Highest weight and boundary states of vertex operator algebra modules from Airy ideals

by

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Abstract

In this thesis we study some aspects of 'Airy structures' first proposed in [39], as an algebraic reformulation of the Chekhov-Eynard-Orantin (CEO) topological recursion initiated in [22] and [23] in order to study the large N expansion of matrix models.

Our primary goal is to engineer new examples of Airy structures taking inspiration from the representation theory of vertex operator algebras. In particular, we construct a highest weight state of a $\mathcal{W}^{-N-1/2}(\mathfrak{sp}_{2N})$-algebra module as a partition function of an Airy structure, following the approach developed in [9]. We do this with the help of an orbifold construction from symplectic fermions that was developed in [18]. Our second key result is the construction of certain Ishibashi boundary states related to affine vertex algebra modules from partition functions of Airy structures. We make use of the Wakimoto free field realizations of affine Lie algebras for this purpose. A novel aspect of both of these examples is that zero modes of the Heisenberg algebra are realized as derivatives instead of variables, and hence the partition functions are vectors that lie in infinite indecomposable extensions of Fock modules of free field algebras.

On the other hand, we also give an alternate formulation of Airy structures as left ideals of $\hbar$-adic completions of Rees Weyl algebras. In particular, we obtain a different proof of the existence and uniqueness of partition functions of Airy structures by realizing 'Airy ideals' as homomorphic images of canoni-

cal left ideals generated by derivatives obtained via certain automorphisms of the Rees Weyl algebra called ‘transvections’.

The thesis contains joint work done with V. Bouchard and T. Creutzig.

Preface

All the chapters in this thesis is joint work with Vincent Bouchard and Thomas Creutzig. In particular, Chapter 4 and Chapter 5 has been submitted for publication. The reference [12] is a preprint of this article and is available online at:

https://arxiv.org/abs/2207.04336

The versions of Chapter 4 and Chapter 5 printed here are almost identical to [12] with the exception of Section 4.I which is not present in the preprint.

Dedicated with thanks to the reader.

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Table of Contents

Chapter 1

Introduction

One of the biggest success of 20th century physics and mathematics is probably quantum field theory (QFT): a mathematical framework to describe much of the natural physical world from the microscopic to cosmological scales. While, the Lagrangian approach to QFT offers a powerful and simple way to describe physical observable and permits a perturbative analysis in powers of the reduced Planck's constant $\hbar$ for a large class of models, it is not sufficient to explain all phenomena. Non-perturbative effects such as instantons and electric charge confinement emerge through elegant explanations using abstract purely mathematical objects.

In fact, interactions between mathematics and physics have always been fruitful and mathematical symmetry as a guiding principle to discover new physical laws has lead to discoveries such as Maxwell's laws of electromagnetism and Einstein's theory of relativity. However starting in the 1980s, the discovery of string theory as an attempt to quantize 4d gravity lead to an unprecedented flurry of new ideas in diverse fields of pure mathematics such as algebraic geometry and topology, representation theory, and modular forms.

One such example is the link between 2d quantum gravity and intersection theory on the compactified moduli space of Riemann surfaces $\overline{\mathcal{M}_{g,n}}$. Two models of 2d quantum gravity were proposed by physicists - the first one is described by the hermitian matrix model and computes an enumeration of triangulations on Riemann surfaces, and the second model that of 'topological gravity' has a cohomological description as an intersection theory on the compactified moduli space of complex curves. In [44], Witten conjectured the equivalence of two models of 2d quantum gravity. Kontsevich proved this conjecture by expressing these intersection numbers in terms of a 'partition function' of a different matrix model. Such functions are governed by an infinite set of differential equations called the KdV equations or equivalently by Virasoro constraints. Such exchanges between physics and mathematics continue to be very fruitful and can be considered a whole new discpline by itself.

Another set of examples are QFTs in two dimensions with conformal symmetries i.e. conformal field theories (CFTs). These have played a crucial role in enhancing our knowledge of the non-perturbative aspects of QFT and string theory (with the two-dimensional space-time playing the role of a string world sheet parameterizing the string evolution). A closely-related algebraic object, the vertex operator algebra (VOA), can be thought of as the symmetry algebra of a CFT. However, the notion of vertex operator algebras (VOAs) was first introduced by Borcherds as the proper formulation for the moonshine module construction for the Monster group. In addition its rich algebraic structure has deepened our understanding of not only conformal field theory but also other purely algebraic fields such as representation theory of the affine Kac-Moody algebras, and ribbon and tensory category theory. On the physics side, it has applications to areas such as condensed matter physics, statistical physics and superstring theory.

In a large number of cases, most of the information of a QFT can be encoded in a single object called it's partition function. While computing partition functions could be a difficult task, sometimes they can be expressed as large N limits of matrix models and can be computed quite explicitly. The area now known as topological recursion was first proposed by Chekhov-Eynard-Orantin (CEO) in [22] and [23] as a means to encode the mathematical structures of matrix models. The 'conventional' CEO topological recursion produces a collection of meromorphic polydifferentials $\omega_{g,n}$ on a spectral curve Σ starting with the "initial data" $\omega_{0,1}, \omega_{0,2}$. However, it has also found many other applications in enumerative geometry, integrable systems, quantization problems, two-dimensional conformal field theory, and in knot theory (See [8] for a partial reference list for related applications). The list of applications is dauntingly vast, but one of the more impressive use-case of topological recursion is in the field of topological string theory and mirror symmetry. In [14], the authors compute recursively all the open and closed B-model amplitudes in closed form, at all genus.

In [39], the authors proposed an algebraic reformulation of this approach in terms of collections of differential operators called Airy structures. More abstractly, Airy structures were constructed as graded deformation quantization modules corresponding to a certain quadratic Lagrangian subvariety in an infinite-dimensional symplectic vector space. This thesis is primarily concerned with deepening our understanding of Airy structures but from a fresh point of view (inspired by the theory of VOAs) building on the work initiated in [9] and [8].

The unexpected power of topological recursion and Airy structures to describe an almost inexhaustible list of mathematical objects as quite simple recursive formulas is very attractive, and could be a powerful tool in understanding old geometric and quantum phenomena and also probe for new ones. In this thesis, we strive to find new examples of collection of differential operators that satisfy the properties to be an Airy structure. In particular we obtain two quite surprising new examples - one from the field of $\mathcal{W}$-algebras of type C, and another from boundary conformal field theory. In the next section, we give an overview of the organization of the thesis and also a quick summary of the original results contained in this cases.

1.A Overview of the thesis

In chapter 2 we give a brief survey of basic results in the theory of VOAs and in particular those relating to the representation theory of $\mathcal{W}$-algebras.

In chapter 3, we begin by presenting the essential ideas behind CEO topological recursion, and a formulation of the notion of 'Higher Airy structures' as originally presented in [9]. The goal of this chapter is to present the traditional viewpoint of Airy structures. In rest of the thesis, we will actually present and use a slightly different approach, that in terms of certain left ideals of the Rees

Weyl algebra. This is the topic of Chapter 4.

Chapters 4 and 5 are contents of the preprint [12] and is joint work with Vincent Bouchard and Thomas Creutzig. In Chapter 4, we propose an alternate way of defining and understanding Airy structures, as ideals of $\hbar$-adic completions of Rees Weyl algebras. The Rees construction (also used in algebraic geometry) is a general construction to convert 'filtered' rings and modules to 'graded' objects. The motivation to apply this construction is that $\mathbb{Z}$-grading on the Weyl algebra used in [9] appears more simply through powers of a parameter $\hbar$. We first reformulate the notion of Airy structures as 'Airy ideals', i.e. left ideals generated by certain collections of differential operators in the $\hbar$-adic completion of the Rees Weyl algebra, which we denote by $\widehat{\mathcal{D}}_A^\hbar$. We introduce automorphisms ϕ of $\widehat{\mathcal{D}}_A^\hbar$ called as transvections and first introduced in [7]. A transvection ϕ is essentially conjugation by an exponential of a power series and acts on $\widehat{\mathcal{D}}_A^\hbar$ as $\phi : (\hbar, \hbar x_a, \hbar\partial_a) \mapsto (\hbar, \hbar x_a, \bar{H}_a)$, for all $a \in A$, with

$$\bar{H}_a = \hbar\partial_a + \sum_{n=0}^{\infty} \hbar^n \partial_a q^{(n+1)}(x_A) \tag{1.1}$$

for some polynomials $q^{(n+1)}(x_A)$ of degree $\leq n+1$. We say that it is *stable* if $q^{(1)} = q^{(2)} = 0$. The main theorem of Chapter 4 is Theorem 4.E.3. It states that given an Airy ideal $\mathcal{I}$, there always exists a stable transvection ϕ such that $\mathcal{I}$ is isomorphic to $\phi(\mathcal{I}_{\text{can}})$, where $\mathcal{I}_{can}$ is the left ideal generated by all the derivatives $\hbar\partial_{x_a}$. A consequence is that the quotient of the completed Rees Weyl algebra by an Airy ideal $\mathcal{I}$ is canonically isomorphic to the completed Rees polynomial module twisted by a 'transvection' automorphism ϕ, which in turn implies the existence of a unique exponential solution Z to the equations

$\mathcal{I} \cdot Z = 0$ after imposing a suitable initial condition. This is precisely the existence and uniqueness statement of partition functions of Airy structures, first proved in [39] from a conceptually different point of view. In the latter part of this chapter we explore how free field realizations of VOAs (such as the Heisenberg) provide a way to construct Rees Weyl algebras through the universal enveloping algebra of modes. A novelty in this chapter is the construction of Airy ideals in which zero modes act as derivatives, instead of multiplication by a variable (which is usually the case). In this case, the partition function has an interpretation as lying in an infinite length indecomposable extension of the Fock module.

In Chapter 5, we present a new Airy ideal constructed from the $\mathcal{W}(\mathfrak{sp}_{2N})$-algebra at level $-N-1/2$. Our starting point is the orbifold construction of this $\mathcal{W}$-algebra from symplectic fermions first proposed in [18]. We use the boson-fermion correspondence to realize the generators in terms of N free bosons. In the spirit of [9], we change to the diagonal basis and perform an automorphism on the Rees Weyl algebra given by a pair of 'dilaton shifts'. The main result (Theorem 5.C.8) is the construction of an Airy ideal where the collection of differential operators are actually infinite linear combination of the generators of the underlying VOA in this case. Interestingly, it also turns out that the partition function lies in an extension of a Fock module, which could play a role in its geometric interpretation.

In Chapter 6, we introduce some concepts from boundary conformal field theory, a fascinating area of physics which has found crucial applications in areas such superstring theory and condensed matter physics! The main result of this chapter is a proof of the existence and uniqueness of Ishibashi states in the ‘bulk Hilbert space’. Ishibashi states are states in the completion of the Bulk Hilbert space that encode the data of certain ’gluing’ conditions on the boundary of the Riemann surface, and can be used to compute correlation functions in the boundary CFT in terms of objects in the bulk CFT. However explicitly computing these is often very hard, and we propose that maybe Airy ideals could be one way to make progress here.

Chapter 7 starts with a brief survey of the construction of free field realizations of modules of affine vertex algebras called Wakimoto modules in terms of pairs of symplectic bosons and a Heisenberg VOA. Our original contribution to this chapter appears in Section 7.E. In this section, we use tensor products of Wakimoto modules with its dual to construct new Airy ideals. The partition function of the Airy ideal can be understood as computing Ishibashi states in a boundary CFT described by an infinite extension of Wakimoto modules.

Finally in Chapter 8, we conclude the thesis by giving the reader further food for thought by presenting several interesting open problems, that have emerged during our studies. During the writing of this thesis, we have discovered several curious objects that have a very good chance to be of immense interest in theoretical physics and geometry. Hence, we implore the reader to pursue these directions.

Chapter 2

Vertex operator algebras

We start by introducing some basic objects in the theory of vertex operator algebras (VOAs). We then present an array of examples that will be essential to the rest of the thesis. This is a very deep and technical subject, and a good reference for the material in this chapter is [2].

2.A Preliminary background

Definition 2.A.1. A vertex algebra is a vector space V with a distinguished vector $\mathbf{1} \in V$ (vacuum vector), together with a linear map,

$$Y(\cdot, z) : V \mapsto \mathrm{End}(V)[[z, z^{-1}]] \tag{2.1}$$

also known as the state-field correspondence. Thus we can write,

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \mathrm{End}(V) \tag{2.2}$$

where the RHS is a formal power series which involves only finitely many negative powers of z when applied to any vector. The vacuum vector satisfies the following properties,

$$a_{(-1)}\mathbf{1} = \mathbf{1}_{(-1)}a = a, \quad a_{(n)}\mathbf{1} = 0, \quad n \geq 0. \tag{2.3}$$

The main axiom is the "Borcherds identity" satisfied by the modes,

$$\begin{aligned} \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \left(a_{m+n-j}(b_{(k+j)}c) - (-1)^n b_{(k+n-j)}(a_{(m+j)}c)\right) \\ = \sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)}b)_{(k+m-j)}c \end{aligned} \tag{2.4}$$

where $a, b, c \in V$. A vertex algebra V is said to be strongly generated by a subset $S \subset V$ if V is linearly spanned by the vacuum $\mathbf{1}$ and all elements of the form,

$$a_{1,n_1} \dots a_{r,n_r}\mathbf{1}, \quad \text{where} \quad r \geq 1, a_i \in S, n_i < 0. \tag{2.5}$$

In addition, V is said to be freely generated if the above spanning set is a basis for the underlying vector space V .

Remark 2.A.2. This definition can be easily generalized to its supersymmetric analogue when we have a $\mathbb{Z}_+$ graded super vector space V. The crucial difference being that (2.4) acquires extra negative signs depending on the parity of the chosen elements.

An important subclass consist of the vertex operator algerbas (VOAs).

Definition 2.A.3. A vertex operator algebra (VOA) is a graded vertex algebra $V = \bigsqcup_{n\in\mathbb{Z}} V_n$ such that $\dim V_n < \infty$ and $V_n = 0$ for n sufficiently small and a distinguished vector $\omega \in V_2$ (conformal vector) satisfying the following conditions,

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c \tag{2.6}$$

for $m, n \in \mathbb{Z}$ and some constant c, where L_n are the modes of ω

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{2.7}$$

and,

$$L_0 v = nv := (\mathrm{wt}\, v)v, \quad \text{for } v \in V_n \tag{2.8}$$

$$\frac{d}{dz}Y(v, z) = Y(L_{-1}v, z). \tag{2.9}$$

In conformal field theory, a field $a(z)$ is thought of as an operator associated to a fixed point z in the 2d complex plane. The limit of two operators approaching each other is described by operator product expansions (OPEs). The following is a consequence of the Borcherds identity (see [36] for further details):

Lemma 2.A.4. *Let V be a vertex algebra and $a, b \in V$. Then,*

$$a(z)b(w) = \sum_{j=0}^{N-1} \left(\iota_{z,w}\frac{1}{(z-w)^{j+1}}\right) c^j(w) + :a(z)b(w): \tag{2.10}$$

where $c^j(w) \in \mathrm{End}\, V[[w, w^{-1}]]$, $\iota_{z,w}R(z, w)$ is the power series expansion of a rational function $R(z, w)$ with poles only at $z = 0, w = 0$ and $|z| = |w|$ in the domain $|z| > |w|$ and $N \geq 1$ is an integer. Finally the colons $:\alpha(z):$ denotes *normal ordering, that is we put all annihilation operators α_n for $n > 0$ to the right of all creation operators α_n for $n < 0$.*

The singular part of (2.10) (called the OPE) encodes the brackets between all the modes of the fields $a(z)$ and $b(z)$ and is sometimes denotes as follows,

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}} \tag{2.11}$$

Definition 2.A.5. The VOA of symplectic bosons is generated by pairs of even fields $a_\alpha(z), a^*_\beta(z)$ for $\alpha, \beta \in S$ of conformal dimension of 1 and 0 respectively and satisfy the OPE,

$$a_\alpha(z)a^*_\beta(w) \sim \frac{\delta_{\alpha,\beta}}{z-w}. \tag{2.12}$$

Hence we have the commutation relations,

$$[a_{\alpha,n}, a^*_{\beta,m}] = \delta_{\alpha,\beta}\delta_{n,-m}\mathbf{1}, \quad [a_{\alpha,n}, a_{\beta,m}] = 0, \quad [a^*_{\alpha,n}, a^*_{\beta,m}] = 0 \tag{2.13}$$

where $\alpha, \beta \in S$ lies in some index set S, $n, m \in \mathbb{Z}$ and $\mathbf{1}$ is a central element. We define a representation of this VOA later in Chapter 7.

Definition 2.A.6. Another example of a free field algebra is the Heisenberg algebra. Let $\mathfrak{a}$ be a finite-dimensional linear space with a scalar product $(\cdot,\cdot)$ with a choice of basis $v_i, i = 1, 2, \ldots, N$. The Heisenberg Lie algebra $\widehat{\mathfrak{a}}$ has generators $b_{i,n}, i = 1, 2, \ldots, N, n \in \mathbb{Z}$ and $\mathbf{1}$ with commutation relations,

$$[b_{i,n}, b_{j,m}] = n(v_i, v_j)\delta_{m+n,0}\mathbf{1}, \quad [\mathbf{1}, b_{i,n}] = 0. \tag{2.14}$$

Hence the OPE of the generating fields is given by,

$$b_i(z)b_j(w) \sim \frac{(v_i, v_j)}{(z-w)^2}. \tag{2.15}$$

The Fock representation π^{ν}_{λ} of $\widehat{\mathfrak{a}}$ is a module freely generated by $b_{i,n}, i = 1, 2 \ldots, N, n < 0$ from a vector v_λ defined by,

$$b_{i,n}v_\lambda = 0, \quad n > 0; \qquad b_{i,0}v_\lambda = \lambda(v_i)v_\lambda, \quad \lambda \in \mathfrak{a}^* \tag{2.16}$$

so that the zero modes $b_i(0)$ act like the character λ and the central element $\mathbf{1}$ acts as ν times the identity. In particular, the vacuum module π^{ν}_0 carries a vertex algebra structure and this is precisely what's called the Heisenberg VOA.

In the following sections we are primarily concerned about twisted modules of $\mathcal{W}$-algebras. For every finite order automorphism σ of a vertex algebra V, we can construct a twisted V-module M^σ, such that when restricted to the σ-invariant subalgebra $V^\sigma \subset V$, M^σ becomes an untwisted module for V^σ. A module of a vertex algebra V is a vector space M with a linear map,

$$Y_M(\cdot, z) : V \mapsto \mathrm{End}\,(M)((z)) \tag{2.17}$$

such that the Borcherds identity (2.4) holds for $a, b \in V$ and $c \in M$. The notion of a twisted-module is a generalization that allows non-integral powers of z in the map $Y(\cdot, z)$.

Definition 2.A.7. Let σ be an automorphism of a VOA V of a finite order h so that it preserves the vacuum and the conformal vector. A σ-twisted module

M of V is a vector space with a linear map $Y(\cdot, z) : V \mapsto \text{End(M)}[[z^{-1/h}, z^{1/h}]]$ such that,

$$Y(a,z) = \sum_{n \in p+\mathbb{Z}} a_{(n)} z^{-n-1}, \quad \text{if } \sigma a = \exp^{-2\pi\sqrt{-1}p} a, \ p \in \frac{1}{h}\mathbb{Z} \tag{2.18}$$

where $a_{(n)} \in \text{End}(M)$. In other words, the monodromy around $z = 0$ is given by the action of σ:

$$Y(\sigma a, z) = Y(a, \exp^{2\pi\sqrt{-1}} z), \quad a = V. \tag{2.19}$$

In addition the modes satisfy the same Borcherds identity stated in (2.4).

We briefly mention the idea of contragredient modules first introduced in [28]. This will be used in Chapter 6 to construct objects in a boundary conformal field theory. Let (M, Y) be a module for a VOA V with grading,

$$M = \bigsqcup_{n \in \mathbb{Q}} M_{(n)} \tag{2.20}$$

and let M' be the graded dual space,

$$M' = \bigsqcup_{n \in \mathbb{Q}} M^*_{(n)}. \tag{2.21}$$

Let $\langle \cdot, \cdot \rangle : M' \times M \mapsto \mathbb{C}$ denote the canonical pairing. The adjoint vertex operator $Y'(v, z) \in \text{End}\,(M')[[z, z^{-1}]]$ is defined by the condition,

$$\langle Y'(v,z)m', m\rangle = \langle m', Y(e^{zL_1}(-z^2)^{L_0}v, z^{-1})m\rangle \tag{2.22}$$

for $v \in V, m' \in M', m \in W$. The result that was proved in [28] is that (M', Y')

carries the structure of a V-module.

In the rest of the chapter we give further important examples of vertex operator algebras (VOAs). In particular we discuss lattice VOAs associated to integral lattices and $\mathcal{W}$-algebras. Lattice VOAs were some of the first examples of VOAs in math literature but more recently, lattice VOAs have also made an appearance in relation to the fractional quantum hall effect such as in [16]. On the other hand, $\mathcal{W}$-algebras were introduced as non-linear extensions of the Virasoro algebra by Alexander Zamolodchikov. We briefly review the different constructions of $\mathcal{W}$-algebras in 2.D.

2.B Lattice vertex algebras

We introduce the notion of a lattice vertex algebra associated to an integral lattice Q of rank l. The interested reader can refer to [5] for a nice review on this topic. We will follow their exposition closely in this section. Let the bilinear form on Q be denoted by $(\cdot,\cdot)$ and we also use the same symbol to denote its extension to the complexification $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{Z}} Q$. We define a bimultiplicative function

$$\epsilon(\alpha,\alpha) = (-1)^{|\alpha^2|(|\alpha^2+1|/2)}, \quad \alpha \in Q. \tag{2.23}$$

The twisted group algebra $\mathbb{C}_\epsilon[Q]$ is spanned by $\{e^\alpha\}_{\alpha\in Q}$ with the multiplication rule,

$$e^\alpha e^\beta = \epsilon(\alpha,\beta)e^{\alpha+\beta}, \quad \alpha,\beta \in Q. \tag{2.24}$$

Let $\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}K$ be the *Heisenberg current algebra* defined by the commutation relations

$$[ht^m, h't^n] = m\delta_{m,-n}(h|h')\mathsf{K}, \quad [ht^m, \mathsf{K}] = 0, \qquad h, h' \in \mathfrak{h}. \tag{2.25}$$

It has a unique irreducible representation of level 1 (i.e., with $\mathsf{K} = 1$) on the Fock space $S = S(\mathfrak{h}[t^{-1}]t^{-1})$ such that $\mathfrak{h}[t^{-1}]t^{-1}$ acts by multiplication and $\mathfrak{h}[t]1 = 0$. This representation extends to the space $V_Q = S \otimes \mathbb{C}_\epsilon[Q]$ by

$$(ht^m)(s \otimes e^\alpha) = \big(ht^m + \delta_{m,0}(h|\alpha)\big)s \otimes e^\alpha \qquad \text{for } \ m \geq 0. \tag{2.26}$$

The left multiplication

$$e^\gamma(s \otimes e^\alpha) = \epsilon(\gamma, \alpha)s \otimes e^{\alpha+\gamma} \tag{2.27}$$

gives rise to a representation in V_Q of the $\mathbb{Z}_2$-graded associative algebra $U(\hat{\mathfrak{h}}) \otimes \mathbb{C}_\epsilon[Q]$. This induces a $\mathbb{Z}_2$ grading on V_Q given by the formula,

$$p(s \otimes e^\alpha) = |\alpha|^2 \mod 2\mathbb{Z} \tag{2.28}$$

For $\alpha \in Q$, the so-called *vertex operator* is given by,

$$\begin{aligned} Y_\alpha(z) &= e^\alpha \ :\exp \int \alpha(z) : \\ &\equiv e^\alpha z^\alpha \exp\Big(\sum_{n<0}(\alpha t^n)\frac{z^{-n}}{-n}\Big) \exp\Big(\sum_{n>0}(\alpha t^n)\frac{z^{-n}}{-n}\Big). \end{aligned} \tag{2.29}$$

It's parity is given by $|\alpha|^2 \mod 2\mathbb{Z}$. The vertex operator $Y_\alpha(z)$ is a field on

V_Q, and it is local with respect to $h(z)$ because

$$[h(z), Y_\alpha(w)] = (h|\alpha) Y_\alpha(w) \delta(z-w), \qquad h \in \mathfrak{h}, \alpha \in Q. \tag{2.30}$$

In addition it can be checked that the fields $Y_\alpha(z)$ are local among themselves. The vertex algebra structure on V_Q is given by the following theorem:

Theorem 2.B.1. *The fields $Y(ht^{-1}, z) = h(z)$ $(h \in \mathfrak{h})$, of parity 0, and $Y(e^\alpha, z) = Y_\alpha(z)$ $(\alpha \in Q)$, of parity $p(e^\alpha) = |\alpha|^2 \mod 2\mathbb{Z}$, generate a vertex algebra structure on $V_Q = S \otimes \mathbb{C}_\epsilon[Q]$ with the vacuum vector $1 \otimes 1$ and the operator T defined by*

$$[T, ht^m] = -m\, ht^{m-1}, \quad Te^\alpha = (\alpha t^{-1}) e^\alpha, \qquad h \in \mathfrak{h}, \alpha \in Q. \tag{2.31}$$

This vertex algebra is conformal of central charge $l := \operatorname{rank} Q$ with the conformal vector

$$\nu = \frac{1}{2} \sum_{i=1}^{l} (a^i t^{-1})(b^i t^{-1}), \tag{2.32}$$

where $\{a^i\}$, $\{b^i\}$ are dual bases of $\mathfrak{h}$.

We now describe another fundamental example of VOAs, those constructed from affine Lie algebras (also sometimes known as affine Kac-Moody algebras). This is an important example due to its connections to classical Lie algebras, and to the theory of $\mathcal{W}$-algebras via the quantum Sokolov-Drinfeld reduction. For further information on these objects relevant to this thesis, the reader can refer to [8] and the references presented there. In the next two sections we adopt and follow the notation and presentation of [8].

2.C Affine vertex algebras

Let $\mathfrak{g}$ be a simple basic complex Lie superalgebra and $\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ a non-degenerate invariant bilinear form on $\mathfrak{g}$, $\langle\cdot,\cdot\rangle$ normalized such that long roots have norm 2. Let

$$\widehat{\mathfrak{g}} = \mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[t^{\pm 1}]\oplus\mathbb{C}.\mathsf{K}\oplus\mathbb{C}.\mathsf{d}$$

be the affinization of $\mathfrak{g}$. Here K is central and d is a derivation. For $x\in\mathfrak{g}$, denote $x\otimes t^n$ by x_n. Then the commutation relations are

$$\forall m,n\in\mathbb{Z},\quad \forall x,y\in\mathfrak{g},\qquad [x_m,y_n]=[x,y]_{m+n}+\delta_{m+n,0}\,m\,\kappa(x,y)K.$$

Let $\mathsf{k}\in\mathbb{C}$ and $\widehat{\mathfrak{g}}_{\mathsf{k}}$ be the Lie superalgebra obtained from $\widehat{\mathfrak{g}}$ by setting $K=\mathsf{k}$ for some constant k. Let $\widehat{\mathfrak{g}}_{\mathsf{k}}^{\pm}$ the subalgebras generated by the positive (resp. negative) modes. The subalgebra of zero-modes is identified with $\mathfrak{g}$ (ignoring the derivation). The universal enveloping algebras of $\widehat{\mathfrak{g}}_{\mathsf{k}}$, $\widehat{\mathfrak{g}}_{\mathsf{k}}^{\pm}$ and $\widehat{\mathfrak{g}}_{\mathsf{k}}^{\pm}\oplus\mathfrak{g}$ are denoted by $\mathcal{A},\mathcal{A}_{\pm},\mathcal{A}_{\geq 0},\mathcal{A}_{\leq 0}$.

Let $\mathcal{B}$ be a basis of $\mathfrak{g}$. For $\mathsf{k}\in\mathbb{C}$, the universal affine vertex superalgebra $V^{\mathsf{k}}(\mathfrak{g})$ of $\mathfrak{g}$ at level k is generated by fields $\{X^x(z)=\sum_{n\in\mathbb{Z}}x_nz^{-n-1}\mid x\in\mathfrak{g}\}$ with OPE

$$X^x(z)X^y(w)\sim\frac{\mathsf{k}\,\kappa(x,y)}{(z-w)^2}+\frac{X^{[x,y]}(w)}{(z-w)}$$

where the bilinear form is now chosen to be the (appropriately normalized) Killing form, denoted by κ. In addition, the set $\{X^x\mid x\in\mathcal{B}\}$ strongly and freely generates $V^{\mathsf{k}}(\mathfrak{g})$.

As $\widehat{\mathfrak{g}}_{\mathsf{k}}$-modules we have $V^{\mathsf{k}}(\mathfrak{g}) \cong \mathcal{A} \otimes_{\mathcal{A}_{\geq 0}} \mathbb{C}.\,|0\rangle$, *i.e.* it is the Verma module induced from the trivial representation $\mathbb{C}.\,|0\rangle$ of $\mathcal{A}_{\geq 0}$. More generally if ρ is a representation of $\mathfrak{g}$, then ρ induces an $\mathcal{A}_{\geq 0}$-module by letting $\mathcal{A}_+$ act trivially. The Verma module of $V^{\mathsf{k}}(\mathfrak{g})$ with top level ρ is then

$$V^{\mathsf{k}}(\rho) = \mathcal{A} \otimes_{\mathcal{A}_{\geq 0}} \rho.$$

Let $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ be as usual a triangular decomposition of $\mathfrak{g}$ into a Cartan subalgebra $\mathfrak{h}$ and positive and negative part. A highest-weight vector v is an eigenvector of $\mathfrak{h}$ that is annihilated by $\mathfrak{g}_+$. If ρ is an irreducible highest-weight representation of $\mathfrak{g}$ of highest weight λ, we write $V^{\mathsf{k}}(\lambda)$ for $V^{\mathsf{k}}(\rho)$. In this case the conformal weight of the top level when $\mathsf{k} + h^\vee \neq 0$ is given by

$$h_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(\mathsf{k} + h^\vee)},$$

with ρ the Weyl vector and $h^\vee$ the dual Coxeter number of $\mathfrak{g}$.

2.D $\mathcal{W}$-algebras from the quantum Drinfeld-Sokolov reduction

The theory of $\mathcal{W}$-algebras can be approached from several directions. One point of view is as follows: given a semisimple Lie algebra $\mathfrak{g}$ with a chosen nilpotent element e, the associated $\mathcal{W}$-algebra is an associative algebra which lies between the algebras $U(\mathfrak{g})$ and $Z(\mathfrak{g})$, in other words a subquotient of $U(\mathfrak{g})$. This construction is the quantum analogue of the classical Drinfeld-Sokolov reduction of Poisson varieties. For a nilpotent element e, let $\mathcal{O}e := G \cdot e$ denote the the nilpotent orbit in the Lie group $G := Lie(\mathfrak{g})$. It has a natural transverse slice called the Slodowy slice that can be expressed as the Hamiltonian reduction of $\mathfrak{g}^*$ of a certain unipotent algebraic group. Finally $\mathcal{W}$-algebras can be interpreted as the quantization of the ring of functions on the Slodowy slice. The general algebraic construction of $\mathcal{W}$-algebras is due to [37]. We sketch it briefly in the special case of a semi-simple Lie algebra $\mathfrak{g}$.

Let (f, h, e) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Here the Cartan subalgebra element h is normalized such that $[h, e] = e$ and $[h, f] = -f$. Then $\mathfrak{g}$ is $\frac{1}{2}\mathbb{Z}$-graded by h-eigenvalues.

There exists a free field vertex superalgebra $C(\mathfrak{g}, f)$ depending on $\mathfrak{g}$ and f and the zero-mode d^f of an odd field in $C(\mathfrak{g}, f, \mathsf{k}) = V^{\mathsf{k}}(\mathfrak{g}) \otimes C(\mathfrak{g}, f)$, such that $(C(\mathfrak{g}, f, \mathsf{k}), \mathrm{d}^f)$ is a complex whose homology is a vertex superalgebra, the *$\mathcal{W}$-algebra of $\mathfrak{g}$ at level k associated to f*:

$$\mathcal{W}^{\mathsf{k}}(\mathfrak{g}, f) := H_*\big(C(\mathfrak{g}, f, \mathsf{k}), \mathrm{d}^f\big).$$

The complex has a $\mathbb{Z}$-grading described by the so-called *ghost number*. One of the main results is that all homologies vanish except in degree zero. Secondly they also prove that as graded vector space, $\mathcal{W}^{\mathsf{k}}(\mathfrak{g}, f) \cong V(\mathfrak{g}^f)$. Furthermore, a set of strong and free generators is associated to a homogeneous basis $\{g_n\}$ of the centralizer of f (denoted by $\mathfrak{g}^f$). The conformal weight of a generator $J_{g_n}(z)$ corresponding to a homogeneous element $g_n(z)$ of h-eigenvalue n is $1-n$. The homology

$$M^{\mathsf{k}}(\lambda, f) = H_*\big(C(\mathfrak{g}, f) \otimes V^{\mathsf{k}}(\lambda), \mathrm{d}^f\big)$$

is a module for $\mathcal{W}^{\mathsf{k}}(\mathfrak{g}, f)$, and it is in fact a Verma module. Thus the quantum reduction construction yields a functor from the category of $\hat{\mathfrak{g}}$-modules to $\mathcal{W}^k(\mathfrak{g}, f)$ modules.

2.D.1 Principal $\mathcal{W}$-algebras at the self-dual level

If f is principal nilpotent, then the $\mathcal{W}$-algebras thus obtained are called principal $\mathcal{W}$-algebras. In this section we give further structural results when the level is chosen to be the so called self-dual level, this appears to be special from the point of view of representation theory and also enumerative geometry.

Let $\mathfrak{g}$ be a semi-simple Lie algebra of type ADE. An important construction of principal $\mathcal{W}$-algebras is given by the coset construction. Let us formulate this result precisely. Let $V_k(\mathfrak{g})$ be the universal affine vertex algebra associated to $\mathfrak{g}$ at level k, and denote by $L_k(\mathfrak{g})$ the unique simple graded quotient of $V_k(\mathfrak{g})$. Suppose that k is an admissible level so that $L_k(\mathfrak{g})$ is an admissible representation. Consider the tensor product vertex algebra $L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})$. We define the commutant subalgebra of a sub-VOA $U \subset V$ by,

$$\mathrm{Com}(U, V) := \{a \in V | a_{(i)} v = 0, \forall v \in U, i \geq 0\}. \tag{2.33}$$

Then in [3], the authors proved that

$$\mathcal{W}^{\mathsf{k}}(\mathfrak{g}) \cong \mathrm{Com}(V^{\ell}(\mathfrak{g}), V^{\ell-1}(\mathfrak{g}) \otimes L_1(\mathfrak{g}))$$

where k and ℓ are related by the formula

$$\frac{1}{\mathsf{k}+h^\vee}+\frac{1}{\ell+h^\vee}=1,$$

with $h^\vee$ the dual Coxeter number of $\mathfrak{g}$. The large level limits, that is $\mathsf{k}\to\infty$ yields orbifolds. More precisely, we have

$$\mathcal{W}^{-h^\vee+1}(\mathfrak{g})\cong L_1(\mathfrak{g})^G,$$

with G the compact Lie group whose Lie algebra is $\mathfrak{g}$. This implies that $W^{-h^\vee+1}(\mathfrak{g})$ is in fact a subalgebra of the Heisenberg subalgebra of $L_1(\mathfrak{g})$. Another fascinating result for principal $\mathcal{W}$-algebras of ADE type called Feigin-Frenkel duality states that,

$$\mathcal{W}^{\mathsf{k}}(\mathfrak{g})\cong\mathcal{W}^{\ell}(\mathfrak{g})$$

where $(\mathsf{k}+h^\vee)(\ell+h^\vee)=1$. Thus if $\ell=-h^\vee+1$ then ℓ is at the Feigin–Frenkel self-dual level.

We denote by R and Q the set of roots and the root lattice respectively of some simple Lie algebra $\mathfrak{g}$. Then the principal $\mathcal{W}$-algebras at the self-dual level can also be described as the intersection of the Fock space $\mathcal{F}\subset V_Q$ and the kernels of all *screening operators*

$$e^\alpha{}_{(0)}=\mathrm{Res}_z Y(e^\alpha,z)\,,\qquad \alpha\in R\,.$$

From this point of view it's clear that $\mathcal{W}$-algebras are realized as subalgebras of a Heisenberg vertex algebra of the same rank as the corresponding Lie algebra. The construction for the general level and arbitrary choice of nilpotent element is carried out in [33].

In this thesis we will actually be interested in $\mathcal{W}$-algebras not of ADE type. In particular for type C, the universal principal $\mathcal{W}$-algebra of type C_N at level $-N-1/2$ is isomorphic to the orbifold of N-pairs of symplectic fermions, the reason is that the coset $\mathrm{Com}(V^k(\mathfrak{sp}_{2N}), V^k(\mathfrak{osp}_{1|2N}))$ is isomorphic to $\mathcal{W}_\ell(\mathfrak{sp}_{2N})$ for generic ℓ with ℓ and k realted via $(\ell+N+1)^{-1}+(k+N+1)^{-1}=2$ by [20, Thm. 4.1]. The limit $k\to\infty$ makes sense and in this limit the coset becomes an orbifold of a free field algebra [19, Thm. 6.10] which in this case is the $Sp(2N)$-orbifold $\mathcal{A}(N)^{Sp(2N)}$ of N-pairs of symplectic fermions $\mathcal{A}(N)$. This fact will be used in Chapter 5 to construct new Airy structures.

From the coset construction it is evident that the subgroup of G that restricts to automorphisms of the Heisenberg subalgebra leaves the $\mathcal{W}$-algebra invariant. Thus twisted modules for the Heisenberg algebra restrict to untwisted $\mathcal{W}$-modules.

Chapter 3

Topological recursion and Airy structures

In this chapter, we change our bearing towards enumerative geometry by introducing the notions of 'topological recursion' and 'Airy structures', the development in these fields have been motivated and influenced by a rich history of interactions between physics and mathematics. The topological recursion formalism was first developed by Eynard and Orantin in [22] as a means to encode the general underlying structure in the solution of various matrix models such as Kontsevich's matrix models. We first introduce Eynard and Orantin's formalism below.

3.A Topological recursion

The formalism of topological recursion takes as input a compact Riemann surface and outputs some differential forms that are related to various enumerative geometry invariants, some examples include (r-spin) intersection theory on the moduli space of curves, semi-simple cohomological field theories (CohFTs), Gromov-Witten theory on toric Calabi-Yau threefolds and the (weighted) projective line, Hurwitz theory, random matrix theory and knot theory.

Definition 3.A.1. A spectral curve is a triple (Σ, x, y) where Σ is a Torelli [1] marked compact Riemann surface and x and y are meromorphic functions on Σ, such that zeroes of dx do not coincide with the zeroes of dy. Hence they must satisfy an irreducible polynomial equation $P(x, y) = 0$.

An example is the "Airy curve" defined by $y^2 = x$ in $\mathbb{C}^2$ and this is precisely the curve connected to the Kontsevich matrix model. Another important object is the Bergmann kernel.

Definition 3.A.2. The Bergmann kernel $B(z_1, z_2)$ is the unique (after normalization over the A-cycles) symmetric bilinear differential on Σ^2 with a double pole along the diagonal $z_1 = z_2$, with leading order term

$$B(z_1, z_2) \to \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \dots \tag{3.1}$$

In this section we will confine ourselves to spectral curves with simple ramification. The generalization to higher ramification was done in [13] and looks a bit more complicated. The essence of topological recursion (TR) is a recursive structure that underlines the loop equations of matrix models. We get as output an infinite set of symmetric meromorphic differentials $\omega_{g,n}(p_1, \dots, p_n)$ on Σ^n for $g, n \in \mathbb{N}$.

Definition 3.A.3. Let (Σ, x, y) be a spectral curve with simple ramification, and $\pi : \Sigma \mapsto \mathbb{P}^1$ a branched covering given by the meromorphic function x,

[1] A Torelli marked compact Riemann surface is a genus g Riemann surface with a choice of symplectic basis of cycles $(A_1, \dots, A_g, B_1, \dots, B_g) \in H_1(\Sigma, \mathbb{Z})$.

and $R \subset \Sigma$ the set of ramification points of π. The initial data is given by,

$$\omega_{0,1}(p) := y(p)dx(p), \quad \omega_{0,2}(p_1, p_2) := B(p_1, p_2) \tag{3.2}$$

for $P = \{p_1, \ldots, p_n\} \in \Sigma^n$. For $n \geq 0, g \geq 0$ and $2g - 2 + n \geq 0$, we uniquely construct symmetric meromorphic differentials $\omega_{g,n}$ on Σ^n with poles along R via the formula,

$$\omega_{g,n+1}(p_0; P) = \sum_{a \in R} Res_{p=a} \left(\frac{\int_\alpha^p B(\cdot, p_0)}{\omega_{0,1}(p) - \omega_{0,1}(\iota_a(p))} \mathcal{R}_{g,n}(p, \iota_\alpha(p); P) \right), \tag{3.3}$$

where α is an arbitrary base point on Σ and ι_α is the locally defined involution around the branch point a. The symbol $\mathcal{R}_{g,n}$ encodes the recursive structure and is defined as,

$$\mathcal{R}_{g,n}(q_1, q_2; P) := \omega_{g-1,n+2}(q_1, q_2; P) + \sum_{\substack{g_1+g_2=g \\ R_1 \cup R_2 = P}}' \omega_{g_1,|R_1|+1}(q_1; R_1)\omega_{g_2,|R_2|+1}(q_2; R_2) \tag{3.4}$$

for $g_1, g_2 \in \mathbb{N}$ and the $'$ denotes that we omit the cases $(g_1, r_1) = (0, \phi)$ and $(g_2, r_2) = (0, \phi)$.

The topological recursion formalism (for arbitrary ramification) lets us compute higher genus data for a large number of enumerative invariants such as Gromov-witten invariants of toric CY threefolds, Hurwitz numbers, r-spin numbers, Weyl-Petersson volumes starting from a choice of a spectral curve and genus 0 data. We give below an example of non-simply ramified spectral curves, that are relevant to the contents of this thesis. We do not treat this point of view any further in this thesis, the interested reader can refer to [9] and [8] for recent work in this area. Consider the family of spectral curves indexed by (r, s) with $r \in \mathbb{Z}_{r\geq 2}$ and $s = \{1, \ldots, r+1\}$ for $r = \pm 1 \mod s$ and parameterised by the equations,

$$x = \frac{z^r}{r},\ y = -\frac{1}{z^{r-s}}. \tag{3.5}$$

These curves are related to examples of Higher Airy structures constructed in [9]. Higher Airy structures are an algebraic reformulation of topological recursion. We will now discuss this in the next section. We abbreviate the topological recursion of Eynard-Orantin discussed above as TR in the following sections.

3.B Higher Airy structures

In [39], Kontsevich and Soibelman sought an algebraic description of the Eynard-Orantin topological recursion, the initial data for which is a set of atmost quadratic differential operators on a vector space V which gives as ouput a formal series of function on V that are simultaenously annihilated by these operators. When V is infinite dimensional we have convergence issues that need to be examined by imposing filtrations or a topology on V. We ignore these technicalities for now and will discuss them in the next section. This was generalized to differential operators of order higher than just quadratic order in [9] to 'Higher Airy structures'. We will state some results of this paper below as a starting point of our study. In Chapter 4 we present an alternate approach to defining and studying Airy structures -as left ideals of graded Rees Weyl algebras obtained as images of certain automorphisms called 'transvections'. However, we first present the conventional approach in existing literature for clarity and completeness.

Let E be an infinite-dimensional $\mathbb{C}$-vector space indexed by a set A. Let $(e_a)_{a\in A}$ be a basis of E and let $(x_a)_{a\in A}$ be the corresponding dual basis of the dual space $E^* = \bigoplus_{k>0} \mathbb{C}\langle x_k \rangle$ indexed by the set E and $\hbar$ be a formal parameter. We make some essential definitions.

Definition 3.B.1. Let $\mathcal{D}_A^\hbar \cong \mathbb{C}[[\hbar, (x_a)_{a\in A}, (\hbar\partial_{x_a})_{a\in A}]]$ be a certain completion of the algebra of differential operators. We define an algebra grading by assigning,

$$\deg(x_l) = \deg(\hbar\partial_{x_l}) = 1, \quad \deg(\hbar) = 2. \tag{3.6}$$

Remark 3.B.2. Firstly, we remark that the $\hbar$-parameter and grading will be introduced in a different way in Chapter 4, and this new convention will be used in rest of the thesis, hence the reader should be careful of not confusing the two notations. We start here by making the more conventional definition appearing in existing literature. The exact relation between the two conventions is outlined in Remark 4.B.8. Secondly, we will define completions of the Weyl Algebra mentioned above more precisely in the next chapter.

We reproduce the definition of a Higher Airy structure introduced in [9]:

Definition 3.B.3. A higher quantum Airy structure on E in the normal form is a family of differential operators $(H_k)_{k\in A}$ of the form,

1. $$H_k = \hbar\partial_{x_k} - P_k, \tag{3.7}$$

   where $P_k \in \mathcal{D}_A^\hbar$ is a sum of terms of degree ≥ 2.

2. Moreover, we require that the left $\mathcal{D}^{\hbar}_A$-ideal generated by the H_k's forms a graded Lie subalgebra i.e. there exists $g^{k_3}_{k_1,k_2} \in \mathcal{D}^{\hbar}$ such that,

$$[H_{k_1}, H_{k_2}] = \hbar \sum_{k_3 \in A} g^{k_3}_{k_1,k_2} H_{k_3}. \tag{3.8}$$

The properties in the above definition are chosen precisely so that the following holds true.

Theorem 3.B.4. *[39] Let E be a finite-dimensional vector space. Given a higher Airy structure $(H_k)_{k\in A}$ in the normal form, the system of equations,*

$$\forall k \in A, \quad H_k \cdot Z = 0, \tag{3.9}$$

has a unique solution of the form,

$$Z = \exp\left(\sum_{\substack{g\geq 0, n\geq 1 \\ 2g-2+n\geq 0}} \frac{\hbar^{g-1}}{n!} F_{g,n} \right), \quad F_{g,n} \in \mathrm{Sym}^n(E^*). \tag{3.10}$$

The formal power series Z is also referred to as the partition function of the Airy structure.

Remark 3.B.5. The above theorem provides a recursive relation on the $F_{g,n}$ with initial data given by certain coefficients of the operators $(H_i)_{i\in A}$. The correspondence between the $F_{g,n}$'s computed from a higher Airy structure and $\omega_{g,n}$'s of Eynard-Orantin topological can be understood as follows. Given a spectral curve with ramification points in the set $r_i \in R$, we can expand the $\omega_{g,n}$'s around the ramification points in a local basis of meromorphic differentials $\{\zeta_{k_i,r_i}\}_{k_i\in\mathbb{N}, r_i\in R}$, to get the scalars $F_{g,n}$ as coefficients. Relevant results and proofs can be found in [9].

A large number of examples of higher Airy structures can be constructed using free field realizations of modules of $\mathcal{W}$-algebras. The partition functions of these are known to be generating functions of enumerative geomety invariants such as Gromov-Witten invariants, r-spin intersection numbers, Hurwitz numbers etc and some of these can also be obtained as partition functions of matrix models. We give below one such example.

Definition 3.B.6. Let $\{J^-_{2n}\}_{n\in\mathbb{Z}}$ and $\{J_{2n+1}\}_{n\in\mathbb{Z}}$ be generators of two Heisenberg algerbas respectively. Consider the set of operators $\{H^2_k, H^3_k\}_{k\in\mathbb{Z}_{\geq 0}}$,

$$H^2_k = J_{2k+1} - \frac{1}{2}\left(\sum_{a+b=k-1} :J^-_{2a}J^-_{2b}: + :J_{2a-1}J_{2b+1}:\right) - \frac{\hbar}{8}\delta_{k,1} \tag{3.11}$$

$$\begin{aligned} H^3_k &= J^-_{2(k+1)} + 2\sum_{k_1+k_2=k-1} J_{2k_1+1}J^-_{2k_2} - \frac{1}{3}\sum_{a+b+k_2=k-2} J^-_{2a}J^-_{2b}J^-_{2k_2} \\ &\quad + \sum_{a+b+k_2=k-1} J_{2a-1}J_{2b+1}J^-_{2k_2} + \frac{\hbar}{4}J^-_{2(k-2)}. \end{aligned} \tag{3.12}$$

We identify negative indexed modes with variables, $J^-_{-2n} := x_{2n}, J_{-2n+1} := x_{2n-1}$ for $n > 0$ and $J_0 := \hbar^{1/2}q$ for some complex constant q. The operators form an Airy structure acting on the vector space $E := \bigoplus_k \mathbb{C}\langle x_k\rangle$.

We remark on the connections to $\mathcal{W}$-algebras, matrix models, and enumerative geometry as promised. The interested reader can refer to [9] for further details.

1. The operators $\{H^2_k, H^3_k\}$ form a representation of the $\mathcal{W}(\mathfrak{sl}_3)$ algebra.

2. The partition function Z is that of the Kontsevich-Penner matrix model

described by the integral,

$$Z = \int [dB] \exp \mathrm{Tr} \left(-\frac{B^3}{3} + \Lambda B + klogB \right) \tag{3.13}$$

and the time variables,

$$t_n = \mathrm{Tr}\Lambda^{-n/3} \quad n \geq 1 \tag{3.14}$$

such that the time $t_n \propto x_n$. The matrix of integration B is an $M \times M$ Hermitian matrix and the measure $[dB]$ is the Lebesgue measure given by the product of the Lebesgue measures of all real components of the matrix B. It is assumed that its size M tends to infinity, and the parameter of deformation k is completely independent of B.

3. The partition function annihilated by the Airy structure constraints generate certain intersection numbers on the moduli space of open Riemann surfaces known as the extended open partition function,

$$\tau_o = \exp \left(\hbar^{2g+b-2} q^b F_{(g,b),n} \right). \tag{3.15}$$

where $F_{(g,b),n}$ are intersection numbers of the open analogues of the ψ classes on the moduli space of Riemann surfaces of genus g with n marked points and b boundaries.

4. In [15], it was proved that τ_0 satisfy the "open KdV integrable hierarchy".

It is an interesting problem to understand the relationship between different enumerative invariants. For instance, the dualities between open and closed string are central in physics. In [32], Gaiotto and Rastelli prove a correspondence between the Kontsevich matrix model and open string field theory. This inspires the following problem.

Problem 3.B.7. Let Z_c and Z_o be the partition function of closed and open intersection numbers respectively. These are known to be partition functions of Airy structures coming from certain modules of $\mathcal{W}(\mathfrak{gl}_2)$ and $\mathcal{W}(\mathfrak{sl}_3)$ respectively. Consider the embedding,

$$\mathcal{W}(\mathfrak{sl}_3) \hookrightarrow \mathcal{W}(\mathfrak{gl}_2) \otimes \mathcal{W}(\mathfrak{gl}_1) \subset \mathcal{H}_3 \tag{3.16}$$

where $\mathcal{H}_3$ is a twisted module of a rank 3 Heisenberg algebra. Is there an operator $\mathcal{A}$ on $\mathcal{H}_3$ such that,

$$\mathcal{A}(Z_c) = Z_o. \tag{3.17}$$

We have some preliminary results with V. Bouchard and T. Creutzig towards such a correpondence, but a definite proof has not been obtained yet. However, this is not the topic of this thesis and we will not discuss it any further.

Remark 3.B.8. Symmetry under the change of integration variables of the matrix integral (3.13) yield an infinite set of differential equations satisfied by Z, named the Schwinger-Dyson (SD) constraints of the matrix model. We have explicitly checked that these SD constraints are indeed the same as the Airy structures coming from twisted modules of the $\mathcal{W}(\mathfrak{sl}_3)$ and $\mathcal{W}(\mathfrak{sl}_4)$ algebras after a re-scaling of the coordinates.

Remark 3.B.9. In [9], the authors construct Airy structures from certain modules of $\mathcal{W}(\mathfrak{g})$-algebra for $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{so}_{2n}, \mathfrak{e}_n$ that is $\mathcal{W}$-algebras of the A-D-

E type. The general method can be summarized as follows: Let $\mathfrak{g}$ be a Lie algebra and σ an element of the Weyl group of $\mathfrak{g}$. Let $\mathcal{W}(\mathfrak{g})$ be the principal $\mathcal{W}$-algerba of $\mathfrak{g}$ at the self-dual level $k = -h^\vee + 1$ (where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

1. Construct a σ- twisted module of the Heisenberg vertex operator algebra. The positive and negative modes of the generators $\{J_n\}_{n\in\mathbb{Z}}$ can be realized as the differential operators that act as derivatives and multiplication by variables respectively on the space on formal series.

2. Upon restriction to $\mathcal{W}(\mathfrak{g})$, we get an untwisted $\mathcal{W}(\mathfrak{g})$-module.

3. Pick a subset of modes generating a left ideal which is a graded Lie subalgebra of the algebra of modes.

4. Conjugate these modes by an operator of the form $\exp(\frac{J_s}{s\hbar})$ for $s > 0$, to bring them in the form of a higher quantum Airy structure.

The exposition in this section sweeps a lot of subtle nuances under the rug. For starters, when considering Weyl algebra in an infinite number of variables one quickly stumbles into issues of convergence of infinite sums. Hence one is forced to define objects such as the Weyl algebra and formal power series more precisely as completions of 'finite variable' objects. Secondly, the $\hbar$ parameter and the associated grading was introduced in an ad-hoc way, which could be a bit puzzling. Finally, the insistence of (3.8) in the definition and its role in the proof of existence and uniqueness of partition functions if a bit opaque. We propose resolutions of these issues in the following chapter by a more technical but precise presentation. In particular we resort to the 'Rees construction' of algebraic geometry to introduce the parameter $\hbar$. More interestingly, we give an alternate definition of the notion of Airy structures as left ideals in a completion of the Rees Weyl algebra (called 'Airy ideals') and obtain a alternative proof of the uniqueness and existence of partition functions. As explained in Chapter 1, the crux of the proof is the realization of Airy ideals as images of canonical left ideals obtained by a family of automorphisms called 'transvections' . This is done with the goal of making the essential features in the definition of Airy structures perhaps more transparent.

Chapter 4

Airy ideals and transvections of Rees algebras

In this chapter we propose a different point of view on the definition of Airy structures, as explained in the introduction. Contents of this chapter are almost identical to [12] except for section 4.I.

4.A Preliminaries

4.A.1 Cyclic modules and twisted modules

We first review basic concepts in the theory of modules that will be needed later on. Let $\mathcal{D}$ be an associative algebra over $\mathbb{C}$,[1] and $\mathcal{M}$ a left $\mathcal{D}$-module. We write $r \cdot m \in \mathcal{M}$ for the action of $r \in \mathcal{D}$ on $m \in \mathcal{M}$.

Definition 4.A.1. The *annihilator* of an element $v \in \mathcal{M}$, which is denoted

[1] We could work over any field $\mathbb{K}$ of characteristic zero instead of $\mathbb{C}$.

by $\mathrm{Ann}_{\mathcal{D}}(v)$, is defined as

$$\mathrm{Ann}_{\mathcal{D}}(v) = \{P \in \mathcal{D} \mid P \cdot v = 0\}. \tag{4.1}$$

It is naturally a left ideal in $\mathcal{D}$.

Definition 4.A.2. A left $\mathcal{D}$-module $\mathcal{M}$ is *cyclic* if it is generated by a single element $v \in \mathcal{M}$.

It is easy to show that a cyclic left $\mathcal{D}$-module $\mathcal{M}$ generated by $v \in \mathcal{M}$ is canonically isomorphic to $\mathcal{D}/\mathrm{Ann}_{\mathcal{D}}(v)$.

We will also need the notion of a twisted module with respect to an automorphism $\phi : \mathcal{D} \to \mathcal{D}$.

Definition 4.A.3. Let $\phi : \mathcal{D} \to \mathcal{D}$ be an automorphism, and $\mathcal{M}$ a left $\mathcal{D}$-module. The twisted module ${}^{\phi}\mathcal{M}$ is given by the same vector space as $\mathcal{M}$, but with the new operation

$$r \cdot_{\phi} m = \phi^{-1}(r) \cdot m. \tag{4.2}$$

It is easy to show that, if $\mathcal{M}$ is a cyclic left $\mathcal{D}$-module generated by $v \in \mathcal{M}$, then the twisted module ${}^{\phi}\mathcal{M}$ is also cyclic and generated by v. Furthermore, the annihilator of $v \in {}^{\phi}\mathcal{M}$, which we denote by ${}^{\phi}\mathrm{Ann}_{\mathcal{D}}(v)$ to avoid confusion with the annihilator $\mathrm{Ann}_{\mathcal{D}}(v)$ of v in $\mathcal{M}$, is:

$${}^{\phi}\mathrm{Ann}_{\mathcal{D}}(v) = \phi\left(\mathrm{Ann}_{\mathcal{D}}(v)\right). \tag{4.3}$$

It then follows that ${}^{\phi}\mathcal{M}$ is canonically isomorphic to $\mathcal{D}/\phi\left(\mathrm{Ann}_{\mathcal{D}}(v)\right)$.

4.A.2 Filtrations, Rees algebras and Rees modules

We now review the Rees construction for filtered algebras and modules. We write $\mathbb{N}$ for the set of non-negative integers, and $\mathbb{N}^*$ for the set of positive integers.

Definition 4.A.4. An *exhaustive ascending filtration* on $\mathcal{D}$ is an increasing sequence of subspaces $F_i\mathcal{D} \subseteq \mathcal{D}$, for $i \in \mathbb{N}$:

$$\{0\} \subseteq F_0\mathcal{D} \subseteq F_1\mathcal{D} \subseteq F_2\mathcal{D} \subseteq \ldots \subseteq \mathcal{D}, \tag{4.4}$$

such that $\cup_{i\in\mathbb{N}} F_i\mathcal{D} = \mathcal{D}$ and $F_i\mathcal{D} \cdot F_j\mathcal{D} \subseteq F_{i+j}\mathcal{D}$ for all $i, j \in \mathbb{N}$. An algebra $\mathcal{D}$ with such a filtration is called a *filtered algebra*.

Filtered modules are defined in a similar way.

Definition 4.A.5. Let $\mathcal{M}$ be a left $\mathcal{D}$-module. An *exhaustive ascending filtration* on $\mathcal{M}$ is given by an increasing sequence of subspace $F_i\mathcal{M} \subset \mathcal{M}$ for $i \in \mathbb{N}$:

$$\{0\} \subseteq F_0\mathcal{M} \subseteq F_1\mathcal{M} \subseteq F_2\mathcal{M} \subseteq \ldots \subseteq \mathcal{M}, \tag{4.5}$$

such that $\cup_{i\in\mathbb{N}} F_i\mathcal{M} = \mathcal{M}$ and $F_i\mathcal{D} \cdot F_j\mathcal{M} \subseteq F_{i+j}\mathcal{M}$ for all $i, j \in \mathbb{N}$. A left $\mathcal{D}$-module $\mathcal{M}$ with such a filtration is called a *filtered module*.

From a filtered algebra, we can construct a graded algebra in a natural way: this is the Rees construction. Note that this construction is different from the standard associated graded algebra $\mathrm{Gr}(\mathcal{D}) = \bigoplus_{n=1}^{\infty} \mathcal{G}_n$ with $\mathcal{G}_n = F_n\mathcal{D}/F_{n-1}\mathcal{D}$.

Definition 4.A.6. Given a filtered algebra $\mathcal{D}$, we define the *Rees algebra* $\mathcal{D}^\hbar$ as:

$$\mathcal{D}^\hbar = \bigoplus_{n\in\mathbb{N}} \hbar^n F_n\mathcal{D}. \tag{4.6}$$

It is a graded algebra, graded by $\hbar$ (with $\deg(\hbar) = 1$). When needed, we write $\mathcal{D}^\hbar_n := \hbar^n F_n\mathcal{D}$ for the subspace of homomegeneous elements of degree n.

It will be very important for us to consider not only Rees algebras, but also their completions with respect to the $\hbar$-adic topology.

Definition 4.A.7. We define the *completed Rees algebra* $\widehat{\mathcal{D}}^\hbar$ as

$$\widehat{\mathcal{D}}^\hbar = \prod_{n\in\mathbb{N}} \hbar^n F_n\mathcal{D}, \tag{4.7}$$

which is the completion with respect to the $\hbar$-adic topology. Explicitly, an element $P \in \widehat{\mathcal{D}}^\hbar$ can be written as a formal power series in $\hbar$:

$$P = \sum_{n=0}^{\infty} \hbar^n P_n, \tag{4.8}$$

for some $P_n \in F_n\mathcal{D}$.

The same Rees construction can be applied to filtered modules.

Definition 4.A.8. Given a filtered $\mathcal{D}$-module $\mathcal{M}$, we define the *Rees module* $\mathcal{M}^\hbar$ as

$$\mathcal{M}^\hbar = \bigoplus_{n\in\mathbb{N}} \hbar^n F_n\mathcal{M}. \tag{4.9}$$

It is a graded left $\mathcal{D}^\hbar$-module, and we write $\mathcal{M}^\hbar_n = \hbar^n F_n\mathcal{M}$ for the subspace of homogeneous elements. We define the completed Rees module $\widehat{\mathcal{M}}^\hbar$ as

$$\widehat{\mathcal{M}}^\hbar = \prod_{n\in\mathbb{N}} \hbar^n F_n\mathcal{M}, \tag{4.10}$$

which is the completion with respect to the $\hbar$-adic topology.

4.B Rees Weyl algebra

We now apply the Rees construction to the Weyl algebra (in either a finite or countably infinite number of variables). Generalization to infinite number of variables requires us to define suitable completions. A brief survey of the categorical notion of completions is undertaken in Appendix A and examples related to the Weyl algebra are expounded on in B.

Let A be an index subset, either finite or countably infinite. We write $x_A = \{x_a\}_{a\in A}$ for the set of variables x_a with $a \in A$, and $\partial_A = \{\partial_a\}_{a\in A}$ for the set of partial derivatives ∂_a with respect to the variables x_a.

Definition 4.B.1. If A is a finite index set, we define the *Weyl algebra* $\mathcal{D}_A = \mathbb{C}[x_A]\langle\partial_A\rangle$ to be the algebra of differential operators over the polynomial ring $\mathbb{C}[x_A]$ in the variables x_A. $\mathcal{D}_A$ is the free associative algebra over $\mathbb{C}$ generated by $\{x_A, \partial_A\}$ modulo the commutation relations

$$[x_a, x_b] = 0, \qquad [\partial_a, \partial_b] = 0, \qquad [\partial_a, x_b] = \delta_{ab}, \qquad \forall a, b \in A. \tag{4.11}$$

In the case where A is countably infinite, we define $\mathcal{D}_A$ to be a particular completion of the Weyl algebra $\mathbb{C}[x_A]\langle\partial_A\rangle$.

Definition 4.B.2. If A is a countably infinite index set, we define the *completed Weyl algebra* $\mathcal{D}_A$ to be the completion of the Weyl algebra $\mathbb{C}[x_A]\langle\partial_A\rangle$ that contains potentially infinite sums in the derivatives, but with polynomial coefficients. Elements of $\mathcal{D}_A$ remain of finite order as differential operators. (A more precise definition appears in Example (B.1) of Appendix (A))[2]

[2]This completion should not be confused with the $\hbar$-adic completion of Rees algebras and modules discussed in the previous section.

In other words, we can write an element $P \in \mathcal{D}_A$ uniquely as

$$P = \sum_{m=0}^{M} \sum_{a_1,\dots,a_m \in A} p_{a_1\cdots a_m}(x_A)\partial_{a_1}\cdots\partial_{a_m}, \tag{4.12}$$

for some $M \in \mathbb{N}$, where the $p_{a_1\cdots a_m}(x_A)$ are polynomials in the variables x_A. If A is a countably infinite index set, we see that the sums over the indices a_i can be infinite, but the coefficients are always polynomial (they cannot include infinite sums of monomials). For example, what this means is that an operator like $\sum_{a\in A}\partial_a$ is in $\mathcal{D}_A$, while $\sum_{a\in A} x_a$ is not.

There is a natural exhaustive ascending filtration on $\mathcal{D}_A$ called the filtration. To construct it, we give degree one to the variables x_a and the partial derivatives ∂_a, and define the subspaces $F_i\mathcal{D}_A$ as containing all operators in $\mathcal{D}_A$ of degree $\leq i$. More precisely:

Definition 4.B.3. The *Bernstein filtration* on $\mathcal{D}_A$ is defined by

$$F_i\mathcal{D}_A = \left\{ \sum_{\substack{m,k\in\mathbb{N}\\ m+k=i}} \sum_{a_1,\dots,a_m\in A} p^{(k)}_{a_1\cdots a_m}(x_A)\partial_{a_1}\cdots\partial_{a_m} \right\}, \tag{4.13}$$

where the $p^{(k)}_{a_1\cdots a_m}(x_A)$ are polynomials of degree $\leq k$. Here, $F_0\mathcal{D}_A = \mathbb{C}$.

From the definition of $\mathcal{D}_A$ and its Bernstein filtration, it is clear that

$$[F_m\mathcal{D}_A, F_n\mathcal{D}_A] \subseteq F_{m+n-2}\mathcal{D}_A. \tag{4.14}$$

As in the previous section, we construct the Rees algebra associated to the filtered algebra $\mathcal{D}_A$ with the Bernstein filtration. (See also Example (B.2) in the Appendix)

Definition 4.B.4. The *Rees Weyl algebra* $\mathcal{D}_A^\hbar$ associated to $\mathcal{D}_A$ with the Bernstein filtration is

$$\mathcal{D}_A^\hbar = \bigoplus_{n\in\mathbb{N}} \hbar^n F_n \mathcal{D}_A, \tag{4.15}$$

which is a graded algebra, graded by $\hbar$ with $\deg(\hbar) = 1$. We write $\mathcal{D}_{A,n}^\hbar = \hbar^n F_n \mathcal{D}_A$ for homogeneous elements of degree n. We define its $\hbar$-adic completion, as in Definition 4.A.7:

$$\widehat{\mathcal{D}}_A^\hbar = \prod_{n\in\mathbb{N}} \hbar^n F_n \mathcal{D}_A, \tag{4.16}$$

From (4.14) we get that

$$[\widehat{\mathcal{D}}_{A,m}^\hbar, \widehat{\mathcal{D}}_{A,n}^\hbar] \subseteq \hbar^2 \widehat{\mathcal{D}}_{A,m+n-2}^\hbar. \tag{4.17}$$

Remark 4.B.5. The Rees Weyl algebra $\mathcal{D}_A^\hbar \subset \widehat{\mathcal{D}}_A^\hbar$ is the subalgebra consisting of differential operators that are polynomials in $\hbar$. Note that we can also think of the Rees Weyl algebra $\mathcal{D}_A^\hbar$ as the free associative algebra over $\mathbb{C}$ generated by $\{\hbar, \hbar x_A, \hbar\partial_A\}$, where $\hbar$ is a central element, and the other generators satisfy the commutation relations

$$[\hbar x_a, \hbar x_b] = 0, \qquad [\hbar\partial_a, \hbar\partial_b] = 0, \qquad [\hbar\partial_a, \hbar x_b] = \hbar^2\delta_{ab}, \qquad \forall a, b \in A. \tag{4.18}$$

Example 4.B.6. To clarify the notation, an operator $P \in \widehat{\mathcal{D}}_A^\hbar$ can be written as

$$P = \sum_{n\in\mathbb{N}} \hbar^n \sum_{\substack{m,k\in\mathbb{N} \\ m+k=n}} \sum_{a_1,\ldots,a_m\in A} p_{a_1\cdots a_m}^{(n,k)}(x_A)\partial_{a_1}\cdots\partial_{a_m}, \tag{4.19}$$

where the $p_{a_1\cdots a_m}^{(n,k)}(x_A)$ are polynomials of degree $\leq k$.

Because of the subtelties arising due to infinite sums when A is a countably infinite index set, we need to define a particular property for collections of operators in $\widehat{\mathcal{D}}^{\hbar}_A$, which we call boundedness (this condition is called "filtered family of operators" in [10] – see Section 2.1.2).

Definition 4.B.7. Let I be a finite or countably infinite index set, and $\{P_i\}_{i\in I}$ be a collection of operators $P_i \in \widehat{\mathcal{D}}^{\hbar}_A$ of the form

$$P_i = \sum_{n\in\mathbb{N}} \hbar^n \sum_{\substack{m,k\in\mathbb{N}\\ m+k=n}} \sum_{a_1,\ldots,a_m\in A} p^{(n,k)}_{i;a_1\cdots a_m}(x_A)\partial_{a_1}\cdots\partial_{a_m}, \tag{4.20}$$

We say that the collection of operators $\{P_i\}_{i\in I}$ is *bounded* if, for all fixed choice of indices $a_1,\ldots,a_m$, n, and k, the polynomials $p^{(n,k)}_{i;a_1\cdots a_m}(x_A)$ vanish for all but finitely many indices $i \in I$. We note that the condition is trivially satisfied if I is a finite index set.

There is a fundamental reason why we consider bounded collection of differential operators. In the following we will study left ideals $\mathcal{I}$ in $\widehat{\mathcal{D}}^{\hbar}_A$ consisting of all $\widehat{\mathcal{D}}^{\hbar}_A$-linear combinations of a collection of operators $\{P_i\}_{i\in I}$; that is, any $Q \in \mathcal{I}$ can be written as

$$Q = \sum_{i\in I} c_i P_i \tag{4.21}$$

for some $c_i \in \widehat{\mathcal{D}}^{\hbar}_A$. If I is a finite index set, this is the left ideal generated by the collection of operators $\{P_i\}_{i\in I}$. However, if I is a countably infinite index set, we will want our ideal $\mathcal{I}$ to contain not only finite $\widehat{\mathcal{D}}^{\hbar}_A$-linear combinations of the P_i, but also infinite ones.[3] But if $\{P_i\}_{i\in I}$ is an arbitrary collection of operators,

[3] We will often abuse notation and still say that this ideal is "generated" by the P_i, even though an ideal generated by a set only contains finite linear combination of the elements in the set, regardless of whether the set is finite or countably infinite. For us, we always include both finite and infinite linear combinations when the generating set is countably infinite.

infinite $\widehat{\mathcal{D}}_A^\hbar$-linear combinations of the P_i may give rise to divergent infinite sums, or to operators whose coefficients are not polynomials in the variables x_A (they may contain infinite sums of monomials). However, if the collection $\{P_i\}_{i\in I}$ is bounded, this cannot happen; in this case, it is straightforward to show that infinite $\widehat{\mathcal{D}}_A^\hbar$-linear combinations of the P_i are always well defined operators whose coefficients are polynomials in the variables x_A (since they are finite sums of polynomials), and therefore in $\widehat{\mathcal{D}}_A^\hbar$. This is the key reason why we consider bounded collection of operators in $\widehat{\mathcal{D}}_A^\hbar$.

Remark 4.B.8. As mentioned in remark (3.B.2), for the readers familiar with the literature on Airy structures, a word of caution is required at this stage. Our $\hbar$ differs from the usual $\hbar$ in the literature on Airy structures. More precisely, as should become clear later, to connect the two approaches, one should start with the traditional definition of Airy structures (for instance in [39, 9]), rescale the variables as $x_i \mapsto \hbar^{1/2} x_i$, and then redefine $\hbar \mapsto \hbar^2$. With this transformation, the grading defined in [39, 9] becomes the natural $\hbar$-grading on the Rees algebra that we introduce here, with $\deg(\hbar) = 1$.

We could also introduce $\hbar$ as in the traditional literature on Airy structures using the Rees construction. What we would need to do then is consider a different filtration on the Weyl algebra, namely the "order filtration" instead of the Bernstein filtration, which is defined by

$$F_i \mathcal{D}_A = \left\{ \sum_{m=0}^{i} \sum_{a_1,\ldots,a_m \in A} p_{a_1\cdots a_m}(x_A) \partial_{a_1} \cdots \partial_{a_m} \right\}, \tag{4.22}$$

where the polynomials $p_{a_1\cdots a_m}(x_A)$ have arbitrary degree. In other words, $F_i\mathcal{D}_A$ consists of differential operators of order at most i, but with polynomial coefficients of arbitrary degree (it corresponds to giving degree one to the partial derivatives ∂_a, but degree zero to the variables x_a). We could then define the corresponding Rees algebra; the result would be the standard $\hbar$-dependent Weyl algebra considered in the literature on Airy structures.

Although the two approaches are ultimately equivalent, we find the introduction of $\hbar$ via the Bernstein filtration instead of the order filtration more natural and transparent, as, among other things, it allows us to work with the Rees polynomial module (i.e. we also introduce $\hbar$ for a $\mathcal{D}^{\hbar}_A$-module via the Rees construction), and we don't need to deal with formal power series in the variables x_A, as will become clear later – we only need to consider the $\hbar$-adic completions for the Weyl algebra and its polynomial module.

4.C Left $\widehat{\mathcal{D}}^{\hbar}_A$-modules

4.C.1 Polynomial $\mathcal{D}_A$-module

A natural left $\mathcal{D}_A$-module is the polynomial algebra $\mathcal{M}_A = \mathbb{C}[x_A]$, where the action is given by the standard action of differential operators on polynomials. In the case where A is countably infinite, one should be a little bit careful here, since our algebra $\mathcal{D}_A$ is the completion of the Weyl algebra, which includes potentially infinite sums over the derivatives. However, since $\mathcal{M}_A$ is a polynomial algebra, the action of differential operators in $\mathcal{D}_A$ on polynomials always collapses the infinite sums to finite sums, and so the action is well defined.[4]

[4]For instance, we need to be careful that we don't encounter situations like the differential operator $\sum_{a\in A} \partial_a$ acting on $\sum_{b\in A} x_b$, since $\sum_{a\in A} \partial_a \left(\sum_{b\in B} x_b\right) = \sum_{a\in A} 1$ which is of course divergent. But, while $\sum_{a\in A} \partial_a$ is in $\mathcal{D}_A$, $\sum_{b\in A} x_b$ is not in $\mathcal{M}_A$ since it is not a polynomial. We never run into this kind of issues because all infinite sums collapse to finite sums, since all elements of $\mathcal{M}_A$ are polynomials, even if we work with infinitely many variables.

The polynomial module $\mathcal{M}_A$ is a cyclic left $\mathcal{D}_A$-module, generated by $1 \in \mathcal{M}_A$. Moreover, the annihilator of 1 is

$$\mathrm{Ann}_{\mathcal{D}_A}(1) = \left\{ \sum_{a \in A} c_a \partial_a \mid c_a \in \mathcal{D}_A \right\}, \tag{4.23}$$

which is the left ideal consisting of $\mathcal{D}_A$-linear combinations of the derivatives. It is clear that $\mathcal{M}_A$ is canonically isomorphic to $\mathcal{D}_A/\mathrm{Ann}_{\mathcal{D}_A}(1)$.

4.C.2 Rees polynomial $\widehat{\mathcal{D}}^{\hbar}_A$-module

Let us now apply the Rees construction to the polynomial $\mathcal{D}_A$-module. We can define many filtrations on the polynomial algebra $\mathcal{M}_A$ that are compatible with the Bernstein filtration on $\mathcal{D}_A$. We will use the following standard filtration.

Definition 4.C.1. We define the *degree filtration* on $\mathcal{M}_A$ as:

$$F_i \mathcal{M}_A = \{\text{polynomials of degree} \leq i\}. \tag{4.24}$$

It is easy to check that $\mathcal{M}_A$ with this filtration is a filtered $\mathcal{D}_A$-module. We then apply the Rees construction for filtered modules.

Definition 4.C.2. We define the Rees polynomial module associated to the degree filtration and its $\hbar$-adic completion:

$$\mathcal{M}^{\hbar}_A = \bigoplus_{n \in \mathbb{N}} \hbar^n F_n \mathcal{M}_A, \qquad \widehat{\mathcal{M}}^{\hbar}_A = \prod_{n \in \mathbb{N}} \hbar^n F_n \mathcal{M}_A. \tag{4.25}$$

Both are graded left $\mathcal{D}^{\hbar}_A$-modules, and the completed module $\widehat{\mathcal{M}}^{\hbar}_A$ is also a left $\widehat{\mathcal{D}}^{\hbar}_A$-module.

Example 4.C.3. To clarify the notation, an element $f \in \widehat{\mathcal{M}}^{\hbar}_A$ is a formal $\hbar$-power series

$$f = \sum_{n=0}^{\infty} \hbar^n f^{(n)}(x_A) \tag{4.26}$$

where the $f^{(n)}(x_A)$ are polynomials in the variables x_A of degree $\leq n$.

In the following we will mostly be interested in the completed module $\widehat{\mathcal{M}}^{\hbar}_A$, realized as a left $\widehat{\mathcal{D}}^{\hbar}_A$-module. It is easy to show that it is a cyclic module, generated by $1 \in \widehat{\mathcal{M}}^{\hbar}_A$. The annihilator of 1 is:

$$\mathrm{Ann}_{\widehat{\mathcal{D}}^{\hbar}_A}(1) = \left\{ \sum_{a \in A} c_a \hbar \partial_a \mid c_a \in \widehat{\mathcal{D}}^{\hbar}_A \right\} =: \mathcal{I}_{\mathrm{can}}, \tag{4.27}$$

which is the left ideal consisting of $\widehat{\mathcal{D}}^{\hbar}_A$-linear combinations of the derivatives $\hbar\partial_a$. As we will refer to this canonical left ideal many times in the following, we introduce the shorthand notation $\mathcal{I}_{\mathrm{can}}$. $\widehat{\mathcal{M}}^{\hbar}_A$ is canonically isomorphic to $\widehat{\mathcal{D}}^{\hbar}_A / \mathcal{I}_{\mathrm{can}}$, which is easy to see.

Since we are working with the Weyl algebra, we can also think of this as solving differential equations. The statement above is that $Z = 1 \in \widehat{\mathcal{M}}^{\hbar}_A$ is a solution to the differential equations $\mathcal{I}_{\mathrm{can}} \cdot Z = 0$. That is, $\hbar\partial_a(1) = 0$ for all $a \in A$, which is obvious. In fact, it is the unique solution to the system $\mathcal{I}_{\mathrm{can}} \cdot Z = 0$ if we impose the initial condition $Z\big|_{x_A=0} = 1$. We call this unique solution $Z = 1$ the *partition function* associated to the left ideal $\mathcal{I}_{\mathrm{can}}$.

4.C.3 $\widehat{\mathcal{D}}^{\hbar}_A$-modules of exponential type

As explained above, the completed Rees polynomial $\widehat{\mathcal{D}}^{\hbar}_A$-module $\widehat{\mathcal{M}}^{\hbar}_A$ is cyclic and generated by 1. In the following we will consider a more general class of $\widehat{\mathcal{D}}^{\hbar}_A$-modules, which we call "of exponential type".

Definition 4.C.4. Let

$$Z = \exp\left(\sum_{n=0}^{\infty} \hbar^{n-1} q^{(n+1)}(x_A)\right) \tag{4.28}$$

for some polynomials $q^{(n+1)}(x_A)$ of degree $\leq n+1$. We define $\widehat{\mathcal{M}}^{\hbar}_A Z$ to be the cyclic left $\widehat{\mathcal{D}}^{\hbar}_A$-module generated by Z, where the action of $\widehat{\mathcal{D}}^{\hbar}_A$ on $\widehat{\mathcal{M}}^{\hbar}_A Z$ is the standard action of differential operators on polynomials and exponentials of polynomials. It is clear that it is a well defined $\widehat{\mathcal{D}}^{\hbar}_A$-module, because of the degree condition on the polynomials $q^{(n+1)}$. We call such modules *of exponential type.*[5]

Note that given a $\widehat{\mathcal{D}}^{\hbar}_A$-module of exponential type, we can always uniquely choose the generator Z to satisfy the property that $Z\big|_{x_A=0} = 1$, i.e. $q^{(n+1)}(0) = 0$.

4.D Transvections, twisted $\widehat{\mathcal{D}}^{\hbar}_A$-modules, and $\widehat{\mathcal{D}}^{\hbar}_A$-modules of exponential type

4.D.1 Transvections

We now define an important class of automorphisms of $\widehat{\mathcal{D}}^{\hbar}_A$, which we call "transvections".[6] Those will play a key role in the story of Airy ideals.

[5] We note here that the argument of the exponential is not in $\widehat{\mathcal{M}}^{\hbar}_A$, since the degree of the polynomials is two more than the power of $\hbar$. But this is fine, as after acting with differential operators on Z using the standard action of differential operators, we get polynomials in $\widehat{\mathcal{M}}^{\hbar}_A$ times Z, as stated.

[6] The name "transvection" comes from Section 4 of [7], suitably generalized to completed Rees Weyl algebras.

Definition 4.D.1. Define the map ϕ that acts on $\widehat{\mathcal{D}}_A^\hbar$ as $\phi : (\hbar, \hbar x_a, \hbar\partial_a) \mapsto (\hbar, \hbar x_a, \bar{H}_a)$, for all $a \in A$, with

$$\bar{H}_a = \hbar\partial_a + \sum_{n=0}^{\infty} \hbar^n \partial_a q^{(n+1)}(x_A) \tag{4.29}$$

for some polynomials $q^{(n+1)}(x_A)$ of degree $\leq n+1$. We call ϕ a *transvection.* We say that it is *stable* if $q^{(1)} = q^{(2)} = 0$.

Remark 4.D.2. We note here that $\partial_a q^{(n+1)}(x_A)$ in (4.29) means the derivative of the polynomial $q^{(n+1)}(x_A)$ with respect to the variable x_a, not the product of ∂_a and $q^{(n+1)}(x_A)$ in the Weyl algebra. Equivalently, we could write $\bar{H}_a$ as

$$\bar{H}_a = \hbar\partial_a + \sum_{n=0}^{\infty} \hbar^n [\partial_a, q^{(n+1)}(x_A)], \tag{4.30}$$

where on the right-hand-side we now mean the commutator with respect to the product in the Weyl algebra.

Lemma 4.D.3. *ϕ is an automorphism of $\widehat{\mathcal{D}}_A^\hbar$.*

Proof. For any $P \in \widehat{\mathcal{D}}_A^\hbar$, it is clear that $\phi(P) \in \widehat{\mathcal{D}}_A^\hbar$. Furthermore, the map ϕ preserves the commutation relations between the generators of the Rees Weyl algebra, since $[\bar{H}_a, \hbar x_b] = \hbar^2\delta_{ab}$ and

$$\begin{aligned} [\bar{H}_a, \bar{H}_b] =& \left[\hbar\partial_a + \sum_{n=0}^{\infty} \hbar^n \partial_a q^{(n+1)}(x_A), \hbar\partial_b + \sum_{n=0}^{\infty} \hbar^n \partial_b q^{(n+1)}(x_A)\right] \\ =& \sum_{n=0}^{\infty} \hbar^{n+1} \left(\partial_a\partial_b q^{(n+1)}(x_A) - \partial_b\partial_a q^{(n+1)}(x_A)\right) \\ =& 0. \end{aligned} \tag{4.31}$$

□

We can think of transvections as conjugations. Indeed, for any $P \in \widehat{\mathcal{D}}^{\hbar}_{A}$, we can think of $\phi(P)$ as being given by

$$\phi(P) = \exp\left(-\sum_{n=0}^{\infty} \hbar^{n-1} q^{(n+1)}(x_A)\right) P \exp\left(\sum_{n=0}^{\infty} \hbar^{n-1} q^{(n+1)}(x_A)\right), \quad (4.32)$$

where multiplication here is understood as multiplication in the Rees Weyl algebra (after formally expanding the exponentials). Using standard properties of derivatives of exponentials, it is clear that this is equivalent to the map specified above.[7]

4.D.2 Twisted polynomial $\widehat{\mathcal{D}}^{\hbar}_{A}$-modules

Now consider the Rees polynomial $\widehat{\mathcal{D}}^{\hbar}_{A}$-module $\widehat{\mathcal{M}}^{\hbar}_{A}$. Given a transvection $\phi : \widehat{\mathcal{D}}^{\hbar}_{A} \to \widehat{\mathcal{D}}^{\hbar}_{A}$, we can construct a twisted left $\widehat{\mathcal{D}}^{\hbar}_{A}$-module ${}^{\phi}\widehat{\mathcal{M}}^{\hbar}_{A}$. Thinking of the transvection as a conjugation, the action on the twisted module is given by

$$\begin{aligned} P \cdot_{\phi} v =& \phi^{-1}(P) \cdot v \\ =& \exp\left(\sum_{n=0}^{\infty} \hbar^{n-1} q^{(n+1)}(x_A)\right) P \exp\left(-\sum_{n=0}^{\infty} \hbar^{n-1} q^{(n+1)}(x_A)\right) \cdot v. \end{aligned} \quad (4.33)$$

Since $\widehat{\mathcal{M}}^{\hbar}_{A}$ is a cyclic $\widehat{\mathcal{D}}^{\hbar}_{A}$-module generated by 1, we know that the twisted module ${}^{\phi}\widehat{\mathcal{M}}^{\hbar}_{A}$ is also cyclic and generated by 1.

Furthermore, the annihilator of 1 in the twisted module is

$${}^{\phi}\mathrm{Ann}_{\widehat{\mathcal{D}}^{\hbar}_{A}}(1) = \phi\left(\mathrm{Ann}_{\widehat{\mathcal{D}}^{\hbar}_{A}}(1)\right) = \phi(\mathcal{I}_{\mathrm{can}}), \quad (4.34)$$

[7] As in footnote 5, we note here that the argument of the exponential is not in $\widehat{\mathcal{M}}^{\hbar}_{A}$, but this is not a problem.

where we used (4.27) for the annihilator of 1 in the polynomial module. In other words, it is the image of the canonical left ideal generated by the derivatives $\hbar\partial_a$ under the automorphism ϕ. From the definition of transvections (Definition 4.D.1), we obtain that

$$ {}^{\phi}\mathrm{Ann}_{\widehat{\mathcal{D}}^{\hbar}_A}(1) = \phi(\mathcal{I}_{\mathrm{can}}) = \left\{ \sum_{a\in A} c_a \bar{H}_a \mid c_a \in \widehat{\mathcal{D}}^{\hbar}_A \right\}, \tag{4.35} $$

with the $\bar{H}_a$ defined in (4.29). If we denote this ideal by $\mathcal{I}$, we conclude that the twisted module ${}^{\phi}\widehat{\mathcal{M}}^{\hbar}_A$ is canonically isomorphic to $\widehat{\mathcal{D}}^{\hbar}_A/\mathcal{I}$.

We can summarize these statements in the following Lemma.

Lemma 4.D.4. *Let $\mathcal{I} \subset \widehat{\mathcal{D}}^{\hbar}_A$ be the left ideal $\mathcal{I} = \left\{\sum_{a\in A} c_a \bar{H}_a \mid c_a \in \widehat{\mathcal{D}}^{\hbar}_A\right\}$, where*

$$ \bar{H}_a = \hbar\partial_a + \sum_{n=0}^{\infty} \hbar^n \partial_a q^{(n+1)}(x_A) \tag{4.36} $$

for some polynomials $q^{(n+1)}(x_A)$ of degree $\leq n+1$. Then $\widehat{\mathcal{D}}^{\hbar}_A/\mathcal{I}$ is a cyclic left module canonically isomorphic to the twisted module ${}^{\phi}\widehat{\mathcal{M}}^{\hbar}_A$, where $\phi : \widehat{\mathcal{D}}^{\hbar}_A \to \widehat{\mathcal{D}}^{\hbar}_A$ is the transvection $\phi : (\hbar, \hbar x_a, \hbar\partial_a) \mapsto (\hbar, \hbar x_a, \bar{H}_a)$.

4.D.3 $\widehat{\mathcal{D}}^{\hbar}_A$-modules of exponential type

As usual, we can think of this result from the point of view of differential equations. The left ideal $\mathcal{I} = \left\{\sum_{a\in A} c_a \bar{H}_a \mid c_a \in \widehat{\mathcal{D}}^{\hbar}_A\right\}$ is the annihilator of 1 in the twisted module ${}^{\phi}\widehat{\mathcal{M}}^{\hbar}_A$. In other words, $Z' = 1$ is a solution to the equations $\mathcal{I} \cdot_{\phi} Z' = 0$. In fact, as before, it is the unique solution if we impose the initial condition $Z'\big|_{x_A=0} = 1$.

However, from the viewpoint of differential equations, this is not so nice, because the action of $\mathcal{I}$ on Z' here is the twisted action $\cdot_{\phi}$, not the standard action of differential operators. Fortunately, since ϕ is a transvection, we can think of it as conjugation, and the action can be written as in (4.33). This means that, instead of thinking of $\widehat{\mathcal{D}}^{\hbar}_A/\mathcal{I}$ as the cyclic twisted module ${}^{\phi}\widehat{\mathcal{M}}^{\hbar}_A$, we can think of it as the unique (untwisted) $\widehat{\mathcal{D}}^{\hbar}_A$-module of exponential type $\widehat{\mathcal{M}}^{\hbar}_A Z$ generated by

$$Z = \exp\left(-\sum_{n=0}^{\infty} \hbar^{n-1} q^{(n+1)}(x_A)\right). \tag{4.37}$$

Furthermore, imposing $Z\big|_{x_A=0} = 1$, we can uniquely choose the generator with $q^{(n+1)}(0) = 0$. $\mathcal{I}$ is of course the annihilator of Z.

In other words, what we have shown is that the Z in (4.37) with $q^{(n+1)}(0) = 0$ is the unique exponential solution to the differential equations $\mathcal{I} \cdot Z = 0$ satisfying the initial condition $Z\big|_{x_A=0} = 1$. This is summarized in the following lemma.

Lemma 4.D.5. *Let $\mathcal{I} \subset \widehat{\mathcal{D}}^{\hbar}_A$ be the left ideal $\mathcal{I} = \left\{\sum_{a\in A} c_a \bar{H}_a \mid c_a \in \widehat{\mathcal{D}}^{\hbar}_A\right\}$, where*

$$\bar{H}_a = \hbar\partial_a + \sum_{n=0}^{\infty} \hbar^n \partial_a q^{(n+1)}(x_A) \tag{4.38}$$

for some polynomials $q^{(n+1)}(x_A)$ of degree $\leq n+1$. Then $\widehat{\mathcal{D}}^{\hbar}_A/\mathcal{I}$ is canonically isomorphic to the module of exponential type $\widehat{\mathcal{M}}^{\hbar}_A Z$ with

$$Z = \exp\left(-\sum_{n=0}^{\infty} \hbar^{n-1} q^{(n+1)}(x_A)\right). \tag{4.39}$$

In other words, Z is a solution to the differential equations $\mathcal{I} \cdot Z = 0$, and if we set $q^{(n+1)}(0) = 0$, it is the unique solution satisfying the initial condition $Z\big|_{x_A=0} = 1$. We call Z the partition function *associated to the left ideal $\mathcal{I}$.*

This is of course a rather trivial statement here as the differential equations are straightforward to solve; it may look like we rewrote something very easy in a very complicated way (but isn't it part of the fun of doing mathematics?). In any case, this result will play an important role in the following, which is why we highlight it.

This can also be interpreted from the point of view of integrability. In classical mechanics, we say that a classical system is "integrable" if there exists a complete set of Poisson commuting observables; in the quantum world this becomes a complete set of commuting operators. Here, we consider a left ideal $\mathcal{I}$ generated by a complete set of commuting first-order differential operators $\bar{H}_a$. Integrable systems are interesting because they can in principle be solved; similarly, we found that there always exists a solution to the differential equations $\mathcal{I} \cdot Z = 0$, and it is unique after imposing an initial condition. The fundamental reason here is because the $\bar{H}_a$ are related to the $\hbar\partial_a$ by an automorphism of the completed Rees Weyl algebra (a transvection).

4.E Airy ideals, Airy modules, and partition functions

In the previous section we saw that, given a transvection ϕ on $\widehat{\mathcal{D}}^{\hbar}_A$, we can construct a twisted polynomial module ${}^{\phi}\widehat{\mathcal{M}}^{\hbar}_A$, which is canonical isomorphic to $\widehat{\mathcal{D}}^{\hbar}_A/\mathcal{I}$ where $\mathcal{I}$ is the ideal generated by a completed set of commuting first-order differential operators of the form (4.29). From the point of view of differential equations, we can instead think of $\widehat{\mathcal{D}}^{\hbar}_A/\mathcal{I}$ as a module of exponential type $\widehat{\mathcal{M}}^{\hbar}_A Z$, with Z the unique exponential solution ((4.37)) to the differential equations $\mathcal{I} \cdot Z = 0$ after imposing a suitable initial condition.

This is nice, but in the end, at least from the point of view of differential equations, this is rather trivial; after all, solving the system of equations $\bar{H}_a \cdot Z = 0$ with $\bar{H}_a$ of the form (4.29) is obvious, and it is clear that (4.37) is the unique solution with $Z\big|_{x_A=0} = 1$. We do not need the fancy ideas of transvections, twisted modules, and modules of exponential type to show this!

The power of the formalism however becomes apparent when we introduce Airy ideals. The idea here is that we introduce a more general class of left ideals $\mathcal{I} \subset \widehat{\mathcal{D}}^{\hbar}_A$, which we call Airy ideals (traditionally called "Airy structures" in the literature). Then, we show that if $\mathcal{I}$ is an Airy ideal, then it is equal to the image of the canonical left ideal $\mathcal{I}_{\text{can}}$ generated by the derivatives $\hbar\partial_a$ for some stable transvection ϕ. This is rather striking, and far from obvious *a priori*. As a result, Lemmas 4.D.4 and 4.D.5 apply; $\widehat{\mathcal{D}}^{\hbar}_A/\mathcal{I}$ is isomorphic to a twisted polynomial module or a module of exponential type, depending on the viewpoint. As a result, there exists a unique solution to the differential equations $\mathcal{I} \cdot Z = 0$ after imposing a suitable initial condition.

We remark that it is absolutely key that we work in the $\hbar$-adic completion of the Rees Weyl algebra here, since the transvection ϕ will generally involve operators $\bar{H}_a$ that are formal power series in $\hbar$. Working within the $\hbar$-adic completion enables us to relate Airy ideals to twisted polynomial modules and modules of exponential types.

4.E.1 Airy ideals

Let us now define the concept of an Airy ideal in $\widehat{\mathcal{D}}^{\hbar}_A$.

Definition 4.E.1. Let $\mathcal{I} \subset \widehat{\mathcal{D}}^{\hbar}_A$ be a left ideal. We say that it is an *Airy ideal* (also called *Airy structure* in the literature) if there exists operators $H_a \in \widehat{\mathcal{D}}^\hbar_A$, for all $a \in A$, such that:

1. The collection of operators $\{H_a\}_{a\in A}$ is bounded (see Definition 4.B.7).

2. The left ideal $\mathcal{I}$ can be written as

$$\mathcal{I} = \left\{ \sum_{a\in A} c_a H_a \mid c_a \in \widehat{\mathcal{D}}^\hbar_A \right\}, \tag{4.40}$$

   which consists as usual of finite and infinite (if A is countably infinite) $\widehat{\mathcal{D}}^\hbar_A$-linear combinations of the H_a.

3. The operators H_a take the form

$$H_a = \hbar\partial_a + O(\hbar^2). \tag{4.41}$$

4. The left ideal $\mathcal{I}$ satisfies the property:

$$[\mathcal{I}, \mathcal{I}] \subseteq \hbar^2 \mathcal{I}. \tag{4.42}$$

Remark 4.E.2. Before we move on, we remark that Condition (4) is non-trivial. First, remark that, trivially, any left ideal $\mathcal{I} \subseteq \widehat{\mathcal{D}}^\hbar_A$ satisfies

$$[\mathcal{I}, \mathcal{I}] \subseteq \mathcal{I}, \tag{4.43}$$

since $PQ - QP \in \mathcal{I}$ for all $P, Q \in \mathcal{I}$. Furthermore, from (4.17), any left ideal $\mathcal{I} \subseteq \widehat{\mathcal{D}}^\hbar_A$ satisfies

$$[\mathcal{I}, \mathcal{I}] \subseteq \hbar^2 \widehat{\mathcal{D}}^\hbar_A. \tag{4.44}$$

Combining the two statements, we conclude that the commutator of any two elements in an arbitrary left ideal $\mathcal{I}$ is an element of the ideal $\mathcal{I}$ that starts at $O(\hbar^2)$. However, this does not mean that it is equal to $\hbar^2$ times an element of the ideal $\mathcal{I}$. That it must be so is the content of Condition (4).

As an example, let $A = \{1, 2\}$, and consider the left ideal $\mathcal{I}$ generated by $H_1 = \hbar\partial_1$ and $H_2 = \hbar\partial_2 + \hbar^2 x_1$. The commutator of H_1 and H_2 is $[H_1, H_2] = \hbar^3$. It is true that $\hbar^3 \in \mathcal{I}$, since

$$\hbar^3 = \hbar\partial_1(\hbar\partial_2 + \hbar^2 x_1) - (\hbar\partial_2 + \hbar^2 x_1)(\hbar\partial_1). \tag{4.45}$$

If we single out a power of $\hbar^2$ on the right-hand-side of the commutator, it is also true that $\hbar \in \widehat{\mathcal{D}}^\hbar_A$. However, $\hbar \notin \mathcal{I}$, as is easy to check. Thus this ideal does not satisfy Condition (4).

4.E.2 Airy ideals, transvections and twisted modules

Since $\mathcal{I} \subset \widehat{\mathcal{D}}^\hbar_A$ is a left ideal, it is clear that $\widehat{\mathcal{D}}^\hbar_A/\mathcal{I}$ is a cyclic left $\widehat{\mathcal{D}}^\hbar_A$-module. What is not clear however is how it relates to the Rees polynomial $\widehat{\mathcal{D}}^\hbar_A$-module $\widehat{\mathcal{M}}^\hbar_A$. This connection is the fundamental theorem in the theory of Airy ideals.

Theorem 4.E.3. *Let $\mathcal{I} \subset \widehat{\mathcal{D}}^\hbar_A$ be an Airy ideal. There there exists a stable transvection $\phi : \widehat{\mathcal{D}}^\hbar_A \to \widehat{\mathcal{D}}^\hbar_A$ (see Definition 4.D.1) such that $\mathcal{I} = \phi(\mathcal{I}_{can})$, where $\mathcal{I}_{can}$ is the left ideal generated by the derivatives $\hbar\partial_a$. As a result, $\widehat{\mathcal{D}}^\hbar_A/\mathcal{I}$ is a cyclic left module canonically isomorphic to the twisted polynomial module ${}^\phi\widehat{\mathcal{M}}^\hbar_A$.*

This is a powerful theorem. What it means is that we can find a complete set of commuting first-order differential operators $\bar{H}_a$ of the form

$$\bar{H}_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n \partial_a q^{(n+1)}(x_A), \tag{4.46}$$

for some polynomials $q^{(n+1)}(x_A)$ of degree $\leq n+1$, such that the Airy ideal $\mathcal{I}$ can be rewritten as

$$\mathcal{I} = \left\{ \sum_{a \in A} c_a \bar{H}_a \mid c_a \in \widehat{\mathcal{D}}_A^\hbar \right\}. \tag{4.47}$$

This is highly non-trivial, as the original Airy ideal $\mathcal{I}$ will not usually be presented in this form.

To prove Theorem 4.E.3 we will first prove a series of lemmas. The first three lemmas do not require Condition (4) in the definition of Airy ideals Definition 4.E.1. The fourth lemma highlights the crucial role played by Condition (4).

Lemma 4.E.4. *Let $\mathcal{I} \subset \widehat{\mathcal{D}}_A^\hbar$ be a left ideal satisfying conditions (1)–(3) of Definition 4.E.1. Then for any $P \in \widehat{\mathcal{D}}_A^\hbar$, we can write*

$$P = \sum_{n=0}^{\infty} \hbar^n p^{(n)}(x_A) + Q \tag{4.48}$$

for some polynomials $p^{(n)}(x_A)$ of degree $\leq n$ and some $Q \in \mathcal{I}$.

Proof. Let $P \in \widehat{\mathcal{D}}_A^\hbar$. We can write

$$P = p^{(0,0)} + \hbar \left(p^{(1,1)} + \sum_{b \in A} p_b^{(1,0)} \partial_b \right) + \hbar^2 \left(p^{(2,2)} + \sum_{b \in A} p_b^{(2,1)} \partial_b + \sum_{b,c \in A} p_{bc}^{(2,0)} \partial_b \partial_c \right) + O(\hbar^3), \tag{4.49}$$

where the $p_{\dots}^{(m,k)}$ are polynomials of degree $\leq k$ (we removed the dependence in x_A for clarity).

The idea is simple. Since $\mathcal{I}$ is an Airy ideal, it is generated by a bounded collection of operators $\{H_a\}_{a\in A}$ of the form $H_a = \hbar\partial_a + O(\hbar^2)$. So for each term in P that is not polynomial, we can replace the right-most derivative $\hbar\partial_a$ by H_a, up to higher order terms in $\hbar$. Applying this procedure recursively order by order in $\hbar$, we will end up rewriting P as a polynomial plus an operator in the ideal $\mathcal{I}$. We see here that it is key that we are working in the $\hbar$-adic completion of the Rees Weyl algebra, otherwise we would not be allowed to keep going order by order in $\hbar$ forever.

More precisely, we start at $O(\hbar)$. We use $\hbar\partial_b = H_b + O(\hbar^2)$ to rewrite

$$\hbar\left(p^{(1,1)} + \sum_{b\in A} p_b^{(1,0)}\partial_b\right) = \hbar p^{(1,1)} + \sum_{b\in A} p_b^{(1,0)} H_b + O(\hbar^2). \tag{4.50}$$

The first term is a polynomial term, and the second term is in $\mathcal{I}$. The procedure however created new terms at the next order, $O(\hbar^2)$, which we must study further. If we write H_a as

$$H_a = \hbar\partial_a + \hbar^2\left(g_a^{(2,2)} + \sum_{b\in A} g_{a;b}^{(2,1)}\partial_b + \sum_{b,c\in A} g_{a;bc}^{(2,0)}\partial_b\partial_c\right) + O(\hbar^3), \tag{4.51}$$

where the $g_{\cdots}^{(2,i)}(x_A)$ are polynomials of degree $\leq i$, then the terms of $O(\hbar^2)$ created by the procedure above take the form

$$-\hbar^2\left(\sum_{b\in A} p_b^{(1,0)} g_b^{(2,2)} + \sum_{b,c\in A} p_b^{(1,0)} g_{b;c}^{(2,1)}\partial_c + \sum_{b,c,d\in A} p_b^{(1,0)} g_{b;cd}^{(2,0)}\partial_c\partial_d\right). \tag{4.52}$$

When the index set A is countably infinite, we need to make sure that the terms in brackets do not involve infinite divergent sums, and are all in $F_2\mathcal{D}_A$

(with respect to the Bernstein filtration, see Definition 4.B.3). We note that because the collection of operators $\{H_a\}_{a\in A}$ is bounded, all polynomials $g_{a;\cdots}^{(2,k)}$ vanish for all but finitely many $a \in A$. Therefore, the sums over $b \in A$ collapse to finite sums and are well defined. The remaining sums over $c, d \in A$ are potentially infinite, but they come with derivatives in these indices, and hence all terms in brackets in (4.52) are in $\mathcal{D}_A$. Furthermore, looking at the degree of the terms with respect to the Bernstein filtration, we see that they are all in $F_2\mathcal{D}_A$.

Now repeat the procedure for all terms at $O(\hbar^2)$, including the newly obtained terms, keeping the polynomial terms and replacing the right-most derivatives in the other terms by H's up to terms of higher order in $\hbar$. The result will be a polynomial term at $O(\hbar^2)$, plus a term that is in the ideal $\mathcal{I}$, plus corrections at higher order. The argument above shows that the corrections are well defined. Then keep applying this procedure recursively, order by order in $\hbar$. In the end, all that remains are polynomial terms plus terms in the ideal $\mathcal{I}$. That is, we conclude that we can write

$$P = \sum_{n=0}^{\infty} \hbar^n p^{(n)}(x_A) + Q \tag{4.53}$$

for some polynomials $p^{(n)}(x_A)$ of degree $\leq n$ and some $Q \in \mathcal{I}$. For instance,

$$p^{(0)} = p^{(0,0)}, \qquad p^{(1)} = p^{(1,1)}, \qquad p^{(2)} = p^{(2,2)} - \sum_{b\in A} p_b^{(1,0)} g_b^{(2,2)}, \tag{4.54}$$

and so on and so forth. □

Lemma 4.E.5. *Let $\mathcal{I} \subset \widehat{\mathcal{D}}_A^\hbar$ be a left ideal satisfying conditions (1)–(3) of Definition 4.E.1. Then there exist operators $\bar{H}_a \in \mathcal{I}$, for all $a \in A$, of the* *form*

$$\bar{H}_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n p_a^{(n)}(x_A) \tag{4.55}$$

for polynomials $p_a^{(n)}(x_A)$ of degree $\leq n$. Furthermore, the collection of operators $\{\bar{H}_a\}_{a\in A}$ is bounded.

Proof. We know that $H_a = \hbar\partial_a + P_a$ for some $P_a \in \widehat{\mathcal{D}}_A^\hbar$ of $O(\hbar^2)$. From Lemma 4.E.4, we can write

$$H_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n p_a^{(n)}(x_A) + Q_a \tag{4.56}$$

for some polynomials $p_a^{(n)}(x_A)$ of degree $\leq n$ and some $Q_a \in \mathcal{I}$. We define

$$\bar{H}_a = H_a - Q_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n p_a^{(n)}(x_A). \tag{4.57}$$

Clearly, $\bar{H}_a \in \mathcal{I}$. Furthermore, since the collection $\{H_a\}_{a\in A}$ is bounded, the polynomials $p_a^{(n)}(x_A)$ must vanish for all but finitely many $a \in A$, and hence the collection $\{\bar{H}_a\}_{a\in A}$ is also bounded. □

Lemma 4.E.6. *Let $\mathcal{I} \subset \widehat{\mathcal{D}}_A^\hbar$ be a left ideal satisfying conditions (1)–(3) of Definition 4.E.1, and $\bar{I} \subseteq I$ be the left ideal generated by the $\bar{H}_a$ of Lemma 4.E.5. Then $\bar{I} = I$.*

In other words, we can think of I as being generated by the $\bar{H}_a$ instead of the H_a: $\mathcal{I} = \{\sum_{a\in A} c_a \bar{H}_a \mid c_a \in \widehat{\mathcal{D}}_A^\hbar\}$.

Proof. By definition, $\bar{H}_a = H_a - Q_a$ for some $Q_a \in \mathcal{I}$. We can write $Q_a = \sum_{b\in A} p_{ab} H_b$ for some $p_{ab} \in \widehat{\mathcal{D}}_A^\hbar$. In fact, from the proof of Lemma 4.E.5, we know that $p_{ab} = O(\hbar)$.

We can write

$$\bar{H}_a = H_a - \sum_{b \in A} p_{ab} H_b \tag{4.58}$$

But then, $H_b = \bar{H}_b + Q_b = \bar{H}_b + \sum_{c \in A} p_{bc} H_c$, and thus we get

$$\bar{H}_a = H_a - \sum_{b \in A} p_{ab} \bar{H}_b - \sum_{b,c \in A} p_{ab} p_{bc} H_c. \tag{4.59}$$

We note here that since both $\{H_a\}_{a \in A}$ and $\{\bar{H}_a\}_{a \in A}$ are bounded, the collection $\{Q_a\}_{a \in A}$ is also bounded, and hence the sum over $b \in A$ in the third term on the right-hand-side is well defined. Furthermore, since $p_{ab} = O(\hbar)$, $p_{ab} p_{bc} = O(\hbar^2)$.

Continuing this process recursively, we end up with the statement that

$$\bar{H}_a = H_a - \bar{Q}_a \tag{4.60}$$

where $\bar{Q}_a$ is an infinite sum of terms that are linear combinations of the $\bar{H}_b$ with coefficients starting at higher and higher order in $\hbar$. Thus, for a finite power of $\hbar$, only a finite number of terms contribute, and the result is that $\bar{Q}_a \in \bar{\mathcal{I}}$. It follows that $H_a \in \bar{\mathcal{I}}$, and hence $\mathcal{I} \subseteq \bar{\mathcal{I}}$. We conclude that $\mathcal{I} = \bar{\mathcal{I}}$. $\square$

So far we have not used at all Condition (4) in the definition of Airy ideals Definition 4.E.1. This condition is crucial; imposing Condition (4) implies that there are no non-zero polynomials in an Airy ideal $\mathcal{I}$. In particular, it implies that the operators $\bar{H}_a$ commute with each other, which in turn implies the existence of a stable transvection that relates $\mathcal{I}$ to the canonical left ideal $\mathcal{I}_{\text{can}}$, as we will see.

Lemma 4.E.7. *Let $\mathcal{I} \subset \widehat{\mathcal{D}}^{\hbar}_A$ be an Airy ideal. Then there are no non-zero polynomials in $\mathcal{I}$.*

Proof. By Lemma 4.E.6, we think of I as $\mathcal{I} = \{\sum_{a\in A} c_a \bar{H}_a \mid c_a \in \widehat{\mathcal{D}}^{\hbar}_A\}$, with $\bar{H}_a$ of the form

$$\bar{H}_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n p_a^{(n)}(x_A). \tag{4.61}$$

We prove the Lemma by induction on the power of $\hbar$. Let $N \geq 2$ be a positive integer. The induction hypothesis is that all polynomials in $\mathcal{I}$ start at least at $O(\hbar^N)$. Then we show that it implies that they must start at least at order $O(\hbar^{N+1})$. By induction on N, this means that all polynomials in $\mathcal{I}$ must vanish.

The base case for the induction is obvious. We need to show that all polynomials in $\mathcal{I}$ must start at least at $O(\hbar^2)$. But since $\mathcal{I}$ is generated by $\bar{H}_a$ of the form (4.61), it is clearly impossible to get a polynomial with a $\hbar^0$ constant term or a $\hbar^1$ linear term as a linear combination of $\bar{H}_a$'s.

Now assume that all polynomials in $\mathcal{I}$ start at least at $O(\hbar^N)$. The commutator of the $\bar{H}_a$ is:

$$\begin{aligned}[\bar{H}_a, \bar{H}_b] &= \left[\hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n p_a^{(n)}(x_A), \hbar\partial_b + \sum_{n=2}^{\infty} \hbar^n p_b^{(n)}(x_A)\right] \\ &= \hbar^2 \sum_{n=2}^{\infty} \hbar^{n-1}\left(\partial_a p_b^{(n)}(x_A) - \partial_b p_a^{(n)}(x_A)\right). \end{aligned} \tag{4.62}$$

By Condition (4) of Definition 4.E.1, we know that

$$\sum_{n=2}^{\infty} \hbar^{n-1}\left(\partial_a p_b^{(n)}(x_A) - \partial_b p_a^{(n)}(x_A)\right) \in \mathcal{I}. \tag{4.63}$$

By assumption, this must start at least at $O(\hbar^N)$, so we must have that

$$\partial_a p_b^{(n)}(x_A) = \partial_b p_a^{(n)}(x_A), \qquad \text{for all } a, b \in A \text{ and } n = 2, \ldots, N. \tag{4.64}$$

Assuming first that A is a finite index set, by Poincare's lemma we conclude that there exists polynomials $q^{(n+1)}(x_A)$ of degree $\leq n+1$ such that

$$p_a^{(n)}(x_A) = \partial_a q^{(n+1)}(x_A), \qquad \text{for all } a \in A \text{ and } n = 2, \ldots, N. \tag{4.65}$$

Moreover, if we require that $q^{(n+1)}(0) = 0$, then the polynomials are uniquely fixed.

If A is a countably infinite index set, then Poincare's lemma still holds, but the $q^{(n+1)}(x_A)$, which will still be of degree $\leq n+1$, could *a priori* involve infinite linear combinations of monomials of the same degree. However, since the collection of operators $\{\bar{H}_a\}_{a\in A}$ is bounded, we know that for a fixed n, the polynomials $p_a^{(n)}(x_A)$ vanish for all but finitely many $a \in A$. This means that for each n the conditions (4.64) become a finite system in a finite numbers of variables, and thus we conclude by Poincare's lemma that the $q^{(n+1)}(x_A)$ will be polynomials.

As result, we conclude that we can write

$$\bar{H}_a = \hbar\partial_a + \sum_{n=2}^{N} \hbar^n \partial_a q^{(n+1)}(x_A) + \sum_{n=N+1}^{\infty} \hbar^n p_n(x_A). \tag{4.66}$$

Now let

$$\phi_N : (\hbar, \hbar x_a, \hbar\partial_a) \mapsto (\hbar, \hbar x_a, \hbar\partial_a - \sum_{n=2}^{N} \hbar^n \partial_a q^{(n+1)}(x_A)). \tag{4.67}$$

It is a stable transvection, and

$$\phi_N(\bar{H}_a) = \hbar\partial_a + \sum_{n=N+1}^{\infty} \hbar^n p_n(x_A). \tag{4.68}$$

Under this automorphism, the ideal $\mathcal{I}$ is mapped to the ideal $\phi_N(\mathcal{I})$ generated by the $\phi_N(\bar{H}_a)$ above. Now suppose that P is a polynomial in $\mathcal{I}$. By definition of the transvection, $\phi_N(P) = P$, and thus P must also be in $\phi_N(\mathcal{I})$. But looking at the form of the generators $\phi_N(\bar{H}_a)$ in (4.68), it is clear that any polynomial in $\phi_N(\mathcal{I})$ must start at least at $O(\hbar^{N+1})$. Therefore, P must start at least at $O(\hbar^{N+1})$, which completes the induction.

□

With these four lemmas under our belt, the proof of Theorem 4.E.3 is straightforward.

Proof of Theorem 4.E.3. By Lemma 4.E.6, we can write $\mathcal{I}$ as $\mathcal{I} = \{\sum_{a\in A} c_a \bar{H}_a \mid c_a \in \widehat{\mathcal{D}}^{\hbar}_A\}$ for some operators $\bar{H}_a \in \mathcal{I}$ of the form

$$\bar{H}_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n p_a^{(n)}(x_A), \tag{4.69}$$

with the $p_a^{(n)}(x_A)$ polynomials of degree $\leq n$. Since there are no non-zero polynomials in $\mathcal{I}$ (Lemma 4.E.7), and that the commutator $[\bar{H}_a, \bar{H}_b] \in \mathcal{I}$ is a polynomial, we must have

$$[\bar{H}_a, \bar{H}_b] = 0 \qquad \text{for all } a, b \in A. \tag{4.70}$$

Using Poincare's Lemma as in the proof of Lemma 4.E.7, we conclude that we can rewrite the differential operators as

$$\bar{H}_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n \partial_a q^{(n+1)}(x_A), \tag{4.71}$$

for polynomials $q^{(n+1)}(x_A)$ of degree $\leq n+1$ with $q^{(n+1)}(0) = 0$. We have thus shown that $\mathcal{I}$ is the image of $\mathcal{I}_{\text{can}} = \left\{ \sum_{a \in A} c_a \hbar \partial_a \mid c_a \in \widehat{\mathcal{D}}_A^{\hbar} \right\}$ under the stable transvection $\phi : (\hbar, \hbar x_a, \hbar \partial_a) \mapsto (\hbar, \hbar x_a, \bar{H}_a)$. By Lemma 4.D.4, it follows that $\widehat{\mathcal{D}}_A^{\hbar}/\mathcal{I}$ is a cyclic left module canonically isomorphic to the twisted polynomial module ${}^{\phi}\widehat{\mathcal{M}}_A^{\hbar}$. □

4.E.3 Partition function

As usual, we can reformulate Theorem 4.E.3 in the language of differential equations. Lemma 4.D.5 directly implies the following corollary, which is the existence and uniqueness theorem at the foundation of the theory of Airy ideals (or Airy structures), first proved by Kontsevich and Soibelman in [39].

Corollary 4.E.8. *Let $\mathcal{I} \subset \widehat{\mathcal{D}}_A^{\hbar}$ be an Airy ideal. Then there exists a unique module of exponential type $\widehat{\mathcal{M}}_A^{\hbar} Z$ that is canonically isomorphic to $\widehat{\mathcal{D}}_A^{\hbar}/\mathcal{I}$. Furthermore, if we impose the initial condition $Z\big|_{x_A=0} = 1$ on the generator, then it is uniquely fixed and takes the form*

$$Z = \exp\left(\sum_{\substack{g \in \frac{1}{2}\mathbb{N}, n \in \mathbb{N}^* \\ 2g-2+n>0}} \hbar^{2g-2+n} F_{g,n}(x_A) \right) \tag{4.72}$$

for some polynomials $F_{g,n}(x_A)$ homogeneous of degree n with $F_{g,n}(0) = 0$.[8] *In other words, Z is the unique solution to the differential equations $\mathcal{I} \cdot Z = 0$ satisfying the initial condition $Z\big|_{x_A=0} = 1$. We call Z the* partition function *of the Airy ideal $\mathcal{I}$.*

[8] The use of homogeneous polynomials $F_{g,n}(x_A)$ of degree n instead of the polynomials $q^{(k+1)}(x_A)$ of degree $\leq k+1$ previously used in Lemma 4.D.5 is simply to connect with the existing literature on the topic, but it is straightforward to show that it is equivalent: the polynomials $q^{(k+1)}(x_A)$ of degree $\leq k+1$ are reconstructed as $q^{(k+1)}(x_A) = \sum_{\substack{g \in \frac{1}{2}\mathbb{N}, n \in \mathbb{N}^* \\ 2g-1+n=k}} F_{g,n}(x_A)$.

Equivalently, Z is the unique solution to the differential equations $H_a Z = 0$, for all $a \in A$, where the H_a generated the Airy ideal $\mathcal{I}$.

Remark 4.E.9. We note here that we could have proven existence and uniqueness of the partition function directly from the definition of Airy ideals; the proof is fairly straightforward, and goes by induction on $\hbar$. This is what is done in [39] (see also [11]). However, our approach of relating Airy ideals to transvections, twisted modules, and modules of exponential type may shed light on what an Airy ideal really is, and why there always exists a unique partition function annihilated by an Airy ideal.

4.E.4 Airy modules

Given an Airy ideal $\mathcal{I} \subset \widehat{\mathcal{D}}^\hbar_A$, we have shown that $\widehat{\mathcal{D}}^\hbar_A/\mathcal{I}$ is a cyclic left module canonically isomorphic to the twisted polynomial module ${}^\phi\widehat{\mathcal{M}}^\hbar_A$. Turning this around, we can define the notion of an Airy left $\widehat{\mathcal{D}}^\hbar_A$-module.

Definition 4.E.10. We say that a cyclic left $\widehat{\mathcal{D}}^\hbar_A$-module is *Airy* if it is generated by an element v whose annihilator $\operatorname{Ann}_{\widehat{\mathcal{D}}^\hbar_A}(v)$ is an Airy ideal.

It is easy to show that all modules of exponential type (see Definition 4.C.4) are Airy modules.

Lemma 4.E.11. *Let $\widehat{\mathcal{M}}^\hbar_A Z$ be a left $\widehat{\mathcal{D}}^\hbar_A$-module of exponential type, with generator*

$$Z = \exp\left(\sum_{n=0}^{\infty} \hbar^{n-1} q^{(n+1)}(x_A)\right) \tag{4.73}$$

for some polynomials $q^{(n+1)}(x_A)$ of degree $\leq n+1$. Then $\widehat{\mathcal{M}}^\hbar_A Z$ is an Airy module.

Proof. This is clear. As we have seen in Lemma 4.D.5, the module of exponential type $\widehat{\mathcal{M}}_A^\hbar Z$ is canonically isomorphic to $\widehat{\mathcal{D}}_A^\hbar/\mathcal{I}$, where

$$\mathcal{I} = \left\{ \sum_{a \in A} c_a \bar{H}_a \mid c_a \in \widehat{\mathcal{D}}_A^\hbar \right\}, \tag{4.74}$$

with

$$\bar{H}_a = \hbar \partial_a - \sum_{n=0}^{\infty} \hbar^n \partial_a q^{(n+1)}(x_A). \tag{4.75}$$

Since $[\bar{H}_a, \bar{H}_b] = 0$ for all $a, b \in A$, it is easy to show that $\mathcal{I}$ satisfies the four conditions in the definition of Airy ideals (Definition 4.E.1).

∎

Similarly, all twisted modules ${}^\phi\widehat{\mathcal{M}}_A^\hbar$ obtained from a transvection ϕ of the completed Rees Weyl algebra are Airy modules.

Lemma 4.E.12. *Let $\phi : \widehat{\mathcal{D}}_A^\hbar \to \widehat{\mathcal{D}}_A^\hbar$ be a transvection, and ${}^\phi\widehat{\mathcal{M}}_A^\hbar$ the ϕ-twisted polynomial module. Then ${}^\phi\widehat{\mathcal{M}}_A^\hbar$ is an Airy module.*

Proof. Same argument as in the proof of the previous Lemma. ∎

4.E.5 $\hbar$-polynomial and $\hbar$-finite Airy ideals

The existence of a partition function becomes particularly interesting when the operators H_a live in the subalgebra $\mathcal{D}_A^\hbar$; that is, the H_a are polynomials (instead of formal series) in $\hbar$. Even more interesting is when all H_a are polynomials of degree less than a certain fixed positive integer N. We thus formulate the following definition.

Definition 4.E.13. Let $\mathcal{I} \subset \widehat{\mathcal{D}}_A^\hbar$ be an Airy ideal. We say that it is *$\hbar$-polynomial* if there exists $H_a \in \mathcal{D}_A^\hbar$ for all $a \in A$ (i.e., they are polynomials in

$\hbar$) satisfying the conditions of Definition 4.E.1.

We say that $\mathcal{I}$ is *$\hbar$-finite* if it is $\hbar$-polynomial, and there exists a positive integer N such that all H_a are polynomials in $\hbar$ of degree $\leq N$. We call the smallest such N the *$\hbar$-degree* of $\mathcal{I}$.

We also say that an Airy left $\widehat{\mathcal{D}}_A^\hbar$-module is $\hbar$-polynomial (resp. $\hbar$-finite) if it is generated by an element v whose annihilator is $\hbar$-polynomial (resp. $\hbar$-finite).

For example, an Airy ideal generated by a collection of operators $\{H_a\}_{a\in\mathbb{N}^*}$ that are polynomials of degree 2 in $\hbar$ is $\hbar$-finite, and has $\hbar$-degree 2. However, an Airy ideal generated by a collection of operators $\{H_a\}_{a\in\mathbb{N}^*}$ that are polynomials of degree a in $\hbar$ is $\hbar$-polynomial, but not $\hbar$-finite (as the $\hbar$-degree of the H_a keeps increasing as a increases).

Why are $\hbar$-finite Airy ideals particularly interesting? We saw in Lemma 4.E.11 that all left $\widehat{\mathcal{D}}_A^\hbar$-modules of exponential type $\widehat{\mathcal{M}}_A^\hbar Z$ are Airy modules. In other words, any exponential of the form

$$Z = \exp\left(\sum_{\substack{g\in\frac{1}{2}\mathbb{N},n\in\mathbb{N}^*\\2g-2+n>0}} \hbar^{2g-2+n}F_{g,n}(x_A)\right) \tag{4.76}$$

is the partition function for some Airy ideal. Indeed, the Airy ideal is given by

$$\mathcal{I} = \left\{\sum_{a\in A} c_a\bar{H}_a \mid c_a \in \widehat{\mathcal{D}}_A^\hbar\right\}, \tag{4.77}$$

with

$$\bar{H}_a = \hbar\partial_a - \sum_{\substack{g\in\frac{1}{2}\mathbb{N},n\in\mathbb{N}^*\\2g-2+n>0}} \hbar^{2g-1+n}\partial_a F_{g,n}(x_A). \tag{4.78}$$

This is a rather trivial statement.

But are all Z of the form (4.76) partition functions for Airy ideals that are $\hbar$-finite? In other words, can we always rewrite the Airy ideal $\mathcal{I}$ above as an ideal generated by operators H_a that are $\hbar$-polynomials of degree $\leq N$ for some N? Equivalently, which partition functions of the form (4.76) satisfy differential equations $H_a Z = 0$ for $\hbar$-polynomial operators H_a of degree $\leq N$ for some N that generate an Airy ideal?

This is a very interesting question. The existence of such operators H_a implies a recursive structure for the polynomials $F_{g,n}(x_A)$, which is the foundation of the story of topological recursion reformulated in the language of Airy structures. See [39, 1, 9] for more on this recursive structure. Traditionally, the usual topological recursion formula is obtained directly by applying the differential operators H_a on the partition function Z, and setting the result to 0. This gives a recursion for the polynomials $F_{g,n}(x_A)$ (or their coefficients) if the operators H_a are $\hbar$-polynomials of degree $\leq N$ for some N. From our point of view, this recursive structure is encapsulated in the recursive construction of the commuting first-order operators $\bar{H}_a$ from the H_a. When the H_a are polynomials in $\hbar$, the infinite set of polynomials (order by order in $\hbar$) in the operators $\bar{H}_a$ will be constructed out of a finite set (in $\hbar$) of polynomials in the H_a, and hence will satisfy some recursive structure determined by initial conditions. The two recursions are of course equivalent, as is easy to show.

4.F Constructing Airy ideals

We are interested in constructing Airy ideals $\mathcal{I} \subset \widehat{\mathcal{D}}^{\hbar}_A$, because as we saw in Corollary 4.E.8 there exists a unique partition function Z such that $\mathcal{I} \cdot Z = 0$.

More often than not, what we will be interested in is the partition function Z, which may be a generating series for some enumerative invariants. Conversely, if the generating series for a particular set of enumerative invariants takes the form of a partition function for a ($\hbar$-finite) Airy ideal, then what this means is that it satisfies a set of differential constraints that uniquely fix the generating series, such as Virasoro constraints, $\mathcal{W}$-constraints, etc.

How can we construct Airy ideals? We need to construct a left ideal $\mathcal{I} \subset \widehat{\mathcal{D}}_A^{\hbar}$ that satisfies the properties of Definition 4.E.1. So what we do is construct a collection of differential operators $\{H_a\}_{a\in A}$ with $H_a \in \widehat{\mathcal{D}}_A^{\hbar}$. To show that the left ideal generated by the H_a is an Airy ideal, what we need to show is:

1. The collection $\{H_a\}_{a\in A}$ is bounded;

2. $H_a = \hbar\partial_a + O(\hbar^2)$;

3. $[H_a, H_b] = \hbar^2 \sum_{c\in A} f_{abc} H_c$ for some $f_{abc} \in \widehat{\mathcal{D}}_A^{\hbar}$.

It is easy to show that the third condition is equivalent to requiring that $[\mathcal{I}, \mathcal{I}] \subseteq \hbar^2 \mathcal{I}$, Condition (4) in Definition 4.E.1.

Concretely, sometimes we may construct a collection of operators $H_a \in \widehat{\mathcal{D}}_A^{\hbar}$ that do not satisfy the condition $H_a = \hbar\partial_a + O(\hbar^2)$, but that satisfy the other two conditions. Does that mean that they do not generate an Airy ideal? Not necessarily. For instance, if I and A are two finite index sets of the same length, then any two collections of differential operators $\{G_i\}_{i\in I}$ and $\{H_a\}_{a\in A}$ related by an invertible $\mathbb{C}$-linear transformation generate the same left ideal in $\widehat{\mathcal{D}}_A^{\hbar}$. This can be generalized to the infinite context as follows (see Definition 2.3 in [10]).

Lemma 4.F.1. *Let I, A be either finite or countably infinite index sets. Let $\{H_i\}_{i\in I}$ be a collection of operators $H_i \in \widehat{\mathcal{D}}^\hbar_A$ such that:*

1. *The collection $\{H_i\}_{i\in I}$ is bounded;*

2. *$[H_i, H_j] = \hbar^2 \sum_{k\in I} f_{ijk} H_k$ for some $f_{ijk} \in \widehat{\mathcal{D}}^\hbar_A$;*

3. $$H_i = \sum_{a\in A} M_{ia} \hbar \partial_a + O(\hbar^2), \tag{4.79}$$

   where the M_{ia} are complex numbers such that for all fixed $a \in A$, the M_{ia} vanish for all but finitely many $i \in I$.

4. *There exists complex numbers N_{bj}, with $b \in A$ and $j \in I$, such that for all fixed $j \in I$, they vanish for all but finitely many $b \in A$, and*

$$\sum_{i\in I} N_{bi} M_{ia} = \delta_{ab} \qquad \forall a, b \in A \qquad \text{and} \qquad \sum_{a\in A} M_{ia} N_{aj} = \delta_{ij} \qquad \forall i, j \in I. \tag{4.80}$$

Then the left ideal $\mathcal{I} = \{\sum_{i\in I} c_i H_i \mid c_i \in \widehat{\mathcal{D}}^\hbar_A\} \subset \widehat{\mathcal{D}}^\hbar_A$ is an Airy ideal.

Moreover, $\mathcal{I}$ can also be written as $\mathcal{I} = \{\sum_{a\in A} c_a \tilde{H}_a \mid c_a \in \widehat{\mathcal{D}}^\hbar_A\}$, where the $\tilde{H}_a$ are defined by

$$\tilde{H}_a = \sum_{i\in I} N_{ai} H_i \in \widehat{\mathcal{D}}^\hbar_A, \tag{4.81}$$

and satisfy $\tilde{H}_a = \hbar\partial_a + O(\hbar^2)$.

Proof. First, we note that the conditions in (4.80) are well defined, since because of the constraints on the coefficients N_{bj} and M_{ia} both sums collapse to finite sums.

Now, by condition (2) it is clear that $\mathcal{I}$ satisfies condition (2) of Definition 4.E.1. Moreover, because of condition (1), it is clear that the $\tilde{H}_a$ are well defined operators in $\widehat{\mathcal{D}}^{\hbar}_{A}$. It is also clear from the properties of the complex numbers M_{ia} and N_{bj} that

$$
\begin{aligned}
\tilde{H}_a =& \sum_{i\in I} N_{ai} H_i && (4.82)\\
=& \sum_{i\in I} N_{ai} \left(\sum_{b\in A} M_{ib} \hbar \partial_b \right) + O(\hbar^2) && (4.83)\\
=& \sum_{b\in A} \left(\sum_{i\in I} N_{ai} M_{ib} \right) \hbar\partial_b + O(\hbar^2) && (4.84)\\
=& \hbar\partial_a + O(\hbar^2), && (4.85)
\end{aligned}
$$

where in the third line we could exchange the order of the sums since the M_{ib} are such that for all fixed $b \in A$, they vanish for all but finitely many $i \in I$. Therefore, to show that $\mathcal{I}$ is an Airy ideal, all that we have to show is that the ideals $\mathcal{I} = \{\sum_{i\in I} c_i H_i \mid c_i \in \widehat{\mathcal{D}}^{\hbar}_{A}\}$ and $\mathcal{J} = \{\sum_{a\in A} c_a \tilde{H}_a \mid c_a \in \widehat{\mathcal{D}}^{\hbar}_{A}\}$ are the same.

First, we note that we can invert the relation between the $\tilde{H}_a$ and the H_i. As $\tilde{H}_a = \sum_{i\in I} N_{ai} H_i$, we have

$$\sum_{a\in A} M_{ia}\tilde{H}_a = \sum_{a\in A} M_{ia} \left(\sum_{j\in I} N_{aj} H_j \right) = \sum_{j\in I} \left(\sum_{a\in A} M_{ia} N_{aj} \right) H_j = H_i, \quad (4.86)$$

where we could exchange the order of the sums since the N_{aj} are such that for all fixed $j \in I$, they vanish for all but finitely many $a \in A$.

Now suppose that $P \in \mathcal{I}$. Then we can write

$$P = \sum_{i\in I} f_i H_i \quad (4.87)$$

for some $f_i \in \widehat{\mathcal{D}}^\hbar_A$. But then

$$P = \sum_{i\in I} f_i H_i = \sum_{i\in I} f_i \left(\sum_{a\in A} M_{ia} \tilde{H}_a \right) = \sum_{a\in A} \left(\sum_{i\in I} f_i M_{ia} \right) \tilde{H}_a. \qquad (4.88)$$

Since the M_{ia} are such that for all fixed $a \in A$, they vanish for all but finitely many $i \in I$, the sums $\sum_{i\in I} f_i M_{ia}$ are finite, and hence certainly produce well defined operators in $\widehat{\mathcal{D}}^\hbar_A$. Therefore $P \in \mathcal{J}$, and hence $\mathcal{I} \subseteq \mathcal{J}$.

Conversely, suppose that $Q \in \mathcal{J}$. Then we can write

$$Q = \sum_{a\in A} g_a \tilde{H}_a = \sum_{a\in A} g_a \left(\sum_{i\in I} N_{ai} H_i \right) = \sum_{i\in I} \left(\sum_{a\in A} g_a N_{ai} \right) H_i, \qquad (4.89)$$

for some $g_a \in \widehat{\mathcal{D}}^\hbar_A$. Again, since the N_{ai} are such that for all fixed $i \in I$, they vanish for all but finitely many $a \in A$, the sums $\sum_{a\in A} g_a N_{ai}$ are finite and hence produce well defined operators in $\widehat{\mathcal{D}}^\hbar_A$. Therefore $Q \in \mathcal{I}$, and hence $\mathcal{J} \subseteq \mathcal{I}$. We conclude that $\mathcal{I} = \mathcal{J}$.

□

Remark 4.F.2. In [10], the authors define "Airy structures in normal form" as being given by collections of differential operators $(H_a)_{a\in A}$ satisfying the condition $H_a = \hbar\partial_a + O(\hbar^2)$, and "Airy structures" as being given by collections of differential operators related to Airy structures in normal forms by transformations as in Lemma 4.F.1. From our point of view, since we define an Airy structure (or rather an Airy ideal) as being the left ideal itself, we do not need to make such a distinction.

4.G Two special cases

We now look at two special cases of the construction, which will be necessary to connect with the Heisenberg algebra in the next section.

Consider the Weyl algebra $\mathcal{D}_A = \mathcal{C}[x_A]\langle\partial_A\rangle$. Let $I \subset A$ be a subset of the index set A, and let $J = A \setminus I \subset A$. There are two subalgebras that can be constructed easily:

1. $\mathcal{D}(x_J, \partial_A) := \mathbb{C}[x_J]\langle\partial_A\rangle \subset \mathcal{D}_A$, which is the subalgebra of differential operators whose coefficients do not depend on the variables x_I;

2. $\mathcal{D}(x_A, \partial_J) := \mathbb{C}[x_A]\langle\partial_J\rangle \subset \mathcal{D}_A$, which is the subalgebra of differential operators in which the derivatives ∂_I do not appear.

It is clear that both of those are subalgebras of the Weyl algebra $\mathcal{D}_A$. In fact, in the second case, we can think of the subalgebra $\mathcal{D}(x_A, \partial_J)$ as being the Weyl algebra $\mathcal{D}_J$ but over the polynomial ring $\mathbb{C}[x_A]$. We will however not take this point of view here, since we want the extra variables x_I to be included in the Bernstein filtration, so it makes perhaps more sense to view $\mathcal{D}(x_A, \partial_J)$ as a subalgebra of $\mathcal{D}_A$.

The Bernstein filtration can be defined on the subalgebras as well, and we can construct the Rees Weyl subalgebras $\mathcal{D}^\hbar(x_J, \partial_A)$ and $\mathcal{D}^\hbar(x_A, \partial_J)$ and their $\hbar$-adic completions $\widehat{\mathcal{D}}^\hbar(x_J, \partial_A)$ and $\widehat{\mathcal{D}}^\hbar(x_A, \partial_J)$, which are subalgebras of $\widehat{\mathcal{D}}^\hbar_A$. The construction of transvections, twisted modules, modules of exponential types, and Airy ideals also goes through the subalgebras. Let us be a bit more precise.

4.G.1 The subalgebra $\widehat{\mathcal{D}}^\hbar(x_A, \partial_J)$

We consider differential operators in $\widehat{\mathcal{D}}^\hbar_A$ whose coefficients are polynomials in all the variables x_A, but that only involve derivatives in ∂_J. Then the Rees polynomial algebra $\widehat{\mathcal{M}}^\hbar_A$ is a left $\widehat{\mathcal{D}}^\hbar(x_A, \partial_J)$-module.

We highlight a few simple statements here, which follow from the construction of the previous sections:

1. Let
$$\mathcal{I}_{\text{can}}(x_A, \partial_J) := \left\{ \sum_{j \in J} c_j \hbar \partial_j \mid c_j \in \widehat{\mathcal{D}}^\hbar(x_A, \partial_J). \right\}. \tag{4.90}$$
Then $\mathcal{I}_{\text{can}}(x_A, \partial_J)$ is the annihilator of $1 \in \widehat{\mathcal{M}}^\hbar_A$, and the Rees polynomial module $\widehat{\mathcal{M}}^\hbar_A$ is canonically isomorphic to $\widehat{\mathcal{D}}^\hbar(x_A, \partial_J)/\mathcal{I}_{\text{can}}(x_A, \partial_J)$.

2. We can define transvections as in Definition 4.D.1, with non-trivial action only on the derivatives $\hbar\partial_J$, but with polynomials $p_a^{(n+1)} = \partial_a q^{(n+1)}$ that depend on all variables x_A. Those transvections are automorphisms of the subalgebra $\widehat{\mathcal{D}}^\hbar(x_A, \partial_J)$. Then, if we define the left ideal
$$\mathcal{I} = \left\{ \sum_{j \in J} c_j \bar{H}_j \mid c_j \in \widehat{\mathcal{D}}^\hbar(x_A, \partial_J) \right\} \tag{4.91}$$
with the $\bar{H}_j$ defined as in Definition 4.D.1, it is clear that $\widehat{\mathcal{D}}^\hbar(x_A, \partial_J)/\mathcal{I}$ is a cyclic left $\widehat{\mathcal{D}}^\hbar(x_A, \partial_J)$-module canonically isomorphic to the twisted module ${}^\phi\widehat{\mathcal{M}}^\hbar_A$.

3. We can also rephrase the statement in terms of modules of exponential type. We get that $\widehat{\mathcal{D}}^\hbar(x_A, \partial_J)/\mathcal{I}$ is canonically isomorphic to the module

of exponential type $\widehat{\mathcal{M}}_A^\hbar Z$ with

$$Z = \exp\left(- \sum_{n=0}^{\infty} \hbar^{n-1} q^{(n+1)}(x_A) \right), \tag{4.92}$$

where the $q^{(n+1)}(x_J)$ are degree $\leq n+1$ in the variables x_A as usual. In terms of differential equations, what this says is that Z is a solution to the differential equations $\mathcal{I} \cdot Z = 0$. A subtelty arises however in the uniqueness statement. Indeed, there are many choices of polynomials $q^{(n+1)}(x_A)$ that give rise to the same transvection, since the transvection only involves the derivatives $\partial_j q^{(n+1)}(x_A)$ with respect to the variables $\partial_j \in \partial_J$. In other words, any two $q^{(n+1)}(x_A)$ that differ by a polynomial $p^{(n+1)}(x_I)$ in the variables x_I give rise to the same transvection in the subalgebra $\widehat{\mathcal{D}}^\hbar(x_A, \partial_J)$. As a result, we can state the uniqueness result as saying that there is a unique solution to the differential equation $\mathcal{I} \cdot Z = 0$ with $Z\big|_{x_J=0}$, which basically amounts to not only requiring that the $q^{(n+1)}(0) = 0$ but also that they do not depend at all on the variables x_I. Or, we could give up on uniqueness, and state that we can construct families of solutions of the form (4.92) parametrized by the variables x_I. The unique solution above would then pick the origin of this family.

As usual, we can define Airy ideals as in Definition 4.E.1, but requiring that the ideal $\mathcal{I} \subset \widehat{\mathcal{D}}^\hbar(x_A, \partial_J)$. That is, we only have differential operators $H_j = \hbar\partial_j + O(\hbar^2)$ for $j \in J \subset A$. Then everything goes through, and Theorem 4.E.3 holds. Corollary 4.E.8 also holds, with the caveat about uniqueness mentioned in point (3) above.

More precisely, if $\mathcal{I} \subset \widehat{\mathcal{D}}^\hbar(x_A, \partial_J)$ is an Airy ideal, then $\widehat{\mathcal{D}}^\hbar(x_A, \partial_J)/\mathcal{I}$ is canonically isomorphic to the twisted module ${}^{\phi}\widehat{\mathcal{M}}^{\hbar}_{A}$ for some stable transvection ϕ of $\widehat{\mathcal{D}}^{\hbar}(x_A, \partial_J)$, and also canonically isomorphic to a module of exponential type $\widehat{\mathcal{M}}^{\hbar}_{J} Z$ with Z of the form of (4.92) with $q^{(1)}(x_A) = q^{(2)}(x_A) = 0$.

From the point of view of differential equations, we conclude that we can construct families of solutions to the differential equations $\mathcal{I} \cdot Z$, parametrized by the extra variables x_I, of the form

$$Z = \exp\left(\sum_{\substack{g\in\frac{1}{2}\mathbb{N}, n\in\mathbb{N}^* \\ 2g-2+n>0}} \hbar^{2g-2+n} F_{g,n}(x_A)\right), \tag{4.93}$$

for some polynomials $F_{g,n}(x_A)$ homogeneous of degree n with $F_{g,n}(0) = 0$. We can also state that there is a unique such solution satisfying the initial condition $Z\big|_{x_J=0} = 1$, which amounts to imposing that the polynomials $F_{g,n}$ do not depend on the variables x_I (this is the origin in the family of solutions parametrized by the x_I).

4.G.2 The subalgebra $\widehat{\mathcal{D}}^{\hbar}(x_J, \partial_A)$

In this case, we consider differential operators in $\widehat{\mathcal{D}}^{\hbar}_{A}$ whose coefficients are polynomials but only in the variables $x_J \subset x_A$. Let $\widehat{\mathcal{M}}^{\hbar}_{J}$ be the $\hbar$-adic completion of the Rees polynomial module in the variables x_J. It is clearly a left $\widehat{\mathcal{D}}^{\hbar}(x_J, \partial_A)$-module, where the derivatives ∂_I act trivially.

1. Let
$$\mathcal{I}_{\mathrm{can}}(x_J, \partial_A) := \left\{ \sum_{a\in A} c_a \hbar\partial_a \mid c_a \in \widehat{\mathcal{D}}^{\hbar}(x_J, \partial_A). \right\}. \tag{4.94}$$
Then $\mathcal{I}_{\mathrm{can}}(x_J, \partial_A)$ is the annihilator of $1 \in \widehat{\mathcal{M}}^{\hbar}_{J}$, and the Rees polynomial module $\widehat{\mathcal{M}}^{\hbar}_{J}$ is canonically isomorphic to $\widehat{\mathcal{D}}^{\hbar}(x_J, \partial_A)/\mathcal{I}_{\mathrm{can}}(x_J, \partial_A)$.

2. We can define transvections as in Definition 4.D.1, but with the polynomials $p_a^{(n+1)} = \partial_a q^{(n+1)}$ depending only on the variables x_J. Those transvections are automorphisms of the subalgebra $\widehat{\mathcal{D}}^\hbar(x_J, \partial_A)$. Then, if we define the left ideal

$$\mathcal{I} = \left\{ \sum_{a \in A} c_a \bar{H}_a \mid c_a \in \widehat{\mathcal{D}}^\hbar(x_J, \partial_A) \right\} \tag{4.95}$$

with the $\bar{H}_a$ defined as in Definition 4.D.1, it is clear that $\widehat{\mathcal{D}}^\hbar(x_J, \partial_A)/\mathcal{I}$ is a cyclic left $\widehat{\mathcal{D}}^\hbar(x_J, \partial_A)$-module canonically isomorphic to the twisted module ${}^\phi\widehat{\mathcal{M}}_J^\hbar$.

3. We can also rephrase the statement in terms of modules of exponential type as usual, but we need to be a little bit careful here. While the polynomials $p_a^{(n+1)} = \partial_a q^{(n+1)}$ do not depend on the variables x_I, this does not mean that the polynomials $q^{(n+1)}$ do not depend on the x_I. What it means is that they can be written as

$$q^{(n+1)}(x_J) + s^{(n+1)}(x_I) \tag{4.96}$$

for linear polynomials $s^{(n+1)}(x_I)$ in the variables x_I. (Here we extend the action of the differential operators on the module by the natural action on the variables x_I.) In other words, we get that $\widehat{\mathcal{D}}^\hbar(x_J, \partial_A)/\mathcal{I}$ is canonically isomorphic to the module of exponential type $\widehat{\mathcal{M}}_J^\hbar Z$ with

$$Z = \exp\left(-\sum_{n=0}^{\infty} \hbar^{n-1} q^{(n+1)}(x_J) - \sum_{n=0}^{\infty} \hbar^{n-1} s^{(n+1)}(x_I) \right), \tag{4.97}$$

where the $q^{(n+1)}(x_J)$ are degree $\leq n+1$ in the variables x_J as usual, but the $s^{(n+1)}(x_I)$ are linear polynomials in the variables x_I. In terms of differential equations, what this says is that Z is a solution to the differential equations $\mathcal{I}\cdot Z = 0$, and if we set $q^{(n+1)}(0) = 0$ and $s^{(n+1)}(0) = 0$, it is the unique solution satisfying $Z\big|_{x_A=0} = 1$.

Then, we can define Airy ideals as in Definition 4.E.1, but requiring that the ideal $\mathcal{I} \subset \widehat{\mathcal{D}}^{\hbar}(x_J, \partial_A)$. That is, the H_a are differential operators with coefficients that do not depend on the x_I. Then everything goes through, and Theorem 4.E.3 holds, with the transvection ϕ being of the form above (i.e. not involving the variables x_I). Corollary 4.E.8 also holds, with the caveat that the partition function takes the form (4.97).

More precisely, if $\mathcal{I} \subset \widehat{\mathcal{D}}^{\hbar}(x_J, \partial_A)$ is an Airy ideal, then $\widehat{\mathcal{D}}^{\hbar}(x_J, \partial_A)/\mathcal{I}$ is canonically isomorphic to the twisted module ${}^{\phi}\widehat{\mathcal{M}}^{\hbar}_J$ for some stable transvection ϕ of the form above, and also canonically isomorphic to a module of exponential type $\widehat{\mathcal{M}}^{\hbar}_J Z$ with Z of the form of (4.97) with $q^{(1)}(x_J) = q^{(2)}(x_J) = s^{(1)}(x_I) = s^{(2)}(x_I) = 0$. Using the standard notation in the literature on Airy structures, we conclude that the unique exponential solution to the differential equations $\mathcal{I} \cdot Z$ with initial condition $Z\big|_{x_A=0}$ takes the form

$$Z = \exp\left(\sum_{\substack{g \in \frac{1}{2}\mathbb{N}, n \in \mathbb{N}^* \\ 2g-2+n>0}} \hbar^{2g-2+n} F_{g,n}(x_J) + \sum_{g \in \frac{1}{2}\mathbb{N}^*} \hbar^{2g-1} F_{g,1}(x_I) \right), \tag{4.98}$$

for some polynomials $F_{g,n}$ homogeneous of degree n in the respective variables, with $F_{g,n}(0) = 0$.

To connect further with standard notation in the literature, it is customary to write down the following expansions for the homogeneous piolynomials $F_{g,n}$:

$$F_{g,n}(x_J) = \frac{1}{n!} \sum_{j_1,\dots,j_n \in J} F_{g,n}[j_1,\dots,j_n] x_{j_1} \cdots x_{j_n}, \tag{4.99}$$

where the $F_{g,n}[j_1,\dots,j_n] \in \mathbb{C}$ are coefficients symmetric under permutations of the entries. What (5.107) says is that, if we were to define the coefficients above for all entries $a_j \in A$ and write down a general partition function, then the coefficients $F_{g,n}[a_1,\dots,a_n]$ would vanish whenever $n \geq 2$ and at least one of the entries is in I.

4.H Airy ideals and the Heisenberg algebra

The Weyl algebra is obviously closely connected to the universal enveloping algebra of the Heisenberg algebra. Thus, not surprisingly, many Airy ideals can be constructed from this vantage point. We now review this construction.

4.H.1 The Heisenberg algebra

Let $\mathfrak{h}$ be the Heisenberg Lie algebra with basis $\{J_n\}_{n\in\mathbb{Z}} \cup \{c\}$ and Lie bracket

$$[J_m, J_n] = m\delta_{m,-n}c, \qquad [J_m, c] = 0, \qquad \forall m,n \in \mathbb{Z}. \tag{4.100}$$

Abusing notation a little bit, we will write $U(\mathfrak{h})$ for the quotient of its universal enveloping algebra by the ideal $c = 1$. It is the free associative algebra over $\mathbb{C}$ generated by $\{J_m\}_{m\in\mathbb{Z}}$ modulo the commutation relations

$$[J_m, J_n] = m\delta_{m,-n}, \qquad \forall m,n \in \mathbb{Z}. \tag{4.101}$$

It is the algebra of modes of the rank one free boson vertex operator algebra (VOA), often denoted by π. For future use, we define $U_+(\mathfrak{h})$ as being the subalgebra generated by the positive modes, $U_{\geq 0}(\mathfrak{h})$ as being the subalgebra generated by the non-negative modes, $U_-(\mathfrak{h})$ as being the subalgebra generated by the negative modes, and $U_0(\mathfrak{h})$ as being the subalgebra generated by the zero mode.

Simple modules of the free boson VOA (or Heisenberg VOA) are Fock modules π_λ, parameterized by a complex weight label λ. They are generated by a highest-weight state $|\lambda\rangle$ satisfying

$$J_n|\lambda\rangle = 0 \quad \text{for } n > 0, \qquad J_0|\lambda\rangle = \lambda|\lambda\rangle \tag{4.102}$$

and the negative modes act freely on the highest-weight state. In particular as a vector space π_λ coincides with the polynomial ring in the negative modes. Not every module of the Heisenberg VOA is completely reducible and in fact there are infinite length indecomposable modules constructed as follows. Consider the polynomial ring $\mathbb{C}[y]$ in one variable. It becomes a module for the abelian Lie algebra $\mathbb{C}J_0$ generated by J_0 under

$$\rho_\lambda : \mathbb{C}J_0 \to \mathbb{C}[y], \qquad J_0 \mapsto \frac{d}{dy} + \lambda. \tag{4.103}$$

The module will be denoted by ρ_λ and it induces firstly to a module of the non-negative modes by demanding that the positive modes act as zero and then to a module π_{ρ_λ} of the free boson VOA by letting all negative modes act freely. In formulas, the induced module is

$$\pi_{\rho_\lambda} = \mathrm{Ind}_{U_{\geq 0}(\mathfrak{h})}^{U(\mathfrak{h})} \rho_\lambda. \tag{4.104}$$

π_{ρ_λ} has the Fock module π_λ as submodule while the quotient is isomorphic to π_{ρ_λ} itself, that is it satisfies the non-split exact sequence

$$0 \to \pi_\lambda \to \pi_{\rho_\lambda} \to \pi_{\rho_\lambda} \to 0. \tag{4.105}$$

As a vector space π_{ρ_λ} is isomorphic to the polynomial ring in the negative modes together with the extra variable y on which the zero-mode acts as $\frac{d}{dy} + \lambda$.

We also want to think of J_0 as a variable. For this consider the representation of $\mathbb{C}J_0$

$$\kappa_\lambda : \mathbb{C}J_0 \to \mathbb{C}[y], \qquad J_0 \mapsto y + \lambda \tag{4.106}$$

and the induced module

$$\pi_{\kappa_\lambda} = \mathrm{Ind}_{U_{\geq 0}(\mathfrak{h})}^{U(\mathfrak{h})} \kappa_\lambda. \tag{4.107}$$

π_{κ_λ} has the Fock module π_λ as homomorphic image (mapping y to zero) while the kernel is isomorphic to π_{κ_λ} itself, that is it satisfies the non-split exact sequence

$$0 \to \pi_{\kappa_\lambda} \to \pi_{\kappa_\lambda} \to \pi_\lambda \to 0. \tag{4.108}$$

As a vector space π_{κ_λ} is isomorphic to the polynomial ring in the negative modes together with the extra variable y on which the zero-mode acts via multiplication by $y + \lambda$. Both modules π_{ρ_λ} and π_{κ_λ} are naturally modules of a slightly larger algebra that we will discuss now.

4.H.2 Adding a conjugate zero-mode

$U(\mathfrak{h})$ is almost the Weyl algebra $\mathcal{D}_{\mathbb{N}^*}$ under the identification

$$J_m = \partial_m, \qquad J_{-m} = m x_m, \qquad m \in \mathbb{N}^*, \tag{4.109}$$

but not quite: the zero mode J_0 is missing. The zero mode is central, as it commutes with all other modes J_m. To take into account the zero mode J_0, we introduce a conjugate zero mode $\tilde{J}_0$ that satisfies the commutation relationsz

$$[J_m, \tilde{J}_0] = \pm\delta_{m,0}. \tag{4.110}$$

The choice of sign here will be dictated by our interpretation of the zero mode J_0 and its conjugate $\tilde{J}_0$. We then consider the free associative algebra over $\mathbb{C}$ generated by the $\{J_m\}_{m\in\mathbb{Z}}$ and $\tilde{J}_0$, which is isomorphic to $U(\mathfrak{h}) \otimes_{\mathbb{C}} \mathbb{C}[\tilde{J}_0]$. Now we have two choices to map the resulting algebra to the Weyl algebra. We set the parameters of the modules of the previous section to $y = x_0$ and $\lambda = 0$.

(π_{κ_0}) The case π_{κ_0} corresponds to a minus sign in (4.110), and, as operators on π_{κ_0}, $J_0 = x_0$ is a variable, and $\tilde{J}_0 = \partial_0$ is a derivative. Via the identification (4.109) we see that π_{κ_0} is $\mathbb{C}[x_0, x_1, \dots]$ the polynomial ring in infinitely many variables including x_0. Modes of fields of the Heisenberg vertex algebra are infinite sums of monomials in the derivatives, excluding ∂_0, whose coefficients are polynomials, that is elements of $\mathbb{C}[x_0, x_1, \dots]$. This means that π_{κ_0} carries an action of $\mathcal{D}_{\mathbb{N}}$ and the algebra of modes of the Heisenberg VOA is contained in the subalgebra $\mathcal{D}(x_{\mathbb{N}}, \partial_{\mathbb{N}^*}) \subset \mathcal{D}_{\mathbb{N}}$ (a special case of the type of subalgebras considered in Section 4.G.1).

(π_{ρ_0}) The case π_{ρ_0} corresponds to a plus sign in (4.110), and $J_0 = \partial_0$ acts as a derivative, and $\tilde{J}_0 = x_0$ as a variable. The module π_{ρ_0} then is also intepreted as $\mathbb{C}[x_0, x_1, \dots]$ and so it again carries an action of $\mathcal{D}_{\mathbb{N}}$, but this time the modes of fields of the Heisenberg vertex algebra are infinite sums of monomials in the derivatives, including ∂_0, but with polynomial coefficients in $\mathbb{C}[x_1, \dots]$. The conjugate zero-mode $\tilde{J}_0 = x_0$ doesn't appear. This means the algebra of modes of the Heisenberg VOA is now contained in the subalgebra $\mathcal{D}(x_{\mathbb{N}^*}, \partial_{\mathbb{N}}) \subset \mathcal{D}_{\mathbb{N}}$ (a special case of the type of subalgebras considered in Section 4.G.2).

In both cases, we identify a completion of the universal enveloping algebra $U(\mathfrak{h})$ with a subalgebra of the Weyl algebra $\mathcal{D}_{\mathbb{N}}$. Note that as modules for $\mathcal{D}_{\mathbb{N}}$ our two modules π_{κ_0} and π_{ρ_0} are isomorphic and are highest-weight modules generated by a highest-weight vector $|x_0\rangle$ on which all positive modes act as zero and all non-negative ones act freely. Let us call this $\mathcal{D}_{\mathbb{N}}$ module M.

Finally, we introduce $\hbar$. In both cases, we implement the Rees construction with respect to the filtration on $U(\mathfrak{h}) \otimes_{\mathbb{C}} \mathbb{C}[\tilde{J}_0]$ defined by

$$F_i \left(U(\mathfrak{h}) \otimes_{\mathbb{C}} \mathbb{C}[\tilde{J}_0] \right) = \{\text{polynomials of degree} \leq i \text{ in the modes } J_m, \tilde{J}_0\}. \tag{4.111}$$

This is of course mapped to the Bernstein filtration as required. We then introduce $\hbar$ via the Rees construction. Ultimately, the result is the free associative algebra over $\mathbb{C}$ generated by $\hbar$, $\hbar\tilde{J}_0$ and the $\{\hbar J_m\}_{m\in\mathbb{Z}}$ modulo their commutation relations, which is isomorphic to the Rees Weyl algebra $\mathcal{D}^{\hbar}_{\mathbb{N}}$ (see Remark 4.B.5). Finally, we consider the $\hbar$-adic completion, which is mapped to $\widehat{\mathcal{D}}^{\hbar}_{\mathbb{N}}$.

As a result of all this, we have identified the $\hbar$-adic completion of the

Rees universal enveloping algebra, which we denote by $\widehat{U}^\hbar(\mathfrak{h})$, with either the subalgebra $\widehat{\mathcal{D}}^\hbar(x_\mathbb{N}, \partial_{\mathbb{N}^*})$, or the subalgebra $\widehat{\mathcal{D}}^\hbar(x_{\mathbb{N}^*}, \partial_\mathbb{N})$, depending on the choice of intepretation for the central zero mode J_0, i.e. the choice of module π_{κ_0} or π_{ρ_0}.

Because of this identification, all the results summarized in Section 4.G.1 and 4.G.2 apply. Let us summarize their meaning in the context of the Heisenberg algebra.

4.H.3 Identifying $\widehat{U}^\hbar(\mathfrak{h})$ with $\widehat{\mathcal{D}}^\hbar(x_\mathbb{N}, \partial_{\mathbb{N}^*})$

The three results highlighted in Section 4.G.1 then have the following interpretation in the case of π_{κ_0}, i.e. $J_0 = x_0$ is a variable:

1. Let $\mathcal{I}_{\text{can}}$ be the left ideal in $\widehat{U}^\hbar(\mathfrak{h})$ generated by the positive modes J_m, $m \in \mathbb{N}^*$. Then $\widehat{U}^\hbar(\mathfrak{h})/\mathcal{I}_{\text{can}}$ is canonically isomorphic to the ($\hbar$-adic completion of the Rees) module M generated by $|x_0\rangle$, which is clear.

2. We define transvections as usual, they take the form $\phi : (\hbar, \hbar J_{-m}, \hbar J_0, \hbar J_m) \mapsto (\hbar, \hbar J_{-m}, \hbar J_0, \bar{H}_m)$, with

$$\bar{H}_m = \hbar J_m + \sum_{n=0}^{\infty} \hbar^n [J_m, q^{(n+1)}(J_0, J_{-1}, J_{-2}, \dots)], \qquad m \in \mathbb{N}^*, \quad (4.112)$$

where the $q^{(n+1)}$ are polynomials of degree $\leq n+1$ in the non-positive modes. Then, if we define $\mathcal{I}$ to be the left ideal generated by the $\bar{H}_m$, $\widehat{U}^\hbar(\mathfrak{h})/\mathcal{I}$ is canonically isomorphic to the ($\hbar$-adic completion of the Rees) module M twisted by the automorphism ϕ.

3. Rephrasing in terms of modules of exponential type, we get that $\widehat{U}^\hbar(\mathfrak{h})/\mathcal{I}$

is canonically isomorphic to the module generated by the state

$$Z|x_0\rangle = \exp\left(-\sum_{n=0}^{\infty} \hbar^{n-1} q^{(n+1)}(J_0, J_{-1}, J_{-2}, \ldots)\right)|x_0\rangle. \tag{4.113}$$

This is of course a family of Fock modules π_{x_0}, parametrized by the choice of highest weight x_0. There is a unique choice if we impose that the highest weight is $x_0 = 0$ (in other words, we set the zero mode J_0 to zero) and the polynomials satisfy $q^{(n+1)}(0) = 0$.

The interesting statements however are for Airy ideals in $\widehat{U}^{\hbar}(\mathfrak{h})$. To construct an Airy ideal, we need to construct a collection of operators $\{H_j\}_{j\in\mathbb{N}^*}$ in $\widehat{U}^{\hbar}(\mathfrak{h})$ of the form

$$H_j = \hbar J_j + O(\hbar^2), \tag{4.114}$$

and satisfying the properties of Definition 4.E.1. Such collections will naturally arise, for instance, from the modes of the strong generators of some algebras, such as $\mathcal{W}$-algebras, which can be constructed as sub-VOAs of the Heisenberg VOA. If we are given such an Airy ideal $\mathcal{I}$, then we know that $\widehat{U}^{\hbar}(\mathfrak{h})/\mathcal{I}$ is canonically isomorphic to a twisted module as above for some stable transvection ϕ, and also canonically isomorphic to a module of exponential type generated by a state as in (4.113) with $q^{(1)} = q^{(2)} = 0$.

In particular, since $q^{(1)} = q^{(2)} = 0$ the argument of the exponential starts at $O(\hbar)$, and expanding the exponential we can think of the state $Z|x_0\rangle$ as living in the $\hbar$-adic completion of the Rees module M generated by $|x_0\rangle$.[9] To summarize, given any Airy ideal, we showed that $\widehat{U}^{\hbar}(\mathfrak{h})/\mathcal{I}$ is canonically

[9] To be precise, we would need to define the Rees Fock module slightly more generally here, since if we expand the exponential the polynomial coefficients will be 3 times the $\hbar$-degree, but this can be done easily without complication.

isomorphic to a cyclic left module generated by a state in the $\hbar$-adic completion of the Rees Fock module.

This is particularly interesting if the modes H_j that generate the Airy ideal actually are a subset of the modes (such as the positive modes) of the strong generators of a sub-VOA, such as a $\mathcal{W}$-algebra. In this case, what we have a constructed is a state $Z|x_0\rangle$ in the $\hbar$-adic completion of the Rees Fock module for $\widehat{U}^\hbar(\mathfrak{h})$ that is annihilated by all the modes H_j in this subset. Considering the action of the other modes of the sub-VOA on this state, we obtain a cyclic module for this sub-VOA, which is generated by the state $Z|x_0\rangle$. Depending on the subset of modes considered, this may be a highest weight module, or a Whittaker module, for the sub-VOA [9, 8].

4.H.4 Identifying $\widehat{U}^\hbar(\mathfrak{h})$ with $\widehat{\mathcal{D}}^\hbar(x_{\mathbb{N}^*}, \partial_{\mathbb{N}})$

The three highlighted results have the following interpretation in the case of π_{ρ_0}, i.e. $J_0 = \partial_0$ is a derivative:

1. Let $\mathcal{I}_{\text{can}}$ be the left ideal in $\widehat{U}^\hbar(\mathfrak{h})$ generated by the non-negative modes J_m, $m \in \mathbb{N}$. Then $\widehat{U}^\hbar(\mathfrak{h})/\mathcal{I}_{\text{can}}$ is canonically isomorphic to the ($\hbar$-adic completion of the Rees) $\widehat{\mathcal{D}}^\hbar(x_{\mathbb{N}^*}, \partial_{\mathbb{N}})$ submodule of M generated by $|x_0\rangle$, which is clear.

2. We define transvections as usual, they take the form $\phi : (\hbar, \hbar J_{-m}, \hbar J_0, \hbar J_m) \mapsto (\hbar, \hbar J_{-m}, \bar{H}_0, \bar{H}_m)$, with

$$\bar{H}_m = \hbar J_m + \sum_{n=0}^{\infty} \hbar^n [J_m, q^{(n+1)}(J_{-1}, J_{-2}, \ldots)], \qquad m \in \mathbb{N}, \qquad (4.115)$$

   where the $q^{(n+1)}$ are polynomials of degree $\leq n+1$ in the negative modes.

Then, if we define $\mathcal{I}$ to be the left ideal generated by the $\bar{H}_m$, $\widehat{U}^\hbar(\mathfrak{h})/\mathcal{I}$ is canonically isomorphic to the ($\hbar$-adic completion of the Rees) submodule of M generated by $|x_0\rangle$ and twisted by the automorphism ϕ.

3. Rephrasing in terms of modules of exponential type, we get that $\widehat{U}^\hbar(\mathfrak{h})/\mathcal{I}$ is canonically isomorphic to the submodule of M generated by the state

$$Z|x_0\rangle = \exp\left(-\sum_{n=0}^{\infty}\hbar^{n-1}q^{(n+1)}(J_{-1},J_{-2},\ldots) - \sum_{n=0}^{\infty}\hbar^{n-1}s^{(n+1)}(\tilde{J}_0)\right)|x_0\rangle, \tag{4.116}$$

where the $s^{(n+1)}(\tilde{J}_0)$ are linear polynomials in the conjugate zero mode $\tilde{J}_0$, which acts on $|x_0\rangle$ as $\tilde{J}_0|x_0\rangle = x_0|x_0\rangle$. There is a unique choice of generator if we impose that the polynomials satisfy $q^{(n+1)}(0) = s^{(n+1)}(0) = 0$.

We can proceed and study Airy ideals in $\widehat{U}^\hbar(\mathfrak{h})$ as usual. To construct an Airy ideal, we need to construct a collection of operators $\{H_a\}_{j\in\mathbb{N}}$ in $\widehat{U}^\hbar(\mathfrak{h})$ of the form

$$H_a = \hbar J_a + O(\hbar^2), \tag{4.117}$$

and satisfying the properties of Definition 4.E.1. Note that there is a H_0 associated to the zero mode J_0 here. Such collections again naturally arise form sub-VOAs such as $\mathcal{W}$-algebras. If we are given an Airy ideal $\mathcal{I}$, then we know that $\widehat{U}^\hbar(\mathfrak{h})/\mathcal{I}$ is canonically isomorphic to the submodule of M generated by $|x_0\rangle$ as above but twisted by some stable transvection ϕ. It is also canonically isomorphic to a module of exponential type generated by a state as in (4.116) with $q^{(1)} = q^{(2)} = s^{(1)} = s^{(2)} = 0$.

What is really interesting here is the appearance of the conjugate zero modes $\tilde{J}_0$ in the exponential in (4.116). Again, for an Airy ideal we must have $q^{(1)} = q^{(2)} = s^{(1)} = s^{(2)} = 0$, so that the argument of the exponential starts at $O(\hbar)$. We can expand the exponential, but the resulting state *does not* live in the $\hbar$-adic completion of the Rees Fock module generated by $|x_0\rangle$ over $\widehat{U}^\hbar(\mathfrak{h})$, because of the appearance of the conjugate modes $\tilde{J}_0$. It instead lives in the submodule of M generated by $|x_0\rangle$ over the $\hbar$-adic completion of the Rees algebra associated to $U(\mathfrak{h}) \otimes_{\mathbb{C}} \mathbb{C}[\tilde{J}_0]$. This is a key distinction between this scenario and the previous one.

In particular, if the H_a form a subset of modes of the strong generators of a sub-VOA, such as a $\mathcal{W}$-algebra, we once again constructed a state $Z|x_0\rangle$ that is annihilated by all the modes H_a in this subset, and this states generates a cyclic module for the sub-VOA, which could be a highest weight module, or a Whittaker module, depending on the subset of modes. However, the state $Z|x_0\rangle$ is not anymore in a Fock module over $U(\mathfrak{h})$ (suitably $\hbar$-adically completed), but rather in the larger module $M = \pi_{\rho_0}$.

Remark 4.H.1. We note that in some cases, $Z|x_0\rangle$ may still live in the $\hbar$-adic completion of the Rees Fock module generated by $|x_0\rangle$ over $\widehat{U}^\hbar(\mathfrak{h})$. This will happen if all the linear polynomials $s^{(n+1)}(\tilde{J}_0)$ vanish. In turn, this will happen if the transvection ϕ does not act on $\hbar J_0$, that is, $\bar{H}_0 = \hbar J_0$. From the point of view of Airy ideals, this means that the operator H_0 is simply equal to $H_0 = \hbar J_0$. This was the case for instance in some of the constructions in [9].

In this particular case, it does not matter what scenario we use to interpret the zero mode. On the one hand, if we think of J_0 as a variable x_0, then since $H_0 = \hbar J_0$ must kill $Z|x_0\rangle$, we must have $x_0 = 0$, i.e. we set the zero mode to zero. On the other hand, if we think of J_0 as a derivative, then $Z|x_0\rangle$ does not include the conjugate modes $\tilde{J}_0$ because the $s^{(n+1)}(\tilde{J}_0)$ vanish, and hence

$H_0 = \hbar J_0$ naturally kills $Z|x_0\rangle$, i.e. we can simply set J_0 to zero as before. In both cases the state $Z|x_0\rangle$ is the same, lives in the Fock module, and we can simply set the zero mode to zero, which is what was done in [9].

However, this is a very particular case; for general Airy ideals, there is no reason why the operator $H_0 = \hbar J_0 + O(\hbar^2)$ that starts with the zero mode should not have terms of $O(\hbar^2)$ or higher. We will see an example of that in the next sections when considering $W(\mathfrak{sp}_{2N})$-algebras.

Remark 4.H.2. We note that even if the operator $H_0 = \hbar J_0 + O(\hbar^2)$ has higher order terms, it does not mean that the linear polynomials $s^{(n+1)}(\tilde{J}_0)$ will be non-zero. H_0 could still be special enough such that all $s^{(n+1)}(\tilde{J}_0) = 0$, in which case $Z|x_0\rangle$ would live in the $\hbar$-adic completion of the Rees module generated by $|x_0\rangle$ over $\widehat{U}^\hbar(\mathfrak{h})$. It appears to be not so easy to determine whether a given Airy ideal will be such that all $s^{(n+1)}(\tilde{J}_0) = 0$; however, what is easy to show is that, if H_0 has polynomial terms at $O(\hbar^2)$, then the linear polynomials $s^{(n+1)}(\tilde{J}_0)$ do not all vanish. So this gives a simple criteria to determine when Z involves the conjugate zero modes $\tilde{J}_0$.

4.H.5 The rank N free boson π^N

In this section we simply note that the construction of the previous section continue to hold if we consider direct sums of Heisenberg algebra $\mathfrak{h} := \bigoplus_{i=1}^N \mathfrak{h}^{(i)}$, with basis $\{J_n^i\}_{i\in\{1,\dots,N\},n\in\mathbb{Z}} \cup \{c\}$ and Lie bracket

$$[J_m^i, J_n^j] = m\delta_{m,-n}\delta_{i,j}c, \qquad [J_m^i, c] = 0, \qquad \forall m,n \in \mathbb{Z}, i,j \in \{1,\dots,N\}. \tag{4.118}$$

The universal enveloping algebra is constructed as usual, quotienting by the ideal $c = 1$. It is the free associative algebra generated by the modes $J_m^{(i)}$ modulo their commutation relations. It is the algebra of modes for the rank N free boson VOA π^N.

The only difference with the previous section is that we now have N zero modes J_0^i. We thus introduce N conjugate zero modes $\tilde{J}_0^i$, and proceed as before with the identification with the Weyl algebra $\mathcal{D}_A$, where we now consider the multi-index set $A = \{(i,n) \mid i \in \{1,\ldots,N\}, n \in \mathbb{N}\}$.

In principle, for each zero mode we can make a choice between the two scenarios of the previous section, i.e. whether we consider J_0^i as a variable or a derivative. It is usually more meaningful to make the same choice for all zero modes. Then we proceed as before, and the results are very similar, so there is no need to re-state them here.

4.H.6 The VOA viewpoint

The last few sections can also be reformulated from the viewpoint of VOAs, which is how the construction of Airy ideals naturally arises. Recall that a VOA is given by the data of a vector space of states V, and a state-field correspondence $Y : V \to \mathrm{End}(V)[[z^{\pm 1}]]$, which satisfies a number of defining axioms. Given a vector $v \in V$, we call $Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ the corresponding field, and the endomorphisms v_n its modes.

A VOA module is another space M, with a maps $Y_M : V \to \mathrm{End}(M)[[z^{\pm 1}]]$. It realizes the modes of $Y_M(v,z)$ as endomorphisms of the space M.

The rank one free boson VOA is generated by a single state $\chi \in V$, with corresponding field $Y(\chi,z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}$. Its modes J_n satisfy the commutation relations of the Heisenberg algebra $\mathfrak{h}$ with $c = 1$:

$$[J_m, J_n] = m\delta_{m,-n}. \tag{4.119}$$

The associative algebra of modes is the universal enveloping algebra $U(\mathfrak{h})$.

To map to $\widehat{\mathcal{D}}^\hbar_{\mathbb{N}}$ and $\widehat{\mathcal{M}}^\hbar_{\mathbb{N}}$ as in the previous sections, we think of the Rees polynomial algebra $\widehat{\mathcal{M}}^\hbar_{\mathbb{N}}$ as a VOA module, with map $Y^\hbar : V \to \mathrm{End}(M)[[z^{\pm 1}]]$ acting on the generating state $\chi \in V$ as

$$Y^\hbar(\chi, z) = \sum_{n\in\mathbb{Z}} \hbar J_n z^{-n-1}. \tag{4.120}$$

We also impose that the module satisfies the property

$$Y^\hbar(Tv, z) = \hbar\partial_z Y^\hbar(v, z) \tag{4.121}$$

for all $v \in V$, where T is the translation endomorphism on V. This turns the algebra of modes into the Rees graded algebra with respect to Li's filtration by conformal weight on the algebra of modes of the free boson [41]. It allows us to identify the algebra of modes with a subalgebra of the Rees Weyl algebra $\widehat{\mathcal{D}}^\hbar_{\mathbb{N}}$ (as in Sections 4.H.3 or 4.H.4, depending on the interpretation of the zero mode J_0), which acts on the module $M = \widehat{\mathcal{M}}^\hbar_{\mathbb{N}}$ (which is also, of course, a left module for the algebra of modes).

Introducing $\hbar$ in this way is in fact very simple. Since it turns the algebra of modes into the Rees graded algebra associated to the filtration by conformal weight, we can simply introduce $\hbar$ at the end of a calculation. Indeed, if $v \in V$ is a state of conformal weight m, then we know that its field in the $\hbar$-deformed module will be given by

$$Y^{\hbar}(v,z) = \hbar^m Y(v,z). \tag{4.122}$$

The rank N free boson VOA is constructed similarly by taking an N-fold tensor product of the rank one free boson VOA. It is generated by n states $\chi^i \in V$, $i = 0, \ldots, N-1$, with corresponding fields $Y(\chi^i, z) = \sum_{n\in\mathbb{Z}} J^i_n z^{-n-1}$. The modes satisfy the commutation relations

$$[J^i_m, J^j_n] = m\delta_{m,-n}\delta_{i,j}, \tag{4.123}$$

as in the previous section with $c = 1$. The algebra of modes is the corresponding universal enveloping algebra, and everything goes through as before.

4.H.7 Twisted modules for the rank N free boson VOA

In many application, the starting point is not quite the algebra of modes of the rank N free boson VOA as in the previous section, but rather the algebra of modes of a twisted module for the rank N free boson VOA.

Roughly speaking, if σ is an automorphism of a VOA V of finite order r, then a σ-twisted VOA module is another space M and a map $Y_\sigma : V \to \mathrm{End}(M)[[z^{\pm 1/r}]]$. The difference of course is that fractional powers of z appear.

In this paper we will only consider the case of the rank N free boson VOA, which is generated by states $\chi^i \in V$, $i = 0, \ldots, N-1$, with the automorphism σ that cyclically permutes the N states:

$$\sigma : \chi^0 \to \chi^1 \to \ldots \to \chi^{N-1} \to \chi^0. \tag{4.124}$$

In this case, we can define a diagonal basis $v^a \in V$, $a = 0, \ldots, N-1$, with

$$v^a = \sum_{j=0}^{N-1} \theta^{-aj} \chi^j, \tag{4.125}$$

with $\theta = e^{2\pi i/N}$. The inverse relation is

$$\chi^i = \frac{1}{N} \sum_{a=0}^{N-1} \theta^{ia} v^a. \tag{4.126}$$

In this diagonal basis, the automorphism σ acts by multiplication by roots of unity:

$$\sigma : v^a \mapsto \theta^a v^a, \qquad a = 0, \ldots, N-1. \tag{4.127}$$

The map $Y_\sigma : V \to \mathrm{End}(M)[[z^{\pm 1/r}]]$ takes the simpler form

$$Y_\sigma(v^a, z) = \sum_{k \in \frac{a}{N} + \mathbb{Z}} J_{kN} z^{-k-1}, \tag{4.128}$$

with the modes satisfying the commutation relations

$$[J_{mN}, J_{nN}] = Nm\delta_{m,-n}. \tag{4.129}$$

In the end, what we found is that, after redefining indices $mN \mapsto k$, the algebra of modes of the σ-twisted module is nothing but the universal enveloping algebra of the Heisenberg algebra $\mathfrak{h}$ with $c = 1$ already studied in Section 4.H.1, which is the algebra of modes for the rank one free boson. It has only one zero mode J_0, not N zero modes as in the untwisted rank N case studied in the previous section.

As the algebra of modes of the twisted modules is identified with the universal enveloping algebra of the Heisenberg algebra $\mathfrak{h}$, all the results of Sections 4.H.3 and 4.H.4 apply, depending on a choice of interpretation for the zero mode.

While we only need to consider the fully cyclic automorphism σ in the rest of the paper, we note that more general automorphisms can certainly be considered, see for instance [9, 10, 6]. For an automorphism σ in the symmetric group S_N that corresponds to a permutation of the N free bosons, the construction works pretty much the same as explained here, applied independently to each cycle in the permutation (see [10, 6]). More precisely, in the end one obtains a set of bosonic modes for each cycle of the permutation σ. The resulting algebra of modes is then naturally identified with the algebra of modes of the untwisted rank M free boson as in the previous section, with M being the number of cycles in σ. It has M distinct zero modes, one for each cycle in the permutation σ.

4.H.8 Boundedness in the VOA setting

Condition (1) in the definition of Airy ideals, see Definition 4.E.1, states that the collection of operators $\{H_a\}_{a\in A}$ must be bounded. If the Airy ideal is generated by a subset of modes of the strong and free generators of a VOA realized as a sub-VOA of the Heisenberg VOA, then the boundedness condition is automatically satisfied. This is what we prove in this section.

Consider a VOA W, that is freely and strongly generated by N-fields $W^1, \ldots, W^N$ and that allows for an embedding in the rank N Heisenberg algebra π^N. Let H be the Virasoro zero-mode of the usual Virasoro field of the Heisenberg algebra. Let

$$W^m(z) = \sum_{k\in\mathbb{Z}} W_k^m z^{-k-1} \tag{4.130}$$

be the mode expansion of the field $W^m(z)$ and we require that W^m has weight $\Delta_m \in \mathbb{Z}_{>0}$ in the sense that

$$[H, W_k^m] = (\Delta_m - k - 1)W_k^m \tag{4.131}$$

for all k. Let

$$W_k^m = \sum_{\substack{0\leq a_1\leq\cdots\leq a_t\\ i_1,\ldots,i_t\in\{0,\ldots,N\}}} A^{i_1,\ldots,i_N}_{a_1,\ldots,a_t}(m,k) J^{i_1}_{a_1}\ldots J^{i_t}_{a_t} \tag{4.132}$$

where the $A^{i_1,\ldots,i_N}_{a_1,\ldots,a_t}(m,k)$ are polynomials in the negative modes. Then the boundedness condition in the VOA setting for the non-negative modes W_k^m is that for any set $0 \leq a_1 \leq \cdots \leq a_t$ one has $A^{i_1,\ldots,i_N}_{a_1,\ldots,a_t}(m,k) = 0$ for all but finitely non-negative modes W_k^m. Interpreting the modes of the Heisenberg algebra as variables and derivatives as before immediately translates to the boundedness condition in the Weyl algebra setting.

Lemma 4.H.3. *The boundedness condtion for non-negative modes W_k^m holds on modules of π^N.*

Proof. Note that $[H, J_k^i] = -kJ_k^i$ for the Heisenberg modes.

Let $I = \{(i_1, k_1), \ldots, (i_r, k_r)\}$ be an ordered index set of length r, that is $i_1, \ldots, i_r \in \{1, \ldots, N\}$ and $k_1, \ldots, k_r \in \mathbb{Z}$ with $k_a \geq k_{a-1}$ and if $k_a = k_{a-1}$ then $i_a \geq_{a-1}$. Set $p_I = J^{i_1}_{k_1} \ldots J^{i_r}_{k_r}$ and $k_I = k_1 + \cdots + k_r$ so that $[H, p_I] = -k_I p_I$.

Let $\mathcal{I}$ be the set of all such index sets of any length. Then W^m_k is of the form

$$W^m_k = \sum_{\substack{I \in \mathcal{I} \\ k_I = k+1-\Delta_m}} a_I p_I \tag{4.133}$$

for certain coefficients a_I. We are interested in the boundedness conditions in the VOA setting. This means for a given ordered monomial $J^{i_s}_{k_s} \dots J^{i_r}_{k_r}$ with $k_s \geq 0$ (and hence all $k_i \geq 0$) we wonder if there exists $\{(i_1, k_1), \dots, (i_{s-1}, k_{s-1})\}$, such that $a_I \neq 0$ for $I = \{(i_1, k_1), \dots, (i_r, k_r)\}$. We have that $k_I \leq k_s + \dots + k_r$ and so we necessarily have $a_I = 0$ if $k + 1 - \Delta_m > k_s + \dots + k_r$. In particular there are only finitely many pairs (m, k) such that a_I can be non-zero, i.e. the boundedness condition holds. $\square$

Let σ be a finite order automorphism of the Heisenberg algebra that leaves W and H invariant. Then the σ-twisted module is still a module for W and still graded by H. Note that with the set-up of the previous setting the twisted modes J_{kN} have H eigenvalue $-k$. As an operator on a σ-twisted module

$$W^m_k = \sum_{0 \leq a_1 \leq \dots \leq a_t} A^\sigma_{a_1, \dots, a_t}(m, k) J_{a_1} \dots J_{a_t} \tag{4.134}$$

with $A^\sigma_{a_1, \dots, a_t}(m, k)$ polynomials in the negative modes.

Lemma 4.H.4. *The boundedness condtion for non-negative modes W^m_k holds on σ-twisted modules after any possible shift of negative modes.*

Proof. The argument is the same as the previous Lemma: $J_{a_1} \dots J_{a_t}$ has H-eigenvalue $-(a_1 + \dots + a_t)/N$ and so $A_{a_1, \dots, a_t}(m, k) = 0$ if $k + 1 - \Delta_m > a_1 + \dots + a_t)/N$, that is for all but finitely many pairs (m, k). Any possible shift of negative modes is nothing but a homomorphism on polynomials in the negative modes and so if $A_{a_1,\ldots,a_t}(m,k) = 0$ then the same remains true after any possible shift of negative modes. $\square$

Remark 4.H.5. The boundedness condition of course holds for subsets of non-negative modes as well. In some cases, see [9], one also wants to include some negative modes W_k^m. The argument for boundedness is still exacly the same. Therefore, the collections of modes considered in [9, 8] are all bounded, and the Airy ideals constructed in these papers are indeed well defined.

4.I Graded Lie subalgebras from modes of $\mathcal{W}$-algebra

In Chapter 3, we mentioned that Airy ideals can be constructed from modules of $\mathcal{W}$-algebras. This is because subalgebras of collection of modes $\{W_m^i\}_{m\in\mathbb{N}}$ of the strong generators of $\mathcal{W}$-algebra generate ideals $\mathcal{I}$ that satisfy the condition $[\mathcal{I},\mathcal{I}] \subseteq \hbar^2\mathcal{I}$. In this section we expound on this example, by sketching the relevant results first proved in [9]. The authors construct a number of left ideals from mode algebras of an arbitrary vertex operator algebra with finitely many strong and free generate, that satisfy the condition $[\mathcal{I},\mathcal{I}] \subseteq \hbar^2\mathcal{I}$ of being an Airy ideal and then one is left with the task to check that the other two conditions are satisfied as well. Further technical details of this construction can also be found in Section 3 of [8]. We will use some results of this section in Chapter 5 to construct new Airy ideals.

Let W be a vertex operator algebra with finitely many strong and free generators $W^1,\ldots,W^n$ of conformal weights $\Delta_1,\ldots,\Delta_n$, and let $\mathcal{A}$ be the suitably completed algebra (also known as the current algebra) of modes of W.

Let F_pW be the subspace of W spanned by elements $W^{i_1}_{-n_1} \cdots W^{i_s}_{-n_s}|0\rangle$ with $\Delta_{i_1} + \cdots + \Delta_{i_s} \le p$, that is "Li's filtration". The algebra of modes $\mathcal{A}$ is a filtered Lie algebra with respect to the commutator $[\cdot, \cdot]$, with a filtration $F_p\mathcal{A}$ induced from F_pW. Let $\widehat{\mathcal{A}}^\hbar$ be its $\hbar$-adic completion,

$$\widehat{\mathcal{A}}^\hbar = \prod_{n \in \mathbb{N}} \hbar^n F_n \mathcal{A}. \tag{4.135}$$

Let S be a given subset of the modes of the strong generators $W^1, \ldots, W^n$ of W, and $\widehat{\mathcal{A}}^\hbar \cdot S \in \widehat{\mathcal{A}}^\hbar$ be the left $\widehat{\mathcal{A}}^\hbar$-ideal generated by S. We say that $\widehat{\mathcal{A}}^\hbar \cdot S$ is a graded Lie subalgebra of $\widehat{\mathcal{A}}^\hbar$ if

$$[\widehat{\mathcal{A}}^\hbar \cdot S, \widehat{\mathcal{A}}^\hbar \cdot S] \subseteq \hbar^2 \widehat{\mathcal{A}}^\hbar \cdot S. \tag{4.136}$$

For notational convenience, we define an ordering such that a mode in S is always greater than a mode not in S. We say that elements of the ideal $\widehat{\mathcal{A}}^\hbar \cdot S$ are good with respect to S. In particular, γ is good if the right-most term of every ordered monomial of γ is in S. We first state a basic observation, that is easily checked.

Lemma 4.I.1. *The left $\widehat{\mathcal{A}}^\hbar$-ideal generated by the modes in S is a graded Lie subalgebra of $\widehat{\mathcal{A}}^\hbar$ if and only if for any two modes $W^i_m, W^j_n \in S$, one has that $[W^i_m, W^j_n] \in \hbar^2 F_{\Delta_i + \Delta_j - 2} \widehat{\mathcal{A}}^\hbar$ is good with respect to S.*

The starting point of construction is a module $\mathcal{M}_\lambda$ generated by a highest weight vector $|\lambda\rangle$ and such that this highest weight vector is annihilated by a mode if and only if this mode is a good mode. Subsequently one needs to

check that the commutator of two modes in S is still good by showing that a basis of $\mathcal{M}_\lambda$ is given by all the ordered monomials that are not good.
We now list two examples of graded Lie subalgebras that exist for all $\mathcal{W}$-algebras at the self dual level. However, one can usually expand this list to further examples depending on the choice of Lie algebra $\mathfrak{g}$. The reader is requested to refer to Section 3 of [9] for the proofs.

Our first example is that of the left ideal generated by all modes of the strong generators of a $\mathcal{W}$ algebra that annihilate the vacuum state $|0\rangle$.

Proposition 4.I.2. *Consider a vertex operator algebra W freely strongly generated by homogeneous states $W^i \in W$ indexed by $i \in \mathcal{I}$ (where $\mathcal{I}$ is a finite set), with respective conformal weights $\Delta_i \in \mathbb{Z}$. Let $\widehat{\mathcal{A}}^\hbar$ be the suitably completed graded algebra of modes of W. Let $S = \{W^i_k\}_{i\in\mathcal{I},k\geq 0}$, and consider the left ideal $\widehat{\mathcal{A}}^\hbar_{\geq 0} := \widehat{\mathcal{A}}^\hbar \cdot S$. Then, $\widehat{\mathcal{A}}^\hbar_{\geq 0}$ is a graded Lie subalgebra of $\widehat{\mathcal{A}}^\hbar$.*

A second example is given by a subset of the vacuum algebra with the starting mode index determined by the conformal weight.

Proposition 4.I.3. *Consider a vertex operator algebra W strongly generated by homogeneous states $W^i \in W$ indexed by $i \in \mathcal{I}$ where $\mathcal{I}$ is a finite set, with respective conformal weights $\Delta_i \in \mathbb{Z}$. Let $\widehat{\mathcal{A}}^\hbar$ denote the suitably completed algebra of modes. Let $S = \{W^i_k\}_{i\in\mathcal{I},k\geq\Delta_i-1}$, and consider the left ideal $\widehat{\mathcal{A}}^\hbar_\Delta := \widehat{\mathcal{A}}^\hbar \cdot S$. Then $\widehat{\mathcal{A}}^\hbar_\Delta$ is a graded Lie subalgebra of $\widehat{\mathcal{A}}^\hbar$.*

Chapter 5

An Airy structure for $\mathcal{W}(\mathfrak{sp}_{2n})$

As stated in 3.B.9, Airy structures (ideals) from $\mathcal{W}$-algebras of type ADE have already been constructed in literature. However, constructions for non-simply laced type Lie algebras still remain to be done. In this chapter, we fill this gap by constructing Airy structures of type C. The starting point for out approach is the orbifold construction for $\mathcal{W}$-algebras at self-dual level as described in 2.D.1. In the process, we exemplify some aspects of Airy structures that were presented in the previous chapter. The contents of this chapter also appear in our preprint [12].

More precisely, we will consider Airy ideals that are generated by the non-negative modes of the strong generators of the principal $\mathcal{W}$-algebra of $\mathfrak{sp}_{2N}$ at level $-N-1/2$, which we denote by $\mathcal{W}^{-N-1/2}(\mathfrak{sp}_{2N})$. To do so, we need to realize the $\mathcal{W}^{-N-1/2}(\mathfrak{sp}_{2N})$-algebras as sub-VOAs of the rank N free boson VOA. In this section we review background notions on the $\mathcal{W}^{-N-1/2}(\mathfrak{sp}_{2N})$-algebras, how they can be realized within the rank N free boson VOA, and how we can construct modules for them from twisted modules for the rank N

free boson VOA.

5.A Generators of $\mathcal{W}(\mathfrak{sp}_{2N})$

We consider the universal principal $\mathcal{W}$-algebra of type C_N at level $-N-1/2$. This algebra is isomorphic to the orbifold of N-pairs of symplectic fermions; the reason is that the coset $\mathrm{Com}\left(V^k(\mathfrak{sp}_{2N}), V^k(\mathfrak{osp}_{1|2N})\right)$ is isomorphic to $\mathcal{W}_\ell(\mathfrak{sp}_{2N})$ for generic ℓ with ℓ and k related via $(\ell+N+1)^{-1}+(k+N+1)^{-1}=2$, by [20, Thm. 4.1] as well as [17, Thm. 3.2]. The limit $k\to\infty$ makes sense; in this limit, the coset becomes an orbifold of a free field algebra [19, Thm. 6.10], which in this case is the $Sp(2N)$-orbifold $\mathcal{A}(N)^{Sp(2N)}$ of N-pairs of symplectic fermions $\mathcal{A}(N)$. For clarity, we write:

$$\mathcal{W}(\mathfrak{sp}_{2N}) := \mathcal{A}(N)^{Sp(2N)} \cong \mathcal{W}^{-N-1/2}(\mathfrak{sp}_{2N}). \tag{5.1}$$

Let us be a little more precise. The symplectic fermion algebra of rank N, $\mathcal{A}(N)$, is strongly and freely generated by N pairs of symplectic fermions $\{e^i(z), f^i(z)\}_{i=1,2,\dots,N}$ by . Their OPEs are given by

$$e^i(z)f^j(w) \sim \frac{\delta_{ij}}{(z-w)^2}. \tag{5.2}$$

Proposition 5.A.1 ([18]). *Let $\{e^i, f^i\}_{i=1,\dots,N}$ be symplectic fermions. Then $\mathcal{W}(\mathfrak{sp}_{2N})$ is freely generated by fields $W^2, W^4, \dots, W^{2N}$ of conformal weights $2, 4, \dots, 2N$ respectively that have the following free field description:*

$$W^m(z) = \frac{1}{(m-2)!}\sum_{i=1}^{N}\left(: e^i(z)\partial_z^{m-2}f^i(z) : + : \partial_z^{m-2}e^i(z)f^i(z) :\right), \tag{5.3}$$

for $m = 2, 4, \ldots, 2N$.

We can express the above result in terms of free bosonic fields after making use of the boson-fermion correspondence [29]. Let $Y(\cdot, z)$ denote the state-operator map for the integral lattice VOA $V_{\mathbb{Z}^N}$ generated by an orthonormal basis $\{\chi^0, \chi^1, \ldots, \chi^{N-1}\}$, and

$$\chi^i(z) = \sum_{n \in \mathbb{Z}} \chi^i_n z^{-n-1} \tag{5.4}$$

be the free bosonic fields, which satisfy the OPE:

$$\chi^i(z)\chi^j(w) \sim \frac{\delta_{ij}}{(z-w)^2}. \tag{5.5}$$

Recall that the free fermion OPE is generated by a pair of odd fields $\psi(z), \psi^*(w)$ with OPE

$$\psi(z)\psi^*(w) \sim \frac{1}{z-w}. \tag{5.6}$$

The boson-fermion correspondence gives a pair of free fermions:

$$\psi_i(z) := Y(\mathbf{e}^{\chi_i}, z), \quad \psi_i^*(z) := Y(\mathbf{e}^{-\chi_i}, z), \tag{5.7}$$

where

$$Y(\mathbf{e}^{\chi_i}, z) = z^{\chi^i_0} U_{\chi^i} \exp\Big(\sum_{n \in \mathbb{Z}_{<0}} \chi^i_n \frac{z^{-n}}{n} \Big) \exp\Big(\sum_{n \in \mathbb{Z}_{>0}} \chi^i_n \frac{z^{-n}}{n} \Big), \tag{5.8}$$

and the shift operators U_{χ^i} satisfy

$$[\chi^i_m, U_{\chi^i}] = \delta_{m,0}\, U_{\chi^i}, \qquad m \in \mathbb{Z}. \tag{5.9}$$

The fields

$$e^i(z) := \psi_i(z), \quad f^i(z) := \partial_z \psi_i^*(z) \tag{5.10}$$

generate a VOA isomorphic to $\mathcal{A}(N)$.

Proposition 5.A.2. *Let $\{W^m(z)\}_{m=2,\dots,2N}$ be the fields defined in (5.3), and define the states:*

$$\nu^m := [\boldsymbol{e}^{\chi^i}_{-1}\boldsymbol{e}^{-\chi^i}_{-m} + \boldsymbol{e}^{\chi^i}_{-m+1}\boldsymbol{e}^{-\chi^i}_{-2}]\boldsymbol{1}. \tag{5.11}$$

Then

$$W^m(z) = \sum_{i=0}^{N-1} Y(\nu^m, z). \tag{5.12}$$

Proof. Translation covariance implies the formula

$$f^i(z) = Y(\mathbf{e}^{-\chi^i}_{-2}\mathbf{1}, z). \tag{5.13}$$

The result then follows directly from (5.7) and application of the reconstruction theorem (See [27, Theorem 4.4.1]) to the VOA $\mathcal{A}(N)$. □

This Proposition gives us an expression for the strong generating fields of $\mathcal{W}(\mathfrak{sp}_{2N})$ within the rank N free boson VOA, which is the starting point to study whether subsets of modes (such as non-negative modes) of the generators of $\mathcal{W}(\mathfrak{sp}_{2N})$ generate an Airy ideal.

5.B Twisted module

In fact, to construct an Airy ideal generated by the modes of the generators of $\mathcal{W}(\mathfrak{sp}_{2N})$, we will need to start with a σ-twisted module for the rank N free boson VOA (see Section 5.B). Upon reduction to the $\mathcal{W}(\mathfrak{sp}_{2N})$ sub-VOA, it will become a normal, untwisted, module for $\mathcal{W}(\mathfrak{sp}_{2N})$.

Let us first review basic properties of twisted modules (studied in detail in [5]) and prove some important formulas. Let Q be an integral lattice with bilinear form $(\cdot,\cdot)$, σ an automorphism of Q, and V_Q be the lattice VOA of Q. The bilinear form can be linearly extended to $\mathbb{C}\otimes_{\mathbb{Z}} Q$. Let M be a σ-twisted V_Q module.

We use the same notation as in Section 5.B. We consider the rank N free boson VOA. Let $\chi^0, \chi^1, \ldots \chi^{N-1}$ be an orthonormal basis for Q. We consider the cyclic automorphism $\sigma : \chi^0 \mapsto \chi^1 \mapsto \cdots \mapsto \chi^{N-1} \mapsto \chi^0$, and the corresponding σ-twisted module with map Y_σ. Let $v^0, v^1, \ldots, v^{N-1}$ be the diagonal basis defined in (4.125), with inverse relation (4.126). The twisted fields are as in (4.128), which we reproduce here for convenience:

$$Y_\sigma(v^a, z) = \sum_{k\in \frac{a}{N}+\mathbb{Z}} J_{kN} z^{-k-1}. \tag{5.14}$$

Using the inverse relation (4.128), we get the twisted fields associated to the original generators χ^i:

$$\begin{aligned} Y_\sigma(\chi^i, z) &= \frac{1}{N} \sum_{m\in\frac{1}{N}\mathbb{Z}} \theta^{imN} J_{mN} z^{-m-1} && (5.15)\\ &= \frac{1}{N}\sum_{m\in\mathbb{Z}}\sum_{a=0}^{N-1} \theta^{ia} J_{a+Nm} z^{-\frac{a}{N}-m-1}, && (5.16)\end{aligned}$$

where $\theta = e^{2\pi i/N}$.

Our goal is to construct the twisted fields $W_\sigma^m(z)$ associated to the $\mathcal{W}(\mathfrak{sp}_{2N})$ generators $W^m(z)$. From Proposition 5.A.2, we know that this involves calculating the twisted fields for products of elements of the form $\mathbf{e}^{\chi^i}_{-a}\mathbf{e}^{-\chi^i}_{-b}$ for $a, b > 0$. The following two formulas do exactly that.

Lemma 5.B.1. *Let $d \geq 1$, $\epsilon \in \{1, -1\}$ and $\{\chi^i\}_{i=0,1,\ldots,N-1}$ be an orthonormal basis of $\mathbb{C}^N$. Then*

$$-Y_\sigma(\boldsymbol{e}^{\epsilon\chi^i}_{-d}\boldsymbol{e}^{-\epsilon\chi^i}_{-1}, z) = \sum_{k=0}^{d} \frac{c_k}{z^k} S_{d-k}(\epsilon\chi^i, z) \tag{5.17}$$

where c_k is the k-th coefficient in the Taylor expansion of the function

$$g(x) = \frac{1}{N} x^{\frac{1-N}{2N}} \prod_{k=1}^{N-1} (x^{1/N} - \theta^k) \tag{5.18}$$

at $x = 1$. In particular, c_k is independent of the basis vectors χ_i and ϵ, and $c_0 = 1$, $c_1 = 0$. Moreover, the $S^{\hbar}_k$ are the Faà di Bruno polynomials defined by

$$S_n(\chi^i, z) := \frac{1}{n!} \left(\partial_z + Y_\sigma(\chi^i, z)\right)^n \cdot 1. \tag{5.19}$$

Proof. The proof is exactly analogous to the proof of Lemma 3.7 in [**?**]. One can check that, $g'(1) = 0$ and thus $c_1 = 0$. $\square$

Remark 5.B.2. From (5.10) and the anticommutativity of free fermions it follows that

$$Y_\sigma(\mathbf{e}^{\chi^i}_{-1}\mathbf{e}^{-\chi^i}_{-d}) = -Y_\sigma(\mathbf{e}^{-\chi^i}_{-d}\mathbf{e}^{\chi^i}_{-1}). \tag{5.20}$$

Lemma 5.B.3. *Let $a, b \in V$ be elements of a VOA and T be the translation operator. Then*

$$\partial_z Y_\sigma(a_{-m}b_{-n}\boldsymbol{1}, z) = Y_\sigma(a_{-m-1}b_{-n}\boldsymbol{1}, z) + Y_\sigma(a_{-m}b_{-n-1}\boldsymbol{1}, z). \tag{5.21}$$

Proof. From [5, equation 3.14] (see also (4.121) in Section 4.H.6),

$$\partial_z Y_\sigma(v, z) = Y_\sigma(Tv, z). \tag{5.22}$$

In addition we know that

$$Tv = v_{-2}\mathbf{1}. \tag{5.23}$$

Applying the above two formulas to the vector $v = a_{-m}b_{-n}\mathbf{1}$ yields (5.21). $\square$

Lemmas (5.17) and (5.B.3), in conjunction with Proposition 5.A.2, allow us to write down explicit expressions for the twisted fields $W^m_\sigma(z)$ associated to the strong generators of $\mathcal{W}(\mathfrak{sp}_{2N})$. At this stage, we now introduce $\hbar$ in our twisted module as in Section 4.H.6. It turns the algebra of modes into the graded Rees algebra associated to Li's filtration by conformal weight. Put simply, since the strong generators $W^m_\sigma(z)$ have conformal weights m, it simply rescales them by $\hbar^m$.

We summarize the result in the following lemma.

Lemma 5.B.4. *Let $\{\chi^i\}_{i=0,1,\dots,N-1}$ be states generating the rank N free boson VOA and σ the fully cyclic automorphism. Then the twisted fields $W^m_{\sigma,\hbar}(z)$, $m = 2, 4, \dots, 2N$ corresponding to the strong generators of $\mathcal{W}(\mathfrak{sp}_{2N})$ in the $\hbar$-deformed σ-twisted module read:*

$$\begin{aligned} W^m_{\sigma,\hbar}(z) = \hbar^m \sum_{i=0}^{N-1} \Bigg[&\sum_{k=0}^{m} \frac{c_k}{z^k} \left(S_{m-k}(-\chi^i, z) + S_{m-k}(\chi^i, z)\right) \\ &+ \sum_{k=1}^{m-1} \frac{kc_k}{z^{k+1}} S_{m-1-k}(\chi^i, z) - \sum_{k=0}^{m-1} \frac{c_k}{z^k} \partial_z S_{m-1-k}(\chi^i, z)\Bigg] \qquad m = 2, 4, \dots, 2N. \end{aligned} \tag{5.24}$$

Proof. Using Proposition 5.A.2 we can write,

$$W^m_\sigma(z) = \sum_{i=0}^{N-1} \left(Y_\sigma(\mathbf{e}^{\chi_i}_{-1}\mathbf{e}^{-\chi_i}_{-m}\mathbf{1}) + Y_\sigma(\mathbf{e}^{\chi_i}_{-m+1}\mathbf{e}^{-\chi_i}_{-2}\mathbf{1}) \right). \tag{5.25}$$

The first term in the sum above can be commuted directly from lemma (5.B.1) with $\epsilon = -1$ and is given by,

$$Y_\sigma(\mathbf{e}^{\chi_i}_{-1}\mathbf{e}^{-\chi_i}_{-m}\mathbf{1}) = \sum_{k=0}^{m} \frac{c_k}{z^k} S_{m-k}(-\chi_i, z) \tag{5.26}$$

where the c_ks are as in Lemma 5.B.1. The second term in (5.25) is given by an application of Lemma (5.B.3) followed by Lemma (5.B.1):

$$\begin{aligned} Y_\sigma(\mathbf{e}^{\chi_i}_{-m+1}\mathbf{e}^{-\chi_i}_{-2}\mathbf{1}) &= \partial_z Y_\sigma(\mathbf{e}^{\chi_i}_{-m+1}\mathbf{e}^{-\chi_i}_{-1}\mathbf{1}, z) - Y_\sigma(\mathbf{e}^{\chi_i}_{-m}\mathbf{e}^{-\chi_i}_{-1}, z) \\ &= \sum_{k=0}^{m-1} \frac{kc_k}{z^{k+1}} S_{m-1-k}(\chi_i, z) - \sum_{k=0}^{m-1} \frac{c_k}{z^k} \partial_z S_{m-1-k}(\chi_i, z) + \sum_{k=0}^{m} \frac{c_k}{z^k} S_{m-k}(\chi_i, z). \end{aligned} \tag{5.27}$$

Adding the above two formulas gives $W^m_\sigma(z)$. Finally, since $W^m_\sigma(z)$ has conformal weight m, we get $W^m_{\sigma,\hbar}(z) = \hbar^m W^m_\sigma(z)$, which yields (5.24). □

Remark 5.B.5. Note that only terms invariant under the automorphism σ will survive after summing over the index i in the above formulas, and hence the expressions will simplify considerably.

5.C Constructing Airy ideals for $\mathcal{W}(\mathfrak{sp}_{2N})$

In the previous section we obtained twisted fields $W^m_{\sigma,\hbar}(z)$ for the strong generators of $\mathcal{W}(\mathfrak{sp}_{2N})$ in a $\hbar$-deformed σ-twisted module for the rank N free boson

VOA. Our goal is to show that the non-negative modes of these fields generate an Airy ideal $\mathcal{I}$ in the $\hbar$-completed Rees algebra of bosonic modes $\widehat{U}^{\hbar}(\mathfrak{h})$. More precisely, this will not be true for the twisted fields $W^{m}_{\sigma,\hbar}(z)$ directly. What we will do is construct an automorphism of $\widehat{U}^{\hbar}(\mathfrak{h})$, which is usually called "dilaton shift" (and is very similar to the transvections studied in Section 4.D.1), such that he image of the fields $W^{m}_{\sigma,\hbar}(z)$ under this automorphism are such that their non-negative modes generate an Airy ideal.

As explained in Section 4.H, to construct an Airy ideal we need to choose a scenario to map $\widehat{U}^{\hbar}(\mathfrak{h})$ into the Rees Weyl algebra: we need to make a choice for the action of the zero mode. In the case of $\mathcal{W}(\mathfrak{sp}_{2N})$, only one choice works: we need to make the zero mode acts as a derivative, and hence we find ourselves in the scenario explored in Section 4.H.4. As far as we are aware, this is the first example of a $\mathcal{W}$-algebra explored in the literature that involves this scenario.

We present first the calculations for the special case of $\mathcal{W}(\mathfrak{sp}_6)$, since the calculations are more explicit. We then move on to the general case.

5.C.1 An Airy ideal for $\mathcal{W}(\mathfrak{sp}_6)$

The strong generators

We use the notation in Section 4.H.6 and Section 5.B. We consider the rank 3 free boson VOA, generated by states $\{\chi^0, \chi^1, \chi^2\}$. We consider the fully cyclic automorphism σ, and the corresponding σ-twisted module with map Y_σ. Let v^0, v^1, v^2 be the diagonal basis. For clarity, we write

$$\chi^i(z) := Y_\sigma(\chi^i, z), \qquad v^i(z) = Y_\sigma(v^i, z), \qquad i = 0, 1, 2. \tag{5.28}$$

We index the modes of the twisted fields in the diagonal basis as usual:

$$\begin{aligned} v^0(z) &= \sum_{n\in\mathbb{Z}} J_{3n} z^{-n-1}, \\ v^1(z) &= \sum_{n\in\mathbb{Z}} J_{3n+1} z^{-n-4/3}, \\ v^2(z) &= \sum_{n\in\mathbb{Z}} J_{3n+2} z^{-n-5/3}. \end{aligned} \tag{5.29}$$

We introduce $\hbar$ in our module as in Section 4.H.6, which turns the algebra of modes into the graded Rees algebra associated to Li's filtration by conformal weight.

We obtained the twisted fields $W^m_{\sigma,\hbar}(z)$ for the strong generators of $\mathcal{W}(\mathfrak{sp}_6)$ in our $\hbar$-deformed σ-twisted module in Lemma 5.B.4. For clarity, we now drop the subscripts $\sigma, \hbar$ on the generators, and denote them simply by $W^m(z)$. In the case of $\mathfrak{sp}_6$, we have three such generators, $W^2(z), W^4(z), W^6(z)$. We define the modes of the strong generators as

$$W^m(z) = \sum_{k\in\mathbb{Z}} W^m_k z^{-k-1}. \tag{5.30}$$

From Lemma 5.B.4 it is easy to see that each of these modes takes the form

$$W^m_k = \hbar^m p^m_k(J_a), \tag{5.31}$$

where $p^m_k(J_a)$ is a polynomial (a sum of normal ordered monomials) in the bosonic modes J_a of degree $\leq m$. As such, the non-negative modes W^m_k with $k \geq 0$ certainly do not generate an Airy ideal (see Definition 4.E.1 and Lemma 4.F.1), as they are homogeneous of degree m in $\hbar$, and $m > 1$. This is not surprising, since $\hbar$ was introduced via Li's filtration by conformal weight, and the fields $W^m(z)$ have conformal weight m.

To obtain an Airy ideal we somehow need to break the $\hbar$ homogeneity of the strong generators of $\mathcal{W}(\mathfrak{sp}_6)$. In order to do this, we introduce an automorphism of the Rees algebra of bosonic modes $\widehat{U}^\hbar(\mathfrak{h})$ called "dilaton shift".

Definition 5.C.1. Let $\phi : \widehat{U}^\hbar(\mathfrak{h}) \to \widehat{U}^\hbar(\mathfrak{h})$ be the automorphism given by

$$\phi : (\hbar, \hbar J_m) \mapsto (\hbar, \hbar J_m + \delta_{m,-3} + \delta_{m,-4}). \tag{5.32}$$

In other words, it simply shifts the modes $\hbar J_{-3} \mapsto \hbar J_{-3} + 1$ and $\hbar J_{-4} \mapsto \hbar J_{-4} + 1$. It can be understood as acting by conjugation; for any $P \in \widehat{U}^\hbar(\mathfrak{h})$, we can think of $\phi(P)$ as being given by

$$\phi(P) = \exp\left(\frac{J_3}{3\hbar} + \frac{J_4}{4\hbar}\right) P \exp\left(-\frac{J_3}{3\hbar} - \frac{J_4}{4\hbar}\right). \tag{5.33}$$

This is very similar to the transvections studied in Section 4.D.1, except that now the non-trivial action is on the coordinates (on the negative modes J_{-3} and J_{-4}) instead of the derivatives.

This is an automorphism of the Rees algebra of bosonic modes. The strong generators $W^m(z)$ of $\mathcal{W}(\mathfrak{sp}_6)$ are mapped to new fields $\phi(W^m(z))$ under this automorphism. We introduce the following notation for the image fields and their modes:

$$H^m(z) := \frac{3^{m-1} m!}{2}\phi(W^m(z)), \qquad H^m_k := \frac{3^{m-1} m!}{2}\phi(W^m_k). \tag{5.34}$$

The rescaling of the generators here is simply for convenience.

Clearly, the action of ϕ breaks homogeneity in $\hbar$, which is what we want. Our goal is to show that the non-negative modes H_k^m, $k \geq 0$, generate an Airy ideal in $\widehat{U}^\hbar(\mathfrak{h})$. To do so, we need to prove Conditions (1)–(4) in Lemma 4.F.1. We first focus on Condition (3), which amounts to studying the $O(\hbar^0)$ and $O(\hbar^1)$ terms in the H_k^m.

Lemma 5.C.2. *The modes H_k^m satisfy, for $k \geq 0$:*

$$H_k^2 = \hbar(2J_{3k} + 2J_{3k+1}) + O(\hbar^2), \tag{5.35}$$

$$H_k^4 = \hbar(4J_{3k} + 12J_{3k+1} + 12J_{3k+2} + 4J_{3k+3}) + O(\hbar^2), \tag{5.36}$$

$$H_k^6 = \hbar(6J_{3k} + 30J_{3k+1} + 60J_{3k+2} + 60J_{3k+3} + 30J_{3k+4} + 6J_{3k+5}) + O(\hbar^2). \tag{5.37}$$

Proof. We start with Lemma 5.B.4 for the strong generators $W^m(z)$, $m = 2, 4, 6$. We mentioned before that the modes take the form

$$W_k^m = \hbar^m p_k^m(J_a), \tag{5.38}$$

where $p_k^m(J_a)$ is a polynomial in the bosonic modes of degree $\leq m$. We note that the automorphism ϕ acts as $\hbar J_{-3} \mapsto \hbar J_{-3} + 1$ and $\hbar J_{-4} \mapsto \hbar J_{-4} + 1$. As such, it can decrease the order in $\hbar$. We are interested in resulting terms of order $O(\hbar^0)$ and $O(\hbar^1)$. Clearly, only the monomials of degree m and $m-1$ in the polynomials $p_k^m(J_a)$ can give rise to terms of order $O(\hbar^0)$ and $O(\hbar^1)$ following the action of ϕ. So we are only interested in these higher degree terms.

From Lemma 5.B.4, we get:

$$W^2(z) = \hbar^2 \sum_{i=0}^{2} \left(\chi^i(z)^2 - \partial_z \chi^i(z) + \frac{1}{27z^2} \right), \tag{5.39}$$

$$W^4(z) = \hbar^4 \sum_{i=0}^{2} \left(\frac{2}{4!} \chi^i(z)^4 - \frac{1}{3!} \partial_z \chi^i(z)^3 + \text{lower degree} \right), \tag{5.40}$$

$$W^6(z) = \hbar^6 \sum_{i=0}^{2} \left(\frac{2}{6!} \chi^i(z)^6 - \frac{1}{5!} \partial_z \chi^i(z)^5 + \text{lower degree} \right), \tag{5.41}$$

where "lower degree" means polynomial terms of degree $\leq m-2$ in the bosonic modes.

These expressions are in terms of the bosonic fields $\chi^i(z)$. We need to rewrite them in terms of the twisted fields $v^i(z)$ in the diagonal basis, since the bosonic modes J_a are defined for the twisted fields $v^i(z)$ (see (5.29)). Recall that

$$\chi^i(z) = \frac{1}{3} \sum_{a=0}^{2} \theta^{ia} v^a(z), \tag{5.42}$$

where $\theta = e^{2\pi i/3}$.

We consider first the highest degree terms in the fields $W^m(z)$, of degree m in the bosonic modes. In terms of the fields $v^i(z)$, the highest degree terms read:

$$\frac{3}{\hbar^2}W^2(z) = (v^0(z))^2 + 2v^1(z)v^2(z) + \dots \tag{5.43}$$

$$\frac{4!3^3}{2\hbar^4}W^4(z) = (v^0(z))^4 + 4v^0(z)(v^1(z))^3 + 4v^0(z)(v^2(z))^3 + 12(v^0(z))^2v^1(z)v^2(z) + 6(v^1(z))^2(v^2(z))^2 + \dots \tag{5.44}$$

$$\frac{6!3^5}{2\hbar^6}W^6(z) = (v_0(z))^6 + (v_1(z))^6 + (v_2(z))^6 + 20(v_0(z))^3(v_2(z))^3 + 20(v_0(z))^3(v_1(z))^3 + 20(v_1(z))^3(v_2(z))^3 + 30v_0(z)v_1(z)(v_2(z))^4 + 30(v_0(z))^4v_1(z)v_2(z) + 30v_0(z)(v_1(z))^4v_2(z) + 90(v_0(z))^2(v_1(z))^2(v_2(z))^2 + \dots \tag{5.45}$$

We can write a general formula as:

$$\frac{m!3^{m-1}}{2\hbar^m}W^m(z) = \sum_{\substack{\alpha_1+2\alpha_2|3\\ \alpha_0+\alpha_1+\alpha_2=m}} \frac{m!}{\alpha_0!\alpha_1!\alpha_2!}(v_0(z))^{\alpha_0}(v_1(z))^{\alpha_1}(v_2(z))^{\alpha_2} + \text{lower degree.} \tag{5.46}$$

In terms of the modes, we get:

$$\frac{m!3^{m-1}}{2\hbar^m}W^m_k = \sum_{\substack{\alpha_1+2\alpha_2|3,\alpha_0+\alpha_1+\alpha_2=m\\ \sum_{p,q,r}\beta^0_p+\beta^1_q+\beta^2_r=3k+3-3m}} \frac{m!}{\alpha_0!\alpha_1!\alpha_2!}\prod_{i=1}^{\alpha_0} : J_{\beta^0_i} : \prod_{i=1}^{\alpha_1} : J_{\beta^1_i} : \prod_{i=1}^{\alpha_2} : J_{\beta^2_i} : \tag{5.47}$$

$$+ \text{lower degree,}$$

where $\beta^j_i \equiv j \pmod 3$.

With these formulae, we can implement the automorphism ϕ from Definition 5.C.1 (the dilaton shift) on the highest degree terms. We see that for the non-negative modes, $k \geq 0$, we obtain precisely the $O(\hbar^1)$ terms in the statement of the Lemma, and no terms of $O(\hbar^0)$.

Next we look at the terms of degree $m-1$ in the bosonic modes in the fields $W^m(z)$. Those could potentially contribute terms of $O(\hbar^1)$ after applying the automorphism ϕ. In terms of the fields $v^i(z)$, the degree $m-1$ terms read:

$$-\frac{1}{\hbar^2}W^2(z) = \partial_z v^0(z) + \dots \tag{5.48}$$

$$-\frac{3!3^2}{\hbar^4}W^4(z) = \partial_z[(v^0(z))^3 + (v^1(z))^3 + (v^2(z))^3 + 6v^0(z)v^1(z)v^2(z)] + \dots \tag{5.49}$$

$$\begin{aligned} -\frac{5!3^4}{\hbar^6}W^6(z) = \partial_z\,[\,&(v^0(z))^5 + 10(v^0(z))^2(v^2(z))^3 + 20(v^0(z))^3v^1(z)v^2(z) \\ &+ 5(v^1(z))^4v^2(z) + 30v^0(z)(v_1(z))^2(v_2(z))^2 + 10(v_0(z))^2(v_2(z))^5 \\ &+ 5v^1(z)(v_2(z))^4\,] + \dots \end{aligned}$$

The action of the automorphism ϕ from Definition 5.C.1 on these degree $m-1$ terms does give rise to $O(\hbar^1)$ terms in the image fields $H^m(z)$. Those terms take the form:

$$H^2(z) = 0 + \dots \tag{5.50}$$

$$H^4(z) = -6\hbar + \dots \tag{5.51}$$

$$H^6(z) = -90\hbar + \dots, \tag{5.52}$$

where we singled out the $O(\hbar)$ terms that arise from applying ϕ to the degree $m-1$ terms in the $W^m(z)$. What is key is that these terms are constants, i.e. do not come with powers of z. As a result, they only appear in the modes H^4_{-1} and H^6_{-1}, and hence do not contribute to the non-negative modes H^m_k with $k \geq 0$. This concludes the proof of the Lemma. □

The Airy ideal

We now prove that the left ideal generated by the modes $\{H_k^2, H_k^4, H_k^6\}$ with $k \geq 0$ in $\widehat{U}^\hbar(\mathfrak{h})$ is an Airy ideal. We find ourselves in the scenario of Section 4.H.4, where the zero mode J_0 of the field $v^0(z)$ (which is the only zero mode, see (5.29)) acts as a derivative ∂_0.

Theorem 5.C.3. *Let $\mathcal{I}$ be the left ideal in $\widehat{U}^\hbar(\mathfrak{h})$ generated by the $\{H_k^2, H_k^4, H_k^6\}$ with $k \geq 0$. Then $\mathcal{I}$ is an Airy ideal.*

Proof. To prove that $\mathcal{I}$ is an Airy ideal, we need to check that Conditions (1)–(4) in Lemma 4.F.1 are satisfied.

Condition (1). The boundedness condition is automatically satisfied for the modes of the fields of a VOA (see Lemma 4.H.4).

Condition (2). It is always satisfied for the subset of non-negative modes of the strong generators of a VOA (see Lemma 4.I.2, also Proposition 3.14 in [9]).

Condition (3). For simplicity, let us re-index our operators H_k^m, $i = 2, 4, 6$, as

$$H_k^m =: L_{3k+\frac{m}{2}-1}. \tag{5.53}$$

Then the operators are indexed by $\{L_i\}_{i\in I}$ with $I = A = \mathbb{N}$. We want to determine whether

$$L_i = \sum_{a\in\mathbb{N}} M_{ia}\hbar J_a + O(\hbar^2) \tag{5.54}$$

for some coefficients M_{ia} such that for all fixed $a \in \mathbb{N}$, they vanish for all but finitely many $i \in \mathbb{N}$. But we have shown in Lemma 5.C.2 that

$$H_k^2 = L_{3k} = \hbar(2J_{3k} + 2J_{3k+1}) + O(\hbar^2), \tag{5.55}$$

$$H_k^4 = L_{3k+1} = \hbar(4J_{3k} + 12J_{3k+1} + 12J_{3k+2} + 4J_{3k+3}) + O(\hbar^2), \tag{5.56}$$

$$H_k^6 = L_{3k+2} = \hbar(6J_{3k} + 30J_{3k+1} + 60J_{3k+2} + 60J_{3k+3} + 30J_{3k+4} + 6J_{3k+5}) + O(\hbar^2). \tag{5.57}$$

As a result, we see that for a fixed $a = 3k + b$ with $b \in \{0, 1, 2\}$, the only non-vanishing coefficients M_{ia} are for $i \leq 3k + 2$. In particular, for all $a \in \mathbb{N}$ they vanish for all but finitely many $i \in \mathbb{N}$, as required.

Condition (4). We need to show that there exists coefficients N_{bj} such that

$$\sum_{i \in \mathbb{N}} N_{bi} M_{ia} = \delta_{ab}, \qquad \sum_{a \in \mathbb{N}} M_{ia} N_{aj} = \delta_{ij}, \tag{5.58}$$

and such that for all fixed $j \in \mathbb{N}$, the coefficients N_{bj} vanish for all but finitely many $b \in \mathbb{N}$. Equivalently, we need to show that we can invert the relations (5.55)–(5.57) to get

$$\sum_{i \in \mathbb{N}} N_{bi} L_i = \hbar J_b + O(\hbar^2), \tag{5.59}$$

with the coefficients such that for all fixed $i \in \mathbb{N}$ they vanish for all but finitely many $b \in \mathbb{N}$.

Let $\mathbf{J}_k := \hbar \begin{pmatrix} J_{3k} \\ J_{3k+1} \\ J_{3k+2} \end{pmatrix}$, $\mathbf{K}_k := \begin{pmatrix} L_{3k} \\ L_{3k+1} - 4\hbar J_{3k+3} \\ L_{3k+2} - 60\hbar J_{3k+3} - 30\hbar J_{3k+4} - 6\hbar J_{3k+5} \end{pmatrix}$ and $\mathbf{M} = \begin{pmatrix} 2 & 2 & 0 \\ 4 & 12 & 12 \\ 6 & 30 & 60 \end{pmatrix}$ for all $k \geq 0$. From (5.55) – (5.57), we have the matrix equations

$$\mathbf{M}\mathbf{J}_k + O(\hbar^2) = \mathbf{K}_k, \qquad \forall k \geq 0. \tag{5.60}$$

As $\mathbf{M}$ is invertible, we can invert the above relation to get

$$\hbar J_{3k} + O(\hbar^2) = \frac{15}{16}L_{3k} - \frac{5}{16}L_{3k+1} + \frac{1}{16}L_{3k+2} - \frac{5}{2}\hbar J_{3k+3} - \frac{15}{8}\hbar J_{3k+4} - \frac{3}{8}\hbar J_{3k+5} \tag{5.61}$$

$$\hbar J_{3k+1} + O(\hbar^2) = -\frac{7}{16}L_{3k} + \frac{5}{16}L_{3k+1} - \frac{1}{16}L_{3k+2} + \frac{5}{2}\hbar J_{3k+3} + \frac{15}{8}\hbar J_{3k+4} + \frac{3}{8}\hbar J_{3k+5} \tag{5.62}$$

$$\hbar J_{3k+2} + O(\hbar^2) = \frac{1}{8}L_{3k} - \frac{1}{8}L_{3k+1} + \frac{1}{24}L_{3k+2} - 2\hbar J_{3k+3} - \frac{5}{4}\hbar J_{3k+4} - \frac{1}{24}\hbar J_{3k+5} \tag{5.63}$$

Substituting back the formulas for $\hbar J_{3k+3}$, $\hbar J_{3k+4}$, and $\hbar J_{3k+5}$ recursively we can write

$$\hbar J_{3k+i} + O(\hbar^2) = \sum_{m\in\mathbb{N}} N_{3k+i,m}L_m \tag{5.64}$$

for $k \geq 0$ and $i \in \{0, 1, 2\}$ and where all $N_{3k+i,m} = 0$ for $m < 3k$. Equivalently, for any fixed $m \in \mathbb{N}$, the only non-vanishing coefficients are $N_{3k+i,m}$ with $3k \leq m$. In particular, for all fixed $m \in \mathbb{N}$ the coefficients vanish for all but finitely many $3k + i \in \mathbb{N}$, and the condition is satisfied.

As all conditions of Lemma 4.F.1 are satisfied, we conclude that the left ideal $\mathcal{I}$ generated by the $\{H_k^2, H_k^4, H_k^6\}$, $k \geq 0$, is an Airy ideal. Furthermore, from the calculation above we see that we can also think left ideal $\mathcal{I}$ as being generated by the differential operators ($k \geq 0$, $i \in \{0, 1, 2\}$):

$$\tilde{L}_{3k+i} = \sum_{m \in \mathbb{N}} N_{3k+i,m} L_m = \hbar J_{3k+i} + O(\hbar^2). \tag{5.65}$$

In particular, we notice that we have an operator $\tilde{L}_0 = \hbar J_0 + O(\hbar^2)$ (and in fact, one can show that the $O(\hbar^2)$ contributions are non-vanishing for this operator). Therefore, we find ourselves in the scenario of Section 4.H.4, where the map from the algebra of modes $\widehat{U}^\hbar(\mathfrak{h})$ to the Rees Weyl algebra goes to the subalgebra $\widehat{\mathcal{D}}^\hbar(x_{\mathbb{N}^*}, \partial_{\mathbb{N}})$ – we need to interpret the zero mode J_0 as a derivative in the Weyl algebra.

□

Now that we know that the left ideal $\mathcal{I}$ generated by the modes of the strong generators of $\mathcal{W}(\mathfrak{sp}_6)$ is an Airy ideal, we obtain an immediate Corollary from Theorem 4.E.3 (see also Section 4.H.4).

Corollary 5.C.4. *Let $\mathcal{I}$ be the left ideal in $\widehat{U}^\hbar(\mathfrak{h})$ generated by the non-negative modes $\{H_k^2, H_k^4, H_k^6\}$, with $k \geq 0$. Then $\widehat{U}^\hbar(\mathfrak{h})/\mathcal{I}$ is a cyclic left module canonically isomorphic to the ($\hbar$-adically completed) submodule of M (see Section 4.H.4) for the rank 3 free boson VOA generated by $|x_0\rangle$, but twisted by some stable transvection on $\widehat{U}^\hbar(\mathfrak{h})$.*

$\widehat{U}^\hbar(\mathfrak{h})/\mathcal{I}$ is also canonically isomorphic to a module of exponential type generated by a state

$$v := Z|x_0\rangle = \exp\left(\sum_{\substack{g \in \frac{1}{2}\mathbb{N}, n \in \mathbb{N}^* \\ 2g-2+n>0}} \hbar^{2g-2+n} F_{g,n}(J_{-1}, J_{-2}, \ldots) + \sum_{g \in \frac{1}{2}\mathbb{N}^*} \hbar^{2g-1} F_{g,1}(\tilde{J}_0) \right) |x_0\rangle, \tag{5.66}$$

for some polynomials $F_{g,n}$ homogeneous of degree n in the respective modes, with $F_{g,n}(0) = 0$. Here the $\tilde{J}_0$ are the modes conjugate to the zero modes J_0.

Furthermore, by construction the state v is annihilated by all non-negative modes $\{H_k^2, H_k^4, H_k^6\}$, with $k \geq 0$:

$$H_k^m v = 0, \qquad m = 2, 4, 6, \quad k \in \mathbb{N}. \tag{5.67}$$

Therefore, the action of the negative modes H_k^m, $k < 0$ on v generates a ($\hbar$-adically completed) Fock module for $\mathcal{W}(\mathfrak{sp}_6)$.

Remark 5.C.5. What is particularly interesting here is that the state v does not live in the $\hbar$-completion of the Fock module generated by $|x_0\rangle$; indeed, the conjugate modes $\tilde{J}_0$ appear in v. This is a direct consequence of the fact that we need to interpret the zero mode J_0 as a derivative instead of a variable – see Section 4.H, and in particular Section 4.H.4.

5.C.2 Airy ideals for $\mathcal{W}(\mathfrak{sp}_{2N})$

In this section we generalize the above construction for all $N \geq 3$. We follow closely the methods and logic of the previous section.

The strong generators

We use the notation in Section 4.H.6 and Section 5.B. We consider the rank N free boson VOA, generated by states $\{\chi^0, \chi^1, \dots, \chi^{N-1}\}$. We consider the fully cyclic automorphism σ, and the corresponding σ-twisted module with map Y_σ. Let $v^0, v^1, \dots, v^{N-1}$ be the diagonal basis. For clarity, we write

$$\chi^i(z) := Y_\sigma(\chi^i, z), \qquad v^i(z) = Y_\sigma(v^i, z), \qquad i = 0, 1, \dots, N-1. \tag{5.68}$$

We index the modes of the twisted fields in the diagonal basis as usual:

$$v^k(z) = \sum_{n\in\mathbb{Z}} J_{Nn+k} z^{-n-1-k/N}, \quad k = 0, 1, \ldots, N-1 \tag{5.69}$$

We introduce $\hbar$ in our module as usual, which turns the algebra of modes into the graded Rees algebra associated to the filtration by conformal weight.

We obtained the twisted fields $W^m_{\sigma,\hbar}(z)$ for the strong generators of $\mathcal{W}(\mathfrak{sp}_{2N})$ in our $\hbar$-deformed σ-twisted module in Lemma 5.B.4. As before, for clarity we drop the subscripts $\sigma, \hbar$ on the generators, and denote them simply by $W^m(z)$. We define the modes of the strong generators as

$$W^m(z) = \sum_{k\in\mathbb{Z}} W^m_k z^{-k-1}. \tag{5.70}$$

From Lemma 5.B.4 it is easy to see that each of these modes takes the form

$$W^m_k = \hbar^m p^m_k(J_a), \tag{5.71}$$

where $p^m_k l(J_a)$ is a polynomial (a sum of normal ordered monomials) in the bosonic modes J_a of degree $\leq m$. As for the $N = 3$ case, the modes W^m_k with $k \geq 0$ certainly do not generate an Airy ideal as they are homogeneous of degree m in $\hbar$, and $m > 1$. To obtain an Airy ideal we introduce an automorphism of $\widehat{U}^\hbar(\mathfrak{h})$ (dilaton shift) that breaks the $\hbar$-homogeneity.

Definition 5.C.6. Let $\phi : \widehat{U}^\hbar(\mathfrak{h}) \to \widehat{U}^\hbar(\mathfrak{h})$ be the automorphism given by

$$\phi : (\hbar, \hbar J_m) \mapsto (\hbar, \hbar J_m + \delta_{m,-N} + \delta_{m,-N-1}). \tag{5.72}$$

In other words, it simply shifts the modes $\hbar J_{-N} \mapsto \hbar J_{-N} + 1$ and $\hbar J_{-N-1} \mapsto$

$\hbar J_{-N-1}+1$. It can be understood as acting by conjugation; for any $P \in \widehat{U}^{\hbar}(\mathfrak{h})$, we can think of $\phi(P)$ as being given by

$$\phi(P) = \exp\left(\frac{J_N}{N\hbar} + \frac{J_{N+1}}{(N+1)\hbar}\right) P \exp\left(-\frac{J_N}{N\hbar} - \frac{J_N}{(N+1)\hbar}\right). \tag{5.73}$$

This is of course a natural generalization of the dilaton shift Definition 5.C.1 for $N = 3$.

The strong generators $W^m(z)$ of $\mathcal{W}(\mathfrak{sp}_{2N})$ are mapped to new fields $\phi(W^m(z))$ under this automorphism. We introduce the following notation for the image fields and their modes:

$$H^m(z) := \frac{N^{m-1}m!}{2}\phi(W^m(z)), \qquad H^m_k := \frac{N^{m-1}m!}{2}\phi(W^m_k). \tag{5.74}$$

Clearly, the action of ϕ breaks homogeneity in $\hbar$, which is what we want. Our goal is to show that the non-negative modes H^m_k, $k \geq 0$, generate an Airy ideal in $\widehat{U}^{\hbar}(\mathfrak{h})$. As in the $N = 3$ case, we need to prove Conditions (1)–(4) in Lemma 4.F.1. We first focus on Condition (3), which amounts to studying the $O(\hbar^0)$ and $O(\hbar^1)$ terms in the H^m_k.

Lemma 5.C.7. *The modes H^m_k satisfy, for $k \geq 0$ and $m = 2, 4, 6, \ldots, 2N$:*

$$H^m_k = \hbar \sum_{i=0}^{m-1} \frac{m!}{(m-i-1)!i!} J_{Nk+i} + O(\hbar^2). \tag{5.75}$$

Proof. We start with Lemma 5.B.4 for the strong generators $W^m(z)$. We mentioned before, the modes take the form

$$W^m_k = \hbar^m p^m_k(J_a), \tag{5.76}$$

where $p_k^m(J_a)$ is a polynomial in the bosonic modes of degree $\leq m$. As in the proof of Lemma 5.C.2 for $N=3$, to study the $O(\hbar^0)$ and $O(\hbar^1)$ terms in the modes H_k^m we are only interested in the monomials of degree m and $m-1$ in the polynomials $p_k^m(J_a)$.

First, Lemma 5.B.4 implies that

$$\frac{1}{\hbar^m} W^m(z) = \sum_{i=0}^{N-1} \left[\frac{2 : \chi^i(z)^m :}{m!} - \frac{1}{(m-1)!} \partial_z : \chi^i(z)^{m-1} : \right] + \text{lower degree}, \tag{5.77}$$

where "lower degree" stands for terms of degree $\leq m-2$ in the bosonic modes.

These expressions are in terms of the bosonic fields $\chi^i(z)$. We need to rewrite them in terms of the twisted fields $v^i(z)$ in the diagonal basis. Recall that

$$\chi^i(z) = \frac{1}{N} \sum_{a=0}^{N-1} \theta^{ia} v^a(z), \tag{5.78}$$

where $\theta = e^{2\pi i/N}$.

We consider the degree m terms in the fields $W^m(z)$. They read:

$$\frac{m! N^{m-1}}{2\hbar^m} W^m(z) = \sum_{\substack{\sum_{i=0}^{N-1} i\alpha_i | N \\ \sum_{i=0}^{N-1} \alpha_i = m}} \frac{m!}{\alpha_0! \alpha_1! \dots \alpha_{N-1}!} \prod_{p=0}^{N-1} v_p(z)^{\alpha_p} + \text{lower degree}. \tag{5.79}$$

In terms of the modes, we get:

$$\frac{m! N^{m-1}}{2\hbar^m} W_k^m = \sum_{\substack{\sum_{i=0}^{N-1} i\alpha_i | N \\ \sum_{i=0}^{N-1} \alpha_i = m}} \frac{m!}{\alpha_0! \alpha_1! \dots \alpha_{N-1}!} \prod_{i=1}^{\alpha_0} : J_{\beta_i^0} : \prod_{i=1}^{\alpha_1} : J_{\beta_i^1} : \dots \prod_{i=1}^{\alpha_{N-1}} : J_{\beta_i^{N-1}} : \tag{5.80}$$

$$+ \text{lower degree}$$

where $\beta_i^j \equiv j \pmod N$ for all i and $\sum_{j=0}^{N-1} \sum_{i=1}^{\alpha_j} \beta_i^j = N(k+1-m)$.

Now we want to implement the dilaton shift (the automorphism ϕ of Definition 5.C.6 on these highest degree terms. From (5.80), the $O(\hbar^0)$ terms can only come from a term proportional to $(J_{-N})^m$, but these terms only come up in the mode expansion of W_{-1}^m. Therefore the dilaton shift does not produce $O(\hbar^0)$ terms in the non-negative modes.

As for the $O(\hbar^1)$ terms, they are essentially determined by the conditions:

$$\sum_{i=0}^{N-1} i\alpha_i | N, \quad \sum_{i=0}^{N-1} \alpha_i = m. \tag{5.81}$$

To get a $O(\hbar^1)$ term we have to shift all modes except for one in the mode expansion given by (5.80), hence we only need to consider terms with $\alpha_i = 0$ for $i = 1, 2, \ldots, N-2$ or $\alpha_i = 1$ for some $i = 1, 2, \ldots, N-2$. For a term of the form $\prod_i J_{\gamma_i}$ we recall that the modes add up as follows,

$$\sum_i \gamma = N(k+1-m). \tag{5.82}$$

The lowest and highest index of the $O(\hbar^1)$ terms produced in H_k^m are due to the terms with $\alpha_0 = m$ and $\alpha_{N-1} = m-1$ respectively, that is terms of the form

$$: J_{Nk}(J_{-N})^{m-1} :, \qquad b_{m,k} : J_{Nk+m-1}(J_{-(N+1)})^{m-1} :, \tag{5.83}$$

and are given by

$$mJ_{Nk}, \qquad d_{n,k} J_{Nk+m-1}, \tag{5.84}$$

respectively, where

$$b_{m,k} = 1, \qquad \text{if } m-1 \equiv N-1 \pmod N \tag{5.85}$$

$$b_{m,k} = m, \qquad \text{otherwise} \tag{5.86}$$

and $d_{m,k} = m$. In between these two extreme cases we have terms of the form,

$$\frac{m!}{(m-a-1)!a!} : J_{Nk+a}(J_{-N})^{m-a-1}(J_{-N-1})^a :, \quad a = 1, 2, \ldots, m-2, \tag{5.87}$$

and after the dilaton shift these yield the following $O(\hbar^1)$ terms:

$$\frac{m!}{(m-a-1)!a!} J_{Nk+a}. \tag{5.88}$$

Finally, we look at the sub-leading degree $m-1$ terms in the fields $W^m(z)$, which may contribute $O(\hbar^1)$ terms after dilaton shift. From (5.77) the degree $m-1$ term in $\frac{1}{\hbar^m}W^m(z)$ is proportional to

$$\sum_{i=0}^{N-1} \partial_z : (\chi^i(z))^{m-1} : . \tag{5.89}$$

After changing to the diagonal basis only terms of the form

$$\partial_z : (v^0(z))^{m-1} :, \quad \partial_z : (v^{N-1}(z))^{m-1} : \delta_{N|m-1}, \quad \partial_z : (v^0(z))^{m-1-a}(v^{N-1}(z))^a : \delta_{N|a}, \tag{5.90}$$

can yield $O(\hbar^1)$ corrections, where $o < a < m-1$. Note that these are terms that are firstly invariant under σ and secondly have as factors only the dilaton shifted fields $v_0(z)$ and $v_{N-1}(z)$. After performing the dilaton shifts $\hbar J_{-N} \mapsto J_{-N} + 1$ and $\hbar J_{-N-1} \mapsto \hbar J_{-N-1} + 1$ it is easy to check that $O(\hbar)$

corrections are only produced for negative modes, and are zero for all non-negative modes, as in the $N = 3$ case. The easiest way to see this is to reformulate the dilaton shift as,

$$v_0(z) \mapsto v_0(z) + \frac{1}{\hbar}, \tag{5.91}$$

$$v_{N-1}(z) \mapsto v_{N-1}(z) + \frac{z^{1/N}}{\hbar}. \tag{5.92}$$

Then the $O(\hbar)$ terms produced by each of the terms mentioned in (5.90) in the operators $W^m(z)$ from this dilaton shift are respectively of the form,

$$\hbar\partial_z(z^0), \quad \hbar\partial_z(z^{\frac{m-1}{N}}), \quad \hbar\partial_z(z^{\frac{a}{N}}). \tag{5.93}$$

As the powers of z in the above expressions are non-negative, the result follows.

$\square$

The Airy ideal

We now prove that the left ideal generated by the modes $\{H_k^m\}_{m=2,4,\ldots,2N}$ with $k \geq 0$ in $\widehat{U}^\hbar(\mathfrak{h})$ is an Airy ideal. We find ourselves in the scenario of Section 4.H.4, where the zero mode J_0 of the field $v^0(z)$ (which is the only zero mode, see (5.29)) acts as a derivative ∂_0.

Theorem 5.C.8. *Let $\mathcal{I}$ be the left ideal in $\widehat{U}^\hbar(\mathfrak{h})$ generated by the $\{H_k^m\}_{m=2,4,\ldots,2N}$ with $k \geq 0$. Then $\mathcal{I}$ is an Airy ideal.*

Proof. To prove that $\mathcal{I}$ is an Airy ideal, we need to check that Conditions (1)–(4) in Lemma 4.F.1 are satisfied.

Condition (1). The boundedness condition is automatically satisfied for the modes of the fields of a VOA (see Lemma 4.H.4).

Condition (2). It is always satisfied for the subset of non-negative modes of the strong generators of a VOA (see Lemma 4.I.2 and also Proposition 3.14 in [9]).

Condition (3). For simplicity, let us re-index our operators H_k^m, $i = 2, 4, \ldots, 2N$, as

$$H_k^m =: L_{Nk+\frac{m}{2}-1}. \tag{5.94}$$

Then the operators are indexed by $\{L_i\}_{i\in I}$ with $I = A = \mathbb{N}$. We want to determine whether

$$L_i = \sum_{a\in\mathbb{N}} M_{ia}\hbar J_a + O(\hbar^2) \tag{5.95}$$

for some coefficients M_{ia} such that for all fixed $a \in \mathbb{N}$, they vanish for all but finitely many $i \in \mathbb{N}$. But we showed in Lemma 5.C.7 that

$$H_k^m = L_{Nk+\frac{m}{2}-1} = \hbar \sum_{i=0}^{m-1} \frac{m!}{(m-i-1)!i!} J_{Nk+i} + O(\hbar^2). \tag{5.96}$$

In particular, we can write (for $k \geq 0$ and $n \in \{0, 1, \ldots, N-1\}$)

$$L_{Nk+n} = \sum_{a\in\mathbb{N}} M_{Nk+n,a}\hbar J_a + O(\hbar^2), \tag{5.97}$$

with $M_{Nk+n,a} = 0$ for all $a < Nk$. In other words, for a fixed a, the only non-vanishing coefficients $M_{Nk+m,a}$ are for $Nk \leq a$. In particular, for any fixed $a \in \mathbb{N}$, the coefficients M_{ia} vanish for all but finitely many $i \in \mathbb{N}$, as required.

Condition (4). We need to show that there exists coefficients N_{bj} such that

$$\sum_{i\in\mathbb{N}} N_{bi} M_{ia} = \delta_{ab}, \qquad \sum_{a\in\mathbb{N}} M_{ia} N_{aj} = \delta_{ij}, \tag{5.98}$$

and such that for all fixed $j \in \mathbb{N}$, the coefficients N_{bj} vanish for all but finitely many $b \in \mathbb{N}$. Equivalently, we need to show that we can invert the relation (5.95) to get

$$\sum_{i \in \mathbb{N}} N_{bi} L_i = \hbar J_b + O(\hbar^2) \tag{5.99}$$

with the coefficients such that for all fixed $i \in \mathbb{N}$ they vanish for all but finitely many $b \in \mathbb{N}$.

We proceed as for the $N = 3$ case. We rewrite (5.96) as

$$L_{Nk+\frac{m}{2}-1} = \sum_{i=0}^{m-1} c_i^m \hbar J_{Nk+i} + O(\hbar^2), \qquad c_i^m = \frac{m!}{(m-i-1)!i!}. \tag{5.100}$$

Let $\mathbf{J}_k := \hbar \begin{pmatrix} J_{Nk} \\ J_{Nk+1} \\ \vdots \\ J_{Nk+N-1} \end{pmatrix}$, $\mathbf{K}_k := \begin{pmatrix} L_{Nk} - \sum_{m>Nk+N+1} c_m^2 \hbar J_{Nk+m} \\ L_{Nk+1} - \sum_{m>Nk+N+1} c_m^4 \hbar J_{Nk+m} \\ \vdots \\ L_{Nk+N-1} - \sum_{m>Nk+N+1} c_m^{2N} \hbar J_{Nk+m} \end{pmatrix}$ and

$$(\mathbf{M})_{ij} = \begin{cases} 0, & j > 2i \\ c_{j-1}^{2i}, & \text{otherwise} \end{cases} \tag{5.101}$$

for $1 \leq i, j \leq N$. From (5.100) we get matrix equations

$$\mathbf{M}\mathbf{J}_k + O(\hbar^2) = \mathbf{K}_k, \qquad k \geq 0. \tag{5.102}$$

One can see that $\mathbf{M}$ is invertible. This can be shown in various ways, such as converting to an upper/lower triangular matrix using Gaussian elimination or

LU decomposition. Note that dividing the ith row by i gives a matrix,

$$(\mathbf{M})_{ij} = \begin{cases} 0, & j > 2i \\ \binom{2i}{j}, & \text{otherwise} \end{cases} \tag{5.103}$$

which is a submatrix of the infinite 'Pascal matrix'. Submatrices of the Pascal matrix with non-zero diagonal entries are invertible, as proved in [38].

As $\mathbf{M}$ is invertible, we can invert the matrix equations to get (for $k \geq 0$ and $i \in \{0, 1, \ldots, N-1\}$):

$$\hbar J_{Nk+i} + O(\hbar^2) = \sum_{j=1}^{N} \delta_j L_{Nk+j-1} + \sum_{j=N}^{2N-1} \epsilon_j \hbar J_{Nk+j} \tag{5.104}$$

for some constants $\delta_j, \epsilon_j \in \mathbb{Q}$. Substituting back recursively the formulae for $\hbar J_{Nk+j}$, we end up with

$$\hbar J_{Nk+i} + O(\hbar^2) = \sum_{m \in \mathbb{N}} N_{Nk+i,m} \pi^{\leq 1}(L_m), \tag{5.105}$$

for $k \geq 0$ and $i \in \{0, 1, 2, N-1\}$ and where all $N_{Nk+i,m} = 0$ for $m < Nk$. Equivalently, for any fixed $m \in \mathbb{N}$, the only non-vanishing coefficients are $N_{Nk+i,m}$ with $Nk \leq m$. In particular, for all fixed $m \in \mathbb{N}$ the coefficients vanish for all but finitely many $Nk + i \in \mathbb{N}$, and the condition is satisfied.

As all conditions of Lemma 4.F.1 are satisfied, we conclude that the left ideal $\mathcal{I}$ generated by the $\{H_k^m\}$, $k \geq 0$, $m \in \{2, 4, \ldots, 2N\}$ is an Airy structure. Furthermore, from the calculation above we see that we can also think of the left ideal as being generated by the differential operators ($k \geq 0$, $i \in$

$\{0, 1, 2, \ldots, N-1\}$):

$$\tilde{L}_{Nk+i} = \sum_{m \in \mathbb{N}} N_{Nk+i,m} L_m = \hbar J_{Nk+i} + O(\hbar^2). \tag{5.106}$$

In particular, as in the $N = 3$ case we have an operator $\tilde{L}_0 = \hbar J_0 + O(\hbar^2)$. Therefore, we are again in the scenario of Section 4.H.4, where we need to intepret the zero mode J_0 as a derivative in the Weyl algebra. $\square$

Now that we know that the left ideal $\mathcal{I}$ generated by the modes of the strong generators of $\mathcal{W}(\mathfrak{sp}_{2N})$ is an Airy ideal, we obtain an immediate Corollary from Theorem 4.E.3 (see also Section 4.H.4).

Corollary 5.C.9. *Let $\mathcal{I}$ be the left ideal in $\widehat{U}^{\hbar}(\mathfrak{h})$ generated by the non-negative modes $\{H_k^m\}_{m=2,4,\ldots,2N}$, with $k \geq 0$. Then $\widehat{U}^{\hbar}(\mathfrak{h})/\mathcal{I}$ is a cyclic left module canonically isomorphic to the ($\hbar$-adically completed) submodule of M (see Section 4.H.4) for the rank N free boson VOA generated by $|x_0\rangle$, but twisted by some stable transvection on $\widehat{U}^{\hbar}(\mathfrak{h})$.*

$\widehat{U}^{\hbar}(\mathfrak{h})/\mathcal{I}$ is also canonically isomorphic to a module of exponential type generated by a state

$$v := Z|0\rangle = \exp\left(\sum_{\substack{g \in \frac{1}{2}\mathbb{N}, n \in \mathbb{N}^* \\ 2g-2+n>0}} \hbar^{2g-2+n} F_{g,n}(J_{-1}, J_{-2}, \ldots) + \sum_{g \in \frac{1}{2}\mathbb{N}^*} \hbar^{2g-1} F_{g,1}(\tilde{J}_0) \right) |0\rangle, \tag{5.107}$$

for some polynomials $F_{g,n}$ homogeneous of degree n in the respective modes, with $F_{g,n}(0) = 0$. Here the $\tilde{J}_0$ are the modes conjugate to the zero modes J_0.

Furthermore, by construction the state v is annihilated by all non-negative *modes* $\{H^m_k\}_{m=2,4,\ldots,2N}$*, with* $k \geq 0$*:*

$$H^m_k v = 0, \qquad m = 2,4,\ldots,2N \quad k \in \mathbb{N}. \tag{5.108}$$

Therefore, the action of the negative modes H^m_k*,* $k < 0$ *on* v *generates a (*$\hbar$*-adically completed) Fock module for* $\mathcal{W}(\mathfrak{sp}_{2N})$*.*

Remark 5.C.10. As in the $N = 3$ case, the state v does not live in the $\hbar$-completion of the Fock module generated by $|x_0\rangle$ for the free boson VOA; indeed, the conjugate modes $\tilde{J}_0$ appear in v. This is a direct consequence of the fact that we need to interpret the zero mode J_0 as a derivative instead of a variable – see Section 4.H, and in particular Section 4.H.4.

Chapter 6

Boundary states

The classical equation of a closed string is described by a conformal field theory (CFT) (i.e. a conformally invariant non-linear sigma model) on the sphere. In terms of the path integral formulation the space of solutions for the theory factorizes into two 'chiral' halves, each of which can be studied independently as chiral algebras. Instead of a sphere, in this chapter we briefly introduce the main defining data of a CFT on an open Riemann surface also known as 'open conformal field theory'. In this case, the two chiral halves are no longer independent. Some of the defining data can be encoded into 'boundary states' that are defined abstractly in terms of the so-called 'Ishibashi states'. The main motivation for this comes from string theory in which ends of open strings give rise to boundaries of the string world sheet. In addition, non-perturbative sectors of string theory called D-branes are a slight generalisation of this setup (See for example [42] and [32]). In the next chapter, we will use the formalism of Airy ideals to construct boundary states associated to affine Lie algebras. The main references for the current chapter are [42] and [35].

6.A Boundary conformal field theory

The starting point for the data of a genus 0 conformal field theory (CFT) are two vertex operator algebras W_L, W_R, which we assume to be the same $\mathcal{W}_L = \mathcal{W}_R = \mathcal{W}$ in the rest of this chapter. We assume that the associated state space $\mathcal{H}^{(P)}$ (where the superscript P is used for a CFT on the plane) decomposes into a sum of $\mathcal{W} \otimes \mathcal{W}$ irreducible representations denoted by $\mathcal{H}_i \otimes \mathcal{H}_{\bar{i}}$ so that,

$$\mathcal{H}^{(P)} = \bigoplus_{i \in \mathcal{I}} \mathcal{H}_i \otimes \mathcal{H}_{\bar{i}}. \tag{6.1}$$

for some indexing set $\mathcal{I}$. However the modules in the sum above cannot be arbitrary. Various additional constraints are required for a CFT to be consistent. In particular we wish that the partition function given by,

$$Z(\tau) := \sum_{i \in \mathcal{I}} \chi_{\mathcal{H}_i}(\tau) \chi_{\mathcal{H}_{\bar{i}}}(-\bar{\tau}) \tag{6.2}$$

be invariant under the action of $SL_2(\mathbb{Z})$ where χ denotes trace of $e^{2\pi i \tau \left(L_0 - \frac{c}{24}\right)}$ Secondly we also require that the OPEs close in $\mathcal{H}^{(P)}$.

An attempt to generalize these notions to a CFT on a Riemann surface with boundary yields significant insights into the global properties of a CFT. The requirement that the CFT should retain all the symmetries required (such as conformal transformations on the boundary) influence global properties (such as the state space) and are encoded as 'gluing conditions' on the boundary and the value of coefficients of one-point functions in the bulk (as these no longer vanish). Conformal symmetries that preserve the boundary can be formulated as the gluing condition,

$$T(z) = \bar{T}(\bar{z}), \quad z = \bar{z}. \tag{6.3}$$

One may also impose extra gluing conditions on the other strong generators W_i,

$$W_i(z) = \Omega(\bar{W})(\bar{z}), \quad z = \bar{z} \tag{6.4}$$

where $\Omega : \mathcal{W} \mapsto \mathcal{W}$ is a local (conformal) automorphism. In addition to the gluing conditions, we need additional data to fix the boundary CFT associated to a given bulk CFT uniquely. As translational invariance is broken, the one-point functions $\langle \phi_i(z, \bar{z}) \rangle$ do not vanish but are given by,

$$\langle \phi_{i,\bar{i}}(z, \bar{z}) \rangle_\alpha = \frac{A_{\alpha,i\bar{i}}}{|z - \bar{z}|^{h_i + \bar{h}_i}} \delta_{\bar{i},\omega^{-1}(i^+)} \tag{6.5}$$

where i^+ denotes the module that is contragredient dual to i, ω is the automorphism induced on the index set $\mathcal{I}$ from Ω and $A_{\alpha,i\bar{i}}$ is a map ,

$$A_{\alpha,i\bar{i}} : \mathcal{H}_{\bar{i}}^{(0)} \mapsto \mathcal{H}_i^{(0)} \tag{6.6}$$

between the grade 0 part of the respective modules.

Let us denote the choice of these coefficients by a continuous label α. Hence the data of a boundary CFT is given by the pair (Ω, α).

6.B Boundary states

In order to compute correlators, and bulk-boundary OPEs of the boundary theory in terms of objects in the bulk theory, Cardy and Ishibashi among other introduced the boundary state formalism. One definition is given in terms of finite temperature correlators of bulk fields in the upper half plane (See Section 4.3.1 of [42]). We give below an equivalent definition in terms of zero-temperature correlators. Consider the change of coordinates,

$$\zeta = \frac{1-iz}{1+iz}, \quad \bar{\zeta} = \frac{1+i\bar{z}}{1-i\bar{z}} \tag{6.7}$$

that maps the half-plane to the complement of the unit disk in the ζ-plane. The boundary state $||\alpha\rangle\rangle$ for a boundary data (Ω, α) is defined by,

$$\langle 0|\phi_i^{(P)}(\zeta_1, \bar{\zeta}_1)\ldots\phi_N^{(P)}(\zeta_N, \bar{\zeta}_N)||\alpha\rangle\rangle_\Omega = \mathcal{J}(\zeta, z)\langle\phi_1^{(H)}(z_1, \bar{z}_1)\ldots\phi_n^{(H)}(z_N, \bar{z}_N)\rangle_\alpha \tag{6.8}$$

where $|0\rangle$ is the vaccum of the CFT on the full plane, $\mathcal{J}$ is the product of Jacobians of the $z-\zeta$ transformation and the subscripts (P) and (H) indicate objects on the full plane and half plane respectively. Let $||\alpha\rangle\rangle_\omega$ be the boundary state for the CFT with boundary data (Ω, α). As (6.3) and (6.4) hold between arbitrary correlators, it implies the relations,

$$\left(L_n^{(P)} - \bar{L}_{-n}^{(P)}\right)||\alpha\rangle\rangle_\omega = 0, \tag{6.9}$$

$$\left(W_n^{(P)} - (-1)^{h_W}\Omega\bar{W}_{-n}^{(P)}\right)||\alpha\rangle\rangle_\omega = 0 \tag{6.10}$$

where h_W is the conformal weight of W. Boundary states can usually be constructed explicitly in terms of certain generalized coherent states associated with the bulk theory (See Section 6.D for some explicit examples). First discovered by Ishibashi in [35], and hence named Ishibashi, the main result is an existence theorem of certain infinite linear combinations for every irreducible representation present in the bulk Hilbert space.

6.C Existence of Ishibashi states

We now prove that (6.9)-(6.10) have an explicit solution in terms of objects in the bulk CFT $\mathcal{H}^{(P)}$. First we introduce some preliminary notation and concepts. Let M be a module of a vertex algebra V (usually a $\mathcal{W}$-algebra), M' be it's contragredient dual and Ω be an automorphism (that preserves the conformal element). We can denote elements of $m \in M$ by the ket $|m\rangle$ and the canonical dual in M' as $\langle m|$.

Remark 6.C.1. In this section we assume that M and M' have countable dimension as vector spaces, but it is easy to generalize the results below for uncountable dimensions (in fact, they are independent of a choice of basis). Let $\{|v_a\rangle\}_{a\in\mathbb{N}}$ and $\{|v'_a\rangle\}_{a'\in\mathbb{N}}$ be orthonormal bases of M and M' respectively and $\langle v_a|$ and $\langle v'_b|$ be the canonical dual vectors to $|v_a\rangle$ and $|v'_b\rangle$ respectively.

We can abbreviate the twisted contragredient dual module by,

$$\widetilde{M} := \Omega^*(M') \tag{6.11}$$

The action on $\widetilde{M}$ is defined by,

$$V \times \widetilde{M} \mapsto \widetilde{M} : x \cdot \Omega^*(m') = \Omega^*(\Omega^{-1}(x) \cdot m'), \quad m' \in M', x \in V. \tag{6.12}$$

We remark that there exists an anti-Lie algebra involution on the algebra of modes θ (See [2, Proposition 3.9.1]). This induces an anti-linear map,

$$U : M \mapsto M' \tag{6.13}$$

such that,

$$UW_n = (-1)^{h_W} \bar{W}_{-n} U. \tag{6.14}$$

and,

$$[\Omega^*, U] = 0 \tag{6.15}$$

Our goal is to construct Ishibashi states $I \in \mathcal{H}^{(B)}$ that satisfy the constraints,

$$(L_n - \bar{L}_{-n}) I = 0 \tag{6.16}$$

$$(W_n - (-1)^{h_W} \Omega \bar{W}_{-n}) I = 0 \tag{6.17}$$

where we have used barred variables to denote the 'right moving copy' of the generators of $\widetilde{M}$. As a matter of fact, in [35] it was shown that there exists a Ishibashi state for every irreducible module $\mathcal{H}_i, i \in \mathcal{I}$ appearing in the bulk

Hilbert space decomposition,

$$\mathcal{H}^{(P)} = \bigoplus_{i\in\mathcal{I}} \mathcal{H}_i \otimes \mathcal{H}_{\bar{i}}. \tag{6.18}$$

Proposition 6.C.2. *There exists a vector I^{M_j} in $M_j \otimes \widetilde{M}_j \subseteq \mathcal{H}^{(P)} := \bigoplus_{j\in J} M_j \otimes \widetilde{M}_j$ for every $j \in J$ that satisfies (6.16)-(6.17). It can be expressed as an infinite sum,*

$$I^{M_j} = \sum_{a\in\mathbb{N}} |v_a^{M_j}\rangle \otimes |\Omega^*(U(v_a^{M_j}))\rangle \tag{6.19}$$

Proof. It is sufficient to prove that for every $j \in J$,

$$\langle v_a^{M_k}| \otimes \langle v_b^{M_l}| \left(L_n - \bar{L}_{-n}\right) I^{M_j} = 0 \tag{6.20}$$

$$\langle v_a^{M_k}| \otimes \langle v_b^{M_l}| \left(W_n - (-1)^{h_W} \Omega \bar{W}_{-n}\right) I^{M_j} = 0 \tag{6.21}$$

for all $a, b \in \mathbb{Z}$ and all $k, l \in J$. We now prove (6.21) of which (6.20) is a special case. Consider the following calculation,

$$\sum_{a\in\mathbb{N}} \langle v_c^{M_k}| \otimes \langle \Omega^* U(v_b^{M_l}))| \left(W_n - (-1)^{h_W} \Omega \bar{W}_{-n}\right) |v_a^{M_j}\rangle \otimes |\Omega^* U v_a^{M_j}\rangle \tag{6.22}$$

$$= \sum_{a\in\mathbb{N}} \left[\delta_{l,j}\delta_{a,b} \langle v_c^{M_k}| W_n |v_a^{M_j}\rangle - \delta_{k,j}\delta_{a,c}(-1)^{h_W} \langle \Omega^* U v_c^{M_k}| \Omega \bar{W}_{-n} |\Omega^* U v_a^{M_j}\rangle\right] \tag{6.23}$$

where we have used the orthonormality of our basis. Now observe that for the second Dirac sandwich above, we obtain

$$\langle \Omega^* U v_c^{M_k} | \, \Omega \bar{W}_{-n} \Omega^* U \, | v_a^{M_j} \rangle = (-1)^{h_W} \langle U v_c^{M_k} | \, U \bar{W}_n \, | v_a^{M_j} \rangle \tag{6.24}$$

from (6.12), (6.15) and (6.14). Substituting this in (6.23) completes the proof.

□

The proof presented above is standard in physics literature and first appeared in [35] (See also [42]). The question of uniqueness of the state given by (6.19) is more subtle, and it appears that it doesn't hold in general. We now prove uniqueness of the Ishibashi state in a special context, those associated to modules of affine Lie algebras. We will dive deeper into the construction of such Ishibashi states in Chapter 7 and relate them to the framework of Airy ideals.

Our starting point is a simple complex Lie algebra, denoted by $\mathfrak{g}$. Let M be an irreducible module of the universal affine vertex superalgebra $V^k(\mathfrak{g})$ and let $\widetilde{M}$ its contragredient dual. Consider two copies of $\mathfrak{g}$, one denoted by $\bar{\mathfrak{g}}$ so that $\mathfrak{g} \cong \bar{\mathfrak{g}}$. We first fix a choice of simple roots Φ and its image $\bar{\Phi} := -\Omega\Phi$ for the two copies $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ respectively. Let $\{x^{\alpha_i}, h^j\}_{i \in I, j \in J}$ and $\{\bar{x}_i^{\alpha}, \bar{h}^j\}_{i \in I, j \in J}$ be the associated Chevalley basis (See 7.A.1 for more on the Chevalley basis) for $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ respectively, where the generators $\{\bar{x}_i^{\alpha}, \bar{h}^j\}$ are the images of $\{x^{\alpha_i}, h^j\}$ under the contragredient duality homomorphism. We also choose an ordering for the index sets I and J. Then $V^k(g)$ is strongly generated by the fields $\{x^{\alpha_j}(z), h_i(z)\}$. Let I^M be the Ishibashi state for the boundary conditions given by,

$$\left(x_n^{\alpha_j} - \Omega \bar{x}_{-n}^{\alpha_j}\right) I^M = 0, \quad j \in J \tag{6.25}$$

$$\left(h_n^i - \Omega \bar{h}_{-n}^i\right) I^M = 0, \quad i \in I. \tag{6.26}$$

for some Lie algebra automorphism Ω.

Proposition 6.C.3. *Let $M \otimes \widetilde{M}$ be as above. If $M \otimes \widetilde{M}$ is also a highest weight module with respect to the choices of simple roots $\Phi, \bar{\Phi}$, then the Ishibashi state $I^M \in M \otimes \widetilde{M}$ given by (6.19) is unique.*

Proof. We denote the tensor product by $\mathcal{M} := M \otimes \widetilde{M}$. It is $\mathbb{C} \times \mathbb{C}$ graded by conformal weight. We denote the decomposition with respect to this grading by,

$$\mathcal{M} = \bigoplus_{n,m \geq 0} \mathcal{M}_{n+\Delta, m+\Delta} \tag{6.27}$$

where Δ is some constant such as,

$$\mathcal{M}_{n,m} < 0, \quad \text{for } n < \Delta \text{ or } m < \Delta$$

and $M_{\Delta,\Delta} \neq 0$. We first prove that if w is a highest weight state in $\mathcal{M}$ and I is any Ishibashi state satisfying (6.25) and (6.26), then $\langle w | v_I \rangle$ is non-vanishing. Let

$$I = \sum_{n,m} I_{m+\Delta, n+\Delta} \tag{6.28}$$

where $I_{n,m}$ denotes the projection of I into $\mathcal{M}_{n+\Delta, m+\Delta}$.

We first prove that $I_{\Delta,\Delta} \neq 0$. Let n' and m' denote constants that are minimal with respect to the conditions $I_{n'+\Delta,\bullet} \neq 0$ and $I_{n'+\Delta,m'+\Delta} \neq 0$ respectively where the index denoted by $\bullet$ could be arbitrary. Substituting (6.28) into (6.25) we can write,

$$x_r^{\alpha_j} I_{n'+\Delta,m'+\Delta} + \sum_{(n,m)\neq(n',m')} x_r^{\alpha_j} I_{n+\Delta,m+\Delta} - \Omega \bar{x}_{-r}^{\alpha_j} I = 0, \quad j \in J, \tag{6.29}$$

and for all $r > 0$. Now note that $x_{r>0}^{\alpha_j}$ lowers the conformal weight, while $x_{-r}^{\alpha_j}$ raises the conformal weight, and similarly for the 'right moving' operators $x_r^{\widetilde{\alpha}_j}$. Due to the minimality condition on n', m' the above equation can hold true only if $x_r^{\alpha_j} I_{n'+\Delta,m'+\Delta} = 0$. Similarly, writing the above equation for $r < 0$, we can conclude that $\bar{x}_{-r}^{\alpha_j} I_{n'+\Delta,m'+\Delta} = 0$ for $r < 0$. This brings us to the first intermediate conclusion that $I_{\Delta,\Delta} \neq 0$.

As $\mathcal{M}$ is also a highest weight module to respect to a choice of roots $\Phi, \bar{\Phi}$ we can write a decomposition,

$$\mathcal{M}_{\Delta,\Delta} = \bigoplus_{\lambda',\mu'} \mathcal{M}_{\Delta,\Delta,\lambda',\mu'}. \tag{6.30}$$

In particular there exist highest weights λ, μ so that,

$$I_{\Delta,\Delta} = \sum_{\beta',\bar{\beta}'\in Q^+} I_{\lambda-\beta',\mu-\bar{\beta}'} \tag{6.31}$$

where Q^+ is the $\mathbb{Z}_{\leq 0}$-span of positive simple roots. Like before we can define a set of extremal indices $\beta, \bar{\beta}$ such that,

$$I_{\lambda-\beta,\bullet} \neq 0,\; I_{\lambda-\beta+\alpha,\bullet} = 0 \quad \forall \alpha \in Q^+ \tag{6.32}$$

$$I_{\lambda-\beta,\mu-\bar{\beta}} \neq 0,\; I_{\lambda-\beta+\alpha,\mu-\bar{\beta}+\bar{\alpha}} = 0 \quad \forall \alpha, \bar{\alpha} \in Q^+ \tag{6.33}$$

We now proceed exactly like in the last paragraph. By substituting (6.31) in (6.17) and using the extremality of $\beta, \bar{\beta}$ we can conclude that,

$$I_{\lambda,\mu} \neq 0. \tag{6.34}$$

Finally let I and I' be two Ishibashi state in $M \otimes \widetilde{M}$ and $|w\rangle := I_{\lambda,\mu}$ be the highest weight state of $M \otimes \widetilde{M}$ then the combination,

$$I'' := I - \frac{\langle w|I\rangle}{\langle w|I'\rangle} I' \tag{6.35}$$

is also an Ishibashi state with,

$$\langle w|I''\rangle = 0 \tag{6.36}$$

which is only possible if $I'' = 0$ and thus I, I' are linearly dependent.

$\square$

Remark 6.C.4. If the top level $I_{\Delta,\Delta}$ is one-dimensional then the first part of the above proof directly gives us uniqueness of the Ishibashi state. This is true in the case when M is a Verma module of a $\mathcal{W}$-algebra for example.

6.D Examples of boundary conditions and Ishibashi states

We give below a few preliminary examples of boundary conditions and the associated Ishibashi states.

Example 6.D.1. 1. Let $\mathcal{H}$ be the rank 1 Heisenberg VOA generated by the modes $\{a_n\}_{n\in\mathbb{Z}}$ and let π_λ be the Fock module generated by a vector $|\lambda\rangle$ so that $a_0|\lambda\rangle = \lambda|\lambda\rangle, \lambda \in \mathbb{C}$. Let $|\lambda, \mu\rangle$ denote the tensor product $|\lambda\rangle \otimes |\mu\rangle$, then the Ishibashi states for the Neumann and Dirichlet boundary conditions (these are the boundary conditions for the compactification of a free boson) are,

$$(a_n \pm \bar{a}_{-n})I_{\lambda,\mu} = 0. \tag{6.37}$$

It is easy to see that this boundary condition is solved by the 'canonical coherent states',

$$I_{\lambda,\mu} = \exp(-\sum_{n=1}^{\infty} \frac{1}{n} a_{-n}\bar{a}_{-n})|\lambda, \mu\rangle \tag{6.38}$$

for the boundary condition with the plus sign (Neumann conditions). We briefly remark that exactly analogous expressions for boundary conditions in terms of exponentials of the 'creation operators' can be obtained for the free fermion VOA as well.

2. Recall that the $N = 2$ superconformal algebra has a free field realization in terms of 4 free fields - 2 free bosons X, Y and 2 free fermions b, c. The

generators take the form,

$$J = -Y - \omega X - :\tilde{b}\tilde{c}: \tag{6.39}$$

$$L = \frac{c\partial b}{2} - \partial Y - XY \tag{6.40}$$

$$G^{-} = \tilde{b}X + \partial\tilde{b} \tag{6.41}$$

$$G^{+} = 2\tilde{c}Y - 2\omega\partial\tilde{c} \tag{6.42}$$

There are 2 possible gluing conditions (referred to as A-type and B-type)given respectively by,

$$\left(L_n - \bar{L}_{-n}\right)|b\rangle = 0, \tag{6.43}$$

$$\left(J_n - \bar{J}_{-n}\right)|b\rangle = 0, \tag{6.44}$$

$$\left(G^{+}_r + i\eta \bar{G^{-}}_{-r}\right)|b\rangle = 0, \tag{6.45}$$

$$\left(G^{-}_r + i\eta \bar{G^{+}}_{-r}\right)|b\rangle = 0 \tag{6.46}$$

and

$$\left(L_n - \bar{L}_{-n}\right)|b\rangle = 0, \tag{6.47}$$

$$\left(J_n + \bar{J}_{-n}\right)|b\rangle = 0, \tag{6.48}$$

$$\left(G^{+}_r + i\eta \bar{G^{+}}_{-r}\right)|b\rangle = 0, \tag{6.49}$$

$$\left(G^{-}_r + i\eta \bar{G^{-}}_{-r}\right)|b\rangle = 0 \tag{6.50}$$

for $\eta = \pm 1$. In [31], the authors construct the most general $\mathcal{N} = 2$ superconformal boundary states from boundary states of the theory of two free uncompactified bosons and fermions, while the treatment of

$\mathcal{N} = 1$ boundary states appears in [30].

Chapter 7

Airy ideals from Wakimoto modules

Free field realizations of vertex operator algebras (VOAs) are broadly speaking embeddings of the VOA into Heisenberg or Clifford algebras. Such embeddings are a ready source of modules from the category $\mathcal{O}$ obtained from suitable Fock representations of the Heisenberg algebra with very explicit formulas for the generators in terms of free fields. In this chapter, we first present a free field realization of the affine Kac-Moody algebras $\hat{\mathfrak{g}}$. Physically this yields a bosonization of the so-called Wess-Zumino-Novikov-Witten (WZNW) model associated to $\hat{\mathfrak{g}}$. The construction is a generalization of the Borel-Weil-Bott (BWB) construction of representations of semi-simple Lie algebras. The BWB theorem gives embeddings of a simple Lie algebra $\mathfrak{g}$ into a Weyl algebra obtained from the infinitesimal action of $\mathfrak{g}$ on the flag manifold. This action can be expressed in terms of differential operators acting on the spaces of global sections of holomorphic line bundles on the flag manifold. We first recall this construction in section 7.B. We then present the essential ingredients for the generalization to the infinite-dimensional case. In this case, we define an embedding of the loop algebra $g((t))$ into the Lie algebra of local vector fields on the formal loop space of the big cell that is then lifted to an embedding of its central extension $\hat{\mathfrak{g}}$ to a subalgebra of the Weyl algebra $\mathcal{A}^{\mathfrak{g}}$. Finally we recast this result as a homomorphism of vertex algebras, and present a family of $\hat{\mathfrak{g}}$-modules $W_{k,\lambda}, k \in \mathbb{C}, \lambda \in \mathfrak{h}^*$. These are the so-called Wakimoto modules and were first constucted by Wakimoto in [43] for $\widehat{\mathfrak{sl}_2}$. This construction was later generalized for all $\hat{\mathfrak{g}}$ by Feigin and Frenkel in [24]. We closely follow the presentation of [26] in this chapter.

A concrete application of free-field realizations of VOAs is the construction of copious number of examples of Airy ideals, almost all of which promise great utility in the study of enumerative geometry. In the second part of this chapter, we propose new examples of Airy ideals by exploiting the Wakimoto realization for every simple Lie algebra $\mathfrak{g}$. The partition functions thus obtained can be interpreted as representations of a special type Ishibashi states that we introduced in chapter 6.

7.A Preliminary background

We recall some basic facts and notation on affine Lie algebras and their representations.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank l. The affine Lie algebra associated to $\mathfrak{g}$ is the universal central extension of the loop algebra $L\mathfrak{g} = g \otimes \mathbb{C}[t, t^{-1}]$ and denoted by $\hat{\mathfrak{g}}$. We denote the central element of $\hat{\mathfrak{g}}$ by

K. The vacuum representation of level k is defined as,

$$V_k = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}K)} \mathbb{C}_k \tag{7.1}$$

where $\mathbb{C}_k$ stands for the trivial one-dimensional representation of the Lie subalgebra $\mathfrak{g}\otimes\mathbb{C}[t]$ of $\hat{\mathfrak{g}}$ on which K acts by multiplication by k and $U(\cdot)$ denotes taking the universal enveloping algebra.

The Lie algebra $\hat{\mathfrak{g}}$ has a Cartan decomposition,

$$\mathfrak{g} = \widetilde{\mathfrak{n}_+} \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}_-} \tag{7.2}$$

where $\widetilde{\mathfrak{n}_\pm} = \mathfrak{n}_\pm \otimes \mathbb{C}\mathbf{1} \oplus \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^\pm]$ and $\widetilde{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}\mathbf{1} \oplus \mathbb{C}K$ where $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is the Cartan decomposition of $\mathfrak{g}$. We are especially interested in modules on which $\widetilde{\mathfrak{n}}_+$ acts locally nilpotently and $\widetilde{\mathfrak{h}}$ acts semi-simply. This is known as the category $\mathcal{O}$ and all irreducible objects in this category can be obtained as quotients of Verma modules. Verma modules are defined as induced representations,

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\widetilde{\mathfrak{n}}_+\oplus\widetilde{\mathfrak{h}})} \mathbb{C}_\lambda \tag{7.3}$$

where $\mathbb{C}_\lambda$ is the one-dimensional $\widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}$-module on which $\widetilde{\mathfrak{n}}_+$ acts as zero and $\widetilde{\mathfrak{h}}$ acts according to the character λ.

7.A.1 Chevalley basis

In this section we sketch the basic ideas behind the construction of a Chevalley basis. Let $\mathfrak{g}$ be a simple Lie algebra with Killing form κ and a root system Φ. Let $\mathfrak{h}$ denote a Cartan subalgebra so that $\dim(\mathfrak{h}) := l = \text{rank}(A) - |\Phi|$. Let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ be a basis of simple roots, so that we have a root space decomposition,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha. \tag{7.4}$$

In addition we set t_α to be the unique element such that,

$$\alpha(h) = \kappa(t_\alpha, h) \quad \forall h \in \mathfrak{h}. \tag{7.5}$$

We define a set of elements $(h_\alpha)_{\alpha \in \Phi}$ given by,

$$h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} \tag{7.6}$$

The starting point of the construction of the Chevalley basis is the following proposition from [34, Section 25.2].

Proposition 7.A.1. *It is possible to choose root vectors $x_\alpha \in \mathfrak{g}_\alpha (\alpha \in \Phi)$ satisfying:*

1. *$[x_\alpha, x_{-\alpha}] = h_\alpha$.*

2. *If $\alpha, \beta, \alpha + \beta \in \Phi$, $[x_\alpha, x_\beta] = c_{\alpha,\beta} x_{\alpha+\beta}$ then $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$.*

We also give below a quick corollary that follows from the symmetry of the structure constants $c_{\alpha,\beta}$ described above.

Corollary 7.A.2. *Let $\Omega : \mathfrak{g} \mapsto \mathfrak{g}$ be the involution given by,*

$$\Omega : x_\alpha \mapsto -x_{-\alpha} \tag{7.7}$$

$$h_\alpha \mapsto -h_\alpha, \tag{7.8}$$

then Ω is a Lie algebra automorphism.

The Proposition 7.A.1 is the precursor to construction of a Chevalley basis (a basis which satisfies the conditions of the above proposition) for $\mathfrak{g}$. It turns out that in this case, all the structure constants are integers (See [34, Section 25.2]).

Theorem 7.A.3. *(Chevalley) Let $x_\alpha, \alpha \in \Phi, h_{\alpha_i}, 1 \leq i \leq l$ be a Chevalley basis of $\mathfrak{g}$. Then the resulting structure constants lie in $\mathbb{Z}$. More precisely:*

1. $[h_{\alpha_j}, h_{\alpha_j}] = 0$.
2. $[h_{\alpha_i}, x_\alpha] = \langle \alpha, \alpha_i \rangle x_\alpha$.
3. $[x_\alpha, x_{-\alpha}] = h_\alpha$ *is a $\mathbb{Z}$-linear combination of* $h_{\alpha_1}, \ldots, h_{\alpha_l}$.

In the rest of this chapter, we will use a choice of Chevalley basis to construct certain modules of affine Lie algebras called Wakimoto modules, and subsequently also Airy ideals from tensor products of these Wakimoto modules.

7.B Finite-dimensional case

Let $\mathfrak{g}$ be a simple Lie algebra of rank l with Cartan decomposition,

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \tag{7.9}$$

and let,

$$\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm \tag{7.10}$$

be the upper and lower Borel subalgebras. Let G be the simply-connected Lie group corresponding to $\mathfrak{g}$, and $N_\pm$ (resp $B_\pm$) the upper and lower unipotent subgroups of G. The homogenous space G/B_- is called the associated flag variety.

Example 7.B.1. For $G = SL_n$, this is the variety of 'full flags' of subspaces of $\mathbb{C}^n : V_1 \subset \ldots \subset V_{n-1} \subset \mathbb{C}^n, \dim V_i = i$. It is easy to check that the group SL_n acts transitively on this variety and the stabilizer of the flag given by $V_i = \mathrm{span}(e_n, \ldots, e_{n-i+1})$ is the subgroup B_-.

We are interested in an embedding of $\mathfrak{g}$ into a Heisenberg algebra, this can be achieved by the infinitesimal action of $\mathfrak{g}$ on an open subspace of G/B_- that is isomorphic to an affine space. An example of a dense open subset is the big cell $\mathcal{U}$.

Definition 7.B.2. Consider the unique open N_+-orbit called the big cell $\mathcal{U} = N_+.[1] \subset G/B_-$ which is isomorphic to N_+. Since N_+ is unipotent it is isomorphic as a vector space to $\mathfrak{n}_+$ and hence the algebra $\mathrm{Fun}\, N_+$ of regular functions on N_+ is a free polynomial algebra. We can choose coordinates $y_\alpha, \alpha \in \Delta_+$, where Δ_+ is the set of positive roots of $\mathfrak{g}$ on $\mathcal{U}$, such that y_α has weight α with respect to the action of the Cartan subgroup of G on N_+, that is

$$h \cdot y_\alpha = \alpha(h) y_\alpha, \quad h \in \mathfrak{h}. \tag{7.11}$$

Given $a \in \mathfrak{g}$, consider the one-parameter subgroup $\gamma(\epsilon) = \exp(\epsilon a)$ in G. We have a Lie algebra homomorphism,

$$\mathfrak{g} \mapsto \operatorname{Vect} N_+, \quad a \mapsto \zeta_a \tag{7.12}$$

where ζ_a is defined as,

$$(\zeta_a f)(x) = \left(\frac{d}{d\epsilon} f(Z_+(\epsilon))\right)|_{\epsilon=0} \tag{7.13}$$

where $Z_+(\epsilon)$ is the projection of the subgroup $\gamma(\epsilon)$ onto N_+.

Example 7.B.3. Let $\mathfrak{g} = \mathfrak{sl}_2$. Then $G/B_- = \mathbb{P}^1$. We take,

$$\mathcal{U} = \{\mathbb{C} \begin{pmatrix} y \\ -1 \end{pmatrix}\} \subset \mathbb{CP}^1. \tag{7.14}$$

The vector field of (7.13) is simply given by,

$$(\zeta_a \cdot f)(d) = \frac{d}{d\epsilon} f(\exp(-\epsilon a) y)|_{\epsilon=0}. \tag{7.15}$$

The above formula allows us to compute the following vector field representation of the standard basis of $\mathfrak{sl}_2$,

$$e \mapsto \frac{\partial}{\partial y}, \quad h \mapsto -2y\frac{\partial}{\partial y}, \quad f \mapsto -y^2\frac{\partial}{\partial y}. \tag{7.16}$$

Let $\mathcal{D}(\mathcal{U})$ be the algebra of differential operators on $\mathcal{U}$. This is nothing but the Weyl algebra generated by $\{y_a, \frac{\partial}{\partial y_\alpha}\}_{\alpha\in\Delta_+}$ with the relations,

$$\left[\frac{\partial}{\partial y_\alpha}, y_\beta\right] = \delta_{\alpha,\beta}, \quad \left[\frac{\partial}{\partial y_\beta}, \frac{\partial}{\partial_{y_\beta}}\right] = [y_\alpha, y_\beta] = 0 \tag{7.17}$$

and let $\{\mathcal{D}_{\leq i}\}$ be the filtration by the order of the differential operator. We have an exact sequence,

$$0 \mapsto \mathrm{Fun}\,\mathcal{U} \mapsto \mathcal{D}_{\leq 1}(\mathcal{U}) \mapsto \mathrm{Vect}\,\mathcal{U} \mapsto 0, \tag{7.18}$$

where $\mathrm{Fun}\,\mathcal{U}$ denotes the ring of regular fections, and $\mathrm{Vect}\,\mathcal{U}$ denotes the Lie algebra of vector fields on $\mathcal{U}$. As $H^2(\mathrm{Vect}\,\mathcal{U}, \mathcal{D}_0) = 0$, this exact sequence splits and the map from $\mathfrak{g} \mapsto \mathrm{Vect}\,\mathcal{U}$ can be lifted to a map $\epsilon : \mathfrak{g} \mapsto \mathcal{D}_1$. The canonical splitting is the one where we lift $\zeta \in \mathrm{Vect}\,\mathcal{U}$ to the unique first order differential operator whose symbol equals ζ and which kills the constant functions. However, such a lifting is not unique but parameterized by $\mathfrak{h}^*$ (See [26]). Hence we infact obtain a family of embeddings $\epsilon_\lambda : \mathfrak{g} \mapsto \mathcal{D}_1 \subset \mathcal{D}$ for $\lambda \in \mathfrak{h}^*$ and a $\mathfrak{g}$-module structure on the space of functions $\mathrm{Fun}\, N_+ = \mathbb{C}[y_a]_{a\in\Delta_+}$. This is a Fock representation of $\mathcal{D}$ generated by y_α from a vector v satisfying $\frac{\partial}{\partial y_\alpha} \cdot v = 0$, $\alpha \in \Delta_+$. This picture can be summarized as follows:

Proposition 7.B.4. *The restriction of $\mathbb{C}[\mathcal{U}]$ to the image of ϵ_λ defines a $\mathfrak{g}$-module isomorphic to $M_\lambda^* := \mathrm{Hom}^{\mathrm{res}}_{U(\mathfrak{b}_-)}(U(\mathfrak{g}), \mathbb{C}_\chi)$ (i.e. the contragredient Verma module with highest weight χ). Here the subscript $U(\mathfrak{b}_-)$ indicates that $U(\mathfrak{g})$ is considered as a $U(\mathfrak{b}_-)$-module under the right action, and the superscript 'res' indicates restriction of homomorphisms to $U(\mathfrak{b}_-)^* \otimes U(\mathfrak{n}_+)^\vee$ where $U(\mathfrak{n}_+)^\vee$ is the restricted dual composed of duals of the degreewise summands with respect to the grading discussed in 7.B.2.*

Example 7.B.5. The module $M^*_\lambda, \lambda \in \mathbb{C}$, over $\mathfrak{sl}_2$ can be realized by,

$$e = \frac{\partial}{\partial y}, \quad h = -2y\frac{\partial}{\partial y} + \lambda, \quad f = -y^2\frac{\partial}{\partial y} + \lambda y. \tag{7.19}$$

7.C Setting for the infinite-dimensional case

The generalization to the case of affine Kac-Moody algebras is a lot more involved for several reasons. Firstly, the definition of formal loop spaces for general algebraic varieties needs to be treated. Secondly, in an affine algebra there are many different Borel subalgebras that are not conjugate to each other. The Lie algebra $\hat{\mathfrak{g}}$ has a Cartan decomposition,

$$\mathfrak{g} = \widetilde{\mathfrak{n}_+} \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}_-} \tag{7.20}$$

where $\widetilde{\mathfrak{n}_\pm} = \mathfrak{n}_\pm \otimes \mathbb{C}\mathbf{1} \oplus \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^\pm]$ and $\widetilde{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}\mathbf{1} \oplus \mathbb{C}K$. However, the one relevant for geometry is the loop decomposition,

$$\hat{\mathfrak{g}} = \widehat{\mathfrak{n}_+} \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}_-} \tag{7.21}$$

where $\widehat{\mathfrak{n}_\pm} = \mathfrak{n}_\pm \otimes \mathbb{C}[t, t^{-1}]$, and $\widehat{\mathfrak{h}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$. The correponding flag manifold $\widetilde{X}$ is the quotient of the loop group LG by the connected component LB^0_- of the loop group of the Borel subgroup B_- of G. The big cell $\widetilde{\mathcal{U}}$ is defined as the orbit of the unit coset under the action of the loop group LN_+ and parameterized by a set of coordinates $y_\alpha(n) := y_\alpha \otimes t^n, \alpha \in \Delta_+, n \in \mathbb{Z}$. The loop algebra acts infinitesimally by vector fields. However , we as we have an infinite number of variables we need to define the suitable completions for

Vect $\widetilde{\mathcal{U}}$ and Fun $\widetilde{\mathcal{U}}$. Concretely, an element of Fun $\widetilde{\mathcal{U}}$ is an infinite series,

$$\sum_{\alpha\in\Delta_+}\sum_{m\leq -M} P_{\alpha,m} y_{\alpha,m}, \tag{7.22}$$

where $P'_{\alpha,m}s$ are polynomials in $y_{\alpha,n}, n \in \mathbb{Z}$. The Lie algebra Vect $\widetilde{\mathcal{U}}$ consists of series of the form,

$$\sum_{\alpha\in\Delta^+}\left[\sum_{n\geq N} P_{\alpha,n}\frac{\partial}{\partial y_{\alpha,n}} + \sum_{m\leq -M} y_{\alpha,m} V_{\alpha,m}\right] \tag{7.23}$$

where the $P_{\alpha,n}$'s are polynomials and the $V_{\alpha,m}$'s are polynomial vector fields. Hence Vect $\widetilde{\mathcal{U}}$ is the completion of the Lie algebra of polynomial vector fields with respect to the topology given by the basis of open neighbourhoods of 0 formed by linear combinations of vector fields given by each of the terms in the above formula. Our goal is to realize the algebra of differential operators (and subsequently vector fields) inside a free field vertex algebra and the space of functions inside a Fock module M. Let us define these objects and the vertex algebra structure.

Let $\mathcal{A}^{\mathfrak{g}}$ be the Weyl algebra with generators,

$$a_{\alpha,n} = \frac{\partial}{\partial y_{\alpha,n}}, \quad a^*_{\alpha,n} = y_{\alpha,-n}, \quad \alpha \in \Delta_+, n \in \mathbb{Z}, \tag{7.24}$$

and relations,

$$[a_{\alpha,n}, a^*_{\beta,m}] = \delta_{\alpha,\beta}\delta_{n,-m}, \quad [a_{\alpha,n}, a_{\beta,m}] = [a^*_{\alpha,n}, a^*_{\beta,m}] = 0. \tag{7.25}$$

Their generating series is defined as,

$$a_\alpha(z) = \sum_{n\in\mathbb{Z}} a_{\alpha,n} z^{-n-1} \tag{7.26}$$

$$a^*_\alpha(z) = \sum_{n\in\mathbb{Z}} a^*_{\alpha,n} z^{-n}. \tag{7.27}$$

Let $M_{\mathfrak{g}}$ be the Fock representation generated by a vector $|0\rangle$ such that,

$$a_{\alpha,n}|0\rangle = 0, n \geq 0; \quad a^*_{\alpha,n}|0\rangle = 0, \quad n > 0. \tag{7.28}$$

$M_{\mathfrak{g}}$ carries the structure of a Z_+-graded vertex algebra (similar to the Fock representation of the free bosons) and is usually referred to as the VOA of symplectic bosons or as alternatively as the $\beta\gamma$ system in physics literature. The completed Weyl algebra $\widetilde{A^{\mathfrak{g}}}$ is the $I_{N,M}$-adic compeletion of $\mathcal{A}^{\mathfrak{g}}$, where $I_{N,M}$ are left ideas generated by $a_{\alpha,n}, n \geq N$ and $a^*_{\alpha,m}, m \geq M$ (See Example (B.3) in the Appendix A for a definition). Hence it consists of arbitrary series of the form,

$$\sum_{n\geq N} P_{\alpha,n} a_{\alpha,n} + \sum_{m\geq M} Q_{\alpha,m} a^*_{\alpha,m}, \quad P_m, Q_m \in \mathcal{A}^{\mathfrak{g}}. \tag{7.29}$$

Let $\mathcal{A}^{\mathfrak{g}}_0$ be the commutative subalgebra of $A^{\mathfrak{g}}$ generated by $a^*_{\alpha,n}, \alpha \in \Delta_+, n \in \mathbb{Z}$ and $\widetilde{\mathcal{A}^{\mathfrak{g}}_0}$ be its completion. From the definitions it is clear that,

$$\widetilde{\mathcal{A}^{\mathfrak{g}}_0} \cong \mathrm{Fun}\widetilde{\mathcal{U}}. \tag{7.30}$$

Let $\mathcal{A}^{\mathfrak{g}}_{\leq 1}$ be the subspace of $\mathcal{A}^{\mathfrak{g}}_{\leq 1}$ spanned by products of elements of $\mathcal{A}^{\mathfrak{g}}_0$ and the generators $a_{\alpha,n}$ and let $\widetilde{\mathcal{A}^{\mathfrak{g}}_{\leq 1}}$ be its completion. There is a short exact sequence of Lie algebras,

$$0 \mapsto \operatorname{Fun}\widetilde{\mathcal{U}} \mapsto \widetilde{\mathcal{A}}^{\mathfrak{g}}_{\leq 1} \mapsto \operatorname{Vect}\widetilde{\mathcal{U}} \mapsto 0. \tag{7.31}$$

Actually we are more interested in 'local' subalgebras of the objects in (7.31). These are elements of the compeleted Weyl algebra that are also obtained as elements of a completed universal enveloping algebra $U(M_{\mathfrak{g}})$. For example, let us denote the local version of $\mathcal{A}^{\mathfrak{g}}{}_0$ by $\mathcal{D}_{0,loc}$. This is the subalgebra spanned by the Fourier co-efficients of all polynomials in the $\partial_z a^*_\alpha(z), n \geq 0$. Restriction of (7.31) to the local versions yields an exact sequence (refer to [26] for their definitions),

$$0 \mapsto \mathcal{D}_{0,loc} \mapsto \mathcal{D}_{1,loc} \mapsto \operatorname{Vect}\widetilde{\mathcal{U}}_{loc} \mapsto 0. \tag{7.32}$$

Like in the finite-dimensional case, the action of $L\mathfrak{g}$ on $\widetilde{\mathcal{U}}$ gives an embedding $\epsilon : L\mathfrak{g} \mapsto \operatorname{Vect}\widetilde{\mathcal{U}}_{loc}$ (See [26] for more details). We are again interested in lifting this to a map to $\mathcal{D}_{1,loc}$. However, in constrast to the finite-dimensional case the exact sequence does not split. This is because that normal ordering distorts commutation relations of elements of $\mathcal{D}_{1,loc}$ and yields an extra term lying in $\mathcal{D}_{0,loc}$ as is apparent from the Wick formula. However, it turns out that it is possible to lift ϵ to a map from the central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ to $\mathcal{D}_{1,loc}$.

Theorem 7.C.1. *[26] There exists a Lie algebra homomorphism $\mathfrak{g} \mapsto \mathcal{D}_{1,loc}$ which maps K to $-h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. The space of homomorphisms is parameterized by $\mathfrak{h} \otimes \mathbb{C}((z))dz$.*

The Fock representation $M_{\mathfrak{g}}$ thus yields a family of modules called Wakimoto modules over $\hat{\mathfrak{g}}$ of level $-h^\vee$ which is called the critical level. We reformulate this statement in the language of VOAs in the next section, and give the generalization to arbitrary levels.

7.D Wakimoto modules

In section (7.B), we obtained a family of Lie algebra homormophisms,

$$\epsilon_\lambda : \mathfrak{g} \mapsto \mathcal{D}_1, \quad \lambda \in \mathfrak{h}^*. \tag{7.33}$$

Let $e_i, h_i, f_i, i = 1, 2, \ldots, l$ be a set of Chevalley generators of $\mathfrak{g}$ for some choice of simple positive roots Δ_+. In the notation of 7.A.1, these are defined as $e_i := x_{\alpha_i}$ and $f_i := x_{-\alpha_i}$. The above representation can be explicitly written as ,

$$\epsilon_\lambda(e_i) = \frac{\partial}{\partial_{y_{\alpha_i}}} + \sum_{\beta \in \Delta^+} P^i_\beta(y_\alpha)\frac{\partial}{\partial y_\beta} \tag{7.34}$$

$$\epsilon_\lambda(h_i) = -\sum_{\beta \in \Delta^+} \beta(h_i) y_\beta \frac{\partial}{\partial y_\beta} + \chi(h_i) \tag{7.35}$$

$$\epsilon_\lambda(f_i) = \sum_{\beta \in \Delta^+} Q^i_\beta(y^\alpha)\frac{\partial}{\partial y_\beta} + \chi(h_i) y_{\alpha_i} \tag{7.36}$$

for some polynomials P^i_β, Q^i_β in $y_\alpha, \alpha \in \Delta^+$ and a fixed map $\beta : \mathfrak{h} \mapsto \mathbb{C}$. The corresponding map for $\epsilon_\lambda : \hat{\mathfrak{g}} \mapsto \mathrm{Vect}\,\widetilde{\mathcal{U}}$ maybe obtained by the substitution,

$$y_\alpha \mapsto \sum_{n \in \mathbb{Z}} y_{\alpha,n} z^n \tag{7.37}$$

$$\frac{\partial}{\partial y_\alpha} \mapsto \sum_{n \in \mathbb{Z}} \frac{\partial}{y_{\alpha,n}} z^{-n-1}. \tag{7.38}$$

Now we are ready to state the main theorem.

Theorem 7.D.1. *There exists a homomorphism of vertex algebras,*

$$w_k : V_k(\mathfrak{g}) \mapsto M_g \otimes \pi_0^{k+h^\vee} \tag{7.39}$$

such that,

$$e_i(z) \mapsto a_{\alpha_i}(z) + \sum_{\beta\in\Delta_+} :P^i_\beta(a^*_\alpha(z)) : \alpha_\beta(z) :, \tag{7.40}$$

$$h_i(z) \mapsto -\sum_{\beta\in\Delta_+} \beta(h_i) : a^*_\beta(z)\alpha_\beta(z) : +b_i(z), \tag{7.41}$$

$$f_i(z) \mapsto \sum_{\beta\in\Delta^+} : Q^i_\beta(a^*_\alpha(z)a_\beta(z)) : +(c_i + (k+h^\vee))\partial_z a^*_{\alpha_i}(z) + b_i(z)a^*_{\alpha_i}(z) \tag{7.42}$$

where P^i_β is a polynomial of degree $\alpha - \beta$ (when graded by the positive part Q_+ of the root lattice) and c_i are some constants.

Any module over the vertex algebra $M_g \otimes \pi_0^{k+h^\vee}$ becomes a $V_k(\mathfrak{g})$-module and hence a $\mathfrak{g}_k$-module (with K acting as 1). Thus,

$$W_{k,\lambda} := M_\mathfrak{g} \otimes \pi_\lambda^{k+h^\vee} \tag{7.43}$$

is a $\mathfrak{g}_k$-module. This is the so-called Wakimoto module of level k and highest weight λ and it lies in category $\mathcal{O}$ as desired. As an example we present the case of $\widehat{\mathfrak{sl}_3}$. We temporarily resort to the physics notation in which we denote the generating fields of a pair symplectic bosons by $\beta(z), \gamma(z)$ for better readability after the identification $a(z) \mapsto \beta(z)$ and $a^*(z) \mapsto \gamma(z)$. The following formulas are reproduced from [25].

Proposition 7.D.2. *Let $\mathfrak{g} := \mathfrak{sl}_3$, $\{\beta^i(z), \gamma^i(z)\}_{i=1,2,3}$ be pairs of symplec-*

tic bosons and $X(z) := (X^1(z), X^2(z))$ *be a pair of free bosons. Let* $\alpha_1 = \frac{1}{\sqrt{2}}(1, \sqrt{3})$, $\alpha_2 = \frac{1}{\sqrt{2}}(1, -\sqrt{3})$. *Then the Wakimoto realization of* (7.39) *at level* k *is given by,*

$$e^1(z) = \beta^1(z) \tag{7.44}$$

$$e^2(z) = \beta^2(z) - \gamma^1(z)\beta^3(z) \tag{7.45}$$

$$e^3(z) = \beta^3(z) \tag{7.46}$$

$$h^1(z) = -\alpha_1(z) \cdot X(z) + 2\gamma^1(z)\beta^1(z) - \gamma^2(z)\beta^2(z) + \gamma^3(z)\beta^3(z) \tag{7.47}$$

$$h^2(z) = -\alpha_2(z) \cdot X(z) - \gamma^1(z)\beta^1(z) + 2\gamma^2(z)\beta^2(z) + \gamma^3(z)\beta^3(z) \tag{7.48}$$

$$\begin{aligned} f^1(z) &= \alpha_1(z) \cdot X(z)\gamma^1(z) - k\partial\gamma^1(z) + \gamma^3(z)\beta^2(z) - (\gamma^1(z))^2\beta^1(z) \\ &\quad + \gamma^1(z)\gamma^2(z)\beta^2(z) - \gamma^1(z)\gamma^3(z)\beta^3(z) \end{aligned} \tag{7.49}$$

$$f^2(z) = (\alpha_2(z) \cdot X(z))\gamma^2(z) - (k+1)\partial\gamma^2(z) - \gamma^3(z)\beta^1(z) - \gamma^2(z)\gamma^2(z)\beta^2(z) \tag{7.50}$$

$$f^3(z) = (\alpha_1(z) + \alpha_2(z)) \cdot X(z)\gamma^3(z) - \alpha_2(z) \cdot X(z)\gamma^1(z)\gamma^2(z) - k\partial\gamma^3(z) \tag{7.51}$$

$$\begin{aligned} &- (k+1)\gamma^1(z)\partial\gamma^2(z)\gamma^1(z)\gamma^3(z)\beta^1(z) - \gamma^2(z)\gamma^3(z)\beta^2(z) \\ &- \gamma^3(z)\gamma^3(z)\beta^3(z) - \gamma^1(z)\gamma^2(z)\gamma^2(z)\beta^2(z) \end{aligned}$$

where products of fields are understood to be normally ordered.

To conclude we have given a functor that maps a module N of the Heisenberg Lie algebra $\widehat{\mathfrak{h}}$ of level k to a module of the Kac-Moody algebra $\hat{\mathfrak{g}}$ but of level shifted by $h^\vee$. This is essentially an example of a semi-infinitely induced module where the module N is extended to a module of $\widehat{\mathfrak{b}}_-$ by 0 and induced to $\hat{\mathfrak{g}}$. This construction can be generalized replacing the Borel subalgebra $\widehat{\mathfrak{b}}_-$

and its Levi quotient $\widehat{\mathfrak{h}}$ by an arbitrary parabolic subalgebra $\mathfrak{p}$ and its Levi quotient $\mathfrak{m}$. The resulting $\hat{\mathfrak{g}}$–modules are called the generalized Wakimoto modules corresponding to $\mathfrak{p}$.

We briefly comment on some of the applications of Wakimoto modules. They have been used to construct chiral correlation functions of the WZW models, in the study of the Drinfeld-Sokolov reduction and $\mathcal{W}$-algebras and in the description of the center of the completed enveloping algebra of an affine Lie algebra. In particular Wakimoto modules can be used to show that, this center at the critical level is isomorphic to functions on the space of ${}^LG$–opers on the formal disc as a Poisson algebra where ${}^LG$ is the Langlands dual of G. The interested reader can refer to [26] for questions related to generalizations and applications. Finally Wakimoto models can be used to obtain free field realizations of $\mathcal{W}$-algebras via the Drinfeld Sokolov reduction of affine Lie algebras.

7.E Constructing Airy ideals from Wakimoto modules

Our goal in this section is to give new Airy ideals associated to modules of the affine Lie algebra $\hat{\mathfrak{g}}$. Let $\mathfrak{g}$ be a simple Lie algebra and $\hat{\mathfrak{g}}$ the corresponding affine Lie algebra. We denote by $\{\alpha_i\}_{i=1,2,\ldots,d}$ a basis of simple roots of $\mathfrak{g}$, and let $\{h_i\}_{i=1,2,\ldots l}$ be generators of the Cartan subalgebra. Then the universal affine vertex algebra $V_k(\mathfrak{g})$ is strongly generated by a set of 'Chevalley fields', denoted by $\{e_i(z), f_j(z), h_k(z)\}$ where $i, j = 1, 2, \ldots, d$ and $k = 1, 2, \ldots l$. Recall that as before, we have slightly modified the notation of Section 7.A.1 by setting $e_i(z) := x_{\alpha_i}(z), f_i(z) := x_{-\alpha_i}(z)$, where $x_{\alpha_i}(z)$ are the affinization of the generators x_{α_i}.

Wakimoto modules are parameterized by a complex number k called the level. In this section, we first define another parameter $\hbar$ by setting $\hbar^2 = (k+h^\vee)^{-1}$. First we rescale the generators (7.40)-(7.42) by $\hbar$ so that now,

$$e^i(z) \mapsto \hbar^2 e^i(z), \qquad h^i(z) \mapsto \hbar^2 h^i(z), \qquad f^i(z) \mapsto \hbar^2 f^i(z). \tag{7.52}$$

We also rescale generators of the underlying Fock module,

$$a_{\alpha_i}(z) \mapsto \hbar^2 a_{\alpha_i}(z), \quad a^*_{\alpha_i}(z) \mapsto a^*_{\alpha_i}(z), \quad b_i(z) \mapsto \hbar^2 b_i(z). \tag{7.53}$$

From now on, we set $W_{\mathfrak{g}}^{\hbar,\lambda} := M_{\mathfrak{g}} \otimes \pi_\lambda^{k+h^\vee}$ and $\widetilde{W}_{\mathfrak{g}}^{\hbar,\lambda}$ be its contragredient dual where k is understood to be expressed in terms of $\hbar$.

In order to furnish Airy ideals out of Wakimoto modules, we are actually forced to consider an infinite extension of these modules as in the case of $\mathcal{W}(\mathfrak{sp}_{2N})$. We first define this extension below, and then present our starting object for the construction. Let $\pi_{\rho_\lambda}^{k+h^\vee}$ denote the Fock module extension of level $k+h^\vee$ defined in (4.104) so that the zero mode acts like a derivative $\frac{d}{dz}+\lambda$ for some formal variable z. Then the Wakimoto module $W_{\mathfrak{g}}^{\hbar,\lambda}$ admits an infinite extension, $W_{\mathfrak{g}}^{\hbar,\rho_\lambda} := M_{\mathfrak{g}} \otimes \pi_{\rho_\lambda}^{k+h^\vee}$ where $M_{\mathfrak{g}}$ is the Fock module of symplectic bosons as before. Finally, consider an involution on the algebra of modes given by,

$$\Omega(e^i_n) = -f^i_n, \quad \Omega(f^i_n) = -e^i_n, \quad \Omega(h^i_n) = -h^i_n. \tag{7.54}$$

The foundation of our construction is the tensor product module $\mathcal{M} := W^{\hbar,\rho_\lambda}_{\mathfrak{g}} \otimes \Omega^*(\widetilde{W}^{\hbar,\rho_{-\lambda}}_{\mathfrak{g}})$.

Consider the set of differential operators $\{A^i_n, B^j_n, C^k_n\}$ for $i, j = 1, 2, \ldots, d$, $k = 1, 2, \ldots l$ given by,

$$A^i_n := e^i_n + a_i \bar{f^i}_{-n}, \quad B^i_n := f^i_n + b_i \bar{e^i}_{-n}, \quad C^i_n := h^i_n + c_i \bar{h^i}_{-n} \tag{7.55}$$

where elements of the second factor are denoted by barred variables and $a_i, b_i, c_i \in \mathbb{C}$ are some non-zero fixed constants. Note that these are precisely the boundary conditions we first encountered in (6.25) and (6.26) when $a_i = b_i = c_i = 1$ for all i and Ω is set to be the involution defined above . In order to construct an Airy ideal, we first need to realize the operators $\{A^i_n, B^j_n, C^k_n\} \in \mathcal{M}$ as elements of a completed Rees Weyl algebra $\widehat{\mathcal{D}^\hbar_A}$ for some index set A. Hence, we make the following definitions: the derivatives of the Weyl algebra are identified with certain linear sums of the free fields,

$$\partial_{x^i_n} := (a_{\alpha_i})_n + a_i n(\bar{a^*}_{\alpha_i})_{-n} \tag{7.56}$$

$$\partial_{y^i_n} := (\bar{a}_{\alpha_i})_n + b_i n(a^*_{\alpha_i})_{-n} \tag{7.57}$$

$$\partial_{z^i_n} := (b_i)_n + c_i(\bar{b}_i)_{-n} \tag{7.58}$$

and the corresponding variables are chosen to be combinations of modes that satisfy the correct commutation relations with the derivatives defined above. More precisely we define,

$$x_n^i := \begin{cases} (a^*_{\alpha_i})_{-n} \quad \text{for} \quad n > 0, \\ \\ \frac{-1}{a_i n}(\bar{a}_{\alpha_i})_n \quad \text{for} \quad n < 0 \\ \\ (a^*_{\alpha_i})_0 \quad \text{for} \quad n = 0 \end{cases} \tag{7.59}$$

$$y_n^i := \begin{cases} (\bar{a}^*_{\alpha_i})_{-n} \quad \text{for} \quad n > 0, \\ \\ \frac{-1}{b_i n}(a_{\alpha_i})_n \quad \text{for} \quad n < 0 \\ \\ (\bar{a}^*_{\alpha_i})_0 \quad \text{for} \quad n = 0 \end{cases} \tag{7.60}$$

$$z_n := \begin{cases} \dfrac{1}{n}(b_i)_{-n} \quad \text{for} \quad n > 0, \\ \\ \frac{-1}{c_i n}(\bar{b}_i)_n \quad \text{for} \quad n < 0 \end{cases} \tag{7.61}$$

Let $\mathcal{D}_A$ be the Weyl algebra generated by $\{\partial_{x_n^i}, \partial_{y_n^j}, \partial_{z_n^k}\}$ and $\{x_n^i, y_n^j, z_n^k\}$ for $n \in \mathbb{Z}$. The corresponding ($\hbar$-graded) Rees Weyl algebra $\mathcal{D}_A^\hbar$ is defined by,

$$\mathcal{D}_A^\hbar = \bigoplus_{n \in \mathbb{N}} \hbar^n F_n \mathcal{D}_A \tag{7.62}$$

where $\{F_n \mathcal{D}_A\}_{n \in \mathbb{Z}}$ is the Berstein filtration of Definition 4.B.3. As before, we can easily construct $\hbar$-adic completion, which we denote by $\widehat{D_A^\hbar}$. Then we have the following theorem:

Theorem 7.E.1. *Let $\mathcal{I} \subset \widehat{D^\hbar_A}$ be the left ideal generated by the collection $\{A^i_n, B^j_n, C^k_n\}$ for $i, j = 1, 2, \ldots, d$, $k = 1, 2, \ldots l$ and $n \in \mathbb{Z}$. Then $\mathcal{I}$ is an Airy ideal.*

Proof. We first note that the relations (7.59)-(7.61) can be inverted as follows,

$$(a_{\alpha_i})_n = \begin{cases} \partial_{x^i_n} - a_i n y^i_n & \text{for} \quad n \geq 0, \\ \\ -b_i n y^i_n & \text{for} \quad n < 0 \end{cases} \tag{7.63}$$

and

$$(a^*_{\alpha_i})_n = \begin{cases} \dfrac{1}{b_i n}\partial_{y^i_{-n}} + \dfrac{a_i}{b_i}x^i_{-n} & \text{for} \quad n > 0, \\ \\ -a_i n x^i_{-n} & \text{for} \quad n \leq 0 \end{cases} \tag{7.64}$$

Similarly for the free boson $b_i(z)$,

$$(b_i)_n = \begin{cases} \hbar\partial_{z^i{}_n} + z_{-n} & \text{for} \quad n > 0, \\ \\ z^i_{-n} & \text{for} \quad n < 0 \\ \\ Q\partial_{z^i_0} + \lambda & \text{for} \quad n = 0 \end{cases} \tag{7.65}$$

where Q is a fixed arbitrary complex number. Analogous expressions hold for the barred variables. For example, the zero mode of the free boson takes the form

$$\bar{b}_0 = \frac{1}{c}[(-Q+1)\partial_{z^i_0} - \lambda]. \tag{7.66}$$

Substituting (7.63)- (7.65) into (7.55) allows us to fully express the constraints $\{A^i_n, B^i_n, C^i_n\}$ in terms of the coordinates $\{x^i_n, y^i_n, z^i_n\}$ and their derivatives. After performing this substitution, the operators $\{A^i_n, B^i_n, C^i_n\} \in \mathcal{M}$ are realized inside the completion $\widehat{\mathcal{D}_A}$. We now check that each of the three conditions for being an Airy ideal are satisfied. It is easy to check that the boundedness and degree 1 conditions are satisfied. In particular, by definition the lowest degree terms take the form of a derivative for each of the constraints $\{A^i_n, B^j_n, C^k_n\}$ and there are no other degree 1 or degree 0 terms. Secondly from (7.55) it follows that the left ideal generated by $\{A^i_n, B^j_n, C^k_n\}$ satisfy the 'graded Lie subalgebra' condition due to the scaling performed in (7.52). This completes the proof. □

Remark 7.E.2. Our primary motivation for considering constraints of the form (7.55) is that they are the defining equations of Ishibashi states in the Wess-Zumino-Witten (WZW) model once we set $a_i = b_i = c_i = 1$ for all i (See also (6.25) and (6.26)). In this case, the partition function of the Airy ideal can be interpreted as a free field representation of a vector satisfying the chosen boundary gluing conditions. However as we start with an infinite extension of Wakimoto modules, the identification to an Ishibashi state I^M corresponding to an irreducible integrable module M, of the affine VOA is more subtle. We briefly remark on how Ishibashi states appear in the WZW model and some of their applications such as computation of string amplitudes in Section 7.F. This is meant to give the reader a taste of some of the contexts in which the operators we constructed above might play a role.

Remark 7.E.3. Let us denote the Wakimoto module of level k and highest weight λ by $W_{k,\lambda}$. Denote by $\mathcal{H}_{k,\lambda}$, the irreducible module of the affine Lie algebra $\hat{\mathfrak{g}}$ with highest weight λ and level k. Let $I^{W_{k,\lambda}}$ and $I^{\mathcal{H}_{k,\lambda}}$ denote the Ishibashi states associated to each of these modules respectively, for some choice of gluing conditions. The two modules are related, in particular $\mathcal{H}_{k,\lambda}$ is an homomorphic image of Wakimoto modules $W_{k,\lambda}$ (See [26, Proposition 6.3.3]). Let $\phi_{k,\lambda}$ denote this homomorphism, so that $\phi_{k,\lambda}(W_{k,\lambda}) = \mathcal{H}_{k,\lambda}$. Since the boundary conditions are preserved under any homomorphism we have the following relationship,

$$\phi_{k,\lambda} \otimes \overline{\phi}_{k,\bar{\lambda}}(I^{W_{k,\lambda}}) = I^{\mathcal{H}_{k,\lambda}} \tag{7.67}$$

where $\bar{\phi}$ is the same automorphism on the second factor and $\bar{\lambda}$ is the highest weight of the contragredient dual. Hence computing Ishibashi states for irreducible modules can be obtained from Ishibashi states of Wakimoto modules, if we can find a explicit representation of ϕ.

7.F D-branes in WZW models

As mentioned in the introduction to the previous chapter, boundary states can be used to describe D-branes in the worldsheet interpretation of string theory. A D-brane can be thought of as the locus of end points of open strings in the target manifold, where the D stands for the 'Dirichlet' boundary conditions that are imposed on the embedding fields in the sigma model. In this section, we give a brief survey of how amplitudes of strings propagating between D-branes in WZW models can be computed using Ishibashi states. Finally we propose some ways in which Airy ideals could help solve some problems in this field.

We follow the exposition of [45] in this section. The amplitude for a string propagating between D-branes with boundary conditions $|\alpha\rangle$ to $|\beta\rangle$ is given by $Z_{\alpha\beta} := \langle\alpha| e^{Ht} |\beta\rangle$. This can also be interpreted via a modular S-transformation $\tau \mapsto -\frac{1}{\tau}$ as a partition function of an open string with end-points on each of these branes. More precisely this duality yields a character expansion,

$$Z_{\alpha\beta}(q) = \langle\alpha| \tilde{q}^{H/2} |\beta\rangle = \sum_{\nu} N^{\nu}_{\alpha\beta}\chi_{v}(q) \tag{7.68}$$

where $N^{0}_{\alpha\beta} = \delta_{\alpha\beta}$ and $N^{\nu}_{\alpha\beta} \in \mathbb{Z}_{+}$ and ν runs over the set of irreducible representations. The integrality conditions on the coefficients are called the Cardy conditions, and the boundary states $|\alpha\rangle, |\beta\rangle$ are said to be Cardy states. In particular, some examples of solutions of $N^{\nu}_{\alpha\beta}$ are given by certain combinations of S-matrix elements.

Recall that strings propogating on a Lie group G are described by the Wess-Zumino-Witten (WZW) action (See [25], [45]),

$$S^{WZW}(g) = \frac{k}{16\pi}\int_{\Sigma} \mathrm{tr}(g^{-1}\partial g)(g^{-1}\bar{\partial} g) + \frac{k}{24\pi}\int_{B} \tilde{g}^{*}\chi \tag{7.69}$$

where $g : \Sigma \mapsto G$, B is a 3-manifold with $\partial B = \Sigma$, $\tilde{g}$ is an extension of g to $B \mapsto G$ and χ is a certain 3-form, (refer to [25, equation 15.19] for further details). String states in this model are constructed as highest weight vectors in representations of the affine Lie algebra $\hat{\mathfrak{g}}$. In this case, the Hilbert space can be decomposed as,

$$\bigoplus_{\lambda \in P_k} H_\lambda \otimes \widetilde{\mathcal{H}_\lambda} \tag{7.70}$$

where P_k denotes the set of irreducible integral highest weights at level k.

If we wish to compute amplitudes of open strings propagating between D-branes in the WZW model, we must first study Ishibashi states. Let $|I_\lambda\rangle$ denote the Ishibashi state associated to the irreducible H_λ. The amplitudes between two Ishibashi states can be computed explicitly in terms of characters and is given by (See [45, Section 3.3.2]),

$$\langle I_\lambda | \, q^{L_0 + \bar{L}_0 - \frac{c}{12}} \, |I_\mu\rangle = \delta_{\lambda,\mu} \chi_\mu(q^2) \tag{7.71}$$

where χ_μ is the character of the irreducible representation of highest weight μ. Secondly we remark that, a representation of Ishibashi states can be used to write a representation of a Cardy-type boundary state such as the ones defined by S-matrix coefficients,

$$|\alpha\rangle = \sum_\lambda \frac{S_{\lambda,\alpha}}{\sqrt{S_{0,\lambda}}} \, |I_\lambda\rangle \, . \tag{7.72}$$

Hence knowledge of Ishibashi states can help us compute physical observables such as the amplitudes of (7.68).

As noted in Remark 7.E.2, the partition functions Z of the Airy ideal constructed in Theorem 7.E.1 satisfy the gluing conditions of Ishibashi states. However, they live in an extension of the bulk Hilbert space defined in (7.70), and hence can't quite be used to compute Cardy states or D-brane amplitudes. On the other hand, as mentioned in (7.E.2), Wakimoto modules are intimately connected to integrable irreducible representations of affine VOAs and thus the partition function given by Theorem (7.E.1) could help us compute the Ishibashi states $|\lambda\rangle$. Hence, we are left with the following question:

Problem 7.F.1. What is the precise relation between the partition function Z and D-brane amplitudes of the WZW model?

We don't have an answer to this question, but we hope that this section can convince the reader as to why this direction is worth pursuing.

Chapter 8

Conclusion

This thesis lies at the intersection of two very influential areas of mathematics - VOAs and enumerative geometry. At this point, it is evident that Airy ideals hold enormous promise due to rich connections with both these areas. One might hope that future research motivated from this thesis could be fruitful due to the inter-disciplinary nature of the subject. We propose below a list of open problems for future research, that the reader might find interesting to tackle.

8.A Some open problems

To begin with, the enumerative geometry aspect of this thesis remains largely unexplored. Hence it is natural to ask:

Problem 8.A.1. Does the partition function of the Airy ideals constructed in Chapter 5 from modules of the $\mathcal{W}(\mathfrak{sp}_{2N})$-algebra have an interpretation as the generating function of any enumerative geometric invariants? Does it have a matrix model realization?

On a similar note:

Problem 8.A.2. Can the partition function of Theorem 7.E.1 be identified with partition functions coming from string theory such as the partition function of D-branes in the WZW model? Is there a geometric interpretation in terms of target space functionals, or perhaps to some kind of enumerative geometry invariants?

In Section 7.F we gave an example of how Airy ideals could be used to compute string amplitudes. In particular we discussed that Ishibashi states in the WZW model are related to characters of modules of affine VOAs (See (7.71)). This motivates the following question:

Problem 8.A.3. Can Airy ideal partition functions such as the one in Theorem 7.E.1 be used to compute characters of affine Lie algebras or even simple Lie algebras?

In this thesis we were only concerned with boundary conditions on affine VOAs, however one could ask:

Problem 8.A.4. Can Airy ideals be constructed from boundary conditions imposed on generators of other VOAs such as $\mathcal{W}$-algebras?

The questions described above are quite concrete and deal with specific examples, in other words they are more suitable for mathematicians who consider themselves to be Dyson's frogs (described in the essay [21]). Mathematicians that consider themselves to be birds might find the following questions more entertaining.

Problem 8.A.5. Can the notion of Airy ideals and partition functions be extended to a context beyond Weyl algebras, or to larger completions of the Weyl algebra?

In this thesis a crucial role was played by the automorphisms of the Weyl algebra called 'transvections'. These seem to be very special in the sense that they somehow capture the essential properties of Airy ideals. However one could also study ideals obtained from other automorphisms of the Weyl algebra,

Problem 8.A.6. Can other automorphisms of the Weyl algebra be used to engineer useful mathematical objects of interest to geometers or algebraists?

Bibliography

[1] Jorgen Ellegaard Andersen, Gaëtan Borot, Leonid O Chekhov, and Nicolas Orantin. The ABCD of topological recursion. *arXiv preprint arXiv:1703.03307*, 2017.

[2] Tomoyuki Arakawa. Introduction to $\mathcal{W}$-algebras and their representation theory. In *Perspectives in Lie theory*, pages 179–250. Springer, 2017.

[3] Tomoyuki Arakawa, Thomas Creutzig, and Andrew R. Linshaw. $\mathcal{W}$-algebras as coset vertex algebras. *Invent. Math.*, 218(1):145–195, 2019.

[4] Michael Atiyah. *Introduction to commutative algebra*. CRC Press, 2018.

[5] Bojko Bakalov and Victor G Kac. Twisted modules over lattice vertex algebras. *arXiv preprint math/0402315*, 2004.

[6] Bojko Bakalov and Todor Milanov. Constraints for the total descendant potential of a simple singularity. *Compositio Mathematica*, 149(5):840–888, 2013.

[7] Alexei Belov-Kanel and Maxim Kontsevich. Automorphisms of the Weyl algebra. *Letters in mathematical physics*, 74(2):181–199, 2005.

[8] Gaëtan Borot, Vincent Bouchard, Nitin K Chidambaram, and Thomas Creutzig. Whittaker vectors for $\mathcal{W}$-algebras from topological recursion. *arXiv preprint arXiv:2104.04516*, 2021.

[9] Gaëtan Borot, Vincent Bouchard, Nitin K. Chidambaram, Thomas Creutzig, and Dmitry Noshchenko. Higher Airy structures, $\mathcal{W}$-algebras and topological recursion. *arXiv preprint arXiv:1812.08738*, 2018.

[10] Gaëtan Borot, Reinier Kramer, and Yannik Schüler. Higher Airy structures and topological recursion for singular spectral curves. *arXiv preprint arXiv: 2010.03512*.

[11] Vincent Bouchard, Paweł Ciosmak, Leszek Hadasz, Kento Osuga, Blazej Ruba, and Piotr Sułkowski. Super Quantum Airy Structures. *Commun. Math. Phys.*, 380(1):449–522, 2020.

[12] Vincent Bouchard, Thomas Creutzig, and Aniket Joshi. Airy ideals, transvections and $\mathcal{W}(\mathfrak{sp}_{2N})$-algebras. *arXiv preprint math/2207.04336*.

[13] Vincent Bouchard, Joel Hutchinson, Prachi Loliencar, Michael Meiers, and Matthew Rupert. A generalized topological recursion for arbitrary ramification. In *Annales Henri Poincaré*, volume 15, pages 143–169. Springer, 2014.

[14] Vincent Bouchard, Albrecht Klemm, Marcos Marino, and Sara Pasquetti. Remodeling the B-model. *Communications in Mathematical Physics*, 287(1):117–178, 2009.

[15] A. Buryak. Open intersection numbers and the wave function of the KdV hierarchy. *Moscow Math. J.*, 16(1):27–44, 2016.

[16] A Cappelli, LS Georgiev, and IT Todorov. A unified conformal field theory description of paired quantum hall states. *Communications in mathematical physics*, 205(3):657–689, 1999.

[17] Thomas Creutzig, Naoki Genra, and Andrew Linshaw. Category $\mathcal{O}$ for vertex algebras of $\mathfrak{osp}(1|2n)$. *arXiv preprint arXiv:2203.08188*, 2022.

[18] Thomas Creutzig and Andrew R. Linshaw. Orbifolds of symplectic fermion algebras. *Trans. Am. Math. Soc.*, 369(1):467–494, 2017.

[19] Thomas Creutzig and Andrew R. Linshaw. Cosets of affine vertex algebras inside larger structures. *J. Algebra*, 517:396–438, 2019.

[20] Thomas Creutzig and Andrew R. Linshaw. Trialities of $\mathcal{W}$-algebras. *Camb. J. Math.*, 10(1):69–194, 2022.

[21] Freeman J Dyson. Birds and frogs in mathematics and physics. *Physics-Uspekhi*, 53(8):825, 2010.

[22] Bertrand Eynard and Nicolas Orantin. Algebraic methods in random matrices and enumerative geometry. 11 2008.

[23] Bertrand Eynard and Nicolas Orantin. Topological recursion in enumerative geometry and random matrices. *J. Phys. A*, 42(29):293001, 2009.

[24] Boris L Feigin and Edward V Frenkel. Representations of affine Kac-Moody algebras, bosonization and resolutions. *Letters in mathematical physics*, 19(4):307–317, 1990.

[25] Philippe Di Francesco, Pierre Mathieu, and Sénéchal David. *Conformal field theory*. Springer, 2011.

[26] Edward Frenkel. *Langlands correspondence for loop groups.* Cambridge University Press, 2010.

[27] Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves*, volume 88 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, second edition, 2004.

[28] Igor Frenkel, Yi-Zhi Huang, and James Lepowsky. *On axiomatic approaches to vertex operator algebras and modules*, volume 494. American Mathematical Soc., 1993.

[29] Daniel Friedan, Emil Martinec, and Stephen Shenker. Conformal invariance, supersymmetry and string theory. *Nuclear Phys. B*, 271(1):93–165, 1986.

[30] M. R. Gaberdiel and A. Recknagel. Conformal boundary states for free bosons and fermions. *JHEP*, 11:016, 2001.

[31] Matthias R. Gaberdiel and Hanno Klemm. N = 2 superconformal boundary states for free bosons and fermions. *Nucl. Phys. B*, 693:281–301, 2004.

[32] Davide Gaiotto and Leonardo Rastelli. A paradigm of open/closed duality Liouville D-branes and the Kontsevich model. *Journal of High Energy Physics*, 2005(07):053, 2005.

[33] Naoki Genra. Screening operators for $\mathcal{W}$-algebras. *Selecta Math. (N.S.)*, 23(3):2157–2202, 2017.

[34] James E Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer Science & Business Media, 2012.

[35] Nobuyuki Ishibashi. The Boundary and Crosscap States in Conformal Field Theories. *Mod. Phys. Lett. A*, 4:251, 1989.

[36] Victor G Kac. *Vertex algebras for beginners*. Number 10. American Mathematical Soc., 1998.

[37] Victor G. Kac and Minoru Wakimoto. Quantum reduction and representation theory of superconformal algebras. *Adv. Math.*, 185(2):400–458, 2004.

[38] Scott N Kersey. Invertibility of submatrices of pascal's matrix and birkhoff interpolation. *arXiv preprint arXiv:1303.6159*, 2013.

[39] Maxim Kontsevich and Yan Soibelman. Airy structures and symplectic geometry of topological recursion. *arXiv preprint arXiv:1701.09137*.

[40] Serge Lang. *Algebra*, volume 211. Springer Science & Business Media, 2012.

[41] Haisheng Li. Vertex algebras and vertex Poisson algebras. *Commun. Contemp. Math.*, 6(1):61–110, 2004.

[42] Andreas Recknagel and Volker Schomerus. *Boundary Conformal Field Theory and the Worldsheet Approach to D-Branes*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 11 2013.

[43] Minoru Wakimoto. Fock representations of the affine lie algebra A_1. *Communications in mathematical physics*, 104(4):605–609, 1986.

[44] Edward Witten. On the Structure of the Topological Phase of Two-dimensional Gravity. *Nucl. Phys. B*, 340:281–332, 1990.

[45] Albrecht Wurtz. *Conformal Field Theory and D-branes*. PhD thesis, 2006.

Appendix A

Preliminary background on completions

In this appendix, we first precisely define what are completions of filtered objects using the category theoretical concept of 'inverse limit'. This is a pure algebraic formulation of completeness which can also be defined in terms of Cauchy sequences for categories with a topological structure (See [4] for example). Secondly, we also give two concrete examples of completions for the Weyl algebra - the Rees Weyl algebra and its $\hbar$-adic completion. Both of these examples appear prominently in Chapters 4, 5 and 7.

A Completions of graded algebras

In this section we introduce the notion of completions of graded algebras. We follow the notation and presentation used in [40].

Definition A.1. Let I be a directed set. For a family of objects A_i, let $\{f_i^j\}_{i\in I}$

be a set of morphisms,

$$f_i^j : A_j \mapsto A_i \quad \forall j \geq i \tag{1.1}$$

satisfying the relation,

$$f_k^i \circ f_i^j = f_k^j \quad \text{and} \quad f_i^i = id, \tag{1.2}$$

if $j \geq i$ and $i \geq k$. Let $\mathcal{C}$ be a category of objects (A, f_i) with $f_i : A \mapsto A_i$ such that for i, j the following diagram is commutative:

$$\begin{tikzcd} & A \arrow[dl, "f_j"'] \arrow[dr, "f_i"] \arrow[d, phantom, "\circlearrowleft"] & \\ A_j \arrow[rr, "f_j^i"'] & & A_i \end{tikzcd}$$

A universal object in this category is called the inverse limit of the system (A_i, f_j^i) and is denoted by $A = \varprojlim A_i$.

Example A.2. A directed set can be obtained by imposing a filtration on the underlying index set. Let A be an index set with an ascending filtration $\mathcal{F}$, $(F_pA)_{p\geq 1}$. Let $i \leq j$ if $j \in F_pA \implies i \in F_pA$ for all p. For a family of objects C_i we call the inverse limit $\varprojlim C_i$ the completion with respect to the filtration $\mathcal{F}$ if the morphisms f_i^j can be defined appropriately. For example, Let E be a vector space with a choice of a countably infinite ordered basis $(e_1, e_2, \ldots)$. Let $F_pE := \text{span}(e_1, \ldots, e_p)$ and $f_i^j : F_jE \mapsto F_iE$ be projection onto F_iE. Then the completion simply consists of infinite sums,

$$v = \sum_{i>0} a_i e_i. \tag{1.3}$$

In this article the category in question are graded algebras.

Definition A.3. Let A be a graded algebra with an ascending exhaustive filtration $\mathcal{F}$ given by a chain $\bigcup_{p>0} F_p A = A$. Then A is said to be complete if,

$$\varprojlim_{p} F_p A = A. \tag{1.4}$$

In particular, we are often interested in completions of graded algebras that have nice behaviour with respect to the grading.

Definition A.4. Let $(A, \mathcal{F})$ be a filtered graded algebra with a $\mathbb{Z}$-grading

$$A = \bigoplus_{d \in \mathbb{Z}} A(d)$$

such that

$$A(d) \cdot A(e) \subset A(d+e). \tag{1.5}$$

Let us set,

$$F'_p A = \bigoplus_{d \leq p} A(d) \tag{1.6}$$

where p is an integer. Then $\{F'_p A\}_{p \geq 1}$ defines a filtration $\mathcal{F}'$ called the "associated filtration". If A is degreewise complete, that is, if each A_d is complete with respect to the filtration $\mathcal{F}$, then A is called a compatible degreewise complete algebra

B Completions of the Weyl algebra

In this thesis, because we are dealing with infinite sums in an infinite number of variables, we are required to define suitable completions of the Weyl algebra, such that the products and module actions are legal. We are primarily interested in two specific examples.

Example B.1. In Definition (4.B.2) the completed Weyl algebra was defined as follows: If A is a countably infinite index set, we define the *completed Weyl algebra* $\mathcal{D}_A$ to be the completion of the Weyl algebra $\mathbb{C}[x_A]\langle\partial_A\rangle$ that contains potentially infinite sums in the derivatives, but with polynomial coefficients. Elements of $\mathcal{D}_A$ remain of finite order as differential operators. In other words, we can write an element $P \in \mathcal{D}_A$ uniquely as

$$P = \sum_{m=0}^{M} \sum_{a_1,\ldots,a_m \in A} p_{a_1\cdots a_m}(x_A)\partial_{a_1}\cdots\partial_{a_m}, \tag{1.7}$$

for some $M \in \mathbb{N}$. This can be defined more precisely using Definition (A.1), let $(\mathbb{C}[x_A]\langle x_A\rangle, \mathcal{F})$ be the filtered Weyl algebra with filtration,

$$F_p\mathcal{D}_A = \left\{ \sum_{m=0}^{i} \sum_{a_1,\ldots,a_m \in F_pA} p_{a_1\cdots a_m}(x_A)\partial_{a_1}\cdots\partial_{a_m} \right\}, \tag{1.8}$$

where $F_pA = \{1, 2, \ldots, p\}$. Then the completed Weyl Algebra is the inverse limit

$$\varprojlim F_p\mathcal{D}_A = \mathcal{D}_A.$$

The second examples is that of the Rees Weyl algebra and its $\hbar$-adic completion, first defined in Definition (4.B.4).

Example B.2. The *Rees Weyl algebra* $\mathcal{D}_A^\hbar$ associated to $\mathcal{D}_A$ with the Bernstein filtration is

$$\mathcal{D}_A^\hbar = \bigoplus_{n\in\mathbb{N}} \hbar^n F_n\mathcal{D}_A, \tag{1.9}$$

where

$$F_i\mathcal{D}_A = \left\{ \sum_{\substack{m,k\in\mathbb{N}\\ m+k=i}} \sum_{a_1,\ldots,a_m\in A} p^{(k)}_{a_1\cdots a_m}(x_A)\partial_{a_1}\cdots\partial_{a_m} \right\}, \tag{1.10}$$

and the $p^{(k)}_{a_1\cdots a_m}(x_A)$s are polynomials of degree $\leq k$. Finally we also wish to consider differential operators that are formal sums in powers of $\hbar$, and hence we define its $\hbar$-adic completion (See Definition 4.A.7):

$$\widehat{\mathcal{D}}^{\hbar}_A = \varprojlim \hbar^n F_n\mathcal{D}_A. \tag{1.11}$$

An element $P \in \widehat{\mathcal{D}}^{\hbar}_A$ can be written as a formal power series in $\hbar$:

$$P = \sum_{n=0}^{\infty} \hbar^n P_n, \tag{1.12}$$

for some $P_n \in F_n\mathcal{D}_A$.

Our final example is more general and is relevant to a variety of contexts in algebraic geometry and number theory.

Example B.3. Let A be a ring and I an ideal of A. We can put a topology on A, where the basis of the topology is given by the sets of the form $x + I^n, x \in A, n \in \mathbb{N}$. The I-adic completion of A is by definition,

$$\widehat{A^I} = \varprojlim A/I^n \tag{1.13}$$

which is a natural A-algebra. I-adic completions of the Weyl algebra were used in (7.29) of Chapter 7 while defining Weyl algebras constructed from the symplectic boson VOA.