Theories that Inform the Use of Variation in Mathematics Pedagogy

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Abstract

A large portion of mathematics education research focuses on elementary and middle year's content with professional development about how to teach mathematics often focused using specific tasks to teach certain topics. This kind of professional development makes it difficult for teachers to develop research based pedagogy that can be applied to the entire mathematics curriculum. The purpose of this paper is to explore the theories that influence the use of variation pedagogy in the teaching of mathematics. Through the use of intentional and structured variation, teachers can design lessons to enhance their students' mathematical understanding while also developing their own mathematical content knowledge.

Theories that Inform the Use of Variation in Mathematics Pedagogy

Research in mathematics education is a growing field with extensive work being written about how students learn and understand mathematics as well as how teachers can effectively teach mathematics. While there is much written about teaching and unpacking elementary and secondary mathematics concepts, there is limited research that explicitly targets how to effectively teach mathematical concepts found in high school mathematics courses, such as pre-calculus and calculus. As a high school math teacher, when I attend conferences and professional development sessions I am often disappointed at the lack of resources that target effectively teaching high school mathematics. I believe effective teaching of high school mathematics involves guiding students to use mathematical skills fluently as well as develop a deeper understanding of concepts being taught. This will allow students to develop knowledge retention as well as enhance their ability to problem solve and think critically. Professional development opportunities generally focus on elementary mathematical concepts or target very specific high school concepts, using what are often described as a "rich mathematical tasks," with little attention given to how this can be applied to the program of studies as a whole. Rich mathematical tasks can provide effective learning opportunities for students, but can be challenging for many teachers to create or replicate on their own. Searching for rich tasks that align with a content heavy program of studies is extremely time consuming and does not always produce successful results.

In Alberta, our current certification system accredits secondary teachers that have taken a minimum number of subject specific advanced courses (Alberta Education, 2020). This means that a mathematics specialist teacher will be certified for having taken advanced mathematics courses such as calculus, linear algebra, and geometry in university. Begle (as cited in Ball & Bass, 2003) investigated the relationship between numbers of courses teachers had taken beyond calculus with how well their own students performed. He found that “taking advanced mathematics courses produced positive main effects on students’ achievement in only 10% of the cases … and negative main effects in 8%” (Ball & Bass, 2003, p. 3). Ball & Bass (2003) link these negative effects to the need for teachers to be able to unpack content knowledge, where as highly skilled mathematicians are used to compressing knowledge. Mathematics teachers who take high level math courses may have experienced more conventional approaches to teaching mathematics and are used to certain “pedagogical images and habits” (p. 3) that may not be effective with young students.

Teachers’ knowledge of mathematics involves linking content with pedagogy and having a fundamental and profound understanding of mathematics concepts and connections (Ball & Bass, 2003; Davis, 2012). Referred to as *specialized content knowledge* (Ball & Bass, 2003) or *mathematical content knowledge* (Charalambos, 2019), this specific knowledge refers to the deep understanding of the concepts specific to mathematics curriculum taught in schools. Ball & Bass (2003) extend the idea of specialized knowledge of mathematics into *pedagogical content knowledge*, which “is a unique kind of knowledge that intertwines content with aspects of teaching and learning” (p. 4). While many teacher education programs in North America separate the teaching of disciplinary knowledge and instructional knowledge (Davis & Simmt, 2006) teachers are often left to develop pedagogical content knowledge through experience and professional development.

High school mathematics teachers often rely on resources, such as approved textbooks, to guide their teaching practice. Textbook resources available for high school mathematics courses that follow the *Mathematics Grade 10 to 12 Program of Studies* (Alberta Education, 2008) generally follow a similar blocking pattern that involves chunking content together by showing examples of work followed by arbitrarily grouped practice questions. Experienced mathematics teachers who recognize the missed opportunities to make connections within the material spend more time creating new materials to better meet the needs of their students. Although exceptional teachers can have different personalities and use many different strategies to engage and interact with students in the classroom, I argue that despite the classroom design, the teaching of mathematics is most effective with purposefully chosen examples, questions and tasks.

The pedagogy of variation within mathematical tasks and lessons has existed as part of the Chinese teaching model for many years (Mok, 2017) becoming increasingly popular among mathematical teaching pedagogy internationally (Pang et al., 2017). While all teachers use some sort of variation and invariance within their teaching, it is how variation can be applied systematically that has become a prominent research focus for effective teaching and learning, especially in mathematics education (Marton & Pang, 2006). Mason (2017) uses the term *variation-pedagogy* to refer to "pedagogic actions used to exploit variation" (p. 430), meaning that even with textbooks and resources constructed using variation principles, it requires an "awakening of teacher awareness" (p. 413) to make appropriate pedagogical choices.

Recent studies have demonstrated how using systematic variation to structure mathematical tasks can enhance a teacher's own mathematical understanding (Kullberg et al., 2017) and cause teachers to "engage differently with the content" (Metz et al., 2016, p. 1256). It is my goal that as teachers learn about how systematic variation can be applied to mathematical examples, questions and tasks, their interaction and engagement with mathematical content will continue to develop. In this paper I will explore the theories that inform a practice of variation pedagogy within the mathematics classroom.

Personal Connection and Positionality

As a high school mathematics teacher, I have always been motivated to do better. I strive to constantly improve my teaching in order to help improve my students' understanding and achievement. My focus was often on using teaching strategies and activities I could do in the classroom to engage students. Not all of my students found math as exciting as I did, so I looked for activities to make math fun and help them enjoy being in class. I would find or create activities that made doing math feel like a game or encourage students to collaborate and speak to each other about math. Through my experiences, I created a tool box of activities that seemed to engage students in working with mathematics. As I used these activities more, I noticed how the different questions I selected within a task could incite particular understandings from my students. I became aware that the questions I used and the order I used them in directed students to notice patterns and build their own connections. I started watching for those AHA! moments among my students as well as attending to how my own understanding of the mathematics was changing. As I began my graduate work in University, I was introduced to Marton's (2015) variation theory of learning and began investigating what it meant to have a deeper mathematical understanding for both teachers and students. I recognized that my work of purposefully choosing questions and examples to make up mathematical tasks largely related to research around the use of variation as part of effective teaching and learning of mathematics.

It took me a decade of teaching secondary mathematics to understand the power of well-chosen tasks and examples. Further research, encouraged by University graduate courses, taught me to continue experimenting with how I structured examples and questions to create effective mathematical tasks and in turn allowed me to realize connections that existed within the program of studies I was teaching. Reading and working with Marton's (2015) variation theory of learning opened the door to my own understanding of the different ways systematic variation can be applied to mathematical topics. As will be explored further throughout this paper, there is not a "right way" of using variation, but it is the act of observation and awareness by teachers that allows for learning opportunities to be had when a systematic, planned form of variation is applied to teaching mathematics. When a teacher focuses their attention on what the intended learning is, they are put in a position where it is necessary to consider their own mathematical understanding and unpack the mathematical content they are hoping to teach to their students. Reflecting on student learning through purposefully structured examples, questions and tasks, possibly through learning studies or professional communities of practice, forces attention onto that mathematical understanding.

I believe that using systematic variation to purposefully choose effective examples and tasks can be applied to all mathematics classrooms, but especially effective in a high school mathematics program of studies. This focus can help connect a high school mathematics teacher's strong mathematical content knowledge with their pedagogical knowledge, transforming their own mathematical understanding. Effectively attending to the use of structured variation can be applied to the teaching of all the content within a high school program of studies.

Literature Review

In this review I will explore theories that influence the use of purposeful, systematic variation within the mathematics classroom. I invite you to engage in the Taxicab geometry exercise written by Krause (as cited in Watson & Mason, 2006) shown in Figure 1 as a way of experiencing the power of *structured variation*. We will refer back to this example throughout the review.

Figure 1

Taxicab Geometry Exercise

$Dt(P, A)$ is the shortest distance from P to A on a two-dimensional coordinate grid, using horizontal and vertical movement only. We call it the taxicab distance.

For this exercise $A = (-2, -1)$. Mark A on a coordinate grid. For each point P in (a) to (h) below calculate $Dt(P, A)$ and mark P on the grid (in the original, they are in a single column so there is no temptation to work across rows instead of in order down the columns):

$(a) P = (1, -1)$	$(e) P = (\frac{1}{2}, -1\frac{1}{2})$
$(b) P = (-2, -4)$	$(f) P = (-1\frac{1}{2}, -3\frac{1}{2})$
$(c) P = (-1, -3)$	$(g) P = (0, 0)$
$(d) P = (0, -2)$	$(h) P = (-2, 2)$

Note. This figure contains a description of Krause's task as cited in "Seeing an exercise as a single mathematical object: Using variation to structure sense making," by A. Watson & J. Mason, 2006, *Mathematical Thinking and Learning, 8*(2), p. 95.

Teaching with Variation: Two Frameworks

As previously noted, most teachers already use some form of variation within their teaching. Explicit variation pedagogy has been part of the Chinese teaching model for many years, while it has been in the past couple of decades that research around the use of variation has become more prominent in European and North American literature. Two groups of researchers have developed separate frameworks around the use of variation and invariance while analyzing mathematics classroom practices (Pang, et al., 2017).

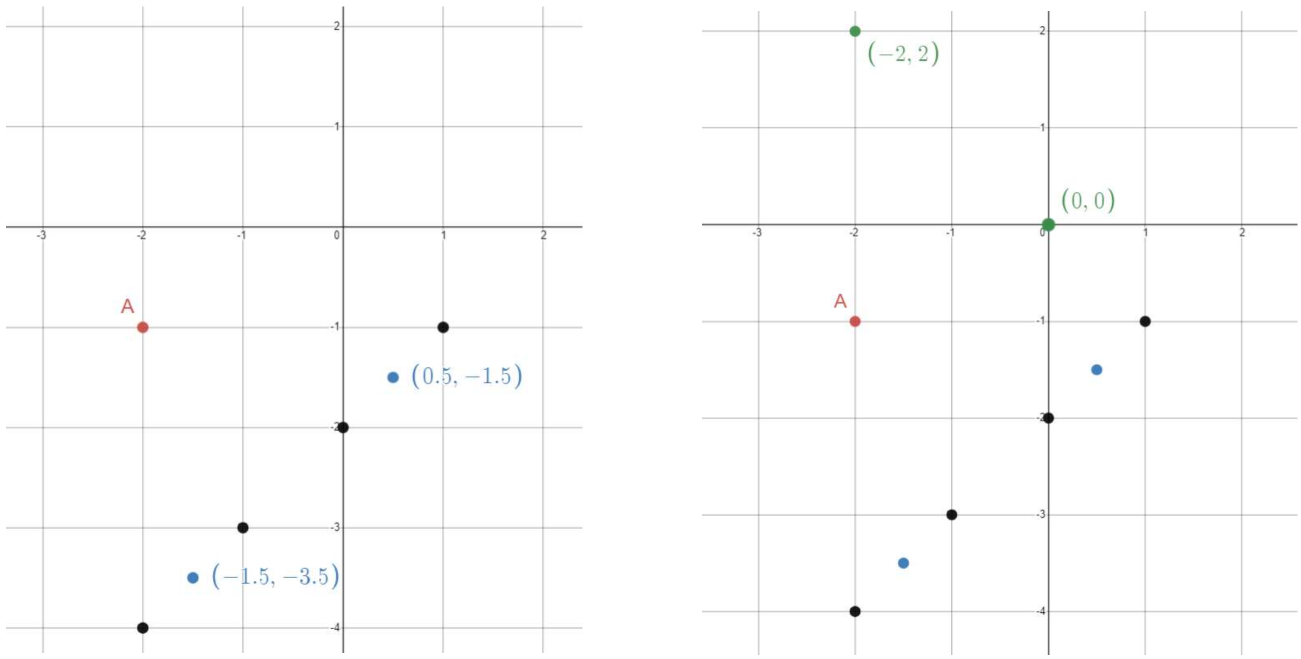
In Sweden and Hong Kong, Ference Marton led a group of researchers to develop the variation theory of learning (Pang et al., 2017). Originating from “a phenomenological interest in differences in how various phenomena appear to people” (Marton & Haggstrom, 2017, p. 383), variation theory focuses on how people identify novel meanings by discerning differences among things that are otherwise the same. For example, if trying to learn about a circle as being a geometric figure, one would need to experience other geometric figures that are not circles and begin to discern how a circle is different. While many analyses and studies using the variation theory framework focus on mathematics education, according to Pang et al., (2017) variation theory “is intended to be a general theory about how people experience and learn to handle certain phenomena” (p. 46). Numerous studies have demonstrated how variation theory can be applied to different subjects and educational levels (Pang & Marton, 2003, 2005, 2013; Lo et al., 2006; Ko, 2013).

Researchers based out of Shanghai, led by Gu Ling-yuan, formalized the use of variation into a theory known as *bianshi* jiaoxue, meaning teaching with variation (Pang et al., 2017). While Marton’s (2015) variation theory is focused on teaching “ways of seeing” across different subjects, the *bianshi* teaching framework is designed to theorize the “effective mathematics teaching practices in Chinese classrooms and extend the Chinese theory of *bianshi*” (Pang et al., 2017, p. 46; Gu, 1991). The *bianshi* teaching framework follows a clearly defined system of instructional principles. Contrary to variation theory, *bianshi* focuses on highlighting essential features by differing the non-essential features. Using the previously mentioned example to teach about circles, when using the *bianshi* framework, a teacher would show students many different circles with various features such as different colours and sizes, helping students to determine the essence of the concept of a circle. *Bianshi* teaching also distinguishes between conceptual variation, which is "concerned with understanding concepts from multiple perspectives" (Mok, 2017, p. 190) and procedural variation which focuses on how these concepts are built upon and developed. Gu describes conceptual variation (as cited in Pang et al., 2017) as allowing students to "learn mathematical concepts from multiple examples highlighting essential features and clarifying the connotation of concepts by initially excluding (and later including) the interference of background and non-essential features of objects" (p. 47). Procedural variation then dictates how mathematical activities progressively unfold to form new concepts and enhance the experience of doing these activities. Procedural variation applied to problem solving activities would "include variation on the problem, different solutions to a problem and one solution applied to different problems" (Pang et al., 2017, p. 47). The *bianshi* framework guides the structure and design of the entire mathematics lesson.

When thinking back to the taxicab distance exercise introduced earlier, consider the points for *P* that were given and the order they were presented in. Watson & Mason (2006) have used this task in an informal study on both teachers and students and noticed that the variation in the list of points prompts two generalizations from learners after plotting the first 4 points. First the distance appears to always equal 3 and second, the new points appear to form a straight line, as shown in Figure 2. The straight line generalization assists learners with plotting the fractional points, $\left(\frac{1}{2},-1\frac{1}{2}\right)$ and $\left(-1\frac{1}{2},-3\frac{1}{2}\right)$. The purpose of $\left(0,0\right)$ followed by $\left(-2,2\right)$ is to break the pattern from the straight line to guide learners into understanding the true shape of the locus being formed.

Figure 2

Straight line Generalization and Pattern Breaking of the Taxicab Distance Exercise



The structure of this task exemplifies how procedural variation as described in the *bianshi* framework, can be applied. Scaffolding the points so that learners create patterns to assist them and then breaking those patterns to engage learners in new understandings, creates a specific, problem solving experience. The learners studied by Watson & Mason (2006) took different approaches to plotting the points and determining distance, but the study showed that, despite prior knowledge or giftedness, most learners reached a similar understanding in the end.

Teachers employing the principles from the *bianshi* framework would likely extend this lesson to include more problems with similar characteristics, such as maintaining a pattern of the points initially forming a line, and beginning to differ non-essential features, such as changing the distance of 3 to be something else. A systematic form of variation would continue to be applied within a single problem, but procedural variation would guide the lesson, drawing attention to the different ways of working through a single problem.

Conflicting Frameworks

Although Marton's (2015) variation theory of learning may appear to contradict Gu's (1991) theory of *bianshi* teaching, multiple analyses of lessons using both frameworks (Pang et al., 2017; Marton & Haggstrom, 2017) show that while the intended practices differ in how they emphasize sameness and difference, the enacted practices look quite similar. Pang et al. (2017) examined a lesson of adding 3 digit numbers that was designed based on the *bianshi* framework, providing an understanding and rationale of the instructional design. The same lesson was then analyzed based on the topics using the variation theory framework. Pang et al. noticed that while differences existed between how examples were used, with *bianshi* emphasizing seeing sameness and variation theory emphasizing seeing differences, the two frameworks are actually highly compatible. While *bianshi* provides a focus on the design of instructional tasks, variation theory explains how teaching can support learning through the use of patterns of variation and invariance. Pang et al. determined the *bianshi* and variation theory frameworks to be mutually exclusive and "useful in developing a collective understanding" (p. 66).

Marton & Haggstrom (2017) compared two lessons, one from Sweden containing elements of the variation theory framework and one from Shanghai which was more representative of the *bianshi* teaching framework. The lessons were analyzed paying attention to what dimensions of variation were opened up and how opportunities to notice critical aspects were provided to students. Marton & Haggstrom found that in both lessons opportunities of discernment were made available, demonstrating the patterns of variation in accordance with the variation theory of learning framework. According to Marton & Haggstrom, both frameworks "agree on the principle that novel and essential aspects of the object of learning in mathematics can only be appropriated by the learners by means of separating those essential aspects from non-essential aspects" (p. 404).

Both Marton's (2015) variation theory of learning and Gu's (1991) theory of *bianshi* teaching provide not only comparable, but complementary frameworks of how systematic variation can be used within the teaching of mathematics. Mason (2017) conjectures that there is no "best sequence" for how and when to apply variation in teaching and Watson & Mason (2006) best explain that "Marton's focus on variation gives us a language" (p. 101) to describe what is possible to be learned. As Marton's (2015) variation theory of learning is frequently cited in studies attempting to demonstrate effective mathematical teaching practices, I feel it is important to further elaborate on the language and terminology used within his variation theory framework.

Variation Theory of Learning

The general framework of variation theory "envisages that for learning to occur, some critical aspects of the object of learning must vary while other aspects remain constant" (Cheng, 2016, p. 284). There are two aspects of the object of learning, the direct and indirect (Marton & Pang, 2006). The direct object of learning focuses on the content, often a concept that students find difficult, while "the indirect object of learning is the capability that we expect the students to develop" (Marton & Pang, 2006, p. 197). The indirect object of learning is focused on not only students having knowledge, but seeing with that knowledge. "Learning to see something in a certain way amounts to discerning certain critical features of that phenomenon and focusing on them simultaneously" (Guo & Pang, 2011). It is through the experience of variation and invariance of particular features that learning and understanding can occur.

Noticing and attending to a feature requires experiencing situations that amount to discernment of that feature. Discernment requires experiencing variation. In order to know what something is, there must be an experience of what it is not. When a critical aspect is varied while other aspects remain the same, the critical aspect is able to be separated and discerned. For example, Marton & Haggstrom (2017) discuss a learner encountering a small, blue circle. The learner must be aware of the features "small", "blue" and "circle" in order to separate and then discern their dimensions of variation.

Marton & Pang (2006) identify four necessary conditions for learning, also known as patterns of variation and invariance, which will allow for discernment of critical aspects. *Contrast* requires seeing what something is not in order to identify what it is, similar to the use of counter examples in mathematical proofs. As a result of seeing contrast, *separation* occurs when a learner is able to discern a certain aspect from the other aspects that remain invariant. Marton (2015) views separation as a resulting characteristic of contrast or as Mok (2017) states "discernment implies separation" (p. 189) and therefore separation is not always mentioned as a necessary condition of learning. Guo & Pang (2011) provide the example of experiencing the colour black. A learner can start to discern the colour black by seeing other colours, such as red or white. The aspect of colour will be separated from other aspects that are kept invariant, such as maintaining the same size, shape or height of an object while only varying colour. *Fusion* takes place when the learner is able to discern multiple critical aspects simultaneously, such as distinguishing between black circles and red squares. The learner would need to discern between the colours and the shapes. *Generalization* occurs when previous discernments can then be applied to multiple and various examples or contexts.

Let's re-examine the taxicab distance exercise through the lens of Marton's (2015) variation theory of learning. Point A is kept constant as well as the distance from point P remaining 3, allowing for other critical aspects to be discerned. Through patterning, learners can accept the distance to be 3 and are now able to discern how determining distance using fractional points is possible. Developing the pattern of the points forming a straight line then introduces the opportunity for points $(0,0)$ and $(-2,2)$ to be different. This contrast now allows learners to discern the shape formed around the locus point A. Allowing learners to think of other points that maintain a fixed distance of 3 from point A can lead to generalization around the idea of locus points. Fusion would require extending the activity to then vary the location of point A or the distance from point A and allow further generalization to be made about locus points.

Kullberg et al. (2017) consider the example of linear functions and their equation in the form $y = mx + b$ to demonstrate the role of contrast, fusion and generalization. If slope is identified as a dimension of variation, a teacher might demonstrate how slopes can affect the linear function by varying the m-value in the equation and keeping the b-value invariant. This would show contrast and separation of the effect of the m-value on the slope of the linear function. Similarly, contrast and separation of the y-intercept can be demonstrated by varying the b-value and keeping the slope invariant. Students may then be allowed to generalize their understanding of how changing the m-value or b-value will impact the linear function. In order to fully understand the effect of the equation on a linear function, there must be simultaneous discernment of several critical aspects, meaning students must experience changes in both the m-value and b-value, for fusion to take place. While many teachers likely teach about the equations and graphs of linear functions by varying the slope and y-intercept in some way, effective systematic variation requires purposeful selection of values with teachers drawing attention to what students are noticing.

Variation theory asserts that “learning implies seeing or experiencing critical aspects of an object of learning” (Kullberg et al., 2017, p. 560). Kullberg et al. (2017) explain how the object of learning should answer the question “what is to be learned?” by defining the content, the educational objective and identifying the critical aspects. Guo & Pang (2011) emphasize that “both the disciplinary knowledge and the students’ understanding should be taken into account when identifying the critical aspects of an object of learning” (p. 298). Kullberg et al. (2017) state that “when guided by variation theory in planning for learning, the teacher must be aware of not just what the critical aspects might be, but how to open then up as dimensions of variation and to determine what values in those dimensions would be critical” (p. 561). Lesson design involves anticipating how students currently view critical aspects and how variation will change students’ understanding.

Marton & Pang (2006) describe 3 forms of the object of learning: the *intended*, the *enacted* and the *lived*. This process of deciding what should be learned represents the *intended object of learning* while considering the students’ current way of seeing, the *lived object of learning* (Marton, 2015). It is necessary that through the lesson the teacher draws attention to the patterns of variation and invariance as this is what can either make learning possible or constrain it (Kullberg et al., 2017; Marton & Pang, 2006). Mason (2017) emphasizes that “teaching is about directing learner attention appropriately” (p. 410). The result of these acts of teaching is the *enacted object of learning* (Marton, 2015). Finally, the outcome of the lesson, what the students have learned, then re-establishes the new *lived object of learning*.

When considering the object of learning in the taxicab distance exercise, Watson & Mason (2006) emphasize the need for discussion in order to allow learners to affirm and consolidate their experience, or what I would argue enhances the fusion and generalization of a learning experience. Watson & Mason claim the enacted object of learning requires the teacher to say "look at this" or recognize what the student is looking or thinking about, noting that "teachers can … aim to constrain the number and nature of the differences they present to learners and thus increase the likelihood that attention will be focused on mathematically crucial variables" (p. 102). The important role of the teacher when enhancing learning through the use of structured variation cannot be overlooked.

Marton (2015) discusses how the use of learning studies, a combination of a traditional Japanese lesson study and design research, demonstrate the impact on teaching by drawing teachers' attention to their use of variation within teaching. Learning studies involve a team of teachers collaborating to "systematically plan, enact, analyze and revise a lesson in order to help the students learn the intended object of learning" (Kullberg et al., 2017, p. 562) by using a specific theory as a guiding framework. It is by observing and analyzing a lesson being taught that the enacted object of learning becomes evident. That is, what aspects of variation and invariance did the teacher bring awareness too and did students become aware of it?

Learning studies are commonly used to demonstrate how variation theory can be applied to classroom teaching and learning as well as how teachers understanding of the object of learning is impacted (Andersson et al., 2018; Carlgren, 2018; Gunnarsoon et al., 2019; Lindberg et al., 2018; Pang & Ling, 2012; Runesson & Kullberg, 2017). Examples of the use of variation within mathematics may be highlighted through a learning study, but due to the extensive nature of the topic, details about how a learning study is enacted will not be covered in this review.

Use of Systematic Variation in the Teaching and Learning of Mathematics

There are numerous studies that demonstrate and re-iterate terminology and techniques of both Marton's (2015) variation theory framework and Gu's (1991) *bianshi* framework (Guo & Pang, 2011; Han et al, 2017; Koichu et al., 2015; Pang et al., 2016; Wong et al., 2009). Though, as Mason (2017) points out when discussing how the issues in variation theory inform pedagogical choices, there is no "single pedagogic-recipe…because effective teaching depends very much on the students and the situation" (p. 422). Rather than focus on studies that continue to reinforce the previously described theories, I have chosen to elaborate on the study by Watson & Mason (2006) that uses the taxicab distance task to demonstrate how a group of exercises can be treated as a singular mathematical object. I will then examine two studies that demonstrate the impact that the use of variation and invariance can have on the teaching and learning of mathematics.

Watson & Mason (2006) claim that "tasks that carefully display constrained variation are generally likely to result in progress in ways that unstructured sets of tasks do not" (p. 92). Recall the thoughts, ideas, and awareness that resulted from the taxicab distance exercise. Would the same engagement have occurred while performing a task involving plotting and determining distances between these seemingly random points? In their study using the taxicab distance exercise, Watson & Mason (2006) found that the variation used in the exercise spurred curiosity among participants and "evoked their natural propensity to look for similarities and to make conjectures" (p. 96). Participants reported being "jolted into thinking mathematically by being offered points that broke their current sense of pattern" (p. 97). Through patterning, the acting of using variation and invariance, learners were motivated but also developed a consistent and similar understanding, despite approaching the task in different ways.

While Marton's (2015) object of learning may be construed as representing a single aspect of a concept, Watson & Mason (2006) treat the structure of the mathematical exercise, be it the group of questions or tasks, as a singular mathematical object, using the word *object* to mean "that which is the focus of attention" (p. 100). They also elaborate on how generalization can be interpreted as "sensing the possible variation in a relationship" (p. 94) expanding that into *abstraction,* where relationships that were seen in a specific situation now have the potential to be seen in similar situations. Considering the previously mentioned example on linear functions and their equations, I interpret the generalization learners might make about the *b*-value and *y*-intercept could become an abstraction as learners experience the idea of the constant value on any equation of a polynomial function as also representing the *y*-intercept.

Kullberg et al. (2017) studied the changes in practice of teachers that had participated in a series of learning studies. Teachers were involved in studies over three semesters which involved a collaborative cycle of planning lessons using ideas from the variation theory framework, teaching, then analyzing the lesson and making revisions. In order to examine the effects on teaching practice, Kullberg et al. also analyzed a separate lesson from each teacher before participating in the learning studies and compared it with a similar lesson taught after.

Kullberg et al. (2017) analyzed examples used by the teacher using Marton's (2015) variation theory framework to discover not only had the intended and enacted object of learning changed, but the teacher's philosophy about teaching to solve linear equations had changed. In the original lesson, the teacher's object of learning was focused on the method and procedure for solving a linear equation. In the follow up lesson, after participating in the learning studies, the teacher's object of learning had changed to be about understanding the structure of the equation.

The teacher's philosophy had shifted from teaching a method resulting in a solution to a philosophy of developing a foundational understanding of an equation in order to then solve it.

In the teacher's first lesson, the dimensions of variation used involved the meaning of the equal sign, representing equations symbolically and numerically, then showing non-solvable equations, followed by solving equations in a context or word problem. In the follow up lesson after the lesson studies, the teacher's design changed. He now created 4 tasks focused on understanding equality and mathematical operations, keeping base equations invariant while systematically using variation to demonstrate the effects of different operations. As shown in Figure 3, one of the teacher's tasks is designed to demonstrate properties of equality. He kept the equation $3+4=7$ invariant while the different operations applied on both sides of the equal sign are varied. This makes it possible for students to discern how the same operation needs to occur on both sides in order to maintain equality.

Figure 3

Properties of Equality Task

1. $3+4=7$
2. $\mathbf{2}+3+4=7+\mathbf{2}$
3. $2+3+4-\mathbf{5}=7+2-\mathbf{5}$
4. $\mathbf{2}\times\mathbf{3}+4\neq 7\times 2$
5. $\mathbf{2}\times\mathbf{3}+4\times 2=7\times 2$

Note. This task shows how systematic variation was used to demonstrate properties of equality. From "What is made possible to learn when using the variation theory of learning in teaching mathematics?" by Kullberg, R., Runesson Kempe, U., & Marton, F, 2017, *ZDM Mathematics Education, 49*, p. 563.

While Kullberg et al. (2017) are unable to conclude how critical this change in lesson philosophy was on student learning, they emphasize that how a teacher decides to attend to the object of learning by incorporating patterns of variation and invariance as well as drawing attention to those patterns can change what is made possible to learn.

Preciado-Babb et al. (2019) demonstrate how variation can be used in mathematics classrooms through their Math Minds Initiative. They consider the use of systematic variation essential in creating and adapting lessons as well as informing in-the moment teaching decisions. Continuous assessment during a lesson as well as teacher's responding appropriately to that feedback is necessary and requires teachers to have a deep mathematical understanding.

The framework for Preciado-Babb et al.'s (2019) lesson design requires the use of four key teaching strategies they have termed *raveling*, *prompting*, *interpreting*, and *deciding*. Raveling involves identifying the critical discernment and decomposition of a concept. The use of macro-raveling involves the long range planning of discernments while micro-ravelling focuses on discernments within an individual lesson, connecting with similar ideas found in *bianshi* teaching (Mok, 2017; Pang et al., 2017). The act of decomposing the concepts links back to the need for teachers to be able to unpack mathematical content (Ball & Bass, 2003). Prompting involves engaging students and drawing attention to each discernment, similar to Mason's (2017) emphasis on teacher awareness when using variation. Interpreting involves sensing the lived object of learning (Marton, 2015) of each student and finally deciding is when the teacher "chooses between stepping back, lingering, or pressing on" (Preciado-Babb et al., 2019, p. 348). Drawing on Watson & Mason's (2006) use of variation, a well-designed set of tasks will allow for similar understandings to be had as well as allowing for ways to extend the task and provide further challenges as learning goals are met.

Through the Math Minds Initiative, Preciado-Babb et al. (2019) have witnessed not only significant improvement in student performance but also improvements in lesson plans and teaching approaches of teachers involved in the study. The teaching strategies used in their lesson design demonstrates how critical teacher's pedagogical content knowledge (Ball & Bass, 2003) is for planning tasks as well as responding to students' needs and abilities effectively. Metz et al. (2016) notes that teachers who engage in professional development as part of the Math Minds Initiative "engage differently with the content" (p. 1256) when it is offered using clearly structured variation. Effective resources that involve purposeful, systematic variation as well as teacher awareness to understand the object of learning and effectively draw attention to it within the classroom have a valuable impact on student engagement and achievement.

Teachers' Choice of Examples in the Mathematics Classroom

Zodik & Zaslavsky (2008) studied five experienced secondary mathematics teachers, analyzing their choice and use of examples within the mathematics classroom. "Three aspects of teacher knowledge strongly relate to exemplification in mathematics education: knowledge of mathematics, knowledge of students' learning, and pedagogical content knowledge" (p. 3). This impacts the decision-making involved in teaching, both in the pre-planned aspects as well as the classroom interactions, or as Mason & Spence (as cited in Zodik & Zaslavsky, 2008) refer to as *knowing to act in the moment*. While there has been increasing research into the mathematical knowledge needed for teaching (Bass & Ball, 2003) and "the critical and multifaceted roles of examples in learning and teaching mathematics" (Zodik & Zaslavsky, 2008, p. 168), Zodik and Zaslavsky focus on teacher knowledge and their use of examples.

Through observations and interviews with the teachers, Zodik and Zaslavsky (2008) categorized the examples used in class as pre-planned or spontaneous. Pre-planned examples often came from resources such as textbooks, whereas "spontaneous examples were often generated in response to students' queries or claims" (p. 172). Examples were also classified into considerations that teachers employed when choosing or constructing the examples. These considerations sometimes reflected pedagogical content knowledge or were sensitive to the needs of their students, but in all cases their choice "relied to a large extent on sound mathematical knowledge of the relevant topics" (p. 173).

The most common consideration in choosing examples was to "start with a simple or familiar case" (Zodik & Zaslavsky, 2008, p. 173) followed by attending to common student errors or difficulties. Teachers also considered how to "draw attention to relevant features" (p. 175) in an attempt to reduce, what Skemp (as cited in Zodik & Zaslavsky, 2008) terms as the *noise* that obstructs the ability to form a concept. Although teachers involved "claimed that they had never articulated how to select and generate examples" (Zodik & Zaslavsky, 2008, p. 173), by choosing examples with the intent of noticing relevant features demonstrates how experienced teachers use aspects of variation, similar to principles of Marton's (2015) variation theory or Watson and Mason's (2006) use of structured variation, within their teaching. Zodik & Zaslavsky (2008) noted that "they [the teachers] had never explicitly thought about these issues. Thus, by asking them to reflect on their work and examine what principles or rules of thumb guide them, they became more aware of their planning and in-the-moment actions" (p. 173). This demonstrates how through experience, teachers develop skills and principles that closely align with aspects of systematic variation. I conjecture that by educating mathematics teachers on the use of systematic variation in teaching mathematics and encouraging reflection on their own teaching practice, teachers will not only enhance their use of purposeful examples within the classroom but learn to articulate their own practice in a professional way.

Conclusion

In this paper I have explored the frameworks of *bianshi* teaching and variation theory to support how a purposeful form of systematic variation can be applied to the teaching of mathematics. The *bianshi* teaching framework guides the use of variation through the entire lesson allowing learners to experience the essence of a concept. Key features of concepts are kept constant while background features are intentionally introduced and varied. Procedural variation is also applied in the types and progression of examples used throughout the lesson. Variation theory of learning (Marton, 2015), on the other hand, focuses on the need to discern critical aspects of the object of learning. Learners focus on contrast by seeing how something is different or changes, which leads to the ability to fuse multiple aspects and generalize topics. While these two frameworks seem to advise different ways to apply variation to mathematics teaching, elements from both frameworks can work to complement each other in guiding teachers to create mathematical lessons and enhance the mathematical understanding of the learners.

The taxicab distance exercise has demonstrated how powerful intentionally structuring a task by attending to how the progression of the task varies can be. A task where learners were given random points would reinforce the skills involved in plotting points on a grid and determining distance, but would not engage learners in making connections to locus points or reaffirm a learner's ability to determine distance when plotting fractional points. If on the other hand, a teacher chose to describe geometric shapes and identify the locus points for students, there is the potential for learning to occur, but at risk of less student engagement and retention when compared to performing the taxicab distance exercise.

The systematic use of variation can be a powerful tool for teachers that can be applied to all mathematical topics within a program of studies. The process of incorporating structured variation into mathematical lessons requires teachers to make intentional choices when developing sets of examples, questions and tasks as well as being aware of what learners should attend to while doing the mathematics. This involves teachers recognizing what the intended object of learning is and keeping that focus throughout the pre-planning of examples to be used in the lesson. Teachers must also be prepared to respond to the enacted object of learning and create spontaneous examples within the learning environment and involve learners in discussion, giving them the opportunity to make their own connections. Finally, teachers must reflect on what learning has occurred in a lesson in order to guide the next learning opportunity. It is through this process of in-depth engagement with the mathematics and consideration of how learners interact with the mathematics that teachers will bridge the gap between their own mathematical knowledge and their pedagogical knowledge.

I believe using elements variation to guide the practice of teaching mathematics provides teachers the opportunity to develop more consistently meaningful examples and tasks as well as enhance the learning opportunities available to students. Teachers already use some form of variation within their teaching. Exploring the theories and research related to the use of variation provides teachers with the language to explain their teaching practice as well as guidelines to continue to make their use of variation more consistently effective. This will ultimately enhance the learning and understanding in the mathematics classroom for both students and teachers.

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