G-Reconstruction and The Hopf Equivariantization Theorem

by

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Abstract

G-structures on fusion categories have been shown to be an important tool to understand orbifolds of vertex operator algebras [24][30][20]. We continue to develop this idea by generalizing Eilenberg-Maclane's notion of an Abelian 3-cocycle to describe G-structures on fusion categories as G-(crossed, ribbon) Abelian 3-cocycles on an algebra H. In particular, we show that a G-(crossed, ribbon) Abelian 3-cocycle on H will induce a G-(crossed braided, ribbon) tensor structure on its category of modules $\mathrm{Mod}(H)$. We then prove that every G-(crossed braided, ribbon) fusion category $\mathcal{C}$ will be equivalent to the category of modules of some finite dimensional algebra H with G-structure induced from a G-(crossed, ribbon) Abelian 3-cocycle. We call this G-reconstruction.

Lastly, we prove that a G-ribbon Abelian 3-cocycle Γ on H allows us to describe the equivariantization $(\mathrm{Mod}(H))^G$ as the category of modules of a ribbon (weak) quasi Hopf algebra $H\#_\Gamma\mathbb{C}[G]$. We call this the Hopf equivariantization theorem.

By G-reconstruction this shows that if $\mathcal{V}$ is a strongly rational vertex operator algebra where G acts faithfully on $\mathcal{V}$ such that $\mathcal{V}^G$ is also strongly rational, then there is an equivalence of modular fusion categories:

$$\mathrm{Mod}\ \mathcal{V}^G \cong \mathrm{Mod}(H\#_\Gamma\mathbb{C}[G]) \tag{0.1}$$

for some finite dimensional algebra H with a G-ribbon Abelian 3-cocycle Γ. This provides a proof of the Dijkgraaf-Witten conjecture, and generalizes it as far as possible in the semi-simple setting.

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Introduction

1.1 Overview

Humans are really only good at linear algebra and counting. As such, an important principle in much of mathematics is to reduce complicated problems to linear algebra. For example, differentiation in analysis and representations in group theory.

Another area where this principle is important is in tensor categories. Tensor categories are a mathematical tool that gives us a uniform way to understand a wide array of mathematical structures. Essentially a tensor category is the categorification of a ring with a unit. That is a tensor category is a category $\mathcal{C}$ where you can add objects, there is a zero object, you can multiply objects through an operation $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product, $\otimes$ has a unit object, and it is associative up to natural isomorphism. If a non-zero object in a tensor category has no non-trivial sub-objects we say it is simple. In this thesis we will only be working with tensor categories where every object is isomorphic to a sum of simple objects. Such tensor categories are called fusion categories. Using the categorification analogy, think of them as semi-simple rings.

To make this categorification analogy even more precise, one can associate to every fusion category $\mathcal{C}$ something called a fusion ring, denoted by $\mathcal{K}^0(\mathcal{C})$. This is defined by letting $\mathcal{O}$ be the set of isomorphism classes of simple objects in $\mathcal{C}$ and then taking the free $\mathbb{Z}$-module generated by $\mathcal{O}$. Multiplication on elements $[X],[Y] \in \mathcal{O}$ is given by writing $[X \otimes_{\mathcal{C}} Y]$ as the direct sum of simple objects.

The simplest example of a fusion category is the category of finite dimensional vector spaces over $\mathbb{C}$, which we denote by Vect. The tensor product is the usual tensor product of vector spaces, and the unit object is $\mathbb{C}$. As every finite dimensional vector space V is isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$ we see that $\mathbb{C}$ is the only simple object of Vect. Therefore, $\mathcal{K}^0(\text{Vect}) \cong \mathbb{Z}$. For this reason Vect is called the trivial fusion category. Throughout this thesis all groups G are assumed to be finite. Two examples that will be particularly important for this thesis are the category of representations of a group G, and the category of G-graded vector spaces. Denote the former by $\text{Rep}(G)$. Objects of $\text{Rep}(G)$ are tuples (V, ρ_V) where V is a finite dimensional vector space over $\mathbb{C}$, and $\rho_V : G \to GL(V)$ is a group homomorphism. The tensor product of $\text{Rep}(G)$ is defined for two objects $(V, \rho_V), (W, \rho_W)$ as $(V \otimes_{\mathbb{C}} W, \rho_{V \otimes W})$ where $\rho_{V\otimes W}(g) := \rho_V(g) \otimes \rho_W(g)$ for $g \in G$. A morphisms in $\text{Rep}(G)$ between $(V, \rho_V), (W, \rho_W)$ is a linear map $f : V \to W$ such that for all $g \in G$:

$$\rho_W(g) \circ f = f \circ \rho_V(g) \tag{1.1}$$

such a morphism called an intertwiner. The fusion ring of $\text{Rep}(G)$ is isomorphic to ring of characters on G. The category of G-graded vector spaces is the category whose objects are finite dimensional vector spaces $V = \bigoplus_{g\in G} V_g$ graded by G. Morphisms $f : V \to W$ in Vect_G are linear maps such that for every $g \in G$ $f(V_g) \subset W_g$. The tensor product is defined as $(\bigoplus_{g\in G} V_g) \otimes (\bigoplus_{g\in G} W_g) = \bigoplus_{g\in G}(V \otimes_{\mathbb{C}} W)_g$ where:

$$(V \otimes_{\mathbb{C}} W)_g := \bigoplus_{\substack{r,k\in G \\ rk=g}} V_r \otimes W_k \tag{1.2}$$

The unit object will be the vector space $(\mathbb{C})_e$ whose g-th component is the zero vector space except when $g = e$ in which case it is $\mathbb{C}$. The set of simple objects of Vect_G are the vector spaces $(\mathbb{C})_g$ for $g \in G$, where $(\mathbb{C})_g$ is defined in a similar way as $(\mathbb{C})_e$. From this one sees then that the set of simple objects, up to isomorphism, of

Vect_G is G and the tensor product on the simple objects is just the group operation. Therefore, $\mathcal{K}^0(\mathrm{Vect}_G) \cong \mathbb{Z}[G]$. For this reason Vect_G is an instance of what is called a *categorical group*. In general a categorical group is a fusion category whose set of simples form a group. That is the fusion ring is a group algebra.

Just as fusion categories are the categorification of unital rings, braided fusion categories are the categorification of commutative unital rings. In [22] Joyal and Street proved that all possible braidings on a categorical group with group of simples G are give by tuples (ω, c) called Abelian 3-cocycles on G. In short an Abelian 3-cocycle is a 3-cocycle ω on G with an ω-twisted G bi-character. For the details we refer the reader to Appendix A.6.1. Just as in usual group cohomology one can define when two Abelian 3-cocycles on G are equivalent. Quotienting out by this relation one obtains the Abelian cohomology group $H^3_{\mathrm{Ab}}(G, \mathbb{C}^\times)$ and this will be in bijection with equivalence classes of braided structures on categorical groups with underlying group G. One of the goals of this thesis is showing that *G-structures* on fusion categories are completely described through a similar cohomological story.

An important example of a G-structure on a fusion category is a categorical group action. Using the ring analogy, a categorical group action is the categorification of a group acting on a ring as ring automorphisms. Another important type of G-structure is a G-crossed braided fusion categories. To motivate this it is best to look at what a G-crossed braided structure is on a categorical group. This was done by Naidu in [31]. In this case if $\mathcal{C}$ is a categorical group with underlying group K, then a G-crossed structure on $\mathcal{C}$ is essentially the categorification of a crossed module (K, G, ∂), that is G acts on K as group automorphisms and $\partial : K \to G$ is a group homomorphism such that for all $g \in G, k_1, k_2 \in K$

$$\partial({}^g k_1) = g\partial(k_1)g^{-1} \tag{1.3}$$

$$^{\partial(k_1)}k_2 = k_1 k_2 k_1^{-1} \tag{1.4}$$

The categorification of a crossed module includes a piece of data called a quasi-Abelian 3-cocycle Γ and this consists of a tuple of cocycles defined on K, G. When $G = \{e\}$ this recovers the usual definition of an Abelian 3-cocycle hence the name.

In general a G-crossed braided fusion category $\mathcal{C}$ will have a G-grading $\mathcal{C} = \bigoplus_{g\in G}\mathcal{C}_g$ and a collection of natural isomorphisms for every $X \in \mathcal{C}_g, Y \in C$

$$c_{X,Y} : X \otimes Y \to T_g(Y) \otimes X \tag{1.5}$$

among many other conditions. This was first defined by Müger in [30] in relation to orbifolds of conformal nets, something that we will explain later. Notice that when $\mathcal{C}$ is a categorical group this just means that $T_g(Y) \cong X \otimes Y \otimes X^{-1}$ which is one of the crossed axioms. A major result of this thesis is giving a cohomological description of G-crossed braided fusion categories similar to Naidu's but for *all* G-crossed braided fusion categories. Additionally, we provide a cohomological description of all G-ribbon fusion categories, these are G-crossed braided fusion categories with the categorical version of a G-equivariant quadratic form.

Joyal and Street's classification of braided structures on categorical groups, and Naidu's classification of crossed braided structures on categorical groups are both examples of the principle of reduction to linear algebra being applied to fusion categories. Another example of this principle in action, that is fundamental to theory developed in this thesis, is the idea of a quasi-triangular weak quasi Hopf algebra.

As a quick reminder, a Hopf algebra H is a unital associative $\mathbb{C}$-algebra with a co-multiplication $\Delta : H \to H \otimes H$, co-unit $\epsilon : H \to \mathbb{C}$ and antipode $S : H \to H$. The co-multiplication is required to be co-associative:

$$(\mathrm{Id}_H \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}_H) \circ \Delta \tag{1.6}$$

and the co-unit must satisfy:

$$(\epsilon \otimes \mathrm{Id}_H) \circ \Delta = (\mathrm{Id}_H \otimes \epsilon) \circ \Delta = \mathrm{Id}_H \tag{1.7}$$

Furthermore, Δ, ϵ must be unital algebra homomorphisms. The reason we want such a structure is it allows us to endow $\mathrm{Mod}(H)$ with the structure of a tensor category by defining the tensor product of two finite dimensional representations $(V, \rho_V), (W, \rho_W)$

as:

$$(V, \rho_V) \otimes (W, \rho_W) = (V \otimes_{\mathbb{C}} W, (\rho_V \otimes \rho_W) \circ \Delta) \tag{1.8}$$

Furthermore, ϵ makes $\mathbb{C}$ a H-module and this will be the tensor unit with this multiplication. Because of Equation 1.6 the tensor product will be associative. Another piece of data included in a Hopf algebra is an anti-pode, this is anti-automorphisms of H and is needed so we can define *dual objects* in $\mathrm{Mod}(H)$. When H is semi-simple $\mathrm{Mod}(H)$ will be a fusion category that is purely defined through linear algebra. An important example to keep in mind is $\mathbb{C}[G]$ where the co-product defined for $g \in G$ as $\Delta(g) = g \otimes g$, the co-unit is $\epsilon(g) = \delta_{g,e}$ and antipode is $S(g) = g^{-1}$. Unfortunately, to apply this technique to all fusion categories we need a more the general notion of a weak quasi Hopf algebra.

Weak quasi Hopf algebras, or wqhfs for short, were first introduced by Drinfeld in [11]. Endowing a unital $\mathbb{C}$-algebra H with the structure of a wqhf will induce tensor category structure on $\mathrm{Mod}(H)$, and essentially reduces the intricacies of tensor categories to linear algebra. When H is semi-simple one should think of H as representing the fusion ring of $\mathrm{Mod}(H)$. The details of a wqhf will be covered in Chapter 2, but briefly a wqhf is a Hopf algebra H with an (partially) invertible element $\Phi \in H \otimes H \otimes H$ called the *Drinfeld Associator* such that Equation (1.6) holds up to conjugation by this element. Φ is required to satisfy some "cohomological" conditions with respect to the co-multiplication Δ and so in some sense is a generalization of 3-cocycles to fusion rings that aren't just group algebras. To give $\mathrm{Mod}(H)$ the structure of a braided tensor category one needs to endow the wqhf H with a quaitriangular structure, qt structure for short. This is an (partially) invertible element $R \in H \otimes H$ called the *R-matrix* and and the R-matrix R will be a sort of "twisted bicharacter" with respect to Φ. As with a wqhf structure, a qt wqhf structure on H reduces the intricacies of braided tensor categories to linear algebra. Furthermore, one can define when two qt wqhfs are similar through something called twist equivalences (see Section 2.7) and it can be shown that two qt wqhf structures on H are twist equivalent if and only if the induced braided tensor category structures on $\mathrm{Mod}(H)$ are equivalent. For these reasons, one can think of the tuple (Φ, R)

as generalizing an Abelian 3-cocycle to a fusion ring. Indeed letting $H = \mathbb{C}^G$ be the algebra of functions on G, one recovers the definition of an Abelian 3-cocycles.

In Chapter 3 we introduce three types of G 3-cocycles on a unital $\mathbb{C}$-algebra H:

1. G Abelian 3-cocycles: These are wqhf structures on H with a compatible "projective" group action of G on H, along with some cocycles defined on H. We show that every G Abelian 3-cocycle on H endows $\mathrm{Mod}(H)$ with the structure of a tensor category and a categorical G-action.

2. G-crossed Abelian 3-cocycles: These are G Abelian 3-cocycles with an element $\overline{c} \in H \otimes H$ called the G-crossed R-matrix. We show that every G-crossed Abelian 3-cocycle endows $\mathrm{Mod}(H)$ with the structure of a G-crossed braided tensor category.

3. G-ribbon Abelian 3-cocycles: These are G-crossed 3-cocycles with a G-equivariant ribbon element $\nu \in H$. We show that every G-ribbon Abelian 3-cocycle endows $\mathrm{Mod}(H)$ with the structure of a G-ribbon tensor category.

We also define when such 3-cocycles are equivalent, and hence obtain the set of equivalence classes $H^3_{G-\mathrm{Ab}}(H), H^3_{G-\mathrm{Crssd}}(H), H^3_{G-\mathrm{Rbbn}}(H)$. We prove that equivalent 3-cocycles induce equivalent G-structure on $\mathrm{Mod}(H)$. In the case that $H := \mathbb{C}^G$ we recover Naidu's definition of a quasi-Abelian 3-cocycle, and hence an Abelian 3-cocycle. Therefore, the material in Chapter 3 is a generalization of Eilenberg-Maclane's Abelian 3-cocycles and Naidu's quasi-Abelian 3-cocycles.

The main reason we have chosen to develop this this story for wqhf is that every (braided) fusion category is equivalent to the category of modules of some (qt) wqhf. This was shown in [21] by Häring, and is an example of a reconstruction result. Essentially reconstruction is the process of determining an algebraic object (up to isomorphism) by it category modules. For example, if you have a Hopf algebra H, then there is a forgetful functor $\mathrm{Forg} : \mathrm{Mod}(H) \to \mathrm{Vect}$. One can show that the set of natural transformations of the functor $\mathrm{Nat}(\mathrm{Forg})$ will be a Hopf algebra and is isomorphic to H. In general if one has a nice fusion category $\mathcal{C}$ and a *fiber functor* $(F, J^F, \phi^F) : \mathcal{C} \to \mathrm{Vect}$, then one can show that $H := \mathrm{Nat}(F)$ is a Hopf algebra and there is an equivalence of fusion categories $\mathcal{C} \cong \text{Mod}(H)$. Think of a fiber functor as the categorical version of a ring homomorphism to $\mathbb{Z}$. In general it is not always possible to find a fiber functor for every fusion category, and instead one must settle for a weak quasi fiber functor $(F, J^F, \phi^F) : \mathcal{C} \to \text{Vect}$. The details of what a weak quasi fiber functor are explained in Section 2.8. In this case $H := \text{Nat}(F)$ will instead be a wqhf. As explained in [21], if $\mathcal{C}$ is a fusion category with set of simples $\mathcal{O}$, then one way to construct a weak quasi fiber functor is by assigning for ever $X \in \mathcal{O}$ a natural number $D(X)$ in a way that is weakly compatible with the tensor structure of $\mathcal{C}$. From this assignment you can create a weak quasi fiber functor by sending $F(X)$ to $\mathbb{C}^{D(X)}$ for simple X and extending by linearity to an arbitrary object. Such a function is called a weak dimension function on $\mathcal{C}$. Häring proved that every (braided) fusion category has a weak dimension function, hence a weak quasi fiber functor and therefore is equivalent to the category of modules of some (qt) wqhf.

In Chapter 4 we prove a G-reconstruction result. That is we show every G-(ribbon,crossed braided) structure on a fusion category is equivalent to the category of modules of some wqhf H with G-structure induced by some G-(ribbon,crossed) Abelian 3-cocycle. Furthermore we show that equivalent G-(ribbon, crossed braided) fusion categories induce equivalent G-(ribbon,crossed) Abelian 3-cocycles on H. The main technical tool we introduce for this is the concept of a *weak quasi G-equivariant fiber functor*. This is a weak quasi fiber functor $F : \mathcal{C} \to \text{Vect}$ such that the functor is G-equivariant up to natural isomorphism. The G-reconstruction theorem implies that G-(ribbon,crossed) Abelian 3-cocycles on a finite dimensional algebra H completely describe all fusion categories with G-(ribbon,crossed) structures. Therefore, the material in Chapter 4 can be thought of as a vast generalization of Joyal and Street's classification to *all* G-structure on fusion categories.

Just as a categorical group action is the categorification of a group action, equivariantization is the categorical version of taking the fixed points. Understanding the equivariantization of G-(ribbon,crossed braided) fusion categories has large implications for understanding orbifolds of vertex operator algebras.

Vertex operator algebras, or VOAs for short, were first defined by Borcherd's in [3]. Being very loose with the details, a VOA is an infinite dimensional vector space V, with an infinite collection of multiplications packaged together in a map $Y : V \otimes_{\mathbb{C}} V \to V[[z]]$ called the state-field correspondence. Here $V[[z]]$ denote the formal power series in z, z^{-1} with coefficients in V. Despite their complicated nature, VOAs play a fundamental role in describing the mathematical structure of a special type of quantum field theories called conformal field theories. Due to this, VOAs has been a very active area of study for the past 30+ years.

Another reason from a mathematical standpoint is there a natural definition of a module of a VOA $\mathcal{V}$, and so one can consider the category Mod $\mathcal{V}$. When $\mathcal{V}$ is *strongly rational* Mod $\mathcal{V}$ will be something called a modular fusion category. Using the categorifictaion analogy a modular fusion category is a commutative ring with an action of the modular group $\mathbf{PSL}(2, \mathbb{Z})$ on it. There is a conjecture [18] that all modular fusion categories are the category of modules of some strongly rational VOA. This is called the *Reconstruction Conjecture* for VOAs, and slowly progress towards solving this conjecture has been made [15].

For this reason among many others it would be extremely valuable to understand the categorical structure of a large family of VOAs called *orbifolds*. An orbifold VOA is a VOA $\mathcal{V}$ with a group G acting on it, we won't define what it means for a group to act on a VOA but see [26] for details, one can take the fixed points $\mathcal{V}^G$ and this will again be a VOA which we call an orbifold. A large amount of work has been done by the VOA community to understand specific cases of the general problem:

> Give a strongly rational VOA $\mathcal{V}$ with a finite group G acting on it. Can you determine the categorical structure of Mod $\mathcal{V}^G$ from Mod $\mathcal{V}$ and the action of G?

The most famous conjecture of this form is the 25+ year old Dijkgraaf-Witten conjecture that says if $\mathcal{V}$ is a strongly rational VOA with a finite group G acting faithfully on it and Mod $\mathcal{V} \cong$ Vect then there exists a 3-cocycle ω on G such that Mod $\mathcal{V}^G \cong$ Mod $D^\omega(G)$ [8]. We won't go into detail of what $D^\omega(G)$ is and instead reference the interested reader to [7], but essentially it is a quasi Hopf algebra with vector space $\mathbb{C}^G \otimes \mathbb{C}[G]$.

Returning to the tensor categorical side, it is known that if $\mathcal{C}$ is a G-(ribbon, crossed braided) fusion category then the equivariantization $\mathcal{C}^G$ will be a (ribbon, braided) fusion category. In Chapter 5 we show that if H is a unital $\mathbb{C}$-algebra and Γ is a G-(ribbon,crossed) Abelian 3-cocycle on H, then the equivariantization of $(\mathrm{Mod}(H))^G$ will be equivalent as a (ribbon, braided) fusion category to the category of modules of $H\#_\Gamma\mathbb{C}[G]$. Here $H\#_\Gamma\mathbb{C}[G]$ is a (ribbon, qt) wqhf defined through the G-(ribbon,crossed) Abelian 3-cocycle Γ on H. In particular, the underlying vector space will be $H \otimes_{\mathbb{C}} \mathbb{C}[G]$. We call this the Hopf equivariantizatio theorem. Combining this with the G-reconstruction theorem from Chapter 4 we obtain a linear algebraic description of the equivariantization of all G-(ribbon,crossed braided) fusion categories. As we will discuss in more detail in the next section this provide a uniform categorical description of *all* strongly rational VOA orbifolds. In particular the Dijkgraaf-Witten theorem is a special case of this description.

Lastly, it should be noted that while the result that every fusion category is equivalent to $\mathrm{Mod}(H)$ for some wqhf H is important, finding the particular finite dimensional algebra H corresponding to a fusion category $\mathcal{C}$ and determining the wqhf structure requires intimate knowledge of $\mathcal{C}$'s fusion ring. When $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ is a G-crossed braided fusion category it is often possible to determine this data simply from $\mathcal{C}_e$ and the action of G[14][1]. We believe that for this reason G-(crossed,ribbon) Abelian 3-cocycles could be a useful tool to understand a wide array of fusion categories.

For the readers convenience we have included a concise description of the thesis. Chapters $1, 2$ are background, while Chapter $3, 4, 5$ are completely original.

- **Chapter 1**: We review the needed G-structures namely, G-tensor categories, G-crossed braided tensor categories and G-ribbon tensor categories. We prove that these structures can be transported along adjoint equivalences.

- **Chapter 2:** We review the various types of wqhf structure on H and how they induce categorical structures on $\mathrm{Mod}(H)$.

- **Chapter 3:** We define various G Abelian 3-cocycles on a unital $\mathbb{C}$-algebra H. Namely:

1. G Abelian 3-cocycles
2. G-Crossed Abelian 3-cocycles
3. G-Ribbon Abelian 3-cocycles

We show that a G Abelian 3-cocycle induces a G-tensor category structure on Mod(H) and a G-(crossed, ribbon) Abelian 3-cocycle on H will induce a G-(crossed braided, ribbon) structure on Mod(H).

- **Chapter 4:** We prove G-reconstruction for fusion categories. That is if $\mathcal{C}$ is a fusion category with one of the previously mentioned G-structures, then there exists a finite dimensional algebra H and corresponding G 3-cocycle $\Gamma_{\mathcal{C}}$ such that $\mathcal{C}$ is equivalent to Mod(H) with G-structure induced by $\Gamma_{\mathcal{C}}$.

- **Chapter 5:** We prove the Hopf equivariantization theorem. That is if H is a unital $\mathbb{C}$-algebra with G-(crossed, ribbon) Abelian 3-cocycle Γ, then we define a (qt, ribbon) wqhf $H\#_{\Gamma}\mathbb{C}[G]$ and we show that there is a equivalence of (braided,ribbon) tensor categories:

$$(\text{Mod}(H))^G \cong \text{Mod}(H\#_{\Gamma}\mathbb{C}[G]) \tag{1.9}$$

where $(\text{Mod}(H))^G$ denotes the equivariantization of Mod(H). In particular by the previous chapter this provides a description of the equivariantization of all G-(crossed braided, ribbon) fusion categories.

If the reader only wants to understand the results from Chapter 5 and does not care about the details, they should skip Chapter 4. Lastly, as the details of some proofs get quite complicated we have moved a majority of the straightforward proofs to the appendix so as to not impede the flow of reading. When a detail has been moved to the appendix the specific section will be referenced.

1.2 Further Discussion and Related Work

While the previous section was pedagogical in this section we discuss in detail the implications the results of this thesis has to the area of VOAs and tensor categories, hence we assume the reader to be knowledgeable in these areas.

As previously stated, the pinnacle of this thesis is forming the following theorem about VOA orbifolds:

Theorem 1.2.1. *Let $\mathcal{V}$ be a strongly rational VOA, and assume that a finite group acts on $\mathcal{V}$ such that $\mathcal{V}^G$ is rational. Then there exists a semi-simple finite dimensional $\mathbb{C}$-algebra H with a G-ribbon 3-cocycle Γ on it such that there is an equivalence of modular fusion categories:*

$$\text{Mod } \mathcal{V}^G \cong \text{Mod}(H \#_\Gamma \mathbb{C}[G]) \tag{1.10}$$

This follows from the main theorem of Chapter 5, Theorem 6.2.2, and utilizes the idea of the "Orbifold Triangle" which we now briefly recall. If $\mathcal{V}, G$ are as as in Theorem 1.2.1 then there are three important actors: $\text{Mod } \mathcal{V}, \text{Mod } \mathcal{V}^G, \text{TwMod}_G(\mathcal{V})$. Here $\text{TwMod}_G(\mathcal{V})$ denotes the category of direct sums of g-twisted $\mathcal{V}$-modules for $g \in G$. By [29] $\text{TwMod}_G \mathcal{V}$ will be a G-ribbon fusion category. These actors will be the vertices of the triangle, are and related to one another through various processes as illustrated below:

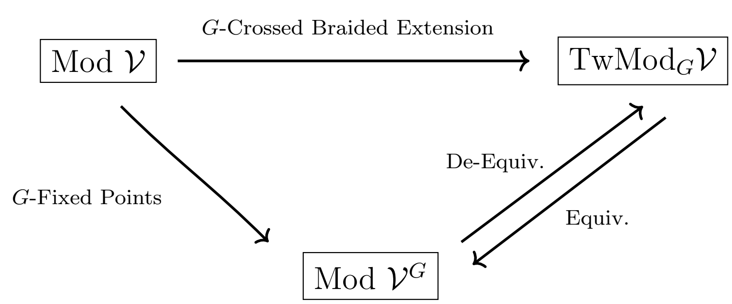

Figure 1.1: The Group Orbifold Triangle

Here Equiv. stands for equivariantization and De-Equiv. stands for de-equivariantization. Think of equivariantization as a restriction functor and de-equivariantization as induction functor. The main principle of the orbifold triangle is that if we already know one of the vertices it is possible to determine or say a lot about the others. For example, if we know what Mod $\mathcal{V}$ is and how G acts on it, then we can say a lot about the G-ribbon structure of $\mathrm{TwMod}_G\ \mathcal{V}$, and therefore the ribbon fusion category Mod $\mathcal{V}^G$. It should also be noted that under the assumption of Theorem 1.2.1 we can apply G-reconstruction to the twisted modules to obtain an equivalence of G-ribbon fusion categories $\mathrm{TwMod}_G\ \mathcal{V} \cong \mathrm{Mod}(H)$. The proof of Theorem 1.2.1 follows from Theorem 6.2.2 combined with the fact proven in [24] that:

$$(\mathrm{TwMod}_G\ \mathcal{V})^G \cong \mathrm{Mod}\ \mathcal{V}^G \tag{1.11}$$

One can think of H as an algebra representing the fusion ring of $\mathrm{TwMod}_G\ \mathcal{V}$, and so Theorem 1.2.1 highlights the role the G-twisted modules play in determining the categorical structure of Mod $\mathcal{V}^G$. A related idea to the orbifold triangle is a generalization called the hypergroup obrifold triangle, which we now briefly explain.

A recent idea in physics has been the idea of a non-invertible symmetries of quantum field theories [16]. Mathematically a non-invertible symmetry is just a fusion ring acting on a quantum field theory. As a VOA describes the chiral half of a conformal field theory there should be a notion of a fusion ring acting on a VOA. The precise mathematical description of what it means for a fusion ring to act on a VOA is currently in development by the author and Terry Gannon [19]. In [32] the author was able to show that for sufficiently nice VOA extension $\mathcal{W} \subset \mathcal{V}$ there exists a fusion ring K (also called a hypergroup) "acting" on $\mathcal{V}$ such that $\mathcal{W} = \mathcal{V}^K$. We say "acting" as the fusion ring action is defined purely through categorical means instead of through the VOA structure of $\mathcal{V}$. One can think of taking the fixed points of a fusion ring as a generalized orbifold, and just as in the group obrifold case there should be a corresponding hypergroup orbifold triangle with the same actors but generalized to the hypergroup setting. The reason we mention this is that the hypergroup orbifold triangle may play a big role in solving Gannon's VOA reconstruction conjecture. For if it is possible to define what it means for a fusion category to act on VOA then if $\mathcal{C}$ acts on a holomorphic VOA $\mathcal{V}$, and we can show the fixed points $\mathcal{V}^{\mathcal{C}}$ is strongly rational, there will be an equivalence of modular fusion categories Mod $\mathcal{V}^{\mathcal{C}} \cong \mathcal{Z}(\mathcal{C})$. Here $\mathcal{Z}(\mathcal{C})$ denotes the Drinfeld double of $\mathcal{C}$. For a more detailed and complete discussion of both the group and hypergroup orbifold triangles see [20][32].

In the categorical direction, Theorem 6.2.2 generalizes the work of Naidu in [31] to all G-crossed fusion categories. More specifically in [31, Definition 3.4] the author defines quasi-Abelian 3-cocycles on a crossed module, and when they are equivalent. After accounting for different conventions, the definition of a G-crossed 3-cocycle Definition 4.2.2 and when they are equivalent Definition 4.2.3 generalizes Naidu's notions to all G-crossed braided fusion categories. Furthermore, one of the main results in [31] is that all G-crossed pointed categories are determined up to equivalence by the cohomology group of quasi-Abelian 3-cocycles [31, Theorem 4.4]. This is a special case of G-reconstruction that we prove in Chapter 4. Lastly, Naidu describes the equivaiantization of all G-crossed pointed categories through a quasi-Hopf algebra. The weak quasi Hopf algebra $H\#_{\Gamma}\mathbb{C}[G]$ and Theorem 6.2.2 recovers this result as a special case.

As mentioned in the last section the Theorem 1.2.1 provides a proof of the Dijkgraaf-Witten conjecture, which concerns holomorphic VOAs. Examples of holomorphic VOAs include lattice VOAs $\mathcal{V}_L$ where the lattice L is self-dual. For example, take the Leech lattice or the E_6, E_8 root lattices. In general, if one has a lattice VOA its category of modules Mod $\mathcal{V}_L$ will be a categorical group, VOAs whose category of modules are a categorical group we call *pointed VOAs*. A more general conjecture then the Dijkgraaf-Witten is the Mason-Ng conjecture. The Mason-Ng conjecture essentially gives a quasi Hopf algebra description for a special type of orbifolds of pointed VOAs. For the details of what the conjectures says we refer the reader to [28]. The detailed proof of the Mason-Ng conjecture is given in [20], but essentially the proof comes down to first showing the category of G-twisted modules is pointed, applying Naidu's equivariantization description and then showing that Mason-Ng's quasi-Hopf algebra in [28] is the same as Naidu's quasi-Hopf algebra in [31]. The last step is a technical issue that comes up because Mason-Ng [28] uses right G-actions while Naidu [31] uses left G-actions. Since Naidu's equivariantization description is a special case of Theorem 6.2.2, we see that Theorem 6.2.2 proves the Mason-Ng conjecture and hence the Dijkgraaf-Witten conjecture.

In the case that $\mathcal{V}$ is strongly rational and its category of modules is pointed with group of simples A, Theorem 1.2.1 can be thought of generalizing the Dijkgraaf-Witten conjecture to as far as possible in the semi-simple case. For one can show in this case that the finite dimensional H in Theorem 1.2.1 can be chosen so that:

$$H \cong \oplus_{g\in G}\mathrm{Mat}([A : A^g])^{\oplus |A^g|} \tag{1.12}$$

Here $A^g := \{a \in A : g \cdot a = a\}$ and $[A : A^g]$ is the index. In the case that $\mathcal{V}$ is holomorphic, that is $A = \{e\}$, one sees that the right hand side of reduces to $\mathbb{C}^G$ and so indeed Theorem 1.2.1 is a generalization of the Dijkgraaf-Witten conjecture.

In [10], Dong and Yamskulna introduced for every finite group G acting on a strongly rational VOA $\mathcal{V}$ an associative algebra $\mathcal{A}_\alpha(G, \mathcal{S}) = \mathbb{C}^{\mathcal{S}} \otimes \mathbb{C}[G]$, where $\mathcal{S}$ is a G-invariant subset of all inequivalent irreducible G-twisted modules of $\mathcal{V}$ and $\alpha \in Z^2(G, (\mathbb{C}^{\mathcal{S}})^\times)$. It is not difficult to show that $H\#_\Gamma\mathbb{C}[G]$ in this case will contain a sub-algebra $H[\mathcal{S}]\#_\Gamma\mathbb{C}[G]$ that is almost identical to $\mathcal{A}_\alpha(G, \mathcal{S})$ except that we have expanded $\mathbb{C}^{\mathcal{S}}$ to matrix algebras indexed by $\mathcal{S}$. Therefore, in this sense $H[\mathcal{S}]\#_\Gamma\mathbb{C}[G]$ is a generalization of $\mathcal{A}_\alpha(G, \mathcal{S})$. The advantage $H\#_\Gamma\mathbb{C}[G]$ has over $\mathcal{A}_\alpha(G, \mathcal{S})$ is it is possible to access the fusion rules of $\mathcal{V}^G$ by defining a co-product on $H\#_\Gamma\mathbb{C}[G]$, while it is not possible to define a co-product on $\mathcal{A}_\alpha(G, \mathcal{S})$ in general. This suggests that $H\#_\Gamma\mathbb{C}[G]$ is the correct algebraic object that should be studied, not $\mathcal{A}_\alpha(G, \mathcal{S})$

Lastly, it is the authors hope that Theorem 1.2.1 will turn out to be a useful tool for calculating the form of the modular data of Mod $\mathcal{V}^G$ when working with specific VOAs. This should be possible through the following process:

1. Determine the fusion ring of $\mathrm{TwMod}_G\ \mathcal{V}$ by using [2], and determine the G-action on the set of simples,

2. Find a small G-equivariant dimension function D.

3. Finding D allows you to determine H as a $\mathbb{C}$-algebra. Furthermore, one can calculate the co-product on H by using the fusion ring data.

4. Using the co-product and product from the previous step it should be possible to parametrize all possible G-ribbon Abelian 3-cocycles on H.

5. Combing this parametrization with the G-action from step 1 we can determine the simple objects of $H\#_\Gamma\mathbb{C}[G]$ and its modular data.

In the case that Mod $\mathcal{V}$ is pointed with group of simples A, we have calculated the fusion ring of $\mathrm{TwMod}_G\,\mathcal{V}$. The simple objects are parametrized by simple characters of inertia subgroups A^g. Furthermore, one can show the fusion rules $N^\tau_{\gamma,\xi}$ for $\gamma \in \widehat{A^g}, \xi \in \widehat{A^k}\ \ \tau \in \widehat{A^{gk}}$ are given by

$$N^\tau_{\gamma,\xi} = \begin{cases} \frac{|A^g\cap A^k|\sqrt{|A|}}{\sqrt{|A^{gk}|\cdot|A^g|\cdot|A^k|}} & \text{if } \mathrm{Res}^{A^{gk}}_{A^g\cap A^k}(\tau) = \gamma\cdot\xi \\ 0 & \text{otherwise} \end{cases} \tag{1.13}$$

Note we are simplifying things for the sake of brevity, for the precise statement we refer the reader to [33]. One can also choose also choose in the this case the weak dimension function given by $d(X) = \frac{|A|}{|A^g|}$ for simple $X \in \mathrm{TwMod}_g\,\mathcal{V}$. In future work we plan to show how to complete steps $3, 4, 5$ for specific VOA orbifolds.

In summary, the results in this thesis provides a uniform description of all strongly rational VOA orbifolds. The results generalize all of Naidu's results in [31], and provides a cohomological description of all G-crossed braided fusion categories and their equivariantizations. Theorem 6.2.2 proves the Mason-Ng conjecture and Dijkgraaf-Witten conjecture [28][9][20], and provide an extension of the conjectures to all pointed VOA orbifolds. The weak quasi Hopf algebras presented in Theorem 1.2.1 also generalize an important object of Dong [10] and suggest that the algebra they have been studying is the incorrect one. Lastly, it seems that it should be possible to use the results to calculate specific examples giving a powerful tool for people interested in understanding orbifold VOAs.

Chapter 2

G-Structures on Tensor Categories

We assume that the reader is familiar with the basics of tensor categories given in [13], and if one is not we briefly review the material in Appendix A. We make the convention that our associators are natural isomorphisms from $(X \otimes Y) \otimes Z$ to $X \otimes (Y \otimes Z)$, and monoidal functors (F, J^F, ϕ^F) will be strong with tensor structures isomorphisms J^F defined from $F(X) \otimes F(Y)$ to $F(X \otimes Y)$, and ϕ^F defined from 1 to $F(1)$. All groups are assumed to be finite.

In this chapter we review the G-structures on tensor categories that will be used extensively throughout the thesis. The first section covers categorical group actions on tensor categories, or G-tensor categories for short, and their associated functors and natural transformations. In the second section we review the notion of a G-crossed braided tensor category and G-ribbon tensor category, their associated functors and equivalences. Section 3 recalls equivariantization and de-equivariantization and we provide a proof that the equivariantization of a G-ribbon tensor category is a ribbon tensor category. For a more detailed overview of the material we refer the reader to the papers [30][23][12][17][13].

2.1 G-Tensor Categories

Just as a group can act on sets, a group can act on a tensor category. In particular, we can consider a category $\underline{G}$ such that its objects are elements of G and the only morphisms are scalar multiplies of the identity morphisms. The group structure of G naturally endows $\underline{G}$ with a strict tensor structure. If $\mathcal{C}$ is a tensor category, then the category of tensor endofuctors $\mathrm{End}_{\otimes}(\mathcal{C})$ is also a tensor category,ecall, see Example A.2.3 for details. If we restrict ourselves to the full sub-category of tensor auto-equivalences $\mathrm{Aut}_{\otimes}(\mathcal{C})$ we obtain a tensor sub-category.

Definition 2.1.1. *G-Tensor Category*

*Let G be a group. A G-*tensor category* is a tensor category $\mathcal{C}$ together with a tensor functor:*

$$(\psi, \gamma, \psi_0) : \underline{G} \to \mathrm{Aut}_{\otimes}(\mathcal{C}) \tag{2.1}$$

where $\psi : \underline{G} \to \mathrm{Aut}_{\otimes}(\mathcal{C})$ is a functor and (γ, ψ_0) is the tensor functor structure. Writing this out explicitly this means for every $g \in G$, $\psi(g) = (T_g, \mu_g, \phi^g)$ $T_g : \mathcal{C} \to \mathcal{C}$ is a tensor functor and (μ_g, ϕ^g) is the tensor structure of T_g. We denote a G-tensor category by a tuple $(\mathcal{C}, (\psi, \gamma, \psi_0))$. If every $\psi(g)$ is a unital tensor functor and ψ itself is a unital tensor functor we say the G-action is unital, or normalized.

See Definition A.2.2 for the definition of a unital tensor functor. From now on we will fix the group G. If $(\mathcal{C}, (\psi, \gamma, \psi_0))$ is a G-tensor category, we will sometimes refer to the tensor functor (ψ, γ, ψ_0) as the G-tensor structure on $\mathcal{C}$.

Definition 2.1.2. *G-Tensor Functors*

Let $(\mathcal{C}, (\psi^1, \gamma^1, \psi^1_0)), (\mathcal{D}, (\psi^2, \gamma^2, \psi^2_0))$ be G-tensor categories, where $\psi^1(g) = (T^1_g, \mu^1_g, \phi^g_1)$ and $\psi^2(g) = (T^2_g, \mu^2_g, \phi^g_2)$. A G-tensor functor from $(\mathcal{C}, (\psi^1, \gamma^1, \psi^1_0))$ to $(\mathcal{D}, (\psi^2, \gamma^2, \psi^2_0))$ is a tensor functor $(F, J^F, \phi^F) : \mathcal{C} \to \mathcal{D}$ and a collection of $\mathbb{C}$-linear monoidal natural isomorphisms $\tau_g : T^2_g \circ F \to F \circ T^1_g$ such that:

$$\tau_e(-) = F((\psi^1_0)_{(-)}) \circ (\psi^2_0)^{-1}_{F(-)} \tag{2.2}$$

and for all $X \in \mathcal{C}$, $g, h \in G$:

$$\tau_{gh}(X) \circ \gamma^2_{g,h}(F(X)) = F(\gamma^1_{g,h}(X)) \circ \tau_g(T^1_h(X)) \circ T^2_g(\tau_h(X)) \tag{2.3}$$

Gathering all the information, we denote a G-tensor functor by $((F, J^F, \phi^F), \{\tau_g\}_{g\in G})$. We refer to $\{\tau_g\}$ as the G-functor structure of (F, J^F, ϕ^F). When the tensor structure of F is clear we simply denote a G-tensor functor by $(F, \{\tau_g\}_{g\in G})$

Definition 2.1.3. *Natural Transformation of G-Tensor Functors*

Let $(F, \{\tau_g\}_{g\in G}), (K, \{\beta_g\}_{g\in G})$ be G-tensor functors from $(\mathcal{C}, (\psi^1, \gamma^1, (\psi^1)_0))$ to $(\mathcal{D}, (\psi^2, \gamma^2, (\psi^2)_0))$, where $\psi^1(g) = (T^1_g, \mu^1_g, \phi^g_1)$ and $\psi^2(g) = (T^2_g, \mu^2_g, \phi^g_2)$. A natural transformation *from $(F, \{\tau_g\}_{g\in G})$ to $(K, \{\beta_g\}_{g\in G})$ is a $\mathbb{C}$-linear monoidal natural transformation $\varphi : F \to K$ such that:*

$$\varphi_{T^1_g(X)} \circ \tau_g(X) = \beta_g(X) \circ T^2_g(\varphi_X) \tag{2.4}$$

Two G-tensor categories are equivalent if there exists a G-tensor functor between them that is an equivalence of categories.

Definition 2.1.4. *G-Fusion Category*

A G-fusion category is a fusion category $\mathcal{C}$ that is also G-tensor category.

We will need to transport G-tensor structures across adjoint equivalences.

Proposition 2.1.1. *Let $(\mathcal{C}, \otimes_\mathcal{C}, \alpha^\mathcal{C}, 1_\mathcal{C}, \ell^\mathcal{C}, r^\mathcal{C}), (\mathcal{D}, \otimes_\mathcal{D}, \alpha^\mathcal{D}, 1_\mathcal{D}, \ell^\mathcal{D}, r^\mathcal{D})$ be tensor categories, with G-tensor structures (ψ, γ, ψ_0) on $\mathcal{C}$ where $\psi(g) = (T_g, \mu_g, \phi^g)$. Suppose that (L, K, η, ϵ) gives an adjoint tensor equivalence where $L : \mathcal{C} \to \mathcal{D}, K : \mathcal{D} \to \mathcal{C}$ are tensor functors, and $\eta : 1_\mathcal{C} \to K \circ L$, $\epsilon : L \circ K \to 1_\mathcal{D}$ are the unit and co-unit respectively. L will induce a G-tensor structure on $\mathcal{D}$ $(\tilde{\psi}, \tilde{\gamma}, (\tilde{\psi}) : \underline{G} \to \mathrm{Aut}_\otimes(\mathcal{D})$, where $\tilde{\psi}(g) := (\tilde{T}, \tilde{\mu}_g, \tilde{\phi}^g)$ is defined by:*

$$\tilde{T}_g(X) := (L \circ T_g \circ K)(X) \qquad \tilde{T}_g(f) := (L \circ T_g \circ K)(f) \tag{Group Action}$$

$$\tilde{\mu}_g(X, Y) := (L(T_g(J^K_{X,Y}))) \circ L(\mu_g(K(X), K(Y))) \circ J^L_{T_g(K(X)),T_g(K(Y))} \tag{Tensor Structure 1}$$

$$\tilde{\phi}^g := L(T_g(\phi^K) \circ \phi^g) \circ \phi^L \qquad \text{(Tensor Structure 2)}$$

$$\tilde{\gamma}_{g,h}(X) := L(\gamma_{g,h}(K(X))) \circ L(T_g(\eta^{-1}_{T_h(K(X))})) \qquad \text{(Group Action Structure 1)}$$

$$\tilde{\psi}_0 := L((\psi_0)_{K(-)}) \circ \epsilon^{-1}_{-} \qquad \text{(Group Action Structure 2)}$$

Furthermore, denote the tensor structure of L, K as $(J^L, \phi^L), (J^K, \phi^K)$ respectively. The tensor functors $(L, J^L, \phi^L), (K, J^K, \phi^K)$ can be upgraded to G-tensor functors from $(\mathcal{C}, (\psi, \gamma, \psi_0))$ to $(\mathcal{D}, (\tilde{\psi}, \tilde{\gamma}, \tilde{\psi}_0))$ with G-functor structure given by:

$$\tau^L_g(X) := (L \circ T_g)(\eta^{-1}_X) \tag{2.5}$$

$$\tau^K_g(X) := \eta^{-1}_{(T_g \circ K)(X)} \tag{2.6}$$

The unit, and co-unit will be natural isomorphisms of G-tensor functors.

Proof. The proof of this is straightforward but tedious, and so we leave it to the studious reader. □

Proposition 2.1.2. *Every G-tensor category is equivalent to a skeletal a G-tensor category such that the left and right unitors are trivial.*

Proof. This is a consequence of the fact that every tensor category is equivalent to a skeletal tensor category with trivial unitors, see Propositions A.3.1 for details, and Proposition 2.1.1. □

The following result was proven in [17]

Proposition 2.1.3. *[17, Proposition 3.1]*

Let $(\mathcal{C}, (\psi, \gamma, \psi_0))$ be a G-tensor category. Then there exists a unital G-tensor structure $(\mathcal{C}, (\psi', \gamma', \psi'_0))$ such that $(\mathrm{Id}, \mathrm{Id}, \mathrm{Id}, \{\beta_g\}_{g \in G}) : (\mathcal{C}, (\psi, \gamma, \psi_0)) \to (\mathcal{C}, (\psi', \gamma', \psi'_0))$ is equivalence of G-tensor categories.

Due to this proposition we assume without loss of generality that all G-tensor categories are normalized.

2.2 G-Crossed Braided Tensor Categories and G-Ribbon Tensor Categories

Definition 2.2.1. *G-Grading*

Let $\mathcal{C}$ be a tensor category. A G-grading on $\mathcal{C}$ is a decomposition into Abelian categories $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ such that for all $g, h \in G$:

$$X \in \mathcal{C}_g,\ Y \in \mathcal{C}_h \Rightarrow X \otimes Y \in \mathcal{C}_{gh} \tag{2.7}$$

$$X \in \mathcal{C}_g \Rightarrow X^* \in \mathcal{C}_{g^{-1}} \tag{2.8}$$

We will be assuming that all our G-gradings are faithful. That is for every $g \in G$, $\mathcal{C}_g$ is non-trivial.

The following was first defined by Müger in [30].

Definition 2.2.2. *G-Crossed Braided Tensor Category*

A G-crossed braided tensor category is a tensor category $\mathcal{C}$, with a G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, and a G-tensor structure $(\psi, \gamma, \psi_0) : G \to \mathrm{Aut}_{\otimes}(\mathcal{C})$ where $\psi_g = (T_g, \mu_g, \psi^g)$. With the additional requirement that:

1. *$T_g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$*

2. *There is a natural collection of isomorphisms referred to as the G-braiding for $X \in \mathcal{C}_g, Y \in \mathcal{C}$:*

$$c_{X,Y} : X \otimes Y \to T_g(Y) \otimes X \tag{2.9}$$

Furthermore, the following diagrams must commute:

For all $g, h \in G$ and $X \in \mathcal{C}_h, Y \in \mathcal{C}$:

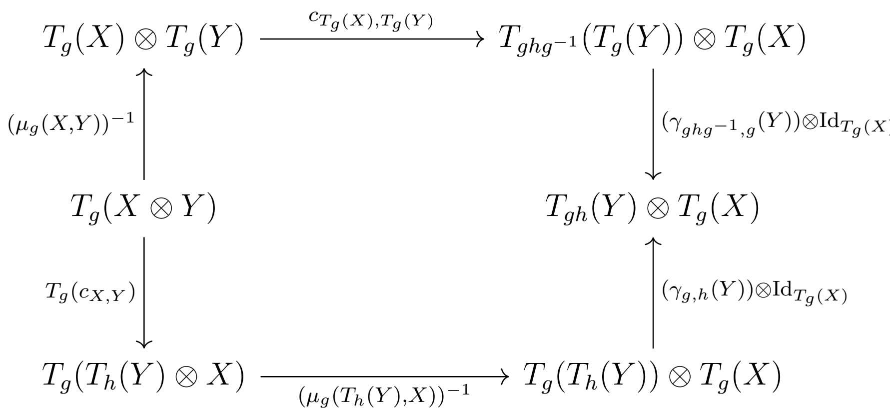

Figure 2.1: Crossed Braiding Axiom I

For all $g \in G$, $X \in \mathcal{C}_g, Y, Z \in \mathcal{C}$*:*

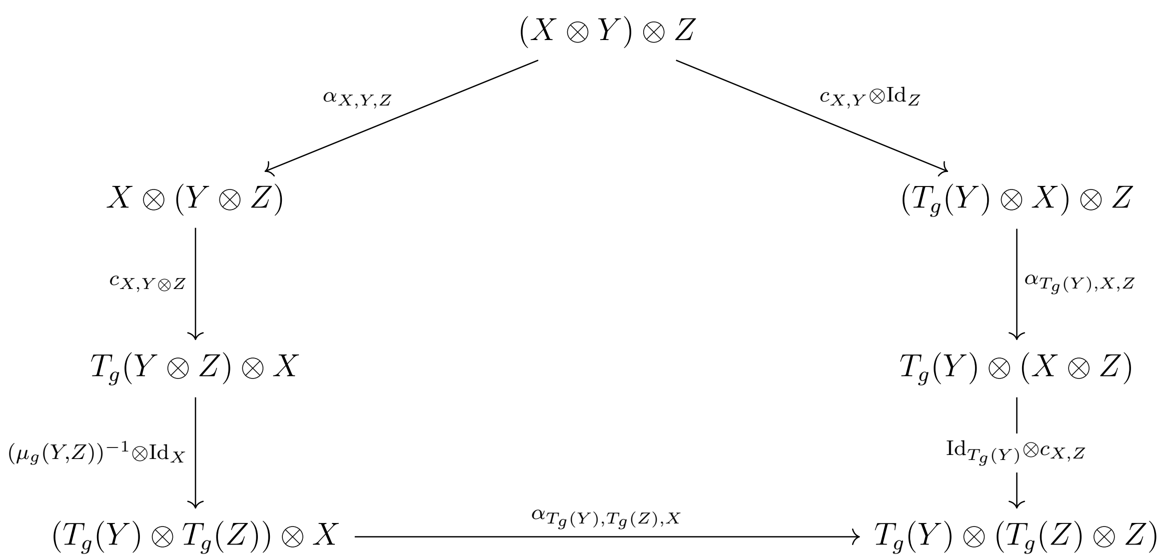

Figure 2.2: Crossed Braiding Axiom II

For all $g, h \in G$ *and* $X \in \mathcal{C}_g, Y \in \mathcal{C}_h, Z \in \mathcal{C}$*:*

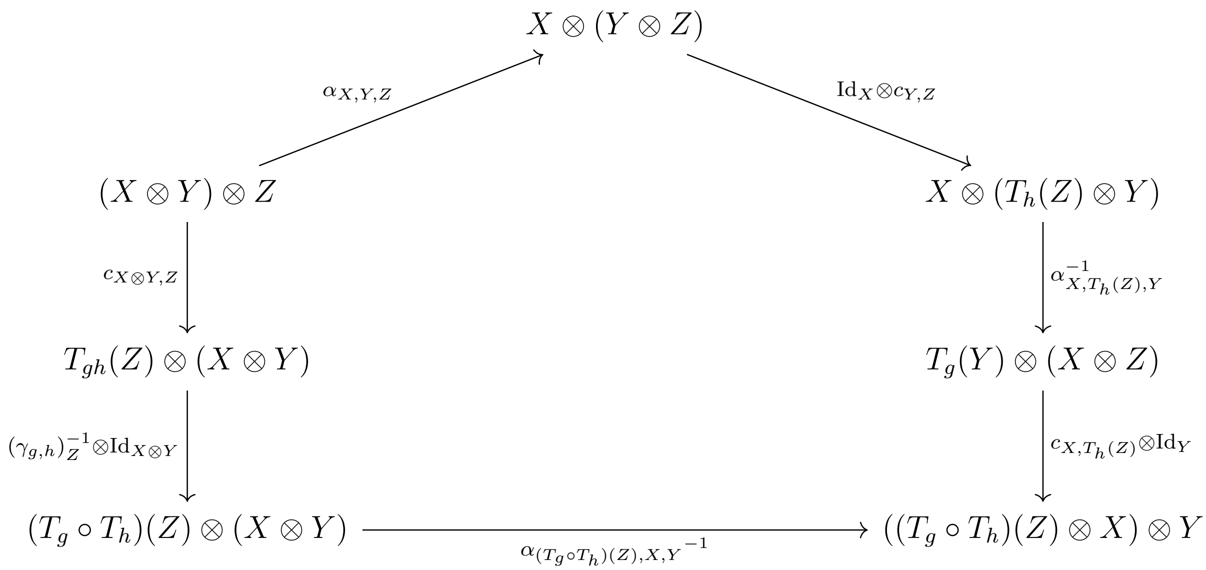

Figure 2.3: Crossed Braiding Axiom III

If $\mathcal{C}$ is a G-crossed braided tensor category we denote the structure by the tuple $(\mathcal{C}, (\psi, \gamma, \psi_0), c)$.

Definition 2.2.3. *G-Crossed Braided Fusion Category*

A G-crossed braided tensor category that is in addition a fusion category is called a G-crossed braided fusion category.

Definition 2.2.4. *G-crossed Braided Functor*

Let $(\mathcal{C}, (\psi^1, \gamma^1, \psi^1_0), c^1)$, $(\mathcal{D}, (\psi^2, \gamma^2, \psi^2_0), c^2)$ be G-crossed braided tensor categories. A G-crossed braided functor $F : (\mathcal{C}, (\psi, \gamma, \psi_0), c) \to (\mathcal{D}, (\psi, \gamma, \psi_0), c)$ is a G-tensor functor $(F, J^F, \phi^F, \{\tau_g\}_{g \in G})$ such that the following diagram commutes for all $g \in G$, $X \in \mathcal{C}_g$, $Y \in \mathcal{C}$:

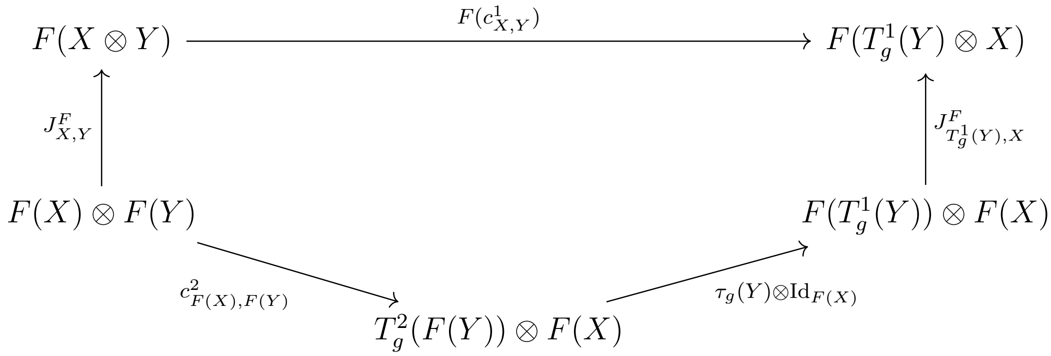

Figure 2.4: G-Crossed Braided Functor Condition I

Additionally, we require that:

$$F(\mathcal{C}_g) \subset \mathcal{D}_g \tag{2.10}$$

If $(F, J^F, \phi^F, \{\tau_g\}_{g \in G}) : \mathcal{C} \to \mathcal{D}$ is a G-crossed braided functor such that F induces an equivalence of categories we say that the G-crossed braided fusion categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent.

Remark 2.2.1. *Notice that if we set* $G = \mathrm{Id}$ *with trivial action, then we recover the notion of a braided fusion category and a braided functor.*

The following was first defined in [23]:

Definition 2.2.5. *G-Ribbon Tensor Category*

A G-ribbon tensor category $(\mathcal{C}, (\psi, \gamma, \psi_0), c)$ *is a G-crossed braided tensor category such that there exists a natural collection of isomorphisms called the G-ribbon twist:*

$$\theta_X : X \to T_g(X) \qquad X \in \mathcal{C}_g \tag{2.11}$$

Such that they satisfy the following axioms. For all $g, h \in G, X \in \mathcal{C}_g, Y \in \mathcal{C}_h$

$$\theta_{X\otimes Y} = \mu_g(X,Y) \circ (\gamma_{(gh)g(gh)^{-1},(gh)g^{-1}}(X) \otimes \gamma_{ghg^{-1},g}(Y)) \circ (\theta_{T_{ghg^{-1}}(X)} \otimes \theta_{T_g(Y)}) \circ c_{T_g(Y),X} \circ c_{X,Y} \tag{2.12}$$

For all $g \in G, X \in \mathcal{C}_g$

$$\theta_{X^*} = T_{g^{-1}}(\theta_X^* \circ d_X^{T_g}) \circ \gamma_{g^{-1},g}(X^*)^{-1} \circ (\psi_0)_{X^*} \tag{2.13}$$

Here $d_X^{T_g} : T_g(X^*) \to T_g(X)^*$ is the canonical isomorphism preserving the dual structure, see section A.4 for details.

For all $g, h \in G, X \in \mathcal{C}_h$:

$$\gamma_{ghg^{-1},g}(X) \circ \theta_{T_g(X)} = \gamma_{g,h}(X) \circ T_g(\theta_X) \tag{2.14}$$

We denote the structure of a G-ribbon tensor category by $(\mathcal{C}, (\psi, \gamma, \psi_0), c, \theta)$.

Definition 2.2.6. *G-Ribbon Functor*

Let $(\mathcal{C}, (\psi^1, \gamma^1, \psi_0^1), c^1, \theta^1)$, $(\mathcal{D}, (\psi^2, \gamma^2, \psi_0^2), c^2, \theta^2)$ be G-ribbon tensor categories. A G-ribbon functor from $\mathcal{C}$ to $\mathcal{D}$ is a G-crossed braided functor $(F, J^F, \phi^F, \{\tau_g\}_{g\in G})$ such that for all $g \in G, X \in \mathcal{C}_g$:

$$\tau_g \circ \theta^2_{F(X)} = F(\theta^1_X) \tag{2.15}$$

We say two G-ribbon tensor categories are equivalent if there is a G-ribbon functor that is also an equivalence. A natural transformations of G-ribbon functors is simply a natural transformation of G-crossed braided functors.

Definition 2.2.7. *G-Ribbon Fusion Category*

A G-ribbon fusion category is a G-tensor category that is also a fusion category.

As with all of the other structures, G-ribbon structures can be transported along adjoint tensor equivalences.

Proposition 2.2.1. *Let $(\mathcal{C}, (\psi, \gamma, \psi_0), c, \theta)$ be a G-ribbon tensor categories, and $\mathcal{D}$ a tensor category. Suppose there is an adjoint equivalence of tensor categories (L, K, η, ϵ)*

where $L : \mathcal{C} \to \mathcal{D}, K : \mathcal{D} \to \mathcal{C}$ are tensor functors, $\eta : 1_{\mathcal{C}} \to K \circ L, \epsilon : L \circ K \to 1_{\mathcal{D}}$ are the unit and co-unit respectively. Then $\mathcal{D}$ can be induced with a G-ribbon tensor category structure given as follows.

1. *The G-tensor structure $(\tilde{\psi}, \tilde{\gamma}, \tilde{\psi}_0)$ is the one induced from Proposition 2.1.1.*

2. *The G-grading is defined by setting:*

$$\mathcal{D}_g := \{X \in \mathcal{D} : X \cong L(Y) \text{ where } Y \in \mathcal{C}_g\} \tag{2.16}$$

3. *The G-crossed braiding is given for $X \in \mathcal{D}_g, Y \in \mathcal{D}$ by:*

$$\tilde{c}_{X,Y} := (\mathrm{Id}_{\tilde{T}_g(Y)} \otimes \epsilon_X) \circ J^L_{T_g(K(Y)),K(X)} \circ L(c_{K(X),K(Y)}) \circ L((J^K_{X,Y})^{-1}) \circ (\epsilon_{X\otimes Y})^{-1} \tag{2.17}$$

4. *The G-ribbon twist is given for $X \in \mathcal{D}_g$ by:*

$$\tilde{\theta}_X := L(\theta_{K(X)}) \circ \epsilon_X^{-1} \tag{2.18}$$

Furthermore, L, K can be upgraded to G-ribbon functors.

Proof. This is straightforward, but tedious and so we leave it to the reader. □

Proposition 2.2.2. *Every G-ribbon tensor category is equivalent as a G-ribbon tensor category to skeletal a G-ribbon tensor category such that the left and right unitors are trivial, and the G-action is unital.*

Proof. This is a consequence of the fact that every tensor category is equivalent to a skeletal tensor category with trivial unitors in combination with Proposition 2.2.1 and Proposition 2.1.3. □

2.3 Equivariantization and De-Equivarianitzation

One of the main goals of this thesis is to describe the modular fusion category of an oribfold VOA. In this section we will review the categorical techniques that allow us to do this: equivariantization and de-dquivariantization.

2.3.1 De-Equivariantization

Definition 2.3.1. *$\mathcal{C}$-Algebras*

Let $\mathcal{C}$ be a ribbon tensor category. A commutative $\mathcal{C}$-algebra *is a tuple (A, μ, ι) where A is an object in $\mathcal{C}$, $\mu : A \otimes_{\mathcal{C}} A \to A$ and $\iota : 1_{\mathcal{C}} \to A$ are maps such that the following hold*

$$\mu \circ (\mathrm{Id}_A \otimes \mu) \circ \alpha_{A,A,A} = \mu \circ (\mu \otimes \mathrm{Id}_A) \tag{Associativity}$$

$$\mu \circ (\iota \otimes \mathrm{Id}_A) \circ (\ell_A)^{-1} = \mu \circ (\mathrm{Id}_A \otimes \iota) \circ (r_A)^{-1} = \mathrm{Id}_A \tag{Unital}$$

$$\mu \circ c_{A,A} = \mu \tag{Commutative}$$

Definition 2.3.2. *$\mathcal{C}$-Algebra Modules*

Let (A, μ_A, ι_A) be a $\mathcal{C}$-algebra. An A-module is an object $M \in \mathcal{C}$, and a morphism $\mu_M : A \otimes M \to M$ such that the following holds:

$$\mu_M \circ (\mu_A \otimes \mathrm{Id}_M) = \mu_M \circ (\mathrm{Id}_A \otimes \mu_M) \circ \alpha_{A,A,M} \tag{Associativity}$$

$$\mu_M \circ (\iota_A \otimes \mathrm{Id}_M) \circ (\ell_M)^{-1} = \mathrm{Id}_M \tag{Unital}$$

We denote a module by (M, μ_M) and when the module structure is clear from context just as M. A morphism of A-modules $f : (M, \mu_M) \to (N, \mu_N)$ is a morphism $f : M \to N$ such that

$$f \circ \mu_M = \mu_N \circ (\mathrm{Id}_A \otimes f) \tag{2.19}$$

If $(M, \mu_M), (N, \mu_N)$ are A-modules we denote the set of A-modules morphisms as

$$\mathrm{Hom}_A((M, \mu_M), (N, \mu_N)) \tag{2.20}$$

With these notions, we can consider the category of A-modules and denote it by $\mathrm{Rep}_{\mathcal{C}}A$. This will have a tensor product structure since the details of this are not necessary for this thesis we refer the interested reader to [25].

Theorem 2.3.1. *De-Equivariantization*

Let $\mathcal{C}$ be a ribbon tensor category, with a ribbon embedding $\mathcal{F} : \mathrm{Rep}(G) \to \mathcal{C}$. Let $\mathbb{C}^G$ denote the algebra of functions on G. Then $\mathbb{C}^G$ is a $\mathrm{Rep}(G)$-algebra, and so $A := \mathcal{F}(\mathbb{C}^G)$ is a $\mathcal{C}$-algebra. $\mathrm{Rep}_{\mathcal{C}}(A)$ will be a G-ribbon tensor category, and is referred to as the de-equivariantization *of $\mathcal{C}$.*

Proof. This was proven in [12] and [24]. □

2.3.2 Equivariantization

Equivariantization is the categorical version of taking the fixed points of a group action.

Definition 2.3.3. *Equivariantization*

Let $(\mathcal{C}, (\psi, \gamma, \psi_0), c, \theta)$ be a G-ribbon tensor category where $\psi_g := (T_g, \mu_g, \psi^g)$. The equivariantization $\mathcal{C}^G$ is the following category:

1. *Objects are pairs $(X, \{u_g\}_{g\in G})$ where $X \in \mathcal{C}$, and $u_g : T_g(X) \to X$ are a collection of isomorphisms such that:*

$$u_{gh} \circ \gamma_{g,h}(X) = u_g \circ T_g(u_h) \tag{2.21}$$

2. *Morphisms $f : (X, \{u_g\}_{g\in G}) \to (Y, \{v_g\}_{g\in G})$ are morphisms $f : X \to Y$ in $\mathcal{C}$ such that:*

$$f \circ u_g = v_g \circ T_g(f) \tag{2.22}$$

This category will form a ribbon tensor category with tensor structure given by:

$$(X, \{u_g\}_{g\in G}) \otimes (Y, \{v_g\}_{g\in G}) := (X \otimes Y, \{(u_g \otimes v_g) \circ (\mu_g)^{-1}\}_{g\in G}) \tag{2.23}$$

$$\alpha_{(X,\{u_g\}),(Y,\{v_g\}),(Z,\{w_g\})} := \alpha_{X,Y,Z} \tag{2.24}$$

$$1_{\mathcal{C}^G} := (1_{\mathcal{C}}, \{(\phi^g)^{-1}\}_{g\in G}) \tag{2.25}$$

$$\ell_{(X,\{u_g\}_{g\in G})} := \ell_X \quad r_{(X,\{u_g\}_{g\in G})} := r_X \tag{2.26}$$

The left duals will be defined by:

$$(X, \{u_g\}_{g\in G})^* := (X^*, \{(u_g^{-1})^* \circ d_X^{T_g}\}_{g\in G}) \tag{2.27}$$

Where $d_X^{T_g} : T_g(X^*) \to T_g(X)^*$ *is the canonical isomorphism between left duals of* $T_g(X)$. *Similarly one can define right duals.*

The braiding of $(X, \{u_g\})_{g\in G}, (Y, \{v_g\})_{g\in G}$ *is defined as the composition of the following morphisms*

$$X\otimes Y \to (\bigoplus_{g\in G} X_g)\otimes Y = \bigoplus_{g\in G} X_g\otimes Y \xrightarrow{c_{X,Y}} \bigoplus_{g\in G} T_g(Y)\otimes X_g \xrightarrow{\oplus v_g \otimes \mathrm{Id}_{X_g}} \bigoplus_{g\in G} Y\otimes X_g \to Y\otimes X \tag{2.28}$$

The ribbon twist $\theta_{(X,\{u_g\}_{g\in G})}$ *of* $(X, \{u_g\}_{g\in G})$ *is defined as the composition of the following morphisms:*

$$X \to \bigoplus_{g\in G} X_g \xrightarrow{\oplus\theta_{X_g}} \bigoplus_{g\in G} T_g(X_g) \xrightarrow{\oplus u_g} \bigoplus_{g\in G} X_g \to X \tag{2.29}$$

Notice that in the definition of the ribbon twist we use the fact that $X \cong \bigoplus_{g\in G} X_g$ implies that every $u_k : \bigoplus_{g\in G} T_k(X_g) \to \bigoplus_{g\in G} X_g$ splits up to a direct sum of isomorphisms $(u_k)_g : T_k(X_g) \to X_{kgk^{-1}}$. In particular this implies that for every $g \in G$ u_g induces an isomorphism $(u_g)_g : T_g(X_g) \to X_g$. We use the short hand u_g in the definition to avoid cumbersome notation.

Now notice if $\mathcal{C}$ is as in Theorem 2.3.1, then $\mathrm{Rep}_{\mathcal{C}}(A)$ will be a G-ribbon tensor category, and so we can take the equivariantization $(\mathrm{Rep}_{\mathcal{C}}(A))^G$. If $\mathcal{D}$ is another G-ribbon tensor category such that $\mathcal{D}$ is equivalent to $\mathrm{Rep}_{\mathcal{C}}(A)$, then it will be true that $\mathcal{D}^G$ is equivalent to $(\mathrm{Rep}_{\mathcal{C}}(A))^G$ as a ribbon tensor category.

Theorem 2.3.2. *Reduction To G-Ribbon Fusion Categories*

If $\mathcal{D}$ *is a* G*-ribbon tensor category such that* $\mathcal{D}$ *is equivalent to* $\mathcal{C}_G$*, then* $\mathcal{D}^G \cong \mathcal{C}$ *as a ribbon tensor category.*

Proof. This was essentially proved by Kirillov in [24] and Drinfeld et. al in [12]. The only thing we need to verify is that the equivariantization of G-ribbon tensor equivalence will be a ribbon equivalence. To that end suppose that $(\mathcal{C}, (\psi^1, \gamma^1, \psi_0^1), c^1, \theta^1)$, $(\mathcal{D}, (\psi^2, \gamma^2, \psi_0^2), c^2, \theta^2)$ are G-ribbon tensor categories, and $(F, J^F, \phi^F, \{\tau_g\})$ is a G-ribbon tensor equivalence. For the sake of brevity we denote the induced functor $(F, J^F, \phi^F, \{\tau_g\})^G$ simply by F^G. On the level of objects we have:

$$F^G(X, \{u_g\}_{g\in G}) := (F(X), \{F(u_g) \circ (\tau_g)_X\}_{g\in G}) \tag{2.30}$$

On the level of morphisms we have:

$$F^G(f) := F(f) \quad \text{where } f : (X, \{u_g\}_{g\in G}) \to (Y, \{v_g\}_{g\in G}) \tag{2.31}$$

Well by definition the twist of $(F(X), \{F(u_g) \circ (\tau_g)_X\}_{g\in G})$ is the composition of the following morphisms:

$$F(X) \to \bigoplus_{g\in G} F(X_g) \xrightarrow{\oplus\theta^2_{F(X_g)}} \bigoplus_{g\in G} (T_g^2 \circ F)(X_g) \xrightarrow{\oplus(F(u_g)\circ(\tau_g)_{X_g})} \bigoplus_{g\in G} X_g \to X \tag{2.32}$$

Since F is a G-ribbon functor we have that $\tau_{X_g} \circ \theta^2_{F(X_g)} = F(\theta^1_{X_g})$. In particular this implies by functorality that:

$$F^G(\theta^1_{(X,\{u_g\})}) = \theta^2_{F^G(X,\{u_g\})} \tag{2.33}$$

Therefore, F^G will be a ribbon functor. This completes the proof. □

Chapter 3

Weak Quasi Hopf Algebras

There is a saying that humans are really only good at linear algebra, and because of this we translate everything into linear algebra. For example, differentiation in analysis and representations of groups. One way to understand fusion categories through linear algebra is by looking at representations of Hopf algebras and their variants. In this chapter we will review the definitions and related concepts from Hopf algebra theory that will be needed in this thesis, and explain how they can be used to describe fusion categories. We do this by starting with the simplest definition, a bi-algebra, and add more structure until we arrive at the definition of ribbon weak quasi Hopf algebras, which we will use later to describe all G-(ribbon,crossed braided) fusion categories. The source material of this chapter is from [13] [5][21][4].

3.1 Bi-algebras

We denote the structure of a unital $\mathbb{C}$-algebra by a tuple $(H, \cdot, 1_H)$ where $\cdot$ denotes the multiplication and 1_H is the multiplicative identity.

If $(H, \cdot, 1_H)$ is a unital $\mathbb{C}$-algebra we know that its categories of finite-dimensional modules $\mathrm{Mod}(H)$ will form an Abelian category. To equip $\mathrm{Mod}(H)$ with a monoidal structure we need a bi-algebra structure on H:

Definition 3.1.1. *Bi-Algebra*

Let $(H, \cdot, 1_H)$ be a unital $\mathbb{C}$-algebra. A bi-algebra structure on H is a tuple (Δ, ϵ) where $\Delta : H \to H \otimes_{\mathbb{C}} H$ and $\epsilon : R \to \mathbb{C}$ are unital $\mathbb{C}$-algebra morphisms such that:

$$\alpha^{\mathrm{Vect}}_{H,H,H} \circ (\mathrm{Id}_H \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}_R) \circ \Delta \tag{3.1}$$

$$r^{\mathrm{Vect}}_H \circ (\mathrm{Id}_H \otimes \epsilon) \circ \Delta = \ell^{\mathrm{Vect}}_H \circ (\epsilon \otimes \mathrm{Id}_H) \circ \Delta = \mathrm{Id}_H \tag{3.2}$$

The map Δ is called the comultiplication, and ϵ is called the counit.

Here $\alpha^{\mathrm{Vect}}, \ell^{\mathrm{Vect}}, r^{\mathrm{Vect}}$ denote the associator, left unitor and right unitor of Vect respectively.

Remark 3.1.1. *Notice here that $H \otimes_{\mathbb{C}} H$ is equipped with the $\mathbb{C}$-algebra structure given by $(r \otimes s) \cdot (t \otimes k) := (r \cdot t) \otimes (s \cdot k)$ and unit $(\eta \otimes_{\mathbb{C}} \eta) \circ (\ell^{\mathrm{Vect}}_{\mathbb{C}})^{-1}$.*

We denote a bi-algebra by a tuple $(H, \cdot, \eta, \Delta, \epsilon)$.

Proposition 3.1.1. *Monoidal Structure Induced by a Bi-algebra*

Let $(H, \cdot, \eta, \Delta, \epsilon)$ be a bi-algebra. Denote an object of $\mathrm{Mod}(H)$ by a tuple (M, ρ_V) where $\rho_V : H \to \mathrm{End}_{\mathbb{C}}(V)$ is the module structure. $\mathrm{Mod}(H)$ can be given the structure of a monoidal category by defining:

$$(M, \rho_M) \otimes (N, \rho_N) := (M \otimes_{\mathbb{C}} N, (\rho_M \otimes \rho_N) \circ \Delta) \tag{3.3}$$

The tensor product of morphisms is just the tensor product of linear maps. The associator of this tensor product is just the associator from Vect, Equation 3.1 guarantees that the associator is an H-module isomorphism. $\mathbb{C}$ is given the structure of an H-module through ϵ, and $\mathbb{C}$ will be the tensor unit. The left unitor is defined by:

$$\ell^{\mathrm{Mod}(H)}_{(V,\rho_V)} := \ell^{\mathrm{Vect}}_V \circ (\epsilon \otimes \mathrm{Id}_V) \tag{3.4}$$

Due to Equation 3.2 this will be an H-module isomorphism. Similarly, the right unitor is defined by:

$$r^{\mathrm{Mod}(H)}_{(V,\mu_V)} := r^{\mathrm{Vect}}_V \circ (\mathrm{Id}_V \otimes \epsilon) \tag{3.5}$$

For the same reasons as the left unitor this will be an H-module isomorphism.

Example 3.1.1. *Let G be a finite group. $\mathbb{C}[G]$ is a unital $\mathbb{C}$-algebra in the obvious way. This will be a bi-algebra with comultiplication $\Delta : \mathbb{C}[G] \to \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C}[G]$ defined by $\Delta(g) := g \otimes g$ and co-unit defined by $\epsilon(g) := \delta_{e,g}$.*

If H has a structure of a bi-algebra, then $H^* := \mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$ will also have the structure of a bi-algebra. This gives a way to produce new monoidal categories. For example, let $\mathbb{C}^G$ denote $(\mathbb{C}[G])^*$, it is not hard to show that as a monoidal category that $\mathrm{Mod}(\mathbb{C}^G)$ is equivalent to Vect_G.

Notation. *Sweedler Notation*

Let $(H, \cdot, 1_H, \Delta, \epsilon)$ be a bi-algebra. When writing formulas for bi-algebras it is often convenient to denote co-multiplication as follows. Let $c \in H$, then there exists $c^i_{(1)}, c^i_{(2)} \in R$ for a finite number of i such that:

$$\Delta(c) := \sum_i c^i_{(1)} \otimes c^i_{(2)} \tag{3.6}$$

Sweedler notation is denoting this simply as:

$$\Delta(c) = c_{(1)} \otimes c_{(2)} \tag{3.7}$$

Sweedler notation allows us to re-write Equation 3.1 as:

$$c_{(1)} \otimes ((c_2)_{(1)} \otimes (c_2)_{(2)}) = ((c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)}) \otimes c_{(2)} \tag{3.8}$$

Notice that we have suppressed the associator.

3.2 Hopf Algebra

To equip $\mathrm{Mod}(H)$ with a rigid monoidal structured, and hence a tensor structure, we need a way to define a module structure on the vector space dual of a module. This is done through a Hopf algebra structure:

Definition 3.2.1. *Hopf Algebra*

A Hopf algebra is a tuple $(H, \cdot, 1_H, \Delta, \epsilon, S)$. *Where* $(H, \cdot, 1_H, \Delta, \epsilon)$ *is a bi-algebra and* $S : H \to H$ *is a* $\mathbb{C}$*-antiautomorphism such that for all* $h \in H$*:*

$$S(h_{(1)})h_{(2)} = h_{(1)}S(h_{(2)}) = \epsilon(h) \tag{3.9}$$

When the Hopf algebra structure is clear from context we denote it simply by H.

Proposition 3.2.1. *Rigid Structure Induced by a Hopf Algebra[13]*

Let H *be a Hopf algebra. Then* $\mathrm{Mod}(H)$ *will be a rigid monoidal structure with left duals defined by:*

$$(V, \rho_V)^* := (V^*, \rho_{V^*}) \qquad \rho_{V^*} := (\rho_V \circ S)^* \tag{3.10}$$

The left evaluation and co-evaluation maps will just be the induced ones from Vect*. Similarly, if one assumes that* S *is invertible, which we always will, then one can define a notion of right duals by:*

$$^*(v, \rho_V) := (V^*, \rho_{^*V}) \qquad \rho_{^*V} := (\rho_V \circ S^{-1})^* \tag{3.11}$$

The right evaluation and co-evaluation maps will just be the induced ones from Vect*.*

Example 3.2.1. *If* G *is a finite group, then* $\mathbb{C}[G]$ *will have an anti-pode given by* $S(g) := g^{-1}$ *and extended linearly to all of* $\mathbb{C}[G]$*. In fact, one should think of anti-podes on Hopf algebras as being some form of inversion.*

3.3 Quasitriangular and Ribbon Hopf Algebras

Fix a Hopf algebra H for this section. To get a braiding on $\mathrm{Mod}(H)$ we need a notion of an R-matrix, or equivalently a quasitriangular structure.

Notation. *In the next few chapters we will use the area standard notation extensively. This was first explained to us in [5]. Let* $n \in \mathbb{N}$, $a = a_1 \otimes \cdots \otimes a_n \in H^{\otimes n}$*, if*

σ is a permutation of n, then we use the following notation:

$$a_{\sigma(1)\cdots\sigma(n)} := a_{\sigma^{-1}(1)} \otimes \cdots a_{\sigma^{-1}(1)} \tag{3.12}$$

If $a = a_1 \otimes \cdots \otimes a_k \in H^{\otimes k}$, $k < n$, and $\sigma : \{1, \cdots, k\} \to \{1, \cdots, n\}$, then let $\overline{a} \in H^n$ be the pure tensor given by tensoring with the unit of H on the right $n - k$ times. Let $\overline{\sigma} \in S_n$ be any the extension of σ. We use the following notation:

$$a_\sigma := \overline{a}_{\overline{\sigma}} \tag{3.13}$$

By linearity we extend these notations to all elements of $H^{\otimes n}, H^{\otimes k}$.

Example 3.3.1. *If $R = \sum_i R_1^i \otimes R_2^i \in H \otimes H$, then $R_{21} = \sum_i R_2^i \otimes R_1^i$. If $\Phi = \sum_i x_1^i \otimes x_2^i \otimes x_3^i \in H^{\otimes 3}$, then:*

$$\Phi_{312} = \sum_i x_2^i \otimes x_3^i \otimes x_1^i \tag{3.14}$$

If $R = \sum_i R_1^i \otimes R_2^i \in H^{\otimes 2}, \Phi \in H^{\otimes 3}$, then:

$$R_{13} \cdot \Phi = (\sum_i R_1^i \otimes 1_H \otimes R_2^i) \cdot \Phi \tag{3.15}$$

Definition 3.3.1. *Quasitriangular Hopf Algebra*

Let H be a Hopf algebra a quasitriangular Hopf algebra is a pair (H, R) where R is an invertible element $R \in H \otimes H$ such that the following relations hold:

$$(\Delta \otimes \mathrm{Id})(R) = R_{13}R_{23} \tag{3.16}$$

$$(\mathrm{Id} \otimes \Delta)(R) = R_{13}R_{12} \tag{3.17}$$

For all $h \in H$ we have:

$$\Delta^{op}(h) = R\Delta(h)R^{-1} \tag{3.18}$$

where $\Delta^{op} = c_{H,H}^{\mathrm{Vect}} \circ \Delta$ and $c_{H \otimes H}^{\mathrm{Vect}} : H \otimes H \to H \otimes H$ is defined for $x, y \in H$ as $c_{H,H}^{\mathrm{Vect}}(x \otimes y) = y \otimes x$.

Proposition 3.3.1. *Braiding Induced by a Quasitriangular Hopf Algebra*

Let (H, R) *be a quasitriangular Hopf algebra.* $\mathrm{Mod}(H)$ *will be a braided tensor category with braiding defined for* $(V, \rho_V), (W, \rho_W)$ *as:*

$$c_{(V,\rho_V),(W,\rho_W)} := c^{\mathrm{Vect}}_{V,W} \circ (\rho_V \otimes \rho_W)(R) \tag{3.19}$$

where $c^{\mathrm{Vect}}_{V,W} : V \otimes_{\mathbb{C}} W \to W \otimes_{\mathbb{C}} V$ *is the standard braiding on* Vect.

Assume now that (H, R) is a quasitriangular Hopf algebra. To get a ribbon structure on $\mathrm{Mod}(H)$ we need the notion of a ribbon Hopf algebra.

Definition 3.3.2. *Ribbon Hopf Algebra*

A ribbon Hopf algebra *is a quasitriangular Hopf algebra* (H, R) *with an element* $v \in H$ *such that* v *is in the centre of* H*, and:*

$$\Delta(\nu) = (\nu \otimes \nu)(R_{21}R) \tag{3.20}$$

$$S(\nu) = \nu \tag{3.21}$$

In this case $\nu \in H$ *is called the ribbon element.*

Proposition 3.3.2. *Ribbon Structure Induced by a Ribbon Hopf Algebra*

Let (H, R, ν) *be a ribbon Hopf algebra. Then* $\mathrm{Mod}(H)$ *has a ribbon structure defined for* $(V, \rho_V), v \in V$ *as:*

$$\theta_{(V,\rho_V)}(v) := \rho_V(\nu)(v) \tag{3.22}$$

Ribbon Hopf algebras will describe all tensor categories with something called a fiber functor:

Definition 3.3.3. *fiber Functor*

Let $\mathcal{C}$ *be a ribbon tensor category. A fiber functor is a tensor functor* $(F, J^F, \phi^F)cC \to$ Vect *that is faithful, and* $F(1_{\mathcal{C}}) = \mathbb{C}$*,* $\phi^F = \mathrm{Id}_{\mathbb{C}}$.

Basically fiber functors allow you to represent objects in your category by vector spaces.

As pointed out in [13, Chapter 5], having a fiber functor is a very strict condition. For example, Vect_G^ω will have a fiber functor if and only if ω is equivalent to the trivial 3-cocycle.

fiber functors give us the first reconstruction result. As this proposition will be described in much greater detail later we briefly state it:

Theorem 3.3.1. *Ribbon Hopf Algebra Reconstruction [13, Proposition 8.11.2]*

Let $(\mathcal{C}, F)$ be a ribbon tensor category and a fiber functor. The space $H := \mathrm{End}(F)$ will have the structure of a ribbon Hopf algebra. Furthermore, F will induce an equivalence of ribbon tensor categories $(L, J^L, \phi^L) : \mathcal{C} \to \mathrm{Mod}(H)$.

3.4 Weak Quasi Bi-Algebra

Most fusion categories won't have a fiber functor, and so to prove a Hopf algebra reconstruction result for all fusion categories we need the notion of a weak quasi bi-algebra.

Definition 3.4.1. *Partially Invertible Element [5]*

Let B be an algebra. Define a linear category whose objects are idempotents p, q of B, and $(p, q) := \{T \in B : qT = T = Tp\}$. If $T \in (p, q)$ then $D(T) := p, R(T) := q$. An element T of B is said to be partially invertible if it is invertible with $D(T) = p, R(T) = q$ if $T \in (p, q)$ and there exists a $T^{-1} \in (q, p)$ such that $T^{-1}T = p, TT^{-1} = q$. T^{-1} is called the partial inverse.

Notice by restricting to this linear category we have uniqueness of partial inverses by the usual arguments.

Definition 3.4.2. *Weak Quasi Bi-Algebra*

Let $(H, \cdot, 1_H)$ be a unital $\mathbb{C}$-algebra. A weak quasi bi-algebra is a tuple $(H, \cdot, 1_H, \Delta, \epsilon, \Phi)$:

1. *Coproduct: A $\mathbb{C}$-algebra homomorphism (not assumed to be unital) $\Delta : H \to H \otimes_{\mathbb{C}} H$*

2. *Counit: A $\mathbb{C}$-algebra homomorphism $\epsilon : H \to \mathbb{C}$ such that:*

$$(\epsilon \otimes 1) \circ \Delta = 1 = (1 \otimes \epsilon) \circ \Delta \tag{3.23}$$

3. *Associator: A partially invertible element $\Phi \in H \otimes H \otimes H$ satisfying:*

$$D(\Phi) = (\Delta \otimes \mathrm{Id}_H) \circ \Delta(1), \qquad R(\Phi) = (\mathrm{Id}_H \otimes \Delta) \circ \Delta(1) \tag{3.24}$$

$$\Phi \cdot (\Delta \otimes \mathrm{Id}_H)(\Delta(h)) = (\mathrm{Id}_H \otimes \Delta)(\Delta(h)) \cdot \Phi, \qquad h \in H \tag{3.25}$$

$$(\mathrm{Id}_H \otimes \mathrm{Id}_H \otimes \Delta)(\Phi) \cdot (\Delta \otimes \mathrm{Id}_H \otimes \mathrm{Id}_H)(\Phi) = (1 \otimes \Phi) \cdot (\mathrm{Id}_H \otimes \Delta \otimes \mathrm{Id}_H)(\Phi) \cdot (\Phi \otimes 1) \tag{3.26}$$

$$(\mathrm{Id}_H \otimes \epsilon \otimes \mathrm{Id}_H)(\Phi) = (\mathrm{Id}_H \otimes \mathrm{Id}_H \otimes \epsilon)(\Phi) = (\mathrm{Id}_H \otimes \mathrm{Id}_H \otimes \epsilon)(\Phi) = \Delta(1) \tag{3.27}$$

Proposition 3.4.1. *Monoidal Structure Induced by a Weak Quasi Bi-Algebra*

Let $(H, \cdot, 1_H, \Delta, \epsilon, \Phi)$ be a weak quasi bi-algebra. Suppose that $(V, \rho_V), (W, \rho_V) \in \mathrm{Mod}(H)$. There is an induced monoidal structure defined for objects as:

$$(V, \rho_V) \otimes (W, \rho_W) := ((\rho_V \otimes \rho_W)(\Delta(1_H))(V \otimes_{\mathbb{C}} W), (\rho_V \otimes \rho_W) \circ \Delta) \tag{3.28}$$

To distinguish $(\rho_V(1_{(1)}) \otimes \rho_W(1_{(2)}))(V \otimes_{\mathbb{C}} W)$ from the usual tensor product of vector spaces we refer to $(\rho_V \otimes \rho_W)(\Delta(1))(V \otimes_{\mathbb{C}} W)$ as the representation space of $V \otimes W$. An important idea we will use throughout this thesis is that a linear map $f : V \otimes_{\mathbb{C}} W \to V \otimes_{\mathbb{C}} W$ that commutes with $(\rho_V \otimes \rho_W)(\Delta(1))$ will restrict to a linear map on the representation space of $V \otimes W$. The tensor product of H-intertwiners is given by taking the usual tensor product of linear maps and then restricting to the corresponding representation spaces. The associator for $(V, \rho_V), (W, \rho_W), (U, \rho_U) \in \mathrm{Mod}(H)$ is given by considering the linear map:

$$(\rho_V \otimes \rho_W \otimes \rho_U)(\Phi) \tag{3.29}$$

and then restricting to the representation spaces. $\mathbb{C}$ is given the structure of a H-module through the counit, and this will be the tensor unit. The left and right unitors *are the same as in Proposition 3.1.1.*

Remark 3.4.1. *Throughout this thesis we will be dealing with linear maps on representation spaces. For the sake of brevity we usually define linear maps on the vector space tensor product with the understanding that they should be restricted to the corresponding representation spaces.*

3.5 Weak Quasi Hopf Algebras

To make the category of modules of a weak quasi bi-algebra a rigid monoidal category we need an anti-pode structure.

Remark 3.5.1. *Sweedler-esque Notation*

If $h \in H^{\otimes n}$ for some n, then write $h = \sum_i h_1^i \otimes h_2^i \otimes h_3^i$. Throughout this thesis we will use the shorthand given by suppressing the summation and index. Therefore, we would write h as:

$$h = h_1 \otimes h_2 \otimes h_3 \tag{3.30}$$

.

Definition 3.5.1. *Weak Quasi Hopf Algebra*

A weak quasi Hopf algebra is a tuple $(H, \cdot, 1_H, \Delta, \epsilon, \Phi, S, \alpha, \beta)$, where $(H, \cdot, 1_H, \Delta, \epsilon, \Phi)$ is a weak quasi bi-algebra, $S : H \to H$ is an anti-automorphism and $\alpha, \beta \in H$ are such that the following hold for all $h \in H$:

$$S(h_{(1)})\alpha h_{(2)} = \epsilon(h)\alpha, \quad h_{(1)}\beta S(h_{(2)}) = \epsilon(h)\beta \tag{3.31}$$

$$x_1\beta S(x_2)\alpha x_3 = 1_H = S(X_1)\alpha X_2\beta S(X_3) \tag{3.32}$$

where

$$\Phi = x_1 \otimes x_2 \otimes x_3 \quad \Phi^{-1} = X_1 \otimes X_2 \otimes X_3 \tag{3.33}$$

Proposition 3.5.1. *Rigid Structure Induced by a Weak Quasi Hopf Algebra*

Let $(H, \cdot, 1_H, \Delta, \epsilon, \Phi, S, \alpha, \beta)$ *be a weak quasi Hopf algebra.* $\mathrm{Mod}(H)$ *will have a rigid monoidal structure given as follows. A left dual of* (V, ρ_V) *is defined as follows:*

$$(V, \rho_V)^* := (V^*, (\rho_V \circ S)^*) \tag{3.34}$$

with left evaluation defined as:

$$\mathrm{ev}_{(V,\rho_V)}^{\mathrm{Mod}(H)}(f \otimes v) := f(\rho_V(\alpha)(v)) \tag{3.35}$$

Following [13], let $\sum_i v_i \otimes f_i$ *be the element of* $V \otimes V^*$ *corresponding to the identity map of* V *through the canonical isomorphism* $V^* \otimes V \to \mathrm{End}_{\mathbb{C}}(V)$*. The left co-evaluation is defined as:*

$$\mathrm{coev}_{(V,\rho_V)}^{\mathrm{Mod}(H)}(1) := \sum_i \rho_V(\beta)(v_i) \otimes f_i \tag{3.36}$$

If S *is invertible, then right duals can be defined in the same way as described in Proposition 3.2.1 and the right evaluation and co-evaluations morphisms similar to how the left evaluation and co-evaluation morphisms were defined.*

For the sake of brevity we refer to a weak quasi Hopf algebra as a wqhf.

3.6 Quasitriangular and Ribbon Weak Quasi Hopf Algebras

Let H be a wqhf. To give $\mathrm{Mod}(H)$ a braiding we need the notion of a quasitriangular wqhf.

Definition 3.6.1. *Quasitriangular WQHF*

A quasitriangular wqhf is a tuple $(H, \cdot, 1_H, \Delta, \epsilon, \Phi, S, \alpha, \beta, R)$ *where* $(H, \cdot, 1_H, \Delta, \epsilon, \Phi, S, \alpha, \beta)$ *is a wqhf and* $R \in H \otimes H$ *is a partially invertible element such that:*

$$D(R) := \Delta(1_H) \quad R(R) := \Delta^{op}(1_H) \tag{3.37}$$

$$\Delta^{op}(a) = R\Delta(a)R^{-1} \tag{3.38}$$

$$(\Delta \otimes \mathrm{Id})(R) = \Phi_{312}R_{13}\Phi_{132}^{-1}R_{23}\Phi_{123} \tag{3.39}$$

$$(\mathrm{Id} \otimes \Delta)(R) = \Phi_{231}^{-1}R_{13}\Phi_{213}R_{12}\Phi_{123}^{-1} \tag{3.40}$$

Remark 3.6.1. *Just as in the quasi Hopf algebra case [4, Section 10.1], one can apply* $(\epsilon \otimes \epsilon \otimes \mathrm{Id})$ *to Equation* (3.39) *and obtain that* $(\mathrm{Id} \otimes \epsilon)(R) = 1_H$. *Similarly, by applying* $(\epsilon \otimes \epsilon \otimes \mathrm{Id})$ *to Equation* (3.40) *one sees that* $(\epsilon \otimes \mathrm{Id})(R) = 1_H$.

Proposition 3.6.1. *Braiding Induced by a Quasitriangular WQHF*

Let $(H, \cdot, \eta, \Delta, \epsilon, \Phi, S, \alpha, \beta, R)$ *be a quasitriangular wqhf.* $\mathrm{Mod}(H)$ *will have the structure of a braided monoidal category. The braiding on the level of objects is defined by first consider the linear map defined on* $(V, \rho_V), (W, \rho_W)$ *as:*

$$c^{\mathrm{Mod}(H)}_{(V,\rho_V),(W,\rho_W)} := c^{\mathrm{Vect}}_{V,W} \circ (\rho_V \otimes \rho_W)(R) \tag{3.41}$$

Where σ *is as in Definition 3.3.1. This will then restrict down to an* H*-intertwiner on the corresponding representation spaces. Similarly the braiding on the level of morphisms is given by sending* $(f \otimes g) \mapsto (g \otimes f)$ *and then restricting to the representation spaces.*

Definition 3.6.2. *Ribbon WQHF*

A ribbon wqhf is a tuple $(H, \cdot, 1_H, \Delta, \epsilon, \Phi, S, \alpha, \beta, R, \nu)$ *where* $(H, \cdot, 1_H, \Delta, \epsilon, \Phi, S, \alpha, \beta, R)$ *is a quasitriangular wqhf and* $v \in H$ *is an invertible central element such that the following holds:*

$$\Delta(\nu) = (\nu \otimes \nu)R_{21}R \tag{3.42}$$

$$S(\nu) = \nu \tag{3.43}$$

In this case ν *is called the ribbon element.*

Remark 3.6.2. *Just as in the quasi Hopf algebra case [4, Remark 13.27], by applying* $(\epsilon \otimes \epsilon)$ *to Equation* (3.42) *one obtains that* $\epsilon(\nu) = 1$.

Proposition 3.6.2. *Ribbon Structure Induced by a Ribbon WQHF*

Let $(H, \cdot, 1_H, \Delta, \epsilon, \Phi, S, \alpha, \beta, R, \nu)$ be a ribbon wqhf. $\mathrm{Mod}(H)$ *will have the structure of a ribbon tensor category with ribbon structure defined for* $(V, \rho_V) \in \mathrm{Mod}(H)$ *as:*

$$\theta_{(V,\rho_V)} := \rho_V(\nu) \tag{3.44}$$

3.7 Twist Equivalences

As we have seen, given a finite dimensional unit algebra $(H, \cdot, 1_H)$ a ribbon wqhf structure $(H, \cdot, 1_H, \Delta, \epsilon, \Phi, S, \alpha, \beta, R, \nu)$ will induce a ribbon tensor structure on $\mathrm{Mod}(H)$. A natural question is when are these ribbon tensor structures equivalent? The answer is given by looking at twists:

Definition 3.7.1. *Weak Quasi Bi-Algebra Twists [5, Definition 5.8]*

Let $(H, \cdot, 1_H, \Delta, \epsilon, \Phi)$ be a weak quasi bi-algebra. A twist is a pair $J, J^{-1} \in H \otimes H$ such that J is partially left invertible:

$$J^{-1}J = \Delta(1) \tag{3.45}$$

Furthermore, J must satisfy the following:

$$(\epsilon \otimes \mathrm{Id}_H)(J) = (\mathrm{Id}_H \otimes \epsilon)(J) = 1_H \tag{3.46}$$

Proposition 3.7.1. *Twists Give Equivalent Structure [13]*

Let $(H, \cdot, 1_H, \Delta, \epsilon, \Phi, S, \alpha, \beta, R, \nu)$ be a ribbon wqhf, and J a twist. This will induce a new ribbon wqhf $(H, \cdot, 1_H, \Delta, \epsilon, \Phi, S, \alpha, \beta, R, \nu)^J = (H, \cdot, 1_H, \Delta^J, \epsilon, \Phi^J, S, \alpha^J, \beta^J, R^J, \nu)$ where:

$$\Delta^J(a) := J\Delta(a)J^{-1} \tag{3.47}$$

$$\Phi^J := (1_H \otimes J) \cdot (\mathrm{Id} \otimes \Delta)(J) \cdot \Phi(\Delta \otimes \mathrm{Id})(J^{-1}) \cdot (J^{-1} \otimes 1_H) \tag{3.48}$$

$$\alpha^J := S(J_1^{-1})\alpha J_2^{-1} \qquad \beta^J := (J)_1 \beta S((J)_2) \tag{3.49}$$

$$R^J := J_{21} \cdot R \cdot J^{-1} \tag{3.50}$$

Here we use the shorthand $J = J_1 \otimes J_2$. Denote this new structure by H^J, as a ribbon tensor category $\mathrm{Mod}(H)$ *is equivalent to* $\mathrm{Mod}(H^J)$. *Conversely, if two ribbon wqhf structure induce the same ribbon tensor category, up to equivalence, then they are twist related.*

Remark 3.7.1. *Note that the definition of a twist depends on the weak quasi-bialgebra, which is why twist equivalent structures define an equivalence relation despite only being left invertible.*

3.8 Reconstruction

Definition 3.8.1. *Weak Quasi fiber Functor*

Let $\mathcal{C}$ be a tensor category. A weak quasi fiber functor is a faithful $\mathbb{C}$-linear functor $F : \mathcal{C} \to \mathrm{Vect}$ and a natural collection of epimorphisms:

$$J^F_{X,Y} : F(X) \otimes_{\mathbb{C}} F(Y) \to F(X \otimes_{\mathcal{C}} Y) \tag{3.51}$$

such that there is an isomorphism $\phi^F : \mathbb{C} \to F(1_{\mathcal{C}})$ where for all $X \in \mathcal{C}$:

$$J^F_{1_{\mathcal{C}},X} \circ (\phi^F \otimes_{\mathbb{C}} \mathrm{Id}_{F(X)}) = F(\ell^{\mathcal{C}}_X)^{-1} \circ \ell^{\mathrm{Vect}}_{F(X)} \tag{3.52}$$

$$J^F_{X,1_{\mathcal{C}}} \circ (\mathrm{Id}_{F(X)} \otimes_{\mathbb{C}} \phi^F) = F(r^{\mathcal{C}}_X)^{-1} \circ r^{\mathrm{Vect}}_{F(X)} \tag{3.53}$$

Furthermore, there must exist a natural collection of isomorphisms:

$$d_X : F(X^*) \cong F(X)^{*_{\mathrm{Vect}}} \tag{3.54}$$

An important concept related to weak quasi fiber functors is that of a weak dimension function.

Definition 3.8.2. *Weak Dimension Function*

Let $\mathcal{C}$ be a fusion category. Denote the isomorphism classes of simple objects by $\mathcal{O}(\mathcal{C})$. A weak dimension function is a function $D : \mathcal{O}(\mathcal{C}) \to \mathbb{N}$ such that for all

$X, Y \in \mathcal{C}$:

$$D(1_{\mathcal{C}}) = 1 \qquad D(X) = D(X^*) \tag{3.55}$$

$$D(X)D(Y) \geq \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{X,Y}^{Z} D(Z) \tag{3.56}$$

The following were proven in [21].

Proposition 3.8.1. *A weak dimension function D on a function category $\mathcal{C}$ induces a weak quasi fiber functor (F, J^F, ϕ^F) as follows.*

On the level of objects:

$$F(X) := \bigoplus_{Y \in \mathcal{O}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(Y, X) \otimes_{\mathbb{C}} \mathbb{C}^{D(Y)} \tag{3.57}$$

On the level of morphisms if $f : X \to Z$ is in $\mathcal{C}$ then for all $Y \in \mathcal{O}(\mathcal{C}), g_Y \in \mathrm{Hom}_{\mathbb{C}}(X, Y), v_Y \in \mathbb{C}^{D(Y)}$:

$$F(f)(\oplus_{Y \in \mathcal{C}} g_Y \otimes v_Y) := \oplus_{Y \in \mathcal{C}} (f \circ g_Y) \otimes v_Y \tag{3.58}$$

The tensor structure J^F is defined by choosing for every $X, Y \in \mathcal{O}(\mathcal{C})$ an epimorphisms:

$$J_{X,Y}^F : \mathbb{C}^{D(X)} \otimes \mathbb{C}^{D(Y)} \to \bigoplus_{Z \in \mathcal{O}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(Z, X \otimes_{\mathcal{C}} Y) \otimes \mathbb{C}^{D(z)} \tag{3.59}$$

and then extending by linearity to all objects of $\mathcal{C}$. The unit object is defined by identifying $\mathrm{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$ *with* $\mathbb{C}$ *and then using the left unitor in* Vect.

Proposition 3.8.2. *[21]*

Every fusion category admits a weak dimension function D defined by:

$$D(X) := \sum_{Y,Z} N_{X,Y}^{Z} \qquad D(1_{\mathcal{C}}) := 1 \tag{3.60}$$

Now that we have all of the definitions we can recall the main reconstruction argument we will need.

Theorem 3.8.1. *[21, Theorem 16]*

Let $\mathcal{C}$ be a ribbon fusion category, and $(F, J^F, \phi^F) : \mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}$ a weak quasi fiber functor. Denote the associator of $\mathcal{C}$ by ϕ. Choose a right inverse of J^F, and denote it by $(J^F)^{-1}$. The following defines a ribbon wqhf:

1. *$H := \mathrm{Nat}(F, F) = \{h_X \in \mathrm{End}_{\mathbb{C}}(F(X)) : \forall f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), h_Y \circ f = f \circ h_X\}$ has the structure of a unital associative algebra, with multiplication defined pointwise as $(h \cdot r)_X := h_X \circ r_x$, and the unit being the trivial natural transformation.*

2. *Using the canonical algebra isomorphism $\mathrm{End}_{\mathbb{C}}(F(X) \otimes_{\mathbb{C}} F(Y)) \cong \mathrm{End}_{\mathbb{C}}(F(X)) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(F(Y))$, the co-product is defined component wise by*

$$\Delta(h)_{X,Y} := (J^F_{X,Y})^{-1} \circ h_{X \otimes_{\mathcal{C}} Y} \circ J^F_{X,Y} \tag{3.61}$$

3. *The co-unit is given by $\epsilon(h) := h_{1_{\mathcal{C}}}$*

4. *The associator is given by*

$$\Phi_{X,Y,Z} := (\mathrm{Id} \otimes (J^F_{Y,Z})^{-1}) \circ (J^F_{X,Y \otimes Z})^{-1} \circ F(\phi_{X,Y,Z}) \circ J^F_{X \otimes Y, Z} \circ (J^F_{X,Y} \otimes \mathrm{Id}) \tag{3.62}$$

5. *The anti-pode structure is given by*:*

$$(Sh)_X := d^*_X \circ (h_{X^*})^* \circ (d^*_X)^{-1} \tag{3.63}$$

Where $d_X : F(X)^* \to F(X^*)$ is the canonical natural transformation, and $d^*_X : F(X^*)^* \to F(X)$ is the dual map combined with the canonical isomorphism $F(X)^{**} \cong F(x)$ for finite dimensional vector spaces. The elements α, β are given component wise by:

$$\alpha_X := (\mathrm{Id} \otimes \mathrm{ev}^{\mathrm{Rep}}_{\rho_X}) \circ (\mathrm{coev}^{\mathrm{Vect}}_{F(X)} \otimes \mathrm{Id}) \tag{3.64}$$

$$\beta_X := (\mathrm{Id} \otimes \mathrm{ev}^{Vect}_{F(X)}) \circ (coev^{\mathrm{Rep}}_{\rho_X} \otimes \mathrm{Id}) \tag{3.65}$$

Where

$$\mathrm{ev}^{\mathrm{Rep}}_{\rho_X} := F(\mathrm{ev}_X) \circ J^F_{X^*,X} \circ (d_X \otimes \mathrm{Id}) \tag{3.66}$$

$$\mathrm{coev}^{\mathrm{Rep}}_{\rho_X} := (\mathrm{Id} \otimes d_X^{-1}) \circ J^{-1}_{X,X^*} \circ F(\mathrm{coev}_X) \tag{3.67}$$

6. *The quasitriangular structure is given by the R-matrix defined for all* $X, Y \in \mathcal{C}$ *by:*

$$R_{X,Y} := (c^{Vect}_{F(X),F(Y)})^{-1} \circ (J^F_{Y,X})^{-1} \circ F(c^{\mathcal{C}}_{X,Y}) \circ J^F_{X,Y} \tag{3.68}$$

7. *The ribbon element* $\nu \in H$ *is defined component-wise for* $X \in \mathcal{C}$*:*

$$\nu_X := F(\theta^{\mathcal{C}}_X) \tag{3.69}$$

This will induce a tensor equivalence $(L, J^F, \phi^F) : \mathcal{C} \to \mathrm{Mod}(H)$ *given by* $L(X) := (F(X), \rho_X)$ *where the* H*-action is defined for* $h \in H$ *as* $\rho_X(h) = h_X$*.*

Remark 3.8.1. *Notice the associator we presented here is different then the one in [21]. This is because of a difference in convention. In [21] the associator is a map* $X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$*, while our convention is the other way.*

Corollary 3.8.1. *Ribbon WQHF Reconstruction*

Every ribbon fusion category is equivalent as a ribbon fusion category to the category of modules of some ribbon WQHF.

We generalize this reconstruction result for G-fusion categories and describe their equivariantizations.

Chapter 4

G Abelian 3-cocycles and Their Variants

In [11] Drinfeld defined quasi Hopf-algebras, and showed that they would give examples of non-strict monoidal categories. We follow a similar approach in this chapter by defining three types of cocycles on a fixed unital algebra H to define a corresponding G-structure:

1. G Abelian 3-cocycles $\Rightarrow$ G-tensor structures on $\mathrm{Mod}(H)$
2. G-Crossed Abelian 3-cocycles $\Rightarrow$ G-crossed braided tensor structures on $\mathrm{Mod}(H)$
3. G-Ribbon Abelian 3-cocycles $\Rightarrow$ G-ribbon tensor structures on $\mathrm{Mod}(H)$

We denote the set of all such G 3-cocycles by $Z^3_{G-\mathrm{Ab}}(H), Z^3_{G-\mathrm{Crd}}(H), Z^3_{G-\mathrm{Rb}}(H)$ respectively. Similar to how twists give a notion of equivalence on the set of all wqhf structures on H, we can define when these G 3-cocycles are equivalent and therefore form equivalence classes: $H^3_{G-\mathrm{Ad}}(H), H^3_{G-\mathrm{Crd}}(H), H^3_{G-\mathrm{Rb}}(H)$. We then show that equivalent G-(ribbon, crossed) Abelian 3-cocycles will induce equivalent G-(ribbon, crossed braided) tensor structures.

4.1 G Abelian 3-Cocycles

Fix throughout this section a unital $\mathbb{C}$-algebra $(H, \cdot, 1_H)$. The following definition of a non-Abelian 2-cocycle will be needed for our description.

Definition 4.1.1. *Non-Abelian 2 Cocycles*

A non-abelian 2 cocycle on a group G with coefficients in H is a tuple $(\Psi, \underline{\gamma})$

- $\Psi : G \to \mathrm{Aut}_{\mathbb{C}-\mathrm{alg}}(H)$
- $\underline{\gamma} : G \times G \to H^{\times}$

Such that they satisfy the following conditions for all $g_1, g_2, g_3 \in G$:

$$\Psi(e) = \mathrm{Id}_K \tag{4.1}$$

$$\Psi(g_1 g_2) = \mathrm{Ad}(\underline{\gamma}_{\ g_1,g_2}) \circ \Psi(g_1) \circ \Psi(g_2) \tag{4.2}$$

$$(\underline{\gamma}_{\ g_1,g_2g_3}) \cdot \Psi(g_1)(\underline{\gamma}_{\ g_2,g_3}) = \underline{\gamma}_{\ g_1g_2,g_3} \cdot \underline{\gamma}_{\ g_1,g_2} \tag{4.3}$$

If $\underline{\gamma}_{\ e,g} = \underline{\gamma}_{\ g,e} = 1_H$ for all $g \in G$, then we say that $(\Psi, \underline{\gamma})$ is normalized.

Notation. *Let $n \in \mathbb{N}$, and B a set. Then $C^n(G, B)$ is the set of all functions from $\prod_{i=1}^n G \to B$. For the sake of brevity we also use the following notation. Let $\Gamma \in H^{\otimes n}$, and $g_1, \cdots, g_n \in G$. Then:*

$$\Gamma^{(g_1,\cdots,g_n)} := (\Psi(g_1^{-1}) \otimes \cdots \otimes \Psi(g_n^{-1}))(\Gamma) \tag{4.4}$$

Definition 4.1.2. *G Abelian 3-cocycle*

A G Abelian 3-cocycle *on $(H, \cdot, 1_H)$ is a tuple $(\epsilon, \Delta, \Phi, S, \alpha, \beta, \mathit{\Psi}, \underline{\gamma}, \overline{\mu})$ where:*

$$\Psi : G \to \mathrm{Aut}_{\mathbb{C}-\mathrm{alg}}(H),\ \underline{\gamma} \in C^2(G, H^{\times}),\ \overline{\mu} \in C^1(G, H \otimes H) \tag{4.5}$$

and $(H, \cdot, 1_H, \Delta, \epsilon, \Phi, S, \alpha, \beta)$ is a weak quasi Hopf algebra. We require the following conditions to be satisfied:

1. *The pair* $(\Psi, \underline{\gamma})$ *is a normalized non-Abelian* 2*-cocycle on* G *with coefficients in* H *such that for all* $g \in G$*:*
$$\epsilon = \epsilon \circ \Psi(g) \tag{4.6}$$

2. *For all* $g \in G$, $\overline{\mu}_g$ *is partially invertible with:*
$$D(\overline{\mu}_g) = \Delta(1_H)^{(g,g)} \quad R(\overline{\mu}_g) = \Delta(1_H) \tag{4.7}$$
That is:
$$\Delta(1_H) \cdot \overline{\mu}_g = \overline{\mu}_g = \overline{\mu}_g \cdot \Delta(1_H)^{(g,g)} \tag{4.8}$$
and there exists a $\overline{\mu}_g^{-1} \in H^{\otimes 2}$ *such that:*
$$\Delta(1_H)^{(g,g)} \cdot \overline{\mu}_g^{-1} = \overline{\mu}_g^{-1} = \overline{\mu}_g^{-1} \cdot \Delta(1_H) \tag{4.9}$$
and:
$$\overline{\mu}_g^{-1} \cdot \overline{\mu}_g = \Delta(1)^{(g,g)} \quad \overline{\mu}_g \cdot \overline{\mu}_g^{-1} = \Delta(1) \tag{4.10}$$

3. *The following equations must hold for all* $g \in G$*:*
$$\overline{\mu}_e = \Delta(1_H) \tag{4.11}$$
$$\mathrm{Ad}(\overline{\mu}_g)(\Delta^{(g,g)}) = \Delta \circ \Psi(g^{-1}) \tag{4.12}$$
$$(\mathrm{Id} \otimes \Delta)(\overline{\mu}_g) \cdot (1_H \otimes \overline{\mu}_g) \cdot \Phi^{(g,g,g)} = \Phi \cdot (\Delta \otimes \mathrm{Id})(\overline{\mu}_g) \cdot (\overline{\mu}_g \otimes 1_H) \tag{4.13}$$
$$(\epsilon \otimes \mathrm{Id})(\overline{\mu}_g) = (\mathrm{Id} \otimes \epsilon)(\overline{\mu}_g) = 1_H \tag{4.14}$$

4. *The follow equation must hold for* $\overline{\mu}, \underline{\gamma}$ $g, k \in G$*:*
$$(\underline{\gamma}_{\ k^{-1},g^{-1}} \otimes \underline{\gamma}_{\ k^{-1},g^{-1}}) = (\overline{\mu}_{gk})^{-1} \cdot \Delta(\underline{\gamma}_{\ k^{-1},g^{-1}}) \cdot \overline{\mu}_k \cdot \Psi(k^{-1})^{\otimes 2}(\overline{\mu}_g) \tag{4.15}$$

Denote the set of G *Abelian* 3*-cocycles by* $Z^3_{G-\mathrm{Ab}}(H)$*.*

Remark 4.1.1. *By applying* $\epsilon \otimes \epsilon$ *to Equation (4.15) one obtains that for* $g, k \in G$

$\epsilon(\underline{\gamma}_{\ k^{-1},g^{-1}})^2 = \epsilon(\underline{\gamma}_{\ k^{-1},g^{-1}})$. *Since* $\underline{\gamma}_{\ k^{-1},g^{-1}} \in H^\times$ *this implies that* $\epsilon(\underline{\gamma}_{\ k^{-1},g^{-1}}) \neq 0$. *Therefore, for all* $g, h \in G$ $\epsilon(\underline{\gamma}_{\ k^{-1},g^{-1}}) = 1$, *or equivalently for all* $g, k \in G$ $\epsilon(\underline{\gamma}_{\ g,k}) = 1$.

Definition 4.1.3. *Equivalent G Abelian 3-cocycles*

Two G Abelian 3-cocycles $(\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu})$, $(\Delta', \epsilon', \Phi', S', \alpha', \beta', \Psi', \underline{\gamma}', \overline{\mu}')$ *are equivalent if there is a twist* J, J^{-1} *and* $\tau \in C^1(G, H^\times)$ *such that:*

$$\Phi^J = \Phi' \qquad \Delta^J = \Delta' \quad \epsilon = \epsilon' \qquad (S, \alpha, \beta)^J = (S', \alpha', \beta') \tag{4.16}$$

(recall Definition 3.7.1) and the following equations hold for all $g, k \in G$*:*

$$\tau_e = 1_H \tag{4.17}$$

$$\mathrm{Ad}(\tau_g)(\Psi'(g)) = \Psi(g) \tag{4.18}$$

$$\Delta(\tau_{g^{-1}}) \cdot \overline{\mu}_g \cdot (\tau_{g^{-1}} \otimes \tau_{g^{-1}})^{-1} = \overline{\mu}_g{}' \tag{4.19}$$

$$\tau_{gk} \cdot \underline{\gamma}'_{\ g,k} = \underline{\gamma}_{\ g,k} \cdot (\Psi(k)(\tau_g \cdot \tau_k)) \tag{4.20}$$

It is straightforward to check that Definition 4.1.3 defines an equivalence relation. We denote the set of equivalence class of G-Abelian 3-cocycles on a $\mathbb{C}$-algebra H by $H^3_{G-\mathrm{Ab}}(H)$

Proposition 4.1.1. *If* $(\Delta, \epsilon, S, \alpha, \beta, \Phi, \Psi, \underline{\gamma}, \overline{\mu})$ *is a G Abelian 3-cocycle, then there is an induced G-tensor structure on* $\mathrm{Mod}(H)$*. The induced G-tensor structure will be normalized in the sense of Definition 2.1.1. Furthermore, equivalent G Abelian 3-cocycles induce equivalent G-tensor category structures in the sense of Definition 2.1.2.*

Proof. This proof is straightforward, but since it is illustrative of many of the ideas in this chapter we go through the translation between the cohomological data and the categorical data step by step.

Define a categorical group action $(\psi, \gamma, \psi_0) : \underline{G} \to \mathrm{Aut}_\otimes(\mathrm{Mod}(H))$ as follows:

$$\psi_g = (T_g, \mu_g, \mathrm{Id}_\mathbb{C}) \tag{4.21}$$

where the functor T_g is given by:

$$T_g(V, \rho_V) := (V, \rho_V \circ \Psi(g^{-1})) \qquad T_g(f) = f \tag{4.22}$$

The tensor structure $\mu_g : T_g(V, \rho_V) \otimes T_g(W, \rho_W) \to T_g((V, \rho_V) \otimes (W, \rho_W))$ is defined by considering the linear map $\mu_g := (\rho_V \otimes \rho_W)(\overline{\mu}_g)$ and restricting to the representation spaces. Notice by Equation (4.6) that $(\mathbb{C}, \epsilon) = T_g(\mathbb{C}, \epsilon)$. Since as a linear map $T_g(\ell^{\mathrm{Mod}(H)}_{(V,\rho_V)}) = \ell^{\mathrm{Vect}}_V$ we see that the triangle axioms for $(V, \rho_V) \in \mathrm{Mod}(H)$ reduce to:

$$(\epsilon \otimes \rho_V)(\overline{\mu}_g) = \mathrm{Id}_V \tag{4.23}$$

$$(\rho_V \otimes \epsilon)(\overline{\mu}_g) \circ (\mathrm{Id}_V \otimes \phi^g_0) = \mathrm{Id}_V \tag{4.24}$$

By Equation (4.14) we see that:

$$(\mathrm{Id}_H \otimes \epsilon)(\overline{\mu}_g) = (\epsilon \otimes \mathrm{Id}_H)(\overline{\mu}_g)\mathrm{Id}_H \tag{4.25}$$

So indeed the triangle axioms will be satisfied. Therefore, we see for each g that $\psi_g \in \mathrm{Aut}_\otimes(\mathrm{Mod}(H))$.

We define γ for $g, k \in G$ as:

$$\gamma_{g,k}(V, \rho_V) : (V, \rho_V \circ \Psi(k^{-1}) \circ \Psi(g^{-1})) \to (V, \rho_V \circ \Psi((gk)^{-1})) \quad \gamma_{g,k}(V, \rho_V)(v) := \rho_V(\underline{\gamma}_{k^{-1},g^{-1}})(v) \tag{4.26}$$

By Equation (4.2) we see that for all $g, k \in G, h \in H$:

$$\Psi((gk)^{-1})(h) \cdot \underline{\gamma}_{k^{-1},g^{-1}} = \underline{\gamma}_{k^{-1},g^{-1}} \cdot (\Psi(k^{-1}) \circ \Psi(g^{-1}))(h) \tag{4.27}$$

So indeed $\gamma_{g,k}$ will be a H-intertwiner, and therefore a natural isomorphism from $T_g \circ T_k \to T_{gk}$. To verify that $\gamma_{g,k}$ gives (ψ, γ, ψ_0) a tensor functor structure we must have:

$$\gamma_{g_1 g_2, g_3} \circ \gamma_{g_1,g_2}(T_{g_3}) = \gamma_{g_1, g_2 g_3} \circ T_{g_1}(\gamma_{g_2,g_3}) \tag{4.28}$$

Since $\underline{\gamma}$ satisfies Equation 4.3 we see that for all $g_1, g_2, g_3 \in G$:

$$\underline{\gamma}_{g_3^{-1}, g_2^{-1}g_1^{-1}} \cdot \Psi(g_3^{-1})(\underline{\gamma}_{g_2^{-1}, g_1^{-1}}) = \underline{\gamma}_{g_3^{-1}g_2^{-1}, g_1^{-1}} \cdot \underline{\gamma}_{g_3^{-1}, g_2^{-1}} \tag{4.29}$$

Expanding we see this is just Equation (4.28). Lastly, since the non-Abelian 2-cocycle is normalized the unitor ψ_0 will just be the identity transformation. Evidently, this G-tensor structure will be unital.

Suppose now that we have two equivalent G Abelian 3-cocycles $((\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu}),$ $(\Delta', \epsilon', \Phi', S', \alpha', \beta', \Psi', \underline{\gamma}', \overline{\mu}')$ with equivalence given by (J, J^{-1}, τ). Denote the G-tensor structures by (Ψ, γ, ψ_0) and $(\Psi', \gamma', \psi_0')$ respectively. I claim that this will induce a G-tensor functor that is also an equivalence of tensor categories. Let $(\mathrm{Id}, (\rho_- \otimes \rho-)(J), \mathrm{Id}_{\mathbb{C}})$ denote the identity functor with tensor structure induced from the twist J. It is routine to verify that this will have the structure of a tensor functor. We give $(\mathrm{Id}, J, \mathrm{Id}_{\mathbb{C}})$ the structure of a G-tensor functor by defining:

$$\tilde{\tau}_g(V, \rho_V) := \rho_V(\tau_{g^{-1}}) \tag{4.30}$$

To check that $(\mathrm{Id}, J, \mathrm{Id}_{\mathbb{C}}, \{\tilde{\tau}_g\}_{g\in G})$ defines a G-tensor functor we need to verify that:

1. $\tilde{\tau}_g$ is an H-intertwiner from $T_g'(V, \rho_V)$ to $T_g(V, \rho_V)$.

2. $\tilde{\tau}_g$ will be a monoidal natural transformation in $\mathrm{Mod}(H)$.

3. $\tilde{\tau}_e = (\psi_0)_- \circ (\psi_0')_-$

4. For all $g, k \in G$: $\tilde{\tau}_{gk}(V, \rho_V) \circ \gamma'_{g,k}(V, \rho_V) = \gamma_{g,k}(V, \rho_V) \circ \tilde{\tau}_g(T_k(V, \rho_V)) \circ T_g'(\tilde{\tau}_k(V, \rho_V))$

$\tilde{\tau}_g$ will be an H-intertwiner by Equation (4.18). $\tilde{\tau}_g$ will be a natural transformation since for H-intertwiners $f : (V, \rho_V) \to (W, \rho_W)$ $f \circ \rho_V(\tau_{g^{-1}}) = \rho_W(\tau_{g^{-1}}) \circ f$. It will be a monoidal natural transformation by Equation (4.19). Since $\psi_0 = \psi_0' = \mathrm{Id}$ $\tilde{\tau}_e = \mathrm{Id}$ we see it will satisfy the third condition. The last condition follows from Equation (4.20). □

4.2 G-Crossed Abelian 3-Cocycles

As before we fix a $\mathbb{C}$-algebra $(H, \cdot, 1_H)$.

Notation. *If G is a finite group, let $\mathbb{C}^G$ denote the algebra of functions on G. This will be an algebra with pointwise multiplication and the unit being the constant 1-function. For $g \in G$ let δ_g denote the function from G to $\mathbb{C}$ that is zero everywhere except at g where it is one. $\mathbb{C}^G$ will be a Hopf algebra with co-multiplication given by:*

$$\Delta(\delta_g) := \sum_{\substack{r,k \in G \\ rk=g}} \delta_r \otimes \delta k \tag{4.31}$$

co-unit given by:

$$\epsilon(\delta_g) = \delta_{g,e} \tag{4.32}$$

where $\delta_{g,e}$ is the Kronecker delta, and antipode given by:

$$S(\delta_g) = \delta_{g^{-1}} \tag{4.33}$$

If H is an algebra, then $Z(H) := \{h \in H : \forall r \in H \; rh = hr\}$.

Definition 4.2.1. *Weak Morphism of WQHFs*

Let $(H, \cdot, 1_H, \Delta_1, \Phi_1, S_1, \alpha_1, \beta_1)$, $(K, \cdot, 1_K, \Delta_2, \Phi_2, S_2, \alpha_2, \beta_2)$ be wqhfs. A weak morphism of wqfhs is a unital algebra homomorphism $f : H \to K$ such that:

$$\Delta_2 \circ f = (f \otimes f) \circ \Delta \tag{4.34}$$

$$S_2 \circ f = f \circ S_1 \tag{4.35}$$

If it additionally satisfies:

$$(f \otimes f \otimes f)(\Phi_1) = \Phi_2 \quad f(\alpha_1) = \alpha_2, f(\beta_1) = \beta_2 \tag{4.36}$$

we say it is just a morphism of wqhfs.

The idea of weak morphisms is we want to look at wqhfs whose fusion rings are isomorphic, but they are not necessarily equivalent as fusion categories. For example, $\mathrm{Vect}_G, \mathrm{Vect}_G^\omega$. To give a G-crossed braided tensor structure on $\mathrm{Mod}(H)$ we need a G-tensor action with a compatible G-grading and a G-crossed braiding. These two extra requirements motivates the following:

Definition 4.2.2. *G-Crossed Abelian 3-Cocycle*

A G-crossed Abelian 3-cocycle is a tuple $(\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu}, \overline{c}, \overline{\partial})$ *where*

$$(\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu}) \in Z^3_{G-\mathrm{Ab}}(H) \tag{4.37}$$

$\overline{\partial} : \mathbb{C}^G \to Z(H)$ *is a linear map, and* $\overline{c} \in H \otimes H$ *is partially invertible such that:*

$$D(\overline{c}) = \Delta(1_H) \qquad R(\overline{c}) = (\Delta^{op}(1_H))^{(e,g)} \tag{4.38}$$

Furthermore, $(\Psi, \underline{\gamma}, \overline{\mu}, \overline{c})$ *must satisfy the following conditions:*

1. $\overline{\partial}$ *is an injective weak morphism of wqhfs.*

2. *For all* $k, g \in G$ *we have:*

$$\psi(k)(\overline{\partial}(\delta_g)) = \overline{\partial}(\delta_{kgk^{-1}}) \tag{4.39}$$

3. *The following equations must hold for* $\overline{\mu}, \underline{\gamma}, \overline{c}$*, and for all* $g \in G \quad h \in H$*:*

$$\overline{c} \cdot \Delta(h) \cdot \overline{c}^{\,-1} \cdot (\overline{\partial}(\delta_g))_1 = (\Delta^{op}(h))^{(e,g)} \cdot (\overline{\partial}(\delta_g))_1 \tag{4.40}$$

For all $g, k \in G$*:*

$$(\underline{\gamma}_{\,g^{-1},gk^{-1}g^{-1}})_2 \cdot \overline{c}^{(g,g)} \cdot (\overline{\mu}_g)^{-1} \cdot (\overline{\partial}(\delta_k))_1 = (\underline{\gamma}_{\,k^{-1},g^{-1}})_2 \cdot ((\overline{\mu}_g)^{-1})_{21}^{(e,k)} \cdot \overline{c} \cdot (\overline{\partial}(\delta_k))_1 \tag{4.41}$$

$$(\overline{\mu_g}^{\,-1})_{23} \cdot (\mathrm{Id}_H \otimes \Delta)(\overline{c}) \cdot (\overline{\partial}(\delta_g))_1 = (\Phi_{231}^{-1})^{(e,g,g)} \cdot \overline{c}_{13} \cdot \Phi_{213}^{(e,g,e)} \cdot \overline{c}_{12} \cdot \Phi_{123}^{-1} \cdot (\overline{\partial}(\delta_g))_1 \tag{4.42}$$

$$(\Delta\otimes\mathrm{Id})(\overline{c})\cdot(\overline{\partial}(\delta_g)\otimes\overline{\partial}(\delta_k))_{12} = (\Phi_{312})^{(e,e,gk)}\cdot(\overline{\gamma}_{g,k})_3\cdot\overline{c}_{13}^{(e,e,k)}\cdot(\Phi_{132}^{-1})^{(e,e,k)}\cdot\overline{c}_{23}\cdot\Phi_{123}\cdot(\overline{\partial}(\delta_g)\otimes\overline{\partial}(\delta_k))_{12} \tag{4.43}$$

Denote the set of G-crossed Abelian 3-cocycles by $Z^3_{G-\mathrm{Crsd}}(H)$.

Notice that when $G = \{e\}$ these axioms will reduce to the axioms of a quasitriangular wqhf in Definition 3.6.1. In particular Equation (4.40) reduces to Equation (3.38), and Equations (4.42), (4.43) reduce to Equations (3.39), (3.40) respectively. Therefore, it is useful to think of these as the G-twisted analogue of a quasitriangular structure.

Remark 4.2.1. *By setting $g = k = e$ in Equation (4.43) and then applying $(\epsilon\otimes\epsilon\otimes\mathrm{Id})$ one obtains that $(\epsilon\otimes\mathrm{Id}_H)(\overline{c}) = 1_H$. Similarly, applying $(\epsilon\otimes\epsilon\otimes Id_H)$ to Equation (4.42) we obtain that $(\mathrm{Id}_H\otimes\epsilon)(\overline{c}) = 1_H$.*

Definition 4.2.3. *Equivalence of G-crossed Abelian 3-cocycles*

Two G-crossed Abelian 3-cocycles $(\Delta,\epsilon,\Phi,S,\alpha,\beta,\Psi,\underline{\gamma},\overline{\mu},\overline{c},\overline{\partial})$, $(\Delta',\epsilon',\Phi',S',\alpha',\beta',\Psi',\underline{\gamma}',\overline{\mu}',\overline{c}',\overline{\partial}')$ are equivalent if there exists a twist J, J^{-1} and a $\tau\in C^1(G,H^\times)$ that induces an equivalence of the underlying G Abelian 3-cocycles and for all $g\in G$:

$$\overline{\partial} = \overline{\partial}' \tag{4.44}$$

$$\overline{c}\cdot J\cdot(1_H\otimes\overline{\partial}(\delta_g)) = (J_{21})^{(e,g)}\cdot(1_H\otimes\tau_g)\cdot\overline{c}'\cdot(1_H\otimes\overline{\partial}(\delta_g)) \tag{4.45}$$

It is straightforward to show that Definition 4.2.3 defines an equivalence relation on $Z^3_{G-\mathrm{Crsd}}(H)$. Denote the set of equivalence classes by $H^3_{G-\mathrm{Crsd}}(H)$.

Proposition 4.2.1. *If $(\Delta,\epsilon,\Phi,S,\alpha,\beta,\Psi,\underline{\gamma},\overline{\mu},\overline{c},\overline{\partial})$ is a G-crossed Abelian 3-cocycle, then there is an induced G-crossed braided tensor category structure on $\mathrm{Mod}(H)$. Furthermore, equivalent G-crossed Abelian 3-cocycles induce equivalent G-crossed braided tensor categories in the sense of Definition 2.2.4.*

Proof. Define the g-th component $\mathrm{Mod}(H)_g$ as all H-modules $(V,\rho_V)\in\mathrm{Mod}(H)$ such that:

$$\rho_V(\overline{\partial}(\delta_g)) = \mathrm{Id}_V \tag{4.46}$$

This will be an Abelian subcategory of $\mathrm{Mod}(H)$. Notice that if $(V, \rho_V) \in \mathrm{Mod}(H)$, then since $\overline{\partial}$ is a central embedding for every $g \in G$ $(V, \rho_V)_g := \rho_V(\overline{\partial}(\delta_g))(V)$ will be a H-submodule of (V, ρ_V) in $\mathrm{Mod}(H)_g$. It is easy to see that for distinct $g, k \in G$ $(V, \rho_V)_g \cap (V \rho_V)_k = \{0\}$. From

$$\sum_{g \in G} \overline{\partial}(\delta_g) = 1_H \tag{4.47}$$

we see that $(V, \rho_V) = \bigoplus_{g \in G}(V, \rho_V)_g$, and hence $\mathrm{Mod}(H) = \bigoplus_{g \in G} \mathrm{Mod}(H)_g$. We now need to check that this G-grading on $\mathrm{Mod}(H)$ is compatible with the tensor structure.

Suppose now that for $g, k \in H$ $(V, \rho_V) \in \mathrm{Mod}(H)_g, (W, \rho_W) \in \mathrm{Mod}(H)_k$, to verify the G-grading is compatible with tensor category structure we need to first verify that:

$$(V, \rho_V) \otimes (W, \rho_W) \in \mathrm{Mod}(H)_{gk} \tag{4.48}$$

To that end recall that since $\overline{\partial}$ is a morphism of wqhf we have for all $r \in G$:

$$(\Delta \circ \overline{\partial})(\delta_r) = ((\overline{\partial} \otimes \overline{\partial}) \circ \Delta)(\delta_r) = \sum_{\substack{t,\ell \in G \\ t\ell = r}} \overline{\partial}(\delta_t) \otimes \overline{\partial}(\delta_\ell) \tag{4.49}$$

Therefore, if $v \in V, w \in W$, then:

$$\begin{aligned}(\rho_V \otimes \rho_W)(\Delta \circ \overline{\partial}(\delta_{gk}))(v \otimes w) &= \sum_{\substack{t,\ell \in G \\ t\ell = gk}} \rho_V(\overline{\partial}(\delta_t))(v) \otimes \rho_W(\overline{\partial}(\delta_\ell))(w) \\ &= \rho_V(\overline{\partial}(\delta_g))(v) \otimes \rho_W(\overline{\partial}(\delta_k))(w) = v \otimes w\end{aligned} \tag{4.50}$$

Thus, we have verified the first condition. For the second we need to check that if $(V, \rho_V) \in \mathrm{Mod}(H)_g$, then $(V, \rho_V)^* \in \mathrm{Mod}(H)_{g^{-1}}$. Well we know that $S(\overline{\partial}(\delta_g)) = \overline{\partial}(S(\delta_g)) = \overline{\partial}(\delta_{g^{-1}})$. Since $(V, \rho_V)^* := (V^*, (\rho_V \circ S)^*)$ we see then that if v^* is the dual pair to $v \in V$ then:

$$(\rho_V(S(\overline{\partial}(\delta_{g^{-1}}))))^*(v^*) = (\rho_V(\overline{\partial}(\delta_g))(v))^* = v^* \tag{4.51}$$

so indeed $(V, \rho_V)^* \in \mathrm{Mod}(H)_{g^{-1}}$. The last condition on the G-grading we need to check is that if $(V, \rho_V) \in \mathrm{Mod}(H)_g$, then $T_k(V, \rho_V) \in \mathrm{Mod}(H)_{kgk^{-1}}$. Well $T_k(V, \rho_V) := (V, \rho_V \circ \Psi(k^{-1}))$ and by Equation (4.39) we see that for $v \in V$:

$$\rho_V(\Psi(k^{-1})(\overline{\partial}(\delta_{kgk^{-1}})))(v) = \rho_V(\overline{\partial}(\delta_g)(v) = v \tag{4.52}$$

Therefore, we have a compatible grading.

For the G-crossed braiding define for $(V, \rho_V), (W, \rho_W) \in \mathrm{Mod}(H)$ a linear map $c_{(V,\rho_V),(W,\rho_W)}$ by:

$$c_{(V,\rho_V),(W,\rho_W)}(v \otimes w) := (\rho_W \otimes \rho_V)(\overline{c}_{21})(w \otimes v) \tag{4.53}$$

Verifying that c will indeed define a G-crossed braiding is tedious, and so we refer the interested reader to Appendix B.1 for their verification. In broad strokes if $(V, \rho_V) \in \mathrm{Mod}(H)_g$, then by Equation (4.40) this will be an H-intertwiner. Therefore, we can restrict to the representation space and we have a H-intertwiner $c_{(V,\rho_V),(W,\rho_W)}$: $(V, \rho_V) \otimes (W, \rho_W) \to T_g(W, \rho_W) \otimes (V, \rho_W)$. This is clearly a natural transformation. To verify the axioms given in Figure 2.1, 2.2, 2.3 simply expand the equations and notice that they follow from Equations (4.41), (4.42), (4.43) respectively.

Suppose $(\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu}, \overline{c}, \overline{\partial}), (\Delta', \epsilon', \Phi', S', \alpha', \beta', \Psi', \underline{\gamma}', \overline{\mu}', \overline{c}', \overline{\partial}')$ are two equivalent G-crossed Abelian 3-cocycles with the equivalence given by (J, J^{-1}, τ). To verify that this induces an equivalence of G-crossed braided tensor categories we only need to verify that the G-functor $(\mathrm{Id}, J, \mathrm{Id}_0, \{\tilde{\tau}_g\}_{g\in G})$ given in the proof of Proposition 4.1.1 satisfies the additional axiom that for $(V, \rho_V) \in \mathrm{Mod}(H)_g, (W, \rho_W) \in \mathrm{Mod}(H)$:

$$J_{T_g(W,\rho_W),(V,\rho_V)} \circ (\tilde{\tau}_g(W, \rho_W) \otimes \mathrm{Id}) \circ c'_{(V,\rho_V),(W,\rho_W)} = c_{(V,\rho_V),(W,\rho_W)} \circ J_{(V,\rho_V),(W,\rho_W)} \tag{4.54}$$

Expanding out the left hand side we obtain:

$$c^{\text{Vect}}_{V,W} \circ (\rho_V \otimes \rho_W)((J_{21})^{(e,g)} \cdot (1_H \otimes \tau_{g^{-1}}) \cdot \overline{c}') \tag{4.55}$$

By Equation (4.45) we see this equals $c^{\text{Vect}}_{V,W} \circ (\rho_V \otimes \rho_W)(\overline{c} \cdot J)$. Here we used the fact that $\rho_V(\overline{\partial}(\delta_g)) = \text{Id}_V$. This completes the proof. □

4.3 G-Ribbon Abelian 3-Cocycles

As before we fix a unital $\mathbb{C}$-algebra $(H, \cdot, 1_H)$. Let $(F, J^F, \phi^F) : \text{Mod}(H) \to \text{Mod}(H)$ be a tensor functor. If $(V, \rho_V) \in \text{Mod}(H)$, then $F((V, \rho_V)^*)$ can be endowed with the structure of a left dual with evaluation and coevaluation given respectively by:

$$\text{ev}^F_{(V,\rho_V)} = (\phi^F)^{-1} \circ F(\text{ev}_{(V,\rho_V)}) \circ J^F_{(V,\rho_V)^*,(V,\rho_V)} \tag{4.56}$$

$$\text{coev}^F_{(V,\rho_V)} := J^{-1}_{(V,\rho_V),(V,\rho_V)^*} \circ F(\text{coev}_{(V,\rho_V)}) \circ \phi^F \tag{4.57}$$

By uniqueness of left duals there exists a unique isomorphism $d^F_{(V,\rho_V)} : F((V, \rho_V)^*) \to F(V, \rho_V)^*$ such that:

$$\text{ev}^F_{(V,\rho_V)} = \text{ev}_{F(V,\rho_V)} \circ (d^F \otimes \text{Id}_V) \tag{4.58}$$

$$(\text{Id} \otimes d^F) \circ \text{coev}^F_{(V,\rho_V)} = \text{coev}_{F(V,\rho_V)} \tag{4.59}$$

It is not difficult to show that $d^F(V, \rho_V)$ can be given explicitly as [13, Section 2.10]:

$$\ell_{F(V,\rho_V)^*} \circ (\text{ev}^F_{(V,\rho_V)} \otimes \text{Id}_{F(V,\rho_V)^*}) \circ \phi^{-1}_{F((V,\rho_V)^*),F(V,\rho_V),F(V,\rho_V)^*} \circ (\text{Id}_{F((V,\rho_V)^*)} \otimes \text{coev}_{F(V,\rho_V)}) \circ (r_{F((V,\rho_V)^*)})^{-1} \tag{4.60}$$

Here ℓ, r denote the left and right unitors of $\text{Mod}(H)$ respectively, and ϕ is the associator of $\text{Mod}(H)$ induced from the Drinfeld associator Φ.

Suppose now that we have a G Abelian 3-cocycle $(\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu})$ on $(H, \cdot, 1_H)$. By Proposition 4.1.1 this will induce a categorical G-action $(\psi, \gamma, \text{Id})$ where $\psi_g := (T_g, \mu_g, 1)$. Write the Drinfeld associator and its partial inverse with the shorthand:

$$\Phi = x_1 \otimes x_2 \otimes x_3 \qquad \Phi^{-1} = X_1 \otimes X_2 \otimes X_3 \tag{4.61}$$

The corresponding rigidity isomorphism d^{T_g} can be written as multiplication by a single element, as it is quite complicated we use the following shorthand:

$$\mathbf{d}^g := S(\Psi(g^{-1})(X_{(1)}))\cdot S((\overline{\mu}_g)_{(1)})\cdot\alpha\cdot(\overline{\mu}_g)_{(2)}\cdot(\Psi(g^{-1})(X_2\cdot\beta))\cdot\Psi(g^{-1})(S(X_{(3)})) \tag{4.62}$$

Notice that the rigidity axioms for a wqhf are captured by Equations (3.31) (3.32) and when you set $g = 1$ you recover these axioms. Therefore, it is best to think of $\mathbf{d}^g$ as the g-twisted version of these equations.

Lemma 4.3.1. *Let $(\Delta, \epsilon, \Phi, \Psi, \underline{\gamma}, \overline{\mu})$ be a G Abelian 3-cocycle on H. Then for $(V, \rho_V) \in \mathrm{Mod}(H)$ we have:*

$$(d^{T_g})(V, \rho_V) = (\rho_V(\mathbf{d}^g))^* \tag{4.63}$$

Proof. Let $v^* \in T_g((V, \rho_V)^*)$. Evaluating $\mathrm{Id}_{T_g(V^*)} \otimes \mathrm{coev}_{T_g(X)} \circ (r_{T_g(X)^*})^{-1}(v)$ we obtain:

$$\sum_i v^* \otimes (\rho_{T_g(V,\rho_V)}(\beta))(v_i) \otimes f_i) = \sum_i v^* \otimes (\rho_V(\Psi(g^{-1}(\beta)))(v_i) \otimes f_i) \tag{4.64}$$

Where $\{v_i\}_{i=1}^{\dim(V)}$ is a basis of $T_g(V)$ and f_i is the dual basis. Applying $\phi^{-1}_{T_g((V,\rho_V)^*),T_g(V,\rho_V),T_g(V,\rho_V)^*}$ to Equation (4.64) we obtain:

$$\sum_i (\rho_{V^*}(\Psi(g^{-1})(X_{(1)}))(v^*) \otimes \rho_V(\Psi(g^{-1})(X_{(2)} \cdot \beta))(v_i)) \otimes \rho_{T_g(V)^*}(X_{(3)})(f_i) =$$
$$\sum_i (\rho_V(S(\Psi(g^{-1})(X_{(1)})))^*(v^*) \otimes \rho_V(\Psi(g^{-1})(X_{(2)}\cdot\beta))(v_i)) \otimes \rho_V(\Psi(g^{-1})(S(X_{(3)})))^*(f_i) \tag{4.65}$$

Applying $\mathrm{ev}^{T_g}_{(X)} \otimes \mathrm{Id}$ to Equation (4.65) we obtain:

$$\sum_i (\rho_V(S((\mu_g)_{(1)}\cdot\Psi(g^{-1})(X_{(1)})))^*(v^*)\left(\rho_V(\alpha\cdot(\overline{\mu}_g)_{(2)}\cdot\Psi(g^{-1})(X_{(2)}\cdot\beta))(v_i)\right)\cdot\rho_V(\Psi(g^{-1})(S(X_{(3)})))^*(f_i) =$$
$$\sum_i v^*\left(\rho_V(S((\mu_g)_{(1)}\cdot\Psi(g^{-1})(X_{(1)})\cdot\alpha\cdot(\overline{\mu}_g)_{(2)}\cdot\Psi(g^{-1})(X_{(2)}\cdot\beta))(v_i)\right)\cdot\rho_V(\Psi(g^{-1})(S(X_{(3)})))^*(f_i) \tag{4.66}$$

To finish this proof, we use the fact that if L, K are linear maps of V, then:

$$\sum_i v^*(L(v_i))K^*(f_i) = (L\circ K)^*(v^*) \tag{4.67}$$

Therefore, we see that Equation (4.66) simplifies to:

$$\left(\rho_V(S((\mu_g)_{(1)}\cdot\Psi(g^{-1})(X_{(1)})\cdot\alpha\cdot(\overline{\mu}_g)_{(2)}\cdot\Psi(g^{-1})(X_{(2)}\cdot\beta)\cdot\Psi(g^{-1})(S(X_{(3)})))\right)^*(v^*) \tag{4.68}$$

This equals $(\rho_V(\mathbf{d}^g))^*(v^*)$, and we are done. □

To give Mod(H) the structure of a G-crossed ribbon tensor category we need a G-ribbon twist.

Definition 4.3.1. *G-Ribbon Abelian 3-cocycle*

A G-Ribbon Abelian 3-cocycle is a tuple $(\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu}, \overline{c}, \overline{\partial}, \nu)$ *such that:*

$$(\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu}, \overline{c}, \overline{\partial}) \in Z^3_{G-\mathrm{Crsd}}(H) \tag{4.69}$$

is a G-crossed Abelian 3-cocycle, and ν satisfies the following for all $g \in G, h \in H$

$$\nu\cdot h\cdot\overline{\partial}(\delta_g) = \Psi(g^{-1})(h)\cdot\nu\cdot\overline{\partial}(\delta_g) \tag{4.70}$$

For all $g, k \in G$:

$$\overline{\mu}_{gk}^{-1}\cdot\Delta(\nu)\cdot(\overline{\partial}(\delta_g)\otimes\overline{\partial}(\delta_k)) = (\underline{\gamma}_{\ gk^{-1}g^{-1},(gk)g^{-1}(gk)^{-1}}\otimes\underline{\gamma}_{\ g^{-1},gk^{-1}g^{-1}})\cdot(\nu^{\otimes 2})^{(gkg^{-1},g)}\cdot(\overline{c}_{21})^{(e,g)}\cdot\overline{c}\cdot(\overline{\partial}(\delta_g)\otimes\overline{\partial}(\delta_k)) \tag{4.71}$$

$$S(\nu)\cdot\overline{\partial}(\delta_g) = S(\underline{\gamma}^{-1}_{\ g^{-1},g})\cdot\mathbf{d}^g\cdot\nu\cdot\overline{\partial}(\delta_g) \tag{4.72}$$

$$\underline{\gamma}_{\ g^{-1},gk^{-1}g^{-1}}\cdot\Psi(g^{-1})(\nu)\cdot\overline{\partial}(\delta_k) = \underline{\gamma}_{\ k^{-1},g^{-1}}\cdot\nu\cdot\overline{\partial}(\delta_k) \tag{4.73}$$

Denote the set of G-ribbon Abelian 3-cocycles on H by $Z^3_{G-\mathrm{Rb}}(H)$.

Remark 4.3.1. *By applying $\epsilon \otimes \epsilon$ to Equation (4.71) one obtains that $\epsilon(\nu) = 1$.*

Definition 4.3.2. *Equivalent G-ribbon Abelian 3-cocycles*

Two G-ribbon Abelian 3-cocycles

$$(\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu}, \overline{c}, \overline{\partial}, \nu) \quad (\Delta', \epsilon', \Phi', S', \alpha', \beta', \Psi', \underline{\gamma}', \overline{\mu}', \overline{c}', \overline{\partial}', \nu')$$

are said to be equivalent if the corresponding G-crossed Abelian 3-cocycles are equivalent through a twist J, J^{-1} and $\tau \in C^1(G, H^\times)$ such that for all $g \in G$:

$$\nu' \cdot \overline{\partial}(\delta_g) = \tau_{g^{-1}} \cdot \nu \cdot \overline{\partial}(\delta_g) \tag{4.74}$$

It is straightforward to show that Definition 4.3.2 defines an equivalence relation on $Z^3_{G-\mathrm{Rb}}(H)$. Denote the set of equivalence classes by $H^3_{G-\mathrm{Rb}}(H)$.

Proposition 4.3.1. *A G-ribbon Abelian 3-cocycle $(\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu}, \overline{c}, \overline{\partial}, \nu)$ induces a G-ribbon tensor category structure on* $\mathrm{Mod}(H)$*. Furthermore, equivalent G-ribbon Abelian 3-cocycles induce equivalent G-ribbon tensor category structures in the sense of Definition 2.2.6.*

Proof. We know from Proposition 4.2.1 that $(\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu}, \overline{c}, \overline{\partial})$ endows $\mathrm{Mod}(H)$ with the structure of a G-crossed braided tensor category. Let $(V, \rho_V) \in \mathrm{Mod}(H)_g$ define a G-ribbon twist for $v \in V$ by:

$$\theta_{(V,\rho_V)}(v) := \rho_V(\nu)(v) \tag{4.75}$$

Equation (4.70) guarantees that $\theta_{(V,\rho_V)}$ is an H-intertwiner, and therefore a natural isomorphism. The first G-ribbon axiom, Equation (2.12), will be satisfied due to Equation (4.71). The second G-ribbon axiom, Equation (2.13) will be satisfied because of Equation (4.72), and the fact that the canonical isomorphism $d^{T_g} : T_g((V, \rho_V)^*) \to T_g(V, \rho_V)^*$ for $v \in V$ equals $d^{T_g}(v) = (\rho_V(\mathbf{d}^{\mathbf{g}}))^*(v)$. The third G-ribbon axiom, Equation (2.14), will be satisfied due to Equation (4.73). For detailed verifications of these conditions see Appendix B.2.

Suppose now that $(\Delta, \epsilon, \Phi, \Psi, \underline{\gamma}, \overline{\mu}, \overline{c}, \overline{\partial}, \nu)$ $(\Delta', \epsilon', \Phi', \Psi', \underline{\gamma}', \overline{\mu}', \overline{c}', \overline{\partial}', \nu')$ are equivalent G-ribbon Abelian 3-cocycles through $(J, J^{-1}, \{\tau_g\}_{g\in G})$. From Proposition 4.2.1 we know that $(\mathrm{Id}, (\rho_- \otimes \rho_-)(J), \mathrm{Id}, \{\tilde{\tau}_g\})$ will induce an equivalence of G-crossed braided tensor functors, so we just need to verify that it is also ribbon. That is we need to check Equation (2.15). Well to that end notice that if $(V, \rho_V) \in \mathrm{Mod}(H)_g$ then for all $v \in V$ we have $\rho_V(\overline{\partial}(\delta)_g)(v) = v$. Therefore, we have $\tau_g \circ \theta_{(V,\rho_V)}(v) = \rho_V(\tau_{g^{-1}} \cdot \nu)(v) = \rho_V(\nu')(v) = \theta'_{(V,\rho_V)}(v)$. So indeed it is a G-ribbon tensor equivalence and we are done. $\square$

Chapter 5

G-Reconstruction and G-Equivariant Dimension Functions

In the previous chapter we defined various G 3-cocycles and showed that they define corresponding G-structures. In this chapter we reverse this process by providing a G-reconstruction argument. The main result of this chapter is the following:

Theorem 5.0.1. *G-Reconstruction*

Every G-(ribbon,crossed braided) fusion category $\mathcal{C}$ is equivalent to the category of modules of some wqhf H with G-structure induced by a G-(ribbon,crossed) Abelian 3*-cocycle $\Gamma_{\mathcal{C}}$. Furthermore, the sets $H^3_{G-\mathrm{Ab}}(H), H^3_{G-\mathrm{Crssd}}(H), H^3_{G-\mathrm{Rbn}}(H)$ are in bijections with the equivalence class of the corresponding G-structures.*

To prove this we first define the notion of a weak quasi G-equivariant fiber functor and show the following more general result:

Theorem 5.0.2. *The maps*

$$(\mathcal{C}, F) \mapsto (H := \mathrm{Nat}(F), \Gamma_{\mathcal{C},F}) \quad (H, \Gamma) \mapsto (\mathrm{Mod}(H), \mathrm{Forg}) \tag{5.1}$$

are inverses between: (1) *The equivalence classes of finite G-(ribbon,crossed braided)* *tensor categories and weak G-equivariant fiber functors.* (2) *The equivalence classes of pairs of finite-dimensional unital algebras H and G-(ribbon,crossed) Abelian* 3-*cocycle on H.*

Notice that this result hold not just for fusion categories, but for all finite tensor categories. If we restrict to fusion categories we get a similar correspondence with semi-simple finite dimensional algebras. We also prove in Lemma 5.1.1 that every G-(ribbon, crossed braided) tensor category with a weak quasi G-equivariant fiber functor is equivalent to the category of modules of some algebra H with a G-(ribbon, crossed) Abelian 3-cocycle.

Theorem 5.0.2 reduces the proof of G-reconstruction for fusion categories to showing that every G-(ribbon, crossed braided) fusion category has a weak quasi G-equivariant fiber functor. To that end we define weak G-equivariant dimension functions, and show that these will induce weak quasi G-equivariant fiber functors. Conversely, every weak quasi G-equivariant fiber functor induces a weak G-equivariant dimension function. We then prove that every G-(ribbon, crossed braided) fusion category will have a weak G-equivariant dimension function which completes the proof of G-reconstruction.

As the wqhf given in Theorem 5.0.2 depends on the chosen fiber functor, and hence the weak G-equivariant dimension function we give a few examples of relatively small weak G-equivariant dimension functions at the end of the chapter.

5.1 G-Reconstruction

Let $(\mathcal{C}, (\psi, \gamma, \psi_0), c, \theta)$ be a G-ribbon tensor category. By Proposition 2.2.2 we can assume without loss of generality that $\mathcal{C}$ is skeletal with trivial left/right unitors and (ψ, γ, ψ_0) is unital. We make this assumption throughout the chapter. Furthermore, it should be noted that since $\mathcal{C}$ is G-graded there exists natural transformations $\pi_g : \mathcal{C} \to \mathcal{C}$ such that for simple $X \in \mathcal{C}_g$, $\pi_g(X) = \mathrm{Id}_X, \pi_k(X) = 0$ for $k \neq g$. These will satisfy the relations $\pi_g \circ \pi_k = \delta_{g,k}\pi_g$ and $\sum_{g\in G} \pi_g = \mathrm{Id}_{\mathcal{C}}$.

Definition 5.1.1. *(weak quasi) G-Equivariant Fiber Functors*

Let $(\mathcal{C},(\psi,\gamma,\psi_0),c,\theta)$ be a G-ribbon tensor category. We say that $(F,J^F,\phi^F,\{R_g\}_{g\in G})$ is a weak quasi G-equivariant fiber functor if (F,J^F,ϕ^F) is a weak quasi fiber and functor, and there exists a collection of natural isomorphisms for $g\in G$:

$$R_g : F \to F \circ T_g \tag{5.2}$$

such that $R_e = \mathrm{Id}$ and $R_g^{1_\mathcal{C}} = \mathrm{Id}_{F(1_\mathcal{C})}$ for every $g \in G$.

The first step to proving G-reconstruction is proving that weak quasi G-equivariant fiber functors define a G-(ribbon, crossed) Abelian 3-cocycle.

Proposition 5.1.1. *Let $(\mathcal{C},(\psi,\gamma,\psi_0),c,\theta)$ be a G-ribbon tensor category. A weak quasi G-equivariant fiber functor $(F,J^F,\phi^F,\{R_g\}_{g\in G})$ defines a G-ribbon Abelian 3-cocycle $(\Delta,\epsilon,\Phi,S,\alpha,\beta,\Psi,\underline{\gamma},\overline{\mu},\overline{c},\overline{\partial},\nu)$ on $H := \mathrm{Nat}(F)$:*

1. *$(\Delta,\epsilon,\Phi,S,\alpha)$ is as described in Theorem 3.8.1.*

2. *$\Psi : G \to \mathrm{Aut}(H)$ is defined for $g \in G$ and $h \in H, X \in \mathcal{C}$ as:*

$$(\Psi(g)(h))_X := (R^X_{g^{-1}})^{-1} \circ h_{T_{g^{-1}}(X)} \circ R^X_{g^{-1}} \tag{5.3}$$

3. *For $g,k \in G$ we have:*

$$\underline{\gamma}_{\ g,k} := (R_{(gk)^{-1}})^{-1} \circ F(\gamma_{k^{-1},g^{-1}}) \circ R_{k^{-1}}^{T_{g^{-1}}} \circ R_{k^{-1}} \tag{5.4}$$

4. *For $g \in G$:*

$$\overline{\mu}_g := (J^F)^{-1} \circ (R_g)^{-1} \circ F(\mu_g) \circ J^F_{T_g,T_g} \circ (R_g \otimes R_g) \tag{5.5}$$

5. *Define $\overline{\partial} : \mathbb{C}^G \to H$ for $g \in G$ by:*

$$\overline{\partial}(\delta_g) = F(\pi_g) \tag{5.6}$$

6. *For $X \in \mathcal{C}_g, Y \in \mathcal{C}$ define:*

$$\overline{c}_{X,Y} := c^{\text{Vect}}_{F(Y),F(X)} \circ ((R^Y_g)^{-1} \otimes \text{Id}_X) \circ (J^F_{T_g(Y),X})^{-1} \circ F(c_{X,Y}) \circ J^F_{X,Y} \tag{5.7}$$

7. *For $X \in \mathcal{C}_g$ define:*

$$\nu_X := (R^X_g)^{-1} \circ F(\theta_X) \tag{5.8}$$

Proof. By Theorem 3.8.1 we know that $(H, \cdot, \eta, \Delta, \epsilon, \Phi, S, \alpha, \beta)$ is a weak quasi Hopf algebra, this proof will hold for tensor categories as well. The proof that the G-ribbon Abelian 3-cocycle axioms are satisfied is straightforward but very tedious, and so we refer the reader to Appendix D for the verifications. □

For readability we split the proof of Theorem 5.0.2 into several lemmas.

Lemma 5.1.1. *Let $(\mathcal{C}, (\psi, \gamma, \psi_0), c, \theta)$ be a G-ribbon tensor category with a weak quasi G-equivariant fiber functor $(F, J^F, \phi^F, \{R_g\}_{g\in G})$. Denote the G-ribbon Abelian 3-cocycle given in Proposition 5.1.1 by $\Gamma_{(\mathcal{C},F)}$. Then the G-ribbon structure induced by $\Gamma_{(\mathcal{C},F)}$ on $\text{Mod}(H)$ is equivalent to $(\mathcal{C}, (\psi, \gamma, \psi_0), c, \theta)$.*

Proof. By [6] and Theorem 3.8.1 we know that there is an equivalence of fusion categories categories[1]:

$$(L, J^F, \phi^F) : \mathcal{C} \to \text{Mod}(H) \tag{5.9}$$

Denote the G-ribbon structure on $\text{Mod}(H)$ induced by $\Gamma_{\mathcal{C},F}$ as $(\text{Mod}(H), (\psi^1, \gamma^1, \psi^1_0), c^1, \theta^1)$ where $\psi^1_g := (T^1_g, \mu^1_g, (\phi^1)^g_0)$. To show that L can be upgraded to a G-ribbon equivalence we need to construct for every $g \in G$ natural isomorphisms:

$$\tau_g : T^1_g \circ L \to L \circ T_g \tag{5.10}$$

[1]It is straightforward to show that if we one knows that $L : \mathcal{C} \to \text{Mod}(H)$ is an equivalence of $\mathbb{C}$-linear locally finite Abelian categories, induced by F, then a weak quasi fibre functor structure on F induces a tensor functor structure on L. By [6, Proposition 2.14] any weak quasi fibre functor will induce an equivalence of Abelian categories and hence tensor categories. Hence Theorem 3.8.1 will hold also for tensor categories.

such that $(L, J^F, \phi^F, \{\tau_g\}_{g\in G})$ is a G-ribbon equivalence. Well let $X \in \mathcal{C}$, then $L(X) = (F(X), \rho_X)$ where the H-action is defined for $h \in H$ as: $\rho_X(h) = h_X$. Therefore,

$$(T_g^1 \circ L)(X) = (F(X), \rho_X \circ \Psi(g^{-1})) \Rightarrow (\rho_X \circ \Psi(g^{-1}))(h) = (R_g^X)^{-1} \circ h_{T_g(X)} \circ R_g^X \quad (5.11)$$

While on the other hand:

$$\rho_{T_g(x)}(h) = h_{T_g(X)} \quad (5.12)$$

Therefore, let $\tau_g = R_g$. We have just proven that it is an H-intertwiner and so τ_g is a collection of natural isomorphisms. Furthermore, verifying that $(L, J^F, \phi^F, \{R_g\}_{g\in G})$ is a G-ribbon equivalence follows from the fact that $\Gamma_{(\mathcal{C},J^F)}$ is defined by conjugating the structure of $((\psi, \gamma, \psi_0), c, \theta)$ by R_g. □

Lemma 5.1.2. *Let H be a finite dimensional unital $\mathbb{C}$-algebra and $\Gamma \in Z^3_{G-\mathrm{Rb}}(H)$. Then the forgetful functor* $\mathrm{Forg} : \mathrm{Mod}(H) \to \mathrm{Vect}$ *is a weak quasi G-equivariant fiber functor with structure given by $J_{(V,\rho_V),(W,\rho_W)} = (\rho_V \otimes \rho_W)(\Delta(1_H))$, $\phi^{\mathrm{Forg}} = \mathrm{Id}_{\mathbb{C}}$, $R_g =$* Id. *Furthermore, the G-ribbon Abelian 3-cocycle induced by* $(\mathrm{Forg}, J, \phi, \{R_g\}_{g\in G})$ *equals Γ on the nose.*

Proof. It is easy to check that (Forg, J, ϕ) is a weak quasi-fiber functor. Denote Γ as $(\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu}, \overline{c}, \overline{\partial}, \nu)$. To see that $R_g = \mathrm{Id}$ will give a natural transformation from $\mathrm{Forg} \to \mathrm{Forg} \circ T_g$ recall that $T_g(V, \rho_V) = (V, \rho_V \circ \Psi(g^{-1}))$ and $T_g(f) = f$ as a linear map, so indeed $\mathrm{Forg} \circ T_g = \mathrm{Forg}$ on the nose.

Let $\Gamma_{\mathrm{Forg},H}$ denote the 3-cocycle induced by $(\mathrm{Forg}, J, \mathrm{Id}_{\mathcal{C}}, \{R_g\}_{g\in G})$. Note that we know $\mathrm{Nat}(\mathrm{Forg}) \cong H$ through the isomorphisms for $h \in H, k \in \mathrm{Nat}(\mathrm{Forg})$, $(V, \rho_V) \in \mathrm{Mod}(H)$:

$$f : H \to \mathrm{Nat}(\mathrm{Forg}) \qquad f(h)_{(V,\rho_V)} = \rho_V(h) \quad (5.13)$$

$$f^{-1} : \mathrm{Nat}(\mathrm{Forg}) \to H \qquad f^{-1}(k) := k_H(1_H) \quad (5.14)$$

From [21] we know that f will induce an isomorphism of wqhfs, and so we only need to verify that:

$$\mathrm{Ad}(f)(\Psi^{\mathrm{Forg}}) = \Psi \tag{5.15}$$

$$f^{-1} \circ \underline{\gamma}^{\mathrm{Forg}} \circ f = \underline{\gamma} \tag{5.16}$$

$$(f^{-1} \otimes f^{-1}) \circ \overline{\mu}^{\mathrm{Forg}} \circ (f \otimes f) = \overline{\mu} \tag{5.17}$$

$$(f^{-1} \otimes f^{-1}) \circ \overline{c}^{\mathrm{Forg}} \circ (f \otimes f) = \overline{c} \tag{5.18}$$

$$f^{-1} \circ \overline{\partial}^{\mathrm{Forg}} = \overline{\partial} \tag{5.19}$$

$$f^{-1} \circ \nu^{\mathrm{Forg}} \circ f = \nu \tag{5.20}$$

Verifying this is straightforward but tedious, and so we leave the proof to the studious reader. □

Definition 5.1.2. *Equivalent Weak Quasi G-Equivariant Fiber Functors*

Let $(\mathcal{C}, (F, J^F, \phi^F, \{R_g\}_{g\in G})), (\mathcal{D}, (K, J^K, \phi^K, \{R^1_g\}_{g\in G}))$ *be two G-tensor categories with weak quasi G-equivariant fiber functors. We say that the two weak quasi G-equivariant fiber functors are equivalent if there exists a G-functor* $(L, J^L, \phi^L, \{\tau_g\}_{g\in G})$ *such that* $L : \mathcal{C} \to \mathcal{D}$ *is an equivalence of categories, and there is a natural isomorphism* $\kappa : F \to K \circ L$

Similarly, one can define equivalence of weak quasi G-equivariant fiber functors on G-crossed braided tensor categories and G-ribbon tensor categories. When we want to be brief we will denote a weak quasi G-equivariant fiber functor $(\mathcal{C}, (F, J^F, \phi^F, \{R_g\}_{g\in G}))$ simply by $(\mathcal{C}, F)$.

Definition 5.1.3. *If* $(H, \Gamma^1), (R, \Gamma^2)$ *are pairs of finite dimensional algebras with G-(ribbon,crossed) Abelian 3-cocycles we say that they are equivalent if there exists a unital algebra isomorphism* $f : H \to R$ *such that* $f(\Gamma^2)$ *is equivalent to* Γ^2*. Here* $f(\Gamma^2)$ *denotes the G-(ribbon,crossed) Abelian 3-cocycle on H defined by pulling back* Γ^2 *through f.*

Theorem 5.0.2. *The maps*

$$(\mathcal{C}, F) \mapsto (H := \mathrm{Nat}(F), \Gamma_{\mathcal{C},F}) \quad (H, \Gamma) \mapsto (\mathrm{Mod}(H), \mathrm{Forg}) \tag{5.21}$$

are inverses between: (1) *The equivalence classes of finite G-(ribbon,crossed braided) tensor categories and weak G-equivariant fiber functors.* (2) *The equivalence classes of pairs of finite dimensional unital algebras H and G-(ribbon,crossed) Abelian* 3-*cocycle on H.*

Proof. By Lemmas 5.1.1, 5.1.2 we only need to show that the maps are well-defined on equivalence classes.

Suppose that $(H,\Gamma^1),(R,\Gamma^2)$ are equivalent with equivalence given by $f : H \to R$. This implies that there exists a twist $J, J^{-1} \in H \otimes H$ and $\xi \in C^1(G, H^\times)$ realizing the equivalence of $\Gamma^1, f(\Gamma^2)$. Define a functor $L : \mathrm{Mod}(H) \to \mathrm{Mod}(R)$ by $L(V,\rho_V) := (V, \rho_V \circ f^{-1}), L(f) = f$. Evidently this will be an equivalence of G-categories when $\mathrm{Mod}(H)$ has the G-structure induced from $f(\Gamma^2)$ and $\mathrm{Mod}(R)$ has the G-structure induced from Γ^2. We can choose κ to simply be the identity natural transformation. It suffices to show then that $\mathrm{Mod}(H)$ endowed the G-structure induced from Γ_1 is equivalent to $\mathrm{Mod}(H)$ endowed with the G-structured induced from $f(\Gamma_2)$. It can be checked that $(\mathrm{Id}, (\rho_- \otimes \rho_-)(J), 1, \{\tau_g\}_{g\in G}) : (\mathrm{Mod}(H), \Gamma^1) \to (\mathrm{Mod}(H), f(\Gamma^2))$ where $\tau_g(V,\rho_V)) := \rho_V(\xi_g)$ will define a G-equivalence and we leave the details to the interested reader. For the same reasons as before we can also choose the natural isomorphism to be the identity in this case.

First, suppose that $(\mathcal{C}, (F_1, J^{F_1}, \phi^{F_1}, \{R^1_g\}_{g\in G}))$ $(\mathcal{D}, (F_2, J^{F_2}, \phi^{F_2}, \{R^2_g\}_{g\in G}))$ are equivalent with equivalence realized by a natural isomorphism $\kappa : F_1 \to F_2 \circ L$ and an adjoint G-(ribbon,crossed braided) tensor equivalence

$$((L, J^L, \phi^L_0, \{\tau^1_g\}_{g\in G}), (E, J^E, \phi^E_0, \{\tau^2_g\}_{g\in G}), \eta, \varepsilon) \tag{5.22}$$

where $\eta : \mathrm{Id}_{\mathcal{C}} \to E \circ L, \varepsilon : L \circ E \to \mathrm{Id}_{\mathcal{D}}$ are the unit and co-unit respectively. Let $H := \mathrm{Nat}(F_1), R := \mathrm{Nat}(F_2)$. Then we have an isomorphism of unital $\mathbb{C}$-algebras $f : H \to R$ given for $h \in H$, by:

$$f(h) := \mathrm{Ad}(F_2(\varepsilon) \circ \kappa_{E(-)})(h_{E(-)}) \tag{5.23}$$

Using the fact that $(L, E, \eta, \varepsilon)$ are an adjoint equivalence it is easy to check that this will have an inverse for $r \in R$:

$$f^{-1}(r) := \mathrm{Ad}(\kappa^{-1})(r_{L(-)}) \tag{5.24}$$

Evidently f is a unital $\mathbb{C}$-algebra isomorphism. It suffices to show that $\Gamma_{\mathcal{C},F_1}$ is equivalent to $f(\Gamma_{\mathcal{D},F_2})$. Well to that end let:

$$J = (\kappa^{-1} \otimes \kappa^{-1}) \circ (J^{F_2}_{L(-),L(-)})^{-1} \circ F_2((J^L)^{-1}) \circ \kappa_{-\otimes-} \circ J^{F_1} \tag{5.25}$$

$$J^{-1} = (J^{F_1})^{-1} \circ \kappa^{-1}_{-\otimes-} \circ F_2(J^L) \circ J^{F_2}_{L(-),L(-)} \circ (\kappa \otimes \kappa) \tag{5.26}$$

One can check that this is a twist of $(H, \Gamma_{\mathcal{C},F_1})$. For every $g \in G$ define:

$$\tau_g = R^1_g \circ \kappa^{-1} \circ F_2(\tau^1_g) \circ ((R^2_g)^{L(-)})^{-1} \circ \kappa \tag{5.27}$$

This will be an invertible element of H and so $\tau \in C^1(G, H^\times)$. It is straightforward, but tedious to verify that (J, τ) makes $(H, \Gamma_{\mathcal{C},F_1}), (H, f(\Gamma_{\mathcal{D},F_2}))$ equivalent and so we leave it to the studious reader. $\square$

5.2 Weak G-Equivariant Dimension Functions

Fix a G-ribbon fusion category $(\mathcal{C}, (\psi, \gamma, \psi_0), c, \theta)$. As mentioned at the start of the chapter we may assume without loss of generality that $\mathcal{C}$ is skeletal with trivial unitors and that the categorical G-action is unital. The key ingredient to proving reconstruction for fusion categories is the notion of a dimension function. For G-fusion categories we will need a similar notion except that it is compatible with the G-action.

Definition 5.2.1. *Weak G-Equivariant Dimension*

For the sake of brevity we denote the set of simples of $\mathcal{C}$ as ∇. A weak G-equivariant dimension *function $D : \nabla \to \mathbb{N}$, is a function such that*

$$D(\mathbf{1}_{\mathcal{C}}) = 1, \quad D(X^*) = D(X), \quad D(T_g(X)) = D(X) \;\; \forall g \in G \tag{5.28}$$

$$D(X)D(Y) \geq \sum_{Z\in\nabla} N_{X,Y}^{Z} D(Z) \tag{5.29}$$

Proposition 5.2.1. *Every fusion category with a categorical G-action has a G-equivariant weak dimension function.*

Proof. Recall from [21] that every fusion category will have a weak dimension function given by:

$$D(1) = 1, D(X) = \sum_{i,j} N_{i,X}^{j} \tag{5.30}$$

I claim that this will also be G-equivariant. To see this, notice that we have:

$$D(T_g(X)) = \sum_{i,j} N_{i,T_g(X)}^{j} = \sum_{i',j'} N_{T_g(i'),T_g(X)}^{T_g(j')}$$

since T_g is a monoidal autoequivalence. On the other hand we know that T_g induces a linear isomorphism from $\mathrm{Hom}(i'\otimes X, j')$ to $\mathrm{Hom}_{\mathcal{C}}(T_g(i'\otimes X), T_g(j')) \cong \mathrm{Hom}_{\mathcal{C}}(T_g(i')\otimes T_g(X), T_g(j'))$. This implies that $N_{T_g(i'),T_g(X)}^{T_g(j')} = N_{i',X}^{j'}$, and so indeed $D(T_g(X)) = D(X)$. □

This result is crucial to prove G-reconstruction, but unfortunately the weak G-equivariant dimension function will in general give very large numbers making the associated weak quasi-Hopf algebra unusable.

In the case that $\mathcal{C}$ is a G-crossed braided fusion category there are many weak G-equivariant dimensions

Lemma 5.2.1. *Let $\mathcal{C}$ be a G-crossed braided fusion category and suppose that $\ell : \mathbb{R} \to \mathbb{R}$ is a function such that $\ell(r) \geq 1$ for $r \geq 1$ and $\ell(1) = 1$. If ℓ is weakly multiplicative, that is for all $r, k \in \mathbb{R}$:*

$$\ell(r \cdot k) \leq \ell(r)\ell(k) \tag{5.31}$$

and $\ell(\mathrm{FPdim}(X)) \in \mathbb{N}$ for all $X \in \mathcal{C}$, then:

$$D(X) := \ell(\mathrm{FPdim}(X)) \tag{5.32}$$

is a weak G-equivariant dimension function.

Proof. Note that $\text{FPdim}(X^*) = \text{FPdim}(X)$ and since we are in a G-crossed braided fusion category we know for each $Y \in \mathcal{C}_g$ and $X \in \mathcal{C}$ that $Y \otimes X \cong T_g(X) \otimes Y$. Therefore, $\text{FPdim}(T_g(X)) = \text{FPdim}(X)$. The rest follows immediately. □

Example 5.2.1. *Define for $x \in R$:*

$$\lceil x \rceil := \min\{n \in \mathbb{N} : n \geq x\} \tag{5.33}$$

This will be weakly multiplicative and so:

$$D(X) = \lceil \text{FPdim}(X) \rceil \tag{5.34}$$

is a weak G-equivariant dimension function.

Under slightly more restrictive conditions on a G-crossed braided fusion category there is a much nicer weak dimension function.

Definition 5.2.2. *Weakly Integral Fusion Category*

A fusion category $\mathcal{C}$ is called weakly integral if:

$$\text{FPdim}(\mathcal{C}) := \sum_{X \in \nabla} \text{FPdim}(X)^2 \in \mathbb{Z} \tag{5.35}$$

Proposition 5.2.2. *Let $\mathcal{C}$ be a weakly integral G-crossed braided fusion category. Then*

$$D(X) := \text{FPdim}(X)^2 \tag{5.36}$$

defines a weak G-equivariant dimension function.

Proof. Since $\mathcal{C}$ is assumed to be weakly integral we know from [13, Proposition 9.6.9 (i)] that $D(X) \in \mathbb{N}$. Lemma 5.2.1 then implies the rest. □

Proposition 5.2.3. *Let $\mathcal{C}$ be a G-(ribbon,crossed braided) fusion category with weak G-equivariant dimension function D. By Theorem 3.8.1 we know that this will induce* *a weak quasi fiber functor* (F, J^F, ϕ^F)*. For every* $g, \in G$ $Y \in \mathcal{C}$ *define a* $\mathbb{C}$*-linear map* $R_g^Y : F(Y) \to (F \circ T_g)(Y)$ *defined for* $h_X \in \mathrm{Hom}_{\mathcal{C}}(X, Y), v_X \in \mathbb{C}^{D(X)}$ *by:*

$$R_g^Y \left(\bigoplus_{X \in \nabla} h_X \otimes v_X \right) := \bigoplus_{X \in \nabla} T_g(h_{T_{g^{-1}}(X)}) \otimes v_{T_{g^{-1}}(X)} \tag{5.37}$$

The following properties will hold:

$$R_g^- \in \mathrm{Nat}(F, F \circ T_g) \tag{5.38}$$

$$R_e^X = \mathrm{Id}_{F(X)} \tag{5.39}$$

$$R_g^{1_{\mathcal{C}}} = \mathrm{Id}_{F(1_{\mathcal{C}})} \tag{5.40}$$

Therefore, $(F, J^F, \phi^F, \{R_g\}_{g \in G})$ *is a weak quasi G-equivariant fiber functor.*

Proof. Since $T_g(h_{T_{g^{-1}}(X)}) \in \mathrm{Hom}_{\mathcal{C}}((T_g \circ T_{g^{-1}})(X), T_g(Y)) = \mathrm{Hom}_{\mathcal{C}}(X, T_g(Y))$ we see that R_g^Y is indeed a map from $F(Y)$ to $(F \circ T_g)(Y)$. To see that $R_g^- \in \mathrm{Nat}(F, F \circ T_g)$ let $Y, Z \in \mathcal{C}$ and $f : Y \to Z$, then it is straightforward to calculate that for all $X \in \nabla$, $h_X \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $v_X \in \mathbb{C}^{D(X)}$:

$$\left(R_g^Z \circ F(f)\right)\left(\oplus_{X \in \nabla} h_X \otimes v_X\right) = \oplus_{X \in \nabla} T_g(f \circ h_{T_{g^{-1}}(X)}) \otimes v_{T_{g^{-1}}(X)} \tag{5.41}$$

$$((F \circ T_g)(f) \circ R_g^Y)(\oplus_{X \in \nabla} h_X \otimes v_X) = \oplus_{X \in \nabla} T_g(f) \circ T_g(h_{T_{g^{-1}}(X)}) \otimes v_{T_{g^{-1}}(X)} \tag{5.42}$$

Since T_g is a functor we see that indeed $R_g^Z \circ F(f) = (F \circ T_g)(f) \circ R_g^Y$.

For the second condition notice that since our G-action is normalized we have $T_e = \mathrm{Id}_{\mathcal{C}}$, and so $R_e^X = \mathrm{Id}_{F(X)}$.

For the third condition, note that since we are in the skeletal normalized setting $T_g(1_{\mathcal{C}}) = 1_{\mathcal{C}}$ and so indeed $R_g^{1_{\mathcal{C}}} = \mathrm{Id}_{F(1_{\mathcal{C}})}$

□

Corollary 5.2.1. *For every G-ribbon fusion category* $\mathcal{C}$ *there exists a finite dimensional* $\mathbb{C}$*-algebra* H *and a G-ribbon Abelian* 3*-cocycle* Γ *such that* $\mathcal{C}$ *is equivalent as a G-ribbon fusion category to* $\mathrm{Mod}(H)$ *with the G-ribbon structure induced by* Γ.

Proof. This follows from Proposition 5.2.1 and Theorem 5.0.2. □

Chapter 6

The Hopf Equivariantizaztion Theorem

In the final chapter of this thesis we use the G-(ribbon, crossed) Abelian 3-cocycle story to describe the equivariantization of any G-(ribbon, crossed braided) fusion category $\mathcal{C}$ as the modules of some ribbon qt wqhf $H\#_\Gamma\mathbb{C}[G]$. In fact, it will describe the equivariantzation of any finite G-(ribbon, crossed braided) tensor category with a weak quasi G-equivariant fiber functor.

The first step in this chapter is a technical result we need to prove that $H\#_\Gamma\mathbb{C}[G]$ is a wqhf. The reader uninterested in the technical details can skip this section. After this we define $H\#_\Gamma\mathbb{C}[G]$ and prove the Hopf equivariantization theorem. That is if Γ is a G-(ribbon, crossed) Abelian 3-cocycle then the equivariantization $(\mathrm{Mod}(H))^G$ of the corresponding G-structure on $\mathrm{Mod}(H)$ will be equivalent as a (ribbon, braided) tensor category to $\mathrm{Mod}(H\#_\Gamma\mathbb{C}[G])$. Combined with the G-reconstruction result for fusion categories this provides a way to describe the equivariantzation of all fusion categories with one of aforementioned G-structures.

As outlined in the introduction, this gives a uniform categorical description of the category of representations of all strongly rational orbifold VOAs and proves the Dijkgraaf-Witten conjecture as a special case.

6.1 A Technical Result

Let $(H, \cdot, 1_H, \epsilon, \Delta, \Phi, S, \alpha, \beta)$ be a wqhf so that $\mathrm{Mod}(H)$ has the structure of a tensor category. By Lemma 4.3.1 we know that $(\rho_V(\mathbf{d}^g))^* : T_g((V, \rho_V)^*) \to T_g(V, \rho_V)^*$ is an H-intertwiner, and so setting $V = H$ we see that for all $h \in H$ we have:

$$(S \circ \Psi(g^{-1}))(h) \cdot \mathbf{d}^g = \mathbf{d}^g \cdot (\Psi(g^{-1}) \circ S)(h) \tag{6.1}$$

Proposition 6.1.1. *Let $(\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu})$ be a G Abelian 3-cocycle, then the following equations hold:*

$$\mathbf{d}^g \cdot \Psi(g^{-1})(\alpha) = S((\overline{\mu}_g)_1) \cdot \alpha \cdot (\overline{\mu}_g)_2 \tag{6.2}$$

$$\Psi(g^{-1})(\beta) \cdot \mathbf{d}^g = (\overline{\mu}_g{}^{-1})_1 \cdot \beta \cdot S((\overline{\mu}_g{}^{-1})_2) \tag{6.3}$$

$$\mathbf{d}^{gk} = S(\underline{\gamma}^{-1}_{k^{-1},g^{-1}}) \cdot \mathbf{d}^k \cdot \Psi(k^{-1})(\mathbf{d}^g) \cdot (\underline{\gamma}^{-1}_{k^{-1},g^{-1}}) \tag{6.4}$$

Proof. Recalling that for $(V, \rho_V), (W, \rho_W) \in \mathrm{Mod}(H)$ the tensor structure of T_g is defined as $\mu_g((V, \rho), (W, \rho_W)) = (\rho_V \otimes \rho_W)(\overline{\mu}_g)$, with trivial unit isomorphism. Recall the shorthand $\overline{\mu}_g = (\overline{\mu}_g)_1 \otimes (\overline{\mu}_g)_2$. Expanding $\mathrm{ev}^{T_g}_{(V,\rho_V)}$ for $f \in V^*, v \in V$ we obtain:

$$\mathrm{ev}^{T_g}_{(V,\rho_V)}((\rho_{V^*} \otimes \rho_V)(\Delta(1_H)^{(g,g)})(f \otimes v)) = \mathrm{ev}_{(V,\rho_V)}((\rho_{V^*} \otimes \rho_V)(\overline{\mu}_g \cdot \Delta(1_H)^{(g,g)})(f \otimes v)) = \tag{6.5}$$

$$\mathrm{ev}_{(V,\rho_V)}((\rho_{V^*} \otimes \rho_V)(\overline{\mu}_g)(f \otimes v)) = \mathrm{ev}_{(V,\rho_V)}(\rho_{V^*}((\overline{\mu}_g)_1)(f) \otimes \rho_V((\overline{\mu}_g)_2)(v)) = \tag{6.6}$$

$$(\rho_{V^*}((\overline{\mu}_g)_1)(f))(\rho_V(\alpha \cdot (\overline{\mu}_g)_2)(v)) = f(\rho_V(S((\overline{\mu}_g)_1) \cdot \alpha \cdot (\overline{\mu}_g)_2)(v)) \tag{6.7}$$

On the other hand expanding we know this equals $\mathrm{ev}_{T_g(V,\rho_V)} \circ (d^{T_g} \otimes \mathrm{Id})$ and expanding for $f \in V^*, v \in V$ we obtain:

$$(\mathrm{ev}_{T_g(V,\rho_V)} \circ (d^{T_g} \otimes \mathrm{Id}))((\rho_{V^*} \otimes \rho_V)(\Delta(1)^{(g,g)})(f \otimes v)) = \tag{6.8}$$

$$(\mathrm{ev}_{T_g(V,\rho_V)} \circ ((\rho_V(\mathbf{d}^g))^* \otimes \mathrm{Id}))((\rho_{V^*} \otimes \rho_V)(\Delta(1)^{(g,g)})(f \otimes v)) = \tag{6.9}$$

$$((\rho_V(S(\Psi(g^{-1}(1_{(1)})) \cdot \mathbf{d}^g))^*)(f)(\rho_V(\Psi(g^{-1})(\alpha) \cdot \Psi(g^{-1})(1_{(2)}))(v)) = \tag{6.10}$$

$$f(\rho_V(S(\Psi(g^{-1}(1_{(1)})) \cdot \mathbf{d}^g \cdot \Psi(g^{-1})(\alpha \cdot 1_{(2)})))(v)) \tag{6.11}$$

But, we know that $S(\Psi(g^{-1})(1_{(1)})) \cdot \mathbf{d}^g = \mathbf{d}^g \cdot \Psi(g^{-1})(S(1_{(1)}))$ by Equation (6.3). Therefore, we may simplify Equation (6.11) to:

$$f(\rho_V(\mathbf{d}^g \cdot \Psi(g^{-1})(S(1_{(1)}) \cdot \alpha \cdot 1_{(2)})))(v)) = f(\rho_V(\mathbf{d}^g \cdot \Psi(g^{-1})(\alpha))(v)) \tag{6.12}$$

as $S(1_{(1)})\alpha \cdot 1_{(2)} = \epsilon(1_H) \cdot \alpha$. Comparing Equation (6.7) to Equation (6.12) we see that indeed $\mathbf{d}^g \cdot \Psi(g^{-1})(\alpha) = S((\overline{\mu}_g)_1) \cdot \alpha \cdot (\overline{\mu}_g)_2$.

The proof that $\Psi(g^{-1})(\beta) \cdot \mathbf{d}^g = (\overline{\mu}_g{}^{-1})_1 \cdot \beta \cdot S((\overline{\mu}_g{}^{-1})_2)$ follows similarly by using the fact that $(\mathrm{Id} \otimes d^{T_g}_{(V,\rho_V)}) \circ \mathrm{coev}^{T_g}_{(V,\rho_V)} = \mathrm{coev}_{T_g(V,\rho_V)}$, and so we omit the proof.

Fpr similar reasons, to prove Equation (6.4) it suffices to show that for all $(V, \rho_V) \in \mathrm{Mod}(H)$:

$$(d^{T_{gk}})(V, \rho_V) = (\gamma_{g,k}(V, \rho_V)^{-1})^* \circ d^{T_g}_{T_k(V,\rho_V)} \circ T_g((d^{T_k})(V, \rho_V)) \circ (\gamma_{g,k}((V, \rho_V)^*))^{-1} \tag{6.13}$$

This follows from the fact that $d^{T_g \circ T_k} = d^{T_g}_{T_k(V,\rho_V)} \circ T_g(d^{T_k}(V, \rho_V))$ and that since $\gamma_{g,k} : T_g \circ T_k \to T_{gk}$ is a natural isomorphism of tensor functors by a general fact of tensor functors (See Proposition A.4.3) we have:

$$(\gamma_{g,k}(V, \rho_V))^* \circ d^{T_{gk}}(V, \rho_V) \circ \gamma_{g,k}((V, \rho_V)^*) = d^{T_g \circ T_k}(V, \rho_V) \tag{6.14}$$

This completes the proof. $\square$

6.2 (Weak Quasi) Hopf Algebra Description of Equivariantization

Let $(\mathcal{C}, (\psi, \gamma, \psi_0), c, \theta)$ be a finite G-ribbon tensor category, and fix a weak G-equivariant fiber functor $(F, J^F, \phi^F, \{R_g\}_{g \in G})$. As we know from the previous chapter this will induce a finite dimensional unital algebra H and a G-ribbon Abelian 3-cocycle $\Gamma_{(\mathcal{C},F)}$, let $(\mathrm{Mod}(H), \Gamma_{(\mathcal{C},F)})$ denote the induced G-ribbon structure on $\mathrm{Mod}(H)$. We know that $(\mathcal{C}, (\psi, \gamma, \psi_0), c, \theta)$ will be equivalent to $(\mathrm{Mod}(H), \Gamma_{(\mathcal{C},F)})$ as a G-ribbon tensor category. By Theorem 2.3.2 we know that equivariantization will send G-ribbon tensor equivalences to ribbon tensor equivalences. Therefore, to describe $\mathcal{C}^G$ it suffices to describe $(\mathrm{Mod}(H))^G$.

Notation. *Denote* $\Gamma_{(\mathcal{C},F)} = (\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu}, \overline{c}, \overline{\partial}, \nu)$.

For ease of notation we will use the following for $g, k \in G$*:*

$$\overline{\gamma}_{g,k} := \underline{\gamma}_{\ k^{-1},g^{-1}} \tag{6.15}$$

Proposition 6.2.1. *Equivariantization of* $\mathrm{Mod}(H)$*:*

With everything as above, an object of $(\mathrm{Mod}(H))^G$ *is an* H*-module* (V, ρ_V) *such that:*

1. V *is a* G*-module with* G*-action* $u_g : V \to V$ *such that as linear maps:*

$$u_{gh} \circ \rho_V(\overline{\gamma}_{g,h}) = u_g \circ u_h \tag{6.16}$$

2. *Each linear map* $u_g : V \to V$ *is an* H*-intertwiner isomorphism from* $(V, \rho_V \circ \Psi(g^{-1}))$ *to* (V, ρ_V)*. That is for all* $h \in H$*:*

$$u_g \circ \rho_V(\Psi(g^{-1})(h)) = \rho_V(h) \circ u_g \tag{6.17}$$

We denote an object of $(\mathrm{Mod}(H))^G$ *as a tuple* $((V, \rho_V), \{u_g\}_{g\in G})$

A morphism $f((V, \rho_V), \{u_g\}_{g\in G}) \to ((W, \rho_V), \{v_g\}_{g\in G})$ *is a* H*-intertwiner* $f : (V, \rho_V) \to (W, \rho_W)$ *that preserves the* G*-action:*

$$f \circ u_g = v_g \circ f \tag{6.18}$$

The tensor product is given by:

$$((V,\rho_V),\{u_g\}_{g\in G})\otimes((W\rho_W),\{w_g\}_{g\in G}) := ((V,\rho_V)\otimes(W,\rho_W),\{(u_g\otimes w_g)\circ(\rho_V\otimes\rho_W)(\overline{\mu_g}^{\ -1})\}_{g\in G}) \tag{6.19}$$

The unit object is $((\mathbb{C},\varepsilon),\{1\}_{g\in G})$ *where* $\{1\}_{g\in G}$ *is the trivial* G*-action. The associator of* $(\mathrm{Mod}(H))^G$ *is just the associator of* $\mathrm{Mod}(H)$*, and the left/right unitors of* $(\mathrm{Mod}(H))^G$ *are just the associators of* $\mathrm{Mod}(H)$*. The braiding of* $(\mathrm{Mod}(H))^G$ *is defined for* $((V,\rho_V),\{u_g\}_{g\in G}),((W,\rho_W),\{v_g\}_{g\in G})$ *by restricting the following map to the representation spaces:*

$$x\otimes y\mapsto c^{\mathrm{Vect}}_{V,W}(\sum_{g\in G}(\mathrm{Id}\otimes v_g)\circ(\rho_V\otimes\rho_W)(\overline{c}\cdot(\overline{\partial(g)}\otimes\mathrm{Id}_H))(x\otimes y)) \tag{6.20}$$

The dual structure of $(\mathrm{Mod}(H))^G$ *is given by:*

$$((V,\rho_V),\{u_g\}_{g\in G})^* = ((V^*,(\rho_V\circ S)^*),\{(\rho_V(\mathbf{d}^g\cdot\overline{\gamma}^{-1}_{g^{-1},g}))\circ u_{g^{-1}})^*\}_{g\in G}) \tag{6.21}$$

The evaluation and co-evaluation maps are just the ones from $\mathrm{Mod}(H)$*.*

The ribbon twist is defined for $((V,\rho_V),\{u_g\}_{g\in G})$ *by:*

$$v\mapsto\sum_{g\in G}u_g(\rho_V(\overline{\theta}\cdot\overline{\partial}(\delta_g))(v)) \tag{6.22}$$

Proof. The proof of this is just substituting in the G-structure define in Chapter 3, and recalling the details of equivariantization (Definition 2.3.3) . The only non-trivial component is the dual structure, which we prove.

First, notice that since we have a choose a normalized G-structure on $\mathcal{C}$ we have that:

$$\overline{\gamma}_{e,g} = (R_g)^{-1}\circ F(\gamma_{e,g})\circ R_e^{T_g}\circ R_g = 1_H \tag{6.23}$$

Therefore, if $((V,\rho_V),\{u_g\}_{g\in G})\in(\mathrm{Mod}(H))^G$, then we see that for all $g\in G$:

$$u_g\circ\rho_V(\overline{\gamma}_{e,g}) = u_g = u_e\circ u_g \Rightarrow u_e = \mathrm{Id}_V \tag{6.24}$$

This also implies that:

$$u_e \circ \rho_V(\overline{\gamma}_{g^{-1},g}) = u_{g^{-1}} \circ u_g = \rho_V(\overline{\gamma}_{g^{-1},g}) \Rightarrow (u_g)^{-1} = \rho_V(\overline{\gamma}^{-1}_{g^{-1},g}) \circ u_{g^{-1}} \tag{6.25}$$

From Definition 2.3.3 we see that:

$$((V, \rho_V), \{u_g\}_{g\in G})^* := ((V^*, (\rho_V \circ S)^*), \{((u_g)^{-1})^* \circ d^{T_g}(V, \rho_V)\}_{g\in G}) \tag{6.26}$$

From Lemma 4.3.1 we know that $d^{T_g}(V, \rho_V) = (\rho_V(\mathbf{d}^g))^*$. Combing all of these observations we see that

$$\begin{gathered}((V, \rho_V), \{u_g\}_{g\in G})^* := ((V^*, (\rho_V \circ S)^*), \{(\rho_V(\overline{\gamma}^{-1}_{g^{-1},g}) \circ u_{g^{-1}})^* \circ (\rho_V)(\mathbf{d}^g))^*)\}_{g\in G}) = \\ ((V^*, (\rho_V \circ S)^*), \{(\rho_V(\mathbf{d}^g \cdot \overline{\gamma}^{-1}_{g^{-1},g}) \circ u_{g^{-1}})^*))\}_{g\in G})\end{gathered} \tag{6.27}$$

This completes the proof. □

Proposition 6.2.2. *Let H be a unital $\mathbb{C}$-algebra, and $\Gamma := (\Delta_H, \epsilon_H, \Phi_H, \Psi, \underline{\gamma}, \overline{\mu}, \overline{c}, \overline{\partial}, \nu)$ a G-ribbon Abelian 3-cocycle. Then define a ribbon wqhf $H\#_\Gamma\mathbb{C}[G]$ as follows:*

As a vector space $H\#_\Gamma\mathbb{C}[G]$ is $H \otimes_{\mathbb{C}} \mathbb{C}[G]$.

1. *For $r, h \in H, g, k \in G$ the product and unit are defined as:*

$$(r \otimes g) \cdot (h \otimes k) := (\overline{\gamma_{g,k}} \cdot (\Psi(k^{-1})(r) \cdot h)) \otimes gk \tag{6.28}$$

$$1_{H\#_\Gamma\mathbb{C}[G]} := 1_H \otimes e \tag{6.29}$$

2. *For every $r \in H, g \in G$ the co-product and co-unit are defined as:*

$$\Delta_{H\#_\Gamma\mathbb{C}[G]}(r \otimes g) := ((\overline{\mu}_g{}^{-1})_1 \cdot r_{(1)} \otimes g) \otimes ((\overline{\mu}_g{}^{-1})_2 \cdot r_{(2)} \otimes g) \tag{6.30}$$

$$\epsilon_{H\#_\Gamma\mathbb{C}[G]}(r \otimes g) := \epsilon_H(r) \tag{6.31}$$

3. The associator is defined as:

$$\Phi_{H\#_\Gamma\mathbb{C}[G]} = (x_1 \otimes \mathrm{Id}) \otimes (x_2 \otimes \mathrm{Id}) \otimes (x_3 \otimes \mathrm{Id}) \tag{6.32}$$

where we use Sweedler notation $\Phi = x_1 \otimes x_2 \otimes x_3$

4. The R-matrix is defined as:

$$R_{H\#_\Gamma\mathbb{C}[G]} := \sum_{g\in G}((\overline{c})_1 \cdot \overline{\partial}(\delta_g) \otimes e) \otimes ((\overline{c})_2 \otimes g) \tag{6.33}$$

$$R^{-1}_{H\#_\Gamma\mathbb{C}[G]} := \sum_{g\in G}((\overline{c}^{\ -1})_1 \cdot \overline{\partial}(\delta_g) \otimes e) \otimes (\Psi(g)((\overline{c}^{\ -1})_2) \cdot (\overline{\gamma}_{g,g^{-1}})^{\ -1} \otimes g^{-1}) \tag{6.34}$$

5. The antipode structure is defined for $r \in H, g \in G$ *as:*

$$S_{H\#_\Gamma\mathbb{C}[G]}(r \otimes g) := \Psi(g)(S(r) \cdot \mathbf{d}^g \cdot (\overline{\gamma}_{g^{-1},g})^{-1}) \otimes g^{-1} \tag{6.35}$$

$$\alpha_{H\#_\Gamma\mathbb{C}[G]} := \alpha_H \otimes e \qquad \beta_{H\#_\Gamma\mathbb{C}[G]} := \beta_H \otimes e \tag{6.36}$$

6. The ribbon element of $H\#_\Gamma\mathbb{C}[G]$ *is defined as:*

$$\nu_{H\#_\Gamma\mathbb{C}[G]} := \sum_{g\in G}(\nu \cdot \overline{\partial}(\delta_g)) \otimes g \tag{6.37}$$

Proof. Checking that this does indeed form a ribbon wqhf is straight forward but extremely tedious. Therefore, we refer the reader to Appendix D. □

Remark 6.2.1. *If* H *is a unital* $\mathbb{C}$*-algebra and* Γ *a* G*-ribbon Abelian* 3*-cocycle, then there exists an embedding of a wqhf* $\iota_{H,\Gamma} : H \to H\#_\Gamma\mathbb{C}[G]$ *given by* $\iota_H(h) := h \otimes e$*. This induces a forgetful functor. When it is clear from context the particular* G*-ribbon Abelian* 3*-cocycle we are working with we denote this forgetful functor by:*

$$\Lambda : \mathrm{Mod}(H\#_\Gamma\mathbb{C}[G]) \to \mathrm{Mod}(H) \tag{6.38}$$

Theorem 6.2.1. *The Hopf Equivariantion Theorem*

Let H be a unital $\mathbb{C}$-algebra with a G-ribbon Abelian 3-cocycle Γ. By Proposition 4.3.1 we know that this induced a G-ribbon structure on $\mathrm{Mod}(H)$. There is an equivalence of ribbon tensor categories given by:

$$(\mathcal{F}, J^{\mathcal{F}}, \phi^{\mathcal{F}}) : \mathrm{Mod}(H\#_\Gamma \mathbb{C}[G]) \to (\mathrm{Mod}(H))^G \tag{6.39}$$

$$\mathcal{F}(V,\rho_V) := (\Lambda(V,\rho_V), \{\rho_V(1_H \otimes g)\}_{g\in G}) \qquad \mathcal{F}(f) := f \tag{6.40}$$

$$J^{\mathcal{F}} := \mathrm{Id} \qquad \phi^F = \mathrm{Id} \tag{6.41}$$

Proof. First, we check that $\mathcal{F}$ is well-defined. Let $(V,\rho_V) \in \mathrm{Mod}(H\#_\Gamma\mathbb{C}[G])$. By Proposition 6.2.1 to show that $(\Lambda(V,\rho_V), \{\rho_V(1_H\otimes g)\}_{g\in G}) \in (\mathrm{Mod}(H))^G$ it suffices to show that $\{\rho_V(1_H\otimes g)\}_{g\in G}$ satisfies Equations (6.16), (6.17). To that end let $v \in V, g, k \in G$ then:

$$\rho_V(1_H\otimes g)(\rho_V(1_H\otimes k)(v)) = \rho_V((1_H\otimes g)\cdot(1_H\otimes k))(v) = \rho_V(\overline{\gamma}_{g,k}\otimes gk)(v) = \rho_V(\overline{\gamma}_{g,k}\otimes e)(\rho_V(1_H\otimes gk)(v)) \tag{6.42}$$

Here we have used the identity that $(\overline{\gamma}_{g,k}\otimes gk) = (\mathrm{Id}_H \otimes gk)\cdot(\overline{\gamma}_{g,k}\otimes e)$. Therefore, Equation (6.16) will be satisfied. To show that Equation (6.17) is satisfied let $g \in G, h \in H$ then:

$$\rho_V(1_H\otimes g)\circ\rho_V(\Psi(g^{-1})(h)\otimes e) = \rho((1_H\otimes g)\cdot(\Psi(g^{-1})(h)\otimes e)) = \rho_V(\Psi(g^{-1})(h)\otimes g) = \rho_V((h\otimes e)\cdot(1_H\otimes g)) = \rho_V(h\otimes e)\circ\rho_V(1_H\otimes g) \tag{6.43}$$

Therefore, Equation (6.17) will be satisfied, and so we see that $\mathcal{F}(V,\rho_V) \in (\mathrm{Mod}(H))^G$. Suppose now that $f : (V,\rho_V) \to (W,\rho_W)$ is a $H\#_\Gamma\mathbb{C}[G]$-intertwiner, then for $h \in G, g \in G$:

$$f\circ\rho_V(h\otimes e) = \rho_W(h\otimes e)\circ f \qquad f\circ\rho_V(1_H\otimes g) = \rho_W(1_H\otimes g)\circ f \tag{6.44}$$

Therefore, f will be a morphism in $(\mathrm{Mod}(H))^G$, and indeed $\mathcal{F}$ is well defined. It is clear that $\mathcal{F}$ is an additive functor and $\mathbb{C}$-linear. Let's verify that $(\mathcal{F}, J^{\mathcal{F}}, \phi^{\mathcal{F}})$ is a tensor functor. Let $(V,\rho_V),(W,\rho_W)\in \mathrm{Mod}(H\#_\Gamma\mathbb{C}[G])$ then:

$$\mathcal{F}((V,\rho_V))\otimes\mathcal{F}((W,\rho_W)) = \tag{6.45}$$

$$(\Lambda(V,\rho_V)\otimes\Lambda(W,\rho_W),\{\rho_V((1_H\otimes g)\cdot((\overline{\mu}_g{}^{-1})_1\otimes e))\otimes\rho_W((1_H\otimes g)\cdot((\overline{\mu}_g{}^{-1})_2\otimes e))\}_{g\in G}) = \tag{6.46}$$

$$(\Lambda(V,\rho_V)\otimes\Lambda(W,\rho_W),\{(\rho_V\otimes\rho_W)(((\overline{\mu}_g{}^{-1})_1\otimes g)\otimes((\overline{\mu}_g{}^{-1})_2\otimes g))\}_{g\in G}) \tag{6.47}$$

On the other hand,

$$\mathcal{F}((V,\rho_V)\otimes(W,\rho_W)) = \tag{6.48}$$

$$\mathcal{F}((\rho_V\otimes\rho_W)(\Delta_{H\#_\Gamma\mathbb{C}[G]}(1))(W\otimes V),(\rho_V\otimes\rho_W)\circ\Delta_{H\#_\Gamma\mathbb{C}[G]}) = \tag{6.49}$$

$$(\Lambda((\rho_V\otimes\rho_W)(\Delta_{H\#_\Gamma\mathbb{C}[G]}(1))(W\otimes V),(\rho_V\otimes\rho_W)\circ\Delta_{H\#_\Gamma\mathbb{C}[G]}),\{(\rho_V\otimes\rho_W)\circ\Delta_{H\#_\Gamma\mathbb{C}[G]}(1_H\otimes g)\}_{g\in G}) \tag{6.50}$$

Since $\iota_{H,\Gamma}: H\to H\#_\Gamma\mathbb{C}[G]$ is a morphism of weak quasi Hopf algebras we see that for $h\in H$:

$$\Delta_{H\#_\Gamma\mathbb{C}[G]}(h\otimes e) = (h_{(1)}\otimes e)\otimes(h_{(2)}\otimes e) \tag{6.51}$$

Therefore,

$$\Lambda((\rho_V\otimes\rho_W)(\Delta_{H\#_\Gamma\mathbb{C}[G]}(1))(W\otimes V)) = \Lambda(V,\rho_V)\otimes\Lambda(W,\rho_W) \tag{6.52}$$

Evaluating $(\rho_V\otimes\rho_W)\circ\Delta_{H\#_\Gamma\mathbb{C}[G]}(1_H\otimes g)$ for $g\in G$ we obtain:

$$(\rho_V\otimes\rho_W)(((\overline{\mu}_g{}^{-1})_1\cdot(1_H)_{(1)}\otimes g)\otimes((\overline{\mu}_g{}^{-1})_2\cdot(1_H)_{(2)}\otimes g)) \tag{6.53}$$

Since $\overline{\mu}_g{}^{-1}\cdot\Delta(1_H) = \overline{\mu}_g{}^{-1}$ we see this coincides with $(\rho_V\otimes\rho_W)(((\overline{\mu}_g{}^{-1})_1\otimes g)\otimes((\overline{\mu}_g{}^{-1})_2\otimes g))$.

The coherence conditions for $J^{\mathcal{F}}$ follows from the fact that H is a wqhf, and we have an embedding $\iota_\Gamma : H \to H\#_\Gamma\mathbb{C}[G]$ of wqhfs. For the unit isomorphism, notice that $\Lambda(\mathbb{C}, \epsilon_{H\#_\Gamma\mathbb{C}[G]}) = (\mathbb{C}, \epsilon_H)$ since $\iota_{H,\Gamma}$ is a morphism of wqhfs. For all $g \in G$ we have $\epsilon_{H\#_\Gamma\mathbb{C}[G]}(1_H \otimes g) = 1$, this shows that:

$$\mathcal{F}(\mathbb{C}, \epsilon_{H\#_\Gamma\mathbb{C}[G]}) = ((\mathbb{C}, \epsilon_H), \{\mathrm{Id}_\mathbb{C}\}_{g\in G}) \tag{6.54}$$

Therefore, one can choose $\phi^\mathcal{F} = \mathrm{Id}_\mathbb{C}$. This will satisfy the required coherence conditions. We have proven that $(\mathcal{F}, J^\mathcal{F}, \phi^\mathcal{F})$ is a tensor functor.

Denote the braiding of $\mathrm{Mod}(H\#_\Gamma\mathbb{C}[G])$ by c^1 and the braiding of $(\mathrm{Mod}(H))^G$ by c^2. To verify that $\mathcal{F}$ is a braided tensor functor we need to verify that for $(V, \rho_V), (W, \rho_W) \in \mathrm{Mod}(H)$ as linear maps we have:

$$c^2_{\mathcal{F}(V,\rho_V),\mathcal{F}(W,\rho_W)} = c^1_{(V,\rho_V),(W,\rho_W)} \tag{6.55}$$

Let $v \in V, w \in W$ then the braiding c^2 is given by restricting the following map to the representation spaces.

$$c^1_{(V,\rho_V),(W,\rho_W)} := c^{\mathrm{Vect}}_{V,W}\circ(\rho_V\otimes\rho_W)(R_{H\#_\Gamma\mathbb{C}[H]}) = c^{\mathrm{Vect}}_{V,W}\circ\sum_{g\in G}(\rho_V\otimes\rho_W)((\overline{c}_1\cdot\overline{\partial}(\delta_g)\otimes e)\otimes(\overline{c}_2\otimes g)) \tag{6.56}$$

On the other hand, we know that the braiding c^2 is given by restricting the following map to the representation spaces.

$$c^2_{\mathcal{F}(V,\rho_V),\mathcal{F}(W,\rho_W)} = \sum_{g\in G}(\mathrm{Id}_W\otimes\rho_V(\overline{\partial}(\delta_g))\circ(\rho_W(1\otimes g)\otimes\mathrm{Id}_V)\circ c^{\mathrm{Vect}}_{V,W}\circ(\rho_V\otimes\rho_W)((\iota^{\otimes 2}_{H,\Gamma})(\overline{c}))\circ(\rho_V(\overline{\partial}(\delta_g))\otimes\mathrm{Id}) \tag{6.57}$$

Evaluating Equation (6.57) we see that:

$$c^1_{(V,\rho_V),(W,\rho_W)} = \mathcal{F}(c^2_{\mathcal{F}(V,\rho_V),\mathcal{F}(W,\rho_W)}) \tag{6.58}$$

Therefore, $(\mathcal{F}, \mathrm{Id}, \mathrm{Id})$ will be braided. To verify that it is ribbon denote the ribbon twist on $\mathrm{Mod}(H\#_\Gamma\mathbb{C}[G])$ by θ^1 and the ribbon twist on $(\mathrm{Mod}(H))^G$ by θ^2. Notice that:

$$\mathcal{F}(\theta^1_{(V,\rho_V)}) = \mathcal{F}(\rho_V(\nu_{H\#_\Gamma\mathbb{C}[G]})) = \mathcal{F}(\sum_{g\in G}(\nu_H \cdot \overline{\partial}(\delta_g) \otimes g)) \tag{6.59}$$

On the other hand we know that:

$$\theta^2_{\mathcal{F}(V,\rho_V)} = \sum_{g\in G} \rho_V((\overline{\partial}(\delta_g) \otimes e) \cdot (1_H \otimes g) \cdot (\nu_H \otimes e) \cdot (\overline{\partial}(\delta_g) \otimes e)) = \tag{6.60}$$

$$\sum_{g\in G} \rho_V(\Psi(g^{-1})(\nu_H) \cdot \overline{\partial}(\delta_g) \otimes g) \tag{6.61}$$

By Equation (4.73) we have:

$$\Psi(g^{-1})(\nu_H) \cdot \overline{\partial}(\delta_g) = \nu_H \cdot \overline{\partial}(\delta_g) \tag{6.62}$$

Therefore, we see that:

$$\mathcal{F}(\theta^1_{(V,\rho_V)}) = \theta^2_{\mathcal{F}(V,\rho_V)} \tag{6.63}$$

The last thing we need to prove is that $\mathcal{F}$ is an equivalence. To that end it is obvious that $\mathcal{F}$ will be faithful. To verify that it is full let

$$f : \mathcal{F}(V, \rho_V) \to \mathcal{F}(W, \rho_W) \tag{6.64}$$

be a morphism in $(\mathrm{Mod}(H))^G$. By definition we must have for all $g \in G$, $h \in H$:

$$\rho_W(1_H \otimes g) \circ f = f \circ \rho_V(1_H \otimes g) \qquad \rho_W(h \otimes e) \circ f = f \circ \rho_V(h \otimes e) \tag{6.65}$$

In particular this implies that:

$$\rho_W(h \otimes g) \circ f = \rho_V(h \otimes g) \circ f \tag{6.66}$$

Therefore, f will be an $H\#_\Gamma\mathbb{C}[G]$-intertwiner and so $\mathcal{F}$ is full.

Lastly, to show that $\mathcal{F}$ is isomorphism dense let $((M, \rho^H_M), \{u_g\}_{g\in G}) \in (\mathrm{Mod}(H))^G$.

Define a $(H\#_\Gamma\mathbb{C}[G])$-module $(M, \rho_M^{H\#_\Gamma\mathbb{C}[G]})$ for $g \in G, h \in H$ by:

$$\rho_M^{H\#_\Gamma\mathbb{C}[G]}(h \otimes g) := u_g \circ \rho_M^H(h) \tag{6.67}$$

To verify that $(M, \rho_M^{H\#_\Gamma\mathbb{C}[G]})$ is a $(H\#_\Gamma\mathbb{C}[G])$-module notice that $u_e = \mathrm{Id}_M, \rho_M^H(1_H) = \mathrm{Id}_M$, so $\rho_M^{H\#_\Gamma\mathbb{C}[G]}(1_H \otimes e) = \mathrm{Id}_M$. If $h, r \in H, g, k \in G$, then:

$$\rho_M^{H\#_\Gamma\mathbb{C}[G]}((h \otimes g) \cdot (r \otimes k)) = \tag{6.68}$$
$$\rho_M^{H\#_\Gamma\mathbb{C}[G]}((\overline{\gamma}_{g,k} \cdot \Psi(k^{-1})(h) \cdot r) \otimes gk) = \tag{6.69}$$
$$u_{gk} \circ \rho_M^H(\overline{\gamma}_{g,k} \cdot \Psi(k^{-1})(h) \cdot r) = u_{gk} \circ \rho_M^H(\overline{\gamma}_{g,k}) \circ \rho_M^H(\Psi(k^{-1})(h)) \circ \rho_M^H(r) = \tag{6.70}$$
$$u_g \circ u_k \circ \rho_M^H(\Psi(k^{-1})(h)) \circ \rho_M^H(r) = \tag{6.71}$$
$$u_g \circ \rho_M^H(h) \circ u_k \circ \rho_M^H(r) = \tag{6.72}$$
$$\rho_M^{H\#_\Gamma\mathbb{C}[G]}(h \otimes g) \circ \rho_M^{H\#_\Gamma\mathbb{C}[G]}(r \otimes k) \tag{6.73}$$

Since:

$$\rho_M^{H\#_\Gamma\mathbb{C}[G]}(h \otimes e) = u_e \circ \rho_M^H(h) = \rho_M^H(h) \tag{6.74}$$

$$\rho_M^{H\#_\Gamma\mathbb{C}[G]}(1_H \otimes g) = u_g \tag{6.75}$$

we see that $\mathcal{F}((M, \rho_M^{H\#_\Gamma\mathbb{C}[G]})) = ((M, \rho_M^H), \{u_g\}_{g\in G})$ right on the nose. This proves that $\mathcal{F}$ will be isomorphism dense and so an equivalence. $\square$

This theorem allows us to prove the main result of the thesis:

Theorem 6.2.2. *For every G-ribbon tensor category $\mathcal{C}$ with a weak quasi G-equivariant fibre functor F there exists a $\mathbb{C}$-algebra H with a G-ribbon Abelian 3-cocycle Γ such that:*

$$\mathcal{C}^G \cong \mathrm{Mod}(H\#_\Gamma\mathbb{C}[G]) \tag{6.76}$$

as ribbon tensor categories. In particular if $\mathcal{C}$ is a finite G-ribbon tensor category then H can be assumed to be finite dimensional.

Corollary 6.2.1. *Let $\mathcal{C}$ be a G-ribbon fusion category. Then there exists a finite* *dimensional semi-simple* $\mathbb{C}$*-algebra* H *with a* G*-ribbon Abelian* 3*-cocycle such that:*

$$\mathcal{C}^G \cong \mathrm{Mod}(H\#_\Gamma\mathbb{C}[G]) \tag{6.77}$$

In particular if D *is a weak* G*-equivariant dimension function on the set of isomorphism classes of simples of* $\mathcal{C}$*, denoted by* ∇*. Then* H *is isomorphic to:*

$$H \cong \bigoplus_{X\in\nabla} \mathrm{Mat}(D(X),\mathbb{C}) \tag{6.78}$$

As we saw at the end of Chapter 4, there are many weak G-equivariant dimension functions we can choose from, some that are relatively small. It is the authors hope that for this reason that in specific cases all of the G-ribbon Abelian 3-cocycles on a fusion ring that admits categroification can be calculated explicitly.

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Appendix A

Category Theory

The material in this appendix will follow the contents of [13].

A.1 Monoidal Categories

Let $\mathcal{C}$ be a category. A monoidal category is a category with a way to multiply things through a tensor product in a coherent way. More precisely:

Definition A.1.1. *Monoidal Category*

A monoidal category is a sextuple $(\mathcal{C}, \otimes, \alpha, 1_{\mathcal{C}}, \ell, r)$ *such that there is a functor called the* tensor product

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \tag{A.1}$$

and there is a natural isomorphism called the associator

$$\alpha : \otimes \circ (\otimes \times \mathrm{Id}) \to \otimes \circ (\mathrm{Id} \times \otimes) \tag{A.2}$$

such that the Pentagon Axiom *is satisfied:*

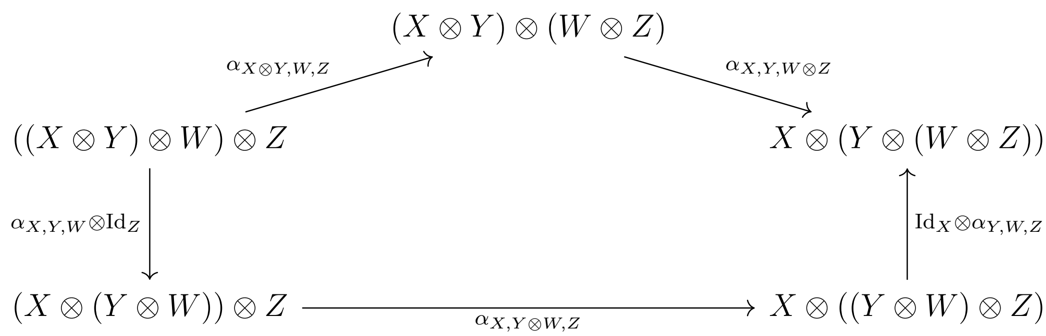

Figure A.1: Pentagon Axiom

In addition we require that there is a unit object $1_{\mathcal{C}}$ *with natural isomorphisms:*

$$\ell_X : 1_{\mathcal{C}} \otimes X \to X \quad r_X : X \otimes 1_{\mathcal{C}} \to X \tag{A.3}$$

such that the triangle axiom *holds:*

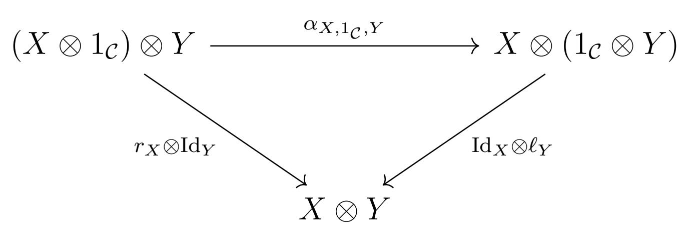

Figure A.2: Triangle Axiom

Definition A.1.2. *Sub-Monoidal Category*

Let $(\mathcal{C}, \otimes, \alpha, 1_{\mathcal{C}}, \ell, r)$ *be a monoidal category and* $\mathcal{D} \subset \mathcal{C}$ *a full sub-category of*

$\mathcal{C}$. If $1_\mathcal{C}$ is an object of $\mathcal{D}$, and $\mathcal{D}$ is closed under the tensor product we say that $(\mathcal{D}, \otimes|_{\mathcal{D}^{\times 3}}, \alpha|_{\mathcal{D}^{\times 3}}, 1_\mathcal{C}, \ell|_\mathcal{D}, r|_\mathcal{D})$ is a sub-monoidal category.

The functor $\otimes$ is referred to as the tensor product, and the natural isomorphisms ℓ, r are called the left and right unitor respectively. When it is clear from context we will usually refer to a monoidal structure $(\mathcal{C}, \otimes, \alpha, 1_\mathcal{C}, \ell, r)$ simply as $\mathcal{C}$. Essentially a monoidal category is a category where you can multiply together in an associative manner, and there is a unit with respect to this unit. The reason the pentagon axiom is required is we don't want any ambiguity when changing brackets. As illustrated by the axiom there are two ways to re-bracket $((X \otimes Y) \otimes W) \otimes Z$ to $X \otimes (Y \otimes (W \otimes Z))$, and the pentagon axiom guarantees that these will be the same.

Example A.1.1. *Vector Spaces*

Let $\mathcal{C}$ be the category of finite dimensional vector spaces over $\mathbb{C}$. This will form a monoidal category with tensor product given by:

$$\otimes(V, W) := V \otimes_\mathbb{C} W, \quad \otimes(f, g) := f \otimes_\mathbb{C} g \tag{A.4}$$

The associator α is the linear isomorphism induced from the associativity of the Cartesian product of vector spaces, the left unitor is the linear isomorphism induced from the bi-linear form $(c, v) \mapsto c \cdot v$, and similarly for the right unitor. Denote this monoidal category simply as Vect

Another example which will be of great interest in this thesis are graded vector spaces

Example A.1.2. *Graded Vector Space*

Let G be a finite group, define the category of G-graded vector spaces Vect_G *to be vector spaces V that are the direct sum of vector spaces labelled by G. That is $V = \bigoplus_{g \in G} V_g$. We refer to V_g as the g-th component of V for $g \in g$. This inherits a monoidal structure from Example A.1.1 with the g-th component of the tensor product defined as:*

$$(V \otimes W)_g := \bigoplus_{\substack{h,k \in G \\ h \cdot k = g}} V_h \otimes_{\mathbb{C}} W_k \tag{A.5}$$

The unital object is defined to be $\mathbb{C}$ with a G-grading given by

$$(\mathbb{C})_g := \begin{cases} \mathbb{C} & if\ g = e \\ 0 & else \end{cases} \tag{A.6}$$

The unitors are the ones induced from Example A.1.1.

Example A.1.3. *Twisted Graded Vector Spaces*

Let G be a group as before. Recall that a 3-cocycle on G with coefficients in $\mathbb{C}^\times$ is a function $\omega : G \times G \times G \to \mathbb{C}^\times$ such that:

$$\omega(g \cdot h, k, r)\omega(g, h, k \cdot r) = \omega(h, k, r)\omega(g, h \cdot k, r)\omega(g, h, k) \tag{A.7}$$

We define Vect_G^ω *to be the category* Vect_G *with the same tensor product, unit object and unitors, but the associator is twisted by ω. Namely, if V, W, U are G-graded vector spaces and $v \in V_g, w \in W_h, u \in U_k$ then:*

$$\alpha_{V,W,U}((v \otimes w) \otimes u) = \omega(g, h, k) \cdot (v \otimes (w \otimes u)) \tag{A.8}$$

One can check that Eq. A.7 ensures the pentagon axiom will hold.

Example A.1.3 suggests that we should think of an associator as some 3-cocycle. This analogy can be useful, but it is limited.

Example A.1.4. *Representations of Finite Group*

Let G be a group, and consider the category of all representations of a group $\text{Rep}(G)$ *over* $\mathbb{C}$*. We denote representations by a tuple (V, ρ_V) where $\rho_V : G \to GL(V)$ is a group homomorphism. This will be a monoidal category with tensor product defined by:*

$$(V, \rho_V) \otimes (W, \rho_W) := (V \otimes_{\mathbb{C}} W, (\rho_V \otimes_{\mathbb{C}} \rho_W) \circ \Delta) \tag{A.9}$$

Here $\Delta : G \to G \times G$ denotes the diagonal morphism $\Delta(g) = g \otimes g$. Notice we are abusing notation, this will be justified in a later chapter, but just ignore it for now. The unital object is the trivial representation $(\mathbb{C}, 1)$ where $1(g) = \text{Id}_{\mathbb{C}}$. The left and right unitors from Example A.1.1 will be G-intertwiners, and so these will give the left and right unitors of $\text{Rep}(G)$*.*

A.2 Monoidal Functors

Definition A.2.1. *Monoidal Functor*

Let $(\mathcal{C}, \otimes, \alpha, 1_{\mathcal{C}}, \ell, r)$, $(\mathcal{D}, \otimes', \alpha', 1_{\mathcal{D}}, \ell', r')$ be a monoidal categories. A monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a tuple (F, J, ϕ) where $F : \mathcal{C} \to \mathcal{D}$ is a functor, $J : \otimes \circ F \times F \to F \circ \otimes$ is a natural isomorphism, and $\phi : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ is an isomorphism such that the following diagram commutes:

$$\begin{array}{ccc}
(F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{\alpha'_{F(X),F(Y),F(Z)}} & F(X) \otimes' (F(Y) \otimes' F(Z)) \\
\Big\downarrow{\scriptstyle J_{X,Y} \otimes' \mathrm{Id}_{F(Z)}} & & \Big\downarrow{\scriptstyle \mathrm{Id}_{F(X)} \otimes J_{Y,Z}} \\
F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' F(Y \otimes Z) \\
\Big\downarrow{\scriptstyle J_{X \otimes Y, Z}} & & \Big\downarrow{\scriptstyle J_{X, Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow[F(\alpha_{X,Y,Z})]{} & F(X \otimes (Y \otimes Z))
\end{array}$$

We also require that:

$$J_{1_{\mathcal{C}},X} \circ (\phi \otimes' \mathrm{Id}_{F(X)}) = F(\ell_X)^{-1} \circ \ell'_{F(X)} \tag{A.10}$$

$$J_{X,1_{\mathcal{C}}} \circ (\mathrm{Id}_{F(X)} \otimes' \phi) = F(r_X)^{-1} \circ r'_{F(X)} \tag{A.11}$$

(J, ϕ) is referred to as the monoidal structure of F.

Remark A.2.1. *Definition A.2.1 is usually referred to as a strong monoidal functor, but in this thesis we will only consider strong monoidal functors and so simply refer to them as monoidal functors.*

Definition A.2.2. *Unital Monoidal Functor*

Let $(F, J, \phi) : \mathcal{C} \to \mathcal{D}$ be a monoidal functor. We say it is unital *if $\phi = \mathrm{Id}_{\mathcal{D}}$.*

Notice if $(F, J^F, \phi^F) : \mathcal{C} \to \mathcal{D}, (G, J^G, \phi^G) : \mathcal{D} \to \mathcal{F}$ are monoidal functors, then $(G \circ F, J^G * J^F, \phi^G * \phi^F)$ is monoidal functor where the isomorphism:

$$(J^G * J^F)_{X,Y} : (G \circ F)(X) \otimes_{\mathcal{F}} (G \circ F)(Y) \to (G \circ F)(X \otimes_{\mathcal{C}} Y) \tag{A.12}$$

is defined by:

$$(J^G * J^F)_{X,Y} := G(J^F_{X,Y}) \circ J^G_{F(X),F(Y)} \tag{A.13}$$

and

$$\phi^G * \phi^F : 1_{\mathcal{F}} \to (G \circ F)(1_{\mathcal{C}}) \tag{A.14}$$

is defined by:

$$\phi^G * \phi^F := G(\phi^F) \circ \phi^G \tag{A.15}$$

We will also need the notion of natural transformations of monoidal functors.

Definition A.2.3. *Natural Transformation of Monoidal Functors*

Let $\mathcal{C}, \mathcal{D}$ be monoidal categories as in Definition A.2.1. Also, let $(F, J^F, \phi^F), (G, J^G, \phi^G)$ be monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation of monoidal fucntors $\tau : (F, J^F, \phi^F) \to (G, J^G, \phi^G)$ is a natural transformation $\tau : F \to G$ such that the following diagrams commute:

$$\begin{array}{ccc} F(X) \otimes' F(Y) & \xrightarrow{J^F_{X,Y}} & F(X \otimes Y) \\ \downarrow{\scriptstyle \tau_X \otimes' \tau_Y} & & \downarrow{\scriptstyle \tau'_{X \otimes Y}} \\ G(X) \otimes' G(Y) & \xrightarrow{J^G_{X,Y}} & G(X \otimes Y) \end{array}$$

$$\begin{array}{ccc} & 1_{\mathcal{D}} & \\ \swarrow{\scriptstyle \phi^F} & & \searrow{\scriptstyle \phi^G} \\ F(1_{\mathcal{C}}) & \xrightarrow{\epsilon_{1_{\mathcal{C}}}} & G(1_{\mathcal{C}}) \end{array}$$

Definition A.2.4. *Equivalence of Monoidal Categories*

Let $\mathcal{C}, \mathcal{D}$ be as before. We say that $\mathcal{C}, \mathcal{D}$ are equivalent as monoidal categories if there exist monoidal functors $(F, J^F, \phi^F) : \mathcal{C} \to \mathcal{D}$, $(G, J^G, \phi^G) : \mathcal{D} \to \mathcal{C}$ such that $(G \circ F, J^G * J^F, \phi^G * \phi^F)$ is naturally isomorphic as a monoidal functor to $(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}, \mathrm{Id}_{1_{\mathcal{C}}})$ and $(F \circ G, J^F * J^G, \phi^F * \phi^G)$ is naturally isomorphic as a monoidal functor to $(\mathrm{Id}_{\mathcal{D}}, \mathrm{Id}, \mathrm{Id}_{1_{\mathcal{D}}})$

The following is claimed in [13, Remark 2.4.10], and we provide a proof for the reader's convenience.

Proposition A.2.1. *Let $\mathcal{C}, \mathcal{D}$ be monoidal categories, and $(F, J^F, \phi^F) : \mathcal{C} \to \mathcal{D}$ a monoidal functor such that $F : \mathcal{C} \to \mathcal{D}$ gives an equivalence of categories. Then (F, J^F, ϕ^F) gives an equivalence of monoidal categories.*

Proof. By assumption there exists an adjoint equivalence (F, G, η, ϵ) with unit $\eta : \mathrm{Id}_{\mathcal{C}} \to G \circ F$ and co-unit $\epsilon : F \circ G \to \mathrm{Id}_{\mathcal{D}}$. Define:

$$J^G_{X,Y} := (\eta_{G(X) \otimes G(Y)} \circ G(J^F_{G(X),G(Y)}) \circ G(\epsilon_X \otimes \epsilon_Y))^{-1} : G(X) \otimes G(Y) \to G(X \otimes Y) \tag{A.16}$$

$$\phi^G := (\epsilon_{1_{\mathcal{C}}} \circ G(\phi^F))^{-1} : 1_{\mathcal{C}} \to G(1_{\mathcal{D}}) \tag{A.17}$$

One can check that this defines a monoidal structure on G such that $(F, J^F, \phi^F), (G, J^G, \phi^G)$ gives an adjoint monoidal equivalence. Checking this is straightforward and so we leave it to the interested reader. □

Example A.2.1. *Let $\mathcal{C} = \mathrm{Vect}^{\omega}_G, \mathcal{D} = \mathrm{Vect}^{\omega'}_G$ be as in Example A.1.3 where ω, ω' are 3-cocycles on G, not necessarily the same. Forgetting the monoidal structure, the categories $\mathcal{C}, \mathcal{D}$ are the exact same, so we have the identity functor $\mathrm{Id} : \mathcal{C} \to \mathcal{D}$. It is* *not difficult to show that there exists a monoidal functor structure on* Id *if and only if* ω, ω' *are co-homologous* 3*-cocycles on* G*.*

Example A.2.2. *Let* G, H *be groups and* $f : G \to H$ *a group homomorphism. This induces a functor* $F : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ *given by* $F(V, \rho_V) := (V, \rho_V \circ f), F(f) := f$ *for* H*-intertwiners* $f : (V, \rho_V) \to (W, \rho_W)$*. We see that* F *is a monoidal functor with monoidal functor structure given by* $(\mathrm{Id}, \mathrm{Id}_{\mathbb{C}})$*.*

Example A.2.3. *Category of Monoidal Endofunctors*

Let $\mathcal{C}$ *be a monoidal functor. Define the category of* endofucntors of $\mathcal{C}$ *as the category* $\mathrm{End}_{\otimes}(\mathcal{C})$ *whose objects are monoidal functors* (F, J) *from* $\mathcal{C}$ *to* $\mathcal{C}$*, and morphisms are natural transformations of monoidal functors.* $\mathrm{End}_{\otimes}(\mathcal{C})$ *will be a monoidal category with tensor product given by composition of monoidal functors. Since composition is associative, the associator of the* $\mathrm{End}_{\otimes}(\mathcal{C})$ *is just the identity, the unit object is the identity functor and the unitors are just the identity.*

A.3 Strict and Skeletal Monoidal Categories

Example A.2.3 is a special example of what is called a *strict category*

Definition A.3.1. *Strict Monoidal Category*

Let $(\mathcal{C}, \otimes, \alpha, 1_{\mathcal{C}}, \ell, r)$ *be a monoidal category. We say that it is strict if*

1. $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ *for all* $X, Y, Z \in \mathcal{C}$*, and* $\alpha = \mathrm{Id}$

2. $X \otimes 1_{\mathcal{C}} = 1_{\mathcal{C}} \otimes X = X$ *for all* $X \in \mathcal{C}$*, and* $\ell_X = r_X = \mathrm{Id}_X$

Essentially, strict categories are monoidal categories where the tensor product is associative on the nose, and we don't have to worry about natural transformations. The following theorem says that every monoidal category is equivalent to a strict one:

Theorem A.3.1. *The MacLane Strictness Theorem*

Every monoidal category is equivalent to a strict monoidal category

Proof. The proof of this is well-known and so we refer the interested reader to [27]. □

Due to this equivalence any statement that can be proven for strict monoidal categories will be true for all monoidal categories. This will be useful when we want to verify identities which otherwise would become cumbersome in the non-strict case. The process of passing from a monoidal category to a strict monoidal category is called strictfying.

A related concept is that of a skeletal category:

Definition A.3.2. *Skeletal Category*

A skeletal category is a category where isomorphic objects are equal.

Since things are rarely equal on the nose in nature skeletal is a unnatural condition, but as we will see it can be useful concept when dealing with some group actions on categories. The following will be crucial to many of the results in this paper, and so we provide a proof despite its elementary nature.

Proposition A.3.1. *Let $(\mathcal{C}, \otimes, \alpha, 1_{\mathcal{C}}, \ell, r)$ be a monoidal category. Suppose $\mathcal{D}$ is a category and there is an adjoint equivalence $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ with unit*

$\eta : 1_{\mathcal{C}} \to G \circ F$ and co-unit $\epsilon : F \circ G \to 1_{\mathcal{D}}$. There is a monoidal structure on $\mathcal{D}$ given by:

$$X \overline{\otimes} Y := F(G(X) \otimes G(Y)) \quad X, Y \in \mathcal{D} \tag{A.18}$$

$$f \overline{\otimes} g := F(G(f) \otimes G(g)) \quad f : X \to Y,\ g : W \to Z \tag{A.19}$$

$$\overline{\alpha}_{X,Y,Z} := F((\mathrm{Id}_{G(X)} \otimes \eta_{G(Y)\otimes G(Z)}) \circ \alpha_{G(X),G(Y),G(Z)} \circ (\eta^{-1}_{G(X)\otimes G(Y)} \otimes \mathrm{Id}_{G(Z)})) \tag{A.20}$$

The unit object is $F(1_{\mathcal{C}})$, with unitors:

$$\overline{\ell}_X := \epsilon_X \circ F(\ell_{G(X)} \circ (\eta^{-1}_{1_{\mathcal{C}}} \otimes \mathrm{Id}_{G(X)})) \tag{A.21}$$

$$\overline{r}_X := \epsilon_X \circ F(r_{G(X)} \circ (\mathrm{Id}_{G(X)} \otimes \eta^{-1}_{1_{\mathcal{C}}})) \tag{A.22}$$

This will upgrade F, G to monoidal functors such that η, ϵ are natural transformations of monoidal functors.

Proof. This proof is straightforward, but very tedious. We leave the details to the reader. □

Proposition A.3.2. *Every monoidal category $\mathcal{C}$ is equivalent to a skeletal monoidal category $\mathcal{C}_0$ such that $\ell_X = \mathrm{Id}_X, r_x = \mathrm{Id}_X$ for all objects in $X \in \mathcal{C}_0$.*

Proof. Following the hint from [13, Exercise 2.8.8] we let I denote the set of isomorphism classes of objects of $\mathcal{C}$. For each isomoprhism class $i \in I$ choose a representative object $X_i \in \mathcal{C}$ such that $X_1 = 1_{\mathcal{C}}$ and for every two isomorphism classes i, j choose isomorphisms $\mu_{i,j} : X_i \otimes X_j \to X_{i \cdot j}$ so that $\mu_{1,j} = \ell_{X_j}, \mu_{i,1} = r_{X_i}$. Let $\mathcal{C}_0$ be the full-subcategory of $\mathcal{C}$ with objects given by $\{X_i\}_{i \in I}$. Define a tensor product on $\mathcal{C}_0$

by:

$$X_i \overline{\otimes} X_j := X_{i\cdot j} \tag{A.23}$$

and if $f : X_i \to X_j, g : X_\ell \to X_r$, then

$$f \overline{\otimes} g = \mu_{j,r} \circ (f \otimes g) \circ \mu_{i,\ell}^{-1} \tag{A.24}$$

This makes $\overline{\otimes} : \mathcal{C}_0 \times \mathcal{C}_0 \to \mathcal{C}_0$ a functor. Define the associator by:

$$\overline{\alpha}_{X_i,X_j,X_k} := \mu_{i,j\cdot k} \circ (\mathrm{Id}_{X_i} \otimes \mu_{j,k}) \circ \alpha_{X_i,X_j,X_k} \circ (\mu_{i,j}^{-1} \otimes \mathrm{Id}_{X_k}) \circ \mu_{i\cdot j,k}^{-1} \tag{A.25}$$

$\mathcal{C}_0$ will have the same unit $1_{\mathcal{C}}$, with left unitor $\overline{\ell}_X = \mathrm{Id}_X$ and right unitor $\overline{r}_X = \mathrm{Id}_X$. Consider the functor $F : \mathcal{C}_0 \to \mathcal{C}$ defined by $F(X_i) = X_i$ and $F(f) = f$. This will evidently be an equivalence. To see that it is a monoidal equivalence, let $J^F_{X_i,X_j} = \mu_{i,j} : X_i \otimes X_j \to F(X_i \overline{\otimes} X_j)$. This will be a natural transformation for if $f : X_i \to X_j, g : X_\ell \to X_r$ are morphisms in $\mathcal{C}_0$, then

$$J^F_{X_j,X_r} \circ (f \otimes g) = \mu_{j,r} \circ (f \otimes g) \circ \mu_{i,\ell}^{-1} \circ \mu_{i,\ell} = (f \overline{\otimes} g) \circ \mu_{i,\ell} = (f \overline{\otimes} g) \circ J^F_{X_i,X_\ell} \tag{A.26}$$

Lastly, let $\phi^F = \mathrm{Id}_{1_{\mathcal{C}}}$. By design (J^F, ϕ^F) will make F a monoidal functor, and so we are done. □

For a monoidal category we call the process of passing to an equivalent skeletal monoidal category as skeletinzation. Also, it is important to note that while we can strictify and skeletinize a monoidal category, we *cannot* necessarily produce an equivalent monoidal category that is both skeletal and strict. In other words, you need to make a choice of which process you will use when proving something.

A.4 Rigid Monoidal Categories

Let $(\mathcal{C}, \otimes, \alpha, 1_{\mathcal{C}}, \ell, r)$ be a monoidal category.

Definition A.4.1. *Left Duals*

If $X \in \mathcal{C}$, then a left dual of X is a tuple $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$ where $X^* \in \mathcal{C}$ and

$$\mathrm{ev}_X : X^* \otimes X \to 1_{\mathcal{C}} \qquad \mathrm{coev}_X : 1_{\mathcal{C}} \to X \otimes X^* \tag{A.27}$$

are morphisms in $\mathcal{C}$ such that the zig-zag relations hold:

$$r_X \circ (\mathrm{Id}_X \otimes \mathrm{ev}_X) \circ \alpha_{X,X^*,X} \circ (\mathrm{coev}_X \otimes \mathrm{Id}_X) \circ \ell_X^{-1} = \mathrm{Id}_X \tag{A.28}$$

$$\ell_X \circ (\mathrm{ev}_X \otimes \mathrm{Id}_X) \circ \alpha^{-1}_{X^*,X,X^*} \circ (\mathrm{Id}_{X^*} \otimes \mathrm{coev}_X) \circ r^{-1}_{X^*} = \mathrm{Id}_{X^*} \tag{A.29}$$

Similarly one can define right dual of an object of X

Definition A.4.2. *Right Duals*

If $X \in \mathcal{C}$, then a right dual of X is a tuple $({}^*X, \mathrm{ev}'_X, \mathrm{coev}'_X)$ where ${}^*X \in \mathcal{C}$ and

$$\mathrm{ev}'_X : X \otimes {}^*X \to 1_{\mathcal{C}} \qquad \mathrm{coev}'_X : 1_{\mathcal{C}} \to {}^*X \otimes X \tag{A.30}$$

are morphisms in $\mathcal{C}$ such that the zig-zag relations hold:

$$\ell_X \circ (\mathrm{ev}'_X \otimes \mathrm{Id}_X) \circ (\alpha^{-1}_{X,{}^*X,X} \circ (\mathrm{Id}_X \otimes \mathrm{coev}'_X) \circ r^{-1}_X = \mathrm{Id}_X \tag{A.31}$$

$$r_{{}^*X} \circ (\mathrm{Id}_X \otimes \mathrm{ev}'_X) \circ (\alpha_{{}^*X,X,{}^*X} \circ (\mathrm{coev}'_X \otimes \mathrm{Id}_{{}^*X}) \circ \ell^{-1}_{{}^*X} = \mathrm{Id}_X \tag{A.32}$$

As one would expect left/right duals will be universal objects:

Proposition A.4.1. *Uniqueness of Duals*

Let $X \in \mathcal{C}$, and suppose $(Y, \mathrm{ev}^1_X, \mathrm{coev}^1_X), (Z, \mathrm{ev}^2_x, \mathrm{coev}^2_X)$ are left duals of X. Then there exists a unique isomorphism $f : Y \to Z$ such that $\mathrm{ev}_Y = \mathrm{ev}_Z \circ (f \otimes \mathrm{Id}_X)$, and $(\mathrm{Id}_X \otimes f) \circ \mathrm{coev}_Y = \mathrm{coev}_Z$. In particular, this isomorphism is:

$$f := \ell_Z \circ (\mathrm{ev}^1_X \otimes \mathrm{Id}_Z) \circ \alpha^{-1}_{Y,X,Z} \circ (\mathrm{Id}_Y \otimes \mathrm{coev}^2_Y) \circ r^{-1}_Y \tag{A.33}$$

A similar statement will hold for right duals.

Proof. See [13, Proposition 2.10.5] for the proof. □

Definition A.4.3. *Rigid Monoidal Category*

A monoidal category $\mathcal{C}$ is called rigid, if every object has a left and right dual.

For the cases we care about left and right duals will more or less be the same thing, and so from now on we only focus on the left dual case.

Definition A.4.4. *Left Dual of a Morphism*

Let $f : X \to Y$ be a morphism in $\mathcal{C}$, the left dual map $f^* : Y^* \to X^*$ is defined *by:*

$$f^* := \ell_{X^*} \circ (\mathrm{ev}_Y \otimes \mathrm{Id}) \circ \alpha^{-1}_{Y^*,Y,X^*} \circ (\mathrm{Id}_{Y^*} \otimes (f \otimes \mathrm{Id}_{X^*})) \circ (\mathrm{Id}_{Y^*} \otimes \mathrm{coev}_X) \circ r^{-1}_{Y^*} \quad \text{(A.34)}$$

One can show that in a rigid monoidal category $\mathcal{C}$ we have a functor $(-)^{**} : \mathcal{C} \to \mathcal{C}$ given by taking the double dual.

Proposition A.4.2. *Let* $(\mathcal{C}, \otimes_{\mathcal{C}}, \alpha^{\mathcal{C}}, 1_{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}}), (\mathcal{D}, \otimes_{\mathcal{D}}, \alpha^{\mathcal{D}}, 1_{\mathcal{D}}, \ell^{\mathcal{D}}, r^{\mathcal{D}})$ *be rigid monoidal categories and* $(F, J, \phi^F) : \mathcal{C} \to \mathcal{D}$ *a monoidal functor. Then* $F(X^*)$ *is a left dual of* X *with evaluation and coevaluation given respectively by:*

$$\mathrm{ev}^F_X := (\phi^F)^{-1} \circ F(\mathrm{ev}_X) \circ J_{X^*,X} \quad \text{(A.35)}$$

$$\mathrm{coev}^F_X := J^{-1}_{X,X^*} \circ F(\mathrm{coev}_X) \circ \phi^F \quad \text{(A.36)}$$

With the same setting as in Proposition A.4.2, by uniqueness of left duals there exists a unique isomorphism $d^F_X : F(X^*) \to F(X)^*$ for every tensor functor (F, J^F, ϕ^F) such that:

$$\mathrm{ev}^F_X = \mathrm{ev}_{F(X)} \circ (d^F_X \otimes \mathrm{Id}) \quad \text{(A.37)}$$

$$(\mathrm{Id} \otimes d^F_X) \circ \mathrm{coev}^F_X = \circ \mathrm{coev}_{F(X)} \quad \text{(A.38)}$$

These isomorphisms will also be compatible with monoidal natural isomorphisms as the next proposition illustrates:

Proposition A.4.3. *Let* $(\mathcal{C}, \otimes_{\mathcal{C}}, \alpha^{\mathcal{C}}, 1_{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}}), (\mathcal{D}, \otimes_{\mathcal{D}}, \alpha^{\mathcal{D}}, 1_{\mathcal{D}}, \ell^{\mathcal{D}}, r^{\mathcal{D}})$ *be rigid monoidal categories and* $(F, J^F, \phi^F), (K, J^K, \phi^K) : \mathcal{C} \to \mathcal{D}$ *monidal functors. If there exists a* *monoidal natural isomorphism $\tau : K \to F$ then:*

$$(\tau_X)^* \circ d_X^F \circ \tau_{X^*} = d_X^K \tag{A.39}$$

Proof. Without loss of generality we may assume that we are in a strict monoidal category.If we can show that:

$$\mathrm{ev}_X^K = \mathrm{ev}_{K(X)} \circ ((\tau_X)^* \circ d_X^F \circ \tau_{X^*}) \otimes \mathrm{Id} \tag{A.40}$$

then this implies that:

$$((\tau_X)^* \circ d_X^F \circ \tau_{X^*}) \otimes \mathrm{Id}) \circ \mathrm{coev}_X^K = \mathrm{coev}_{K(X)} \tag{A.41}$$

To see this denote for the sake of brevity $f = (\tau_X)^* \circ d_X^F \circ \tau_{X^*})$. Then note that:

$$(\mathrm{ev}_X^K \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \mathrm{coev}_{K(X)}) = (\mathrm{ev}_{K(X)} \otimes \mathrm{Id}) \circ (f \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \mathrm{coev}_{K(X)}) = \tag{A.42}$$

$$(\mathrm{ev}_{K(X)} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \mathrm{coev}_{K(X)}) \circ f = f \tag{A.43}$$

Therefore, this implies that:

$$(\mathrm{Id} \otimes f) \circ \mathrm{coev}_X^K = (\mathrm{Id} \otimes \mathrm{ev}_X^K \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{coev}_{K(X)}) \circ \mathrm{coev}_X^K = \tag{A.44}$$

$$(\mathrm{Id} \otimes \mathrm{ev}_X^K \otimes \mathrm{Id}) \circ (\mathrm{coev}_X^K \otimes \mathrm{Id}) \circ \mathrm{coev}_{K(X)} = \mathrm{coev}_{K(X)} \tag{A.45}$$

By uniqueness of isomorphisms this means to prove the Proposition we only need to show that Equation (A.40) will hold. Well to that end since we are in a strict category note that:

$$(\tau_X)^* = (\mathrm{ev}_{F(X)} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau_X \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \mathrm{coev}_{K(X)}) \tag{A.46}$$

so we have that $\mathrm{ev}_{K(X)} \circ (f \otimes \mathrm{Id})$ equals:

$$\mathrm{ev}_{K(X)} \circ ((\mathrm{ev}_{F(X)} \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau_X \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \mathrm{coev}_{K(X)}) \otimes \mathrm{Id}) \circ (d_X^F \otimes \mathrm{Id}) \circ (\tau_{X^*} \otimes \mathrm{Id}) = \tag{A.47}$$

$$\mathrm{ev}_{K(X)} \circ ((\mathrm{ev}_X^F \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau_X \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \mathrm{coev}_{K(X)} \otimes Id) \circ (\tau_{X^*} \otimes \mathrm{Id}) \tag{A.48}$$

Notice that

$$\mathrm{ev}_X^F \circ (\mathrm{Id} \otimes \tau_X) = (\phi^F)^{-1} \circ F(\mathrm{ev}_X) \circ J^F_{X^*,X} \circ (\mathrm{Id} \otimes \tau_X) = \tag{A.49}$$

$$(\phi^F)^{-1} \circ F(\mathrm{ev}_X) \circ \tau_{X^* \otimes X} \circ J^K_{X^*,X} \circ (\tau_{X^*}^{-1} \otimes \mathrm{Id}) = \tag{A.50}$$

$$(\phi^F)^{-1} \circ \tau_1 \circ K(\mathrm{ev}_X) \circ \circ J^K_{X^*,X} \circ (\tau_{X^*}^{-1} \otimes \mathrm{Id}) = \tag{A.51}$$

$$(\phi^K)^{-1} \circ K(\mathrm{ev}_X) \circ \circ J^K_{X^*,X} \circ (\tau_{X^*}^{-1} \otimes \mathrm{Id}) = \mathrm{ev}_X^K \circ (\tau_{X^*}^{-1} \otimes \mathrm{Id}) \tag{A.52}$$

Combing Equation (A.52) with Equation (A.48) we obtain:

$$\mathrm{ev}_{K(X)} \circ (\mathrm{ev}_X^K \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \mathrm{coev}_{K(X)} \otimes Id) = \mathrm{ev}_X^K \circ (\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{ev}_{K(X)}) \circ (\mathrm{Id} \otimes \mathrm{coev}_{K(X)} \otimes \mathrm{Id}) = \mathrm{ev}_X^K \tag{A.53}$$

Completing the proof. □

A.5 Finite Tensor Categories and Fusion Categories

The monoidal categories we will be interested are *finite tensor categories* over $\mathbb{C}$:

Definition A.5.1. *Finite Tensor Category [13, Definition 1.8.6, Definition 4.1.1]*

A finite tensor category $(\mathcal{C}, \otimes, \alpha, 1_{\mathcal{C}}, \ell, r)$, is a monoidal category such that:

1. *$\mathcal{C}$ is a $\mathbb{C}$-linear Abelian category.*

2. *For every $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a finite dimensional vector space over $\mathbb{C}$.*

3. *Every $X \in \mathcal{C}$ has a composition series. That is there exists some $n \in \mathbb{N}$ and objects $X_0, \cdots, X_n$ such that each X_i is a sub-object of X_{i+1} for $0 \leq i \leq n-1$, $X_0 = 0, X = X_n$, and for each $0 \leq i \leq n-1$, X_{i+1}/X_i is simple.*

4. *For every simple object X there exists a projective object P and an epimorphism $p : P \to X$. That is every simple object has a projective cover.*

5. *There are finitely many simple objects up to isomorphism.*

6. *The tensor product is $\mathbb{C}$-bilinear on morphisms.*

7. *$(\mathcal{C}, \otimes, \alpha, 1_{\mathcal{C}}, \ell, r)$ is a rigid monoidal category.*

8. *There is an isomorphism of vector spaces $\mathrm{End}_{\mathbb{C}}(1) \cong \mathbb{C}$.*

If a monoidal category only satisfies conditions $1, 2, 3, 6, 7, 8$ then we say it is a tensor category.

Definition A.5.2. *Fusion Category*

If $(\mathcal{C}, \otimes, \alpha, 1_{\mathcal{C}}, \ell, r)$ *is a finite tensor category, such that every object is isomorphic to the direct sum of simple objects, then we say it is a* fusion category*.*

A useful way to think of these definitions is that just as a monoidal category is the categorification of a monoid, a finite tensor category is the categorificcation of a finite dimensional unital $\mathbb{C}$-algebra, and a fusion category is the categorification of a finite dimensional semi-simple unital $\mathbb{C}$-algebra.

Definition A.5.3. *Tensor Functor*

Let $\mathcal{C}, \mathcal{D}$ *be finite tensor categories. A tensor functor* $(F, J^F, \phi^F) : \mathcal{C} \to \mathcal{D}$ *is a monoidal functor that is also* $\mathbb{C}$*-linear and additive. If there exists a tensor functor that is also an equivalence of categories we say that* $\mathcal{C}, \mathcal{D}$ *are equivalent as finite tensor categories.*

All of the results from Section A.1 to A.4 will hold for finite tensor categories when the appropriate adjustments are made.

Notation. *If* $\mathcal{C}$ *is a finite tensor category and* $X, Y, Z \in \mathcal{C}$ *such that* Z *is simple then let* $N_{X,Y}^{Z}$ *denote the number of times* Z *occurs in a composition series of a* $X \otimes_{\mathcal{C}} Y$*.*

Note that the Jordan-Hölder theorem will hold for Abelian categories, and so $N_{X,Y}^{Z}$ is well-defined. Furthermore, if $\mathcal{C}$ is a fusion category then

$$N_{X,Y}^{Z} = \dim_{\mathbb{C}} \operatorname{Hom}(X \otimes_{\mathcal{C}} Y, Z) \tag{A.54}$$

Definition A.5.4. *Gorthendieck Ring of a Finite Tensor Category*

Let $\mathcal{C}$ be a finite tensor category. If $X \in \mathcal{C}$ let $[X]$ denote the isomorphism class of X. The Gorthendieck ring of $\mathcal{C}$ is the finite dimensional $\mathbb{C}$-algebra $\mathcal{K}^0(\mathcal{C}) := \mathbb{C}[\mathcal{O}(\mathcal{C})]$ where multiplication is defined on objects as:

$$[X] \cdot [Y] := \sum_{[Z] \in \mathcal{O}(\mathcal{C})} N_{X,Y}^{Z} [Z] \tag{A.55}$$

and extended by linearity to $\mathcal{K}^0(\mathcal{C})$. $\mathcal{K}^0(\mathcal{C})$ will be a unital associative algebra with this operation.

In the case that $\mathcal{C}$ is a fusion category, the Gorthendieck ring is also called the fusion ring of $\mathcal{C}$. We need the following to defined Frobenius-Perron Dimension

Proposition A.5.1. *[13, Theorem 3.2.1]*

Let A be a square matrix with non-negative real entries. If A has strictly positive entries, then there exists a largest positive eigenvalue $\lambda(A)$.

Definition A.5.5. *Frobenius-Perron Dimension*

Let $\mathcal{C}$ be a fusion category. Choose a set of representatives from the isomorphism classes in $\mathcal{O}(\mathcal{C})$ $X_1, \cdots, X_r$. For $X \in \mathcal{C}$ the Frobenius-Perron dimension of X denoted by $\mathrm{FPdim}(X)$ is defined to be the largest positive eigenvalue of the matrix:

$$(N_{X,X_i}^{X_j})_{i,j=1}^{r} \tag{A.56}$$

A.6 Braided Monoidal Categories

Just as a monoidal category is the categorification of a monoid, a braided monoidal category is the categorification of an Abelian monoid.

Definition A.6.1. *Braided Monoidal Category*

Let $(\mathcal{C}, \otimes, \alpha, 1_{\mathcal{C}}, \ell, r)$ *be a monoidal category. A braiding on* $\mathcal{C}$ *is a natural isomorphism*

$$c : \otimes \to \otimes^{op} \tag{A.57}$$

where $\otimes^{\mathrm{op}}$ *is the functor with inputs switched, in other words* $\otimes^{\mathrm{op}}(X, Y) = Y \otimes X$ *for all* $X, Y \in \mathcal{C}$ *and similarly with morphisms. The braiding is required to satisfy the* Hexagon Axioms*:*

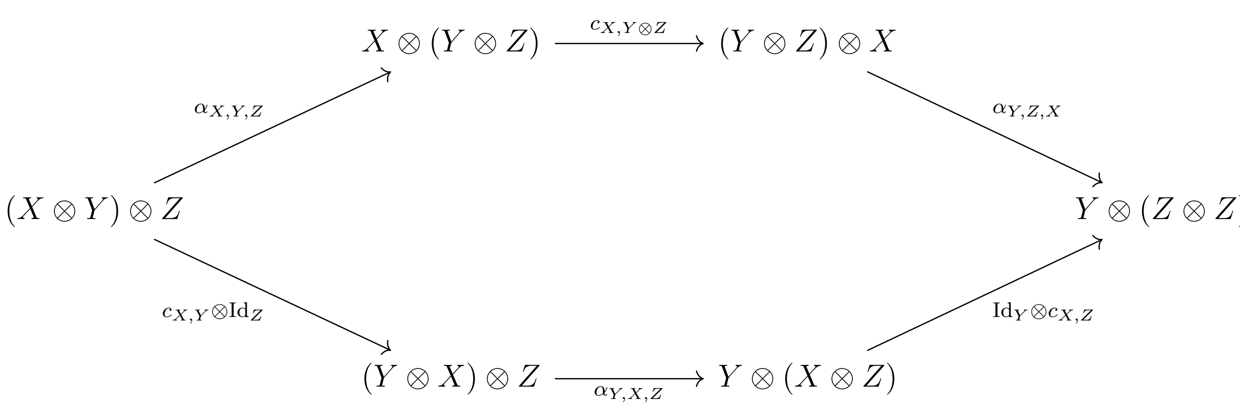

Figure A.3: Hexagon Axiom 1

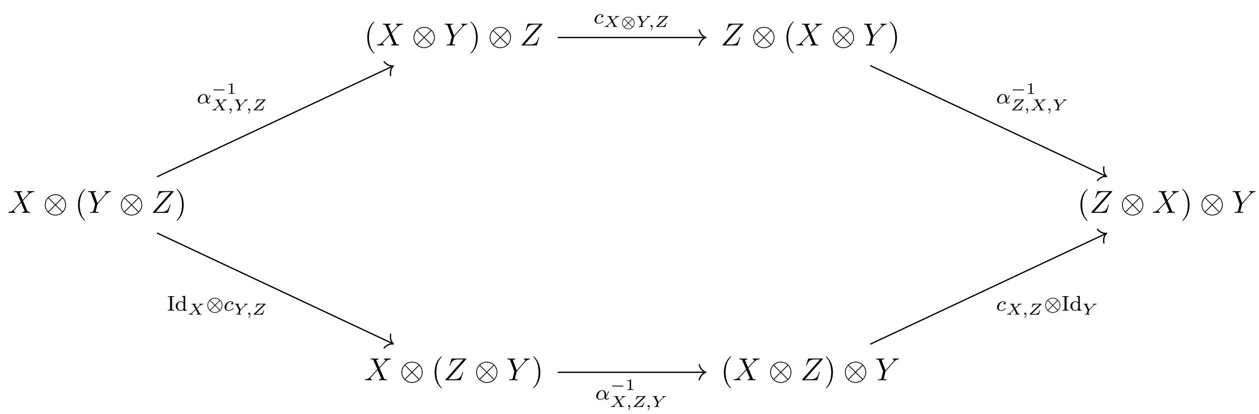

Figure A.4: Hexagon Axiom 2

When the monoidal structure of a braided category is clear we will simply denote it by the tuple $(\mathcal{C}, c)$*.*

Definition A.6.2. *Braided Monoidal Functor*

Let $(\mathcal{C}, \otimes^{\mathcal{C}}, \alpha^{\mathcal{C}}, 1_{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}}, c^1), (\mathcal{D}, \otimes^{\mathcal{D}}, \alpha^{\mathcal{D}}, 1_{\mathcal{D}}, \ell^{\mathcal{D}}, r^{\mathcal{D}}, c^2)$ *be braided monoidal categories. A monoidal functor* (F, J^F, ϕ^F) *from* $\mathcal{C}$ *to* $\mathcal{D}$ *is a braided monoidal functor if:*

$$J^F_{Y,X} \circ c^2_{F(X),F(Y)} = F(c^1_{X,Y}) \circ J^F_{X,Y} \tag{A.58}$$

Two braided monoidal categories are equivalent if there is a braided monoidal functor that induces an equivalence of categories.

As expected, if we have two monoidal categories $\mathcal{C}, \mathcal{D}$ such that $\mathcal{C}$ is braided, and there is an adjoint monoidal equivalence $F : \mathcal{C} \to \mathcal{D}$, then $\mathcal{D}$ can be endowed with a braided monoidal structure such that F is a braided equivalence. Since this result will be a special case of a more general proposition, we omit it.

Definition A.6.3. *Braided Tensor Category*

A braided monoidal category that is also a tensor category is called a braided tensor category. Finite braided tensor categories, and braided fusion categories are defined similarly.

Definition A.6.4. *Braided Tensor Functors*

If $(F, J^F, \phi^F) : \mathcal{C} \to \mathcal{D}$ *is a tensor functor between tenor categories that is also a braided monoidal functor, then we say it is a braided tensor functor.*

A.6.1 Braidings on Vect_G^ω and Abelian 3-cocycles

An key component of this thesis is describing certain categorical structure in linear algebraic terms. Braidings on Vect_G^ω and Abelian 3-cocycles will be a very special case of this description, and so following [13, 8.4] we review this in the hope that it will make the general result more digestible.

As mentioned in Example A.1.3 a finite group G and a 3-cocycle ω will form a fusion category by considering the finite dimensional vector spaces graded by G, taking the usual tensor product and letting the associator be twisted by ω. It is easy to show that

$$\mathcal{O}(\mathrm{Vect}_G^\omega) = \{[\mathbb{C}_g] : g \in G\} \tag{A.59}$$

where $\mathbb{C}_g$ is the one-dimensional vector space with G grading:

$$(\mathbb{C}_g)_k := \begin{cases} \mathbb{C} & \text{if } g = k \\ 0 & \text{otherwise} \end{cases} \tag{A.60}$$

By taking a skeleton of Vect_G^ω we can find an equivalent monoidal category $\mathcal{C}$ whose simple objects are $\{\delta_g : g \in G\}$ and we have:

$$\delta_g \otimes \delta_h = \delta_{gh} \tag{A.61}$$

for all $g, h \in G$. From now on, whenever we say Vect_G^ω we will be referring to such a skeleton. Notice that if Vect_G^ω has a braiding on it, then necessarily G is an Abelian group.

Since $\text{End}_{\mathcal{C}}(\delta_g) \cong \mathbb{C}\text{Id}_{\delta_g}$, we see that any braiding is determines a complex function $c : G \times G \to \mathbb{C}^\times$ such that it satisfies the following equations for all $g_1, g_2, g_3 \in G$:

$$\omega(g_2, g_3, g_1)c(g_1, g_2g_3)\omega(g_1, g_2, g_3) = c(g_1, g_3)\omega(g_2, g_1, g_3)c(g_1, g_2) \tag{A.62}$$

$$\omega(g_3, g_1, g_2)^{-1}c(g_1g_2, g_3)\omega(g_1, g_2, g_3)^{-1} = c(g_1, g_3)\omega(g_1, g_3, g_2)^{-1}c(g_2, g_3) \tag{A.63}$$

Any pair of functions (ω, c) that satisfy Equations A.7, A.62, A.63 are called a *Abelian 3-cocycle*. Denote the set of all such pairs as $Z^3_{Ab}(G, \mathbb{C}^\times)$. This will form an Abelian group by multiplication.

We say an Abelian 3-cocycle (ω, c) is an Abelian coboundary if there exists a function $k : G \times G \to \mathbb{C}^\times$ such that for all $g_1, g_2, g_3 \in G$:

$$\omega(g_1, g_2, g_3) := \frac{k(g_2, g_3)k(g_1, g_2g_3)}{k(g_1g_2, g_3)k(g_1, g_2)} \tag{A.64}$$

$$c(g_1, g_2) = \frac{k(g_1, g_2)}{k(g_2, g_1)} \tag{A.65}$$

We denote the set of Abelian coboundaries by $B^3_{ab}(G, \mathbb{C}^\times)$ and this will be a subgroup of $Z^3_{ab}(G, \mathbb{C}^\times)$. The quotient group $H^3_{ab}(G, \mathbb{C}^\times) := \frac{Z^3_{ab}(G,\mathbb{C}^\times)}{B^3_{ab}(G,\mathbb{C}^\times)}$ is called the Abelian 3-Cohomology group of G.

Theorem A.6.1. *Elements of $H^3_{ab}(G, \mathbb{C}^\times)$ are in bijection with equivalence classes of braided monoidal structures on the category of G-graded vector spaces.*

Proof. See [13, Theorem 8.4.9] for the proof. □

A.7 Ribbon Categories, Spherical Structures and Trace

Definition A.7.1. *Ribbon Twist*

Let $\mathcal{C}$ be a braided tensor category. A ribbon twist is a monoidal natural isomorphism $\theta : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$ such that the following hold for all $X, Y \in \mathcal{C}$:

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y} \tag{A.66}$$

$$(\theta_X)^* = \theta_{X^*} \tag{A.67}$$

If a braided tensor category $\mathcal{C}$ has a ribbon structure, we refer to $\mathcal{C}$ as a ribbon tensor category.

Definition A.7.2. *[22]*

Let $(\mathcal{C}, \theta), (\mathcal{D}, \theta')$ be ribbon tensor categories. We say that a braided tensor functor

$F : \mathcal{C} \to \mathcal{D}$ is ribbon if:

$$F(\theta) = \theta' \tag{A.68}$$

We say that two ribbon tensor categories are equivalent if there exists a braided ribbon tensor functor between them that is also an equivalence of categories.

Example A.7.1. *Let $\mathcal{C} = \mathrm{Vect}_G^{(\omega,c)}$ where (ω, c) is an Abelian 3-cocycle. We can define a ribbon twist on simple $\delta_g \in \mathcal{C}$ by:*

$$\theta_{\delta_g} := (c(g,g))^2 \mathrm{Id}_{\delta_g} \tag{A.69}$$

and extend by linearity to all of $\mathcal{C}$.

Definition A.7.3. *Trace [13]*

Let $\mathcal{C}$ be a rigid monoidal category, $X \in \mathcal{C}$ and $f : X \to X^{**}$, the trace of f is defined as:

$$\mathrm{Tr}(f) := \mathrm{ev}_{X^*} \circ (f \otimes \mathrm{Id}) \circ \mathrm{coev}_V \tag{A.70}$$

Notice that we can consider $\mathrm{Tr}(f) \in \mathbb{C}$ by using the identity $\mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}}) = \mathbb{C}1_{\mathcal{C}}$. For finite dimensional vector spaces there is a canonical isomorphism from a vector space to its double dual. The generalization of this to braided tensor categories is called a pivotal structure:

Definition A.7.4. *Pivotal Structure*

A pivotal structure is a monoidal natural isomorphism $\delta : \mathrm{Id}_{\mathcal{C}} \to (-)^{**}$.

We will call a fusion category with a pivotal structure a pivotal fusion category.

Definition A.7.5. *Categorical Dimension*

Let $\mathcal{C}$ be a pivotal tensor category with pivotal structure δ. The categorical dimension of $X \in \mathcal{C}$ is:

$$\dim_{\mathcal{C}}(X) := \mathrm{Tr}(\delta_X) \tag{A.71}$$

Definition A.7.6. *Spherical Structure*

A pivotal structure such that for every $X \in \mathcal{C}$ we have:

$$\dim_{\mathcal{C}}(X) = \dim_{\mathcal{C}}(X^*) \tag{A.72}$$

is called a spherical structure. A spherical tensor category is a tensor category equipped with a spherical structure. Spherical finite tensor categories and spherical fusion categories are similarly defined.

Definition A.7.7. *Quantum Trace*

Let $\mathcal{C}$ be a spherical tensor category, and δ the spherical structure. If $X \in \mathcal{C}$ and $f : X \to X$ is a morphism, then the trace of f with respect to the pivotal structure δ is defined as:

$$\mathrm{Tr}(f) := \mathrm{Tr}(\delta_X \circ f) \tag{A.73}$$

Lastly, a ribbon twist on a braided rigid tensor category will induce a spherical structure:

Proposition A.7.1. *[13, Proposition 8.10.12]*

Let $\mathcal{C}$ be a ribbon fusion category, with ribbon structure θ. Then $\mathcal{C}$ is also a *spherical fusion category with spherical structure:*

$$\delta_X := \theta_X \circ r_X \circ (\mathrm{Id}_X \otimes \mathrm{ev}_{X^*}) \circ (\mathrm{Id}_X \otimes c^{-1}_{X^{**},X^*}) \circ \alpha_{X^*,X,X^{**}} \circ (\mathrm{coev}_X \otimes \mathrm{Id}_{X^{**}}) \circ (\ell_{X^{**}})^{-1} \tag{A.74}$$

A.8 Modular Fusion Categories

Definition A.8.1. *S-matrix*

Let $\mathcal{C}$ be a ribbon fusion category. Choose representatives $X_1, \cdots, X_r$ from each isomorphism class in $\mathcal{O}(\mathcal{C})$. The S-matrix of $\mathcal{C}$ is:

$$S := (s_{X_i,X_j})^r_{i,j=1} \quad s_{X_i,X_j} := \mathrm{Tr}(c_{X_j,X_i} \circ c_{X_i,X_j}) \tag{A.75}$$

Definition A.8.2. *Modular Fusion Category*

A modular fusion category is a ribbon fusion category with non-degenerate S-matrix.

For the sake of brevity we will denote modular fusion category by MFC.

Definition A.8.3. *Modular Data of a Modular Fusion Category*

Let $\mathcal{C}$ be a MFC. The modular data of $\mathcal{C}$ are:

1. *The set of representatives of isomorphism classes in $\mathcal{O}(\mathcal{C})$ $X_1, \cdots, X_r$.*

2. *The S-matrix.*

3. The T-matrix defined for $1 \leq i, j \leq r$:

$$T_{i,j} = \delta_{i,j}\theta_{X_i}^{-1} \tag{A.76}$$

The modular data of a MFC will be an invariant.

Appendix B

G-Crossed Cocycles: The Proofs

B.1 Verifying The Crossed Braiding Axioms Induced by a G-crossed Abelian 3-cocycle

Notation. *For $g, k \in G$:*

$$\overline{\gamma}_{g,k} := \underline{\gamma}_{\,k^{-1},g^{-1}} \tag{B.1}$$

If $\tau \in S_n$ let σ_τ denote the map defined on $V_1 \otimes \cdots \otimes V_n$ by $\sigma_\tau(v_1 \otimes \cdots \otimes v_n) = v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)}$.

Let's verify the first equation for the crossed braiding axioms. We need to check that if $(V, \rho_V) \in \mathrm{Mod}(H)_g, (W, \rho_W) \in \mathrm{Mod}(H)$, then $\sigma \circ (\rho_V \otimes \rho_W)(\overline{c})$ is an H-intertwiner from $(V, \rho_V) \otimes (W, \rho_W)$ to $(W, \rho_W \circ \Psi(g^{-1})) \otimes (V, \rho_V)$. Let $h \in H$. Then:

$$\sigma \circ (\rho_V \otimes \rho_W)(\overline{c}) \circ (\rho_V \otimes \rho_W)(\Delta(h)) = \tag{B.2}$$

$$\sigma \circ (\rho_V \otimes \rho_W)(\overline{c}) \circ (\rho_V \otimes \rho_W)(\Delta(h) \cdot (\overline{\partial}(\delta_g) \otimes 1_H)) = \tag{B.3}$$

$$\sigma \circ (\rho_V \otimes \rho_W)(\overline{c} \cdot \Delta(h) \cdot (\overline{\partial}(\delta_g) \otimes 1_H)) \tag{B.4}$$

By Equation (4.40) we can simplify Equation (B.4) to:

$$\sigma \circ (\rho_V \otimes \rho_W)((\mathrm{Id} \otimes \Psi(g^{-1}))(\Delta^{op}(h)) \cdot \overline{c} \cdot (\overline{\partial}(\delta_g) \otimes 1_H)) = \tag{B.5}$$

$$\sigma \circ (\rho_V \otimes \rho_{T_g(W,\rho_W)})(\Delta^{op}(h)) \circ (\rho_V \otimes \rho_W)(\overline{c} \cdot (\overline{\partial}(\delta_g) \otimes 1_H)) = \tag{B.6}$$

$$(\rho_{T_g(W,\rho_W)} \otimes \rho_V)(\Delta(h)) \circ \sigma \circ (\rho_V \otimes \rho_W(\overline{c} \cdot (\overline{\partial}(\delta_g) \otimes 1_H)) = \tag{B.7}$$

$$(\rho_{T_g(W,\rho_W)} \otimes \rho_V)(\Delta(h)) \circ \sigma \circ (\rho_V \otimes \rho_W)(\overline{c}) \tag{B.8}$$

Therefore, it is indeed an H-intertwiner.

Next, we need to verify the first crossed braiding axiom. That is we need to show Figure 2.1 will hold. Writing this out explicitly we need to show that if $g, k \in G$ $(V, \rho_V) \in \mathrm{Mod}(H)_k, (W, \rho_W) \in \mathrm{Mod}(H)$, then:

$$(\gamma_{g,k}(W, \rho_W) \otimes \mathrm{Id}_{T_g(V,\rho)}) \circ (\mu_g(T_h(W, \rho_W), (V, \rho_V)))^{-1} \circ T_g(c_{(V,\rho_V),(W,\rho_W)}) = \tag{B.9}$$

$$(\gamma_{gkg^{-1},g}(W, \rho_W) \otimes \mathrm{Id}_{T_g(V,\rho_V)}) \circ c_{T_g(V,\rho_V),T_g(W,\rho_W)} \circ (\mu_g((V, \rho_V), (W, \rho_W)))^{-1} \tag{B.10}$$

To that end we expand Equation (B.9) to obtain:

$$(\rho_W \otimes \rho_V)(\overline{\gamma}_{g,k} \otimes 1_H) \circ (\rho_{T_k(W,\rho_W)} \otimes \rho_V)((\overline{\mu}_g)^{-1}) \circ \sigma \circ (\rho_V \otimes \rho_W)(\overline{c}) = \tag{B.11}$$

$$(\rho_W \otimes \rho_V)((\overline{\gamma}_{g,k} \otimes 1_H) \cdot (\Psi(k^{-1}) \otimes \mathrm{Id}_H)(\overline{\mu}_g)^{-1}) \circ \sigma(\rho_V \otimes \rho_W)(\overline{c} \cdot (\overline{\partial}(\delta_k)) \otimes 1_H)) = \tag{B.12}$$

$$\sigma \circ (\rho_V \otimes \rho_W)((1_H \otimes \overline{\gamma}_{g,k}) \cdot (\mathrm{Id}_H \otimes \Psi(k^{-1}))(\overline{\mu}_g)^{-1}_{21} \cdot \overline{c} \cdot (\overline{\partial}(\delta_k)) \otimes 1_H)) = \tag{B.13}$$

$$\sigma \circ (\rho_V \otimes \rho_W)((\overline{\gamma}_{g,k})_2 \cdot ((\overline{\mu}_g)^{-1})_{21}^{(e,k)} \cdot \overline{c} \cdot (\overline{\partial}(\delta_k))_1) \tag{B.14}$$

By Equation (4.41) we can simplify Equation (B.14) to:

$$\sigma \circ (\rho_V \otimes \rho_W)((\overline{\gamma}_{gkg^{-1},g})_2 \cdot \overline{c}^{(g,g)} \cdot (\overline{\mu}_g)^{-1} \cdot (\overline{\partial}(\delta_k))_1) = \tag{B.15}$$

$$(\rho_W \otimes \rho_V)((\overline{\gamma}_{gkg^{-1},g})_1) \circ \sigma \circ (\rho_V \otimes \rho_W)(\overline{c}^{(g,g)}) \circ (\rho_V \otimes \rho_W)((\overline{\mu}_g)^{-1} \cdot (\overline{\partial}(\delta_k))_1) = \tag{B.16}$$

$$(\rho_W \otimes \rho_V)((\overline{\gamma}_{gkg^{-1},g})_1) \circ \sigma \circ (\rho_{T_g(V,\rho_V)} \otimes \rho_{T_g(W,\rho_W)})(\overline{c}) \circ (\rho_V \otimes \rho_W)((\overline{\mu}_g)^{-1} \cdot (\overline{\partial}(\delta_k))_1) = \tag{B.17}$$

$$(\rho_W \otimes \rho_V)((\overline{\gamma}_{gkg^{-1},g})_1) \circ \sigma \circ (\rho_{T_g(V,\rho_V)} \otimes \rho_{T_g(W,\rho_W)})(\overline{c}) \circ (\rho_V \otimes \rho_W)((\overline{\mu}_g)^{-1}) \tag{B.18}$$

But, Equation (B.10) is just the expanded version of Eqnation B.9. Therefore, we see the first corssed braiding axiom will hold.

Next, we need to verify the second crossed braiding axiom. That is we need to show that Figure 2.2 will hold. Writing this out explicitly we need to show that if $g \in G$, $(V, \rho_V) \in \mathrm{Mod}(H)_g$ $(W, \rho_V), (U, \rho_V) \in \mathrm{Mod}(H)$, then:

$$\alpha_{T_g(W,\rho_W),T_g(U,\rho_U),(V,\rho_V)} \circ (\mu_g((W, \rho_W), (U, \rho_U)))^{-1} \otimes \mathrm{Id}_{(V,\rho_V)} \circ \cdots \tag{B.19}$$

$$\cdots c_{(V,\rho_V),(W,\rho_W)\otimes(U,\rho_U)} \circ \alpha_{(V,\rho_V),(W,\rho_W),(U,\rho_U)} =$$

$$(\mathrm{Id}_{T_g(W,\rho_W)} \otimes c_{(V,\rho_V),(U,\rho_U)}) \circ \alpha_{T_g(W,\rho_W),(V,\rho_V),(U,\rho_U)} \circ (c_{(V,\rho_V),(W,\rho_W)} \otimes \mathrm{Id}_{(U,\rho_U)}) \tag{B.20}$$

Expanding Equation (B.19) we obtain:

$$(\rho_{T_g(W,\rho_W)} \otimes \rho_{T_g(U,\rho_U)} \otimes \rho_V)(\Phi) \circ (\rho_W \otimes \rho_U \otimes \rho_V)((\overline{\mu_g})_{12}^{-1}) \circ \sigma_{(132)} \cdots \tag{B.21}$$

$$\cdots \circ (\rho_V \otimes \rho_{(W,\rho_W)\otimes(U,\rho_U)})(\overline{c}) \circ (\rho_V \otimes \rho_W \otimes \rho_U)(\Phi) =$$

$$(\rho_W \otimes \rho_U \otimes \rho_V)(\Phi^{(g,g,e)} \cdot (\overline{\mu_g})_{12}^{-1}) \circ \sigma_{(132)} \circ (\rho_V \otimes \rho_{(W,\rho_W)\otimes(U,\rho_U)})(\overline{c}) \circ (\rho_V \otimes \rho_W \otimes \rho_U)(\Phi) = \tag{B.22}$$

$$(\rho_W \otimes \rho_U \otimes \rho_V)(\Phi^{(g,g,e)} \cdot (\overline{\mu_g})_{12}^{-1}) \circ \sigma_{(132)} \circ (\rho_V \otimes \rho_W \otimes \rho_U)((\mathrm{Id} \otimes \Delta)(\overline{c}) \cdot \Phi) = \tag{B.23}$$

$$\sigma_{(132)} \circ (\rho_V \otimes \rho_W \otimes \rho_U)(\Phi_{231}^{(e,g,g)} \cdot (\overline{\mu}_g)_{23}^{-1} \cdot (\mathrm{Id} \otimes \Delta)(\overline{c}) \cdot \Phi) \tag{B.24}$$

Notice that on the representation space Φ and $\overline{\mu}_g$ will be invertible. Furthermore, since $(V, \rho_V) \in \mathrm{Mod}(H)_g$ we have that $\rho_V(\overline{\partial}(\delta_g)) = \mathrm{Id}_V$. Therefore, restricting to the representation space see that by Equation (4.42) we can simplify Equation (B.24) to:

$$\sigma_{(132)} \circ (\rho_V \otimes \rho_W \otimes \rho_U)(\overline{c}_{13} \cdot \Phi_{213}^{(e,g,e)} \cdot \overline{c}_{12}) = \tag{B.25}$$

$$\sigma_{(23)} \circ \sigma_{(12)} \circ (\rho_V \otimes \rho_W \otimes \rho_U)(\overline{c}_{13} \cdot \Phi_{213}^{(e,g,e)} \cdot \overline{c}_{12}) = \tag{B.26}$$

$$\sigma_{(23)} \circ (\rho_W \otimes \rho_V \otimes \rho_U)(\overline{c}_{23}) \circ (\rho_W \otimes \rho_V \otimes \rho_U)(\Phi_{123}^{(g,e,e)}) \circ \sigma_{(12)} \circ (\rho_V \otimes \rho_W \otimes \rho_U)(\overline{c}_{12}) \tag{B.27}$$

But, Equation (B.27) is just the expanded form of Equation (B.20). Therefore, the second crossed braiding axiom will indeed hold.

Lastly, we need to verify that the third crossed braiding axiom will hold. That is we need to show Figure 2.3 will hold. Writing this out explicitly we need to show if $g,k \in G$ $(V,\rho_V) \in \mathrm{Mod}(H)_g, (W,\rho_W) \in \mathrm{Mod}(H)_k, (U,\rho_U) \in \mathrm{Mod}(H)$, then:

$$\alpha^{-1}_{(T_g \circ T_k)(U,\rho_U),(V,\rho_V),(W,\rho_W)} \circ (\gamma_{g,k}(U,\rho_U)^{-1} \otimes \mathrm{Id}) \circ c_{(V,\rho_V)\otimes(W,\rho_W),(U,\rho_U)} = \tag{B.28}$$

$$(c_{(V,\rho_V),T_k(U,\rho_U)} \otimes Id) \circ \alpha^{-1}_{(V,\rho_V),T_k(U,\rho_U),(W,\rho_W)} \circ (\mathrm{Id} \otimes c_{(W,\rho_W),(U,\rho_U)}) \circ \alpha_{(V,\rho_V),(W,\rho_W),(U,\rho_U)} \tag{B.29}$$

Expanding Equation (B.28) we obtain:

$$(\rho_{(T_g \circ T_k)(U,\rho_U)} \otimes \rho_V \otimes \rho_W)(\Phi^{-1}) \circ (\rho_U \otimes \rho_V \otimes \rho_W)((\overline{\gamma}_{g,k})_1^{-1}) \circ \cdots \tag{B.30}$$

$$\cdots \sigma_{(123)} \circ (\rho_{(V,\rho_V)\otimes(W,\rho_W)} \otimes \rho_U)(\overline{c}) =$$

$$(\rho_U \otimes \rho_V \otimes \rho_W)((\overline{\gamma}_{g,k})_1 \cdot (\Phi^{-1})^{(gk,e,e)}) \circ \sigma_{(123)} \circ (\rho_V \otimes \rho_W \otimes \rho_U)((\Delta \otimes \mathrm{Id})(\overline{c})) \tag{B.31}$$

Notice that we obtained Equation (B.31) by using the fact that:

$$\Psi((gk)^{-1}) = \mathrm{Ad}(\overline{\gamma}_{g,k})(\Psi(k^{-1}) \circ \Psi(g^{-1})) \tag{B.32}$$

We can simplify Equation (B.31) to:

$$\sigma_{(123)} \circ (\rho_V \otimes \rho_W \otimes \rho_U)((\overline{\gamma}_{g,k})_3^{-1}(\Phi^{-1})_{312}^{(e,e,gk)} \cdot (\Delta \otimes \mathrm{Id})(\overline{c})) \tag{B.33}$$

Since $(V,\rho_V) \in \mathrm{Mod}(H)_g, (W,\rho_W) \in \mathrm{Mod}(H)_k$ we have that $(\rho_V \otimes \rho_W)(\overline{\partial}(\delta_g) \otimes \overline{\partial}(\delta_k)) = \mathrm{Id}$. Furthermore, $\overline{\gamma}_{g,k}$ and Φ will be invertible on the representation spaces. Therefore, by restricting to the representation spaces we can use Equation (4.43) to simplify B.33 to:

$$\sigma_{(123)} \circ (\rho_V \otimes \rho_W \otimes \rho_U)(\overline{c}_{13}^{(e,e,k)} \cdot (\Phi^{-1})_{132}^{(e,e,k)} \cdot \overline{c}_{23} \cdot \Phi) = \tag{B.34}$$

$$\sigma_{(12)} \circ \sigma_{(23)} \circ (\rho_V \otimes \rho_W \otimes \rho_U)(\overline{c}_{13}^{(e,e,k)} \cdot (\Phi^{-1})_{132}^{(e,e,k)} \cdot \overline{c}_{23} \cdot \Phi) = \tag{B.35}$$

$$\sigma_{(12)} \circ (\rho_V \otimes \rho_W \otimes \rho_U)(\overline{c}_{12}^{(e,k,e)}) \circ (\rho_V \otimes \rho_W \otimes \rho_U)((\Phi^{-1})^{(e,k,e)}) \circ \cdots \tag{B.36}$$

$$\cdots \sigma_{(23)} \circ (\rho_V \otimes \rho_U \otimes \rho_V)(\overline{c}_{23}) \circ (\rho_V \otimes \rho_U \otimes \rho_V)(\Phi)$$

But, Equation (B.36) is just the expanded form of Equation (B.29). Therefore, the third crossed braiding axiom will indeed hold.

B.2 Verifying the G-Ribbon Axioms Induced by a G-Ribbon Abelian 3-cocycle

Fix a G-Ribbon Abelian 3-cocycle on H $(\Delta, \epsilon, \Phi, S, \alpha, \beta, \Psi, \underline{\gamma}, \overline{\mu}, \overline{c}, \overline{\partial}, \nu)$. By previous work we know this will induce a G-crossed braided tensor structure on $\mathrm{Mod}(H)$. Therefore, we only need to check that $\theta_{(V,\rho_V)} = \rho_V(\nu)$ is a G-ribbon element.

θ is an H-intertwiner

Let $g \in G$, and suppose $(V, \rho_V) \in \mathrm{Mod}(H)_g$. We need to verify that $\theta_{(V,\rho_V)} : (V, \rho_V) \to T_g(V, \rho_V)$ is an H-intertwiner. To that end let $h \in H$, then:

$$\rho_{T_g(V,\rho_V)}(h) \cdot \theta_{(V,\rho_V)} = \rho_{T_g(V,\rho_V)}(h) \cdot \rho_V(\nu) = \rho_V(\Psi(g^{-1}(h) \cdot \nu) = \rho_V(\Psi(g^{-1})(h) \cdot \nu \cdot \overline{\partial}(\delta_g)) \tag{B.37}$$

By Equation (4.70) we can simplify Equation (B.37) to:

$$\rho_V(\nu \cdot h \cdot \overline{\partial}(\delta_g)) = \rho_V(\nu \cdot h) = \rho_V(\nu) \cdot \rho_V(h) = \theta_{(V,\rho_V)} \cdot \rho_V(h) \tag{B.38}$$

Therefore, it is an H-intertwiner.

θ Statisfies Equation (2.12)

Next, we need to verify that for all $g, k \in G$, $(V, \rho_V) \in \mathrm{Mod}(H)_g$, $(W, \rho_W) \in \mathrm{Mod}(H)_k$

$$\theta_{(V,\rho_V)\otimes(W,\rho_W)} = \tag{B.39}$$

$$\mu_g((V,\rho_V),(W,\rho_V)) \circ (\gamma_{(gk)g(gk)^{-1},(gk)g^{-1}}(V,\rho_V) \otimes \gamma_{gkg^{-1},g}(W,\rho_W)) \circ \cdots \tag{B.40}$$

$$(\theta_{T_{gkg^{-1}}(V,\rho_V)} \otimes \theta_{T_g(W,\rho_W)}) \circ c_{T_g(W,\rho_W),(V,\rho_V)} \circ c_{(V,\rho_V),(W,\rho_W)}$$

Expanding the right hand side of this equation we obtain:

$$(\rho_V \otimes \rho_W)(\overline{\mu}_g) \circ (\rho_V(\underline{\gamma}_{\ gk^{-1}g^{-1},(gk)g^{-1}(gk)^{-1}}) \otimes \rho_W(\underline{\gamma}_{\ g^{-1},gk^{-1}g})) \circ \cdots$$

$$(\rho_{T_{gkg^{-1}}(V,\rho_V)}(\nu) \otimes \rho_{T_g(W,\rho_W)}(\nu)) \circ c^{\mathrm{Vect}}_{W,V} \circ (\rho_{T_g(W,\rho_W)} \otimes \rho_V)(\overline{c}) \circ c^{\mathrm{Vect}}_{V,W} \circ (\rho_V \otimes \rho_W)(\overline{c}) = \tag{B.41}$$

$$(\rho_V \otimes \rho_W)(\overline{\mu}_g \cdot (\underline{\gamma}_{\ gk^{-1}g^{-1},(gk)g^{-1}(gk)^{-1}} \otimes \underline{\gamma}_{\ g^{-1},gk^{-1}g}) \circ (\nu^{\otimes 2})^{(gkg^{-1},g)} \cdot (\overline{c}_{21}^{(e,g)}) \cdot \overline{c}) = \tag{B.42}$$

$$(\rho_V \otimes \rho_W)(\overline{\mu}_g \cdot (\underline{\gamma}_{\ gk^{-1}g^{-1},(gk)g^{-1}(gk)^{-1}} \otimes \underline{\gamma}_{\ g^{-1},gk^{-1}g}) \circ (\nu^{\otimes 2})^{(gkg^{-1},g)} \cdot (\overline{c}_{21}^{(e,g)}) \cdot \overline{c} \cdot (\overline{\partial}(\delta_g) \otimes \overline{\partial}(\delta_k))) \tag{B.43}$$

By Equation (4.71), and the fact that $\overline{\mu}_g \cdot \overline{\mu}_g^{-1} = \Delta(1_H)$ we see that this reduces to:

$$(\rho_V \otimes \rho_W)(\Delta(\nu) \cdot (\overline{\partial}(\delta_g) \otimes \overline{\partial}(\delta_k))) = (\rho_V \otimes \rho_W)(\Delta(\upsilon)) = \theta_{(V,\rho_V)\otimes(W,\rho_W)} \quad \text{(B.44)}$$

θ Satisfies Equation (2.13)

Next, we need to verify that θ satisfies for all $g \in G, (V, \rho_V) \in \mathrm{Mod}(H)_g$:

$$\theta_{(V,\rho_V)^*} = T_{g^{-1}}(\theta^*_{(V,\rho_V)} \circ d^{T_g}_{(V,\rho_V)}) \circ \gamma_{g^{-1},g}((V, \rho_V)^*)^{-1} \quad \text{(B.45)}$$

Using Lemma 4.3.1 we can expand the right hand side to:

$$(\rho_V(\nu))^* \circ (\rho_V(\mathbf{d}^g))^* \circ \rho_{V^*}(\underline{\gamma}^{-1}_{g^{-1},g}) = (\rho_V(\nu))^* \circ (\rho_V(\mathbf{d}^g))^* \circ \rho_V(S(\underline{\gamma}^{-1}_{g^{-1},g}))^* = \quad \text{(B.46)}$$

$$(\rho_V(S(\underline{\gamma}^{-1}_{g^{-1},g}) \cdot \mathbf{d}^g \cdot \nu))^* = (\rho_V(S(\underline{\gamma}^{-1}_{g^{-1},g}) \cdot \mathbf{d}^g \cdot \nu \cdot \overline{\partial}(\delta_g)))^* \quad \text{(B.47)}$$

By Equation (4.72) this equals:

$$(\rho_V(S(\nu) \cdot \overline{\partial}(\delta_g)))^* = (\rho_V(S(\nu)))^* = \rho_{V^*}(\nu) = \theta_{(V,\rho_V)^*} \quad \text{(B.48)}$$

θ Satisfies Equation (2.14)

Next, we need to verify that θ satisfies for all $g, k \in G$, $(V, \rho_V) \in \mathrm{Mod}(H)_k$:

$$\gamma_{gkg^{-1},g}(V, \rho_V) \circ \theta_{T_g(V,\rho_V)} = \gamma_{g,k}(V, \rho_V) \circ T_g(\theta_{(V,\rho_V)}) \quad \text{(B.49)}$$

Expanding the left hand side we obtain:

$$\rho_V(\underline{\gamma}_{g^{-1},gk^{-1}g^{-1}} \cdot \Psi(g^{-1})(\nu)) = \rho_V(\underline{\gamma}_{g^{-1},gk^{-1}g^{-1}} \cdot \Psi(g^{-1})(\nu) \cdot \overline{\partial}(\delta_k)) \tag{B.50}$$

By Equation (4.73) this equals:

$$\rho_V(\underline{\gamma}_{k^{-1},g^{-1}} \cdot \nu \cdot \overline{\partial}(\delta_k)) = \rho_V(\underline{\gamma}_{k^{-1},g^{-1}} \cdot \nu) = \gamma_{g,k}(V,\rho_V) \circ T_g(\theta_{(V,\rho_V)}) \tag{B.51}$$

Appendix C

G-Crossed Reconstruction: The Proofs

C.1 Proof of the Non-Abelian 2-cocycle Condition

To prove the crossed condition, that is Equation 4.2, first recall that:

$$\Psi(g) = \mathrm{Ad}(R_{g^{-1}}^{-1})((-)_{T_{g^{-1}}}) \tag{C.1}$$

In particular, we see that if $h \in H$, then:

$$\mathrm{Ad}((\underline{\gamma}_{\ g,k})(\Psi(g) \circ \Psi(k))(h)) = \underline{\gamma}_{\ g,k}(\Psi(g) \circ \Psi(k))(h)(\underline{\gamma}_{\ g,k})^{-1} = \tag{C.2}$$

$$\underline{\gamma}_{\ g,k} \circ \Psi(g)(R_{k^{-1}}^{-1} \circ h_{T_{k^{-1}}} \circ R_{k^{-1}}) \circ (\underline{\gamma}_{\ g,k})^{-1} = \tag{C.3}$$

$$\underline{\gamma}_{\ g,k} \circ (R_{g^{-1}}^{-1} \circ (R_{k^{-1}}^{T_{g^{-1}}})^{-1} \circ h_{T_{k^{-1}} \circ T_{g^{-1}}} \circ R_{k^{-1}}^{T_g^{-1}} \circ R_{g^{-1}}) \circ (\underline{\gamma}_{\ g,k})^{-1} \tag{C.4}$$

Since:

$$\underline{\gamma}_{\,g,k} := (R_{(gk)^{-1}})^{-1} \circ F(\gamma_{k^{-1},g^{-1}}) \circ R_{k^{-1}}^{T_{g^{-1}}} \circ R_{g^{-1}} \tag{C.5}$$

Therefore, we may simplify Equation C.4 to:

$$(R_{(gk)^{-1}})^{-1} \circ F(\gamma_{k^{-1},g^{-1}}) \circ h_{T_{k^{-1}} \circ T_{g^{-1}}} \circ F(\gamma^{-1}_{k^{-1},g^{-1}}) \circ R_{(gk)^{-1}} = \tag{C.6}$$

$$(R_{(gk)^{-1}})^{-1} \circ h_{T_{(gk)^{-1}}} \circ R_{(gk)^{-1}} = \tag{C.7}$$

$$\Psi(gk)(h) \tag{C.8}$$

Here we used the fact that $F(\gamma_{k^{-1},g^{-1}}) \circ h_{T_{k^{-1}} \circ T_{g^{-1}}} \circ F(\gamma^{-1}_{k^{-1},g^{-1}}) = h_{T_{(gk)^{-1}}}$ Next, we verify the cocycle condition, that is Equation 4.3. In particular, for every $g_1, g_2, g_3 \in G$ we need to show that:

$$\underline{\gamma}_{\,g_1,g_2g_3} \cdot \Psi(g_1)(\underline{\gamma}_{\,g_2,g_3}) = \underline{\gamma}_{\,g_1g_2,g_3} \cdot \underline{\gamma}_{\,g_1,g_2} \tag{C.9}$$

Expanding the left hand side of Equation C.9 and using naturality we obtain:

$$\Big((R_{(g_1g_2g_3)^{-1}})^{-1} \circ F(\gamma_{(g_2g_3)^{-1},g_1^{-1}}) \circ R_{(g_2g_3)^{-1}}^{T_{(g_1)^{-1}}} \circ R_{(g_1)^{-1}}\Big) \circ \cdots \tag{C.10}$$

$$\Big(R^{-1}_{g_1^{-1}} \circ (R^{T_{g_1^{-1}}}_{(g_2g_3)^{-1}})^{-1} \circ F(\gamma_{g_3^{-1},g_2^{-1}}(T_{g_1^{-1}})) \circ R_{g_3^{-1}}^{T_{g_2^{-1}} \circ T_{g_1^{-1}}} \circ R_{g_2^{-1}}^{T_{g_1^{-1}}} \circ R_{g_1^{-1}}\Big) =$$

$$(R_{(g_1g_2g_3)^{-1}})^{-1} \circ F(\gamma_{(g_2g_3)^{-1},g_1^{-1}} \circ \gamma_{g_3^{-1},g_2^{-1}}(T_{g_1^{-1}})) \circ R_{g_3^{-1}}^{T_{g_2^{-1}} \circ T_{g_1^{-1}}} \circ R_{g_2^{-1}}^{T_{g_1}^{-1}} \circ R_{g_1^{-1}} \tag{C.11}$$

We know that the following will hold:

$$\gamma_{g_3^{-1}g_2^{-1},g_1^{-1}} \circ \gamma_{g_3^{-1},g_2^{-1}}(T_{g_1^{-1}}) = \gamma_{g_3^{-1},g_2^{-1}g_1^{-1}} \circ T_{g_3^{-1}}(\gamma_{g_2^{-1},g_1^{-1}}) \tag{C.12}$$

Therefore, using Equation C.12 and naturality we may simplify Equation C.11 to:

$$(R_{(g_1g_2g_3)^{-1}})^{-1} \circ F(\gamma_{g_3^{-1},g_2^{-1}g_1^{-1}} \circ T_{g_3^{-1}}(\gamma_{g_2^{-1},g_1^{-1}})) \circ R_{g_3^{-1}}^{T_{g_2^{-1}} \circ T_{g_1^{-1}}} \circ R_{g_2^{-1}}^{T_{g_1}^{-1}} \circ R_{g_1^{-1}} = \tag{C.13}$$

$$(R_{(g_1g_2g_3)^{-1}})^{-1} \circ F(\gamma_{g_3^{-1},g_2^{-1}g_1^{-1}}) \circ F(T_{g_3^{-1}}(\gamma_{g_2^{-1},g_1^{-1}})) \circ R_{g_3^{-1}}^{T_{g_2^{-1}} \circ T_{g_1^{-1}}} \circ R_{g_2^{-1}}^{T_{g_1}^{-1}} \circ R_{g_1^{-1}} = \tag{C.14}$$

$$(R_{(g_1g_2g_3)^{-1}})^{-1} \circ F(\gamma_{g_3^{-1},g_2^{-1}g_1^{-1}}) \circ R_{g_3^{-1}}^{T_{(g_1g_2)^{-1}}} \circ F(\gamma_{g_2^{-1},g_1^{-1}}) \circ R_{g_2^{-1}}^{T_{g_1}^{-1}} \circ R_{g_1^{-1}} = \tag{C.15}$$

$$\begin{aligned} (R_{(g_1g_2g_3)^{-1}})^{-1} \circ F(\gamma_{g_3^{-1},g_2^{-1}g_1^{-1}}) \circ R_{g_3^{-1}}^{T_{(g_1g_2)^{-1}}} \circ R_{(g_1g_2)^{-1}} \circ \cdots \\ \circ (R_{(g_1g_2)^{-1}})^{-1} \circ F(\gamma_{g_2^{-1},g_1^{-1}}) \circ R_{g_2^{-1}}^{T_{g_1}^{-1}} \circ R_{g_1^{-1}} = \end{aligned} \tag{C.16}$$

$$\underline{\gamma}_{\,g_1g_2,g_3} \cdot \underline{\gamma}_{\,g_1,g_2} \tag{C.17}$$

To see that $(\Psi, \underline{\gamma})$ is normalized, note that we have choosen a unital categorical group action and so for all $g \in G$

$$\underline{\gamma}_{\,e,g} = (R_{g^{-1}})^{-1} \circ F(\gamma_{g^{-1},e}) \circ R_{g^{-1}} = 1_H \tag{C.18}$$

Similarly we see that $\underline{\gamma}_{\,g,e} = \mathrm{Id}_H$.

$(\Psi, \underline{\gamma})$ Satisfies Equation (4.6)

Let $h \in h$, $g \in G$. Then:

$$\epsilon(h) = h_{\mathcal{C}}, (\epsilon \circ \Psi(g))(h) = (\Psi(g))(h)_{1_{\mathcal{C}}} = (R_{g^{-1}}^{1_{\mathcal{C}}})^{-1} \circ h_{T_{g^{-1}}(1_{\mathcal{C}})} \circ R_{g^{-1}}^{1_{\mathcal{C}}} \tag{C.19}$$

As we have choosen a normalized action we have that $T_g(1_{\mathcal{C}}) = 1_{\mathcal{C}}$ on the nose, and note that $F(1_{\mathcal{C}}) \cong \mathbb{C}$. This implies that $R_{g^{-1}}^{1_{\mathcal{C}}} : F(1_{\mathcal{C}}) \to F(1_{\mathcal{C}})$ commutes with $h_{1_{\mathcal{C}}}$.

Therefore, Equation (C.19) reduces to $h_{1_{\mathcal{C}}}$.

C.2 Proof of $\overline{\mu}_g$ Axioms

Recall that for $g \in G, X, Y \in \mathcal{C}$ we have:

$$\overline{\mu}_g(X,Y) := (J^F_{X,Y})^{-1} \circ (R_g^{X\otimes Y})^{-1} \circ F(\mu_g(X,Y)) \circ J^F_{T_g(X),T_g(Y)} \circ (R_g^X \otimes R_g^Y) \quad \text{(C.20)}$$

$\overline{\mu}_g$ is Partially Invertible

To show that it is partially invertible recall that $\Delta(1)_{X,Y} = (J^F_{X,Y})^{-1} \circ J^F_{X,Y}$. Therefore, define for g, X, Y:

$$(\overline{\mu}_g{}^{-1})_{X,Y} := (R_g^X \otimes R_g^Y)^{-1} \circ (J^F_{T_g(X),T_g(Y)})^{-1} \circ F(\mu_g(X,Y)^{-1}) \circ R_g^{X\otimes Y} \circ J^F_{X,Y} \quad \text{(C.21)}$$

Calculating we see that for X, Y:

$$\Delta(1_H)_{X,Y} \cdot (\overline{\mu}_g)_{X,Y} = \quad \text{(C.22)}$$

$$(J^F_{X,Y})^{-1} \circ J^F_{X,Y} \circ (J^F_{X,Y})^{-1} \circ (R_g^{X\otimes Y})^{-1} \circ F(\mu_g(X,Y)) \circ J^F_{X,Y} \circ (R_g^X \otimes R_g^Y) = \quad \text{(C.23)}$$

$$(J^F_{X,Y})^{-1} \circ (R_g^{X\otimes Y})^{-1} \circ F(\mu_g(X,Y)) \circ J^F_{T_g(X),T_g(Y)} \circ (R_g^X \otimes R_g^Y) = \overline{\mu}_g \quad \text{(C.24)}$$

as $(J^F)^{-1}$ was chosen to be a left inverse of $J^F_{X,Y}$. Furthermore,

$$(\overline{\mu}_g)_{X,Y} \circ (\Delta(1_H)^{(g,g)})_{X,Y} = \tag{C.25}$$

$$(J^F_{X,Y})^{-1} \circ (R_g^{X\otimes Y})^{-1} \circ F(\mu_g(X,Y)) \circ J^F_{T_g(X),T_g(Y)} \circ \cdots = \tag{C.26}$$

$$\cdots (R_g^X \otimes R_g^Y) \circ (R_g^X \otimes R_g^Y)^{-1} \Delta(1_H)_{T_g(X),T_g(Y)} \circ (R_g^X \otimes R_g^Y)$$

$$(J^F_{X,Y})^{-1} \circ (R_g^{X\otimes Y})^{-1} \circ F(\mu_g(X,Y)) \circ J^F_{T_g(X),T_g(Y)} \cdots = \tag{C.27}$$

$$\cdots \circ (J^F_{T_g(X),T_g(Y)})^{-1} \circ J^F_{T_g(X),T_g(y)} \circ (R_g^X \otimes R_g^Y)$$

$$(J^F_{X,Y})^{-1} \circ (R_g^{X\otimes Y})^{-1} \circ F(\mu_g(X,Y)) \circ J^F_{T_g(X),T_g(y)} \circ (R_g^X \otimes R_g^Y) = \tag{C.28}$$

$$(\overline{\mu}_g)_{X,Y} \tag{C.29}$$

Similarly, one can check that $\overline{\mu}_g{}^{-1} \cdot \Delta(1_H) = \Delta(1_H)^{(g,g)} \cdot \overline{\mu}_g{}^{-1} = \overline{\mu}_g{}^{-1}$.

Expanding $(\overline{\mu}_g \cdot \overline{\mu}_g{}^{-1})_{X,Y}$ for $X, Y \in \mathcal{C}$ we obtain:

$$(J^F_{X,Y})^{-1} \circ (R_g^{X\otimes Y})^{-1} \circ F(\mu_g(X,Y)) \circ J^F_{T_g(X),T_g(Y)} \circ (R_g^X \otimes R_g^Y) \circ (R_g^X \otimes R_g^Y)^{-1} \cdots = \tag{C.30}$$

$$\cdots \circ (J^F_{T_g(X),T_g(Y)})^{-1} \circ F(\mu_g(X,Y)^{-1}) \circ R_g^{X\otimes Y} \circ J^F_{X,Y}$$

$$(J^F_{X,Y})^{-1} \circ J^F_{X,Y} = (\Delta(1_H))_{X,Y} \tag{C.31}$$

This shows that $\overline{\mu}_g \cdot \overline{\mu}_g{}^{-1} = \Delta(1_H)$. Similarly, by expanding $\overline{\mu}_g{}^{-1} \cdot \overline{\mu}_g$ one sees that it equals $\Delta(1_H)^{(g,g)}$.

$\overline{\mu}_g$ Satisfies Equation (4.11)

We need to verify that $\overline{\mu}_e = \Delta(1_H)$.This follows immedietly from the fact that $\mu_e = \mathrm{Id}$ and $R_e = \mathrm{Id}$.

$\overline{\mu}_g$ Satisfies Equation (4.12)

We need to verify that for all $h \in H$

$$\overline{\mu}_g \cdot \Delta(h)^{(g,g)} \cdot \overline{\mu}_g^{-1} = \Delta(\Psi(g^{-1})(h)) \tag{C.32}$$

Expanding the left hand side at $X, Y \in \mathcal{C}$ we obtain:

$$\begin{aligned}(J^F_{X,Y})^{-1} \circ (R_g^{X\otimes Y})^{-1} \circ F(\mu_g(X,Y)) \circ J^F_{T_g(X),T_g(Y)} \circ (R_g^X \otimes R_g^Y) \circ (R_g^X \otimes R_g^Y)^{-1} \circ \cdots \\ \cdots (J^F_{T_g(X),T_g(Y)})^{-1} \circ h_{T_g(X)\otimes T_g(Y)} \circ J^F_{T_g(X),T_g(Y)} \circ (R_g^X \otimes R_g^Y) \circ (R_g^X \otimes R_g^Y)^{-1} \circ \cdots \\ \cdots (J^F_{T_g(X),T_g(Y)})^{-1} \circ F(\mu_g(X,Y)^{-1}) \circ R_g^{X\otimes Y} \circ J^F_{X,Y}\end{aligned} \tag{C.33}$$

Cancelling out all the terms we obtain:

$$(J^F_{X,Y})^{-1} \circ (R_g^{X\otimes Y})^{-1} \circ F(\mu_g(X,Y)) \circ h_{T_g(X)\otimes T_g(Y)} \circ F(\mu_g(X,Y)^{-1}) \circ R_g^{X\otimes Y} \circ J^F_{X,Y} = \tag{C.34}$$

$$(J^F_{X,Y})^{-1} \circ (R_g^{X\otimes Y}) \circ h_{T_g(X\otimes Y)} \circ R_g^{X\otimes Y} \circ J^F_{X,Y} = \Delta(\Psi(g^{-1})(h))_{X,Y} \tag{C.35}$$

This proves the desired equality.

$\overline{\mu}_g$ Satisfies Equation (4.13)

It is easy to show that:

$$(\Phi)^{(g,g,g)} \cdot ((\overline{\mu}_g)^{-1} \otimes 1_H) \cdot (\Delta \otimes \mathrm{Id})((\overline{\mu}_g)^{-1}) = (1_H \otimes (\overline{\mu}_g)^{-1}) \cdot (\mathrm{Id} \otimes \Delta)((\overline{\mu}_g)^{-1}) \cdot \Phi \tag{C.36}$$

is equivalent to Equation (4.13) and so we prove this version. Well, recall that:

$$\Phi_{X,Y,Z} := (\mathrm{Id} \otimes (J^F_{Y,Z})^{-1}) \circ (J^F_{X,Y\otimes Z})^{-1} \circ F(\Phi_{X,Y,Z}) \circ J^F_{X\otimes Y,Z} \circ (J^F_{X,Y} \otimes \mathrm{Id}) \quad (C.37)$$

$$(\overline{\mu}_g)^{-1} = (R_g^{-1} \otimes R_g^{-1}) \circ (J^F_{T_g,T_g})^{-1} \circ F(\mu_g^{-1}) \circ R_g \circ J^F \quad (C.38)$$

Therefore, expanding $(\Phi)^{(g,g,g)} \cdot ((\overline{\mu}_g)^{-1} \otimes 1_H)$ at $X, Y, Z \in \mathcal{C}$ we obtain:

$$(R_g^X \otimes R_g^Y \otimes R_g^Z)^{-1} \circ (\mathrm{Id} \otimes (J^F_{T_g(Y),T_g(Z)})^{-1}) \circ (J^F_{T_g(X),T_g(Y)\otimes T_g(Z)})^{-1} \circ F(\Phi_{T_g(X),T_g(Y),T_g(Z)}) \circ J^F_{T_g(X)\otimes T_g(Y),T_g(Z)} \cdots$$
$$\cdots \circ (J^F_{T_g(X),T_g(Y)} \otimes \mathrm{Id}) \circ (R_g^X \otimes R_g^Y \otimes R_g^Z) \circ ((R_g^X)^{-1} \otimes (R_g^Y)^{-1} \otimes \mathrm{Id}) \circ ((J^F_{T_g(X),T_g(Y)})^{-1} \otimes \mathrm{Id}) \circ \cdots$$
$$(F(\mu_g^{-1}(X,Y)) \otimes \mathrm{Id}) \circ (R_g^{X\otimes Y} \otimes \mathrm{Id}) \circ (J^F_{X,Y} \otimes \mathrm{Id}) = \quad (C.39)$$

$$(R_g^X \otimes R_g^Y \otimes R_g^Z)^{-1} \circ (\mathrm{Id} \otimes (J^F_{T_g(Y),T_g(Z)})^{-1}) \circ (J^F_{T_g(X),T_g(Y)\otimes T_g(Z)})^{-1} \circ F(\Phi_{T_g(X),T_g(Y),T_g(Z)}) \circ J^F_{T_g(X)\otimes T_g(Y),T_g(Z)} \cdots$$
$$\cdots \circ (\mathrm{Id} \otimes \mathrm{Id} \otimes R_g^Z) \circ (F(\mu_g^{-1}(X,Y)) \otimes \mathrm{Id}) \circ (R_g^{X\otimes Y} \otimes \mathrm{Id}) \circ (J^F_{X,Y} \otimes \mathrm{Id}) = \quad (C.40)$$

$$(R_g^X \otimes R_g^Y \otimes R_g^Z)^{-1} \circ (\mathrm{Id} \otimes (J^F_{T_g(Y),T_g(Z)})^{-1}) \circ (J^F_{T_g(X),T_g(Y)\otimes T_g(Z)})^{-1} \circ F(\Phi_{T_g(X),T_g(Y),T_g(Z)} \circ (\mu_g^{-1}(X,Y) \otimes \mathrm{Id})) \circ \cdots$$
$$\cdots J^F_{T_g(X\otimes Y),T_g(Z)} \circ (\mathrm{Id} \otimes \mathrm{Id} \otimes R_g^Z) \circ (R_g^{X\otimes Y} \otimes \mathrm{Id}) \circ (J^F_{X,Y} \otimes \mathrm{Id}) = \quad (C.41)$$

On the otherhand notice that $(\Delta \otimes \mathrm{Id})((\overline{\mu}_g)^{-1})_{X,Y,Z}$ equals:

$$((J^F_{X,Y})^{-1} \otimes \mathrm{Id}_Z) \circ ((R_g^{X\otimes Y})^{-1} \otimes (R_g^Z)^{-1}) \circ (J^F_{T_g(X\otimes Y),T_g(Z)})^{-1} \circ F(\mu_g^{-1}(X\otimes Y, Z)) \circ R_g^{(X\otimes Y)\otimes Z} \circ J^F_{X\otimes Y,Z} \circ (J^F_{X,Y} \otimes \mathrm{Id}) \quad (C.42)$$

Therefore, composing Equation (C.41) we obtain:

$$(R_g^X \otimes R_g^Y \otimes R_g^Z)^{-1} \circ (\mathrm{Id} \otimes (J^F_{T_g(Y),T_g(Z)})^{-1}) \circ (J^F_{T_g(X),T_g(Y)\otimes T_g(Z)})^{-1} \circ \cdots$$

$$\cdots F(\Phi_{T_g(X),T_g(Y),T_g(Z)} \circ (\mu_g^{-1}(X,Y) \otimes \mathrm{Id}) \circ \mu_g^{-1}(X \otimes Y, Z)) \circ R_g^{(X\otimes Y)\otimes Z} \circ J^F_{X\otimes Y,Z} \circ (J^F_{X,Y} \otimes \mathrm{Id}) \tag{C.43}$$

Since μ_g is a tensor functor we that for all $X, Y, Z \in \mathcal{C}$:

$$\Phi_{T_g(X),T_g(Y),T_g(Z)} \circ (\mu_g^{-1}(X,Y) \otimes \mathrm{Id}) \circ \mu_g^{-1}(X \otimes Y, Z) = (\mathrm{Id} \otimes \mu_g^{-1}(Y,Z)) \circ \mu_g^{-1}(X, Y \otimes Z) \circ \Phi_{X,Y,Z} \tag{C.44}$$

Therefore, we may reduce Equation (C.43) to:

$$(R_g^X \otimes R_g^Y \otimes R_g^Z)^{-1} \circ (\mathrm{Id} \otimes (J^F_{T_g(Y),T_g(Z)})^{-1}) \circ (J^F_{T_g(X),T_g(Y)\otimes T_g(Z)})^{-1} \circ \cdots \tag{C.45}$$

$$\cdots F((\mathrm{Id} \otimes \mu_g^{-1}(Y,Z)) \circ \mu_g^{-1}(X, Y \otimes Z) \circ \Phi_{X,Y,Z}) \circ R_g^{(X\otimes Y)\otimes Z} \circ J^F_{X\otimes Y,Z} \circ (J^F_{X,Y} \otimes \mathrm{Id}) = \tag{C.46}$$

$$(R_g^X \otimes R_g^Y \otimes R_g^Z)^{-1} \circ (\mathrm{Id} \otimes (J^F_{T_g(Y),T_g(Z)})^{-1}) \circ \mathrm{Id} \otimes F(\mu_g^{-1}(Y,Z)) \circ \cdots \tag{C.47}$$

$$\cdots (J^F_{T_g(X),T_g(Y\otimes Z)})^{-1} \circ F(\mu_g^{-1}(X, Y \otimes Z) \circ \Phi_{X,Y,Z}) \circ R_g^{(X\otimes Y)\otimes Z} \circ J^F_{X\otimes Y,Z} = \tag{C.48}$$

$$(R_g^X \otimes R_g^Y \otimes R_g^Z)^{-1} \circ (\mathrm{Id} \otimes (J^F_{T_g(Y),T_g(Z)})^{-1}) \circ \mathrm{Id} \otimes F(\mu_g^{-1}(Y,Z)) \circ \cdots \tag{C.49}$$

$$\cdots (J^F_{T_g(X),T_g(Y\otimes Z)})^{-1} \circ F(\mu_g^{-1}(X, Y\otimes Z)) \circ R_g^{X\otimes(Y\otimes Z)} \circ F(\Phi_{X,Y,Z}) \circ J^F_{X\otimes Y,Z} \circ (J^F_{X,Y}\otimes \mathrm{Id}) \tag{C.50}$$

But notice that $(\mathrm{Id}\otimes(\overline{\mu}_g)^{-1})_{X,Y,Z}$ equals:

$$(\mathrm{Id}\otimes(R_g^Y)^{-1}\otimes(R_g^Z)^{-1})\circ(\mathrm{Id}\otimes(J^F_{T_g(Y),T_g(Z)})^{-1})\circ(\mathrm{Id}\otimes F(\mu_g^{-1}(Y,Z)))\circ(\mathrm{Id}\otimes R_g^{Y\otimes Z})\circ(\mathrm{Id}\otimes J^F_{Y,Z}) \tag{C.51}$$

Additionally, $(\mathrm{Id}\otimes\Delta)((\overline{\mu}_g)^{-1})_{X,Y,Z}$ equals:

$$(\mathrm{Id}\otimes(J^F_{Y,Z})^{-1})\circ((R_g^X)^{-1}\otimes(R_g^{Y\otimes Z})^{-1})\circ(J^F_{T_g(X),T_g(Y\otimes Z)})^{-1}\circ F(\mu_g^{-1}(X,Y\otimes Z))\circ R_g^{X\otimes(Y\otimes Z)}\circ J^F_{X,Y\otimes Z}\circ(\mathrm{Id}\otimes J^F_{Y,Z}) \tag{C.52}$$

Therefore, we see that that by composing Equations (C.51), (C.52), (C.37) together we will obtain Equation (C.50). So indeed Equation (C.36) holds.

$\overline{\mu}_g$ Satisfies Equation (4.14)

We need to verify that:

$$(\epsilon\otimes \mathrm{Id}_H)(\overline{\mu}_g) = (\mathrm{Id}_H\otimes\epsilon)(\overline{\mu}_g) = 1_H \tag{C.53}$$

Expanding $(\epsilon\otimes \mathrm{Id}_H)(\overline{\mu}_g)$ we obtain:

$$(J^F_{1_{\mathcal{C}},Y})^{-1} \circ (R_g^{1_{\mathcal{C}}\otimes Y})^{-1} \circ F(\mu_g(1_{\mathcal{C}}, Y)) \circ J^F_{1_{\mathcal{C}},T_g(Y)} \circ (R_g^{1_{\mathcal{C}}} \otimes R_g^Y) = \tag{C.54}$$

But, $J^F_{1_\mathcal{C},Y} = \ell^{\text{Vect}}_{F(Y)} \circ ((\phi^F)^{-1} \otimes \text{Id}_{F(Y)})$, and so we obtain:

$$(\phi^F \otimes \text{Id}_{F(Y)}) \circ (\ell^{\text{Vect}}_{F(Y)})^{-1} \circ (R^Y_g)^{-1} \circ \ell^{\text{Vect}}_{F(Y)} \circ ((\phi^F)^{-1} \otimes \text{Id}_{F(Y)})(R^{1_\mathcal{C}}_g \otimes R^Y_g) = \tag{C.55}$$

$$(\phi^F \otimes \text{Id}_{F(Y)}) \circ (\text{Id}_\mathbb{C} \otimes (R^Y_g)^{-1}) \circ ((\phi^F)^{-1} \otimes \text{Id}_{F(Y)})(R^{1_\mathcal{C}}_g \otimes R^Y_g) = \tag{C.56}$$

$$(R^{1_\mathcal{C}}_g \otimes \text{Id}_{F(Y)}) = \text{Id}_{F(1_\mathcal{C})} \otimes \text{Id}_{F(Y)} \tag{C.57}$$

This proves that $(\epsilon \otimes \text{Id}_H)(\overline{\mu}_g) = 1_H$ (note we are slightly abusing notation). Similarly, one can prove that $(\text{Id}_H \otimes \epsilon)(\overline{\mu}_g) = 1_H$.

$\overline{\mu}_g, \underline{\gamma}$ Satisfy Equation (4.15)

Recall that We need to verify that for all $g, k \in G$

$$(\underline{\gamma}_{\ k^{-1},g^{-1}} \otimes \underline{\gamma}_{\ k^{-1},g^{-1}}) = (\overline{\mu}_{gk})^{-1} \cdot \Delta(\underline{\gamma}_{\ k^{-1},g^{-1}}) \cdot \overline{\mu}_k \cdot \Psi(k^{-1})^{\otimes 2}(\overline{\mu}_g) \tag{C.58}$$

Expanding $\overline{\mu}_{gk} \cdot (\underline{\gamma}_{\ k^{-1},g^{-1}} \otimes \underline{\gamma}_{\ k^{-1},g^{-1}})$ at $X, Y \in \mathcal{C}$ we obtain:

$$(J^F_{X,Y})^{-1} \circ (R^{X\otimes Y}_{gk})^{-1} \circ F(\mu_{gk}(X,Y)) \circ J^F_{T_{gk}(X),T_{gk}(Y)} \circ (R^X_{gk} \otimes R^Y_{gk}) \circ \tag{C.59}$$

$$\cdots (R^X_{gk} \otimes R^Y_{gk})^{-1} \circ (F(\gamma_{g,k}(X)) \otimes F(\gamma_{g,k}(Y))) \circ (R^{T_k(X)}_g \otimes R^{T_k(Y)}_g) \circ (R^X_k \otimes R^Y_k) =$$

$$(J^F_{X,Y})^{-1} \circ (R^{X\otimes Y}_{gk})^{-1} \circ F(\mu_{gk}(X,Y)) \circ (\gamma_{g,k}(X) \otimes \gamma_{g,k}(Y))) \circ \cdots \tag{C.60}$$

$$\cdots J^F_{(T_g\circ T_k)(X),(T_g\circ T_k)(Y)} \circ (R^{T_k(X)}_g \otimes R^{T_k(Y)}_g) \circ (R^X_k \otimes R^Y_k) =$$

Since $\gamma_{g,k} : T_g \circ T_k \to T_{gk}$ is a monoidal natural isomorphism we have that:

$$\mu_{gk}(X,Y) \circ (\gamma_{g,k}(X) \otimes \gamma_{g,k}(Y)) = \gamma_{g,k}(X \otimes Y) \circ T_g(\mu_k(X,Y)) \circ \mu_g(T_k(X), T_k(Y)) \tag{C.61}$$

Therefore, we may simplify Equation (C.60) to:

$$(J^F_{X,Y})^{-1} \circ (R^{X\otimes Y}_{gk})^{-1} \circ F(\gamma_{g,k}(X \otimes Y) \circ T_g(\mu_k(X,Y)) \circ \mu_g(T_k(X), T_k(Y))) \circ \cdots$$
$$\cdots J^F_{(T_g\circ T_k)(X),(T_g\circ T_k)(Y)} \circ (R^{T_k(X)}_g \otimes R^{T_k(Y)}_g) \circ (R^X_k \otimes R^Y_k) = \tag{C.62}$$

$$(J^F_{X,Y})^{-1} \circ (R^{X\otimes Y}_{gk})^{-1} \circ F(\gamma_{g,k}(X \otimes Y)) \circ R^{T_k(X\otimes Y)}_g \circ R^{X\otimes Y}_k \circ \cdots$$
$$\cdots (R^{X\otimes Y}_k)^{-1} \circ (R^{T_k(X\otimes Y)}_g)^{-1} F(T_g(\mu_k(X,Y))) \circ F(\mu_g(T_k(X), T_k(Y))) \circ \cdots$$
$$\cdots J^F_{(T_g\circ T_k)(X),(T_g\circ T_k)(Y)} \circ (R^{T_k(X)}_g \otimes R^{T_k(Y)}_g) \circ (R^X_k \otimes R^Y_k) = \tag{C.63}$$

$$(J^F_{X,Y})^{-1} \circ (R^{X\otimes Y}_{gk})^{-1} \circ F(\gamma_{g,k}(X \otimes Y)) \circ R^{T_k(X\otimes Y)}_g \circ R^{X\otimes Y}_k \circ \cdots$$
$$\cdots (R^{X\otimes Y}_k)^{-1} \circ F(\mu_k(X,Y)) \circ (R^{T_k(X)\otimes T_k(Y)}_g)^{-1} \circ F(\mu_g(T_k(X), T_k(Y))) \circ \cdots$$
$$\cdots J^F_{(T_g\circ T_k)(X),(T_g\circ T_k)(Y)} \circ (R^{T_k(X)}_g \otimes R^{T_k(Y)}_g) \circ (R^X_k \otimes R^Y_k) = \tag{C.64}$$

$$(J^F_{X,Y})^{-1} \circ (R^{X\otimes Y}_{gk})^{-1} \circ F(\gamma_{g,k}(X \otimes Y)) \circ R^{T_k(X\otimes Y)}_g \circ R^{X\otimes Y}_k \circ \cdots$$

$$\cdots (R^{X\otimes Y}_k)^{-1} \circ F(\mu_k(X,Y)) \circ J^F_{T_k(X),T_k(Y)} \circ (R^X_k \otimes R^Y_k) \circ \cdots$$

$$\cdots (R^X_k \otimes R^Y_k)^{-1} \circ (J^F_{T_k(X),T_k(Y)})^{-1} \circ (R^{T_k(X)\otimes T_k(Y)}_g)^{-1} \circ \cdots$$

$$\cdots F(\mu_g(T_k(X), T_k(Y))) \circ J^F_{(T_g\circ T_k)(X),(T_g\circ T_k)(Y)} \circ (R^{T_k(X)}_g \otimes R^{T_k(Y)}_g) \circ (R^X_k \otimes R^Y_k) = \tag{C.65}$$

$$\Delta(\underline{\gamma}_{\ k^{-1},g^{-1}})_{X,Y} \circ (\overline{\mu}_k)_{X,Y}((\overline{\mu}_g)^{(k,k)})_{X,Y} \tag{C.66}$$

Therefore, we are done.

C.3 Verifying G-Crossed Axioms and G-Ribbon Axioms

As the G-crossed axioms are verified in the exact same way as the last section, we leave their explicit verification out of this thesis and leave the details to the interested reader. For the G-ribbon axioms we prove the most technically challenging axiom, Equation (4.72), and then leave the detailed verifications of the other axioms to the interested reader.

Recall that for $X \in \mathcal{C}_g$

$$\nu_X := (R^X_g)^{-1} \circ F(\theta_X) \tag{C.67}$$

ν **Satsifies** (4.72)

First, it can be checked that for $X \in \mathcal{C}_g$:

$$(\mathbf{d}_g)_X = d_X^* \circ (R_g^{X^*})^* \circ (F((d_X^{T_g}))^*) \circ (d_{T_g(X)}^*)^{-1} \circ R_g^X \tag{C.68}$$

We need to verify that for all $g \in G$

$$S(\nu) \cdot \overline{\partial}(\delta_g) = S(\underline{\gamma}^{-1}_{\ g^{-1},g}) \cdot \mathbf{d}^g \cdot \nu \cdot \overline{\partial}(\delta_g) \tag{C.69}$$

Expanding the left hand side at $X \in \mathcal{C}_g$ we obtain:

$$d_X^* \circ F(\theta_{X^*})^* \circ ((R_{g^{-1}}^{X^*})^{-1})^* \circ (d_X^*)^{-1} \circ F(\pi_g(X)) \tag{C.70}$$

On the other hand notice that $S(\underline{\gamma}_{\ g^{-1},g})_X$ equals:

$$d_X^* \circ F(\gamma_{g^{-1},g}(X^*)^{-1})^* \circ ((R_{g^{-1}}^{T_g(X^*)})^{-1})^* \circ ((R_g^{X^*})^{-1})^* \circ (d_X^*)^{-1} \tag{C.71}$$

Combining this with $(\mathbf{d}_g)_X$ we obtain:

$$d_X^* \circ F(\gamma_{g^{-1},g}(X^*)^{-1})^* \circ ((R_{g^{-1}}^{T_g(X^*)})^{-1})^* \circ ((R_g^{X^*})^{-1})^* \circ (d_X^*)^{-1} \circ d_X^* \circ \cdots \tag{C.72}$$

$$\cdots (R_g^{X^*})^* \circ (F(d_X^{T_g})^*) \circ (d_{T_g(X)}^*)^{-1} \circ (R_g^X) =$$

$$d_X^* \circ F(\gamma_{g^{-1},g}(X^*)^{-1})^* \circ ((R_{g^{-1}}^{T_g(X^*)})^{-1})^* \circ (F(d_X^{T_g})^*) \circ (d_{T_g(X)}^*)^{-1} \circ (R_g^X) \tag{C.73}$$

Composing this with ν_X we obtain:

$$d_X^* \circ F(\gamma_{g^{-1},g}(X^*)^{-1})^* \circ ((R_{g^{-1}}^{T_g(X^*)})^{-1})^* \circ (F(d_X^{T_g})^*) \circ (d_{T_g(X)}^*)^{-1} \circ (R_g^X) \circ (R_g^X)^{-1} \circ F(\theta_X) = \tag{C.74}$$

$$d_X^* \circ F(\gamma_{g^{-1},g}(X^*)^{-1})^* \circ ((R_{g^{-1}}^{T_g(X^*)})^{-1})^* \circ (F(d_X^{T_g})^*) \circ (d_{T_g(X)}^*)^{-1} \circ F(\theta_X) = \tag{C.75}$$

$$d_X^* \circ F(\gamma_{g^{-1},g}(X^*)^{-1})^* \circ ((R_{g^{-1}}^{T_g(X^*)})^{-1})^* \circ (F(d_X^{T_g})^*) \circ F(\theta_X^*)^* \circ (d_X^*)^{-1} = \tag{C.76}$$

$$d_X^* \circ F(\gamma_{g^{-1},g}(X^*)^{-1})^* \circ ((R_{g^{-1}}^{T_g(X^*)})^{-1})^* \circ (F(\theta_X^* \circ d_X^{T_g})^*) \circ (d_X^*)^{-1} = \tag{C.77}$$

$$d_X^* \circ F(\gamma_{g^{-1},g}(X^*)^{-1})^* \circ F(T_{g^{-1}}(\theta_X^* \circ d_X^{T_g}))^* \circ ((R_{g^{-1}}^{X^*})^{-1})^*(d_X^*)^{-1} = \tag{C.78}$$

$$d_X^* \circ F(T_{g^{-1}}(\theta_X^* \circ d_X^{T_g}) \circ \gamma_{g^{-1},g}(X^*)^{-1})^* \circ ((R_{g^{-1}}^{X^*})^{-1})^*(d_X^*)^{-1} = \tag{C.79}$$

By assumption we have that:

$$\theta_{X^*} = T_{g^{-1}}(\theta_X^* \circ d_X^{T_g}) \circ \gamma_{g^{-1},g}(X^*)^{-1} \tag{C.80}$$

Therefore, we may simplify to:

$$d_X^* \circ ((R_{g^{-1}}^{X^*})^{-1} \circ F(\theta_{X^*}))^* \circ (d_X^*)^{-1} = S(\nu)_X \tag{C.81}$$

for $X \in \mathcal{C}_g$. Hence ν will satisfy Equation (4.72).

Appendix D

The Hopf Equivariantization Theorem: The Proofs

Notation. *For $g, k \in G$:*

$$\overline{\gamma}_{g,k} := \underline{\gamma}_{\,k^{-1},g^{-1}} \tag{D.1}$$

Multiplication is Associative and Unital

First, let's verify that multiplication is associative. Let $r, h, t \in H, g, k, \ell \in G$ then:

$$((r \otimes g) \cdot (h \otimes k)) \cdot (t \otimes \ell) = (\overline{\gamma}_{g,k} \cdot \Psi(k^{-1})(r) \cdot h \cdot t \otimes gk) \cdot (t \otimes \ell) = \tag{D.2}$$

$$(\overline{\gamma}_{gk,\ell} \cdot \Psi(\ell^{-1})(\overline{\gamma}_{g,k} \cdot \Psi(k^{-1})(r) \cdot h) \cdot t \otimes (gk)\ell) \tag{D.3}$$

On the other hand we know from Equation (4.3) that:

$$\overline{\gamma}_{gk,\ell} \cdot \Psi(\ell^{-1})(\overline{\gamma}_{g,k}) = \overline{\gamma}_{g,k\ell} \cdot \overline{\gamma}_{k,\ell} \tag{D.4}$$

Therefore, we may simplify Equation (D.3) to:

$$((\overline{\gamma}_{g,k\ell} \cdot \overline{\gamma}_{k,\ell} \cdot (\Psi(\ell^{-1}) \circ \Psi(k^{-1})(r) \cdot \Psi(\ell^{-1})(h) \cdot t) \otimes (gk)\ell) \tag{D.5}$$

By Equation (4.2) we know that:

$$(\Psi(\ell^{-1}) \circ \Psi(k^{-1}))(r) = \overline{\gamma}_{k,\ell}^{-1} \cdot (\Psi((k\ell)^{-1})(r) \cdot \overline{\gamma}_{k,\ell} \tag{D.6}$$

Therefore, we can simplify Equation (D.5) to:

$$(\overline{\gamma}_{g,k\ell} \cdot \Psi((k\ell)^{-1})(r) \cdot \overline{\gamma}_{k,\ell} \cdot \Psi(\ell^{-1})(h) \cdot t) \otimes (gk\ell) = \tag{D.7}$$

$$(r \otimes g) \cdot (\overline{\gamma}_{k,\ell} \cdot \Psi(\ell^{-1})(h) \cdot t) \otimes (k\ell)) = \tag{D.8}$$

$$(r \otimes g) \cdot ((h \otimes k) \cdot (t \otimes \ell)) \tag{D.9}$$

Therefore, multiplication is associative. Since we have chosen a normalzied cocycle, we see that $1_H \otimes e$ will indeed by the unit with respect to this multiplication.

Co-multiplication is an Algebra Homomorphism

Next, we need to verify that Δ is a $\mathbb{C}$-algebra homomorphism. To that end let $h, r \in H, g, k \in G$, then:

$$\Delta_{H\#_\Gamma\mathbb{C}[G]}(h \otimes g) \cdot \Delta_{H\#_\Gamma\mathbb{C}[G]}(r \otimes k) = \tag{D.10}$$

$$((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)} \otimes g) \otimes ((\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)} \otimes g) \cdot ((\overline{\mu}_k{}^{-1})_1 \cdot r_{(1)} \otimes k) \otimes ((\overline{\mu}_k{}^{-1})_2 \cdot r_{(2)} \otimes k) = \tag{D.11}$$

$$(\overline{\gamma}_{g,k} \cdot \Psi(k^{-1})((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)}) \cdot (\overline{\mu}_k{}^{-1})_1 \cdot r_{(1)} \otimes gk) \otimes (\overline{\gamma}_{g,k} \cdot \Psi(k^{-1})((\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)}) \cdot (\overline{\mu}_k{}^{-1})_2 \cdot r_{(2)} \otimes gk) \tag{D.12}$$

On the other hand by Equation (4.12), the fact that Δ is an algebra homorphism, and since for all $k \in G$ $\Psi(k^{-1})$ is a unital automorphism that for all $k \in G, h \in H$ the following holds:

$$\overline{\mu}_k \cdot (\Psi(k^{-1})^{\otimes 2})(\Delta(h)) = (\Delta(\Psi(k^{-1})(h))) \cdot \overline{\mu}_k \Rightarrow \tag{D.13}$$

$$\Delta(1) \cdot (\Psi(k^{-1})^{\otimes 2})(\Delta(h)) \cdot \overline{\mu}_k{}^{-1} = \overline{\mu}_k{}^{-1} \cdot (\Delta(\Psi(k^{-1})(h))) \cdot \Delta(1) \Rightarrow \tag{D.14}$$

$$(\Psi(k^{-1})^{\otimes 2})(\Delta(h)) \cdot \overline{\mu}_k{}^{-1} = \overline{\mu}_k{}^{-1} \cdot (\Delta(\Psi(k^{-1})(h))) \tag{D.15}$$

Therefore, by using Equation (D.15) we may simplify Equation (D.12) to:

$$(\overline{\gamma}_{g,k} \cdot \Psi(k^{-1})((\overline{\mu}_g{}^{-1})_1) \cdot (\overline{\mu}_k{}^{-1})_1 \cdot (\Psi(k^{-1})(h))_{(1)} \cdot r_{(1)} \otimes gk) \otimes \cdots \tag{D.16}$$

$$\cdots (\overline{\gamma}_{g,k} \cdot \Psi(k^{-1})((\overline{\mu}_g{}^{-1})_2) \cdot (\overline{\mu}_k{}^{-1})_2 \cdot (\Psi(k^{-1})(h))_{(2)} \cdot r_{(2)} \otimes gk) \tag{D.17}$$

By Equation (4.15) we know that for all $g, k \in G$:

$$(\overline{\gamma}_{g,k} \otimes \overline{\gamma}_{g,k}) = (\overline{\mu}_{gk})^{-1} \cdot \Delta(\overline{\gamma}_{g,k}) \cdot \overline{\mu}_k \cdot \Psi(k^{-1})(\overline{\mu}_g) \tag{D.18}$$

We can simplify Equation (D.18) to:

$$(\overline{\gamma}_{g,k} \otimes \overline{\gamma}_{g,k}) \cdot \Psi(k^{-1})((\overline{\mu}_g)^{-1}) \cdot (\overline{\mu}_k)^{-1} = (\overline{\mu}_{gk})^{-1} \cdot \Delta(\overline{\gamma}_{g,k}) \tag{D.19}$$

Using Equation (D.19) we may simplify Equation (D.17) to:

$$((\overline{\mu}_{gk}^{-1})_1 \cdot (\overline{\gamma}_{g,k})_{(1)} \cdot (\Psi(k^{-1})(h))_{(1)} \cdot r_{(1)} \otimes gk) \otimes ((\overline{\mu}_{gk}^{-1})_2 \cdot (\overline{\gamma}_{g,k})_{(2)} \cdot (\Psi(k^{-1})(h))_{(2)} \cdot r_{(2)} \otimes gk) = \tag{D.20}$$

$$((\overline{\mu}_{gk}^{-1})_1 \cdot (\overline{\gamma}_{g,k} \cdot (\Psi(k^{-1})(h) \cdot r)_{(1)} \otimes gk) \otimes ((\overline{\mu}_{gk}^{-1})_2 \cdot (\overline{\gamma}_{g,k} \cdot (\Psi(k^{-1})(h)) \cdot r)_{(2)} \otimes gk) = \tag{D.21}$$

$$\Delta_{H\#_\Gamma \mathbb{C}[G]}(\overline{\gamma}_{g,k} \cdot (\Psi(k^{-1})(h) \cdot r) \otimes gk) = \Delta_{H\#_\Gamma \mathbb{C}[G]}((h \otimes g) \cdot (r \otimes k)) \tag{D.22}$$

Therefore, Δ will indeed by an algebra homomorphism.

Co-unit Axioms

Next, we need to verify that $\epsilon_{H\#_\Gamma \mathbb{C}[G]}$ is an algebra-homomorphism. By Remark 4.1.1 we know for $g, k \in G$:

$$\epsilon(\overline{\gamma}_{g,k}) = 1 \tag{D.23}$$

Therefore, we see that for $g, k \in G, h, r \in H$:

$$\epsilon_{H\#_\Gamma \mathbb{C}[G]}((h \otimes g) \cdot (r \otimes k)) = \epsilon_{H\#_\Gamma \mathbb{C}[G]}((\overline{\gamma}_{g,k} \cdot \Psi(k^{-1})(h) \cdot r) \otimes gk) = \tag{D.24}$$

$$\epsilon_H((\overline{\gamma}_{g,k} \cdot \Psi(k^{-1})(h) \cdot r)) = \epsilon_H(\Psi(k^{-1})(h) \cdot r) = \epsilon_H(\Psi(k^{-1})(h))\epsilon_H(r) = \tag{D.25}$$

$$\epsilon_H(h)\epsilon_H(r) = \epsilon_{H\#_\Gamma\mathbb{C}[G]}(h \otimes g)\epsilon_{H\#_\Gamma\mathbb{C}[G]}(r \otimes k) \tag{D.26}$$

Next, we need to verify that:

$$(\epsilon_{H\#_\Gamma\mathbb{C}[G]} \otimes \mathrm{Id}) \circ \Delta_{H\#_\Gamma\mathbb{C}[G]} = (\mathrm{Id} \otimes \epsilon_{H\#_\Gamma\mathbb{C}[G]}) \circ \Delta_{H\#_\Gamma\mathbb{C}[G]} = \mathrm{Id} \tag{D.27}$$

To that end let $g \in G, h \in H$:

$$(\epsilon_{H\#_\Gamma\mathbb{C}[G]} \otimes \mathrm{Id}) \circ \Delta_{H\#_\Gamma\mathbb{C}[G]}(h \otimes g) = \tag{D.28}$$

$$(\epsilon_{H\#_\Gamma\mathbb{C}[G]} \otimes \mathrm{Id})(((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)} \otimes g) \otimes ((\overline{\mu}\ ^{-1})_2 \cdot h_{(2)} \otimes g)) = \tag{D.29}$$

$$\epsilon_H((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)})((\overline{\mu}\ ^{-1})_2 \cdot h_{(2)} \otimes g)) = \tag{D.30}$$

$$((\epsilon_H((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)})(\overline{\mu}\ ^{-1})_2 \cdot h_{(2)}) \otimes g = \tag{D.31}$$

$$h \otimes g \tag{D.32}$$

Notice in the last line we use the fact that $(\epsilon_H \otimes \mathrm{Id}) \circ \Delta_H = \mathrm{Id}_H$, and the fact that we have a normalized 3-cocycle so:

$$\epsilon_H((\overline{\mu}_g{}^{-1})_1)(\overline{\mu}_g{}^{-1})_2 = (\epsilon_H((\overline{\mu}_g)_1)(\overline{\mu}_g)_2)\ ^{-1} = 1_H \tag{D.33}$$

Similarly, one can check that $(\mathrm{Id} \otimes \epsilon_{H\#_\Gamma\mathbb{C}[G]}) \circ \Delta_{H\#_\Gamma\mathbb{C}[G]} = \mathrm{Id}_H$.

Drinfeld Associator Identities

Now, we need to verify the associator identities. First, notice that we have a wqhf morphism $\iota_{H,\Gamma} : H \to H\#_\Gamma\mathbb{C}[G]$ given by $\iota_{H,\Gamma}(h) = h \otimes e$.

The first things to verify is that $\Phi_{H\#_\Gamma\mathbb{C}[G]}$ is partially invertible with:

$$D(\Phi_{H\#_\Gamma\mathbb{C}[G]}) = (\Delta_{H\#_\Gamma\mathbb{C}[G]}\otimes\mathrm{Id})\circ\Delta_{H\#_\Gamma\mathbb{C}[G]}(1) \quad R(\Phi_{H\#_\Gamma\mathbb{C}[G]}) = (\mathrm{Id}\otimes\Delta_{H\#_\Gamma\mathbb{C}[G]})\circ\Delta_{H\#_\Gamma\mathbb{C}[G]}(1) \tag{D.34}$$

To that end, notice that:

$$\Phi_{H\#_\Gamma\mathbb{C}[G]} = (\iota_{H,\Gamma}^{\otimes 3})(\Phi_H) \tag{D.35}$$

Therefore, let $\Phi^{-1}_{H\#_\Gamma\mathbb{C}[G]} := (\iota_{H,\Gamma}^{\otimes 3})(\Phi_H^{-1})$. Then:

$$\Phi^{-1}_{H\#_\Gamma\mathbb{C}[G]} \cdot \Phi_{H\#_\Gamma\mathbb{C}[G]} = \tag{D.36}$$

$$(\iota_{H,\Gamma}^{\otimes 3})(\Phi_H^{-1} \cdot \Phi_H) = \tag{D.37}$$

$$(\iota_{H,\Gamma}^{\otimes 3})((\Delta_H \otimes \mathrm{Id}) \circ \Delta(1_H)) \tag{D.38}$$

Since $\iota_{H,\Gamma}$ is a morphism of weak quasi Hopf algebras we have that this equals:

$$(\Delta_{H\#_\Gamma\mathbb{C}[G]} \otimes \mathrm{Id}) \circ \Delta_{H\#_\Gamma\mathbb{C}[G]}(1_{H\#_\Gamma\mathbb{C}[G]}) \tag{D.39}$$

Similarly, one can check that:

$$\Phi_{H\#_\Gamma\mathbb{C}[G]} \cdot \Phi^{-1}_{H\#_\Gamma\mathbb{C}[G]} = (\mathrm{Id} \otimes \Delta_{H\#_\Gamma\mathbb{C}[G]}) \circ \Delta_{H\#_\Gamma\mathbb{C}[G]}(1_{H\#_\Gamma\mathbb{C}[G]}) \tag{D.40}$$

Next, we need to verify that for all $h \in H, g \in G$ that:

$$\Phi_{H\#_\Gamma \mathbb{C}[G]} \cdot (\Delta_{H\#_\Gamma \mathbb{C}[G]} \otimes \mathrm{Id})(\Delta_{H\#_\Gamma \mathbb{C}[G]}(h \otimes g)) = (\mathrm{Id} \otimes \Delta_{H\#_\Gamma \mathbb{C}[G]})(\Delta_{H\#_\Gamma \mathbb{C}[G]}(h \otimes g)) \cdot \Phi_{H\#_\Gamma \mathbb{C}[G]} \tag{D.41}$$

Expanding the left hand side of Equation (D.41) we obtain:

$$\Phi_{H\#_\Gamma \mathbb{C}[G]} \cdot ((\overline{\mu}_g{}^{-1})_1 \cdot ((\overline{\mu}_g{}^{-1})_1 \cdot r_{(1)})_{(1)} \otimes g) \otimes ((\overline{\mu}_g{}^{-1})_2 \cdot ((\overline{\mu}_g{}^{-1})_1 \cdot r_{(1)})_{(2)} \otimes g)) \otimes \cdots$$
$$\cdots \otimes ((\overline{\mu}_g{}^{-1})_2 \cdot r_{(2)} \otimes g) \tag{D.42}$$

By manipulating Equation (4.13) we know that:

$$(\Psi(g^{-1})^{\otimes 3})(\Phi_H) \cdot ((\overline{\mu}_g)^{-1} \otimes 1_H) \cdot (\Delta_H \otimes \mathrm{Id})((\overline{\mu}_g)^{-1}) = (1_H \otimes (\overline{\mu}_g)^{-1}) \cdot (\mathrm{Id} \otimes \Delta_H)((\overline{\mu}_g)^{-1}) \cdot \Phi_H \tag{D.43}$$

If we expand $\Phi_{H\#_\Gamma \mathbb{C}[G]}$ in Equation (D.42) and multiply, then Φ_H will pick up the action $(\Psi(g^{-1})^{\otimes 3})$ by definition. Therefore, we may use Equation (D.43) to simplify Equation (D.42) to:

$$((\overline{\mu}_g{}^{-1})_1 \cdot x_{(1)} \cdot (r_{(1)})_{(1)} \otimes g) \otimes ((\overline{\mu}_g{}^{-1})_1 \cdot ((\overline{\mu}_g{}^{-1})_2)_{(1)} \cdot x_{(2)} \cdot (r_{(1)})_{(2)} \otimes g)) \otimes ((\overline{\mu}_g{}^{-1})_2 \cdot ((\overline{\mu}_g{}^{-1})_2)_{(2)} \cdot x_{(3)} \cdot r_{(2)} \otimes g) \tag{D.44}$$

By Equation (3.25) we know that:

$$(x_{(1)} \cdot (r_{(1)})_{(1)}) \otimes (x_{(2)} \cdot (r_{(1)})_{(2)}) \otimes (x_{(3)} \cdot r_{(2)}) = (r_{(1)} \cdot x_{(1)}) \otimes ((r_{(2)})_{(1)} \cdot x_{(2)}) \otimes ((r_{(2)})_{(2)} \cdot x_{(3)}) \tag{D.45}$$

Therefore, we may simplify Equation (D.44) to:

$$((\overline{\mu}_g{}^{-1})_1 \cdot r_{(1)} \cdot x_{(1)} \otimes g) \otimes ((\overline{\mu}_g{}^{-1})_1 \cdot ((\overline{\mu}_g{}^{-1})_2)_{(1)} \cdot (r_{(2)})_{(1)} \cdot x_{(2)} \otimes g)) \otimes \cdots \tag{D.46}$$

$$\cdots ((\overline{\mu}_g{}^{-1})_2 \cdot ((\overline{\mu}_g{}^{-1})_2)_{(2)} \cdot (r_{(2)})_{(2)} \cdot x_{(3)} \otimes g) =$$

$$((\overline{\mu}_g{}^{-1})_1 \cdot r_{(1)} \cdot x_{(1)} \otimes g) \otimes ((\overline{\mu}_g{}^{-1})_1 \cdot ((\overline{\mu}_g{}^{-1})_2 \cdot r_{(2)})_{(1)} \cdot x_{(2)} \otimes g)) \otimes \cdots \tag{D.47}$$

$$\cdots ((\overline{\mu}_g{}^{-1})_2 \cdot ((\overline{\mu}_g{}^{-1})_2 \cdot r_{(2)})_{(2)} \cdot x_{(3)} \otimes g) =$$

$$(\mathrm{Id} \otimes \Delta_{H\#_\Gamma \mathbb{C}[G]}) \circ \Delta_{H\#_\Gamma \mathbb{C}[G]}(r \otimes g) \cdot \Phi_{H\#_\Gamma \mathbb{C}[G]} \tag{D.48}$$

Therefore, Equation (D.41) will hold. The other axioms invovlving the associator, that is Equations (3.26) (3.27), will follow form the fact that they hold for Φ_H and that $\iota_{H,\Gamma}$ is an embedding of wqhf.

To summarize so far we have shown that for a nomralized G-ribbon Abelian 3-cocycle Γ that $H\#_\Gamma \mathbb{C}[G]$ is a weak quasi bi-algebra.

Next, we will verify that it is a weak quasi Hopf algebra.

First, we need to verify that for $h \in H, g \in G$:

$$S_{H\#_\Gamma \mathbb{C}[G]}((h \otimes g)_{(1)}) \cdot \alpha_{H\#_\Gamma \mathbb{C}[G]} \cdot (h \otimes g)_{(2)} = \epsilon_{H\#_\Gamma \mathbb{C}[G]}(h \otimes g)\alpha_{H\#_\Gamma \mathbb{C}[G]} \tag{D.49}$$

To that end notice that:

$$(h \otimes g)_{(1)} = ((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)}) \otimes g \tag{D.50}$$

$$(h \otimes g)_{(2)} = ((\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)}) \otimes g \tag{D.51}$$

Substituting this in Equation (D.49) and expanding we obtain:

$$S_{H\#_\Gamma \mathbb{C}[G]}(((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)}) \otimes g) \cdot \alpha_{H\#_\Gamma \mathbb{C}[G]} \cdot)(((\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)}) \otimes g) = \tag{D.52}$$

$$(\Psi(g)(S((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)}) \cdot \mathbf{d}^g \cdot (\overline{\gamma}_{g^{-1},g})^{-1}) \otimes g^{-1}) \cdot (\alpha_H \otimes e) \cdot (((\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)}) \otimes g) = \tag{D.53}$$

$$(\Psi(g)(S((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)}) \cdot \mathbf{d}^g \cdot (\overline{\gamma}_{g^{-1},g})^{-1}) \otimes g^{-1}) \cdot (\Psi(g^{-1})(\alpha_H) \cdot (\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)}) \otimes g) = \tag{D.54}$$

$$\overline{\gamma}_{g^{-1},g} \cdot (\Psi(g^{-1}) \circ \Psi(g)) \left(S((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)}) \cdot \mathbf{d}^g \cdot (\overline{\gamma}_{g^{-1},g})^{-1}\right) \cdot \Psi(g^{-1})(\alpha_H) \cdot ((\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)}) \otimes e \tag{D.55}$$

Since $(\Psi(g^{-1}) \circ \Psi(g)) = \mathrm{Ad}(\overline{\gamma}^{-1}_{g^{-1},g})$ we see that Equation (D.55) simplifies to:

$$\overline{\gamma}_{g^{-1},g} \cdot \overline{\gamma}^{-1}_{g^{-1},g} \cdot S((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)}) \cdot \mathbf{d}^g \cdot (\overline{\gamma}_{g^{-1},g})^{-1} \cdot \overline{\gamma}_{g^{-1},g} \cdot \Psi(g^{-1})\,(\alpha_H) \cdot ((\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)}) \otimes e = \tag{D.56}$$

$$S((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)}) \cdot \mathbf{d}^g \cdot \Psi(g^{-1})\,(\alpha_H) \cdot ((\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)}) \otimes e = \tag{D.57}$$

$$S(h_{(1)}) \cdot S((\overline{\mu}_g{}^{-1})_1) \cdot \mathbf{d}^g \cdot \Psi(g^{-1})(\alpha_H) \cdot (\overline{\mu}_g{}^{-1})_2 \cdot S(h_{(2)}) \otimes e \tag{D.58}$$

Since we have a rigid G-ribbon 3-cocycle we know that:

$$\mathbf{d}^g \cdot \Psi(g^{-1})(\alpha_H) = S((\overline{\mu}_g)_1) \cdot \alpha_H \cdot (\overline{\mu}_g)_2 \tag{D.59}$$

Therefore, Equation (D.58) simplifies to:

$$S(h_{(1)}) \cdot S((\overline{\mu}_g{}^{-1})_1) \cdot S((\overline{\mu}_g)_1) \cdot \alpha_H \cdot (\overline{\mu}_g)_2 \cdot (\overline{\mu}_g{}^{-1})_2 \cdot S(h_{(2)}) \otimes e = \tag{D.60}$$

$$S(h_{(1)}) \cdot \alpha_H \cdot h_{(2)} \otimes e = \epsilon(h)(\alpha_H \otimes e) = \tag{D.61}$$

$$\epsilon_{H\#_\Gamma \mathbb{C}[G]}(h \otimes g)(\alpha_H \otimes e) \tag{D.62}$$

Now, we need to check that for $h \in H, g \in G$:

$$(h \otimes g)_{(1)} \cdot \beta_{H\#_\Gamma \mathbb{C}[G]} \cdot S_{H\#_\Gamma \mathbb{C}[G]}((h \otimes g)_{(2)}) = \epsilon_{H\#_\Gamma \mathbb{C}[G]}(h \otimes g) \cdot \beta_{H\#_\Gamma \mathbb{C}[G]} \tag{D.63}$$

Expanding this we obtain:

$$((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)} \otimes g) \cdot (\beta_H \otimes e) \cdot (\Psi(g)(S((\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)}) \cdot \mathbf{d}^g \cdot (\overline{\gamma}^{-1}_{g^{-1},g})) \otimes g^{-1}) = \tag{D.64}$$

$$((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)} \cdot \beta_H \otimes g) \cdot (\Psi(g)(S((\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)}) \cdot \mathbf{d}^g \cdot (\overline{\gamma}^{-1}_{g^{-1},g})) \otimes g^{-1}) = \tag{D.65}$$

$$\overline{\gamma}_{g,g^{-1}} \cdot \Psi(g)((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)} \cdot \beta_H \cdot S((\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)}) \cdot \mathbf{d}^g \cdot (\overline{\gamma}^{-1}_{g^{-1},g})) \otimes e = \tag{D.66}$$

$$\overline{\gamma}_{g,g^{-1}} \cdot \Psi(g)((\overline{\mu}_g{}^{-1})_1 \cdot h_{(1)} \cdot \beta_H \cdot S(h_{(2)}) \cdot S((\overline{\mu}_g{}^{-1})_2) \cdot \mathbf{d}^g \cdot (\overline{\gamma}^{-1}_{g^{-1},g})) \otimes e = \tag{D.67}$$

$$\overline{\gamma}_{g,g^{-1}} \Psi(g)((\overline{\mu}_g{}^{-1})_1 \cdot \epsilon(h)\beta_H \cdot S((\overline{\mu}_g{}^{-1})_2) \cdot \mathbf{d}^g \cdot (\overline{\gamma}^{-1}_{g^{-1},g})) \otimes e = \tag{D.68}$$

$$\epsilon(h)\overline{\gamma}_{g,g^{-1}} \cdot \Psi(g)(\Psi(g^{-1})(\beta_H) \cdot \overline{\gamma}^{-1}_{g^{-1},g}) \otimes e \tag{D.69}$$

Notice that because of the 2-cocycle relation we have that:

$$\Psi(g)(\overline{\gamma}_{g^{-1},g}) = \overline{\gamma}_{g,g^{-1}} \tag{D.70}$$

This implies that Equation (D.69) reduces to:

$$\epsilon_{H\#_\Gamma \mathbb{C}[G]}(h \otimes g) \cdot (\beta_H \otimes e) \tag{D.71}$$

Lastly, to verify that $(S_{H\#_\Gamma\mathbb{C}[G]}, \alpha_{H\#_\Gamma\mathbb{C}[G]}, \beta_{H\#_\Gamma\mathbb{C}[G]})$ satisfies Equation (3.32), simply notice that $\iota_{H,\Gamma} : H \to H\#_\Gamma\mathbb{C}[G]$ is a wqhf morphism and $(\iota_{H,\Gamma}^{\otimes 3})(\Phi_H) = \Phi_{H\#_\Gamma\mathbb{C}[G]}$. Therefore, since Equation(3.32) is satisfied for (S_H, α_H, β_H) we see it will also be satisfied for $H\#_\Gamma\mathbb{C}[G]$. This proves that $H\#_\Gamma\mathbb{C}[G]$ is a weak quasi Hopf algebra.

D.1 Verifying Quasitriangular Structure

Now, we need to verify that $H\#_\Gamma\mathbb{C}[G]$ is a quasitriangular wqhf. To that end let us first verify that:

$$R_{H\#_\Gamma\mathbb{C}[G]} := \sum_{g\in G}(\bar{c}_1 \cdot \overline{\partial}(\delta_g) \otimes e) \otimes (\bar{c}_2 \otimes g) \tag{D.72}$$

$$R^{-1}_{H\#_\Gamma\mathbb{C}[G]} := \sum_{g\in G}((\bar{c}^{\ -1})_1 \cdot \overline{\partial}(\delta_g) \otimes e) \otimes ((\Psi(g)((\bar{c}^{\ -1})_2) \cdot \overline{\gamma}^{\ -1}_{g,g^{-1}}) \otimes g^{-1}) \tag{D.73}$$

is indeed partially invertible.

To that end we expand $R^{-1}_{H\#_\Gamma\mathbb{C}[G]} \cdot R_{H\#_\Gamma\mathbb{C}[G]}$ to obtain:

$$\sum_{g\in G}((\bar{c}^{\ -1})_1 \cdot \overline{\partial}(\delta_g) \otimes e) \otimes ((\Psi(g)((\bar{c}^{\ -1})_2) \cdot \overline{\gamma}^{\ -1}_{g,g^{-1}}) \otimes g^{-1}) \cdot (\bar{c}_1 \cdot \overline{\partial}(\delta_g) \otimes e) \otimes (\bar{c}_2 \otimes g) = \tag{D.74}$$

$$\sum_{g\in G}((\bar{c}^{\ -1})_1 \cdot \bar{c}_1 \cdot \overline{\partial}(\delta_g) \otimes e) \otimes (\overline{\gamma}_{g^{-1},g} \cdot (\Psi(g^{-1}) \circ \Psi(g))\left((\bar{c}^{-1})_2\right) \cdot \Psi(g^{-1})(\overline{\gamma}^{-1}_{g,g^{-1}}) \cdot \bar{c}_2 \otimes e) = \tag{D.75}$$

$$\sum_{g\in G}((1)_{(1)} \cdot \overline{\partial}(\delta_g) \otimes e) \otimes (\overline{\gamma}_{g^{-1},g} \cdot (\Psi(g^{-1}) \circ \Psi(g))\left((\bar{c}^{-1})_2\right) \cdot \overline{\gamma}^{-1}_{g^{-1},g} \cdot \bar{c}_2 \otimes e) = \tag{D.76}$$

$$\sum_{g\in G}((1)_{(1)}\cdot\overline{\partial}(\delta_g)\otimes e)\otimes((\overline{c}^{\ -1})_2\cdot\overline{c}_2\otimes e)= \tag{D.77}$$

$$\Delta_{H\#_\Gamma\mathbb{C}[G]}(1_{H\#_\Gamma\mathbb{C}[g]}) \tag{D.78}$$

Here we have used the fact that $\Psi(g^{-1})(\overline{\gamma}_{g,g^{-1}}^{-1}) = \overline{\gamma}_{g^{-1},g}^{-1}$, $\overline{c}^{-1}\cdot\overline{c} = \Delta(1_H)$, and $\mathrm{Ad}(\overline{\gamma}_{g,g^{-1}})\circ\Psi(g)\circ\Psi(g^{-1}) = \mathrm{Id}$.

Expanding $R_{H\#_\Gamma\mathbb{C}[G]}\cdot R^{-1}_{H\#_\Gamma\mathbb{C}[G]}$ we obtain:

$$\sum_{g\in G}(\overline{c}_1\cdot\overline{\partial}(\delta_g)\otimes e)\otimes(\overline{c}_2\otimes g)\cdot((\overline{c}^{\ -1})_1\cdot\overline{\partial}(\delta_g)\otimes e)\otimes((\Psi(g)((\overline{c}^{\ -1})_2)\cdot\overline{\gamma}_{g,g^{-1}}^{-1})\otimes g^{-1})= \tag{D.79}$$

$$\sum_{g\in G}(\overline{c}_1\cdot(\overline{c}^{\ -1})_1\cdot\overline{\partial}(\delta_g)\otimes e)\otimes(\overline{\gamma}_{g,g^{-1}}\cdot\Psi(g)(\overline{c}_2\cdot(\overline{c}^{\ -1})_2)\cdot\overline{\gamma}_{g,g^{-1}}^{-1}\otimes e)= \tag{D.80}$$

$$\sum_{g\in G}((1)_{(2)}\cdot\overline{\partial}(\delta_g)\otimes e)\otimes(\overline{\gamma}_{g,g^{-1}}\cdot(\Psi(g)\circ\Psi(g^{-1}))((1)_{(1)})\cdot\overline{\gamma}_{g,g^{-1}}^{-1}\otimes e)= \tag{D.81}$$

$$\sum_{g\in G}((1)_{(2)}\cdot\overline{\partial}(\delta_g)\otimes e)\otimes((1)_{(1)}\otimes e)=\Delta^{op}_{H\#_\Gamma\mathbb{C}[G]}(1_{H\#_\Gamma\mathbb{C}[G]}) \tag{D.82}$$

Here we used the fact that $\overline{c}\cdot\overline{c}^{-1} = (\mathrm{Id}\otimes\Psi(g^{-1}))(\Delta^{op}(1_H))$, and that $\mathrm{Ad}(\overline{\gamma}_{g,g^{-1}})\circ\Psi(g)\circ\Psi(g^{-1}) = \mathrm{Id}$.

Therefore, it will indeed be partially invertible.

Next, we need to verify that for all $h\in H, g\in G$ that:

$$\Delta^{op}_{H\#_\Gamma\mathbb{C}[G]}(h\otimes g) = R_{H\#_\Gamma\mathbb{C}[G]}\cdot\Delta_{H\#_\Gamma\mathbb{C}[G]}(h\otimes g)\cdot R^{-1}_{H\#_\Gamma\mathbb{C}[G]} \tag{D.83}$$

Expanding the right hand side out we obtain:

$$\sum_{k,t\in G}(\overline{c}_1\cdot\overline{\partial}(\delta_k)\otimes e)\otimes(\overline{c}_2\otimes k)\cdot\Delta_{H\#_\Gamma\mathbb{C}[G]}(h\otimes g)\cdots$$
$$\cdots((\overline{c}^{\,-1})_1\cdot\overline{\partial}(\delta_t)\otimes e)\otimes((\Psi(t)((\overline{c}^{\,-1})_2)\cdot\overline{\gamma}^{-1}_{t,t^{-1}})\otimes t^{-1})= \tag{D.84}$$

$$\sum_{k,t\in G}(\overline{c}_1\cdot\overline{\partial}(\delta_k)\otimes e)\otimes(\overline{c}_2\otimes k)\cdot(((\overline{\mu}_g{}^{-1})_1\cdot h_{(1)})\otimes g)\otimes((\overline{\mu}_g{}^{-1})_2\cdot h_{(2)})\otimes g)\cdots$$
$$\cdots((\overline{c}^{\,-1})_1\cdot\overline{\partial}(\delta_t)\otimes e)\otimes((\Psi(t)((\overline{c}^{\,-1})_2)\cdot\overline{\gamma}^{-1}_{t,t^{-1}})\otimes t^{-1})= \tag{D.85}$$

$$\sum_{k,t\in G}(\Psi(g^{-1})(\overline{c}_1)\cdot(\overline{\mu}_g{}^{-1})_1\cdot h_{(1)}\cdot\overline{\partial}(\delta_{g^{-1}kg})\otimes g)\otimes(\overline{\gamma}_{k,g}\cdot\Psi(g^{-1})(\overline{c}_2)\cdot(\overline{\mu}_g{}^{-1})_2\cdot h_{(2)})\otimes kg)\cdots$$
$$\cdots((\overline{c}^{\,-1})_1\cdot\overline{\partial}(\delta_t)\otimes e)\otimes((\Psi(t)((\overline{c}^{\,-1})_2)\cdot\overline{\gamma}^{-1}_{t,t^{-1}})\otimes t^{-1})= \tag{D.86}$$

$$\sum_{k,t\in G}(\Psi(g^{-1})(\overline{c}_1)\cdot(\overline{\mu}_g{}^{-1})_1\cdot h_{(1)}\cdot\overline{\partial}(\delta_{g^{-1}kg})\cdot\overline{\partial}(\delta_t)\cdot(\overline{c}^{\,-1})_1\otimes g)\otimes$$
$$(\overline{\gamma}_{kg,t^{-1}}\cdot\Psi(t)\left(\overline{\gamma}_{k,g}\cdot\Psi(g^{-1})(\overline{c}_2)\cdot(\overline{\mu}_g{}^{-1})_2\cdot h_{(2)}\right)\cdot(\Psi(t)((\overline{c}^{\,-1})_2)\cdot\overline{\gamma}^{-1}_{t,t^{-1}})\otimes kgt^{-1}) \tag{D.87}$$

Notice that $\overline{\partial}(\delta_{g^{-1}kg})\cdot\overline{\partial}(\delta_t)=\delta_{t,g^{-1}kg}\overline{\partial}(\delta_t)$. Therefore, the non-zero terms will be indexed by the elements of $(t,k)\in G^2$ where $k=gtg^{-1}$. This means Equation (D.87) equals:

$$\sum_{t\in G}(\Psi(g^{-1})(\overline{c}_1)\cdot(\overline{\mu}_g{}^{-1})_1\cdot h_{(1)}\cdot\overline{\partial}(\delta_t)\cdot(\overline{c}\ ^{-1})_1\otimes g)\otimes$$
$$(\overline{\gamma}_{gt,t^{-1}}\cdot\Psi(t)\left(\overline{\gamma}_{gtg^{-1},g}\cdot\Psi(g^{-1})(\overline{c}_2)\cdot(\overline{\mu}_g{}^{-1})_2\cdot h_{(2)}\right)\cdot(\Psi(t)((\overline{c}\ ^{-1})_2)\cdot\overline{\gamma}^{-1}_{t,t^{-1}})\otimes(gtg^{-1}g)t^{-1}= \tag{D.88}$$

Since we know that:

$$(\overline{\gamma}_{gtg^{-1},g})_2\cdot(\Psi(g^{-1})^{\otimes 2})(\overline{c})\cdot(\overline{\mu}_g)^{-1}\cdot(\overline{\partial}(\delta_t))_1=(\overline{\gamma}_{g,t})_2\cdot(\mathrm{Id}\otimes\Psi(t^{-1}))((\overline{\mu}_g)_{21})\cdot\overline{c}\cdot(\overline{\partial}(\delta_t))_1 \tag{D.89}$$

We can simplify Equation (D.87) to:

$$\sum_{t\in G}((\overline{\mu}_g{}^{-1})_2\cdot\overline{c}_1\cdot h_{(1)}\cdot(\overline{c}\ ^{-1})_1\cdot\overline{\partial}(\delta_t)\otimes g)\otimes \tag{D.90}$$
$$(\overline{\gamma}_{gt,t^{-1}}\cdot\Psi(t)\left(\overline{\gamma}_{g,t}\cdot\Psi(t^{-1})((\overline{\mu}_g^{-1})_1)\cdot\overline{c}_2\cdot h_{(2)}\cdot(\overline{c}\ ^{-1})_2\right)\cdot\overline{\gamma}^{-1}_{t,t^{-1}})\otimes g)$$

But by assumption:

$$\overline{c}\cdot\Delta(h)\cdot\overline{c}^{-1}\cdot(\overline{\partial}(\delta)_t)_1=(\mathrm{Id}\otimes\Psi(t^{-1}))\Delta^{op}(h)\cdot(\overline{\partial}(\delta_t))_1 \tag{D.91}$$

Therefore, Equation (D.90) reduces to:

$$\sum_{t\in G}((\overline{\mu}_g{}^{-1})_2\cdot h_{(2)}\cdot\overline{\partial}(\delta_t)\otimes g)\otimes(\overline{\gamma}_{gt,t^{-1}}\cdot\Psi(t)(\overline{\gamma}_{g,t})\cdot\Psi(t)\left(\Psi(t^{-1})((\overline{\mu}_g^{-1})_1\cdot h_{(1)})\right)\cdot\overline{\gamma}^{-1}_{t,t^{-1}})\otimes g)= \tag{D.92}$$

$$\sum_{t\in G}((\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)} \cdot \overline{\partial}(\delta_t) \otimes g) \otimes (\overline{\gamma}_{t,t^{-1}} \cdot \Psi(t)\left(\Psi(t^{-1})((\overline{\mu}_g^{-1})_1 \cdot h_{(1)})\right) \cdot \overline{\gamma}_{t,t^{-1}}^{-1}) \otimes g) = \tag{D.93}$$

$$\sum_{t\in G}((\overline{\mu}_g{}^{-1})_2 \cdot h_{(2)} \cdot \overline{\partial}(\delta_t) \otimes g) \otimes ((\overline{\mu}_g^{-1})_1 \cdot h_{(1)}) \otimes g) = \tag{D.94}$$

$$\Delta^{op}_{H\#_\Gamma \mathbb{C}[G]}(h \otimes g) \tag{D.95}$$

Here we used the fact that $\overline{\gamma}_{gt,t^{-1}} \cdot \Psi(t)(\overline{\gamma}_{g,t}) = \overline{\gamma}_{t,t^{-1}}$.

Next, we need to verify that:

$$(\Delta_{H\#_\Gamma \mathbb{C}[G]} \otimes \mathrm{Id})(R_{H\#_\Gamma \mathbb{C}[G]}) = (\Phi_{H\#_\Gamma \mathbb{C}[G]})_{312} \cdot (R_{H\#_\Gamma \mathbb{C}[G]})_{13} \cdot (\Phi^{-1}_{H\#_\Gamma \mathbb{C}[G]})_{123} \cdot (R_{H\#_\Gamma \mathbb{C}[G]})_{23} \cdot (\Phi_{H\#_\Gamma \mathbb{C}[G]})_{123} \tag{D.96}$$

Well to that end notice that:

$$(\Delta \otimes \mathrm{Id})(R_{H\#_\Gamma \mathbb{C}[G]}) = \sum_{g\in G}(\left((\overline{c}_1)_{(1)} \cdot (\overline{\partial}(\delta_g))_{(1)}\right) \otimes e) \otimes (\left((\overline{c}_1)_{(2)} \cdot (\overline{\partial}(\delta_g))_{(2)}\right) \otimes e) \otimes (\overline{c}_2 \otimes g) = \tag{D.97}$$

$$\sum_{\substack{g,k,t\in G \\ k\cdot t=g}} (\left((\overline{c}_1)_{(1)} \cdot \overline{\partial}(\delta_k)\right) \otimes e) \otimes (\left((\overline{c}_1)_{(2)} \cdot \overline{\partial}(\delta_t)\right) \otimes e) \otimes (\overline{c}_2 \otimes g) \tag{D.98}$$

On the other hand we know that:

$$(\Delta\otimes\mathrm{Id})(\overline{c})\cdot(\overline{\partial}(\delta_k)\otimes\overline{\partial}(\delta_t))_{12} = (\Phi_{312})^{(e,e,kt)}\cdot(\overline{\gamma}_{k,t})_3\cdot\overline{c}_{13}^{(e,e,t)}\cdot(\Phi_{132}^{-1})^{(e,e,t)}\cdot\overline{c}_{23}\cdot\Phi_{123}\cdot(\overline{\partial}(\delta_k)\otimes\overline{\partial}(\delta_t))_{12} \tag{D.99}$$

Suppress the double summation for the sake of brevity, then Equation (D.98) equals:

$$((x_2\cdot\overline{c}_1\cdot X_1\cdot x_1\cdot\overline{\partial}(\delta_k)\otimes e)\otimes(x_3\cdot X_3\cdot\overline{c}_1\cdot x_2\cdot\overline{\partial}(\delta_t)\otimes e)\cdot(\Psi(g^{-1})(x_1)\cdot\overline{\gamma}_{k,t}\cdot\Psi(t^{-1})(\overline{c}_2\cdot X_2)\cdot\overline{c}_2\cdot x_3\otimes g)) \tag{D.100}$$

Factoring out the multiplication we see this is just:

$$(\Phi_{H\#_\Gamma\mathbb{C}[G]})_{312}\cdot((\overline{c}_1\cdot X_1\cdot x_1\cdot\overline{\partial}(\delta_k)\otimes e)\otimes(X_3\cdot\overline{c}_1\cdot x_2\cdot\overline{\partial}(\delta_t)\otimes e)\cdot(\overline{\gamma}_{k,t}\cdot\Psi(t^{-1})(\overline{c}_2\cdot X_2)\cdot\overline{c}_2\cdot x_3\otimes g)) = \tag{D.101}$$

$$(\Phi_{H\#_\Gamma\mathbb{C}[G]})_{312}\cdot(\overline{c}_1\otimes e)\otimes(1_H\otimes e)\otimes(\overline{c}_2\otimes k)\cdot$$
$$((X_3\cdot x_1\cdot\overline{\partial}(\delta_k)\otimes e)\otimes(X_1\cdot\overline{c}_1\cdot x_2\cdot\overline{\partial}(\delta_t)\otimes e)\otimes(\Psi(t^{-1})(X_2)\cdot\overline{c}_2\cdot x_3\otimes t)) = \tag{D.102}$$

$$(\Phi_{H\#_\Gamma\mathbb{C}[G]})_{312}\cdot(R_{H\#_\Gamma\mathbb{C}[G]})_{13}\cdot$$
$$((X_1\cdot x_1\otimes e)\otimes(X_3\cdot\overline{c}_1\cdot x_2\cdot\overline{\partial}(\delta_t)\otimes e)\otimes(\Psi(t^{-1})(X_2)\cdot\overline{c}_2\cdot x_3\otimes t)) = \tag{D.103}$$

$$(\Phi_{H\#_\Gamma\mathbb{C}[G]})_{312}\cdot(R_{H\#_\Gamma\mathbb{C}[G]})_{13}\cdot((X_1\otimes e)\otimes(X_3\otimes e)\otimes(X_2\otimes e))\cdot$$
$$((x_1\otimes e)\otimes(\overline{c}_1\cdot x_2\cdot\overline{\partial}(\delta_t)\otimes e)\otimes(\overline{c}_2\cdot x_3\otimes t)) = \tag{D.104}$$

$$(\Phi_{H\#_\Gamma\mathbb{C}[G]})_{312} \cdot (R_{H\#_\Gamma\mathbb{C}[G]})_{13} \cdot (\Phi^{-1}_{H\#_\Gamma\mathbb{C}[G]})_{132} \cdot$$

$$((x_1 \otimes e) \otimes (\overline{c}_1 \cdot x_2 \cdot \overline{\partial}(\delta_t) \otimes e) \otimes (\overline{c}_2 \cdot x_3 \otimes t)) = \tag{D.105}$$

$$(\Phi_{H\#_\Gamma\mathbb{C}[G]})_{312} \cdot (R_{H\#_\Gamma\mathbb{C}[G]})_{13} \cdot (\Phi^{-1}_{H\#_\Gamma\mathbb{C}[G]})_{132} \cdot (R_{H\#_\Gamma\mathbb{C}[G]})_{23} \cdot (\Phi_{H\#_\Gamma\mathbb{C}[G]})_{123} \tag{D.106}$$

This proves the first braiding axiom for $R_{H\#_\Gamma\mathbb{C}[G]}$.

Lastly, we need to prove that:

$$(\mathrm{Id}\otimes\Delta)(R_{H\#_\Gamma\mathbb{C}[G]}) = (\Phi^{-1}_{H\#_\Gamma\mathbb{C}[G]})_{231}(R_{H\#_\Gamma\mathbb{C}[G]})_{13}(\Phi_{H\#_\Gamma\mathbb{C}[G]})_{213}(R_{H\#_\Gamma\mathbb{C}[G]})_{12}(\Phi^{-1}_{H\#_\Gamma\mathbb{C}[G]})_{123} \tag{D.107}$$

Expanding the left hand side of this we obtain:

$$\sum_{g\in G}(\overline{c}_1 \cdot \overline{\partial}(\delta_g) \otimes e) \otimes ((\overline{\mu}_g{}^{-1})_{(1)} \cdot (\overline{c}_2)_{(2)} \otimes g) \otimes ((\overline{\mu}_g{}^{-1})_{(1)} \cdot (\overline{c}_2)_{(2)} \otimes g) \tag{D.108}$$

On the other hand we know that:

$$(\overline{\mu_g}\ {}^{-1})_{23} \cdot (\mathrm{Id}_H \otimes \Delta)(\overline{c}) \cdot (\overline{\partial}(\delta_g))_1 = (\Phi^{-1}_{231})^{(e,g,g)} \cdot \overline{c}_{13} \cdot \Phi^{(e,g,e)}_{213} \cdot \overline{c}_{12} \cdot \Phi^{-1}_{123} \cdot (\overline{\partial}(\delta_g))_1 \tag{D.109}$$

Expanding this out in shorthand we obtain:

$$(\overline{c}_1 \cdot \overline{\partial}(\delta_g)) \otimes ((\overline{\mu}_g{}^{-1})_1 \cdot (\overline{c}_2)_{(1)}) \otimes ((\overline{\mu}_g{}^{-1})_2 \cdot (\overline{c}_2)_{(2)}) = \tag{D.110}$$

$$(X_3 \cdot \overline{c}_1 \cdot x_2 \cdot \overline{c}_1 \cdot X_1 \cdot \overline{\partial}(\delta_g)) \otimes (\Psi(g^{-1})(X_1 \cdot x_1) \cdot \overline{c}_2 \cdot X_2) \otimes (\Psi(g^{-1})(X_2) \cdot \overline{c}_2 \cdot x_3 \cdot X_3) \tag{D.111}$$

Just as in the previous proof, we suppress the summation for the sake of brevity. We may simplify Equation (D.108) to:

$$(X_3 \cdot \overline{c}_1 \cdot x_2 \cdot \overline{c}_1 \cdot X_1 \cdot \overline{\partial}(\delta_g) \otimes e) \otimes (\Psi(g^{-1})(X_1 \cdot x_1) \cdot \overline{c}_2 \cdot X_2 \otimes g) \otimes (\Psi(g^{-1})(X_2) \cdot \overline{c}_2 \cdot x_3 \cdot X_3 \otimes g) = \tag{D.112}$$

$$(X_3 \otimes e) \otimes (X_1 \otimes e) \otimes (X_2 \otimes e) \cdot$$
$$(\overline{c}_1 \cdot x_2 \cdot \overline{c}_1 \cdot X_1 \cdot \overline{\partial}(\delta_g) \otimes e) \otimes (\Psi(g^{-1})(x_1) \cdot \overline{c}_2 \cdot X_2 \otimes g) \otimes (\overline{c}_2 \cdot x_3 \cdot X_3 \otimes g) = \tag{D.113}$$

$$(\Phi^{-1}_{H\#_\Gamma \mathbb{C}[G]})_{231} \cdot ((\overline{c}_1 \cdot \overline{\partial}(\delta_g) \otimes e) \otimes (1_H \otimes e) \otimes (\overline{c}_2 \otimes g)) \cdot$$
$$(x_2 \cdot \overline{c}_1 \cdot X_1 \otimes e) \otimes (\Psi(g^{-1})(x_1) \cdot \overline{c}_2 \cdot X_2 \otimes g) \otimes (x_3 \cdot X_3 \otimes e) = \tag{D.114}$$

$$(\Phi^{-1}_{H\#_\Gamma \mathbb{C}[G]})_{231} \cdot (R_{H\#_\Gamma \mathbb{C}[G]})_{13} \cdot ((x_2 \otimes e) \otimes (x_1 \otimes e) \otimes (x_3 \otimes e))$$
$$(\overline{c}_1 \cdot X_1 \otimes e) \otimes (\overline{c}_2 \cdot X_2 \otimes g) \otimes (X_3 \otimes e) = \tag{D.115}$$

$$(\Phi^{-1}_{H\#_\Gamma \mathbb{C}[G]})_{231} \cdot (R_{H\#_\Gamma \mathbb{C}[G]})_{13} \cdot (\Phi_{H\#_\Gamma \mathbb{C}[G]})_{213} \cdot ((\overline{c}_1 \otimes e) \otimes (\overline{c}_2 \otimes g) \otimes (1_H \otimes e)) \cdot$$
$$((X_1 \otimes e) \otimes (X_2 \otimes e) \otimes (X_3 \otimes e)) = \tag{D.116}$$

$$(\Phi^{-1}_{H\#_\Gamma \mathbb{C}[G]})_{231} \cdot (R_{H\#_\Gamma \mathbb{C}[G]})_{13} \cdot (\Phi_{H\#_\Gamma \mathbb{C}[G]})_{213} \cdot (R_{H\#_\Gamma \mathbb{C}[G]})_{12} \cdot (\Phi^{-1}_{H\#_\Gamma \mathbb{C}[G]})_{123} \tag{D.117}$$

This varifies the last braiding axiom, so indeed $R_{H\#_\Gamma \mathbb{C}[G]}$ will give a quasitriangular structure on $H\#_{\Gamma}\mathbb{C}[G]$.

D.2 Verifying Ribbon Structure

First, we need to verify that:

$$\Delta_{H\#_{\Gamma}\mathbb{C}[G]}(\nu) = (\nu_{H\#_{\Gamma}\mathbb{C}[G]} \otimes \nu_{H\#_{\Gamma}\mathbb{C}[G]}) \cdot (R_{H\#_{\Gamma}\mathbb{C}[G]})_{21} \cdot R_{H\#_{\Gamma}\mathbb{C}[G]} \tag{D.118}$$

Expanding the right hand side we have:

$$(\sum_{g,k\in G}(\nu \cdot \overline{\partial}(\delta_g) \otimes g) \otimes (\nu \cdot \overline{\partial}(\delta_k) \otimes k)) \cdot (\sum_{t\in G}(\overline{c}_2 \otimes t) \otimes (\overline{c}_1 \cdot \overline{\partial}(\delta_t) \otimes e)) \cdot (\sum_{p\in G}(\overline{c}_1 \cdot \overline{\partial}(\delta_p) \otimes e) \otimes (\overline{c}_2 \otimes p)) = \tag{D.119}$$

$$(\sum_{g,k\in G}(\nu \cdot \overline{\partial}(\delta_g) \otimes g) \otimes (\nu \cdot \overline{\partial}(\delta_k) \otimes k)) \cdot (\sum_{t,p\in G}(\overline{c}_2 \cdot \overline{c}_1 \cdot \overline{\partial}(\delta_p) \otimes t) \otimes (\Psi(p^{-1})(\overline{c}_1 \cdot \overline{\partial}(\delta_t)) \cdot \overline{c}_2 \otimes p)) = \tag{D.120}$$

$$\sum_{g,k,t,p\in G}(\overline{\gamma}_{g,t} \cdot \Psi(t^{-1})\left(\nu \cdot \overline{\partial}(\delta_g)\right) \cdot \overline{c}_2 \cdot \overline{c}_1 \cdot \overline{\partial}(\delta_p) \otimes gt) \otimes (\overline{\gamma}_{k,p} \cdot \Psi(p^{-1})\left(\nu \cdot \overline{\partial}(\delta_k) \cdot \overline{c}_1 \cdot \overline{\partial}(\delta_t)\right) \cdot \overline{c_2} \otimes kp) = \tag{D.121}$$

$$\sum_{\substack{g,k,p\in G \\ g=kpk^{-1}}}(\overline{\gamma}_{g,k} \cdot \Psi(k^{-1})(\nu) \cdot \overline{c}_2 \cdot \overline{c}_1 \cdot \overline{\partial}(\delta_p) \otimes kp) \otimes (\overline{\gamma}_{k,p} \cdot \Psi(p^{-1})(\nu \cdot \overline{c}_1) \cdot \overline{\partial}(\delta_{p^{-1}kp}) \cdot \overline{c_2} \otimes kp) \tag{D.122}$$

On the otherhand, we know that:

$$\overline{\mu}_{kp}^{-1}\cdot\Delta(\nu)\cdot(\overline{\partial}(\delta_p)\otimes\overline{\partial}(\delta_{p^{-1}kp})) = (\overline{\gamma}_{kpk^{-1},k}\otimes\overline{\gamma}_{k,p})\cdot(\nu^{\otimes 2})^{(k,p)}\cdot(\overline{c}_{21})^{(e,p)}\cdot\overline{c}\cdot(\overline{\partial}(\delta_p)\otimes\overline{\partial}(\delta_{p^{-1}kp})) \tag{D.123}$$

Therefore, we see that Equation (D.122) will equal:

$$\sum_{k,p\in G}((\overline{\mu}_{kp}^{-1})_1\cdot(\nu)_{(1)}\cdot\overline{\partial}(\delta_p)\otimes kp)\otimes((\overline{\mu}_{kp}^{-1})_2\cdot(\nu)_{(2)}\cdot\overline{\partial}(\delta_{p^{-1}kp})\otimes kp) = \tag{D.124}$$

$$\sum_{\ell\in G}\sum_{\substack{k,p\in G\\ kp=\ell}}((\overline{\mu}_{\ell}^{-1})_1\cdot(\nu)_{(1)}\cdot\overline{\partial}(\delta_p)\otimes \ell)\otimes((\overline{\mu}_{\ell}^{-1})_2\cdot(\nu)_{(2)}\cdot\overline{\partial}(\delta_{p^{-1}kp})\otimes \ell) = \tag{D.125}$$

$$\sum_{\ell\in G}\Delta((\nu\cdot\overline{\partial}(\delta_\ell)\otimes\ell)) = \Delta_{H\#_\Gamma\mathbb{C}[G]}(\nu_{H\#_\Gamma\mathbb{C}[G]}) \tag{D.126}$$

The next thing we need to prove is that:

$$S_{H\#_\Gamma\mathbb{C}[G]}(\nu_{H\#_\Gamma\mathbb{C}[G]}) = \nu_{H\#_\Gamma\mathbb{C}[G]} \tag{D.127}$$

To that end we expand the LHS to obtain:

$$\sum_{g\in G}\Psi(g)\left(S(\nu\cdot\overline{\partial}(\delta_g))\cdot\mathbf{d}^g\cdot\overline{\gamma}_{g^{-1},g}^{-1}\right)\otimes g^{-1}) = \tag{D.128}$$

$$\sum_{g\in G}\Psi(g)\left(S(\nu)\cdot\overline{\partial}(\delta_{g^{-1}})\cdot\mathbf{d}^g\cdot\overline{\gamma}_{g^{-1},g}^{-1}\right)\otimes g^{-1}) \tag{D.129}$$

We know that:

$$S(\nu)\cdot\overline{\partial}(\delta_{g^{-1}}) = S(\overline{\gamma}_{g,g^{-1}}^{-1})\cdot\mathbf{d}^{(g^{-1})}\cdot\nu\cdot\overline{\partial}(\delta_{g^{-1}}) \tag{D.130}$$

Therefore, Equation (D.129) simplifies to:

$$\sum_{g\in G}\Psi(g)(S(\overline{\gamma}_{g,g^{-1}}^{-1}))\cdot\Psi(g)(\mathbf{d}^{(g^{-1})})\cdot\Psi(g)(\nu)\cdot\overline{\partial}(\delta_{g^{-1}})\cdot\Psi(g)(\mathbf{d}^{g})\cdot\Psi(g)(\overline{\gamma}_{g^{-1},g}^{-1})\otimes g^{-1} \tag{D.131}$$

On the other hand we also know that:

$$\Psi(g)(\nu)\cdot\overline{\partial}(\delta_{g^{-1}})=\nu\cdot\overline{\partial}(\delta_{g^{-1}}) \tag{D.132}$$

and for all $h\in H$:

$$\Psi(g)(h)\cdot\nu\cdot\overline{\partial}(\delta_{g^{-1}})=\nu\cdot h\cdot\overline{\partial}(\delta_{g^{-1}}) \tag{D.133}$$

Therefore, Equation (D.131) simplifies to:

$$\sum_{g\in G}\nu\cdot\overline{\partial}(\delta_{g^{-1}})\cdot S(\overline{\gamma}_{g,g^{-1}}^{-1})\cdot\mathbf{d}^{(g^{-1})}\cdot\Psi(g)(\mathbf{d}^{g})\cdot\Psi(g)(\overline{\gamma}_{g^{-1},g}^{-1})\otimes g^{-1} \tag{D.134}$$

Since $\Psi(g)(\overline{\gamma}_{g^{-1},g})=\overline{\gamma}_{g,g^{-1}}$ we obtain:

$$\sum_{g\in G}\nu\cdot\overline{\partial}(\delta_{g^{-1}})\cdot S(\overline{\gamma}_{g,g^{-1}}^{-1})\cdot\mathbf{d}^{(g^{-1})}\cdot\Psi(g)(\mathbf{d}^{g})\cdot\overline{\gamma}_{g,g^{-1}}^{-1}\otimes g^{-1} \tag{D.135}$$

But, we our by assumption Γ is rigid, which means:

$$d^{e}=S(\overline{\gamma}_{g,g^{-1}}^{-1})\cdot\mathbf{d}^{(g^{-1})}\cdot\Psi(g)(\mathbf{d}^{g})\cdot\overline{\gamma}_{g,g^{-1}}^{-1}=1_{H} \tag{D.136}$$

Therefore, Equation (D.135) becomes:

$$\sum_{g \in G} \nu \cdot \overline{\partial}(\delta_{g^{-1}}) \otimes g^{-1} = \nu_{H \#_{\Gamma} \mathbb{C}[G]} \tag{D.137}$$

This confirms that $\nu_{H \#_{\Gamma} \mathbb{C}[G]}$ gives a ribbon structure on $H \#_{\Gamma} \mathbb{C}[G]$.