, T]$ but is discounted by δ at time $t \in [\tau, T]$. We make the following assumptions about the stochastic process I:
c1) θ is a positive constant.
c2) $\mathcal{M} = \{\mathcal{M}(t); t \in [0, T]\}$ is a Poisson process with a deterministic intensity process $\rho(t) \geq 0$, $t \in [0, T]$. If the frequency of exposures is high, then $\rho(\cdot)$ is large.
c3) $\{Y_i\}_{i=1,2,3,\cdots}$ is a sequence of i.i.d. random variables and independent of $\mathcal{M}$. We suppose they are the images of a random variable Y that is positive and finite almost surely. $Y_0 > 0$ is a constant known at time 0. We denote $\mu = E[Y]$.
c4) $\{\tau_i\}_{i=1,2,3,\cdots}$ is a sequence of non-decreasing stopping times. In the above equation, $\tau_0 = 0$, and for every $i \in \{1, 2, \cdots, \mathcal{M}(t)\}$: $\tau_i \leq t$.
c5) δ is a positive constant.

Applications of Cox processes with shot noise intensity to insurance can be found in Albrecher and Asmussen [2006], Macci and Torrisi [2011], Schmidt [2014], and Zhu [2013]. The number of claims from catastrophic events depends on the stochastic intensity of natural disasters. The above intensity process I measures the frequency of external risky events (by $\mathcal{M}$), their magnitude (by Y_i), and their time (by τ_i) to determine the effect of catastrophic events. As time passes, the magnitude decreases (by δ). We consider a probability space $(\Omega, \mathcal{F}_1, \mathbb{P})$ together with a filtration $\mathbb{F}_1 := \{\mathcal{F}_{1,t}, t \in [0,T]\}$ that is the $\mathbb{P}$-augmentation of the natural filtration

$$\sigma\left(\mathcal{N}(u), \mathcal{M}(u), u \in [0,t]; l_i, i \in \{0,1,\cdots,\mathcal{N}(t)\}; Y_j, \tau_j, j \in \{0,1,\cdots,\mathcal{M}(t)\}\right).$$

If the insured affects the risky external environment, the total claim amount process $\Psi = \{\Psi(t); t \in [0,T]\}$ is given by

$$\Psi(t) = \sum_{i=1}^{\mathcal{N}^a(t)} l_i = l_1 + l_2 + \cdots + l_{\mathcal{N}^a(t)},$$

where the number of claims process $\mathcal{N}^a = \{\mathcal{N}^a(t); t \in [0,T]\}$ is a Cox process with stochastic intensity rate $\pi = \{\pi(t); t \in [0,T]\}$ given by

$$\pi(t) := \theta\left(\sum_{i=0}^{\mathcal{M}(t)} Y_i e^{\delta(\tau_i - t)}\right)\left(1 - e^{-t}(\bar{a} + \int_0^t a_u r_u e^u du)\right). \tag{2.1.2}$$

Here, the process $a = \{a_t; t \in [0,T]\}$ represents the actions to reduce the magnitude of external risk events and $\bar{a}$ is a constant that represents the measures to reduce the magnitude of risk events taken before the contract is implemented. We assume that a is adapted to the filtration $\mathbb{F}_1$. We also assume that $0 \leq a_t \leq K$ for $t \in [0,T]$ and $\bar{a} \in [0,K]$, where $K \in [0,1]$ is a constant that represents the proportion of the intensity that can be cleared through actions. The remaining $1-K$ proportion of the intensity is not avoidable through actions. r_u is the effectiveness of action a_u. The process $r = \{r_t; t \in [0,T]\}$ is called the productivity of action in the principal-agent problem (Williams [2009]). Demarzo and Sannikov [2017] and Cvitanić and Zhang [2013] also introduce the coefficient r_u to adjust for the action a_u. For example, the precaution against flood is more effective in the rainy season than in the dry season. Correspondingly, in flood insurance, r_u is generally larger in rainy seasons. We assume that $r_u \in [0,1]$ for every $u \in [0,T]$. If the action and the effective rate take their highest values K and 1 respectively at every time in

$[0,T]$, then (2.1.2) becomes $\theta\left(\sum_{i=0}^{\mathcal{M}(t)} Y_i e^{\delta(\tau_i - t)}\right)(1-K)$. Under the conditions that $0 \le \bar{a} \le 1$, $0 \le a_u \le 1$, and $0 \le r_u \le 1$ for $u \in [0,T]$, we have that $\pi(t)$ is nonnegative for $t \in [0,T]$. In other words, the intensity of the random variable $\mathcal{N}^a(t)$ is nonnegative for $t \in [0,T]$. In the special case where $\bar{a} = 0$ and $a_u = 0$ for every $u \in [0,T]$, we have for every $t \in [0,T]$: $I(t) = \pi(t)$. Hence, the case in which the insured affects the external risk environment is more general than the case in which the insured does not affect the external risk environment.

Therefore, we assume that the insured can affect the external risk environment. In other words, we assume that the number of claims is represented by the stochastic process $\mathcal{N}^a = \{\mathcal{N}^a(t); t \in [0,T]\}$, which is a Cox process with the stochastic intensity rate $\pi = \{\pi(t); t \in [0,T]\}$ defined in (2.1.2).

We can understand the actions $a = \{a_t; t \in [0,T]\}$ in the intensity process from the following four aspects. First, the more actions inserted, the smaller the intensity is. Second, a_u has an effect on $\pi(t)$ for every $t \in [u,T]$. Thus, an earlier action can play a role for a long time while a late action plays a role only for a short time. Particularly, a_T is effective for almost zero duration. Third, the ratio between the weights of $a_{u'}$ and a_u in (2.1.2) is $e^{(u'-u)}$ if $0 < u', u \le t$. If it is closer to time t when an action is implemented, the action is more effective at time t. Fourth, the action a_u is made at time u. As time passes by, the contribution of a_u shrinks by e^{u-t} at time $t \in (u,T]$.

In the case of flood insurance, the insured is a property owner and the risk event is a flood. We denote by Y_i the magnitude of the i-th flood. The risk events can affect the frequency of claims, so we represent them in the intensity rate process $\pi = \{\pi(t); t \in [0,T]\}$. The effect of each risk event lasts for some time, but it is discounted (by δ) as time passes. For instance, the destructive power of a flood lasts from the time of flood rising to the time of cleaning up. However, the effect of the flood is weaker as time goes by. The process a represents actions, like using flood-resistance materials, that the property owner is required to take to reduce the frequency of claims.

2.2 The Insurance Problem

We assume symmetric information, in the sense that all the information is transparent and accessible to both insurer and insured. The information structure is denoted by $\mathbb{F}_1 = \{\mathcal{F}_{1,t}, t \in [0,T]\}$ and the model is constructed on the probability space $(\Omega, \mathcal{F}_1, \mathbb{P})$. Following the principal-agent literature that considers a representative principal and a representative agent (see, for instance, Section 4.1 of Bolton and Dewatripont [2005], Cadenillas et al. [2007] and Sannikov [2008]), we consider a representative insurer and a representative insured.

The insurer selects the premium rate and the compensation. During the contract period, the client will pay the premium continuously. The company commits to compensate the insured immediately after he faces a loss. The compensation can cover partially or completely the loss. The insurer observes all the information, in particular, the insured's actions. The insurer requires the amount of action in the contract, and that must be followed by the insured. That is consistent with many papers on optimal contract theory. Under the full information case, Cvitanić and Zhang [2013] points out that the principal offers the contract and dictates the agent's actions. In the full information section, Williams [2015] also said the principal decides the actions. The first-best models in Chapter 4 of Bolton and Dewatripont [2005] expressed the same ideas. In practice, to reduce losses, the insurance company may write down provisions that require the insured to take designated actions in catastrophe and other insurance contracts. For example, the catastrophe insurance policy may require the insured to do necessary maintenance on the property. Otherwise, the insurer is entitled to deny compensation for the loss directly or indirectly caused by the lack of maintenance. See, for instance, Flex Insurance Company (2022). Thus, we suppose the actions are taken to maximize the insurer's utility. On the other hand, the insured will sign the contract if his participation constraint is satisfied. We denote by

$$(a, d, D) = \{(a_t, d_t, D_i); t \in [0,T] \text{ and } i = 1, 2, \cdots\}$$

the contract offered by the insurer. After signing the contract, the insured pays continuously the premium rate d_t and takes action a_t at time t. When the i-th loss happens, the insurer compensates the insured with the amount D_i. We do not assume that D_i is equal to l_i.

We assume that the insured and the insurer have Von Neumann-Morgenstern utility functions $U_1 : \mathbb{R} \mapsto \mathbb{R}$ and $U_2 : \mathbb{R} \mapsto \mathbb{R}$, respectively. U_2 is concave and U_1 is strictly concave. These utility functions are also supposed to be strictly increasing, twice differentiable, and

$$\begin{aligned} U_2'(-\infty) &= \lim_{y \to -\infty} U_2'(y) = +\infty, \quad U_2'(+\infty) = \lim_{y \to +\infty} U_2'(y) = 0, \\ U_1'(-\infty) &= \lim_{y \to -\infty} U_1'(y) = +\infty, \quad U_1'(+\infty) = \lim_{y \to +\infty} U_1'(y) = 0. \end{aligned} \tag{2.2.3}$$

Only in this chapter, there is an extra assumption on the utility functions. That is,

$$U_1(0), U_2(0) \leq 0.$$

The insurer's expected total utility for a policy (a, d, D) is

$$\mathcal{J}(d, D, a) := E\left[\int_0^T U_2(d_t)dt + \sum_{i=1}^{\mathcal{N}^a(T)} U_2(-D_i)\right]. \tag{2.2.4}$$

The cost function of action is denoted by V_1, and is assumed to be positive, increasing, differentiable, strictly convex and satisfying $V_1(0) = V_1'(0) = 0$. Next, we present the participation constraint. We denote the reservation utility by $R \in \mathbb{R}$. R is the expected total utility that the insured can obtain from outside options. The insurer wants to offer a contract that gives an expected total utility greater than or equal to R to the insured. Otherwise, the insured will prefer outside options, and will not accept the contract offer.

The income rate of the insured is represented by $\{w_t, t \geq 0\}$. We assume that $w_t > 0$ is deterministic for every $t \geq 0$. We denote by $\mathcal{A}_1$ the class of admissible controls. These are the controls (a, d, D) that are adapted to the filtration $\mathbb{F}_1$.

Problem 1. *The insurer wants to select the policy* $(\hat{a}, \hat{d}, \hat{D}) \in \mathcal{A}_1$ *that solves the problem*

$$\max_{(d,D,a)\in\mathcal{A}_1} \quad \mathcal{J}(d, D, a)$$

$$s.t. \quad E\left[\int_0^T U_1(w_t - d_t)dt + \sum_{i=1}^{\mathcal{N}^a(T)} U_1(D_i - l_i) - \int_0^T V_1(a_t)dt\right] \geq R, \tag{2.2.5}$$

$$0 \leq a_t \leq K, \ \ for\ all\ t \in [0, T]. \tag{2.2.6}$$

In this chapter, we assume the utility functions are negative when the variables are negative. The insurer loses some amount of utility if a compensation is made and the insured loses some amount of utility if he encounters the loss from an accident. From the terms $\sum_{i=1}^{\mathcal{N}^a(T)} U_2(-D_i)$ in (2.2.4) and $\sum_{i=1}^{\mathcal{N}^a(T)} U_1(D_i - l_i)$ in (2.2.5), we observe that the total loss of utility due to the claims can be reduced by taking actions.

2.3 The Optimal Insurance Contract

An extended generator on Markov processes consisting of random jumps is explicitly calculated in Theorem 5.5 in Davis [1984]. Following this theorem, we will present a generator of the process $\{(I(t), t), t \geq 0\}$. The generator helps with our calculation of the expectation of $\mathcal{N}^a(T)$. We denote the cumulative distribution function of the jump Y by F_Y. We assume that F_Y and the intensity ρ defined in Section 2.1 are Riemann integrable.

Suppose a function $f(\cdot, \cdot)$ belongs to the domain of the generator denoted by $\mathbb{A}$. Then $\mathbb{A}$ acting on $f(I,t)$ is defined by

$$\mathbb{A}f(I,t) := \frac{\partial f}{\partial t} - \delta I \frac{\partial f}{\partial I} + \rho(t)\int_0^\infty f(I+\theta y,t)dF_Y(y) - \rho(t)f(I,t). \quad (2.3.7)$$

Theorem 5.5 of Davis [1984] describes the domain of the generator, and Dassios and Jang [2003] give sufficient conditions under which f is in the domain of $\mathbb{A}$. In our case, $f : [0,\infty) \times [0,T] \mapsto \mathbb{R}$ belongs to the domain of $\mathbb{A}$ if $f \in C^1([0,\infty)\times[0,T];\mathbb{R})$ and

$$\Big| \int_0^\infty f(I+\theta y,t)dF_Y(y) - f(I,t) \Big| < \infty.$$

As stated by Proposition 1 in Dassios and Embrechts [1989], $\{f(I_t,t), t \geq 0\}$ is a martingale if $\mathbb{A}f(I,t) = 0$. See also Davis [1984]. Therefore, we have the following result.

Lemma 1. *The stochastic process*

$$\sum_{i=0}^{\mathcal{M}(t)} Y_i e^{\delta \tau_i} - \mu \int_0^t e^{\delta u}\rho(s)ds$$

is a martingale.

Proof. We denote

$$f(I,t) := \frac{1}{\theta} I e^{\delta t} - \mu \int_0^t e^{\delta u}\rho(u)du.$$

It is obvious that $f(I,t)$ is differentiable with respect to each I and t.

$$\begin{aligned}
&\Big| \int_0^\infty f(I+\theta y,t)dF_Y(y) - f(I,t) \Big| \\
&= \Big| \int_0^\infty \left(\frac{1}{\theta}(I+\theta y)e^{\delta t} - \mu \int_0^t e^{\delta u}\rho(u)du \right) dF_Y(y) - \left(\frac{1}{\theta} I e^{\delta t} - \mu \int_0^t e^{\delta u}\rho(u)du \right) \Big| \\
&= \left| e^{\delta t}\mu \right| < \infty
\end{aligned}$$

Applying (2.3.7), we obtain

$$
\begin{aligned}
\mathbb{A}f(I,t) &= \frac{1}{\theta}I\delta e^{\delta t} - \mu e^{\delta t}\rho(t) - \frac{1}{\theta}I\delta e^{\delta t} + \rho(t)\int_0^{\infty}\left(\frac{1}{\theta}(I+\theta y)e^{\delta t} - \mu\int_0^t e^{\delta u}\rho(u)du\right)dF_Y(y) \\
&\quad -\rho(t)\left(\frac{1}{\theta}Ie^{\delta t} - \mu\int_0^t e^{\delta u}\rho(u)du\right) \\
&= -\mu e^{\delta t}\rho(t) + \rho(t)\frac{1}{\theta}Ie^{\delta t} + \rho(t)\mu e^{\delta t} - \rho(t)\mu\int_0^t e^{\delta u}\rho(u)du \\
&\quad -\rho(t)\frac{1}{\theta}Ie^{\delta t} + \rho(t)\mu\int_0^t e^{\delta u}\rho(u)du \\
&= 0.
\end{aligned}
$$

According to Proposition 1 in Dassios and Embrechts [1989], we obtain that the stochastic process defined by

$$
f(I(t),t) = \frac{1}{\theta}I(t)e^{\delta t} - \mu\int_0^t e^{\delta u}\rho(u)du
$$

is a martingale. From (2.1.1), we can get the required statement. □

Now we can obtain the expected number of claims.

Proposition 1. *The expected number of claims corresponding to actions* $a = \{a_u, u \in [0,T]\}$ *is*

$$
\begin{aligned}
E[\mathcal{N}^a(T)] =& \theta\int_0^T (1-e^{-t}\bar{a})e^{-\delta t}\left(Y_0 + \mu\int_0^t \rho(s)e^{\delta s}ds\right)dt \\
& - \theta\int_0^T e^{-(1+\delta)t}E\left[\int_0^t a_u r_u e^u\left(\mu\int_u^t \rho(s)e^{\delta s}ds + \sum_{i=0}^{\mathcal{M}(u)} Y_i e^{\delta\tau_i}\right)du\right]dt.
\end{aligned}
\tag{2.3.8}
$$

Proof. $\mathcal{N}^a = \{\mathcal{N}^a(t); t \geq 0\}$ is a Cox process with intensity process $\pi(\cdot)$. From Lemma 3a of Grandell [1976] or Theorem 2.7 of Dassios and Jang [2003], we have

$$
E[\mathcal{N}^a(T)] = \int_0^T E[\pi(t)]dt. \tag{2.3.9}
$$

According to the equation (2.1.2),

$$
\begin{aligned}
&E[\pi(t)] \\
&= \theta\Bigg((1 - e^{-t}\bar{a}) E\Big[\sum_{i=0}^{\mathcal{M}(t)} Y_i e^{\delta(\tau_i - t)}\Big] - e^{-(1+\delta)t} E\Big[\Big(\sum_{i=0}^{\mathcal{M}(t)} Y_i e^{\delta\tau_i}\Big)\int_0^t a_u r_u e^u du\Big]\Bigg).
\end{aligned}
$$

According to Lemma 1,

$$
E\Big[\sum_{i=0}^{\mathcal{M}(t)} Y_i e^{\delta(\tau_i - t)}\Big] = e^{-\delta t} E\Big[\sum_{i=0}^{\mathcal{M}(t)} Y_i e^{\delta\tau_i}\Big] = e^{-\delta t}\left(Y_0 + \mu\int_0^t \rho(u) e^{\delta u} du\right)
$$

and

$$
\begin{aligned}
E & \Big[\Big(\sum_{i=0}^{\mathcal{M}(t)} Y_i e^{\delta\tau_i}\Big)\int_0^t a_u r_u e^u du\Big] \\
= & \int_0^t E\left[a_u r_u e^u \Big(\sum_{i=0}^{\mathcal{M}(t)} Y_i e^{\delta\tau_i}\Big)\right] du \\
= & E\left[\int_0^t a_u r_u e^u E\Big[\sum_{i=0}^{\mathcal{M}(t)} Y_i e^{\delta\tau_i} | \mathcal{F}_{1,u}\Big] du\right] \\
= & E\left[\int_0^t a_u r_u e^u \left(\mu\int_u^t \rho(u) e^{\delta u} du + \sum_{i=0}^{\mathcal{M}(u)} Y_i e^{\delta\tau_i}\right) du\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E[\pi(t)] =& \theta(1 - e^{-t}\bar{a}) e^{-\delta t}\left(Y_0 + \mu\int_0^t \rho(u) e^{\delta u} du\right) \\
& - \theta e^{-(1+\delta)t} E\left[\int_0^t a_u r_u e^u \left(\mu\int_u^t \rho(u) e^{\delta u} du + \sum_{i=0}^{\mathcal{M}(u)} Y_i e^{\delta\tau_i}\right) du\right].
\end{aligned} \tag{2.3.10}
$$

We replace $E[\pi(t)]$ in (2.3.9) by (2.3.10) to obtain (2.3.8).

□

Changing the order of integration, we can obtain another way to express (2.3.8).

$$
\begin{aligned}
E[\mathcal{N}^a(T)] =& \theta \int_0^T (1 - e^{-t}\bar{a})e^{-\delta t} \left(Y_0 + \mu \int_0^t \rho(s)e^{\delta s} ds \right) dt \\
& - \theta E \left[\int_0^T a_u r_u e^u \int_u^T e^{-(1+\delta)t} \left(\mu \int_u^t \rho(s)e^{\delta s} ds + \sum_{i=0}^{\mathcal{M}(u)} Y_i e^{\delta \tau_i} \right) dt du \right].
\end{aligned}
$$

The role actions $a = \{a_u, u \in [0, T]\}$ play can also be observed through the expression above. The integration following a_u is from time u to T. It indicates that the effect of a_u lasts in the time period $[u, T]$. The action exerted at different moments makes different contributions in the remaining period.

We denote

$$
\begin{aligned}
\bar{B} &:= \int_0^T (1 - e^{-t}\bar{a})e^{-\delta t} \left(Y_0 + \mu \int_0^t \rho(s)e^{\delta s} ds \right) dt, \\
B_t &:= r_t e^t \int_t^T e^{-(1+\delta)u} \left(\mu \int_t^u \rho(s)e^{\delta s} ds + \sum_{i=0}^{\mathcal{M}(t)} Y_i e^{\delta \tau_i} \right) du.
\end{aligned}
$$

Now, we can write $E[\mathcal{N}^a(T)]$ as

$$
E[\mathcal{N}^a(T)] = \theta \bar{B} - \theta E \left[\int_0^T a_t B_t dt \right]. \tag{2.3.11}
$$

Since $r_t \geq 0$, $\rho(t) \geq 0$ for $t \in [0, T]$, and $Y_i > 0$ for $i = 0, 1, 2, \cdots$, it immediately follows that $B_t \geq 0$ for each $\omega \in \Omega$ and $t \in [0, T]$. Recalling that $\pi(t)$ is nonnegative for $t \in [0, T]$, we derive that $E[\mathcal{N}^a(T)] \geq 0$. Let $a_u = 1$ almost surely for $u \in [0, T]$, we can see $E[\mathcal{N}^a(T)] = \theta \left(\bar{B} - E \left[\int_0^T B_t dt \right] \right)$ from (2.3.11). Further, let $\bar{a} = 1$ and $r_t = 1$ almost surely for $t \in [0, T]$, then $\pi(t) = 0$ almost surely for $t \in [0, T]$ and it results in $E[\mathcal{N}^a(T)] = 0$. It follows that $\bar{B} = E \left[\int_0^T B_t dt \right]$. Otherwise, $\bar{B} > E \left[\int_0^T a_t B_t dt \right]$. $\bar{B}$ can be understood as the expected number of claims if actions are not involved. B_t is the intensity rate of accidents that can be removed by one unit of action at time t.

To find the solution of the model, we use the Lagrangian method and define the functional $\mathcal{L}_1$ by

$$\begin{aligned}\mathcal{L}_1(d,D,a;\Lambda_1,\Lambda_2) :=& E\left[\int_0^T U_2(d_t)dt + \sum_{i=1}^{\mathcal{N}^a(T)} U_2(-D_i)\right] \\ &+ \Lambda_1 E\left[\int_0^T U_1(w_t-d_t)dt + \sum_{i=1}^{\mathcal{N}^a(T)} U_1(D_i-l_i) - \int_0^T V_1(a_t)dt\right] \\ &+ E\left[\int_0^T \Lambda_2^t a_t dt\right],\end{aligned} \tag{2.3.12}$$

where Λ_1 and Λ_2^t, adapted to $\mathbb{F}_1$, $t \in [0,T]$ are Lagrangian multipliers. The first order conditions for d and D are

$$U_2'(-D_i) - \Lambda_1 U_1'(D_i - l_i) = 0 \quad \text{and} \quad U_2'(d_t) - \Lambda_1 U_1'(w_t - d_t) = 0. \tag{2.3.13}$$

Since Λ_1 is constant, the solution of D_i from the equations above is dependent of l_i only. Hence, the sequences $\{U_2(D_i)\}_{i=1,2,\cdots}$ and $\{U_1(D_i - l_i)\}_{i=1,2,\cdots}$ are i.i.d. and independent of the process $\mathcal{N}^a$. Thus, the Lagrangian (2.3.12) can be rewritten as

$$\begin{aligned}\mathcal{L}_1(d,D,a;\Lambda_1,\Lambda_2) =& E\left[\int_0^T U_2(d_t)dt\right] + E[\mathcal{N}^a(T)]E[U_2(-D) + \Lambda_1 U_1(D-l)] \\ &+ \Lambda_1 E\left[\int_0^T U_1(w_t-d_t)dt\right] - \Lambda_1 E\left[\int_0^T V_1(a_t)dt\right] \\ &+ E\left[\int_0^T \Lambda_2^t a_t dt\right].\end{aligned} \tag{2.3.14}$$

Derive the first order condition from (2.3.14) for a_t to obtain

$$\Lambda_1 V_1'(a_t) - \Lambda_2^t = -\theta E[U_2(-D) + \Lambda_1 U_1(D-l)]B_t \tag{2.3.15}$$

for each $t \in [0,T]$ and $\omega \in \Omega$. The values of the Lagrangian multipliers can show important information of the solutions. Consider Λ_1 first. To ensure the first order condition (2.3.13) valid, Λ_1 must be positive. If $\Lambda_1 = 0$, we can get $D_i = -\infty$ and $d_t = \infty$ from (2.3.13). However, this causes a contradiction to constraint (2.2.5). From (2.2.3), we have $\lim_{D_i \to -\infty} U_1(D_i - l_i) = -\infty$ for $i = 1, 2, \cdots$ and $\lim_{d_t \to \infty} U_1(w_t - d_t) = -\infty$ for $t \in [0, T]$. Then, the left-hand-side of (2.2.5) is going to $-\infty$. Since R is finite, (2.2.5) cannot be satisfied. Hence, $\Lambda_1 > 0$. Consider Λ_2^t now. If $\Lambda_2^t = 0$ for some $t \in [0, T]$ and some $\omega \in \Omega$, it means the constraint (2.2.6) is not binding. The action we obtain from (2.3.15),

$$a_t = V_1'^{-1}\left(-\frac{\theta}{\Lambda_1}E[U_2(-D) + \Lambda_1 U_1(D - l)]B_t\right),$$

satisfies (2.2.6). If $\Lambda_2^t < 0$ for some $t \in [0, T]$ and some $\omega \in \Omega$, it means the right-hand-side of (2.3.15) is big enough such that

$$\Lambda_1 V_1'(K) < -\theta E[U_2(-D) + \Lambda_1 U_1(D - l)]B_t,$$

which shows the marginal cost of action is always smaller than the marginal benefit. Inserting actions more than K will bring the company more utility, but this preference is prevented by the upper bound of a_t. The constraint $a_t \leq K$ binds and the optimal action is just K. If $\Lambda_2^t > 0$ for some $t \in [0, T]$ and some $\omega \in \Omega$, it means the right-hand-side of (2.3.15) is negative such that

$$\Lambda_1 V_1'(0) > -\theta E[U_2(-D) + \Lambda_1 U_1(D - l)]B_t,$$

which shows the marginal cost of action is always bigger than the marginal benefit. Less action is required but the constraint $0 \leq a_t$ binds. The optimal action is just 0.

We define the function $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(y_1, y_2) := \frac{U_2'(y_1)}{U_1'(w - y_1 - y_2)}.$$

Recalling that the utility functions are strictly increasing, we have $U_2'(y_1) > 0$ and $U_1'(w - y_1 - y_2) > 0$. Hence, g is a positive function. Recalling that U_2 is concave and U_1 is strictly concave, we get that $U_2'(y_1)$ is a decreasing function of y_1 and $U_1'(w-y_1-y_2)$ is a strictly increasing function of y_1. Hence, $g(y_1,y_2)$ is a strictly decreasing function of y_1. Thus, $g(\cdot,y_2)$ is invertible. The inverse function is denoted by $g^{-1}(\cdot,y_2)$ and it is also a strictly decreasing function. Similarly, we obtain that $g(y_1,y_2)$ is a strictly decreasing function of y_2.

Consider the function $\mathbb{U}_1$ defined by

$$\begin{aligned}\mathbb{U}_1(\Lambda_1) \ :=\ & \int_0^T U_1(w_t-d_t^{\Lambda_1})dt + E\left[U_1(D^{\Lambda_1}-l)\right]\theta\left(\bar{B}-E[\int_0^T a_t^{\Lambda_1}B_t dt]\right)\\ & - E\left[\int_0^T V_1(a_t^{\Lambda_1})dt\right], \end{aligned} \tag{2.3.16}$$

where

$$\begin{aligned}
&D^{\Lambda_1} = g^{-1}(\Lambda_1,l),\\
&d_t^{\Lambda_1} = -g^{-1}(\Lambda_1,-w_t),\\
&a_t^{\Lambda_1} = V_1'^{-1}\left(-\theta E\left[\frac{1}{\Lambda_1}U_2(-g^{-1}(\Lambda_1,l))+U_1(g^{-1}(\Lambda_1,l)-l)\right]B_t\right)\\
&\qquad\qquad \text{if}\ \ 0\le -\theta E\left[\frac{1}{\Lambda_1}U_2(-g^{-1}(\Lambda_1,l))+U_1(g^{-1}(\Lambda_1,l)-l)\right]B_t \le V_1'(K),\\
&a_t^{\Lambda_1} = K\ \text{if}\ \ V_1'(K) < -\theta E\left[\frac{1}{\Lambda_1}U_2(-g^{-1}(\Lambda_1,l))+U_1(g^{-1}(\Lambda_1,l)-l)\right]B_t,\\
&a_t^{\Lambda_1} = 0\ \ \text{if}\ -\theta E\left[\frac{1}{\Lambda_1}U_2(-g^{-1}(\Lambda_1,l))+U_1(g^{-1}(\Lambda_1,l)-l)\right]B_t < 0.
\end{aligned} \tag{2.3.17}$$

The controls in (2.3.17) are the solution of equations (2.3.13) and (2.3.15). $\mathbb{U}_1(\Lambda_1)$ is the customer's expected total utility corresponding to the controls $(d^{\Lambda_1}, D^{\Lambda_1}, a^{\Lambda_1})$. We know that $g(y_1,y_2)$ is an increasing function of y_1, so the inverse function is also an increasing function. Thus, $D_i^{\Lambda_1}$ increases and $d_t^{\Lambda_1}$ decreases when Λ_1 increases. That is, the customer can get more compensation and pay less premium at the same time. The customer's utility from the contract may also increase. It inspires us to think that $\mathbb{U}_1(\Lambda_1)$ may be an increasing function of Λ_1. The obstacle is we are not sure how $a_t^{\Lambda_1}$ moves according to Λ_1. From (2.3.17), we can see $a_t^{\Lambda_1}$ is closely related to

$$\mathcal{U}(\Lambda_1) \;:=\; -\frac{1}{\Lambda_1}U_2(-g^{-1}(\Lambda_1, l)) - U_1(g^{-1}(\Lambda_1, l) - l). \tag{2.3.18}$$

Here, $\theta E[\mathcal{U}(\Lambda_1)]B_t$ can be recognized as the marginal benefit of the action. $V_1'(a_t)$ can be recognized as the marginal cost of the action. When $\Lambda_2^t = 0$ for some t, (2.3.15) becomes $V_1'(a_t) = -\theta E[\mathcal{U}(\Lambda_1)]B_t$. It illustrates that the optimal action is reached when its marginal benefit equals its marginal cost. To explore more connections between $a_t^{\Lambda_1}$ and Λ_1, we consider the derivative

$$\begin{aligned}\mathcal{U}'(\Lambda_1) =& \frac{1}{\Lambda_1^2}U_2(-g^{-1}(\Lambda_1, l)) + \frac{1}{\Lambda_1}U_2'(-g^{-1}(\Lambda_1, l))g^{-1'}(\Lambda_1, l)\\ &- U_1'(g^{-1}(\Lambda_1, l) - l)g^{-1'}(\Lambda_1, l).\end{aligned} \tag{2.3.19}$$

From (2.3.13), we have $\dfrac{U_2'(-D_i)}{\Lambda_1} = U_1'(D_i - l_i)$. Here, $D_i^{\Lambda_1} = g^{-1}(\Lambda_1, l_i)$, so we obtain $\dfrac{1}{\Lambda_1}U_2'(-g^{-1}(\Lambda_1, l)) = U_1'(g^{-1}(\Lambda_1, l) - l)$. Now, we rewrite (2.3.19) to get

$$\mathcal{U}'(\Lambda_1) = \frac{1}{\Lambda_1^2}U_2(-g^{-1}(\Lambda_1, l)). \tag{2.3.20}$$

Theorem 1. $\mathbb{U}_1(\Lambda_1)$ *is an increasing function of* Λ_1 *for* $\Lambda_1 \in (0, \infty)$.

Proof. We split $\bar{B}$ as $\bar{B} = E\left[\int_0^T KB_t dt\right] + \bar{B} - E\left[\int_0^T KB_t dt\right]$. Then, we rewrite (2.3.16) to obtain

$$\mathbb{U}_1(\Lambda_1) = \int_0^T U_1(w_t - d_t^{\Lambda_1})dt \tag{2.3.21}$$

$$+ E\left[U_1(D^{\Lambda_1} - l)\right]\theta\left(\bar{B} - E\left[\int_0^T KB_t dt\right]\right) \tag{2.3.22}$$

$$+ E\left[U_1(D^{\Lambda_1} - l)\right]\theta\left(E\left[\int_0^T KB_t dt\right] - E\left[\int_0^T a_t^{\Lambda_1}B_t dt\right]\right) - E\left[\int_0^T V_1(a_t^{\Lambda_1})dt\right] \tag{2.3.23}$$

From (2.3.17), $d_t^{\Lambda_1}$ is a decreasing function of Λ_1 for every $t \in [0, T]$. Thus, the term (2.3.21) is an increasing function of Λ_1. Recalling from (2.3.11) that

$\bar{B} - E\left[\int_0^T a_t B_t dt\right] \geq 0$, we obtain $\bar{B} - E\left[\int_0^T K B_t dt\right] \geq 0$. Also recalling that D^{Λ_1} is an increasing function of Λ_1 for every $l \in \mathcal{R}_l$, we see that the term (2.3.22) is an increasing function of Λ_1. Next, we will analyze the remaining terms in (2.3.23). For each $\omega \in \Omega$ and each $t \in [0, T]$, consider

$$\varphi_1(\Lambda_1) := \theta E\left[U_1(D^{\Lambda_1} - l)\right]\left(K - a_t^{\Lambda_1}\right) B_t - V_1(a_t^{\Lambda_1}).$$

We will show $\varphi_1(\Lambda_1)$ is an increasing function of Λ_1. $a_t^{\Lambda_1}$ takes different values for different Λ_1, so we will discuss the following three cases.
(i) If Λ_1 is such that $a_t^{\Lambda_1} = 0$, then we have

$$\varphi_1(\Lambda_1) = \theta E\left[U_1(D^{\Lambda_1} - l)\right] K B_t - V_1(0) = E\left[U_1(D^{\Lambda_1} - l)\right] K B_t. \quad (2.3.24)$$

Recalling $K \geq 0$, $B_t \geq 0$, and D^{Λ_1} is an increasing function of Λ_1 for every $l \in \mathcal{R}_l$, we get (2.3.24) is an increasing function of Λ_1.
(ii) If Λ_1 is such that $a_t^{\Lambda_1} = K$, then we have

$$\varphi_1(\Lambda_1) = \theta E\left[U_1(D^{\Lambda_1} - l)\right](K - K) B_t - V_1(K) = -V_1(K). \quad (2.3.25)$$

(2.3.25) is constant.
(iii) If Λ_1 is such that

$$a_t^{\Lambda_1} = V_1'^{-1}\left(- \theta E[\frac{1}{\Lambda_1} U_2(-g^{-1}(\Lambda_1, l)) + U_1(g^{-1}(\Lambda_1, l) - l)] B_t \right),$$

then we have

$$\begin{aligned} \varphi_1'(\Lambda_1) &= \theta E\left[U_1'(D^{\Lambda_1} - l)\frac{\partial D^{\Lambda_1}}{\partial \Lambda_1}\right](K - a_t^{\Lambda_1}) B_t \\ &\quad + \theta E\left[U_1(D^{\Lambda_1} - l)\right] B_t(-\frac{\partial a_t^{\Lambda_1}}{\partial \Lambda_1}) - V_1'(a_t^{\Lambda_1})\frac{\partial a_t^{\Lambda_1}}{\partial \Lambda_1}. \end{aligned}$$

Here, $D^{\Lambda_1} = g^{-1}(\Lambda_1, l)$ and

$$V_1'(a_t^{\Lambda_1}) = -\theta E[\frac{1}{\Lambda_1} U_2(-g^{-1}(\Lambda_1, l)) + U_1(g^{-1}(\Lambda_1, l) - l)] B_t.$$

Now we have

$$\begin{aligned}
\varphi_1'(\Lambda_1) &= \theta E\left[U_1'\left(g^{-1}(\Lambda_1,l)-l\right)\frac{\partial g^{-1}(\Lambda_1,l)}{\partial\Lambda_1}\right](K-a_t^{\Lambda_1})B_t \\
&\quad +\theta E\left[U_1\left(g^{-1}(\Lambda_1,l)-l\right)\right]B_t\left(-\frac{\partial a_t^{\Lambda_1}}{\partial\Lambda_1}\right) \\
&\quad +\theta E\left[\frac{1}{\Lambda_1}U_2(-g^{-1}(\Lambda_1,l))+U_1(g^{-1}(\Lambda_1,l)-l)\right]B_t\frac{\partial a_t^{\Lambda_1}}{\partial\Lambda_1} \\
&= \theta E\left[U_1'\left(g^{-1}(\Lambda_1,l)-l\right)\frac{\partial g^{-1}(\Lambda_1,l)}{\partial\Lambda_1}\right](K-a_t^{\Lambda_1})B_t \\
&\quad +\theta E\left[\frac{1}{\Lambda_1}U_2(-g^{-1}(\Lambda_1,l))\right]B_t\frac{\partial a_t^{\Lambda_1}}{\partial\Lambda_1}. \qquad (2.3.26)
\end{aligned}$$

Recalling the definition of $\mathcal{U}(\Lambda_1)$ in (2.3.18), we can see $a_t^{\Lambda_1}=V_1'^{-1}\left(\theta E[\mathcal{U}(\Lambda_1)]B_t\right)$. From (2.3.20), we obtain

$$\begin{aligned}
\frac{\partial a_t^{\Lambda_1}}{\partial\Lambda_1} &= V_1^{'-1'}\left(\theta E[\mathcal{U}(\Lambda_1)]B_t\right)\theta B_t E[\mathcal{U}'(\Lambda_1)] \\
&= V_1^{'-1'}\left(\theta E[\mathcal{U}(\Lambda_1)]B_t\right)\theta B_t E\left[\frac{1}{\Lambda_1^2}U_2(-g^{-1}(\Lambda_1,l))\right].
\end{aligned}$$

We rewrite (2.3.26) to get

$$\begin{aligned}
\varphi_1'(\Lambda_1) &= \theta E\left[U_1'\left(g^{-1}(\Lambda_1,l)-l\right)\frac{\partial g^{-1}(\Lambda_1,l)}{\partial\Lambda_1}\right](K-a_t^{\Lambda_1})B_t \qquad (2.3.27)\\
&\quad +\theta^2\frac{1}{\Lambda_1^3}\left(E[U_2(-g^{-1}(\Lambda_1,l))]\right)^2 B_t^2 V_1^{'-1'}\left(\theta E[\mathcal{U}(\Lambda_1)]B_t\right) \qquad (2.3.28)
\end{aligned}$$

U_1 is an increasing function and $g^{-1}(\Lambda_1,l)$ is an increasing function of Λ_1, meaning

$$U_1'\left(g^{-1}(\Lambda_1,l)-l\right)\frac{\partial g^{-1}(\Lambda_1,l)}{\partial\Lambda_1}>0.$$

We also know that $K-a_t^{\Lambda_1}>0$ and $B_t\geq 0$ for every $t\in[0,T]$ and $\omega\in\Omega$. Therefore, (2.3.27) is non-negative. V_1' is an increasing function, so its inverse $V_1^{'-1}$ must also be an increasing function. We can state that $V_1^{'-1'}\left(\theta E[\mathcal{U}(\Lambda_1)]B_t\right)\geq 0$ and therefore (2.3.28) is non-negative.

To summarize, we have shown that $\varphi_1(\Lambda_1)$ is an increasing function of Λ_1 in each case. It is obvious that $\varphi_1(\Lambda_1)$ is continuous, so we state that $\varphi_1(\Lambda_1)$ is an increasing function of Λ_1 in the interval $\Lambda_1 \in (0, \infty)$.

Taking the integration of $\varphi_1(\Lambda_1)$ from 0 to T and then taking the expectation on the integration, we obtain (2.3.23). So (2.3.23) increases when Λ_1 increases. Recalling that (2.3.21) and (2.3.22) also increase when Λ_1 increases, we conclude that $\mathbb{U}_1(\Lambda_1)$ is an increasing function of $\Lambda_1 \in (0, \infty)$. □

We define $\hat{\Lambda}_1$ by the following equation,

$$\mathbb{U}_1(\hat{\Lambda}_1) = R. \tag{2.3.29}$$

Then we have

Theorem 2. *If there exists $\hat{\Lambda}_1 > 0$ such that (2.3.29) holds, then the optimal insurance contract $(\hat{d}, \hat{D}, \hat{a}) = (d^{\hat{\Lambda}_1}, D^{\hat{\Lambda}_1}, a^{\hat{\Lambda}_1})$ is given by*

$$\hat{d}_t = -g^{-1}(\hat{\Lambda}_1, -w_t), \tag{2.3.30}$$

$$\hat{D}_i = g^{-1}(\hat{\Lambda}_1, l_i), \tag{2.3.31}$$

$$\hat{a}_t = \begin{cases} 0 & \text{if} \quad -\theta E\left[\frac{1}{\hat{\Lambda}_1} U_2(-g^{-1}(\hat{\Lambda}_1, l)) + U_1(g^{-1}(\hat{\Lambda}_1, l) - l)\right] B_t < 0, \\ V_1'^{-1}\left(-\theta E\left[\frac{1}{\hat{\Lambda}_1} U_2(-g^{-1}(\hat{\Lambda}_1, l)) + U_1(g^{-1}(\hat{\Lambda}_1, l) - l)\right] B_t \right) & \\ \quad \text{if} \quad 0 \le -\theta E\left[\frac{1}{\hat{\Lambda}_1} U_2(-g^{-1}(\hat{\Lambda}_1, l)) + U_1(g^{-1}(\hat{\Lambda}_1, l) - l)\right] B_t \le V_1'(K), & \\ K & \text{if} \quad V_1'(K) < -\theta E\left[\frac{1}{\hat{\Lambda}_1} U_2(-g^{-1}(\hat{\Lambda}_1, l)) + U_1(g^{-1}(\hat{\Lambda}_1, l) - l)\right] B_t. \end{cases} \tag{2.3.32}$$

Proof. First, we want to verify that the process a defined by (2.3.32) satisfies the constraint (2.2.6).

We consider three possibilities for

$$-\theta E\left[\frac{1}{\hat{\Lambda}_1} U_2(-g^{-1}(\hat{\Lambda}_1, l)) + U_1(g^{-1}(\hat{\Lambda}_1, l) - l)\right] B_t.$$

If

$$-\theta E\left[\frac{1}{\hat{\Lambda}_1}U_2(-g^{-1}(\hat{\Lambda}_1,l))+U_1(g^{-1}(\hat{\Lambda}_1,l)-l)\right]B_t<0,$$

then $\hat{a}_t=0$ and the constraint (2.2.6) is trivially satisfied. If

$$-\theta E\left[\frac{1}{\hat{\Lambda}_1}U_2(-g^{-1}(\hat{\Lambda}_1,l))+U_1(g^{-1}(\hat{\Lambda}_1,l)-l)\right]B_t>V_1'(K),$$

then $\hat{a}_t=K$ and the constraint (2.2.6) is trivially satisfied. If

$$0\le-\theta E\left[\frac{1}{\hat{\Lambda}_1}U_2(-g^{-1}(\hat{\Lambda}_1,l))+U_1(g^{-1}(\hat{\Lambda}_1,l)-l)\right]B_t\le V_1'(K),$$

then the strict convexity of V_1 and the condition $V_1'(0)=0$ imply that

$$0\le V_1'^{-1}\left(-\theta E\left[\frac{1}{\hat{\Lambda}_1}U_2(-g^{-1}(\hat{\Lambda}_1,l))+U_1(g^{-1}(\hat{\Lambda}_1,l)-l)\right]B_t\right)\le K,$$

which is equivalent to $0\le\hat{a}_t\le K$. Hence, $\hat{a}_t\in[0,K]$ for each $t\in[0,T]$ and (2.3.32) satisfies the condition (2.2.6).

Let a be a fixed admissible action process that satisfies (2.2.6). Then we find the first order conditions similar to (2.3.13) for D_i and d_t,

$$U_2'(-D_i)-\Lambda^aU_1'(D_i-l_i)=0,\quad U_2'(d_t)-\Lambda^aU_1'(w_t-d_t)=0, \tag{2.3.33}$$

where Λ^a is the Lagrangian multiplier. Since U_1 and U_2 are increasing functions, Λ^a must be positive to make the equations above meaningful. The solution of the first order conditions is

$$D_i^a=g^{-1}(\Lambda^a,l_i),\quad d_t^a=-g^{-1}(\Lambda^a,-w_t).$$

We define

$$\mathbb{U}_a(\Lambda^a) := E\left[\int_0^T U_1(w_t + g^{-1}(\Lambda^a, -w_t))dt \right.$$
$$\left. + \sum_{i=1}^{\mathcal{N}^a(T)} U_1(g^{-1}(\Lambda^a, l_i) - l_i) - \int_0^T V_1(a_t)dt\right].$$

We denote the root of $\mathbb{U}_a(\Lambda^a) = R$ by $\hat{\Lambda}^a$ and correspondingly we define $\hat{D}_i^a := g^{-1}(\hat{\Lambda}^a, l_i)$ and $\hat{d}_t^a := -g^{-1}(\hat{\Lambda}^a, -w_t)$. Next, we discuss the existence of $\hat{\Lambda}^a$ for a fixed process a. We will show that $\hat{\Lambda}^a$ exists if for the fixed process a, there are compensation and premium processes such that (2.2.5) holds. For the fixed process a, let $D = \{D_i; i = 1, 2, \cdots\}$ and $d = \{d_t; t \in [0, T]\}$ be any adapted compensation sequence and premium process that satisfy (2.2.5). When $\Lambda^a \to \infty$,

$$g^{-1}(\Lambda^a, l_i) \to \infty, \quad g^{-1}(\Lambda^a, -w_t) \to \infty,$$

which yields $g^{-1}(\Lambda^a, l_i) \geq D_i$ for $i = 1, 2, \cdots$ and $g^{-1}(\Lambda^a, -w_t) \geq -d_t$ for $t \in [0, T]$. Recalling that U_1 is an increasing function, we have

$$\lim_{\Lambda^a \to \infty} \mathbb{U}_a(\Lambda^a) \geq E\left[\int_0^T U_1(w_t - d_t)dt + \sum_{i=1}^{\mathcal{N}^a(T)} U_1(D_i - l_i) - \int_0^T V_1(a_t)dt\right] \geq R$$

from (2.2.5). When $\Lambda^a \to 0^+$,

$$g^{-1}(\Lambda^a, l_i) \to -\infty, \quad g^{-1}(\Lambda^a, -w_t) \to -\infty,$$

resulting in $\mathbb{U}_a(\Lambda^a) \to -\infty$ and consequently $\mathbb{U}_a(\Lambda^a) < R$. Due to the continuity of $\mathbb{U}_a(\Lambda^a)$, we see that there exists $\hat{\Lambda}^a \in (0, \infty)$ such that $\mathbb{U}_a(\hat{\Lambda}^a) = R$ holds.

We will prove Theorem 2 in two steps. First, we will show $\mathcal{J}(\hat{d}^a, \hat{D}^a, a) \geq \mathcal{J}(d, D, a)$ for any fixed action process a that satisfies (2.2.6). Afterwards, we will show $\mathcal{J}(\hat{d}, \hat{D}, \hat{a}) \geq \mathcal{J}(\hat{d}^a, \hat{D}^a, a)$. We need some preparation before starting the steps.

Lemma 2.

$$\theta E\left[\frac{1}{\hat{\Lambda}_1}U_2(-\hat{D})+U_1(\hat{D}-l)\right]B_t(a_t-\hat{a}_t)\ \geq\ V_1(\hat{a}_t)-V_1(a_t).$$

Proof. According to (2.3.30) and (2.3.31), it is sufficient to prove that

$$\theta E\left[\frac{1}{\hat{\Lambda}_1}U_2(-g^{-1}(\hat{\Lambda}_1,l))+U_1(g^{-1}(\hat{\Lambda}_1,l)-l)\right]B_t(a_t-\hat{a}_t)\ \geq\ V_1(\hat{a}_t)-V_1(a_t).$$

If $0<\hat{a}_t<K$,

$$V_1'(\hat{a}_t)(\hat{a}_t-a_t)=\theta E[\frac{1}{\hat{\Lambda}_1}U_2(-g^{-1}(\hat{\Lambda}_1,l))+U_1(g^{-1}(\hat{\Lambda}_1,l)-l)]B_t(a_t-\hat{a}_t).$$

If $\hat{a}_t=K$, then

$$\hat{a}_t-a_t=K-a_t\geq 0$$

and

$$V_1'(\hat{a}_t)=V_1'(K)<-\theta E[\frac{1}{\hat{\Lambda}_1}U_2(-g^{-1}(\hat{\Lambda}_1,l))+U_1(g^{-1}(\hat{\Lambda}_1,l)-l)]B_t,$$

which yields

$$V_1'(\hat{a}_t)(\hat{a}_t-a_t)\leq\theta E[\frac{1}{\hat{\Lambda}_1}U_2(-g^{-1}(\hat{\Lambda}_1,l))+U_1(g^{-1}(\hat{\Lambda}_1,l)-l)]B_t(a_t-\hat{a}_t).$$

If $\hat{a}_t=0$, then

$$\hat{a}_t-a_t=0-a_t\leq 0$$

and

$$V_1'(\hat{a}_t)=V_1'(0)=0>-\theta E[\frac{1}{\hat{\Lambda}_1}U_2(-g^{-1}(\hat{\Lambda}_1,l))+U_1(g^{-1}(\hat{\Lambda}_1,l)-l)]B_t,$$

which yields

$$V_1'(\hat{a}_t)(\hat{a}_t - a_t) \leq \theta E[\frac{1}{\hat{\Lambda}_1} U_2(-g^{-1}(\hat{\Lambda}_1, l)) + U_1(g^{-1}(\hat{\Lambda}_1, l) - l)] B_t(a_t - \hat{a}_t).$$

Due to the convexity of V_1, we have $V_1'(\hat{a}_t)(\hat{a}_t - a_t) \geq V_1(\hat{a}_t) - V_1(a_t)$. The required statement follows. □

Step 1. Since U_1 and U_2 are both concave functions, we obtain the inequality

$$\int_0^T U_1(w_t - d_t)dt + \sum_{i=1}^{\mathcal{N}^a(T)} U_1(D_i - l_i) - \left(\int_0^T U_1(w_t - \hat{d}_t^a)dt + \sum_{i=1}^{\mathcal{N}^a(T)} U_1(\hat{D}_i^a - l_i) \right)$$
$$\leq \int_0^T U_1'(w_t - \hat{d}_t^a)(\hat{d}_t^a - d_t)dt + \sum_{i=1}^{\mathcal{N}^a(T)} \left(U_1'(\hat{D}_i^a - l_i)(D_i - \hat{D}_i^a) \right). \quad (2.3.34)$$

Furthermore, (2.2.4) implies

$$\mathcal{J}(\hat{d}^a, \hat{D}^a, a) - \mathcal{J}(d, D, a)$$
$$= E\left[\int_0^T \left(U_2(\hat{d}_t^a) - U_2(d_t) \right) dt + \sum_{i=1}^{\mathcal{N}^a(T)} \left(U_2(-\hat{D}_i^a) - U_2(-D_i) \right) \right],$$

which yields

$$\mathcal{J}(\hat{d}^a, \hat{D}^a, a) - \mathcal{J}(d, D, a)$$
$$\geq E\left[\int_0^T U_2'(\hat{d}_t^a)(\hat{d}_t^a - d_t)dt + \sum_{i=1}^{\mathcal{N}^a(T)} \left(U_2'(-\hat{D}_i^a)(D_i - \hat{D}_i^a) \right) \right]. \quad (2.3.35)$$

According to (2.3.33), we can replace $U_2'(\hat{d}_t^a)$ by $\hat{\Lambda}^a U_1'(w_t - \hat{d}_t^a)$ and replace $U_2'(-\hat{D}_i^a)$ by $\hat{\Lambda}^a U_1'(\hat{D}_i^a - l_i)$ in (2.3.35). Comparing (2.3.34) and (2.3.35), we obtain

$$
\begin{aligned}
&\mathcal{J}(\hat{d}^a, \hat{D}^a, a) - \mathcal{J}(d, D, a) \\
\geq\ & E\left[\int_0^T \hat{\Lambda}^a U_1'(w_t - \hat{d}_t^a)(\hat{d}_t^a - d_t)dt + \hat{\Lambda}^a \sum_{i=1}^{\mathcal{N}^a(T)} \Big(U_1'(\hat{D}_i^a - l_i)(D_i - \hat{D}_i^a)\Big)\right] \\
\geq\ & \hat{\Lambda}^a E\left[\int_0^T U_1(w_t - d_t)dt + \sum_{i=1}^{\mathcal{N}^a(T)} U_1(D_i - l_i)\right. \\
& \qquad\qquad \left. - \left(\int_0^T U_1(w_t - \hat{d}_t^a)dt + \sum_{i=1}^{\mathcal{N}^a(T)} U_1(\hat{D}_i^a - l_i)\right)\right].
\end{aligned}
$$

According to (2.2.5), we obtain

$$
\begin{aligned}
&\mathcal{J}(\hat{d}^a, \hat{D}^a, a) - \mathcal{J}(d, D, a) \\
&\geq \hat{\Lambda}^a \left((R + E\left[\int_0^T V_1(a_t)dt\right]) - (R + E\left[\int_0^T V_1(a_t)dt\right])\right) = 0.
\end{aligned}
$$

Therefore, $\hat{d}^a$ and $\hat{D}^a$ are the optimal controls when a is the fixed action process.

Step 2. As a Lagrangian multiplier, $\hat{\Lambda}^a$ is a constant. The randomness of $\hat{D}_i^a$ depends on l_i only, so $\hat{D}_i^a$ is independent of $\mathcal{N}^a(t)$ for $i = 1, 2, \cdots$, and we get the following equations for any a satisfying (2.2.5) and (2.2.6).

$$
\begin{aligned}
E\left[\sum_{i=1}^{\mathcal{N}^a(T)} U_1(\hat{D}_i^a - l_i)\right] &= E[\mathcal{N}^a(T)]E\left[U_1(\hat{D}^a - l)\right], \\
E\left[\sum_{i=1}^{\mathcal{N}^a(T)} U_2(-\hat{D}_i^a)\right] &= E[\mathcal{N}^a(T)]E\left[U_2(-\hat{D}^a)\right],
\end{aligned} \tag{2.3.36}
$$

where $E[\mathcal{N}^a(T)] = \theta E\left[\bar{B} - \int_0^T a_t B_t dt\right]$ from (2.3.11). Similarly, we obtain

$$\begin{aligned} E\left[\sum_{i=1}^{\mathcal{N}^{\hat{a}}(T)} U_1(\hat{D}_i - l_i)\right] &= E[\mathcal{N}^{\hat{a}}(T)] E\left[U_1(\hat{D} - l)\right], \\ E\left[\sum_{i=1}^{\mathcal{N}^{\hat{a}}(T)} U_2(-\hat{D}_i)\right] &= E[\mathcal{N}^{\hat{a}}(T)] E\left[U_2(-\hat{D})\right], \end{aligned} \tag{2.3.37}$$

where $E[\mathcal{N}^{\hat{a}}(T)] = \theta E\left[\bar{B} - \int_0^T \hat{a}_t B_t dt\right]$. Hence, the difference between $\mathcal{J}(\hat{d}, \hat{D}, \hat{a})$ and $\mathcal{J}(\hat{d}^a, \hat{D}^a, a)$ is

$$\begin{aligned} &\mathcal{J}(\hat{d}, \hat{D}, \hat{a}) - \mathcal{J}(\hat{d}^a, \hat{D}^a, a) \\ = \; & \int_0^T \left(U_2(\hat{d}_t) - U_2(\hat{d}^a_t)\right) dt + E\left[\sum_{i=1}^{\mathcal{N}^{\hat{a}}(T)} U_2(-\hat{D}_i) - \sum_{i=1}^{\mathcal{N}^a(T)} U_2(-\hat{D}^a_i)\right] \\ = \; & \int_0^T \left(U_2(\hat{d}_t) - U_2(\hat{d}^a_t)\right) dt \\ & +\theta E\left[\bar{B} - \int_0^T \hat{a}_t B_t dt\right] E\left[U_2(-\hat{D})\right] - \theta E\left[\bar{B} - \int_0^T a_t B_t dt\right] E\left[U_2(-\hat{D}^a)\right] \\ = \; & \int_0^T \left(U_2(\hat{d}_t) - U_2(\hat{d}^a_t)\right) dt + \theta E\left[\int_0^T (a_t - \hat{a}_t) B_t dt\right] E\left[U_2(-\hat{D})\right] \\ & +\theta E\left[\bar{B} - \int_0^T a_t B_t dt\right] E\left[U_2(-\hat{D}) - U_2(-\hat{D}^a)\right]. \end{aligned}$$

Recalling $E\left[\bar{B} - \int_0^T a_t B_t dt\right] \geq 0$ and the concavity of the utility function U_2, we obtain

$$\begin{aligned} &\mathcal{J}(\hat{d}, \hat{D}, \hat{a}) - \mathcal{J}(\hat{d}^a, \hat{D}^a, a) \\ &\geq \int_0^T U_2'(\hat{d}_t)(\hat{d}_t - \hat{d}^a_t) dt + \theta E\left[\int_0^T (a_t - \hat{a}_t) B_t dt\right] E\left[U_2(-\hat{D})\right] \\ &\quad + \theta E\left[\bar{B} - \int_0^T a_t B_t dt\right] E\left[U_2'(-\hat{D})(\hat{D}^a - \hat{D})\right]. \end{aligned}$$

According to (2.3.13), this inequality can be rewritten as

$$\begin{aligned}
&\mathcal{J}(\hat{d},\hat{D},\hat{a})-\mathcal{J}(\hat{d}^a,\hat{D}^a,a)\\
&\geq \int_0^T \hat{\Lambda}_1 U_1'(w_t-\hat{d}_t)(\hat{d}_t-\hat{d}_t^a)dt+\theta E\left[\int_0^T (a_t-\hat{a}_t)B_t dt\right]E\left[U_2(-\hat{D})\right]\\
&\quad +\theta E\left[\bar{B}-\int_0^T a_t B_t dt\right]E\left[\hat{\Lambda}_1 U_1'(\hat{D}-l)(\hat{D}^a-\hat{D})\right].
\end{aligned}$$

Due to the concavity of the utility function U_1, we have

$$\begin{aligned}
&\mathcal{J}(\hat{d},\hat{D},\hat{a})-\mathcal{J}(\hat{d}^a,\hat{D}^a,a)\\
&\geq \hat{\Lambda}_1\int_0^T \Big(U_1(w_t-\hat{d}_t^a)-U_1(w_t-\hat{d}_t)\Big)dt\\
&\quad +\hat{\Lambda}_1\theta E\left[\int_0^T (a_t-\hat{a}_t)B_t dt\right]E\left[\frac{1}{\hat{\Lambda}_1}U_2(-\hat{D})+U_1(\hat{D}-l)-U_1(\hat{D}-l)\right]\\
&\quad +\hat{\Lambda}_1\theta E\left[\bar{B}-\int_0^T a_t B_t dt\right]E\left[U_1(\hat{D}^a-l)-U_1(\hat{D}-l)\right].
\end{aligned}$$

Applying Lemma 2, we obtain

$$\begin{aligned}
&\frac{1}{\hat{\Lambda}_1}\Big(\mathcal{J}(\hat{d},\hat{D},\hat{a})-\mathcal{J}(\hat{d}^a,\hat{D}^a,a)\Big)\\
&\geq \int_0^T \Big(U_1(w_t-\hat{d}_t^a)-U_1(w_t-\hat{d}_t)\Big)dt+E\left[\int_0^T \Big(V_1(\hat{a}_t)-V_1(a_t)\Big)dt\right]\\
&\quad +\theta E\left[\int_0^T (\hat{a}_t-a_t)B_t dt\right]E\left[U_1(\hat{D}-l)\right]\\
&\quad +\theta E\left[\bar{B}-\int_0^T a_t B_t dt\right]E\left[U_1(\hat{D}^a-l)-U_1(\hat{D}-l)\right]\\
&= \int_0^T \Big(U_1(w_t-\hat{d}_t^a)-U_1(w_t-\hat{d}_t)\Big)dt+E\left[\int_0^T \Big(V_1(\hat{a}_t)-V_1(a_t)\Big)dt\right]\\
&\quad +\theta E\left[\bar{B}-\int_0^T a_t B_t dt\right]E\left[U_1(\hat{D}^a-l)\right]\\
&\quad -\theta E\left[\bar{B}-\int_0^T \hat{a}_t B_t dt\right]E\left[U_1(\hat{D}-l)\right].
\end{aligned}$$

Applying (2.3.36) and (2.3.37)to the expression above, we obtain

$$\begin{aligned}
&\frac{1}{\hat{\Lambda}_1}\Big(\mathcal{J}(\hat{d},\hat{D},\hat{a})-\mathcal{J}(\hat{d}^a,\hat{D}^a,a)\Big)\\
&\geq E\left[\int_0^T U_1(w_t-\hat{d}_t^a)dt+\sum_{i=1}^{\mathcal{N}^a(T)}U_1(\hat{D}_i^a-l_i)-\int_0^T V_1(a_t)dt\right]\\
&\quad -E\left[\int_0^T U_1(w_t-\hat{d}_t)dt+\sum_{i=1}^{\mathcal{N}^{\hat{a}}(T)}U_1(\hat{D}_i-l_i)-\int_0^T V_1(\hat{a}_t)dt\right]\\
&=R-R=0.
\end{aligned}$$

Therefore, $\mathcal{J}(\hat{d},\hat{D},\hat{a}) \geq \mathcal{J}(d,D,a)$ for every admissible control (d,D,a) that satisfies the constraints of Problem 1. If $\hat{\Lambda}_1 > 0$ satisfies (2.3.29), we conclude that $(\hat{d},\hat{D},\hat{a})$ is the optimal solution. □

Remark 1. *There is Λ_1 such that $\mathbb{U}_1(\Lambda_1) < R$ whatever R is* [1]*. It must be smaller than $\hat{\Lambda}_1$ according to Theorem 1 if $\hat{\Lambda}_1$ exists. However, the existence of $\hat{\Lambda}_1$ depends on the value of R. In Theorem 4 of the next section, we will show the existence and uniqueness of $\hat{\Lambda}_1$ with an appropriate value of R.*

Remark 2. *The optimal action $\hat{a}_t$ is an increasing function of B_t. We can explain it in three ways. First, if r_t is high, actions at this moment are more effective. The insured wants to take this opportunity to insert more actions. Second, the insured prefers to insert more actions earlier if we neglect the uncertainty elements r_t, Y_i, and τ_i. For example, if $r_t = r_0$ for every $t \in [0,T]$ and $Y = 0$ almost surely, then*

$$B_t = r_0 e^t \int_t^T e^{-(1+\delta)u}\left(Y_0 e^{\delta\tau_0}\right)du = \frac{r_0 Y_0}{1+\delta}(e^{-\delta t} - e^{t-(1+\delta)T}),$$

which is a decreasing function of t. Thus, $\hat{a}_t$ is also a decreasing function of t. Especially, $B_t = 0$ when $t = T$, resulting in $a_T = 0$. The action taken at an earlier time is effective for a longer period. It can reduce the intensity of the accidents throughout the whole period. The insured is motivated to act as much

[1] We showed $\lim_{\Lambda_1\to 0^+}\mathbb{U}_1(\Lambda_1) = -\infty$ when we discussed Lagrangian multipliers in (2.3.15).

as possible at the beginning. The action taken at maturity is only effective at the moment T. It makes almost no contribution to lowering the intensity. The insured does not want to waste his action, thus takes zero action at time T. Third, the bigger $\sum_{i=0}^{\mathcal{M}(t)} Y_i e^{\delta \tau_i}$ is, the bigger B_t is. Thus more actions should be inserted when the accumulated external exposure is more. Note that the same amount of action deducts the same proportion of the intensity of claims. When the exposure is high, the same amount of action can remove more intensity. The actions are therefore more valuable and the insured will choose to execute more actions at these moments.

2.4 The Reservation Utility

In this section, we calculate the reservation utility R of (2.2.5), which is the utility of the potential insured if he does not purchase insurance. The participation constraint (2.2.5) means that the expected total utility from purchasing insurance is greater than or equal to the expected total utility from not purchasing insurance. In this section, we will (i) calculate the reservation utility R when the potential insured does not purchase insurance, (ii) compare the actions taken when the potential insured does and does not enter the insurance market, and (iii) show that $\hat{\Lambda}_1$ of Theorem 2 exists uniquely.

If the potential insured does not enter the insurance market, then he will not pay a premium and, as a consequence, will not receive any compensation. However, he will select the action to maximize his expected total utility.

We denote by $\mathcal{A}_R$ the class of stochastic processes $a : [0, T] \times \Omega \mapsto \mathbb{R}$ that are adapted to the filtration $\mathbb{F}_1$.

Problem 2. *If the potential insured does not purchase insurance, he wants to* *obtain the control $a^* \in \mathcal{A}_R$ that solves the problem*

$$\max_{a \in \mathcal{A}_R} \ E\left[\int_0^T U_1(w_t)dt + \sum_{i=1}^{\mathcal{N}^a(T)} U_1(-l_i) - \int_0^T V_1(a_t)dt\right]$$
$$s.t. \ \ 0 \le a_t \le K, \ \ \textit{for all } t \in [0,T].$$

According to (2.3.11), $E\left[\sum_{i=1}^{\mathcal{N}^a(T)} U_1(-l_i)\right]$ can be rewritten as

$$E\left[\sum_{i=1}^{\mathcal{N}^a(T)} U_1(-l_i)\right] = E[U_1(-l)]\left(\theta\bar{B} - \theta E\left[\int_0^T a_t B_t dt\right]\right). \tag{2.4.38}$$

We define the Lagrangian function

$$\mathcal{L}_2(a;\Lambda_3) := \int_0^T U_1(w_t)dt + E[U_1(-l)]\left(\theta\bar{B} - \theta E\left[\int_0^T a_t B_t dt\right]\right) - E\left[\int_0^T V_1(a_t)dt\right] + E\left[\int_0^T \Lambda_3^t a_t dt\right],$$

where Λ_3^t, $t \in [0,T]$, adapted to $\mathbb{F}_1$, are Lagrangian multipliers. We take the differentiation of the Lagrangian function with respect to a_t and obtain the first order conditions

$$V_1'(a_t) - \Lambda_3^t = -\theta B_t E[U_1(-l)] \tag{2.4.39}$$

for $t \in [0,T]$ and $\omega \in \Omega$. $U_1(-l) < 0$ for $l \in \mathcal{R}_l$, then $-\theta B_t E[U_1(-l)] \ge 0$ for each $t \in [0,T]$ and $\omega \in \Omega$. If $0 \le -\theta B_t E[U_1(-l)] \le V_1'(K)$ for some $t \in [0,T]$ and $\omega \in \Omega$, $\Lambda_3^t(\omega) = 0$. The solution of (2.4.39) for a_t satisfies the constraint, so the constraint does not bind. If $-\theta B_t E[U_1(-l)] \ge V_1'(K)$ for some $t \in [0,T]$ and $\omega \in \Omega$, $\Lambda_3^t(\omega) < 0$. In this case, the marginal benefit of the action is always bigger than its marginal cost. However, the constraint $a_t \le K$ binds, so the optimal action is just K.

Proposition 2. *The optimal control of Problem 2 is given by*

$$a_t^* = \begin{cases} V_1'^{-1}\left(-\theta B_t E[U_1(-l)]\right) & \textit{if } V_1'(K) \geq -\theta B_t E[U_1(-l)] \\ K & \textit{if } V_1'(K) < -\theta B_t E[U_1(-l)]. \end{cases} \tag{2.4.40}$$

Proof. Let $\{a_t\}_{t\in[0,T]}$ be any action process that satisfies the constraints of Problem 2. We will compare the utilities from implementing the two action processes a^* and a. We denote by $\mathcal{D}(a^*, a)$ the difference of the expected total utilities associated with a^* and a. That is,

$$\begin{aligned} \mathcal{D}(a^*, a) : &= E\left[\int_0^T U_1(w_t)dt + \sum_{i=1}^{\mathcal{N}^{a^*}(T)} U_1(-l_i) - \int_0^T V_1(a_t^*)dt\right] \\ &\quad - E\left[\int_0^T U_1(w_t)dt + \sum_{i=1}^{\mathcal{N}^{a}(T)} U_1(-l_i) - \int_0^T V_1(a_t)dt\right]. \end{aligned}$$

According to (2.4.38), we have

$$\begin{aligned} &\mathcal{D}(a^*, a) \\ &= E\left[U_1(-l)\right]\left(\theta\bar{B} - \theta E\left[\int_0^T a_t^* B_t dt\right]\right) \\ &\quad - E\left[U_1(-l)\right]\left(\theta\bar{B} - \theta E\left[\int_0^T a_t B_t dt\right]\right) + E\left[\int_0^T \left(V_1(a_t) - V_1(a_t^*)\right)dt\right] \\ &= \theta E[U_1(-l)]\left(E\left[\int_0^T (a_t - a_t^*)B_t dt\right]\right) + E\left[\int_0^T \left(V_1(a_t) - V_1(a_t^*)\right)dt\right]. \end{aligned}$$

The convexity of V_1 implies

$$\begin{aligned} &\mathcal{D}(a^*, a) \\ &\geq \theta E[U_1(-l)]\left(E\left[\int_0^T (a_t - a_t^*)B_t dt\right]\right) + E\left[\int_0^T V_1'(a_t^*)(a_t - a_t^*)dt\right] \\ &= E\left[\int_0^T \left(V_1'(a_t^*) + \theta E[U_1(-l)]B_t\right)(a_t - a_t^*)dt\right]. \end{aligned}$$

Next, we consider the two cases described in equation (2.4.40). If $a_t^* = K$, from

(2.4.40), we have

$$a_t - a_t^* = a_t - K \leq 0 \ \text{ and } \ V_1'(a_t^*) = V_1'(K) \leq -\theta E[U_1(-l)]B_t$$

which yields

$$\left(V_1'(a_t^*) + \theta E[U_1(-l)]B_t\right)(a_t - a_t^*) \geq 0.$$

Otherwise, if $a_t^* = V_1'^{-1}\left(-\theta B_t E[U_1(-l)]\right)$, we have $V_1'(a_t^*) = -\theta E[U_1(-l)]B_t$, which yields

$$\left(V_1'(a_t^*) + \theta E[U_1(-l)]B_t\right)(a_t - a_t^*) = 0.$$

Now we can obtain $\mathcal{D}(a^*, a) \geq 0$ and conclude that the action process a^* is the optimal control of Problem 2. □

We recall a^{Λ_1} defined in (2.3.17). Comparing the two action processes a^{Λ_1} and a^*, we have the following relation.

Theorem 3. *For every $t \in [0, T]$:*

$$V_1'(a_t^{\Lambda_1}) \leq V_1'(a_t^*) - \frac{1}{\Lambda_1} U_2(0) B_t \theta.$$

Proof. Since $U_2(0) \leq 0$ and $\Lambda_1 > 0$, we have $-\frac{1}{\Lambda_1} U_2(0)\theta B_t \geq 0$. We will consider three cases for $a_t^{\Lambda_1}$.

(i) Consider $a_t^{\Lambda_1} = 0$. Then, $V_1'(a_t^{\Lambda_1}) = V_1'(0) = 0$. Noting $a_t^* > 0$, we know $V_1'(a_t^*) > 0$. It follows that

$$V_1'(a_t^{\Lambda_1}) \leq V_1'(a_t^*) \leq V_1'(a_t^*) - \frac{1}{\Lambda_1} U_2(0)\theta B_t.$$

(ii) Consider $a_t^{\Lambda_1} = K$. From (2.3.17), we have

$$V_1'(a_t^{\Lambda_1}) = V_1'(K) < -\theta E\left[\frac{1}{\Lambda_1} U_2(-g^{-1}(\Lambda_1, l)) + U_1(g^{-1}(\Lambda_1, l) - l)\right] B_t.$$

If $a_t^* = V_1'^{-1}\left(-\theta B_t E[U_1(-l)]\right)$, we have

$$V_1'(a_t^*) = -\theta B_t E[U_1(-l)].$$

It follows that

$$\begin{aligned}
&V_1'(a_t^{\Lambda_1}) - V_1'(a_t^*) \\
&\leq -\theta B_t E\left[\frac{1}{\Lambda_1}U_2(-g^{-1}(\Lambda_1, l)) + U_1(g^{-1}(\Lambda_1, l) - l) - U_1(-l)\right].
\end{aligned}$$

The concavity of the utility functions implies

$$U_1(g^{-1}(\Lambda_1, l) - l) - U_1(-l) \geq g^{-1}(\Lambda_1, l)U_1'(g^{-1}(\Lambda_1, l) - l)$$

and

$$U_2(0) - U_2(-g^{-1}(\Lambda_1, l)) \leq g^{-1}(\Lambda_1, l)U_2'(-g^{-1}(\Lambda_1, l))$$

for every $l \in \mathcal{R}_l$, so we have

$$\begin{aligned}
&V_1'(a_t^{\Lambda_1}) - V_1'(a_t^*) \\
&\leq -\theta B_t E\left[\frac{1}{\Lambda_1}U_2(-g^{-1}(\Lambda_1, l)) + g^{-1}(\Lambda_1, l)U_1'(g^{-1}(\Lambda_1, l) - l)\right] \\
&= -\theta B_t E\left[\frac{1}{\Lambda_1}U_2(-g^{-1}(\Lambda_1, l)) + g^{-1}(\Lambda_1, l)\frac{1}{\Lambda_1}U_2'(-g^{-1}(\Lambda_1, l))\right] \\
&\leq -\frac{1}{\Lambda_1}\theta U_2(0)B_t. \qquad (2.4.41)
\end{aligned}$$

If $a_t^* = K$, then

$$V_1'(a_t^{\Lambda_1}) - V_1'(a_t^*) = V_1'(K) - V_1'(K) = 0 \leq -\frac{1}{\Lambda_1}\theta U_2(0)B_t.$$

(iii) Consider $a_t^{\Lambda_1} = V_1'^{-1}\left(-\theta E[\frac{1}{\Lambda_1}U_2(-g^{-1}(\Lambda_1, l)) + U_1(g^{-1}(\Lambda_1, l) - l)]B_t\right)$.

If $a_t^* = V_1'^{-1}\left(-\theta B_t E[U_1(-l)]\right)$, we have

$$\begin{aligned} & V_1'(a_t^{\Lambda_1}) - V_1'(a_t^*) \\ & = -\theta B_t E\left[\frac{1}{\Lambda_1} U_2(-g^{-1}(\Lambda_1, l)) + U_1(g^{-1}(\Lambda_1, l) - l) - U_1(-l)\right]. \end{aligned}$$

Now we can repeat (2.4.41) to get $V_1'(a_t^{\Lambda_1}) - V_1'(a_t^*) \leq -\frac{1}{\Lambda_1}\theta U_2(0) B_t$.

If $a_t^* = K$, it is obvious that

$$V_1'(a_t^{\Lambda_1}) - V_1'(a_t^*) = V_1'(a_t^{\Lambda_1}) - V_1'(K) < 0 \leq -\frac{1}{\Lambda_1}\theta U_2(0) B_t.$$

As a summary of all the cases discussed above, the required statement is proved. □

We observe that if $U_2(0) = 0$, then $V_1'(a_t^{\Lambda_1}) \leq V_1'(a_t^*)$ as a consequence of Theorem 3. Since $V_1'(\cdot)$ is an increasing function, we have the following relation between the two action processes.

Corollary 1. *If $U_2(0) = 0$, then for every $t \in [0, T]$:*

$$a_t^{\Lambda_1} \leq a_t^*.$$

Theorem 3 shows that a^{Λ_1} is constrained by a^*. This constraint is more evident when $U_2(0) = 0$.

Taking a^* into the objective function of Problem 2, we obtain the reservation utility

$$R = \int_0^T U_1(w_t)dt + \theta E[U_1(-l)]\left(\bar{B} - E\left[\int_0^T a_t^* B_t dt\right]\right) - E\left[\int_0^T V_1(a_t^*)dt\right]. \tag{2.4.42}$$

We define $\underline{\Lambda_1}$ by the equation

$$E[U_1(g^{-1}(\underline{\Lambda_1}, l) - l)] = E[U_1(-l)]. \tag{2.4.43}$$

Lemma 3. $\underline{\Lambda_1}$ *exists uniquely. Furthermore,* $\mathbb{U}_1(\underline{\Lambda_1}) < R$*, where* R *is the reservation utility defined by (2.4.42).*

Proof. We consider $\varphi_2(\underline{\lambda}) := E[U_1(g^{-1}(\underline{\lambda}, l) - l)]$ as a function of $\underline{\lambda} \in (0, \infty)$. From the definition of the function g, we have $g(0, l) = \dfrac{U_2'(0)}{U_1'(0-l)}$. It follows that $g^{-1}\left(\dfrac{U_2'(0)}{U_1'(-l)}, l\right) = 0$. When $\underline{\lambda} = \dfrac{U_2'(0)}{U_1'(-\inf \mathcal{R}_l)}$, $\underline{\lambda} \geq \dfrac{U_2'(0)}{U_1'(-l)}$ for every $l \in \mathcal{R}_l$ due to the concavity of U_1. Since $g^{-1}(\cdot, y_2)$ is an increasing function, $g^{-1}(\underline{\lambda}, l) \geq 0$ for every $l \in \mathcal{R}_l$. It results in $\varphi_2(\underline{\lambda}) \geq E\left[U_1(-l)\right]$. When $\underline{\lambda} = \dfrac{U_2'(0)}{U_1'(-\sup \mathcal{R}_l)}$, $\underline{\lambda} \leq \dfrac{U_2'(0)}{U_1'(-l)}$ for every $l \in \mathcal{R}_l$. Then we have $g^{-1}(\underline{\lambda}, l) \leq 0$ for every $l \in \mathcal{R}_l$ and $\varphi_2(\underline{\lambda}) \leq E\left[U_1(-l)\right]$. $\varphi_2(\underline{\lambda})$ is continuous and monotone because g^{-1} and U_1 are continuous and monotone functions. Using the Mean Value Theorem, we can conclude there is a unique $\underline{\Lambda_1}$ such that $\varphi_2(\underline{\Lambda_1}) = E\left[U_1(-l)\right]$ and $\underline{\Lambda_1} \in \left[\dfrac{U_2'(0)}{U_1'(-\sup \mathcal{R}_l)}, \dfrac{U_2'(0)}{U_1'(-\inf \mathcal{R}_l)}\right]$.

Noting $D^{\underline{\Lambda_1}} = g^{-1}(\underline{\Lambda_1}, l)$, we have $E[U_1(D^{\underline{\Lambda_1}} - l)] = E[U_1(-l)]$ according to (2.4.43). From (2.3.16),

$$\begin{aligned} \mathbb{U}_1(\underline{\Lambda_1}) = & \int_0^T U_1(w_t - d_t^{\underline{\Lambda_1}})dt + E\left[U_1(-l)\right]\theta\left(\bar{B} - E[\int_0^T a_t^{\underline{\Lambda_1}} B_t dt]\right) \\ & - E\left[\int_0^T V_1(a_t^{\underline{\Lambda_1}})dt\right]. \end{aligned}$$

Comparing (2.4.42) and the expression above, we obtain

$$\begin{aligned} R - \mathbb{U}_1(\underline{\Lambda_1}) \ = \ & \int_0^T \left(U_1(w_t) - U_1(w_t - d_t^{\underline{\Lambda_1}})\right) dt \\ & + \theta E[U_1(-l)] E\left[\int_0^T (a_t^{\underline{\Lambda_1}} - a_t^*) B_t dt\right] \\ & + E\left[\int_0^T \left(V_1(a_t^{\underline{\Lambda_1}}) - V_1(a_t^*)\right) dt\right]. \end{aligned}$$

The range of $\underline{\Lambda_1}$ indicates that $\underline{\Lambda_1} < \dfrac{U_2'(0)}{U_1'(0)} < \dfrac{U_2'(0)}{U_1'(w_t)}$. It yields $d_t^{\underline{\Lambda_1}} > 0$ and $U_1(w_t) - U_1(w_t - d_t^{\underline{\Lambda_1}}) > 0$ for $t \in [0, T]$. Thus, the equation above implies

$$\begin{aligned} &R - \mathbb{U}_1(\underline{\Lambda_1}) \\ &> \theta E[U_1(-l)]E\left[\int_0^T (a_t^{\underline{\Lambda_1}} - a_t^*)B_t dt\right] + E\left[\int_0^T \left(V_1(a_t^{\underline{\Lambda_1}}) - V_1(a_t^*)\right) dt\right]. \end{aligned}$$

Since $V_1(\cdot)$ is a convex function, $V_1(a_t^{\underline{\Lambda_1}}) - V_1(a_t^*) \geq V_1'(a_t^*)\left(a_t^{\underline{\Lambda_1}} - a_t^*\right)$. Hence,

$$R - \mathbb{U}_1(\underline{\Lambda_1}) > E\left[\int_0^T \left(\theta E[U_1(-l)]B_t + V_1'(a_t^*)\right)(a_t^{\underline{\Lambda_1}} - a_t^*)dt\right]. \tag{2.4.44}$$

Next, we consider the two cases described in (2.4.40). If $a_t^* = K$, then from (2.4.40),

$$V_1'(a_t^*) < -\theta E[U_1(-l)]B_t \quad \text{and} \quad a_t^{\underline{\Lambda_1}} \leq a_t^*,$$

which yield $\left(\theta E[U_1(-l)]B_t + V_1'(a_t^*)\right)(a_t^{\underline{\Lambda_1}} - a_t^*) \geq 0$. If $a_t^* = V_1'^{-1}(-\theta E[U_1(-l)]B_t)$, then

$$V_1'(a_t^*) = -\theta E[U_1(-l)]B_t,$$

which yields $\left(\theta E[U_1(-l)]B_t + V_1'(a_t^*)\right)(a_t^{\underline{\Lambda_1}} - a_t^*) = 0$. Then, from (2.4.44), we obtain $R - \mathbb{U}_1(\underline{\Lambda_1}) > 0$. □

Theorem 4. *There exists a unique $\hat{\Lambda}_1$ such that (2.3.29) holds and $\hat{\Lambda}_1 \in (\underline{\Lambda_1}, \infty)$.*

Proof. Our first objective is to show that $\mathbb{U}_1(\Lambda_1) \geq R$ when $\Lambda_1 \to \infty$. Here R is presented in (2.4.42). Since $\lim_{\Lambda_1 \to \infty} \dfrac{1}{\Lambda_1} U_2(0)\theta B_t = 0$ almost surely for each $t \in [0, T]$, we have $\lim_{\Lambda_1 \to \infty} a_t^{\Lambda_1} \leq a_t^*$ almost surely for $t \in [0, T]$ according to Theorem 3. From the definition of D^{Λ_1} and $d_t^{\Lambda_1}$ in (2.3.17), we have

$$\frac{U_2'(-D^{\Lambda_1})}{U_1'(D^{\Lambda_1} - l)} = \Lambda_1 \quad \text{and} \quad \frac{U_2'(d_t^{\Lambda_1})}{U_1'(w_t - d_t^{\Lambda_1})} = \Lambda_1.$$

When $\Lambda_1 \to \infty$, we obtain $D^{\Lambda_1} \to \infty$ and $d_t^{\Lambda_1} \to -\infty$, which means $D^{\Lambda_1} > 0$ for every $l \in \mathcal{R}_l$ and $d_t^{\Lambda_1} < 0$ for every $t \in [0, T]$. To simplify the notation, we rewrite $\bar{B}$ as $\bar{B} = \displaystyle\int_0^T b_t dt$, where

$$b_t := (1 - e^{-t}\bar{a})e^{-\delta t}\left(Y_0 + \mu \int_0^t \rho(u)e^{\delta u}du\right).$$

If $\lim\limits_{\Lambda_1\to\infty} a_t^{\Lambda_1} = K$, then $a_t^* = K$ and

$$\begin{aligned}
&\theta E\left[U_1(D^{\Lambda_1} - l)\right]\left(b_t - a_t^{\Lambda_1}B_t\right) - V_1(a_t^{\Lambda_1}) - \theta E[U_1(-l)]\,(b_t - a_t^* B_t) + V_1(a_t^*) \\
&= \left(\theta E\left[U_1(D^{\Lambda_1} - l)\right](b_t - KB_t) - V_1(K)\right) \\
&\quad - \left(\theta E[U_1(-l)]\,(b_t - KB_t) - V_1(K)\right) \\
&= \theta E\left[U_1(D^{\Lambda_1} - l) - U_1(-l)\right](b_t - a_t^* B_t)
\end{aligned} \tag{2.4.45}$$

almost surely when $\Lambda_1 \to \infty$. If $\lim\limits_{\Lambda_1\to\infty} a_t^{\Lambda_1} < K$, then from (2.3.17), we have

$$\begin{aligned}
\lim_{\Lambda_1\to\infty} V_1'(a_t^{\Lambda_1}) &\geq \lim_{\Lambda_1\to\infty} -\theta E\left[\frac{1}{\Lambda_1}U_2(-g^{-1}(\Lambda_1, l)) + U_1(g^{-1}(\Lambda_1, l) - l)\right] B_t \\
&= \lim_{\Lambda_1\to\infty} -\theta E\left[\frac{1}{\Lambda_1}U_2(-D^{\Lambda_1}) + U_1(D^{\Lambda_1} - l)\right] B_t.
\end{aligned}$$

Noting $\lim\limits_{\Lambda_1\to\infty} D^{\Lambda_1} > 0$ and noting U_2 is negative with negative variables, we get

$$\lim_{\Lambda_1\to\infty} V_1'(a_t^{\Lambda_1}) \geq \lim_{\Lambda_1\to\infty} -\theta E\left[U_1(D^{\Lambda_1} - l)\right] B_t.$$

Hence,

$$\begin{aligned}
V_1(a_t^*) - \lim_{\Lambda_1\to\infty} V_1(a_t^{\Lambda_1}) &\geq \lim_{\Lambda_1\to\infty} V_1'(a_t^{\Lambda_1})(a_t^* - a_t^{\Lambda_1}) \\
&\geq \lim_{\Lambda_1\to\infty} -\theta E\left[U_1(D^{\Lambda_1} - l)\right] B_t(a_t^* - a_t^{\Lambda_1})
\end{aligned}$$

almost surely, and consequently

$$
\begin{aligned}
&\Big(\theta E\left[U_1(D^{\Lambda_1}-l)\right]\left(b_t-a_t^{\Lambda_1}B_t\right)-V_1(a_t^{\Lambda_1})\Big)-\Big(\theta E[U_1(-l)]\left(b_t-a_t^{*}B_t\right)-V_1(a_t^{*})\Big)\\
&\geq \theta E\left[U_1(D^{\Lambda_1}-l)\right]\left(b_t-a_t^{\Lambda_1}B_t\right)-\theta E[U_1(-l)]\left(b_t-a_t^{*}B_t\right)\\
&\quad -\theta E\left[U_1(D^{\Lambda_1}-l)\right]B_t(a_t^{*}-a_t^{\Lambda_1})\\
&=\theta E\left[U_1(D^{\Lambda_1}-l)-U_1(-l)\right]\left(b_t-a_t^{*}B_t\right) \qquad (2.4.46)
\end{aligned}
$$

almost surely when $\Lambda_1\to\infty$. From (2.4.45) and (2.4.46), we see that it is almost surely that

$$
\begin{aligned}
&\Big(\theta E\left[U_1(D^{\Lambda_1}-l)\right]\left(b_t-a_t^{\Lambda_1}B_t\right)-V_1(a_t^{\Lambda_1})\Big)-\Big(\theta E[U_1(-l)]\left(b_t-a_t^{*}B_t\right)-V_1(a_t^{*})\Big)\\
&\quad\geq \theta E\left[U_1(D^{\Lambda_1}-l)-U_1(-l)\right]\left(b_t-a_t^{*}B_t\right)
\end{aligned}
$$

for each case when $\Lambda_1\to\infty$. Integrating and taking expectation on both sides of the above inequality, we obtain

$$
\begin{aligned}
&\left(\theta E\left[U_1(D^{\Lambda_1}-l)\right]E\left[\int_0^T b_t dt-\int_0^T a_t^{\Lambda_1}B_t dt\right]-E\left[\int_0^T V_1(a_t^{\Lambda_1})dt\right]\right)\\
&\qquad -\left(\theta E\left[U_1(-l)\right]E\left[\int_0^T b_t dt-\int_0^T a_t^{*}B_t dt\right]-E\left[\int_0^T V_1(a_t^{*})dt\right]\right)\\
&\geq\ \theta E\left[U_1(D^{\Lambda_1}-l)-U_1(-l)\right]E\left[\int_0^T b_t dt-\int_0^T a_t^{*}B_t dt\right],
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\left(\theta E\left[U_1(D^{\Lambda_1}-l)\right]E\left[\bar{B}-\int_0^T a_t^{\Lambda_1}B_t dt\right]-E\left[\int_0^T V_1(a_t^{\Lambda_1})dt\right]\right)\\
&\qquad -\left(\theta E\left[U_1(-l)\right]E\left[\bar{B}-\int_0^T a_t^{*}B_t dt\right]-E\left[\int_0^T V_1(a_t^{*})dt\right]\right)\\
&\geq\ \theta E\left[U_1(D^{\Lambda_1}-l)-U_1(-l)\right]E\left[\bar{B}-\int_0^T a_t^{*}B_t dt\right]. \qquad (2.4.47)
\end{aligned}
$$

Recalling $\bar{B}-E\left[\int_0^T a_t^{*}B_t dt\right]\geq 0$ and $\lim\limits_{\Lambda_1\to\infty} D^{\Lambda_1}>0$ for every $l\in\mathcal{R}_l$, we obtain that the right-hand-side of (2.4.47) is non-negative. Thus,

$$\begin{aligned}
&\theta E\left[U_1(D^{\Lambda_1}-l)\right] E\left[\bar{B}-\int_0^T a_t^{\Lambda_1} B_t dt\right]-E\left[\int_0^T V_1(a_t^{\Lambda_1})dt\right] \\
&\quad \geq \theta E\left[U_1(-l)\right] E\left[\bar{B}-\int_0^T a_t^* B_t dt\right]-E\left[\int_0^T V_1(a_t^*)dt\right] \qquad (2.4.48)
\end{aligned}$$

when $\Lambda_1 \to \infty$. Recalling that $\lim_{\Lambda_1 \to \infty} d_t^{\Lambda_1} < 0$ for $t \in [0, T]$, we have

$$\lim_{\Lambda_1 \to \infty} U_1(w_t - d_t^{\Lambda_1}) > U_1(w_t) \qquad (2.4.49)$$

for $t \in [0, T]$. Combining (2.4.48) and (2.4.49), we obtain

$$\begin{aligned}
&\int_0^T U_1(w_t - d_t^{\Lambda_1})dt + \theta E\left[U_1(D^{\Lambda_1}-l)\right]\left(\bar{B}-E\left[\int_0^T a_t^{\Lambda_1} B_t dt\right]\right) \\
&\qquad -E\left[\int_0^T V_1(a_t^{\Lambda_1})dt\right] \\
&\quad > \int_0^T U_1(w_t)dt + \theta E[U_1(-l)]\left(\bar{B}-E\left[\int_0^T a_t^* B_t dt\right]\right)-E\left[\int_0^T V_1(a_t^*)dt\right]
\end{aligned}$$

when $\Lambda_1 \to \infty$. This is equivalent to $\lim_{\Lambda_1 \to \infty} \mathbb{U}_1(\Lambda_1) > R$. Lemma 3 states that $\mathbb{U}_1(\underline{\Lambda_1}) < R$. $\mathbb{U}_1(\Lambda_1)$ is a continuous function of Λ_1. From Theorem 1, we also know that $\mathbb{U}_1(\Lambda_1)$ is an increasing function of Λ_1. Therefore, there is a unique $\hat{\Lambda}_1$ such that (2.3.29) holds and $\hat{\Lambda}_1 \in (\underline{\Lambda_1}, \infty)$. □

Thus, Theorem 4 completes the solution of Problem 1.

We define the highest income rate by $w_{sup} := \sup\{w_t : t \in [0, T]\}$. We also define $\bar{\Lambda}_1 := \dfrac{U_2'(0)}{U_1'(w_{sup})}$. Then we have the following constraint for $\hat{\Lambda}_1$.

Corollary 2. *If $U_2(0) = 0$, then there exists a unique $\hat{\Lambda}_1$ such that (2.3.29) holds and $\hat{\Lambda}_1 \in (\underline{\Lambda_1}, \bar{\Lambda}_1)$.*

Proof. From (2.3.17), we see that $d_t^{\Lambda_1} = -g^{-1}(\Lambda_1, -w_t) = 0$ when $\Lambda_1 = g(0, -w_t) = \dfrac{U_2'(0)}{U_1'(w_t)}$ for each $t \in [0, T]$. Noting that $\bar{\Lambda}_1 = \dfrac{U_2'(0)}{U_1'(w_{sup})} \geq$

$\dfrac{U_2'(0)}{U_1'(w_t)}$ and that $d_t^{\Lambda_1}$ is a decreasing function of Λ_1 for $t \in [0,T]$, we have $d_t^{\bar{\Lambda}_1} \le 0$ for $t \in [0,T]$.

From (2.3.17), we see that $D^{\Lambda_1} = g^{-1}(\Lambda_1, l) = 0$ when $\Lambda_1 = g(0,l) = \dfrac{U_2'(0)}{U_1'(-l)}$ for each $l \in \mathcal{R}_l$. Noting that $\bar{\Lambda}_1 = \dfrac{U_2'(0)}{U_1'(w_{sup})} > \dfrac{U_2'(0)}{U_1'(-l)}$ and that D^{Λ_1} is an increasing function of Λ_1 for $l \in \mathcal{R}_l$, we have $D^{\bar{\Lambda}_1} > 0$ for $l \in \mathcal{R}_l$.

From (2.3.16) and (2.4.42), we obtain

$$\begin{aligned}
\mathbb{U}_1(\bar{\Lambda}_1) - R = &\int_0^T \left(U_1(w_t - d_t^{\bar{\Lambda}_1}) - U_1(w_t) \right) dt \\
&+ \theta E\left[U_1(D^{\bar{\Lambda}_1} - l) \right] \left(\bar{B} - E\left[\int_0^T a_t^{\bar{\Lambda}_1} B_t dt \right] \right) \\
&- \theta E[U_1(-l)] \left(\bar{B} - E\left[\int_0^T a_t^* B_t dt \right] \right) \\
&+ E\left[\int_0^T \left(V_1(a_t^*) - V_1(a_t^{\bar{\Lambda}_1}) \right) dt \right].
\end{aligned}$$

In the above equation, we have $U_1(w_t - d_t^{\bar{\Lambda}_1}) - U_1(w_t) \ge 0$ for $t \in [0,T]$ because $d_t^{\bar{\Lambda}_1} \le 0$ for $t \in [0,T]$. Since $D^{\bar{\Lambda}_1} > 0$ for $l \in \mathcal{R}_l$, we have

$$\begin{aligned}
&- \theta E[U_1(-l)] \left(\bar{B} - E\left[\int_0^T a_t^* B_t dt \right] \right) \\
&\ge -\theta E[U_1(D^{\bar{\Lambda}_1} - l)] \left(\bar{B} - E\left[\int_0^T a_t^* B_t dt \right] \right).
\end{aligned}$$

From (2.3.17), we also have

$$\begin{aligned}
V_1(a_t^*) - V_1(a_t^{\bar{\Lambda}_1}) \ge &V_1'(a_t^{\bar{\Lambda}_1})(a_t^* - a_t^{\bar{\Lambda}_1}) \\
= &- \theta E\left[\frac{1}{\bar{\Lambda}_1} U_2(-D^{\bar{\Lambda}_1}) + U_1(D^{\bar{\Lambda}_1} - l) \right] B_t (a_t^* - a_t^{\bar{\Lambda}_1}).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\mathbb{U}_1(\bar{\Lambda}_1) - R &\geq \theta E\left[U_1(D^{\bar{\Lambda}_1} - l)\right]\left(\bar{B} - E\left[\int_0^T a_t^{\bar{\Lambda}_1} B_t dt\right]\right) \\ &\quad -\theta E[U_1(D^{\bar{\Lambda}_1} - l)]\left(\bar{B} - E\left[\int_0^T a_t^* B_t dt\right]\right) \\ &\quad -\theta E\left[\frac{1}{\bar{\Lambda}_1}U_2(-D^{\bar{\Lambda}_1}) + U_1(D^{\bar{\Lambda}_1} - l)\right] E\left[\int_0^T B_t(a_t^* - a_t^{\bar{\Lambda}_1})dt\right] \\ &= -\theta E\left[\frac{1}{\bar{\Lambda}_1}U_2(-D^{\bar{\Lambda}_1})\right] E\left[\int_0^T B_t(a_t^* - a_t^{\bar{\Lambda}_1})dt\right]. \qquad (2.4.50)\end{aligned}$$

Here, $E\left[U_2(-D^{\bar{\Lambda}_1})\right] \leq 0$ because $D^{\bar{\Lambda}_1} \geq 0$ for each $l \in \mathcal{R}_l$. Corollary 1 shows that $a_t^* - a_t^{\bar{\Lambda}_1} \geq 0$ for every $t \in [0, T]$ when $U_2(0) = 0$. Now we can get $\mathbb{U}_1(\bar{\Lambda}_1) - R \geq 0$ from (2.4.50). Because $\mathbb{U}_1(\Lambda_1)$ is an increasing function of Λ_1, $\hat{\Lambda}_1 < \bar{\Lambda}_1$. Theorem 4 shows that $\hat{\Lambda}_1 > \underline{\Lambda_1}$, so we can conclude the unique $\hat{\Lambda}_1$ is located in the interval $(\underline{\Lambda_1}, \bar{\Lambda}_1)$. □

2.5 The Exponential Utility and the Quadratic Cost

In this section, we apply the theory developed in Sections 2.3 and 2.4 to the case

$$U_1(y) = -e^{-\gamma_1 y}, \quad U_2(y) = -e^{-\gamma_2 y}, \quad V_1(y) = my^2, \quad K = 1, \quad w_t = 0,$$

where $\gamma_1 > \gamma_2 > 0$ and $m > 0$ are constant parameters. Then, g is given by

$$g(y_1, y_2) = \frac{U_2'(-y_1)}{U_1'(y_1 - y_2)} = \frac{\gamma_2 e^{\gamma_2 y_1}}{\gamma_1 e^{-\gamma_1(y_1 - y_2)}}.$$

For a fixed y_2, the inverse function $g^{-1}(\cdot, y_2)$ is given by

$$g^{-1}(y, y_2) = \frac{\ln(y) + \ln(\frac{\gamma_1}{\gamma_2}) + \gamma_1 y_2}{\gamma_1 + \gamma_2}.$$

From (2.3.30) and (2.3.31), we obtain

$$\hat{d}_t = -g^{-1}(\hat{\Lambda}_1, -w_t) = -g^{-1}(\hat{\Lambda}_1, 0) = -\frac{\ln(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})}{\gamma_1+\gamma_2} \text{ for } t \in [0,T]; \quad (2.5.51)$$

$$\hat{D}_i = g^{-1}(\hat{\Lambda}_1, l_i) = \frac{\gamma_1 l_i + \ln(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})}{\gamma_1+\gamma_2} \text{ for } i = 1, 2, 3, \cdots . \quad (2.5.52)$$

We have

$$\begin{aligned} &-\theta E[\frac{1}{\hat{\Lambda}_1}U_2(-g^{-1}(\hat{\Lambda}_1, l)) + U_1(g^{-1}(\hat{\Lambda}_1, l) - l)]B_t \\ &= \theta E\left[\frac{1}{\hat{\Lambda}_1}e^{\frac{\gamma_2}{\gamma_1+\gamma_2}(\gamma_1 l + \ln(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2}))} + e^{-\frac{\gamma_1}{\gamma_1+\gamma_2}(\ln(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2}) - \gamma_2 l)}\right]B_t \\ &= \theta B_t(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{\gamma_1}{\gamma_1+\gamma_2}}E[e^{\frac{\gamma_1\gamma_2 l}{\gamma_1+\gamma_2}}](1 + \frac{\gamma_1}{\gamma_2}), \end{aligned}$$

which is positive for every $t \in [0,T]$. In this example, $V_1'(y) = 2my$, so $V_1'(K) = V_1'(1) = 2m$ and $V_1'^{-1}(y) = \frac{y}{2m}$. Hence,

$$\hat{a}_t = \begin{cases} \frac{\theta}{2m}B_t(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{\gamma_1}{\gamma_1+\gamma_2}}E[e^{\frac{\gamma_1\gamma_2 l}{\gamma_1+\gamma_2}}](1+\frac{\gamma_1}{\gamma_2}) & \text{if } \theta B_t(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{\gamma_1}{\gamma_1+\gamma_2}}E[e^{\frac{\gamma_1\gamma_2 l}{\gamma_1+\gamma_2}}](1+\frac{\gamma_1}{\gamma_2}) \leq 2m \\ 1 & \text{if } \theta B_t(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{\gamma_1}{\gamma_1+\gamma_2}}E[e^{\frac{\gamma_1\gamma_2 l}{\gamma_1+\gamma_2}}](1+\frac{\gamma_1}{\gamma_2}) > 2m. \end{cases} \quad (2.5.53)$$

Since $-\theta B_t E[U_1(-l)] = \theta B_t E\left[e^{\gamma_1 l}\right]$, applying (2.4.40), we obtain

$$a_t^* = \begin{cases} \frac{\theta}{2m}B_t E\left[e^{\gamma_1 l}\right] & \text{if } \theta B_t E\left[e^{\gamma_1 l}\right] \leq 2m \\ 1 & \text{if } \theta B_t E\left[e^{\gamma_1 l}\right] > 2m. \end{cases}$$

$\hat{\Lambda}_1$ in (2.5.51)-(2.5.53) is the solution of $\mathbb{U}_1(\hat{\Lambda}_1) = R$. We denote $C_t :=$ $\theta B_t(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{\gamma_1}{\gamma_1+\gamma_2}}E[e^{\frac{\gamma_1\gamma_2 l}{\gamma_1+\gamma_2}}](1+\frac{\gamma_1}{\gamma_2})$. Recalling (2.3.16), we have

$$\begin{aligned}
\mathbb{U}_1(\hat{\Lambda}_1) =& \int_0^T U_1\left(0+\frac{\ln(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})}{\gamma_1+\gamma_2}\right)dt \\
&+E\left[U_1\left(\frac{\gamma_1 l_i+\ln(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})}{\gamma_1+\gamma_2}-l\right)\right]\theta\left(\bar{B}-E[\int_0^T \hat{a}_t B_t dt]\right)-E\left[\int_0^T m\hat{a}_t^2 dt\right] \\
=&-(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{\gamma_1}{\gamma_1+\gamma_2}}T-E\left[e^{\frac{\gamma_1\gamma_2 l}{\gamma_1+\gamma_2}}\right](\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{\gamma_1}{\gamma_1+\gamma_2}}\theta \\
&\left(\bar{B}-E\left[\int_0^T\left(\frac{\theta}{2m}B_t^2(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{\gamma_1}{\gamma_1+\gamma_2}}E[e^{\frac{\gamma_1\gamma_2 l}{\gamma_1+\gamma_2}}](1+\frac{\gamma_1}{\gamma_2})\mathbb{I}_{\{C_t\le 2m\}}+B_t\mathbb{I}_{\{C_t>2m\}}\right)dt\right]\right) \\
&-E\left[\int_0^T m\left(\frac{\theta^2}{4m^2}B_t^2(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{2\gamma_1}{\gamma_1+\gamma_2}}\left(E[e^{\frac{\gamma_1\gamma_2 l}{\gamma_1+\gamma_2}}]\right)^2(1+\frac{\gamma_1}{\gamma_2})^2\mathbb{I}_{\{C_t\le 2m\}}+\mathbb{I}_{\{C_t>2m\}}\right)dt\right] \\
=&-(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{\gamma_1}{\gamma_1+\gamma_2}}T-E\left[e^{\frac{\gamma_1\gamma_2 l}{\gamma_1+\gamma_2}}\right](\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{\gamma_1}{\gamma_1+\gamma_2}}\theta\left(\bar{B}-E\left[\int_0^T B_t\mathbb{I}_{\{C_t>2m\}}dt\right]\right) \\
&+\frac{\theta^2}{2m}\left(E[e^{\frac{\gamma_1\gamma_2 l}{\gamma_1+\gamma_2}}]\right)^2\left(1+\frac{\gamma_1}{\gamma_2}-\frac{1}{2}(1+\frac{\gamma_1}{\gamma_2})^2\right)E\left[\int_0^T B_t^2\mathbb{I}_{\{C_t\le 2m\}}dt\right](\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{2\gamma_1}{\gamma_1+\gamma_2}} \\
&-mE\left[\int_0^T\mathbb{I}_{\{C_t>2m\}}dt\right] \\
=&\frac{\theta^2}{4m}\left(E[e^{\frac{\gamma_1\gamma_2 l}{\gamma_1+\gamma_2}}]\right)^2(1-\frac{\gamma_1^2}{\gamma_2^2})E\left[\int_0^T B_t^2\mathbb{I}_{\{C_t\le 2m\}}dt\right](\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{2\gamma_1}{\gamma_1+\gamma_2}} \\
&-\left(T+\theta E[e^{\frac{\gamma_1\gamma_2 l}{\gamma_1+\gamma_2}}]\left(\bar{B}-E\left[\int_0^T B_t\mathbb{I}_{\{C_t>2m\}}dt\right]\right)\right)(\frac{\hat{\Lambda}_1\gamma_1}{\gamma_2})^{-\frac{\gamma_1}{\gamma_1+\gamma_2}} \\
&-mE\left[\int_0^T\mathbb{I}_{\{C_t>2m\}}dt\right].
\end{aligned}$$

According to (2.4.42), we have

$$
\begin{aligned}
R =& \int_0^T U_1(0)dt + \theta E[U_1(-l)] \left(\bar{B} - E\left[\int_0^T a_t^* B_t dt \right] \right) - E\left[\int_0^T m(a_t^*)^2 dt \right] \\
=& -(T + \theta \bar{B} E[e^{\gamma_1 l}]) \\
&+ \theta E[e^{\gamma_1 l}] E\left[\int_0^T \left(\frac{\theta}{2m} B_t E[e^{\gamma_1 l}] \mathbb{I}_{\{\theta B_t E[e^{\gamma_1 l}] \le 2m\}} + \mathbb{I}_{\{\theta B_t E[e^{\gamma_1 l}] > 2m\}} \right) B_t dt \right] \\
&- mE\left[\int_0^T \left(\frac{\theta^2}{4m^2} (E[e^{\gamma_1 l}])^2 B_t^2 \mathbb{I}_{\{\theta B_t E[e^{\gamma_1 l}] \le 2m\}} + \mathbb{I}_{\{\theta B_t E[e^{\gamma_1 l}] > 2m\}} \right) dt \right] \\
=& \frac{\theta^2}{4m} (E[e^{\gamma_1 l}])^2 E\left[\int_0^T B_t^2 \mathbb{I}_{\{\theta B_t E[e^{\gamma_1 l}] \le 2m\}} dt \right] \\
&+ E\left[\int_0^T (\theta B_t E[e^{\gamma_1 l}] - m) \mathbb{I}_{\{\theta B_t E[e^{\gamma_1 l}] > 2m\}} dt \right] - (T + \theta \bar{B} E[e^{\gamma_1 l}]).
\end{aligned}
$$

Even though the equation $\mathbb{U}_1(\hat{\Lambda}_1) = R$ looks complicated, the monotonicity of $\mathbb{U}_1(\Lambda_1)$ and the uniqueness of $\hat{\Lambda}_1$ allow us to use the bisection method to find $\hat{\Lambda}_1$ in the following numerical analysis.

Equation (2.5.52) shows that $\hat{D}_i \neq l_i$, so full compensation is not optimal.

Example 1. *To consider a numerical example, assume that the magnitude Y of the external risky events has exponential distribution, and the intensity ρ is constant: $\rho(t) \equiv \rho \in [0, \infty)$.*

We will investigate how the solution depends on the parameters θ, ρ, $E[l]$, μ, γ_1, γ_2, and the variance of $I(t)$ for $t \in [0, T]$. We fix the other parameters as $T = 1$, $Y_0 = 1$, $m = 5$, $\delta = 1$, $r_t = 1$, and $\bar{a} = 0$.

The benchmark parameter values are $\theta = 1$, $\rho = 1$, $\mu = 1$, $\gamma_1 = 2$, $\gamma_2 = 1$, and l has probability distribution $P\{l = 2\} = 0.5$, $P\{l = 2.2\} = 0.3$, $P\{l = 2.4\} = 0.2$. Then, $\hat{\Lambda}_1 = 0.0108$ and the optimal insurance contract (for *these parameter values) is given by*

$$\hat{d}_t = 1.2794;$$

$$\hat{D}_i = \begin{cases} 0.0539 & \text{if } l_i = 2.0 \\ 0.1873 & \text{if } l_i = 2.2 \\ 0.3206 & \text{if } l_i = 2.4; \end{cases}$$

$$\hat{a}_t = \min\{16.2321 B_t, 1\}.$$

Since $\hat{a}$ is a stochastic process, we will consider $E[\hat{a}_t]$. Figures 2.1 to 2.4 show that $\hat{d}_t$, $\hat{D}_i$, and $E[\hat{a}_t]$ increase when the parameters μ, θ, ρ, and $E[l]$ increase. These four parameters reflect the risk in different aspects. Thus, when the risk increases, the insurer requires a higher premium, pays less compensation, and requires the insured to increase his expected action.

Figures 2.1 to 2.4 also show that the expected insured's action decreases when time passes, and that the insured is required to take no action when maturity approaches. This is consistent with Remark 2 of Theorem 2.

Figure 2.5 shows that when the insured's risk aversion γ_1 increases, the premium increases, the compensation decreases, and the expected action increases.

Figure 2.6 shows how the solution depends on the insurer's risk aversion γ_2. We recall that the insured's reservation utility presented in Section 2.4 is not affected by the insurer's risk aversion γ_2. Figure 2.6 shows that the premium and the compensation decrease when the insurer's risk aversion increases. This makes sense because, as the risk aversion increases, the insurer avoids risk by paying less compensation in exchange for receiving less premium.

We have also studied the situation in which the mean remains the same but the variance changes. Figure 2.7 shows that the variance does not affect much the optimal premium d or compensation D when the mean is fixed. However, the optimal expected action $E[\hat{a}]$ decreases when the variance of $I(t)$ increases. Since it is impossible to list the variances of $I(t)$ for all $t \in [0, T]$ in the figure, *we use the variance of $I(T)$ as a representation.*

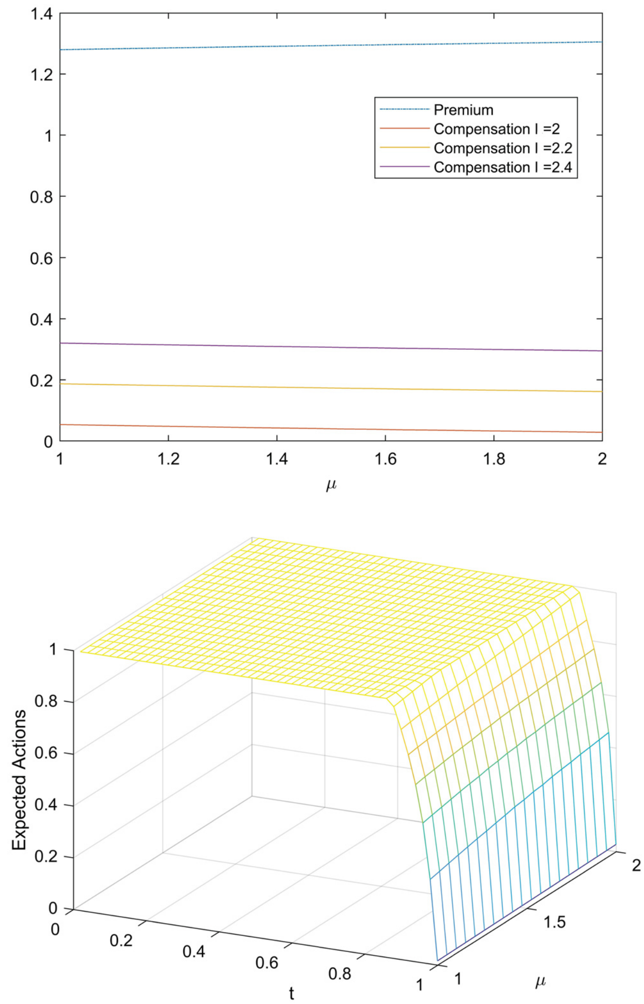

Figure 2.1: The other parameters are $\theta = 1$, $\rho = 1$, $\gamma_1 = 2$, and $\gamma_2 = 1$. Furthermore, $P\{l = 2.0\} = 0.5$, $P\{l = 2.2\} = 0.3$, $P\{l = 2.4\} = 0.2$.

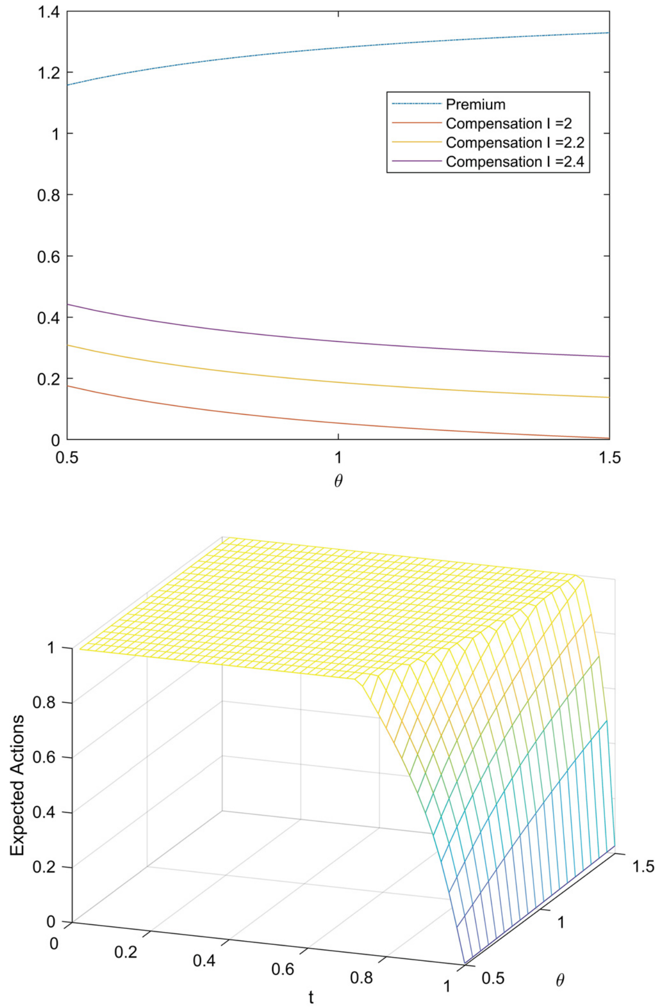

Figure 2.2: The other parameters are $\rho = 1$, $\mu = 1$, $\gamma_1 = 2$, and $\gamma_2 = 1$. Furthermore, $P\{l = 2.0\} = 0.5$, $P\{l = 2.2\} = 0.3$, $P\{l = 2.4\} = 0.2$.

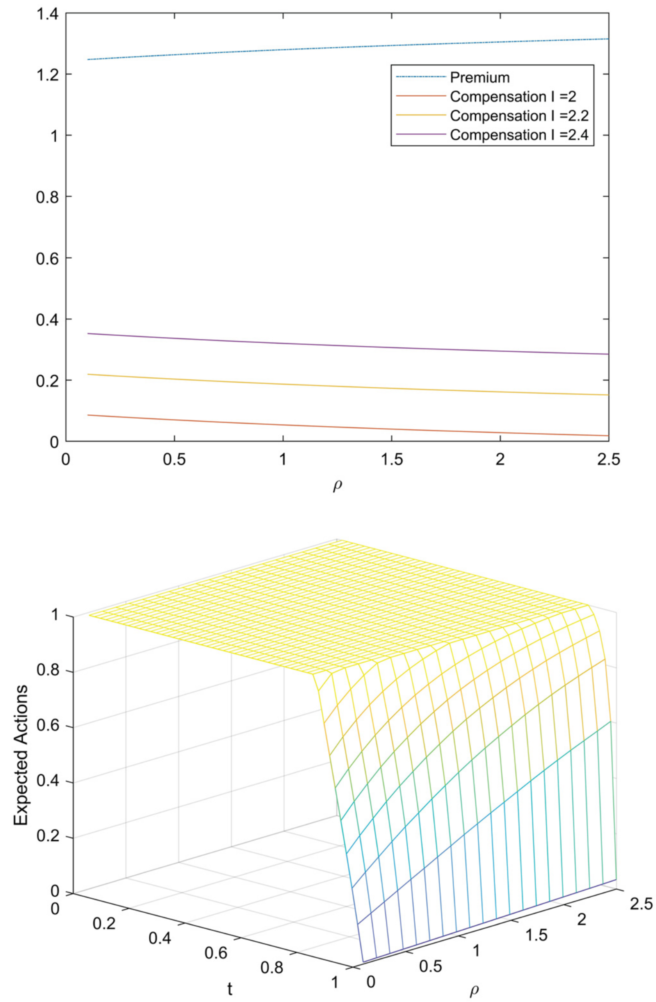

Figure 2.3: The other parameters are $\theta = 1$, $\mu = 1$, $\gamma_1 = 2$, and $\gamma_2 = 1$. Furthermore, $P\{l = 2.0\} = 0.5$, $P\{l = 2.2\} = 0.3$, $P\{l = 2.4\} = 0.2$.

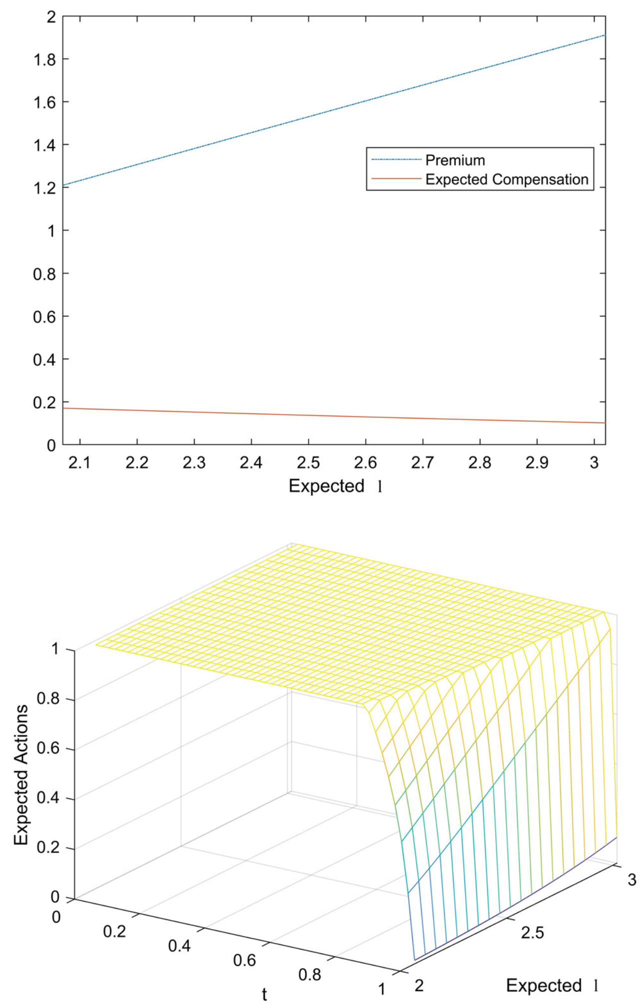

Figure 2.4: The other parameters are $\theta = 1$, $\rho = 1$, $\mu = 1$, $\gamma_1 = 2$, and $\gamma_2 = 1$.

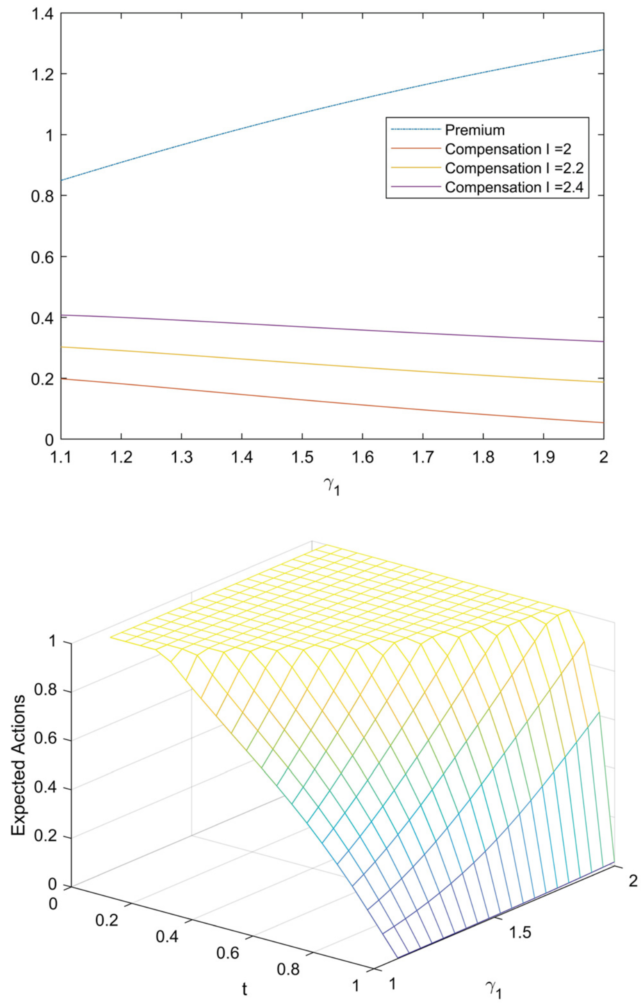

Figure 2.5: The other parameters are $\theta = 1$, $\mu = 1$, $\rho = 1$, and $\gamma_2 = 1$. Furthermore, $P\{l = 2.0\} = 0.5$, $P\{l = 2.2\} = 0.3$, $P\{l = 2.4\} = 0.2$.

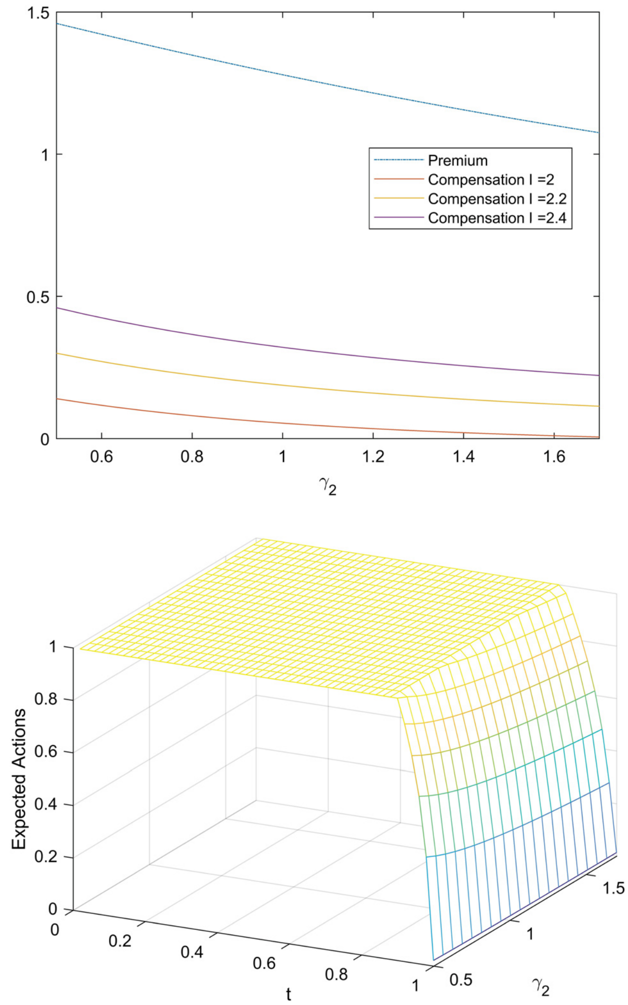

Figure 2.6: The other parameters are $\theta = 1$, $\mu = 1$, $\rho = 1$, and $\gamma_1 = 2$. Furthermore, $P\{l = 2.0\} = 0.5$, $P\{l = 2.2\} = 0.3$, $P\{l = 2.4\} = 0.2$.

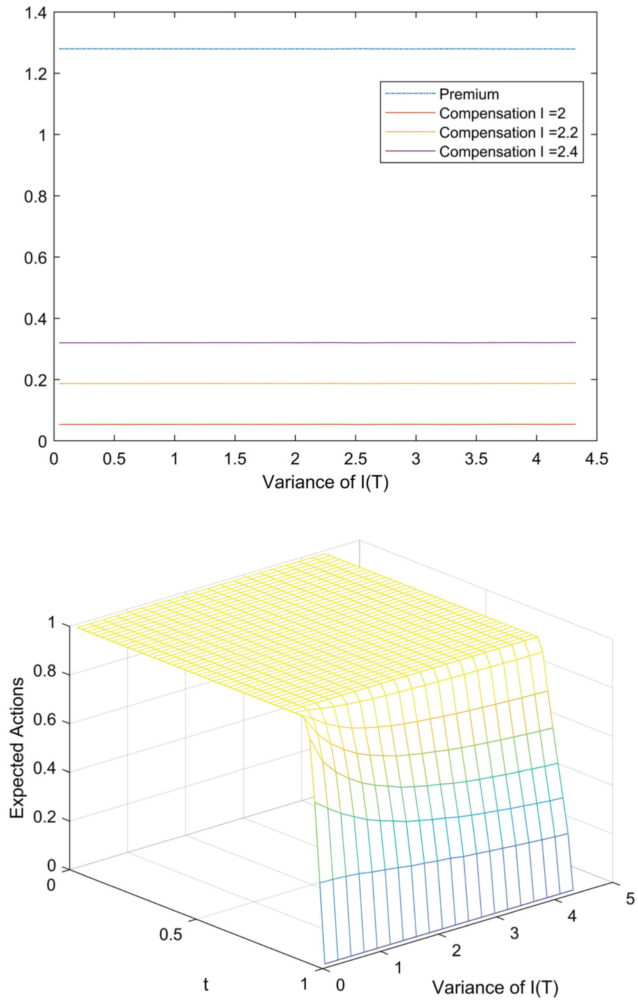

Figure 2.7: The other parameters are $\theta = 1$, $\gamma_1 = 2$, and $\gamma_2 = 1$. Furthermore, $P\{l = 2.0\} = 0.5$, $P\{l = 2.2\} = 0.3$, $P\{l = 2.4\} = 0.2$.

Chapter 3

Adverse Selection and Multiple Claim States

This chapter consists of three sections. The problem is constructed in Section 3.1 and the solutions to the problem are presented in Section 3.2. We explain why the boundaries on the premium and compensation are necessary.

3.1 The Model

Let $S \subset [0, \infty)$ be the loss set. For every $s \in S$, s is a possible loss amount at one period. Suppose there are two types of insureds in the insurance market. The low-risk type possesses a high probability of encountering a small or zero loss and a low probability of encountering a big loss in each period. The high-risk type possesses a low probability of encountering a small or zero loss and a high probability of encountering a big loss in each period. Let L and H denote the risk levels for the low-risk and high-risk insured respectively. At the n-th period, the loss amount is a random variable X_n. So, for each ω belonging to the sample space Ω, $X_n(\omega) \in S$. Since the type is innate and will not change over time, we suppose the probability of each type having some amount of loss at each period is fixed. That is, we suppose $X_1, X_2, X_3, \cdots, X_N$ are identically and independently distributed for either type, where N is the total number of periods. The information revealed up to the n-th period from the loss is the filtration $\mathbb{F}_2 := \{\mathcal{F}_{2,n}, n = 1, 2, \cdots, N\}$, where

$$\mathcal{F}_{2,n} := \sigma(X_1, X_2, \cdots, X_n)$$

is a σ-field generated by random variables. On the space $(\Omega, \mathcal{F})$, we denote

$$P := \text{probability measure of the low-risk insured}$$
$$Q := \text{probability measure of the high-risk insured}$$

and we suppose P and Q are equivalent. Let X be a random variable that has the same probability distribution as $X_1, X_2, \cdots, X_N$. We denote

$$F_P := \text{distribution function of } X \text{ under } P$$
$$F_Q := \text{distribution function of } X \text{ under } Q.$$

We define for every $s \in S$,

$$M(s) \;:=\; \frac{dF_Q(s)}{dF_P(s)}. \tag{3.1.1}$$

In the special case in which X is a discrete random variable under P and under Q, we can write

$$M(s) \;:=\; \frac{dF_Q(s)}{dF_P(s)} = \frac{\widetilde{p}_Q(s)}{\widetilde{p}_P(s)},$$

where $\widetilde{p}_Q$ is the probability mass function of X under Q and $\widetilde{p}_P$ is the probability mass function of X under P. In the special case in which X is a continuous random variable under P and under Q, we can write

$$M(s) \;:=\; \frac{dF_Q(s)}{dF_P(s)} = \frac{f_Q(s)}{f_P(s)},$$

where f_Q is the probability density function of X under Q and f_P is the probability density function of X under P. To show how the two types behave differently, we suppose $M(s)$ is a strictly increasing function of s. It means that the high-risk insured is more likely to encounter a big loss while is less likely to encounter a small loss compared to the low-risk insured. We denote $M_n := M(X_n)$ for $n = 1, 2, \cdots, N$. It is obvious that

$$E^P[M_n] = 1 \text{ for } n = 1, 2, \cdots, N \tag{3.1.2}$$

where E^P means the expectation under the measure P. Let E^Q denote the expectation under the measure Q. The performance of the insured in previous periods is crucial for the insurer to tell the type and determine the contract in the following periods, so we need to record the path of loss. We define

$$S^n := \{(x_1, x_2, \cdots, x_n) : x_1, x_2, \cdots, x_n \in S\}.$$

Every $x \in S^n$ is n-dimensional. x_i is the loss amount at the i-th period for $i = 1, 2, \cdots, n$. The probability of walking path $x = (x_1, x_2, \cdots, x_n)$ is

$dF_P^{(n)}(x) = dF_P(x_1)dF_P(x_2)\cdots dF_P(x_n)$ for the low-risk insured and
$dF_Q^{(n)}(x) = dF_Q(x_1)dF_Q(x_2)\cdots dF_Q(x_n)$ for the high-risk insured.

If X is a discrete random variable under P and under Q, the probability of walking path $x = (x_1, x_2, \cdots, x_n)$ is

$dF_P^{(n)}(x) = \widetilde{p}_P(x_1)\widetilde{p}_P(x_2)\cdots\widetilde{p}_P(x_n)$ for the low-risk insured and
$dF_Q^{(n)}(x) = \widetilde{p}_Q(x_1)\widetilde{p}_Q(x_2)\cdots\widetilde{p}_Q(x_n)$ for the high-risk insured.

If X is a continuous random variable under P and under Q, the probability of walking path $x = (x_1, x_2, \cdots, x_n)$ is

$dF_P^{(n)}(x) = f_P(x_1)f_P(x_2)\cdots f_P(x_n)(dx)$ for the low-risk insured and
$dF_Q^{(n)}(x) = f_Q(x_1)f_Q(x_2)\cdots f_Q(x_n)(dx)$ for the high-risk insured.

Lemma 4. *Let $f : S \to \mathbb{R}$ and $h : S \to \mathbb{R}$ be two functions. Suppose there exists $s' \in S$ such that $f(s) \geq h(s)$ when $s < s'$ and $f(s) \leq h(s)$ when $s > s'$.*
(i) If $E^P[f(X)] = E^P[h(X)]$, then $E^Q[f(X)] \leq E^Q[h(X)]$.

(ii) If $E^Q[f(X)] = E^Q[h(X)]$, then $E^P[f(X)] \geq E^P[h(X)]$.

Proof. When $X < s'$, $M(X) < M(s')$ because $M(s)$ is strictly increasing in s. Meanwhile, $f(X) - h(X) \geq 0$. Thus,

$$M(X)\left(f(X) - h(X)\right) \leq M(s')\left(f(X) - h(X)\right).$$

When $X > s'$, $M(X) > M(s')$. Meanwhile, $f(X) - h(X) \leq 0$. Thus,

$$M(X)\left(f(X) - h(X)\right) \leq M(s')\left(f(X) - h(X)\right).$$

When $X = s'$,

$$M(X)\left(f(X) - h(X)\right) = M(s')\left(f(X) - h(X)\right).$$

(i) Therefore,

$$\begin{aligned} E^Q[f(X)] - E^Q[h(X)] &= E^P[M(X)(f(X) - h(X))] \\ &\leq M(s')E^P[f(X) - h(X)] = 0. \end{aligned}$$

The required statement follows immediately.

(ii) Therefore,

$$\begin{aligned} 0 &= E^Q[f(X)] - E^Q[h(X)] \\ &= E^P[M(X)(f(X) - h(X))] \leq M(s')E^P[f(X) - h(X)]. \end{aligned}$$

The required statement follows. □

Consider a decreasing function $f : S \to \mathbb{R}$ and a constant function $h : S \to \mathbb{R}$ that satisfy the condition $E^P[f(X)] = h(X)$. Then, the s' described in the lemma must exist. The relation $E^Q[f(X)] \leq h(X)$ can be obtained through the lemma. It follows that

Corollary 3. *If $f : S \to \mathbb{R}$ is a decreasing function, then $E^P[f(X)] \geq E^Q[f(X)]$.*

It is natural to derive the next corollary.

Corollary 4. *If $f : S \to \mathbb{R}$ is a strictly decreasing function, then $E^P[f(X)] > E^Q[f(X)]$.*

Lemma 5. *If $f : \mathbb{R} \to \mathbb{R}$ is a decreasing function, then*

$$\int_{x\in S^n} f\left(\frac{dF_Q^{(n)}(x)}{dF_P^{(n)}(x)}\right) dF_P^{(n)}(x) > \int_{x\in S^n} f\left(\frac{dF_Q^{(n)}(x)}{dF_P^{(n)}(x)}\right) dF_Q^{(n)}(x)$$

for $n = 1, 2, \cdots, N$.

Proof. $dF_Q^{(n)}(x) > dF_P^{(n)}(x)$ implies $\dfrac{dF_Q^{(n)}(x)}{dF_P^{(n)}(x)} > 1$. Since f is a decreasing function, we have $f\left(\dfrac{dF_Q^{(n)}(x)}{dF_P^{(n)}(x)}\right) < f(1)$. Thus,

$$(dF_Q^{(n)}(x) - dF_P^{(n)}(x))f\left(\frac{dF_Q^{(n)}(x)}{dF_P^{(n)}(x)}\right) < (dF_Q^{(n)}(x) - dF_P^{(n)}(x))f(1). \quad (3.1.3)$$

$dF_Q^{(n)}(x) \le dF_P^{(n)}(x)$ implies $\dfrac{dF_Q^{(n)}(x)}{dF_P^{(n)}(x)} \le 1$. Since f is a decreasing function, we have $f\left(\dfrac{dF_Q^{(n)}(x)}{dF_P^{(n)}(x)}\right) \ge f(1)$. Thus,

$$(dF_Q^{(n)}(x) - dF_P^{(n)}(x))f\left(\frac{dF_Q^{(n)}(x)}{dF_P^{(n)}(x)}\right) \le (dF_Q^{(n)}(x) - dF_P^{(n)}(x))f(1). \quad (3.1.4)$$

From (3.1.3) and (3.1.4), we obtain

$$\int\limits_{x\in S^n} f\left(\frac{dF_Q^{(n)}(x)}{dF_P^{(n)}(x)}\right)\left(dF_Q^{(n)}(x) - dF_P^{(n)}(x)\right) < \int\limits_{x\in S^n} f(1)\left(dF_Q^{(n)}(x) - dF_P^{(n)}(x)\right)$$

for $n = 1, 2, \cdots, N$. Noting that $\int\limits_{x\in S^n} dF_Q^{(n)}(x) = \int\limits_{x\in S^n} dF_P^{(n)}(x) = 1$, we continue the inequality above to obtain

$$\int_{x\in S^n} f\left(\frac{dF_Q^{(n)}(x)}{dF_P^{(n)}(x)}\right)\Big(dF_Q^{(n)}(x) - dF_P^{(n)}(x)\Big)$$
$$< f(1)\int_{x\in S^n}\Big(dF_Q^{(n)}(x) - dF_P^{(n)}(x)\Big) = 0.$$

The required statement follows immediately. □

After observing the loss path $x \in S^n$, the insurer provides the contracts in the next period with premium denoted by $d^x_{L,n+1}$ for the low-risk insured, premium denoted by $d^x_{H,n+1}$ for the high-risk insured, compensation denoted by $D^x_{L,n+1}$ for the low-risk insured, and compensation denoted by $D^x_{H,n+1}$ for the high-risk insured. The amount of compensation will be determined also according to the loss amount at the $(n+1)$-th period. Now we have the contract for type L

$$\mathcal{C}_L = \{(d_{L,n}, D_{L,n}); n = 1, 2, \cdots, N\}$$

and the contract for type H

$$\mathcal{C}_H = \{(d_{H,n}, D_{H,n}); n = 1, 2, \cdots, N\}.$$

The external income of the insured at each period is assumed to be constant and denoted by w. The proportions of type L and H among the insured are denoted by p_L and p_H respectively and $p_L + p_H = 1$.

Given $x \in S^n$, at the $(n+1)$-th period, the insurer's utility is thus $U_2(d^x_{L,n+1} - D^x_{L,n+1})$ from type L and $U_2(d^x_{H,n+1} - D^x_{H,n+1})$ from type H. We denote $\mathcal{J}_1(d_L, D_L, d_H, D_H)$ as the expected total utility of the insurer during all the periods. Then we have

$$\begin{aligned}\mathcal{J}_1(d_L, D_L, d_H, D_H) :&= p_L E^P \left\{ \sum_{n=1}^{N} \int_{S^{n-1}} U_2(d^x_{L,n} - D^x_{L,n}) dF_P^{(n-1)}(x) \right\} \\ &\quad + p_H E^Q \left\{ \sum_{n=1}^{N} \int_{S^{n-1}} U_2(d^x_{H,n} - D^x_{H,n}) dF_Q^{(n-1)}(x) \right\}.\end{aligned}$$

If the low-risk insured chooses $\mathcal{C}_i, i = L, H$, his expected total utility is

$$\mathcal{V}(L, i) := E^P \left\{ \sum_{n=1}^{N} \int_{S^{n-1}} U_1(w - d^x_{i,n} + D^x_{i,n} - X_n) dF_P^{(n-1)}(x) \right\}$$

during the contracting periods. If the high-risk insured chooses $\mathcal{C}_i, i = L, H$, his expected total utility is

$$\mathcal{V}(H, i) := E^Q \left\{ \sum_{n=1}^{N} \int_{S^{n-1}} U_1(w - d^x_{i,n} + D^x_{i,n} - X_n) dF_Q^{(n-1)}(x) \right\}$$

during the contracting periods. From the relation of measures presented in (3.1.1), we show the last expectation under measure P.

$$\begin{aligned}\mathcal{V}(H, i) &= \left\{ \sum_{n=1}^{N} \int_{S^{n-1}} E^Q U_1(w - d^x_{i,n} + D^x_{i,n} - X_n) dF_Q^{(n-1)}(x) \right\} \\ &= \left\{ \sum_{n=1}^{N} \int_{S^{n-1}} E^P M_n U_1(w - d^x_{i,n} + D^x_{i,n} - X_n) dF_Q^{(n-1)}(x) \right\} \\ &= E^P \left\{ \sum_{n=1}^{N} \int_{S^{n-1}} M_n U_1(w - d^x_{i,n} + D^x_{i,n} - X_n) dF_Q^{(n-1)}(x) \right\}. \quad (3.1.5)\end{aligned}$$

We denote by $\mathcal{A}_2$ the class of admissible controls. These are the controls (d_L, D_L, d_H, D_H) that are adapted to the filtration $\mathbb{F}_2 := \{\mathcal{F}_{2,1}, \mathcal{F}_{2,2}, \cdots, \mathcal{F}_{2,N}\}$. Under some conditions, the insurer aims to maximize her utility by controlling the premiums and compensations. We construct the following problem.

Problem 3.

$$\begin{aligned}
\max_{(d_L,D_L,d_H,D_H)\in\mathcal{A}_2} \quad & \mathcal{J}_1(d_L,D_L,d_H,D_H) \\
s.t. \quad & \mathcal{V}(L,L)\ge R_L, && (3.1.6)\\
& \mathcal{V}(H,H)\ge R_H, && (3.1.7)\\
& \mathcal{V}(L,L)\ge \mathcal{V}(L,H), && (3.1.8)\\
& \mathcal{V}(H,H)\ge \mathcal{V}(H,L), && (3.1.9)\\
& d_{L,n}\le \bar{d}, D_{L,n}\ge \underline{D} \ \ for \ \ n=1,2,\cdots,N. && (3.1.10)
\end{aligned}$$

R_L and R_H are reservation utilities of type L and type H respectively. We define them as utilities the insureds obtain when the insureds do not enter the insurance market.

$$R_L := E^P\left\{\sum_{n=1}^{N} U_1(w-X_n)\right\}, \qquad R_H := E^Q\left\{\sum_{n=1}^{N} U_1(w-X_n)\right\},$$

and we assume $E^P\left\{\sum_{n=1}^{N} U_1(w-X_n)\right\}, E^Q\left\{\sum_{n=1}^{N} U_1(w-X_n)\right\} > -\infty$. Constraints (3.1.6), (3.1.7) show that the insured gets more utility from the insurance contract than that from not entering the insurance market. Therefore, the insured will enter the insurance market and take the contract offer. We call (3.1.6)and (3.1.7) reservation constraints. Since the insureds' types are not observable to the insurer, the low-risk insured may choose $\mathcal{C}_H$ if he could get more utility from it. Constraint (3.1.8) ensures that type L will choose $\mathcal{C}_L$ because it brings him more utility than $\mathcal{C}_H$. Constraint (3.1.9) illustrates the same idea for type H. We call (3.1.8) and (3.1.9) incentive compatibility constraints. Constraint (3.1.10) is the main difference we have from models of the literature in the same area. In this constraint, $\bar{d}$ and $\underline{\mathrm{D}}$ are constant boundaries for the premium and compensation of type L. Since $\bar{d}$ is the highest premium that the insured will pay and $\underline{\mathrm{D}}$ is the lowest compensation that the insured will receive, we suppose that $\bar{d} > \underline{\mathrm{D}}$. With more than two states, there could be $M(s)\to\infty$ when s increases. Under this circumstance, we will demonstrate that the traditional model is inappropriate and it is crucial to introduce the boundaries for $d_{L,n}$ and $D_{L,n}$. If $M(s)$ is bounded, we will show the conditions under which the traditional model is inappropriate and (3.1.10) is necessary and the conditions under which the traditional model is still valid and (3.1.10) is not necessary.

3.2 The Solutions

To solve for the optimal controls, we will, first, temporarily ignore constraints (3.1.7) and (3.1.8) in the problem. With the remaining constraints, we will then apply the Lagrangian method to the problem and derive the candidate solutions. The candidate solutions will be proved to be optimal to the problem without (3.1.7) and (3.1.8). Next, it will be shown that these candidate solutions satisfy the ignored constraint (3.1.8). If the constraint (3.1.7) is also satisfied by the candidate solutions, then the candidate solutions are optimal to the original problem. At last, if (3.1.7) is not satisfied, we will show other solutions to the original problem.

Let λ_1^R and λ_1^I be the Lagrangian multipliers for the reservation constraint (3.1.6) and the incentive compatibility constraint (3.1.9) respectively. The first order conditions of the problem are

$$\begin{aligned} &p_L U_2'(d_{L,n}^x - D_{L,n}^x) - \lambda_1^R U_1'(w - d_{L,n}^x + D_{L,n}^x - X_n) \\ &+ \lambda_1^I M_n \frac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} U_1'(w - d_{L,n}^x + D_{L,n}^x - X_n) = 0, \\ &p_H U_2'(d_{H,n}^x - D_{H,n}^x) - \lambda_1^I U_1'(w - d_{H,n}^x + D_{H,n}^x - X_n) = 0. \end{aligned}$$

We rewrite the first order conditions and obtain

$$\begin{aligned} g(d_{L,n}^x - D_{L,n}^x, X_n) &= \frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, \\ g(d_{H,n}^x - D_{H,n}^x, X_n) &= \frac{\lambda_1^I}{p_H}. \end{aligned} \tag{3.2.11}$$

From the equations above, we see that λ_1^I has to be positive because g is a positive function. We also need $\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} > 0$ for the same reason. However, if $M(\cdot)$ is unbounded, then there is always $\omega \in \Omega$ such that

$$\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \leq 0 \tag{3.2.12}$$

for every $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. In this case, the first order conditions are invalid and the solutions cannot be derived from the first order conditions. That is the reason we need to take the boundaries $\bar{d}$ and $\underline{\mathrm{D}}$ into consideration. In this case, $\bar{d}$ will be the optimal premium and $\underline{\mathrm{D}}$ will be the optimal compensation for type L. We will show more about the presence of $\bar{d}$ and $\underline{\mathrm{D}}$ in the solutions later. For $x \in S^{n-1}$ where $n = 1, 2, \cdots, N$, we define

$$A_1^x := \left\{ s \in S; g(\bar{d} - \underline{\mathrm{D}}, s) \leq \frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M(s) dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \right\}.$$

If $X_n \in A_1^x$ for some $\omega \in \Omega$, then

$$g(\bar{d} - \underline{\mathrm{D}}, X_n) \leq \frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}.$$

For $X_n \in A_1^x$, we see that $\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} > 0$ because g is a positive function. That means the first order condition for type L in (3.2.11) is valid. (3.2.11) also shows that $g(\bar{d} - \underline{\mathrm{D}}, X_n) \leq g(d_{L,n}^x - D_{L,n}^x, X_n)$. Because $g(y_1, y_2)$ is a decreasing function of y_1, we obtain $\bar{d} - \underline{\mathrm{D}} \geq d_{L,n}^x - D_{L,n}^x$ for $X_n \in A_1^x$. Constraint (3.1.10) requires $d_{L,n}^x - D_{L,n}^x \leq \bar{d} - \underline{\mathrm{D}}$ for each $n = 1, 2, \cdots, N$ and each $x \in S^{n-1}$. So, for $X_n \in A_1^x$, the value of $d_{L,n}^x - D_{L,n}^x$ derived from (3.2.11) locates inside the boundaries. Then, we will use the $d_{L,n}^x - D_{L,n}^x$ derived from (3.2.11) as the solution for type L. Otherwise, if $X_n \notin A_1^x$, the result $d_{L,n}^x - D_{L,n}^x$ from (3.2.11) may exceed $\bar{d} - \underline{\mathrm{D}}$ or (3.2.11) may not be valid. Then, for $X_n \in S - A_1^x$, we just let

$$d_{L,n}^x - D_{L,n}^x = \bar{d} - \underline{\mathrm{D}}. \tag{3.2.13}$$

The solution (3.2.13) can be reached only if the premium equals $\bar{d}$ and the compensation equals $\underline{\mathrm{D}}$. The premium $d_{L,n}^x$ is determined and paid at the beginning of period n, so $d_{L,n}^x = \bar{d}$ not only when (3.2.13) occurs but for every $X_n \in S$. Correspondingly, the value of $D_{L,n}^x$ is obtained in different cases through (3.2.11) and (3.2.13). We will derive the solution for type H through the first order condition. We now present the solutions.

$$d_{L,n}^x = \bar{d}, \tag{3.2.14}$$

$$D_{L,n}^x = \begin{cases} \bar{d} - g^{-1}\left(\dfrac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\right), & \text{if } X_n \in A_1^x \\ \underline{\mathrm{D}}, & \text{if } X_n \in S - A_1^x \end{cases} \tag{3.2.15}$$

$$d_{H,n}^x - D_{H,n}^x = g^{-1}\left(\frac{\lambda_1^I}{p_H}, X_n\right). \tag{3.2.16}$$

Here, with $\mathbb{I}$ denoting the indicator function, λ_1^R and λ_1^I are such that

$$E^P\Bigg\{\sum_{n=1}^{N}\int_{S^{n-1}}\Bigg[U_1\Bigg(w-g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)},X_n\Big)-X_n\Bigg)\mathbb{I}_{X_n\in A_1^x} + U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-A_1^x}\Bigg]dF_P^{(n-1)}(x)\Bigg\}=R_L, \tag{3.2.17}$$

$$E^Q\Bigg\{\sum_{n=1}^{N}\int_{S^{n-1}}U_1\Bigg(w-g^{-1}\Big(\frac{\lambda_1^I}{p_H},X_n\Big)-X_n\Bigg)dF_Q^{(n-1)}(x)\Bigg\}$$
$$=E^Q\Bigg\{\sum_{n=1}^{N}\int_{S^{n-1}}\Bigg[U_1\Bigg(w-g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{n-1}(x)},X_n\Big)-X_n\Bigg)\mathbb{I}_{X_n\in A_1^x} + U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-A_1^x}\Bigg]dF_Q^{(n-1)}(x)\Bigg\}. \tag{3.2.18}$$

In (3.2.14)–(3.2.15), when $X_n\in A_1^x$, as discussed above, the solution $d_{L,n}^x - D_{L,n}^x \le \bar{d}-\underline{\mathrm{D}}$. Noting that $d_{L,n}^x=\bar{d}$, we get $D_{L,n}^x\ge\underline{\mathrm{D}}$ for $X_n\in A_1^x$. If for some $\omega\in\Omega$, $x\in S^{n-1}$, and $n=1,2,\cdots,N$, we have $\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}>0$, then

$$g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)},X_n\Big) \tag{3.2.19}$$

is well defined on $S-A_1^x$. Then,

$$g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)},X_n\Big)\ge d_{L,n}^x-D_{L,n}^x=\bar{d}-\underline{\mathrm{D}} \tag{3.2.20}$$

when $X_n\in S-A_1^x$.

Supposing $y,z\in\mathbb{R}^+$, we define the notation S_1. $S_1\subseteq S$ and for every $s\in S_1$, $g^{-1}\left(\frac{ydF_P^{(n-1)}(x)-zM(s)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)},s\right)$ exists.

Lemma 6. *Let $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. Then,*

$$-g^{-1}\left(\frac{ydF_P^{(n-1)}(x) - zM(s)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, s\right) - s$$

is a decreasing function of $s \in S_1$.

Proof. We denote

$$\Gamma(x, y, z, s) := g^{-1}\left(\frac{ydF_P^{(n-1)}(x) - zM(s)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, s\right). \tag{3.2.21}$$

It is sufficient to illustrate $-\Gamma(x, y, z, s) - s$ decreases as s increases. From the definition of g, (3.2.21) is equivalent to the equation

$$\frac{U_2'(\Gamma)}{U_1'(w - \Gamma - s)} = \frac{ydF_P^{(n-1)}(x) - zM(s)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}. \tag{3.2.22}$$

We will prove by contradiction. Suppose $-\Gamma(x, y, z, s) - s$ does not decrease when some $s \in S_1$ increases, then we have Γ decreases for some $s \in S_1$. $U_2'(\Gamma)$ will not decrease because U_2 is a concave function. Meanwhile, we have $U_1'(w - \Gamma - s)$ does not increase because U_1 is a concave function too. Recall that $U_2'(\Gamma)$ and $U_1'(w - \Gamma - s)$ are both positive. So, we obtain the left-hand-side of (3.2.22) does not decrease as s increases. However, $M(s)$ is a strictly increasing function of s and $z > 0$, so we have the right-hand-side of (3.2.22) decreases as s increases. That is a contradiction. Thus, the required statement follows. □

Proposition 3. *There exist $\lambda_1^R > 0$ and $\lambda_1^I > 0$ such that (3.2.17) and (3.2.18) hold.*

Proof. **Step 1.** We will show that for any fixed $\lambda_1^I \in (0, \infty)$, we have $\lambda_1^R > 0$ such that (3.2.17) holds. Let $y \in \mathbb{R}^+$. Similar to A_1^x, we define the set $B_1^x(y) := \left\{s; g(\bar{d} - \underline{\mathrm{D}}, s) \leq \frac{ydF_P^{(n-1)}(x) - \lambda_1^I M(s)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}\right\}$. We denote $\phi_1(y) := g^{-1}\left(\frac{ydF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\right)$.

When $y \to 0$, $\dfrac{y dF_P^{(n-1)}(x) - \lambda_1^I M(s) dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \leq 0$ for each $s \in S$, $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. Noticing g is a function that takes only positive values, we have $B_1^x(y) = \emptyset$ for every $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. Therefore,

$$E^P \left\{ \sum_{n=1}^{N} \int_{x \in S^{n-1}} \left[U_1\big(w - \phi_1(y) - X_n\big) \mathbb{I}_{X_n \in B_1^x(y)} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right) \mathbb{I}_{X_n \in S - B_1^x(y)} \right] dF_P^{(n-1)}(x) \right\}$$

$$= E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right) dF_P^{(n-1)}(x) \right].$$

Recalling that $\bar{d} > \underline{\mathrm{D}}$, we obtain that

$$U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right) < U_1\left(w - X_n\right),$$

which yields

$$\begin{aligned} & E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right) dF_P^{(n-1)}(x) \right] \\ & < E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_1(w - X_n) dF_P^{(n-1)}(x) \right] = R_L. \end{aligned} \qquad (3.2.23)$$

When $y \to \infty$, $\dfrac{y dF_P^{(n-1)}(x) - \lambda_1^I M(s) dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \to \infty$ for every $s \in S$, $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. Noting $g(\bar{d} - \underline{\mathrm{D}}, s)$ is finite for each $s \in S$, we obtain $B_1^x(y) = S$ for every $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. Therefore,

$$E^P \Bigg\{ \sum_{n=1}^{N} \int_{x \in S^{n-1}} \big[U_1\big(w - \phi_1(y) - X_n\big) \mathbb{I}_{X_n \in B_1^x(y)} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right) \mathbb{I}_{X_n \in S - B_1^x(y)} \big] dF_P^{(n-1)}(x) \Bigg\}$$

$$= E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_1\big(w - \phi_1(y) - X_n\big) dF_P^{(n-1)}(x) \right].$$

According to the definition of g, we have $\phi_1(y) \to -\infty$ for each $X_n \in S$ and $x \in S^{n-1}$ when $y \to \infty$. So,

$$U_1\left(w - \phi_1(y) - X_n\right) > U_1(w - X_n)$$

and it yields

$$\begin{aligned} & E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_1\left(w - \phi_1(y) - X_n\right) dF_P^{(n-1)}(x) \right] \\ & > E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_1(w - X_n) dF_P^{(n-1)}(x) \right] = R_L. \end{aligned} \tag{3.2.24}$$

The expression

$$E^P \Bigg\{ \sum_{n=1}^{N} \int_{x \in S^{n-1}} \big[U_1\big(w - \phi_1(y) - X_n\big) \mathbb{I}_{X_n \in B_1^x(y)} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right) \mathbb{I}_{X_n \in S - B_1^x(y)} \big] dF_P^{(n-1)}(x) \Bigg\}$$

is continuous of y. Due to the Mean Value Theorem, as a result of (3.2.23) and

(3.2.24), we can always find $y \in (0,\infty)$ for a fixed λ_1^I such that

$$E^P\Bigg\{\sum_{n=1}^{N}\int_{x\in S^{n-1}}\big[U_1\big(w-\phi_1(y)-X_n\big)\mathbb{I}_{X_n\in B_1^x(y)} + U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-B_1^x(y)}\big]dF_P^{(n-1)}(x)\Bigg\} = R_L.$$

Equivalently speaking, we have a $\lambda_1^R \in (0,\infty)$ for any fixed $\lambda_1^I \in (0,\infty)$ such that (3.2.17) holds.

Step 2. We will show that among the pairs $(\lambda_1^R, \lambda_1^I)$ that satisfy (3.2.17), we can always find a pair that satisfies (3.2.18). Let $z \in \mathbb{R}^+$. We start with the pair of variables (y, z) satisfying

$$E^P\Bigg\{\sum_{n=1}^{N}\int_{x\in S^{n-1}}\big[U_1\big(w-\phi_2(y,z)-X_n\big)\mathbb{I}_{X_n\in B_2^x(y,z)} + U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-B_2^x(y,z)}\big]dF_P^{(n-1)}(x)\Bigg\} = R_L, \tag{3.2.25}$$

where

$$\phi_2(y,z) := g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - zM_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big), \tag{3.2.26}$$

$$B_2^x(y,z) := \Big\{s; g(\bar{d}-\underline{\mathrm{D}}, s) \le \frac{ydF_P^{(n-1)}(x) - zM(s)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}\Big\}. \tag{3.2.27}$$

Equation (3.2.25) is the same as (3.2.17) except that $(\lambda_1^R, \lambda_1^I)$ in (3.2.17) is replaced by (y, z) in (3.2.25). From Lemma 6, we derive that

$$-g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - zM(s)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, s\Big) - s$$

is a decreasing function of $s \in S$ for every $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. It is obvious that both

$$U_1\Bigg(w - g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - zM(s)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, s\Big) - s\Bigg) \text{ and } U_1(w - \bar{d} + \underline{\mathrm{D}} - s)$$

are decreasing functions of s. According to Corollary 3, we obtain that

$$E^Q\Bigg[U_1\Bigg(w - \phi_2(y,z) - X_n\Bigg)\mathbb{I}_{X_n \in B_2^x(y,z)} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - B_2^x(y,z)}\Bigg] \tag{3.2.28}$$

$$\leq E^P\Bigg[U_1\Bigg(w - \phi_2(y,z) - X_n\Bigg)\mathbb{I}_{X_n \in B_2^x(y,z)} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - B_2^x(y,z)}\Bigg] \tag{3.2.29}$$

for every $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. Recalling $g^{-1}(\cdot, y_2)$ is a decreasing function, we get the term (3.2.28) is a decreasing function of $\dfrac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)}$. According to Lemma 5, we have

$$\int\limits_{x \in S^{n-1}} E^Q\Bigg[U_1\Bigg(w - \phi_2(y,z) - X_n\Bigg)\mathbb{I}_{X_n \in B_2^x(y,z)} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - B_2^x(y,z)}\Bigg] dF_Q^{(n-1)}(x)$$

$$< \int\limits_{x \in S^{n-1}} E^Q\Bigg[U_1\Bigg(w - \phi_2(y,z) - X_n\Bigg)\mathbb{I}_{X_n \in B_2^x(y,z)} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - B_2^x(y,z)}\Bigg] dF_P^{(n-1)}(x).$$

Noticing the relation between (3.2.28) and (3.2.29), from the inequality above, we obtain

$$\begin{aligned}
&\int\limits_{x\in S^{n-1}} E^{Q}\Big[U_1\Big(w-\phi_2(y,z)-X_n\Big)\mathbb{I}_{X_n\in B_2^x(y,z)} \\
&\qquad\qquad\qquad + U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-B_2^x(y,z)}\Big]dF_Q^{(n-1)}(x) \\
&< \int\limits_{x\in S^{n-1}} E^{P}\Big[U_1\Big(w-\phi_2(y,z)-X_n\Big)\mathbb{I}_{X_n\in B_2^x(y,z)} \\
&\qquad\qquad\qquad + U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-B_2^x(y,z)}\Big]dF_P^{(n-1)}(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
&E^{Q}\Big\{\sum_{n=1}^{N}\int\limits_{x\in S^{n-1}}\Big[U_1\Big(w-\phi_2(y,z)-X_n\Big)\mathbb{I}_{X_n\in B_2^x(y,z)} \\
&\qquad\qquad\qquad + U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-B_2^x(y,z)}\Big]dF_Q^{(n-1)}(x)\Big\} \\
&< E^{P}\Big\{\sum_{n=1}^{N}\int\limits_{x\in S^{n-1}}\Big[U_1\Big(w-\phi_2(y,z)-X_n\Big)\mathbb{I}_{X_n\in B_2^x(y,z)} \\
&\qquad\qquad\qquad + U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-B_2^x(y,z)}\Big]dF_P^{(n-1)}(x)\Big\} \\
&= R_L. \qquad (3.2.30)
\end{aligned}$$

When $z\to 0$, to keep (3.2.25) holding, y must not approach 0. We will show that by contradiction. Suppose $y,z\to 0$, then $B_2^x(y,z)=\emptyset$ for each $x\in S^{n-1}$ and $n=1,2,\cdots,N$. The left-hand-side of (3.2.25) becomes

$$E^{P}\left[\sum_{n=1}^{N}\int\limits_{x\in S^{n-1}} U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)dF_P^{(n-1)}(x)\right].$$

From (3.2.23), we see the expression above is smaller than R_L, which means

(3.2.25) does not hold. Therefore, when $z \to 0$, there exists $y^* \in \mathbb{R}^+$ such that $y > y^* > 0$ and consequently,

$$\frac{y dF_P^{(n-1)}(x) - zM_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} = \frac{y}{p_L} > \frac{z}{p_H}$$

for every $\omega \in \Omega$, $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$. Then, $\phi_2(y,z) < g^{-1}\left(\frac{z}{p_H}, X_n\right)$ for every $X_n \in B_2^x(y,z)$ and $x \in S^{n-1}$ because $g^{-1}(\cdot, y_2)$ is a decreasing function. We get $g^{-1}(\frac{z}{p_H}, X_n) \to \infty$ from the definition of g when $z \to 0$, so $\bar{d} - \underline{\mathrm{D}} < g^{-1}(\frac{z}{p_H}, X_n)$ for every $X_n \in S - B_2^x(y,z)$. That gives us

$$\begin{aligned} &U_1\left(w - g^{-1}\left(\frac{z}{p_H}, X_n\right) - X_n\right) \\ &< U_1\big(w - \phi_2(y,z) - X_n\big)\mathbb{I}_{X_n \in B_2^x(y,z)} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - B_2^x(y,z)}, \end{aligned}$$

which yields

$$\begin{aligned} &E^Q\left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_1\left(w - g^{-1}\left(\frac{z}{p_H}, X_n\right) - X_n\right) dF_Q^{(n-1)}(x)\right] \\ &< E^Q\Bigg\{\sum_{n=1}^{N} \int_{x \in S^{n-1}} \Bigg[U_1\left(w - g^{-1}\left(\frac{y dF_P^{(n-1)}(x) - zM_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\right) - X_n\right)\mathbb{I}_{X_n \in B_2^x(y,z)} \\ &\qquad + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - B_2^x(y,z)}\Bigg] dF_Q^{(n-1)}(x)\Bigg\}. \end{aligned} \tag{3.2.31}$$

When $z \to \infty$, $g^{-1}\left(\frac{z}{p_H}, X_n\right) \to -\infty$ from the definition of g for every $X_n \in S$. Then,

$$U_1\left(w - g^{-1}\left(\frac{z}{p_H}, X_n\right) - X_n\right) > U_1(w)$$

for each $X_n \in S$ and $x \in S^{n-1}$. So,

$$E^Q\left[U_1\Big(w - g^{-1}\Big(\frac{z}{p_H}, X_n\Big) - X_n\Big)\right] > U_1(w) > E^P\left[U_1(w - X_n)\right].$$

Continue the inequality above to obtain

$$\begin{aligned} &E^Q\left[\int\limits_{x \in S^{n-1}} U_1\Big(w - g^{-1}\Big(\frac{z}{p_H}, X_n\Big) - X_n\Big) dF_Q^{(n-1)}(x)\right] \\ &> E^P\left[\int\limits_{x \in S^{n-1}} U_1(w - X_n) dF_Q^{(n-1)}(x)\right] = E^P\left[U_1(w - X_n)\right], \end{aligned}$$

which yields

$$\begin{aligned} &E^Q\left[\sum_{n=1}^{N}\int\limits_{x \in S^{n-1}} U_1\Big(w - g^{-1}\Big(\frac{z}{p_H}, X_n\Big) - X_n\Big) dF_Q^{(n-1)}(x)\right] \\ &> E^P\left[\sum_{n=1}^{N} U_1(w - X_n)\right] = R_L. \end{aligned} \tag{3.2.32}$$

Combining (3.2.30) and (3.2.32), we obtain

$$\begin{aligned} &E^Q\left[\sum_{n=1}^{N}\int\limits_{x \in S^{n-1}} U_1\Big(w - g^{-1}\Big(\frac{z}{p_H}, X_n\Big) - X_n\Big) dF_Q^{(n-1)}(x)\right] \\ &> E^Q\Bigg\{\sum_{n=1}^{N}\int\limits_{x \in S^{n-1}} \Bigg[U_1\Big(w - g^{-1}\Big(\frac{y dF_P^{(n-1)}(x) - z M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Big)\mathbb{I}_{X_n \in B_2^x(y,z)} \\ &\qquad + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - B_2^x(y,z)}\Bigg] dF_Q^{(n-1)}(x)\Bigg\} \end{aligned} \tag{3.2.33}$$

when $z \to \infty$.

From (3.2.31) and (3.2.33), due to Mean Value Theorem, we state that among the pairs $(\lambda_1^R, \lambda_1^I)$ satisfying (3.2.17), there is a pair that satisfies (3.2.18). In summary, there exist $\lambda_1^R > 0$ and $\lambda_1^I > 0$ such that (3.2.17) and (3.2.18) hold. □

Now we can show the optimality of the candidate solutions.

Proposition 4. *Without the constraints (3.1.7) and (3.1.8), the optimal solutions are given by (3.2.14)–(3.2.16).*

Proof. Let c_L, C_L, c_H, and C_H be the premium process for type L, compensation process for type L, premium process for type H, and compensation process for type H respectively. Suppose $(c_L, C_L, c_H, C_H) \in \mathcal{A}_2$ and the constraints (3.1.6) and (3.1.9) are satisfied. We will compare the insurer's utilities from (3.2.14)–(3.2.16) and from (c_L, C_L, c_H, C_H).

$$\begin{aligned}
&\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H) \\
&= p_L E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} \left(U_2(d_{L,n}^x - D_{L,n}^x) - U_2(c_{L,n}^x - C_{L,n}^x) \right) dF_P^{(n-1)}(x) \right] \\
&\quad + p_H E^Q \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} \left(U_2(d_{H,n}^x - D_{H,n}^x) - U_2(c_{H,n}^x - C_{H,n}^x) \right) dF_Q^{(n-1)}(x) \right].
\end{aligned}$$

Since U_2 is a concave function, we obtain

$$\begin{aligned}
&\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H) \\
&\geq p_L E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_2'(d_{L,n}^x - D_{L,n}^x) \left(d_{L,n}^x - D_{L,n}^x - c_{L,n}^x + C_{L,n}^x \right) dF_P^{(n-1)}(x) \right] \\
&\quad + p_H E^Q \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_2'(d_{H,n}^x - D_{H,n}^x) \left(d_{H,n}^x - D_{H,n}^x - c_{H,n}^x + C_{H,n}^x \right) dF_Q^{(n-1)}(x) \right].
\end{aligned} \tag{3.2.34}$$

(3.2.15) shows two cases of the solution for type L. When $X_n \in A_1^x$, from

(3.2.11),

$$\frac{U_2'(d_{L,n}^x - D_{L,n}^x)}{U_1'(w - d_{L,n}^x + D_{L,n}^x - X_n)} = \frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}. \tag{3.2.35}$$

When $X_n \in S - A_1^x$,

$$\frac{U_2'(d_{L,n}^x - D_{L,n}^x)}{U_1'(w - d_{L,n}^x + D_{L,n}^x - X_n)} = \frac{U_2'(\bar{d} - \underline{\mathrm{D}})}{U_1'(w - \bar{d} + \underline{\mathrm{D}} - X_n)}.$$

From the definition of A_1^x, when $X_n \in S - A_1^x$,

$$g(\bar{d} - \underline{\mathrm{D}}, X_n) = \frac{U_2'(\bar{d} - \underline{\mathrm{D}})}{U_1'(w - \bar{d} + \underline{\mathrm{D}} - X_n)} > \frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}.$$

In this case, $d_{L,n}^x - D_{L,n}^x = \bar{d} - \underline{\mathrm{D}} \geq c_{L,n}^x - C_{L,n}^x$. So,

$$\begin{aligned} &\frac{U_2'(d_{L,n}^x - D_{L,n}^x)\left(d_{L,n}^x - D_{L,n}^x - c_{L,n}^x + C_{L,n}^x\right)}{U_1'(w - d_{L,n}^x + D_{L,n}^x - X_n)} \\ &\geq \frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}\left(d_{L,n}^x - D_{L,n}^x - c_{L,n}^x + C_{L,n}^x\right). \end{aligned} \tag{3.2.36}$$

The inequality (3.2.36) is true not only for $X_n \in S - A_1^x$ but also true for $X_n \in A_1^x$ as a result of (3.2.35). For type H, it follows (3.2.16) that

$$\frac{U_2'(d_{H,n}^x - D_{H,n}^x)}{U_1'(w - d_{H,n}^x + D_{H,n}^x - X_n)} = \frac{\lambda_1^I}{p_H}. \tag{3.2.37}$$

Applying (3.2.36) and (3.2.37) to (3.2.34), we have

$$
\begin{aligned}
&\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H) \\
&\geq E^P\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1'(w - d_{L,n}^x + D_{L,n}^x - X_n)\left(d_{L,n}^x - D_{L,n}^x - c_{L,n}^x + C_{L,n}^x\right) \\
&\qquad\qquad\qquad\qquad \Big(\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)\Big)\Bigg] \\
&\quad + E^Q\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1'(w - d_{H,n}^x + D_{H,n}^x - X_n) \\
&\qquad\qquad\qquad\qquad \lambda_1^I\left(d_{H,n}^x - D_{H,n}^x - c_{H,n}^x + C_{H,n}^x\right) dF_Q^{(n-1)}(x)\Bigg].
\end{aligned}
$$

Since U_1 is a concave function, we obtain

$$
\begin{aligned}
&\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H) \\
&\geq E^P\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \left(U_1(w - c_{L,n}^x + C_{L,n}^x - X_n) - U_1(w - d_{L,n}^x + D_{L,n}^x - X_n)\right) \\
&\qquad\qquad\qquad\qquad \Big(\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)\Big)\Bigg] \\
&\quad + E^Q\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \lambda_1^I\Big(U_1(w - c_{H,n}^x + C_{H,n}^x - X_n) \\
&\qquad\qquad\qquad\qquad - U_1(w - d_{H,n}^x + D_{H,n}^x - X_n)\Big) dF_Q^{(n-1)}(x)\Bigg].
\end{aligned}
$$

Applying (3.1.5) to the inequality above, we obtain

$$
\begin{aligned}
&\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H) \\
&\geq \lambda_1^R E^P \Bigg[\sum_{n=1}^{N} \int_{x \in S^{n-1}} \Big(U_1(w - c_{L,n}^x + C_{L,n}^x - X_n) \\
&\qquad\qquad\qquad\qquad - U_1(w - d_{L,n}^x + D_{L,n}^x - X_n) \Big) dF_P^{(n-1)}(x) \Bigg] \\
&\quad - \lambda_1^I E^Q \Bigg[\sum_{n=1}^{N} \int_{x \in S^{n-1}} \Big(U_1(w - c_{L,n}^x + C_{L,n}^x - X_n) \\
&\qquad\qquad\qquad\qquad - U_1(w - d_{L,n}^x + D_{L,n}^x - X_n) \Big) dF_Q^{(n-1)}(x) \Bigg] \\
&\quad + \lambda_1^I E^Q \Bigg[\sum_{n=1}^{N} \int_{x \in S^{n-1}} \Big(U_1(w - c_{H,n}^x + C_{H,n}^x - X_n) \\
&\qquad\qquad\qquad\qquad - U_1(w - d_{H,n}^x + D_{H,n}^x - X_n) \Big) dF_Q^{(n-1)}(x) \Bigg].
\end{aligned}
$$

Recall that $\lambda_1^R, \lambda_1^I > 0$. Constraint (3.1.6) implies

$$
\begin{aligned}
&E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_1(w - c_{L,n}^x + C_{L,n}^x - X_n) dF_P^{(n-1)}(x) \right] \\
&\geq R_L = E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_1(w - d_{L,n}^x + D_{L,n}^x - X_n) dF_P^{(n-1)}(x) \right].
\end{aligned}
$$

Constraint (3.1.9) implies

$$
\begin{aligned}
&E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1(w - c^x_{H,n} + C^x_{H,n} - X_n)dF_Q^{(n-1)}(x)\right]\\
&\quad\geq E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1(w - c^x_{L,n} + C^x_{L,n} - X_n)dF_Q^{(n-1)}(x)\right],\\
&E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1(w - d^x_{H,n} + D^x_{H,n} - X_n)dF_Q^{(n-1)}(x)\right]\\
&\quad= E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1(w - d^x_{L,n} + D^x_{L,n} - X_n)dF_Q^{(n-1)}(x)\right].
\end{aligned}
$$

Then, we obtain

$$\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H) \geq 0.$$

The required statement follows. □

Searching for the solutions, we have been ignoring the reservation constraint (3.1.7) and the incentive compatibility constraint (3.1.8). The candidate solutions (3.2.14)–(3.2.16) are obtained without these constraints. However, the proposition below shows that the candidate solutions also satisfy (3.1.8). We prepare for the proposition with the definition of G^*. $G^* \in \mathbb{R}$ is defined by

$$
\begin{aligned}
&E^Q\Bigg[U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_1^R - \lambda_1^I M(X)G^*}{p_L}, X\Big) - X\Bigg)\mathbb{I}_{X\in A_{G^*}}\\
&\qquad\qquad\qquad + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X\right)\mathbb{I}_{X\in S-A_{G^*}}\Bigg]\\
&= E^Q\left[U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X\Big) - X\Bigg)\right]. \qquad (3.2.38)
\end{aligned}
$$

Here, $A_{G^*} := \Big\{s \in S; g(\bar{d} - \underline{\mathrm{D}}, s) \leq \dfrac{\lambda_1^R - \lambda_1^I M(s)G^*}{p_L}\Big\}$.

Lemma 7. *There exists a unique $G^* \in \mathbb{R}$ such that (3.2.38) holds.*

Proof. Consider the following expression as a function of $G \in \mathbb{R}$,

$$E^Q\Bigg[U_1\Big(w - g^{-1}\Big(\frac{\lambda_1^R - \lambda_1^I M(X)G}{p_L}, X\Big) - X\Big)\mathbb{I}_{X \in A_G} \\ + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X\right)\mathbb{I}_{X \in S - A_G}\Bigg]. \tag{3.2.39}$$

Here, $A_G := \Big\{s; g(\bar{d} - \underline{\mathrm{D}}, s) \leq \dfrac{\lambda_1^R - \lambda_1^I M(s)G}{p_L}\Big\}$.

When $G \to -\infty$, then $\dfrac{\lambda_1^R - \lambda_1^I M(X)G}{p_L} \to \infty$ for every $\omega \in \Omega$. According to the definition of g, we obtain that $-g^{-1}\Big(\dfrac{\lambda_1^R - \lambda_1^I M(X)G}{p_L}, X\Big) \to \infty$ for each $\omega \in \Omega$. At the same time, $g(\bar{d} - \underline{\mathrm{D}}, s) \leq \dfrac{\lambda_1^R - \lambda_1^I M(s)G}{p_L}$ for every $s \in S$, so $A_G = S$ when $G \to -\infty$. Therefore,

$$E^Q\Bigg[U_1\Big(w - g^{-1}\Big(\frac{\lambda_1^R - \lambda_1^I M(X)G}{p_L}, X\Big) - X\Big)\mathbb{I}_{X \in A_G} \\ + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X\right)\mathbb{I}_{X \in S - A_G}\Bigg]$$

$$= E^Q\left[U_1\Big(w - g^{-1}\Big(\frac{\lambda_1^R - \lambda_1^I M(X)G}{p_L}, X\Big) - X\Big)\right].$$

Because $-g^{-1}\Big(\dfrac{\lambda_1^I}{p_H}, X\Big) < -g^{-1}\Big(\dfrac{\lambda_1^R - \lambda_1^I M(X)G}{p_L}, X\Big)$ for each $\omega \in \Omega$ when

$G \to -\infty$, we obtain

$$E^Q\bigg[U_1\Big(w - g^{-1}\Big(\frac{\lambda_1^R - \lambda_1^I M(X)G}{p_L}, X\Big) - X\Big)\mathbb{I}_{X\in A_G} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X\right)\mathbb{I}_{X\in S-A_G}\bigg]$$

$$> E^Q\left[U_1\Big(w - g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X\Big) - X\Big)\right]$$

when $G \to -\infty$.

For $X_n \in A_1^x$, the solution $d_{L,n}^x - D_{L,n}^x \le \bar{d} - \underline{\mathrm{D}}$, so

$$g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{n-1}(x)}, X_n\Big) \le \bar{d} - \underline{\mathrm{D}}.$$

Next we will show for some $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$, the P probability of X_n such that

$$g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{n-1}(x)}, X_n\Big) < \bar{d} - \underline{\mathrm{D}} \tag{3.2.40}$$

is bigger than 0. We will show it by contradiction. Suppose for every $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$, the P probability of X_n such that (3.2.40) holds is 0. Then the left-hand-side of (3.2.17) becomes $E^P\left[\sum_{n=1}^{N} U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\right]$. However, since $\bar{d} - \underline{\mathrm{D}} > 0$, it follows that

$$E^P\left[\sum_{n=1}^{N} U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\right] < E^P\left[\sum_{n=1}^{N} U_1\left(w - X_n\right)\right] = R.$$

Now (3.2.17) is not satisfied and the contradiction appears. Since $P \ll Q$, we obtain that for some $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$, the Q probability of X_n

such that (3.2.40) holds is bigger than 0. That shows

$$E^Q\Bigg\{\sum_{n=1}^{N}\int_{S^{n-1}}\bigg[U_1\bigg(w-g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{n-1}(x)},X_n\Big)-X_n\bigg)\mathbb{I}_{X_n\in A_1^x}$$

$$+U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-A_1^x}\bigg]dF_Q^{(n-1)}(x)\Bigg\}$$

$$> E^Q\bigg[\sum_{n=1}^{N}\int_{S^{n-1}}U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)dF_Q^{(n-1)}(x)\bigg].$$

Then, from (3.2.18), we obtain

$$E^Q\left\{\sum_{n=1}^{N}\int_{S^{n-1}}U_1\bigg(w-g^{-1}\Big(\frac{\lambda_1^I}{p_H},X_n\Big)-X_n\bigg)dF_Q^{(n-1)}(x)\right\}$$
$$> E^Q\bigg[\sum_{n=1}^{N}\int_{S^{n-1}}U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)dF_Q^{(n-1)}(x)\bigg]$$

which is equivalent to

$$E^Q\left[U_1\bigg(w-g^{-1}\Big(\frac{\lambda_1^I}{p_H},X\Big)-X\bigg)\right]>E^Q\Big[U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X\right)\Big]. \quad (3.2.41)$$

When $G\to\infty$, then $\dfrac{\lambda_1^R-\lambda_1^I M(X)G}{p_L}\to-\infty$ for every $\omega\in\Omega$. According to the definition of A_G, we see that $A_G=\emptyset$. Therefore,

$$E^Q\bigg[U_1\bigg(w-g^{-1}\Big(\frac{\lambda_1^R-\lambda_1^I M(X)G}{p_L},X\Big)-X\bigg)\mathbb{I}_{X\in A_G}$$

$$+U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X\in S-A_G}\bigg]$$

$$= E^Q\left[U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X\right)\right].$$

From (3.2.41), we obtain

$$E^Q\Bigg[U_1\Big(w-g^{-1}\Big(\frac{\lambda_1^R-\lambda_1^I M(X)G}{p_L},X\Big)-X\Big)\mathbb{I}_{X\in A_G} + U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X\right)\mathbb{I}_{X\in S-A_G}\Bigg]$$

$$< E^Q\left[U_1\Big(w-g^{-1}\Big(\frac{\lambda_1^I}{p_H},X\Big)-X\Big)\right]$$

when $G\to\infty$.

Recalling $-g^{-1}(\cdot,y_2)$ and U_1 are increasing functions, we see that (3.2.39) is a decreasing function of G. It is obvious that (3.2.39) is a continuous function of G. Due to the Mean Value Theorem, the required statement follows. □

Proposition 5. *The controls (3.2.14)–(3.2.16) satisfy the constraint (3.1.8).*

Proof. We denote $\underline{\mathrm{s}} := \inf S$. We also define $X^*\in S$ such that $g(\bar{d}-\underline{\mathrm{D}},X^*)$ is closest to $\dfrac{\lambda_1^I}{p_H}$.

$$X^* := \operatorname*{argmin}_{s\in S}\left|g(\bar{d}-\underline{\mathrm{D}},s)-\frac{\lambda_1^I}{p_H}\right|.$$

$g(y_1,y_2)$ is a decreasing function of y_2, so X^* is unique. Plug (3.2.14)–(3.2.16) into (3.1.8) and obtain

$$E^P\Bigg\{\sum_{n=1}^{N}\int_{x\in S^{n-1}}\Bigg[U_1\Big(w-g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)},X_n\Big)-X_n\Big)\mathbb{I}_{X_n\in A_1^x} + U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-A_1^x}\Bigg]dF_P^{(n-1)}(x)\Bigg\}$$

$$\geq E^P\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}}U_1\Big(w-g^{-1}\Big(\frac{\lambda_1^I}{p_H},X_n\Big)-X_n\Big)dF_P^{(n-1)}(x)\right]. \qquad (3.2.42)$$

For the simplification of the notaiton, we denote

$$\Delta := U_1\Big(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Big)\mathbb{I}_{X_n \in A_1^x}$$
$$+ U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - A_1^x} - U_1\Big(w - g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) - X_n\Big).$$

(3.2.42) is thus rewritten as $E^P\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \Delta dF_P^{(n-1)}(x)\right] \geq 0$. From (3.1.1), the inequality is equivalent to

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}\Delta dF_Q^{(n-1)}(x)\right] \geq 0. \tag{3.2.43}$$

It is sufficient to prove (3.2.43) is true. According to the relation between λ_1^R and λ_1^I, we separate the analysis into two cases.

Case 1: $\dfrac{\lambda_1^R}{p_L} \leq \dfrac{\lambda_1^I}{p_H}$.

It is obvious that $\dfrac{\lambda_1^R}{p_L} - \dfrac{\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \leq \dfrac{\lambda_1^R}{p_L}$ for every $\omega \in \Omega$, $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$. Recalling that $g^{-1}(\cdot, y_2)$ is a decreasing function, we have

$$-g^{-1}\left(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\right) \leq -g^{-1}\left(\frac{\lambda_1^R}{p_L}, X_n\right) \leq -g^{-1}\left(\frac{\lambda_1^I}{p_H}, X_n\right) \tag{3.2.44}$$

for every $\omega \in \Omega$, $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$.

1.1

For any $n = 1, 2, \cdots, N$, if $X_n < X^*$, then $g(\bar{d} - \underline{\mathrm{D}}, X_n) > g(\bar{d} - \underline{\mathrm{D}}, X^*)$ because $g(y_1, y_2)$ is a decreasing function of y_2. Thus, $g(\bar{d} - \underline{\mathrm{D}}, X_n) > \dfrac{\lambda_1^I}{p_H}$. Since $g(y_1, y_2)$ is a decreasing function of y_1, we have $g^{-1}\left(\dfrac{\lambda_1^I}{p_H}, X_n\right) > \bar{d} - \underline{\mathrm{D}}$. From (3.2.44), we obtain for $X_n < X^*$,

$$-g^{-1}\left(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\right) \leq -g^{-1}\left(\frac{\lambda_1^I}{p_H}, X_n\right) < -\bar{d} + \underline{\mathrm{D}}. \quad (3.2.45)$$

It means that

$$g(\bar{d} - \underline{\mathrm{D}}, X_n) > \frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}.$$

Recalling the definition of A_1^x, we see if $X_n < X^*$, then $X_n \in S - A_1^x$ for every $x \in S^{n-1}$. Thus, as a result of (3.2.45),

$$\begin{aligned}
&U_1\left(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\right)\mathbb{I}_{X_n \in A_1^x} \\
&\quad + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - A_1^x} \\
&= U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right) \\
&> U_1\left(w - g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) - X_n\right).
\end{aligned}$$

It means $\Delta > 0$. At the same time, because $M(s)$ is a strictly increasing function, $\frac{1}{M_n} > \frac{1}{M(X^*)}$ for $X_n \leq X^*$. So, for any $n = 1, 2, \cdots, N$, if $X_n < X^*$, then $\frac{1}{M_n}\Delta > \frac{1}{M(X^*)}\Delta$.

1.2

For any $n = 1, 2, \cdots, N$, if $X_n > X^*$, $g(\bar{d} - \underline{\mathrm{D}}, X_n) < g(\bar{d} - \underline{\mathrm{D}}, X^*)$ because $g(y_1, y_2)$ is a decreasing function of y_2. Thus, $g(\bar{d} - \underline{\mathrm{D}}, X_n) < \frac{\lambda_1^I}{p_H}$. Since $g(y_1, y_2)$ is a decreasing function of y_1, we have $g^{-1}\left(\frac{\lambda_1^I}{p_H}, X_n\right) < \bar{d} - \underline{\mathrm{D}}$. Considering

$g^{-1}\left(\frac{\lambda_1^I}{p_H}, X_n\right) < \bar{d} - \underline{\mathrm{D}}$ and (3.2.44) together, we obtain for $X_n > X^*$,

$$
\begin{aligned}
&U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Bigg)\mathbb{I}_{X_n \in A_1^x} \\
&\quad + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - A_1^x} \\
&< U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) - X_n\Bigg),
\end{aligned}
$$

which means $\Delta < 0$. At the same time, $\frac{1}{M_n} < \frac{1}{M(X^*)}$ for $X_n > X^*$. So, for any $n = 1, 2, \cdots, N$, if $X_n > X^*$, then $\frac{1}{M_n}\Delta > \frac{1}{M(X^*)}\Delta$.

Consider

$$
\begin{aligned}
&E^Q\left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} \frac{1}{M_n} \frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)} \Delta dF_Q^{(n-1)}(x)\right] \\
&= E^Q\Bigg[\sum_{n=1}^{N} \int_{x \in S^{n-1}} \frac{1}{M_n} \frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)} \\
&\qquad\qquad \left(\Delta \mathbb{I}_{\{X_n < X^*\}} + \Delta \mathbb{I}_{\{X_n > X^*\}} + \Delta \mathbb{I}_{\{X_n = X^*\}}\right) dF_Q^{(n-1)}(x)\Bigg].
\end{aligned}
$$

Recalling that $\frac{1}{M_n}\Delta > \frac{1}{M(X^*)}\Delta$ when $X_n < X^*$ and $X_n > X^*$, and $\frac{1}{M_n}\Delta =$

$\frac{1}{M(X^*)}\Delta$ when $X_n = X^*$, we obtain

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}\Delta dF_Q^{(n-1)}(x)\right]$$
$$\geq E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \frac{1}{M(X^*)}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}\right.$$
$$\left.\left(\Delta\mathbb{I}_{\{X_n<X^*\}}+\Delta\mathbb{I}_{\{X_n>X^*\}}+\Delta\mathbb{I}_{\{X_n=X^*\}}\right)dF_Q^{(n-1)}(x)\right]$$
$$= \sum_{n=1}^{N}\int_{x\in S^{n-1}} \frac{1}{M(X^*)}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}E^Q[\Delta]dF_Q^{(n-1)}(x). \tag{3.2.46}$$

From the discussion in Lemma 7, we know that (3.2.39) is a decreasing function of G. So, when $\frac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} \geq G^*$, we obtain $E^Q[\Delta] \leq 0$ and $\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)} \leq \frac{1}{G^*}$. That yields $\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}E^Q[\Delta] \geq \frac{1}{G^*}E^Q[\Delta]$. When $\frac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} < G^*$, we obtain $E^Q[\Delta] > 0$ and $\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)} > \frac{1}{G^*}$. That yields $\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}E^Q[\Delta] \geq \frac{1}{G^*}E^Q[\Delta]$ too. We continue (3.2.46) to obtain

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}\Delta dF_Q^{(n-1)}(x)\right]$$
$$\geq \sum_{n=1}^{N}\int_{x\in S^{n-1}} \frac{1}{M(X^*)}\frac{1}{G^*}E^Q[\Delta]dF_Q^{(n-1)}(x)$$
$$= \frac{1}{M(X^*)}\frac{1}{G^*}E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \Delta dF_Q^{(n-1)}(x)\right], \tag{3.2.47}$$

which is 0 according to (3.2.18). Now (3.2.43) is reached.

Case 2: $\dfrac{\lambda_1^R}{p_L} > \dfrac{\lambda_1^I}{p_H}$.

We will discuss this case under the following possible conditions.

2.1

$g\left(\bar{d} - \underline{\mathrm{D}}, \underline{\mathrm{s}}\right) < \dfrac{\lambda_1^I}{p_H}$.

Observing $X_n \geq \underline{\mathrm{s}}$, we have $g\left(\bar{d} - \underline{\mathrm{D}}, X_n\right) < \dfrac{\lambda_1^I}{p_H}$ for every $\omega \in \Omega$ and $n = 1, 2, \cdots, N$. $g(y_1, y_2)$ is a decreasing function of y_1, so $-g^{-1}\left(\dfrac{\lambda_1^I}{p_H}, X_n\right) > -\bar{d} + \underline{\mathrm{D}}$ for every $\omega \in \Omega$ and $n = 1, 2, \cdots, N$.

2.1.1

For any $n = 1, 2, \cdots, N$ and any $x \in S^{n-1}$, if $\dfrac{M_n dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} < (\dfrac{\lambda_1^R}{p_L} - \dfrac{\lambda_1^I}{p_H})\dfrac{p_L}{\lambda_1^I}$, then there will be

$$\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} > \frac{\lambda_1^I}{p_H} > g\left(\bar{d} - \underline{\mathrm{D}}, X_n\right) \tag{3.2.48}$$

for every $\omega \in \Omega$ and $n = 1, 2, \cdots, N$. So, from the definition of A_1^x, we see that $X_n \in A_1^x$ for every $\omega \in \Omega$ and $n = 1, 2 \cdots, N$. Noting $g^{-1}(\cdot, y_2)$ is a decreasing function, we obtain from (3.2.48),

$$-g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) > -g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big).$$

Thus, we derive that

$$\begin{aligned}
&U_1\bigg(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\bigg)\mathbb{I}_{X_n \in A_1^x} \\
&\quad + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - A_1^x} \\
&= U_1\bigg(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\bigg) \\
&> U_1\bigg(w - g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) - X_n\bigg).
\end{aligned}$$

This is equivalent to $\Delta > 0$. Since $\frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)} > \frac{1}{(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}}$, we obtain

$$\frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}\Delta > \frac{1}{(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}}\Delta. \tag{3.2.49}$$

2.1.2

For any $n = 1, 2, \cdots, N$ and any $x \in S^{n-1}$, if $\frac{M_n dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} \geq (\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}$, then there will be

$$\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \leq \frac{\lambda_1^I}{p_H},$$

which yields

$$-g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) \geq -g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big)$$

for every $\omega \in \Omega$ and $n = 1, 2, \cdots, N$. Because $-g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) > -\bar{d} + \underline{\mathrm{D}}$ too, we state that

$$\begin{aligned}
&- g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) \\
&\geq -g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big)\mathbb{I}_{X_n \in A_1^x} + (-\bar{d} + \underline{\mathrm{D}})\mathbb{I}_{X_n \in S - A_1^x},
\end{aligned}$$

which yields

$$\begin{aligned}
&U_1\Big(w - g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) - X_n\Big) \\
&\geq U_1\Big(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Big)\mathbb{I}_{X_n \in A_1^x} \\
&\quad + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - A_1^x}.
\end{aligned}$$

This is equivalent to $\Delta \leq 0$. Since $\frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)} \leq \frac{1}{(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}}$, we obtain

$$\frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}\Delta \geq \frac{1}{(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}}\Delta. \tag{3.2.50}$$

From (3.2.49) and (3.2.50), we obtain

$$\begin{aligned}
&E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}\Delta dF_Q^{(n-1)}(x)\right] \\
&\geq \frac{1}{(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}} E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \Delta dF_Q^{(n-1)}(x)\right],
\end{aligned}$$

which is 0 according to (3.2.18). Now (3.2.43) is reached.

2.2
$g\left(\bar{d} - \underline{\mathrm{D}}, \underline{\mathrm{s}}_1\right) \geq \frac{\lambda_1^I}{p_H}$.
In (3.2.14)–(3.2.15), $d_{L,n}^x - D_{L,n}^x \leq \bar{d} - \underline{\mathrm{D}}$ when $X_n \in A_1^x$, thus

$$\begin{aligned}
&-g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big)\mathbb{I}_{X_n\in A_1^x} + (-\bar{d} + \underline{\mathrm{D}})\mathbb{I}_{X_n\in S-A_1^x} \\
&\geq -\bar{d} + \underline{\mathrm{D}}
\end{aligned} \tag{3.2.51}$$

for every $\omega \in \Omega$, $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$. From (3.2.20), we obtain if (3.2.19) exists for every $\omega \in \Omega$, $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$, then

$$\begin{aligned}
&-g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big)\mathbb{I}_{X_n\in A_1^x} + (-\bar{d} + \underline{\mathrm{D}})\mathbb{I}_{X_n\in S-A_1^x} \\
&\geq -g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big)
\end{aligned} \tag{3.2.52}$$

for every $\omega \in \Omega$, $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$. We will consider the following situations.

2.2.1

Consider $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$ such that

$$\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M(X^*) dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \leq \frac{\lambda_1^I}{p_H}. \tag{3.2.53}$$

2.2.1.1

For $X_n < X^*$, $g(\bar{d} - \underline{\mathrm{D}}, X_n) > g(\bar{d} - \underline{\mathrm{D}}, X^*)$ because $g(y_1, y_2)$ is a decreasing function of y_2. From the definition of X^*, we obtain that $g(\bar{d} - \underline{\mathrm{D}}, X_n) > \dfrac{\lambda_1^I}{p_H}$ and then $-\bar{d} + \underline{\mathrm{D}} > -g^{-1}\Big(\dfrac{\lambda_1^I}{p_H}, X_n\Big)$ for $X_n < X^*$. That yields, as a result of (3.2.51),

$$\begin{aligned} &- g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big)\mathbb{I}_{X_n \in A_1^x} + (-\bar{d} + \underline{\mathrm{D}})\mathbb{I}_{X_n \in S - A_1^x} \\ &> -g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) \end{aligned}$$

for $X_n < X^*$. Therefore,

$$\begin{aligned} &U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Bigg)\mathbb{I}_{X_n \in A_1^x} \\ &\quad + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - A_1^x} \\ &> U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) - X_n\Bigg) \end{aligned}$$

for $X_n < X^*$. This is equivalent to $\Delta > 0$. At the same time, $\dfrac{1}{M_n} > \dfrac{1}{M(X^*)}$ since $M(s)$ is a strictly increasing function of s. So, for $X_n < X^*$, we have $\dfrac{1}{M_n}\Delta > \dfrac{1}{M(X^*)}\Delta$.

2.2.1.2

For $X_n > X^*$, we have from (3.2.53),

$$\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} < \frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M(X^*) dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \leq \frac{\lambda_1^I}{p_H}.$$

Because $-g^{-1}(\cdot, y_2)$ is an increasing function, we have

$$-g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) < -g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) \qquad (3.2.54)$$

for $X_n > X^*$. $g\left(\bar{d} - \underline{\mathrm{D}}, X_n\right) < g\left(\bar{d} - \underline{\mathrm{D}}, X^*\right)$ if $X_n > X^*$, so we obtain $g\left(\bar{d} - \underline{\mathrm{D}}, X_n\right) < \frac{\lambda_1^I}{p_H}$ for $X_n > X^*$ according to the definition of X^*. Then, for $X_n > X^*$,

$$-g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) > -\bar{d} + \underline{\mathrm{D}}. \qquad (3.2.55)$$

As a summary of (3.2.54) and (3.2.55),

$$\begin{aligned} &- g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big)\mathbb{I}_{X_n \in A_1^x} + (-\bar{d} + \underline{\mathrm{D}})\mathbb{I}_{X_n \in S - A_1^x} \\ &< -g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big). \end{aligned}$$

for $X_n > X^*$. Therefore,

$$\begin{aligned} &U_1\bigg(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\bigg)\mathbb{I}_{X_n \in A_1^x} \\ &\quad + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - A_1^x} \\ &< U_1\bigg(w - g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) - X_n\bigg) \end{aligned}$$

for $X_n > X^*$. This is equivalent to $\Delta < 0$. At the same time, $\frac{1}{M_n} < \frac{1}{M(X^*)}$. So, for $X_n > X^*$, we have $\frac{1}{M_n}\Delta > \frac{1}{M(X^*)}\Delta$.

Similar to (3.2.46), we obtain, under the condition (3.2.53),

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}}\frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}\Delta dF_Q^{(n-1)}(x)\right]$$

$$\geq E^Q\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}}\frac{1}{M(X^*)}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}$$

$$\left(\Delta\mathbb{I}_{\{X_n<X^*\}}+\Delta\mathbb{I}_{\{X_n>X^*\}}+\Delta\mathbb{I}_{\{X_n=X^*\}}\right)dF_Q^{(n-1)}(x)\Bigg]$$

$$=\sum_{n=1}^{N}\int_{x\in S^{n-1}}\frac{1}{M(X^*)}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}E^Q[\Delta]dF_Q^{(n-1)}(x). \qquad (3.2.56)$$

2.2.2

Consider $x\in S^{n-1}$ and $n=1,2,\cdots,N$ such that

$$\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M(X^*)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}>\frac{\lambda_1^I}{p_H}. \qquad (3.2.57)$$

Now (3.2.19) always exists because $\frac{\lambda_1^I}{p_H}>0$.

2.2.2.1

If $\frac{M_n dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)}\leq(\frac{\lambda_1^R}{p_L}-\frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}$, then $\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}\geq\frac{\lambda_1^I}{p_H}$. So, if $\frac{M_n dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)}\leq(\frac{\lambda_1^R}{p_L}-\frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}$,

$$-g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)},X_n\Big)\geq -g^{-1}\Big(\frac{\lambda_1^I}{p_H},X_n\Big).$$

It follows that, as a result of (3.2.52), if $\dfrac{M_n dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} \leq (\dfrac{\lambda_1^R}{p_L} - \dfrac{\lambda_1^I}{p_H})\dfrac{p_L}{\lambda_1^I}$,

$$\begin{aligned}&- g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big)\mathbb{I}_{X_n \in A_1^x} + (-\bar{d} + \underline{\mathrm{D}})\mathbb{I}_{X_n \in S - A_1^x} \\ &\geq -g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big).\end{aligned}$$

Therefore, if $\dfrac{M_n dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} \leq (\dfrac{\lambda_1^R}{p_L} - \dfrac{\lambda_1^I}{p_H})\dfrac{p_L}{\lambda_1^I}$,

$$\begin{aligned}&U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Bigg)\mathbb{I}_{X_n \in A_1^x} \\ &\quad + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n \in S - A_1^x} \\ &\geq U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) - X_n\Bigg).\end{aligned}$$

This is equivalent to $\Delta \geq 0$. At the same time, since $\dfrac{dF_P^{(n-1)}(x)}{M_n dF_Q^{(n-1)}(x)} \geq \dfrac{1}{(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}}$, we have $\dfrac{dF_P^{(n-1)}(x)}{M_n dF_Q^{(n-1)}(x)}\Delta \geq \dfrac{1}{(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}}\Delta$.

2.2.2.2

If $\dfrac{M_n dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} > (\dfrac{\lambda_1^R}{p_L} - \dfrac{\lambda_1^I}{p_H})\dfrac{p_L}{\lambda_1^I}$, then $\dfrac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} < \dfrac{\lambda_1^I}{p_H}$. So, if $\dfrac{M_n dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} > (\dfrac{\lambda_1^R}{p_L} - \dfrac{\lambda_1^I}{p_H})\dfrac{p_L}{\lambda_1^I}$,

$$-g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) < -g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big). \quad (3.2.58)$$

Recalling (3.2.57), we also have, if $\frac{M_n dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} > (\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}$,

$$\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M(X^*) dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} > \frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}.$$

The inequality above shows that $M_n > M(X^*)$ and thus $X_n > X^*$. From the definition of X^*, we see when $X_n > X^*$, $g(\bar{d} - \underline{\mathrm{D}}, X_n) < \frac{\lambda_1^I}{p_H}$. Considering $g(\bar{d} - \underline{\mathrm{D}}, X_n) < \frac{\lambda_1^I}{p_H}$ and (3.2.58) together, we obtain that

$$\begin{aligned} &- g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big)\mathbb{I}_{X_n \in A_1^x} + (-\bar{d} + \underline{\mathrm{D}})\mathbb{I}_{X_n \in S - A_1^x} \\ &< -g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) \end{aligned}$$

which will yield $\Delta < 0$. At the same time, since $\frac{dF_P^{(n-1)}(x)}{M_n dF_Q^{(n-1)}(x)} < \frac{1}{(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}}$, we have $\frac{dF_P^{(n-1)}(x)}{M_n dF_Q^{(n-1)}(x)}\Delta > \frac{1}{(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}}\Delta$.

Therefore, under the condition (3.2.57), we have

$$\begin{aligned} &E^Q\left[\sum_{n=1}^{N}\int_{x \in S^{n-1}} \frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}\Delta dF_Q^{(n-1)}(x)\right] \\ &\geq \frac{1}{(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}} E^Q\left[\sum_{n=1}^{N}\int_{x \in S^{n-1}} \Delta dF_Q^{(n-1)}(x)\right]. \end{aligned} \tag{3.2.59}$$

Combining the result (3.2.56) under the condition (3.2.53)and the result

(3.2.59) under the condition (3.2.57), we obtain

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}}\frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}\Delta dF_Q^{(n-1)}(x)\right]$$
$$\geq \sum_{n=1}^{N}\int_{x\in S^{n-1}}\frac{1}{M(X^*)}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}$$
$$E^Q[\Delta]\,\mathbb{I}_{\{\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M(X^*)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}\leq\frac{\lambda_1^I}{p_H}\}}dF_Q^{(n-1)}(x)$$
$$+\frac{1}{(\frac{\lambda_1^R}{p_L}-\frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}}\sum_{n=1}^{N}\int_{x\in S^{n-1}}E^Q[\Delta]\,\mathbb{I}_{\{\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M(X^*)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}>\frac{\lambda_1^I}{p_H}\}}dF_Q^{(n-1)}(x). \quad (3.2.60)$$

Note that (3.2.53) is equivalent to

$$\frac{1}{M(X^*)}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}-\frac{1}{(\frac{\lambda_1^R}{p_L}-\frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}}\leq 0. \quad (3.2.61)$$

Also note that Δ decreases when $\dfrac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)}$ increases, so does $E^Q[\Delta]$.

If $E^Q[\Delta]\leq 0$ when $\dfrac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)}=(\dfrac{\lambda_1^R}{p_L}-\dfrac{\lambda_1^I}{p_H})\dfrac{p_L}{\lambda_1^I M(X^*)}$, then $E^Q[\Delta]\leq 0$ when $\dfrac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)}\geq(\dfrac{\lambda_1^R}{p_L}-\dfrac{\lambda_1^I}{p_H})\dfrac{p_L}{\lambda_1^I M(X^*)}$ which is a transformation of $\dfrac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M(X^*)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}\leq\dfrac{\lambda_1^I}{p_H}$. That says, when (3.2.53) happens, $E^Q[\Delta] \leq 0$ and (3.2.61) is true. We continue (3.2.60) to obtain

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}\Delta dF_Q^{(n-1)}(x)\right]$$

$$\geq \frac{1}{(\frac{\lambda_1^R}{p_L}-\frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}}E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \Delta dF_Q^{(n-1)}(x)\right]$$

$$+\sum_{n=1}^{N}\int_{x\in S^{n-1}} \left(\frac{1}{M(X^*)}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}-\frac{1}{(\frac{\lambda_1^R}{p_L}-\frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}}\right)$$

$$E^Q\left[\Delta\right]\mathbb{I}_{\{\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M(X^*)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}\leq\frac{\lambda_1^I}{p_H}\}}dF_Q^{(n-1)}(x).$$

Because $E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \Delta dF_Q^{(n-1)}(x)\right]=0$ according to (3.2.18), we have

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}\Delta dF_Q^{(n-1)}(x)\right]$$

$$\geq\sum_{n=1}^{N}\int_{x\in S^{n-1}} \left(\frac{1}{M(X^*)}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}-\frac{1}{(\frac{\lambda_1^R}{p_L}-\frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}}\right)$$

$$E^Q\left[\Delta\right]\mathbb{I}_{\{\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M(X^*)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}\leq\frac{\lambda_1^I}{p_H}\}}dF_Q^{(n-1)}(x)$$

$$\geq 0.$$

Now (3.2.43) is reached.

If $E^Q[\Delta] > 0$ when $\frac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} = (\frac{\lambda_1^R}{p_L}-\frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I M(X^*)}$, then $G^* > (\frac{\lambda_1^R}{p_L}-\frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I M(X^*)}$. It also means $E^Q[\Delta] > 0$ when $\frac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} < (\frac{\lambda_1^R}{p_L}-\frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I M(X^*)}$ which is a transformation of $\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M(X^*)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} > \frac{\lambda_1^I}{p_H}$. That says, when (3.2.57) happens, $E^Q[\Delta] > 0$ and $\dfrac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} < G^*$. When (3.2.53) happens, split it into two situations: one is $\dfrac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} < G^*$ and it follows that $E^Q[\Delta] > 0$; the other one is $\dfrac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} \geq G^*$ and it follows that $E^Q[\Delta] \leq 0$.

$$
\begin{aligned}
&E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} \frac{1}{M_n}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}\Delta dF_Q^{(n-1)}(x)\right] \\
&\geq \sum_{n=1}^{N}\int_{x\in S^{n-1}} \frac{1}{M(X^*)}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)} \\
&\qquad\qquad E^Q\left[\Delta\right]\mathbb{I}_{\left\{\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M(X^*)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}\leq\frac{\lambda_1^I}{p_H} \text{ and } \frac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)}\geq G^*\right\}} dF_Q^{(n-1)}(x) \\
&\quad+\sum_{n=1}^{N}\int_{x\in S^{n-1}} \frac{1}{M(X^*)}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)} \\
&\qquad\qquad E^Q\left[\Delta\right]\mathbb{I}_{\left\{\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M(X^*)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}\leq\frac{\lambda_1^I}{p_H} \text{ and } \frac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)}< G^*\right\}} dF_Q^{(n-1)}(x) \\
&\quad+\sum_{n=1}^{N}\int_{x\in S^{n-1}} \frac{1}{(\frac{\lambda_1^R}{p_L}-\frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}} E^Q\left[\Delta\right]\mathbb{I}_{\left\{\frac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M(X^*)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}>\frac{\lambda_1^I}{p_H}\right\}} dF_Q^{(n-1)}(x).
\end{aligned}
\tag{3.2.62}
$$

When $\dfrac{\lambda_1^R dF_P^{(n-1)}(x)-\lambda_1^I M(X^*)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \leq \dfrac{\lambda_1^I}{p_H}$ and $\dfrac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} \geq G^*$,

$$
\frac{1}{M(X^*)}\frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)}E^Q[\Delta] \geq \frac{1}{M(X^*)}\frac{1}{G^*}E^Q[\Delta].
$$

When $\dfrac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M(X^*) dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \leq \dfrac{\lambda_1^I}{p_H}$ and $\dfrac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)} < G^*$,

$$\frac{1}{M(X^*)} \frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)} E^Q[\Delta] > \frac{1}{M(X^*)} \frac{1}{G^*} E^Q[\Delta].$$

When $\dfrac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M(X^*) dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} > \dfrac{\lambda_1^I}{p_H}$,

$$\frac{1}{(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}} > \frac{1}{M(X^*)} \frac{1}{G^*}$$

because $G^* > (\dfrac{\lambda_1^R}{p_L} - \dfrac{\lambda_1^I}{p_H}) \dfrac{p_L}{\lambda_1^I M(X^*)}$. It results that $\dfrac{1}{(\frac{\lambda_1^R}{p_L} - \frac{\lambda_1^I}{p_H})\frac{p_L}{\lambda_1^I}} E^Q[\Delta] >$ $\dfrac{1}{M(X^*)} \dfrac{1}{G^*} E^Q[\Delta]$. Therefore, from (3.2.62),

$$E^Q \left[\sum_{n=1}^{N} \int\limits_{x \in S^{n-1}} \frac{1}{M_n} \frac{dF_P^{(n-1)}(x)}{dF_Q^{(n-1)}(x)} \Delta dF_Q^{(n-1)}(x) \right]$$
$$\geq \frac{1}{M(X^*)} \frac{1}{G^*} \sum_{n=1}^{N} \int\limits_{x \in S^{n-1}} E^Q[\Delta] dF_Q^{(n-1)}(x) = 0.$$

Now, (3.2.43) is reached. We have traversed all the possible circumstances and proved in each circumstance, (3.1.8) is satisfied by (3.2.14)–(3.2.16). □

The reservation constraint (3.1.7) is the only constraint we have not considered. If the candidate solutions satisfy (3.1.7) too, they are automatically the optimal solutions to the original problem. This statement is expressed by the following theorem.

Theorem 5. *If (3.2.14)–(3.2.16) satisfy the constraint (3.1.7), then (3.2.14)–(3.2.16) are optimal solutions to the problem.*

However, we cannot guarantee the reservation constraint for type H is always satisfied by the candidate solutions. In the remaining part of this section, we will discuss the optimal solutions if (3.1.7) is not satisfied by (3.2.14)–(3.2.16).

For $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$, we define

$$A_2^x := \Big\{ s \in S; g(\bar{d} - \underline{\mathrm{D}}, s) \leq \frac{\lambda_2^R dF_P^{(n-1)}(x) - \lambda_2^I M(s) dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \Big\}.$$

The candidate solutions are

$$d_{L,n}^x = \bar{d}, \tag{3.2.63}$$

$$D_{L,n}^x = \begin{cases} \bar{d} - g^{-1}\left(\dfrac{\lambda_2^R dF_P^{(n-1)}(x) - \lambda_2^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n \right), & \text{if } X_n \in A_2^x \\ \underline{\mathrm{D}}, & \text{if } X_n \in S - A_2^x \end{cases} \tag{3.2.64}$$

$$d_{H,n}^x - D_{H,n}^x = g^{-1}\left(\frac{\lambda_3}{p_H}, X_n \right). \tag{3.2.65}$$

Here, λ_2^R and λ_2^I are such that

$$E^P\Bigg\{ \sum_{n=1}^N \int_{x \in S^{n-1}} \Bigg[U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_2^R dF_P^{(n-1)}(x) - \lambda_2^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n \Big) - X_n \Bigg) \mathbb{I}_{X_n \in A_2^x} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n \right) \mathbb{I}_{X_n \in S - A_2^x} \Bigg] dF_P^{(n-1)}(x) \Bigg\} = R_L, \tag{3.2.66}$$

$$E^Q\Bigg\{ \sum_{n=1}^N \int_{x \in S^{n-1}} \Bigg[U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_2^R dF_P^{(n-1)}(x) - \lambda_2^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n \Big) - X_n \Bigg) \mathbb{I}_{X_n \in A_2^x} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n \right) \mathbb{I}_{X_n \in S - A_2^x} \Bigg] dF_Q^{(n-1)}(x) \Bigg\} = R_H, \tag{3.2.67}$$

and λ_3 is such that

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w-g^{-1}\Big(\frac{\lambda_3}{p_H},X_n\Big)-X_n\Big)dF_Q^{(n-1)}(x)\right]=R_H. \quad (3.2.68)$$

As one of the equations to define λ_2^R and λ_2^I, (3.2.66) is not different from (3.2.17) which is an equation to define λ_1^R and λ_1^I. This similarity will help us simplify the proof of the following proposition.

Proposition 6. *Suppose that (3.2.14)–(3.2.16) do not satisfy (3.1.7). Then, there exist* $\lambda_2^R \in (0, \lambda_1^R)$ *and* $\lambda_2^I \in (0, \lambda_1^I)$ *such that (3.2.66) and (3.2.67) hold. Also, there exists* $\lambda_3 \in (\lambda_1^I, \infty)$ *such that (3.2.68) holds.*

Proof. From the discussion in Step 1 of the proof of Proposition 3, we know that for any fixed $\lambda_2^I \in (0,\infty)$, there exists $\lambda_2^R > 0$ such that (3.2.66) holds. Next we will show that among the pairs $(\lambda_2^R, \lambda_2^I)$ that satisfy (3.2.66), there exists a pair that satisfy (3.2.67). Similar to Step 2 in the proof of Proposition 3, we again start with the pair of variables (y, z) satisfying (3.2.25).

When $z = \lambda_1^I$, we have $y = \lambda_1^R$ to keep (3.2.25) holding. Because (3.2.14)–(3.2.16) do not satisfy (3.1.7), there is

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w-g^{-1}\Big(\frac{z}{p_H},X_n\Big)-X_n\Big)dF_Q^{(n-1)}(x)\right]<R_H.$$

Noticing (3.2.18), we have when $z = \lambda_1^I$,

$$E^Q\left\{\sum_{n=1}^{N}\int_{x\in S^{n-1}}\Big[U_1\left(w-\phi_2(y,z)-X_n\right)\mathbb{I}_{X_n\in B_2^x(y,z)} + U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-B_2^x(y,z)}\Big]dF_Q^{(n-1)}(x)\right\}<R_H. \quad (3.2.69)$$

Here, $\phi_2(y,z)$ and $B_2^x(y,z)$ are defined by (3.2.26) and (3.2.27). When $z\to 0$,

$\phi_2(y,z) = g^{-1}\Big(\dfrac{y}{p_L}, X_n\Big)$ almost surely under the probability measure P. So, (3.2.25) becomes

$$E^P\Bigg\{\sum_{n=1}^{N}\int_{x\in S^{n-1}}\Bigg[U_1\Bigg(w-g^{-1}\Big(\frac{y}{p_L},X_n\Big)-X_n\Bigg)\mathbb{I}_{X_n\in B_2^x(y,0)}$$
$$+U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-B_2^x(y,0)}\Bigg]dF_P^{(n-1)}(x)\Bigg\}=R_L. \tag{3.2.70}$$

Realizing that

$$E^P\Bigg[U_1\Bigg(w-g^{-1}\Big(\frac{y}{p_L},X_n\Big)-X_n\Bigg)\mathbb{I}_{X_n\in B_2^x(y,0)}+U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-B_2^x(y,0)}\Bigg]$$

is the same for each $n=1,2,\cdots,N$ and $x\in S^{n-1}$, we rewrite (3.2.70) as

$$NE^P\Bigg[U_1\Bigg(w-g^{-1}\Big(\frac{y}{p_L},X\Big)-X\Bigg)\mathbb{I}_{X\in B_2^x(y,0)}+U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X\right)\mathbb{I}_{X\in S-B_2^x(y,0)}\Bigg]$$
$$=NE^P[U_1(w-X)]. \tag{3.2.71}$$

Here, we derived the right-hand-side of the equation based on the definition of R_L. We define $X'\in\mathbb{R}$ by

$$X' := \operatorname*{argmin}_{s\in S}\left|g(0,s)-\frac{y}{p_L}\right|.$$

For $X<X'$, $g(0,X)>\dfrac{y}{p_L}$ and it yields $-g^{-1}(\dfrac{y}{p_L},X)<0$. On the other hand, for $X>X'$, $g(0,X)<\dfrac{y}{p_L}$ and it yields $-g^{-1}(\dfrac{y}{p_L},X)>0$. Because $g(\bar{d}-\underline{\mathrm{D}},X)<g(0,X)<\dfrac{y}{p_L}$, from the definition of $B_2^x(y,z)$, we say $X\in B_2^x(y,0)$ when $X>X'$. From the discussion above, we obtain the summary that

$$U_1\Bigg(w-g^{-1}\Big(\frac{y}{p_L},X\Big)-X\Bigg)\mathbb{I}_{X\in B_2^x(y,0)}+U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X\in S-B_2^x(y,0)}$$
$$\le U_1(w-X)$$

when $X < X'$ and

$$U_1\left(w - g^{-1}\Big(\frac{y}{p_L}, X\Big) - X\right)\mathbb{I}_{X\in B_2^x(y,0)} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X\in S-B_2^x(y,0)}$$
$$\geq U_1(w - X)$$

when $X > X'$. With (3.2.71), according to Lemma 4, we obtain

$$NE^Q\left[U_1\left(w - g^{-1}\Big(\frac{y}{p_L}, X\Big) - X\right)\mathbb{I}_{X\in B_2^x(y,0)} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X\in S-B_2^x(y,0)}\right]$$
$$\geq NE^Q[U_1(w - X)],$$

It is equivalent to

$$E^Q\left\{\sum_{n=1}^{N}\int_{x\in S^{n-1}}\left[U_1\left(w - g^{-1}\Big(\frac{y}{p_L}, X_n\Big) - X_n\right)\mathbb{I}_{X_n\in B_2^x(y(0),0)} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right)\mathbb{I}_{X_n\in S-B_2^x(y(0),0)}\right]dF_Q^{(n-1)}(x)\right\} \geq R_H. \tag{3.2.72}$$

Now we obtain (3.2.72)when $z \to 0$. Combining (3.2.69) and (3.2.72), we can state that there exist $\lambda_2^I \in (0, \lambda_1^I)$ and the corresponding λ_2^R such that (3.2.67) holds. Therefore, there is a pair of $(\lambda_2^R, \lambda_2^I)$ such that (3.2.66) and (3.2.67) hold.

From the discussion above, it is clear that $0 < \lambda_2^I < \lambda_1^I$ and $0 < \lambda_2^R$. Next we will illustrate that $\lambda_2^R < \lambda_1^R$. We know that when $y = \lambda_1^R$ and $z = \lambda_1^I$, (3.2.25) holds and (3.2.25) is not different from (3.2.17). If z decreases and becomes less than λ_1^I but y stays the same as λ_1^R, the left-hand-side of (3.2.25) becomes larger because $-g^{-1}(\cdot, y_2)$ and U_1 are increasing functions. Therefore, to keep (3.2.25) holding when $z < \lambda_1^I$, there must be $y < \lambda_1^R$ too.

Next we will prove the statement about λ_3. Consider the function of z,

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w-g^{-1}\Big(\frac{z}{p_H},X_n\Big)-X_n\Big)dF_Q^{(n-1)}(x)\right].$$

When $z=\lambda_1^I$, then

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w-g^{-1}\Big(\frac{z}{p_H},X_n\Big)-X_n\Big)dF_Q^{(n-1)}(x)\right]<R_H$$

because (3.2.16) does not satisfy (3.1.7). When $z\to\infty$, then $-g^{-1}\Big(\dfrac{z}{p_H},X_n\Big)\to\infty$ according to the definition of g. So,

$$\begin{aligned}&E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w-g^{-1}\Big(\frac{z}{p_H},X_n\Big)-X_n\Big)dF_Q^{(n-1)}(x)\right]\\&>E^Q\left[\sum_{n=1}^{N}U_1\Big(w-X_n\Big)\right]=R_H\end{aligned}$$

when $z\to\infty$. Therefore, there exists $\lambda_3\in(\lambda_1^I,\infty)$ such that

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w-g^{-1}\Big(\frac{\lambda_3}{p_H},X_n\Big)-X_n\Big)dF_Q^{(n-1)}(x)\right]=R_H.$$

□

Now, we will show that the candidate solutions, (3.2.63)–(3.2.65), satisfy all the constraints. (3.2.66) shows the contract (3.2.63)–(3.2.64) for type L is such that

$$E^P\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1(w-d_{L,n}^x+D_{L,n}^x-X_n)dF_P^{(n-1)}(x)\right]=R_L.$$

Therefore, the reservation constraint (3.1.6) for type L is satisfied. (3.2.68) shows the contract for type H from (3.2.65) is such that

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1(w-d^x_{H,n}+D^x_{H,n}-X_n)dF_Q^{(n-1)}(x)\right]=R_H. \tag{3.2.73}$$

Therefore, the reservation constraint (3.1.7) for type H is satisfied. (3.2.67) and (3.2.68) together show the solutions from (3.2.63)–(3.2.65) are such that

$$\begin{aligned}&E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1(w-d^x_{H,n}+D^x_{H,n}-X_n)dF_Q^{(n-1)}(x)\right]\\&=E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1(w-d^x_{L,n}+D^x_{L,n}-X_n)dF_Q^{(n-1)}(x)\right].\end{aligned} \tag{3.2.74}$$

Therefore, the incentive compatibility constraint (3.1.9) for type H is satisfied. The proposition below will show that the incentive compatibility constraint for type L is also satisfied by the candidate solutions.

Proposition 7. *The controls (3.2.63)–(3.2.65) satisfy (3.1.8).*

Proof. Plugging (3.2.63)–(3.2.65) into (3.1.8), we need to prove

$$\begin{aligned}E^P\Bigg\{\sum_{n=1}^{N}\int_{x\in S^{n-1}}&\Bigg[U_1\Bigg(w-g^{-1}\Big(\frac{\lambda_2^R dF_P^{(n-1)}(x)-\lambda_2^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)},X_n\Big)-X_n\Bigg)\mathbb{I}_{X_n\in A_2^x}\\&+U_1\left(w-\bar{d}+\underline{\mathrm{D}}-X_n\right)\mathbb{I}_{X_n\in S-A_2^x}\Bigg]dF_P^{(n-1)}(x)\Bigg\}\\\geq E^P&\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Bigg(w-g^{-1}\Big(\frac{\lambda_3}{p_L},X_n\Big)-X_n\Bigg)dF_P^{(n-1)}(x)\right].\end{aligned}$$

According to the relation presented by (3.2.66), we see it is sufficient to show

$$R_L\geq E^P\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Bigg(w-g^{-1}\Big(\frac{\lambda_3}{p_L},X_n\Big)-X_n\Bigg)dF_P^{(n-1)}(x)\right].$$

Note $U_1\bigg(w - g^{-1}\Big(\frac{\lambda_3}{p_L}, X_n\Big) - X_n\bigg)$ is the same for every $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. Also recalling the definition of R_L, we can rewrite the inequality above as

$$NE^P[U_1(w - X)] \geq NE^P\left[U_1\bigg(w - g^{-1}\Big(\frac{\lambda_3}{p_L}, X\Big) - X\bigg)\right].$$

Define

$$X'' := \underset{s \in S}{\operatorname{argmin}} \left| g(0, s) - \frac{\lambda_3}{p_H} \right|.$$

Since $g(y_1, y_2)$ is a decreasing function of y_2, $g(0, X) > \frac{\lambda_3}{p_H}$ when $X < X''$. Since $-g^{-1}(\cdot, y_2)$ is an increasing function, $-g^{-1}(\frac{\lambda_3}{p_H}, X) < 0$ when $X < X''$. So,

$$U_1(w - X) > U_1\bigg(w - g^{-1}\Big(\frac{\lambda_3}{p_L}, X\Big) - X\bigg)$$

when $X < X''$. Similarly, we have

$$U_1(w - X) < U_1\bigg(w - g^{-1}\Big(\frac{\lambda_3}{p_L}, X\Big) - X\bigg)$$

when $X < X''$. (3.2.68) is equivalent to

$$NE^Q[U_1(w - X)] = NE^Q\left[U_1\bigg(w - g^{-1}\Big(\frac{\lambda_3}{p_L}, X\Big) - X\bigg)\right].$$

Therefore, according to Lemma 4, we conclude that

$$NE^P[U_1(w - X)] \geq NE^P\left[U_1\bigg(w - g^{-1}\Big(\frac{\lambda_3}{p_L}, X\Big) - X\bigg)\right]$$

and (3.1.8) is satisfied. □

The optimality of the candidate solutions is proved by the following theorem.

Theorem 6. *Suppose that (3.2.14)–(3.2.16) do not satisfy (3.1.7). Then,* *(3.2.63)–(3.2.65) are the optimal solutions to the problem.*

Proof. The beginning of the proof is not much different from that of Proposition 4, so we will skip some details here. Let c_L, C_L, c_H, and C_H be the premium process for type L, compensation process for type L, premium process for type H, and compensation process for type H respectively. Suppose $(c_L, C_L, c_H, C_H) \in \mathcal{A}_2$ and satisfy all the constraints in the problem. We will compare the insurer's utilities from (3.2.63)–(3.2.65) and from (c_L, C_L, c_H, C_H).

Since U_2 is a concave function, we obtain

$$
\begin{aligned}
&\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H) \\
&\geq p_L E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} dF_P^{(n-1)}(x) U_2'(d_{L,n}^x - D_{L,n}^x) \left(d_{L,n}^x - D_{L,n}^x - c_{L,n}^x + C_{L,n}^x \right) \right] \\
&\quad + p_H E^Q \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} dF_Q^{(n-1)}(x) U_2'(d_{H,n}^x - D_{H,n}^x) \left(d_{H,n}^x - D_{H,n}^x - c_{H,n}^x + C_{H,n}^x \right) \right].
\end{aligned}
$$

From (3.2.63)–(3.2.64), we derive that when $X_n \in A_2^x$,

$$
\frac{U_2'(d_{L,n}^x - D_{L,n}^x)}{U_1'(w - d_{L,n}^x + D_{L,n}^x - X_n)} = \frac{\lambda_2^R dF_P^{(n-1)}(x) - \lambda_2^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}
$$

and when $X_n \in S - A_2^x$,

$$
\begin{aligned}
&\frac{U_2'(d_{L,n}^x - D_{L,n}^x) \left(d_{L,n}^x - D_{L,n}^x - c_{L,n}^x + C_{L,n}^x \right)}{U_1'(w - d_{L,n}^x + D_{L,n}^x - X_n)} \\
&\geq \frac{\lambda_2^R dF_P^{(n-1)}(x) - \lambda_2^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \left(d_{L,n}^x - D_{L,n}^x - c_{L,n}^x + C_{L,n}^x \right).
\end{aligned}
$$

From (3.2.65), we derive that

$$
\frac{U_2'(d_{H,n}^x - D_{H,n}^x)}{U_1'(w - d_{H,n}^x + D_{H,n}^x - X_n)} = \frac{\lambda_3}{p_H}. \tag{3.2.75}
$$

Then we obtian

$$
\begin{aligned}
&\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H) \\
&\geq E^P\left[\sum_{n=1}^{N} \int_{x\in S^{n-1}} U_1'(w - d_{L,n}^x + D_{L,n}^x - X_n)\left(d_{L,n}^x - D_{L,n}^x - c_{L,n}^x + C_{L,n}^x\right)\right. \\
&\qquad\qquad \left.\left(\lambda_2^R dF_P^{(n-1)}(x) - \lambda_2^I M_n dF_Q^{(n-1)}(x)\right)\right] \\
&\quad + E^Q\left[\sum_{n=1}^{N} \int_{x\in S^{n-1}} U_1'(w - d_{H,n}^x + D_{H,n}^x - X_n)\lambda_3\right. \\
&\qquad\qquad \left.\left(d_{H,n}^x - D_{H,n}^x - c_{H,n}^x + C_{H,n}^x\right) dF_Q^{(n-1)}(x)\right].
\end{aligned}
$$

Since U_1 is a concave function, we obtain

$$
\begin{aligned}
&\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H) \\
&\geq E^P\left[\sum_{n=1}^{N} \int_{x\in S^{n-1}} \left(U_1(w - c_{L,n}^x + C_{L,n}^x - X_n) - U_1(w - d_{L,n}^x + D_{L,n}^x - X_n)\right)\right. \\
&\qquad\qquad \left.\lambda_2^R dF_P^{(n-1)}(x)\right] \\
&\quad - E^P\left[\sum_{n=1}^{N} \int_{x\in S^{n-1}} \left(U_1(w - c_{L,n}^x + C_{L,n}^x - X_n) - U_1(w - d_{L,n}^x + D_{L,n}^x - X_n)\right)\right. \\
&\qquad\qquad \left.\lambda_2^I M_n dF_Q^{(n-1)}(x)\right] \\
&\quad + E^Q\left[\sum_{n=1}^{N} \int_{x\in S^{n-1}} \left(U_1(w - c_{H,n}^x + C_{H,n}^x - X_n) - U_1(w - d_{H,n}^x + D_{H,n}^x - X_n)\right)\right. \\
&\qquad\qquad \left.\lambda_3 dF_Q^{(n-1)}(x)\right].
\end{aligned}
$$

Applying (3.1.5) to the inequality above, we obtain

$$
\begin{aligned}
&\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H) \\
&\geq \lambda_2^R E^P \Bigg[\sum_{n=1}^{N} \int_{x \in S^{n-1}} \big(U_1(w - c_{L,n}^x + C_{L,n}^x - X_n) - U_1(w - d_{L,n}^x + D_{L,n}^x - X_n) \big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad dF_P^{(n-1)}(x) \Bigg] \\
&\quad - E^Q \Bigg[\sum_{n=1}^{N} \int_{x \in S^{n-1}} \lambda_2^I \big(U_1(w - c_{L,n}^x + C_{L,n}^x - X_n) - U_1(w - d_{L,n}^x + D_{L,n}^x - X_n) \big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad dF_Q^{(n-1)}(x) \Bigg] \\
&\quad + E^Q \Bigg[\sum_{n=1}^{N} \int_{x \in S^{n-1}} \lambda_3 \big(U_1(w - c_{H,n}^x + C_{H,n}^x - X_n) - U_1(w - d_{H,n}^x + D_{H,n}^x - X_n) \big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad dF_Q^{(n-1)}(x) \Bigg].
\end{aligned}
$$

Recalling that $\lambda_2^R > 0$ and

$$
\begin{aligned}
E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_1(w - c_{L,n}^x + C_{L,n}^x - X_n) dF_P^{(n-1)}(x) \right] &\geq R_L, \\
E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_1(w - d_{L,n}^x + D_{L,n}^x - X_n) dF_P^{(n-1)}(x) \right] &= R_L,
\end{aligned}
$$

we have

$$\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H)$$
$$\geq -E^Q\Bigg[\sum_{n=1}^{N} \int_{x\in S^{n-1}} \lambda_2^I \big(U_1(w - c_{L,n}^x + C_{L,n}^x - X_n) - U_1(w - d_{L,n}^x + D_{L,n}^x - X_n)\big) dF_Q^{(n-1)}(x)\Bigg]$$
$$+ E^Q\Bigg[\sum_{n=1}^{N} \int_{x\in S^{n-1}} \lambda_3 \big(U_1(w - c_{H,n}^x + C_{H,n}^x - X_n) - U_1(w - d_{H,n}^x + D_{H,n}^x - X_n)\big) dF_Q^{(n-1)}(x)\Bigg].$$

Rearrange the terms to obtain

$$\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H)$$
$$\geq \lambda_2^I E^Q\Bigg[\sum_{n=1}^{N} \int_{x\in S^{n-1}} \big(U_1(w - c_{H,n}^x + C_{H,n}^x - X_n) - U_1(w - c_{L,n}^x + C_{L,n}^x - X_n)\big) dF_Q^{(n-1)}(x)\Bigg] \tag{3.2.76}$$
$$+ \lambda_2^I E^Q\Bigg[\sum_{n=1}^{N} \int_{x\in S^{n-1}} \big(U_1(w - d_{L,n}^x + D_{L,n}^x - X_n) - U_1(w - d_{H,n}^x + D_{H,n}^x - X_n)\big) dF_Q^{(n-1)}(x)\Bigg] \tag{3.2.77}$$
$$+ E^Q\Bigg[\sum_{n=1}^{N} \int_{x\in S^{n-1}} \big(U_1(w - c_{H,n}^x + C_{H,n}^x - X_n) - U_1(w - d_{H,n}^x + D_{H,n}^x - X_n)\big) (\lambda_3 - \lambda_2^I) dF_Q^{(n-1)}(x)\Bigg].$$

Noting Constraint (3.1.9) and $\lambda_2^I > 0$, we obtain the term (3.2.76) is nonnegative. Also noting the equation (3.2.74), we obtain the term (3.2.77) is 0. So,

$$\begin{aligned}
&\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H) \\
&\geq (\lambda_3 - \lambda_2^I)E^Q\left[\sum_{n=1}^{N} \int_{x\in S^{n-1}} \Big(U_1(w - c_{H,n}^x + C_{H,n}^x - X_n) \right. \\
&\qquad\qquad \left. - U_1(w - d_{H,n}^x + D_{H,n}^x - X_n)\Big) dF_Q^{(n-1)}(x)\right].
\end{aligned}$$

Because $\lambda_3 > \lambda_1^I$ but $\lambda_2^I < \lambda_1^I$, then $\lambda_3 - \lambda_2^I > 0$. Constraint (3.1.7) tells us that

$$E^Q\left[\sum_{n=1}^{N} \int_{x\in S^{n-1}} U_1(w - c_{H,n}^x + C_{H,n}^x - X_n) dF_Q^{(n-1)}(x)\right] \geq R_H.$$

Comparing the expression above and (3.2.73), we obtain

$$\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H) \geq 0.$$

The proof is finished. □

3.3 The Boundaries of Premium and Compensation

Different from the traditional model, we introduce the boundary $\bar{d}$ for premium and the boundary $\underline{\mathrm{D}}$ for compensation. In this section, we will see how $\bar{d}$ and $\underline{\mathrm{D}}$ affect the insurer's utility. From there, we will explain why these boundaries are necessary for our model. To achieve our goal, we will first demonstrate the conditions under which $\bar{d}$ and $\underline{\mathrm{D}}$ are involved in the the insurer's utility. Then we discuss the changes it will cause on the insurer's utility when $\bar{d}$ and $\underline{\mathrm{D}}$ change.

From the solutions presented in the previous section, we see that the solutions involve $\bar{d}$ and $\underline{\mathrm{D}}$. But in the problem, the premium and compensation always appear together in the form of $d^x_{L,n} - D^x_{L,n}$. If we plug the solutions into the objective of the problem, the insurer's utility involves $\bar{d}$ and $\underline{\mathrm{D}}$ only when $S - A^x_1$ and $S - A^x_2$ are not empty for some $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. If $S - A^x_1$ and $S - A^x_2$ are empty sets for every $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$, then the boundaries will not appear in the objective, not even affect it. If the boundaries do appear in the objective, they are also in the form of $\bar{d} - \underline{\mathrm{D}}$. Thus, our analysis will focus on how $\bar{d} - \underline{\mathrm{D}}$ affects the insurer's utility. Note the definitions of sets A^x_1 and A^x_2 and note that $g(y_1, y_2)$ is a decreasing function of y_1. In intuition, $S - A^x_1$ and $S - A^x_2$ will be empty sets if $\bar{d} - \underline{\mathrm{D}}$ is big enough. We will discuss the conditions under which $S - A^x_1$ and $S - A^x_2$ can be empty sets for every $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$.

3.3.1 The set $S - A^x_1$

In the previous section, we showed two circumstances under which we obtain different solutions to the problem. We start with the first circumstance where (3.2.14)–(3.2.16) present the solutions. The solutions involve $S - A^x_1$ and we will discuss when we can have $S - A^x_1 = \emptyset$ for every $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. Consider (3.2.17)–(3.2.18) with $S - A^x_1 = \emptyset$ for every $x \in S^{n-1}$ and $n =$

$1, 2, \cdots, N$.

$$E^P\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w - g^{-1}\big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\big) - X_n\Big) dF_P^{(n-1)}(x)\right] = R_L, \tag{3.3.78}$$

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w - g^{-1}\big(\frac{\lambda_1^I}{p_H}, X_n\big) - X_n\Big) dF_Q^{(n-1)}(x)\right]$$
$$= E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w - g^{-1}\big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\big) - X_n\Big) dF_Q^{(n-1)}(x)\right]. \tag{3.3.79}$$

Unlike (3.2.17)–(3.2.18), there is not a general statement that there always exist $\lambda_1^R > 0$ and $\lambda_1^I > 0$ such that (3.3.78)–(3.3.79) hold. We define $\bar{M} := \lim_{s\to\sup S} M(s)$. Because $M(s)$ strictly increases as s increases, we know $\bar{M} \geq M(X)$ for every $\omega \in \Omega$. We categorize $\bar{M}$ to the following cases and analyze if there are $\lambda_1^R > 0$ and $\lambda_1^I > 0$ such that (3.3.78)–(3.3.79) hold.

• $\bar{M}$ is finite but $P\{M(X) = \bar{M}\} \neq 0$ and $Q\{M(X) = \bar{M}\} \neq 0$.
In this case, there always exist $\lambda_1^R > 0$ and $\lambda_1^I > 0$ such that (3.3.78)–(3.3.79) hold. The complete illustration of this property is shown below.

Property 1. *If $\bar{M}$ is finite but $P\{M(X) = \bar{M}\} \neq 0$ and $Q\{M(X) = \bar{M}\} \neq 0$, then there exist $\lambda_1^R > 0$ and $\lambda_1^I > 0$ such that (3.3.78)–(3.3.79) hold.*

Proof. **Step 1.** We will show that for any fixed $\lambda_1^I \in (0, \infty)$, we have $\lambda_1^R \in (\lambda_1^I(\bar{M})^N, \infty)$ such that (3.3.78) holds.

When $y = \lambda_1^I(\bar{M})^N$ for every $\omega \in \Omega$, $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$,

$$\begin{aligned}
&\frac{ydF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \\
&= \frac{\lambda_1^I(\bar{M})^N}{p_L} - \frac{\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \\
&= \frac{\lambda_1^I(\bar{M})^N}{p_L} - \frac{\lambda_1^I M_N dF_Q(x_1) dF_Q(x_2) \cdots dF_Q(x_{N-1})}{p_L dF_P(x_1) dF_P(x_2) \cdots dF_P(x_{N-1})} \\
&= \frac{\lambda_1^I(\bar{M})^N}{p_L} - \frac{\lambda_1^I M_N M(x_1) M(x_2) \cdots M(x_{N-1})}{p_L} \\
&\geq 0.
\end{aligned}$$

So, the expression $g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big)$ always exists. When $y = \lambda_1^I(\bar{M})^N$ for every $\omega \in \Omega$, $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$,

$$\frac{ydF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} = \frac{\lambda_1^I(\bar{M})^N}{p_L} - \frac{\lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \leq \frac{\lambda_1^I(\bar{M})^N}{p_L}.$$

So, noting $-g^{-1}(\cdot, y_2)$ is an increasing function and noting Lemma 6, we have for every $\omega \in \Omega$, $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$,

$$\begin{aligned}
&U_1\left(w - g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\right) \\
&\leq U_1\left(w - g^{-1}\Big(\frac{\lambda_1^I(\bar{M})^N}{p_L}, X_n\Big) - X_n\right) \\
&\leq U_1\left(w - g^{-1}\Big(\frac{\lambda_1^I(\bar{M})^N}{p_L}, 0\Big)\right) < \infty. \qquad (3.3.80)
\end{aligned}$$

However, with the specific path $x_1, x_2, \cdots, x_{N-1} = \sup S$, and the specific value of random variable $X_N = \sup S$, we have

$$\begin{aligned}\frac{ydF_P^{(N-1)}(x) - \lambda_1^I M_N dF_Q^{(N-1)}(x)}{p_L dF_P^{(N-1)}(x)} &= \frac{\lambda_1^I(\bar{M})^N}{p_L} - \frac{\lambda_1^I M_N M(x_1)M(x_2)\cdots M(x_{N-1})}{p_L} \\ &= \frac{\lambda_1^I(\bar{M})^N}{p_L} - \frac{\lambda_1^I(\bar{M})^N}{p_L} \\ &= 0,\end{aligned}$$

and it yields

$$g^{-1}\Big(\frac{ydF_P^{(N-1)}(x) - \lambda_1^I M_N dF_Q^{(N-1)}(x)}{p_L dF_P^{(N-1)}(x)}, X_N\Big) \to \infty$$

if $x_1, x_2, \cdots, x_{N-1} = \sup S$, and $X_N = \sup S$. Consequently,

$$U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(N-1)}(x) - \lambda_1^I M_N dF_Q^{(N-1)}(x)}{p_L dF_P^{(N-1)}(x)}, X_N\Big) - X_N\Bigg) \to -\infty.$$

Noting $P(M = \bar{M}) \neq 0$, we have $P(X = \sup S) \neq 0$. Further, we have $dF_P^{(N-1)}(x) = P(X_1 = \sup S)P(X_2 = \sup S)\cdots P(X_{N-1} = \sup S) > 0$ and $P(X_N = \sup S) > 0$. So, with (3.3.80), we obtain

$$E^P\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Bigg(w - g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Bigg) dF_P^{(n-1)}(x)\Bigg] \to -\infty$$

and consequently,

$$E^P\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Bigg(w - g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Bigg) dF_P^{(n-1)}(x)\Bigg] < R_L \tag{3.3.81}$$

when $y = \lambda_1^I (\bar{M})^N$.

When $y \to \infty$, then $\dfrac{y dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \to \infty$ and $g^{-1}\Big(\dfrac{y dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) \to -\infty$ for every $x \in S^{n-1}$, $\omega \in \Omega$, and $n = 1, 2, \cdots, N$. Then,

$$U_1\bigg(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\bigg) > U_1(w - X_n)$$

for every $\omega \in \Omega$, $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$. So,

$$E^P\Bigg[\sum_{n=1}^{N} \int\limits_{x \in S^{n-1}} U_1\bigg(w - g^{-1}\Big(\frac{y dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\bigg) dF_P^{(n-1)}(x)\Bigg]$$

$$> E^P\left[\sum_{n=1}^{N} U_1(w - X_n)\right] = R_L. \tag{3.3.82}$$

From (3.3.81) and (3.3.82), we get for any $\lambda_1^I \in (0, \infty)$, there exists $\lambda_1^R \in (\lambda_1^I (\bar{M})^N, \infty)$ such that (3.3.78) holds.

Step 2. We will show that among the pairs $(\lambda_1^R, \lambda_1^I)$ that satisfy (3.3.78), we can always find one pair that satisfies (3.3.79). Let $z \in (0, \infty)$ and $y \in (z(\bar{M})^N, \infty)$ be the pairs that satisfy

$$E^P\Bigg[\sum_{n=1}^{N} \int\limits_{x \in S^{n-1}} U_1\bigg(w - g^{-1}\Big(\frac{y dF_P^{(n-1)}(x) - z M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\bigg) dF_P^{(n-1)}(x)\Bigg] = R_L. \tag{3.3.83}$$

When $z = 0$, it is obvious that

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w - g^{-1}\Big(\frac{z}{p_H}, X_n\Big) - X_n\Big)dF_Q^{(n-1)}(x)\right] \to -\infty. \quad (3.3.84)$$

Since $z = 0$, then (3.3.83) becomes

$$E^P\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w - g^{-1}\Big(\frac{y}{p_L}, X_n\Big) - X_n\Big)dF_P^{(n-1)}(x)\right] = R_L,$$

which is equivalent to

$$NE^P\left[U_1\Big(w - g^{-1}\Big(\frac{y}{p_L}, X\Big) - X\Big)\right] = NE^P\left[U_1\Big(w - X\Big)\right]. \quad (3.3.85)$$

Reminding that $g(y_1, y_2)$ is a strictly decreasing function of y_1 and y_2, we get $g^{-1}\Big(\frac{y}{p_L}, X\Big)$ decreases strictly when X increases. That means

$$U_1\Big(w - X\Big) - U_1\Big(w - g^{-1}\Big(\frac{y}{p_L}, X\Big) - X\Big)$$

decreases strictly when X increases. According to Corollary 4, we obtain

$$\begin{aligned}&E^P\left[U_1\Big(w - X\Big) - U_1\Big(w - g^{-1}\Big(\frac{y}{p_L}, X\Big) - X\Big)\right]\\ &> E^Q\left[U_1\Big(w - X\Big) - U_1\Big(w - g^{-1}\Big(\frac{y}{p_L}, X\Big) - X\Big)\right].\end{aligned}$$

Comparing (3.3.85) and the inequality above, we have

$$E^Q\left[U_1\Big(w - g^{-1}\Big(\frac{y}{p_L}, X\Big) - X\Big)\right] > E^Q\left[U_1\Big(w - X\Big)\right],$$

which yields

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w-g^{-1}\Big(\frac{y}{p_L},X_n\Big)-X_n\Big)dF_Q^{(n-1)}(x)\right]$$
$$>E^Q\left[\sum_{n=1}^{N}U_1\Big(w-X\Big)\right].$$

Thus, when $z=0$, we obtain

$$E^Q\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w-g^{-1}\Big(\frac{ydF_P^{(n-1)}(x)-zM_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)},X_n\Big)-X_n\Big)$$
$$dF_Q^{(n-1)}(x)\Bigg]>R_H. \tag{3.3.86}$$

Recalling (3.3.84), we obtain

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w-g^{-1}\Big(\frac{z}{p_H},X_n\Big)-X_n\Big)dF_Q^{(n-1)}(x)\right]$$
$$<E^Q\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w-g^{-1}\Big(\frac{ydF_P^{(n-1)}(x)-zM_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)},X_n\Big)-X_n\Big)$$
$$dF_Q^{(n-1)}(x)\Bigg] \tag{3.3.87}$$

when $z=0$.

When $z\to\infty$, it is obvious that $g^{-1}\Big(\frac{z}{p_H},X_n\Big)\to-\infty$ for every $\omega\in\Omega$ and $n=1,2,\cdots,N$. From Lemma 6, we get that

$$-g^{-1}\left(\frac{ydF_P^{(n-1)}(x)-zM(s)dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)},s\right)-s$$

is a decreasing function of s for $s \in S$. So, according to Corollary 3, we obtain

$$E^P\left[U_1\Big(w - g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - zM_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Big)\right] \geq E^Q\left[U_1\Big(w - g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - zM_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Big)\right]$$

for $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. It yields

$$E^P\left[\int_{x\in S^{n-1}} U_1\Big(w - g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - zM_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Big) dF_Q^{(n-1)}(x)\right] \geq E^Q\left[\int_{x\in S^{n-1}} U_1\Big(w - g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - zM_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Big) dF_Q^{(n-1)}(x)\right]. \tag{3.3.88}$$

$E^P\left[U_1\Big(w - g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - zM_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Big)\right]$ is a decreasing function of $\frac{dF_Q^{(n-1)}(x)}{dF_P^{(n-1)}(x)}$, so according to Lemma 5, we have

$$E^P\left[\int_{x\in S^{n-1}} U_1\Big(w - g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - zM_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Big) dF_P^{(n-1)}(x)\right] > E^P\left[\int_{x\in S^{n-1}} U_1\Big(w - g^{-1}\Big(\frac{ydF_P^{(n-1)}(x) - zM_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Big) dF_Q^{(n-1)}(x)\right]. \tag{3.3.89}$$

Considering (3.3.88) and (3.3.89) together, we obtain

$$E^P\left[\int_{x\in S^{n-1}} dF_P^{(n-1)}(x)U_1\bigg(w-g^{-1}\Big(\frac{ydF_P^{(n-1)}(x)-zM_ndF_Q^{(n-1)}(x)}{p_LdF_P^{(n-1)}(x)},X_n\Big)-X_n\bigg)\right]$$
$$> E^Q\Bigg[\int_{x\in S^{n-1}} dF_Q^{(n-1)}(x)U_1\bigg(w-g^{-1}\Big(\frac{ydF_P^{(n-1)}(x)-zM_ndF_Q^{(n-1)}(x)}{p_LdF_P^{(n-1)}(x)},X_n\Big)$$
$$-X_n\bigg)\Bigg]$$

for $n=1,2,\cdots,N$. It follows that

$$E^Q\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\bigg(w-g^{-1}\Big(\frac{ydF_P^{(n-1)}(x)-zM_ndF_Q^{(n-1)}(x)}{p_LdF_P^{(n-1)}(x)},X_n\Big)-X_n\bigg)$$
$$dF_Q^{(n-1)}(x)\Bigg]$$
$$< E^P\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\bigg(w-g^{-1}\Big(\frac{ydF_P^{(n-1)}(x)-zM_ndF_Q^{(n-1)}(x)}{p_LdF_P^{(n-1)}(x)},X_n\Big)-X_n\bigg)$$
$$dF_P^{(n-1)}(x)\Bigg]=R_L.$$

It is evident that

$$R_L < E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} dF_Q^{(n-1)}(x)U_1\bigg(w-g^{-1}\Big(\frac{z}{p_H},X_n\Big)-X_n\bigg)\right]$$

when $z \to \infty$. So,

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w - g^{-1}\Big(\frac{z}{p_H}, X_n\Big) - X_n\Big) dF_Q^{(n-1)}(x)\right]$$
$$> E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w - g^{-1}\Big(\frac{y dF_P^{(n-1)}(x) - z M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Big) dF_Q^{(n-1)}(x)\right] \tag{3.3.90}$$

when $z \to \infty$. (3.3.87) and (3.3.90) together show that there exist $\lambda_1^I \in (0, \infty)$ and the corresponding λ_1^R such that

$$E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w - g^{-1}\Big(\frac{\lambda_1^I}{p_H}, X_n\Big) - X_n\Big) dF_Q^{(n-1)}(x)\right]$$
$$= E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Big(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Big) dF_Q^{(n-1)}(x)\right].$$

We state now there exist $\lambda_1^R > 0$ and $\lambda_1^I > 0$ such that (3.3.78) and (3.3.79) hold. □

• $\bar{M}$ is infinite.

g is a function that takes only positive values, so $\dfrac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}$ has to be positive to make the term $g^{-1}\left(\dfrac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\right)$ exist. However, in this case, there is always some $\omega \in \Omega$ such that

$$\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} < 0$$

for each $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$ despite the values of λ_1^R and λ_1^I. Automatically, (3.3.78)–(3.3.79) do not hold.

• $\bar{M}$ is finite but $P\{M(X) = \bar{M}\} = 0$ and $Q\{M(X) = \bar{M}\} = 0$.
Depending on the values of parameters and utility functions, there may not may not $\lambda_1^R > 0$ and $\lambda_1^I > 0$ such that (3.3.78)–(3.3.79) hold.

We will show that if there exist $\lambda_1^R > 0$ and $\lambda_1^I > 0$ such that (3.3.78)–(3.3.79) hold, then the traditional adverse selection model with self selection can still sort out the types effectively. In (3.2.15), when $X_n \in A_1^x$, the optimal premium is part of the optimal compensation. It does not matter what the premium and compensation are individually on A_1^x but their difference matters. If λ_1^R and λ_1^I are such that (3.3.78)–(3.3.79) hold, then $S - A_1^x = \emptyset$ for every $s \in S$, every $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$. We consider a special case of (3.2.14)–(3.2.15) where $S - A_1^x = \emptyset$. We take

$$d_{L,n}^x - D_{L,n}^x = g^{-1}\left(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\right) \tag{3.3.91}$$

as the solution for type L and keep (3.2.16) as the solution for type H. (3.3.78)–(3.3.79) can also be considered as a special case of (3.2.17)–(3.2.18) where $S - A_1^x = \emptyset$ for every $s \in S$, every $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$. Thus, (3.3.91) and (3.2.16) are the optimal solutions for the traditional model. The insureds are distinguished through their selection of contracts. As a result, the boundaries are not needed.

If there are not λ_1^R and λ_1^I such that (3.3.78)–(3.3.79) hold, there will not be $S - A_1^x = \emptyset$ for every $s \in S$, every $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$. Then the boundaries will appear in the insurer's utility and the value of $\bar{d} - \underline{\mathrm{D}}$ affects the insurer's utility.

3.3.2 The set $S - A_2^x$

Next, we consider the second circumstance where the solutions are presented by (3.2.63)–(3.2.65). The solutions involve the set $S - A_2^x$ and we will discuss when we can have $S - A_2^x = \emptyset$ for every $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. With $S - A_2^x = \emptyset$ for every $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$, (3.2.66)–(3.2.67) become

$$E^P\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_2^R dF_P^{(n-1)}(x) - \lambda_2^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Bigg) dF_P^{(n-1)}(x)\Bigg] = R_L, \tag{3.3.92}$$

$$E^Q\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_2^R dF_P^{(n-1)}(x) - \lambda_2^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Bigg) dF_Q^{(n-1)}(x)\Bigg] = R_H. \tag{3.3.93}$$

It is important to remind that the solutions (3.2.63)–(3.2.65) are taken because the solutions (3.2.14)–(3.2.16) do not match (3.1.7). This is the condition when we consider the solutions (3.2.63)–(3.2.65) and the set $S - A_2^x$ in them. This condition means (3.2.69) is true. We rewrite (3.2.69) as

$$E^Q\Bigg\{\sum_{n=1}^{N}\int_{x\in S^{n-1}} \Bigg[U_1\Bigg(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\Bigg) \mathbb{I}_{X_n\in A_1^x} + U_1\left(w - \bar{d} + \underline{\mathrm{D}} - X_n\right) \mathbb{I}_{X_n\in S-A_1^x}\Bigg] dF_Q^{(n-1)}(x)\Bigg\} < R_H.$$

If (3.3.78)–(3.3.79) hold, then $S - A_1^x = \emptyset$. The expression above becomes

$$E^Q\left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_1\left(w - g^{-1}\Big(\frac{\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\Big) - X_n\right) dF_Q^{(n-1)}(x)\right] < R_H. \tag{3.3.94}$$

Keeping the inequality (3.3.94) in mind, we can show

Lemma 8. *If there exist $\lambda_1^R > 0$ and $\lambda_1^I > 0$ such that (3.3.78)–(3.3.79) hold and if (3.3.94) is true, then there are $\lambda_2^R \in (0, \lambda_1^R)$ and $\lambda_2^I \in (0, \lambda_1^I)$ such that (3.3.92)–(3.3.93) hold.*

Proof. Some of the proof is similar to that of Property 1, so we will skip some details. We will show that there exist λ_2^R and λ_2^R such that (3.3.92) and (3.3.93) are true. Since $\mathbb{D}_1 \neq \emptyset$, there exist λ_1^R and λ_1^I such that (3.3.78) and (3.3.79) hold.

Like Step 1 in the proof of Property 1, we know that for any fixed $\lambda_2^I \in (0, \lambda_1^I)$, there exists $\lambda_2^R > 0$ such that (3.3.92) holds. Next we will show that among the pairs $(\lambda_2^R, \lambda_2^I)$ that satisfy (3.3.92), there is a pair that satisfies (3.3.93). Similar to Step 2 in the proof of Property 1, we let $z \in (0, \infty)$ and $y \in (z(\bar{M})^N, \infty)$ be the pairs that satisfy (3.3.83).

When $z = 0$, then we obtain (3.3.86) like in the proof of Property 1. When $z = \lambda_1^I$, then $y = \lambda_1^R$ to keep (3.3.83) holding. Then we have (3.3.94). Considering (3.3.86) and (3.3.94) together, we state that there exist $\lambda_2^I \in (0, \lambda_1^I)$ and the corresponding λ_2^R such that (3.3.93) holds. Therefore, there is a pair of $(\lambda_2^R, \lambda_2^I)$ such that (3.3.92) and (3.3.93) hold.

From the discussion above, it is clear that $0 < \lambda_2^I < \lambda_1^I$ and $0 < \lambda_2^R$. Next we will illustrate that $\lambda_2^R < \lambda_1^R$. We know when $y = \lambda_1^R$ and $z = \lambda_1^I$, (3.3.83) holds. If z decreases and becomes less than λ_1^I but y stays the same as λ_1^R, the left-hand-side of (3.3.83) becomes larger because $-g^{-1}(\cdot, y_2)$ and U_1 are both increasing functions. Thus, to keep (3.3.83) holding when z decreases below λ_1^I, there must be $y < \lambda_1^R$. □

• $\bar{M}$ is finite but $P\{M(X) = \bar{M}\} \neq 0$ and $Q\{M(X) = \bar{M}\} \neq 0$.
In this case, as the result of Lemma 8, there are $\lambda_2^R \in (0, \lambda_1^R)$ and $\lambda_2^I \in (0, \lambda_1^I)$ such that (3.3.92)–(3.3.93) hold.

• $\bar{M}$ is infinite.
In this case, there are not $\lambda_2^R \in (0, \lambda_1^R)$ and $\lambda_2^I \in (0, \lambda_1^I)$ such that (3.3.92)–(3.3.93) hold. The reason is the same as that when we discuss $S - A_1^x$ in the case of infinite $\bar{M}$.

• $\bar{M}$ is finite but $P\{M(X) = \bar{M}\} = 0$ and $Q\{M(X) = \bar{M}\} = 0$.
Depending on the values of parameters and utility functions, there may or may not $\lambda_2^R \in (0, \lambda_1^R)$ and $\lambda_2^I \in (0, \lambda_1^I)$ such that (3.3.92)–(3.3.93) hold.

Similar to the discussion in Subsection 4.1, if there exist λ_2^R and λ_2^I such that (3.3.78)–(3.3.79) hold, then we will show the traditional adverse selection model with self selection can still sort out the types effectively. Suppose λ_2^R and λ_2^I satisfy (3.3.78)–(3.3.79), then $S - A_2^x = \emptyset$ for every $s \in S$, every $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$. We take

$$d_{L,n}^x - D_{L,n}^x = g^{-1}\left(\frac{\lambda_2^R dF_P^{(n-1)}(x) - \lambda_2^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\right) \tag{3.3.95}$$

as the solution for type L. The solution for type H is still (3.2.65). Then, (3.3.95) and (3.2.65) are the optimal solutions to the traditional model. The insureds' types are distinguished. As a result, the boundaries are not needed.

If there are not λ_2^R and λ_2^I such that (3.3.78)–(3.3.79) hold, there will not be $S - A_2^x = \emptyset$ for every $s \in S$, every $x \in S^{n-1}$, and $n = 1, 2, \cdots, N$. Then the boundaries will appear in the insurer's utility and the value of $\bar{d} - \underline{\mathrm{D}}$ affects the insurer's utility.

Theorem 7. *(i) If $S - A_1^x \neq \emptyset$ for some $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$, then the insurer's optimal utility from (3.2.14)–(3.2.16) increases when $\bar{d} - \underline{D}$ increases. (ii) If $S - A_2^x \neq \emptyset$ for some $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$, then the insurer's optimal utility from (3.2.63)–(3.2.65) increases when $\bar{d} - \underline{D}$ increases.*

Proof. (i) The proof has considerable similarities with that of Proposition 4, so we will skip some details. Let $\bar{d}'$ be an upper bound of the premium and let $\underline{\mathrm{D}}'$ be a lower bound of the compensation. Suppose that $\bar{d} - \underline{\mathrm{D}} > \bar{d}' - \underline{\mathrm{D}}'$. The solutions are (3.2.14)–(3.2.16) when the boundaries are $\bar{d}$ and $\underline{\mathrm{D}}$. We suppose $S - A_1^x \neq \emptyset$ for some $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. When the boundaries are $\bar{d}'$ and $\underline{\mathrm{D}}'$, like (3.2.14)–(3.2.16), we denote the solutions by

$$d'^x_{L,n} = \bar{d}',$$

$$D'^x_{L,n} = \begin{cases} \bar{d}' - g^{-1}\left(\dfrac{\lambda'^R_1 dF_P^{(n-1)}(x) - \lambda'^I_1 M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\right), & \text{if } X_n \in A'^x_1 \\ \underline{\mathrm{D}}', & \text{if } X_n \in S - A'^x_1 \end{cases}$$

$$d'^x_{H,n} - D'^x_{H,n} = g^{-1}\left(\frac{\lambda'^I_1}{p_H}, X_n\right).$$

The definition of λ'^R_1, λ'^I_1, and A'^x_1 are in the same logic as that of λ^R_1, λ^I_1, and A^x_1 respectively. For simplification, we do not write down the definitions of them. We suppose $S - A'^x_1 \neq \emptyset$ for some $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. Comparing the insurer's utilities from the solutions above and the solutions (3.2.14)–(3.2.16), we obtain

$$\begin{aligned} &\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(d'_L, D'_L, d'_H, D'_H) \\ &= p_L E^P\left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} \left(U_2(d^x_{L,n} - D^x_{L,n}) - U_2(d'^x_{L,n} - D'^x_{L,n})\right) dF_P^{(n-1)}(x)\right] \\ &\quad + p_H E^Q\left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} \left(U_2(d^x_{H,n} - D^x_{H,n}) - U_2(d'^x_{H,n} - D'^x_{H,n})\right) dF_Q^{(n-1)}(x)\right]. \end{aligned}$$

Since U_2 is a concave function, we have

$$\begin{aligned}
&\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(d'_L, D'_L, d'_H, D'_H) \\
&\geq p_L E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U'_2(d^x_{L,n} - D^x_{L,n}) \left(d^x_{L,n} - D^x_{L,n} - d'^x_{L,n} + D'^x_{L,n} \right) dF_P^{(n-1)}(x) \right] \\
&\quad + p_H E^Q \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U'_2(d^x_{H,n} - D^x_{H,n}) \left(d^x_{H,n} - D^x_{H,n} - d'^x_{H,n} + D'^x_{H,n} \right) \right. \\
&\qquad \left. dF_Q^{(n-1)}(x) \right].
\end{aligned}$$

When $X_n \in A^x_1$, from (3.2.14)–(3.2.15),

$$\frac{U'_2(d^x_{L,n} - D^x_{L,n})}{U'_1(w - d^x_{L,n} + D^x_{L,n} - X_n)} = \frac{\lambda^R_1 dF_P^{(n-1)}(x) - \lambda^I_1 M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}.$$

When $X_n \in S - A^x_1$, from the definition of A^x_1, we know

$$\frac{U'_2(d^x_{L,n} - D^x_{L,n})}{U'_1(w - d^x_{L,n} + D^x_{L,n} - X_n)} = g(\bar{d} - \underline{\mathrm{D}}, X_n) > \frac{\lambda^R_1 dF_P^{(n-1)}(x) - \lambda^I_1 M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}.$$

In this case, $d^x_{L,n} - D^x_{L,n} = \bar{d} - \underline{\mathrm{D}} > \bar{d}' - \underline{\mathrm{D}}' \geq d'^x_{L,n} - D'^x_{L,n}$. So,

$$\begin{aligned}
&\frac{U'_2(d^x_{L,n} - D^x_{L,n}) \left(d^x_{L,n} - D^x_{L,n} - d'^x_{L,n} + D'^x_{L,n} \right)}{U'_1(w - d^x_{L,n} + D^x_{L,n} - X_n)} \\
&> \frac{\lambda^R_1 dF_P^{(n-1)}(x) - \lambda^I_1 M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \left(d^x_{L,n} - D^x_{L,n} - d'^x_{L,n} + D'^x_{L,n} \right).
\end{aligned}$$

For type H, it follows (3.2.16) that

$$\frac{U'_2(d^x_{H,n} - D^x_{H,n})}{U'_1(w - d^x_{H,n} + D^x_{H,n} - X_n)} = \frac{\lambda^I_1}{p_H}.$$

Then,

$$\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H)$$
$$> E^P\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1'(w - d_{L,n}^x + D_{L,n}^x - X_n)\left(d_{L,n}^x - D_{L,n}^x - c_{L,n}^x + C_{L,n}^x\right)$$
$$\left(\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)\right)\Bigg]$$
$$+ E^Q\Bigg[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1'(w - d_{H,n}^x + D_{H,n}^x - X_n)$$
$$\lambda_1^I\left(d_{H,n}^x - D_{H,n}^x - c_{H,n}^x + C_{H,n}^x\right) dF_Q^{(n-1)}(x)\Bigg].$$

The demonstration from here share the procedures with the demonstration following (3.2.37) in the proof of Proposition 4. Analogously, we obtain

$$\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(d_L', D_L', d_H', D_H') > 0.$$

(ii) The proof has considerable similarities with that of Theorem 6, so we will skip some details. Let $\bar{d}'$ be an upper bound of the premium and let $\underline{\mathrm{D}}'$ be a lower bound of the compensation. Suppose that $\bar{d} - \underline{\mathrm{D}} > \bar{d}' - \underline{\mathrm{D}}'$. The solutions are (3.2.63)–(3.2.65) when the boundaries are $\bar{d}$ and $\underline{\mathrm{D}}$. We suppose $S - A_2^x \neq \emptyset$ for some $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. When the boundaries are $\bar{d}'$ and $\underline{\mathrm{D}}'$, like (3.2.63)–(3.2.65), we denote the solutions by

$$d_{L,n}'^x = \bar{d}',$$
$$D_{L,n}'^x = \begin{cases} \bar{d}' - g^{-1}\left(\dfrac{\lambda_2'^R dF_P^{(n-1)}(x) - \lambda_2'^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)}, X_n\right), & \text{if } X_n \in A_2'^x \\ \underline{\mathrm{D}}', & \text{if } X_n \in S - A_2'^x \end{cases}$$
$$d_{H,n}'^x - D_{H,n}'^x = g^{-1}\left(\frac{\lambda_3'}{p_H}, X_n\right).$$

The definition of $\lambda_2'^R$, $\lambda_2'^I$, λ_3', and $A_3'^x$ are in the same logic as that of λ_2^R, λ_2^I, λ_3, and A_3^x respectively. For simplification, we do not write down the definitions of them. We suppose $S - A_2'^x \neq \emptyset$ for some $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. Compare the utilities from the solutions to obtain

$$\begin{aligned}
&\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(d'_L, D'_L, d'_H, D'_H) \\
&\geq p_L E^P \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_2'(d_{L,n}^x - D_{L,n}^x) \left(d_{L,n}^x - D_{L,n}^x - d_{L,n}'^x + D_{L,n}'^x \right) dF_P^{(n-1)}(x) \right] \\
&\quad + p_H E^Q \left[\sum_{n=1}^{N} \int_{x \in S^{n-1}} U_2'(d_{H,n}^x - D_{H,n}^x) \left(d_{H,n}^x - D_{H,n}^x - d_{H,n}'^x + D_{H,n}'^x \right) dF_Q^{(n-1)}(x) \right].
\end{aligned}$$

For the same reason illustrated in (i), we have

$$\begin{aligned}
&\frac{U_2'(d_{L,n}^x - D_{L,n}^x) \left(d_{L,n}^x - D_{L,n}^x - d_{L,n}'^x + D_{L,n}'^x \right)}{U_1'(w - d_{L,n}^x + D_{L,n}^x - X_n)} \\
&\geq \frac{\lambda_2^R dF_P^{(n-1)}(x) - \lambda_2^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \left(d_{L,n}^x - D_{L,n}^x - d_{L,n}'^x + D_{L,n}'^x \right)
\end{aligned}$$

for $X_n \in A_2'^x$,

$$\begin{aligned}
&\frac{U_2'(d_{L,n}^x - D_{L,n}^x) \left(d_{L,n}^x - D_{L,n}^x - d_{L,n}'^x + D_{L,n}'^x \right)}{U_1'(w - d_{L,n}^x + D_{L,n}^x - X_n)} \\
&> \frac{\lambda_2^R dF_P^{(n-1)}(x) - \lambda_2^I M_n dF_Q^{(n-1)}(x)}{p_L dF_P^{(n-1)}(x)} \left(d_{L,n}^x - D_{L,n}^x - d_{L,n}'^x + D_{L,n}'^x \right)
\end{aligned}$$

for $X_n \in S - A_2'^x$, and

$$\frac{U_2'(d_{H,n}^x - D_{H,n}^x)}{U_1'(w - d_{H,n}^x + D_{H,n}^x - X_n)} = \frac{\lambda_3}{p_H}.$$

Thus,

$$\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(c_L, C_L, c_H, C_H) >$$

$$E^P\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1'(w - d_{L,n}^x + D_{L,n}^x - X_n)\left(d_{L,n}^x - D_{L,n}^x - c_{L,n}^x + C_{L,n}^x\right)\right.$$

$$\left.\left(\lambda_1^R dF_P^{(n-1)}(x) - \lambda_1^I M_n dF_Q^{(n-1)}(x)\right)\right]$$

$$+ E^Q\left[\sum_{n=1}^{N}\int_{x\in S^{n-1}} U_1'(w - d_{H,n}^x + D_{H,n}^x - X_n)\right.$$

$$\left.\lambda_1^I\left(d_{H,n}^x - D_{H,n}^x - c_{H,n}^x + C_{H,n}^x\right) dF_Q^{(n-1)}(x)\right].$$

The demonstration from here share the procedures with the demonstration following (3.2.75) in the proof of Theorem 6. Analogously, we obtain

$$\mathcal{J}_1(d_L, D_L, d_H, D_H) - \mathcal{J}_1(d_L', D_L', d_H', D_H') > 0.$$

□

3.3.3 Necessity of the boundaries

Consider a case where there are not λ_1^R and λ_1^I such that (3.3.78)-(3.3.79) hold in the first circumstance, for example, a case where $\bar{M}$ is infinity. If there are no boundaries on the premium and compensation, or equivalently $\bar{d} - \underline{\mathrm{D}} = \infty$, we will demonstrate there are not feasible optimal contracts.

In the first circumstance where (3.2.14)–(3.2.16) present the solutions, we start from a finite $\bar{d} - \underline{\mathrm{D}}$. According to (3.2.14)–(3.2.15), the optimal premium is $\bar{d}$ and the optimal compensation is $\underline{\mathrm{D}}$ for $X_n \in S - A_1^x$ and $n = 1, 2, \cdots, N$. We emphasize that $S - A_1^x \neq \emptyset$ for some $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. When $\bar{d} - \underline{\mathrm{D}}$ increases, the optimal premium and compensation change accordingly. $S - A_1^x \neq \emptyset$ and on $S - A_1^x$, the difference between the optimal premium and the optimal compensation is bigger. With the bigger $\bar{d} - \underline{\mathrm{D}}$, according to

Theorem 7, the objective of the problem increases. The insurer prefers a bigger gap between premium and compensation on $S - A_1^x$. Imagine $\bar{d} - \underline{\mathrm{D}}$ keeps increasing to infinity. Still, $S - A_1^x \neq \emptyset$ for some $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. To get the most utility, the insurer will let the difference between premium and compensation be infinity on $S - A_1^x$. When there are no boundaries, the insurer will charge an infinite premium, pay a negatively infinite compensation, or both, on a nonempty set $S - A_1^x$.

Consider a case where there are not λ_2^R and λ_2^I such that (3.3.92)-(3.3.93) hold in the second circumstance, for example, a case where $\bar{M}$ is infinity. We will also show that there are not feasible optimal contracts if there are no boundaries on the premium and compensation.

In the second circumstance where (3.2.63)–(3.2.65) present the solutions, according to (3.2.63)–(3.2.64), the optimal premium is $\bar{d}$ and the optimal compensation is $\underline{\mathrm{D}}$ for $X_n \in S - A_2^x$ and $n = 1, 2, \cdots, N$. Again, $S - A_2^x \neq \emptyset$ for some $x \in S^{n-1}$ and $n = 1, 2, \cdots, N$. When $\bar{d} - \underline{\mathrm{D}}$ increases, the gap between premium and compensation on $S - A_2^x$ increases. From Theorem 7, we know the objective of the problem increases. The insurer will choose a larger $\bar{d} - \underline{\mathrm{D}}$ to obtain more utility. When $\bar{d} - \underline{\mathrm{D}}$ increases to infinity, still $S - A_2^x$ is not empty. The insurer will let the difference between premium and compensation be infinite on $S - A_2^x$. That is an infinite premium, a negatively infinite compensation, or both.

For summarizing, there is not a feasible optimal premium and a feasible optimal compensation if there is not an upper bound for the premium or a lower bound for the compensation. Therefore, the constraint (3.1.10) is necessary for the model with more than two outcome states.

Chapter 4

Conclusion and Future Work

We have studied two models of optimal insurance contracts that an insurer should propose to a potential insured.

Motivated by climate change and catastrophic events, we have created a new full information model with persistent efforts. The number of claims is assumed to be a shot-noise Cox process. However, this model for the number of claims can be applied to many other risk management problems. To the best of our knowledge, we have obtained the first analytical solution for the optimal premium, the optimal compensation, and the optimal actions of the insured when the number of the claims process is a Cox process. The solution shows that the optimal expected action decreases over time. It also shows that the amount of action decided by the insurer is restricted by the amount of action the potential insured selects when he is not in the insurance market. An example with exponential utilities allows us to see how the solution depends on the parameters of the model.

With two risk types of potential insureds, we construct a new model with adverse selection. Different from the traditional self-selection model with adverse selection, our model adds an upper boundary for the premium and a lower boundary for the compensation. The main results we obtained are the optimal premium and compensation shown by Theorem 5 and Theorem 6. Under two circumstances, the solutions are in different forms. To verify the circumstance and then select the suitable form, we cannot bypass the first form shown by (3.2.14)-(3.2.16). If the controls (3.2.14)-(3.2.16) satisfy (3.1.7), then they are the optimal solutions. Otherwise, (3.2.63)-(3.2.65) show the right form of the optimal controls. The traditional adverse selection model with self-selection has only two outcomes: accident and no accident. However, our model allows general outcome states. With the general outcome states, we will obtain infinite premiums or compensations as the optimal solutions for some states from the traditional model. The boundaries we add make us avoid infeasible solutions.

In the previous literature focusing on principle-agent problems, the researchers invent different models to deal with adverse selection and distinguish the types of agents to the contract. Some advocate the contracts are offered to all the agents, like the second model in this thesis. But some literature advocates that contracts are offered only to the "best" type, for instance, the low-risk type in an insurance market and the high-productivity type in an employment market. I would like to compare these different ideas and try to find out which is more effective.

Bibliography

Federal Emergency Management Agency, National Flood Insurance Program, Summary of Coverage. `https://agents.floodsmart.gov/sites/default/files/fema_nfip-summary-of-coverage-brochure-11-2022.pdf`. November 11, 2022.

Flex Insurance Company Residential Strata Plan Insurance Product Disclosure Statement and Policy Wording. `https://www.flexinsurance.com.au/assets/Documents/FlexInsurance_ResiStrataPDS_Jan2022_online.pdf`. October 29, 2022.

H. Albrecher and S. Asmussen. Ruin Probabilities and Aggregate Claims Distributions for Shot Noise Cox Processes. *Scandinavian Actuarial Journal*, 2006:86–110, 2006.

D. P. Baron and R. B. Myerson. Regulating a Monopolist with Unknown Costs. *Econometrica*, 50:911–930, 1982.

R. E. Beard, T. Pentikäinen, and E. Pesonen. *Risk Theory*. Chapman and Hall Ltd., London, 1984.

P. Bolton and M. Dewatripont. *Contract Theory*. The MIT Press, 2005.

M. J. Browne. Evidence of adverse selection in the individual health insurance market. *Journal of Risk and Insurance*, 59:13–33, 1992.

H. Bühlmann. *Mathematical Methods in Risk Theory*. Springer-Verlag Berlin Heidelberg, 1970.

A. Cadenillas, J. Cvitanić, and F. Zapatero. Optimal risk-sharing with effort and project choice. *Journal of Economic Theory*, 133:403–440, 2007.

H. Chade and E. Schlee. Optimal insurance with adverse selection. *Theoretical Economics*, 7:571–607, 2012.

A. Cohen and P. Siegelman. Testing for adverse selection in insurance markets. *Journal of Risk and insurance*, 77:39–84, 2010.

R. Cooper and B. Hayes. Multi-period insurance contracts. *International Journal of Industrial Organization*, 5:211–231, 1987.

J. Cvitanić and J. Zhang. Optimal compensation with adverse selection and dynamic actions. *Mathematics and Financial Economics*, 1:21–55, 2007.

J. Cvitanić and J. Zhang. *Contract Theory in Continuous-Time Model*. Springer, 2013.

A. Dassios and P. Embrechts. Martingales and Insurance Risk. *Communications in Statistics. Stochastic Models*, 5:181–217, 1989.

A. Dassios and J. Jang. Pricing of Catastrophe Reinsurance and Derivatives Using the Cox Process with Shot Noise Intensity. *Finance and Stochastics*, 7:73–95, 2003.

A. Dassios, J. Jang, and H. Zhao. A Risk Model with Renewal Shot-Noise Cox Process. *Insurance: Mathematics and Economics*, 65:55–65, 2015.

M. H. A. Davis. Piecewise-Deterministic Markov Processes: A General Class of Non-Diffusion Stochastic Models. *Journal of the Royal Statistical Society. Series B (Methodological)*, 46:353–388, 1984.

P. M. Demarzo and Y. Sannikov. Learning, Termination, and Payout Policy in Dynamic Incentive Contracts. *The Review of Economic Studies*, 84:182–236, 2017.

G. Dionne. Adverse selection and repeated insurance contracts. *Geneva Papers on Risk and Insurance*, 8:316–333, 1983.

G. Dionne and P. Lasserre. Adverse selection, repeated insurance contracts and announcement strategy. *The Review of Economic Studies*, 52:719–723, 1985.

G. Dionne and P. Lasserre. Adverse selection and finite-horizon insurance contracts. *European Economic Review*, 31:843–861, 1987.

G. Dionne, N. Doherty, and N. Fombaron. *Adverse Selection in Insurance Markets*, volume 22, chapter 7. Spinger, Dordrecht, 2000.

J. Grandell. *Doubly Stochastic Poisson Processes*. Springer-Verlag Berlin Heidelberg, 1976.

M. Halac, N. Kartik, and Q. Liu. Optimal contracts for experimentation. *Review of Economic Studies*, 83:1040–1091, 2016.

M. F. Hellwig. Incentive problems with unidimensional hidden characteristics: A unified approach. *Econometrica*, 78:1201–1237, 2010.

F. Hoffmann, R. Inderst, and M. Opp. Only Time Will Tell: A Theory of Deferred Compensation. *The Review of Economic Studies*, 88:1253–1278, 2021.

F. Hoffmann, R. Inderst, and M. Opp. The Economics of Deferral and Clawback Requirements. *The Journal of Finance*, 77:2423–2470, 2022.

H. Hopenhayn and A. Jarque. Unobservable Persistent Productivity and Long Term Contracts. *Review of Economic Dynamics*, 13:333–349, 2010.

A. Jarque. Repeated Moral Hazard with Effort Persistence. *Journal of Economic Theory*, 145:2412–2423, 2010.

M. Jeleva and B. Villeneuve. Insurance contracts with imprecise probabilities and adverse selection. *Economic theory*, 23:777–794, 2004.

B. Jullien. Participation constraints in adverse selection models. *Journal of Economic Theory*, 93:1–47, 2000.

F. Lindskog and A. J. McNeil. Common Poisson Shock Models: Applications to Insurance and Credit Risk Modelling. *ASTIN Bulletin*, 33:209–238, 2003.

B. Ma, J. Ye, Y. Huang, and M. F. Bashir. Research of two-period insurance contract model with a low compensation period under adverse selection. *Managerial and Decision Economics*, 41:293–307, 2020.

C. Macci and G. L. Torrisi. Risk Processes with Shot Noise Cox Claim Number Process and Reserve Dependent Premium Rate. *Insurance: Mathematics and Economics*, 48:134–145, 2011.

E. Maskin and J. Riley. Monopoly with Incomplete Information. *The RAND Journal of Economics*, 15:171–196, 1984.

J. Medhi. *Stochastic Processes*. New Age International(P) Limited, Publishers, 1982.

K. S. Moore and V. R. Young. Optimal Insurance in a Continuous-Time Model. *Insurance: Mathematics and Economics*, 39:47–68, 2006.

T. Mukoyama and A. Şahin. Repeated Moral Hazard with Persistence. *Economic Theory*, 25:831–854, 2005.

C. M. Ramsay, V. I. Oguledo, and P. Pathak. Pricing high-risk and low-risk insurance contracts with incomplete information and production costs. *Insurance: Mathematics and Economics*, 52:606–614, 2013.

Y. Sannikov. A Continuous- Time Version of the Principal: Agent Problem. *The Review of Economic Studies*, 75:957–984, 2008.

T. Schmidt. Catastrophe Insurance Modeled by Shot-Noise Processes. *Risks*, 2:3–24, 2014.

P. Shi, W. Zhang, and E. A. Valdez. Testing adverse selection with two-dimensional information: evidence from the Singapore auto insurance market. *Journal of Risk and Insurance*, 79:1077–1114, 2012.

J. E. Stiglitz. Monopoly, Non-linear Pricing and Imperfect Information: The Insurance Market. *The Review of Economic Studies*, 44:407–430, 1977.

N. Williams. On Dynamic Principal-Agent Problems in Continuous Time. Working paper, University of Wisconsin-Madison, 2009.

N. Williams. A Solvable Continuous Time Dynamic Principal–Agent Model. *Journal of Economic Theory*, 159:989–1015, 2015.

L. Zhu. Ruin Probabilities for Risk Processes with Non-stationary Arrivals and Subexponential Claims. *Insurance: Mathematics and Economics*, 53: 544–550, 2013.

B. Zou and A. Cadenillas. Optimal Investment and Risk Control Policies for an Insurer: Expected Utility Maximization. *Insurance: Mathematics and Economics*, 58:57–67, 2014.

B. Zou and A. Cadenillas. Optimal Investment and Liability Ratio Policies in a Multidimensional Regime Switching Model. *Risks*, 5:6, 2017.