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AMENABLE ACTIONS OF LOCALLY COMPACT GROUPS ON COSET SPACES

by

MAHATHEVA SKANTHARAJAH

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To: My Parents

ABSTRACT

Let G be a locally compact group, H a closed subgroup, and G/H the homogeneous space of left cosets of H in G with the usual quotient topology. Let $L^\infty(G/H,\nu)$ be the Banach space of all essentially bounded measurable functions on G/H with the essential supremum norm, where ν is a quasi-invariant measure on G/H. Let F be a norm closed, conjugate closed subspace of $L^\infty(G/H,\nu)$ containing the constants. A linear functional m on F with $m(1) = \|m\| = 1$ is called a mean. If F is left invariant, then a mean m on F is a left invariant mean [LIM] if $m(\delta_g \square f) = m(f)$ for all $g \in G$, $f \in F$. If F is topologically left invariant, then a mean m on F is topologically left invariant if $m(\phi \square f) = m(f)$ for all $f \in F$, $\phi \in P(G)$. G acts amenably on the coset space G/H if there is a LIM on $L^\infty(G/H,\nu)$. G is amenable if there is a LIM on $L^\infty(G)$. The notion of amenable action on coset space is due to F. Greenleaf and has been studied extensively by, among others, F. Greenleaf and P. Eymard. In this thesis, we examine some additional consequences of the amenable action of G on G/H.

The first part of the thesis deals with a number of characterizations of the existence of a LIM on $L^\infty(G/H,\nu)$. Analogues of Dixmier's criteria, a theorem of M. Day, Day's fixed point theorem, and Silverman's extension theorem are given. It is proved that the amenable action of G on G/H is equivalent to two G-invariant complemented subspace properties. These theorems have been proved by A. Lau for the case of amenable groups. A new proof of a theorem of P. Eymard that the Reiter-Glicksberg geometric property is equivalent to the existence of

a LIM on $L^{\infty}(G/H,\nu)$ is presented. Also given in this chapter is a partial generalization of a characterization of amenable groups due to W. Emerson.

In the second part of the thesis, we establish the existence of a unique LIM on the space WAP(G/H) of all weakly almost periodic functions on G/H and then on AP(G/H), the space of all almost periodic functions on G/H.

In the last part of the thesis, we give a new proof of a characterization of permanently positive sets, essentially due to W. Rudin, based on some ideas of D. Stafney. We apply this result to prove a well known result, proved independently by E. Granirer and W. Rudin, that if G is a non discrete locally compact group which is amenable as a discrete group, then there is a LIM on $L^{\infty}(G)$ which is not topologically left invariant. A similar theorem has been proved by D. Stafney for a non discrete second countable locally compact abelian group.

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CHAPTER I

INTRODUCTION

Let G be a locally compact group, H a closed subgroup, and G/H the homogeneous space of left cosets of H in G with the usual quotient topology. Let $L^{\infty}(G/H,\nu)$ be the Banach space of all essentially bounded measurable functions on G/H with the essential supremum norm, where ν is a quasi-invariant measure on G/H. Let F be a norm closed, conjugate closed subspace of $L^{\infty}(G/H,\nu)$ which contains the constant functions. A linear functional m on F with $m(1) = \|m\| = 1$ is called a mean. If F is left translation invariant, then a mean m on F is a left invariant mean [LIM] if $m(\delta_g \square f) = m(f)$ for all $g \in G$, $f \in F$. If F is topologically left invariant, then a mean m on F is a topological left invariant mean if $m(\phi \square f) = m(f)$ for all $f \in F$, $\phi \in P(G)$. G acts amenably on the coset space G/H, or, the homogeneous space G/H is amenable, if there is a LIM on $L^{\infty}(G/H,\nu)$. G is called amenable if there is a LIM on $L^{\infty}(G)$. If G is amenable when it has the discrete topology, then G is amenable as a discrete group. The notion of amenable actions of a locally compact group G on coset space G/H is due to F. Greenleaf [20] and has been studied extensively by, among others, F. Greenleaf in [20] and P. Eymard in [13]. In this thesis, we examine some additional consequences of the amenable action of G on G/H.

The second chapter contains a summary of the definitions and notations used throughout the thesis. The main contribution of our work is contained in Chapte~~rs III, IV,~~ and V. Each of these chapters begins with an introduction which briefly describes the contents of the chapter.

In Chapter III, we obtain some new characterizations of the amenable action of G on G/H. In particular, we prove analogues of a theorem of J. Dixmier [22, Theorem 17.4], a theorem of M. Day [22, Theorem 17.15], Day's fixed point theorem [6, Theorem 4], and Silverman's extension theorem [41, Theorem 15]. Our fixed point theorem is an improvement of a theorem of P. Eymard [13, p. 12]. A. Lau proved in [26] that a locally compact group G is amenable if and only if any dual Banach space has the weak*G-invariant complemented subspace property [26, Theorem 4.1] and that this is equivalent to the invariantly complemented subspace property (C) [26, Theorem 4.3]. We give generalizations of these two results. A new proof of an interesting theorem of P. Eymard [13, p. 29] that the Reiter-Glicksberg geometric property is equivalent to the existence of a LIM on $L^{\infty}(G/H,\nu)$, and a partial generalization of a theorem of W. Emerson [12, Theorem 1.7] involving geometric and algebraic properties are also presented. The last section contains a number of well known examples of amenable homogeneous spaces and a summary of important results of this chapter.

In Chapter IV, we establish the existence of a unique LIM on the space WAP(G/H) of all weakly almost periodic functions on G/H and hence on the space AP(G/H) of all almost periodic functions on G/H. The necessary background material for this chapter can be found at the beginning of Section 4.2.

In Chapter V, we give a proof of a result of W. Rudin [39, Theorem 2.4] based on some ideas of D. Stafney [42, Section 3] that a

σ-compact locally compact nondiscrete group contains a permanently positive set of finite measure whose complement is also permanently positive. We discuss the relations between permanently positive sets $E \subset G$ and left invariant means on $L^{\infty}(G)$ for a locally compact nondiscrete group which is amenable as a discrete group, and then apply these results to prove a well known result of E. Granirer [18] and W. Rudin [39] that if G is a nondiscrete locally compact group which is amenable as a discrete group, then there is a left invariant mean on $L^{\infty}(G)$ which is not a topological left invariant mean.

CHAPTER II

PRELIMINARIES AND NOTATIONS

2.1 Functions on coset spaces and actions

Let G be a locally compact Hausdorff group, H a closed subgroup, and G/H the homogeneous space of left cosets of H in G with the usual quotient topology. Let π be the natural mapping from G to G/H so that $\pi(g) = gH$. Let λ_G, λ_H be fixed left Haar measures on G, H, respectively, and Δ_G, Δ_H denote the corresponding modular functions. We write $d\lambda_G(g) = dg$, $d\lambda_H(h) = dh$ for brevity. Let $C_{oo}(G/H)$, $C_{oo}(G)$ be the spaces of continuous functions with compact support, $C_o(G/H)$, $C_o(G)$ the spaces of continuous functions vanishing at infinity, and $M(G/H) \cong C_o(G/H)^*$, $M(G) \cong C_o(G)^*$ the Banach spaces of bounded regular Borel measures with the total variation norm. A Borel measure ν on G/H is called *quasi-invariant* if $\nu(E) = 0$ implies $\nu(gE) = 0$ for every measurable subset E of G/H and for every g in G. A Borel measure ν on G/H is *invariant* if $\nu(E) = \nu(gE)$ for every measurable subset E of G/H and for every g in G. Let ν be a quasi-invariant regular measure on G/H determined by a locally λ_G-integrable everywhere positive function ρ on G which satisfies the following relation:

$$\rho(gh) = \rho(g)\Delta_G(h^{-1})\Delta_H(h) \quad \text{for all} \quad g \in G,\ h \in H.$$

Thus,

$$\int_{G/H} d\nu(gH)\int_H f(gh)dh = \int_G f(g)\rho(g)dg, \quad f \in C_o(G)$$

Such a measure always exists and is unique up to equivalence of null sets in the sense that any two quasi-invariant measures on G/H are absolutely continuous with respect to each other. G/H supports an invariant measure if and only if $\Delta_{G|H} = \Delta_H$. If $H = \{e\}$, we always take $\nu = \lambda_G$. For more details on quasi-invariant and invariant measures see [15, p. 257-270], [22, p. 203-215] and [28, p. 237-246].

Let CB(G/H)[CB(G)] denote the Banach space of bounded continuous complex valued functions on G/H [G] with the supremum norm $\|\cdot\|_\infty$. Let $L^\infty(G/H,\nu)\,[L^\infty(G)]$ be the Banach space of all essentially bounded Borel measurable complex valued functions on G/H [G] with the essential supremum norm $\|\cdot\|_{\nu,\infty}[\|\cdot\|_\infty]$. $L'(G/H,\nu)\,[L'(G)]$ will denote the Banach space of integrable Borel functions on G/H [G] with the L'-norm $\|\cdot\|_{\nu,1}[\|\cdot\|_1]$. The usual action of G on G/H induces actions of G and M(G) on certain function spaces on G/H. The action of M(G) on CB(G/H) is defined by

$$\mu \,\square\, f(\xi) = \int_G f(g\xi)d\mu(g), \quad \mu \in M(G), \quad f \in CB(G/H) \tag{1}$$

(see [20, p.296]). The usual action $M(G) \times M(G/H) \to M(G/H)$ is given by

$$< \mu * \eta, f > = \int_{G\times G/H} f(g\xi)d\mu \times \eta(g,\xi), \ \mu \in M(G), \ \eta \in M(G/H), \ f \in C_o(G/H).$$

Then we have

$$(\mu_1 * \mu_2) * \eta = \mu_1 * (\mu_2 * \eta),$$

$$\|\mu * \eta\| \leq \|\mu\| \ \|\eta\| \tag{2}$$

and $M(G/H)$ is a Banach module over $M(G)$. Furthermore, $L'(G/H,\nu)$ becomes a closed $M(G)$-submodule if we embed $L'(G/H,\nu)$ in $M(G/H)$ (see [21, p. 151-173]). The action $M(G) \times L'(G/H,\nu) \to L'(G/H,\nu)$ induces an action of $M(G)$ on $L^\infty(G/H,\nu)$ given by

$$<\mu \,\square\, f,\phi>=<f,\mu * \phi>, \; \mu \in M(G), \; f \in L^\infty(G/H,\nu), \; \phi \in L'(G/H,\nu). \qquad (3)$$

It is shown in [20, p. 298] that formulae (1), (3) are compatible when we embed $CB(G/H)$ in $L^\infty(G/H,\nu)$ via the canonical injection $j : CB(G/H) \to L^\infty(G/H,\nu)$ and that if $f \in L^\infty(G/H,\nu)$ then $\delta_g \,\square\, f(\xi) = f(g\xi)$ locally ν almost everywhere (a.e.) for each $g \in G$, where δ_g is the point mass at $g \in G$. The following properties can be easily verified:

$$\begin{aligned} &(\mu_1 * \mu_2) \,\square\, f = \mu_2 \,\square\, (\mu_1 \,\square\, f), \\ &\|\mu \,\square\, f\|_\infty \le \|\mu\| \; \|f\|_\infty, \quad f \in CB(G/H), \\ &\|\mu \,\square\, f\|_{\nu,\infty} \le \|\mu\| \; \|f\|_{\nu,\infty}, \quad f \in L^\infty(G/H,\nu) \end{aligned} \qquad (4)$$

A function $f \in CB(G/H)$ is called *bounded left uniformly continuous* if the map $g \to \delta_g \,\square\, f$ from G to $(CB(G/H), \|\cdot\|_\infty)$ is continuous. Let $UCB_\ell(G/H)$ denote all such functions. Then $UCB_\ell(G/H)$ includes $C_o(G/H)$ (see [20, p. 299]).

2.2 Invariant means on subspaces of $L^\infty(G/H,\nu)$

Let X be a norm closed conjugate closed linear subspace of $L^\infty(G/H,\nu)$ which contains the constant functions. Then a linear functional m on X is called a *mean* if

(a) $m(\overline{f}) = \overline{m(f)}$ for all $f \in X$

(b) ess inf $f \leq m(f) \leq$ ess sup f for all real-valued $f \in X$

If $X \subseteq CB(G/H)$ then (b) may be replaced by

(b') $\inf f \leq m(f) \leq \sup f$ for all real-valued $f \in X$.

Note that the condition (b) [(b')] is equivalent to:

(c) [(c')] $f \geq 0$ loc. ν - a.e. implies $m(f) \geq 0$ and $m(1) = 1$

[$f \geq 0$ implies $m(f) \geq 0$ and $m(1) = 1$].

If $X \subseteq CB(G/H)$, then each $\xi \in G/H$ gives a mean p_ξ on X defined by

$$p_\xi(f) = f(\xi), \quad f \in X.$$

A mean p of the form $p = \sum_{i=1}^{N} c_i p_{\xi_i}$, where $\xi_i \in G/H$, $c_i \geq 0$, and $\sum_{i=1}^{N} c_i = 1$ is called a *finite mean*. Denote the set of all finite means on X by $\Sigma(X)$. The next proposition is easily proved by slightly modifying the arguments in [34, p. 23-27].

<u>Proposition 2.2.1</u>. Let G be a locally compact group, H a closed subgroup, and X a norm closed, conjugate closed linear subspace of $L^\infty(G/H,\nu)$ which contains the constant functions. Then a linear functional m on X is a mean if and only if $m(1) = \|m\| = 1 \cdot \Sigma(X)$ is a weak*-compact convex set in X^*. Let $P(\nu) = \{\phi \in L'(G/H,\nu) : \phi \geq 0, \|\phi\|_{\nu,1} = 1\}$. Then the means corresponding to $P(\nu)$ form a weak*-dense convex subset of $\Sigma(X)$. Also the finite means on X are weak*-dense in

$\Sigma(X)$ if $X \subseteq CB(G/H)$.

A subspace X of $L^\infty(G/H,\nu)$ is called *left-invariant* if $\delta_g \square f \in X$ for all $g \in G$, $f \in X$. X is called *topologically left invariant* if $\phi \square f \in X$ for all $f \in X$, $\phi \in P(G) = \{\phi \in L'(G) : \phi \geq 0, \|\phi\|_1 = 1\}$. The subspaces $CB(G/H)$, $UCB_\ell(G/H)$ are sup. norm closed, conjugate closed containing the constant functions. Also $L'(G) \square L^\infty(G/H,\nu) \subseteq UCB_\ell(G/H)$ (see [20, p. 306]). For each $g \in G$ we define the left [right] [illegible]tion operator $\ell_g [r_g]$ on $L^\infty(G)$ by

$$\ell_g\, f(a) = f(ga) \; [r_g\, f(a) = f(ag)].$$

A linear subspace X of $L^\infty(G)$ is *left (right) invariant* if $\ell_g f \in X$ $[r_g f \in X]$ for all $g \in G$, $f \in X$.

Definition 2.2.2. Let X be a norm closed conjugate closed left invariant subspace of $L^\infty(G/H,\nu)$ which contains the constant functions. A mean m on X is a *left invariant mean* [LIM] if $m(\delta_g \square f) = m(f)$ for all $g \in G$, $f \in X$. If X is a norm closed, conjugate closed topologically invariant subspace of $L^\infty(G/H,\nu)$ which contains the constant functions, then a mean m on X is a *topological left invariant mean* [TLIM] if $m(\phi \square f) = m(f)$ for all $\phi \in P(G)$, $f \in X$.

Remark 2.2.3. (i) The notion of topological invariance was first introduced by A. Hulanicki in [23]. In [20], P. Greenleaf introduced topological left invariant means on subspaces of $L^\infty(G/H,\nu)$.

(ii) If $H = \{e\}$, then $\phi \square f = (1/\Delta_G)\tilde{\phi} * f$ for $\phi \in P(G)$, $f \in L^\infty(G)$, where $\tilde{\phi}$ is defined by $\tilde{\phi}(g) = \phi(g^{-1})$, $g \in G$. Since $(1/\Delta_G)\tilde{\phi} \in P(G)$ for each $\phi \in P(G)$, (see [46, p. 352]), X is topologically left invariant if and only if $\phi * f \in X$ for all $\phi \in P(G)$, $f \in X$, as defined in [19, p.24].

Theorem 2.2.4 (Greenleaf [20]). Let G be a locally compact group, H a closed subgroup. Then the following six statements are equivalent

1. [1A] There exists a LIM[TLIM] on $UCB_{\ell}(G/H)$
2. [2A] There exists a LIM[TLIM] on $CB(G/H)$
3. [3A] There exists a LIM[TLIM] on $L^{\infty}(G/H,\nu)$.

Remark 2.2.5. Any TLIM on X is a LIM on X for $X = L^{\infty}(G/H,\nu)$, $CB(G/H)$, $UCB_{\ell}(G/H)$ and any LIM on $UCB_{\ell}(G/H)$ is a TLIM (see [20, p.303]). A LIM on a larger space need not be a TLIM (see Section 5.3).

Definition 2.2.6. A net $\{\phi_{\alpha}\} \subset P(\nu)$ *converges weakly* [*strongly*] *to left invariance* if $\delta_g * \phi_{\alpha} - \phi_{\alpha}$ converges to zero for each $g \in G$, in the $\sigma(L^{\infty}(G/H,\nu)^*, L^{\infty}(G/H,\nu))$-topology [$\|\cdot\|_{\nu,1}$-topology]. It is *weakly* [*strongly*] *convergent to topological left invariance* if $\phi * \phi_{\alpha} - \phi_{\alpha}$ converges to zero, in the $\sigma(L^{\infty}(G/H,\nu)^*, L^{\infty}(G/H,\nu)$-topology [$\|\cdot\|_{\nu,1}$-topology], for each $\phi \in P(G)$.

The next three results are stated in [20, p.307] and can be proved exactly as in [19, Sections 2.4 and 3.2]. For the sake of completeness we give the proofs here.

Lemma 2.2.7. Let G be a locally compact group, H a closed subgroup. Then there is a net in $P(\nu)$ weakly convergent to [topological] left invariance if and only if there is a net in $P(\nu)$ strongly convergent to [topological] left invariance.

Proof: If a net in $P(\nu)$ converges strongly to topological left invariance, then trivially it converges weakly to topological left invariance.

Conversely, if $\{\phi_\alpha\} \subset P(\nu)$ converges weakly to topological left invariance, following an idea of Namioka [32], let $E = \pi\{L'(G/H,\nu) : \phi \in P(G)\}$ be the locally convex product space with the product of norm topologies. Define the linear map $T : L'(G/H,\nu) \to E$ by

$$Tf(\phi) = \phi * f - f \quad \text{for all} \quad \phi \in P(G), \quad f \in L^\infty(G/H,\nu).$$

Now the weak topology on E is the product of weak topologies on $L'(G/H,\nu)$, (see [24, p.160]). Since $\phi * \phi_\alpha - \phi_\alpha$ converges to zero in the weak*-topology of $L'(G/H,\nu)$ for each $\phi \in P(G)$, zero lies in the weak closure of $T(P(\nu)) \subseteq E$. Since E is locally convex and $T(P(\nu))$ is a convex set, the weak and norm closures of $T(P(\nu))$ coincide. Therefore, there is some net $\{\psi_\beta\} \subset P(\nu)$ such that $T(\psi_\beta)$ converges to zero in E. That is, $\| \phi * \psi_\beta - \psi_\beta \|_{\nu,1}$ converges to zero for all $\phi \in P(G)$.

The other version is similar. □

Theorem 2.2.8. Let G be a locally compact group, H a closed subgroup. There is a net in $P(\nu)$ weakly convergent to [topological] left invariance if and only if there is a [topological] left invariant mean on $L^\infty(G/H,\nu)$.

Proof: If $\{\phi_\alpha\} \subset P(\nu)$ converges weakly to left invariance, $\{\phi_\alpha\}$ lies within the weak* compact convex set Σ of all means on $L^\infty(G/H,\nu)$.

We may assume, taking a subnet if necessary, that $\{\phi_\alpha\}$ converges to a mean m in the weak*-topology. But then m is a left invariant mean, since $<m,\delta_g \square f> - <m,f> = \lim_\alpha <\phi_\alpha,\delta_g \square f> - \lim_\alpha <\phi_\alpha,f> = \lim_\alpha <\delta_g * \phi_\alpha - \phi_\alpha,f> = 0$ for all $g \in G$. On the other hand, if m is any left invariant mean on $L^\infty(G/H,\nu)$, by Proposition 2.2.1, there is a net $\{\phi_\alpha\} \subset P(\nu)$ such that $\{\phi_\alpha\}$ converges to m in the weak*-topology. Now if $f \in L^\infty(G/H,\nu)$ and $g \in G$ we have $<m,\delta_g \square f> = <m,f>$, so

$$\begin{aligned}\lim_\alpha <\delta_g * \phi_\alpha - \phi_\alpha,f> &= \lim_\alpha <\delta_g * \phi_\alpha,f> - \lim_\alpha <\phi_\alpha,f> \\ &= \lim_\alpha <\phi_\alpha,\delta_g \square f> - \lim_\alpha <\phi_\alpha,f> \\ &= <m,\delta_g \square f> - <m,f> \\ &= <m,f> - <m,f> \\ &= 0.\end{aligned}$$

Thus the net $\{\phi_\alpha\}$ converges weakly to left invariance. The topological version can be proved in a similar way. □

Corollary 2.2.9. Let G be a locally compact group, H a closed subgroup. If $\{\phi_\alpha\} \subset P(\nu)$ converges weakly to [topological] left invariance, then any weak*-limit point of $\{\phi_\alpha\}$ in the set Σ of all means on $L^\infty(G/H,\nu)$ is a [topological] left invariant mean on $L^\infty(G/H,\nu)$. □

Theorem 2.2.10. Let G be a locally compact group, H a closed subgroup. Then there is a net in $P(\nu)$ strongly convergent to [topological] left invariance if and only if there is a [topological] left invariant mean on $L^\infty(G/H,\nu)$.

Proof: Follows from Lemma 2.2.7 and Theorem 2.2.8. $\square$

We close this section with a generalization of a result of H. Reiter [35, p.697]. We need the following definition.

Definition 2.2.11. Let G be a locally compact group, H a closed subgroup. We say that the pair $(G : H)$ *satisfies Reiter's condition* (P_1) if given any $\varepsilon > 0$ and a compact set $K \subset G$ then there is some $\phi \in P(\nu)$ such that $\| \delta_g * \phi - \phi \|_{\nu,1} < \varepsilon$ for all $g \in K$.

The following theorem is stated in [20, p.307] without proof. Our proof follows an idea of A. Hulanicki [23, p.95-97].

Theorem 2.2.12. Let G be a locally compact group, H a closed subgroup. Then there is a LIM on $L^\infty(G/H,\nu)$ if and only if the pair $(G : H)$ has property (P_1).

Proof: If the pair $(G : H)$ has property (P_1), let $D = \{\alpha = (K,\varepsilon) :$ K is a compact set in G and $\varepsilon > 0\}$. Direct D so that $(K,\varepsilon) > (K',\varepsilon')$ if $K \supseteq K'$ and $0 < \varepsilon < \varepsilon'$. For each $\alpha = (K,\varepsilon)$ choose $\phi_\alpha \in P(\nu)$ such that

$$\| \delta_g * \phi_\alpha - \phi_\alpha \|_{\nu,1} < \varepsilon \quad \text{for all} \quad g \in K.$$

Then the net $\{\phi_\alpha\}$ clearly converges strongly to left invariance and hence, by Theorem 2.2.10, there is a LIM on $L^\infty(G/H,\nu)$.

Conversely, if there is a LIM on $L^\infty(G/H,\nu)$, by Theorems 2.2.4 and 2.2.10, there is a net $\{\phi_\alpha\} \subset P(\nu)$ strongly convergent to topological left invariance. Let $\varepsilon > 0$ and compact set $K \subset G$ be given and let β be a fixed element in $P(G)$. Then there exists a compact neighbourhood E of the identity element e in G such that

$$\begin{aligned} &\|\phi_E * \beta - \beta\|_1 < \varepsilon \text{ and} \\ &\|\delta_g * \beta - \beta\|_1 < \varepsilon \text{ for all } g \in E, \end{aligned} \tag{1}$$

where $\phi_E \in P(G)$ is the normalized characteristic function of E (see [22, Theorems 20.4 and 20.27]). Now choose $\{g_1,\ldots,g_N\} \subset G$ so that $K \subseteq \bigcup_{k=1}^{N} g_k E$ and set $\psi_k = \phi_{g_K E} [= \delta_g * \phi_E]$ for $k = 1,\ldots,N$ (assume $g_1 = e$). Since $\{\phi_\alpha\}$ converges strongly to topological left invariance there exists some ϕ_α such that

$$\begin{aligned} &\|\psi_k * \phi_\alpha - \phi_\alpha\|_{\nu,1} < \varepsilon \text{ for } k = 1,\ldots,N \text{ and} \\ &\|\beta * \phi_\alpha - \phi_\alpha\|_{\nu,1} < \varepsilon \end{aligned} \tag{2}$$

Let $\phi = \beta * \phi_\alpha$. Then clearly $\phi \in P(\nu)$. We claim that ϕ is the element we need in (P_1). We will show $\|\delta_{g_i t} * \phi - \phi\|_{\nu,1} < 5\varepsilon$ for $i = 1,\ldots,N$ and $t \in E$. For $t \in E$ we have,

$$\begin{aligned} \|\phi_E * \phi - \phi_t * \phi\|_{\nu,1} &\le \|\phi_E * \phi - \phi\|_{\nu,1} + \|\phi - \delta_t * \phi\|_{\nu,1} \\ &= \|\phi_E * (\beta*\phi_\alpha) - (\beta*\phi_\alpha)\|_{\nu,1} - \|(\beta*\phi_\alpha) - \delta_t * (\beta*\phi_\alpha)\|_{\nu,1} \\ &\le \|\phi_E * \beta - \beta\|_1 + \|\beta - \delta_t * \beta\|, \text{ by (2), Section 2.1} \\ &\quad 2\varepsilon, \text{ by (1).} \end{aligned}$$

This implies that if $t \in E$ and $i = 1,\ldots,N$, then

$$\|\phi_{g_iE} * \phi - \delta_{g_it} * \phi\|_{\nu,1} = \|\delta_{g_i} * (\phi_E * \phi) - \delta_{g_i} * (\delta_t * \phi)\|_{\nu,1}$$

$$= \|\phi_E * \phi - \delta_t * \phi\|_{\nu,1} < 2\varepsilon.$$

Therefore,

$$\|\delta_{g_it} * \phi - \phi\|_{\nu,1} \leq \|\delta_{g_it} * \phi - \phi_{g_iE} * \phi\|_{\nu,1} + \|\phi_{g_iE} * \phi - \phi\|_{\nu,1}$$

$$< 2\varepsilon + \|\phi_{g_iE} * \phi - \phi\|_{\nu,1}$$

$$= 2\varepsilon + \|\phi_{g_iE} * (\beta * \phi_\alpha) - (\beta * \phi_\alpha)\|_{\nu,1}$$

$$\leq 2\varepsilon + \|\phi_{g_iE} * (\beta * \phi_\alpha) - \phi_{g_iE} * \phi_\alpha\|_{\nu,1}$$

$$+ \|\phi_{g_iE} * \phi_\alpha - \phi_\alpha\|_{\nu,1} + \|\phi_\alpha - \beta * \phi_\alpha\|_{\nu,1}$$

$$\leq 2\varepsilon + 2\|\beta * \phi_\alpha - \phi_\alpha\|_{\nu,1} + \|\phi_{g_iE} * \phi_\alpha - \phi_\alpha\|_{\nu,1}$$

$$< 5\varepsilon, \text{ by } (2). \qquad \square$$

CHAPTER III

AMENABLE ACTIONS OF LOCALLY COMPACT GROUPS ON COSET SPACES

3.1 Introduction

In this chapter, we deal with some of the characterizations of amenable actions of locally compact groups on coset spaces. We begin with some basic characterizations of the existence of a LIM on $L^{\infty}(G/H,\nu)$. Theorems similar to 17.4 and 17.15 of [22, p.231 and p.235] are proved in Section 3.2. Also obtained in this section is a partial extension of a result of W. Emerson [12, Theorem 1.7]. In Section 3.3, we prove an analogue of Day's fixed point theorem [6, Theorem 4]. In Section 3.4, an analogue of Silverman's extension theorem [41, Theorem 15] is proved. Section 3.5 deals with invariantly complemented subspaces of $L^{\infty}(G/H,\nu)$ relating to the amenable action of G on G/H. The results obtained in this section are due to A. Lau [26] for the case of amenable locally compact groups. In Section 3.6, we give a new proof of a theorem of P. Eymard [13, p.21-39] that the Reiter-Glicksberg property is equivalent to the existence of a LIM on $L^{\infty}(G/H,\nu)$. The last section contains a number of well known examples of amenable homogeneous spaces and a summary of the results that we obtained in Sections 3.2 through 3.6.

3.2 Invariant means on $L^{\infty}(G/H,\nu)$

Definition 3.2.1. Let G be a locally compact group, H a closed subgroup. We say G *acts amenably on the coset space* G/H, or following [34, p.364], G/H *is amenable,* if there is a left invariant mean on $L^\infty(G/H,\nu)$. Thus G/H is amenable if and only if any one of the six statements of Theorem 2.2.4 holds. G is called *amenable* if $G/\{e\}$ is amenable. G is *amenable as a discrete group* if G is amenable when G has the discrete topology. By Remark 2.2.3(ii), G is amenable if and only if any one of the five properties of [19, Theorem 2.2.1] is satisfied.

Proposition 3.2.2. Let G be a locally compact group, H a closed subgroup, and X a conjugate closed subspace of $L^\infty(G/H,\nu)$. Then the following statements are equivalent.

(i) There is a mean m on $L^\infty(G/H,\nu)$ such that $m(X) = 0$.

(ii) ess inf $h \leq 0$ for all real-valued $h \in X$.

(iii) $\mathrm{dist}(1,X) = 1$.

Proof: (i) $\Rightarrow$ (ii): If m is a mean on $L^\infty(G/H,\nu)$ such that $m(X) = 0$, then $0 = m(h) \geq$ ess inf h for all real-valued $h \in X$.

(ii) $\Rightarrow$ (iii): If $\mathrm{dist}(1,X) < 1$, there is an $f \in X$ such that $\|1 - f\|_{\nu,\infty} < 1$. Then $\|1 - \mathrm{Re}f\|_{\nu,\infty} < 1$. On the other hand, $\|1 - \mathrm{Re}f\|_{\nu,\infty} = \text{ess sup } |1 - \mathrm{Re}f| \geq \text{ess sup}(1 - \mathrm{Re}f) = 1 - \text{ess inf } \mathrm{Re}f \geq 1$, as $\mathrm{Re}f \in X$. This contradiction proves (iii).

(iii) $\Rightarrow$ (i): By the Hahn Banach theorem there exists $m \in L^\infty(G/H,\nu)^*$ such that $m(1) = \|m\| = 1$, and $m(X) = 0$. Then, by Proposition 2.2.1, m is the required mean. $\square$

Definition 3.2.3. Let G be a locally compact group, H a closed subgroup. $\pi_0[\pi_1]$ will denote the subspace of $L^\infty(G/H,\nu)$ spanned by $[\delta_g \square f - f : g \in G,\ f \in L^\infty(G/H,\nu)][\{\phi \square f - f : \phi \in P(G),\ f \in L^\infty(G/H,\nu)\}]$.

Corollary 3.2.4. Let G be a locally compact group, H a closed subgroup. Then the following statements are equivalent

(i) There is a [topological] left invariant mean on $L^\infty(G/H,\nu)$.

(ii) ess inf $h \leq 0$ for all real valued $h \in \pi_0[\pi_1]$.

(iii) $\mathrm{dist}(1,\pi_0) = 1$ $[\mathrm{dist}(1,\pi_1) = 1]$.

Theorem 3.2.5. Let G be a locally compact group, H a closed subgroup. Then there is a [topological] left invariant mean on $L^\infty(G/H,\nu)$ if and only if $\pi_0[\pi_1]$ is not uniformly dense in $L^\infty(G/H,\nu)$.

Proof: If there is a left invariant mean on $L^\infty(G/H,\nu)$, then by Corollary 3.2.4, $\mathrm{dist}(1,\pi_0) = 1$, so π_0 is not uniformly dense in $L^\infty(G/H,\nu)$.

Conversely, if π_0 is not uniformly dense in $L^\infty(G/H,\nu)$, then by the Hahn Banach theorem there exist $\phi \in L^\infty(G/H$ and $f_0 \in L^\infty(G/H,\nu)$ such that $\phi(f_0) = 1$, $\phi(\pi_0) = 0$. Following an idea of Lau [26, p.230], let $\psi = 1/2(\phi + \phi^*)$ where $\phi^*(f) = \overline{\phi(\bar{f})}$ for $f \in L^\infty(G/H,\nu)$. Then ψ is non-zero (since $\psi(f_0) = 1$), self adjoint, and left translation invariant. Write $\psi = \psi^+ - \psi^-$, $\|\psi\| = \|\psi^+\| + \|\psi^-\|$ uniquely (see [44, Theorem 4.2]). Then for each $a \in G$, $\delta_a^*\psi^+$ and $\delta_a^*\psi^-$ are non-negative and have the same norm as ψ^+ and ψ^- respectively. Consequently, assuming without loss of generality, that $\psi^+ \neq 0$, let $m = \psi^+/\psi^+(1)$. Note that $\psi^+(1) \neq 0$ since $\psi^+(f_0) \leq \psi^+(\|f_0\|_{\nu,\infty}1) = \|f_0\|_{\nu,\infty}\, \psi^+(1)$. Then m is a left invariant mean on $L^\infty(G/H,\nu)$ since $\psi^+ = \delta_a^*\psi^+$ for each $a \in G$. Similar arguments apply to the topological version. □

Remark 3.2.6. (i) The above results remain true for the spaces

$UCB_{\ell}(G/H)$, $CB(G/H)$ with ess inf. replaced by inf.

(ii) The equivalence of (i) and (ii) of Proposition 3.2.2 is due to W. Emerson [12, Proposition 2.1] for the case when $H = \{e\}$. The equivalence of (i) and (ii) of Corollary 3.2.4 is an analogue of the von Neumann-Dixmier criteria for amenable groups (see [19, p.25]). Corollary 3.2.4 and Theorem 3.2.5 can be found in [22, Theorem 17.4 and Theorem 17.15] for the case when $H = \{e\}$ and G has the discrete topology. Theorem 17.15 is due to M. Day (see [22, p.245]).

The remainder of this chapter is essentially due to W. Emerson [12] for the case when $H = \{e\}$. Our methods of proofs also follow ideas of his.

<u>Proposition 3.2.7</u>. Let G be a locally compact group, H a closed subgroup. Then $L^{\infty}(G/H,\nu)$ has a subspace π satisfying the following two properties if and only if there is a [topological] left invariant mean on $L^{\infty}(G/H,\nu)$.

(i) $\delta_g \square f - f \in \pi$ for all $f \in L^{\infty}(G/H,\nu)$. $g \in G$.

[(i') $\phi \square f - f \in \pi$ for all $f \in L^{\infty}(G/H,\nu)$, $\phi \in P(G)$].

(ii) ess inf $f \leq 0$ for all real-valued $f \in \pi$.

<u>Proof</u>: If there is a LIM on $L^{\infty}(G/H,\nu)$, let $\pi = \ker m = \{f \in L^{\infty}(G/H,\nu) : m(f) = 0\}$. Then, since $m(\delta_g \square f) = m(f)$ for all $g \in G$, $f \in L^{\infty}(G/H,\nu)$, we have $m(\delta_g \square f - f) = 0$ and consequently $\delta_g \square f - f \in \pi$. Moreover, if $f \in \pi$ is real-valued then ess inf $f \leq m(f) = 0$.

Conversely, assume that there is a subspace π satisfying (i) and (ii). Since (i) gives $\pi_0 \subset \pi$ while (ii) guarantees that ess inf $h \leq 0$ for all real-valued $h \in \pi_0$, there is a LIM on

$L^\infty(G/H,\nu)$ by Corollary 3.2.4. The topological version is similar. □

Definition 3.2.8. Let G be a locally compact group, H a closed subgroup. Let $\pi_p(G/H) = \{f \in L^\infty(G/H,\nu) : \inf\{\|\phi \,\square\, f\|_{\nu,\infty} : \phi \in P(G)\} = 0\}$. Thus, $f \in \pi_p(G/H)$ if and only if zero lies in the norm (weak) closure of the convex set $P(G) \,\square\, f$. Let $\pi_p(G) =$ $\{f \in L^\infty(G) : \inf\{\|\phi * f\|_\infty : \phi \in P(G)\} = 0\} = \{f \in L^\infty(G) : \inf\{\|\phi \,\square\, f\|_\infty : \phi \in P(G)\} = 0\}$ by Remark 2.2.3(ii).

Proposition 3.2.9. Let G be a locally compact group, H a closed subgroup. Then $\pi_p(G/H)$ is closed under multiplication and satisfies criteria (i) and (ii) of Proposition 3.2.7.

Proof: First the closure of $\pi_p(G/H)$ under scalar multiplication is trivial since $\|\phi \,\square\, (cf)\|_{\nu,\infty} = |c|\|\phi \,\square\, f\|_{\nu,\infty}$. Moreover, if $f \in L^\infty(G/H,\nu)$ and $\delta = \text{ess inf } f > 0$, then for any $\phi \in P(G)$

$$\phi \,\square\, f(\xi) = \int_G \phi(g)f(g\xi)dg \geq \int_G \phi(g)\delta dg = \delta \quad \text{so}$$

$$\inf\{\|\phi \,\square\, f\|_{\nu,\infty} : \phi \in P(G)\} \geq \delta > 0, \text{ verifying (ii).}$$

Finally, for any fixed positive integer n, $a \in G$, and any subset E of G of finite positive Haar measure $\lambda_G(E)$, let $\phi_n =$ $\frac{1}{n\,\lambda_G(E)} \sum_{k=1}^{n} 1_{a^kE}$, where 1_A is the characteristic function of $A \subset G$ and consider $f_n = \phi_n \,\square\, (\delta_a \,\square\, f - f)$. Then,

$$f_n(\xi) = \int_G \phi_n(g)(\delta_a \,\square\, f - f)(g\xi)dg$$

$$= \int_G \phi_n(g)(f(ag\xi) - f(g\xi))dg$$

$$= \frac{1}{n\lambda_G(G)} \sum_{k=1}^{n} \int_{a^kE} (f(ag\xi) - f(g\xi))dg$$

$$= \frac{1}{n\lambda_G(E)} (\int_{a^{n+1}E} f(g\xi)dg - \int_{aE} f(g\xi)dg),$$

since $\int_E f(ag\xi)dg = \int_{aE} f(g\xi)dg$. Therefore, $\|f_n\|_{\nu,\infty} \leq (2/n)\|f\|_{\nu,\infty}$, so $\lim \|f_n\|_{\nu,\infty} = 0$. Hence $\delta_a \square f - f \in \pi_p(G/H)$ and (i) is also verified. □

Corollary 3.2.10. The coset space G/H is amenable if $\pi_p(G/H)$ is closed under addition.

Remark 3.2.11. By proposition 3.2.2, $L^\infty(G/H,\nu)$ has a conjugate closed subspace π satisfying (i) [(i')] and (ii) of Proposition 3.2.7 if and only if there is a [topological] left invariant mean m on $L^\infty(G/H,\nu)$ such that $m(\pi) = 0$. Thus, since $\pi_p(G/H)$ is conjugate closed, there is LIM m on $L^\infty(G/H,\nu)$ such that $m(\pi_p(G/H)) = 0$ if $\pi_p(G/H)$ is closed under addition.

Definition 3.2.12. Let G be a locally compact group, H a closed subgroup. The pair (G : H) is said to have the property (B): if $d(\phi_1 * P(\nu),\phi_2 * P(\nu)) = \inf\{\|\phi_1 * \psi_1 - \phi_2 * \psi_2\|_{\nu,1} : \psi_1,\psi_2 \in P(\nu)\} = 0$ for any $\phi_1,\phi_2 \in P(G)$. If $H = \{e\}$, property (B) becomes $d(\phi_1 * P(G),\phi_2 * P(G)) = \inf\{\|\phi_1 * \psi_1 - \phi_2 * \psi_2\|_1 : \psi_1,\psi_2 \in P(G)\} = 0$ for any $\phi_1,\phi_2 \in P(G)$.

Proposition 3.2.13. If G/H is amenable, then the pair (G : H) has property (B).

Proof: If G/H is amenable, then by Theorem 2.2.10, there is a net $\{\psi_\alpha\}$ in $P(\nu)$ strongl convergent to topological left invariance. This implies that $\lim_\alpha \|\phi_1 * \psi_\alpha - \phi_2 * \psi_\alpha\|_{\nu,1} = 0$ for any $\phi_1, \phi_2 \in P(G)$, so $d(\phi_1 * P(\nu), \phi_2 * P(\nu)) = 0$. That is, $(G : H)$ has the property (B). □

Proposition 3.2.14. Let G be a locally compact group, H a closed subgroup. If $d(\phi_1 * P(G), \phi_2 * P(G)) = 0$ for any $\phi_1, \phi_2 \in P(G)$, then $\pi_p(G/H)$ is closed under addition.

Proof: Fix f_1, $f_2 \in \pi_p(G/H)$ and choose $\phi_{i,n} \in P(G)$ $(i = 1,2)$ such that $\|\phi_{i,n} \Box f_i\|_{\nu,\infty} < 1/n$ $(i = 1,2)$ for each positive integer n. Next choose $\psi_{1,n}$ and $\psi_{2,n}$ in $P(G)$ such that

$$\|\phi_{1,n} * \psi_{1,n} - \phi_{2,n} * \psi_{2,n}\|_{\nu,1} < 1/n.$$

Let $\Phi_n = \phi_{1,n} * \psi_{1,n}$. Then, $\Phi_n \in P(G)$ and

$$\begin{aligned} \Phi_n \Box (f_1 + f_2) &= (\phi_{1,n} * \psi_{1,n}) \Box (f_1 + f_2) \\ &= (\phi_{1,n} * \psi_{1,n}) \Box f_1 + (\phi_{1,n} * \psi_{1,n}) \Box f_2 \\ &= \psi_{1,n} \Box (\phi_{1,n} \Box f_1) + \psi_{1,n} \Box (\phi_{1,n} \Box f_2) \end{aligned}$$

(see Equations 4, Section 2.1). Therefore,

$$\begin{aligned} \phi_n \Box (f_1 + f_2) = \psi_{1,n} \Box (\phi_{1,n} \Box f_1) &+ (\phi_{1,n} * \psi_{1,n} - \phi_{2,n} * \psi_{2,n}) \Box f_2 \\ &+ \psi_{2,n} \Box (\phi_{2,n} \Box f_2). \end{aligned}$$

So, $\|\Phi_n \Box (f_1 + f_2)\|_{\nu,\infty} \le \frac{1}{n}(2 + \|f_2\|_{\nu,\infty})$. Thus, $\lim_{n\to\infty} \|\Phi_n \Box (f_1 + f_2)\|_{\nu,\infty} = 0$, and consequently $f_1 + f_2 \in \pi_p(G/H)$. □

Theorem 3.2.15 (Emerson [12]. If G is a locally compact group, then the following statements are equivalent.

(a) G is amenable.

(b) $\pi_p(G)$ is closed under addition.

(c) $d(\phi_1 * P(G), \phi_2 * P(G)) = 0$ for any $\phi_1, \phi_2 \in P(G)$. That is, G has property (B).

Proof: (c) $\Rightarrow$ (b) is Proposition 3.2.14; (b) $\Rightarrow$ (a) is Corollary 3.2.10; and (a) $\Rightarrow$ (c) is Proposition 3.2.13. □

Remark 3.2.16. (i) We do not know if the converse of Proposition 3.2.13 or Corollary 3.2.10 is true when $H \neq \{e\}$.

(ii) Results 3.2.7 through 3.2.15 remain true for the spaces $UCB_\ell(G/H)$, $CB(G/H)$ with ess inf replaced by inf etc.

(iii) Let S be the subspace of $L^\infty(G/H,\nu)$ consisting of all simple functions on G/H, and π_o' the subspace spanned by $\{\delta_g \square f - f ; g \in G, f \in S\}$. Then any LIM on S has an extension to a LIM on $L^\infty(G/H,\nu)$. Thus by Proposition 3.2.2, there is a LIM on $L^\infty(G/H,\nu)$ if and only if ess inf $h \leq 0$ for all real-valued $h \in \pi_o'$. Therefore, Proposition 3.2.7 remains true even if we consider a subspace π of S instead of a subspace of $L^\infty(G/H,\nu)$. This implies that the results which follow Proposition 3.2.7 remain valid if we define $\pi_p(G/H) = \{f \in S : \inf\{\|\phi \square f\|_{\nu,\infty} : \phi \in P(G)\} = 0\}$. However, we cannot replace ess inf by inf in Proposition 3.2.7 as in [12, Propositional 1.1], since we identify two measurable functions when they coincide off a locally null set.

3.3 Fixed point property

Definition 3.3.1. Let G be a locally compact group, H a closed subgroup, and S a topological space. We say that G *acts on* S if there is a map: $G \times S \to S$, denoted by $(g,s) \to g \cdot s$, such that

(i) $g_1 \cdot (g_2 \cdot s) = (g_1 g_2) \cdot s$ for all $g_1, g_2 \in G$, $s \in S$.

(ii) if e is the identity element of G, then $e \cdot s = s$ for each $s \in S$.

(iii) the map $s \to g \cdot s$ of S into itself is continuous for each $g \in G$.

Note that $s \to g \cdot s$ is a homeomorphism of S into S for each $g \in G$. By a fixed point for $G[H]$ we mean a point $s_0 \in S$ such that $g \cdot s_0 = s_0$ for each $g \in G[h \cdot s_0 = s_0$ for each $h \in H]$. If S is a convex subset of a locally convex linear topological space $[1 \cdot c \cdot s]$, we say G *acts affinely on* S if G acts on S and the map $s \to g \cdot s$ is affine for each $g \in G$.

The fixed point theorem of P. Eymard [13, p.12] and Theorem 2.2.4 imply amenability of a coset space G/H is equivalent to each of the following fixed point properties.

(F_1): if whenever G acts affinely on a convex compact set S in a $1 \cdot c \cdot s$ E with the map $(g,s) \to g \cdot s$ jointly continuous and there is a fixed point for H then there is a fixed point for G.

(F_2): if whenever G acts affinely on a convex compact set S in a $1 \cdot c \cdot s$ E with the map $(g,s) \to g \cdot s$ separately continuous and there is a fixed point for H, then there is a fixed point for G.

The following theorem is an analogue of Day's fixed point theorem [6, Theorem 4].

Theorem 3.3.2. A coset space G/H is amenable if and only if the pair $(G:H)$ has the fixed point property (F_3): if whenever G acts affinely on a compact convex set S in a $l \cdot c \cdot s$ E and there is a fixed point $s_o \in S$ for H with the map $g \to g \cdot s_o$ continuous, then there is a fixed point for G.

Proof: If the pair $(G:H)$ has the fixed point property (F_3), then it has the fixed point property (F_1). Hence G/H is amenable.

Conversely, if G acts affinely on a compact convex set S in a $l \cdot c \cdot s$ E and there is a fixed point $s_o \in S$ for H with the map $g \to g \cdot s_o$ continuous, let m be any mean on $CB(G/H)$. Let $A(S)$ be the Banach space of all affine continuous functions on S. Thus $\{\phi|_S : \phi \in E^*\} \subseteq A(S)$. For each $\phi \in A(S)$, define a map $f_\phi : G/H \to \mathbb{C}$ by

$$f_\phi(gH) = \langle \phi, g \cdot s_o \rangle$$

Then clearly f_ϕ is continous. Since S is compact f_ϕ is bounded Next define a function $m(s_o)$ on $A(S)$ by

$$\langle m(s_o), \phi \rangle = \langle m, f_\phi \rangle, \quad \phi \in A(S).$$

Let $\alpha, \beta \in \mathbb{C}$, $\phi, \psi \in A(S)$. Then, $f_{\alpha\phi+\beta\psi}(gH) = \langle \alpha\phi + \beta\psi, g \cdot s_o \rangle = \alpha \langle \phi, g \cdot s_o \rangle + \beta \langle \psi, g \cdot s_o \rangle = (\alpha f_\phi + \beta f_\psi)(gH)$. Thus, $f_{\alpha\phi+\beta\psi} = \alpha f_\phi + \beta f_\psi$. Therefore $\langle m(s_o), \alpha\phi + \beta\psi \rangle = \langle m, f_{\alpha\phi+\beta\psi} \rangle = \langle m, \alpha f_\phi + \beta f_\psi \rangle = \alpha \langle m, f_\phi \rangle + \beta \langle m, f_\psi \rangle = \alpha \langle m(s_o), \phi \rangle + \beta \langle m(s_o), \psi \rangle$, so $m(s_o)$ is linear. Also $|\langle m(s_o), \phi \rangle| = |\langle m, f_\phi \rangle| \leq \sup_{g \in G} |f_\phi(gH)| = \sup_{g \in G} |\langle \phi, g \cdot s_o \rangle| \leq \sup_{s \in S} |\langle \phi, s \rangle| < \infty$, as S compact. Hence $m(s_o)$ is bounded. Therefore the restriction map $m(s_o)|_{E^*}$ is in E**. Let Q be the cannonical mapping of E into E** given by $Q(x)(\phi) = \langle\phi,x\rangle$, $x \in E$, $\phi \in E^*$. First we show that there exists $s_{oo} \in S$ such that $Q(s_{oo}) = m(s_o)|_{E^*}$. Then we will prove that s_{oo} is a fixed point for G if we take m to be a left invariant mean on CB(G/H). Let $p = \sum_{i=1}^{N} c_i p_{\xi_i}$ be a finite mean, where $\xi_i = g_iH \in G/H$, $c_i \geq 0$ and $\sum_{i=1}^{N} c_i = 1$. Then $\langle p(s_o),\phi\rangle = \langle p,f_\phi\rangle = \langle \sum_{i=1}^{N} c_i p_{\xi_i}, f_\phi\rangle = \sum_{i=1}^{N} c_i f_\phi(g_iH) = \sum_{i=1}^{N} c_i \langle\phi, g_i \cdot s_o\rangle = \langle \phi, \sum_{i=1}^{N} c_i g_i \cdot s_o\rangle$ for each $\phi \in A(S)$. Hence, if $\phi \in E^*$, then $\langle p(s_o),\phi\rangle = \langle \phi, \sum_{i=1}^{N} c_i g_i \cdot s_o\rangle = \langle Q(\sum_{i=1}^{N} c_i g_i \cdot s),\phi\rangle$ so $p(s_o)|_{E^*} = Q(\sum_{i=1}^{N} c_i g_i \cdot s_o) \in Q(S)$. Let $\{p_j\}$ be a net of finite means converging to m in the weak* topology. Such a net exists by Proposition 2.2.1. Then for each j there exists $s_j \in S$ such that $p_j(s_o)|_{E^*} = Q(S_j) \in Q(S)$. We may assume, passing through a subnet if necessary, that $\{S_j\}$ converges to some $s_{oo} \in S$, as S is compact. Then $\langle m(s_o),\phi\rangle = \langle m,f_\phi\rangle = \lim_j \langle p_j,f_\phi\rangle = \lim_j \langle p_j(s_o),\phi\rangle = \lim_j \langle\phi,S_j\rangle = \langle\phi,s_{oo}\rangle = \langle Q(s_{oo}),\phi\rangle$ for all $\phi \in A(S)$, in particular for all $\phi \in E^*$. Thus $Q(s_{oo}) = m(s_o)|_{E^*}$. If $\phi \in E^*$ write $\phi \,\square\, a(s) = \langle\phi,a \cdot s\rangle$ for $s \in S$, $a \in G$. Then $\phi \,\square\, a$ is clearly in A(S). For each $\phi \in A(S)$, $a \in G$ we have $f_{\phi \square a}(gH) = \langle\phi \,\square\, a, g \cdot s_o\rangle = \langle\phi, (ag) \cdot s_o\rangle = f_\phi(agH) = \delta_a \,\square\, f(gH)$, so $f_{\phi \square a} = \delta_a \,\square\, f$. Thus, if m is a LIM, then $\langle\phi,g \cdot s_{oo}\rangle = \langle \phi \,\square\, g,s_{oo}\rangle = \langle m(s_o),\phi \,\square\, g\rangle = \langle m,f_{\phi \square g}\rangle = \langle m,\delta_g \,\square\, f_\phi\rangle = \langle m,f_\phi\rangle = \langle m(s_o),\phi\rangle = \langle Q(s_o),\phi\rangle = \langle\phi,s_{oo}\rangle$ for all $g \in G$, $\phi \in E^*$. Hence, $g \cdot s_{oo} = s_{oo}$ for each $g \in G$, so s_{oo} is a fixed point for G. □

3.4 Hahn Banach Extension Property

Definition 3.4.1. Let G be a locally compact group, H a closed subgroup. A *representation* T of G on a vector space E is a homomorphism of G into the semigroup of linear transformations on E. *An anti representation* T of G on E is an anti homomorphism of G into the semigroup of linear transformations on E. We usually denote a representation or an anti representation by its image $\{T_g : g \in G\}$.

The pair $(G : H)$ is said to have the *Hahn Banach Extension Property (HBEP)* if whenever $\{T = T_g : g \in G\}$ is an anti representation of G on a vector space E, p a seminorm of E, F a subspace of E, and ϕ a linear functional on F such that

(i) $T_g F \subseteq F$ for each $g \in G$

(ii) $p(T_g\ x) \leq p(x)$ for all $g \in G,\ x \in E$

(iii) $\phi(T_g\ y) = \phi(y)$ for all $y \in F,\ g \in G$

(iv) $|\phi(y)| \leq p(y)$ for each $y \in F$

(v) ϕ has an extension to a linear functional ψ on E such that $|\psi(x)| \leq p(x)$ for each $x \in E$, $\psi(T_h(x)) = \psi(x)$ for all $x \in E,\ h \in H$, and the function $\psi_x : G/H \to \mathbb{C}$ defined by $\psi_x(gH) = \psi(T_g\ x)$ is continuous for each $x \in E$ then ϕ has an extension to a linear functional Φ on E such that (a) $|\Phi(x)| \leq p(x)$ for each $x \in E$ (b) $\Phi(T_g\ x) = \Phi(x)$ for all $g \in G,\ x \in E$.

The following theorem is an analogue of a result of R.J. Silverman [41, Theorem 15]. For a generalized version of Silverman's extension theorem see [14, Theorem 3.2.3].

Theorem 3.4.2. Let G be a locally compact group, H a closed subgroup. Then the coset space G/H is amenable if and only if the pair $(G : H)$

has the property (HBEP).

Proof: If G/H is amenable, let m be a left invariant mean on $CB(G/H)$. Let E be a vector space, p a seminorm on E, $T = \{T_g : g \in G\}$ an anti representation of G on E, F a subspace and ϕ a linear functional on F such that conditions (i)-(v) of Definition 3.4.1 hold. Let $E^{\#}$ be the algebraic dual of E. Let $\Sigma = \{\psi \in E^{\#} : \psi|_F = \phi,$ $|\psi(x)| \leq p(x)$ for each $x \in E\}$. Then Σ is nonempty by (v), and is a convex $\sigma(E^{\#},E)$-compact set in $E^{\#}$ (see [9, V.4.1]). Define $g \cdot \psi$ so $g \cdot \psi(x) = \langle \psi, T_{g^{-1}} x \rangle$, $\psi \in \Sigma$, $g \in G$, $x \in E$. Then $G \times \Sigma \to \Sigma$ is an affine action. Let $\Psi \in E^{\#}$ satisfy the hypothesis (v). For each $h \in H$ we have $h \cdot \Psi(x) = \langle \Psi, T_{h^{-1}} x \rangle = \Psi(x)$, so $\Psi \in \Sigma$ is a fixed point for H. If $\{g_\alpha\}$ is a net in G converging to g in G then $\langle g_\alpha \cdot \Psi, x \rangle = \langle \Psi, T_{g_\alpha^{-1}} x \rangle$, and $\lim_\alpha \langle \Psi, T_{g_\alpha^{-1}} x \rangle = \langle \Psi, T_{g^{-1}} x \rangle =$ $\langle g \cdot \Psi, x \rangle$ for each $x \in E$ by (v). Hence the map $g \to g \cdot \Psi$ is continuous. Therefore, by Theorem 3.3.2, there exists $\Phi \in \Sigma$ such that $g \cdot \Phi = \Phi$ for each $g \in G$. Hence Φ is an extension of ϕ such that

(a) $|\Phi(x)| \leq p(x)$ for each $x \in E$.

(b) $\Phi(T_g x) = \Phi(x)$ for all $g \in G$, $x \in E$.

Conversely, if the pair $(G : H)$ has the property (HBEP), let $E = CB(G/H)$, $F = \{\alpha 1 : \alpha \in \mathbb{C}\}$, $T = \{\delta_g : g \in G\}$, and $p = \|\cdot\|_\infty$. Define a linear functional ϕ on E by $\phi(\alpha 1) = \alpha$. Let ψ be the linear functional on E defined by $\psi(f) = f(\xi_o)$ for $f \in E$, where $\xi_o = H$. Then ψ extends ϕ, $\|\psi\| = \|\phi\| = \phi(1) = 1$, and $\langle \psi, \delta_h \square f \rangle = \langle \psi, f \rangle$ for all $h \in H$, $f \in E$. If $\{g_\alpha\}$ is a net in G converging to g in G, then $\psi_f(g_\alpha H) = \psi(\delta_{g_\alpha} \square f) = \delta_{g_\alpha} \square f(\xi_o) = f(g_\alpha \xi_o)$ and $\lim_\alpha f(g_\alpha \xi_o) =$

$f(g\xi_o) = <\psi, \delta_g \square f> = \psi_f(gH)$ for $f \in E$. Thus the mapping ψ_f is continuous for each $f \in E$. Thus, the conditions (i) through (v) of the (HBEP) are satisfied. Therefore, ϕ extends to a linear functional Φ on E such that $|\Phi(f)| \leq \|f\|$ for each $f \in E$ and $\Phi(\delta_g \square f) = \Phi(f)$ for all $g \in G$, $f \in E$. Then Φ is a left invariant mean on $E = CB(G/H)$, so G/H is amenable. □

Remark 3.4.3. Let E be a vector space, p a seminorm on E, $T = \{T_g : g \in G\}$ an anti representation of G on E, F a subspace of E and ϕ a linear functional on F such that the conditions of the (HBEP) are satisfied.

(i) If E is a linear topological space and p is continuous, then Φ is continuous.

(ii) If E is a normed linear space and $p(x) = \|\phi\|\|x\|$ for all $x \in E$, then there exists a linear extension Φ of ϕ such that $\|\Phi\| = \|\phi\|$ and $\Phi(T_g\ x) = \Phi(x)$ for all $x \in E$, $g \in G$.

3.5 Invariantly complemented subspace property

Definition 3.5.1. Let G be a locally compact group, H a closed subgroup. Let A be a closed left invariant subspace of $L^\infty(G/H,\nu)$ and X a weak*-closed left invariant subspace of $L^\infty(G/H,\nu)$ contained in A. We say X *is H-invariantly complemented in* A if there exists a continuous projection P from A onto X such that $P\delta_h = \delta_h P$ for each $h \in H$. X is *invariantly complemented* if $P\delta_g = \delta_g P$ for each $g \in G$.

H. Rosenthal proved in [38, Theorem 1.1] that if G is abelian and X is complemented in $L^\infty(G)$, then X is invariantly complemented. Actually, Rosenthal's proof is valid for any locally compact group which is amenable as a discrete group. Recently A. Lau extended this result to a large class of locally compact groups in [26, Theorem 4.3] The next theorem is a generalization of Lau's result.

Theorem 3.5.2. Let G be a locally compact group, H a closed subgroup. Then G/H is amenable if and only if the pair (G : H) has the following property (C). Whenever $T = \{T_g : g \in G\}$ is a representation of G as weak*-weak* continuous linear isometries from a dual Banach space E onto E and A is a closed left invariant subspace of E such that the map $(g,x) \to T_g x$ is a continuous linear map of the product subspace $G \times A$ into A, if X is a weak*-closed left invariant subspace of E contained in A and if there exists a continuous projection Q from A onto X such that $QT_h = T_hQ$ for each $h \in H$ with $\|Q\| \leq \alpha$, then there is a continuous projection P from A onto X such that $PT_g = T_gP$ for each $g \in G$ with $\|P\| \leq \alpha$.

Proof: If the coset space G/H is amenable, let m be a left invariant mean on CB(G/H). Let $T = \{T_g : g \in G\}$ be a representation of G as weak*-weak* continuous linear isometries from a dual Banach space E onto itself and A a closed left invariant subspace of E such that the map $(g,x) \to T_g x$ is a continuous linear map of the product subspace $G \times A$ into A. Let X be a weak*-closed left invariant subspace of E contained in A and Q a continuous projection from A onto X such that $QT_h = T_hQ$ for each $h \in H$ with $\|Q\| \leq \alpha$. Let E_* be a fixed predual of E. Now fix $x \in E$. Define a mapping

$f_{x,\phi} : G/H \to \mathbb{C}$ by $f_{x,\phi}(gH) = \langle T_g Q T_{g^{-1}} x, \phi \rangle$ for each $\phi \in E_*$. Then $f_{x,\phi}$ is clearly bounded since $|f_{x,\phi}(gH)| \leq |\langle T_g Q T_{g^{-1}} x, \phi \rangle| \leq \alpha \|x\| \|\phi\|$. Let $\{g_\alpha\}$ be a net in G converging to g in G. Then $\lim_\alpha |f_{x,\phi}(g_\alpha H) - f_{x,\phi}(gH)| \leq \|\phi\| \lim_\alpha \|T_{g_\alpha} Q T_{g_\alpha^{-1}} x - T_g Q T_{g^{-1}} x\| = 0$. So $f_{x,\phi}$ is continuous, and therefore $f_{x,\phi} \in CB(G/H)$. Now, for $\alpha, \beta \in \mathbb{C}$, $\phi, \psi \in E_*$ we have $f_{x,\alpha\phi+\beta\psi} = \alpha f_{x,\phi} + \beta f_{x,\psi}$. Thus $\phi \to \langle m, f_{x,\phi} \rangle$ is a bounded linear functional on E_*, of norm $\leq \alpha \|x\|$. Hence, there is a unique $Px \in E$ such that $\langle Px, \phi \rangle = \langle m, f_{x,\phi} \rangle$ for all $x \in A$, $\phi \in E_*$. P is a bounded linear operator from A into E with $\|P\| \leq \alpha$. To show that P is a projection from A onto X it suffices to show that (i) range $P \subseteq X$, and that (ii) if $x \in X$, then $Px = x$. (i) range $P \subseteq X$: Let $X_\perp = \{\phi \in E_* : \langle \phi, x \rangle = 0$ for all $x \in X$ Then $(X_\perp)^\perp = \{x \in E : \langle \phi, x \rangle = 0$ for each $\phi \in X_\perp\} = X$, since X is weak*-closed. Suppose $\phi \in X_\perp$. Then $f_{x,\phi} = 0$, as $T_g Q T_{g^{-1}} x \in X$ for all $g \in G$, $x \in X$. Thus $\langle Px, \phi \rangle = 0$ for each $\phi \in X_\perp$, so $Px \in X$. (ii) $Px = x$ if $x \in X$: If $x \in X$, then $T_g Q T_{g^{-1}} x = x$. Hence, $\langle Px, \phi \rangle = \langle m, f_{x,\phi} \rangle = \langle m, \langle \phi, x \rangle 1 \rangle = \langle \phi, x \rangle$ for all $\phi \in E_*$. Thus $Px = x$. To show P is invariant, fix $g \in G$, $x \in A$, $\phi \in E_*$. Since T_g is weak*-weak* continuous, there is a bounded linear operator S_g from E_* into itself such that $T_g = S_g^*$ (see [42, Lemma 5.14]). $\langle T_g Px, \phi \rangle = \langle Px, S_g \phi \rangle$, so $\langle T_g Px, \phi \rangle = \langle m, f_{x,S_g\phi} \rangle$. But $f_{x,T_g\phi}(aH) = \langle T_a Q T_{a^{-1}} x, S_g \phi \rangle = \langle T_g T_a Q T_{a^{-1}} x, \phi \rangle = \langle T_{ga} Q T_{a^{-1}g^{-1}} (T_g x), \phi \rangle = \langle T_{ga} Q T_{(ga)^{-1}} x, \phi \rangle = f_{T_g x, \phi}(gaH) = \delta_g \square f_{T_g x, \phi}(aH)$. Therefore, by the invariance of m, $\langle T_g Px, \phi \rangle = \langle m, f_{x,S_g\phi} \rangle = \langle m, \delta_g \square f_{T_g x,\phi} \rangle = \langle m, f_{T_g x, \phi} \rangle = \langle PT_g x, \phi \rangle$, so $PT_g = T_g P$ for all $g \in G$.

Conversely, if the pair $(G:H)$ has the property (C), consider the representation $= \{\delta_{g-1} : g \in G\}$ of G on $L^\infty(G/H,\nu)$ and the closed subspaces $A = UCB_\ell(G/H)$, $X = \{\lambda 1 : \lambda \in \mathbb{C}\}$. Let $\{g_\alpha\}$ be a net in G converging to $g \in G$, and $\{f_\alpha\}$ a net in $UCB_\ell(G/H)$ converging to $f \in UCB_\ell(G/H)$. Then, $\lim_\alpha \|\delta_{g_\alpha} \square f_\alpha - \delta_g \square f\|_\infty =$ $\lim_\alpha \|\delta_{g_\alpha} \square f_\alpha - \delta_{g_\alpha} \square f + \delta_{g_\alpha} \square f - \delta_g \square f\|_\infty \leq \lim_\alpha \|f_\alpha - f\|_\infty +$ $\lim_\alpha \|\delta_{g_\alpha} \square f - \delta_g \square f\|_\infty = 0$. Hence A and X satisfy the conditions of (C). Let Q be the projection of A onto X defined by $Q(f) = f(\xi_o)1$, where $\xi_o = H$. Then $\|Q\| = 1$ and $Q\delta_h = \delta_h Q$ for each $h \in H$. Hence there is a continuous projection P from $UCB\,(G/H)$ onto X commuting with each δ_g, $g \in G$, and $\|P\| \leq 1$. Define $m(f) = P(f)(\xi_o)$ for each $f \in UCB\,(G/H)$. Then, $m(1) = \|m\| = 1$ and $m(\delta_g \square f) = m(f)$ for all $f \in A$, $g \in G$. Thus G/H is amenable by Theorem 2.2.4. □

Corollary 3.5.3. Let G be a locally compact group, H a closed subgroup. Then G/H is amenable if and only if every weak* closed left invariant subspace of $L^\infty(G/H,\nu)$ which is contained and H-invariantly complemented in $UCB_\ell(G/H)$ is invariantly complemented in $UCB_\ell(G/H)$.

Definition 3.5.4. Let E be a dual Banach space with a fixed predual E_*. We say that E has the *weak* G-invariant complemented subspace property with respect to* H if the following property holds:

Whenver $T = \{T_g : g \in G\}$ is a representation of G as linear isometries from E onto E such that the map $(g,x) \to T_g x$ is a separately continuous map from $G \times E$ into E when E has the weak* topology if X is a weak*-closed left invariant subspace of E and if there exists a [illegible]*-weak* continuous projection Q form E onto

X such that $QT_h = T_hQ$ for each $h \in H$ with $\|Q\| \leq \alpha$, then there exists a continuous projection P of E onto X such that $PT_g = T_gP$ for each $g \in G$ with $\|P\| \leq \alpha$.

The following theorem is due to A. ~~Lau~~ [26, Theorem 4.1] for the case when $H = \{e\}$.

Theorem 3.5.5. Let G be a locally compact group, H a closed subgroup. If the coset space G/H is amenable, then any dual Banach space has the weak* G-invariant complemented subspace property with respect to H. Conversely, if $L^\infty(G/H,\nu)$ has the weak* G-invariant complemented subspace property with respect to H and there is a $\phi \in P(\nu)$ such that $\delta_h * \phi = \phi$ for each $h \in H$, then G/H is amenable.

Proof: If G/H is amenable, let m be a LIM on $CB(G/H)$. Let $T = \{T_g : g \in G\}$ be a representation of G as linear isometries from a dual Banach space E onto itself such that the map $(g,x) \to T_g x$ is a separately continuous linear map when E has the weak*-topology. Let X be a weak*-closed left invariant subspace of E and Q a weak*-weak* continuous projection of E onto X such that $QT_h = T_hQ$ for each $h \in H$ with $\|Q\| \leq \alpha$. Fix $x \in E$. For each $\phi \in E_*$ define the mapping $f_{x,\phi} : G/H \to \mathbb{C}$ by $f_{x,\phi}(gH) = \langle T_gQT_{g^{-1}}x,\phi\rangle$. Then $f_{x,\phi}$ is clearly bounded since $|f_{x,\phi}(gH)| = |\langle T_gQT_{g^{-1}}x,\phi\rangle| \leq \alpha\|x\|\|\phi\|$. Let $\Phi : G \times B \to B$ be the mapping given by $(g,y) \to T_g\, y$, where $B = \{y \in E : \|y\| \leq \alpha\|x\|\}$. Since Φ is separately continuous and B is weak*-compact, by Ellis' theorem [11, Theorem 1], Φ is jointly continuous. If $\{g_\alpha\}$ is a net in G converging to g in G, then

$\{QT_{g_\alpha^{-1}}x\}$ is a net in B converging in the weak*-topology to $QT_g\,x$ in B. Thus, $\{T_{g_\alpha}QT_{g_\alpha^{-1}}x\}$ converges to $T_gQT_{g^{-1}}x$ in the weak*-topology also. Hence $f_{x,\phi}$ is in CB(G/H). The function $\phi \to \langle m, f_{x,\phi}\rangle$ is a bounded linear functional on E_*, of norm $\leq \alpha\|x\|$. The remainder of the proof is exactly the same as Theorem 3.5.2. We safely omit the details.

Conversely, if $L^\infty(G/H,\nu)$ has the weak* G-invariant complemented subspace property with respect to H and there is a $\phi \in P(\nu)$ such th $\delta_h * \phi = \phi$ for each $h \in H$, let $E = L^\infty(G/H,\nu)$, $T = \{\delta_{g^{-1}} : g \in G\}$, and $X = \{\lambda 1 : \lambda \in \mathbb{C}\}$. Let Q be the projection of E onto X defined by $Q(f) = \langle f,\phi\rangle$. Then Q is weak*-weak* continuous, $\delta_hQ = Q\delta_h$ for each $h \in H$ and $\|Q\| = 1$. Hence there exists a continuous projection P from $L^\infty(G/H,\nu)$ onto X such that $P\delta_g = \delta_gP$ for each $g \in G$ with $\|P\| \leq 1$. Define $\langle m,f\rangle = \langle P(f),\phi\rangle$ for each $f \in L^\infty(G/H,\nu)$. Then $m(1) = \|m\| = 1$, and $m(\delta_g \,\square\, f) = m(f)$ for all $g \in G$, $f \in L^\infty(G/H,\nu)$. Hence G/H is amenable.

A weak*-closed subspace X of $L^\infty(G/H,\nu)$ is weak* complemented if there exists a weak*-weak* continuous projection from $L^\infty(G/H,\nu)$ onto X.

<u>Corollary 3.5.6</u>. Let G be a locally compact group, H a closed subgroup. If the coset space G/H is amenable, then any H-invariantly weak* complemented left invariant subspace of $L^\infty(G/H,\nu)$ is invariantly complemented. The converse is also true, if there is a $\phi \in P(\nu)$ such that $\delta_h * \phi = \phi$ for each $h \in H$.

Remark 3.5.7. (i) If G/H supports an invariant measure, then the converse of Theorem 3.5.5 or Corollary 3.5.6 is true.

(ii) We could use Theorem 3.3.2 to prove the above results exactly the same way as in [26, Theorem 4.1 and Theorem 4.3].

(iii) A. Lau proved in [26, Theorem 3.3] that G is amenable if and only if every left invariant W^*-subalgebra of $L^\infty(G)$ is complemented. We do not know if G/H is amenable whether a weak* closed self adjoint subalgebra which contains the constant functions is necessarily complemented for the case when $H \neq \{e\}$. However, the converse is true. Lau's proof works in our case.

(iv) A. Lau and V. Losert recently proved in [27, Corollary 2] that G is amenable if and only if every weak*-closed complemented invariant subspace of $L^\infty(G/H,\nu)$, H is a closed subgroup of G, is the range of a projection on $L^\infty(G/H,\nu)$ which commutes with translations.

3.6 Reiter-Glicksberg property

Definition 3.6.1. Let G be a locally compact group, H a closed subgroup, and let E,F be normed linear spaces, σ a norm decreasing linear mapping of E into F, and $T = \{T_g : g \in G\}$ an anti representation of G on E with $\|T_g\| \leq 1$ for each $g \in G$. Let C_x be the closed convex hull of $\{\sigma \circ T_g\, x : g \in G\}$ in F for each $x \in E$, and J the closed subspace of E spanned by $\{T_g\, x - x : g \in G, x \in E\}$. T is said to be continuous [weakly continuous] if the map $g \to T_g\, x$ is continuous [weakly continuous] for each $x \in E$. We say that the pair $(G : H)$ has the *Reiter-Glicksberg Property* (RG) [(RG_s)] if whenever E,F are normed linear spaces, σ a norm decreasing linear mapping of E

into F, and $T = \{T_g : g \in G\}$ a continuous [weakly continuous] anti representation of G on E such that $\|T_g\| \leq 1$ for each $g \in G$ and $\sigma \circ T_h = \sigma$ for each $h \in H$, then $\text{dist}_F(0,C_x) \leq \text{dist}_E(x,J)$ for each $x \in E$.

P. Eymard proved in [13, p. 21-25] using his fixed point theorem that ~~if~~ there is a LIM on $UCB_\ell(G/H)$ [$CB(G/H)$], then the pair $(G : H)$ has the Reiter-Glicksberg property (RG)[(RG_s)]. We give a new proof of this result here. The next lemma is originally due to H. Reiter [35, Lemma B] for the case when $H = \{e\}$.

<u>Lemma 3.6.2</u>. Let G/H be an amenable homogeneous space, and let E,F be normed linear spaces, $T = \{T_g : g \in G\}$ a weakly continuous anti representation of G on E with $\|T_g\| \leq 1$ for every $g \in G$, and a norm decreasing linear mapping of E into F such that $\sigma \circ T_h = \sigma$ for each $h \in H$. If $x \in E$ satisfies $\text{dist}_F(0,C_x) = \delta > 0$, then there exists $\phi \in E^*$ such that $\|\phi\| \leq 1$, $\phi(T_g\, y) = \phi(y)$ for all $g \in G$, $y \in E$, and $\text{Re}\langle\phi,x\rangle \geq \delta$. In addition, if σ is an isometry of E into F, then ϕ also satisfies $\|\phi\| = 1$, and $\langle\phi,x\rangle = \delta$.

<u>Proof</u>: Let m be a LIM on $CB(G/H)$. By the Hahn Banach theorem, there exists $\psi \in F^*$ such that

$$\begin{aligned}&\text{Re}\langle\psi,y\rangle \geq \delta \quad \text{for each } y \in C_x \text{ and} \\ &\|\psi\| = 1 \quad \text{(see [35, Lemma A])}\end{aligned} \tag{1}$$

For each $y \in E$, define the function $f_y : G/H \to \mathbb{C}$ by

$$f_y(gH) = \langle\psi,\sigma \circ Tg\, y\rangle$$

For $y \in E$, we have $|f_y(gH)| = |\langle\psi,\sigma\circ T_g\, y\rangle| \leq \|y\|$, so f_y is bounded. Also f_y is continuous, as the mapping $g \to T_g\, y$ is continuous when E has the weak topology. Hence, $f_y \in CB(G/H)$. Now, $f_{\alpha y+\beta z} = \alpha f_y + \beta f_z$ for $y,z \in E$, α, $\beta \in \mathbb{C}$, and $\|f_y\|_\infty \leq \|y\|$. Therefore, the mapping $y \to \langle m,f_y\rangle$ defines a bounded linear functional on $CB(G/H)$. Hence, there exists $\phi \in E^*$ such that

$$\langle\phi,y\rangle = \langle m,f_y\rangle \text{ for every } y \in E, \text{ and } \|\phi\| \leq 1 \tag{2}$$

For $a \in G$, $y \in E$, we have $f_{T_a y}(gH) = \langle\psi,\sigma\circ T_g(T_a y)\rangle = \langle\psi,\sigma\circ T_{ag}y\rangle = f_y(agH) = \delta_a \,\square\, f_y(gH)$. Hence, $\langle\phi,T_a y\rangle = \langle m,f_{T_a y}\rangle = \langle m,\delta_a \,\square\, f_y\rangle = \langle m,f_y\rangle = \langle\phi,y\rangle$ for all $a \in G$, $y \in E$, so ϕ is invariant. Let Q_x be the closed convex hull of $\{T_g\, x : g \in G\}$ in E. If $y \in Q_x$, then $\sigma\circ T_g\, y \in C_x$ for each $g \in G$, so $\|y\| \geq |\langle\phi,y\rangle| \geq \mathrm{Re}\langle\phi,y\rangle = \mathrm{Re}\langle m,f_y\rangle = \langle m, \mathrm{Re}\, f_y\rangle \geq \delta$ by (1) and (2). If σ is an isometry, then $\sigma Q_x = C_x$, so $\|y\|$ can be arbitrarily close to δ, while $\langle\phi,y\rangle$ is constant on Q_x. It follows that $|\langle\phi,y\rangle| = \mathrm{Re}\langle\phi,y\rangle = \delta$ for $y \in Q_x$ and $\|Q\| = 1$. This completes the proof of the lemma since $x \in Q_x$. □

Corollary 3.6.3. If G/H is an amenable homogeneous space, then the pair $(G : H)$ has property (RG_s).

Proof: Let E,F be normed linear space, $T = \{T_g : g \in G\}$ a weakly continuous anti representation of G on E with $\|T_g\| \leq 1$ for every $g \in G$, and σ a norm decreasing linear mapping of E into F such that $\sigma\circ T_h = \sigma$ for each $h \in H$. By the previous lemma, choose $\phi \in E^*$ so that $\|\phi\| \leq 1$, $\phi(T_g\, y) = \phi(y)$ for all $g \in G$, $y \in E$. Then for

$x \in E$, $\mathrm{dist}_E(x,J) = \inf_{y\in J} \|x - y\| \geq \inf_{y\in J} |<\phi,x - y>| = |<\phi,x>| \geq$ $\mathrm{dist}_F(0,C_x)$. □

Remark 3.6.4. Suppose the conditions of Corollary 3.6.3 hold. Then equality holds in the Reiter-Glicksberg condition in the following cases (i) σ is an isometry: since $C_x \subseteq \sigma(x) + \sigma(J)$, we have $\mathrm{dist}_F(0,C_x) \geq \mathrm{dist}_F(\sigma(x),\sigma(J)) = \mathrm{dist}_E(x,J)$.
(ii) $F = E/N$, where N is a subspace of E contained in J and the natural mapping from E to F (see [13, p.25]).

Remark 3.6.5. Let $E = L'(G)$, $F = L'(G/H,\nu)$, and define $\sigma : E \to F$ by $\sigma f(gH) = \int_H \frac{f(gh)}{\rho(gh)} dh$ for $f \in L'(G)$. For $f \in L'(G)$, $g \in G$, let $T_g f = f * \delta_g$. Then

(i) $\|\sigma\| \leq 1$.

(ii) $\sigma \circ T_h = \sigma$ for every $h \in H$.

(iii) $T = \{T_g : g \in G\}$ is a continuous anti representation of G on $L'(G)$ with $\|T_g\| = 1$ for every $g \in G$ (see [13, p.31-34]).

The following lemma is in [13, p.32].

Lemma 3.6.6. Let G be a locally compact group, H a closed subgroup. Suppose the pair $(G : H)$ satisfies the Reiter-Glicksberg property (RG). Let $f_1,\ldots,f_N \in L'(G)$ satisfy $\int_G f_j(g)dg = 0$ for $j = 1,\ldots,N$. Then, for any $\varepsilon > 0$, there are finitely many numbers $c_i > 0$, with $\Sigma c_i = 1$, and elements $g_i \in G$ such that $\| \sigma (\Sigma_i c_i f_j * \delta_{g_i})\|_{\nu,1} < \varepsilon$ for $j = 1,\ldots,N$.

P. Eymard utilized Lemma 3.6.6 to show that if the pair $(G : H)$ has property (RG), then it has property (P_1). It follows then that the converse of Corollary 3.6.3 is also true (see the diagram in [13, p.29]). As a consequence of Lemma 3.6.6, we have the following corollary.

Corollary 3.6.7. Let G be a locally compact group, H a closed subgroup. If the pair $(G : H)$ has property (RG), then there is a net in $P(\nu)$ strongly convergent to left invariance.

Proof: Let $D = \{\alpha = (g_1,\ldots,g_N;\varepsilon) : \varepsilon > 0,\ g_i \in G, \text{ and } N < \infty\}$. Direct D so $\alpha > \alpha'$ if $\{g_i\} \supset \{g_i'\}$ and $0 < \varepsilon < \varepsilon'$. Fix $\phi \in P(G)$. Then, for each $\alpha \in D$, since $<\delta_{g_i} * \phi - \phi, 1> = 0$ for $i = 1,\ldots,N$, by Lemma 3.6.6, there exist finitely many numbers $c_j > 0$, with $\Sigma c_j = 1$, and elements $a_j \in G$ such that $\|\sigma(\sum_j(\delta_{g_i} * \phi - \phi) * \delta_{a_j})\|_{\nu,1} < \varepsilon$, $1 \leq i \leq N$. Let $\phi_\alpha = \sigma(\sum_j c_j\phi * \delta_{a_j})$. Then $\{\phi_\alpha\}$ converges strongly to left invariance since $\delta_g * \phi_\alpha = \sigma(\sum_j c_j(\delta_g * \phi) * \delta_{a_j})$ for each $g \in G$ (see [21, p.167]). □

We now summarize the main results of this section in the next theorem.

Theorem 3.6.8. Let G be a locally compact group, H a closed subgroup Then the following statements are equivalent

(i) G/H is amenable

(ii) $(G: H)$ has property (RG)

(iii) $(G : H)$ has property (RG_s).

3.7 Examples and Summary

In this section, we give a number of well known examples of amenable homogeneous spaces and a summary of important results which are either derived in Sections 3.2 through 3.6, or are well known. Some open problems are also stated.

Examples 3.7.A: 1. [13, p.15]. Let G be a locally compact group and H, K closed subgroups with $K \subset H$. Then

(a) If the coset space G/K is amenable, so is G/H: Let m_1 be a LIM on $CB(G/K)$. Define $\Psi : CB(G/H) \to CB(G/K)$ by $\Psi(f)(gK) = f(gH)$, $f \in CB(G/H)$. If $g_1K = g_2K$ then $g_1^{-1}g_2 \in K \subset H$, so $g_1H = g_2H$. Thus $\Psi(f)$ is a well defined function in $CB(G/K)$. Next define a function $m : CB(G/H) \to \mathbb{C}$ by $\langle m,f\rangle = \langle m_1,\Psi(f)\rangle$ for $f \in CB(G/H)$. Then m is a LIM on $CB(G/H)$. □

(b) If the coset spaces H/K and G/H are amenable, then G/K is amenable:

Let m_1, m_2 be left invariant means on $CB(H/K)$, $CB(G/H)$, respectively. Define $\Psi : CB(G/K) \to CB(G/H)$ by $\Psi(f)(gH) = \langle m_1, (\delta_g \square f)|_{H/K}\rangle$, $f \in CB(G/K)$. If $g_1H = g_2H$ then $g_2 = g_1h$ for some $h \in H$. Thus $\langle m_1, (\delta_{g_2} \square f)|_{H/K}\rangle = \langle m_1, (\delta_{g_1h} \square f)|_{H/K}\rangle = \langle m_1, (\delta_h \square (\delta_{g_1} \square f))|_{H/K}\rangle = \langle m_1 (\delta_{g_1} \square f)|_{H/K}\rangle$, for $f \in CB(G/K)$ by left invariance of m_1. Hence, Ψ is well defined. Next define $m : CB(G/K) \to \mathbb{C}$ by $\langle m,f\rangle = \langle m_2,\Psi(f)\rangle$, $f \in CB(G/K)$. Then m is a LIM on $CB(G/H)$. □

2 (a). [20, p.306]. A locally compact group G is amenable if and only if G acts amenably on every coset space G/H, where H is a closed subgroup:

If G is amenable, then by Example 1(a) G acts amenably on G/H. The converse is trivial. □

(b) Let H be a closed subgroup of a locally compact group G. Then H is amenable and G acts amenably on G/H if and only if G is amenable:

If G acts amenably on G/H and H is amenable then by Example 1(b) G is amenable. The converse follows from Example 2(a) and [19, Theorem 2.3.2].

3. [34, p.112]. Any compact group G is amenable. The normalized Haar measure is the unique TLIM on $L^\infty(G)$. Also it is the unique LIM on CB(G).

4. [13, p.17]. Let G be a locally compact group, and H a closed subgroup with G/H compact. If $\Delta_G|_H = \Delta_H$, then G/H supports a (unique) normalized invariant measure ν. This measure is the unique TLIM [LIM] on $L^\infty(G/H,\nu)$ [CB(G/H)].

5. [19, p.5]. Any abelian group is amenable as a discrete group. In fact, every solvable group is amenable as a discrete group by Example 2(b) (see also [19, p.9]).

6. A group which contains a free subgroup on two generators is not amenable as a discrete group (see [19, p.9]).

7. A locally compact group which is amenable as a discrete group is amenable. The converse is not true:

The orthogonal group $SO(3,\mathbb{R})$ contains a free subgroup on two generators, so $SO(3,\mathbb{R})$ is not amenable as a discrete group (see [19, p.6 and p.9-12]).

However, since $SO(3,\mathbf{R})$ is compact in the usual topology, it is amenable by Example 3.

8. [20, p.304]. Let $G = SO(3,\mathbf{R})$ with its usual topology, $Z = S^2$ the unit sphere in $\mathbf{R}^3$. Fix a point x in Z, and let $H = \{g \in G : gx = x\}$ where G has the usual action as proper rotations of Z. Then H is a closed subgroup of G. Since G is amenable, G acts amenably on G/H. In fact, G/H is homeomorphic to Z and G/H supports a finite invariant measure (see [15, p.269]).

9. The linear groups $SL(2,\mathbb{C})$ and $SL(2,\mathbf{R})$ with the usual topology are nonamenable since they admit a discrete (closed) free subgroup on two generators (see [34, p.125]). Since there is a finite invariant measure on $SL(2,\mathbf{R})/SL(2,\mathbb{Z})$, $SL(2,\mathbf{R})$ acts amenably on $SL(2,\mathbf{R})/SL(2,\mathbb{Z})$. Nc [illegible] $SL(2,\mathbb{Z})$ is not amenable by Example 2(b) (see [13, p.18-19]).

For more examples on locally compact amenable groups we refer the reader to [34, Chapter 3].

3.7B Summary

Let G be a locally compact group, and H a closed subgroup. We give a summary of important results relating to the existence of a LIM on $L^\infty(G/H,\nu)$ here for the reader's convenience.

Each of the following statements is equivalent to the amenable action of G on G/H (Definition 3.2.1).

1. [1A] Ess inf $h \leq 0$ for all real valued $h \in \pi_0[\pi_1]$ (Corollary 3.2.4).

2. [2A] $\pi_o[\pi_1]$ is not uniformly dense in $L^\infty(G/H,\nu)$ (Theorem 3.2.5).

3. [3A] $L^\infty(G/H,\nu)$ contains a subspace π satisfying the following two properties:

(i) $\delta_g \square f - f \in \pi$ for all $f \in L^\infty(G/H,\nu)$, $g \in G$.

[(i) $\phi \square f - f \in \pi$ for all $\phi \in P(G)$, $f \in L^\infty(G/H,\nu)$]

(ii) ess inf $f \leq 0$ for all real valued $f \in \pi$ (Proposition 3.2.7).

4. [4A] There is a net in $P(\nu)$ strongly convergent to [topological] left invariance (Theorem 2.2.10).

5. The pair $(G : H)$ satisfies Reiter's condition (P_1) . (Theorem 2.2.12).

6. [6A, 6B] The pair $(G : H)$ has the fixed point property $F_3[F_2,F_1]$ (Theorem 3.3.2, see also [13, p.12]).

7. The pair $(G : H)$ has the Hahn-Banach extension property (HBEP) (Theorem 3.4.2).

8. The pair $(G : H)$ has the invariantly complemented subspace property (C) (Theorem 3.5.2).

9. [9A] The pair $(G : H)$ has the Reiter-Glicksberg property $RG[RG_s]$ (Theorem 3.6.8; see also [13, p.21-29].

If G acts amenably on G/H then the following statements are true.

10. The pair $(G : H)$ has property (B) (Proposition 3.2.13).

11. Any dual Banach space has the G-invariant weak* complemented

subspace property with respect to H (Theorem 3.5.5).

Each of the following statements implies the amenable action of G on G/H.

12. $\pi_p(G/H)$ is closed under addition (Corollary 3.2.10).

13. Every left invariant W*-subalgebra of $L^\infty(G/H,\nu)$ is invariantly complemented (Remark 3.5.6 (iii)).

14. (Converse of 11). G/H supports an invariant measure ν and $L^\infty(G/H,\nu)$ has the G-invariant weak* complemented subspace property with respect to H (Theorem 3.5.5).

3.7C Some open problems

Let G be a locally compact group, H a closed subgroup. Does the amenable action of G on G/H imply any of the following statements?

1. $\pi_p(G/H)$ is closed under addition.
2. Every left invariant W*-subalgebra of $L^\infty(G/H,\nu)$ is invariantly complemented.

Is the amenable action of G on G/H implied by any of the following statements?

3. The pair (G : H) has property (B).

4. Any dual Banach space has the G-invariant weak* complemented subspace property with respect to H (see 3.7C:14).

5. Is the amenable action of G on G/H characterized by the following

property? Every weak* closed left invariant H-invariantly complemented subspace of $L^\infty(G/H,\nu)$ is invariantly complemented.

6. Does the following local property imply the existence of a topological left invariant mean on $L^\infty(G/H,\nu)$? For each f in $L^\infty(G/H,\nu)$ there is a mean m_f in $L^\infty(G/H,\nu)^*$ such that $m_f(\phi \,\square\, f) = m_f(f)$ for all ϕ in $P(G)$ (see [12, Theorem 2.3] and [46, Theorem 5.2]).

CHAPTER IV

WEAKLY ALMOST PERIODIC FUNCTIONS ON COSET SPACES

4.1 Introduction

A function $f \in CB(G/H)$ is almost periodic [weakly almost periodic] if the orbit $O_L(f)$ is relatively compact in the norm [weak] topology of $CB(G/H)$. $AP(G/H)$ $[WAP(G/H)]$ is the space of all almost periodic [weakly almost periodic] functions on G/H.

In this chapter, we prove that $WAP(G/H) \subset UCB_\ell(G/H)$ and that if H is compact than $C_o(G/H) \subset WAP(G/H)$. A characterization of weakly almost periodic functions on G/H is obtained. Finally, we establish the existence of a unique left invariant mean on $WAP(G/H)$ and then on $AP(G/H)$.

4.2 Invariant means on weakly almost periodic functions on coset spaces

Let G be a locally compact group, H a closed subgroup. A function $f \in CB(G)$ is called almost periodic [weakly almost periodic] if the left orbit $O_L(f) = \{\ell_g f : g \in G\}$ is relatively compact in the norm [weak] topology of $CB(G)$. We denote the corresponding classes of functions by $AP(G)$, $WAP(G)$. A function f on G is said to be *left (right) uniformly continuous* if for each $\varepsilon > 0$, there is a neighbourhood U of the identity element e in G such that $|f(g) - f(ga)| < \varepsilon$ $[|f(g) - f(ag)| < \varepsilon]$ for all $a \in U$, $g \in G$. We denote the space of left [right] uniformly continuous functions on G by $UCB_\ell(G)$ $[UCB_r(G)]$. Note that our definition of $UCB_\ell(G/H)$ coincides with the definition of $UCB_r(G)$ when $H = \{e\}$ (see footnote in [20, p.299]). $UCB(G) = UCB_r(G) \cap UCB_\ell(G)$ is called the space of uniformly continuous functions on G.

As is well known, the orbit $O_L(f)$ is relatively norm [weak] compact in CB(G) if and only if the right orbit $O_R(f) = \{r_g f : g \in G\}$ is relatively norm [weak] compact in CB(G). Furthermore, AP(G), WAP(G) are norm closed, conjugate closed two sided invariant [that is, both left and right] subalgebras of CB(G) (see [22, Theorem 18.1] and [2, Corollary 1.1.1]), and $C_o(G)$, $AP(G) \subseteq WAP(G) \subseteq UCB(G)$ (see [7, Section 5] or [10, Sections 10-16]). It is also well known that there is a unique two-sided invariant mean on WAP(G) and hence on AP(G) (see [19, Section 3.1] and also [22, Theorems 18.8 and 18.9]).

J. von Neumann originally established the existence of a unique invariant mean on AP(G) in [45]. Ryll-Nardzewski proved the existence of a unique invariant mean on WAP(G) using his fixed point theorem in [40]. There are many proofs for Ryll-Nardzewski's fixed point theorem. His original proof was probabilistic, using the Martingale Convergence Theorem (see [40]). A geometric proof given by Asplund-Namioka is given in [19, p.97-99]. We refer the reader to a proof of Namioka [33] and a very recent proof of Dugundji and Andrzej Granas [8].

<u>Definition 4.2.1.</u> Let G be a locally compact group, H a closed subgroup. A function $f \in CB(G/H)$ is called *almost periodic (weakly almost periodic)* if the orbit $O_L(f) = \{\delta_g \square f : g \in G\}$ is relatively compact in the norm [weak] topology of CB(G/H). We denote the space of all almost periodic [weakly almost periodic] functions on G/H by AP(G/H) [WAP(G/H)].

<u>Lemma 4.2.2.</u> Let G be a locally compact group, H a closed subgroup.

Then $AP(G/H)$ is a norm closed conjugate closed left translation invariant subalgebra of $CB(G/H)$ which contains the constant functions.

Proof: Left translation invariance follows from the fact that $O_L(f) = O_L(\delta_a \square f)$ for all $f \in CB(G/H)$, $a \in G$. Clearly $AP(G/H)$ contains the constant functions. If $f \in AP(G/H)$, then $\bar{f} \in AP(G/H)$ since the mapping $f \to \bar{f}$ is norm continuous and $O_L(\bar{f}) = \overline{O_L(f)} = \{\overline{\delta_g \square f} : g \in G\}$. Let $f_1, f_2 \in AP(G/H)$. If $\{a_n\}$ is a sequence in G, choose a subsequence $\{a_{n_k}\} \subset \{a_n\}$ and $F_1, F_2 \in CB(G/H)$ such that $\lim_{k\to\infty} \|\delta_{a_{n_k}} \square f_1 - F_1\|_\infty = 0$, $\lim_{k\to\infty} \|\delta_{a_{n_k}} \square f_2 - F_2\|_\infty = 0$. Then, $\lim_{k\to\infty} \|\delta_{a_{n_k}} \square (f_1 + f_2) - (F_1 + F_2)\|_\infty = 0$, and $\lim_{k\to\infty} \|\delta_{a_{n_k}} \square (f_1 f_2) - F_1 F_2\|_\infty \leq \lim_{k\to\infty} \|\delta_{a_{n_k}} \square f_1 - F_1\|_\infty \|f_2\|_\infty + \lim_{k\to\infty} \|\delta_{a_{n_k}} \square f_2 - F_2\|_\infty \|F_1\|_\infty = 0$. Hence $f_1 + f_2$, $f_1 f_2 \in AP(G/H)$.

To see that $AP(G/H)$ is norm closed, let $\{f_n\}$ be a sequence in $AP(G/H)$ converging to $f \in CB(G/H)$. If $\{a_m\}$ is a sequence in G, by the diagonal process, choose a subsequence $\{a_{m_k}\} \subset \{a_m\}$ such that $\{\delta_{a_{m_k}} \square f_i\}$ converges for each i, say to $F_i \in CB(G/H)$. Then for $j > i$, $\|F_j - F_i\|_\infty \leq \|\delta_{a_{m_k}} \square f_i - F_i\|_\infty + \|f_i - f_j\|_\infty + \|\delta_{a_{m_k}} \square f_j - F_j\|_\infty$. This implies, since $\{f_n\}$ is norm Cauchy in $CB(G/H)$, that $\{F_n\}$ is Cauchy in $CB(G/H)$. Therefore, there exists $F \in CB(G/H)$ such that $\lim_{n\to\infty} \|F_n - F\|_\infty = 0$. Now, for any i, we have, $\|\delta_{a_{m_k}} \square f - F\|$. $\|\delta_{a_{m_k}} \square f_i - F_i\|_\infty + \|F_2 - F\|_\infty + \|\delta_{a_{m_k}} \square f - \delta_{a_{m_k}} \square f_i\|_\infty$. This implies that $\lim_{k\to\infty} \|\delta_{a_{m_k}} \square f - F\|_\infty = 0$, so $f \in AP(G/H)$. This finishes the proof of the lemma. $\square$

Lemma 4.2.3. Let G be a locally compact group, H a closed subgroup. Then $WAP(G/H)$ is a norm closed conjugate closed left translation invariant subalgebra of $CB(G/H)$ containing $AP(G/H)$.

Proof: Left translation invariance follows from the fact that $O_L(f) = O_L(\delta_a \Box f)$ for all $a \in G$, $f \in CB(G/H)$. Trivially $AP(G/H) \subseteq WAP(G/H)$. If $f \in WAP(G/H)$, then $\bar{f} \in WAP(G/H)$ as $f \to \bar{f}$ is (norm) continuous and $O_L(\bar{f}) = \overline{O_L(f)}$. Let $f_1, f_2 \in WAP(G/H)$. If $\{a_n\}$ is a sequence in G, choose a subsequence $\{a_{n_k}\} \subset \{a_n\}$ and $F_1, F_2 \in CB(G/H)$ such that $\lim_{k\to\infty} \delta_{a_{n_k}} \Box f_1 = F_1$ weakly and $\lim_{k\to\infty} \delta_{a_{n_k}} \Box f_2 = F_2$ weakly (see [9, V.6.1]). Then, $\lim_{k\to\infty} \delta_{a_{n_k}} \Box (f_1 + f_2) = F_1 + F_2$ weakly, so $f_1 + f_2 \in WAP(G/H)$. To show $f_1 f_2 \in WAP(G/H)$, write $CB(G/H) = C(\Omega)$, where Ω is the spectrum of the C*-algebra $CB(G/H)$. Note that Ω is compact as $CB(G/H)$ is unital. $\{\delta_{a_{n_k}} \Box (f_1 f_2)\}$ is clearly uniformly bounded by $\|f_1\|_\infty \|f_2\|_\infty$. We will show that $\{\delta_{a_{n_k}} \Box (f_1 f_2)\}$ converges pointwise to $F_1 F_2$. Then, since weak sequential convergence in $C(\Omega)$ is equivalent to pointwise convergence and uniform boundedness (see [10, Theorem 1.3]); it follows that $f_1 f_2 \in WAP(G/H)$. But pointwise convergence of $\{\delta_{a_{n_k}} \Box (f_1 f_2)\}$ follows immediately from the fact that both $\{\delta_{a_{n_k}} \Box f_1\}$ and $\{\delta_{a_{n_k}} \Box f_2\}$ converge pointwise to F_1, F_2 respectively. Next, let $\{f_n\}$ be a sequence in $WAP(G/H)$ converging to $f \in CB(G/H)$ in the norm topology. If $\{a_m\}$ is a sequence in G, by the diagonal process, we can find a subsequence $\{a_{m_k}\} \subset \{a_m\}$ such that $\lim_{k\to\infty} \delta_{a_{m_k}} \Box f_i = F_i \in CB(G/H)$ weakly, say, for each i. Then, for $j > i$,

$$\begin{aligned}\|F_j - F_i\|_\infty &= \sup_{\substack{\|\phi\| \leq 1 \\ \phi \in CB(G/H)^*}} |<\phi,F_j> - <\phi,F_i>| \\ &= \sup_{\substack{\|\phi\| \leq 1 \\ \phi \in CB(G/H)^*}} [\lim_{k\to\infty} |<\phi,\delta_{a_{m_k}} \square f_j> - <\phi,\delta_{a_{m_k}} \square f_i>|] \\ &= \|f_j - f_i\|_\infty .\end{aligned}$$

Therefore, since $\{f_n\}$ is sup norm Cauchy, so is $\{F_n\}$. Hence, there exists $F \in CB(G/H)$ such that $\lim_{n\to\infty} \|F_n - F\|_\infty = 0$. Now, for each i and $\phi \in CB(G/H)^*$, $|\phi(\delta_{a_{m_k}} \square f) - \phi(F)| \leq |\phi(F) - \phi(F_i)| + |\phi(F_i) - \phi(\delta_{a_{m_k}} \square f_i)| + |\phi(\delta_{a_{m_i}} \square f_i) - \phi(\delta_{a_{m_k}} \square f)| \leq \|\phi\| \|f_i - f_j\|_\infty + \|\phi\| \|F_i - F\|_\infty + |\phi(F_i) - \phi(\delta_{a_{m_k}} \square f_i)|$. This implies that $\{\delta_{a_{m_k}} \square f\}$ converges weakly to F in $CB(G/H)$, so by [9, V.6.1] $f \in WAP(G/H)$. Hence, we have the lemma. $\square$

Lemma 4.2.4. Let G be a locally compact group; H a closed subgroup. Then $WAP(G/H) \subseteq UCB_\ell(G/H)$.

Proof: Let $f \in WAP(G/H)$. We first show that the map $g \to \delta_g \square f$ is continuous when $CB(G/H)$ has the weak topology. If $\{a_\alpha\}$ is a net in G converging to a in G, then $\{\delta_{a_\alpha} \square f\}$ converges pointwise to $\delta_a \square f$. Hence, the net $\{\delta_{a_\alpha} \square f\}$ has at most one weak (norm) cluster point $\delta_a \square f$. By the relative weak compactness of $O_L(f)$ this net must converge weakly to $\delta_a \square f$, and consequently the mapping $g \to \delta_g \square f$ is weakly continuous. Now, by a result of Lau (see the proposition in [25, p.151]), the mapping $g \to \delta_g \square f$ is norm continuous, and consequently $f \in UCB_\ell(G/H)$. For the sake of completeness we give details of this fact.

Following an idea of Mitchell in [31, Theorem 1], let Σ be the weak* compact convex set of all means on WAP(G/H). Define $a \cdot m$ so $\langle a \cdot m, f \rangle = \langle m, \delta_{a^{-1}} \square f \rangle$ for $a \in G$, $f \in WAP(G/H)$, $m \in \Sigma$. The mapping $(a,m) \to a \cdot m$ is separately continuous when $WAP(G/H)^*$ has the weak* topology and hence Ellis' result [11, Theorem 1] implies that the map $(a,m) \to a \cdot m$ is jointly continuous. Suppose $f \notin UCB_\ell(G/H)$. Then there exist $\delta > 0$ and nets $\{a_\alpha\}$, $\{b_\alpha\}$ with $\{a_\alpha\}$ converging to a in G such that $|f(a_\alpha b_\alpha H) - f(ab_\alpha H)| \geq \delta$ for each α. Choose a subnet $\{\delta_{b_\beta H}\}$ of $\{\delta_{b_\alpha H}\}$ such that $\{\delta_{b_\beta H}\}$ converges to m in the weak* topology. Hence, $0 < \delta \leq \lim_\beta |\langle \delta_{b_\beta H}, \delta_{a_\beta} \square f \rangle - \langle \delta_{b_\beta H}, \delta_a \square f \rangle| =$ $\lim_\beta | \langle a_\beta^{-1} \delta_{b_\beta H}, f \rangle - \langle a^{-1} \delta_{b_\beta H}, f \rangle | = 0$, a contradiction. So $f \in UCB_\ell(G/H)$, and hence $WAP(G/H) \subset UCB_\ell(G/H)$. □

Lemma 4.2.5. Let G be a locally compact group, H a closed subgroup, and $\pi : G \to G/H$ the natural map. If $\tilde{\pi} : CB(G/H) \to CB(G)$ is the induced map given by $\tilde{\pi}(f) = f \circ \pi$ for $f \in CB(G/H)$, then $\tilde{\pi}(AP(G/H)) =$ $AP(G) \cap \tilde{\pi}(CB(G/H))$, $\tilde{\pi}(WAP(G/H)) = WAP(G) \cap \tilde{\pi}(CB(G/H))$.

Proof: $\tilde{\pi}$ is an order preserving isometry which preserves the constant functions. $\tilde{\pi}(\delta_a \square f) = \ell_a \tilde{\pi}(f)$ for all $a \in G$, $f \in CB(G/H)$, so $\tilde{\pi}(O_L(f)) =$ $\tilde{\pi}(\{\delta_a \square f : a \in G\}) = \{\ell_a \tilde{\pi}(f) : a \in G\} = O_L(\tilde{\pi}(f))$. Also $\tilde{\pi}$ is norm-norm continuous and hence weak-weak continuous. If $f \in AP(G/H)[WAP(G/H)]$, let $\{a_n\}$ be a sequence in G. Then we can find a subsequence $\{a_{n_k}\}$ and $F \in CB(G/H)$ such that $\{\delta_{a_{n_k}} \square f\}$ converges to $F \in CB(G/H)$ in the norm (weak) topology. Hence, $\{\tilde{\pi}(\delta_{a_{n_k}} \square f)\} = \{\ell_{a_{n_k}} \tilde{\pi}(f)\}$ converges to $\pi(F) \in CB(G)$ in the norm (weak) topology. Therefore,

$\tilde{\pi}(f) \in AP(G)[WAP(G)] \cdot \tilde{\pi}(CB(G/H))$ is a norm closed subspace of $CB(G)$. Hence, the weak topology on $\tilde{\pi}(CB(G/H))$ is the relative weak topology of $CB(G)$. Therefore, by considering $\tilde{\pi}^{-1} : \tilde{\pi}(CB(G/H)) \to CB(G)$, a similar argument as above shows that if $f \in CB(G/H)$, then $\tilde{\pi}(f) \in AP(G)[WAP(G)]$ implies $f \in AP(G/H)[WAP(G/H)]$. Hence $\tilde{\pi}(AP(G/H)) = AP(G) \cap \tilde{\pi}(CB(G/H))$, $\tilde{\pi}(WAP(G/H)) = WAP(G) \cap \tilde{\pi}(CB(G/H))$. □

The following two results were suggested by B. Forrest (personal communications). Our proofs are different from his and employ Lemma 4.2.5. His methods of proof follow ideas from [2, Theorem 3.6] and [1, Theorem 8.2], and do not need any of the properties of $WAP(G)$.

Lemma 4.2.6. Let G be a locally compact group, H a compact subgroup. Then $C_o(G/H) \subset WAP(G/H)$.

Proof: Let $f \in C_o(G/H)$ and $\varepsilon > 0$. Then there is a compact subset $K \subset G/H$ such that $|f(\xi)| < \varepsilon$ for all $\xi \in G/H \setminus K$. $\pi^{-1}(K)$ is compact in G (see [22, p.39]) and $|\tilde{\pi}(f)(g)| < \varepsilon$ for all $g \in G \setminus \pi^{-1}(K)$, so $\tilde{\pi}(f) \in C_o(G)$. But $C_o(G) \subseteq WAP(G)$, hence $\tilde{\pi}(f) \in WAP(G)$. Therefore, by the above lemma $f \in WAP(G/H)$. Hence, $C_o(G/H) \subset WAP(G/H)$. □

Proposition 4.2.7. Let G be a locally compact group, H a closed subgroup, and $f \in CB(G/H)$. Then $f \in WAP(G/H)$ if and only if $\lim_m \lim_n f(a_n b_m H) = \lim_n \lim_m f(a_n b_m H)$ whenever $\{a_n\}$, $\{b_m\}$ are sequences in G such that all relevant limits exist.

Proof: Let $\tilde{\pi} : CB(G/H) \to CB(G)$ be as in Lemma 4.2.5. Then

$\tilde{\pi}(f) \in WAP(G)$ if and only if $f \in WAP(G/H)$. But, by [1, Theorem 8.2], $\tilde{\pi}(f) \in WAP(G)$ if and only if whenever $\{a_n\}$, $\{b_m\}$ are sequences in G such that all relevant limits in $\lim\limits_m \lim\limits_n \tilde{\pi}(f)(a_n b_m)$ and $\lim\limits_n \lim\limits_m \tilde{\pi}(f)(a_n b_m)$ exist then they are equal. Since $\lim\limits_m \lim\limits_n \tilde{\pi}(f)(a_n b_m) = \lim \lim f(a_n b_m H)$ and $\lim\limits_n \lim\limits_m \tilde{\pi}(f)(a_n b_m) = \lim\limits_n \lim\limits_m f(a_n b_m H)$, the proof of the proposition follows immediately. □

Remark 4.2.8. (i) Lemma 4.2.4 could be deduced from Lemma 4.2.5 using the fact $WAP(G) \subset UCB(G)$:

Let $f \in WAP(G/H)$. Then $\tilde{\pi}(f) \in WAP(G) \subseteq UCB(G)$. If $\{a_\alpha\}$ is a net in G converging to $a \in G$, then $\lim\limits_\alpha \|\delta_{a_\alpha} \square f - \delta_a \square f\|_\infty = \lim\limits_\alpha \|\tilde{\pi}(\delta_{a_\alpha} \square f - \delta_a \square f)\|_\infty = \lim\limits_\alpha \|\ell_a \tilde{\pi}(f) - \ell_a \tilde{\pi}(f)\|_\infty = 0$, so $f \in UCB_\ell(G/H)$.

(ii) We do not know if $C_o(G/H) \subseteq WAP(G/H)$ when H is a closed non compact subgroup.

The next lemma is in [2, p.36] for the case where $H = \{e\}$. This is originally proved by W.F. Eberlein in [10] for an abelian locally compact group.

Lemma 4.2.9. Let G be a locally compact group, H a closed subgroup, and $T = \{T_g : g \in G\}$ a weakly continuous anti representation of G on a normed linear space E with $\|T_g\| \leq 1$ for each $g \in G$. If $\{T_g x : g \in G\}$ is relatively norm [weak] compact in E for some $x \in E$ and $\Phi \in E^*$ satisfies $<\Phi, T_h y> = <\Phi, y>$ for all $y \in E$, $h \in H$, then the function $f : G/H \to \mathbb{C}$ defined by $f(gH) = <\Phi, T_g x>$ is almost periodic [weakly almost periodic].

Proof: Let $O(x) = \{T_g x : g \in G\}$. Define a linear transformation $\Psi : E \to CB(G/H)$ by $\Psi y(gH) = \langle \Phi, T_g y \rangle$ for $y \in E$. Then, Ψ is a bounded linear transformation, of norm $\leq \|\Phi\|$. Therefore Ψ is also weak-weak continuous. Hence, if $O(x)$ is relatively norm [weak] compact in E, then $\Psi(O(x))$ is relatively norm [weak] compact in $CB(G/H)$. But $\Psi(O(x)) = \{aH \to \langle \Phi, T_{ga} x \rangle : g \in G\} = \{aH \to \delta_g \square f(aH) : g \in G\} = \{\delta_g \square f : g \in G\}$. Hence, if $O(x)$ is relatively norm [weak] compact, then f is almost periodic [weakly almost periodic]. Hence we have the lemma. □.

We now state and prove the main theorem of this chapter. The proofs below were suggested by A. Lau. They are much easier than our original proof. See Remark 4.2.12 for our original proof.

Theorem 4.2.10. Let G be a locally compact group, H a closed subgroup. Then there is a unique LIM on $WAP(G/H)$.

Proof: Let $\tilde{\pi} : CB(G/H) \to CB(G)$ be as in Lemma 4.2.5. Let m be the invariant (two-sided) mean on $WAP(G)$. Define a function m_1 on $WAP(G/H)$ by $\langle m_1, f \rangle = \langle m, \tilde{\pi}(f) \rangle$, $f \in WAP(G/H)$. Then m_1 is a LIM on $WAP(G/H)$. If m_2 is another LIM on $WAP(G/H)$, consider the representation $T = \{\ell_{g^{-1}} : g \in G\}$ on the subspace $\tilde{\pi}(WAP(G/H))$ of $WAP(G)$. Then by the usual Hahn Banach Theorem, m_2 extends to a mean Ψ on $WAP(G)$. By Lemma 4.2.9, the function $\Psi_f : G \to \mathbb{C}$ defined by $\Psi_f(g) = \langle \Psi_f, \ell_{g^{-1}} f \rangle$ is weakly almost periodic. Hence, the conditions (i) through (iv) of (HBEP2) of [14, Definition 3.2.1] are satisfied.

Therefore, by [14, Theorem 3.2.3], m_2 extends to a LIM m' on WAP(G). Therefore, $m' = m$, by the uniqueness of m. In particular, we have $m_1 = m_2$. Hence, the theorem follows. □

Theorem 4.2.11. Let G be a locally compact group, H a closed subgroup. Then there is a unique LIM on AP(G/H).

Proof: The existence immediately follows from Theorem 4.2.10. The uniqueness can be proved exactly as above. We safely omit the details.

Remark 4.2.12. We give our original proof of Theorems 4.2.10 and 4.2.11 here.

Let f WAP(G/H). Let $\tilde{\pi}: CB(G/H) \to CB(G)$ be as in Lemma 4.2.5. Then $\tilde{\pi}(C_L(f)) = C_L(\tilde{\pi}(f))$, where $C_L(f)$, $C_L(\tilde{\pi}(f))$ denote the (weak) closed convex hulls of the orbits $O_L(f)$, $O_L(\tilde{\pi}(f))$ respectively, as $\tilde{\pi}$ is an isometry. The weak closure $C_L(\tilde{\pi}(f))$ of the convex hull of the orbit $O_L(\tilde{\pi}(f))$ is weakly compact (see [9, V.6.4]). Define $g \cdot F = \ell_{g-1}F$ for $g \in G$, $F \in C_L(\tilde{\pi}(f))$. Then $G \times C_L(\tilde{\pi}(f)) \to C_L(\tilde{\pi}(f))$ is an affine distal action. Therefore, by the Ryll-Nardzewski's fixed point theorem [19, Theorem 3.1.1], $C_L(\tilde{\pi}(f))$ has a fixed point, say $\lambda(\tilde{\pi}(f))$, for G. Then, $\lambda(\tilde{\pi}(f))$ is a unique constant function in $C_L(\tilde{\pi}(f))$, (see [19, p.39-42]). Hence, $C_L(f)$ has a unique constant function, say $\mu(f)$, so that $\tilde{\pi}(\mu(f)) = \lambda(\tilde{\pi}(f))$. Define a function m on WAP(G/H) by $m(f) = \mu(f)(H) = \lambda(\tilde{\pi}(f))(e)$, $f \in$ WAP(G/H). Then m is a left invariant mean on WAP(G/H). Let $\bar{m}$ be another LIM on WAP(G/H). Since $C_L(f)$ is the norm closure of $O_L(f)$, we can find a net $S_\alpha(f)$ of finite convex sums of left translates of such that

$$\lim_{\alpha} \|S_{\alpha}(f) - \mu(f)\|_{\infty} = 0 \tag{1}$$

So, $|\tilde{m}(f) - \mu(f)(H)| = \lim_{\alpha} |\tilde{m}(S_{\alpha}(f) - \tilde{m}(\mu(f))| \leq \lim_{\alpha} \|S_{\alpha}(f) - \mu(f)\|_{\infty} = 0$, since $\tilde{m}$ is a LIM. Hence, $\tilde{m}(f) = \mu(f)(H) = m(f)$. This finishes the proof of Theorem 4.2.10. Existence of a LIM on AP(G/H) is trivial as AP(G/H) $\subseteq$ WAP(G/H). Uniqueness follows from (1). Hence, we have Theorem 4.2.11.

Problem 4.2.13. Let G be a locally compact group, H a closed subgroup Is it true that $C_o(G/H) \subseteq$ WAP(G/H)? (See Lemma 4.2.6.)

CHAPTER V

INVARIANT MEANS ON $L^\infty(G)$ FOR A LOCALLY COMPACT NONDISCRETE GROUP WHICH IS AMENABLE AS A DISCRETE GROUP

5.1 Introduction

A Borel measurable set $E \subset G$ is permanently positive if $\bigcap_{i=1}^{N} g_iE$ is not locally λ_G-null for any finite subset $\{g_1,\ldots,g_N\}$ of G.

In Section 5.2, we prove that if G is a nondiscrete σ-comp[illegible] locally compact group then it contains a permanently positive set o[illegible] finite measure whose complement is also permanently positive. [illegible] proof uses ideas from [42, Section 3].

In Section 5.3, we prove that if G is a nondiscrete locally compact group which is amenable as a discrete group and X is a conjugate closed subspace of $L^\infty(G)$, then there is a left invariant mean m on $L^\infty(G)$ such that $m(X) = 0$ if and only if $\mathrm{dist}(1,X) = 1$. The proof of this theorem follows an idea of D. Stafney [42, Lemma 3.2]. As a corollary we obtain a result of W. Rudin [39] that if G is a locally compact nondiscrete group which is amenable as a discrete group, then a Borel set $E \subset G$ is permanently positive if and only if there is a left invariant mean m on $L^\infty(G)$ such that $m(1_E) = 1$. A consequence of this corollary is a well known result of W. Rudin [39] and E. Granirer [18] that if G is a nondiscrete group, then there is a left invariant mean on $L^\infty(G)$ which is not topologically left invariant.

5.2 Permanently positive sets

W. Rudin introduced permanently positive sets in [39, Section II] and proved that every infinite compact group contains a permanently positive set whose complement is also permanently positive. Later J. Rosenblatt proved in [36, Proposition 3.4] that a nondiscrete σ-compact locally compact group contains a strictly positive set of finite measure whose complement is also strictly positive. Nearly ten years before Rudin's paper, D. Stafney proved in [42, Theorem 3.13] that if G is a nondiscrete abelian second countable locally compact group, then Borel subsets E of G such that $\lambda_G(E) < \infty$ and both E and its complement E^c satisfy a weaker form of the notion of strict positiveness (see [42, Lemma 3.2]) [illegible] ery numerous. In this section, we adapt his arguments in [42, Section 3] to show that if G is a nondiscrete σ-compact locally compact group then it contains a Borel subset E of finite measure so that both E and E^c are permanently positive in G.

Definition 5.2.1. Let G be a locally compact group. A Borel set $E \subset G$ is *permanently positive* [PP] if $\bigcap_{i=1}^{N} g_iE$ is not locally λ_G-null for any finite subset $\{g_1,\ldots,g_N\} \subset G$. A Borel set $E \subset G$ is *strictly positive* if $\bigcap_{i=1}^{N} g_iE \cap V$ is not locally λ_G-null for any non-empty open set V and finite subset $\{g_1,\ldots,g_N\} \subset G$. Thus a strictly positive set is always PP and dense open sets are strictly positive. If G is compact a Borel set $E \subset G$ is strictly positive if and only if E is PP If G is non-compact then there are permanently positive open sets which are not strictly positive (see [36, Lemma 3.3]).

Let G be a locally compact group. For each pair A,B of Borel measurable sets in G, define the equivalence relation $A \sim B$ to mean $\lambda_G(A \Delta B) = 0$. Let M be the collection of all equivalence classes of Borel measurable subsets of G, metrized by $\rho(A,B) = \arctan \lambda_G(A \Delta B)$, $A, B \in M$. Then (M,ρ) is a complete metric space (see [9, III.7]). Let N denote the sets (equivalence classes) of finite measure. Then N is a closed subset of M and therefore (M,ρ) is also a complete metric space. Fix a compact set D and an open set V of finite positive measure. For each Borel measurable set E and each $u = (g_1,\ldots,g_m) \in D \times \ldots \times D$, the direct product of D, m times, where m is a positive integer, let

$$H_m(u,E) = \lambda_G\left(\bigcap_{j=1}^{m} g_jE \cap V\right) \quad \text{and}$$

$$K_m(E) = \inf\{H_m(u,E) : u \in D \times \ldots \times D\}.$$

<u>Lemma 5.2.2</u>. Let G be a locally compact group. Then the mapping $E \to K_m(E)$ from M to $\mathbf{R}$ is continuous.

<u>Proof</u>: We first show that if E,F are Borel measurable sets in G and $u = (g_1,\ldots,g_m) \in D \times \ldots \times D$, then

$$|H_m(u,E) - H_m(u,F)| \leq m(\sup\{\lambda_G(g^{-1}V \cap (E \setminus F)) : g \in D\} + \sup\{\lambda_G(g^{-1}V \cap (F \setminus E)) : g \in D\}) \qquad (1)$$

For any measurable sets A and B of finite measure we have,

$$|\lambda_G(A) - \lambda_G(B)| \leq \lambda_G(A \cap B^c) + \lambda_G(B \cap A^c).$$

Hence, $|H_m(u,E) - H_m(u,F)| \le \lambda_G((\bigcap_{i=1}^{m} g_iE \cap V) \cap (\bigcup_{j=1}^{m} (g_jF \cap V)^c))$ (2)

$$+ \lambda_G((\bigcap_{j=1}^{m} g_jF \cap V) \cap (\bigcup_{i=1}^{m} (g_iE \cap V)^c))$$

Considering the first term of the right hand side of (2) we have

$$\begin{aligned}
\lambda_G((\bigcap_{i=1}^{m} g_iE \cap V) \cap (\mathop{}_{j=1}^{m} (g_iF \cap V)^c)) &= \lambda_G(\bigcup_{j=1}^{m} ((\bigcap_{i=1}^{m} g_iE \cap V) \cap (g_jF \cap V)^c)) \\
&\le \lambda_G(\bigcap_{j=1}^{m} ((g_jE \cap V) \cap (g_jF \cap V)^c)) \\
&\le m\lambda_G((g_jE \cap V) \cap (g_jF \cap V)) \text{ for some } 1 \le j \le m \\
&= m\lambda_G(g_j^{-1}V \cap (E \setminus F)) \\
&\le m(\sup\{\lambda_G(g^{-1}V \cap (E \setminus F)) : g \in D\}), \\
&\qquad \text{since } g_j \in D.
\end{aligned}$$

The inequality (1) now follows by applying the same argument to the second term of the right hand side of (2). It follows from (1) that if $\{E_k\}$ is a sequence of elements from M converging to $E \in M$, then $H_m(u,E_k)$ converges to $H_m(u,E)$ uniformly for $u \in D \times \ldots \times D$, so $K_m(E_k)$ converges to $K_m(E)$. Hence, the mapping $E \to K_m(E)$ is continuous. □

Lemma 5.2.3. Let G be a locally compact group, A, B Borel measurable sets, and B have finite measure. Then the mappings $g \to \lambda_G(B \cap (A \setminus gA))$ and $g \to \lambda_G(B \cap (gA \setminus A))$ from G to $\mathbb{R}$ are continuous.

Proof: $B \cap (A \setminus gA) = (B \cap A) \setminus ((B \cap A) \cap gA)$. Now, $\lambda_G((B \cap A) \cap gA) =$

$1_{B\cap A} * 1_{A^{-1}}(g)$ is the convolution of an $L^1(G)$-function and an $L^\infty(G)$-function, and therefore the mapping $g \to \lambda_G((B \cap A) \cap gA)$ is continuous (see [19, Lemma 2.12]). $\lambda_G(B \cap (A \setminus gA)) = \lambda_G(B \cap A) - \lambda_G((B \cap A) \cap gA)$. Hence, the mapping $g \to \lambda_G(B \cap (gA \setminus A))$ is continuous. A similar argument shows the mapping $g \to \lambda_G(B \cap (A \setminus gA))$ is continuous. □

Lemma 5.2.4. Let G be a locally compact group. Then the mapping $u \to H_m(u,E)$ from $D \times \ldots \times D$ to $\mathbf{R}$ is continuous, where E is a Borel measurable subset of G.

Proof: Let $u = (g_1,\ldots,g_m)$ and $v = (a_1,\ldots,a_m)$ be elements in $D \times \ldots \times D$. Then,

$$\begin{aligned}
|H_m(u,E) - H_m(v,E)| &\leq \lambda_G((\bigcap_{i=1}^{m} (g_iE \cap V)) \cap (\bigcup_{j=1}^{m} (a_jE \cap V)^c)) \\
&\quad + \lambda_G((\bigcap_{j=1}^{m} (a_jE \cap V)) \cap (\bigcup_{i=1}^{m} (g_iE \cap V)^c)) \\
&\leq \lambda_G(\bigcup_{j=1}^{m} ((g_jE \cap V) \setminus (a_jE \cap V))) \\
&\quad + \lambda_G(\bigcup_{j=1}^{m} ((a_jE \cap V) \setminus (g_jE \cap V))) \\
&\leq \sum_{j=1}^{m} \lambda_G(g_j^{-1}V \cap (E \setminus g_j^{-1}a_jE)) \\
&\quad + \sum_{j=1}^{m} \lambda_G(g_j^{-1}V \cap (g_j^{-1}a_jE \setminus E)).
\end{aligned}$$

Since $g_j^{-1}V$ has finite measure for $j = 1,2,\ldots,m$ it follows from

Lemma 5.2.3 that $H_m(u,E)$ converges to $H_m(v,E)$ as u tends to v. Hence, the mapping $u \to H_m(u,E)$ from $D \times \ldots \times D$ to $\mathbb{R}$ is continuous. □

Corollary 5.2.5. Let G be a locally compact group. If $H_m(u,E)$ is positive for each $u \in D \times \ldots \times D$, then $K_m(E) > 0$, where E is a Borel measurable set in G. In particular, if E is a dense open set then $K \quad) > 0$ for $m = 1,2,\ldots$.

Proof: Since $D \times \ldots \times D$ is compact and the mapping $u \to H_m(u,E)$ is continuous, $H_m(u,E)$ assumes its infimum at some point in $D \times \ldots \times D$. Hence, the first part of the corollary follows. If E is dense open, then $H_m(u,E) > 0$ for $u \in D \times \ldots \times D$ and $n = 1,2,\ldots$. Hence, $K_m(E) > 0$ for $m = 1,2,\ldots$. □

In what follows we assume that G is a locally compact σ-compa nondiscrete group. Let $\{D_n\}$ be an increasing sequence of compact sets such that $G = \bigcup_{n=1}^{\infty} D_n$. Let V be a fixed nonempty open set of finite measure. For each Borel measurable set E and each $u = (g_1,\ldots,g_m) \in D_n \times \ldots \times D_n$, m times, where m,n are positive integers, let

$$H_{n,m}(u,E) = \lambda_G\Big(\bigcap_{j=1}^{m} g_i E \cap V\Big) \text{ and } K_{n,m}(E) = \inf\{H_{n,m}(u,E) : u \in D_n \times \ldots \times D_n\}.$$

The next lemma is in [18] and [19]. For the sake of completeness we give the proof.

Lemma 5.2.6. Let G be a σ-compact locally compact nondiscrete group. Then for each $\varepsilon > 0$, there exists a dense open set W such that $\lambda_G(W) < \varepsilon$.

Proof: Since G is not discrete, there exists an open neighbourhood U_n of the identity element, such that $\lambda_G(U_n) < 1/n$, where n is a positive integer. It follows from [22, Theor[illegible].7] that G has a compact normal subgroup N such that $N \subset \bigcap_{n=1}^{\infty} U_n$ and G/N is separable. Let $\{g_i\}_{i=1}^{\infty}$ be a subset of G with $\{g_iN\}_{i=1}^{\infty}$ dense in G/N. Then $\lambda_G(g_iN) = \lambda_G(N) = 0$ for each i. Hence, if $\varepsilon > 0$, is given we can find open sets $W_i \subset G$ such that $g_iN \subset W_i$ and $\lambda_G(W_i) < \varepsilon/2^i$ for each i. Let $W = \bigcup_{i=1}^{\infty} W_i$. Then clearly W is open and $\lambda_G(W) < \varepsilon$. To see that W is dense in G, let U be open set in G, and $\pi : G \to G/N$ the natural map. Then $\pi(U)$ is open in G/N, so $\pi(U) \cap \{g_iN\}_{i=1}^{\infty} \neq \phi$. Therefore, $U \cap (\bigcup_{i=1}^{\infty} g_iN) \neq \phi$. Thus, [illegible] $\bigcup_{i=1}^{\infty} g_iN$ and hence W is dense in G. □

Lemma 5.2.7. Let G be a locally compact σ-compact nondiscrete group, E a Borel measurable set of finite measure, and m,n positive integers. Then, for each $\varepsilon > 0$, there is a closed no where dense set F such that $\rho(E,F) < \varepsilon$ and $K_{n,m}(F) > 0$.

Proof: Let δ be a positive real number with $\tan^{-1} 2\delta < \varepsilon/3$. By Lemma 5.2.6 choose a dense open set U_1 so that ~~$\lambda_G(U_1)$~~ $< \delta$ and by regularity of λ_G choose an open set U_2 such that $\lambda_G(E) + \delta > \lambda_G(U_2)$ and $U_2 \supseteq E$. Let $U = U_1 \cup U_2$. Then U is a dense open set of finite measure containing E. $\lambda_G(U \Delta E) = \lambda_G(U) - \lambda_G(E) \leq \lambda_G(U_1) + \lambda_G(U_2) - \lambda_G(E) < 2\delta$, so $\rho(E,U) < \varepsilon/3$. Also by Corollary 5.2.5, we have $K_{n,m}(U) > 0$. Again by regularity choose an increasing sequence $\{A_k\}$ of compact sets such that $A_k \subset U$ and $\lim_{k\to\infty} \lambda_G(A_k) = \lambda_G(U)$ since $\{A_k\}$ converges to U in M and $K_{n,m}(U) > 0$, by Lemma 5.2.2, we can find a compact set $A \subset U$ $(A = A_k$ for some $k)$ such that $K_{n,m}(A) > 0$ and $\rho(U,A) < \varepsilon/3$. By Lemma 5.2.6 choose a sequence $\{W_k\}$ of dense open sets in G with $\lim_{k\to\infty} \lambda_G(W_k) = 0$. Then $\{A \setminus W_k\}$ converges to A in M. Since $k_{n,m}(A) > 0$, by Lemma 5.2.2, there is a dense open set W $(W = W_k$ for some $k)$ of arbitrarily small measure such that $\rho(A \setminus W, A) < \varepsilon/3$ and $K_{n,m}(A \setminus W) > 0$. Finally set $F = A \setminus W$. Then F is closed, nowhere dense, $K_{n,m}(F) > 0$, and $\rho(E,F) < \varepsilon$. □

We are now in a position to prove our main theorem of this section.

Theorem 5.2.8. Let G be a locally compact σ-compact nondiscrete group. Then there is a Borel measurable set $E \subset G$ with $\lambda_G(E) < \infty$ such that both E and E^c are permanently positive.

Proof: For each pair of positive integers m,n, let $Q(n,m) = \{E \in N : K_{n,m}(E) = 0\}$ and $Q'(n,m) = \{E \in N : K_{n,m}(E^c) = 0\}$. Then $Q(n,m)$ and $Q'(n,m)$ are the kernels of the maps $E \to K_{n,m}(E)$ and $E \to K_{n,m}(E^c)$ respectively. The first map is continuous on M by Lemma 5.2.2. The second map is a composition of the first map and the map $E \to E^c$ which is clearly continuous. Therefore, the sets $Q(n,m)$ and $Q'(n,m)$ are closed in N. Let $E \in N$ and $\varepsilon > 0$. Then, by Lemma 5.2.7, there is a closed nowhere dense set F such that $\rho(E,F) < \varepsilon$ and $K_{n,m}(F) > 0$. Since F^c is open and dense, by Corollary 5.2.5, we also have $K_{n,m}(F^c) > 0$. This implies that both $Q(n,m)$ and $Q'(n,m)$ have empty interior. Hence, by the Baire category theorem there is a set $E \in N$ such that E is not in any of the $Q(n,m)$ or $Q'(n,m)$ for $m, n = 1,2,\ldots$. That is, $K_{n,m}(E) = \inf\{\lambda_G(\bigcap_{j=1}^{m} g_j E \cap V) : u = (g_1,\ldots,g_m) \in D_n \times \cdots \times D_n\} > 0$ and

$K_{n,m}(E^c) = \inf\{\lambda_G(\bigcap_{j=1}^{m} g_j E^c \cap V) : u = (g_1,\ldots,g_m) \in D_n \times \cdots \times D_n\} > 0$ for $m,n = 1,2,\ldots$. This implies, in particular that both E and E^c are PP. ☐

Remark 5.2.9. (i) In fact, by Lemma 3.3 in [36], both E and E^c are strictly positive.

(ii) By Corollary 5.2.5, $Q(n,m) = \{E \in N : \lambda_G(\bigcap_{j=1}^{m} g_j E \cap V) = 0$ for some $g_1,\ldots,g_m \in D_n\}$ and $Q'(n,m) = \{E \in N : \lambda_G(\bigcap_{j=1}^{m} g_j E^c \cap V) = 0$ for some $g_1,\ldots,g_m \in D_n\}$. Hence, by the Baire category theorem the Borel sets E of finite measure such that both E and E^c are strictly positive are dense and second category in N.

(iii) Theorem 5.2.8 is already known (see [36, Proposition 3.4]) and the proof uses similar types of arguments as above. ☐

5.3 Invariant means on $L^\infty(G)$

In this section we prove that if G is a locally compact non-discrete group which is amenable as a discrete group, then there is a left invariant mean on $L^\infty(G)$ which is not topologically left invariant.

Theorem 5.3.1. Let G be a locally compact group which is amenable as a discrete group. If X is a conjugate closed left invariant subspace of $L^\infty(G)$, then the following statements are equivalent.

(i) There is a left invariant mean m on $L^\infty(G)$ such that $m(X) = 0$.

(ii) ess inf $h \leq 0$ for all real valued $h \in X$.

(iii) $\mathrm{dist}(1,X) = 1$.

Proof: By Proposition 3.2.2 we only have to prove (iii) ⇒ (i).

Following an idea of D. Stafney [42, Lemma 3.2], consider the anti representation $T = \{\ell_a : a \in G\}$, the subspace $F = \{\lambda 1 : \lambda \in \mathbb{C}\}$, and the linear functional ϕ on F given by $\phi(\lambda 1) = \lambda$, $\lambda \in \mathbb{C}$. Define a seminorm p on $L^\infty(G)$ by $p(f) = \inf\{\|f - h\|_\infty : h \in X\}$, $f \in L^\infty(G)$. Then, (i) clearly $\ell_a F \subseteq F$ for each $a \in G$.

(ii) $p(\ell_a f) \leq p(f)$ for all $a \in G$, $f \in L(G)$, since X is left invariant.

(iii) $\phi(\ell_a f) = \phi(f)$ for all $a \in G$, $f \in X$.

(iv) $|\phi(\lambda 1)| \leq p(\lambda 1)$ for each $\lambda 1 \in F$, as $\operatorname{dist}(1,X) = 1$.

By the usual Hahn Banach theorem ϕ extends to a linear functional ψ on $L^\infty(G)$ such that $|\psi(f)| \leq p(f)$ for each $f \in L_\prime(G)$. Hence, the condition (v) of (HBEP) (see Definition 3.4.1) is satisfied when the group G has the discrete topology. Therefore, since G is amenable as discrete, the linear functional ϕ extends to a linear functional m on $L^\infty(G)$ such that

(a) $m(\ell_a f) = m(f)$ for all $a \in G$, $f \in L^\infty(G)$

(b) $|m(f)| \leq p(f)$ for each $f \in L^\infty(G)$.

It follows then that m is a left invariant mean on $L^\infty(G)$ such that $m(X) = 0$. □

Remark 5.3.2. Let G be a locally compact group, H a closed subgroup, and G/H support an invariant measure ν. Suppose there is a left invariant mean on B(G/H), the bounded functions on G/H. If X is a conjugate closed left invariant subspace of $L^\infty(G/H,\nu)$ then the following statements are equivalent.

(i) There is a LIM m on $L^\infty(G/H,)$ such that $m(X) = 0$.

(ii) ess inf $h \leq 0$ for all real valued $h \in X$.

(iii) $\text{dist}(1,X) = 1$.

The proof is exactly as above. Note that by the usual Hahn Banach theorem and by the invariance of ν, the condition (v) of (HBEP) is satisfied. ☐

Corollary 5.3.3. Let G be a locally compact group which is amenable as a discrete group. If J is a left invariant ideal in $L^\infty(G)$ then there is a LIM m on $L^\infty(G)$ such that $m(J) = 0$ if and only if $J \neq L^\infty(G)$.

Proof: If $J \neq L^\infty(G)$ then there is a multiplicative linear functional ϕ on $L^\infty(G)$ such that $\phi(J) = 0$. Then $\text{dist}(1,J) = 1$. Hence, by Theorem 5.3.1, there is a LIM m on $L^\infty(G)$ such that $m(J) = 0$. The converse is trivial since $1 \notin J$. ☐

Corollary 5.3.4. Let G be a locally compact group which is amenable as a discrete group. Then a Borel measurable set $E \subset G$ is permanently positive if and only if there is a LIM m on $L^\infty(G)$ such that $m(1_E) = 1$.

Proof: If $J = \{f \in L^\infty(G) : f = 0$ on $\bigcap_{i=1}^{N} g_iE$ for some $g_1,\ldots,g_N \in G$, and $N < \infty\}$ then J is a left invariant ideal in $L^\infty(G)$. E is PP if and only if $\bigcap_{i=1}^{N} g_iE$ is not locally null for any finite subset $\{g_1,\ldots,g_N\} \subset G$. Therefore, E is PP is and only if $1 \notin J$, and hence if and only if $J \neq L^\infty(G)$. Now the corollary follows from Corollary 5.3.3. ☐

Corollary 5.3.5. Let G be a locally compact group which is amenable as a discrete group. Then a Borel set $E \subset G$ is PP if and only if $\text{dist}(1_E, \pi_o) \geq 1$, where π_o is the linear subspace of $L^\infty(G)$ spanned by $\{\ell_a f - f : a \in G,\ f \in L^\infty(G)\}$.

Proof: If E is PP, then by Corollary 5.3.4 there exists a left invariant mean m on $L^\infty(G)$ such that $m(1_E) = 1$, so $\text{dist}(1_E, \pi_o) \geq 1$. Conversely, if $\text{dist}(1_E, \pi_o) \geq 1$, by the Hahn Banach theorem there exists a linear functional m on $L^\infty(G)$ such that $m(1_E) = 1$, $m(\pi_o) = 0$, and $\|m\| \leq 1$. Then m is a LIM on $L^\infty(G)$ with $m(1_E) = 1$. Thus E is PP by Corollary 5.3.4. □

Remark 5.3.6. (i) Let G be a σ-compact locally compact group. If for each permanently positive set $E \subset G$, $\text{dist}(1_E, \pi_o) \geq 1$, then G is amenable as a discrete group (see [3, p.46-50]).

(ii) For a discrete group G, a set $E \subset G$ is PP is and only if it is left thick in the sense of Mitchell [30]. Therefore, Corollary 5.3.4 can be considered as a generalization of his result [30, Theorem 7] (see [3, p.49]).

The following theorem was proved independently by W. Rudin [39] and E. Granirer [18]. See also Lemma 3.2 of D. Stafney [42]. We present Rudin's proof here.

Theorem 5.3.7. Let G be a locally compact nondiscrete group which is amenable as a discrete group. Then there is a left invariant mean on $L^\infty(G)$ which is not topologically left invariant.

Proof: Let $\phi \in P(G) \cap C_{oo}(G)$. Then by absolute continuity there is a $\delta > 0$ such that

$$\int_G \phi(g)dg < 1/2 \quad \text{whenever} \quad \lambda_G(E) < \delta \cdot \tag{1}$$

Let $U = \{g \in G : (g) \neq 0\}$. Then U is open and $\bar{U}$ is compact. We assume that U has the identity element e. Let G_o be the subgroup of G generated by $\bar{U}$ so that $G_\sigma = \bigcup_{n=1}^{\infty} (\bar{U} \cup (\bar{U})^{-1})^n = \bigcup_{n=1}^{\infty} (U \cup U^{-1})^n$. Then G_o is open (and hence closed). Since G_o is a locally compact, compactly generated nondiscrete group, by Lemma 5.2.6, G_o contains an open dense set A_o with $\lambda_G(A_o) < \delta$ (see Remarks in [18, p.619]). Write $G = \bigcup_{\alpha \in I} g_\alpha G_o$, $g_\alpha G_o \cap g_\beta G_o = \phi$ if $\alpha \neq \beta$. Let $A = \bigcup_{\alpha \in I} g_\alpha A_o$. Then clearly $E = A \cap A^{-1}$ is open and dense in G. By Corollary 5.3.4, there is a LIM m on $L^\infty(G)$ such that $m(1_E) = 1$, as E is PP.

Let $f = 1 - 1_E$. Then, $m(f) = 0$ (2)

$$\phi * f(a) = \int_G \phi(g)f(g^{-1}a)dg$$

$$= 1 - \int_G \phi(g)1_E(g^{-1}a)dg$$

since $E = E^{-1}$, $1_E(g^{-1}a) = 1_{aE}(g)$. Thus,

$$\phi * f(a) = 1 - \int_{aE} \phi(g)dg.$$

Since $\lambda_G(G_o \cap aE) < \delta$, it follows from (1) that $\phi * f(a) > 1/2$ for $a \in G$. Therefore, $m(\phi * f) \geq 1/2$, because m is a mean. (3)

m is not topologically left invariant by (2) and (3). This finishes the proof. □

<u>Remark 5.3.8</u>. (i) Let G be a non-discrete σ-compact locally compact

group which is amenable as a discrete group. Then the set of all left invariant means on $L^{\infty}(G)$ which are not topologically left invariant is not norm-separable (see [36, Theorem 3.6.3]).

(ii) Recall that every abelian and hence solvable group is amenable as a discrete group (see Section 3.7A). Also the orthogonal groups $SO(n,\mathbb{R})$, $n \geq 3$, are compact and hence amenable. However, since they contain a free subgroup on two generators, they are not amenable as discrete groups (see [34, Section 14]). It is known that $L^{\infty}(G)$ has a unique LIM for $G = SO(n,\mathbb{R})$, $n \geq 5$. (See [29], [36] and [43]).

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