University of Alberta

STOCHASTIC CONTROL WITH REGIME SWITCHING AND ITS APPLICATIONS TO FINANCIAL ECONOMICS

by

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To Mami.

ABSTRACT

We consider problems of stochastic control with regime switching: maximization problems for stochastic models that are modulated by an observable finite-state continuous-time Markov chain. We develop the theory required to solve those problems, and we apply that theory to important problems in Financial Economics.

We present new results on the theory of classical control with regime switching, and on the theory of singular control with regime switching. We also develop the theory of impulse control with regime switching. In fact, we obtain the first version of the Hamilton-Jacobi-Bellman equation for a problem of classical stochastic control with regime switching in random time horizon and for a utility function or cost function dependent on the regime. We obtain as well the first verification theorem for a problem of stochastic impulse control with regime switching.

Furthermore, we apply our results to solve the consumption-investment problem in financial markets with regime switching, and the dividend policy problem for a company that presents business cycles. The first problem is solved explicitly, while the second problem is solved analytically for bounded dividend rates, for unbounded dividend rates and for the case in which there are dividend taxes and a fixed cost associated with each dividend payment.

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Chapter 1

Introduction

Traditionally, continuous-time financial models are built on Brownian motion: they focus on capturing the uncertainty generated by the continuous and minuscule movements of the financial market. Examples of this include the Black-Scholes model for option pricing, consumption-investment models (as in Merton (1969)), dividend policy models (as in Jeanblanc-Piqué and Shiryaev (1995), and Asmussen and Taksar (1997)), stochastic volatility models (as in Hull and White (1987)) and interest rate models (as in Vasicek (1977), among many others). However, the financial market also presents more steady long-term movements. In fact, the behavior of the financial market is affected by long-standing macroeconomic conditions.

The macroeconomic effect on the financial market has been well documented in the empirical Finance literature. Chen, Roll, and Ross (1986), Chen (1991), Fama (1981,1990), Huang and Kracaw (1984), Maheu and McCurdy (2000), Guerra and Tabak (2002), Pearce and Roley (1988), Schaller and Van Norden (1997), Schwert (1987,1989), and Wei and Wong (1992), among others, have studied the relationship between the financial market and macroeconomic activities by considering, for instance, factors such as asset prices, production rates, productivity, growth rate, unemployment, yield spread, interest rates, inflation, and dividend yields. Evidence that the financial market follows the economy is found in all of these studies.

Regime switching models capture the long-term movements in the financial modeling by including a finite-state continuous-time Markov chain that represents the uncertainty generated by the more steady market conditions. Regime switching models were originally proposed in financial modeling by Hamilton (1989) to model stock return time series, and showed to be a more accurate representation of the financial reality than the usual models with deterministic coefficients.

Financial models with regime switching have been recently studied in different contexts. Buffington and Elliott (2002), Di Graziano and Rogers (2006), Guo (2001), Guo and Shepp (2001), Guo and Zhang (2004), Jobert and Rogers (2006), Mamon and Rodrigo (2005), and Yao, Zhang and Zhou (2004) solve option pricing problems in financial markets with regime switching. Optimal selling rules in a regime switching framework are studied by Zhang (2001), and Guo and Zhang (2005).

Portfolio management problems have been studied using techniques of classical stochastic control with regime switching by Zhang and Yin (2004), and Zhou and Yin (2003). Honda (2003), Nagai and Runggaldier (2005), and Sass and Haussmann (2004) also study portfolio optimization problems, but under the assumption of partial information: the stock prices are observable while the Markov process that models the market regime is hidden. All the above mentioned papers in classical stochastic control with regime switching consider a finite horizon formulation and none of them consider the utility function dependent on the Markov chain. Moreover, only Zhou and Yin (2003) consider all the parameters of the model to depend on the Markov chain, although they only succeed in giving an explicitly solution for independent interest rates. Stockbridge (2002) presents an infinite horizon portfolio optimization problem with regime switching, but with a different approach: he establishes an equivalent linear programming formulation of the problem.

Zariphopoulou (1992) considers an infinite horizon investment-consumption problem in which the rate of return of the stock is a continuous-time Markov chain. She formulates a verification theorem for this problem of singular stochastic control with regime switching, but does not provide a proof of the theorem. Guo, Miao and Morellec (2005) also present a verification theorem for a stochastic singular control with regime switching problem of irreversible investment, but fail as well to provide a rigorous mathematical proof.

In this thesis, we present new results on the theory of stochastic control with regime switching. We generalize stochastic control techniques for random time horizon problems to include regime switching models in which the regime is modeled by an observable continuous-time finite-state Markov chain. Furthermore, we formalize the generalized techniques through rigorously proved verification theorems. In the formulation, we consider the dependence of the utility functions (or cost functions) on the Markov chain.

In the case of classical stochastic control with regime switching, we present the first version of the Hamilton-Jacobi-Bellman equation for problems with random time horizon and with a regime-dependent utility function. The theory of classical stochastic control with regime switching is presented in Chapter 2.

For singular stochastic control with regime switching, we present the first version of the variational inequalities for those type of problems, that is, for problems with random time horizon and with a regime-dependent utility function. In addition, we give the first rigorous mathematical proof of a verification theorem for singular stochastic control with regime switching problems. The theory of singular stochastic control with regime switching is presented in Chapter 5.

In Chapter 7, we develop the theory of stochastic impulse control with regime switching and, as a result, we obtain the first verification theorem for those kind of problems. The use of stochastic impulse control techniques for solving problems in a regime switching formulation is a novel concept in the Mathematical Finance literature.

The generalized techniques for stochastic control with regime switching are later applied to solve important problems in Financial Economics.

In chapter 3, we use the classical stochastic control with regime switching techniques to solve explicitly a consumption-investment problem in a financial market with regime switching. The problem consists of a single investor that selects an investment policy and a consumption rate to maximize his total expected utility of consumption until bankruptcy. All the market parameters and the investor's utility of consumption are dependent on the regime of the market. We obtain explicit optimal consumption and investment policies for specific HARA utility functions. Our solutions show that the optimal policy depends on the regime. We include an economical analysis of the solutions, which shows how both the optimal proportion to allocate in a stock and the optimal consumption to wealth ratio behave in terms of the economical conditions and in terms of the investor's level of risk aversion.

In Chapter 4, a second application of the techniques for classical stochastic control with regime switching is presented. We consider a company whose cash reservoir is affected by macroeconomic conditions (a company that presents business cycles) and model the cash reservoir as a Brownian motion with both drift and volatility modulated by the Markov chain representing the regime of the economy. The problem consists in selecting the bounded dividend rate that maximizes the expected total discounted dividend payments to be received by the shareholders. We succeed in solving analytically the problem and our solution shows that the optimal dividend policy depends strongly on the macroeconomic conditions.

In Chapter 6, we use the generalized techniques for singular stochastic control with regime switching to analytically solve a different type of dividend payment problem. In this problem, the cash reservoir of the company again varies according to macroeconomic conditions and is represented by a Brownian motion with Markov-modulated drift and volatility. However, in this case we consider that the dividend rates are not bounded and hence the problem consists of finding the unbounded dividend policy that maximizes the expected total discounted dividend payments received by the shareholders. Also in this problem, our solution shows a strong dependence on macroeconomic conditions.

In Chapter 8, we apply the techniques for stochastic impulse control with regime switching to solve a third type of dividend payment problem. The management of a company, whose cash reservoir is affected by business cycles, wants to optimize the expected total dividends that will be paid to their shareholders. We consider the presence of a fixed cost associated with each dividend payment and the presence of dividend taxes. The problem consists of selecting the optimal times and optimal amounts of dividends to be paid by the company to its shareholders. We succeed in obtaining the optimal dividend policy for this problem. To the best of our knowledge, this is the first application of stochastic impulse control with regime switching to a Financial Economics problem.

The three problems of optimal dividend policy that are presented in Chapters 4, 6 and 8 are the only mathematical formulations that allow business cycles (generated by macroeconomic conditions) to affect the cash reservoir of a company. As pointed out by Ho and Wu (2001): "Firm's earnings are usually correlated with the overall market movements and are often influenced by business cycles" (p.973). In the dividend problems presented in this thesis, the earnings of a company that are used to pay dividends are represented by the cash reservoir process. The empirical evidence of the effect of business cycles on a company's cash reservoir leads us to actually suggest the use of regime switching for modeling the dynamics of cash reservoir processes, to provide a more accurate representation of the financial reality.

We finish this thesis by giving a review of our main results, both mathematical and financial. In Appendix A, we present the derivation of the Itô's formula for Markov modulated processes. That version of the Itô's formula is used in the different verification theorems in this thesis. The proofs of some Lemmas are provided in the other appendices as they are significantly long and do not particularly contribute to the research process adhered to by the main text.

The contributions of this thesis are both mathematical and financial. The main contributions in Mathematics are the generalized stochastic control techniques that include the regime switching characteristic, and the formalization of the corresponding verification theorems. An explicit solution to a problem of classical stochastic control with regime switching is also given, along with analytical solutions to a different problem of classical stochastic control with regime switching, a problem of singular stochastic control with regime switching, and finally to a problem of stochastic impulse control with regime switching.

From the financial point of view, we consider a type of modeling that grants a more accurate representation of the financial reality by including the macroeconomic effect on the behavior of the financial market. We also find the first explicit optimal consumption-investment policies for certain HARA utility functions when the effect of the macroeconomic conditions are taking into account. Furthermore, we present the first mathematical model in the literature that considers the cash reservoir of a company and their dividend policies dependent on macroeconomic conditions.

Chapter 2

Classical stochastic control with regime switching

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider also a N-dimensional standard Brownian motion $W = \{W_t,\, t \geq 0\}$; and an observable continuous-time, stationary, finite-state Markov chain $\epsilon = \{\epsilon_t,\, t \geq 0\}$. Let us denote by $\mathcal{S}$ the state-space of this Markov chain; that is, for every $t \in [0, \infty)$: $\epsilon_t = \epsilon(t) \in \mathcal{S} = \{1, 2, \ldots, S\}$, where $S \geq 2$. We assume that the stochastic processes W and ϵ are independent. Furthermore, we assume that the Markov chain has a strongly irreducible generator $Q = [\,q_{ij}\,]_{S\times S}$, where $q_{ii} := -\lambda_i < 0$ and $\sum_{j\in\mathcal{S}} q_{ij} = 0$ for every regime $i \in \mathcal{S}$. We denote by $\mathbb{F} = \{\mathcal{F}_t,\, t \geq 0\}$ the $\mathbb{P}$-augmentation of the filtration $\{\mathcal{F}_t^{W,\epsilon},\, t \geq 0\}$ generated by the stochastic processes W and ϵ. Here, $\mathcal{F}_t^{W,\epsilon} := \sigma\{W_s, \epsilon_s : 0 \leq s \leq t\}$ for every $t \in [0, \infty)$.

Let U be a closed convex subset of $\mathbb{R}^d$ and $u : [0, \infty) \times \Omega \to U$ be an $\mathbb{F}$-adapted control process. Moreover, let $\mathcal{O}$ be an open, nonempty, convex subset of $\mathbb{R}^M$ (the *solvency region*). Consider an $\mathbb{F}$-adapted process $X = \{X_t,\, t \geq 0\}$ that satisfies the stochastic integral equation

$$X_t \;=\; x + \int_0^t f(X_s, \epsilon_s, u_s)\, ds + \int_0^t g(X_s, \epsilon_s, u_s)\, dW_s, \tag{2.1}$$

with initial value $X_0 = x \in \mathcal{O}$ and initial state $\epsilon_0 = \epsilon(0) = i \in \mathcal{S}$. We assume here that $f : \mathbb{R}^M \times \mathcal{S} \times U \to \mathbb{R}^M$ and $g : \mathbb{R}^M \times \mathcal{S} \times U \to \mathbb{R}^{M \times N}$. We define the function $G : \mathbb{R}^M \times \mathcal{S} \times U \to \mathbb{R}^{M \times M}$ as $G(x,i,u) := g(x,i,u) \cdot g(x,i,u)^T$, where A^T denotes the transpose of the matrix A.

Let us denote the Euclidean norm in $\mathbb{R}^p$, $p \in \mathbb{N}$, by $\|\cdot\|_p$. Also, for any A, $B \in \mathbb{R}^{p\times q}$, $p, q \in \mathbb{N}$, we define

$$A \bullet B \;:=\; \text{trace}\,\left(A \cdot B^T\right) \;=\; \sum_{j=1}^{p} \sum_{k=1}^{q} A_{jk}\, B_{jk}.$$

We note then that $\|A\|_{p\times q} = (A \bullet A)^{1/2}$ for every $A \in \mathbb{R}^{p\times q}$, $p, q \in \mathbb{N}$.

The following definitions are the standard definitions of existence and uniqueness of strong solutions for the stochastic integral equation (2.1).

Definition 2.1. A strong solution of the stochastic integral equation (2.1) is an $\mathbb{F}$-adapted process $X = \{X_t, t \geq 0\}$ such that $X_0 = x$, $\mathbb{P}$-a.s., that satisfies (2.1), and such that, for every $t \in [0, \infty)$:

$$\mathbb{P}\left\{ \int_0^t \| f(X_s, \epsilon_s, u_s)\|_M \, ds \;<\; +\infty \right\} \;=\; 1 \tag{2.2}$$

$$\mathbb{P}\left\{ \int_0^t (g \bullet g)(X_s, \epsilon_s, u_s) \, ds \;<\; +\infty \right\} \;=\; 1. \tag{2.3}$$

Here, $(g \bullet g)(x,i,u) \;:=\; g(x,i,u) \bullet g(x,i,u)$.

Definition 2.2. The stochastic integral equation (2.1) has a unique strong solution if, for any two strong solutions X and Y of (2.1), $\mathbb{P}\{X_t = Y_t,\, t \in [0,\infty)\} = 1$ holds.

Consider the first time when the process X leaves the solvency region. Define such stopping time as

$$\Theta \;=\; \Theta_X \;:=\; \inf\{\, t \geq 0 : X_t \notin \mathcal{O}\,\}$$

and impose $X_t = X_\Theta$ for every $t \in [\Theta, \infty)$.

Let $H : \mathbb{R}^M \times \mathcal{S} \times U \longrightarrow \mathbb{R}$ be a given function. Define then the functional

$$(2.4) \qquad J(x,i;u) \;:=\; E_{x,i}\left[\int_0^\Theta e^{-\delta t} H(X_t, \epsilon_t, u_t)\, dt\right],$$

where $E_{x,i}$ represents the expectation conditioned to the initial values $X_0 = x$ and $\epsilon_0 = i$. The parameter $\delta > 0$ is the discount rate for H. In order for the functional (2.4) to be well defined, we need that

$$(2.5) \qquad \mathbb{P}\left\{\int_0^\Theta e^{-\delta s} |H(X_s, \epsilon_s, u_s)|\, ds \;<\; +\infty\right\} \;=\; 1.$$

Definition 2.3. For every $x \in \mathcal{O}$ and $i \in \mathcal{S}$, we define an *admissible control process* as an adapted control process $u : [0, \infty) \times \Omega \longrightarrow U$ such that the trajectory $X = X^u$ is the unique strong solution of (2.1), and such that condition (2.5) is satisfied. The set of such admissible controls will be denoted by $\mathcal{A}(x,i)$.

Problem 2.1. The stochastic control problem related to this setup is to select, for every $x \in \mathcal{O}$ and $i \in \mathcal{S}$, an optimal admissible control $u^* \in \mathcal{A}(x,i)$ that maximizes the functional (2.4), and define the value function

$$(2.6) \qquad V(x,i) \;:=\; J(x,i;u^*) \;=\; \sup_{u \in \mathcal{A}(x,i)} J(x,i;u).$$

Problem 2.1 is a problem of classical stochastic control with regime switching. In order to solve it, we state the adequate verification theorem that gives the sufficient conditions for the solution of the problem.

2.1 The verification theorem

Consider $\psi : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}$ and define the operators $L_i(u)$, for each $i \in \mathcal{S}$, by

$$\begin{aligned} L_i(u)\,\psi(x,i) &:= \frac{1}{2}\,G(x,i,u) \bullet \psi_{xx}(x,i) + f(x,i,u)^T\,\psi_x(x,i) \;-\; \delta\,\psi(x,i) \\ &= \frac{1}{2}\sum_{m=1}^{M}\sum_{p=1}^{M} G_{mp}(x,i,u)\,\frac{\partial^2\psi}{\partial x_m \partial x_p}(x,i) \\ &\qquad\qquad + \sum_{m=1}^{M} f_m(x,i,u)\,\frac{\partial\psi}{\partial x_m}(x,i) \;-\; \delta\,\psi(x,i). \end{aligned}$$

Here, ψ_x denotes the gradient of ψ with respect to x, and ψ_{xx} denotes the Jacobian matrix of ψ with respect to x.

Following the notation given by Yin and Zhang (1998), we define as well, for each $i \in \mathcal{S}$,

$$Q\,\psi(x,\cdot)(i) \;:=\; \sum_{j\in\mathcal{S}} q_{ij}\,\psi(x,j) \;=\; -\lambda_i\,\psi(x,i) + \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij}\,\psi(x,j).$$

Furthermore, we use $\mathcal{B}(A)$ and $\mathcal{C}l(A)$ to denote, respectively, the border and the closure of the set A.

Theorem 2.1. Let $v(\cdot,i) \in C^1(\mathcal{O}) \cap C^2(\mathcal{O}\backslash N_i)$, $i \in \mathcal{S}$, be a real function in $\mathcal{O}$, where N_i, $i \in \mathcal{S}$, are finite subsets of $\mathcal{O}$. Consider $\vartheta : \mathcal{S} \to \mathbb{R}$, and assume that for every $y \in \mathcal{B}(\mathcal{O})$, $\lim_{x\to y} v(x,i) = \vartheta(i) < \infty$, $i \in \mathcal{S}$. Moreover, assume that $v(\cdot,i)$ has polynomial growth, for each $i \in \mathcal{S}$. Suppose that the function $v(\cdot,i)$, $i \in \mathcal{S}$, satisfies the Hamilton-Jacobi-Bellman equation

$$\sup_{u\in U}\{L_i(u)\,v(x,i) + H(x,i,u)\} \;+\; Q\,(v(x,\cdot) - \vartheta(\cdot))(i) \;=\; 0, \tag{2.7}$$

for every $x \in \mathcal{O}$, and define the control process $\hat{u}$ by

$$\hat{u}(t) := \arg\sup_{\alpha \in U}\{L_{\epsilon(t)}(\alpha)\, v(X_t, \epsilon_t) + H(X_t, \epsilon_t, \alpha)\} \tag{2.8}$$

for every $t \in [0, \widehat{\Theta})$, and by $\hat{u}(t) := 0$ for every $t \in [\widehat{\Theta}, \infty)$, where $\widehat{\Theta} = \Theta_{X^{\hat{u}}}$. Then, if the control $\hat{u}$ is admissible, it is an optimal control for Problem 2.1. Moreover, the value function is given, for $i \in \mathcal{S}$, by

$$\begin{aligned} V(x,i) &= E_{x,i}\left[\int_0^{\widehat{\Theta}} e^{-\delta s} H(X_s, \epsilon_s, \hat{u}_s)\, ds\right] \\ &= v(x,i) - \vartheta(i) + \delta\, E_{x,i}\left[\int_0^{\widehat{\Theta}} e^{-\delta s}\vartheta(\epsilon_s)\, ds\right] \end{aligned}$$

for every $x \in \mathcal{O}$, and by $V(x,i) = 0$ for every $x \notin \mathcal{O}$.

Proof. Consider the function $\varphi(\cdot,\cdot,i)$, $i \in \mathcal{S}$, defined as $\varphi(t,x,i) := e^{-\delta t}(v(x,i) - \vartheta(i))$ and consider an arbitrary admissible control u. Using the Itô's formula for Markov-modulated processes (the derivation of the formula is given in Appendix A), we obtain

$$\begin{aligned} d\varphi(t, X_t, \epsilon_t) &= \varphi_t(t, X_t, \epsilon_t)\, dt + \sum_{m=1}^{M} \frac{\partial \varphi}{\partial x_m}(t, X_t, \epsilon_t)\, d(X_m)_t \\ &\quad + \frac{1}{2}\sum_{m=1}^{M}\sum_{p=1}^{M} \frac{\partial^2 \varphi}{\partial x_m \partial x_p}(t, X_t, \epsilon_t)\, d\langle X_m, X_p\rangle_t \\ &\quad + Q\,\varphi(t, X_t, \cdot)(\epsilon_t)\, dt + dM_t^{\varphi}, \\ &= \varphi_t(t, X_t, \epsilon_t)\, dt + \sum_{m=1}^{M} f_m(X_t, \epsilon_t, u_t)\frac{\partial \varphi}{\partial x_m}(t, X_t, \epsilon_t)\, dt \\ &\quad + \sum_{m=1}^{M}\sum_{n=1}^{N} g_{mn}(X_t, \epsilon_t, u_t)\frac{\partial \varphi}{\partial x_m}(t, X_t, \epsilon_t)\, d(W_n)_t \\ &\quad + \frac{1}{2}\sum_{m=1}^{M}\sum_{p=1}^{M} G_{mp}(X_t, \epsilon_t, u_t)\, \frac{\partial^2 \varphi}{\partial x_m \partial x_p}(t, X_t, \epsilon_t)\, dt \\ &\quad + Q\,\varphi(t, X_t, \cdot)(\epsilon_t)\, dt + dM_t^{\varphi}, \end{aligned}$$

where X_m and X_p denote de m-th and the p-th component of X, respectively, and W_n denotes the n-th component of W.

The process $\{M_t^\varphi, t \geq 0\}$ is a real-valued, square integrable martingale, with $M_0^\varphi = 0$ $\mathbb{P}$-a.s., when $\varphi(\cdot,\cdot,i)$, $i \in \mathcal{S}$, is bounded. In fact, $\{M_t^\varphi, t \geq 0\}$ is a martingale when $v(\cdot, i)$, $i \in \mathcal{S}$, is bounded because $\vartheta(i)$ is finite for every $i \in \mathcal{S}$. We have then,

$$
\begin{aligned}
& d\varphi(t, X_t, \epsilon_t) \\
&\quad = -\delta e^{-\delta t}(v(X_t,\epsilon_t) - \vartheta(\epsilon_t))dt + e^{-\delta t}\sum_{m=1}^{M} f_m(X_t,\epsilon_t,u_t)\frac{\partial v}{\partial x_m}(X_t,\epsilon_t)dt \\
&\qquad + e^{-\delta t}\sum_{m=1}^{M}\sum_{n=1}^{N} g_{mn}(X_t,\epsilon_t,u_t)\,\frac{\partial v}{\partial x_m}(X_t,\epsilon_t)\,d(W_n)_t \\
&\qquad + \frac{1}{2}\,e^{-\delta t}\sum_{m=1}^{M}\sum_{p=1}^{M} G_{mp}(X_t,\epsilon_t,u_t)\,\frac{\partial^2 v}{\partial x_m \partial x_p}(X_t,\epsilon_t)\,dt \\
&\qquad + e^{-\delta t} Q(v(X_t,\cdot) - \vartheta(\cdot))(\epsilon_t)\,dt + dM_t^\varphi \\
&\quad = e^{-\delta t}\big(L_{\epsilon(t)}(u_t)\,v(X_t,\epsilon_t) + Q(v(X_t,\cdot) - \vartheta(\cdot))(\epsilon_t)\big)dt + \delta\, e^{-\delta t}\vartheta(\epsilon_t)\,dt \\
&\qquad + e^{-\delta t}\sum_{m=1}^{M}\sum_{n=1}^{N} g_{mn}(X_t,\epsilon_t,u_t)\,\frac{\partial v}{\partial x_m}(X_t,\epsilon_t)\,d(W_n)_t + dM_t^\varphi.
\end{aligned}
$$

Recall that $v(\cdot, i)$, $i \in \mathcal{S}$, satisfies the HJB equation (2.7). Therefore,

$$
\begin{aligned}
d\varphi(t, X_t, \epsilon_t) \;\leq\; & -e^{-\delta t} H(X_t,\epsilon_t,u_t)\,dt + \delta\, e^{-\delta t}\vartheta(\epsilon_t)\,dt \\
& + e^{-\delta t}\sum_{m=1}^{M}\sum_{n=1}^{N} g_{mn}(X_t,\epsilon_t,u_t)\,\frac{\partial v}{\partial x_m}(X_t,\epsilon_t)\,d(W_n)_t + dM_t^\varphi.
\end{aligned}
$$

Consider $\mathcal{U} = \{U_k,\, k \geq 1\}$ to be an open cover of $\mathcal{O}$. Let $k \geq 1$ be such that $X_0 = x \in \cup_{j=1}^k U_j \subset \mathcal{O}$ and such that $\cup_{j=1}^k U_j$ is bounded, and define the stopping time $\tau_k := \inf\{\, t \geq 0 : X_t \notin \cup_{j=1}^k U_j \,\}$. For every time $t \in [\,0, \infty)$, we get

$$
\begin{aligned}
&\varphi(t \wedge \tau_k, X_{t\wedge\tau_k}, \epsilon_{t\wedge\tau_k}) - \varphi(0, X_0, \epsilon_0) \\
&\quad \le \ -\int_0^{t\wedge\tau_k} e^{-\delta s} H(X_s, \epsilon_s, u_s)\, ds \ + \ \delta \int_0^{t\wedge\tau_k} e^{-\delta s} \vartheta(\epsilon_s)\, ds \\
&\qquad + \int_0^{t\wedge\tau_k} e^{-\delta s} \sum_{m=1}^{M} \sum_{n=1}^{N} g_{mn}(X_s, \epsilon_s, u_s) \frac{\partial v}{\partial x_m}(X_s, \epsilon_s)\, d(W_n)_s + M^{\varphi}_{t\wedge\tau_k}.
\end{aligned}
$$

By taking the conditional expectation to both sides of the inequality above given $X_0 = x$ and $\epsilon_0 = i$, we have

$$
\begin{aligned}
&E_{x,i}\left[\varphi(t \wedge \tau_k, X_{t\wedge\tau_k}, \epsilon_{t\wedge\tau_k})\right] - (v(x,i) - \vartheta(i)) \\
&\quad = \ E_{x,i}\left[\varphi(t \wedge \tau_k, X_{t\wedge\tau_k}, \epsilon_{t\wedge\tau_k})\right] - E_{x,i}\left[v(X_0, \epsilon_0) - \vartheta(\epsilon_0)\right] \\
&\quad \le \ -E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s} H(X_s, \epsilon_s, u_s)\, ds\right] + \delta\, E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s} \vartheta(\epsilon_s)\, ds\right] \\
&\qquad + E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s} \sum_{m=1}^{M} \sum_{n=1}^{N} g_{mn}(X_s, \epsilon_s, u_s) \frac{\partial v}{\partial x_m}(X_s, \epsilon_s)\, d(W_n)_s\right] \\
&\qquad + E_{x,i}\left[M^{\varphi}_{t\wedge\tau_k}\right]. \qquad (2.9)
\end{aligned}
$$

We note that inequality (2.9) is well defined because u is an admissible process, which implies that (2.5) is satisfied. We also note that $X_s \in \mathcal{C}l(\cup_{j=1}^{k} U_j)$ when $s \in [0, t \wedge \tau_k]$. Then, for every $m = 1, \ldots, M$,

$$
\frac{\partial v}{\partial x_m}(X_s, \epsilon_s) \text{ is bounded for every } s \in [0, t \wedge \tau_k],
$$

due to the continuity of $\partial v / \partial x_m(\cdot, i)$, $i \in \mathcal{S}$. Define

$$
M := \max\left\{ \frac{\partial v}{\partial x_m}(x,i) : \ x \in \mathcal{C}l\left(\cup_{j=1}^{k} U_j\right),\ i \in \mathcal{S} \right\} < +\infty.
$$

Then,

$$E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-2\delta s} g_{mn}^2(X_s,\epsilon_s,u_s)\left(\frac{\partial v}{\partial x_m}(X_s,\epsilon_s)\right)^2 ds\right]$$

$$\leq M^2 E_{x,i}\left[\int_0^{t\wedge\tau_k} g_{mn}^2(X_s,\epsilon_s,u_s)\,ds\right]$$

$$\leq M^2 E_{x,i}\left[\int_0^{t} g_{mn}^2(X_s,\epsilon_s,u_s)\,ds\right]. \tag{2.10}$$

Recall that u is an admissible process. Hence, $X = X^u$ is a strong solution for (2.1) and, therefore, condition (2.3) is satisfied. Thus,

$$\int_0^t g_{mn}^2(X_s,\epsilon_s,u_s)\,ds \;\leq\; \int_0^t (g\bullet g)(X_s,\epsilon_s,u_s)\,ds \;<\; +\infty, \quad \mathbb{P}-\text{a.s.},$$

for every $t \in [0,\infty)$, and every $m = 1,\ldots,M$ and $n = 1,\ldots,N$. Then, inequality (2.10) implies that

$$E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-2\delta s} g_{mn}^2(X_s,\epsilon_s,u_s)\left(\frac{\partial v}{\partial x_m}(X_s,\epsilon_s)\right)^2 ds\right] \;<\; +\infty,$$

for every $t \in [0,\infty)$, which further implies that

$$E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s} g_{mn}(X_s,\epsilon_s,u_s)\frac{\partial v}{\partial x_m}(X_s,\epsilon_s)\,d(W_n)_s\right] = 0,$$

for every $m = 1,\ldots,M$ and every $n = 1,\ldots,N$. Furthermore, we note that $v(x,i)$ is bounded for every $x \in \mathcal{C}l\left(\cup_{j=1}^k U_j\right)$, for every $i \in \mathcal{S}$, because $v(\cdot,i)$ is continuous. Thus, $v(X_s,\epsilon_s)$ is bounded for every $s \in [0, t\wedge\tau_k]$. Then, $\{M_{t\wedge\tau_k}^{\varphi}, t \geq 0\}$ is a square integrable martingale and hence $E_{x,i}[M_{t\wedge\tau_k}^{\varphi}] =$

$E_{x,i}[\,M_0^{\varphi}\,] = 0$. Therefore, we can write inequality (2.9) as

$$E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s} H(X_s, \epsilon_s, u_s)\, ds\right]$$
$$\leq\ -E_{x,i}\left[\varphi(t\wedge\tau_k, X_{t\wedge\tau_k}, \epsilon_{t\wedge\tau_k})\right] + v(x,i) - \vartheta(i) + \delta E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s}\vartheta(\epsilon_s)\, ds\right].$$

We note that the inequality above is well defined because u is admissible and, hence, (2.5) is satisfied. Letting $k \to +\infty$, we get that $\tau_k \to \Theta$. Hence,

$$\begin{aligned} E_{x,i}\left[\int_0^{t\wedge\Theta} e^{-\delta s} H(X_s, \epsilon_s, u_s)\, ds\right] & \\ \leq\ & -E_{x,i}\left[e^{-\delta(t\wedge\Theta)}(v(X_{t\wedge\Theta}, \epsilon_{t\wedge\Theta}) - \vartheta(\epsilon_{t\wedge\Theta}))\, I_{\{\Theta<+\infty\}}\right] \\ & -E_{x,i}\left[e^{-\delta t}(v(X_t, \epsilon_t) - \vartheta(\epsilon_t))\, I_{\{\Theta=+\infty\}}\right] \\ & + v(x,i) - \vartheta(i) + \delta E_{x,i}\left[\int_0^{t\wedge\Theta} e^{-\delta s}\vartheta(\epsilon_s)\, ds\right]. \end{aligned}$$

Letting $t \to \infty$, if $\Theta < +\infty$, we have that $X_{t\wedge\Theta} \to X_\Theta$. Thus,

$$\lim_{t\to\infty}(v(X_{t\wedge\Theta}, \epsilon_{t\wedge\Theta}) - \vartheta(\epsilon_{t\wedge\Theta})) \;=\; \lim_{x\to X_\Theta} v(x, \epsilon_\Theta) - \vartheta(\epsilon_\Theta) \;=\; \vartheta(\epsilon_\Theta) - \vartheta(\epsilon_\Theta) \;=\; 0$$

because $X_\Theta(\omega) \in \mathcal{B}(\mathcal{O})$ for every $\omega \in \Omega$ such that $\Theta(\omega) < +\infty$. Moreover, the polynomial growth condition of $v(\cdot, i)$ and the fact that $\vartheta(i)$ is finite, $i \in \mathcal{S}$, implies that

$$\lim_{t\to\infty} E_{x,i}\left[e^{-\delta t}(v(X_t, \epsilon_t) - \vartheta(\epsilon_t))\, I_{\{\Theta=+\infty\}}\right] \;=\; 0.$$

Hence, recalling that (2.5) is satisfied and applying the Dominated Convergence Theorem, we obtain

$$E_{x,i}\left[\int_0^{\Theta} e^{-\delta s} H(X_s,\epsilon_s,u_s)\,ds\right] \;\le\; -E_{x,i}\left[e^{-\delta\Theta}(\vartheta(\epsilon_\Theta)-\vartheta(\epsilon_\Theta))\,I_{\{\Theta<+\infty\}}\right] + v(x,i) - \vartheta(i) + \delta E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}\vartheta(\epsilon_s)\,ds\right].$$

Equivalently,

$$J(x,i;u) \;\le\; v(x,i) - \vartheta(i) \,+\, \delta\, E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}\vartheta(\epsilon_s)\,ds\right].$$

Note that if $u = \hat{u}$ as in (2.8) then

$$\begin{aligned} & v(x,i) - \vartheta(i) + \delta\, E_{x,i}\left[\int_0^{\hat\Theta} e^{-\delta s}\vartheta(\epsilon_s)\,ds\right] \\ (2.11)\qquad & = E_{x,i}\left[\int_0^{\hat\Theta} e^{-\delta s}\, H(X_s,\epsilon_s,\hat{u}_s)\,ds\right] \;=\; V(x,i). \end{aligned}$$

□

The following Corollary gives a different way to express the value function of Problem 2.1, which might come specially handy if $\mathbb{P}\{\Theta = +\infty\} = 1$ or $Q\,\vartheta(\cdot)(i) = 0$ for every $i \in \mathcal{S}$.

Corollary 2.1. The value function of Problem 2.1 is given, for $i \in \mathcal{S}$, by

$$V(x,i) \;=\; v(x,i) - E_{x,i}\left[e^{-\delta\hat\Theta}\,\vartheta(\epsilon_{\hat\Theta})\right] + E_{x,i}\left[\int_0^{\hat\Theta} e^{-\delta s}\, Q\,\vartheta(\cdot)(\epsilon_s)\,ds\right]$$

for every $x \in \mathcal{O}$ and by $V(x,i) = 0$ for every $x \notin \mathcal{O}$.

Proof. Using the Itô formula for $\phi(t,\epsilon_t) = -e^{-\delta t}\vartheta(\epsilon_t)$, we get

$$d\,\phi(t,\epsilon_t) \;=\; \delta\, e^{-\delta t}\vartheta(\epsilon_t)\,dt \;-\; e^{-\delta t}\, Q\,\vartheta(\cdot)(\epsilon_t)\,dt \;+\; dM_t^{\phi}$$

where $\{M_t^\phi, t \geq 0\}$ is a square integrable martingale with $E_{x,i}[\, M_{\widehat{\Theta}}^\phi\,] = E_{x,i}[\, M_0^\phi\,] = 0$. Equivalently,

$$\delta \int_0^{\widehat{\Theta}} e^{-\delta s} \vartheta(\epsilon_s)\, ds \;=\; -e^{-\delta \widehat{\Theta}} \vartheta(\epsilon_{\widehat{\Theta}}) + \vartheta(\epsilon_0) + \int_0^{\widehat{\Theta}} e^{-\delta s} Q\, \vartheta(\cdot)(\epsilon_s)\, ds - M_{\widehat{\Theta}}^\phi .$$

Applying conditional expectation with respect to $X_0 = x$ and $\epsilon_0 = i$ to the expression above, we obtain

$$\begin{aligned} \delta\, E_{x,i}\left[\int_0^{\widehat{\Theta}} e^{-\delta s}\, \vartheta(\epsilon_s)\, ds \right] \;&=\; -\, E_{x,i}[\, e^{-\delta \widehat{\Theta}}\, \vartheta(\epsilon_{\widehat{\Theta}})\,] \,+\, \vartheta(i) \\ &\quad +\, E_{x,i}\left[\int_0^{\widehat{\Theta}} e^{-\delta s}\, Q\, \vartheta(\cdot)(\epsilon_s)\, ds \right]. \end{aligned} \tag{2.12}$$

Replacing (2.12) in equation (2.11), we obtain

$$V(x,i) \;=\; v(x,i) -\, E_{x,i}[\, e^{-\delta \widehat{\Theta}}\, \vartheta(\epsilon_{\widehat{\Theta}})\,] \,+\, E_{x,i}\left[\int_0^{\widehat{\Theta}} e^{-\delta s}\, Q\, \vartheta(\cdot)(\epsilon_s)\, ds \right].$$

□

Remark 2.1. The admissibility conditions given in Definition 2.3 can be modified in different ways. For instance, we might define the admissible set $\mathcal{A}(x,i)$ as the set of $\mathbb{F}$-adapted controls u such that, for $\phi(x,i,u) = f(x,i,u)$, $g(x,i,u)$, $H(x,i,u)$: $\|\, \phi(x,i,u) - \phi(y,i,u)\| \leq K_\phi \|\, x - y\|_M$, for every $x, y \in \mathcal{O}$ and $i \in \mathcal{S}$, and $\|\, \phi(0,i,u)\| \leq L_\phi$ for every $i \in \mathcal{S}$, for some constants K_ϕ and L_ϕ. In fact, these previous conditions are sufficient conditions for the existence and uniqueness of a strong solution $X = X^u$ for (2.1), and for (2.5).

Remark 2.2. We can avoid the condition that $v(\cdot, i)$, $i \in \mathcal{S}$, is $C^1(\mathcal{O})$ in Theorem 2.1 if we use instead other conditions that guarantee $v(\cdot, i)$ and $v_x(\cdot, i)$, $i \in \mathcal{S}$, to be bounded in every closed bounded subset of $\mathcal{O}$.

In the following two chapters, we present examples of problems of classical stochastic control with regime switching. In chapter 3, we solve explicitly a consumption-investment problem when the financial market presents different regimes and the utility function of the investor is affected by those regimes. In that problem we consider $\vartheta(i) := U(0,i)/\delta$ for every $i \in \mathcal{S}$, where $U(\cdot, i)$ is the utility function of the investor in regime i.

In chapter 4, we solve a dividend policy problem when the dividend rate is bounded and the cash reservoir of the company is affected by macroeconomic conditions. That problem uses $\vartheta(i) := 0$ for every $i \in \mathcal{S}$.

Chapter 3

Consumption-investment in a financial market with regime switching

The initial contribution to consumption-investment problems in continuous-time was done by Merton (1969). Other important models of consumption-investment can be found in Karatzas, Lehoczky, Sethi and Shreve (1986), Karatzas and Shreve (1998), Sethi (1997), et cetera.

In financial markets with regime switching, the maximization of expected utility from consumption and/or terminal wealth has been studied by some authors. Zariphopoulou (1992) considers a financial market in which the riskless asset is deterministic and the price of the risky asset depends only on a continuous-time Markov chain. Zhang and Yin (2004) study nearly-optimal strategies in a financial market with regime switching. Stockbridge (2002) establishes an equivalent linear programming formulation of the portfolio optimization problem with regime switching. Sass and Haussmann (2004) consider

[1] A version of the results shown in this chapter are presented in: Sotomayor, L.R. and A. Cadenillas, Explicit solutions of consumption-investment problems in financial markets with regime switching, *to appear in Mathematical Finance* (2008).

a finite time horizon and solve numerically the problem of maximizing the investors expected utility of terminal wealth. They consider the case of partial information: the investor observes the prices of the stocks but the Markov process that models the market regime is hidden. In their model only the rates of return of the risky assets are dependent of the market regime. Nagai and Runggaldier (2005) also consider the maximization of the investors expected utility of terminal wealth but for a power utility function. The problem setting is similar to the one presented by Sass and Haussmann (2004) and the solution is computed as an expected value. The authors do not obtain explicit solutions, but suggest the use of Monte Carlo simulation to obtain numerical solutions. Honda (2003) considers a risk-averse investor that wants to maximize his expected total discounted utility of consumption plus his expected utility of terminal wealth, in a market with a riskless asset and one risky asset. Both utilities are considered as the same utility function and not dependent of the regime of the market. Honda considers hidden Markov chain to represent the market mood (partial information) and, therefore, only past and present stock prices are observed by the investor. Moreover, Honda considers the stock volatility to be constant (independent of the regime of the market). Honda computes the optimal investment and consumption policies numerically and only for power utility functions.

Portfolio optimization problems with regime switching have also been studied under a different criterion. Zhou and Yin (2003) propose a continuous-time Markowitzs mean-variance portfolio selection model with regime switching. They assume the market parameters to be regime and time-dependent. Their objective is to obtain the efficient portfolio that minimizes the risk of terminal wealth given a fixed expected terminal wealth. They characterize the efficient frontier in terms of the solution of two systems of ordinary differential equations. When the interest-rate process is not affected by the regime of the market, they can express the optimal portfolio in terms of the solution of one system of ordinary differential equations. This system can be solved explicitly in some cases, for instance, when all the coefficients of the market are independent of time. One of the drawbacks of their approach is that they allow the wealth process to take negative values: they explain, in the introduction of their paper, that requiring a nonnegative wealth process would give rise to a very difficult problem from the stochastic control point of view[2].

In this chapter, we consider a consumption-investment problem that consists of a single investor who invests continuously in time in one riskless asset and N risky assets. He also consumes continuously in time. We consider a financial market with regime switching in which all the coefficients of the market depend on the regime. We also allow the utility function to depend on the regime. The objective of the investor is to maximize his expected total discounted utility from consumption until bankruptcy.

We present in this chapter the first explicit solutions for a problem on maximization of expected utility from consumption and/or terminal wealth in financial markets with regime switching[3]. The two verification theorems of this chapter are the first versions of the Hamilton-Jacobi-Bellman equation in the literature on classical stochastic control with regime switching in random control horizon. One of them allows the utility function to depend on the regime, which is a complete new idea in classical stochastic control. Another mathematical contribution of this chapter is the explicit solution of systems of nonlinear ordinary differential equations.

Our explicit solutions show that the optimal policy depends on the regime. We also make an economic analysis of the optimal policy in a financial market with two assets (a riskless asset and a risky asset) and two regimes (a

[2]In our problem we introduce a stopping time (the time of bankruptcy) that allows us to avoid the difficulty of a state-constrained control problem.

[3]As mentioned before, Zhou and Yin (2003) have also found explicit solutions, but under a different criterion.

"bull market" and a "bear market"). Combining our mathematical results with those in the literature on empirical finance, we show that the optimal proportion to invest in the risky asset is always greater in a bull market than in a bear market. This result is independent of the investors risk tolerance. On the other hand, the optimal consumption to wealth ratio depends not only on the regime, but also on the investors risk tolerance. We show that a very risk-averse investor will consume proportionally more in a bull market than in a bear market, and the opposite occurs for a low risk-averse investor.

3.1 The financial market with regime switching

Let the $\mathbb{F}$-adapted processes $P_0 = \{P_0(t), t \geq 0\}$ and $P = \{P(t) = (P_1(t), \ldots, P_N(t)), t \geq 0\}$ represent the price of the riskless asset and the vector of prices of the N risky assets, respectively. These processes satisfy the following Markov-modulated stochastic differential equations:

$$\begin{aligned} dP_0(t) &= r_{\epsilon(t)}\, P_0(t)\, dt \\ dP_n(t) &= \mu^n_{\epsilon(t)}\, P_n(t)\, dt + \sigma^n_{\epsilon(t)}\, P_n(t)\, dW_t^T, \qquad 1 \leq n \leq N, \end{aligned}$$

with initial prices $P_0(0) = 1$ and $P_n(0) = p_n > 0$, and initial regime $\epsilon(0) = \epsilon_0$.

The rates of return for the riskless asset (r_i, $i \in \mathcal{S}$) and the expected rate of returns on the risky assets (μ_i^n, $1 \leq n \leq N$, $i \in \mathcal{S}$) are positive constants. Moreover, σ_i, $i \in \mathcal{S}$, are $N \times N$ matrices that represent the volatility of the risky assets in the regimes $i \in \mathcal{S}$. We assume that the matrices $\Sigma_i := \sigma_i \sigma_i^T$, $i \in \mathcal{S}$, are positive definite. Here, σ_i^n denotes the n-th row of σ_i. We are assuming that the coefficients of the market (i.e. r, μ and σ) depend on the regime of the economy.

The investor chooses a portfolio $\pi = \{\pi(t) = (\pi_1(t), \ldots, \pi_N(t)), t \geq 0\}$, representing the fraction of wealth invested in each risky asset. We will assume that the portfolio vector process is unconstrained. As a consequence, short-selling is allowed in the market. However, we need a technical condition to be satisfied.

Definition 3.1. A *portfolio* vector process is an $\mathbb{F}$-adapted stochastic vector process π such that $E_{x,i}[\int_0^t \pi(s)\Sigma_{\epsilon(s)}\pi(s)^T ds] < +\infty$ for all $t \in [0, \infty)$.

The fraction of wealth invested in the riskless asset at time $t \in [0, \infty)$ is then $1 - \sum_{n=1}^N \pi_n(t)$. The investor also chooses a consumption rate process $c = \{c(t), t \geq 0\}$.

Definition 3.2. A *consumption rate* process is a nonnegative $\mathbb{F}$-adapted stochastic process such that $E_{x,i}[\int_0^t c(s)\, ds] < +\infty$ for all $t \in [0, \infty)$.

The adapted process $X = \{X(t),\, t \geq 0\}$ represents the investor's wealth process determined by a specific portfolio and consumption rate. The stochastic differential equation for this process is then[4]

(3.1)

$$dX(t) = (\mu_{\epsilon(t)} - r_{\epsilon(t)}\mathbf{1})\, \pi^T(t)\, X(t)\, dt + (r_{\epsilon(t)}X(t) - c(t))dt + \pi(t)\, \sigma_{\epsilon(t)}\, X(t)\, dW_t^T$$

with initial wealth $X(0) = x > 0$ and initial state $\epsilon(0) = i \in \mathcal{S}$. Here $\mu_i = (\mu_i^1, \ldots, \mu_i^N)$, and $\mathbf{1}$ is the N-dimensional vector of ones. This linear stochastic differential equation has an explicit unique solution for every portfolio vector π

[4]Karatzas, Lehoczky, Sethi and Shreve (1986) prove that "the change in wealth is due only to capital gains from price changes in the assets and to consumption". That is,

$$dX_t = \sum_{n=0}^{N} \eta_n(t)\, dP_n(t) - c_t\, dt,$$

where $\eta_n(t)$ denotes the number of shares held of asset n at time t. This equation holds also in the presence of regime switching. Hence, using $\pi_n(t) = \eta_n(t)P_n(t)/X_t$, $1 \leq n \leq N$ and $1 - \mathbf{1}\pi^T = \eta_0 P_0(t)/X_t$, and the equations of the prices P_0 and P_n, $1 \leq n \leq N$, we obtain equation (3.1).

and every consumption rate process c. Indeed, by applying Itô's differentiation rule on $X(t)/Z(t)$,

$$X(t) = Z(t)\left(x - \int_0^t c(s)\, Z(s)^{-1}\, ds\right) \qquad \text{for every } t \in [0,\infty), \tag{3.2}$$

where

$$dZ(t) = \left((\mu_{\epsilon(t)} - r_{\epsilon(t)}\mathbf{1})\,\pi(t)^T + r_{\epsilon(t)}\right) Z(t)\,dt + \pi(t)\,\sigma_{\epsilon(t)} Z(t)\,dW_t^T$$

or equivalently

$$Z(t) = \exp\left\{\int_0^t \left((\mu_{\epsilon(s)} - r_{\epsilon(s)}\mathbf{1})\,\pi(s)^T + r_{\epsilon(s)} - \frac{1}{2}\pi(s)\,\Sigma_{\epsilon(s)}\,\pi(s)^T\right) ds + \int_0^t \pi(s)\,\sigma_{\epsilon(s)}\,dW_s^T\right\}.$$

We note that the process $Z = \{Z(t), t \geq 0\}$ can be interpreted as the wealth process without consumption and with initial wealth $Z(0) = 1$.

The investor will be able to consume and invest only if his wealth is positive. Thus, we need to consider the stopping time of bankruptcy

$$\Theta := \inf\{t \geq 0 : X(t) \leq 0\}$$

and impose $X(t) = 0$ for all $t \in [\Theta, \infty)$.

Definition 3.3. A *utility function* is a function $U : (0,\infty) \times \mathcal{S} \to \mathbb{R}$ such that, for each regime $i \in \mathcal{S}$, $U(\cdot, i)$ is $C^3(0,\infty)$, strictly increasing, and strictly concave. We consider only utility functions for which there exists a constant $K \in (0,\infty)$ such that for every $y \in (0,\infty), i \in \mathcal{S}$:

$$U(y,i) \leq K(1+y). \tag{3.3}$$

We note that, for a fixed $i \in \mathcal{S}$, it is possible to define $U(0,i) := \lim_{y \downarrow 0} U(y,i)$ and $U'(0,i) := \lim_{y \downarrow 0} U'(y,i)$, and extend the utility function to $[0,\infty) \times \mathcal{S}$. Moreover, we impose the condition $\lim_{y \to \infty} U'(y,i) = 0$ for every $i \in \mathcal{S}$.

Thus, we are considering only risk-averse investors in our analysis. This assumption is not far from reality as risk-aversion is the attitude of most investors in the financial markets.

We denote by $\delta > 0$ the discount rate.

Definition 3.4. An *admissible control* process is a stochastic process $u : [0,\infty) \times \Omega \to \mathbb{R}^N \times [0,\infty)$ defined by $u(t,\omega) := (\pi(t,\omega), c(t,\omega))$, where π is a portfolio vector process, and c is a consumption rate process such that

$$(3.4) \qquad E_{x,i}\left[\int_0^{\Theta} e^{-\delta t}\, U^-(c(t), \epsilon(t))\, dt\right] < +\infty,$$

where we denote $U^-(\cdot, i) = \max\{0, -U(\cdot, i)\}$, $i \in \mathcal{S}$. The set of all admissible controls is denoted by $\mathcal{A}$.

We note that if the investor continues consuming after bankruptcy, his consumption must be zero. That is, after bankruptcy, the investor's total discounted utility of consumption is given by

$$Q(\Theta) := \int_{\Theta}^{\infty} e^{-\delta t} U(0, \epsilon(t))\, dt.$$

The term $Q(\Theta)$ is already presented by Karatzas, Lehoczky, Sethi and Shreve (1986) for the case in which the utility function does not depend on the regime of the market. In fact, if $U(0,i) = U(0,j) = U(0)$ for every $i,j \in \mathcal{S}$, then $Q(\Theta) = e^{-\delta\Theta} U(0)/\delta$. That is, $Q(\Theta)$ becomes the discounted value of the natural payment at bankruptcy of $U(0)/\delta$.

We must note that, in the case where $U(0,j) = -\infty$ for certain $j \in \mathcal{S}$ and

$U(0,i) > -\infty$ for every $i \in \mathcal{S}$, $i \neq j$,

$$E_{x,i}[\,Q(\Theta)\,] \;=\; E_{x,i}\left[\int_{\Theta}^{\infty} e^{-\delta t} U(0,\epsilon_t)\, I_{\{\epsilon(t)\neq j\}} dt\right] \\ +\; U(0,j)\, E_{x,i}\left[\int_{\Theta}^{\infty} e^{-\delta t} I_{\{\epsilon(t)=j\}} dt\right].$$

The first term in the right-hand side of the equation above is finite. Then, if $E_{x,i}\left[\int_{\Theta}^{\infty} e^{-\delta t} I_{\{\epsilon(t)=j\}} dt\right] > 0$, we have that $E_{x,i}[\,Q(\Theta)\,] := -\infty$. Otherwise, if $E_{x,i}\left[\int_{\Theta}^{\infty} e^{-\delta t} I_{\{\epsilon(t)=j\}} dt\right] = 0$, then we define

$$E_{x,i}[\,Q(\Theta)\,] \;:=\; E_{x,i}\left[\int_{\Theta}^{\infty} e^{-\delta t} U(0,\epsilon_t) I_{\{\epsilon(t)\neq j\}} dt\right] \;>\; -\infty.$$

In particular, if $E_{x,i}[\,e^{-\delta\Theta}\,] = 0$, then

$$E_{x,i}[\,Q(\Theta)\,] \;=\; E_{x,i}\left[\int_{\Theta}^{\infty} e^{-\delta t} U(0,\epsilon_t) I_{\{\epsilon(t)\neq j\}} dt\right] \;=\; 0.$$

The investor wants to solve the following problem:

Problem 3.1. For each $i \in \mathcal{S}$, select an admissible control $\hat{u} = (\hat{\pi}, \hat{c})$ that maximizes

$$J(x,i;u) \;:=\; E_{x,i}\left[\int_{0}^{\Theta} e^{-\delta t} U(c(t),\epsilon(t))\, dt \;+\; Q(\Theta)\right],$$

and find the corresponding value function $V(\cdot,i)$, $i \in \mathcal{S}$, defined by

$$V(x,i) \;:=\; J(x,i;\hat{u}) \;=\; E_{x,i}\left[\int_{0}^{\widehat{\Theta}} e^{-\delta t} U(\hat{c}(t),\epsilon(t))\, dt \;+\; Q(\widehat{\Theta})\right].$$

The assumption that the utility function depends on the market regime is supported by the literature on Financial Economics. As Karni (1993) points out, the regime of the market is relevant for the definition of preferences. Hence, investors' decisions are given by their preferences under the specific market regime. Melino and Yang (2003) suggest that regime dependent preferences are necessary to explain the equity premium puzzle. According to Melino and Yang (2003), the investor acts as if he is more risk averse during recessions than during economic booms. This result implies a counter-cyclical risk aversion. The same counter-cyclical pattern for risk aversion is empirically obtained by Gordon and St-Amour (2003). Bosch-Domènech and Silvestre (1999) show further evidence that risk aversion varies with the level of the income at risk. Their results imply that risk aversion is pro-cyclical: investors have higher risk aversion when greater consumption growth and higher consumption levels exist, that is, during good economic conditions. Danthine et al. (2004) also emphasize that the investor's coefficient of risk aversion varies with the economy's regime.

3.2 Verification theorems

Let $\psi : (0,\infty)\times\mathcal{S} \to \mathbb{R}$ be a function and define the operators $L_i(u) = L_i(\pi, c)$, for each $i \in \mathcal{S}$, by

$$L_i(\pi, c)\,\psi \;:=\; \frac{1}{2}\,\pi\Sigma_i\pi^T x^2\psi'' + (\mu_i - r_i\mathbf{1})\pi^T x\psi' + (r_i x - c)\psi' - \delta\psi.$$

The following verification theorem is the first one in the literature on classical stochastic control that allows the regime to enter into the utility (or cost) function.

Theorem 3.1. Suppose that $U(0,i)$ is finite for every $i \in \mathcal{S}$. Let $v(\cdot,i) \in C^2(0,\infty)$, $i \in \mathcal{S}$, be a concave and increasing function in $(0,\infty)$ such that $v(0+,i) = U(0,i)/\delta$, $i \in \mathcal{S}$. If the function $v(\cdot,i)$, $i \in \mathcal{S}$, satisfies the Hamilton-

Jacobi-Bellman equation

$$\sup_{(\pi,c)\in(-\infty,\infty)^N\times[0,\infty)} \{L_i(\pi,c)\,v(x,i)+U(c,i)\}$$

$$= \lambda_i\left(v(x,i)-\frac{U(0,i)}{\delta}\right)-\sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij}\left(v(x,j)-\frac{U(0,j)}{\delta}\right), \tag{3.5}$$

for every $x>0$, then the admissible control $\hat{u}=(\hat{\pi},\hat{c})$ defined by

(3.6)

$$\hat{u}(t)=(\hat{\pi}(t),\hat{c}(t))=\arg\sup_{(\pi,c)\in(-\infty,\infty)^N\times[0,\infty)}\{L_{\epsilon(t)}((\pi,c))\,v(X(t),\epsilon(t))+U(c,\epsilon(t))\}I_{\{t\in[0,\Theta)\}}$$

is an optimal solution to Problem 3.1. Moreover, the value function is given by

$$\begin{aligned} V(x,i) &= E_{x,i}\left[\int_0^{\widehat{\Theta}} e^{-\delta s}U(\hat{c}(s),\epsilon(s))\,ds + Q(\widehat{\Theta})\right] \\ &= v(x,i)+\frac{1}{\delta}E_{x,i}\left[\int_0^{\infty} e^{-\delta s}\,dU(0,\epsilon(s))\right], \end{aligned}$$

where we are denoting

$$dU(0,\epsilon(s)) := -\lambda_{\epsilon(s)}U(0,\epsilon(s))\,ds + \sum_{j\neq\epsilon(s)} q_{\epsilon(s)j}\,U(0,j)\,ds.$$

Proof. Consider the function $f(\cdot,\cdot,i)$, $i\in\mathcal{S}$, such that $f(t,X_t,\epsilon_t)=e^{-\delta t}(v(X_t,\epsilon_t)$ $-\,U(0,\epsilon_t)/\delta)$ and consider every admissible control $u=(\pi,c)$. Using the Itô formula for Markov-modulated processes, we get

$$\begin{aligned}
& df(t, X_t, \epsilon_t) \\
&= \frac{1}{2}\pi_t \Sigma_{\epsilon(t)} \pi_t^T X_t^2 e^{-\delta t} v''(X_t, \epsilon_t) dt + (\mu_{\epsilon(t)} - r_{\epsilon(t)} \mathbf{1}) \pi_t^T X_t e^{-\delta t} v'(X_t, \epsilon_t) dt \\
&\quad + (r_{\epsilon(t)} X_t - c_t)\, e^{-\delta t} v'(X_t, \epsilon_t) dt - \delta e^{-\delta t} \left(v(X_t, \epsilon_t) - \frac{U(0, \epsilon_t)}{\delta} \right) dt \\
&\quad + \pi_t\, \sigma_{\epsilon(t)} X_t e^{-\delta t} v'(X_t, \epsilon_t) dW_t^T + Qf(t, X_t, \cdot)(\epsilon_t) dt + dM_t^f \\
&= e^{-\delta t} \left(\frac{1}{2}\pi_t \Sigma_{\epsilon(t)} \pi_t^T X_t^2\, v''(X_t, \epsilon_t) + (\mu_{\epsilon(t)} - r_{\epsilon(t)} \mathbf{1}) \pi_t^T X_t\, v'(X_t, \epsilon_t) \right. \\
&\quad \left. + (r_{\epsilon(t)} X_t - c_t)\, v'(X_t, \epsilon_t) - \delta v(X_t, \epsilon_t) \right) dt + e^{-\delta t} U(0, \epsilon_t)\, dt \\
&\quad + e^{-\delta t} \left(-\lambda_{\epsilon(t)} \left(v(X_t, \epsilon_t) - \frac{U(0, \epsilon_t)}{\delta} \right) + \sum_{j \neq \epsilon(t)} q_{\epsilon(t)j} \left(v(X_t, j) - \frac{U(0, j)}{\delta} \right) \right) dt \\
&\quad + \pi_t\, \sigma_{\epsilon(t)} X_t\, e^{-\delta t} v'(X_t, \epsilon_t)\, dW_t^T + dM_t^f,
\end{aligned}$$

where $q_{ii} = -\lambda_i$, for each $i \in \mathcal{S}$. Also, the process $\{M_t^f, t \geq 0\}$ is a real-valued, square integrable martingale, with $M_0^f = 0$ $\mathbb{P}$-a.s., when $f(\cdot, \cdot, i)$, $i \in \mathcal{S}$, is bounded, i.e., when $v(\cdot, i)$, $i \in \mathcal{S}$, is bounded since $U(0, i)$ is finite for every $i \in \mathcal{S}$ (we will discuss below that $v(X_s, \epsilon_s)$ is bounded for some values of s). Therefore,

$$\begin{aligned}
df(t, X_t, \epsilon_t) &= e^{-\delta t} \left(L_{\epsilon(t)}(u(t)) v(X_t, \epsilon_t) - \lambda_{\epsilon(t)} \left(v(X_t, \epsilon_t) - \frac{U(0, \epsilon_t)}{\delta} \right) \right. \\
&\qquad \left. + \sum_{j \neq \epsilon(t)} q_{\epsilon(t)j} \left(v(X_t, j) - \frac{U(0, j)}{\delta} \right) \right) dt \\
&\qquad + e^{-\delta t} U(0, \epsilon_t)\, dt + \pi_t\, \sigma_{\epsilon(t)} X_t\, e^{-\delta t}\, v'(X_t, \epsilon_t)\, dW_t^T + dM_t^f.
\end{aligned}$$

Moreover, from condition (3.5),

$$df(t, X_t, \epsilon_t) \leq -e^{-\delta t}\,(U(c_t, \epsilon_t) - U(0, \epsilon_t))\,dt + \pi_t\,\sigma_{\epsilon(t)} X_t\,e^{-\delta t}\,v'(X_t, \epsilon_t)\,dW_t^T + dM_t^f.$$

Let a and b satisfy $0 < a < X_0 = x < b < +\infty$, and define the stopping times $\tau_a := \inf\{\, t \geq 0 : X_t = a \,\}$ and $\tau_b := \inf\{\, t \geq 0 : X_t = b \,\}$. Then, for every time $t \in [0, \infty)$, we get

$$\begin{aligned} & f(t \wedge \tau_a \wedge \tau_b, X_{t\wedge\tau_a\wedge\tau_b}, \epsilon_{t\wedge\tau_a\wedge\tau_b}) \\ & \qquad \leq \quad v(X_0, \epsilon_0) - \frac{U(0, \epsilon_0)}{\delta} - \int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s}(U(c_s, \epsilon_s) - U(0, \epsilon_s))\,ds \\ & \qquad\qquad + \int_0^{t\wedge\tau_a\wedge\tau_b} \pi_s\,\sigma_{\epsilon(s)} X_s\,e^{-\delta s}\,v'(X_s, \epsilon_s)\,dW_s^T \;+\; M^f_{t\wedge\tau_a\wedge\tau_b}. \end{aligned}$$

By taking the conditional expectation to both sides of the inequality above given $X_0 = x$ and $\epsilon_0 = i$, we have

$$\begin{aligned} & E_{x,i}[f(t \wedge \tau_a \wedge \tau_b, X_{t\wedge\tau_a\wedge\tau_b}, \epsilon_{t\wedge\tau_a\wedge\tau_b})] \\ & \qquad \leq \; v(x, i) - \frac{U(0, i)}{\delta} \; - \; E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s}\,(U(c_s, \epsilon_s) - U(0, \epsilon_s))\,ds\right] \\ & \qquad\qquad + \; E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} \pi_s\,\sigma_{\epsilon(s)} X_s e^{-\delta s}\,v'(X_s, \epsilon_s)\,dW_s^T\right] \;+\; E_{x,i}[\,M^f_{t\wedge\tau_a\wedge\tau_b}\,]. \end{aligned} \tag{3.7}$$

Inequality (3.7) is well defined because the utility function U is strictly increasing. We note that $U(c_s, \epsilon_s) \geq U(0, \epsilon_s)$ for every $s \geq 0$ implies

$$0 \;\leq\; E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s}\,(U(c_s, \epsilon_s) - U(0, \epsilon_s))\,ds\right].$$

Moreover, the growth condition (3.3) and the fact that c is a consumption rate process guarantee also that

$$E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s}(U(c_s,\epsilon_s)-U(0,\epsilon_s))ds\right] \;\leq\; K_1\, E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s}(1+c_s)ds\right]$$
$$<\; K_1 E_{x,i}\left[\int_0^{t}(1+c_s)ds\right] \;<\; +\infty,$$

where $K_1 = \max\{K, K-\min\{U(0,i), i\in\mathcal{S}\}\}$. Note that u is admissible and that $v'(X_s,\epsilon_s)$ is bounded for every $s\in[\,0, t\wedge\tau_a\wedge\tau_b\,]$ ($v(\cdot,i)$, $i\in\mathcal{S}$, is concave and $\max\{v'(a,\epsilon_{\tau_a}), v'(b,\epsilon_{\tau_b})\} < M$, where M is a finite constant). Then,

$$E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} \pi_s\Sigma_{\epsilon(s)}\pi_s^T X_s^2 e^{-2\delta s}(v'(X_s,\epsilon_s))^2 ds\right] \;<\; M^2b^2E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} \pi_s\Sigma_{\epsilon(s)}\pi_s^T ds\right]$$
$$<\; M^2b^2E_{x,i}\left[\int_0^{t} \pi_s\Sigma_{\epsilon(s)}\pi_s^T ds\right] \;<\; +\infty,$$

for all $t\in[\,0,\infty)$. This implies

$$E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} \pi_s\,\sigma_{\epsilon(s)}X_s e^{-\delta s}\,v'(X_s,\epsilon_s)\,dW_s^T\right] \;=\; 0.$$

We note that $v(a,\epsilon_{\tau_a})$ and $v(b,\epsilon_{\tau_b})$ are finite. Hence, since for every $i\in\mathcal{S}$, $v(\cdot,i)$ is increasing, $v(X_s,\epsilon_s)$ is bounded for every $s\in[\,0, t\wedge\tau_a\wedge\tau_b\,]$. Then, $\{M^f_{t\wedge\tau_a\wedge\tau_b},\, t\geq 0\}$ is a square integrable martingale and hence $E_{x,i}[\,M^f_{t\wedge\tau_a\wedge\tau_b}\,] = E_{x,i}[\,M^f_0\,] = 0$. Therefore, we can write inequality (3.7) as

$$v(x,i) \;\geq\; E_{x,i}[\,f(t\wedge\tau_a\wedge\tau_b, X_{t\wedge\tau_a\wedge\tau_b}, \epsilon_{t\wedge\tau_a\wedge\tau_b})\,]$$
$$+\,E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s}\,(U(c_s,\epsilon_s)-U(0,\epsilon_s))\,ds\right] + \frac{U(0,i)}{\delta}.$$

Then, letting $a\downarrow 0$ and $b\uparrow +\infty$, we get that $\tau_a\to\Theta$ and $\tau_b\to\infty$. Recall that $v(0+,i) = U(0,i)/\delta$, $i\in\mathcal{S}$, that $U(\cdot,i)$ has polynomial growth for every $i\in\mathcal{S}$, and that $U(\cdot,i)-U(0,i)\geq 0$ also for every $i\in\mathcal{S}$. Then, letting $t\to\infty$ and applying the Monotone Convergence Theorem, we have

$$\begin{aligned}(3.8)\ v(x,i) &\geq 0 + E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}\left(U(c_s,\epsilon_s) - U(0,\epsilon_s)\right) ds\right] + \frac{U(0,i)}{\delta} \\ &= J(x,i;u) - E_{x,i}\left[\int_{\Theta}^{\infty} e^{-\delta s}\, U(0,\epsilon_s)\, ds\right] \\ &\qquad - E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}\, U(0,\epsilon_s)\, ds\right] + \frac{U(0,i)}{\delta} \\ &= J(x,i;u) - E_{x,i}\left[\int_0^{\infty} e^{-\delta s}\, U(0,\epsilon_s)\, ds\right] + \frac{U(0,i)}{\delta}.\end{aligned}$$

Using the Itô formula for $g(t,\epsilon_t) = -e^{-\delta t}U(0,\epsilon_t)/\delta$, we get

$$dg(t,\epsilon_t) = e^{-\delta t}U(0,\epsilon_t)\, dt - \frac{1}{\delta}\, e^{-\delta t}\left(-\lambda_{\epsilon(t)}U(0,\epsilon_t) + \sum_{j\neq\epsilon(t)} q_{\epsilon(t)j}\, U(0,j)\right) dt + dM_t^g$$

where $\{M_t^g, t \geq 0\}$ is a square integrable martingale with $\lim_{t\to\infty} E_{x,i}[\,M_t^g\,] = E_{x,i}[\,M_0^g\,] = 0$. Equivalently,

$$\int_0^{\infty} e^{-\delta s}U(0,\epsilon_s)\, ds = \frac{1}{\delta}\, U(0,\epsilon_0) + \frac{1}{\delta}\int_0^{\infty} e^{-\delta s} dU(0,\epsilon_s) - \lim_{t\to\infty} M_t^g.$$

Applying conditional expectation with respect to $X_0 = x$ and $\epsilon_0 = i$ to the expression above, we obtain

$$(3.9)\ E_{x,i}\left[\int_0^{\infty} e^{-\delta s}\, U(0,\epsilon_s)\, ds\right] = \frac{1}{\delta}\, U(0,i) + \frac{1}{\delta}\, E_{x,i}\left[\int_0^{\infty} e^{-\delta s}\, dU(0,\epsilon_s)\right].$$

Then, from equations (3.8) and (3.9), we get

$$v(x,i) \geq J(x,i;u) - \frac{1}{\delta}\, E_{x,i}\left[\int_0^{\infty} e^{-\delta s} dU(0,\epsilon_s)\right],$$

where (π, c) is an arbitrary admissible pair. Note that if $u = \hat{u} = (\hat{\pi}, \hat{c})$ as in

(3.6), then

$$v(x,i) + \frac{1}{\delta} E_{x,i}\left[\int_0^{\infty} e^{-\delta s} dU(0,\epsilon_s)\right]$$
$$= E_{x,i}\left[\int_0^{\widehat{\Theta}} e^{-\delta s} U(\hat{c}_s,\epsilon_s)\,ds + Q(\widehat{\Theta})\right] = V(x,i).$$

□

Now, we consider the special case in which the utility function does not depend on the regime of the market.

Corollary 3.1. Suppose that $U(\cdot,i) = U(\cdot,j)$ for every $i,j \in \mathcal{S}$. Moreover, suppose that the conditions of Theorem 3.1 are satisfied. If the function $v(\cdot,i)$, $i \in \mathcal{S}$, satisfies the Hamilton-Jacobi-Bellman equation

(3.10)
$$\sup_{(\pi,c)\in(-\infty,\infty)^N\times[0,\infty)} \{L_i(\pi,c)\,v(x,i) + U(c,i)\} = \lambda_i\, v(x,i) - \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij}\, v(x,j),$$

then the admissible policy $(\hat{\pi},\hat{c})$ defined in (3.6) is optimal. Furthermore, the value function is given by

$$V(x,i) = v(x,i).$$

Proof. We know from Theorem 3.1 that the Hamilton-Jacobi-Bellman equation is given by (3.5). When $U(\cdot,i) = U(\cdot,j)$, for every $i,j \in \mathcal{S}$, we obtain

$$\sup_{(\pi,c)\in(-\infty,\infty)^N\times[0,\infty)} \{L_i(\pi,c)\,v(x,i) + U(c,i)\}$$
$$= \lambda_i\, v(x,i) - \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij}\, v(x,j) - \lambda_i \frac{U(0,i)}{\delta} + \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij} \frac{U(0,i)}{\delta}$$
$$= \lambda_i\, v(x,i) - \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij}\, v(x,j) + \Big(\sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij} - \lambda_i\Big) \frac{U(0,i)}{\delta},$$

where $\lambda_i = -q_{ii}$. Therefore,

$$\sup_{(\pi,c)\in(-\infty,\infty)^N\times[0,\infty)} \{L_i(\pi,c)\, v(x,i) + U(c,i)\}$$
$$= \lambda_i\, v(x,i) - \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij}\, v(x,j) + \sum_{j\in\mathcal{S}} q_{ij}\, \frac{U(0,i)}{\delta},$$

and from properties of the generator matrix, $\sum_{j\in\mathcal{S}} q_{ij} = 0$, which implies equation (3.10). According to Theorem 3.1, the value function is given by

$$V(x,i) = v(x,i) + \frac{1}{\delta}\, E_{x,i}\left[\int_0^\infty e^{-\delta s} dU(0,\epsilon(s))\right].$$

We note that

$$dU(0,\epsilon(s)) = U(0,i)\left\{-\lambda_{\epsilon(s)} + \sum_{j\neq\epsilon(s)} q_{\epsilon(s)j}\right\} ds = U(0,i) \sum_{j\in\mathcal{S}} q_{\epsilon(s)j}\, ds = 0.$$

Therefore, $V(x,i) = v(x,i)$ for every $x > 0$ and $i \in \mathcal{S}$. □

The following verification theorem considers the case in which $U(0,i) = -\infty$ for every $i \in \mathcal{S}$. We note that in such case

$$E_{x,i}[\,Q(\Theta)\,] = E_{x,i}\left[\int_\Theta^\infty \delta P\, e^{-\delta s} ds\right] = P\, E_{x,i}[\,e^{-\delta\Theta}\,],$$

where $P := U(0,i)/\delta = -\infty$. Hence, we define for this case

$$E_{x,i}\,[\,Q(\Theta)\,] := \begin{cases} 0 & \text{if} \quad E_{x,i}[\,e^{-\delta\Theta}\,] = 0 \\ -\infty & \text{if} \quad E_{x,i}[\,e^{-\delta\Theta}\,] > 0 \end{cases}.$$

Theorem 3.2. Suppose that the utility function does not depend on the market regime, and $U(0,i) = -\infty$ for every $i \in \mathcal{S}$. Let $v(\cdot,i) \in C^2(0,\infty)$, $i \in \mathcal{S}$, be a concave and increasing function in $(0,\infty)$ with $\lim_{x\to 0} v(x,i) = U(0,i) = -\infty$ for every $i \in \mathcal{S}$. Suppose that the function $v(\cdot,i)$, $i \in \mathcal{S}$, satisfies the Hamilton-Jacobi-Bellman equation

(3.11)

$$\sup_{(\pi, c)\in(-\infty,\infty)^N\times[0,\infty)} \{L_i(\pi, c)\, v(x,i) + U(c,i)\} \;=\; \lambda_i\, v(x,i) - \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij}\, v(x,j),$$

for every $x > 0$. Then, the admissible policy

(3.12)

$$\hat{u}(t) = (\hat{\pi}(t), \hat{c}(t)) = \arg \sup_{(\pi,c)\in(-\infty,\infty)^N\times[0,\infty)} \{L_{\epsilon(t)}((\pi, c))\, v(X(t), \epsilon(t)) + U(c, \epsilon(t))\}\cdot 1_{\{t\in[0,\Theta)\}},$$

is optimal solution to Problem 3.1. Moreover,

$$V(x,i) \;=\; v(x,i) \;=\; E_{x,i}\left[\int_0^{\widehat{\Theta}} e^{-\delta s} U(\hat{c}(s), \epsilon(s))\, ds \;+\; Q(\widehat{\Theta})\right].$$

Proof. Consider for this case the function $h(\cdot,\cdot,i)$, $i \in \mathcal{S}$, such that $h(t, X_t, \epsilon_t) = e^{-\delta t} v(X_t, \epsilon_t)$ and consider an arbitrary admissible control $u = (\pi, c)$. Similarly to the proof in Theorem 3.1, we use the Itô formula for Markov-modulated processes and condition (3.11) to obtain

$$dh(t, X_t, \epsilon_t) \;\leq\; -e^{-\delta t}\, U(c_t, \epsilon_t)\, dt + \pi_t\, \sigma_{\epsilon(t)} X_t\, e^{-\delta t}\, v'(X_t, \epsilon_t)\, dW_t^T \;+\; dM_t^h,$$

where the equality holds for the admissible control process $\hat{u}$ defined by (3.12).

The process $\{M_t^h, t \geq 0\}$ is a real-valued, square integrable martingale, with $M_0^h = 0$ $\mathbb{P}$-a.s., when $v(\cdot, i)$, $i \in \mathcal{S}$, is bounded (in this case too, we will discuss later in the proof that $v(X_s, \epsilon_s)$ is bounded for some values of s). Let a and b satisfy $0 < a < X_0 = x < b < +\infty$, and define the stopping times $\tau_a := \inf\{\, t \geq 0 : X_t = a \,\}$ and $\tau_b := \inf\{\, t \geq 0 : X_t = b \,\}$. Then, for every time $t \in [0, \infty)$, we get

$$
\begin{aligned}
& h(t \wedge \tau_a \wedge \tau_b, X_{t\wedge\tau_a\wedge\tau_b}, \epsilon_{t\wedge\tau_a\wedge\tau_b}) \\
& \qquad \leq \quad v(X_0, \epsilon_0) - \int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s} U(c_s, \epsilon_s)\, ds \\
& \qquad\quad + \int_0^{t\wedge\tau_a\wedge\tau_b} \pi_s\, \sigma_{\epsilon(s)} X_s\, e^{-\delta s}\, v'(X_s, \epsilon_s)\, dW_s^T \; + \; M^h_{t\wedge\tau_a\wedge\tau_b}.
\end{aligned}
$$

By taking conditional expectation to both sides of the inequality above given $X_0 = x$ and $\epsilon_0 = i$, we have

$$
\begin{aligned}
& E_{x,i}[\, h(t \wedge \tau_a \wedge \tau_b, X_{t\wedge\tau_a\wedge\tau_b}, \epsilon_{t\wedge\tau_a\wedge\tau_b})\,] \\
& \quad \leq \; v(x,i) - E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s}\, U(c_s, \epsilon_s)\, ds\right] \\
& \qquad + E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} \pi_s\, \sigma_{\epsilon(s)} X_s e^{-\delta s}\, v'(X_s, \epsilon_s)\, dW_s^T\right] + E_{x,i}[\, M^h_{t\wedge\tau_a\wedge\tau_b}\,] \\
& \quad = \; v(x,i) - E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s} U^+(c_s, \epsilon_s) ds\right] + E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s} U^-(c_s, \epsilon_s) ds\right] \\
& \qquad + E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} \pi_s\, \sigma_{\epsilon(s)} X_s\, e^{-\delta s} v'(X_s, \epsilon_s)\, dW_s^T\right] + E_{x,i}[\, M^h_{t\wedge\tau_a\wedge\tau_b}\,].
\end{aligned}
\tag{3.13}
$$

Inequality (3.13) is well defined. In fact, the growth condition (3.3) and the fact that c is a consumption rate process imply that

$$
\begin{aligned}
0 \leq E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s} U^+(c_s, \epsilon_s) ds\right] \; &= \; E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s} U(c_s, \epsilon_s) I_{\{U(c_s)>0\}} ds\right] \\
&\leq \; K\, E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s} (1 + c_s) I_{\{U(c_s)>0\}} ds\right] \\
&\leq \; K\, E_{x,i}\left[\int_0^{t} (1 + c_s) ds\right] \; < \; +\infty.
\end{aligned}
$$

Moreover, due to the admissibility condition (3.4),

$$0 \leq E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s}U^-(c_s,\epsilon_s)ds\right] \leq E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}U^-(c_s,\epsilon_s)ds\right] < +\infty.$$

Furthermore, due to the concavity of $v(\cdot,i)$, $i \in \mathcal{S}$, to the fact that $v'(X_s,\epsilon_s) \leq \max\{v'(a,i), i \in \mathcal{S}\}$ for every $s \in [0, t\wedge\tau_a\wedge\tau_b]$ and to the admissibility condition of the portfolio process π, we obtain that for every $t \in [0,\infty)$,

$$E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} \pi_s\,\sigma_{\epsilon(s)}X_s e^{-\delta s}\,v'(X_s,\epsilon_s)\,dW_s^T\right] = 0.$$

In addition, since $v(X_s,\epsilon_s)$ is bounded in the region $s \in [0, t\wedge\tau_a\wedge\tau_b]$, the process $\{M^f_{t\wedge\tau_a\wedge\tau_b},\ t \geq 0\}$ is a square integrable martingale and hence $E_{x,i}[\,M^f_{t\wedge\tau_a\wedge\tau_b}\,] = E_{x,i}[\,M^f_0\,] = 0$. Hence, we rewrite inequality (3.13) as

$$\begin{aligned} v(x,i) \;\geq\; & E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s}U^+(c_s,\epsilon_s)ds\right] - E_{x,i}\left[\int_0^{t\wedge\tau_a\wedge\tau_b} e^{-\delta s}U^-(c_s,\epsilon_s)ds\right] \\ & + E_{x,i}\left[e^{-\delta(t\wedge\tau_a\wedge\tau_b)}v(X_{t\wedge\tau_a\wedge\tau_b},\epsilon_{t\wedge\tau_a\wedge\tau_b})\right]. \end{aligned}$$

Then, letting $a \downarrow 0$ and $b \uparrow +\infty$, we get that $\tau_a \to \Theta$ and $\tau_b \to \infty$. Recall that $v(0+,i) = P$ and that $v(\cdot,i)$ is concave and increasing for every $i \in \mathcal{S}$. Moreover, recall that $U(\cdot,i)$, $i \in \mathcal{S}$, satisfies the growth condition (3.3) and that the admissibility condition (3.4) holds for the control $u = (\pi, c)$. Then, letting $t \to \infty$ and applying both the Dominated Convergence Theorem and the Monotone Convergence Theorem, we have that

$$\begin{aligned} v(x,i) &\geq E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}U^{+}(c_s,\epsilon_s)ds - \int_0^{\Theta} e^{-\delta s}U^{-}(c_s,\epsilon_s)ds\right] + E_{x,i}[P\,e^{-\delta\Theta}] \\ &= E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}U(c_s,\epsilon_s)\,ds + P\,e^{-\delta\Theta}\right] \\ &= E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}U(c_s,\epsilon_s)\,ds + Q(\Theta)\right]. \end{aligned}$$

We note that the equality holds if $u = \hat{u} = (\hat{\pi}, \hat{c})$ as in (3.12). □

The verification theorems that we have presented in this section assume that the solution of the Hamilton-Jacobi-Bellman equation is smooth. That is good enough for our objective of finding explicit solutions for the consumption-investment problems of this chapter. However, the solution of the Hamilton-Jacobi-Bellman equation would not be smooth in the case of portfolio constraints, and it would be necessary to apply a non-smooth version of the above verification theorems (see, for instance, Yong and Zhou (1999) for the theory of non-smooth value functions for classical stochastic control problems without regime switching).

Remark 3.1. We note that if $U(0,i)$ is finite for every $i \in \mathcal{S}$, the admissibility condition (3.4) is immediately satisfied by every consumption rate process. In fact, for a finite $U(0,i)$, the increasing property of $U(\cdot,i)$ already implies that $U^{-}(\cdot,i)$ is bounded by above and below.

Remark 3.2. The verification theorems and corollary of this section can be easily generalized to the case in which the coefficients of the market (μ, r, and σ) depend not only on the regime but also on time. However, the results presented in this section are exactly what we need to obtain the explicit solutions of the next section.

3.3 Construction of the solution and examples of explicit solutions

In this section, we obtain explicit solutions for consumption-investment problems in financial markets with regime switching. We first find a candidate for optimal consumption-investment and then we prove rigorously that such candidate is indeed optimal.

We want to find a function $v(\cdot, i)$, $i \in \mathcal{S}$, that satisfies the conditions of the verification theorems. We note that equation (3.5) is equivalent to

$$\sup_{\pi\in(-\infty,\infty)^N} \{h(\pi, x, i)\} + \sup_{c\geq 0}\{g(c, x, i)\} - \delta v(x, i)$$

$$= \lambda_i \left(v(x,i) - \frac{U(0,i)}{\delta}\right) - \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij} \left(v(x,j) - \frac{U(0,j)}{\delta}\right),$$

where

$$h(\pi, x, i) := \frac{1}{2}\pi\,\Sigma_i\,\pi^T x^2\, v''(x,i) + (\mu_i - r_i\mathbf{1})\,\pi^T x v'(x,i)$$

and

$$g(c, x, i) := (r_i x - c)\, v'(x,i) + U(c,i).$$

We conjecture that $v(\cdot, i)$, $i \in \mathcal{S}$, is strictly concave, i.e., $v''(x,i) < 0$ for all $x > 0$, $i \in \mathcal{S}$. Then, for every $x > 0$ such that $v''(x,i) < 0$, we have

$$\text{(3.14)} \qquad \widehat{\Pi}(x,i) := \arg\sup_{\pi\in\mathbb{R}^N} \{h(\pi, x, i)\} = -\frac{v'(x,i)}{xv''(x,i)}(\mu_i - r_i\mathbf{1})\Sigma_i^{-1}$$

and

$$\text{(3.15)} \qquad h(\widehat{\Pi}, x, i) = -\frac{1}{2}\frac{v'(x,i)^2}{v''(x,i)}(\mu_i - r_i\mathbf{1})\Sigma_i^{-1}(\mu_i - r_i\mathbf{1})^T = -\frac{v'(x,i)^2}{v''(x,i)}\gamma_i > 0,$$

where we are denoting

$$\gamma_i \;:=\; \frac{1}{2}\,(\mu_i - r_i\mathbf{1})\,\Sigma_i^{-1}\,(\mu_i - r_i\mathbf{1})^T \;\geq\; 0, \qquad i \in \mathcal{S}.$$

We will also denote $\vec{\gamma} = (\gamma_1, \ldots, \gamma_S)$ and $\vec{r} = (r_1, \ldots, r_S)$.

Furthermore, we conjecture that $v(\cdot, i)$, $i \in \mathcal{S}$, is strictly increasing, i.e., $v'(x,i) > 0$ for all $x > 0$, $i \in \mathcal{S}$. Then,

$$\widehat{C}(x,i) \;:=\; \arg\sup_{c\geq 0}\{\, g(c,x,i)\,\} \;=\; I(v'(x,i),i) \tag{3.16}$$

where $I(\cdot, i) = (U'(\cdot,i))^{-1}$, $i \in \mathcal{S}$, is the inverse function of $U'(\cdot, i)$, $i \in \mathcal{S}$. Note that for HARA utility functions, such inverse always exists on $(0, \infty)$. Also,

$$g(\widehat{C},x,i) \;=\; (r_i x - I(v'(x,i),i))\,v'(x,i) + U\left(I(v'(x,i),i),i\right). \tag{3.17}$$

Then, replacing (3.15) and considering (3.16) in equation (3.5), we have

$$\begin{aligned}\gamma_i\,\frac{v'(x,i)^2}{v''(x,i)} - (r_i x - \widehat{C})v'(x,i) - U(\widehat{C},i) + (\delta+\lambda_i)v(x,i) - \lambda_i\frac{U(0,i)}{\delta} \\ = \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij}\left(v(x,j) - \frac{U(0,j)}{\delta}\right).\end{aligned} \tag{3.18}$$

Note that in the case when the utility function does not depend on the market regime, Corollary 3.1 and Theorem 3.2 tell us that we need equation (3.10), or equivalently (3.11), to be satisfied. Following a similar procedure as above and noticing that $\sum_{j\in\mathcal{S}} q_{ij} = 0$, we see that equations (3.10) and (3.11) are equivalent to

$$\gamma_i\,\frac{v'(x,i)^2}{v''(x,i)} - (r_i x - \widehat{C})v'(x,i) - U(\widehat{C},i) + (\delta+\lambda_i)v(x,i) \;=\; \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij}\,v(x,j). \tag{3.19}$$

In order to find an explicit function $v(\cdot, i)$, $i \in \mathcal{S}$, that satisfies either equation (3.18) or equation (3.19), we need to specify the utility function $U(\cdot, i)$, $i \in \mathcal{S}$. We will consider four utility functions: the first three have been selected because of their importance in financial economics, and the last one to show how our method can be applied when a utility function depends on the regime. The first three cases assume that the utility function does not depend on the market regime:

1. $U(x, i) = \ln x$, $x > 0$,

2. $U(x, i) = \kappa - x^{\alpha}$, $x > 0$, where $\alpha < 0$ and $\kappa \geq 0$,

3. $U(x, i) = x^{\alpha}$, $x > 0$, where $0 < \alpha < 1$,

for every $i \in \mathcal{S}$. The fourth case considers two regimes of the economy, and that the utility function varies on them. Indeed, the utility function maintains the same power form in both regimes but changes its parameters according to them. Specifically,

4. $U(x, i) = \beta_i x^{1/2}$, $x > 0$, where $\beta_i > 0$, $i = 1, 2$.

It is possible to verify that these four functions satisfy all the requirements for being utility functions. For a fixed i, they are $C^3(0, \infty)$, strictly increasing and strictly concave in $(0, \infty)$, and also $\lim_{x \to \infty} U'(x, i) = 0$ in the four cases. Indeed, for a fixed i, each $U(\cdot, i)$ belongs to the Hyperbolic Absolute Risk Aversion (HARA) class of utility functions. Moreover, these four utility functions satisfy a polynomial growth condition of the type $U(x, i) \leq K(1 + x)$ for suitable constants $K > 0$. In fact, $K = 1$ for the first and third utility functions and $K = \kappa + 1$ for the second one. The last utility function uses $K = \max\{\beta_1, \beta_2\}$.

For each of the selected HARA utility functions, Problem 3.1 changes into the following particular problems:

Problem 3.2. Select an admissible policy $\hat{u} = (\hat{\pi}, \hat{c}) \in \mathcal{A}$ that maximizes

$$J_1(x, i; u) := \mathrm{E}_{x,i}\left[\int_0^{\Theta} e^{-\delta t}\ln(c_t)\, dt \,+\, P\, e^{-\delta\Theta}\right],$$

where $P = -\infty$.

Problem 3.3. Select an admissible policy $\hat{u} = (\hat{\pi}, \hat{c}) \in \mathcal{A}$ that maximizes

$$J_2(x, i; u) \;:=\; \mathrm{E}_{x,i}\left[\int_0^{\Theta} e^{-\delta t}\,(\kappa - c(t)^{\alpha})\, dt \,+\, P\, e^{-\delta\Theta}\right],$$

where $\alpha < 0$, $\kappa \geq 0$ and $P = -\infty$.

Problem 3.4. Select an admissible policy $\hat{u} = (\hat{\pi}, \hat{c}) \in \mathcal{A}$ that maximizes

$$J_3(x, i; u) \;:=\; \mathrm{E}_{x,i}\left[\int_0^{\Theta} e^{-\delta t}\, c(t)^{\alpha}\, dt\right],$$

where $0 < \alpha < 1$.

Problem 3.5. Select an admissible policy $\hat{u} = (\hat{\pi}, \hat{c}) \in \mathcal{A}$ that maximizes

$$J_4(x, i; u) \;:=\; \mathrm{E}_{x,i}\left[\int_0^{\Theta} e^{-\delta t}\, \beta_{\epsilon(t)}\, c(t)^{1/2}\, dt\right].$$

In order to have a well defined consumption-investment problem, we need to impose an additional condition over any power utility function $U(\cdot, i)$, $i \in \mathcal{S}$. If we consider $U(x, i) = x^{\alpha}$ such that $0 < \alpha < 1$, $i \in \mathcal{S}$, as in Problem 4.3, the parameter α must also satisfy the inequalities

$$\delta \,>\, r_i\,\alpha + \frac{1}{2}\,\frac{\alpha}{1-\alpha}\,(\mu_i - r_i\mathbf{1})\,\Sigma_i^{-1}\,(\mu_i - r_i\mathbf{1})^T \tag{3.20}$$

for every $i \in \mathcal{S}$. We will see later that this condition guarantees the existence of a concave increasing value function for this specific utility function. For

Problem 4.4, the condition that we need is

$$\delta > \frac{1}{2} r_i + \frac{1}{2} (\mu_i - r_i \mathbf{1}) \Sigma_i^{-1} (\mu_i - r_i \mathbf{1})^T, \qquad i = 1, 2. \tag{3.21}$$

3.3.1 $U(x,i) = \ln x$ for every $i \in \mathcal{S}$.

In this case we have $U'(x,i) = 1/x$ and therefore $I(x,i) = 1/x$. Hence, from equation (3.16), we have

$$\widehat{C}(x,i) = \frac{1}{v'(x,i)} > 0$$

for $x > 0$. Replacing this value into equation (3.19), we get the following system of S *nonlinear* ordinary differential equations:

$$\gamma_i \frac{v'(x,i)^2}{v''(x,i)} - r_i x\, v'(x,i) + 1 + \ln v'(x,i) + (\delta + \lambda_i)\, v(x,i) = \sum_{j \in \mathcal{S} \setminus \{i\}} q_{ij}\, v(x,j),$$

$i = 1, 2, \ldots, S$. We find that the solution for this system is

$$v(x,i) = \frac{1}{\delta} \ln \delta x + A_i, \qquad i \in \mathcal{S},$$

where the vector $\vec{A} = (A_1, \ldots, A_S)$ satisfies the equation

$$\vec{A} \cdot (\delta\, \mathbb{I} - Q^T) = \frac{1}{\delta} (\vec{\gamma} + \vec{r} - \delta \mathbf{1}), \tag{3.22}$$

and $\mathbb{I}$ represents the $S \times S$ identity matrix.

Note that $v'(x,i) = 1/\delta x > 0$ for all $x > 0$ and also $v''(x,i) = -1/\delta x^2 < 0$ for all $x > 0$, for every $i \in \mathcal{S}$. That is, $v(\cdot, i)$, $i \in \mathcal{S}$, is a strictly increasing and strictly concave function in $(0, \infty)$. Note also that $v(0,i) = -\infty$, for every

$i \in \mathcal{S}$. Thus, from equations (3.14) and (3.16),

$$\widehat{\Pi}(x,i) \;=\; (\mu_i - r_i \mathbf{1})\,\Sigma_i^{-1} \qquad \text{and} \qquad \widehat{C}(x,i) \;=\; \delta x.$$

Hence, the candidate for optimal policy $\hat{u}$ is given by

$$\hat{u}(t) \;=\; (\hat{\pi}(t), \hat{c}(t)) \;=\; \Big(\big(\mu_{\epsilon(t)} - r_{\epsilon(t)}\mathbf{1}\big)\; \Sigma_{\epsilon(t)}^{-1}\,,\; \delta \hat{X}(t) \Big),$$

for $t \in [0, \Theta)$. We note that

$$E_{x,i}\left[\int_0^t \hat{\pi}(s)\Sigma_{\epsilon(s)}\hat{\pi}(s)^T\,ds\right] \;=\; 2\,E_{x,i}\left[\int_0^t \gamma_{\epsilon(s)}\,ds\right] \;<\; 2\bar{\gamma}\mathbf{1}^T t \;<\; +\infty$$

for all $t \in [0, \infty)$. Moreover, replacing this policy $\hat{u}$ in (3.1) we get

$$dX^{\hat{u}}(t) \;=\; (2\gamma_{\epsilon(t)} + r_{\epsilon(t)} - \delta)\,X^{\hat{u}}(t)\,dt + (\mu_{\epsilon(t)} - r_{\epsilon(t)}\mathbf{1})(\sigma_{\epsilon(t)}^{-1})^T\,X^{\hat{u}}(t)\,dW_t^T.$$

That is,

(3.23)

$$\hat{X}(t) \;=\; x\,\exp\left\{\int_0^t (\gamma_{\epsilon(s)} + r_{\epsilon(t)} - \delta)\,ds + \int_0^t (\mu_{\epsilon(s)} - r_{\epsilon(t)}\mathbf{1})(\sigma_{\epsilon(s)}^{-1})^T\,dW_s^T\right\}.$$

We observe that $\hat{X}$ is similar to a geometric Brownian motion and, when $x > 0$, $\hat{X}(t) > 0$ for every $t \in [0, \infty)$. Hence, $\widehat{\Theta} = \Theta^{\hat{X}} = +\infty$ $\mathbb{P}$-a.s. in this case. In addition, for every $t \in [0, \infty)$,

$$E_{x,i}\left[\int_0^t \hat{c}(s)\,ds\right] \;\leq\; \delta\,x \int_0^t \exp\{(2\bar{\gamma}\,\mathbf{1}^T + \bar{r}\,\mathbf{1}^T)s\}\,ds \;<\; +\infty.$$

Furthermore,

$$
\begin{aligned}
E_{x,i}\left[\int_0^{\widehat{\Theta}} e^{-\delta t}|\ln(\hat{c}(t))|\,dt\right] &\le E_{x,i}\left[\int_0^{\widehat{\Theta}} e^{-\delta t}\left(|ln(\delta x)| + \int_0^t |r_{\epsilon(s)} - \delta|ds\right)dt\right] \\
&\le \frac{1}{\delta}|\ln(\delta x)| + \frac{1}{\delta^2}\max\{|r_i - \delta|, i \in \mathcal{S}\} \;<\; +\infty.
\end{aligned}
$$

Thus, $\hat{u}(t) = (\hat{\pi}(t), \hat{c}(t))$ is an admissible policy. Now, we are going to prove rigorously that the function $v(\cdot, i)$, $i \in \mathcal{S}$, and the policy $\hat{u}$ are, respectively, the value function and optimal control for Problem 3.2.

Proposition 3.1. The function $v(\cdot, i)$, $i \in \mathcal{S}$, given by

$$
v(x,i) \;=\; \begin{cases} \frac{1}{\delta}\ln \delta x + A_i, & x > 0 \\ -\infty, & x = 0 \end{cases}
$$

where $\vec{A} = (A_1, \ldots, A_S)$ satisfies the equation (3.22), is the value function of Problem 3.2. Furthermore, the policy $\hat{u}$ given by

$$
\hat{u}(t) \;=\; (\hat{\pi}(t), \hat{c}(t)) \;=\; \Big(\, (\mu_{\epsilon(t)} - r_{\epsilon(t)}\mathbf{1})\,\Sigma^{-1}_{\epsilon(t)}\,,\; \delta\hat{X}(t)\,\Big) \tag{3.24}
$$

is an optimal solution of Problem 3.2.

Proof. In this case $U(0,i) = -\infty$, $i \in \mathcal{S}$, and $\widehat{\Theta} = +\infty$ $\mathbb{P}$-a.s. as shown above. Then it is enough to show that the function $v(\cdot, i)$, $i \in \mathcal{S}$ defined above satisfies the conditions of Theorem 3.2. It is easy to see that $v(\cdot, i) \in C^2(0,\infty)$, $i \in \mathcal{S}$. Moreover, we showed previously that $v(\cdot, i)$, $i \in \mathcal{S}$, is strictly increasing and strictly concave in $(0,\infty)$. By definition, $\lim_{x\to 0} v(x,i) = U(0,i)/\delta = -\infty$, $i \in \mathcal{S}$. We also note that by construction, the equality

$$
L_i(\widehat{\Pi}(x,i), \widehat{C}(x,i))v(x,i) + U(\widehat{C}(x,i),i) \;=\; \lambda_i\, v(x,i) - \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij}\, v(x,j)
$$

holds for every $x > 0$ and every $i \in \mathcal{S}$, where $\widehat{\Pi}(x,i) = (\mu_i - r_i\mathbf{1})\Sigma_i^{-1}$ and $\widehat{C}(x,i) = \delta x$. Hence,

$$L_i(\hat{u}_t)v(X_t,\epsilon_t) + U(\hat{c}_t,\epsilon_t) = \lambda_{\epsilon(t)}\, v(X_t,\epsilon_t) - \sum_{j\in\mathcal{S}\setminus\{\epsilon(t)\}} q_{\epsilon(t)j}\, v(X_t,j)$$

holds for every $t \in [0,\infty)$ and $\omega \in \Omega$, where $\hat{u}$ is given by (3.24) for $t \in [0,\infty)$. We have shown above that this policy $\hat{u}$ is admissible. Therefore, from Theorem 3.2, $\hat{u}$ is an optimal solution to Problem 3.2 and $v(\cdot,i)$, $i \in \mathcal{S}$, is the corresponding value function. □

Corollary 3.2. In the special case when $S = 2$, we have

$$v(x,i) = \begin{cases} \frac{1}{\delta}\ln\delta x + A_i, & x > 0 \\ 0, & x = 0 \end{cases} \qquad i = 1,2,$$

and

$$A_i = \frac{(\gamma_i + r_i - \delta)(\lambda_{3-i} + \delta) + (\gamma_{3-i} + r_{3-i} - \delta)\lambda_i}{\delta^2(\delta + \lambda_1 + \lambda_2)}, \qquad i = 1,2.$$

Proof. When we only have two states $i = 1,2$, the generator matrix satisfies $q_{12} = \lambda_1$ and $q_{21} = \lambda_2$. Hence, $\det(\delta\mathbb{I} - Q^T) = \delta(\delta + \lambda_1 + \lambda_2) \neq 0$ and equation (3.22) gives

$$\vec{A} = \frac{1}{\delta}(\gamma_1 + r_1 - \delta, \gamma_2 + r_2 - \delta)\begin{pmatrix} \delta + \lambda_1 & -\lambda_2 \\ -\lambda_1 & \delta + \lambda_2 \end{pmatrix}^{-1}.$$

A few more simple calculations lead to the expression shown above. □

3.3.2 $U(x,i) = \kappa - x^\alpha$ for every $i \in \mathcal{S}$, where $\alpha < 0$ and $\kappa \geq 0$.

For the second HARA utility function, we have $U'(x,i) = -\alpha\, x^{-(1-\alpha)}$ and therefore $I(x,i) = (-\alpha/x)^{1/(1-\alpha)}$, $x > 0$. Hence, from equation (3.16), we obtain

$$\widehat{C}(x,i) = (-\alpha)^{1/(1-\alpha)} v'(x,i)^{-1/(1-\alpha)} > 0$$

whenever $v'(x,i) > 0$. Replacing this value into equation (3.19), we get the following system of *nonlinear* ordinary differential equations:

$$\gamma_i \frac{v'(x,i)^2}{v''(x,i)} - r_i x v'(x,i) + \left((-\alpha)^{1/(1-\alpha)} + (-\alpha)^{\alpha/(1-\alpha)}\right) v'(x,i)^{-\alpha/(1-\alpha)} + (\delta + \lambda_i)\, v(x,i) - \kappa = \sum_{j \in \mathcal{S}\setminus\{i\}} q_{ij}\, v(x,j),$$

$i = 1, 2, \ldots, S$. We find that the solution of the system is given by

$$v(x,i) = -\breve{A}_i^{1-\alpha} x^\alpha + \frac{\kappa}{\delta}, \qquad i \in \mathcal{S},$$

where the coefficients $\breve{A}_i$, $i \in \mathcal{S}$, satisfy the following system of nonlinear equations

$$\left(\delta + \lambda_i - r_i\,\alpha - \gamma_i\,\frac{\alpha}{1-\alpha}\right) \breve{A}_i^{1-\alpha} - (1-\alpha)\,\breve{A}_i^{-\alpha} = \sum_{j \in \mathcal{S}\setminus\{i\}} q_{ij}\,\breve{A}_j^{1-\alpha}, \tag{3.25}$$

$i = 1, 2 \ldots, S$. Let us denote, for each $i \in \mathcal{S}$,

$$\eta_i := \delta + \lambda_i - r_i\alpha - \gamma_i\,\frac{\alpha}{1-\alpha}.$$

We note that for $\alpha < 0$, condition (3.20) is always satisfied because $\delta > 0$. Then, $\eta_i > \lambda_i$ for every $i \in \mathcal{S}$.

Lemma 3.1. *The system* (3.25) *has a unique real solution* $\breve{A}_i$, $i \in \mathcal{S}$, *such that* $\breve{A}_i \geq (1-\alpha)/\max_{k\in\mathcal{S}}\{\eta_k - \lambda_k\} > 0$, $i \in \mathcal{S}$.

Proof. The proof of this lemma is presented in Appendix B. □

We will only consider the unique positive solution described in Lemma 3.1. Thus, the function $v(\cdot, i)$, $i \in \mathcal{S}$, is strictly increasing and strictly concave in $(0, \infty)$. In fact, for each $i \in \mathcal{S}$, $v'(x,i) = -\alpha\, \breve{A}_i^{1-\alpha}/x^{1-\alpha} > 0$ for all $x > 0$, and $v''(x,i) = \alpha(1-\alpha)\breve{A}_i^{1-\alpha}/x^{2-\alpha} < 0$ for all $x > 0$, since $\alpha > 0$.

Furthermore, from equations (3.14) and (3.16),

$$\widehat{\Pi}(x,i) = \frac{1}{1-\alpha}(\mu_i - r_i\mathbf{1})\,\Sigma_i^{-1} \qquad \text{and} \qquad \widehat{C}(x,i) = \frac{1}{\breve{A}_i}\,x.$$

Thus, a candidate for optimal policy $\hat{u}$ is given by

$$\hat{u}(t) = (\hat{\pi}(t), \hat{c}(t)) = \left(\frac{1}{1-\alpha}(\mu_{\epsilon(t)} - r_{\epsilon(t)}\mathbf{1})\Sigma_{\epsilon(t)}^{-1},\ \frac{1}{\breve{A}_{\epsilon(t)}}\hat{X}(t) \right),$$

for $t \in [0, \widehat{\Theta})$. We note that

$$E_{x,i}\left[\int_0^t \hat{\pi}(s)\Sigma_{\epsilon(s)}\hat{\pi}(s)^T ds\right] = \frac{2}{(1-\alpha)^2}E_{x,i}\left[\int_0^t \gamma_{\epsilon(s)}ds\right] < 2\frac{\bar{\gamma}\mathbf{1}^T}{(1-\alpha)^2}t < +\infty$$

for all $t \in [0, \infty)$. Moreover, replacing the policy $\hat{u}$ in (3.1) we get that the wealth process generated by $\hat{u}$ is given, for $t \in [0, \widehat{Theta})$, by

$$\begin{aligned}\hat{X}(t) = x \exp\Bigg\{ &\int_0^t \left(\frac{1-2\alpha}{(1-\alpha)^2}\gamma_{\epsilon(s)} + r_{\epsilon(s)} - \frac{1}{\breve{A}_{\epsilon(s)}} \right) ds \\ &+ \int_0^t \frac{1}{1-\alpha}(\mu_{\epsilon(s)} - r_{\epsilon(s)}\mathbf{1})(\sigma_{\epsilon(s)}^{-1})^T dW_s^T \Bigg\}.\end{aligned}$$

We observe that $\widehat{\Theta} = +\infty$ $\mathbb{P}$-a.s., because $\hat{X}(t) > 0$ for all $t \in [0, \infty)$. Then, for all $t \in [0, \infty)$,

$$E_{x,i}\left[\int_0^t \hat{c}(s)\, ds\right] \le x\breve{A}\int_0^t \exp\left\{\left(2\frac{\bar{\gamma}\mathbf{1}^T}{1-\alpha} + \bar{r}\,\mathbf{1}^T\right)s\right\} ds < +\infty,$$

where $\breve{A} := \sum_{i=1}^{S} \breve{A}_i^{-1}$. Moreover,

$$E_{x,i}\left[\int_0^{\hat{\Theta}} e^{-\delta t}|\kappa - \hat{c}^{\alpha}(t)|dt\right] \le \frac{\kappa}{\delta} + \tilde{K}\int_0^{\infty} \exp\left\{\left(-\delta + 2\frac{\alpha\bar{\gamma}\,\mathbf{1}^T}{1-\alpha} + \alpha\bar{r}\,\mathbf{1}^T\right)t\right\} dt$$
$$< +\infty,$$

where $\tilde{K} := x^{\alpha}\sum_{i=1}^{S} \breve{A}_i^{-\alpha}/(2\frac{\alpha\bar{\gamma}\mathbf{1}^T}{1-\alpha} + \alpha\bar{r}\,\mathbf{1}^T)$, because $\bar{\gamma}\mathbf{1}^T \ge 0$, $\bar{r}\mathbf{1}^T \ge 0$ and $\alpha < 0$. Hence, the policy $\hat{u}$ is admissible. Now we are going to prove rigorously that the function v and the control $\hat{u}$ are the value function and optimal policy for Problem 3.3.

Proposition 3.2. Let $\breve{A}_i$, $i \in \mathcal{S}$, be the unique positive solution of the system of equations (3.25) described by Lemma 3.1. Then, the function $v(\cdot, i)$, $i \in \mathcal{S}$, defined by

$$v(x,i) = \begin{cases} -\breve{A}_i^{1-\alpha}x^{\alpha} + \frac{\kappa}{\delta}, & x > 0 \\ -\infty, & x = 0 \end{cases}$$

is the value function $V(\cdot, i)$, $i \in \mathcal{S}$, of Problem 3.3. Moreover, the policy $\hat{u}$ defined by

$$\text{(3.26)} \quad \hat{u}(t) = (\hat{\pi}(t), \hat{c}(t)) = \left(\frac{1}{1-\alpha}(\mu_{\epsilon(t)} - r_{\epsilon(t)}\mathbf{1})\Sigma_{\epsilon(t)}^{-1},\ \frac{1}{\breve{A}_{\epsilon(t)}}\hat{X}(t)\right),$$

for $t \in [0, \infty)$, is an optimal solution of Problem 3.3.

Proof. By definition $v(\cdot, i) \in C^2(0, \infty)$, $i \in \mathcal{S}$, and $\lim_{x\to 0} v(x,i) = U(0,i) = -\infty$, $i \in \mathcal{S}$. Moreover, $v(\cdot, i)$, $i \in \mathcal{S}$, is strictly increasing and strictly concave in $(0, \infty)$ because $\breve{A}_i > 0$ for every $i \in \mathcal{S}$, as shown above. Also by construction, the equality

$$L_i(\widehat{\Pi}(x,i),\widehat{C}(x,i))v(x,i) + U(\widehat{C}(x,i),i) \;=\; \lambda_i\, v(x,i) - \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij}\, v(x,j)$$

holds for every $x > 0$ and every $i \in \mathcal{S}$, where $\widehat{\Pi}(x,i) = (1-\alpha)^{-1}(\mu_i - r_i\mathbf{1})\Sigma_i^{-1}$ and $\widehat{C}(x,i) = \breve{A}_i^{-1}x$. Hence,

$$L_i(\hat{u}_t)v(\hat{X}_t,\epsilon_t) + U(\hat{c}_t,\epsilon_t) \;=\; \lambda_{\epsilon(t)}\, v(\hat{X}_t,\epsilon_t) - \sum_{j\in\mathcal{S}\setminus\{\epsilon(t)\}} q_{\epsilon(t)j}\, v(\hat{X}_t,j)$$

holds for every $t \in [0,\infty)$ and $\omega \in \Omega$, where $\hat{u}$ is given by (3.26) for $t \in [0,\infty)$. It has been proved above that this policy is in fact admissible and $\widehat{\Theta} = +\infty$ a.s. Therefore, from Theorem 3.2, $\hat{u}$ defined by (3.26) is an optimal solution to Problem 3.3 and $v(\cdot,i)$, $i \in \mathcal{S}$, is the corresponding value function. □

Corollary 3.3. In the special case when $S = 2$, the value function is defined by $V(x,i) = v(x,i) = -\breve{A}_i^{1-\alpha}x^\alpha + \frac{\kappa}{\delta}$ for $x > 0$ and $i = 1,2$, where

$$\text{(3.27)}\qquad \begin{aligned} \breve{A}_1^{1-\alpha} \;&=\; \left(\frac{\delta+\lambda_2-r_2\alpha}{\lambda_2} - \frac{\gamma_2}{\lambda_2}\frac{\alpha}{1-\alpha}\right)\breve{A}_2^{1-\alpha} - \frac{1-\alpha}{\lambda_2}\,\breve{A}_2^{-\alpha} \\ \breve{A}_2^{1-\alpha} \;&=\; \left(\frac{\delta+\lambda_1-r_1\alpha}{\lambda_1} - \frac{\gamma_1}{\lambda_1}\frac{\alpha}{1-\alpha}\right)\breve{A}_1^{1-\alpha} - \frac{1-\alpha}{\lambda_1}\,\breve{A}_1^{-\alpha}. \end{aligned}$$

Proof. It follows straightforward from the system (3.25), where $q_{12} = \lambda_1$ and $q_{21} = \lambda_2$. □

3.3.3 $U(x,i) = x^\alpha$ for every $i \in \mathcal{S}$, where $0 < \alpha < 1$.

For the third HARA utility function, we have $U'(x,i) = \alpha\, x^{-(1-\alpha)}$ and $I(x,i) = (\alpha/x)^{1/(1-\alpha)}$ for $x > 0$. Hence, from equation (3.16), we obtain

$$\widehat{C}(x,i) \;=\; \alpha^{1/(1-\alpha)}v'(x,i)^{-1/(1-\alpha)} > 0$$

whenever $v'(x,i) > 0$. Replacing this value into equation (3.19), we get the following system of *nonlinear* ordinary differential equations:

$$\gamma_i \frac{v'(x,i)^2}{v''(x,i)} - r_i x v'(x,i) - \left(\alpha^{\alpha/(1-\alpha)} - \alpha^{1/(1-\alpha)}\right) v'(x,i)^{-\alpha/(1-\alpha)}$$
$$+(\delta+\lambda_i)v(x,i) \;=\; \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij}\, v(x,j),$$

$i = 1, 2, \ldots, S$. We find that the solution for this system is

$$v(x,i) \;=\; \widetilde{A}_i^{1-\alpha}\, x^{\alpha}, \qquad i \in \mathcal{S},$$

where the coefficients $\widetilde{A}_i$, $i \in \mathcal{S}$, satisfy the system of nonlinear equations

$$(3.28)\quad \left(\delta + \lambda_i - r_i\alpha - \gamma_i\frac{\alpha}{1-\alpha}\right) \widetilde{A}_i^{1-\alpha} - (1-\alpha)\, \widetilde{A}_i^{-\alpha} \;=\; \sum_{j\in\mathcal{S}\setminus\{i\}} q_{ij}\, \widetilde{A}_j^{1-\alpha},$$

$i = 1, 2\ldots, S$, which is similar to the system of equations (3.25) (however, $0 < \alpha < 1$ in this case). Recall that condition (3.20) must be satisfied in this case. We define η_i as in subsection 3.3.2.

Lemma 3.2. *The system* (3.28) *has a unique nonzero real solution* $\widetilde{A}_i$, $i \in \mathcal{S}$, *such that* $\widetilde{A}_i > (1-\alpha)/\eta_i$, $i \in \mathcal{S}$.

Proof. The proof of this lemma is presented in Appendix B. □

We will only use the unique positive solution $\tilde{A}_i$, $i \in \mathcal{S}$, described in Lemma 3.2. Hence, $v'(x,i) = \widetilde{A}_i^{1-\alpha}\alpha/x^{1-\alpha} > 0$ for all $x > 0$, and also $v''(x,i) = -\widetilde{A}_i^{1-\alpha}\alpha(1-\alpha)/x^{2-\alpha} < 0$ for all $x > 0$. That is, $v(\cdot, i)$, $i \in \mathcal{S}$, is strictly increasing and strictly concave in $(0, \infty)$.

Hence, from equations (3.14) and (3.16),

$$\widehat{\Pi}(x,i) \;=\; \frac{1}{(1-\alpha)}\,(\mu_i - r_i\mathbf{1})\,\Sigma_i^{-1} \qquad \text{and} \qquad \widehat{C}(x,i) \;=\; \frac{1}{\widetilde{A}_i}\, x.$$

Then, a candidate for an optimal policy $\hat{u}$ is given by

$$\hat{u}(t) \;=\; (\hat{\pi}(t), \hat{c}(t)) \;=\; \left(\frac{1}{(1-\alpha)} (\mu_{\epsilon(t)} - r_{\epsilon(t)}\mathbf{1})\, \Sigma^{-1}_{\epsilon(t)}\,,\; \frac{1}{\widetilde{A}_{\epsilon(t)}} \hat{X}(t) \right),$$

for $t \in [0, \widehat{\Theta})$. We note that this policy is similar to (3.26). Following a similar procedure, we can show that the policy $\hat{u}$ above is admissible for Problem 3.4. Thus, for $t \in [0, \widehat{\Theta})$,

$$\begin{aligned}\widehat{X}(t) \;=\; x \exp\Bigg\{ & \int_0^t \left(\frac{1-2\alpha}{(1-\alpha)^2}\gamma_{\epsilon(s)} + r_{\epsilon(s)} - \frac{1}{\widetilde{A}_{\epsilon(s)}} \right) ds \\ & + \int_0^t \frac{1}{1-\alpha} (\mu_{\epsilon(s)} - r_{\epsilon(s)}\mathbf{1})(\sigma^{-1}_{\epsilon(s)})^T dW_s^T \Bigg\}.\end{aligned}$$

Since $\hat{X}(t) > 0$ for all $t \in [0, \infty)$, $\widehat{\Theta} = +\infty$ $\mathbb{P}$-a.s. in this case too.

Now we prove rigorously that the function v and the policy $\hat{u}$ are the value function and optimal policy for Problem 3.4.

Proposition 3.3. Let $\tilde{A}_i$, $i \in \mathcal{S}$, be the unique positive solution of the system of equations (3.28) described by Lemma 3.2. Then the function $v(\cdot, i)$, $i \in \mathcal{S}$, defined by

$$v(x, i) \;=\; \tilde{A}_i^{1-\alpha}\, x^{\alpha}, \qquad x \geq 0,$$

is the value function $V(\cdot, i)$, $i \in \mathcal{S}$, of Problem 3.4. Moreover, the policy $\hat{u}$ defined by

$$\text{(3.29)}\quad \hat{u}(t) \;=\; (\hat{\pi}(t), \hat{c}(t)) \;=\; \left(\frac{1}{(1-\alpha)} (\mu_{\epsilon(t)} - r_{\epsilon(t)}\mathbf{1})\, \Sigma^{-1}_{\epsilon(t)}\,,\; \frac{1}{\widetilde{A}_{\epsilon(t)}} \hat{X}(t) \right),$$

for $t \in [0, \infty)$, is an optimal solution of Problem 3.4.

Proof. The proof of this Proposition is similar to the proof of Proposition 3.2, but considering $v(0, i) = U(0, i)/\delta = 0$. It is based on the fact that the coefficients $\tilde{A}_i$ are positive and, therefore, the conditions of Corollary 3.1 are satisfied. Hence, $\hat{u}$ is an optimal solution to Problem 3.4 and $V(\cdot,i) = v(\cdot,i)$, $i \in \mathcal{S}$, in $[0,\infty)$. $\square$

Corollary 3.4. In the special case when $S = 2$, the value function is defined by $V(x,i) = v(x,i) = \tilde{A}_i^{1-\alpha} x^\alpha$, for $x \geq 0$ and $i = 1, 2$, where

$$\tilde{A}_1^{1-\alpha} = \left(\frac{\delta + \lambda_2 - r_2\alpha}{\lambda_2} - \frac{\gamma_2}{\lambda_2} \frac{\alpha}{1-\alpha} \right) \tilde{A}_2^{1-\alpha} - \frac{1-\alpha}{\lambda_2} \tilde{A}_2^{-\alpha} \tag{3.30}$$

$$\tilde{A}_2^{1-\alpha} = \left(\frac{\delta + \lambda_1 - r_1\alpha}{\lambda_1} - \frac{\gamma_1}{\lambda_1} \frac{\alpha}{1-\alpha} \right) \tilde{A}_1^{1-\alpha} - \frac{1-\alpha}{\lambda_1} \tilde{A}_1^{-\alpha}.$$

Proof. It follows straightforward from the system (3.28), where $q_{12} = \lambda_1$ and $q_{21} = \lambda_2$. $\square$

3.3.4 $U(x,i) = \beta_i\, x^{1/2}$ where $\beta_i > 0$ for $i = 1, 2$.

In this subsection we show how our method can be applied to solve consumption-investment problems in which the utility function depends on the regime. To simplify the notation, we assume that there are only two regimes.

In this case we have $U'(x,i) = 1/2\,\beta_i\, x^{-1/2}$ and $I(x,i) = 1/4\,\beta_i^2\, x^{-2}$ for $x > 0$. Hence, from equation (3.16), we obtain

$$\widehat{C}(x,i) = \frac{\beta_i^2}{4}\, v'(x,i)^{-2} > 0$$

whenever $v'(x,i) > 0$. Replacing this value into equation (3.18), we get the following system of *nonlinear* ordinary differential equations:

$$\gamma_1 \frac{v'(x,1)^2}{v''(x,1)} - r_1 x v'(x,1) - \frac{\beta_1^2}{4}\, v'(x,1)^{-1} + (\delta + \lambda_1)\, v(x,1) = \lambda_1\, v(x,2)$$

$$\gamma_2 \frac{v'(x,2)^2}{v''(x,2)} - r_2 x v'(x,2) - \frac{\beta_2^2}{4}\, v'(x,2)^{-1} + (\delta + \lambda_2)\, v(x,2) = \lambda_2\, v(x,1).$$

We find that the solution for the system above is

$$v(x,i) \;=\; (\bar{A}_i\, x)^{1/2}, \qquad i = 1,2,$$

where the coefficients $\bar{A}_i$, $i = 1,2$, satisfy the following system of nonlinear equations

$$\begin{aligned} (\bar{A}_1\bar{A}_2)^{1/2} &= \frac{1}{\lambda_1}\left(\delta + \lambda_1 - \frac{r_1}{2} - \gamma_1\right)\bar{A}_1 - \frac{\beta_1^2}{2\lambda_1} \\ (\bar{A}_1\bar{A}_2)^{1/2} &= \frac{1}{\lambda_2}\left(\delta + \lambda_2 - \frac{r_2}{2} - \gamma_2\right)\bar{A}_2 - \frac{\beta_2^2}{2\lambda_2}. \end{aligned} \tag{3.31}$$

Let us denote, for each $i = 1,2$,

$$\eta_i \;:=\; \frac{1}{\lambda_i}\,(\delta + \lambda_i - \frac{r_i}{2} - \gamma_i).$$

Lemma 3.3. *The system* (3.31) *has a unique real solution* $\bar{A}_i$, $i = 1,2$, *such that* $\bar{A}_i \geq \beta_i^2/(2\lambda_i\eta_i) > 0$, $i = 1,2$.

Proof. The proof of this lemma is presented in Appendix B. □

Lemma 3.3 implies that $v(\cdot, i)$, $i = 1,2$, is strictly increasing and strictly concave in $(0,\infty)$.

From equations (3.14) and (3.16),

$$\widehat{\Pi}(x,i) \;=\; 2\,(\mu_i - r_i\mathbf{1})\,\Sigma_i^{-1} \qquad \text{and} \qquad \widehat{C}(x,i) \;=\; \frac{\beta_i^2}{\bar{A}_i}\,x.$$

Thus, the candidate for an optimal policy $\hat{u}$ is given, for $t \in [0, \widehat{\Theta})$, by

$$\hat{u}(t) \;=\; (\hat{\pi}(t), \hat{c}(t)) \;=\; \left(\; 2\,(\mu_{\epsilon(t)} - r_{\epsilon(t)}\mathbf{1})\,\Sigma_{\epsilon(t)}^{-1}\,,\; \frac{\beta_{\epsilon(t)}^2}{\bar{A}_{\epsilon(t)}}\,\hat{X}(t)\;\right).$$

We observe that for all $t \in [0, \infty)$,

$$E_{x,i}\left[\int_0^t \hat{\pi}(s)\Sigma_{\epsilon(s)}\hat{\pi}(s)^T\,ds\right] \;=\; 8\,E_{x,i}\left[\int_0^t \gamma_{\epsilon(s)}\,ds\right] \;<\; 8\,\bar{\gamma}\,\mathbf{1}^T t \;<\; +\infty.$$

Moreover, replacing the policy $\hat{u}$ in (3.1) we get

$$dX^{\hat{u}}(t) \;=\; \left(4\,\gamma_{\epsilon(t)} + r_{\epsilon(t)} - \frac{\beta^2_{\epsilon(t)}}{\bar{A}_{\epsilon(t)}}\right) X^{\hat{u}}(t)\,dt + 2\,(\mu_{\epsilon(t)} - r_{\epsilon(t)}\mathbf{1})(\sigma^{-1}_{\epsilon(t)})^T\,X^{\hat{u}}(t)\,dW_t^T.$$

Hence, the wealth process generated by $\hat{u}$ is given, for $t \in [0, \widehat{\Theta}]$, by

$$\hat{X}(t) \;=\; x\exp\left\{\int_0^t \left(r_{\epsilon(s)} - \frac{\beta^2_{\epsilon(t)}}{\bar{A}_{\epsilon(s)}}\right) ds + 2\int_0^t (\mu_{\epsilon(s)} - r_{\epsilon(s)}\mathbf{1})(\sigma^{-1}_{\epsilon(s)})^T\,dW_s^T\right\}.$$

We note that $\widehat{\Theta} = +\infty$ a.s. because $\hat{X}(t) > 0$ for all $t \in [0, \infty)$ (recall that $x > 0$). Then,

$$E_{x,i}\left[\int_0^t \hat{c}(s)\,ds\right] \;\leq\; x\left(\frac{\beta_1^2}{\bar{A}_1} + \frac{\beta_2^2}{\bar{A}_2}\right)\int_0^t \exp\left\{\left(4\,\bar{\gamma}\,\mathbf{1}^T + \bar{r}\,\mathbf{1}^T\right)s\right\}\,ds \;<\; +\infty,$$

for all $t \in [0, \infty)$. Thus, the policy $\hat{u}$ is admissible.

Proposition 3.4. Let $\bar{A}_1$ and $\bar{A}_2$ be the unique positive solution of the system of equations (3.31). Then, the function $v(\cdot, i)$, $i = 1, 2$, defined by

$$v(x, i) \;=\; (\bar{A}_i\,x)^{1/2}, \qquad x \geq 0,$$

is the value function $V(\cdot, i)$, $i = 1, 2$, for Problem 3.5. Moreover, the policy $\hat{u}$ defined by

$$\hat{u}(t) \;=\; (\hat{\pi}(t), \hat{c}(t)) \;=\; \left(2\,(\mu_{\epsilon(t)} - r_{\epsilon(t)}\mathbf{1})\,\Sigma^{-1}_{\epsilon(t)}\,,\; \frac{\beta^2_{\epsilon(t)}}{\bar{A}_{\epsilon(t)}}\,\hat{X}(t)\right) \tag{3.32}$$

is an optimal solution of Problem 3.5.

Proof. By definition $v(\cdot,i) \in C^2(0,\infty)$, $i \in \mathcal{S}$, and $v(0,i) = 0$, $i \in \mathcal{S}$. Moreover, $v(\cdot,i)$, $i \in \mathcal{S}$, is strictly increasing and strictly concave in $(0,\infty)$ as shown above. Also by construction, the equality

$$
\begin{aligned}
&L_i(\widehat{\Pi}(x,i),\widehat{C}(x,i))v(x,i) + U(\widehat{C}(x,i),i) \\
&\qquad = \lambda_i\left(v(x,i) - \frac{U(0,i)}{\delta}\right) - \lambda_i\left(v(x,3-i) - \frac{U(0,3-i)}{\delta}\right)
\end{aligned}
$$

holds for every $i \in \mathcal{S}$, where $\widehat{\Pi}(x,i) = 2(\mu_i - r_i 1)\Sigma_i^{-1}$ and $\widehat{C}(x,i) = (\beta_i^2/\bar{A}_i)x$. Hence,

$$
\begin{aligned}
&L_i(\hat{u}_t)v(\hat{X}_t,\epsilon_t) + U(\hat{c}_t,\epsilon_t) \\
&\qquad = \lambda_{\epsilon(t)}\left(v(\hat{X}_t,\epsilon_t) - \frac{U(0,\epsilon_t)}{\delta}\right) - \lambda_{\epsilon(t)}\left(v(x,3-\epsilon_t) - \frac{U(0,3-\epsilon_t)}{\delta}\right)
\end{aligned}
$$

holds for every $t \in [0,\infty)$ and $\omega \in \Omega$, where $\hat{u}$ is given by (3.32). It was proved above that this policy is admissible. Therefore, from Theorem 3.1, $\hat{u}$ is an optimal solution to Problem 3.5 and

$$
v(x,i) + \frac{1}{\delta}E_{x,i}\left[\int_0^{\widehat{\Theta}} e^{-\delta s} dU(0,\epsilon_s)\right],
$$

$i = 1,2$, is the corresponding value function for $x \geq 0$. However, in this case $dU(0,\epsilon_s) = 0$ for every $s > 0$. Then, $V(\cdot,i) = v(\cdot,i)$, $i = 1,2$, in $[0,\infty)$. □

Proposition 3.4 shows the first explicit solution in the literature on classical stochastic control with regime switching in which the utility or cost function depends on the regime.

3.4 Economic Analysis.

We have considered in this chapter that the behavior of the financial market can be modeled by a finite number $S \geq 2$ of regimes, each with its specific parameters. To simplify the economic interpretation and analysis, in this section we will consider only two regimes of the market. We will consider regime 1 to represent a market with good economic conditions. For instance, it might be a market with low inflation, a market in economic boom, or a market in which security prices are rising or are expected to rise. On the other hand, regime 2 will represent a market with bad economic conditions. It might be a market with high inflation, a market in economic recession, or a market in which security prices are falling or are expected to fall. We will call these two regimes of the market "bull market" and "bear market", respectively.

We consider only one risky stock with expected rate of return μ_i and volatility σ_i, in the regime $i = 1, 2$. Following the analysis done by Fama and French (1989) and French, Schwert and Stambaugh (1987), we assume that $r_i < \mu_i$, for every $i = 1, 2$. For comparative purposes, and because of their importance in financial economics, we will consider only the first three utility functions described in the last section. One of our objectives is to analyze how the regime switching together with the level of risk aversion affect the decisions of the investor.

According to our previous analysis, the optimal investment portfolio for each of the first three cases considered can be expressed as

$$\widehat{\Pi}(x, i) \;=\; \kappa\, \frac{\mu_i - r_i}{\sigma_i^2}, \tag{3.33}$$

where κ is a known positive constant that depends on the investor's utility function.

Equation (3.33) shows that the optimal portfolio is constant in every regime of the market, that is, it does not vary in time and does not depend on the investor's wealth. The optimal portfolio will only change if the market changes its regime. This result is consistent with the results of Karatzas, Lehoczky, Sethi and Shreve (1986), and Merton (1969), that show that under constant market coefficients and just one regime of the market, the optimal portfolio is constant (it does not depend on time or investor's wealth).

Nevertheless, in our model the optimal portfolio does depend on the regime of the market: it is directly proportional to the expected excess return on the stock in such regime $(\mu_i - r_i)$, and inversely proportional to the variance of the stock return in such regime (σ_i^2). Fama and French (1989) show using empirical data that positive expected excess returns are lower during a bull market and higher during a bear market. We observe as well that the optimal portfolio depends also on the stock volatility. Schwert (1989) shows empirical evidence that stock volatility is higher during a bear market and that, in general, stock market volatility is related to the regime of the economy. Moreover,in their seminal paper, French, Schwert and Stambaugh (1987) find evidence of a positive relationship between the excess return on common stocks and the volatility of stock returns. However, this positive relationship between expected excess return and volatility is different according to the current regime of the market. Even though both the expected excess return and the risk on the stock are higher during a bear market than during a bull market, the effect of the volatility in the given ratio is much stronger and offsets the effect of the high expected excess return. Therein, the ratio 'expected excess return/return variance' is greater when the market conditions are strong. French, Schwert and Stambaugh (1987) confirm this property when regressing the stock expected excess return to the variance of the stock return. They propose an ARIMA model that can be expressed as

$$\frac{\text{Expected excess return}}{\text{return variance}} = \frac{a}{\text{return variance}} + b.$$

Moreover, they show evidence that the estimates of the model parameters a and b are reliably positive. Such evidence implies that the ratio 'expected excess return/return variance' is inversely proportional to the variance of the stock return.

Therefore, if we regard the optimal portfolio (3.33) only as a proportion of the ratio 'expected excess return/return variance', we realize that $\hat{\Pi}(x, 2) < \hat{\Pi}(x, 1)$. This means that the investor prefers to invest a smaller fraction of wealth in the risky asset when the market presents weak conditions. If market conditions are strong, the investor will assign a greater fraction of his wealth to invest in the stock. This is due to the dominant effect of the stock volatility in the bear market.

From equation (3.26), we see that when the investor has low risk tolerance ($\alpha < 0$), the fraction of wealth invested in the stock is even smaller in each regime but the relation $\hat{\Pi}(x, 2) < \hat{\Pi}(x, 1)$ remains the same. *In other words, every investor (independent of the level of risk aversion) should allocate a higher proportion of his wealth in a risky asset during a bull market than during a bear market.*

The optimal consumption process in the three cases depends on the wealth process. We will consider the consumption to wealth process ratio for an appropriate analysis of the investor's consumption according to the market regime. The consumption to wealth process ratio, or "relative consumption", is given by

$$\frac{\hat{c}(t)}{\hat{X}(t)} = \delta, \quad \text{for the logarithmic utility function;}$$
$$\frac{\hat{c}(t)}{\hat{X}(t)} = \frac{1}{\breve{A}_{\epsilon(t)}}, \quad \text{for the power utility function with } \alpha < 0; \text{ and}$$
$$\frac{\hat{c}(t)}{\hat{X}(t)} = \frac{1}{\tilde{A}_{\epsilon(t)}}, \quad \text{for the power utility function with } 0 < \alpha < 1.$$

The constants $\breve{A}_i$ and $\tilde{A}_i$, $i = 1, 2$, are the ones derived from the system of equations (3.27) and (3.30), respectively. Since they are all positive constants, the consumption rate process is directly proportional to the wealth process at every time $t \in [0, \infty)$. Therefore, the investor increases his consumption whenever his wealth increases, regardless of the market regime. This agrees with previous studies on the wealth effect on consumption. See for instance Modigliani (1971) and Poterba (2000) for a discussion of the positive effect of wealth on consumer spending.

Since the constants $\breve{A}_i$ and $\widetilde{A}_i$, $i = 1, 2$, depend on the level of risk-aversion of the investor, we are going to analyze separately the optimal consumption to wealth process ratio for each of the utility functions.

When the investor has a moderate risk tolerance and uses a logarithmic utility function as in the first case, the optimal consumption to wealth is given by $\widehat{C}(x, i)/x = \delta$. Hence, the consumption to wealth ratio is constant (it does not depend on the regime of the market). Therefore, $\widehat{C}(x, 2)/x = \widehat{C}(x, 1)/x$ and a moderate risk-averse investor will consume the same proportion of his wealth in a bear market as in a bull market.

When the investor has a low risk tolerance and uses a power utility function with $\alpha < 0$ as in the second case, the optimal consumption to wealth ratio is given by $\widehat{C}(x, i)/x = 1/\breve{A}_i$. The positive constants $\breve{A}_1$ and $\breve{A}_2$ are given by the system of equations (3.27). We solve numerically that system for the market coefficients[6] $\mu_1 = 0.15$, $\mu_2 = 0.25$, $\sigma_1 = 0.25$, $\sigma_2 = 0.6$, $r_1 = 0.05$ and $r_2 = 0.01$, the discount rate $\delta = 0.07$, and the exponents $\alpha = -1$, -2, -10 and -15. We consider a fixed $\lambda_1 = 6.04$ and different values for λ_2. The results are shown in Table 3.1. We note that $\eta_1 - \lambda_1 > \eta_2 - \lambda_2$ is satisfied for every value of λ_2. Then, we can see that the relation $\breve{A}_2 > \breve{A}_1$ is also satisfied in every case, as expected from the first paragraph of the proof of Lemma 3.1. Hence, $\widehat{C}(x,1)/x > \widehat{C}(x,2)/x$. Therefore, a very risk-averse investor will consume proportionally more in a bull market when the economic conditions are strong, and consume proportionally less in a bear market in case the recession lasts more than expected. However, this greater consumption to wealth rate in the bull market is still lower than the consumption to wealth rate of a less risk-averse investor under the same conditions. We observe that behavior also from Table 3.1: for a fixed λ_2, the optimal consumption to wealth ratios $\widehat{C}(x,1)/x$ and $\widehat{C}(x,2)/x$ decrease as α becomes more negative, that is, as the investor becomes more risk-averse. Therefore, we can say that a very risk-averse investor does not consume much during a bull market and consumes even less during a bear market. In addition, despite the decrease in consumption, the optimal consumption to wealth ratio becomes more sensitive to changes of regime as the risk aversion increases. That is, the absolute value of the difference between $\widehat{C}(x,1)/x$ and $\widehat{C}(x,2)/x$ becomes greater in percentage as α becomes more negative.

We can also notice from Table 3.1 that, for a fixed α, the optimal consumption to wealth ratios increase as λ_2 increases, that is, as the rate at which there is a transition into a bull market increases. Hence, a high risk-averse investor will increase his current consumption rate when the rate of transition into good economic conditions increases. The opposite behavior is found when the

[6] For arbitrary market coefficients, it is necessary to verify that the market conditions $0 < \mu_1 - r_1 < \mu_2 - r_2$, $0 < \sigma_1 < \sigma_2$ and $(\mu_2 - r_2)/\sigma_2^2 < (\mu_1 - r_1)/\sigma_1^2$ are satisfied. These conditions are discussed in the beginning of this section.

rate of transition into a bear market rises, that is when λ_1 increases. A high risk-averse investor, hence, will reduce his current consumption as the market's rate of transition into bad economic conditions increases.

We can also observe from Table 3.1 that, in every market regime, the optimal consumption to wealth ratio is small for an investor with high risk-aversion ($\alpha < 0$). Indeed, the ratio is always lower than 0.10. Hence, a change of wealth does not have a strong effect on the investor's consumption rate.

When the investor has a high risk tolerance and uses a power utility function with $0 < \alpha < 1$ as in the third case, the optimal consumption to wealth ratio is given by $\widehat{C}(x,i)/x = 1/\tilde{A}_i$. The difference with the second case is that now $\tilde{A}_1 > \tilde{A}_2$, as shown in Table 3.2. In fact, Table 3.2 exhibits the results of solving numerically the system of equations (3.30) for $\mu_1 = 0.15$, $\mu_2 = 0.25$, $\sigma_1 = 0.25$, $\sigma_2 = 0.6$, $r_1 = 0.05$, $r_2 = 0.01$ and $\alpha = 0.1$, 0.3, 0.5 and 0.7. We consider a fixed $\lambda_1 = 6.04$ and different values for λ_2. The discount rate $\delta = 0.8$ was selected in such a way that the condition (3.20) is satisfied for all the given α's and λ_2's. From these results we can observe that $\widehat{C}(x,2)/x > \widehat{C}(x,1)/x$ when the investor has a high risk tolerance. Therefore, when an investor has high risk tolerance, his optimal consumption to wealth ratio is greater under weak market conditions than under strong conditions. We can also observe that the absolute value of the difference between the optimal consumption to wealth ratios increases in percentage as α increases. Hence, the lower the level of risk aversion of the investor, the more sensitive is his optimal consumption to wealth ratio to changes in the market regime.

Table 3.2 also shows that, for a fixed $\alpha \in (0,1)$, the optimal consumption to wealth ratios increase as λ_1 increases and decrease as λ_2 increases. Hence, a low risk-averse investor increases his relative consumption when the rate of transition into a bear market increases. On the other hand, if the rate of transition into a bull market increases, then the investor decreases his relative consumption.

Table 3.2 also tells us that, in every market regime, the optimal consumption to wealth ratios are in general big for investors with low risk aversion. These ratios are even greater than 1 for some values of $\alpha \in (0, 1)$. Hence, a change of wealth has a strong impact on the investor's optimal consumption rate when he has high risk tolerance.

λ_2	$\breve{A}_1$	$\breve{A}_2$	$\widehat{C}(x,1)/x$	$\widehat{C}(x,2)/x$
For $\alpha = -1$				
1.9	15.41018	15.44898	0.06489	0.06473
6.4	14.21908	14.24183	0.07033	0.07022
10.4	13.76023	13.77692	0.07267	0.07259
23.4	13.17473	13.18366	0.07590	0.07585
100.4	12.67981	12.68219	0.07887	0.07885
For $\alpha = -2$				
1.9	18.43385	18.49528	0.05425	0.05407
6.4	16.25535	16.29007	0.06152	0.06139
10.4	15.46656	15.49159	0.06466	0.06455
23.4	14.49812	14.51124	0.06897	0.06891
100.4	13.71155	13.71499	0.07293	0.07291
For $\alpha = -10$				
1.9	32.73348	32.88119	0.03055	0.03041
6.4	24.66638	24.73838	0.04054	0.04042
10.4	22.27145	22.32079	0.04490	0.04480
23.4	19.64588	19.67021	0.05090	0.05084
100.4	17.74307	17.74914	0.05636	0.05634
For $\alpha = -15$				
1.9	37.01913	37.19033	0.02701	0.02689
6.4	26.78448	26.86512	0.03734	0.03722

Table 3.1: Optimal consumption to wealth ratio for an investor with high risk-aversion ($\alpha < 0$).

λ_2	$\tilde{A}_1$	$\tilde{A}_2$	$\widehat{C}(x,1)/x$	$\widehat{C}(x,2)/x$
For $\alpha = 0.1$				
1.9	1.14090	1.14032	0.87650	0.87694
6.4	1.14224	1.14186	0.87547	0.87576
10.4	1.14285	1.14255	0.87501	0.87523
23.4	1.14372	1.14355	0.87434	0.87447
100.4	1.14455	1.14450	0.87371	0.87374
For $\alpha = 0.3$				
1.9	0.92258	0.92083	1.08392	1.08598
6.4	0.92584	0.92466	1.08010	1.08147
10.4	0.92734	0.92643	1.07836	1.07941
23.4	0.92951	0.92951	1.07584	1.07584
100.4	0.93161	0.93146	1.07341	1.07358
For $\alpha = 0.5$				
1.9	0.70631	0.70330	1.41580	1.42187
6.4	0.71051	0.70846	1.40743	1.41150
10.4	0.71249	0.71089	1.40353	1.40668
23.4	0.71539	0.71447	1.39783	1.39964
100.4	0.71827	0.71800	1.39224	1.39276
For $\alpha = 0.7$				
1.9	0.50391	0.49920	1.98448	2.00322
6.4	0.50843	0.50515	1.96685	1.97963
10.4	0.51063	0.50804	1.95838	1.96834
23.4	0.51395	0.51242	1.94573	1.95152
100.4	0.51734	0.51689	1.93298	1.93464

Table 3.2: Optimal consumption to wealth ratio for an investor with low risk-aversion ($0 < \alpha < 1$).

Chapter 4

Bounded dividend policy in the presence of business cycles

The selection of an optimal dividend policy is one of the most important decisions of a firm. For instance, according to Poterba (2004), the corporate sector of the U.S.A. paid out more than $350 billion on payouts in 2002 alone. Miller and Modigliani (1961) initiated the research on dividend policy, and since then there has been an extensive literature dedicated to this topic. The dividend policy has indeed become one of the most important areas of research in corporate finance.

During recent years, there has been a growing literature on papers that apply advanced methods of stochastic control to model and study the optimal dividend policy problem. This literature was initiated by Asmussen and Taksar (1997), Jeanblanc-Picqué and Shiryaev (1995), and Radner and Shepp (1996). Other important papers that develop stochastic control models to study optimal dividend policies include Asmussen, Høјgaard and Taksar (2000), Cadenillas, Choulli, Taksar and Zhang (2006), Cadenillas, Sarkar and

[7] The results shown in this chapter are presented in sections 3 and 5 in: Sotomayor, L.R. and A. Cadenillas, Optimal dividend policy in the presence of business cycles, *submitted for publication* (2008).

Zapatero (2007), Choulli, Taksar and Zhou (2001), Choulli, Taksar and Zhou (2003), Décamps and Villeneuve (2007), Gerber and Shiu (2004), Højgaard and Taksar (1999), Højgaard and Taksar (2001), Ly Vath, Pham and Villeneuve (2006), Taksar (2000), and Taksar and Zhou (1998). All these papers assume that there is only one source of uncertainty, which is modeled by a standard Brownian motion. However, there is evidence that a company's cash reservoir is also affected by long-term economic conditions. For instance, Ho and Wu (2001) explain that the income of a company depends on the conditions of the markets, which generate the business cycles. In fact, they claim that "firms'earnings are usually correlated to overall market movements and are often influenced by business cycles."

Further evidence that macroeconomic conditions affect the cash reservoir of a company is given by Hackbart, Miao and Morellec (2006), who observe that macroeconomic conditions affect for instance the credit risk and dynamic capital structure choice of a company. Moreover, Perez-Quiros and Timmermann (2000) explain how credit risk is reflected in the company's cash flows. Hu and Schiantarelli (1998) show empirical evidence of a relationship between macroeconomic conditions and the financing constraints of a company. Indeed, they agree that economic conditions determine two investment regimes in the company: high-premium and low-premium. Monetary policy (and financing constraints therein) affects the cash reservoir of the company since external cash flow (or its cost) depends on this policy. Furthermore, Driffill, Raybaudi and Sola (2003) show that a company's profit function changes during economic growth and economic recession (varying the cash reservoir according to the regime of the economy). Driffill and Sola (2001) also show empirically that the cash earnings generated by investment projects (in a company) follow regime switching and that it is inappropriate to neglect this modeling for flow of investment dividends.

Empirical evidence not only supports that macroeconomic conditions affect the cash reservoir (and the earnings) of a company, but also that they influence the dividend policies. Indeed, Gertler and Hubbard (1993) predict that "macroeconomic conditions should influence payments to equityholders, even independently of the firm's earnings performance" (p.269), and show empirically that dividend policies behave according to macroeconomic conditions. Such behavior cannot be explained by conventional dividend models that do not consider the macroeconomic effect.

In this chapter, we consider a dividend payment model with regime switching where the cash reservoir depends on the regime of the economy. We consider, for sake of simplicity, that the economy only shifts between two different regimes: economic growth and economic recession. That is, for this problem, $\mathcal{S} = \{1, 2\}$. The objective of the company's management is to select the dividend policy that maximizes the total expected discounted cumulative amount of dividends to be paid out to the shareholders. In this chapter, we consider the case when the dividend rates are bounded. In Chapter 6, we will consider the case when the dividend rates are unbounded; and in Chapter 8, we will consider the case when there exists a fixed cost associated to the dividend payments.

We show that the optimal dividend payment policy is different from the case in which there is only one regime. When paying dividends in the regime switching case, the company has to be aware not only of its cash reservoir level, but also of the regime of the economy at the time. This dependence on the economic conditions is not considered in one-regime models.

4.1 The dividend model with regime switching

Let the adapted process $X = \{X_t,\, t \geq 0\}$ represent the cash reservoir of the company. We assume that X satisfies the stochastic differential equation:

$$dX_t \;=\; \mu_{\epsilon(t)}\, dt + \sigma_{\epsilon(t)}\, dW_t - dZ_t$$

with initial level of the cash reservoir $X_0 = x > 0$ and initial state $\epsilon(0) = i$. The drift coefficients (μ_1 and μ_2) and the volatility parameters (σ_1 and σ_2) are positive constants. The adapted process $Z = \{Z_t, t \geq 0\}$ represents the cumulative amount of dividends paid-out by the company up to time t. In general, Z is considered to be a nonnegative and nondecreasing stochastic process with sample paths which are left-continuous with right limits. In this chapter we consider the model with bounded dividend rates, hence, the dividend process $Z = \{Z_t, t \geq 0\}$ is a path-continuous process that satisfies the equation

$$dZ_t \;=\; u(t)\, dt,$$

where $u : [0,\infty) \times \Omega \to [0,\infty)$ is an $\mathbb{F}$-adapted process that represents the rate at which the dividends are paid. Let us denote by $K > 0$ the bound for the dividend rate. In addition, let us denote by $\delta > 0$ the discount rate for these dividend payments.

We note that the company will not be able to pay dividends unless the cash reservoir is positive. Thus, we need to consider the stopping time of bankruptcy

$$\Theta := \inf\{t \geq 0 : X_t \leq 0\}$$

and impose $X_t = 0$ for every $t \in [\Theta, \infty)$.

Definition 4.1. An *admissible stochastic control* is an $\mathbb{F}$-adapted control process $u : [0,\infty) \times \Omega \to [0,\infty)$ that satisfies $u(t,\omega) \in [0,K]$ for every $(t,\omega) \in [0,\infty) \times \Omega$, and $u(t,\omega) = 0$ for every $(t,\omega) \in [\Theta,\infty) \times \Omega$. The set of all admissible controls is denoted by $\mathcal{A}$.

We note that for an admissible dividend rate process u, the cash reservoir is given, for every $t \in [0,\Theta)$, by

$$X_t^u = x + \int_0^t \left(\mu_{\epsilon(s)} - u(s)\right) ds + \int_0^t \sigma_{\epsilon(s)}\, dW_s. \tag{4.1}$$

. The management of the company wants to solve the following problem:

Problem 4.1. For each $i = 1,2$, select an admissible dividend rate u that maximizes

$$J(x,i;u) := \mathrm{E}_{x,i}\left[\int_0^\Theta e^{-\delta t} u(t)\, dt\right].$$

We note that the value function for Problem 4.1 satisfies $V(0,i) = 0$ for both $i = 1,2$, because if the initial level of the cash reservoir is $x = 0$, then the company is already bankrupt and, hence,

$$V(0,i) = \sup_{u\in\mathcal{A}} E_{x,i}\left[\int_0^0 e^{-\delta s} u(s) ds\right] = 0.$$

Furthermore, if $\hat{u}$ denotes the optimal control for Problem 4.1, then the admissibility of $\hat{u}$ implies that $\hat{u}(t,\omega) \in [0,K]$ for every $(t,\omega) \in [0,\infty) \times \Omega$. Thus, for each $i = 1,2$,

$$V(x,i) = E_{x,i}\left[\int_0^\Theta e^{-\delta s}\hat{u}(s)\, ds\right] \leq E_{x,i}\left[\int_0^\Theta e^{-\delta s} K ds\right] = \frac{K}{\delta} E_{x,i}\left[1 - e^{-\delta\Theta}\right] \leq \frac{K}{\delta}.$$

Hence, the value function $V(\cdot,i)$, $i = 1,2$, is a bounded function.

4.2 Verification theorem

Let $\psi : (0,\infty) \times \{1,2\} \to \mathbb{R}$ be a function and define the operators $L_i(u)$, for each $i = 1,2$, in the following way

$$L_i(u)\,\psi \;:=\; \frac{1}{2}\,\sigma_i^2\,\psi'' + (\mu_i - u)\,\psi' - \delta\,\psi.$$

Theorem 4.1. Let $v(\cdot,i) \in C^2([0,\infty)\backslash N_i)$, $i = 1,2$, where N_i are finite subsets of $(0,\infty)$. Let $v(\cdot,i)$ and $v'(\cdot,i)$, $i = 1,2$, be bounded on $[0,\infty)$ and let $v(0,i) = 0$, $i = 1,2$. If the function $v(\cdot,i)$, $i = 1,2$, satisfies the equation

$$\sup_{u_t \in [0,K]} \{L_{\epsilon(t)}(u_t)\,v(X_t,\epsilon_t) + u_t\} \;=\; \lambda_{\epsilon(t)}(v(X_t,\epsilon_t) - v(X_t, 3-\epsilon_t)), \tag{4.2}$$

for every $t \in [0,\Theta)$, then the control $\hat{u}$ defined by

$$\hat{u}(t) \;=\; \arg \sup_{u_t \in [0,K]} \{L_{\epsilon(t)}(u_t)\,v(X_t,\epsilon_t) + u_t\}$$

for $t \in [0,\Theta)$ and $\hat{u}(t) = 0$ for $t \in [\Theta,\infty)$, is optimal solution of Problem 4.1. Moreover, $v(\cdot,i)$, $i = 1,2$, is the value function for Problem 4.1.

Proof. Consider the function $f(\cdot,\cdot,i)$, $i = 1,2$, such that $f(t,x,i) = e^{-\delta t}v(x,i)$ and an admissible control u. Using the Itô formula for Markov-modulated processes, we get

$$\begin{aligned} df(t,X_t,\epsilon_t) \;&=\; \Big(\frac{1}{2}\,\sigma_{\epsilon(t)}^2\,f_{xx}(t,X_t,\epsilon_t) + (\mu_{\epsilon(t)} - u_t)\,f_x(t,X_t,\epsilon_t) + f_t(t,X_t,\epsilon_t)\Big)dt \\ &\quad + \big(-\lambda_{\epsilon(t)}\,f(t,X_t,\epsilon_t) + \lambda_{\epsilon(t)}\,f(t,X_t,3-\epsilon_t)\big)\,dt \\ &\quad + f_x(t,X_t,\epsilon_t)\,\sigma_{\epsilon(t)}\,dW_t \;+\; dM_t^f. \end{aligned}$$

The process $M^f = \{M_t^f, t \geq 0\}$ is a square integrable martingale when

$f(\cdot,\cdot,i)$, $i=1,2$, is bounded. Therefore,

$$df(t,X_t,\epsilon_t) = e^{-\delta t}\big(L_{\epsilon(t)}(u_t)v(X_t,\epsilon_t) - \lambda_{\epsilon(t)}\Delta_t\big)dt + \sigma_{\epsilon(t)}e^{-\delta t}v'(X_t,\epsilon_t)dW_t + dM^f_t,$$

where $\Delta_t := v(X_t,\epsilon_t) - v(X_t, 3-\epsilon_t)$. Then, for every time $t \in [0,\infty)$, we get

$$\begin{aligned} e^{-\delta(t\wedge\Theta)}v(X_{t\wedge\Theta},\epsilon_{t\wedge\Theta}) &= v(X_0,\epsilon_0) + \int_0^{t\wedge\Theta} e^{-\delta s}\left(L_{\epsilon(s)}(u_s)v(X_s,\epsilon_s) - \lambda_{\epsilon(s)}\Delta_s\right)ds \\ &\quad + \int_0^{t\wedge\Theta} \sigma_{\epsilon(s)}e^{-\delta s}v'(X_s,\epsilon_s)dW_s + M^f_{t\wedge\Theta} - M^f_0. \end{aligned}$$

Applying conditional expectation to the equation above with respect to $X_0 = x$ and $\epsilon_0 = i$, we have

$$\begin{aligned} E_{x,i}\left[e^{-\delta(t\wedge\Theta)}v(X_{t\wedge\Theta},\epsilon_{t\wedge\Theta})\right] &= v(x,i) \\ &\quad + E_{x,i}\left[\int_0^{t\wedge\Theta} e^{-\delta s}(L_{\epsilon(s)}(u_s)v(X_s,\epsilon_s) - \lambda_{\epsilon(s)}\Delta_s)\,ds\right] \\ &\quad + E_{x,i}\left[\int_0^{t\wedge\Theta} \sigma_{\epsilon(s)}e^{-\delta s}v'(X_s,\epsilon_s)\,dW_s\right] + E_{x,i}\left[M^f_{t\wedge\Theta} - M^f_0\right]. \end{aligned} \tag{4.3}$$

We note that $\sigma_{\epsilon(s)}e^{-\delta s}\,v'(X_s,\epsilon_s)$ is bounded for every $s \in [0, t\wedge\Theta]$. Thus,

$$E_{x,i}\left[\int_0^{t\wedge\Theta} \sigma_{\epsilon(s)}e^{-\delta s}v'(X_s,\epsilon_s)dW_s\right] = 0.$$

Moreover, since $v(\cdot,i)$, $i=1,2$, is bounded, the function $f(t,\cdot,i) = e^{-\delta t}v(\cdot,i)$, $i=1,2$, is bounded for every $t\in[0,\infty)$. Thus, M^f is a square integrable martingale and, hence, $E_{x,i}[\,M^f_{t\wedge\Theta}\,] = E_{x,i}[\,M^f_0\,]$. Then, from equations (4.2) – (4.3), we get

$$v(x,i) \geq E_{x,i}\left[e^{-\delta(t\wedge\Theta)}v(X_{t\wedge\Theta},\epsilon_{t\wedge\Theta})\right] + E_{x,i}\left[\int_0^{t\wedge\Theta} e^{-\delta s}u(s)ds\right]. \tag{4.4}$$

Taking $t\to\infty$ and using $v(0,i)=0$, we observe that the conditional expectation $E_{x,i}[\, e^{-\delta(t\wedge\Theta)} v(X_{t\wedge\Theta}, \epsilon_{t\wedge\Theta})\,] \;\to\; 0$. Therefore, for every arbitrary admissible control u, $v(x,i) \geq J(x,i;u)$. In particular, inequality (4.4) becomes an equality if $u = \hat{u}$, and hence $v(x,i) = J(x,i;\hat{u})$. □

4.3 Construction of the solution

We need to find a function $v(x,i)$, $i = 1,2$, such that the conditions in Theorem 4.1 are satisfied. In particular, we want equation (4.2) to be satisfied. We note that (4.2) is equivalent to

$$\frac{1}{2}\sigma^2_{\epsilon(t)} v''(X_t,\epsilon_t) + \mu_{\epsilon(t)} v'(X_t,\epsilon_t) - \delta v(X_t,\epsilon_t) + \sup_{u_t\in[0,K]} \{u_t(1 - v'(X_t,\epsilon_t))\} = \lambda_{\epsilon(t)} \left(v(X_t,\epsilon_t) - v(X_t, 3-\epsilon_t)\right).$$

We note also that for $t \in [\,0,\Theta)$,

$$\hat{u}(t) = \arg \sup_{u_t\in[\,0,K\,]} \{u_t\,(1 - v'(X_t,\epsilon_t))\} = \begin{cases} 0 & \text{if} \quad v'(X_t,\epsilon_t) > 1 \\ K & \text{if} \quad v'(X_t,\epsilon_t) \leq 1. \end{cases}$$

Hence, the candidate for optimal control $\hat{u}$ has the form $\hat{u}(t) = \varphi(X_t,\epsilon_t)$ for $t \in [\,0,\Theta)$, where $\varphi(\cdot,i)$, $i = 1,2$, is a measurable function such that

$$\varphi(x,i) = \begin{cases} 0 & \text{if} \quad v'(x,i) > 1 \\ K & \text{if} \quad v'(x,i) \leq 1 \end{cases}$$

for $x \geq 0$. Thus, to find a function $v(x,i)$, $i = 1,2$, that satisfies equation (4.2) is equivalent to solve the equation

$$\frac{1}{2}\sigma_i^2\, v''(x,i) + \mu_i\, v'(x,i) - \delta\, v(x,i) + \varphi(x,i)\,(1 - v'(x,i)) = \lambda_i\,(v(x,i) - v(x,3-i))$$

for each $i = 1, 2$; or equivalently, to solve

$$\frac{1}{2}\sigma_i^2 v''(x,i) + \mu_i v'(x,i) - \delta v(x,i) = \lambda_i (v(x,i) - v(x, 3-i))$$

when $v'(x,i) > 1$, and to solve

$$\frac{1}{2}\sigma_i^2 v''(x,i) + (\mu_i - K) v'(x,i) - \delta v(x,i) + K = \lambda_i (v(x,i) - v(x, 3-i))$$

when $v'(x,i) \leq 1$, for each $i = 1, 2$.

We consider three possible cases: $v'(0,i) > 1$ for both $i = 1, 2$; $v'(0, i_0) \leq 1$ and $v'(0, 3-i_0) > 1$ for some $i_0 \in \{1, 2\}$; and $v'(0,i) \leq 1$ for both $i = 1, 2$. For the three cases we need the following lemma, which has been adapted from Remark 2.1 by Guo (2001).

Lemma 4.1. *For $i = 1, 2$, consider the real function $\phi_i(z) = -\frac{1}{2}\sigma_i^2 z^2 - \tilde{\mu}_i z + (\lambda_i + \delta)$ where $\tilde{\mu}_i$ is a function of μ_i. Since σ_1, σ_2, λ_1 and λ_2 are positive, the equation $\phi_1(z)\,\phi_2(z) = \lambda_1\lambda_2$ has four real roots such that $z_1 < z_2 < 0 < z_3 < z_4$.*

Proof. The quadratic equation $\phi_1(z) = 0$ has two real roots because $\tilde{\mu}_1^2 + 2\sigma_1^2(\lambda_1 + \delta) > 0$. Let us denote them by θ_1 and θ_2. In fact, $\theta_1 < 0 < \theta_2$ because $\phi_1(0) = \lambda_1 + \delta > 0$ and $\phi_1(+\infty) = \phi_1(-\infty) < 0$. Consider the polynomial $p(z) = \phi_1(z)\,\phi_2(z) - \lambda_1\lambda_2$. Then, $p(\theta_1) = p(\theta_2) = -\lambda_1\lambda_2 < 0$. We also note that $p(0) = (\lambda_1 + \delta)(\lambda_2 + \delta) - \lambda_1\lambda_2 > 0$. Moreover, $p(+\infty) = p(-\infty) > 0$. Then, from the Weierstrass Intermediate Value Theorem, there exist four real roots for $p(z)$ such that $z_1 < \theta_1 < z_2 < 0 < z_3 < \theta_2 < z_4$. □

Our objective in the remainder of this section is to find a candidate for value function and a candidate for optimal control. For the three cases we conjecture $v'(\cdot, i)$, $i = 1, 2$, to be continuous, positive and bounded in x such that $\lim_{x\to\infty} v'(x,i) = 0$ for both $i = 1, 2$. We also conjecture $v'(\cdot, i)$, $i = 1, 2$, to be non-increasing in order to have $v(\cdot, i)$, $i = 1, 2$, concave in x. We will prove later (in section 4.4) that these conjectures are satisfied by the value function.

Case 1: $v'(0,i) > 1$ **for both** $i = 1, 2$.

In this case, due to our conjecture that $v'(\cdot, i)$, $i = 1, 2$, is non-increasing, there exists a threshold $\tilde{x}_i \in (0, \infty)$ such that $v'(\tilde{x}_i, i) = 1$. Hence, for $x \in [0, \tilde{x}_i)$, we have

$$\frac{1}{2}\sigma_i^2 v''(x,i) + \mu_i v'(x,i) - \delta v(x,i) = \lambda_i(v(x,i) - v(x, 3-i)),$$

and for $x \in [\tilde{x}_i, \infty)$,

$$\frac{1}{2}\sigma_i^2 v''(x,i) + (\mu_i - K)v'(x,i) - \delta v(x,i) + K = \lambda_i(v(x,i) - v(x, 3-i)).$$

The relation between $\tilde{x}_1$ and $\tilde{x}_2$ depends on the relations among the drift coefficients, the volatility parameters and the rates λ_1 and λ_2. We will only consider $\tilde{x}_1 < \tilde{x}_2$; the case $\tilde{x}_1 > \tilde{x}_2$ has a similar treatment. Thus, we have to consider three possibilities for the initial level of the cash reservoir: $x \in [0, \tilde{x}_1)$, $x \in [\tilde{x}_1, \tilde{x}_2)$ and $x \in [\tilde{x}_2, \infty)$.

When $x \in [0, \tilde{x}_1)$, the solution for the problem will satisfy the following system of ordinary differential equations:

$$\begin{aligned} -(\lambda_1 + \delta)\, v(x,1) + \mu_1\, v'(x,1) + \frac{1}{2}\sigma_1^2\, v''(x,1) + \lambda_1\, v(x,2) &= 0 \\ -(\lambda_2 + \delta)\, v(x,2) + \mu_2\, v'(x,2) + \frac{1}{2}\sigma_2^2\, v''(x,2) + \lambda_2\, v(x,1) &= 0. \end{aligned} \tag{4.5}$$

Consider the characteristic function for (4.5), $\phi_1^1(\beta)\ \phi_2^1(\beta) = \lambda_1\lambda_2$, where

$$\phi_i^1(\beta) := -\frac{1}{2}\sigma_i^2\beta^2 - \mu_i\,\beta + (\lambda_i + \delta), \qquad i = 1, 2.$$

Lemma 4.1 proves that this characteristic function has 4 real roots: $\beta_1 < \beta_2 < 0 < \beta_3 < \beta_4$. Then, the solution for the system of equations (4.5) is

$$(4.6)\quad v(x,1) \;=\; A_1\, e^{\beta_1(x-\tilde{x}_1)} + A_2\, e^{\beta_2(x-\tilde{x}_1)} + A_3\, e^{\beta_3(x-\tilde{x}_1)} + A_4\, e^{\beta_4(x-\tilde{x}_1)}$$

$$(4.7)\quad v(x,2) \;=\; B_1\, e^{\beta_1(x-\tilde{x}_1)} + B_2\, e^{\beta_2(x-\tilde{x}_1)} + B_3\, e^{\beta_3(x-\tilde{x}_1)} + B_4\, e^{\beta_4(x-\tilde{x}_1)},$$

where, for each $j = 1,2,3,4$,

$$(4.8)\qquad B_j \;=\; \frac{\phi_1^1(\beta_j)}{\lambda_1}\, A_j \;=\; \frac{\lambda_2}{\phi_2^1(\beta_j)}\, A_j.$$

When $x \in [\,\tilde{x}_1, \tilde{x}_2)$, the required solution will satisfy the following system of linear differential equations:

(4.9)

$$\begin{aligned} -(\lambda_1+\delta)\, v(x,1) + (\mu_1 - K)\, v'(x,1) + \frac{1}{2}\,\sigma_1^2\, v''(x,1) + K + \lambda_1\, v(x,2) &= 0 \\ -(\lambda_2+\delta)\, v(x,2) + \mu_2\, v'(x,2) + \frac{1}{2}\,\sigma_2^2\, v''(x,2) + \lambda_2\, v(x,1) &= 0. \end{aligned}$$

Consider the characteristic function for (4.9), $\phi_1^2(\alpha)\;\phi_2^2(\alpha) \;=\; \lambda_1\lambda_2$, where

$$\begin{aligned} \phi_1^2(\alpha) &:= -\frac{1}{2}\,\sigma_1^2\alpha^2 - (\mu_1 - K)\,\alpha + (\lambda_1+\delta) \\ \phi_2^2(\alpha) &:= -\frac{1}{2}\,\sigma_2^2\alpha^2 - \mu_2\,\alpha + (\lambda_2+\delta). \end{aligned}$$

From Lemma 4.1, this characteristic function has 4 real roots $\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$. Then, we find that the solution for the system of equations (4.9) is:

$$(4.10)\quad v(x,1) = \widetilde{A}_1\, e^{\alpha_1(x-\tilde{x}_2)} + \widetilde{A}_2\, e^{\alpha_2(x-\tilde{x}_2)} + \widetilde{A}_3\, e^{\alpha_3(x-\tilde{x}_2)} + \widetilde{A}_4\, e^{\alpha_4 x} + F_1$$

$$(4.11)\quad v(x,2) = \widetilde{B}_1\, e^{\alpha_1(x-\tilde{x}_2)} + \widetilde{B}_2\, e^{\alpha_2(x-\tilde{x}_2)} + \widetilde{B}_3\, e^{\alpha_3(x-\tilde{x}_2)} + \widetilde{B}_4\, e^{\alpha_4 x} + F_2,$$

where

$$F_i \;:=\; \frac{(\,\lambda_2 + (2-i)\,\delta\,)\;K}{(\lambda_1+\delta)(\lambda_2+\delta) - \lambda_1\lambda_2}. \tag{4.12}$$

Moreover, for each $j = 1, 2, 3, 4$, the condition for the coefficients is

$$\widetilde{B}_j \;=\; \frac{\phi_1^2(\alpha_j)}{\lambda_1}\,\widetilde{A}_j \;=\; \frac{\lambda_2}{\phi_2^2(\alpha_j)}\,\widetilde{A}_j \tag{4.13}$$

Finally, when $x \in [\,\tilde{x}_2, \infty)$, the function $v(\cdot, i)$, $i = 1, 2$, is the solution of the system

(4.14)

$$\begin{aligned} -(\lambda_1+\delta)v(x,1) + (\mu_1 - K)v'(x,1) + \frac{1}{2}\,\sigma_1^2 v''(x,1) + K + \lambda_1 v(x,2) &= 0 \\ -(\lambda_2+\delta)v(x,2) + (\mu_2 - K)v'(x,2) + \frac{1}{2}\,\sigma_2^2 v''(x,2) + K + \lambda_2 v(x,1) &= 0. \end{aligned}$$

Consider the characteristic function for (4.14), $\phi_1^3(\gamma)\;\phi_2^3(\gamma) \;=\; \lambda_1\lambda_2$, where

$$\phi_i^3(\gamma) \;:=\; -\frac{1}{2}\,\sigma_i^2\gamma^2 - (\mu_i - K)\,\gamma + (\lambda_i + \delta).$$

From Lemma 4.1, this function has 4 real roots $\gamma_1 < \gamma_2 < 0 < \gamma_3 < \gamma_4$. Then, the solution for the system of linear differential equations (4.14) is given by:

$$\begin{aligned} v(x,1) &= \widehat{A}_1 e^{\gamma_1 x} + \widehat{A}_2 e^{\gamma_2 x} + \widehat{A}_3 e^{\gamma_3 x} + \widehat{A}_4 e^{\gamma_4 x} + \frac{K}{\delta} \\ v(x,2) &= \widehat{B}_1 e^{\gamma_1 x} + \widehat{B}_2 e^{\gamma_2 x} + \widehat{B}_3 e^{\gamma_3 x} + \widehat{B}_4 e^{\gamma_4 x} + \frac{K}{\delta}, \end{aligned}$$

where the condition for the coefficients, for each $j = 1, 2, 3, 4$, is

$$\widehat{B}_j \;=\; \frac{\phi_1^3(\gamma_j)}{\lambda_1}\,\widehat{A}_j \;=\; \frac{\lambda_2}{\phi_2^3(\gamma_j)}\,\widehat{A}_j. \tag{4.15}$$

Recall that we are conjecturing the functions v and v' to be bounded, which necessarily makes $\widehat{A}_3 = \widehat{A}_4 = \widehat{B}_3 = \widehat{B}_4 = 0$. Therefore,

$$v(x,1) = \widehat{A}_1 e^{\gamma_1 x} + \widehat{A}_2 e^{\gamma_2 x} + \frac{K}{\delta} \tag{4.16}$$

$$v(x,2) = \widehat{B}_1 e^{\gamma_1 x} + \widehat{B}_2 e^{\gamma_2 x} + \frac{K}{\delta}, \tag{4.17}$$

is the solution for the system (4.14), where (4.15) is satisfied for $j = 1, 2$.

Now, in order to find the thresholds $\tilde{x}_1$ and $\tilde{x}_2$, and the coefficients in functions (4.6), (4.10) and (4.16), we conjecture that the *smooth-fit condition* holds. We also want $v'(\tilde{x}_i, i) = 1$ for each $i = 1, 2$. Thus, we need to solve the following equations

$$\begin{aligned} v(0,i) &= 0 \\ v(\tilde{x}_i-, i) &= v(\tilde{x}_i+, i) \\ v(\tilde{x}_{3-i}-, i) &= v(\tilde{x}_{3-i}+, i) \\ v'(\tilde{x}_i-, i) &= 1 \\ v'(\tilde{x}_i+, i) &= 1 \\ v'(\tilde{x}_{3-i}-, i) &= v'(\tilde{x}_{3-i}+, i), \end{aligned} \tag{4.18}$$

for each $i = 1, 2$. The solution of the system of equations (4.18) will give us the values for $\tilde{x}_1$ and $\tilde{x}_2$, and also the values for A_j, $\widetilde{A}_j$, $j = 1, 2, 3, 4$, and $\widehat{A}_j$, $j = 1, 2$. The values for the corresponding B_j, $\widetilde{B}_j$ and $\widehat{B}_j$ will be found from equations (4.8), (4.13), and (4.15).

Recall we are assuming $v'(0, i) > 1$ for both $i = 1, 2$. This situation is satisfied if the coefficients found through the system of equations (4.18) satisfy

$$\begin{aligned} A_1\beta_1 + A_2\beta_2 + A_3\beta_3 + A_4\beta_4 &> 1, \\ A_1\phi_1^1(\beta_1)\beta_1 + A_2\phi_1^1(\beta_2)\beta_2 + A_3\phi_1^1(\beta_3)\beta_3 + A_4\phi_1^1(\beta_4)\beta_4 &> \lambda_1, \end{aligned} \tag{4.19}$$

and also if $\tilde{x}_2 > \tilde{x}_1 > 0$ is satisfied.

Case 2: There exists $i_0 \in \{1,2\}$ such that $v'(0,i_0) \leq 1$ and $v'(0,3-i_0) > 1$.

Now we assume that $v'(0,i_0) \leq 1$ for a certain fix $i_0 \in \{1,2\}$. Recall the conjecture that $v'(\cdot,i)$ is non-increasing, which implies $v'(x,i_0) \leq v'(0,i_0) \leq 1$ for every $x > 0$. Hence, $v'(x,i_0) \leq 1$ for every $x \geq 0$. Then, equation (4.2) is equivalent to

$$\frac{1}{2}\sigma_{i_0}^2 v''(x,i_0) + (\mu_{i_0} - K)v'(x,i_0) - \delta v(x,i_0) + K = \lambda_i(v(x,i_0) - v(x,3-i_0))$$

for every $x \geq 0$. Moreover, since $v'(0,3-i_0) > 1$, an analysis similar to the one for Case 1 implies that there exists a threshold $\tilde{x}_{3-i_0} > 0$ such that $v'(\tilde{x}_{3-i_0}, 3-i_0) = 1$. Thus, for every $x \in [0, \tilde{x}_{3-i_0})$,

$$\frac{1}{2}\sigma_{3-i_0}^2 v''(x,3-i_0) + \mu_{3-i_0} v'(x,3-i_0) - \delta v(x,3-i_0) = \lambda_{3-i_0}(v(x,3-i_0) - v(x,i_0)),$$

and, for every $x \in [\tilde{x}_{3-i_0}, \infty)$,

$$\begin{aligned} \frac{1}{2}\sigma_{3-i_0}^2 v''(x,3-i_0) + (\mu_{3-i_0} - K)v'(x,3-i_0) - \delta v(x,3-i_0) + K & \\ = \quad \lambda_{3-i_0}(v(x,3-i_0) - v(x,i_0)). \end{aligned}$$

Hence, when $x \in [0, \tilde{x}_{3-i_0})$, $v(\cdot,i)$, $i = 1,2$, satisfies the following system of differential equations:

(4.20)

$$\begin{aligned} \frac{1}{2}\sigma_{i_0}^2 v''(x,i_0) + (\mu_{i_0} - K)v'(x,i_0) - (\lambda_{i_0} + \delta)v(x,i_0) + K &= -\lambda_{i_0} v(x,3-i_0) \\ \frac{1}{2}\sigma_{3-i_0}^2 v''(x,3-i_0) + \mu_{3-i_0} v'(x,3-i_0) - (\lambda_{3-i_0} + \delta)v(x,3-i_0) & \\ &= -\lambda_{3-i_0} v(x,i_0). \end{aligned}$$

Consider the associated characteristic function for (4.20), $\phi^4_{i_0}(\theta)\,\phi^4_{3-i_0}(\theta) = \lambda_1\lambda_2$, where

$$\begin{aligned}\phi^4_{i_0}(\theta) &:= -\frac{1}{2}\sigma^2_{i_0}\theta^2 - (\mu_{i_0} - K)\theta + (\lambda_{i_0} + \delta)\\ \phi^4_{3-i_0}(\theta) &:= -\frac{1}{2}\sigma^2_{3-i_0}\theta^2 - \mu_{3-i_0}\theta + (\lambda_{3-i_0} + \delta).\end{aligned}$$

From Lemma 4.1, this characteristic function has 4 real roots $\theta_1 < \theta_2 < 0 < \theta_3 < \theta_4$. We note that if actually $i_0 = 1$, then we have the system (4.9) of differential equations, and hence $\phi^4_1 = \phi^2_1$, $\phi^4_2 = \phi^2_2$, and $\theta_j = \alpha_j$, $j = 1, 2, 3, 4$. In general, the solution for the system of equations (4.20) is:

$$\text{(4.21)} \qquad v(x, i_0) = C_1 e^{\theta_1 x} + C_2 e^{\theta_2 x} + C_3 e^{\theta_3 x} + C_4 e^{\theta_4 x} + F_{i_0}$$

$$\text{(4.22)} \qquad v(x, 3-i_0) = D_1 e^{\theta_1 x} + D_2 e^{\theta_2 x} + D_3 e^{\theta_3 x} + D_4 e^{\theta_4 x} + F_{3-i_0},$$

where

$$\text{(4.23)} \qquad F_i := \frac{\left(\lambda_{3-i_0} + \delta\, I_{\{i=i_0\}}\right) K}{(\lambda_1 + \delta)(\lambda_2 + \delta) - \lambda_1\lambda_2}.$$

Moreover, the condition for the coefficients is, for each $j = 1, 2, 3, 4$,

$$\text{(4.24)} \qquad D_j = \frac{\phi^4_{i_0}(\theta_j)}{\lambda_{i_0}} C_j = \frac{\lambda_{3-i_0}}{\phi^4_{3-i_0}(\theta_j)} C_j.$$

On the other hand, when $x \in [\tilde{x}_{3-i_0}, \infty)$, the function $v(\cdot, i)$, $i = 1, 2$, is the solution of the system

$$\begin{aligned}\frac{1}{2}\sigma^2_{i_0} v''(x, i_0) + (\mu_{i_0} - K)v'(x, i_0) - (\lambda_{i_0} + \delta)v(x, i_0) + K &= -\lambda_{i_0} v(x, 3-i_0)\\ \frac{1}{2}\sigma^2_{3-i_0} v''(x, 3-i_0) + (\mu_{3-i_0} - K)v'(x, 3-i_0) - (\lambda_{3-i_0} + \delta)v(x, 3-i_0) + K &\\ &= -\lambda_{3-i_0} v(x, i_0).\end{aligned}$$

This system of equations is similar to (4.14). Hence, the solution for the above system of differential equations is given by:

$$\begin{aligned} v(x, i_0) &= \widetilde{C}_1 e^{\gamma_1 x} + \widetilde{C}_2 e^{\gamma_2 x} + \widetilde{C}_3 e^{\gamma_3 x} + \widetilde{C}_4 e^{\gamma_4 x} + \frac{K}{\delta} \\ v(x, 3-i_0) &= \widetilde{D}_1 e^{\gamma_1 x} + \widetilde{D}_2 e^{\gamma_2 x} + \widetilde{D}_3 e^{\gamma_3 x} + \widetilde{D}_4 e^{\gamma_4 x} + \frac{K}{\delta}, \end{aligned}$$

where the condition for the coefficients is

$$\widetilde{D}_j = \frac{\phi^3_{i_0}(\gamma_j)}{\lambda_{i_0}} \widetilde{C}_j = \frac{\lambda_{3-i_0}}{\phi^3_{3-i_0}(\gamma_j)} \widetilde{C}_j \tag{4.25}$$

for each $j = 1, 2, 3, 4$. Recall that we are conjecturing that the functions $v(\cdot, i)$, $i = 1, 2$, and $v'(\cdot, i)$, $i = 1, 2$, are bounded, which necessarily makes $\widetilde{C}_3 = \widetilde{C}_4 = \widetilde{D}_3 = \widetilde{D}_4 = 0$. Hence,

$$v(x, i_0) = \widetilde{C}_1 e^{\gamma_1 x} + \widetilde{C}_2 e^{\gamma_2 x} + \frac{K}{\delta} \tag{4.26}$$

$$v(x, 3-i_0) = \widetilde{D}_1 e^{\gamma_1 x} + \widetilde{D}_2 e^{\gamma_2 x} + \frac{K}{\delta}, \tag{4.27}$$

is the solution for the system above, where (4.25) is satisfied for $j = 1, 2$.

Now, in order to find the threshold $\tilde{x}_{3-i_0}$ and the coefficients in (4.21) and (4.26), we conjecture that the *smooth-fit condition* holds. Also, we want

$v'(\tilde{x}_{3-i_0}, 3-i_0) = 1$. Hence, we need to solve the following system of equations

$$
(4.28)\qquad \begin{aligned}
v(0, i_0) &= 0 \\
v(0, 3-i_0) &= 0 \\
v(\tilde{x}_{3-i_0}-, i_0) &= v(\tilde{x}_{3-i_0}+, i_0) \\
v(\tilde{x}_{3-i_0}-, 3-i_0) &= v(\tilde{x}_{3-i_0}+, 3-i_0) \\
v'(\tilde{x}_{3-i_0}-, 3-i_0) &= 1 \\
v'(\tilde{x}_{3-i_0}+, 3-i_0) &= 1 \\
v'(\tilde{x}_{3-i_0}-, i_0) &= v'(\tilde{x}_{3-i_0}+, i_0).
\end{aligned}
$$

The solution of the system of equations (4.28) will give us the value for $\tilde{x}_{3-i_0}$ and also the values for C_j, $j = 1, 2, 3, 4$, and $\widetilde{C}_j$, $j = 1, 2$. The values for the corresponding D_j and $\widetilde{D}_j$ will be found from equations (4.24) and (4.25).

Recall we are considering the case in which $v'(0, i_0) \le 1$ and $v'(0, 3-i_0) > 1$. In this case, the coefficients in (4.28) satisfy

$$
(4.29)\qquad \begin{aligned}
0 \;&\le\; C_1\,\theta_1 + C_2\,\theta_2 + C_3\,\theta_3 + C_4\,\theta_4 \;\le\; 1, \\
\lambda_{i_0} \;&<\; C_1\,\phi^4_{i_0}(\theta_1)\,\theta_1 + C_2\,\phi^4_{i_0}(\theta_2)\,\theta_2 + C_3\,\phi^4_{i_0}(\theta_3)\,\theta_3 + C_4\,\phi^4_{i_0}(\theta_4)\,\theta_4.
\end{aligned}
$$

In addition, $\tilde{x}_{3-i_0} > 0$.

Case 3: $v'(0, i) \le 1$ for both $i = 1, 2$.

If $v'(0, i) \le 1$ for both $i = 1, 2$, then $v'(x, i) \le 1$ for every $x \ge 0$ and $\varphi(x, i) = K$ for every $x \ge 0$, for both $i = 1, 2$ as well. Hence, the function $v(\cdot, i)$, $i = 1, 2$, will satisfy the system of differential equations:

$$
\begin{aligned}
-(\lambda_1 + \delta)\, v(x, 1) + (\mu_1 - K)\, v'(x, 1) + \frac{1}{2}\,\sigma_1^2\, v''(x, 1) + K + \lambda_1\, v(x, 2) &= 0 \\
-(\lambda_2 + \delta)\, v(x, 2) + (\mu_2 - K)\, v'(x, 2) + \frac{1}{2}\,\sigma_2^2\, v''(x, 2) + K + \lambda_2\, v(x, 1) &= 0.
\end{aligned}
$$

for every $x \geq 0$, which is equivalent to the system of equations (4.14). Thus, the solution for this system of differential equations is given by

$$\begin{aligned} v(x,1) &= \widehat{C}_1 e^{\gamma_1 x} + \widehat{C}_2 e^{\gamma_2 x} + \widehat{C}_3 e^{\gamma_3 x} + \widehat{C}_4 e^{\gamma_4 x} + \frac{K}{\delta} \\ v(x,2) &= \widehat{D}_1 e^{\gamma_1 x} + \widehat{D}_2 e^{\gamma_2 x} + \widehat{D}_3 e^{\gamma_3 x} + \widehat{D}_4 e^{\gamma_4 x} + \frac{K}{\delta}, \end{aligned}$$

where the condition for the coefficients is

$$\widehat{D}_j = \frac{\phi_1^3(\gamma_j)}{\lambda_1} \widehat{C}_j = \frac{\lambda_2}{\phi_2^3(\gamma_j)} \widehat{C}_j \tag{4.30}$$

for each $j = 1,2,3,4$. The conjecture that $v(\cdot,i)$, $i = 1,2$, and $v'(\cdot,i)$, $i = 1,2$, are bounded functions implies that $\widehat{C}_3 = \widehat{C}_4 = \widehat{D}_3 = \widehat{D}_4 = 0$. Hence,

$$v(x,1) = \widehat{C}_1 e^{\gamma_1 x} + \widehat{C}_2 e^{\gamma_2 x} + \frac{K}{\delta} \tag{4.31}$$

$$v(x,2) = \widehat{D}_1 e^{\gamma_1 x} + \widehat{D}_2 e^{\gamma_2 x} + \frac{K}{\delta}, \tag{4.32}$$

is the solution for the system of equations where (4.30) is satisfied for $j = 1,2$.

In order to find the coefficients in (4.31) we have the initial conditions $v(0,1) = v(0,2) = 0$. This gives us the coefficients

$$\begin{aligned} \widehat{C}_1 &= -\frac{K}{\delta} \frac{\phi_1^3(\gamma_2) - \lambda_1}{\phi_1^3(\gamma_2) - \phi_1^3(\gamma_1)} \\ \widehat{C}_2 &= -\frac{K}{\delta} \frac{\lambda_1 - \phi_1^3(\gamma_1)}{\phi_1^3(\gamma_2) - \phi_1^3(\gamma_1)}. \end{aligned} \tag{4.33}$$

Moreover, the coefficients $\widehat{D}_1$ and $\widehat{D}_2$ are given by condition (4.30). One additional condition that needs to be satisfied in order to have this situation is related to the assumption $v'(0,i) \leq 1$ for both $i = 1,2$. That assumption is equivalent to

$$(4.34)\quad 0 \;\le\; -\frac{K}{\delta}\,\frac{\phi_1^3(\gamma_2)\gamma_1 + \lambda_1(\gamma_2-\gamma_1) - \phi_1^3(\gamma_1)\gamma_2}{\phi_1^3(\gamma_2)-\phi_1^3(\gamma_1)} \;\le\; 1,$$

$$0 \;\le\; -\frac{K}{\lambda_1\,\delta}\,\frac{\phi_1^3(\gamma_1)\phi_1^3(\gamma_2)(\gamma_1-\gamma_2) + \lambda_1(\phi_1^3(\gamma_1)\gamma_1 - \phi_1^3(\gamma_2)\gamma_2)}{\phi_1^3(\gamma_2)-\phi_1^3(\gamma_1)} \;\le\; 1.$$

4.4 Verification of the solution

In section 4.3, we made some conjectures to find a candidate for value function. In this section, we will prove that the function $v(\cdot,i)$, $i=1,2$, is actually the value function $V(\cdot,i)$, $i=1,2$, of Problem 4.1. We will also present the optimal dividend policy.

Theorem 4.2. Let A_j, $\widetilde{A}_j$, $j=1,2,3,4$, and $\widehat{A}_j$, $j=1,2$, be the solution of the system of equations (4.18) and suppose that they satisfy condition (4.19). Let B_j, $\widetilde{B}_j$, $j=1,2,3,4$, and $\widehat{B}_j$, $j=1,2$, be defined by (4.8), (4.13) and (4.15); and F_i, $i=1,2$, defined by (4.12). Suppose, without loss of generality, that $\tilde{x}_1<\tilde{x}_2$. Then, the function $v(\cdot,i)$, $i=1,2$, given by

$$v(x,1)=\begin{cases} A_1e^{\beta_1(x-\tilde{x}_1)}+A_2e^{\beta_2(x-\tilde{x}_1)}+A_3e^{\beta_3(x-\tilde{x}_1)}+A_4e^{\beta_4(x-\tilde{x}_1)}, & x\in[0,\widetilde{x}_1),\\ \widetilde{A}_1e^{\alpha_1(x-\tilde{x}_2)}+\widetilde{A}_2e^{\alpha_2(x-\tilde{x}_2)}+\widetilde{A}_3e^{\alpha_3(x-\tilde{x}_2)}+\widetilde{A}_4e^{\alpha_4(x-\tilde{x}_2)}+F_1, & x\in[\widetilde{x}_1,\widetilde{x}_2),\\ \widehat{A}_1e^{\gamma_1x}+\widehat{A}_2e^{\gamma_2x}+\frac{K}{\delta}, & x\in[\widetilde{x}_2,\infty),\end{cases}$$

and

$$v(x,2)=\begin{cases} B_1e^{\beta_1(x-\tilde{x}_1)}+B_2e^{\beta_2(x-\tilde{x}_1)}+B_3e^{\beta_3(x-\tilde{x}_1)}+B_4e^{\beta_4(x-\tilde{x}_1)}, & x\in[0,\widetilde{x}_1),\\ \widetilde{B}_1e^{\alpha_1(x-\tilde{x}_2)}+\widetilde{B}_2e^{\alpha_2(x-\tilde{x}_2)}+\widetilde{B}_3e^{\alpha_3(x-\tilde{x}_2)}+\widetilde{B}_4e^{\alpha_4(x-\tilde{x}_2)}+F_2, & x\in[\widetilde{x}_1,\widetilde{x}_2),\\ \widehat{B}_1e^{\gamma_1x}+\widehat{B}_2e^{\gamma_2x}+\frac{K}{\delta}, & x\in[\widetilde{x}_2,\infty),\end{cases}$$

is the value function $V(\cdot,i)$, $i=1,2$, of Problem 4.1. Furthermore, the control

$\hat{u}$ defined by

$$\hat{u}(t) = \begin{cases} 0 & \text{if} \quad \epsilon_t = i \quad \text{and} \quad X_t \in [0, \widetilde{x}_i), \\ K & \text{if} \quad \epsilon_t = i \quad \text{and} \quad X_t \in [\widetilde{x}_i, \infty), \end{cases}$$

for $t \in [0, \Theta)$ and $\hat{u}(t) = 0$ for $t \in [\Theta, \infty)$, is optimal dividend policy for Problem 4.1.

Consider the case in which A_j, $\widetilde{A}_j$, $j = 1, 2, 3, 4$, and $\widehat{A}_j$, $j = 1, 2$, do not satisfy condition (4.19). Instead, suppose that condition (4.29) is satisfied for a fixed $i_0 \in \{1, 2\}$, where C_j, $j = 1, 2, 3, 4$, and $\widetilde{C}_j$, $j = 1, 2$, are the solution of the system of equations (4.28). Moreover, let D_j, $j = 1, 2, 3, 4$, and $\widetilde{D}_j$, $j = 1, 2$, be defined by (4.24) and (4.25); and let F_{i_0} and F_{3-i_0} be defined by (4.23). Then, the function $v(\cdot, i)$, $i = 1, 2$, given by

$$v(x, i_0) = \begin{cases} C_1 e^{\theta_1 x} + C_2 e^{\theta_2 x} + C_3 e^{\theta_3 x} + C_4 e^{\theta_4 x} + F_{i_0}, & x \in [0, \widetilde{x}_{3-i_0}), \\ \widetilde{C}_1 e^{\gamma_1 x} + \widetilde{C}_2 e^{\gamma_2 x} + \frac{K}{\delta}, & x \in [\widetilde{x}_{3-i_0}, \infty), \end{cases}$$

and

$$v(x, 3-i_0) = \begin{cases} D_1 e^{\theta_1 x} + D_2 e^{\theta_2 x} + D_3 e^{\theta_3 x} + D_4 e^{\theta_4 x} + F_{3-i_0}, & x \in [0, \widetilde{x}_{3-i_0}), \\ \widetilde{D}_1 e^{\gamma_1 x} + \widetilde{D}_2 e^{\gamma_2 x} + \frac{K}{\delta}, & x \in [\widetilde{x}_{3-i_0}, \infty), \end{cases}$$

is the value function $V(\cdot, i)$, $i = 1, 2$, of Problem 4.1. Furthermore, the control $\hat{u}$ defined by

$$\hat{u}(t) = \begin{cases} K & \text{if} \quad \epsilon_t = i_0, \\ 0 & \text{if} \quad \epsilon_t = 3 - i_0 \quad \text{and} \quad X_t \in [0, \widetilde{x}_{3-i_0}), \\ K & \text{if} \quad \epsilon_t = 3 - i_0 \quad \text{and} \quad X_t \in [\widetilde{x}_{3-i_0}, \infty), \end{cases}$$

for $t \in [0, \Theta)$ and $\hat{u}(t) = 0$ for $t \in [\Theta, \infty)$, is optimal dividend policy for

Problem 4.1.

Consider the case in which neither condition (4.19) nor condition (4.29) are satisfied. Thus, condition (4.34) is satisfied. Moreover, let $\widehat{C}_j$, $j = 1, 2$, be defined by (4.33), and $\widehat{D}_j$, $j = 1, 2$, by (4.30). Then the function $v(\cdot, i)$, $i = 1, 2$, given by

$$\begin{aligned} v(x,1) &= \widehat{C}_1 e^{\gamma_1 x} + \widehat{C}_2 e^{\gamma_2 x} + \frac{K}{\delta}, \\ v(x,2) &= \widehat{D}_1 e^{\gamma_1 x} + \widehat{D}_2 e^{\gamma_2 x} + \frac{K}{\delta}, \end{aligned}$$

is the value function $V(\cdot, i)$, $i = 1, 2$, of Problem 4.1. In this case, the control $\hat{u}$ defined by

$$\hat{u}(t) = \begin{cases} K & \text{if} \quad t \in [0, \Theta), \\ 0 & \text{if} \quad t \in [\Theta, \infty), \end{cases}$$

is optimal dividend policy for Problem 4.1.

Proof. In order to prove that the function $v(\cdot, i)$, $i = 1, 2$, defined above is solution for Problem 4.1, we need to show that it satisfies the conditions of Theorem 4.1. We have three different cases to consider depending on which condition holds ((4.19), (4.29) or (4.34)). Since the proofs for each of the three cases are similar, we will consider only the first case, that is, the case in which condition (4.19) holds.

It is easy to see that by definition $v(\cdot, i) \in C^2([0, \infty) - \{\tilde{x}_1, \tilde{x}_2\})$, $i = 1, 2$. We recall that continuity at $\tilde{x}_1$ and $\tilde{x}_2$ is given by equations (4.18). Moreover, for $x \in [\tilde{x}_2, \infty)$,

$$\begin{aligned} v(x,1) &= \widehat{A}_1 e^{\gamma_1 x} + \widehat{A}_2 e^{\gamma_2 x} + \frac{K}{\delta}, \\ v(x,2) &= \widehat{B}_1 e^{\gamma_1 x} + \widehat{B}_2 e^{\gamma_2 x} + \frac{K}{\delta} \end{aligned}$$

where $\gamma_1 < \gamma_2 < 0$. Thus, $\lim_{x\to\infty} v(x,1) = \lim_{x\to\infty} v(x,2) = K/\delta < +\infty$. This together with the continuity of $v(\cdot,i)$, $i = 1,2$, on $[0,\infty)$ imply that $v(\cdot,i)$, $i = 1,2$, is bounded on $[0,\infty)$. Similarly, $v'(\cdot,i)$, $i = 1,2$, is bounded on $[0,\infty)$ because it is continuous on $[0,\infty)$ (given again by equations (4.18)) and because $\lim_{x\to\infty} v'(x,1) = \lim_{x\to\infty} v'(x,2) = 0$. Moreover, the system of equations (4.18) guarantees that $v(0,1) = v(0,2) = 0$.

By construction and the initial condition (4.19), we have

$$\varphi(x,i) = \begin{cases} 0 & \text{if } x \in [0,\tilde{x}_i), \\ K & \text{if } x \in [\tilde{x}_i,\infty), \end{cases}$$

which implies $\varphi(x,i) \in [0,K]$, for every $x \in [0,\infty)$, $i = 1,2$. Also by construction,

$$\hat{u}(t) := \arg \sup_{u_t \in [0,K]} \{L_{\epsilon(t)}(u_t)v(X_t,\epsilon_t) + u_t\} = \varphi(X_t,\epsilon_t)$$

for $t \in [0,\Theta)$, and hence

$$\hat{u}(t) = \begin{cases} 0 & \text{if } \epsilon_t = i \text{ and } X_t \in [0,\widetilde{x}_i), \\ K & \text{if } \epsilon_t = i \text{ and } X_t \in [\widetilde{x}_i,\infty), \end{cases}$$

for $t \in [0,\Theta)$ and $\hat{u}(t) = 0$ for $t \in [\Theta,\infty)$. Moreover, the equation

$$L_i(\varphi(x,i))v(x,i) + \varphi(x,i) = \lambda_i(v(x,i) - v(x,3-i)) \tag{4.35}$$

holds for both $i = 1,2$. Indeed, for $x \in [0,\tilde{x}_1)$, the function $v(x,i)$, $i = 1,2$, is solution of the system of differential equations (4.5); for $x \in [\tilde{x}_1,\tilde{x}_2)$, it is solution of system (4.9); and for $x \in [\tilde{x}_2,\infty)$, of system (4.14). That is, the function $v(\cdot,i)$, $i = 1,2$, satisfies equation (4.2) because equality (4.35) holds also when we replace x by X_t, i by ϵ_t, and $\varphi(x,i)$ by $\varphi(X_t,\epsilon_t) = \hat{u}(t)$ for

$t \in [0, \Theta)$. Therefore, from Theorem 4.1, the control $\hat{u}$ is optimal solution of Problem 4.1, and $v(\cdot, i)$, $i = 1, 2$, is the value function for Problem 4.1. $\square$

4.5 Comparison to the one-regime case

Asmussen and Taksar (1997) show, in a model that allows only one regime, that the optimal dividend policy is to pay no dividends while the process $X = \{X_t, t \geq 0\}$ is below a certain level $\hat{x}$, and pay dividends when the process reaches or exceeds $\hat{x}$. This result is comparable to the one presented in Theorem 4.2. The difference is the existence of two different thresholds, $\widetilde{x}_1$ and $\widetilde{x}_2$, where one of them is used in the case of economic recession and the other one in the case of economic growth. In the regime switching model, the solution for the dividend payment problem is more complex because it involves the new process ϵ that was not considered in the one-regime model. However, this solution is also more precise, because the new process is relevant for the financial analysis.

We will use a numerical example to compare the results of the regime switching model presented in this chapter to the one-regime model presented by Asmussen and Taksar (1997).

Let us consider two companies, Company AA and Company BB, that are not affected by changes of the macroeconomic conditions. Their cash reservoir behaves the same in periods of economic growth as in periods of economic recession, that is, these companies only present one regime. Suppose that Company AA has a regime with parameters $\mu_1 = 0.05$ and $\sigma_1 = 0.70$, and that Company BB has a regime with $\mu_2 = 0.15$ and $\sigma_2 = 0.45$. We notice that the macroeconomic conditions for Company BB are better than those for Company AA. Consider finally a Company AB that follows a regime switching model for its cash reservoir (with both regimes of growth and recession) with parameters $\lambda_1 = 0.06$ and $\lambda_2 = 0.04$, $\mu_1 = 0.05$ and $\mu_2 = 0.15$, and $\sigma_1 = 0.70$ and $\sigma_2 = 0.45$. Moreover, we assume that the market has a discount rate of $\delta = 0.12$. Assume that Company AA, Company BB and Company AB cannot pay dividend rates greater than $K = 5$.

Following Asmussen and Taksar (1997), we obtain the optimal thresholds $\hat{x}_{AA} = 0.3617$ and $\hat{x}_{BB} = 0.9463$ for Company AA and Company BB, respectively. The optimal dividend policy for each company is to pay no dividends until the cash reservoir reaches the respectively threshold and to pay dividends at a rate K when the threshold is reached. We show below that the optimal dividend policy is different for Company AB.

Using the model parameters for Company AB and solving the system of equations (4.18), we obtain the thresholds $\tilde{x}_1 = 0.4670$ and $\tilde{x}_2 = 0.9037$. We also obtain the coefficients A_j, B_j, $\widetilde{A}_j$, $\widetilde{B}_j$, $j = 1, 2, 3, 4$, and $\widehat{A}_j$ and $\widehat{B}_j$, $j = 1, 2$, that satisfy conditions (4.19). Hence, Theorem 4.2 allow us to construct the value function $V_{AB}(\cdot, i)$, $i = 1, 2$, for Company AB. Figure 4.1 and Figure 4.2 show that value function and also the derivative of the value function for the company. Furthermore, also from Theorem 4.2 we have that the optimal dividend rate for Company AB is

$$\hat{u}(t) = \begin{cases} 0 & \text{if} \quad \epsilon_t = i \quad \text{and} \quad X_t \in [0, \tilde{x}_i), \\ K & \text{if} \quad \epsilon_t = i \quad \text{and} \quad X_t \in [\tilde{x}_i, \infty), \end{cases}$$

for any instant $t \in [0, \infty)$ before bankruptcy. That is, the optimal dividend policy for Company AB is the following: whenever the company is in regime 1, it should not pay dividends if the level of its cash reservoir is lower than $\tilde{x}_1 = 0.4670$ and pay the rate K otherwise; if the company is in regime 2, it should also pay the rate K as dividends only when the level of the cash reservoir is greater or equal to $\tilde{x}_2 = 0.9037$. We notice, then, that the optimal dividend policy for Company AB depends strongly on the regime of the economy. Indeed, to pay dividends, the company needs to consider first the present regime, and then verify that the level of its cash reservoir is larger or equal to the threshold corresponding to that regime.

It is obvious that Company BB has better economic conditions than Company AA does, so shareholders should receive on average a higher amount of total discounted dividend payments. Figure 4.3 shows, as expected, that for every $x \in [0, \infty)$: $V_{BB}(x) \geq V_{AA}(x)$. Company AB is in a economy that presents periods of growth and periods of recession. Thus, Company AB can be considered to be better off than Company AA but not as good as Company BB. Figure 4.3 shows that, indeed, $V_{AB}(x, i)$, $i = 1, 2$, is always greater than $V_{AA}(x)$ and is always lower than $V_{BB}(x)$, independently of the initial regime i.

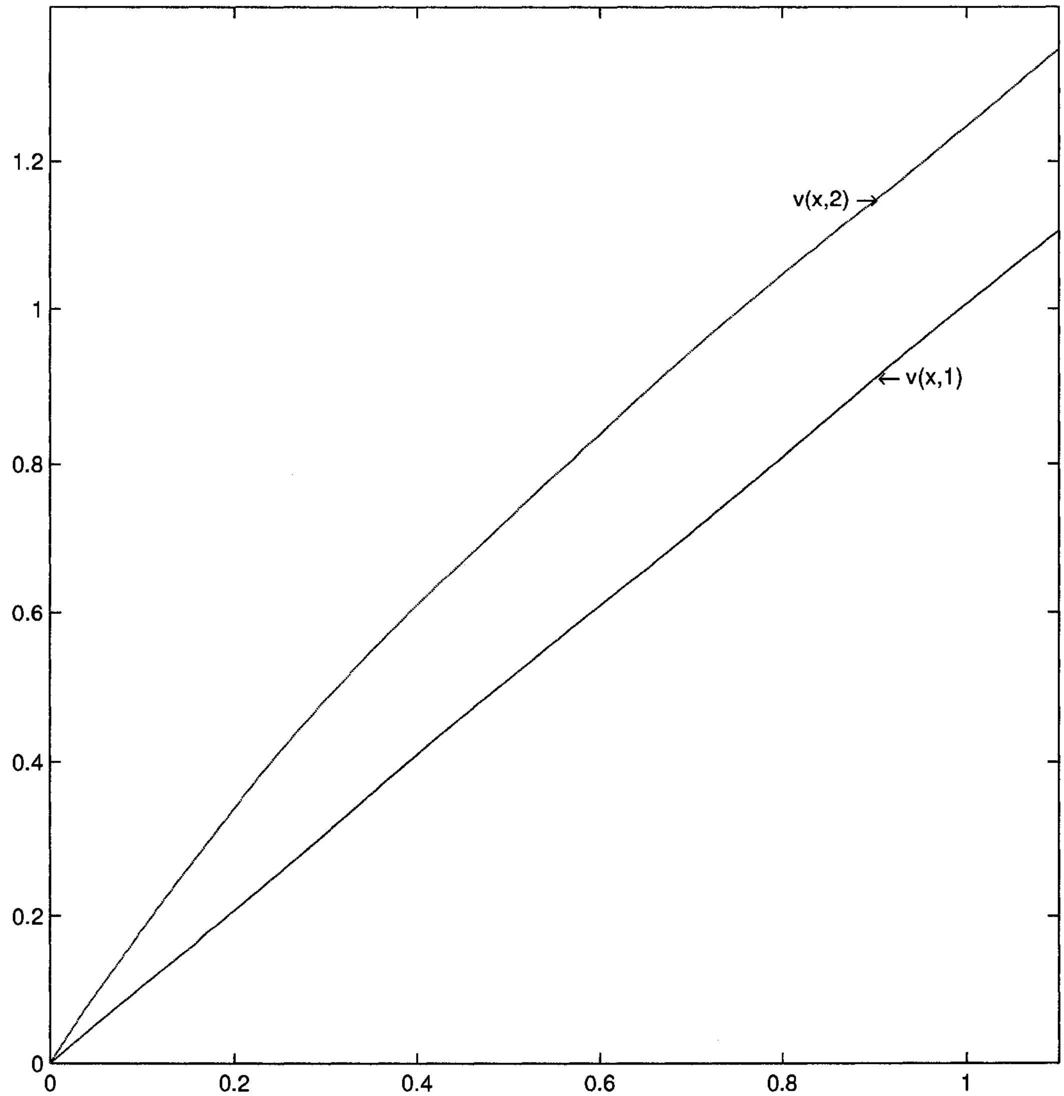

Figure 4.1: Value function for Company AB when the dividend rates are bounded: $V_{AB}(\cdot, i)$, $i = 1, 2$, for model parameters $\lambda_1 = 0.06$, $\lambda_2 = 0.04$, $\mu_1 = 0.05$, $\mu_2 = 0.15$, $\sigma_1 = 0.70$, $\sigma_2 = 0.45$, $\delta = 0.12$ and $K = 5$.

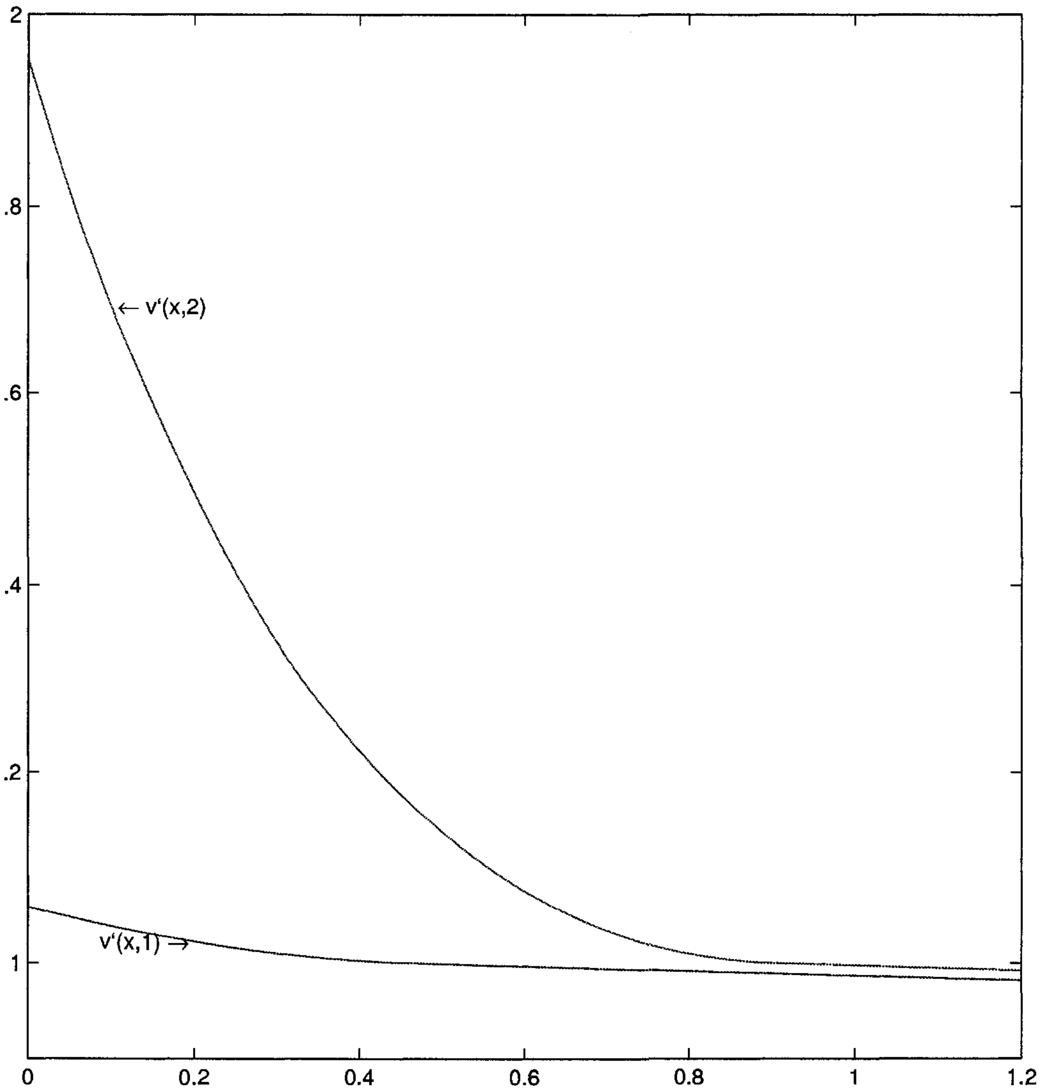

Figure 4.2: Derivative of the value function for Company AB when the dividend rates are bounded: $V'_{AB}(\cdot, i)$, $i = 1, 2$, for model parameters $\lambda_1 = 0.06$, $\lambda_2 = 0.04$, $\mu_1 = 0.05$, $\mu_2 = 0.15$, $\sigma_1 = 0.70$, $\sigma_2 = 0.45$, $\delta = 0.12$ and $K = 5$.

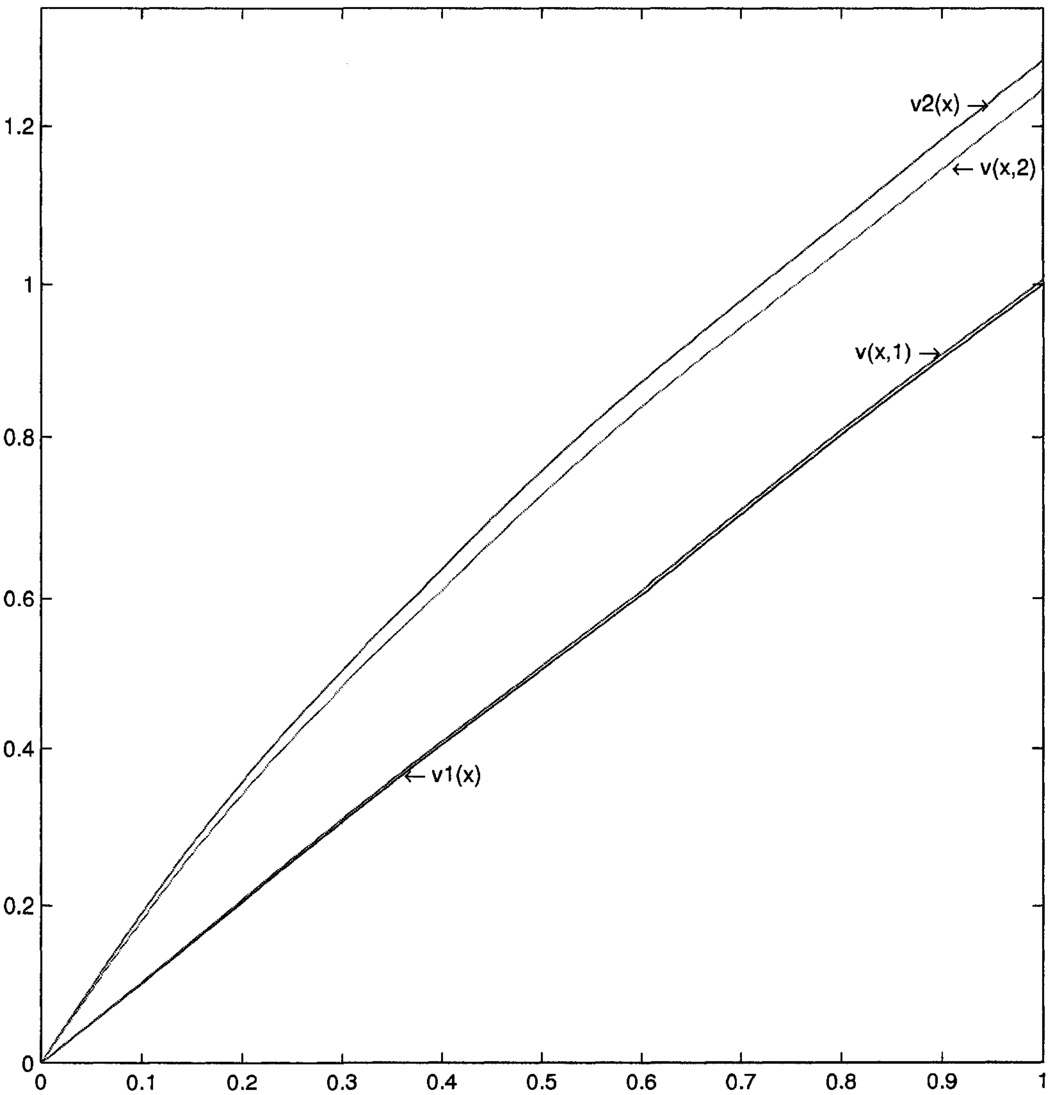

Figure 4.3: Comparison of the value functions for Companies AA, BB and AB when the dividend rates are bounded (comparison with one-regime model). From the top to the bottom: $V_{BB}(\cdot)$, $V_{AB}(\cdot, 2)$, $V_{AB}(\cdot, 1)$ and $V_{AA}(\cdot)$.

Chapter 5

Singular stochastic control with regime switching

The initial setup for singular stochastic control with regime switching problems is similar to the one for classical stochastic control with regime switching problems. We have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a N-dimensional standard Brownian motion W, an observable continuous-time, stationary, finite-state Markov chain ϵ with strongly irreducible generator Q and the $\mathbb{P}$-augmented filtration $\mathbb{F}$. This time, however, the $\mathbb{F}$-adapted process $X = \{X_t,\, t \geq 0\}$ satisfies a different stochastic differential equation.

Let $Z = \{Z_t,\, t \geq 0\}$ be a d-dimensional $\mathbb{F}$-adapted control process of finite variation. We assume that Z is a nonnegative and nondecreasing stochastic process.

Let the open, nonempty, convex set $\mathcal{O} \subset \mathbb{R}^M$ be the solvency region. Consider an $\mathbb{F}$- adapted process $X = \{X_t, t \geq 0\}$ that satisfies the stochastic integral equation

$$(5.1) \qquad X_t \;=\; x + \int_0^t f(X_s, \epsilon_s)\, ds + \int_0^t g(X_s, \epsilon_s)\, dWs + \int_0^t z(\epsilon_s)\, dZ_s,$$

with initial value $X_0 = x \in \mathcal{O}$ and initial state $\epsilon_0 = \epsilon(0) = i \in \mathcal{S}$. We assume that the functions $f : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}^M$ and $g : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}^{M\times N}$ are Lipschitz continuous functions in x. We assume that $z : \mathcal{S} \to \mathbb{R}^{M\times d}$. We define $G : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}^{M\times M}$ as $G(x,i) := g(x,i) \cdot g(x,i)^T$. We note that the Lipschitz conditions implies that $f(\cdot,i)$ and $g(\cdot,i)$ are bounded in every closed subset of $\mathbb{R}^M$, for every $i \in \mathcal{S}$. Hence, $G(\cdot,i)$ is bounded in every closed subset of $\mathbb{R}^M$, for every $i \in \mathcal{S}$.

Definition 5.1. A strong solution of the stochastic integral equation (5.1) is an $\mathbb{F}$-adapted process $X = \{X_t, t \geq 0\}$ such that $X_0 = x$, $\mathbb{P}$-a.s., that satisfies (5.1), and such that, for every $t \in [0,\infty)$:

$$\mathbb{P}\left\{ \sum_{\ell=1}^{d} \int_0^t \| z_{\cdot\ell}(\epsilon_s)\|_M \, d(Z_\ell)_s \; < \; +\infty \right\} \; = \; 1, \tag{5.2}$$

where $z_{\cdot\ell}(i)$ denotes the ℓ-th column of $z(i)$.

Consider the first time when the process X leaves the solvency region. Define such stopping time as

$$\Theta \; = \; \Theta_X \; := \; \inf\{\, t \geq 0 : X_t \notin \mathcal{O}\,\}$$

and impose $X_t = X_\Theta$ for every $t \in [\Theta, \infty)$.

Let $H : \mathcal{S} \to \mathbb{R}^d$ be a given function. Define the functional

$$J(x,i;Z) \; := \; E_{x,i}\left[\int_0^\Theta e^{-\delta t} H(\epsilon_t)^T \, dZ_t \right]. \tag{5.3}$$

The parameter $\delta > 0$ is the discount rate for H. We note that in order for the functional (5.3) to be well defined, we need to require:

$$\mathbb{P}\left\{ \sum_{\ell=1}^{d} \int_0^\Theta e^{-\delta s} | H_\ell(\epsilon_s)| \; d(Z_\ell)_s \; < \; +\infty \right\} \; = \; 1. \tag{5.4}$$

Definition 5.2. For every $x \in \mathcal{O}$ and $i \in \mathcal{S}$, we define an *admissible control process* as a nonnegative nondecreasing adapted control process $Z : [0, \infty) \times \Omega \to \mathbb{R}^d$ with sample paths which are left-continuous with right limits, such that $dZ(t, w) = 0$ for every $(t, w) \in [\Theta, \infty) \times \Omega$, such that the trajectory $X = X^Z$ is the unique strong solution of (5.1), and such that (5.4) is satisfied. The set of such admissible controls will be denoted by $\mathcal{A}(x, i)$.

Problem 5.1. The stochastic control problem related to this setup is then to select, for every $x \in \mathcal{O}$ and $i \in \mathcal{S}$, an optimal admissible control $Z^* \in \mathcal{A}(x, i)$ that maximizes the functional (5.3), and define the value function

$$V(x, i) \;:=\; J(x, i; Z^*) \;=\; \sup_{Z \in \mathcal{A}(x,i)} J(x, i; Z). \tag{5.5}$$

We note that the admissible control processes Z do not necessarily have continuous sample paths; we only need them to have sample paths that are left-continuous with right limits. Hence, we define the set of times when Z has a discontinuity by

$$\Lambda \;:=\; \{\, t \geq 0 : \, Z_{t+} \neq Z_t \,\} .$$

The set Λ is a countable set because Z can jump only a countable number of times during a period $[0, t)$, $t \geq 0$. We denote by Z^d the discontinuous part of Z. That is,

$$Z_t^d \;:=\; \sum_{s \in [0,\, t),\, s \in \Lambda} (Z_{s+} - Z_s),$$

for every $t \in [0, \infty)$. In addition, we denote by Z^c the continuous part of Z. That is, $Z_t^c := Z_t - Z_t^d$, for every $t \in [0, \infty)$.

5.1 The verification theorem

Consider $\psi : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}$ and define the operators L_i, for each $i \in \mathcal{S}$, by

$$\begin{aligned} L_i\,\psi(x,i) \;&:=\; \frac{1}{2}\,G(x,i) \bullet \psi_{xx}(x,i) + f(x,i)^T\,\psi_x(x,i) \;-\; \delta\,\psi(x,i) \\ &=\; \frac{1}{2}\sum_{m=1}^{M}\sum_{p=1}^{M} G_{mp}(x,i)\,\frac{\partial^2\psi}{\partial x_m \partial x_p}(x,i) \\ &\qquad\qquad + \sum_{m=1}^{M} f_m(x,i)\,\frac{\partial\psi}{\partial x_m}(x,i) \;-\; \delta\,\psi(x,i). \end{aligned}$$

We specify now some notation that we need for the following definitions. For a function $v : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}$, we recall that we denote the gradient of v with respect to x by v_x. We also use $\{e_\ell\,,\ \ell = 1,\dots,d\}$ to represent the standard basis of $\mathbb{R}^d$. For a function $v : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}$, we define the continuation region

$$\mathcal{C}(i) := \Big\{x : \big(H(i)^T + v_x(x,i)^T z(i)\big)\cdot e_\ell < 0, \ \text{ for every } \ell = 1,\dots,d\Big\},$$

for each $i \in \mathcal{S}$. We define as well the control process associated with the function $v(\cdot,i)$, $i \in \mathcal{S}$.

Definition 5.3. An $\mathbb{F}$-adapted, nonnegative and nondecreasing control process Z^v is associated with the function $v(\cdot,i)$, $i \in \mathcal{S}$, above if

(i) $X_t^{Z^v} = x + \int_0^t f(X_s^{Z^v},\epsilon_s)\,ds + \int_0^t g(X_s^{Z^v},\epsilon_s)\,dW_s + \int_0^t z(\epsilon_s)\,dZ_s^v,$
for all $t \in [0,\Theta)$ and $X_t^{Z^v} = X_\Theta^{Z^v}$ for all $t \in [\Theta,\infty)$.

(ii) $X_t^{Z^v} \in \mathcal{C}(\epsilon_t)$, Leb. a.e. $t \in [0,\Theta)$, $\mathbb{P}-a.s.$,

(iii) $\int_0^t e^{-\delta s}\big(H(\epsilon_s)^T + v_x(X_s^{Z^v},\epsilon_s)^T z(\epsilon_s)\big)\,dZ_s^v = 0,$ for all $t \in [0,\Theta)$, $\mathbb{P}-a.s.$

(iv) $v(X_{t+}^{Z^v},\epsilon_t) - v(X_t^{Z^v},\epsilon_t) = -H(\epsilon_t)^T(Z_{t+}^v - Z_t^v),$ for all $t \in [0,\Theta)$, $\mathbb{P}-a.s.$

Now we present a Verification Theorem that gives sufficient conditions for a function to be the value function for Problem 5.1.

Theorem 5.1. Let $v(\cdot, i) \in C^1(\mathcal{O}) \cap C^2(\mathcal{O} \backslash N_i)$, $i \in \mathcal{S}$, be a real function in $\mathcal{O}$, where N_i, $i \in \mathcal{S}$, are finite subsets of $\mathcal{O}$. Let $v(x, i) = 0$ for every $x \notin \mathcal{O}$, $i \in \mathcal{S}$. Moreover, assume that $v(\cdot, i)$ has polynomial growth, $i \in \mathcal{S}$. Suppose that the function $v(\cdot, i)$, $i \in \mathcal{S}$, satisfies for every $x \in \mathcal{C}(i)$, the Hamilton-Jacobi-Bellman equation

$$L_i v(x,i) + Q v(x,\cdot)(i) = 0. \tag{5.6}$$

Moreover, suppose that $v(\cdot, i)$, $i \in \mathcal{S}$, satisfies for every $x \in \mathcal{O}$:

$$\big(H(i)^T + v_x(x,i)^T z(i) \big) \cdot e_\ell \le 0, \tag{5.7}$$

for every $\ell = 1, \ldots, d$, and

$$L_i v(x,i) + Q v(x,\cdot)(i) \le 0. \tag{5.8}$$

Then, $v(x, i) \geq J(x, i; Z)$ for every admissible control process Z.

Proof. Consider an arbitrary admissible control Z and the trajectory $X = X^Z$ generated by Z. First of all, we note that $X_{t+} \neq X_t$ if and only if $Z_{t+} \neq Z_t$. Hence, $\{X_{t+} \neq X_t\} = \{t \in \Lambda\}$. Consider then the function $\varphi(\cdot, \cdot, i)$, $i \in \mathcal{S}$, defined as $\varphi(t, x, i) = e^{-\delta t} v(x, i)$. Then, using the Itô's formula for Markov-modulated processes, we get

$$
\begin{aligned}
d\varphi(t,X_t,\epsilon_t) &= \varphi_t(t,X_t,\epsilon_t)\,dt \;+\; \sum_{m=1}^{M}\frac{\partial\varphi}{\partial x_m}(t,X_t,\epsilon_t)\,d(X_m^c)_t \\
&\quad + \frac{1}{2}\sum_{m=1}^{M}\sum_{p=1}^{M}\frac{\partial^2\varphi}{\partial x_m\partial x_p}(t,X_t,\epsilon_t)\,d\!<\!X_m,X_p\!>_t \\
&\quad + \Big(\varphi(t,X_{t+},\epsilon_t)-\varphi(t,X_t,\epsilon_t)\Big)\; I_{\{X_{t+}\neq X_t\}} \\
&\quad + Q\,\varphi(t,X_t,\cdot)(\epsilon_t)\,dt \;+\; dM_t^{\varphi} \\
&= \varphi_t(t,X_t,\epsilon_t)\,dt \;+\; \sum_{m=1}^{M} f_m(X_t,\epsilon_t)\,\frac{\partial\varphi}{\partial x_m}(t,X_t,\epsilon_t)\,dt \\
&\quad + \sum_{m=1}^{M}\sum_{n=1}^{N} g_{mn}(X_t,\epsilon_t)\,\frac{\partial\varphi}{\partial x_m}(t,X_t,\epsilon_t)\,d(W_n)_t \\
&\quad + \sum_{m=1}^{M}\sum_{\ell=1}^{d} z_{m\ell}(\epsilon_t)\,\frac{\partial\varphi}{\partial x_m}(t,X_t,\epsilon_t)\,d(Z_\ell^c)_t \\
&\quad + \frac{1}{2}\sum_{m=1}^{M}\sum_{p=1}^{M} G_{mp}(X_t,\epsilon_t)\,\frac{\partial^2\varphi}{\partial x_m\partial x_p}(t,X_t,\epsilon_t)\,dt \\
&\quad + \Big(\varphi(t,X_{t+},\epsilon_t)-\varphi(t,X_t,\epsilon_t)\Big)\; I_{\{\,t\in\Lambda\,\}} \\
&\quad + Q\,\varphi(t,X_t,\cdot)(\epsilon_t)\,dt \;+\; dM_t^{\varphi} \\
&= -\delta e^{-\delta t}v(X_t,\epsilon_t)\,dt \;+\; e^{-\delta t}\sum_{m=1}^{M} f_m(X_t,\epsilon_t)\frac{\partial\,v}{\partial x_m}\,(X_t,\epsilon_t)\,dt \\
&\quad + e^{-\delta t}\sum_{m=1}^{M}\sum_{n=1}^{N} g_{mn}(X_t,\epsilon_t)\,\frac{\partial\,v}{\partial x_m}\,(X_t,\epsilon_t)\,d(W_n)_t \\
&\quad + e^{-\delta t}\sum_{m=1}^{M}\sum_{\ell=1}^{d} z_{m\ell}(\epsilon_t)\frac{\partial\,v}{\partial x_m}(X_t,\epsilon_t)\,d(Z_\ell^c)_t \\
&\quad + \frac{1}{2}\,e^{-\delta t}\sum_{m=1}^{M}\sum_{p=1}^{M} G_{mp}(X_t,\epsilon_t)\,\frac{\partial^2\,v}{\partial x_m\partial x_p}\,(X_t,\epsilon_t)\,dt \\
&\quad + e^{-\delta t}(v(X_{t+},\epsilon_t)-v(X_t,\epsilon_t))I_{\{t\in\Lambda\}} + e^{-\delta t}Qv(X_t,\cdot)(\epsilon_t)dt + dM_t^{\varphi},
\end{aligned}
$$

where X_m^c denotes the continuous part of the m-th component X_m. The process $\{M_t^\varphi, t \geq 0\}$ is a real-valued, square integrable martingale, with $M_0^\varphi = 0$ $\mathbb{P}$-a.s., when $v(\cdot, i)$, $i \in \mathcal{S}$, is bounded. We have then,

$$\begin{aligned} d\varphi(t, X_t, \epsilon_t) &= e^{-\delta t}\left(L_{\epsilon(t)}\, v(X_t, \epsilon_t) + Qv(X_t, \cdot)(\epsilon_t)\right) dt \\ &\quad + e^{-\delta t} \sum_{m=1}^{M} \sum_{n=1}^{N} g_{mn}(X_t, \epsilon_t)\, \frac{\partial v}{\partial x_m}\,(X_t, \epsilon_t)\, d(W_n)_t \\ &\quad + e^{-\delta t} \sum_{m=1}^{M} \sum_{\ell=1}^{d} z_{m\ell}(\epsilon_t) \frac{\partial v}{\partial x_m}(X_t, \epsilon_t)\, d(Z_\ell^c)_t \\ &\quad + e^{-\delta t}(v(X_{t+}, \epsilon_t) - v(X_t, \epsilon_t))\, I_{\{t \in \Lambda\}} \,+\, dM_t^\varphi. \end{aligned}$$

Consider $\mathcal{U} = \{U_k,\, k \geq 1\}$ to be an open cover of $\mathcal{O}$. Let $k \geq 1$ be such that $X_0 = x \in \cup_{j=1}^k U_j \subset \mathcal{O}$ and such that $\cup_{j=1}^k U_j$ is bounded, and define the stopping time $\tau_k = \inf\{\, t \geq 0 : X_t \notin \cup_{j=1}^k U_j\,\}$. For every time $t \in [0, \infty)$, we obtain

$$\begin{aligned} & e^{-\delta(t\wedge\tau_k)} v(X_{(t\wedge\tau_k)+}, \epsilon_{t\wedge\tau_k}) - v(X_0, \epsilon_0) \\ &\qquad = \int_0^{t\wedge\tau_k} e^{-\delta s}\left(L_{\epsilon(s)}\, v(X_s, \epsilon_s) + Q\, v(X_s, \cdot)(\epsilon_s)\right) ds \\ &\qquad\quad + \int_0^{t\wedge\tau_k} e^{-\delta s} \sum_{m=1}^{M} \sum_{n=1}^{N} g_{nm}(X_s, \epsilon_s) \frac{\partial v}{\partial x_m}\,(X_s, \epsilon_s)\, d(W_n)_s \\ &\qquad\quad + \int_0^{t\wedge\tau_k} e^{-\delta s} \sum_{m=1}^{M} \sum_{\ell=1}^{d} z_{m\ell}(\epsilon_s) \frac{\partial v}{\partial x_m}(X_s, \epsilon_s)\, d(Z_\ell^c)_s \\ &\qquad\quad + \sum_{s \in [0,\, t\wedge\tau_k),\, s \in \Lambda} e^{-\delta s}(v(X_{s+}, \epsilon_s) - v(X_s, \epsilon_s)) \,+\, M_{t\wedge\tau_k}^\varphi. \end{aligned}$$

By taking the conditional expectation to both sides of the inequality above given $X_0 = x$ and $\epsilon_0 = i$, we have

$$\begin{aligned}
&E_{x,i}\left[e^{-\delta(t\wedge\tau_k)} v(X_{(t\wedge\tau_k)+}, \epsilon_{t\wedge\tau_k})\right] - v(x,i) \\
&\quad= E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s}\left(L_{\epsilon(s)}\, v(X_s,\epsilon_s) + Qv(X_s,\cdot)(\epsilon_s)\right) ds\right] \\
&\qquad + E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s} \sum_{m=1}^{M}\sum_{n=1}^{N} g_{mn}(X_s,\epsilon_s)\frac{\partial v}{\partial x_m}(X_s,\epsilon_s)\, d(W_n)_s\right] \\
&\qquad + E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s} \sum_{m=1}^{M}\sum_{\ell=1}^{d} z_{m\ell}(\epsilon_s)\frac{\partial v}{\partial x_m}(X_s,\epsilon_s)\, d(Z_\ell^c)_s\right] \\
&\qquad + E_{x,i}\left[\sum_{s\in[0,\,t\wedge\tau_k),\, s\in\Lambda} e^{-\delta s}(v(X_{s+},\epsilon_s) - v(X_s,\epsilon_s))\right] + E_{x,i}\left[M^{\varphi}_{t\wedge\tau_k}\right]. \qquad (5.9)
\end{aligned}$$

We note that $X_s \in Cl\big(\cup_{j=1}^k U_j\big)$ when $s \in [0, t\wedge\tau_k)$, and that, for every $m = 1,\ldots,M$, $\partial v/\partial x_m(X_s,\epsilon_s)$ is bounded for every $s \in [0, t\wedge\tau_k)$ due to the continuity of $\partial v/\partial x_m(\cdot,i)$, $i \in \mathcal{S}$. Let

$$M := \max\left\{\frac{\partial v}{\partial x_m}(x,i),\ x \in Cl\big(\cup_{j=1}^k U_j\big), i \in \mathcal{S}\right\} < +\infty.$$

Then,

$$\begin{aligned}
&E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-2\delta s}\, v_x(X_s,\epsilon_s)^T G(X_s,\epsilon_s)\, v_x(X_s,\epsilon_s)\, ds\right] \\
&\qquad\qquad \le M^2 E_{x,i}\left[\int_0^{t\wedge\tau_k} \mathbf{1}^T G(X_s,\epsilon_s)\mathbf{1}\, ds\right] \\
&\qquad\qquad \le M^2 E_{x,i}\left[\int_0^{t} \mathbf{1}^T G(X_s,\epsilon_s)\mathbf{1}\, ds\right]. \qquad (5.10)
\end{aligned}$$

We recall that $G(\cdot,i)$ is bounded in every closed subset of $\mathbb{R}^M$ for every $i \in \mathcal{S}$.

We recall as well that $X_s \in \mathcal{Cl}\big(\cup_{j=1}^k U_j\big)$ when $s \in [0, t\wedge\tau_k)$. Thus, from (5.10),

$$E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-2\delta s}\, v_x(X_s,\epsilon_s)^T G(X_s,\epsilon_s)\, v_x(X_s,\epsilon_s)\, ds\right] \;<\; \infty,$$

for every $t \in [0,\infty)$, which implies that

$$E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s}\, v_x(X_s,\epsilon_s)^T g(X_s,\epsilon_s)\, dW_s\right] \;=\; 0,$$

or equivalently,

$$E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s} \sum_{m=1}^{M}\sum_{n=1}^{N} g_{mn}(X_s,\epsilon_s)\frac{\partial v}{\partial x_m}(X_s,\epsilon_s)\, d(W_n)_s\right] \;=\; 0.$$

Furthermore, we note that $v(x,i)$ is bounded for every $x \in \mathcal{Cl}(\cup_{j=1}^k U_j)$, for every $i \in \mathcal{S}$, because $v(\cdot,i)$ is continuous. Thus, $v(X_s,\epsilon_s)$ is bounded for every $s \in [0, t\wedge\tau_k)$. Then, $\{M^{\varphi}_{t\wedge\tau_k}, t \geq 0\}$ is a square integrable martingale and hence $E_{x,i}[\, M^{\varphi}_{t\wedge\tau_k}\,] = E_{x,i}[\, M^{\varphi}_0\,] = 0$. Therefore, equation (5.9) becomes

$$\begin{aligned}
&E_{x,i}\left[e^{-\delta(t\wedge\tau_k)} v(X_{(t\wedge\tau_k)+},\epsilon_{t\wedge\tau_k})\right] - v(x,i)\\
&\qquad = \; E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s}\left(L_{\epsilon(s)}\, v(X_s,\epsilon_s) + Q\, v(X_s,\cdot)(\epsilon_s)\right) ds\right]\\
&\qquad\quad + E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s}\sum_{m=1}^{M}\sum_{\ell=1}^{d} z_{m\ell}(\epsilon_s)\frac{\partial v}{\partial x_m}(X_s,\epsilon_s)\, d(Z^c_\ell)_s\right]\\
&\qquad\quad + E_{x,i}\left[\sum_{s\in[0,\,t\wedge\tau_k),\, s\in\Lambda} e^{-\delta s}\big(v(X_{s+},\epsilon_s) - v(X_s,\epsilon_s)\big)\right].
\end{aligned} \tag{5.11}$$

Recall that $v(x,i)$, $i \in \mathcal{S}$, satisfies (5.7) for every $x \in \mathcal{O}$. Thus, for every $s \in [0, t\wedge\tau_k)$,

$$\sum_{m=1}^{M} z_{m\ell}(\epsilon_s)\frac{\partial v}{\partial x_m}(X_s,\epsilon_s) \;\leq\; -H_\ell(\epsilon_s), \tag{5.12}$$

for every $\ell = 1, \ldots, d$. Hence,

$$E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s}\sum_{m=1}^{M}\sum_{\ell=1}^{d} z_{m\ell}(\epsilon_s)\frac{\partial v}{\partial x_m}(X_s,\epsilon_s)\,d(Z^c_\ell)_s\right]$$

$$\leq \; -E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s}\sum_{\ell=1}^{d} H_\ell(\epsilon_s)\,d(Z^c_\ell)_s\right]$$

$$= \; -E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s} H(\epsilon_s)^T dZ^c_s\right]. \tag{5.13}$$

We note that the inequality (5.13) is well defined because Z is admissible and, hence, (5.4) holds. Also, from the Mean Value Theorem, we have

$$v(X_{s+},\epsilon_s) - v(X_s,\epsilon_s) \;=\; \int_0^1 v_x(X_s + \lambda(X_{s+}-X_s),\epsilon_s)^T\,(X_{s+}-X_s)\,d\lambda,$$

for every $s \in [\,0, t\wedge\tau_k)$. We note that $Y_s(\lambda) := X_s + \lambda(X_{s+}-X_s) \in \mathcal{O}$ for every $s \in [\,0, t\wedge\tau_k)$ and every $\lambda \in [\,0,1\,]$, since $\mathcal{O}$ is convex and X_s, $X_{s+} \in \mathcal{O}$. Moreover, we note that, for $s \in [\,0, t\wedge\tau_k) \cap \Lambda \subseteq \Lambda$,

$$X_{s+} - X_s \;=\; z(\epsilon_s)(Z_{s+}-Z_s) \;=\; \sum_{\ell=1}^{d} z_{\cdot\ell}(\epsilon_s)\,(Z_{s+}-Z_s)_\ell.$$

Hence, for every $s \in [\,0, t\wedge\tau_k) \cap \Lambda$,

$$v(X_{s+},\epsilon_s) - v(X_s,\epsilon_s) \;=\; \int_0^1 \sum_{\ell=1}^{d} v_x(Y_s(\lambda),\epsilon_s)^T z_{\cdot\ell}(\epsilon_s)\,(Z_{s+}-Z_s)_\ell\,d\lambda$$

$$\leq \; -\int_0^1 \sum_{\ell=1}^{d} H_\ell(\epsilon_s)\,(Z_{s+}-Z_s)_\ell\,d\lambda$$

$$= \; -H(\epsilon_s)^T(Z_{s+}-Z_s), \tag{5.14}$$

because (5.7) is satisfied for every $x \in \mathcal{O}$ and in special for $Y_s(\lambda) \in \mathcal{O}$.

Therefore, using inequalities (5.8), (5.13) and (5.14) in equation (5.11), we obtain

$$E_{x,i}\left[e^{-\delta(t\wedge\tau_k)}v(X_{(t\wedge\tau_k)+},\epsilon_{t\wedge\tau_k})\right]-v(x,i)$$

$$\leq -E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s}H(\epsilon_s)^T dZ_s^c\right]$$

$$- E_{x,i}\left[\sum_{s\in[0,\,t\wedge\tau_k),\,s\in\Lambda} e^{-\delta s}H(\epsilon_s)^T(Z_{s+}-Z_s)\right]. \tag{5.15}$$

We note that $(Z_{s+}-Z_s)I_{\{s\in\Lambda\}} = dZ_s^d$ and that $Z_s^c+Z_s^d = Z_s$ for every $s\in[0,\infty)$ and in special for every $s\in[0,t\wedge\tau_k)$. Then,

$$E_{x,i}\left[e^{-\delta(t\wedge\tau_k)}v(X_{(t\wedge\tau_k)+},\epsilon_{t\wedge\tau_k})\right]-v(x,i) \;\leq\; -E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s}H(\epsilon_s)^T dZ_s\right].$$

Letting $k\to+\infty$, we get that $\tau_k\to\Theta$. Thus,

$$E_{x,i}\left[\int_0^{t\wedge\Theta} e^{-\delta s}H(\epsilon_s)^T dZ_s\right] \;\leq\; v(x,i)$$

$$- E_{x,i}\left[e^{-\delta(t\wedge\Theta)}v(X_{(t\wedge\Theta)+},\epsilon_{t\wedge\Theta})\,I_{\{\Theta<+\infty\}}\right]$$

$$- E_{x,i}\left[e^{-\delta t}v(X_{t+},\epsilon_t)\,I_{\{\Theta=+\infty\}}\right]. \tag{5.16}$$

We note that the polynomial growth condition of $v(\cdot,i)$, $i\in\mathcal{S}$, implies that

$$\lim_{t\to\infty} E_{x,i}\left[e^{-\delta t}v(X_{t+},\epsilon_t)\,I_{\{\Theta=+\infty\}}\right] \;=\; 0.$$

Hence, letting $t\to\infty$ and applying the Dominated Convergence Theorem in equation (5.16), we get

$$E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}H(\epsilon_s)^T dZ_s\right] \;\leq\; v(x,i) \;-\; E_{x,i}\left[e^{-\delta\Theta}v(X_{\Theta+},\epsilon_\Theta)\,I_{\{\Theta<+\infty\}}\right].$$

Recall that $X_t = X_\Theta$ for every $t\in[\Theta,\infty)$, and that $v(x,i) = 0$ for every

$x \notin \mathcal{O}$, $i \in \mathcal{S}$. Hence, we obtain that $J(x,i;Z) \leq v(x,i)$ for the arbitrary control process Z. $\square$

We know now the sufficient conditions for obtaining a bound for the functional $J(x,i;Z)$. However, we still need to find an admissible control process Z^* such that $J(x,i;Z^*) = v(x,i)$, which will be the optimal control for Problem 5.1. The following theorem gives us the sufficient conditions for such optimal control process.

Theorem 5.2. Consider the function $v(\cdot,i)$, $i \in \mathcal{S}$ given by Theorem 5.1, and consider the stochastic control associated with v, $Z^v = \{Z_t^v, t \geq 0\}$. Define the control process $\hat{Z} = \{\hat{Z}_t, t \geq 0\}$ by

$$\hat{Z}_t = \begin{cases} Z_t^v & \text{if} \quad t \in [0, \widehat{\Theta}) \\ Z_{\widehat{\Theta}}^v & \text{if} \quad t \in [\widehat{\Theta}, \infty), \end{cases}$$

where $\widehat{\Theta} := \Theta^{X^{Z^v}}$. Then, if the control $\hat{Z}$ is admissible, it is the optimal control for Problem 5.1. Furthermore, $v(\cdot,i)$, $i \in \mathcal{S}$, is the value function for Problem 5.1.

Proof. Assume that the control process Z^v (and hence control $\hat{Z}$) is admissible. From (ii) in Definition 5.3 and the HJB equation (5.6) we have that

$$E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s}\left(L_{\epsilon(s)}\, v(X_s^{Z^v},\epsilon_s) + Q\, v(X_s^{Z^v},\cdot)(\epsilon_s)\right) ds\right] = 0.$$

for every $t \in [0,\infty)$, where $k \geq 1$ such that $X_0 = x \in \cup_{j=1}^k U_j \subset \mathcal{O}$ and such that $\cup_{j=1}^{k} U_j$ is bounded. Hence, from (5.11), for every $t \in [0, \infty)$,

$$
\begin{aligned}
&E_{x,i}\left[e^{-\delta(t\wedge\tau_k)} v\big(X^{Z^v}_{(t\wedge\tau_k)+}, \epsilon_{t\wedge\tau_k}\big)\right] - v(x,i) \\
&\quad= E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s} v_x\big(X^{Z^v}_s, \epsilon_s\big)^T z(\epsilon_s)\, d(Z^v)^c_s\right] \\
&\qquad + E_{x,i}\left[\sum_{s\in[0,\,t\wedge\tau_k),\, s\in\Lambda} e^{-\delta s}\left(v\big(X^{Z^v}_{s+}, \epsilon_s\big) - v\big(X^{Z^v}_s, \epsilon_s\big)\right)\right] \\
&\quad= E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s} v_x\big(X^{Z^v}_s, \epsilon_s\big)^T z(\epsilon_s)\, dZ^v_s\right] \\
&\qquad - E_{x,i}\left[\sum_{s\in[0,\,t\wedge\tau_k),\, s\in\Lambda} e^{-\delta s} v_x(X^{Z^v}_s, \epsilon_s)^T z(\epsilon_s)({Z^v}_{s+} - Z^v_s)\right] \\
&\qquad + E_{x,i}\left[\sum_{s\in[0,\,t\wedge\tau_k),\, s\in\Lambda} e^{-\delta s}\left(v\big(X^{Z^v}_{s+}, \epsilon_s\big) - v\big(X^{Z^v}_s, \epsilon_s\big)\right)\right].
\end{aligned}
$$

Using (iii) and (iv) from Definition 5.3 and (5.7), we get

$$
\begin{aligned}
&E_{x,i}\left[e^{-\delta(t\wedge\tau_k)} v\big(X^{Z^v}_{(t\wedge\tau_k)+}, \epsilon_{t\wedge\tau_k}\big)\right] - v(x,i) \\
&\quad= -E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s} H(\epsilon_s)^T dZ^v_s\right] \\
&\qquad - E_{x,i}\left[\sum_{s\in[0,\,t\wedge\tau_k),\, s\in\Lambda} e^{-\delta s} v_x(X^{Z^v}_s, \epsilon_s)^T z(\epsilon_s)(Z^v_{s+} - Z^v_s)\right] \\
&\qquad - E_{x,i}\left[\sum_{s\in[0,\,t\wedge\tau_k),\, s\in\Lambda} e^{-\delta s} H(\epsilon_s)(Z^v_{s+} - Z^v_s)\right] \\
&\quad\geq -E_{x,i}\left[\int_0^{t\wedge\tau_k} e^{-\delta s} H(\epsilon_s)^T dZ^v_s\right].
\end{aligned}
$$

Letting $k \to +\infty$ and $t \to \infty$, recalling the polynomial growth property of

$v(\cdot,i)$, $i \in \mathcal{S}$, and applying the Dominated Convergence Theorem, we obtain

$$E_{x,i}\left[e^{-\delta\widehat{\Theta}}v\Big(X^{Z^v}_{\widehat{\Theta}+},\epsilon_{\widehat{\Theta}}\Big)\; I_{\{\widehat{\Theta}<+\infty\}}\right] - v(x,i) \;\; \geq \;\; -E_{x,i}\left[\int_0^{\widehat{\Theta}} e^{-\delta s}H(\epsilon_s)^T dZ^v_s\right].$$

By using $v(X^{Z^v}_{\widehat{\Theta}+},\epsilon_{\widehat{\Theta}}) = 0$, we obtain that $v(x,i) \leq J(x,i;Z^v)$. Since we assume that Z^v is admissible, Theorem 5.1 tells us that $v(x,i) \geq J(x,i;Z^v)$. Hence, $J(x,i;\hat{Z}) = J(x,i;Z^v) = v(x,i)$ is satisfied under the assumption of admissibility of Z^v. □

Remark 5.1. For this case too, we can avoid that $v(\cdot,i)$, $i \in \mathcal{S}$, is $\mathcal{C}^1(\mathcal{O})$ in Theorem 5.1 if instead we use another condition that guarantees that $v(\cdot,i)$, $i \in \mathcal{S}$, and $v_x(\cdot,i)$, $i \in \mathcal{S}$, are bounded in every closed bounded subset of $\mathcal{O}$.

In chapter 6 we present an example of a problem of singular stochastic control with regime switching. We consider the dividend policy problem presented in chapter 4 but under the assumption that the dividend rate is unbounded.

Chapter 6

Unbounded dividend policy in the presence of business cycles

In this chapter, as in chapter 4, we consider a dividend payment model with regime switching. Following empirical evidence, we assume that the cash reservoir of a company is affected by business cycles that are generated by macroeconomic conditions. We consider that the economy shifts only between a regime of economic growth and a regime of economic recession. That is, in this case, $\mathcal{S} = \{1, 2\}$. The company's management pays dividends to their shareholders using certain unbounded dividend rate. In fact, the objective of the management is to select the dividend policy that maximizes the total expected discounted cumulative amount of dividends to be paid out to the shareholders.

For this problem, we consider the financial market that was presented in chapter 4. We let the adapted process X represent the cash reservoir of the

[8] The results shown in this chapter are presented in sections 4 and 5 in: Sotomayor, L.R. and A. Cadenillas, Optimal dividend policy in the presence of business cycles, *submitted for publication* (2008).

company. We assume that X satisfies the stochastic differential equation

$$dX_t \;=\; \mu_{\epsilon(t)}dt \;+\; \sigma_{\epsilon(t)}dW_t \;-\; dZ_t$$

with initial level of the cash reservoir $X_0 = x > 0$ and initial state $\epsilon_0 = \epsilon(0) = i$. The process Z represents the cumulative amount of dividends paid-out by the company up to a certain time. We consider in this problem that the company pays dividends using an unbounded dividend rate. Hence, the dividend process Z satisfies the equation

$$dZ_t \;=\; u(t)\,dt,$$

where $u(t)$ represents the dividend rate at time t. We denote the stopping time of bankruptcy by Θ, and impose $X_t = 0$ for every $t \in [\Theta, \infty)$.

Under this setup, and because we require u to be unbounded, we define the admissible controls for the problem in a more general way than $\mathbb{F}$-adapted processes $u : [0, \infty) \times \Omega \to [0, \infty)$.

Definition 6.1. An *admissible stochastic control* is an $\mathbb{F}$-adapted, nonnegative and nondecreasing control process $Z : [0, \infty) \times \Omega \to [0, \infty)$, with sample paths that are left-continuous with right limits, and such that $Z(t, \omega) = 0$ for every $(t, \omega) \in [\Theta, \infty) \times \Omega$. The set of all admissible controls is denoted by $\mathcal{A}$.

Under these conditions, the management of the company wants to solve the following problem .

Problem 6.1. For each $i = 1, 2$, solve the optimization problem

$$V(x, i) := \sup_{Z \in \mathcal{A}} E_{x,i}\left[\int_0^{\Theta} e^{-\delta s} dZ_s\right].$$

6.1 Verification theorem

Let $\psi : (0,\infty) \times \{1,2\} \to \mathbb{R}$ be a function and define the operator $\widetilde{L}_i$ for each $i = 1,2$, in the following way:

$$\widetilde{L}_i \psi := \frac{1}{2} \sigma_i^2 \psi'' + \mu_i \psi' - \delta \psi.$$

For a function $v : (0,\infty) \times \{1,2\} \to \mathbb{R}$, consider the Hamilton-Jacobi-Bellman equation

$$\max \left\{ \widetilde{L}_i v(x,i) - \lambda_i (v(x,i) - v(x,3-i)),\ 1 - v'(x,i) \right\} = 0, \tag{6.1}$$

where $x \geq 0$ and $i = 1,2$. We observe that, for each $i = 1,2$, the continuation region is defined for this problem as

$$\mathcal{C}(i) := \left\{ x \geq 0 : \widetilde{L}_i v(x,i) - \lambda_i (v(x,i) - v(x,3-i)) = 0,\ 1 - v'(x,i) < 0 \right\},$$

and the intervention region is defined as

$$\Sigma(i) := \left\{ x \geq 0 : \widetilde{L}_i v(x,i) - \lambda_i (v(x,i) - v(x,3-i)) < 0,\ 1 - v'(x,i) = 0 \right\}.$$

Definition 6.2. An $\mathbb{F}$-adapted, nonnegative and nondecreasing process Z^v is a *control process associated* with the function $v(\cdot,i)$, $i = 1,2$, above if

$$(i) \quad X_t^{Z^v} = \begin{cases} x + \int_0^t \mu_{\epsilon(s)}\, ds + \int_0^t \sigma_{\epsilon(s)}\, dW_s - Z_t^v, & \text{for all} \quad t \in [0,\Theta) \\ 0 & \text{for all} \quad t \in [\Theta, \infty) \end{cases}$$

$$(ii) \quad X_t^{Z^v} \in \mathcal{C}l(\mathcal{C}(\epsilon_t)), \quad \text{for all } t \in [0,\infty), \quad \mathbb{P}-a.s.,$$

$$(iii) \quad \int_0^\infty I_{\{X_s^{Z^v} \in \mathcal{C}(\epsilon_s)\}}\, dZ_s^v = \int_0^\Theta I_{\{X_s^{Z^v} \in \mathcal{C}(\epsilon_s)\}}\, dZ_s^v = 0, \quad \mathbb{P}-a.s.$$

We will present the verification theorem that gives the sufficient conditions for the solution of Problem 6.1, but we need to state first the following Lemma.

Lemma 6.1. *Let $Z = Z^v$ be the control process associated with a nondecreasing and differentiable function $v(\cdot, i)$, $i = 1, 2$, and let τ be any stopping time. If $v'(X_s^Z, \epsilon_s)$ is bounded for every $s \in [0, \tau]$ then*

$$E_{x,i}\left[\int_0^\tau e^{-\delta s}\, v'(X_s^Z, \epsilon_s)\, dZ_s^c\right] - E_{x,i}\left[\sum_{0\leq s\leq \tau,\, s\in\Lambda} e^{-\delta s}\left(v(X_{s+}^Z, \epsilon_s) - v(X_s^Z, \epsilon_s)\right)\right]$$

$$= \quad E_{x,i}\left[\int_0^\tau e^{-\delta s}\, dZ_s\right],$$

where Λ denotes the set of times when Z has a discontinuity.

Proof. The proof of the Lemma is presented in Appendix C. □

Theorem 6.1. Let $v(\cdot, i) \in C^2([0, \infty)\backslash N_i)$, $i = 1, 2$, where N_i are finite subsets of $(0, \infty)$. Let $v(\cdot, i)$, $i = 1, 2$, be an increasing and concave function on $[0, \infty)$ with $v(0, i) = 0$, $i = 1, 2$. Suppose that the function $v(\cdot, i)$, $i = 1, 2$, satisfies the Hamilton-Jacobi-Bellman equation (6.1) for every $x \geq 0$, $i = 1, 2$, and consider the stochastic control Z^v associated with v. Then, the process $\hat{Z} = \{\hat{Z}_t, t \geq 0\}$ defined by $\hat{Z}_t = Z_t^v$ for $t \in [0, \widehat{\Theta})$ and $d\hat{Z}_t = 0$ for $t \in [\widehat{\Theta}, \infty)$, is the optimal dividend policy for Problem 6.1. Here, $\widehat{\Theta} := \Theta^{X^{Z^v}}$. Furthermore, $v(\cdot, i)$, $i = 1, 2$, is the value function for Problem 6.1.

Proof. Consider an admissible control Z and the corresponding semimartingale

$$X_t = x + \int_0^t \mu_{\epsilon(s)} ds + \int_0^t \sigma_{\epsilon(s)} dW_s - Z_t^c - Z_t^d.$$

Consider also the function $f(\cdot, \cdot, i)$, $i = 1, 2$, defined by $f(t, x, i) = e^{-\delta t} v(x, i)$.

Then,

$$
\begin{aligned}
df(t, X_t, \epsilon_t) &= \left(\frac{1}{2}\sigma^2_{\epsilon(t)}\, f_{xx}(t, X_t, \epsilon_t) + \mu_{\epsilon(t)}\, f_x(t, X_t, \epsilon_t) + f_t(t, X_t, \epsilon_t)\right) dt \\
&\quad + f_x(t, X_t, \epsilon_t)\, \sigma_{\epsilon(t)}\, dW_t \;-\; f_x(t, X_t, \epsilon_t)\, dZ^c_t \\
&\quad + \Big(f(t, X_{t+}, \epsilon_t) - f(t, X_t, \epsilon_t)\Big)\, I_{\{t \in \Lambda\}} \\
&\quad + \big(-\lambda_{\epsilon(t)} f(t, X_t, \epsilon_t) + \lambda_{\epsilon(t)} f(t, X_t, 3 - \epsilon_t)\big)\, dt \;+\; dM^f_t \\
&= e^{-\delta t}\left(\widetilde{L}_{\epsilon(t)} v(X_t, \epsilon_t) - \lambda_{\epsilon(t)}\Delta_t\right) dt \;+\; \sigma_{\epsilon(t)}\, e^{-\delta t} v'(X_t, \epsilon_t)\, dW_t \\
&\quad - e^{-\delta t} v'(X_t, \epsilon_t)\, dZ^c_t + e^{-\delta t}(v(X_{t+}, \epsilon_t) - v(X_t, \epsilon_t)) I_{\{t \in \Lambda\}} + dM^f_t,
\end{aligned}
$$

where $\Delta_t := v(X_t, \epsilon_t) - v(X_t, 3 - \epsilon_t)$, and Λ is the set of times when Z is discontinuous. We observe that $v(\cdot, i)$, $i = 1, 2$, and $v'(\cdot, i)$, $i = 1, 2$, are not necessarily bounded. However, we are assuming that $v(\cdot, i)$, $i = 1, 2$, is concave and increasing. Let a and b be real numbers satisfying $0 < a < X_0 = x < b < +\infty$, and define $\tau_a := \inf\{t \geq 0 : X_t = a\}$, $\tau_b := \inf\{t \geq 0 : X_t = b\}$ and $\tau := \tau_a \wedge \tau_b$. Then, for every time $t \in [0, \infty)$,

$$
\begin{aligned}
& e^{-\delta(t\wedge\tau)} v(X_{(t\wedge\tau)+}, \epsilon_{t\wedge\tau}) - v(X_0, \epsilon_0) \\
&\qquad = \int_0^{t\wedge\tau} e^{-\delta s}\left(\widetilde{L}_{\epsilon(s)} v(X_s, \epsilon_s) - \lambda_{\epsilon(s)}\Delta_s\right) ds \\
&\qquad\quad + \int_0^{t\wedge\tau} \sigma_{\epsilon(s)} e^{-\delta s} v'(X_s, \epsilon_s)\, dW_s - \int_0^{t\wedge\tau} e^{-\delta s} v'(X_s, \epsilon_s)\, dZ^c_s \\
&\qquad\quad + \sum_{0 \leq s < t\wedge\tau,\, s \in \Lambda} e^{-\delta s}\left(v(X_{s+}, \epsilon_s) - v(X_s, \epsilon_s)\right) + M^f_{t\wedge\tau} - M^f_0.
\end{aligned} \tag{6.2}
$$

Taking conditional expectation on both sides of equation (6.2) given $X_0 = x$

and $\epsilon_0 = i$, we have

$$E_{x,i}\left[e^{-\delta(t\wedge\tau)}v(X_{(t\wedge\tau)+},\epsilon_{t\wedge\tau})\right] - v(x,i)$$

$$\begin{aligned}
&= E_{x,i}\left[\int_0^{t\wedge\tau} e^{-\delta s}\left(\widetilde{L}_{\epsilon(s)}v(X_s,\epsilon_s) - \lambda_{\epsilon(s)}\Delta_s\right)ds\right] \\
&\quad + E_{x,i}\left[\int_0^{t\wedge\tau}\sigma_{\epsilon(s)}e^{-\delta s}v'(X_s,\epsilon_s)dW_s\right] - E_{x,i}\left[\int_0^{t\wedge\tau}e^{-\delta s}v'(X_s,\epsilon_s)dZ_s^c\right] \\
&\quad + E_{x,i}\left[\sum_{0\le s<t\wedge\tau,\, s\in\Lambda} e^{-\delta s}\left(v(X_{s+},\epsilon_s) - v(X_s,\epsilon_s)\right)\right] + E_{x,i}\left[M^f_{t\wedge\tau} - M^f_0\right]. \qquad (6.3)
\end{aligned}$$

Equation (6.1) guarantees that $\widetilde{L}_{\epsilon(t)}v(X_t,\epsilon_t) - \lambda_{\epsilon(t)}\Delta_t \le 0$. Moreover, $v'(x,i) \ge 1$ for $x \in (0,\infty)$ and the Mean Value theorem imply that $v(y_1,i) - v(y_2,i) \ge y_1 - y_2$ for every $y_1, y_2 \in (0,\infty)$, $y_1 > y_2$, and for both $i = 1,2$. Hence, replacing $i = \epsilon_t$, $y_1 = X_t$ and $y_2 = X_{t+}$, we obtain that $v(X_{t+},\epsilon_t) - v(X_t,\epsilon_t) \le X_{t+} - X_t$. We also note that $X_{s+} - X_s = Z_s - Z_{s+}$. Then, from equation (6.3),

$$E_{x,i}\left[e^{-\delta(t\wedge\tau)}v(X_{(t\wedge\tau)+},\epsilon_{t\wedge\tau})\right] - v(x,i)$$

$$\begin{aligned}
&\le E_{x,i}\left[\int_0^{t\wedge\tau}\sigma_{\epsilon(s)}e^{-\delta s}v'(X_s,\epsilon_s)dW_s\right] - E_{x,i}\left[\int_0^{t\wedge\tau}e^{-\delta s}dZ_s^c\right] \\
&\quad - E_{x,i}\left[\sum_{0\le s<t\wedge\tau,\, s\in\Lambda} e^{-\delta s}\left(Z_{s+} - Z_s\right)\right] + E_{x,i}\left[M^f_{t\wedge\tau} - M^f_0\right] \\
&= E_{x,i}\left[\int_0^{t\wedge\tau}\sigma_{\epsilon(s)}e^{-\delta s}v'(X_s,\epsilon_s)\,dW_s\right] - E_{x,i}\left[\int_0^{t\wedge\tau}e^{-\delta s}dZ_s\right] \\
&\quad + E_{x,i}\left[M^f_{t\wedge\tau} - M^f_0\right].
\end{aligned}$$

We note that this inequality becomes an equality for the stochastic control Z^v associated with v and, hence, for the admissible $\hat{Z}$. Indeed, condition *(ii)* in Definition 6.2 implies that $X_s^{Z^v} \in \mathcal{C}(\epsilon_s)$ Leb a.e. $s \in [0, t\wedge\tau)$ $\mathbb{P}$-a.e. and, hence,

$$E_{x,i}\left[\int_0^{t\wedge\tau} e^{-\delta s}\left(\widetilde{L}_{\epsilon(s)}v(X_s^{Z^v},\epsilon_s) - \lambda_{\epsilon(s)}\left(v(X_s^{Z^v},\epsilon_s) - v(X_s^{Z^v},3-\epsilon_s)\right)\right) ds\right] = 0.$$

Furthermore, from Lemma 6.1 we obtain

$$\begin{aligned} E_{x,i}\left[\int_0^{t\wedge\tau} e^{-\delta s}v'(X_s^{Z^v},\epsilon_s)d(Z^v)_s^c\right] - E_{x,i}\left[\sum_{0\le s<t\wedge\tau,\, s\in\Lambda} e^{-\delta s}\left(v(X_{s+}^{Z^v},\epsilon_s) - v(X_s^{Z^v},\epsilon_s)\right)\right] \\ = E_{x,i}\left[\int_0^{t\wedge\tau} e^{-\delta s} dZ_s^v\right]. \end{aligned}$$

We note that $v(X_s,\epsilon_s)$ and $v'(X_s,\epsilon_s)$ are bounded when $s\in[0,t\wedge\tau)$. Then, $\{M^f_{t\wedge\tau}, t\ge 0\}$ is a square integrable martingale and, hence, $E_{x,i}[M^f_{t\wedge\tau}] = E_{x,i}[M^f_0]$ for every $t\in[0,\infty)$. Furthermore, $\sigma^2_{\epsilon(s)}\, e^{-2\delta s}(v'(X_s,\epsilon_s))^2$ is bounded when $s\in[0,t\wedge\tau)$, and hence

$$E_{x,i}\left[\int_0^{t\wedge\tau} \sigma_{\epsilon(s)}\, e^{-\delta s}\, v'(X_s,\epsilon_s)\, dW_s\right] = 0.$$

Letting $a\downarrow 0$ and $b\uparrow +\infty$, we get $\tau_a\to\Theta$ and $\tau_b\to+\infty$. Then, $\tau\to\Theta$. Also, taking $t\to\infty$ and using that $X_t=0$ for every $t\in[\Theta,\infty)$, and that $v(0,i)=0$ for both $i=1,2$, we get

$$\begin{aligned} v(x,i) &\ge E_{x,i}\left[e^{-\delta\Theta}v(X_{\Theta+},\epsilon_\Theta)\right] + E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}dZ_s\right] \\ &= E_{x,i}\left[e^{-\delta\Theta}v(0,\epsilon_\Theta)\right] + E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}\, dZ_s\right] \\ &= E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}\, dZ_s\right] \quad = \quad J(x,i;Z). \end{aligned}$$

In particular, for $Z=\hat{Z}$ we have $v(x,i)=J(x,i;\hat{Z})$. □

To the best of our knowledge, Guo, Miao and Morellec (2005), Sotomayor and Cadenillas (2008b), and Zariphopoulou (1992) are the only papers that present verification theorems for stochastic singular control problems with regime switching. Guo, Miao and Morellec (2005) study an irreversible investment problem, but do not provide a rigorous mathematical proof of their verification theorem. Zariphopoulou (1992) studies a consumption-investment problem, but without providing the proof of the verification theorem.

6.2 Construction of the solution

We want to find a function $v(\cdot, i)$, $i = 1, 2$, that satisfies the conditions of Theorem 6.1. In particular, we want equation (6.1) to be satisfied.

We conjecture $v'(\cdot, i)$ to be continuous, positive and non-increasing. Denote $\tilde{x}_i := \inf\{x \geq 0 : v'(x, i) \leq 1\}$ for each $i = 1, 2$ and suppose $\min\{\tilde{x}_1, \tilde{x}_2\} > 0$. When $x \in [0, \tilde{x}_i)$, $v'(x, i) > 1$. Hence, in view of (6.1), $v(\cdot, i)$ satisfies

$$\frac{1}{2}\sigma_i^2 v''(x, i) + \mu_i v'(x, i) - \delta v(x, i) \;=\; \lambda_i(v(x, i) - v(x, 3 - i)).$$

When $x \in (\tilde{x}_i, \infty)$, $v'(x, i) \leq 1$. The concavity of $v(\cdot, i)$ and the fact that $v'(x, i) \geq 1$ (see equation (6.1)), imply that $v'(x, i) = 1$ for every $x \in (\tilde{x}_i, \infty)$.

The relation between the thresholds $\tilde{x}_1$ and $\tilde{x}_2$ depends on the relations among the drift coefficients, the volatility parameters and the rates λ_1 and λ_2. We will just consider the case $\tilde{x}_1 < \tilde{x}_2$; the case $\tilde{x}_1 > \tilde{x}_2$ has a similar treatment. Thus, we have to consider three options for the initial level of the cash reservoir: $x \in [0, \tilde{x}_1)$, $x \in [\tilde{x}_1, \tilde{x}_2)$ and $x \in [\tilde{x}_2, \infty)$.

When $x \in [0, \tilde{x}_1)$, $v(\cdot, i)$, $i = 1, 2$, satisfies the following system of differential equations:

$$(6.4)\quad \begin{aligned} -(\lambda_1+\delta)\,v(x,1)+\mu_1\,v'(x,1)+\frac{1}{2}\,\sigma_1^2\,v''(x,1)+\lambda_1\,v(x,2) &= 0 \\ -(\lambda_2+\delta)\,v(x,2)+\mu_2\,v'(x,2)+\frac{1}{2}\,\sigma_2^2\,v''(x,2)+\lambda_2\,v(x,1) &= 0. \end{aligned}$$

This system is equivalent to the system (4.5). Hence, the solution is

$$(6.5)\quad v(x,1) = A_1\,e^{\beta_1(x-\tilde{x}_1)} + A_2\,e^{\beta_2(x-\tilde{x}_1)} + A_3\,e^{\beta_3(x-\tilde{x}_1)} + A_4\,e^{\beta_4(x-\tilde{x}_1)}$$

$$(6.6)\quad v(x,2) = B_1\,e^{\beta_1(x-\tilde{x}_1)} + B_2\,e^{\beta_2(x-\tilde{x}_1)} + B_3\,e^{\beta_3(x-\tilde{x}_1)} + B_4\,e^{\beta_4(x-\tilde{x}_1)},$$

where, for each $j=1,2,3,4$,

$$(6.7)\qquad B_j = \frac{\phi_1^1(\beta_j)}{\lambda_1}\,A_j = \frac{\lambda_2}{\phi_2^1(\beta_j)}\,A_j.$$

The real values $\beta_1 < \beta_2 < 0 < \beta_3 < \beta_4$ above are the real roots of the characteristic function $\phi_1^1(\beta)\;\phi_2^1(\beta) = \lambda_1\lambda_2$, where

$$\phi_i^1(\beta) := -\frac{1}{2}\,\sigma_i^2\beta^2 - \mu_i\,\beta + (\lambda_i+\delta), \qquad i=1,2.$$

We note that if, for some $i_0 \in \{1,2\}$, $v''(x,i_0)=0$ on an open interval of $[0,\tilde{x}_1)$, then the solution to the system of differential equations (6.4) on this interval would be given by $v(x,i) = C_i\exp(\kappa_1 x) + D_i\exp(\kappa_2 x)$, $i=1,2$, where κ_1 and κ_2 are the two positive roots of $\mu_1\mu_2\kappa^2 - (\mu_1(\lambda_2+\delta)+\mu_2(\lambda_1+\delta))\kappa + (\lambda_1+\delta)(\lambda_2+\delta)-\lambda_1\lambda_2 = 0$. We note that the function $v(\cdot,i)$, $i=1,2$, obtained in that case is not a concave function. Hence, the conjecture of concavity for $v(\cdot,i)$ implies that $v''(\cdot,i)$, $i=1,2$, does not vanish on $[0,\tilde{x}_1)$.

Proposition 6.1. Suppose that the solution $v(\cdot,i)$, $i=1,2$, of the HJB equation (6.1) is such that $v'(\cdot,i)$, $i=1,2$, is continuous, positive and non-increasing. Suppose that $\tilde{x}_1$ is the first point where $v''(\cdot, 1)$ vanishes and that $v'(\tilde{x}_1, 1) = 1$. Then, $v'(x, 1) = 1$ for every $x \in [\tilde{x}_1, \tilde{x}_2)$.

Proof. We want to prove first that $v''(x, 1) = 0$ for every $x \in [\tilde{x}_1, \tilde{x}_2)$. Suppose there exists $a \in (\tilde{x}_1, \tilde{x}_2)$ such that $v''(a, 1) < 0$. Define $b := \sup\{x < a : v''(x, 1) = 0\}$. Then, on the interval (b, a), $v'(x, 1) \neq 1$ and, hence, $v(\cdot, i)$, $i = 1, 2$, satisfies the system (6.4) (recall that also $v'(x, 2) > 1$ on this interval). Thus, $v(x, 1) = A'_1 e^{\beta_1(x-a)} + A'_2 e^{\beta_2(x-a)} + A'_3 e^{\beta_3(x-a)} + A'_4 e^{\beta_4(x-a)}$ on (b, a). We note that $v'''(x, 1) > 0$ on the interval (b, a) because $v'(x, 1) > 0$. Then, $v''(x, 1)$ is strictly increasing on (b, a) and, hence, $v''(x, 1) > v''(b, 1) = 0$, which contradicts concavity. This proves that $v''(x, 1) = 0$ for every $[\tilde{x}_1, \tilde{x}_2)$, and therefore, $v'(x, 1) = v'(\tilde{x}_1, 1) = 1$ for every $x \in [\tilde{x}_1, \tilde{x}_2)$. □

When $x \in (\tilde{x}_1, \tilde{x}_2)$, we want $v'(x, 1) = 1$ to be satisfied. Thus, we consider $v(x, 1) = x + K_1$ and, hence,

$$(6.8) \quad -(\lambda_2 + \delta)\, v(x, 2) + \mu_2\, v'(x, 2) + \frac{1}{2}\,\sigma_2^2\, v''(x, 2) \;=\; -\lambda_2\,(x + K_1).$$

Solving the ordinary differential equation, we find that

$$(6.9) \quad v(x, 1) \;=\; x + K_1$$

$$(6.10) \quad v(x, 2) \;=\; \widetilde{A}_1\, e^{\alpha_1(x-\tilde{x}_2)} + \widetilde{A}_2\, e^{\alpha_2(x-\tilde{x}_2)} + \frac{\lambda_2}{\delta + \lambda_2}\,(x + K_1) + \frac{\lambda_2 \mu_2}{(\delta + \lambda_2)^2}$$

where $\alpha_1 < 0 < \alpha_2$ are the real roots of the characteristic function

$$\phi(\alpha) \;:=\; \frac{1}{2}\,\sigma_2^2\,\alpha^2 + \mu_2\,\alpha - (\lambda_2 + \delta) \;=\; 0.$$

From (6.8), we observe that if $v''(x, 2) = 0$ on an open interval, then the solution of (6.8) on that interval would be given by

$$v(x,2) = C_2\, e^{(\delta+\lambda_2)/\mu_2\, x} + \frac{\lambda_2}{\delta+\lambda_2}\left(x + K_1 + \frac{\mu_2}{\delta+\lambda_2}\right),$$

which is not a concave function. Thus, the conjecture of concavity for $v(\cdot,2)$ prevents $v''(\cdot,2)$ from vanishing on $[\tilde{x}_1, \tilde{x}_2)$.

Proposition 6.2. Suppose that the solution $v(\cdot,i)$, $i = 1,2$, of the HJB equation (6.1) is such that $v'(\cdot,i)$, $i = 1,2$, is continuous, positive and non-increasing. Suppose that $\tilde{x}_2$ is the first point where $v''(\cdot,2)$ vanishes and that $v'(\tilde{x}_2,1) = v'(\tilde{x}_2,2) = 1$. Then $v'(x,i) = 1$ for every $x \in [\tilde{x}_2,\infty)$, for both $i = 1,2$.

Proof. We want to prove first that $v''(x,1) = v''(x,2) = 0$ on the interval $(\tilde{x}_2,\infty)$. If we suppose that there exists $a \in (\tilde{x}_2,\infty)$ such that $v''(a,1) < 0$ and $v''(a,2) < 0$, we obtain a similar case to the one presented in the proof of Proposition 6.1, which also leads to contradiction. We suppose instead that there exists $a \in (\tilde{x}_2,\infty)$ such that $v''(a,1) = 0$ and $v''(a,2) < 0$; the case $v''(a,1) < 0$ and $v''(a,2) = 0$ is analogous. Let $b := \sup\{x < x' : v''(x,2) = 0\}$. Then, on the interval (b,a), $v(x,2)$ satisfies the differential equation (6.8) and, hence, it has the form $v(x,2) = \widetilde{A}_1' e^{\alpha_1(x-a)} + \widetilde{A}_2' e^{\alpha_2(x-a)} + C_1 x + C_2$. We note that $v'''(x,2) > 0$ on the interval (b,a) because $v'(x,2) > 0$. Then, $v''(x,2)$ is strictly increasing on (b,a) and $v''(x,2) > v''(b,2) = 0$, which contradicts concavity. Hence, $v''(x,1) = v''(x,2) = 0$ for every $x \in (\tilde{x}_2,\infty)$. Finally, it is clear that $v'(x,1) = v'(\tilde{x}_2,1) = 1$ and $v'(x,2) = v'(\tilde{x}_2,2) = 1$ for every $x \in [\tilde{x}_1,\tilde{x}_2)$. □

According to Proposition 6.2, if $\tilde{x}_2$ is the first point where $v''(\cdot,2)$ vanishes and $v'(\tilde{x}_2, 1) = v'(\tilde{x}_2, 2) = 1$, then for every $x \in [\tilde{x}_2, \infty)$:

$$v(x,1) = x + K_1 \tag{6.11}$$

$$v(x,2) = x + K_2. \tag{6.12}$$

In order to find the thresholds $\tilde{x}_1$ and $\tilde{x}_2$, and the coefficients and constants in equations (6.5) – (6.6), (6.9) – (6.10), and (6.11) – (6.12), we conjecture that the *smooth-fit condition* holds. We also want $v'(\tilde{x}_i, i) = 1$ and $v''(\tilde{x}_i, i) = 0$ for each $i = 1, 2$. Thus, we need to solve the following system of equations

$$\begin{aligned}
v(0,i) &= 0, && \text{for both } i = 1,2 \\
v(\tilde{x}_1-, i) &= v(\tilde{x}_1+, i), && \text{for both } i = 1,2 \\
v(\tilde{x}_2-, 2) &= v(\tilde{x}_2+, 2) && \\
v'(\tilde{x}_1-, 1) &= 1 && \\
v'(\tilde{x}_2-, 2) &= 1 && \\
v'(\tilde{x}_1-, 2) &= v'(\tilde{x}_1+, 2) && \\
v''(\tilde{x}_i-, i) &= 0, && \text{for both } i = 1,2.
\end{aligned} \tag{6.13}$$

The solution of the system of equations (6.13) gives us the values for $\tilde{x}_1$ and $\tilde{x}_2$, and also the values for the coefficients A_j, $j = 1,2,3,4$, and $\widetilde{A}_j$, $j = 1,2$, and the constants K_1 and K_2. The values for B_j, $j = 1,2,3,4$, are found from equation (6.7).

6.3 Verification of the solution

In the previous section, we made conjectures to find a candidate for value function $v(\cdot, i)$, $i = 1, 2$. In this section, we will prove that $v(\cdot, i)$, $i = 1, 2$, is indeed the value function of Problem 6.1.

Theorem 6.2. Let $\tilde{x}_i$, $i = 1, 2$, A_j, $j = 1, 2, 3, 4$, $\widetilde{A}_j$, $j = 1, 2$, and K_i, $i = 1, 2$, be the solution of the system of equations (6.13). Let B_j, $j = 1,2,3,4$, be defined by (6.7). Then, the function $v(\cdot, i)$, $i = 1,2$, given by

$$v(x,1) = \begin{cases} A_1\, e^{\beta_1(x-\tilde{x}_1)} + A_2\, e^{\beta_2(x-\tilde{x}_1)} + A_3\, e^{\beta_3(x-\tilde{x}_1)} + A_4\, e^{\beta_4(x-\tilde{x}_1)}, & x \in [0, \tilde{x}_1), \\ x + K_1, & x \in [\tilde{x}_1, \infty), \end{cases}$$

and

$$v(x,2) = \begin{cases} B_1\, e^{\beta_1(x-\tilde{x}_1)} + B_2\, e^{\beta_2(x-\tilde{x}_1)} + B_3\, e^{\beta_3(x-\tilde{x}_1)} + B_4\, e^{\beta_4(x-\tilde{x}_1)}, & x \in [0, \tilde{x}_1), \\ \widetilde{A}_1\, e^{\alpha_1(x-\tilde{x}_2)} + \widetilde{A}_2\, e^{\alpha_2(x-\tilde{x}_2)} + \frac{\lambda_2}{\delta+\lambda_2}\,(x+K_1) + \frac{\lambda_2\mu_2}{(\delta+\lambda_2)^2}, & x \in [\tilde{x}_1, \tilde{x}_2), \\ x + K_2, & x \in [\tilde{x}_2, \infty), \end{cases}$$

is the value function $V(\cdot, i)$, $i = 1,2$, of Problem 6.1.

Proof. In order to prove that the function $v(\cdot, i)$, $i = 1,2$, defined above is solution for Problem 6.1, it is enough to show that it satisfies the conditions of Theorem 6.1.

It is easy to see that $v(\cdot,1) \in C^2([0,\infty))$ and $v(\cdot,2) \in C^2([0,\infty)-\{\tilde{x}_1\})$, by definition, the smooth-fit conditions, and condition $v''(\tilde{x}_i-, i) = 0$ in (6.13). Recall that $\tilde{x}_i := \inf\{x \geq 0 : v'(x,i) \leq 1\}$. Then, for every $x \in [0, \tilde{x}_i)$: $v'(x,i) > 1$. Also, by definition, $v'(x,i) = 1$ for every $x \in [\tilde{x}_i, \infty)$. Hence, $v(\cdot, i)$, $i = 1,2$, is increasing on $[0,\infty)$.

Now, we want to verify that $v(\cdot, i)$, $i = 1,2$, is concave. For $i = 1$, we see that $v'''(x,1) > 0$ for every $x \in [0, \tilde{x}_1)$, because $v'(\cdot,1) > 0$ and

$$v'''(x,1) = \beta_1^2 \cdot A_1\beta_1 e^{\beta_1(x-\tilde{x}_1)} + \beta_2^2 \cdot A_2\beta_2 e^{\beta_2(x-\tilde{x}_1)} + \beta_3^2 \cdot A_3\beta_3 e^{\beta_3(x-\tilde{x}_1)} + \beta_4^2 \cdot A_4\beta_4 e^{\beta_4(x-\tilde{x}_1)}.$$

Thus, $v''(\cdot,1)$ is strictly increasing on the interval $[0, \tilde{x}_1)$ and, hence, $v''(x,1) < v''(\tilde{x}_1, 1) = 0$ for every $x \in [0, \tilde{x}_1)$. The same argument applies to show that

$v'''(\cdot, 2) > 0$ on $[0, \tilde{x}_1)$. On $[\tilde{x}_1, \tilde{x}_2)$,

$$v'(x,2) = \widetilde{A}\alpha_1 e^{\alpha_1(x-\tilde{x}_2)} + \widetilde{A}_2\alpha_2 e^{\alpha_2(x-\tilde{x}_2)} + \frac{\lambda_2}{\delta+\lambda_2} > 1$$

which implies that $\widetilde{A}\alpha_1 e^{\alpha_1(x-\tilde{x}_2)} + \widetilde{A}_2\alpha_2 e^{\alpha_2(x-\tilde{x}_2)} > \delta/(\delta+\lambda_2) > 0$. Hence,

$$v'''(x,2) = \alpha_1^2 \cdot \widetilde{A}\alpha_1 e^{\alpha_1(x-\tilde{x}_2)} + \alpha_2^2 \cdot \widetilde{A}_2\alpha_2 e^{\alpha_2(x-\tilde{x}_2)} > 0$$

and $v''(\cdot, 2)$ is also strictly increasing on $[\tilde{x}_1, \tilde{x}_2)$. Then, $v''(x,2) < v''(\tilde{x}_2, 2) = 0$ for every $x \in [0, \tilde{x}_2)$. Hence, $v(\cdot, i)$, $i = 1, 2$, is indeed concave because $v''(x,i) < 0$ when $x \in [0, \tilde{x}_i)$ and $v''(x,i) = 0$ when $x \in [\tilde{x}_i, \infty)$.

In addition, the system (6.13) guarantees that $v(0,1) = v(0,2) = 0$.

It remains to verify that equation (6.1) holds for both $i = 1, 2$. For $x \in [0, \tilde{x}_1)$, the function $v(x,i)$, $i = 1, 2$, is solution of the system of differential equations (6.4). Also, $v'(x,1) > 1$ and $v'(x,2) > 1$ because $x \in [0, \tilde{x}_1) \subset [0, \tilde{x}_2)$. For $x \in [\tilde{x}_1, \tilde{x}_2)$, the conditions (6.13) and Proposition 6.1 tell us that $v(x,1) = x + K_1$ satisfies (6.1), and $v(x,2)$ is solution of the system (6.8) such that $v'(x,2) > 1$ (recall that $x \in [\tilde{x}_1, \tilde{x}_2) \subset [0, \tilde{x}_2)$). Finally, for $x \in [\tilde{x}_2, \infty)$, the conditions (6.13) and Proposition 6.2 prove that $v(x,i) = x + K_i$ satisfy (6.1). Hence, equation (6.1) is satisfied for every $x \geq 0$ and $i = 1, 2$. □

6.4 The optimal dividend payment policy

In the previous sections we characterized and constructed the value function for Problem 6.1. Moreover, we rewrote the continuation and intervention regions in terms of the thresholds $\tilde{x}_1$ and $\tilde{x}_2$. In this section, we will rewrite the optimal control $\hat{Z}$ also in terms of $\tilde{x}_1$ and $\tilde{x}_2$.

Consider the following Skorohod problem for one-dimensional diffusions with regime switching on $t \in [0, \infty)$:

$$
\begin{aligned}
X_t^* &= x + \int_0^t \mu_{\epsilon(s)}\, ds + \int_0^t \sigma_{\epsilon(s)}\, dW_s - Z_t^* \\
X_t^* &\leq \tilde{x}_{\epsilon(t)} \\
\int_0^\infty I_{\{X_s^* < \tilde{x}_{\epsilon(s)}\}}\, dZ_s^* &= 0.
\end{aligned} \tag{6.14}
$$

The above Skorohod system determines the process X^* reflected at either $\tilde{x}_1$ or $\tilde{x}_2$ via the process Z^*. Indeed, if $\epsilon(t) = 1$, then X^* is reflected at $\tilde{x}_1$, while if $\epsilon(t) = 2$, then X^* is reflected at $\tilde{x}_2$.

It is obvious from Definition 6.2 and section 4.2 that when we restrict the process Z^* (solution to the Skorohod problem (6.14)) to $[0, \Theta]$, we obtain the control process associated with the function $v(\cdot, i)$, $i = 1, 2$, given in Theorem 4.2. Moreover, we see that $\hat{X} = X^*$ on $[0, \Theta]$.

Theorem 6.3. Let (X^*, Z^*) be the solution of the Skorohod problem (6.14). Then, the process $\hat{Z} = \{\hat{Z}_t, t \geq 0\}$ defined by $\hat{Z}_t = Z_t^*$ for $t \in [0, \Theta)$ and $d\hat{Z}_t = 0$ for $t \in [\Theta, \infty)$, is the optimal dividend policy for Problem 6.1.

Therefore, the optimal dividend policy for the company when the regime of the economy is i, is the following: (a) do not pay dividends when the level of the cash reservoir is below the threshold $\tilde{x}_i$, and (b) whenever the level of the cash reservoir is equal or larger than $\tilde{x}_i$, everything in excess of $\tilde{x}_i$ should be distributed as dividend payments. It is very interesting to observe that dividend payments can also occur just because of a change in regime: those payments happen when the level of the cash reservoir falls in the interval $(\tilde{x}_1, \tilde{x}_2)$, and the economic regime changes from economic growth ($i = 2$) to economic recession ($i = 1$).

6.5 Comparison to the one-regime case

Following the comparison to the one-regime case done in chapter 4 for bounded dividend rates, we will use a numerical example to compare the results of the regime switching model presented in this chapter to the results of the one-regime model presented in Asmussen and Taksar (1997) for unbounded dividend rates. In fact, we will use the same model parameters that were used in the example in chapter 4.

We consider a Company AA that has only one regime with parameters $\mu_1 = 0.05$ and $\sigma_1 = 0.70$, and a Company BB that has only one regime with $\mu_2 = 0.15$ and $\sigma_2 = 0.45$. We also consider a Company AB that follows a regime switching model for its cash reservoir (with regimes of economic growth and economic recession) with parameters $\lambda_1 = 0.06$ and $\lambda_2 = 0.04$, $\mu_1 = 0.05$ and $\mu_2 = 0.15$, and $\sigma_1 = 0.70$ and $\sigma_2 = 0.45$. We assume as well that the market has a discount rate of $\delta = 0.12$.

Following the method used by Asmussen and Taksar (1997) for unbounded dividend rates, we see that the optimal thresholds for Company AA and Company BB are $\hat{x}_{AA} = 0.4109$ and $\hat{x}_{BB} = 0.9668$, respectively. The optimal dividend policy for these companies is to pay no dividends while the level of their cash reservoirs is less than the respectively threshold and to pay dividends when that level is greater or equal to the threshold.

Company AB has a different optimal dividend policy. Using the model parameters (persistence in regimes, drift coefficients, volatility parameters and discount rate) and solving the system of equations (6.13), we obtain the thresholds $\tilde{x}_1 = 0.5166$ and $\tilde{x}_2 = 0.9249$. Hence, Theorem 6.2 tells us that the optimal dividend policy for Company AB consists on not paying dividends when the regime of the economy is 1 and the level of the cash reservoir is less than $\tilde{x}_1$, and when the regime of the economy is 2 and the level of the cash reservoir is less than $\tilde{x}_2$; and on paying dividends otherwise. Furthermore, we obtain the value function $V_{AB}(\cdot, i)$, $i = 1, 2$. Figure 6.1 and Figure 6.2 show us the value function and the derivative of the value function for Company AB.

In this case we can also compare the value functions for Company AA, Company BB, and Company AB. The results are similar to the case of bounded dividend rates: $V_{AB}(x, i)$, $i = 1, 2$, is always greater than $V_{AA}(x)$ and is always lower than $V_{BB}(x)$, independently of the initial regime i (Figure 6.3).

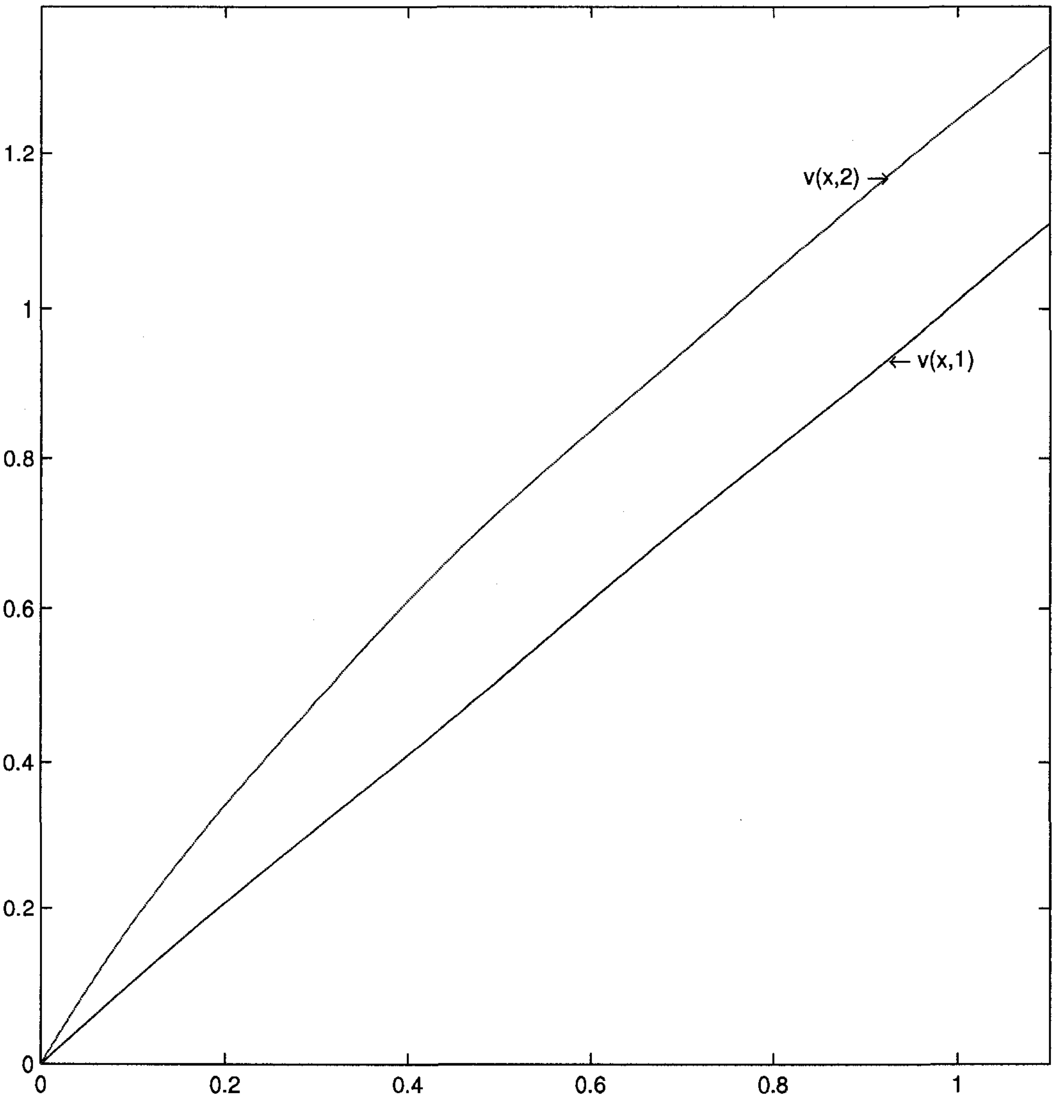

Figure 6.1: Value function for Company AB when the dividend rates are unbounded: $V_{AB}(\cdot, i)$, $i = 1, 2$, for model parameters $\lambda_1 = 0.06$, $\lambda_2 = 0.04$, $\mu_1 = 0.05$, $\mu_2 = 0.15$, $\sigma_1 = 0.70$, $\sigma_2 = 0.45$, $\delta = 0.12$ and $K = 5$.

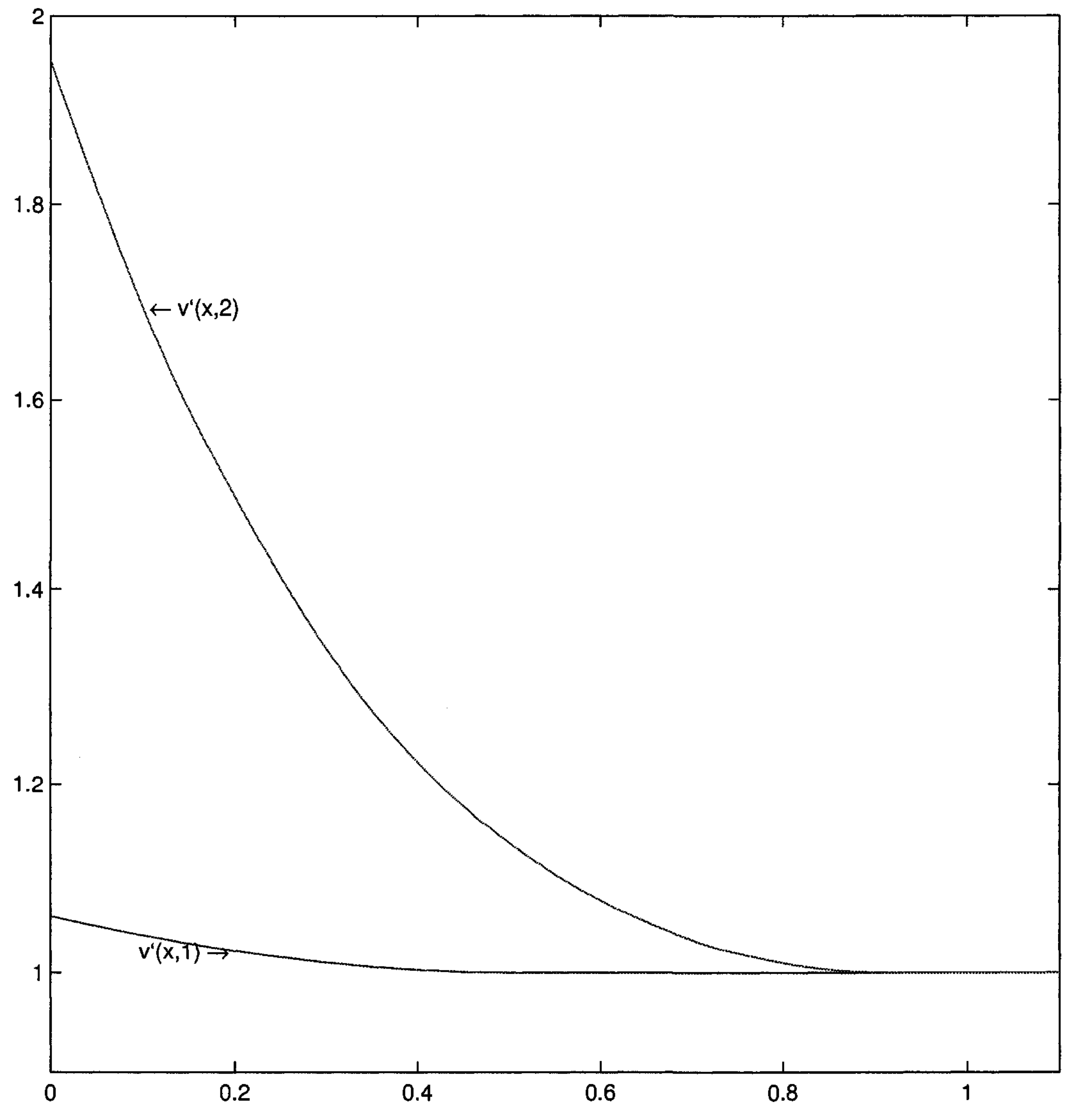

Figure 6.2: Derivative of the value function for Company AB when the dividend rates are unbounded: $V'_{AB}(\cdot, i)$, $i = 1, 2$, for model parameters $\lambda_1 = 0.06$, $\lambda_2 = 0.04$, $\mu_1 = 0.05$, $\mu_2 = 0.15$, $\sigma_1 = 0.70$, $\sigma_2 = 0.45$, $\delta = 0.12$ and $K = 5$.

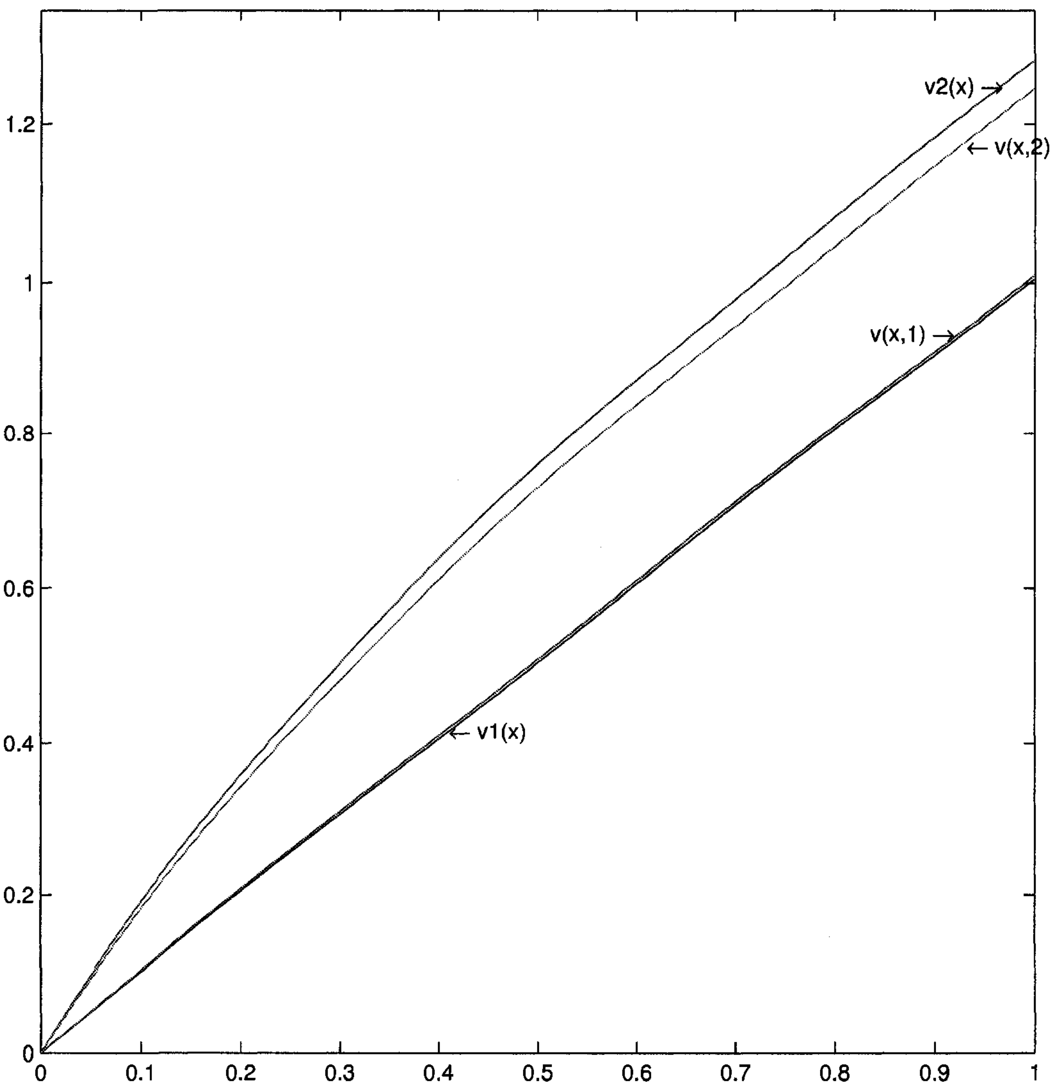

Figure 6.3: Comparison of the value functions for Companies AA, BB and AB when the dividend rates are unbounded (comparison with one-regime model). From the top to the bottom: $V_{BB}(\cdot)$, $V_{AB}(\cdot, 2)$, $V_{AB}(\cdot, 1)$ and $V_{AA}(\cdot)$.

Chapter 7

Stochastic impulse control with regime switching

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider also a N-dimensional standard Brownian motion W, and an observable continuous-time, stationary, finite-state Markov chain ϵ with strongly irreducible generator Q, independent of W. Moreover, consider the $\mathbb{P}$-augmented filtration $\mathbb{F}$.

Let $T = \{\tau_k, k \geq 1\}$ be an increasing sequence of stopping times, and denote $\tau_0 := 0$. We call the stopping times τ_k, $k \geq 1$, the intervention times. Moreover, let $\xi = \{\xi_k, k \geq 1\}$ be a sequence of random variables such that $\xi_k : \Omega \to \mathbb{R}^M$ is $\mathcal{F}_{\tau_k}$-measurable, $k \geq 1$. We use the notation $\xi_0 := 0$, and call the variables ξ_k, $k \geq 1$, the impulses at the intervention times.

Let the open, nonempty, convex set $\mathcal{O} \subset \mathbb{R}^M$ be the solvency region. Consider an $\mathbb{F}$-adapted process $X = \{X_t, t \geq 0\}$ that satisfies the following stochastic integral equation

$$X_t \;=\; x + \int_0^t f(X_s, \epsilon_s)\, ds + \int_0^t g(X_s, \epsilon_s)\, dW_s + \sum_{k \geq 1} h(\xi_k)\, I_{\{\tau_k < t\}}, \tag{7.1}$$

with initial value $X_0 = x \in \mathcal{O}$ and initial state $\epsilon_0 = \epsilon(0) = i \in \mathcal{S}$. We assume that the functions $f : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}^M$ and $g : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}^{M\times N}$ are Lipschitz continuous functions in x. Moreover, we assume that the function $h : \mathbb{R}^M \to \mathbb{R}^M$ is linear. We define $G : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}^{M\times M}$ as $G(x,i) := g(x,i)\cdot g(x,i)^T$. We note that the Lipschitz conditions implies that $f(\cdot,i)$ and $g(\cdot,i)$ are bounded in every closed subset of $\mathbb{R}^M$, for every $i \in \mathcal{S}$. Hence, $G(\cdot,i)$ is bounded in every closed subset of $\mathbb{R}^M$, for every $i \in \mathcal{S}$.

Consider the first time when the process X leaves the solvency region. Define such stopping time as

$$\Theta \;=\; \Theta_X \;:=\; \inf\{\, t \geq 0 : X_t \notin \mathcal{O} \,\}$$

and impose $X_t = X_\Theta$ for every $t \in [\Theta, \infty)$.

Let $H : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}$ be a function such that, for every $y \in \mathbb{R}^M$ and $i \in \mathcal{S}$,

$$| \, H(y,i) \, | \;\leq\; \kappa_i \, (\, 1 + \| \, y \, \|_M \,)\,, \tag{7.2}$$

where κ_i, $i \in \mathcal{S}$, are positive real numbers. Define the functional

$$J(x,i;T,\xi) \;:=\; E_{x,i}\left[\, \sum_{k\geq 1} e^{-\delta\tau_k} H(\xi_k, \epsilon_{\tau_k})\, I_{\{\tau_k<\Theta\}} \,\right]. \tag{7.3}$$

The parameter $\delta > 0$ is the discount rate for H.

Recall that h is a linear function and that H satisfies (7.2). Then, in order for the trajectory $X^{(T,\xi)}$ given by (7.1) and the functional (7.3) to be well defined, we only need

$$E_{x,i}\left[\, \sum_{k\geq 1} \; e^{-\delta\tau_k} I_{\{\tau_k<\Theta\}} \,\right] \;<\; +\infty \tag{7.4}$$

and

$$E_{x,i}\left[\sum_{k\geq 1} e^{-\delta\tau_k}\|\,\xi_k\|_M\, I_{\{\tau_k<\Theta\}}\right] \;<\; +\infty. \tag{7.5}$$

We note that condition (7.4) is equivalent to

$$\mathbb{P}\left\{\lim_{k\to\infty}\tau_k \leq t\wedge\Theta\right\} \;=\; 0, \qquad \text{for every } t\in[0,\infty). \tag{7.6}$$

We observe as well that, since h is a linear function, conditions (7.4) and (7.5) imply that

$$\lim_{t\to\infty} E_{x,i}\left[e^{-\delta t}X_{t+}\right] \;<\; +\infty. \tag{7.7}$$

Taking into consideration the previous analysis, we define admissibility for intervention times and impulses in the following way.

Definition 7.1. For every $x\in\mathcal{O}$ and $i\in\mathcal{S}$, we define an *admissible impulse control* as a stochastic impulse control process u defined by

$$u=(T,\xi)=(\tau_1,\ldots,\tau_k,\ldots;\xi_1,\ldots,\xi_k,\ldots)$$

that satisfies conditions (7.5) and (7.6). The set of all admissible controls will be denoted by $\mathcal{A}(x,i)$.

Problem 7.1. The stochastic control problem related to this setup is to select an optimal admissible control $u^*=(T^*,\xi^*)\in\mathcal{A}(x,i)$ that maximizes the functional (7.3) for every $x\in\mathcal{O}$ and $i\in\mathcal{S}$. We define the value function by

$$V(x,i) \;:=\; J(x,i;u^*) \;=\; \sup_{u\in\mathcal{A}(x,i)} J(x,i;u).$$

7.1 The verification theorem

Consider $\psi : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}$ and define the operators L_i, for each $i \in \mathcal{S}$, by

$$\begin{aligned} L_i\,\psi(x,i) \;&:=\; \frac{1}{2}G(x,i) \bullet \psi_{xx}(x,i) \;+\; f(x,i)^T\psi_x(x,i) \;-\; \delta\,\psi(x,i) \\ &=\; \frac{1}{2}\sum_{m=1}^{M}\sum_{p=1}^{M} G_{mp}(x,i)\,\frac{\partial^2\psi}{\partial x_m \partial x_p}(x,i) \\ &\qquad\qquad + \sum_{m=1}^{M} f_m(x,i)\,\frac{\partial\psi}{\partial x_m}(x,i) \;-\; \delta\,\psi(x,i). \end{aligned}$$

For every $x \in \mathcal{O}$ and $i \in \mathcal{S}$, we define the set of admissible impulses by

$$\Xi(x,i) \;:=\; \Big\{\eta \in \mathbb{R}^M :\; H(\eta,i) \in \mathbb{R},\;\; x + h(\eta) \in \mathcal{O}\Big\}.$$

For $\psi : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}$, we define then the maximum utility operators M_i by

$$M_i\,\psi(x,j) \;\;:=\;\; \sup\Big\{\, H(\eta,i) + \psi(x+h(\eta),j) \,:\; \eta \in \Xi(x,i)\Big\},$$

If the set $\Xi(x,i)$ is empty for some $(x,i) \in \mathcal{O}\times\mathcal{S}$, we denote $M_i\,\psi(x,i) = -\infty$ by convention.

For a function $v : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}$, we define for each $i \in \mathcal{S}$, the continuation region

$$\mathcal{C}(i) \;:=\; \Big\{x \in \mathcal{O} :\; M_i\,v(x,i) - v(x,i) < 0\Big\}.$$

Furthermore, we define the following quasi-variational inequalities.

Definition 7.2. A function $v : \mathbb{R}^M \times \mathcal{S} \to \mathbb{R}$ satisfies the quasi-variational inequalities (QVI) of Problem 7.1 if, for every $i \in \mathcal{S}$, it satisfies the Hamilton-Jacobi-Bellman equation

$$L_i\,v(x,i) \;+\; Q\,v(x,\cdot)(i) \;\;=\;\; 0, \tag{7.8}$$

for every $x \in \mathcal{C}(i)$, and if it also satisfies

$$L_i v(x,i) + Q\, v(x,\cdot)(i) \leq 0, \tag{7.9}$$

$$M_i v(x,i) - v(x,i) \leq 0, \tag{7.10}$$

for every every $x \in \mathcal{O}$.

Then, when v satisfies the QVI of Problem 7.1, we can define a control process $u^v = (T^v, \xi^v)$ associated with v in the following way.

Definition 7.3. The control process

$$u^v = (T^v, \xi^v) = (\tau_1^v, \ldots, \tau_k^v, \ldots; \xi_1^v, \ldots, \xi_k^v, \ldots)$$

is called the QVI-control associated with v if

$$\tau_1^v := \inf\left\{ t \geq 0 : v(X_t^v, \epsilon_t) = M_{\epsilon(t)} v(X_t^v, \epsilon_t) \right\} \tag{7.11}$$

$$\xi_1^v := \arg\sup\left\{ v\Big(X_{\tau_1^v}^v + h(\eta), \epsilon_{\tau_1^v}\Big) + H\big(\eta, \epsilon_{\tau_1^v}\big) : \eta \in \Xi\Big(X_{\tau_1^v}^v, \epsilon_{\tau_1^v}\Big) \right\}, \tag{7.12}$$

and for every $k \geq 2$,

$$\tau_k^v := \inf\left\{ t > \tau_{k-1} : v\,(X_t^v, \epsilon_t) = M_{\epsilon(t)} v(X_t^v, \epsilon_t) \right\} \tag{7.13}$$

$$\xi_k^v := \arg\sup\left\{ v\Big(X_{\tau_k^v}^v + h(\eta), \epsilon_{\tau_k^v}\Big) + H\big(\eta, \epsilon_{\tau_k^v}\big) : \eta \in \Xi\Big(X_{\tau_k^v}^v, \epsilon_{\tau_k^v}\Big) \right\}, \tag{7.14}$$

where X^v denotes the trajectory given by (7.1) for the control process $u^v = (T^v, \xi^v)$. In addition, we denote $\tau_0^v := 0$ and $\xi_0^v := 0$.

Theorem 7.1. Let $v(\cdot, i) \in C^1(\mathcal{O}) \cap C^2(\mathcal{O} \backslash N_i)$, $i \in \mathcal{S}$, be a real function in $\mathcal{O}$, where N_i, $i \in \mathcal{S}$, are finite subsets of $\mathcal{O}$. Let $v(x,i) = 0$ for every $x \notin \mathcal{O}$, $i \in \mathcal{S}$. Suppose that the function $v(\cdot, i)$, $i \in \mathcal{S}$, satisfies the quasi-variational inequalities (7.8) – (7.10). Moreover, for each $i \in \mathcal{S}$, let $D_i \subset \mathcal{O}$ be an open set, and suppose that

$$(7.15) \qquad |\, v(x,i)\,| \;\leq\; \alpha_i\,(\,1 + \|\, x\,\|_M)\,, \qquad \text{for every } x \in \mathcal{O}\backslash D_i,$$

where $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{S}$. If the QVI-control $u^v = (T^v, \xi^v)$ associated with v is admissible, then it is the optimal control for Problem 7.1. Furthermore, $v(\cdot, i)$, $i \in \mathcal{S}$, is the value function for Problem 7.1.

Proof. We note first that for each $i \in \mathcal{S}$, both $v(\cdot, i)$ and $v_x(\cdot, i)$ are continuous in $\mathcal{C}l(D_i)$, and hence they are bounded in $\mathcal{C}l(D_i)$.

Consider an arbitrary admissible control $u = (T, \xi)$ and the trajectory $X = X^u$ determined by u. Consider that X has the stopping time of bankruptcy $\Theta = \Theta_X$. Let us denote $\Lambda := \{t \geq 0 : X_{t+} \neq X_t\} = \{\tau_k, k \geq 1\}$.

We note that condition (7.5) is satisfied for $u = (T, \xi)$, which implies (7.7). Then, using the fact that $v(\cdot, i)$ is bounded in $\mathcal{C}l(D_i)$ and that satisfies (7.15) in $\mathcal{O}\backslash D_i$, using condition (7.7) and using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{t\to\infty} E_{x,i}\left[e^{-\delta(t\wedge\Theta)} v(X_{(t\wedge\Theta)+}, \epsilon_{t\wedge\Theta})\right] \;=\; E_{x,i}\left[e^{-\delta\Theta} v(X_{\Theta+}, \epsilon_\Theta)\right]$$

Recalling then that $X_t = X_\Theta \notin \mathcal{O}$ for every $t \in [\Theta, \infty)$, and that $v(x,i) = 0$ for every $x \notin \mathcal{O}$, $i \in \mathcal{S}$, we get

$$(7.16) \quad \lim_{t\to\infty} E_{x,i}\left[e^{-\delta(t\wedge\Theta)} v(X_{(t\wedge\Theta)+}, \epsilon_{t\wedge\Theta})\right] \;=\; E_{x,i}\left[e^{-\delta\Theta} v(X_\Theta, \epsilon_\Theta)\right] \;=\; 0.$$

Consider $\mathcal{U} = \{U_j,\, j \geq 1\}$ to be an open cover of $\mathcal{O}$. Let $J \geq 1$ be such that $X_0 = x \in \cup_{j=1}^J U_j \subset \mathcal{O}$ and such that $\cup_{j=1}^J U_j$ is bounded, and define the stopping time $\mathrm{s}_J = \inf\{\, t \geq 0 : X_t \notin \cup_{j=1}^J U_j\,\}$. Then, for every time $t \in [0, \infty)$

and every $K \geq 1$ and $J \geq 1$, we get

$$
\begin{aligned}
&e^{-\delta(t\wedge\tau_K\wedge \mathbf{s}_J)} v(X_{(t\wedge\tau_K\wedge \mathbf{s}_J)+}, \epsilon_{t\wedge\tau_K\wedge \mathbf{s}_J}) - v(X_0, \epsilon_0) \\
&\qquad = \sum_{k=1}^{K} \Big(e^{-\delta(t\wedge\tau_k\wedge \mathbf{s}_J)} v(X_{t\wedge\tau_k\wedge \mathbf{s}_J}, \epsilon_{t\wedge\tau_k\wedge \mathbf{s}_J}) \\
&\qquad\qquad - e^{-\delta(t\wedge\tau_{k-1}\wedge \mathbf{s}_J)} v(X_{(t\wedge\tau_{k-1}\wedge \mathbf{s}_J)+}, \epsilon_{t\wedge\tau_{k-1}\wedge \mathbf{s}_J}) \Big) \\
&\qquad\quad + \sum_{s\in[0,\, t\wedge\tau_K\wedge \mathbf{s}_J)\cap\Lambda} e^{-\delta s} \left(v(X_{s+}, \epsilon_s) - v(X_s, \epsilon_s) \right).
\end{aligned}
$$

We observe that $s \in [0, t\wedge\tau_K\wedge \mathbf{s}_J)\cap\Lambda$ if and only if $s = \tau_k$ and $\tau_k < t\wedge \mathbf{s}_J \wedge\Theta$, for some $k \in \{1, \ldots, K\}$. Hence,

$$
\begin{aligned}
&e^{-\delta(t\wedge\tau_K\wedge \mathbf{s}_J)} v(X_{(t\wedge\tau_K\wedge \mathbf{s}_J)+}, \epsilon_{t\wedge\tau_K\wedge \mathbf{s}_J}) - v(X_0, \epsilon_0) \\
&\qquad = \sum_{k=1}^{K} \Big(e^{-\delta(t\wedge\tau_k\wedge \mathbf{s}_J)} v(X_{t\wedge\tau_k\wedge \mathbf{s}_J}, \epsilon_{t\wedge\tau_k\wedge \mathbf{s}_J}) \\
&\qquad\qquad - e^{-\delta(t\wedge\tau_{k-1}\wedge \mathbf{s}_J)} v(X_{(t\wedge\tau_{k-1}\wedge \mathbf{s}_J)+}, \epsilon_{t\wedge\tau_{k-1}\wedge \mathbf{s}_J}) \Big) \\
&\qquad\quad + \sum_{k=1}^{K} e^{-\delta\,\tau_k} \left(v(X_{\tau_k+}, \epsilon_{\tau_k}) - v(X_{\tau_k}, \epsilon_{\tau_k}) \right) I_{\{\tau_k \leq t\wedge \mathbf{s}_J\wedge\Theta\}}.
\end{aligned}
\tag{7.17}
$$

Consider the function $\varphi(\cdot,\cdot,i)$, $i \in \mathcal{S}$, defined as $\varphi(t, x, i) = e^{-\delta t} v(x, i)$. We note that X is a continuous semimartingale in any interval $(\tau_{k-1}, \tau_k\,]$, $k \geq 1$. Then, we use the Itô's formula for Markov-modulate processes for $\varphi(t, X_t, \epsilon_t)$

in the intervals $(t \wedge \tau_{k-1} \wedge \mathrm{s}_J, t \wedge \tau_k \wedge \mathrm{s}_J\,]$, $k \geq 1$, to obtain

$$
\begin{aligned}
d\varphi(t, X_t, \epsilon_t) &= \varphi_t(t, X_t, \epsilon_t)\, dt \;+\; \sum_{m=1}^{M} f_m(X_t, \epsilon_t) \frac{\partial \varphi}{\partial x_m}(t, X_t, \epsilon_t)\, dt \\
&\quad + \sum_{m=1}^{M} \sum_{n=1}^{N} g_{mn}(X_t, \epsilon_t) \frac{\partial \varphi}{\partial x_m}(t, X_t, \epsilon_t)\, d(W_n)_t \\
&\quad + \frac{1}{2} \sum_{m=1}^{M} \sum_{p=1}^{M} G_{mp}(X_t, \epsilon_t) \frac{\partial^2 \varphi}{\partial x_m \partial x_p}(t, X_t, \epsilon_t)\, dt \\
&\quad + Q\, \varphi(t, X_t, \cdot)(\epsilon_t)\, dt \;+\; dM_t^{\varphi}.
\end{aligned}
$$

The process $\{M_t^{\varphi}, t \geq 0\}$ is a real-valued, square integrable martingale, with $M_0^{\varphi} = 0$ $\mathbb{P}$-a.s., when $v(\cdot, i)$, $i \in \mathcal{S}$, is bounded. We have then that, in the intervals $(t \wedge \tau_{k-1} \wedge \mathrm{s}_J, t \wedge \tau_k \wedge \mathrm{s}_J\,]$, $k \geq 1$,

$$
\begin{aligned}
d\varphi(t, X_t, \epsilon_t) &= -\delta e^{-\delta t} v(X_t, \epsilon_t)\, dt \;+\; e^{-\delta t} \sum_{m=1}^{M} f_m(X_t, \epsilon_t) \frac{\partial\, v}{\partial x_m}(X_t, \epsilon_t)\, dt \\
&\quad + e^{-\delta t} \sum_{m=1}^{M} \sum_{n=1}^{N} g_{mn}(X_t, \epsilon_t) \frac{\partial\, v}{\partial x_m}(X_t, \epsilon_t)\, d(W_n)_t \\
&\quad + \frac{1}{2}\, e^{-\delta t} \sum_{m=1}^{M} \sum_{p=1}^{M} G_{mp}(X_t, \epsilon_t) \frac{\partial^2\, v}{\partial x_m \partial x_p}(X_t, \epsilon_t)\, dt \\
&\quad + e^{-\delta t} Q\, v(X_t, \cdot)(\epsilon_t)\, dt + dM_t^{\varphi} \\
&= e^{-\delta t} \big(L_{\epsilon(t)}\, v(X_t, \epsilon_t) + Q\, v(X_t, \cdot)(\epsilon_t) \big)\, dt \\
&\quad + e^{-\delta t} \sum_{m=1}^{M} \sum_{n=1}^{N} g_{mn}(X_t, \epsilon_t) \frac{\partial\, v}{\partial x_m}(X_t, \epsilon_t)\, d(W_n)_t \;+\; dM_t^{\varphi}.
\end{aligned}
$$

Hence,

$$e^{-\delta(t\wedge\tau_k\wedge \mathbf{s}_J)}v(X_{t\wedge\tau_k\wedge \mathbf{s}_J},\epsilon_{t\wedge\tau_k\wedge \mathbf{s}_J}) - e^{-\delta(t\wedge\tau_{k-1}\wedge \mathbf{s}_J)}v(X_{(t\wedge\tau_{k-1}\wedge \mathbf{s}_J)+},\epsilon_{t\wedge\tau_{k-1}\wedge \mathbf{s}_J})$$

$$= \int_{t\wedge\tau_{k-1}\wedge \mathbf{s}_J}^{t\wedge\tau_k\wedge \mathbf{s}_J} e^{-\delta s}\left(L_{\epsilon(s)}\, v(X_s,\epsilon_s) + Q\, v(X_s,\cdot)(\epsilon_s)\right) ds$$

$$+ \int_{t\wedge\tau_{k-1}\wedge \mathbf{s}_J}^{t\wedge\tau_k\wedge \mathbf{s}_J} e^{-\delta s}\sum_{m=1}^{M}\sum_{n=1}^{N} g_{mn}(X_s,\epsilon_s)\,\frac{\partial v}{\partial x_m}(X_s,\epsilon_s)\, d(W_n)_s$$

$$+ M^{\varphi}_{t\wedge\tau_k\wedge \mathbf{s}_J} - M^{\varphi}_{(t\wedge\tau_{k-1}\wedge \mathbf{s}_J)+}. \tag{7.18}$$

The function $v(\cdot,i)$, $i\in\mathcal{S}$, satisfies the QVI (7.9). Then, for every $k\geq 1$,

$$\int_{t\wedge\tau_{k-1}\wedge \mathbf{s}_J}^{t\wedge\tau_k\wedge \mathbf{s}_J} e^{-\delta s}\left(L_{\epsilon(s)}\, v(X_s,\epsilon_s) + Q\, v(X_s,\cdot)(\epsilon_s)\right) ds \;\leq\; 0. \tag{7.19}$$

We note that the equality holds if (T,ξ) is the QVI-control associated with v. Moreover, $v(\cdot,i)$, $i\in\mathcal{S}$, also satisfies the QVI (7.10). Hence, for every $k\geq 1$,

$$e^{-\delta\tau_k}\left(v(X_{\tau_k+},\epsilon_{\tau_k}) - v(X_{\tau_k},\epsilon_{\tau_k})\right) I_{\{\tau_k< t\wedge \mathbf{s}_J\wedge\Theta\}}$$

$$= e^{-\delta\tau_k}\left(v(X_{\tau_k}+h(\xi_k),\epsilon_{\tau_k}) - v(X_{\tau_k},\epsilon_{\tau_k})\right) I_{\{\tau_k< t\wedge \mathbf{s}_J\wedge\Theta\}}$$

$$\leq -e^{-\delta\tau_k}H(\xi_k,\epsilon_{\tau_k})\, I_{\{\tau_k< t\wedge \mathbf{s}_J\wedge\Theta\}}. \tag{7.20}$$

Equality holds for the QVI-control associated with v (if admissible).

Combining equations (7.17), (7.18), (7.19) and (7.20), and taking conditional expectation given $X_0 = x$ and $\epsilon_0 = i$, we obtain

$$
\begin{aligned}
& E_{x,i}\left[e^{-\delta(t\wedge\tau_K\wedge \mathbf{s}_J)}\, v(X_{(t\wedge\tau_K\wedge \mathbf{s}_J)+}, \epsilon_{t\wedge\tau_K\wedge \mathbf{s}_J}) \right] - v(x,i) \\
&\qquad \leq \; E_{x,i}\left[\sum_{k=1}^{K} \int_{t\wedge\tau_{k-1}\wedge \mathbf{s}_J}^{t\wedge\tau_k\wedge \mathbf{s}_J} e^{-\delta s} \sum_{m=1}^{M}\sum_{n=1}^{N} g_{mn}(X_s,\epsilon_s)\, \frac{\partial v}{\partial x_m}(X_s,\epsilon_s)\, d(W_n)_s \right] \\
&\qquad\quad - E_{x,i}\left[\sum_{k=1}^{K} e^{-\delta\,\tau_k} H(\xi_k,\epsilon_{\tau_k})\, I_{\{\tau_k < t\wedge \mathbf{s}_J\wedge\Theta\}} \right] \\
&\qquad\quad + E_{x,i}\left[\sum_{k=1}^{K} \left(M^{\varphi}_{t\wedge\tau_k\wedge \mathbf{s}_J} - M^{\varphi}_{(t\wedge\tau_{k-1}\wedge \mathbf{s}_J)+} \right) \right] \\
&\qquad = \; E_{x,i}\left[\int_{0}^{t\wedge\tau_K\wedge \mathbf{s}_J} e^{-\delta s} \sum_{m=1}^{M}\sum_{n=1}^{N} g_{mn}(X_s,\epsilon_s)\, \frac{\partial v}{\partial x_m}(X_s,\epsilon_s)\, d(W_n)_s \right] \\
&\qquad\quad - E_{x,i}\left[\sum_{k=1}^{K} e^{-\delta\,\tau_k} H(\xi_k,\epsilon_{\tau_k})\, I_{\{\tau_k < t\wedge \mathbf{s}_J\wedge\Theta\}} \right] \\
&\qquad\quad + E_{x,i}\left[M^{\varphi}_{t\wedge\tau_K\wedge \mathbf{s}_J} - M^{\varphi}_{0} \right]. \qquad (7.21)
\end{aligned}
$$

We recall that $v_x(\cdot,i)$ is continuous in $\mathcal{O}$ and, hence, bounded in a closed subset of $\mathcal{O}$, $i \in \mathcal{S}$. We recall as well that $G(\cdot,i)$ is bounded in every closed subset of $\mathbb{R}^M$, for every $i \in \mathcal{S}$. We recall finally that $X_s \in \mathcal{C}l(\cup_{j=1}^{J} U_j)$ when $s \in [0, t\wedge\tau_K\wedge \mathbf{s}_J]$. Thus,

$$
E_{x,i}\left[\int_{0}^{t\wedge\tau_K\wedge \mathbf{s}_J} e^{-2\delta s} v_x(X_s,\epsilon_s)^T G(X_s,\epsilon_s)\, v_x(X_s,\epsilon_s)\, ds \right] \; < \; +\infty,
$$

which implies

$$
E_{x,i}\left[\int_{0}^{t\wedge\tau_K\wedge \mathbf{s}_J} e^{-\delta s} v_x(X_s,\epsilon_s)^T g(X_s,\epsilon_s)\, dW_s \right] \; = \; 0,
$$

or equivalently,

$$E_{x,i}\left[\int_0^{t\wedge\tau_K\wedge \mathbf{s}_J} e^{-\delta s}\sum_{m=1}^{M}\sum_{n=1}^{N} g_{mn}(X_s,\epsilon_s)\frac{\partial v}{\partial x_m}(X_s,\epsilon_s)\,d(W_n)_s\right] = 0.$$

We note as well that, for every $i \in \mathcal{S}$, $v(\cdot,i)$ is bounded in $Cl(\cup_{j=1}^{J} U_j)$ because of the continuity of $v(\cdot,i)$. Also, $X_s \in Cl(\cup_{j=1}^{J} U_j)$ for every $s \in [\,0, t\wedge\tau_K\wedge \mathbf{s}_J\,]$. Thus, $v(X_s,\epsilon_s)$ is bounded for every $s \in [\,0, t\wedge\tau_K\wedge \mathbf{s}_J\,]$. Then, the process $\{M^{\varphi}_{t\wedge\tau_K\wedge \mathbf{s}_J}, t \geq 0\}$ is a square integrable martingale and $E_{x,i}[\,M^{f}_{t\wedge\tau_K\wedge \mathbf{s}_J}\,] = E_{x,i}[\,M^{\varphi}_0\,]$. Thus, from (7.21),

$$\begin{aligned} v(x,i) \;&\geq\; E_{x,i}\left[e^{-\delta(t\wedge\tau_K\wedge \mathbf{s}_J)}v(X_{(t\wedge\tau_K\wedge \mathbf{s}_J)+},\epsilon_{t\wedge\tau_K\wedge \mathbf{s}_J})\right] \\ &\qquad + E_{x,i}\left[\sum_{k=1}^{K} e^{-\delta\,\tau_k}H(\xi_k,\epsilon_{\tau_k})\,I_{\{\tau_k < t\wedge \mathbf{s}_J\wedge\Theta\}}\right], \end{aligned}$$

where the equality holds for the QVI-control associated with v (if admissible).

Letting $J \to +\infty$, we get that $\mathbf{s}_J \to \Theta$. Also, since (T,ξ) is admissible, condition (7.6) holds and, hence,

$$\begin{aligned} v(x,i) \;&\geq\; \lim_{K\to\infty} E_{x,i}\left[e^{-\delta(t\wedge\tau_K\wedge\Theta)}v(X_{(t\wedge\tau_K\wedge\Theta)+},\epsilon_{t\wedge\tau_K\wedge\Theta})\right] \\ &\qquad + \lim_{K\to\infty} E_{x,i}\left[\sum_{k=1}^{K} e^{-\delta\,\tau_k}H(\xi_k,\epsilon_{\tau_k})\,I_{\{\tau_k < t\wedge\Theta\}}\right] \\ &= E_{x,i}\left[e^{-\delta(t\wedge\Theta)}v(X_{(t\wedge\Theta)+},\epsilon_{t\wedge\Theta})\right] + E_{x,i}\left[\sum_{k\geq 1} e^{-\delta\,\tau_k}H(\xi_k,\epsilon_{\tau_k})\,I_{\{\tau_k < t\wedge\Theta\}}\right]. \end{aligned}$$

Letting $t \to \infty$, using (7.16), and applying the Dominated Convergence Theorem, we get

$$v(x,i) \;\geq\; 0 + E_{x,i}\left[\sum_{k\geq 1} e^{-\delta\,\tau_k}H(\xi_k,\epsilon_{\tau_k})\,I_{\{\tau_k < \Theta\}}\right],$$

with equality for the QVI-control associated with v (if admissible). Therefore, for every admissible control (T, ξ), $v(x,i) \geq J(x,i;T,\xi)$. In particular, if (T^v, ξ^v) is admissible, $v(x,i) = J(x,i;T^v,\xi^v)$. $\square$

In the next chapter we present an example of a problem of stochastic impulse control with regime switching. We consider an optimal dividend problem for a company whose cash reservoir is affected by business cycles. The company's management has to select the optimal times and optimal amounts of dividends to pay to their shareholders. We consider dividend taxes and a fixed cost that is incurred every time that dividends are paid.

Chapter 8

Optimal dividend policy in the presence of a fixed dividend cost, dividend taxes and business cycles

Research on dividend policy started with Miller and Modigliani (1961) under the assumption of a perfect market, and has continued since then assuming different market conditions. Currently, dividend policy is one of the most important research areas in corporate finance. Although, its importance is not limited to research but it extends to practice. Annually, billions of dollars are paid out by the corporate sector in dividends (see Allen and Michaely (2003), and Poterba (2004) for actual amounts). Moreover, a company's dividend policy has a fundamental role in their financing and investing decisions.

Taksar (2000) presents a survey of different stochastic models for the optimal dividend policy. Standard dividend optimization problems are solved

[9]The results shown in this chapter are presented in: Sotomayor, L.R. and A. Cadenillas, Stochastic impulse control with regime switching for the optimal dividend policy when there are business cycles, *in preparation* (2008).

using classical and singular stochastic control techniques. Examples of this are the papers by Asmussen and Taksar (1997), Asmussen, Høјgaard and Taksar (2000), Cadenillas, Choulli, Taksar and Zhang (2006), Choulli, Taksar and Zhou (2001), Choulli, Taksar and Zhou (2003), Høјgaard and Taksar (2001), Jeanblanc-Piqué and Shiryaev (1995), Radner and Shepp (1996), and Taksar and Zhou (1998).

Jeanblanc-Piqué and Shiryaev (1995) initiated the use of stochastic impulse control for solving optimal dividend policy problems. They considered the problem of discrete dividend payments with fixed dividend fee (payment cost), and maximized the expected total amount of dividends to be paid to shareholders. Other papers that use stochastic impulse control techniques for dividend problems are Cadenillas, Sarkar and Zapatero (2007), and Cadenillas, Choulli, Taksar and Zhang (2006). However, none of these papers has considered the effect of macroeconomic conditions on dividend policies.

As mentioned in chapter 4 (for bounded dividend rates), empirical evidence supports that macroeconomic conditions influence the dividend policies. Gertler and Hubbard (1993) show in fact that dividend policies behave according to macroeconomic conditions. Furthermore, Ho and Wu (2001) explain that the earnings (and cash reservoir) of a company depend on the conditions of the market. Dividends are paid from the company's cash reservoir; hence a macroeconomic effect on the cash reservoir implies a macroeconomic effect on the dividend policies. Further empirical evidence of the macroeconomic effect is given by Driffill, Raybaudi and Sola (2003), Driffill and Sola (2001), Hackbart, Miao and Morellec (2006), and Hu and Schiantarelli (1998).

We consider an economy that shifts between two different regimes: economic growth and economic recession. We consider as well a company whose cash reservoir is affected by the regime of the economy. The objective of the management of the company is to select the times and amounts of dividends that maximize the total expected discounted cumulative amount of dividends to be paid out to their shareholders. We consider that the payment of dividends creates both a fixed dividend cost and a variable cost which is given by the dividend tax rate. Both costs are paid by the shareholders.

8.1 The dividend model with regime switching

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider also a standard Brownian motion $W = \{W_t, t \geq 0\}$, and a (finite-state) continuous-time Markov chain $\epsilon = \{\epsilon_t, t \geq 0\}$, such that $\epsilon_t = \epsilon(t) \in \{1, 2\}$, for every $t \in [0, \infty)$. The state $\epsilon(t)$ of the Markov chain represents the regime of the economy at time t. We assume that the processes ϵ and W are independent. Furthermore, we assume that the Markov chain has a strongly irreducible generator $Q = [\, q_{ij} \,]_{2\times 2}$ where $q_{ii} = -\lambda_i$ and $q_{i(3-i)} = \lambda_i$, with $\lambda_1, \lambda_2 > 0$. Then, we can write

$$d\epsilon_t \;=\; \left(\lambda_1\, I_{\{\epsilon_t=1\}} - \lambda_2\, I_{\{\epsilon_t=2\}}\right) dt \;+\; dM_t,$$

where $\{M_t, t \geq 0\}$ is a square integrable martingale (see Buffington and Elliott (2002), Elliott and Swishchuk (2004), Guo (2001), and Yin and Zhang (1998)) with $M_0 = 0$ $\mathbb{P}$-a.s. Consider $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the $\mathbb{P}$-augmentation of the filtration $\{\mathcal{F}_t^{W,\epsilon}, t \geq 0\}$ generated by the Brownian motion and the Markov chain, where $\mathcal{F}_t^{W,\epsilon} = \sigma\{W_s, \epsilon_s : 0 \leq s \leq t\}$ for every $t \in [0, \infty)$.

Let the $\mathbb{F}$-adapted process $X = \{X_t, t \geq 0\}$ represent the cash reservoir of the company. Assume that X satisfies the stochastic differential equation:

$$dX_t \;=\; \mu_{\epsilon(t)}\, dt + \sigma_{\epsilon(t)}\, dW_t - dZ_t$$

with initial level of the cash reservoir $X_0 = x > 0$ and initial regime $\epsilon_0 =$

$\epsilon(0) = i$. The drift coefficients (μ_1 and μ_2) and the volatility parameters (σ_1 and σ_2) are all positive constants.

The adapted process $Z = \{Z_t, t \geq 0\}$ represents the cumulative amount of dividends that are paid out by the company up to time t. Suppose dividends are paid at instants τ_n, $n \geq 1$, where $T = \{\tau_n, n \geq 1\}$ is an increasing sequence of stopping times. In addition, the amount of dividends paid at instant τ_n is ξ_n, $n \geq 1$, where $\xi = \{\xi_n, n \geq 1\}$ is a sequence of nonnegative random variables such that $\xi_n : \Omega \to [0, \infty)$ is $\mathcal{F}_{\tau_n}$-measurable, $n \geq 1$. We denote $\tau_0 := 0$ and $\xi_0 := 0$. Hence,

$$X_t \;=\; x + \int_0^t \mu_{\epsilon(s)}\, ds + \int_0^t \sigma_{\epsilon(s)}\, dW_s - \sum_{n\geq 1} \xi_n\, I_{\{\tau_n < t\}}$$

for every $t \in [0, \infty)$. Consider $\delta > 0$ the discount rate for the dividend payments.

We observe that the company will not be able to pay dividends if it runs out of cash, i.e. if its cash reservoir becomes zero (or negative). Thus, we need to define the stopping time of bankruptcy

$$\Theta = \Theta_X := \inf\Big\{t \geq 0 : \; X_t \leq 0\Big\}$$

and impose $X_t = 0$ for every $t \in [\Theta, \infty)$. Thus, we redefine the process $X = \{X_t, t \geq 0\}$ as

(8.1)

$$X_t \;=\; \begin{cases} x + \int_0^t \mu_{\epsilon(s)}\, ds + \int_0^t \sigma_{\epsilon(s)}\, dW_s - \sum_{n\geq 1} \xi_n\, I_{\{\tau_n < t\}} & \text{if } t \in [0, \Theta) \\ 0 & \text{if } t \in [\Theta, \infty). \end{cases}$$

For every cash reservoir process $X = X^{(T,\xi)}$, it is possible to prove that

$$\lim_{t\to\infty} E_{x,i}\left[e^{-\delta t} X_{t+}\right] \;=\; 0, \tag{8.2}$$

where $E_{x,i}[\cdot]$ represents the expectation conditioned to $X_0 = x$ and $\epsilon_0 = \epsilon(0) = i$. Indeed,

$$
\begin{aligned}
0 \;\le\; & \lim_{t\to\infty} E_{x,i}\left[e^{-\delta t}X_{t+}\right] \\
= \;& \lim_{t\to\infty} E_{x,i}\left[e^{-\delta t}X_{t+}I_{\{\Theta\le t\}}\right] + \lim_{t\to\infty} E_{x,i}\left[e^{-\delta t}X_{t+}I_{\{\Theta> t\}}\right] \\
= \;& \lim_{t\to\infty} E_{x,i}\left[e^{-\delta t}X_{t+}I_{\{\Theta> t\}}\right] \\
\le \;& \lim_{t\to\infty} E_{x,i}\left[e^{-\delta t}\left(x + \int_0^{t+}\mu_{\epsilon(s)}ds + \int_0^{t+}\sigma_{\epsilon(s)}dW_s\right)I_{\{\Theta> t\}}\right] \\
\le \;& \lim_{t\to\infty} e^{-\delta t}E_{x,i}\left[x + \int_0^{t}\mu_{\epsilon(s)}ds + \int_0^{t}\sigma_{\epsilon(s)}dW_s\right] \\
\le \;& \lim_{t\to\infty} e^{-\delta t}\left(x + \max\{\mu_1,\mu_2\}\cdot t\right) \quad = \quad 0.
\end{aligned}
$$

Then, following a similar analysis to Cadenillas and Zapatero (1999), we note that (8.2) implies that, for every pair (T,ξ),

$$E_{x,i}\left[\sum_{n\ge 1} e^{-\delta\tau_n}\xi_n\, I_{\{\tau_n<\Theta\}}\right] < \infty. \tag{8.3}$$

The company selects the dividend policy by deciding when to pay dividends and the amount of dividends to be paid each time. That is, the company selects both the sequence of stopping times T and the sequence of random variables ξ. We will require, however, certain technical condition.

Definition 8.1. The stochastic impulse control process u defined by

$$u = (T,\xi) = (\tau_1,\ldots,\tau_n,\ldots;\xi_1,\ldots,\xi_n,\ldots)$$

that satisfies the condition

$$\mathbb{P}\left\{\lim_{n\to\infty}\tau_n \le t\wedge\Theta\right\} = 0 \quad \text{for every } t\in[0,\infty), \tag{8.4}$$

is called *admissible impulse control.* The set of all admissible controls is denoted by $\mathcal{A}$.

Remark 8.1. The dividend policy that consists in never paying dividends is obtained through any control (T, ξ) such that $\mathbb{P}\{\tau_1 = \infty\} = 1$. By definition, these controls satisfy (8.4). Hence, to not pay dividends at all is an admissible strategy.

The management of the company wants to find the dividend policy that is optimal for the shareholders. For this reason, the management should consider the shareholders' preferences and the costs that receiving dividend payments generates to them.

We assume that the shareholders pay both a fixed cost and a variable cost for receiving a dividend payment. The fixed cost is the amount $K \in (0, \infty)$ that is charged every time a dividend is paid independently of its amount. The variable cost is the dividend tax charged on the dividend amount. We denote the dividend tax rate as $1 - k$, where $k \in (0, 1]$. If dividend income is not taxable in the prevalent economy, we use $k = 1$.

Since the management of the company is supposed to be looking after the interests of the shareholders, the shareholders' preferences are measured through the management's utility function. Following Jeanblanc-Piqué and Shiryaev (1995), we consider risk neutral shareholders and, hence, a risk neutral management. That is, we assume that the management has the linear utility function $g : [0, \infty) \to [0, \infty)$ defined by

$$g(x) := kx - K.$$

Thus, in order to optimize the operation of the company and find the optimal dividend policy for the shareholders, the company's management wants to solve the following problem.

Problem 8.1. Select an admissible control $u = (T, \xi)$ that maximizes

$$J(x,i;u) \;=\; J(x,i;T,\xi) \;:=\; E_{x,i}\left[\sum_{n\geq 1} e^{-\delta\tau_n}\, g(\xi_n)\, I_{\{\tau_n<\Theta\}}\right],$$

and find the value function $V : [0,\infty) \times \{1,2\} \to [0,\infty)$ defined by

$$V(x,i) \;:=\; \sup_{u\in\mathcal{A}} J(x,i;u).$$

We note that the admissibility condition (8.4) implies that

$$E_{x,i}\left[\sum_{n\geq 1} e^{-\delta\tau_n} I_{\{\tau_n<\Theta\}}\right] \;<\; \infty,$$

for every admissible control (T, ξ). Moreover, we note that

$$0 \;\leq\; J(x,i;T,\xi) \;\;\leq\;\; k\, E_{x,i}\left[\sum_{n\geq 1} e^{-\delta\tau_n}\, \xi_n\, I_{\{\tau_n<\Theta\}}\right].$$

Hence, equation (8.3) guarantees that the right-hand side of the inequality above is finite. Thus, the functional $J(x,i;u)$ is well defined and finite for every admissible control $u = (T, \xi)$.

Jeanblanc-Picqué and Shiryaev (1995) study Problem 8.1 for the special case in which the economy has only one regime, that is, ϵ takes only one possible value. Also, they did not consider the presence of dividend taxes although the utility function assumed for the shareholders is also linear.

8.2 The value function

In this section, we present some properties of the value function $V(\cdot, i)$, $i = 1, 2$, defined in Problem 8.1. We also give the conditions that characterize both the value function and the optimal control for this problem.

Proposition 8.1. The value function $V(\cdot, i)$, $i = 1, 2$, in Problem 8.1 is a real function such that $V(0, i) = 0$ for both $i = 1, 2$; such that

$$V(y,i) \geq V(x,i) + g(y-x) \tag{8.5}$$

for every $x \in (0, \infty)$ and $y \in [x, \infty)$.

Proof. Consider the initial capital $x > 0$ and the initial regime $i \in \{1, 2\}$, and let $u = (T, \xi)$ be an admissible control. For every $y \geq x$, we can follow the control process $\bar{u} = (\bar{T}, \bar{\xi})$ such that $\bar{\tau}_1 = 0$, $\bar{\tau}_n = \tau_{n-1}$ for $n \geq 2$, $\bar{\xi}_1 = y - x$ and $\bar{\xi}_n = \xi_{n-1}$ for $n \geq 2$. Let $X^u(a)$ denote the controlled process X^u such that $X_0^u = a$. Then,

$$\begin{aligned} X_t^{\bar{u}}(y) &= y + \int_0^t \mu_{\epsilon(s)}\, ds + \int_0^t \sigma_{\epsilon(s)}\, dW_s - \sum_{n\geq 1} \bar{\xi}_n\, I_{\{\bar{\tau}_n < t\}} \\ &= y + \int_0^t \mu_{\epsilon(s)}\, ds + \int_0^t \sigma_{\epsilon(s)}\, dW_s - \sum_{n\geq 1} \xi_n\, I_{\{\tau_n < t\}} - (y-x) \;=\; X_t^u(x), \end{aligned}$$

for every $t \in (0, \infty)$. We note that $\bar{u}$ is an admissible control since

$$\mathbb{P}\left\{\lim_{n\to\infty} \bar{\tau}_n \leq t \wedge \Theta_{X^{\bar{u}}}\right\} = \mathbb{P}\left\{\lim_{n\to\infty} \tau_n \leq t \wedge \Theta_{X^u}\right\} = 0,$$

and $\lim_{t\to\infty} E_{y,i}[\, e^{-\delta t} X_{t+}^{\bar{u}}\,] = \lim_{t\to\infty} E_{x,i}[\, e^{-\delta t} X_{t+}^{u}\,] = 0$. Therefore,

$$V(y,i) \geq J(y,i;\bar{T},\bar{\xi}) = J(x,i;T,\xi) + g(y-x),$$

and, hence, $V(y,i) \geq V(x,i) + g(y-x)$. Moreover, if the initial level of the cash reservoir is $x = 0$, the time of bankruptcy is $\Theta = 0$ and, hence, $V(0,i) = \sum_{n\geq 0} e^{-\delta\tau_n} g(\xi_n) \cdot 0 = 0$, for every initial regime i. □

In order to solve Problem 8.1, we shall define the following operators L_i for each $i = 1, 2$. Let $\psi : [0, \infty) \times \{1, 2\} \to \mathbb{R}$ be a real function and define

$$L_i \psi(x,i) \;:=\; \frac{1}{2}\,\sigma_i^2\,\frac{\partial^2 \psi}{\partial x^2}(x,i) \,+\, \mu_i\,\frac{\partial \psi}{\partial x}(x,i) \,-\, \delta\,\psi(x,i).$$

Also, we define the maximum utility operator M by

$$M\psi(x,i) \;:=\; \sup\Big\{ g(y) + \psi(x-y,i) : \, y \in [0,\, x] \Big\},$$

where $x \in [0, \infty)$ and $i = 1, 2$. The function $MV(\cdot, i)$ represents, when the regime of the economy is i, the expected utility generated by the dividend payment strategy that consists on paying first the best possible dividend amount and then selecting the optimal times and optimal amounts of the following dividend payments.

When the level of the cash reservoir is x and the regime of the economy is i, the expected utility associated with the optimal dividend policy is $V(x, i)$. The expected utility associated to any other dividend policy that is not optimal cannot be greater than $V(x, i)$. In particular, $MV(x, i)$, the expected utility associated with paying the best dividend amount and then following an optimal policy, cannot be greater than $V(x, i)$. This discussion motivates the following definition.

Definition 8.2. A function $v : [0, \infty) \times \{1, 2\} \to [0, \infty)$ satisfies the quasi-variational inequalities (QVI) of Problem 8.1 if, for each $i = 1, 2$, and every $x \in [0, \infty)$,

$$L_i v(x,i) - \lambda_i (v(x,i) - v(x, 3-i)) \;\leq\; 0, \tag{8.6}$$

$$Mv(x,i) - v(x,i) \;\leq\; 0, \tag{8.7}$$

$$\Big(L_i v(x,i) - \lambda_i (v(x,i) - v(x,3-i)) \Big) \Big(Mv(x,i) - v(x,i) \Big) \;=\; 0. \tag{8.8}$$

We note that the conditions (8.6) – (8.8) define two disjoint regions for each $i = 1, 2$, in the interval $(0, \infty)$: the continuation region

$$\mathcal{C}(i) := \Big\{ x > 0 : L_i v(x,i) - \lambda_i(v(x,i) - v(x,3-i)) = 0,\ Mv(x,i) - v(x,i) < 0 \Big\},$$

and the intervention region

$$\Sigma(i) := \Big\{ x > 0 : L_i v(x,i) - \lambda_i(v(x,i) - v(x,3-i)) < 0,\ Mv(x,i) - v(x,i) = 0 \Big\}.$$

Then, when v satisfies the QVI of Problem 8.1, we can define a control process $u^v = (T^v, \xi^v)$ associated with v in the following way.

Definition 8.3. The control process $u^v = (T^v, \xi^v) = (\tau_1^v, \ldots, \tau_n^v, \ldots; \xi_1^v, \ldots, \xi_n^v, \ldots)$ is called the QVI-control associated with v if

$$(8.9)\quad \tau_1^v := \inf\{\, t \geq 0 : v(X_t^v, \epsilon_t) = Mv(X_t^v, \epsilon_t) \,\}$$

$$(8.10)\quad \xi_1^v := \arg\sup\{\, v(X_{\tau_1^v}^v - \eta, \epsilon_{\tau_1^v}) + g(\eta) : \eta \in [\,0, \infty), X_{\tau_1^v}^v \in [\,\eta, \infty) \,\},$$

and for every $n \geq 2$,

$$(8.11)\quad \tau_n^v := \inf\{\, t > \tau_{n-1} : v(X_t^v, \epsilon_t) = Mv(X_t^v, \epsilon_t) \,\}$$

$$(8.12)\quad \xi_n^v := \arg\sup\{\, v(X_{\tau_n^v}^v - \eta, \epsilon_{\tau_n^v}) + g(\eta) : \eta \in [\,0, \infty), X_{\tau_n^v}^v \in [\,\eta, \infty) \,\},$$

where X^v denotes the cash reservoir process given by (8.1) generated by the control process $u^v = (T^v, \xi^v)$. In addition, we denote $\tau_0^v := 0$ and $\xi_0^v := 0$.

The following theorem is, to the best of our knowledge, the first verification theorem for stochastic impulse control with regime switching.

Theorem 8.1 (Verification Theorem). Let $v(\cdot, i) \in C^1([\,0, \infty)) \cap C^2([\,0, \infty) - N_i)$, $i = 1, 2$, where N_i are finite subsets of $(0, \infty)$. Let $v(0, i) = 0$, $i = 1, 2$ and suppose that $v(\cdot, i)$, $i = 1, 2$, is solution of the QVI (8.6) – (8.8). Moreover, for each $i = 1, 2$, let $\eta_i \in (0, \infty)$ and suppose that

$$(8.13) \qquad v(x, i) \;=\; a_i \,+\, k\,(x - d_i) - K, \qquad \text{for every } x \in [\,\eta_i, \infty),$$

where $a_i \in \mathbb{R}$ and $d_i \in (0, \eta_i)$. If the QVI-control $u^v = (T^v, \xi^v)$ associated with v is admissible, then it is the optimal control solution of Problem 8.1. Moreover, $v(\cdot, i)$, $i = 1, 2$, is the value function for Problem 8.1.

Proof. We note first that for each $i = 1, 2$, both $v(\cdot, i)$ and $v'(\cdot, i)$ are continuous functions in $[\,0,\, \eta_i\,]$, such that $v(\eta_i, i) = a_i + k(\eta_i - d_i) - K$ and $v'(\eta_i, i) = k$. Hence, for each $i = 1, 2$, $v(\cdot, i)$ and $v'(\cdot, i)$ are bounded in $[\,0,\, \eta_i\,]$. Moreover, $v'(\cdot, i)$ is constant in $[\,\eta_i,\, \infty)$, so $v'(\cdot, i)$ is also bounded in $[\,\eta_i, \infty)$, $i = 1, 2$.

Consider an arbitrary admissible control (T, ξ) and the corresponding process $X = X^{(T,\xi)}$ with stopping time of bankruptcy Θ. We note that X is a continuous semimartingale in any interval $(\tau_{n-1}, \tau_n\,]$, $n \geq 1$. Let a be a real number satisfying $0 < X_0 = x < a < +\infty$, and define the stopping times $\tau_a := \inf\{t \geq 0 : X_t = a\}$ and $\mathrm{s}_n := \tau_n \wedge \tau_a \wedge \Theta$, $n \geq 1$. Let us denote $\Delta_\epsilon v(x, i) := v(x, i) - v(x, 3 - i)$, and $\Lambda := \{t \geq 0 : X_{t+} \neq X_t\} = \{\tau_n, n \geq 1\}$. Hence, $s \in \Lambda$ if and only if $s = \tau_m$ for some $m \geq 1$. Consider the function $f(\cdot, \cdot, i)$, $i = 1, 2$, defined by $f(t, x, i) := e^{-\delta t} v(x, i)$. Then, for every $t \in [\,0, \infty)$ and $n \geq 1$, we apply the Itô formula for Markov modulated processes for

$f(t, X_t, \epsilon_t)$, to obtain

$$
\begin{aligned}
e^{-\delta(t\wedge s_n)} v(X_{(t\wedge s_n)+}, \epsilon_{t\wedge s_n}) - v(X_0, \epsilon_0) & \\
= \int_0^{t\wedge s_n} e^{-\delta s} \left(L_{\epsilon(s)} v(X_s, \epsilon_s) - \lambda_{\epsilon(s)} \Delta_\epsilon v(X_s, \epsilon_s) \right) ds & \\
+ \int_0^{t\wedge s_n} \sigma_{\epsilon(s)} e^{-\delta s} v'(X_s, \epsilon_s)\, dW_s & \\
+ \sum_{s\in[0,\, t\wedge s_n)\cap\Lambda} e^{-\delta s} \left(v(X_{s+}, \epsilon_s) - v(X_s, \epsilon_s) \right) + M^f_{t\wedge s_n} - M^f_0. & \qquad (8.14)
\end{aligned}
$$

We observe that $s \in [0, t\wedge s_n)\cap\Lambda$ if and only if $s = \tau_m$ and $\tau_m < t\wedge\tau_a\wedge\Theta$, for some $m = 1, \ldots, n$. Moreover, the process $\{M^f_t, t \geq 0\}$ is a square integrable martingale when $v(\cdot, i)$, $i = 1, 2$, is bounded (see Björk (1980)).

The function $v(\cdot, i)$, $i = 1, 2$, satisfies the QVI (8.6). Then, for every $n \geq 1$,

$$
\int_0^{t\wedge s_n} e^{-\delta s} \left(L_{\epsilon(s)} v(X_s, \epsilon_s) - \lambda_{\epsilon(s)} \Delta_\epsilon v(X_s, \epsilon_s) \right) ds \;\leq\; 0. \qquad (8.15)
$$

We note that the equality holds if (T, ξ) is the QVI-control associated with v. Moreover, $v(\cdot, i)$, $i = 1, 2$, also satisfies (8.7). Hence, for every $m \geq 1$,

(8.16)

$$
e^{-\delta\tau_m} \left(v(X_{\tau_m+}, \epsilon_{\tau_m}) - v(X_{\tau_m}, \epsilon_{\tau_m}) \right) I_{\{\tau_m < t\wedge\tau_a\wedge\Theta\}} \;\leq\; -e^{-\delta\tau_m} g(\xi_m)\, I_{\{\tau_m < t\wedge\tau_a\wedge\Theta\}}.
$$

Equality holds for the QVI-control associated with v (if admissible).

Combining equations (8.14), (8.15) and (8.16), and taking conditional expectation given $X_0 = x$ and $\epsilon_0 = i$, we obtain

$$
\begin{aligned}
& E_{x,i}\left[e^{-\delta(t\wedge \mathbf{s}_n)}\, v(X_{(t\wedge \mathbf{s}_n)+}, \epsilon_{t\wedge \mathbf{s}_n})\right] - v(x,i) \\
& \qquad \leq \; E_{x,i}\left[\int_0^{t\wedge \mathbf{s}_n} \sigma_{\epsilon(s)}\, e^{-\delta s}\, v'(X_s, \epsilon_s)\, dW_s\right] \\
& \qquad\quad - E_{x,i}\left[\sum_{m=1}^{n} e^{-\delta\,\tau_m} g(\xi_m)\, I_{\{\tau_m < t\wedge\tau_a\wedge\Theta\}}\right] + E_{x,i}\left[M^f_{t\wedge \mathbf{s}_n} - M^f_0\right].
\end{aligned}
\tag{8.17}
$$

We recall that $v'(X_s, \epsilon_s)$ is bounded when $s \in [0, \infty)$. Then, $\sigma^2_{\epsilon(s)}\, e^{-2\delta s}(v'(X_s, \epsilon_s))^2$ is bounded when $s \in [0,\, t\wedge \mathbf{s}_n]$ and therefore, for every $n \geq 1$,

$$
E_{x,i}\left[\int_0^{t\wedge \mathbf{s}_n} \sigma_{\epsilon(s)}\, e^{-\delta s}\, v'(X_s, \epsilon_s)\, dW_s\right] \;=\; 0.
$$

We note that $v(X_s, \epsilon_s)$ is bounded in $[0,\, t\wedge\tau_a\wedge\Theta]$. Then the process $\{M_{t\wedge\tau_a\wedge\Theta}, t \geq 0\}$ is a square integrable martingale, and $E_{x,i}[\, M^f_{t\wedge\tau_n\wedge\tau_a\wedge\Theta}\,] = E_{x,i}[\, M^f_{t\wedge \mathbf{s}_n}\,] = E_{x,i}[\, M^f_0\,]$. Thus, from (8.17),

$$
v(x,i) \;\geq\; E_{x,i}\left[e^{-\delta(t\wedge \mathbf{s}_n)} v(X_{(t\wedge \mathbf{s}_n)+}, \epsilon_{t\wedge \mathbf{s}_n})\right] + E_{x,i}\left[\sum_{m=1}^{n} e^{-\delta\,\tau_m} g(\xi_m)\, I_{\{\tau_m < t\wedge\tau_a\wedge\Theta\}}\right],
$$

where the equality holds for the QVI-control associated with v (if admissible). Letting $a \uparrow +\infty$, we get that $\tau_a \to +\infty$. Then, $\mathbf{s}_n \to \tau_n \wedge \Theta$. Also, since (T, ξ) is admissible, condition (8.4) holds and, hence,

$$
\begin{aligned}
v(x,i) \;\; &\geq \;\; \lim_{n\to\infty} E_{x,i}\left[e^{-\delta(t\wedge\tau_n\wedge\Theta)} v(X_{(t\wedge\tau_n\wedge\Theta)+}, \epsilon_{t\wedge\tau_n\wedge\Theta})\right] \\
&\qquad\qquad + \lim_{n\to\infty} E_{x,i}\left[\sum_{m=1}^{n} e^{-\delta\,\tau_m} g(\xi_m)\, I_{\{\tau_m < t\wedge\Theta\}}\right] \\
&= \;\; E_{x,i}\left[e^{-\delta(t\wedge\Theta)} v(X_{(t\wedge\Theta)+}, \epsilon_{t\wedge\Theta})\right] + E_{x,i}\left[\sum_{m\geq 1} e^{-\delta\,\tau_m} g(\xi_m)\, I_{\{\tau_m < t\wedge\Theta\}}\right].
\end{aligned}
$$

Letting $t \to \infty$ and recalling that $v(0,i) = 0$ for both $i = 1, 2$, we obtain

$$\begin{aligned} v(x,i) &\geq E_{x,i}\left[e^{-\delta\Theta} v(X_\Theta, \epsilon_\Theta)\right] + E_{x,i}\left[\sum_{m\geq 1} e^{-\delta\tau_m} g(\xi_m)\, I_{\{\tau_m<\Theta\}}\right] \\ &= E_{x,i}\left[e^{-\delta\Theta} v(0, \epsilon_\Theta)\right] + E_{x,i}\left[\sum_{m\geq 1} e^{-\delta\tau_m} g(\xi_m)\, I_{\{\tau_m<\Theta\}}\right] \\ &= E_{x,i}\left[\sum_{m\geq 1} e^{-\delta\tau_m} g(\xi_m)\, I_{\{\tau_m<\Theta\}}\right], \end{aligned} \tag{8.18}$$

with equality for the QVI-control associated with v (if admissible). Therefore, for every admissible control (T, ξ), $v(x,i) \geq J(x,i;T,\xi)$. In particular, if (T^v, ξ^v) is admissible, $v(x,i) = J(x,i;T^v,\xi^v)$. □

8.3 Construction of the solution.

We conjecture that the continuation region for the value function $V(\cdot, i)$, $i = 1, 2$, is given by $\mathcal{C}(i) = (0, b_i)$, for some $b_i \in [0, \infty)$, $i = 1, 2$. Thus, we conjecture that the optimal control (associated with V) $(\hat{T}, \hat{\xi}) = (T^V, \xi^V)$ satisfies, for every $n \geq 1$,

$$\hat{\tau}_n = \inf\left\{t \geq \hat{\tau}_{n-1} : X_t \notin (0, b_{\epsilon(t)})\right\}, \tag{8.19}$$

where we denote $\hat{\tau}_0 := 0$. That is, we conjecture that it is optimal for the cash reservoir process to remain in the band $(0, b_i)$ when the regime of the economy is i, and that it is optimal to pay dividends when the cash reservoir process is above the threshold b_i and the regime is i.

We conjecture as well that, for every $n \geq 1$, the optimal amount of dividends paid at $\hat{\tau}_n$ is

$$(8.20) \qquad \hat{\xi}_n = X_{\hat{\tau}_n} - X_{\hat{\tau}_n+} = X_{\hat{\tau}_n} - \beta_{\epsilon(\hat{\tau}_n)} I_{\{X_{\hat{\tau}_n} \geq b_{\epsilon(\hat{\tau}_n)}\}},$$

where β_i, $i = 1, 2$, are positive numbers such that $\beta_i < b_i$. That is, we conjecture that it is optimal for the cash reservoir process to jump to the level β_i when the regime of the economy is i and the level of the cash reservoir is equal or larger than b_i.

We observe that, on an interval of time $[t_1, t_2]$, if the regime of the economy is constant and $X_{t_1} \in [0, b_{\epsilon(t_1)})$, then it is optimal to pay dividends only when the cash reservoir process reaches the threshold $b_{\epsilon(t_1)}$. Whenever that happens, the optimal amount of dividends to pay is the same and equal to $b_{\epsilon(t_1)} - \beta_{\epsilon(t_1)}$. Hence, in this case, dividend payments only occur optimally due to changes in the continuous part of the cash reservoir process. In general, however, dividend payments can also occur due only to changes in the regime of the economy. That occurs at time t if the regime of the economy changes, for instance, ϵ changes from 2 to 1, and $b_2 > b_1$ and $X_t \in [b_1, b_2)$.

In addition, if the initial regime of the economy is i and the initial level of the cash reservoir is high enough, i.e. $x \in (b_i, \infty)$, we would expect to be optimal to pay an initial dividend of $x - \beta_i$ and, in that way, make the cash reservoir process jump to the level β_i. Hence, we conjecture that the value function $V(\cdot, i)$, $i = 1, 2$, satisfies

$$(8.21) \quad V(x, i) = V(\beta_i, i) + k(x - \beta_i) - K \quad \text{for every } x \in [b_i, \infty),\ i = 1, 2.$$

We need to find, then, a function $v(\cdot, i) \in C^1([0, \infty)) \cap C^2([0, \infty) - N_i)$, $i = 1, 2$, for some finite $N_i \subset (0, \infty)$, solution of the QVI (8.6) – (8.8), such that $v(0, i) = 0$, $i = 1, 2$, and that satisfies the conjectures above. Define, for each $i = 1, 2$,

$$\beta_i := \inf\Big\{ x > 0 : \; v'(x,i) \le k \Big\} \tag{8.22}$$

$$b_i := \inf\Big\{ x > \beta_i : \; v'(x,i) \ge k \Big\}. \tag{8.23}$$

We note that always $\beta_i < b_i$ for both $i = 1, 2$; however, the relation between the thresholds b_1 and b_2 depends on the relations among the model parameters. Without loss of generality we will consider $b_1 < b_2$: the case $b_1 > b_2$ has a similar treatment. Thus, we will consider three possible regions for the initial level of the cash reservoir: $x \in [0, b_1)$, $x \in [b_1, b_2)$ and $x \in [b_2, \infty)$.

When $x \in [0, b_1)$, we want $v(\cdot, i)$, $i = 1, 2$, to satisfy the following system of ordinary differential equations:

$$\begin{aligned} \frac{1}{2}\sigma_1^2 v''(x,1) + \mu_1 v'(x,1) - (\delta + \lambda_1) v(x,1) + \lambda_1 v(x,2) &= 0 \\ \frac{1}{2}\sigma_2^2 v''(x,2) + \mu_2 v'(x,2) - (\delta + \lambda_2) v(x,2) + \lambda_2 v(x,1) &= 0. \end{aligned} \tag{8.24}$$

This system of differential equations is solved by Sotomayor and Cadenillas (2007). Consider $\phi_i(\alpha) := \frac{1}{2}\sigma_i^2\alpha^2 + \mu_i \alpha - (\delta + \lambda_i)$, $i = 1, 2$, and the real roots $\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$ of the characteristic function $\phi_1(\alpha)\,\phi_2(\alpha) = \lambda_1\lambda_2$. The solution of the system (8.24) is

$$F(x,1) = A_1 e^{\alpha_1 x} + A_2 e^{\alpha_2 x} + A_3 e^{\alpha_3 x} + A_4 e^{\alpha_4 x} \tag{8.25}$$

$$F(x,2) = B_1 e^{\alpha_1 x} + B_2 e^{\alpha_2 x} + B_3 e^{\alpha_3 x} + B_4 e^{\alpha_4 x}, \tag{8.26}$$

where, for each $j = 1, 2, 3, 4$,

$$B_j = \frac{\phi_1(\alpha_j)}{\lambda_1} A_j = \frac{\lambda_2}{\phi_2(\alpha_j)} A_j. \tag{8.27}$$

When $x \in [b_1, b_2)$, $v(x,1)$ has to satisfy (8.21) for $i = 1$. We note that $\beta_1 \in (0, b_1)$ and, hence, $v(\beta_1, 1) = F(\beta_1, 1)$. Then,

$$v(x,1) = v(\beta_1,1) + k(x-\beta_1) - K = F(\beta_1,1) + k(x-\beta_1) - K. \tag{8.28}$$

Also, $v(\cdot, 2)$ should satisfy the differential equation

$$\frac{1}{2}\sigma_2^2 v''(x,2) + \mu_2 v'(x,2) - (\delta+\lambda_2)v(x,2) + \lambda_2 F(\beta_1,1) + \lambda_2\left(k(x-\beta_1) - K\right) = 0. \tag{8.29}$$

The solution for equation (8.29) is given by

$$G(x) = \bar{A}_1 e^{\bar{\alpha}_1 x} + \bar{A}_2 e^{\bar{\alpha}_2 x} + \frac{\lambda_2}{\delta+\lambda_2}(k(x-\beta_1) - K) + \frac{\lambda_2}{\delta+\lambda_2}F(\beta_1,1) + \frac{\lambda_2\mu_2 k}{(\delta+\lambda_2)^2}, \tag{8.30}$$

where $\bar{\alpha}_1 < 0 < \bar{\alpha}_2$ are the real roots of the characteristic function $\phi_2(\bar{\alpha}) = \frac{1}{2}\sigma_2^2\bar{\alpha}^2 + \mu_2\bar{\alpha} - (\delta+\lambda_2) = 0$.

Finally, when $x \in [b_2, \infty)$, equation (8.21) has to be satisfied for both $i = 1, 2$. We note that $v(\cdot, 2)$ varies if $\beta_2 \in (0, b_1)$ or $\beta_2 \in [b_1, b_2)$. If $\beta_2 \in (0, b_1)$, then $v(\beta_2, 2) = F(\beta_2, 2)$; and if $\beta_2 \in [b_1, b_2)$, then $v(\beta_2, 2) = G(\beta_2)$. We denote, hence,

$$H(\beta_2) := F(\beta_2, 2)\, I_{\{\beta_2 \in (0,b_1)\}} + G(\beta_2)\, I_{\{\beta_2 \in [b_1,b_2)\}}. \tag{8.31}$$

Thus, when $x \in [b_2, \infty)$,

$$v(x,1) = v(\beta_1,1) + k(x-\beta_1) - K = F(\beta_1,1) + k(x-\beta_1) - K \tag{8.32}$$

$$v(x,2) = v(\beta_2,2) + k(x-\beta_2) - K = H(\beta_2) + k(x-\beta_2) - K. \tag{8.33}$$

We conjecture that the function $v(\cdot, i)$, $i = 1, 2$, constructed above satisfies the *smooth-fit conditions.* Thus, $v(\cdot,i)$, $i=1,2$, would satisfy

$$(8.34)\qquad \begin{aligned} v(0,i) &= 0, && \text{for both } i=1,2\\ v(b_1-,i) &= v(b_1+,i), && \text{for both } i=1,2\\ v(b_2-,2) &= v(b_2+,2) && \\ v'(b_1-,i) &= v'(b_1+,i), && \text{for both } i=1,2\\ v'(b_2-,2) &= v'(b_2+,2) && \\ v'(\beta_i,i) &= k, && \text{for both } i=1,2. \end{aligned}$$

The solution of the system of equations (8.34) will give us the thresholds b_1 and b_2, and β_1 and β_2, the coefficients A_j, $j=1,2,3,4$, for the function (8.25), and the coefficients $\bar{A}_1$ and $\bar{A}_2$ for the function (8.30). The coefficients B_j, $j=1,2,3,4$, for (8.26) will be found from condition (8.27).

We note that the continuity condition given in (8.34) for $v(\cdot,2)$ implies that $F(b_1,2)=G(b_1)$ and, hence, $H(\beta_2)=F(\beta_2,2)=G(\beta_2)$ when $\beta_2=b_1$.

8.4 Verification of the solution.

In the previous section, we constructed a candidate for optimal control (see equations (8.19)-(8.20)) and a candidate for value function $v(\cdot,i)$, $i=1,2$. To prove that in fact the function $v(\cdot,i)$, $i=1,2$, is the value function of Problem 8.1, we need to show that it satisfies all the conditions of Theorem 8.1. In order to do so, we need to state first some properties of $v(\cdot,i)$, $i=1,2$.

We conjecture that the following inequalities are satisfied for every solution of the system of equations (8.34), where also $b_1<b_2$. The numerical examples in Section 6 show that in fact this is an appropriate conjecture as all of them

satisfy the inequalities below.

(8.35)

$$-(\delta+\lambda_1)(F(\beta_1,1)-\beta_1 k)+\lambda_1(H(\beta_2)-\beta_2 k)-\delta(k\,b_1-K)+\mu_1 k \;\leq\; -\lambda_1 K$$

(8.36)

$$\lambda_2(F(\beta_1,1)-\beta_1 k)-(\delta+\lambda_2)(H(\beta_2)-\beta_2 k)-\delta(k\,b_2-K)+\mu_2 k \;\leq\; 0.$$

Proposition 8.2. The functions $F(\cdot,i)$, $i=1,2$, and G satisfy:

(a) For all $x\in[0,\beta_1)$: $F'(x,1)>k$; and for all $x\in(\beta_1,b_1)$: $F'(x,1)<k$.

(b) If $\beta_2\in(0,b_1)$, then for all $x\in[0,\beta_2)$: $F'(x,2)>k$; and for all $x\in(\beta_2,b_1]$: $F'(x,2)<k$.

If $\beta_2\in[b_1,b_2)$, then for all $x\in[0,b_1]$: $F'(x,2)>k$.

(c) If $\beta_2\in(0,b_1)$, then for all $x\in[b_1,b_2)$: $G'(x)<k$.

If $\beta_2\in[b_1,b_2)$, then for all $x\in[b_1,\beta_2)$: $G'(x)>k$; and for all $x\in(\beta_2,b_2)$: $G'(x)<k$.

Proof. The proof of this Proposition comes straightforward from the definition of β_i and the fact that $v'(x,i)>k$ for every $x\in[0,\beta_i)$; and from the definition of b_i and the fact that $v'(x,i)<k$ for every $x\in(\beta_i,b_i)$. For (*b*) and (*c*), we also use the continuity of $v'(\cdot,2)-k$. □

We will use the properties given in Proposition 8.2 to calculate $Mv(x,i)$ for $x\in[0,\infty)$ and $i=1,2$, in the Lemma 8.1. We also need the following Proposition.

Proposition 8.3. Let the inequalities (8.35)–(8.36) hold. Then, the functions $F(\cdot,i)$, $i=1,2$, and G satisfy the following inequalities. For every $x\in[b_1,b_2)$,

$$-(\delta+\lambda_1)\,(k(x-\beta_1)-K)+\lambda_1\,G(x)-(\delta+\lambda_1)\,F(\beta_1,1)+\mu_1 k \;\leq\; 0;$$

and for every $x \in [b_2, \infty)$,

$$-(\delta+\lambda_1)(k(x-\beta_1)-K)+\lambda_1(k(x-\beta_2)-K)-(\delta+\lambda_1)F(\beta_1,1) \\ +\lambda_1 H(\beta_2)+\mu_1 k \leq 0,$$

$$-(\delta+\lambda_2)(k(x-\beta_2)-K)+\lambda_2(k(x-\beta_1)-K)+\lambda_2 F(\beta_1,1) \\ -(\delta+\lambda_2)H(\beta_2)+\mu_2 k \leq 0.$$

Proof. Define the function

$$\Lambda(x) := -(\delta+\lambda_1)(k(x-\beta_1)-K)+\lambda_1 G(x)-(\delta+\lambda_1)F(\beta_1,1)+\mu_1 k.$$

We note that, in order to prove the first inequality of the Proposition, we need to prove that $\Lambda(x) \leq 0$ for every $x \in [b_1, b_2)$.

If $\beta_2 \in (0, b_1)$, Proposition 8.2(c) tells us that Λ is strictly decreasing in $[b_1, b_2)$. Then, for every $x \in [b_1, b_2)$,

(8.37)

$$\Lambda(x) \leq \Lambda(b_1) = -(\delta+\lambda_1)(k(b_1-\beta_1)-K)+\lambda_1 G(b_1)-(\delta+\lambda_1)F(\beta_1,1)+\mu_1 k.$$

We note that $v(\cdot, 2)$ is continuous at b_1 (from (8.34)) and, hence, $F(b_1, 2) = G(b_1)$. By replacing and rearranging in (8.37), we obtain that, for every $x \in [b_1, b_2)$,

$$\begin{aligned}
\Lambda(x) &\leq -(\delta+\lambda_1)(k(b_1-\beta_1)-K)+\lambda_1 F(b_1,2)-(\delta+\lambda_1)F(\beta_1,1)+\mu_1 k \\
&= -\delta b_1 k - \lambda_1 b_1 k + (\delta+\lambda_1)\beta_1 k + (\delta+\lambda_1)K + \lambda_1 F(b_1,2) \\
&\qquad -(\delta+\lambda_1)F(\beta_1,1)+\mu_1 k \\
&= -\delta b_1 k + (\delta+\lambda_1)K + \lambda_1(F(b_1,2)-b_1 k) - (\delta+\lambda_1)(F(\beta_1,1)-\beta_1 k) + \mu_1 k.
\end{aligned}$$

Proposition 8.2(b) tells us that $F(x,2) - kx \leq F(\beta_2, 2) - \beta_2 k$ for every $x \in$

$[0, b_1]$. This inequality holds in particular for $x = b_1$. Moreover, if $\beta_2 \in (0, b_1)$, then $H(\beta_2) = F(\beta_2, 2)$. This means that, for every $x \in [b_1, b_2)$,

$$\begin{aligned}\Lambda(x) &\leq -\delta b_1 k+(\delta+\lambda_1)K+\lambda_1(F(\beta_2,2)-\beta_2 k)-(\delta+\lambda_1)(F(\beta_1,1)-\beta_1 k)+\mu_1 k \\ &= -\delta b_1 k+(\delta+\lambda_1)K+\lambda_1(H(\beta_2)-\beta_2 k)-(\delta+\lambda_1)(F(\beta_1,1)-\beta_1 k)+\mu_1 k.\end{aligned}$$

If $\beta_2 \in [b_1, b_2)$, Proposition 8.2(c) tells us that, in the interval $[b_1, b_2)$, $G(x) - kx$ attains its maximum at $x = \beta_2$. In addition, $H(\beta_2) = G(\beta_2)$. Thus, for every $x \in [b_1, b_2)$,

$$\begin{aligned}\Lambda(x) &= -\delta kx-\lambda_1 kx+(\delta+\lambda_1)\beta_1 k+(\delta+\lambda_1)K+\lambda_1 G(x)-(\delta+\lambda_1)F(\beta_1,1)+\mu_1 k \\ &= -\delta(kx-K) + \lambda_1 K + \lambda_1(G(x)-kx)-(\delta+\lambda_1)(F(\beta_1,1)-\beta_1 k) + \mu_1 k \\ &\leq -\delta\, k\, x + (\delta+\lambda_1)K+\lambda_1(G(\beta_2)-\beta_2 k)-(\delta+\lambda_1)(F(\beta_1,1)-\beta_1 k) + \mu_1 k \\ &= -\delta\, k\, x + (\delta+\lambda_1)K+\lambda_1(H(\beta_2)-\beta_2 k)-(\delta+\lambda_1)(F(\beta_1,1)-\beta_1 k) + \mu_1 k \\ &\leq -\delta\, b_1\, k + (\delta+\lambda_1)K+\lambda_1(H(\beta_2)-\beta_2 k)-(\delta+\lambda_1)(F(\beta_1,1)-\beta_1 k)+\mu_1 k.\end{aligned}$$

In both cases, when $\beta_2 \in (0, b_1)$ and $\beta_2 \in [b_1, b_2)$, we can use (8.35) to obtain that, for every $x \in [b_1, b_2)$,

$$\begin{aligned}\Lambda(x) &\leq -(\delta+\lambda_1)(F(\beta_1,1)-\beta_1 k)+\lambda_1(H(\beta_2)-\beta_2 k)-\delta(b_1\, k-K)+\lambda_1 K+\mu_1 k \\ &\leq \lambda_1 K - \lambda_1 K \quad = \quad 0.\end{aligned}$$

This finishes the proof of the first inequality of the Proposition. The other two inequalities can be written as

$$\begin{aligned}\eta_1(x) &:= -\delta kx + \delta K + \lambda_1(H(\beta_2)-\beta_2 k)-(\delta + \lambda_1)(F(\beta_1,1)-\beta_1 k) + \mu_1\, k \leq 0, \\ \eta_2(x) &:= -\delta kx + \delta K + \lambda_2(F(\beta_1,1)-\beta_1 k)-(\delta + \lambda_2)(H(\beta_2)-\beta_2 k) + \mu_2\, k \leq 0,\end{aligned}$$

for $x \in [b_2, \infty)$. Then, using (8.35) we get that, for every $x \in [b_2, \infty)$,

$$\eta_1(x) \;\leq\; -\lambda_1 K + \delta(k\, b_1 - K) - \delta k\, x + \delta K \;=\; -\lambda_1 K - \delta(x - b_1)\, k \;\leq\; 0.$$

Moreover, using (8.36) we obtain, for every $x \in [b_2, \infty)$,

$$\eta_2(x) \;\leq\; \delta(k\, b_2 - K) - \delta k\, x + \delta K \;=\; -\delta(x - b_2)\, k \;\leq\; 0.$$

This completes the proof of the Proposition. □

Lemma 8.1. *Let b_i, β_i, $i = 1, 2$, A_j, $j = 1, 2, 3, 4$, and $\bar{A}_j$, $j = 1, 2$, be the solution of the system of equations (8.34) and suppose that $b_i \in [0, \infty)$ and $\beta_i \in (0, b_i)$ for both $i = 1, 2$. Let B_j, $j = 1, 2, 3, 4$, be defined by (8.27). Suppose without lost of generality that $b_1 < b_2$. Define the functions $F(\cdot, 1)$ and $F(\cdot, 2)$ by (8.25) and (8.26), respectively; and define the function G by (8.30). Define also the value $H(\beta_2)$ by (8.31). Define also the function $v(\cdot, i)$, $i = 1, 2$, by*

$$v(x,1) = \begin{cases} F(x,1) & \text{if } x \in [0, b_1), \\ F(\beta_1, 1) + k(x - \beta_1) - K & \text{if } x \in [b_1, \infty), \end{cases} \tag{8.38}$$

and

$$v(x,2) = \begin{cases} F(x,2) & \text{if } x \in [0, b_1), \\ G(x) & \text{if } x \in [b_1, b_2), \\ H(\beta_2) + k(x - \beta_2) - K & \text{if } x \in [b_2, \infty). \end{cases} \tag{8.39}$$

Then, the function $Mv(\cdot, i)$, $i = 1, 2$, *is given by*

$$Mv(x,i) = \begin{cases} v(x,i) - K & \text{if } x \in [0, \beta_i), \\ v(\beta_i, i) + k(x - \beta_i) - K & \text{if } x \in [\beta_i, b_i), \\ v(x,i) & \text{if } x \in [b_i, \infty). \end{cases}$$

Proof. In order to prove this Lemma, we need to find the $y \in [0, x]$ that maximizes $v(y,i) + g(x-y)$ for $x \in [0, \infty)$ and for each $i = 1, 2$.

Consider first $i = 1$. For $x \in [0, \beta_1)$, $v(y,1) + g(x-y) = F(y,1) + k(x - y) - K$ is increasing in $[0,x] \subset [0, \beta_1)$ according to Proposition 8.2. Thus, $Mv(x,1) = F(x,1) - K = v(x,1) - K$. For $x \in [\beta_1, b_1)$, $v(y,1) + g(x-y) = F(y,1) + k(x-y) - K$ which is maximal at $y = \beta_1$ (Proposition 8.2). Thus, $Mv(x,1) = F(\beta_1,1) + k(x-\beta_1) - K = v(\beta_1,1) + k(x-\beta_1) - K$. For $x \in [b_1, \infty)$,

$$v(y,1) + g(x-y) = \begin{cases} F(y,1) + k(x-y) - K & \text{if } y \in [0, b_1), \\ F(\beta_1,1) + k(x-\beta_1) - 2K & \text{if } y \in [b_1, x]. \end{cases}$$

We see then that $v(y,1) + g(x-y)$ attains a global maximum at $y = \beta_1$ (Proposition 8.2). Thus, $Mv(x,1) = F(\beta_1,1) + k(x-\beta_1) - K = v(x,1)$.

When $i = 2$, we have to consider two cases: $\beta_2 \in (0, b_1)$ and $\beta_2 \in [b_1, b_2)$. We will only prove the Lemma for the case when $\beta_2 \in (0, b_1)$; the proof for $\beta_2 \in [b_1, b_2)$ is similar. Then, for $x \in [0, \beta_2)$, $v(y,2) + g(x-y) = F(y,2) + k(x-y) - K$ which is increasing in $[0,x]$ (Proposition 8.2). Thus, $Mv(x,2) = F(x,2) - K = v(x,2) - K$. For $x \in [\beta_2, b_1)$, $v(y,2) + g(x-y) = F(y,2) + k(x-y) - K$ which is maximal at $y = \beta_2$ (Proposition 8.2). Thus, $Mv(x,2) = F(\beta_2,2) + k(x - \beta_2) - K = v(\beta_2,2) + k(x-\beta_2) - K$. For $x \in [b_1, b_2)$,

$$v(y,2) + g(x-y) = \begin{cases} F(y,2) + k(x-y) - K & \text{if } y \in [0, b_1), \\ G(y) + k(x-y) - K & \text{if } y \in [b_1, x]. \end{cases}$$

We see that $v(y,2)+g(x-y)$ is maximal at $y=\beta_2$ (Proposition 8.2). Thus, $Mv(x,2)=F(\beta_2,2)+k(x-\beta_2)-K=v(\beta_2,2)+k(x-\beta_2)-K$. For $x\in[b_2,\infty)$,

$$v(y,2)+g(x-y) = \begin{cases} F(y,2)+k(x-y)-K & \text{if } y\in[0,b_1), \\ G(y)+k(x-y)-K & \text{if } y\in[b_1,b_2), \\ F(\beta_2,2)+k(x-\beta_2)-2K & \text{if } y\in[b_2,x]. \end{cases}$$

Then, $v(y,2)+g(x-y)$ is maximal at $y=\beta_2$ (Proposition 8.2). Thus, $Mv(x,2)=F(\beta_2,2)+k(x-\beta_2)-K=v(x,2)$. □

The following Theorem proves rigorously that the function $v(\cdot,i)$, $i=1,2$, constructed in the previous section is the value function of Problem 8.1.

Theorem 8.2. Let b_i, β_i, $i=1,2$, A_j, $j=1,2,3,4$, and $\bar{A}_j$, $j=1,2$, be the solution of the system of equations (8.34) and suppose that $b_i\in[0,\infty)$ and $\beta_i\in(0,b_i)$ for both $i=1,2$, and that $b_1<b_2$. Let B_j, $j=1,2,3,4$, be defined by (8.27). Suppose without lost of generality that $b_1<b_2$. Define the functions $F(\cdot,1)$ and $F(\cdot,2)$ by (8.25) and (8.26), respectively; and define the function G by (8.30). Define also the value $H(\beta_2)$ by (8.31). Assume that the inequalities (8.35) – (8.36) are satisfied. Then, the function $v(\cdot,i)$, $i=1,2$, given by (8.38) and (8.39), is the value function $V(\cdot,i)$, $i=1,2$, of Problem 8.1. Furthermore, the optimal strategy is given by (8.19) – (8.20).

Proof. We need to prove that the function $v(\cdot,i)$, $i=1,2$, defined above satisfies the conditions of Theorem 8.1. It is easy to see that $v(\cdot,1)\in C^1([0,\infty))\cap C^2([0,\infty)-\{b_1\})$ and $v(\cdot,2)\in C^1([0,\infty))\cap C^2([0,\infty)-\{b_1,b_2\})$, by definition. Moreover, for each $i=1,2$, $v(\cdot,i)$ has the form (8.13) in $[b_i,\infty)$, where $a_1=F(\beta_1,1)$, $a_2=H(\beta_2)$ and $d_i=\beta_i\in(0,b_i)$. In addition, the system (8.34) guarantees that $v(0,1)=v(0,2)=0$.

In the following steps we will prove that $v(\cdot,i)$, $i=1,2$, satisfies the QVI of Problem 8.1.

For $x \in [0, b_1)$, $v(\cdot, i)$, $i = 1, 2$, is solution of the system of differential equations (8.24) and, hence, (8.6) and (8.8) are satisfied for $i = 1, 2$. For $x \in [b_1, b_2)$, (8.6) holds for $i = 1$ because of properties given in Proposition 8.3. Indeed, for $x \in [b_1, b_2)$,

$$\begin{aligned} &L_1 v(x,1) - \lambda_1(v(x,1) - v(x,2)) \\ &\quad = \mu_1 k - (\delta + \lambda_1)(F(\beta_1, 1) + k(x - \beta_1) - K) + \lambda_1 G(x) \\ &\quad = -(\delta + \lambda_1)(k(x - \beta_1) - K) + \lambda_1 G(x) - (\delta + \lambda_1)F(\beta_1, 1) + \mu_1 k \le 0. \end{aligned}$$

Moreover, both (8.6) and (8.8) hold for $i = 2$ because $v(\cdot, 2)$ is the solution of the differential equation (8.29). Finally, for $x \in [b_2, \infty)$, Proposition 8.3 guarantees that (8.6) is satisfied for $i = 1, 2$. Indeed, for $x \in [b_2, \infty)$,

$$\begin{aligned} &L_i v(x,i) - \lambda_i(v(x,i) - v(x, 3-i)) \\ &\quad = \mu_i k - (\delta + \lambda_i)(H(\beta_i) + k(x - \beta_i) - K) + \lambda_i(H(\beta_{3-i}) + k(x - \beta_{3-i}) - K) \\ &\quad = -(\delta + \lambda_i)(k(x - \beta_i) - K) + \lambda_i(k(x - \beta_{3-i}) - K) - (\delta + \lambda_i)H(\beta_i) \\ &\qquad + \lambda_i H(\beta_{3-i}) + \mu_i k, \end{aligned}$$

for $i = 1, 2$, where we are denoting $H(\beta_1) := F(\beta_1, 1)$. Thus, by Proposition 8.3, for every $x \in [b_2, \infty)$, $L_i v(x,i) - \lambda_i(v(x,i) - v(x, 3-i)) \le 0$, $i = 1, 2$.

Lemma 8.1 tells us that, for $i = 1, 2$,

$$Mv(x,i) - v(x,i) = \begin{cases} -K & \text{if } x \in [0, \beta_i), \\ -v(x,i) + v(\beta_i, i) + k(x - \beta_i) - K & \text{if } x \in [\beta_i, b_i), \\ 0 & \text{if } x \in [b_i, \infty). \end{cases}$$

It is obvious that $Mv(x,i) - v(x,i) \le 0$ for $x \in [0, \beta_i)$. Moreover, $Mv(x,i) - v(x,i)$ is increasing for $x \in [\beta_i, b_i)$, $i = 1, 2$, because of Proposition 8.2. Hence,

$Mv(x,i)-v(x,i) < Mv(b_i,i)-v(b_i,i) = 0$ for $x \in [\beta_i, b_i)$, $i = 1, 2$. Then, (8.7) is satisfied for $x \in [0, \infty)$ and $i = 1, 2$. Moreover, (8.8) holds for $x \in [b_1, b_2)$ and $i = 1$, and it holds for $x \in [b_2, \infty)$ and $i = 1, 2$. Thus, the function $v(\cdot, i)$, $i = 1, 2$, satisfies the QVI of Problem 8.1.

We still need to show that $(\hat{T}, \hat{\xi})$ is an admissible control and that it is the QVI-control associated with v. First of all, we note that the trajectory generated by $(\hat{T}, \hat{\xi})$ behaves like a Brownian motion with drift in each random interval (τ_{n-1}, τ_n), $n \geq 1$. Moreover, $\mathbb{P}\left\{\text{for all } t \in (0, \infty) : X_t \in [0, b_{\epsilon(t)}]\right\} = 1$. This means that, for a fixed $t \in [0, \infty)$, the trajectory generated by $(\hat{T}, \hat{\xi})$ will reach b_i when the state of the Markov chain is i only a countable number of times before $t \wedge \Theta$. In addition, this trajectory will reach 0 at most once before $t \wedge \Theta$. Hence, condition (8.4) is satisfied and $(\hat{T}, \hat{\xi})$ is admissible.

Finally, we will show that $(\hat{T}, \hat{\xi})$ is indeed the QVI-control associated with v. We note from Lemma 8.1 that $Mv(X_t, \epsilon_t) = v(X_t, \epsilon_t)$ if and only if $X_t \in [b_{\epsilon(t)}, \infty)$. Hence, $\tau_n^v = \inf\{t \geq \tau_{n-1} : X_t \in [b_{\epsilon(t)}, \infty)\}$, $n \geq 1$. Thus, $\tau_n^v = \hat{\tau}_n$ for every $n \geq 1$. Moreover, we note that for $x \in [b_i, \infty)$ and $y \in [0, x]$, the function (in y) $v(y, i) + g(x - y)$ always attains its maximum at $y = \beta_i$ (see Lemma 8.1). Hence, $v(x - y, i) + g(y)$ is maximal at $y = x - \beta_i$ for $x \in [b_i, \infty)$ and $y \in [0, x]$. Thus, for every $n \geq 1$,

$$\begin{aligned} \xi_n^v &= \arg\sup\left\{v(X_{\hat{\tau}_n} - \eta, \epsilon_{\hat{\tau}_n}) + g(\eta), \eta \in [0, X_{\hat{\tau}_n}],\, X_{\hat{\tau}_n} \in [b_{\epsilon(\hat{\tau}_n)}, \infty)\right\} \\ &= (X_{\hat{\tau}_n} - \beta_{\epsilon(\hat{\tau}_n)})\, I_{\{X_{\hat{\tau}_n} \geq b_{\epsilon(\hat{\tau}_n)}\}} \quad = \quad \hat{\xi}_n. \end{aligned}$$

□

8.5 Numerical examples and analysis of results.

In the previous sections we solved Problem 8.1 analytically by constructing the function $v(\cdot, i)$, $i = 1, 2$, that satisfies the system of equations (8.34). The solution of (8.34) is not explicit since the functions involved are not linear, but it can easily be found using the Newton's method. In this section we present some numerical examples of the solution.

We consider that the economy has two regimes: economic recession ($i = 1$) and economic growth ($i = 2$). We consider as well that, during the regime of recession, the cash reservoir drift coefficient is $\mu_1 = 0.08$ and the volatility parameter is $\sigma_1 = 0.70$; and during the regime of growth, the drift coefficient is $\mu_2 = 0.12$ and the volatility is $\sigma_2 = 0.50$. Moreover, the persistence parameters for the regimes are $\lambda_1 = 0.06$ and $\lambda_2 = 0.03$, the discount rate is $\delta = 0.12$, the dividend tax rate is $1 - k = 0.2$, and the fixed set-up cost is $K = 0.005$. The solution of (8.34) for these model parameters gives us the thresholds $b_1 = 0.9590$ and $b_2 = 1.0720$ (the level of the cash reservoir at which it is optimal to pay dividends), and the thresholds $\beta_1 = 0.4203$ and $\beta_2 = 0.6407$ (the level to which it is optimal to bring down the cash reservoir when paying dividends). We also obtain the coefficients A_j, B_j, $j = 1, 2, 3, 4$, and $\bar{A}_j$, $j = 1, 2$, for the value function. Furthermore, the inequalities (8.35)$-$(8.36) are satisfied. Figure 8.1 and Figure 8.2 show the value function and the derivative of the value function for this example.

We can compare the results above to the results—modified to include dividend taxes—presented by Jeanblanc-Piqué and Shiryaev (1995) for the one-regime case. If the economy is always in economic recession, the model by Jeanblanc-Piqué and Shiryaev gives us the thresholds[8] $b^1_{JPS} = 0.9162$ and

[8] We will use the subindex JPS from now on to denote one-regime models (and their results).

$\beta^1_{JPS} = 0.3786$. On the other hand, if the economy is always in economic growth, we obtain $b^2_{JPS} = 1.0881$ and $\beta^2_{JPS} = 0.6577$. As expected, we see that $b^1_{JPS} < b_1 < b_2 < b^2_{JPS}$ and $\beta^1_{JPS} < \beta_1 < \beta_2 < \beta^2_{JPS}$. These relationships among the thresholds b and the thresholds β are expected because a model with only economic growth represents better economic conditions than a regime switching model with both growth and recession, and such regime switching models represents better economic conditions than a model with only economic recession. Another expected result is that, in a economy of only growth, shareholders should receive on average a higher amount of total discounted dividend payments than what they receive in a economy of with both growth and recession periods. Besides, shareholders in a economy of only recession should receive on average a lower amount of total discounted dividends than what they receive in a economy with regimes. Figure 8.3 confirms our expectations by showing that $V^1_{JPS}(x) \leq V(x,1) \leq V(x,2) \leq V^2_{JPS}(x)$ for every initial level of the cash reservoir $x \in [0, \infty)$.

Further comparison of results can be done if we vary the model parameters. Table 8.1 and Table 8.2 present the thresholds b_i and β_i, $i = 1, 2$, for fixed $\lambda_1 = 0.06$, $\mu_1 = 0.08$ and $\sigma_1 = 0.70$, and different values of λ_2, μ_2, σ_2, δ, k and K. They also present the size of the optimal amount of dividends to be paid in each regime ($b_1 - \beta_1$ and $b_2 - \beta_2$). The results presented in Table 8.1 and Table 8.2 show how the different parameters of the model affect the optimal thresholds and the size of the optimal dividend.

The persistence parameter λ_1 positively affects the thresholds b_i and β_i and the size of the optimal amount of dividends $b_i - \beta_i$ (although the effect in $b_2 - \beta_2$ is barely noticeable): b_i, β_i, $i = 1, 2$, and $b_1 - \beta_1$ increase as λ_1 increases. The effect of the persistence parameter λ_2 on the thresholds b_i and β_i is the opposite: they decrease as λ_2 increases. On the other hand, the optimal dividend size $b_2 - \beta_2$ increases as λ_2 increases. The effect of

λ_2 on $b_1 - \beta_1$ is not significant. These relationships imply that, for shorter economic recession periods, it is optimal to pay larger amount of dividends during recession. Moreover, it is optimal to pay dividends when the level of the cash reservoir is higher, despite the economic regime. In addition, for shorter economic growth periods, it is optimal to pay larger amount of dividends during economic growth. Also, it is optimal to pay dividends when the level of the cash reservoir is lower, despite of the economic regime in this case too.

The drift parameter μ_2 (and hence the difference between the drift coefficients $\mu_2 - \mu_1$) positively affects b_1, b_2, β_1 and β_2, but their effect on $b_i - \beta_i$ depends on the regime. In fact, the size of the optimal amount of dividends to be paid in a regime of economic recession increases as μ_2 increases, that is, as the expected growth rate of the cash reservoir in a regime of economic growth increases. On the other hand, the size of the optimal amount of dividends to be paid in a regime of economic growth decreases as the expected growth rate μ_2 increases.

The effect of σ_2 on b_i and β_i also depends on the regime. The greater is the volatility of the cash reservoir in a period of economic growth, the higher is the level of the cash reservoir needed to optimally pay dividends during recession, and the lower is the level of the cash reservoir needed to optimally pay dividends during economic growth. On the other hand, the effect of σ_2 on $b_i - \beta_i$ is the same for both economic regimes: a decrease in the difference between volatilities (or, equivalently, an increase in σ_2) implies an increase in the size of the dividends $b_i - \beta_i$, that is, it is optimal to pay larger amount of dividends in both economic regimes.

The effect of δ is the same for b_i, β_i and $b_i - \beta_i$ (independently of the regime of the economy): higher δ implies lower b_1, b_2, β_1, β_2, $b_1 - \beta_1$ and $b_2 - \beta_2$.

Cadenillas, Sarkar and Zapatero (2007) found that, in a one-regime mean-reverting cash-reservoir model, the optimal size of the dividend payment is directly proportional to the fixed disutility K and to the dividend tax rate: the size of the payout $b - \beta$ decreases as K decreases and as the tax rate $1 - k$ decreases. In fact, the relationship between the optimal dividend size and the tax rate is a surprising property and one of the main results in Cadenillas, Sarkar and Zapatero (2007). For our regime switching dividend model, we obtain similar results. One of them is that the size of the dividend payout increases in both regimes with the fixed cost K.

We obtain as well that in a regime switching model, the size of the dividend payout decreases in both regimes as the tax rate decreases. Table 8.3 shows that in fact this relationship is also a property of the one-regime model presented by Jeanblanc-Piqué and Shiryaev (1995). This result is surprising, as pointed out by Cadenillas, Sarkar and Zapatero (2007), and seems to contradict the "old view" (or "traditional view") literature in the effects of taxation on dividend payments (see Auerbach (2003) for a survey on "old view" and "new view" literature). The old view on dividend taxation predicts that increases in dividend taxes decrease dividend payout rate. This prediction is supported by empirical evidence by Chetty and Saez (2005), Poterba (2004), and Poterba and Summers (1985), among others. Our result does not contradict though the empirical evidence. Even though in our model the optimal dividend payout increases with an increase in the tax rate, a decrease in the dividend payment frequency might still lead to an overall decrease of the dividends paid out (or of the annualized dividend amount, as calculated by Cadenillas, Sarkar and Zapatero (2007)).

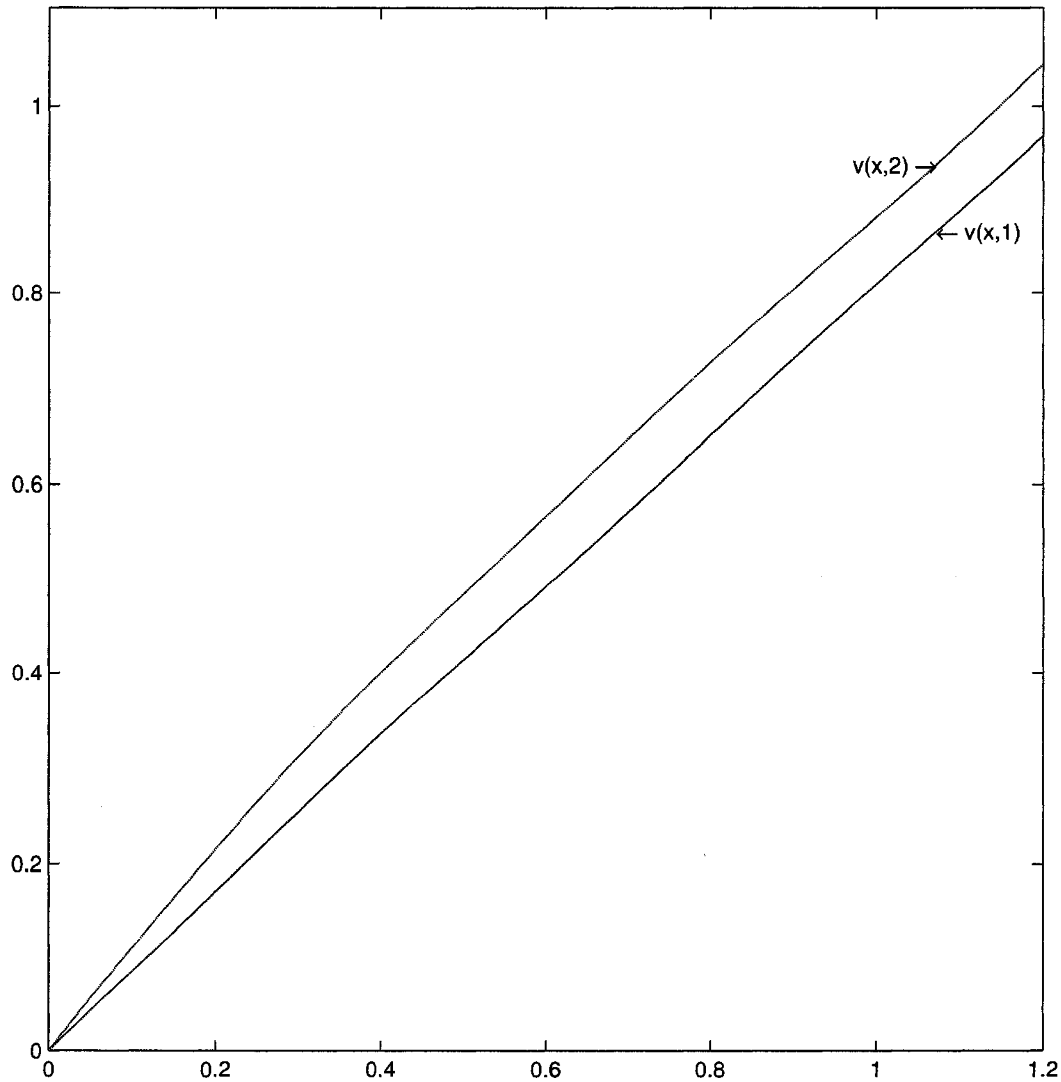

Figure 8.1: Example of value function $V(\cdot, i)$, $i = 1, 2$, for model parameters $\lambda_1 = 0.06$, $\lambda_2 = 0.03$, $\mu_1 = 0.08$, $\sigma_1 = 0.70$, $\mu_2 = 0.12$, $\sigma_2 = 0.50$, $\delta = 0.12$, $k = 0.8$ and $K = 0.005$.

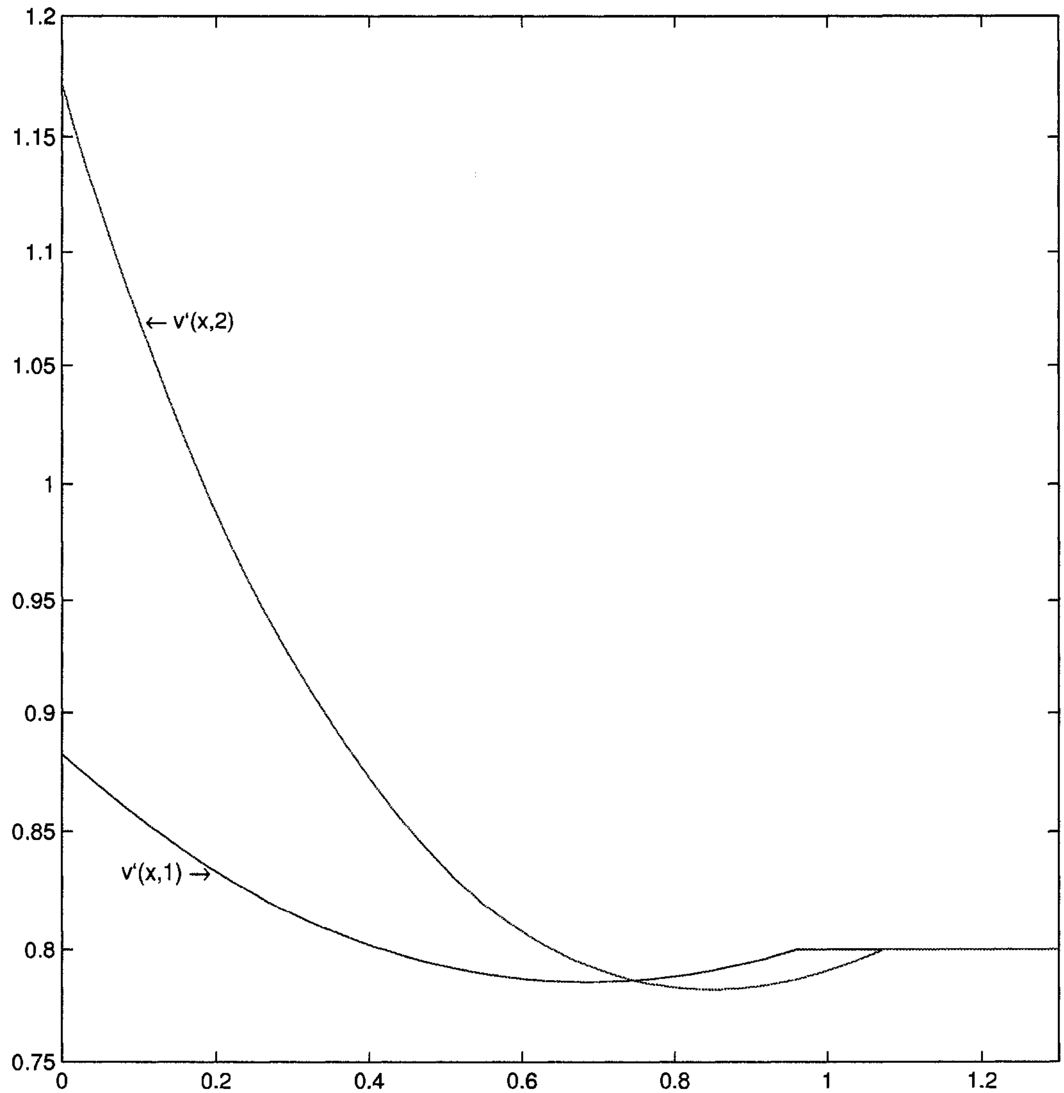

Figure 8.2: Example of the derivative of the value function $V(\cdot, i)$, $i = 1, 2$, for model parameters $\lambda_1 = 0.06$, $\lambda_2 = 0.03$, $\mu_1 = 0.08$, $\sigma_1 = 0.70$, $\mu_2 = 0.12$, $\sigma_2 = 0.50$, $\delta = 0.12$, $k = 0.8$ and $K = 0.005$.

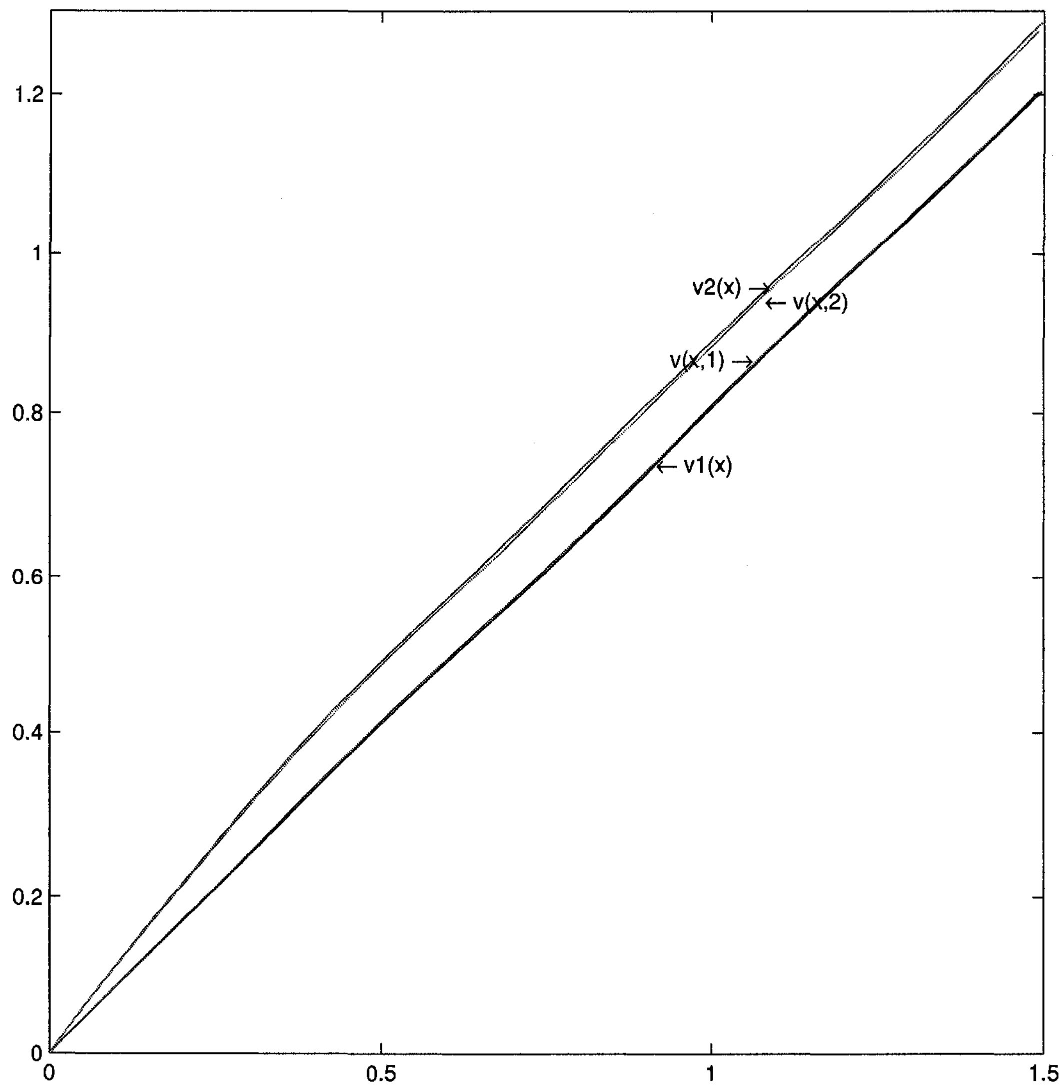

Figure 8.3: Comparison of the value functions for one-regime economies of economic recession and of economic growth, and for a regime switching economy. From the top to the bottom: $V^2_{JPS}(\cdot)$, $V(\cdot,2)$, $V(\cdot,1)$ and $V^1_{JPS}(\cdot)$.

λ_1	λ_2	μ_2	σ_2	δ	k	K	β_1	b_1	$b_1-\beta_1$
0.06	0.03	0.12	0.50	0.12	0.80	0.005	0.4203	0.9590	0.5387
0.02	0.03	0.12	0.50	0.12	0.80	0.005	0.3928	0.9309	0.5381
0.04	0.03	0.12	0.50	0.12	0.80	0.005	0.4067	0.9452	0.5385
0.08	0.03	0.12	0.50	0.12	0.80	0.005	0.4335	0.9724	0.5389
0.10	0.03	0.12	0.50	0.12	0.80	0.005	0.4463	0.9852	0.5389
0.06	0.01	0.12	0.50	0.12	0.80	0.005	0.4222	0.9610	0.5388
0.06	0.05	0.12	0.50	0.12	0.80	0.005	0.4186	0.9572	0.5386
0.06	0.07	0.12	0.50	0.12	0.80	0.005	0.4170	0.9555	0.5385
0.06	0.09	0.12	0.50	0.12	0.80	0.005	0.4156	0.9540	0.5384
0.06	0.03	0.10	0.50	0.12	0.80	0.005	0.4008	0.9385	0.5377
0.06	0.03	0.11	0.50	0.12	0.80	0.005	0.4099	0.9481	0.5382
0.06	0.03	0.13	0.50	0.12	0.80	0.005	0.4318	0.9712	0.5394
0.06	0.03	0.14	0.50	0.12	0.80	0.005	0.4444	0.9845	0.5401
0.06	0.03	0.12	0.40	0.12	0.80	0.005	0.4412	0.9792	0.5380
0.06	0.03	0.12	0.60	0.12	0.80	0.005	0.4069	0.9459	0.5390
0.06	0.03	0.12	0.70	0.12	0.80	0.005	0.3981	0.9372	0.5391
0.06	0.03	0.12	0.50	0.10	0.80	0.005	0.5523	1.1245	0.5722
0.06	0.03	0.12	0.50	0.11	0.80	0.005	0.4796	1.0340	0.5544
0.06	0.03	0.12	0.50	0.13	0.80	0.005	0.3712	0.8958	0.5246
0.06	0.03	0.12	0.50	0.14	0.80	0.005	0.3299	0.8418	0.5119
0.06	0.03	0.12	0.50	0.12	0.60	0.005	0.3932	0.9867	0.5936
0.06	0.03	0.12	0.50	0.12	0.70	0.005	0.4080	0.9715	0.5635
0.06	0.03	0.12	0.50	0.12	0.90	0.005	0.4306	0.9484	0.5178
0.06	0.03	0.12	0.50	0.12	1.00	0.005	0.4396	0.9393	0.4998
0.06	0.03	0.12	0.50	0.12	0.80	0.003	0.4624	0.9161	0.4537
0.06	0.03	0.12	0.50	0.12	0.80	0.004	0.4396	0.9393	0.4998
0.06	0.03	0.12	0.50	0.12	0.80	0.006	0.4034	0.9762	0.5729
0.06	0.03	0.12	0.50	0.12	0.80	0.007	0.3883	0.9917	0.6034

Table 8.1: Results for model parameters $\lambda_1 = 0.06$, $\mu_1 = 0.08$ and $\sigma_1 = 0.70$, and different values of λ_2, μ_2, σ_2, δ, k and K.

λ_1	λ_2	μ_2	σ_2	δ	k	K	β_2	b_2	$b_2-\beta_2$
0.06	0.03	0.12	0.50	0.12	0.80	0.005	0.6407	1.0720	0.4313
0.02	0.03	0.12	0.50	0.12	0.80	0.005	0.6404	1.0716	0.4312
0.04	0.03	0.12	0.50	0.12	0.80	0.005	0.6405	1.0718	0.4313
0.08	0.03	0.12	0.50	0.12	0.80	0.005	0.6409	1.0722	0.4313
0.10	0.03	0.12	0.50	0.12	0.80	0.005	0.6410	1.0724	0.4314
0.06	0.01	0.12	0.50	0.12	0.80	0.005	0.6518	1.0825	0.4307
0.06	0.05	0.12	0.50	0.12	0.80	0.005	0.6304	1.0623	0.4319
0.06	0.07	0.12	0.50	0.12	0.80	0.005	0.6207	1.0533	0.4326
0.06	0.09	0.12	0.50	0.12	0.80	0.005	0.6116	1.0450	0.4334
0.06	0.03	0.10	0.50	0.12	0.80	0.005	0.5320	0.9635	0.4315
0.06	0.03	0.11	0.50	0.12	0.80	0.005	0.5883	1.0197	0.4314
0.06	0.03	0.13	0.50	0.12	0.80	0.005	0.6893	1.1206	0.4313
0.06	0.03	0.14	0.50	0.12	0.80	0.005	0.7342	1.1655	0.4313
0.06	0.03	0.12	0.40	0.12	0.80	0.005	0.6081	0.9809	0.3728
0.06	0.03	0.12	0.60	0.12	0.80	0.005	0.6527	1.1390	0.4863
0.06	0.03	0.12	0.70	0.12	0.80	0.005	0.6529	1.1914	0.5385
0.06	0.03	0.12	0.50	0.10	0.80	0.005	0.7663	1.2247	0.4584
0.06	0.03	0.12	0.50	0.11	0.80	0.005	0.6987	1.1427	0.4440
0.06	0.03	0.12	0.50	0.13	0.80	0.005	0.5904	1.0104	0.4199
0.06	0.03	0.12	0.50	0.14	0.80	0.005	0.5465	0.9562	0.4097
0.06	0.03	0.12	0.50	0.12	0.60	0.005	0.6201	1.0958	0.4757
0.06	0.03	0.12	0.50	0.12	0.70	0.005	0.6314	1.0827	0.4513
0.06	0.03	0.12	0.50	0.12	0.90	0.005	0.6486	1.0630	0.4144
0.06	0.03	0.12	0.50	0.12	1.00	0.005	0.6554	1.0552	0.3999
0.06	0.03	0.12	0.50	0.12	0.80	0.003	0.6727	1.0355	0.3628
0.06	0.03	0.12	0.50	0.12	0.80	0.004	0.6554	1.0552	0.3999
0.06	0.03	0.12	0.50	0.12	0.80	0.006	0.6279	1.0868	0.4589
0.06	0.03	0.12	0.50	0.12	0.80	0.007	0.6164	1.1001	0.4837

Table 8.2: Results for model parameters $\lambda_1 = 0.06$, $\mu_1 = 0.08$ and $\sigma_1 = 0.70$, and different values of λ_2, μ_2, σ_2, δ, k and K (contd.)

$1-k$	0.00	0.10	0.20	0.30	0.40
Regime switching model					
$b_1-\beta_1$	0.4998	0.5178	0.5387	0.5635	0.5936
$b_2-\beta_2$	0.3999	0.4144	0.4313	0.4513	0.4757
One-regime model for economic recession					
β^1_{JPS}	0.3975	0.3888	0.3786	0.3666	0.3521
b^1_{JPS}	0.8962	0.9055	0.9162	0.9289	0.9444
$b^1_{JPS}-\beta^1_{JPS}$	0.4987	0.5167	0.5376	0.5623	0.5923
One-regime model for economic growth					
β^2_{JPS}	0.6723	0.6655	0.6577	0.6484	0.6373
b^2_{JPS}	1.0714	1.0791	1.0881	1.0987	1.1117
$b^2_{JPS}-\beta^2_{JPS}$	0.3991	0.4136	0.4304	0.4503	0.4744

Table 8.3: Optimal size of dividend payments for different dividend taxes, for the regime switching model and one-regime models. The model parameters are $\lambda_1 = 0.06$, $\lambda_2 = 0.03$, $\mu_1 = 0.08$, $\mu_2 = 0.12$, $\sigma_1 = 0.70$, $\sigma_2 = 0.50$, $\delta = 0.12$ and $K = 0.005$.

Chapter 9

Conclusions

In this thesis, we present the theory of stochastic control with regime switching for random time horizon problems in which the regime is modeled by an observable continuous-time finite-state Markov chain. We introduce the novel idea of utility functions (or cost functions) that depend on the regime. Furthermore, we formalize the techniques of stochastic control with regime switching through rigourously proved verification theorems.

We present the first Hamilton-Jacobi-Bellman equations for problems of classical stochastic control with regime switching in random time horizon and with regime-dependent utility functions. In addition, the first versions of variational inequalities for problems of singular stochastic control with regime switching also in a random time horizon are presented. Furthermore, we develop the theory of stochastic impulse control with regime switching and, as a result, give the first verification theorem for stochastic impulse control with regime switching.

Furthermore, we apply the theory of stochastic control with regime switching to solve important problems in Financial Economics. We solve explicitly the consumption-investment problem in a financial market with various regimes. The solution includes the explicit solutions of systems of nonlinear ordinary differential equations. We also give an economic analysis of our solutions by combining our mathematical results with the results presented in the empirical finance literature. We conclude that every investor should optimally invest a higher proportion of his/her wealth in the risky asset during a bull market (a market in economic growth) than during a bear market (a market in economic recession). The level of risk aversion of the investor does not affect our result. Conversely, the qualitative behaviour of the optimal consumption to wealth ratio depends on both the regime of the market and the investor's level of risk aversion. Our analysis shows that: i) investors with high risk tolerance have a greater consumption to wealth ratio in a bear market than in a bull market; ii) investors with moderate risk tolerance present an optimal consumption to wealth ratio indifferent to the regimes of the market; and iii) investors with low risk tolerance prefer to consume proportionally more in a bull market than in a bear market.

We also apply the techniques for classical stochastic control with regime switching to solve analytically a dividend payment problem for a company whose cash reservoir is affected by macroeconomic conditions which generate periods of economic growth and periods of economic recession. In this problem, the objective of the company's management is to maximize the expected discounted cumulative amount of dividends paid out to the shareholders, under the assumption of using bounded dividend rates. Our result shows that the optimal dividend policy depends strongly on the regime of the economy. Indeed, to pay optimal dividends, the company needs to consider first which regime the economy is in, and then verify that the level of cash reservoir is larger than or equal to a certain threshold that is regime-dependent.

Afterwards, we formulate a similar dividend payment problem, but under the assumption that the dividend rates are no longer bounded. The solution requires the use of the generalized techniques for singular stochastic control with regime switching. As with the bounded assumption, our results show that the optimal dividend policy also varies strongly with the macroeconomic conditions. However, under this formulation the optimal dividend payments are not given at a constant rate, but rather consist of the excess of cash reservoir that is distributed at the time the level of the cash reservoir is larger than or equal to a regime-dependent threshold. In order to follow the optimal dividend policy, the company again needs to consider if the economy is in a growth or recession period before opting for the payment of dividends.

We present a final application of the theory of stochastic control with regime switching by studying a third type of dividend payment problem. A company whose cash reservoir is affected by business cycles (generated by macroeconomic conditions) is considered and the analytical solution of the optimal dividend problem with dividend taxes and a fixed cost associated with each dividend payment is given. The solution uses the techniques of stochastic impulse control with regime switching that are also developed in this thesis. As far as we know, this is the first analytical solution to a problem of stochastic impulse control with regime switching. Furthermore, to the best of our knowledge, this is the first problem of that type in the Mathematical Finance literature. Our solution shows that the optimal size and times of the dividend payments vary according to the business cycles. An analysis of our results also shows how the model parameters affect the optimal size of the dividend payments, and that the size of the optimal dividend payout increases with the dividend taxes in every regime of the economy.

We find a similar interesting result in the three problems of optimal dividend policy that are presented in this thesis. Particularly, optimal dividend payments can occur because of an increase in the level of the cash reservoir (as expected), but optimal dividend payments can also occur due only to changes in the regime of the economy. This last situation happens when the level of the cash reservoir falls in between the two regime-dependent thresholds (that are obtained in the solution of the problems) and the regime of the economy changes from economic growth to economic recession.

As mentioned in the Introduction, the three problems of optimal dividend policy that are studied in this thesis are the only mathematical models in the literature that allow the cash reservoir of the company (and hence their dividend policies) to depend on the regime of the economy. The fact that we propose the use of regime switching models for modeling the dynamics of a company's cash reservoir process is just one of the many significant contributions of this thesis.

Bibliography

[1] Allen, F., Michaely, R. Payout policy, in *Handbook of the Economics of Finance*, volume 1, G.Constantinides, M.Harris and R.Stulz, eds. Elsevier, Amsterdam (2003).

[2] Asmussen, S., Højgaard, B., Taksar, M. Optimal risk control and dividend distribution policies. Example of excess-of loss reinsurance for an insurance corporation. *Finance and Stochastics*, 4, 299-324 (2000).

[3] Asmussen, S., Taksar, M. Controlled diffusion models for optimal dividend pay-out. *Insurance: Mathematics and Economics*, 20, 1-15 (1997).

[4] Auerbach, A. Taxation and corporate financial policy, in *Handbook of Public Economics*, volume 3, A.Auerbach and M.Feldstein, eds. Elsevier, Amsterdam (2003).

[5] Björk, T. Finite dimensional optimal filters for a class of Itô-processes with jumping parameters. *Stochastics*, 4, 167-183 (1980).

[6] Boguslavskaya, E. On optimization of dividend flow for a company in the presence of liquidation value. *Working paper* (2003).

[7] Bosch-Domènech, A., Silvestre, J. Does risk aversion or attraction depend on income? An experiment. *Economics Letters*, 65, 265-273 (1999).

[8] Buffington, J., Elliott, R.J. American Options with regime switching. *International Journal of Theoretical and Applied Finance*, 5:5, 497-514 (2002).

[9] Cadenillas, A., Choulli, T., Taksar, M., Zhang, L. Classical and impulse stochastic control for the optimization of the dividend and risk policies of an insurance firm. *Mathematical Finance*, 16:1, 181-202 (2006).

[10] Cadenillas, A., Haussmann, U. The stochastic maximum principle for a singular control problem. *Stochastics and Stochastics Reports*, 49, 211-237 (1994).

[11] Cadenillas, A., Sarkar, S., Zapatero, F. Optimal dividend policy with mean-reverting cash reservoir. *Mathematical Finance*, 17:1, 81-109 (2007).

[12] Cadenillas, A., Zapatero, F. Optimal central bank intervention in the foreign exchange market. *Journal of Economic Theory*, 87, 218-242 (1999).

[13] Cadenillas, A., Zapatero, F. Classical and impulse stochastic control of the exchange rate using interest rates and reserves. *Mathematical Finance*, 10:2, 141-156 (2000).

[14] Chen, N. Financial investment opportunities and the macroeconomy. *Journal of Finance*, 46, 1467-1484 (1991).

[15] Chen, N., Roll, R., Ross, S. Economic forces and the stock market. *Journal of Business*, 59:3, 383-403 (1986).

[16] Chetty, R., Saez, E. Dividend taxes and corporate behavior: evidence from the 2003 dividend tax cut. *Quaterly Journal of Economics*, 120:3, 791-833 (2005).

[17] Choulli, T., Taksar, M., Zhou, X.Y. Excess-of-loss reinsurance for a company with debt liability and constraints on risk reduction. *Quantitative Finance*, 1, 573-596 (2001).

[18] Choulli, T., Taksar, M., Zhou, X.Y. A diffusion model for optimal dividend distribution for a company with constraints on risk control. *SIAM Journal of Control and Optimization*, 41:6, 1946-1979 (2003).

[19] Choulli, T., Taksar, M., Zhou, X.Y. Interplay between dividend rate and business constraints for a financial corporation. *Annals of Applied Probability*, 14, 1810-1837 (2004).

[20] Danthine, J.P., Donaldson, J.B., Giannikos, C., Guirguis, H. On the consequences of state dependent preferences for the pricing of financial assets. *Financial Research Letters*, 1, 143-153 (2004).

[21] Décamps, J.P., Villeneuve, S. Optimal dividend policy and growth option. *Finance and Stochastics*, 11:1, 3-27 (2007).

[22] Di Graziano, G., Rogers, L.C.G. Barrier Option Pricing for Assets with Markov-Modulated Dividends. *Journal of Computational Finance*, 9:4 (2006).

[23] Driffill, J., Raybaudi, M., Sola, M. Investment under uncertainty with stochastically switching profit streams: entry and exit over the business cycle. *Studies in Nonlinear Dynamics and Econometrics*, 7:1, 1131-1131 (2003).

[24] Driffill, J., Sola, M. Irreversible investment and changes in regime. *Working paper*, University of London (2001).

[25] Elliott, R., Swishchuk, A. Pricing options and variance swaps in Markov-modulated Brownian and fractional Brownian markets. *Working paper* (2004).

[26] Fama, E. Stock returns, real activity, inflation and money. *American Economic Review*, 71:4, 545-565 (1981).

[27] Fama, E. Stock returns, expected returns, and real activity. *Journal of Finance*, 45, 1089-1108 (1990).

[28] Fama, E., French, K. Business Conditions and Expected Returns on Stock and Bonds. *Journal of Financial Economics*, 25, 23-49 (1989).

[29] Fleming, W., Rishel, R. *Deterministic and Stochastic Optimal Control.* Springer-Verlag, New York (1975).

[30] French, K., Schwert G. and Stambaugh, R. Expected Stock Returns and Volatility. *Journal of Financial Economics*, 19, 3-29 (1987).

[31] Gerber, H.U., Shiu, E.S.W. Optimal dividends: analysis with Brownian motion. *North American Actuarial Journal*, 8:1, 1-20 (2004).

[32] Gertler, M., Hubbard, R.G. Corporate financial policy, taxation, and macroeconomic risk. *The RAND Journal of Economics*, 24:2, 286-303 (1993).

[33] Gordon, S., St-Amour, P. Asset returns and state-dependent risk preferences. *Journal of Business & Economic Statistics*, 22, 241-252 (2004).

[34] Guerra, S.M., Tabak, B. Stock returns and volatility. *Working Papers Series*, 54, Central Bank of Brazil, Research Department (2002).

[35] Guo, X. An explicit solution to an optimal stopping problem with regime switching. *Journal of Applied Probability*, 38:2, 464-481 (2001).

[36] Guo, X., Miao, J., Morellec, E. Irreversible investment with regime shifts. *Journal of Economic Theory*, 122:1, 37-59 (2005).

[37] Guo, X., Shepp, L. Some optimal stopping problem with non-trivial boundaries for pricing exotic options. *Journal of Applied Probability*, 38, 1-12 (2001).

[38] Guo, X., Zhang, Q. Closed-form solutions for perpetual American put options with regime switching. *SIAM Journal on Control and Optimization*, 64:6, 2034-2049 (2004).

[39] Guo, X., Zhang, Q. Optimal selling rules in a regime switching model. *IEEE Transactions on Automatic Control*, 50:9, 1450-1455 (2005).

[40] Hackbarth, D., Miao, J., Morellec, E. Capital structure, credit risk, and macroeconomic conditions. *Journal of Financial Economics*, 82,519-550 (2006).

[41] Hamilton, J. A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica*, 57, 357-384 (1989).

[42] Hardy, M.R. A regime-switching model of long-term stock returns. *North American Actuarial Journal*, 5:2, 41-53 (2001).

[43] Ho, S.K., Wu, C. The earnings information content of dividend initiations and omissions. *Journal of Business Finance and Accounting*, 28, 963-1081 (2001).

[44] Højgaard, B., Taksar, M. Controlling risk exposure and dividends payout schemes: insurance company example. *Mathematical Finance*, 9:2, 153-182 (1999).

[45] Højgaard, B., Taksar, M. Optimal risk control for a large corporation in the presence of returns on investments. *Finance and Stochastics*, 5, 527547 (2001).

[46] Honda, T. Optimal portfolio choice for unobservable and regime-switching mean returns. *Journal of Economic Dynamics and Control*, 28, 45-78.

[47] Hu, X., Schiantarelli, F. Investment and capital market imperfections: a switching regression approach using U.S. firm panel data. *The Review of Economics and Statistics*, 80:3, 466-479 (1998).

[48] Huang, R., Kracaw, W. Stock market returns and real activity: A note. *Journal of Finance*, 39, 267-273 (1984).

[49] Hull, J., White A. The pricing of options on assets with stochastic volatilities. *Journal of Finance*, 42, 281-300 (1987).

[50] Jeanblanc-Picqué, M., Shiryaev, A.N. Optimization of the flow of dividends. *Russian Mathematical Surveys*, 50:2, 257-277 (1995).

[51] Jobert, A., Rogers, L.C.G. Option pricing with Markov-modulated dynamics. *SIAM Journal on Control and Optimization*, 44:6, 2063-2078 (2006).

[52] Karatzas, I., Lehoczky, J., Sethi, S., Shreve, S. Explicit solution of a general consumption/investment problem. *Mathematics of Operations Research*, 11:2, 261-294 (1986).

[53] Karatzas, I., Shreve, S. *Methods of Mathematical Finance*. New York: Springer (1998).

[54] Karni, E. Subjective expected utility theory with state-dependent preferences. *Journal of Economic Theory*, 60, 428-438 (1993).

[55] Korn, R. *Optimal Portfolios. Stochastic Models for Optimal Investment and Risk Management in Continuous Time.* Singapore: World Scientific (1997).

[56] Lions, P.L., Sznitman, A.S. Stochastic differential equations with reflecting boundary conditions.*Communications on Pure and Applied Mathematics*, 37, 511-537 (1984).

[57] Ly Vath, V., Pham, H., Villeneuve, S. A mixed singular/switching control problem for a dividend policy with reversible technology investment. *Working paper* (2006).

[58] Maheu, J., McCurdy, T. Identifying bull and bear markets in stock returns. *Journal of Business and Economic Statistics*, 18:1, 100-112 (2000).

[59] Mamon, R.S., Rodrigo, M.R. Explicit solutions to European options in a regime-switching economy. *Operations Research Letters*, 33, 581-586 (2005).

[60] Melino, A., Yang, A.X. State-dependent preferences can explain the equity premium puzzle. *Review of Economic Dynamics*, 6, 806-830 (2003).

[61] Merton, R. Lifetime portfolio selection under uncertainty: the continuous-time case. *Review of Economics and Statistics*, 51, 247-257 (1969).

[62] Miller, M., Modigliani, F. Dividend policy, growth and the valuation of shares. *Journal of Business*, 34, 411433 (1961).

[63] Modigliani, F. Consumer spending and monetary policy: the linkages. *Federal Reserve Bank of Boston Conference Series*, 5, 9-84 (1971).

[64] Nagai, H., Runggaldier, W.J. PDE approach to utility maximization for market models with hidden Markov factors. To appear in *5th Seminar on Stochastic Analysis, Random Fields and Applications*, R.C.Dalang, M.Dozzi, F.Russo, eds. Progress in Probability, Birkhuser Verlag (2005).

[65] Pearce, D., Roley, V. Firm characteristics, unanticipated inflation, and stock returns. *Journal of Finance*, 43:4, 965-981 (1988).

[66] Perez-Quiros, G., Timmermann, A. Firm size and cyclical variations in stock returns. *The Journal of Finance*, 55:3, 1229-1262 (2000).

[67] Poterba, J. Stock market wealth and consumption. *Journal of Economic Perspectives*, 14:2, 99-118 (2000).

[68] Poterba, J. Taxation and corporate payout policy. *The American Economic Review*, 94:2, 171-175 (2004).

[69] Poterba, J., Summers, L. The economic effects of dividend taxation, in *Recent Advances in Corporate Finance*, E.Altman and M.Subrahmanyam, eds. Dow Jones-Irwin, Homewood, IL (1985).

[70] Radner, R., Shepp, L. Risk vs. profit potential: a model for corporate strategy. *Journal of Economic Dynamics and Control*, 20, 1373-1393 (1996).

[71] Sass, J., Haussmann, U.G. Optimizing the terminal wealth under partial information: the drift process as a continuous time Markov chain. *Finance and Stochastics*, 8, 553-577 (2004).

[72] Schaller, H., van Norden, S. Regime switching in stock market returns. *Applied Financial Economics*, 7, 177-191 (1997).

[73] Schwert, G.W. Effects of model specification on tests for unit roots in macroeconomic data. *Journal of Monetary Economics*, 20, 73-103 (1987).

[74] Schwert, G.W. Why does stock market volatility change over time? *The Journal of Finance*, 44:5, 1115-1153 (1989).

[75] Sethi, S.P. *Optimal Consumption and Investment with Bankruptcy.* Boston: Kluwer Academic Publishers (1997).

[76] Sotomayor, L.R., Cadenillas, A. Explicit solutions of consumption-investment problems in financial markets with regime switching. *To appear in Mathematical Finance* (2008).

[77] Sotomayor, L.R., Cadenillas, A. Optimal dividend policy in the presence of business cycles. *Preprint* (2008).

[78] Sotomayor, L.R., Cadenillas, A. Stochastic impulse control with regime switching for the optimal dividend policy when there are business cycles. *Preprint* (2008).

[79] Stockbridge, R. Portfolio optimization in markets having stochastic rates; in *Lecture Notes in Control and Information Sciences: Stochastic Theory and Control*, vol 280, B. Pasik-Duncan, ed. Berlin: Springer-Verlag, 447-458 (2002).

[80] Taksar, M. Optimal risk and dividend distribution control models for an insurance company. *Mathematical Methods of Operations Research*, 51, 1-42 (2000).

[81] Taksar, M., Zhou, X.Y. Optimal risk and dividend control for a company with a debt liability. *Insurance: Mathematics and Economics*, 22, 105-122 (1998).

[82] Vasicek, O. An equilibrium characterisation of the term structure. *Journal of Financial Economics*, 5, 177-188 (1977).

[83] Wei, K.C.J., Wong, K.M. Tests of inflation and industry portfolio stock returns. *Journal Of Economics and Business*, 44, 77-94 (1992).

[84] Yao, D.D., Zhang, Q., Zhou, X.Y. Regime-switching model for European options, in *Stochastic Processes, Optimization, and Control Theory. Applications in Financial Engineering, Queueing Networks, and Manufacturing Systems*. H.Yan, G.Yin and Q.Zhang, eds. Springer (2006).

[85] Yin, G., Zhang Q. *Continuous-time Markov Chains and Applications. A Singular Perturbation Approach*. Springer-Verlag, New York (1998).

[86] Yin, G., Zhou, X.Y. Markowitz's mean-variance portfolio selection with regime switching: from discrete-time models to their continuous-time limits. *IEEE Transactions on Automatic Control*, 49, 349-360 (2004).

[87] Yong, J., Zhou, X.Y. *Stochastic Controls: Hamiltonian Systems and HJB Equations*. New York: Springer (1999).

[88] Zariphopoulou, T. Investment-consumption models with transaction fees and Markov-chain parameters. *SIAM Journal on Control and Optimization*, 30:3, 613-636 (1992).

[89] Zhang, Q. Stock Trading: An optimal selling rule. *SIAM Journal on Control and Optimization*, 40:1, 64-87 (2001).

[90] Zhang, Q., Yin, G. Nearly-optimal asset allocation in hybrid stock investment models. *Journal of Optimization Theory and Applications*, 121:2, 197-222 (2004).

[91] Zhou, X.Y., Yin, G. Markowitz's mean-variance portfolio selection with regime switching: a continuous time model. *SIAM Journal on Control and Optimization*, 42:4, 1466-1482 (2003).

Appendix A

Itô's formula for Markov modulated processes

For each $i \in \mathcal{S}$, consider the stochastic process $\delta(i) = \{\delta_t(i), t \geq 0\}$ defined by $\delta_t(i) := I_{\{\epsilon_t = i\}}$ for every $t \in [0, \infty)$. Then,

$$\begin{aligned} \delta_t(i) &= \delta_0(i) + \int_0^t \Big(-\lambda_i \, \delta_s(i) + \sum_{j \in \mathcal{S} \setminus \{i\}} q_{ij} \, \delta_s(j) \Big) \, ds \, + \, m_t^i \\ &= \delta_0(i) + \int_0^t Q \, \delta_s(\cdot)(i) \, ds \, + \, m_t^i, \end{aligned} \tag{A.1}$$

where $m^i = \{m_t^i, t \geq 0\}$ is a square integrable martingale with $m_0^i = 0$ $\mathbb{P}$-a.s. (see Björk (1980), and for further reference see Buffington and Elliott (2002), and Elliott and Swishchuk (2004)).

Theorem A.1. Let $X = \{X_t, t \geq 0\}$ be a Markov modulated process that follows the stochastic differential equation

$$dX_t \; = \; \mu(t, \epsilon_t, \omega) \, dt \, + \, \sigma(t, \epsilon_t, \omega) \, dW_t, \tag{A.2}$$

where $\mu : [0, \infty) \times \mathcal{S} \times \Omega \to \mathbb{R}^M$ and $\sigma : [0, \infty) \times \mathcal{S} \times \Omega \to \mathbb{R}^{M \times N}$. Here W is a

N-dimensional standard Brownian motion. Let $\varphi(\cdot,\cdot,i) \in C^{1,2}([0,\infty) \times \mathbb{R}^M)$ be a real function for every $i \in \mathcal{S}$. Define $Y_t := \varphi(t, X_t, \epsilon_t)$. Then, Y is also a Markov modulated process and

$$
\begin{aligned}
dY_t \;=\;& \varphi_t(t, X_t, \epsilon_t)\, dt \;+\; \mu(t, \epsilon_t, \omega)^T \varphi_x(t, X_t, \epsilon_t)\, dt \\
& + \frac{1}{2}\,(\sigma\sigma^T)(t, \epsilon_t, \omega) \bullet \varphi_{xx}(t, X_t, \epsilon_t)\, dt \\
& + \varphi_x(t, X_t, \epsilon_t)^T \sigma(t, \epsilon_t, \omega)\, dW_t \;+\; Q\,\varphi(t, X_t, \cdot)(\epsilon_t)\, dt \;+\; dM_t^{\varphi},
\end{aligned}
$$

where the process $M^{\varphi} = \{M_t^{\varphi}, t \geq 0\}$ is a square integrable martingale with $M_0^{\varphi} = 0$ $\mathbb{P}$-a.s., when $\varphi(\cdot,\cdot,i)$, $i \in \mathcal{S}$, is bounded.

Proof. First of all, we note that

$$
Y_t \;=\; \varphi(t, X_t, \epsilon_t) \;=\; \sum_{i \in \mathcal{S}} \varphi(t, X_t, i) \cdot \delta_t(i).
$$

Using the product rule for each $\alpha_t(i) := \varphi(t, X_t, i) \cdot \delta_t(i)$, $i \in \mathcal{S}$, applying the Itô's formula to $\varphi(t, X_t, i)$, and using $(A.1)$ and $(A.2)$, we obtain

$$
\begin{aligned}
d\,\alpha_t(i) \;=\;& \delta_t(i)\, d\varphi(t, X_t, i) \;+\; \varphi(t, X_t, i)\, d\delta_t(i) + d{<}\varphi(\cdot, X, i), \delta_{\cdot}(i){>}_t \\
=\;& \delta_t(i)\Big(\varphi_t(t, X_t, i)\, dt + \varphi_x(t, X_t, i)^T dX_t + \frac{1}{2}\,\varphi_{xx}(t, X_t, i) \bullet d{<}X{>}_t\Big) \\
& + \varphi(t, X_t, i)\, d\delta_t(i) \;+\; \varphi_x(t, X_t, i)^T d{<}X, \delta_{\cdot}(i){>}_t \\
=\;& \delta_t(i)\, \varphi_t(t, X_t, i)\, dt \;+\; \delta_t(i)\, \mu(t, \epsilon_t, \omega)^T \varphi_x(t, X_t, i)\, dt \\
& + \delta_t(i)\, \varphi_x(t, X_t, i)^T \sigma(t, \epsilon_t, \omega)\, dW_t \\
& + \frac{1}{2}\, \delta_t(i)\, \varphi_{xx}(t, X_t, i) \bullet (\sigma_i \sigma^T)(t, \epsilon_t, \omega)\, dt \\
& + \varphi(t, X_t, i)\, Q\, \delta_t(\cdot)(i)\, dt \;+\; \varphi(t, X_t, i)\, dm_t^i \;+\; 0. \qquad \text{(A.3)}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
dY_t = \sum_{i \in \mathcal{S}} d\,\alpha_t(i)) &= \sum_{i \in \mathcal{S}} \delta_t(i)\,\varphi_t(t, X_t, i)\,dt + \sum_{i \in \mathcal{S}} \delta_t(i)\,\mu(t, \epsilon_t, \omega)^T \varphi_x(t, X_t, i)\,dt \\
&\quad + \sum_{i \in \mathcal{S}} \delta_t(i)\,\varphi_x(t, X_t, i)^T \sigma(t, \epsilon_t, \omega)\,dW_t \\
&\quad + \frac{1}{2} \sum_{i \in \mathcal{S}} \delta_t(i)\,(\sigma\sigma^T)(t, \epsilon_t, \omega) \bullet \varphi_{xx}(t, X_t, i)\,dt \\
&\quad + \sum_{i \in \mathcal{S}} \varphi(t, X_t, i)\,Q\,\delta_t(\cdot)(i)\,dt \;+\; \sum_{i \in \mathcal{S}} \varphi(t, X_t, i)\,dm_t^i \\
&= \varphi_t(t, X_t, \epsilon_t)\,dt \;+\; \mu(t, \epsilon_t, \omega)^T \varphi_x(t, X_t, \epsilon_t)\,dt \\
&\quad + \frac{1}{2}\,(\sigma\sigma^T)(t, \epsilon_t, \omega) \bullet \varphi_{xx}(t, X_t, \epsilon_t)\,dt \\
&\quad + \varphi_x(t, X_t, \epsilon_t)^T \sigma(t, \epsilon_t, \omega)\,dW_t \;+\; \sum_{i \in \mathcal{S}} \varphi(t, X_t, i)\,Q\,\delta_t(\cdot)(i)\,dt \\
&\quad + \sum_{i \in \mathcal{S}} \varphi(t, X_t, i)\,dm_t^i. \qquad \text{(A.4)}
\end{aligned}
$$

We note that

$$
\begin{aligned}
\sum_{i \in \mathcal{S}} \varphi(t, X_t, i)\,Q\,\delta_t(\cdot)(i)\,dt &= \sum_{i \in \mathcal{S}} \varphi(t, X_t, i) \sum_{j \in \mathcal{S}} q_{ij}\,\delta_t(j)\,dt \\
&= \sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}} q_{ij}\,\varphi(t, X_t, i)\,\delta_t(j)\,dt \\
&= \sum_{j \in \mathcal{S}} \sum_{i \in \mathcal{S}} q_{ji}\,\varphi(t, X_t, i)\,\delta_t(j)\,dt \\
&= \sum_{i \in \mathcal{S}} q_{\epsilon(t) i}\,\varphi(t, X_t, i)\,dt \\
&= Q\,\varphi(t, X_t, \cdot)(\epsilon_t)\,dt. \qquad \text{(A.5)}
\end{aligned}
$$

Let us define the stochastic process $M^\varphi = \{M_t^\varphi, t \geq 0\}$ by

$$M_t^\varphi := \int_0^t \sum_{i \in \mathcal{S}} \varphi(s, X_s, i)\, dm_s^i \tag{A.6}$$

for every $t \in [0, \infty)$. We note that M^φ is a square integrable martingale when $\varphi(\cdot, \cdot, i)$, $i \in \mathcal{S}$, is bounded. Then, using (A.5) and (A.6) in (A.4), we finally obtain

$$\begin{aligned} dY_t &= \varphi_t(t, X_t, \epsilon_t)\, dt + \mu(t, \epsilon_t, \omega)^T \varphi_x(t, X_t, \epsilon_t)\, dt \\ &\quad + \frac{1}{2}\, (\sigma\sigma^T)(t, \epsilon_t, \omega) \bullet \varphi_{xx}(t, X_t, \epsilon_t)\, dt \\ &\quad + \varphi_x(t, X_t, \epsilon_t)^T \sigma(t, \epsilon_t, \omega)\, dW_t + Q\, \varphi(t, X_t, \cdot)(\epsilon_t)\, dt + dM_t^\varphi. \end{aligned}$$

□

Appendix B

B.1 Proof of Lemma 3.1

It is easy to see that $\breve{A}_i \equiv 0$, $i \in \mathcal{S}$, is a trivial solution of the system, because $1-\alpha$ and $-\alpha$ are both positive constants. However, we are interested in proving the existence and uniqueness of the solution described in this Lemma. To facilitate the presentation, we will prove this Lemma for the case $S = 2$. We will consider first the case $\eta_1 - \lambda_1 > \eta_2 - \lambda_2$. Let us define

$$\tilde{x}_1 := \frac{1-\alpha}{\eta_1 - \lambda_1}$$

and

$$\tilde{x}_2 := -\frac{\alpha\left(\eta_1 + \eta_2 + \sqrt{(\eta_1-\eta_2)^2 + 4\lambda_1\lambda_2}\right)}{2(\eta_1\eta_2 - \lambda_1\lambda_2)},$$

and consider the interval $(\tilde{x}, +\infty)$, where $\tilde{x} := \max\{\tilde{x}_1, \tilde{x}_2\}$. Consider also the relation

(B.1)

$$y = y(x) = \sqrt[1-\alpha]{\frac{\eta_1}{\lambda_1}x^{1-\alpha} - \frac{(1-\alpha)}{\lambda_1}x^{-\alpha}} = \sqrt[1-\alpha]{\frac{1}{\lambda_1}\left(\eta_1 x - (1-\alpha)\right)x^{-\alpha}},$$

on its domain $x \in ((1-\alpha)/\eta_1, +\infty)$. We note that for a given $x > (1-\alpha)/\eta_1$, relation $(B.1)$ defines a unique value for $y > 0$. We also note that $y < (\sqrt[1-\alpha]{\eta_1/\lambda_1})\, x$ whenever $x > (1-\alpha)/\eta_1$. Define the function $f : ((1-\alpha)/\eta_1, +\infty) \to \mathbb{R}$ by

$$f(x) := (\eta_1\eta_2 - \lambda_1\lambda_2)\, x^{1-\alpha} - \eta_2\, (1-\alpha)\, x^{-\alpha} - \lambda_1(1-\alpha)\, y^{-\alpha},$$

where $y = y(x) > 0$ is given by $(B.1)$. We observe that to find a zero of f restricted to $(B.1)$ is equivalent to find a solution for the system of equations (3.25) when $S = 2$.

We note that f is strictly increasing in $(\tilde{x}, +\infty)$. Indeed, $y(\tilde{x}_1) = (1-\alpha)/(\eta_1 - \lambda_1)$, and $y > x$ when $x > \tilde{x}_1$. Then, if $x > \tilde{x}_1$,

$$\begin{aligned}
\frac{f'(x)}{1-\alpha} &= (\eta_1\eta_2 - \lambda_1\lambda_2)\, x^{-\alpha} + \alpha\, \eta_2\, x^{-\alpha-1} + \alpha\, (\eta_1 x + \alpha)\, x^{-\alpha-1}\, \frac{1}{y} \\
&> (\eta_1\eta_2 - \lambda_1\lambda_2)\, x^{-\alpha} + \alpha\, \eta_2\, x^{-\alpha-1} + \alpha(\eta_1 x + \alpha) x^{-\alpha-2} \\
&= \left\{(\eta_1\eta_2 - \lambda_1\lambda_2)\, x^2 + \alpha\, (\eta_1 + \eta_2)\, x + \alpha^2\right\}\, x^{-\alpha-2}.
\end{aligned}$$

We remember that $\eta_1 > \lambda_1 > 0$ and $\eta_2 > \lambda_2 > 0$. Thus, $\tilde{x}_2$ is a zero for $g(x) := (\eta_1\eta_2 - \lambda_1\lambda_2)\, x^2 + \alpha(\eta_1 + \eta_2)\, x + \alpha^2$, such that $g(x) > 0$ for every $x > \tilde{x}_2$. Hence, $f'(x) > 0$ for every $x \in (\tilde{x}, +\infty)$.

We also note that $f(x) > (\eta_1\eta_2 - \lambda_1\lambda_2)\, x^{1-\alpha} - (1-\alpha)(\eta_2 + \lambda_1^{1/1-\alpha}\eta_1^{-\alpha/1-\alpha})\, x^{-\alpha}$ for every $x \in ((1-\alpha)/\eta_1, +\infty)$. Since $\eta_1 > \lambda_1 > 0$ and $\eta_2 > \lambda_2 > 0$, we have $f(x) \to +\infty$ when $x \to +\infty$.

Now, we are going to consider two cases: $x = \tilde{x}_1$ and $x = \tilde{x}_2$. When $\tilde{x} = \tilde{x}_1$, we have that $y(\tilde{x}) = \tilde{x}_1$, and hence

$$f(\tilde{x}) = f(\tilde{x}_1) = -\lambda_1(\eta_1 - \eta_2 - \lambda_1 + \lambda_2)(1-\alpha)^{1-\alpha}/(\eta_1 - \lambda_1)^{1-\alpha} < 0.$$

Applying the Intermediate Value Theorem, we get that there exists $x_0 \in (\tilde{x}, +\infty)$ such that $f(x_0) = 0$. Also, the corresponding $y_0 = y(x_0)$ given by $(B.1)$ is in $((1-\alpha)/(\eta_1 - \lambda_1), +\infty)$, because $y > x$ when $x > \tilde{x} = (1-\alpha)/(\eta_1 - \lambda_1)$. That is, there exists a real solution (x_0, y_0) for the system of equations (3.25) such that $x_0 > \tilde{x} = (1-\alpha)/(\eta_1 - \lambda_1)$ and $y_0 > (1-\alpha)/(\eta_1 - \lambda_1)$. This solution is unique in $(\tilde{x}, +\infty)$ as f is strictly increasing in $(\tilde{x}, +\infty)$. We consider now the case $\tilde{x} = \tilde{x}_2$. When $\tilde{x} = \tilde{x}_2$ and $f(\tilde{x}_2) < 0$, the result is exactly the same as the one above. When $\tilde{x} = \tilde{x}_2$ and $f(\tilde{x}_2) \geq 0$, there is still a positive solution $x_0 \in (\tilde{x}_1, \tilde{x}_2]$ and no other solution in $(\tilde{x}, +\infty)$.

This proves that if $\eta_1 - \lambda_1 > \eta_2 - \lambda_2$, then there exists a unique solution $(\breve{A}_1, \breve{A}_2)$ for the system of equations (3.25) such that $\breve{A}_1 > (1-\alpha)/(\eta_1 - \lambda_1)$ and $\breve{A}_2 > (1-\alpha)/(\eta_1 - \lambda_1)$.

If $\eta_1 - \lambda_1 = \eta_2 - \lambda_2$, we define $\tilde{x}_1$, $\tilde{x}_2$, $y : ((1-\alpha)/\eta_1, +\infty) \to \mathbb{R}$ and $f : ((1-\alpha)/\eta_1, +\infty) \to \mathbb{R}$ as it was done above for the case $\eta_1 - \lambda_1 > \eta_2 - \lambda_2$. Note that now $\eta_1 - \eta_2 = \lambda_1 - \lambda_2$, $\eta_1 + \lambda_2 = \eta_2 + \lambda_1$ and $\eta_1 > \lambda_1 > 0$. Thus,

$$
\begin{aligned}
\tilde{x}_2 &= -\frac{\alpha\left(\eta_1 + \eta_2 + \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_1\lambda_2}\right)}{2(\eta_1(\eta_2 + \lambda_1) - \lambda_1(\eta_1 + \lambda_2))} \\
&= -\frac{\alpha\,(\eta_1 + \eta_2 + \lambda_1 + \lambda_2)}{2(\eta_1 - \lambda_1)(\eta_2 + \lambda_1)} \\
&= -\frac{\alpha\,(2\eta_2 + 2\lambda_1)}{2(\eta_1 - \lambda_1)(\eta_2 + \lambda_1)} \\
&= -\frac{\alpha}{\eta_1 - \lambda_1} \\
&< \frac{1-\alpha}{\eta_1 - \lambda_1} \quad = \quad \tilde{x}_1.
\end{aligned}
$$

Following the analysis done for the case $\eta_1 - \lambda_1 > \eta_2 - \lambda_2$, it is easy to verify that f is strictly increasing in $(\tilde{x}_1, +\infty)$ and also $f(\tilde{x}_1) = 0$. This implies that $\tilde{x}_1 = (1-\alpha)/(\eta_1 - \lambda_1)$ is the only zero for f in $[(1-\alpha)/(\eta_1 - \lambda_1), +\infty)$.

Also, $y(\tilde{x}_1) = (1-\alpha)/(\eta_1 - \lambda_1) = (1-\alpha)/(\eta_2 - \lambda_2)$. Hence, there exists a unique solution $(\breve{A}_1, \breve{A}_2)$ for the system of equations (3.25) such that $\breve{A}_1 = (1-\alpha)/(\eta_1 - \lambda_1)$ and $\breve{A}_2 = (1-\alpha)/(\eta_1 - \lambda_1)$.

Similarly, it can be proved that if $\eta_1 - \lambda_1 < \eta_2 - \lambda_2$, then there exists a unique solution $(\breve{A}_1, \breve{A}_2)$ for the system of equations (3.25) such that $\breve{A}_1 > (1-\alpha)/(\eta_2 - \lambda_2)$ and $\breve{A}_2 > (1-\alpha)/(\eta_2 - \lambda_2)$. □

B.2 Proof of Lemma 3.2

To facilitate the presentation, we will prove this Lemma for the case $S = 2$. Consider the relation

$$\text{(B.2)} \qquad \lambda_1 \, y^{1-\alpha} = \eta_1 \, x^{1-\alpha} - \frac{(1-\alpha)}{x^\alpha} = x^{-\alpha} \left(\eta_1 x - (1-\alpha) \right)$$

for $x \in ((1-\alpha)/\eta_1, +\infty)$. We observe that $x > (1-\alpha)/\eta_1$ implies that the right-hand-side of $(B.2)$ is positive. Thus, the relation $(B.2)$ defines a unique $y \in (0, \infty)$.

For $x \in ((1-\alpha)/\eta_1, +\infty)$, we define the function f by

$$f(x) := (\eta_1\eta_2 - \lambda_1\lambda_2) \, x^{1-\alpha} - \frac{\eta_2(1-\alpha)}{x^\alpha} - \frac{\lambda_1(1-\alpha)}{y^\alpha},$$

where y is given in terms of x by $(B.2)$. We note that to find a zero for f restricted to $(B.2)$ is equivalent to find a solution for system (3.28) when $S = 2$.

We observe from $(B.2)$ that when $x \to (1-\alpha)/\eta_1$, $y^{1-\alpha} \to 0$. Hence, $f(x) \to -\infty$ when $x \to (1-\alpha)/\eta_1$. Moreover, when $x \to +\infty$, $y^{1-\alpha} \to +\infty$ and, hence, $f(x) \to +\infty$. Thus, applying the Intermediate Value Theorem, there exists $x_0 \in ((1-\alpha)/\eta_1, +\infty)$ such that $f(x_0) = 0$. Also, the correspondent $y_0 = y(x_0)$ given by $(B.2)$ is in $(0, +\infty)$. In addition, $y_0 > (1-\alpha)/\eta_2$, because

$$f(x_0) = \lambda_1(\eta_2 y_0 - (1-\alpha))/y_0^\alpha - \lambda_1\lambda_2 x_0^{1-\alpha} = 0$$

leads to $\eta_2 y_0 - (1-\alpha) = \lambda_2 x_0^{1-\alpha} y_0^\alpha > 0$. That is, there exists a real solution (x_0, y_0) for the system of equations (3.28) such that $x_0 > (1-\alpha)/\eta_1$ and $y_0 > (1-\alpha)/\eta_2$. Moreover, it is also possible to prove that this solution is unique in such region. In fact,

$$f'(x) = \frac{(\eta_1\eta_2 - \lambda_1\lambda_2)(1-\alpha)}{x^\alpha} + \frac{\eta_2\,\alpha(1-\alpha)}{x^{1+\alpha}} + \frac{\alpha(1-\alpha)}{y}\left(\frac{\eta_1}{x^\alpha} + \frac{\alpha}{x^{1+\alpha}}\right).$$

Since $\eta_1\eta_2 - \lambda_1\lambda_2 \geq 0$, we have that $f'(x) > 0$ for every $x \in ((1-\alpha)/\eta_1, +\infty)$. Then, the function f is strictly increasing and has only one zero in $((1-\alpha)/\eta_1, +\infty)$. Therefore, the pair (x_0, y_0) is the unique real solution for the system of equations (3.28), such that $x_0 > (1-\alpha)/\eta_1$ and $y_0 > (1-\alpha)/\eta_2$.

In the general case where $S \geq 2$, we define $f : ((1-\alpha)/\eta_1, +\infty) \to \mathbb{R}$ in a similar way. In fact,

$$f(x) \;:=\; x^{1-\alpha} - \rho\,(1-\alpha)\,x^{-\alpha} - \sum_{k=1}^{S-1} \rho_k\,(1-\alpha)\,y_k^{-\alpha},$$

where ρ and ρ_k, $k = 1, \ldots, S-1$ are factors given by the market parameters q_{ij}, $i, j \in \mathcal{S}$, and η_i, $i \in \mathcal{S}$, and obtained from the system of equations (3.28). We can see that in this case f will depend not only on x and y but on x and other $S-1$ variables y_k, $k = 1, \ldots, S-1$, that are actually dependent on x themselves, as they satisfy the system of equations (3.28). In fact, these dependencies are unique. The function f in that way defined will be strictly increasing in $((1-\alpha)/\eta_1, +\infty)$ and such that $f(x) \to -\infty$ as $x \to (1-\alpha)/\eta_1$ and $f(x) \to +\infty$ as $x \to +\infty$. Hence, similar arguments to the case $S = 2$ apply. $\square$

B.3 Proof of Lemma 3.3

Bye equaling the right sides of (3.31), we obtain

$$\eta_2 \bar{A}_2 = \eta_1 \bar{A}_1 - \left(\frac{\beta_1^2}{2\lambda_1} - \frac{\beta_2^2}{2\lambda_2}\right).$$

Recall that condition (3.21) must be satisfied for this case, so $\eta_i > 1$ for every $i = 1, 2$. Then, replacing $\bar{A}_2$ in the original equations, we get that

$$\frac{\eta_1}{\eta_2} \bar{A}_1^2 - \left(\frac{\beta_1^2}{2\lambda_1\eta_2} - \frac{\beta_2^2}{2\lambda_1\eta_2}\right) \bar{A}_1 = \eta_1^2 \bar{A}_1^2 - \frac{\beta_1^2\eta_1}{\lambda_1} \bar{A}_1 + \frac{\beta_1^4}{4\lambda_1^2}.$$

That is,

$$\frac{\eta_1}{\eta_2} (1 - \eta_1\eta_2) \bar{A}_1^2 - \left(\frac{\beta_1^2}{2\lambda_1\eta_2} - \frac{\beta_2^2}{2\lambda_1\eta_2} - \frac{\beta_1^2\eta_1}{\lambda_1}\right) \bar{A}_1 - \frac{\beta_1^4}{4\lambda_1^2} = 0.$$

We note then that $\bar{A}_1$ satisfies an equation of the form $a\bar{A}_1^2 + b\bar{A}_1 + c = 0$ where $b^2 - 4ac > 0$. Indeed,

$$b^2 - 4ac = \frac{(\beta_1^2 - \beta_2^2)^2}{4\lambda_1^2\eta_2^2} + \frac{\beta_1^2\beta_2^2\eta_1}{\lambda_1^2\eta_2} > 0.$$

Moreover, $\bar{A}_1 > 0$, as $(\bar{A}_1\bar{A}_2)^{1/2} \geq 0$ implies that $\eta_1\bar{A}_1 - 1/2\beta_1^2/\lambda_1 \geq 0$.

The same analysis can be done for $\bar{A}_2$ to obtain, as well, a unique positive solution for (3.31). □

Appendix C

C.1 Proof of Lemma 6.1

First of all we will prove for $X := X^Z$ that

$$E_{x,i}\left[\int_0^\tau e^{-\delta s}\, v'(X_s, \epsilon_s)\, I_{\{X_s \in \mathcal{C}(\epsilon_s)\}}\, dZ_s\right] \;=\; 0. \tag{C.1}$$

Since $v(\cdot, i)$, $i = 1, 2$, is a nondecreasing function in x and Z is a nondecreasing process, it is enough to prove that

$$E_{x,i}\left[\int_0^\tau e^{-\delta s}\, v'(X_s, \epsilon_s)\, I_{\{X_s \in \mathcal{C}(\epsilon_s)\}}\, dZ_s\right] \;\leq\; 0.$$

Let $M \in (0, \infty)$ be such that for every $s \in [0, \tau]$: $0 \leq v'(X_s, \epsilon_s) < M$. Then, since $Z = Z^v$ satisfies (iii) in Definition 6.2,

$$E_{x,i}\left[\int_0^\tau e^{-\delta s} v'(X_s, \epsilon_s) I_{\{X_s \in \mathcal{C}(\epsilon_s)\}} dZ_s\right] \;\leq\; M \cdot E_{x,i}\left[\int_0^\tau I_{\{X_s \in \mathcal{C}(\epsilon_s)\}} dZ_s\right] \;\leq\; 0.$$

This proves $(C.1)$. Now, let us denote by $\mathcal{B}(i) := \mathcal{B}(\mathcal{C}(i))$, the border of the continuation region $\mathcal{C}(i)$, $i = 1, 2$. We note that (ii) in Definition 6.2 holds and $X_t \in \mathcal{C}(\epsilon_t) \cup \mathcal{B}(\epsilon_t)$ for every $t \in [0, \infty)$. Then,

$$E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} v'(X_s, \epsilon_s)\, dZ_s\right] = E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} v'(X_s, \epsilon_s) I_{\{X_s \in \mathcal{B}(\epsilon_s)\}} dZ_s\right] + E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} v'(X_s, \epsilon_s) I_{\{X_s \in \mathcal{C}(\epsilon_s)\}} dZ_s\right].$$

Moreover, $v'(x, i) = 1$ for every $x \in \mathcal{B}(i)$, $i = 1, 2$. Hence, from $(C.1)$ and the above equation, we get

(C.2)

$$E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} v'(X_s, \epsilon_s)\, dZ_s\right] = E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} I_{\{X_s \in \mathcal{B}(\epsilon_s)\}} dZ_s\right] \leq E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} dZ_s\right].$$

Furthermore, from (ii) and (iii) in Definition 6.2, we have

$$\begin{aligned} E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} I_{\{X_s \in \mathcal{B}(\epsilon_s)\}} dZ_s\right] &= E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} dZ_s\right] - E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} I_{\{X_s \in \mathcal{C}(\epsilon_s)\}} dZ_s\right] \\ &\geq E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} dZ_s\right] - E_{x,i}\left[\int_0^{\mathcal{T}} I_{\{X_s \in \mathcal{C}(\epsilon_s)\}} dZ_s\right] \\ &= E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} dZ_s\right]. \end{aligned} \tag{C.3}$$

Thus, from (C.2) and (C.3),

$$E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} v'(X_s, \epsilon_s)\, dZ_s\right] = E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} dZ_s\right]. \tag{C.4}$$

Furthermore, following Asmussen and Taksar (1997)–section 3– we have

$$\begin{aligned} E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} v'(X_s, \epsilon_s)\, dZ_s{}^c\right] - E_{x,i}\left[\sum_{0 \leq s \leq \mathcal{T},\, s \in \Lambda} e^{-\delta s} \left(v(X_{s^+}, \epsilon_s) - v(X_s, \epsilon_s)\right)\right] \\ = E_{x,i}\left[\int_0^{\mathcal{T}} e^{-\delta s} v'(X_s, \epsilon_s)\, dZ_s\right]. \end{aligned} \tag{C.5}$$

The Lemma then follows from $(C.4)$-$(C.5)$.