I believe as I did as a child, that life has meaning, direction, and value;
that no suffering is lost, that each drop of blood and every tear counts;
and that the meaning of the world is to be found in St. John's
"Deus Caritas Est" – God Is Love.
Francois Mauriac

University of Alberta

Imperfect hedging and risk management of equity-linked life insurance contracts

by

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A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematical Finance

Department of Mathematical and Statistical Sciences

Edmonton, Alberta
Fall 2006

Your file Votre référence
ISBN: 978-0-494-23103-6
Our file Notre référence
ISBN: 978-0-494-23103-6

Abstract

Pricing of insurance contracts has generated much interest among researchers and practitioners in the last two decades. Rapid mortality decline in developed nations calls for methodologies that properly assess and price risks entailed in insurance contracts. To address this problem, we propose the use of two types of imperfect hedging techniques – quantile and efficient hedging. We show that they are effective tools for managing both financial and insurance risk elements inherent in equity-linked life insurance contracts. Financial risk comes from the volatility of the financial instruments underlying the contract, while insurance risk arises from the dependence of the payoff on the client's survival to maturity of the contract.

First, we introduce the two hedging methodologies and show why they are attractive for pricing of equity-linked life insurance contracts. We give explicit theoretical results for the price of a contract paying the maximum of two risky asset values at maturity, provided the contract buyer survives to this date. We also prove a result which allows straightforward generalization of our approach to payoffs with n risky assets. Using numerical examples, we demonstrate risk management possibilities for the seller of the contract and the advantages of applying quantile or efficient hedging. These methods are computationally inexpensive, intuitive, and flexible in terms of risk management yet precise in quantifying financial and insurance risks.

Next, we study modern mortality trends in the context of imperfect hedging. We analyze the classical mortality models of Gompertz and Makeham, and the recently developed approach of Lee and Carter for fitting and forecasting mortality. Thus we extend the topic of quantile and efficient hedging beyond financial mathematics into actuarial science. By performing a comparative study between the United States, Sweden and Japan, we show that mortality model selection carries significant implications for risk management in equity-linked life insurance.

Acknowledgements

I am very grateful to my supervisor, Dr. Alexander Melnikov, for suggesting such an interesting area of research for my graduate studies, for acquainting me with the world of mathematical finance in graduate lectures and research-related discussions, for patient guidance throughout my academic career and support of post-graduate plans, and for encouragement both at happy times and in moments of adversity.

I am very thankful to Dr. Felipe Aguerrevere for introducing me to a number of different topics and fascinating perspectives in finance and economics during courses at the University of Alberta School of Business and in communications outside the classroom, and for continual support and encouragement of my interests in finance as well as career plans.

I am grateful to my professors and colleagues for the knowledge and insight shared with me during lectures and otherwise.

I am also eternally thankful to my family and friends for being in my life and encouraging me, throughout my graduate experiences and always.

To Nadiya, Olha, Boyan and LittleBun

Please observe an alternative way to spell the author's name: Yulia V. Romaniuk

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1 Introduction

1.1 General idea and goal of the dissertation

Equity-linked insurance contracts have been studied since the middle of the 1970s. The payoff in such contracts depends on two factors: the value of some underlying financial instrument(s) (hence the term *equity-linked*), and some insurance-type event in the life of the owner of the contract (death, retirement, survival to a certain date etc.). As such, the payoff contains both financial and insurance risk elements, which have to be priced so that the resulting premium is fair to both the seller and the buyer of the contract. The famous result of Black and Scholes (1973) and Merton (1973) tells us that, in an idealized market setting, as long as the seller obtains a price equal to the expectation under the risk-neutral probability measure of the discounted payoff, the seller can hedge this payoff perfectly – with probability of successful hedging equal to 1. Perfect hedging relies on the ability to trade the financial asset underlying the option and the option itself in a particular manner so as to offset any movement in the values of the underlying and the option. However, mortality risk cannot be offset in the same manner, since mortality is not (yet) traded, which makes the insurance market incomplete and renders perfect hedging of equity-linked life insurance contracts impossible (see section 2.3.2 for more details).

The main goal of this dissertation is to address the problem of appropriate pricing of equity-linked life insurance contracts and hedging of the risks involved. As will be discussed in section 2.1, a number of imperfect hedging techniques have been applied to price equity-linked insurance agreements. We choose to use two imperfect hedging techniques: quantile and efficient hedging, developed respectively in Foellmer and Leukert (1999) and Foellmer and Leukert (2000). Quantile hedging allows the hedger to maximize the probability of successful hedging, while efficient hedging seeks to minimize the expected shortfall risk, which is the expected potential loss from the hedging strategy, weighted by some loss function reflecting the hedger's risk preferences. Developed from sophisticated statistical testing techniques, quantile and efficient hedging are powerful tools which allow for many quantitative risk management possibilities; at the same time, these methods are computationally practical, understandable, and justifiable not only to academics, but to practitioners in the insurance industry.

We consider a single-premium equity-linked life insurance contract which enables its holder to receive the greater of the values of two risky assets (such as stocks) at maturity of the contract, provided the policyholder survives to this date. We solve the question of optimal pricing and hedging of such a contract in the context of quantile and efficient hedging and illustrate the idea numerically. We also prove a theoretical result which allows our methodology to be generalized to a contract involving n risky assets in a straightforward manner, with the only difference in derivations of pricing formulas between two and n risky assets being the time spent on calculations. In this, we extend the application of quantile and efficient hedging for a budget-constrained investor from a setting with a single risky asset (see

Melnikov and Romaniuk (2006)) to a multi-asset Black-Scholes-Merton-type setting.

We also extend our studies to investigate some relevant problems in the field of actuarial science. The modelling of mortality is a hot topic today, in light of changes in survival probability patterns over the last century. It has been noted (see, for example, Horiuchi and Wilmoth (1998), Koissi et al. (2006) or Tuljapurkar et al. (2000)) that mortality rates in developed countries have been declining, where as increases in life expectancy have been underestimated. Such trends call for the assessment of presently used mortality models to investigate whether these models reflect the current mortality patterns adequately. To this purpose, we study three mortality models of Gompertz, Makeham and Lee-Carter (see Lee and Carter (1992)); the first two are classical actuarial models, widely used today by practitioners, and the third is the newly developed model which treats mortality as a stochastic process. Such study directly affects the risk implications for the type of equity-linked insurance contract considered here, since survival probability is one of the factors on which the contract's payoff depends.

Below we give motivation for the problem studied in this dissertation, namely, proper pricing and risk management of equity-linked life insurance contracts, as well as reasons behind considering the particular area of mortality modelling as an extension of the main results and setting.

1.2 Motivation

Insurance industry has been growing at a tremendous pace in the last decade, especially with the development of new markets in Europe and Asia. Equity-linked and unit-linked (contracts paying one unit of some risky asset) business has been especially successful. For example, Swiss Re reported about world insurance growth in 1997 "high growth in the life business in Europe, North America and the emerging markets in Western Europe, life business grew after adjustment for inflation by 10.5 %, and in North America by 6.9 % ... The high growth rates were spurred in particular by dynamic business in unit-linked and index-linked insurance products" (Swiss Re online (1997)). National Association for Variable Annuities notes that total industry sales of equity-indexed annuities grew from .2 to 12.6 billion dollars from 1995 to 2003 (NAVA (2004)). Additionally, in Spain in 2000, the "greatest growth occurred in unit-linked [insurance products] (81 % in comparison with 21 % in other types)" (see Spanish Institute for Foreign Trade (2002)), and Winterthur Life achieved a "strong and remarkable growth in unit-linked business" in Hong Kong in 2003 and launched markets for unit-linked insurance in 2001 and 2002 in Japan and Taiwan respectively (Winterthur Life online (2004)).

While growth in equity-linked (unit-linked, or equity-indexed) business is great, it is not clear whether insurance companies are developing risk measures and hedging strategies necessary to deal with all the risks the company undertakes when it sells equity-linked contracts. In the insurance industry, the effects of failing to adopt adequate risk management models can be devastating. If the company overestimates and overprices its risks, the consumer will bear the financial burden of excessive insurance premiums, which is likely to lead to governmental inquiries and regulation, as evident in Canada today. On the other hand, if the company undervalues its risks, it may face million-dollar losses and lose investors' and shareholders' confidence. While some events causing damage and deaths and calling for massive insurance claims cannot be predicted (terrorist activities, hurricanes, tsunami etc.), price movements in the underlying assets, as well as long-term mortality patterns (deceleration of mortality at older ages, increasing life expectancy etc.) can be analyzed qualitatively and quantitatively. Therefore, the question of finding hedging methodologies that can account for and value the financial and insurance risk elements, and provide appropriate risk management strategies is of great interest and importance from both theoretical and practical perspectives.

One of the impediments to risk pricing arises from the ever-increasing market demand for flexible and personalized insurance products. To respond to this demand and compete with financial institutions (banks, trusts, mutual funds etc.), insurance firms quickly develop and advertise, along with traditional life and health insurance, comparative products for investment and wealth management. The latter instruments are attractive to investors, as they tend to have shorter maturities and more exposure to financial market risk than traditional insurance contracts. However, it is often impossible to design and implement pricing tools that adequately measure and valuate both financial and insurance risks before equity-linked products hit the market, resulting in mispriced portfolios and potentially negative repercussions in terms of losses and governmental regulations. Now, there has been a number of suggestions for hedging methods in connection to insurance. These include perfect and mean-variance hedging, as well as numerical and simulation techniques (Brennan and Schwartz (1976), Brennan and Schwartz (1979), Boyle and Schwartz (1977), Delbaen (1986), Bacinello and Ortu (1993), Aase and Persson (1994), Ekern and Persson (1996), Boyle and Hardy (1997), Bacinello (2001)), utility-based indifference pricing (see Hodges and Neuberger (1989), Young (2003)), and risk-minimization strategies (Moeller (1998), Moeller (2001), Cvitanic and Karatzas (1999) and Cvitanic (2000)). While these approaches have their merits, we feel that quantile and efficient hedging are particularly suited for equity-linked insurance, because these hedging techniques are very intuitive, easily implementable, demand little in terms of computing power, as well as allow the hedger to calculate explicitly all the necessary premiums, risks, and hedging strategies.

The need for correct mortality modelling arises from the general pattern of decline in mortality in many developed countries in the last century (see, for example, Wilmoth and Horiuchi (1999), Tuljapurkar et al. (2000), Lynch and Brown (2001), Yashin et al. (2001)). Frequently, the rates of decline of mortality for older ages and increase of life expectancy have been underestimated (Koissi et al. (2006)). The rectangularization of the survival probability curve, together with lower mortality at older ages and higher life expectancy are particularly troublesome developments for insurance firms: the changing mortality profile demands appropriate assessment methods and adequate actuarial tools to manage mortality risks. Recently, the topic of mortality risk has become the focus of attention of actuarial and insurance companies (Pitacco (2003)) and the topic of numerous conferences and symposiums dealing specifically with the question of measurement and pricing of mortality risk. Once the company selects a particular mortality model, the decision will be directly reflected in the prices of and risk management strategies for insurance portfolios, since survival patterns affect basically all products in the insurance industry: life and health, accident and disability, as well as investment and wealth management. We investigate mortality modelling to demonstrate how survival trends displayed by different categories of insured clients and transmitted by the choice of a mortality model can be taken into account when designing risk management strategies for the equity-linked contracts in consideration.

1.3 Outline of the thesis

In chapter 1, we introduced general ideas behind the research topics discussed in this dissertation. Section 1.2 gave motivation behind the particular topics studied: imperfect hedging, its applications to life insurance, and effects of mortality modelling on risk management with insurance contracts.

In chapter 2, quantile and efficient hedging are discussed in detail. First, we review the statistical origins behind the two hedging methodologies (section 2.2.1). Next, we describe the methods as presented in Foellmer and Leukert (1999) and Foellmer and Leukert (2000) (sections 2.2.3 and 2.2.4) and show how they apply in our setting with two risky assets (section 2.3.3). The financial and insurance settings in which we study the problem of optimal pricing and hedging under budget constraints are presented formally in section 2.3. We proceed to present one of the main contributions of this thesis: the explicit pricing formulas for equity-linked life insurance contracts based on quantile and efficient hedging (section 2.4); for efficient hedging, there are three cases, one for each of the risk preference scenarios of the hedger: risk-aversion, risk-taking and risk-indifference. The results are given as theorems and are followed by their respective proofs.

Next, we discuss how to generalize the application of quantile and efficient hedging to contracts involving n risky assets (section 2.5). We state the theorem (which we label the *multi-asset theorem*) that is required for such generalization. The proof of the theorem is given separately in Appendix 2 due to its technicality. Then we illustrate how the risk management strategies resulting from quantile and efficient hedging can be applied in practice (section 2.6). We also examine how large the expected losses resulting from the application of imperfect hedging can get (section 2.7) and provide a numerical example to support the theory (section 2.7.1).

Chapter 3 deals with the effects of mortality modelling on risk management with equity-linked life insurance contracts. Section 3.1 provides background on mortality modelling; section 3.2 describes in more detail the three mortality models (Gompertz, Makeham and Lee-Carter) studied in this dissertation and provides the definitions of the necessary actuarial concepts (section 3.2.1). The setting for the problem is described in sections 3.3 and 3.4, and derivation of the pricing formulas is discussed in section 3.5.1. After this, we present numerical illustrations to highlight the implications of insurance and financial risk elements for risk management with equity-linked life insurance contracts (section 3.6); the estimation of the parameters used by the three mortality models is described in section 3.6.1. We suggest several directions for future studies in the area of imperfect hedging and mortality modelling in section 3.7.

The dissertation concludes with bibliography and Appendices 1, 2 and 3, in which the derivation of the explicit formula for the density of the risk-neutral probability measure, the proof of the multi-asset theorem, and tables with the estimated parameters for the Lee-Carter model are given respectively.

2 Quantile and efficient hedging for pricing of equity-linked life insurance contracts

2.1 Background on imperfect hedging in insurance

Soon after the celebrated papers by Black and Scholes (1973) and Merton (1973) on the pricing of call options, the topic of pricing of equity-linked insurance contracts became rather popular. As mentioned above and explained in detail in section 2.3.2, equity-linked contracts incorporate financial and insurance risk elements, and perfect hedging in the sense of Black, Scholes and Merton does not work: mortality risk of the option holder cannot be offset by trading in the insurance market, as mortality is not a traded asset.[1] This section will review some of the research that has been done in the area of pricing risks entailed in equity-linked insurance products.

Brennan and Schwartz (1976), Brennan and Schwartz (1979) consider an equity-linked life insurance policy with an asset value guarantee and determine the value of such policy using the economic concept of equilibrium pricing; the calculated value corresponds to the perfectly competitive price. Note that the payoff is just the greater of some guaranteed fixed amount or the value of the underlying risky fund at maturity of the policy. The authors also propose a strategy which would eliminate financial risk inherent in the payoff of the contract, but indicate that the insurance company would have to hold reserves to offset mortality risk, or sell a large number of contracts in hopes of eliminating mortality risk (this is known as pooling). In this sense, financial risk is hedged perfectly, but no strategies other than the traditional ones have been suggested to deal with insurance risk.

Boyle and Schwartz (1977) work out a similar solution for death benefit and maturity benefit guarantee contracts, which pay the larger of a fixed guarantee or value of some risky fund at expiration of the contract or upon the death of the policyholder. Delbaen (1986) extends previous articles by proposing Monte-Carlo simulation to price fixed term equity-linked contracts with guarantee, for which premiums have to be paid periodically and survival probability of the client is factored into the value of the contract. Monte-Carlo methodology is also applied to calculate the number of shares of the underlying risky fund to be included in the policyholder's benefit (with periodic premiums, this number no longer equals unity); in Brennan and Schwartz (1976) and Brennan and Schwartz (1979), the number of shares in the policy benefit for contracts with periodic premiums is determined numerically as a solution to the partial differential equation arising from the problem setup.

Bacinello and Ortu (1993) further build on the above papers by considering the case of equity-linked contracts where guarantees are determined endogenously based on the premiums paid, as opposed to being specified exogenously, as in Brennan

[1] However, it appears that market for mortality is slowly developing. Special thanks to the anonymous referee of our submission (Melnikov and Romaniuk (2006)) for this interesting and useful observation.

and Schwartz (1976), Brennan and Schwartz (1979) and Delbaen (1986). Aase and Persson (1994) work in a setting where the number of shares of the underlying risky fund included in the benefit is non-random, which is different from the setting of Brennan and Schwartz (1979), Boyle and Schwartz (1977) and Delbaen (1986) (see above). Although such setting is simpler, it allows the authors to derive analytic solutions for the premiums of contracts in consideration and avoid simulations or numerical solutions. Note, however, that certain unclear assumptions are made in order to obtain explicit pricing formulas in Aase and Persson (1994), for example, "risk-neutrality with respect to mortality" (Theorem 1 on p. 37).

Ekern and Persson (1996) calculate premiums for a large variety of equity-linked contracts, including those with payoffs where the contract owner chooses the larger of the values of two risky assets (and possibly a guaranteed amount) at maturity of the contract, similar to the payoffs we consider in this dissertation. However, the authors completely disregard the pricing of mortality risk, calling it "unsystematic risk" for which "the insurer does not receive any compensation" (see p. 39 in Ekern and Persson (1996)). The justification provided is the traditional argument that mortality risk can be eliminated by selling a large number of equity-linked contracts. Note, however, that such justification is not acceptable in the current insurance research.[2]

Boyle and Hardy (1997) examine the pricing of and reserving for maturity guarantees for policies where the policyholders' premiums are invested in a specified portfolio which is guaranteed not to fall below a certain level at maturity. The authors compare two approaches, stochastic simulation and options pricing, to price and calculate reserves, and find that relative merits of each of the methods depend, among other factors, on the nature of the guarantee. Bacinello (2001) analyzes the pricing of one of the most common life insurance policies sold in Italy in the last two decades; the contract involves a bonus rate, whose value depends on the performance of some reference fund and is not allowed to fall below some minimum interest rate guaranteed.

Moeller (1998) looks at a portfolio of equity-linked contracts, thus incorporating financial risk and group mortality risk into the setting. The author defines risk process as conditional expectation with respect to the risk-neutral measure of squared errors in future costs (the squared difference between the cost of the hedging strategy at maturity of the contract and the current cost). The hedging strategy (the number of shares in the underlying risky asset) is then determined uniquely by minimizing the squared errors described above and the squared error from time 0 to maturity T. Moeller (1998) also shows that for a portfolio of contracts, at any time the number of shares of risky asset underlying the payoff will depend on the expectation about the number of individuals surviving to that point in time. In his later paper (see Moeller (2001)), the author examines a portfolio of equity-linked life insurance contracts and determines risk-minimizing strategies in a discrete-time

[2]Among many useful comments on our paper (Melnikov and Romaniuk (2006)), the referee has indicated that studies have shown that pooling mechanisms do not work.

setting for the Cox-Ross-Rubinstein model.

Utility-based indifference pricing approach was introduced in Hodges and Neuberger (1989); in this pricing methodology, the premium for the contract is calculated in such a way as to make the hedger (the insurance company, in our case) indifferent between including and not including a specified number of contracts in his/her portfolio (see also Argesanu (2004), Kuehn (2002)). The method has been extended to equity-linked insurance contracts by Young and Zariphopoulou (2002a) and Young and Zariphopoulou (2002b), who look at the utility-based pricing when insurance risks are independent of the underlying financial asset, and Young (2003), where the death benefit payable to the insured client depends on the value evolution of the underlying.

Spivak and Cvitanic (1999) consider the problem of maximizing the probability of an agent's wealth at maturity being no less than the value of a contingent claim with the same expiration date; duality method from utility maximization literature is used to analyze this problem. Applying the same methodology, Cvitanic and Karatzas (1999) study dynamic measures of risk that look at the "worst-case" scenario: the measures consider the largest (worst) possible, minimized over all real-world probability measures, shortfall (see also Kirch (2001)). Shortfall is defined as the expectation of the positive part of the difference between the value of the contingent claim at maturity of the contract and terminal value of the hedging strategy. Cvitanic (2000) shows that in incomplete markets, as long as all equivalent martingale measures are included in the set of possible real-world probability measures, the "max-min" quantity, which is the maximal minimized shortfall, will coincide from the perspective of both the seller and the buyer of the contract. Although we do not know of explicit applications of the above approaches to equity-linked insurance, the ideas are similar to those of quantile hedging (Foellmer and Leukert (1999)), which aims at maximizing the probability of successful hedging, and efficient hedging (Foellmer and Leukert (2000)), whose goal is to minimize the expected shortfall, weighted by the hedger's risk preference. These two methods will be described in detail in the next section.

Building on the papers mentioned in the previous paragraph, Nakano (2004) considers the minimization of shortfall risk in a jump-diffusion setting, but unlike Foellmer and Leukert (2000), who impose a nonnegativity constraint on the wealth process, the author only requires it to be integrable. Based on this, the optimal portfolio and the optimal terminal wealth are derived explicitly. Kirch and Runggaldier (2004) study the applications of efficient hedging when asset prices follow a geometric Poisson process, where price changes occur at random points in time (this is a natural generalization of the Cox-Ross-Rubinstein model), with intensities constant in time, but not necessarily known to the investor.

Finally, quantile and efficient hedging methods have been applied to price equity-linked insurance products on various occasions. Krutchenko and Melnikov (2001), Melnikov (2004a), and Melnikov and Skornyakova (2005) apply quantile hedging in the context of diffusion and jump-diffusion models. The latter paper examines equity-linked life insurance contracts with flexible guarantees. Melnikov (2004b) studies optimal pricing utilizing an efficient hedging approach in a diffusion setting, and Kirch and Melnikov (2005) use efficient hedging in a jump-diffusion framework with perfectly correlated Wiener processes to price equity-linked life insurance contracts with fixed guarantees. Melnikov et al. (2005) extend these results to correlated Wiener processes. Melnikov and Romaniuk (2006) show how quantile hedging can be applied to price and hedge financial and mortality risk elements inherent in equity-linked life insurance contracts with deterministic guarantees, as well as examine mortality modelling and its effects on the resulting risk management strategies. The results presented in Melnikov and Romaniuk (2006) are also a part of this dissertation and will be given in chapter 3.

2.2 Quantile and efficient hedging

Here we describe the two imperfect hedging approaches used in the thesis: quantile and efficient hedging. We begin with a brief discussion of the Neyman-Pearson lemma (as summarized in Korn and Korn (2000)), since both hedging methods are based on this important statistical result.

2.2.1 Neyman-Pearson lemma

Suppose that we want to test the null hypothesis h_0 with probability measure P_0 on the space $(\Omega, \mathcal{F}, P_0)$ against an alternative h_1, with probability measure P_1 on $(\Omega, \mathcal{F}, P_1)$. There are four possible outcomes of the test:
1. accept h_0 when it is true,
2. accept h_0 when it is false: this is called *Type II error*, denoted here β,
3. reject h_0 when it is true: this is called *Type I error*, denoted here α,
4. reject h_0 when it is indeed false (this is usually the desired outcome for the test).
We reject h_0 based on whether the value of some test statistic falls into the rejection region, which we denote R.

Note that for outcome (2), the 'true' probability measure is P_1, as h_1 is the hypothesis that holds in the real world. Thus in (2), the test statistic falls into the acceptance region, R^C, and

$$\beta = P_1(R^C). \tag{2.1}$$

For outcome (3), the 'true' probability measure is P_0, as h_0 is the correct hypothesis, thus when we reject h_0, we get the Type I error

$$\alpha = P_0(R). \tag{2.2}$$

Generally, the probability measure corresponding to the alternate hypothesis is taken as the known, or real-world probability measure, and the aim of the test is to make the correct decision as given in outcome (4). That is, we want to reject h_0 in favor of h_1 when h_1 (and thus P_1) holds; the probability for outcome (4) is called *power of the test*, and we have

$$\text{power of the test} = 1 - \beta = P_1(R). \tag{2.3}$$

When testing the two hypotheses, we usually want to control the size of Type I error while minimizing the Type II error, or, equivalently, fix α and maximize the power of the test $1-\beta$. Then the test is referred to as being conducted at $1-\alpha$ *significance level.*

The Neyman-Pearson lemma provides the structure of the set on which, for a given significance level α, the power of the test $1-\beta$ is maximized.

Neyman-Pearson lemma

Define

$$\tilde{A} = \left\{ \frac{dP_1}{dP_0} > \tilde{k} \right\}, \text{ where } \tilde{k} = \inf \left\{ k : P_0 \left(\frac{dP_1}{dP_0} > k \right) \leq \alpha \right\}. \tag{2.4}$$

Suppose $P_0(\tilde{A}) = \alpha$. Then for all $A \in \mathcal{F}$ such that $P_0(A) \leq P_0(\tilde{A})$,

$$P_1(A) \leq P_1(\tilde{A}). \tag{2.5}$$

For us, $\tilde{A}$ is the 'optimal' rejection region that maximizes the probability of making the correct decision: rejecting the false h_0 in favor of h_1 and working with the real-world probability measure P_1. The conclusion of the lemma and the definition of $\tilde{A}$ in (2.4) are the elements underlying the quantile and efficient hedging results, as shown in Foellmer and Leukert (1999), Foellmer and Leukert (2000). Keeping this in mind, let us look at each hedging approach in more detail.

2.2.2 Setting

Below we discuss quantile hedging, following the main ideas and, for the most part, notation in Foellmer and Leukert (1999). Suppose that the discounted price process $X = (X_t)_{t\in[0,T]}$ of the underlying risky asset is a semimartingale on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t\in[O,T]}$. There are no arbitrage opportunities in the market, that is, the set $\mathcal{P}$ of all equivalent martingale measures is nonempty. Note that Foellmer and Leukert (1999) conduct calculations assuming that the risk-free interest rate r equals zero.

A trading strategy (V_0, ξ) with initial capital $V_0 \geq 0$ and a predictable process ξ, corresponding to the number of shares of the risky asset X, is called *self-financing* if the capital generated by the strategy satisfies

$$V_t = V_0 + \int_0^t \xi_s dX_s \quad \forall t \in [0,T], \quad P - a.s.. \tag{2.6}$$

A self-financing strategy is *admissible* if

$$V_t \geq 0 \quad \forall t \in [0,T], \quad P - a.s.. \tag{2.7}$$

Only admissible hedging strategies are allowed.

From option pricing theory, we know that in the complete case, the equivalent martingale measure $P^* \in \mathcal{P}$ is unique (see, for example, Melnikov et al. (2002)). Consider a contingent claim whose payoff is an $\mathcal{F}_T$-measurable nonnegative random variable H satisfying $H \in L^1(P^*)$. The payoff H can be hedged perfectly (in a complete market), that is, there exists a unique (admissible) hedging strategy with (minimal) cost V_0 such that

$$P(V_T \geq H) = 1. \tag{2.8}$$

We also know that the cost of this perfect hedging strategy is given by

$$V_0 = H_0 = E^*(He^{-rT}). \tag{2.9}$$

Above, and for the remainder of the dissertation, e denotes the exponent function, T the maturity of the contract, and E^* the expectation with respect to the equivalent martingale measure P^*.

Therefore, as long as the contract seller receives H_0, he/she will generate sufficient funds to pay the buyer at maturity of the contract. However, what if the hedger is unable or unwilling to provide the initial capital required for the perfect hedge, and all he/she has available is the amount $\tilde{V}_0 < H_0$? Quantile and efficient hedging provide different answers to this problem.

2.2.3 Quantile hedging

Quantile hedging proposes to solve the problem of $\tilde{V}_0 < H_0$ by maximizing the probability of a successful hedge. That is, Foellmer and Leukert seek an admissible strategy (V_0, ξ) that will

$$\begin{aligned} &\text{maximize} \quad && P\left(V_T = V_0 + \int_0^T \xi_s dX_s \geq H\right) \\ &\text{under the constraint} \quad && V_0 \leq \tilde{V}_0 < H_0. \end{aligned} \tag{2.10}$$

Now, define *success set* corresponding to the admissible strategy (V_0, ξ) and its terminal wealth V_T as $\{V_T \geq H\}$. Proposition (2.8) on pp. 254-255 in Foellmer and Leukert (1999) states that
if $\tilde{A} \in \mathcal{F}_T$ maximizes $P(A)$ under the constraint $E^*(HI_A) \leq \tilde{V}_0$, and if $\tilde{\xi}$ is a perfect hedge for the modified contingent claim with payoff $\tilde{H} = HI_{\tilde{A}} \in L^1(P^*)$, then $(\tilde{V}_0, \tilde{\xi})$ solves the optimization problem given in (2.10). Moreover, the corresponding success set coincides almost surely with $\tilde{A}$. Refer to Foellmer and Leukert (1999) for further details (such as existence of $\tilde{\xi}$) and proof of the proposition. Also note that by this proposition, $\tilde{A}$ is the set on which the payoff H can be hedged with maximal probability of success.

Next, Foellmer and Leukert utilize the Neyman-Pearson lemma to provide the explicit structure of the success set $\tilde{A}$. The implicit hypotheses are

h_0: the hedge will fail

h_1: the hedge will be successful.

We want to maximize the probability of h_1 being the correct situation, that is, maximize the power of the test as described in (2.3) for the Neyman-Pearson lemma. Introduce a probability measure Q^* with density

$$\frac{dQ^*}{dP^*} = \frac{H}{H_0}. \tag{2.11}$$

Note that Q^* corresponds to P_0 (the probability measure of the hypothesis we want to reject), where as P corresponds to P_1, the real-world probability measure.

Under Q^*, the budget constraint $E^*(HI_A) \leq \tilde{V}_0$ becomes

$$Q^*(A) \leq \frac{\tilde{V}_0}{H_0}. \tag{2.12}$$

In terms of notation, E^* still refers to the expectation with respect to P^*, where as E^{Q^*} will denote expectation under Q^* (such notation will be used in this and all subsequent sections of the dissertation).

Let

$$\alpha = \frac{\tilde{V}_0}{H_0}; \tag{2.13}$$

this α is the Type I error in the Neyman-Pearson lemma (see (2.2)). Define

$$\tilde{a} = \inf\left\{a : Q^*\left(\frac{dP}{dP^*} > a \cdot H\right) \leq \alpha\right\}; \tag{2.14}$$

this $\tilde{a}$ is equivalent to $\tilde{k}$ in the Neyman-Pearson lemma (see (2.4)). Define the set $\tilde{A}$ corresponding to $\tilde{a}$ as

$$\tilde{A} = \left\{\frac{dP}{dP^*} > \tilde{a} \cdot H\right\}. \tag{2.15}$$

Here, as usual, dP^*/dP denotes the density of the equivalent martingale measure P^*.

Theorem (2.22) on p. 256 in Foellmer and Leukert (1999) says that if $Q^*(\tilde{A}) = \alpha$ holds, then the optimal hedging strategy solving (2.10) is actually $(\tilde{V}_0, \tilde{\xi})$, where $\tilde{\xi}$ is the perfect hedge for the modified payoff $\tilde{H} = HI_{\tilde{A}}$ and $\tilde{V}_0$ its cost. Therefore, using the result of the Neyman-Pearson lemma to derive the structure of $\tilde{A}$ (see details in (2.62)), Foellmer and Leukert show that the probability of successful hedging, or power of the test, is maximized for a given level of Type I error $Q^*(\tilde{A}) = \alpha = \tilde{V}_0/H_0$. Setting Type I error at this level is equivalent to requiring $\tilde{V}_0$ for the initial capital of the optimal hedge $\tilde{\xi}$ (as shown in (2.58)). Also, knowing the structure of the success set $\tilde{A}$ (2.15) allows us to calculate explicit formulas for $\tilde{\xi}$ and its cost $\tilde{V}_0$. These ideas are discussed in more detail in section 2.3.3.

Foellmer and Leukert (1999) point out (see pp. 252 and 261) that there are two risk management possibilities based on the quantile hedging methodology. First, the investor could face some given initial capital (that is smaller than the amount necessary for a perfect hedge) and would need to maximize the probability of successful hedging based on this given capital. Or, if the investor has some control over the initial capital, he/she could fix the acceptable probability of successful hedging and then calculate the amount necessary to hedge the contingent claim in consideration to guarantee the chosen probability of success. We analyze both of these risk management perspectives and demonstrate how they can be applied in practice (see section 2.6).

2.2.4 Efficient hedging

In this section, whenever possible, we follow the notation of the previous section (it coincides with that in Foellmer and Leukert (2000) with some exceptions). Efficient hedging addresses the issue of insufficient initial capital ($\tilde{V}_0 < H_0$) by minimizing the expected hedging losses, while taking into account the hedger's risk preferences through some loss function. The *loss function* $l(x)$ is an increasing function defined on $[0, \infty)$ with $l(0) = 0$ and $E(l(H)) < \infty$. Note that, in general, the loss function is concave up, as the investor (hedger) is assumed to be risk-averse, meaning that the larger is the loss, the less willing is the investor to bear it. However, it is possible that some investors may be the atypical risk-takers (these could be addictive gambler types, for whom it is more difficult to stop gambling as the game goes on and they lose more and more money). The loss function for risk-takers is concave down.

Foellmer and Leukert work with the loss function $l(x) = x^p$, and distinguish and analyze three possible cases for the value of p:
1. $p = 1$: risk-neutral investor,
2. $p > 1$: risk-averse investor,
3. $0 < p < 1$: risk-taker.

We will discuss the solution for each case above, but let us first formulate the optimization problem which is to be solved.

Define *shortfall risk* as the expectation of loss from the hedging strategy affected by risk preference of the hedger:

$$E(l((H - V_T)^+)). \tag{2.16}$$

Efficient hedging aims at finding an admissible strategy (V_0, ξ) that minimizes the shortfall risk and costs no more than $\tilde{V}_0$. That is,

$$\begin{aligned} \text{minimize} \quad & E(l((H - V_T)^+)) \\ \text{under the constraint} \quad & V_0 \leq \tilde{V}_0 < H_0. \end{aligned} \tag{2.17}$$

Similar to the hypothesis testing techniques used to develop quantile hedging, Foellmer and Leukert show in Proposition 3.1 (p. 121 in Foellmer and Leukert (2000)) that there exists an $\mathcal{F}_T$-measurable function $\tilde{\varphi} :\longrightarrow [0,1]$ that minimizes $E(l((1-\varphi)H))$ under the constraint $E^*(\varphi H) \leq \tilde{V}_0$. Such $\tilde{\varphi}$ is unique $(P - a.s.)$

on $\{H > 0\}$ for loss functions that are strictly concave up. Theorem 3.2 on p. 123 (Foellmer and Leukert (2000)) shows that the strategy $(\tilde{V}_0, \tilde{\xi})$ that hedges the modified claim $\tilde{H} = \tilde{\varphi}H$ also solves the optimization problem (2.17). Moreover, the success ratio, defined in general as $\varphi(V_0, \xi) = I_{\{V_T \geq H\}} + \frac{V_T}{H} I_{\{V_T < H\}}$, for $\tilde{H}$ coincides $P - a.s.$ with $\tilde{\varphi}$. Therefore, as in the case of quantile hedging, knowing the structure of $\tilde{\varphi}$, we solve the optimization problem (2.17) by finding the perfect hedge for the modified claim $\tilde{H} = \tilde{\varphi}H$.

Note that the requirement that $\tilde{\varphi}$ is unique on $\{H > 0\}$ for concave up loss functions is not restrictive to us. Later in their article (see pp. 125 and 129), Foellmer and Leukert show that $\tilde{\varphi}$ is unique on $\{H > 0\}$ for $p > 1$ and $0 < p < 1$. In particular, for $l(x) = x^p$, in the case of risk-aversion, $p > 1$ and

$$\tilde{\varphi} = 1 - \left(\frac{I\left(\tilde{a}\frac{dP^*}{dP}\right)}{H} \wedge 1 \right), \tag{2.18}$$

where $I = (l')^{-1}$ denotes the inverse of l'. In the situation of risk-taking, $0 < p < 1$ and

$$\tilde{\varphi} = I_{\left\{1 > \tilde{a}H^{1-p}\frac{dP^*}{dP}\right\}}. \tag{2.19}$$

In the special case of risk-indifference, $p = 1$ and

$$\tilde{\varphi} = I_{\left\{\frac{dP}{dP^*} > \tilde{a}\right\}}, \tag{2.20}$$

provided that $P^*\left(\{dP/dP^* = \tilde{a}\} \cap \{H > 0\}\right) = 0$. In all above cases, $\tilde{a}$ is calculated from the constraint on the initial capital $E^*(\tilde{\varphi}H) = \tilde{V}_0$ ((2.12) in the case of quantile hedging; see also the definition of $\tilde{k}$ in the Neyman-Pearson lemma (2.4)).

In our setting with one or several risky assets, the density dP^*/dP of the risk-neutral measure P^* will always be continuous, since it will be a function of a linear combination of one or more Wiener processes; thus $P^*\{dP/dP^* = \tilde{a}\} = 0$. Also, we work with contracts that pay the larger of the values of two or more risky assets (or a single risky asset and some positive guarantee) at maturity, so the payoff H will always be positive. Therefore, all conditions required to obtain a unique solution for the structure of the optimal hedge are satisfied. Now we proceed to describe our setting and to derive explicit formulas, based on the results of quantile and efficient hedging given above, that will enable us to calculate the cost of the optimal hedge and find strategies to manage financial and insurance risks inherent in equity-linked life insurance contracts.

2.3 Our setting

2.3.1 Financial setting

We work in a financial market with interest rate $r > 0$, riskless asset (bank account, for instance) $B = (B_t)_{t \in [0,T]}$, and two risky assets S^1 and S^2 (such as stocks),

$S^i = (S^i_t)_{t\in[0,T]}$, with price evolutions

$$\begin{aligned} dB_t &= rB_t dt \quad \Leftrightarrow \quad B_t = B_0 e^{rt}, \quad B_0 := 1; \\ dS^i_t &= S^i_t(\mu_i dt + \sigma_i dW^i_t) \quad \Leftrightarrow \quad S^i_t = S^i_0 e^{\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W^i_t}, \quad i = 1, 2, \end{aligned} \tag{2.21}$$

where constants $\mu_i \in \mathbb{R}$, $\sigma_i > 0$ are return and volatility of the instantaneous return $\frac{dS^i}{S^i}$ of the risky asset S^i. Note that S^1 and S^2 are based on two different Wiener processes, $W^1 = (W^1_t)_{t\in[0,T]}$ and $W^2 = (W^2_t)_{t\in[0,T]}$. All processes are given on a standard stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ and are adapted to the filtration $\mathbb{F}$, generated by W^i. The correlation between W^1 and W^2 is ρ, and we make the standard assumption that $\rho^2 < 1$, that is, the risks underlying the assets cannot be perfectly (positively or negatively) correlated.

Define a *trading strategy*, or *portfolio*, as a predictable process π such that

$$\pi = (\pi_t)_{t\in[0,T]} = (\beta_t, \gamma^1_t, \gamma^2_t)_{t\in[0,T]}. \tag{2.22}$$

Here, β represents the money invested in the riskless asset, B, while γ^i_t is the number of shares of S^i_t held in the portfolio at the instant of time t. As before, we suppose that all contracts to be discussed mature at time T.

The *capital* of π is given by

$$V^\pi_t = \beta_t B_t + \gamma^1_t S^1_t + \gamma^2_t S^2_t. \tag{2.23}$$

The strategies whose discounted capital satisfies

$$\frac{V^\pi_t}{B_t} = V^\pi_0 + \sum_{i=1}^{2} \int_0^t \gamma^i_u d\left(\frac{S^i_u}{B_u}\right) \tag{2.24}$$

are called *self-financing*. Only self-financing strategies with nonnegative capital are *admissible*.

In our financial market setting, there are two sources of risk, W^1 and W^2, and two risky assets, thus the financial market is complete (and arbitrage-free; see Melnikov et al. (2002)), and there exists a unique equivalent martingale measure P^* with density Z such that

$$Z_t = \frac{dP^*}{dP}|_{\mathcal{F}_t}, \quad t \in [0, T]. \tag{2.25}$$

Using general methodology for finding martingale measures (as presented in Melnikov and Shiryaev (1996), Melnikov et al. (2002)), we calculate the expression for Z explicitly (derivation details are given in Appendix 1):

$$Z_t = e^{\phi_1 W^1_t + \phi_2 W^2_t - \frac{\sigma^2_\phi}{2} t}, \tag{2.26}$$

where

$$\begin{aligned}\phi_1 &= \frac{r(\sigma_2 - \sigma_1\rho) + \rho\mu_2\sigma_1 - \mu_1\sigma_2}{\sigma_1\sigma_2(1-\rho^2)}, \\ \phi_2 &= \frac{r(\sigma_1 - \sigma_2\rho) + \rho\mu_1\sigma_2 - \mu_2\sigma_1}{\sigma_1\sigma_2(1-\rho^2)},\end{aligned} \tag{2.27}$$

and

$$\sigma_\phi^2 = \phi_1^2 + \phi_2^2 + 2\rho\phi_1\phi_2. \tag{2.28}$$

Under P^*, the evolutions of S^1 and S^2 can be rewritten as

$$dS_t^i = S_t^i(rdt + \sigma_i dW_t^{i*}) \quad \Leftrightarrow \quad S_t^i = S_0^i e^{\left(r - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_t^{i*}}, \quad i = 1, 2. \tag{2.29}$$

Here, $W^{i*} = (W_t^{i*})_{t\in[0,T]}$ are Wiener processes with correlation ρ under P^* that satisfy (see, for example, Melnikov et al. (2002))

$$W_t^{i*} = W_t^i + \theta_i t \tag{2.30}$$

with

$$\theta_i = \frac{\mu_i - r}{\sigma_i}. \tag{2.31}$$

It follows that discounted S^1 and S^2 are martingales w. r. to P^*. Using (2.26) and (2.30), we can rewrite the expression for Z under P^*:

$$Z_t = e^{\phi_1 W_t^{1*} + \phi_2 W_t^{2*} - \left(\frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)t}. \tag{2.32}$$

Note that in derivations of pricing formulas and hedging strategies, we will use the formulas for S^i and Z under both the original and the risk-neutral probability measures.

Now, consider θ_i given above in (2.31). In financial literature, θ is referred to as the market price of risk: this is the additional 'reward' per unit volatility investors receive to compensate them for the willingness to bear risk when putting money in risky as opposed to riskless assets. We require

$$\theta_i > 0 \quad \Leftrightarrow \quad \mu_i > r, \tag{2.33}$$

otherwise an arbitrage strategy with zero initial outlay and positive expected return could be constructed (short a stock and invest the proceeds in a bond, then use the principal and interest from the bond to buy back the stock).

As in the setting of quantile and efficient hedging (see section 2.2.2), we have an $\mathcal{F}_T$-measurable random variable H denoting the contingent claim with payoff H. In the setting considered here, the payoff depends on the values of two risky assets at maturity of the contract, at which point the contract holder may choose the larger of the two values:

$$H = \max\{S_T^1, S_T^2\} = S_T^1 I_{\{S_T^1 \geq S_T^2\}} + S_T^2 I_{\{S_T^1 < S_T^2\}}. \tag{2.34}$$

As such, H is the payoff for a purely financial contract. From option-pricing theory, we know that its fair price H_0 is

$$H_0 = E^*\left(\frac{H}{e^{rT}}\right), \tag{2.35}$$

and as long as the seller of the contract receives H_0, the liability H can be hedged perfectly (see, for instance, Melnikov et al. (2002)).

Since H_0 will appear frequently in subsequent analysis and calculations, we give the explicit form of (2.35), with derivations following the result. The perfect hedge price of the contract with payoff H (2.34) is

$$H_0 = E^*\left(\frac{\max\{S_T^1, S_T^2\}}{e^{rT}}\right) = S_0^1 \cdot \Psi^1(\tilde{y}_1) + S_0^2 \cdot \Psi^1(\tilde{y}_2), \tag{2.36}$$

where Ψ^1 denotes one-dimensional cumulative normal distribution: for $u \sim N(0,1)$,

$$\Psi^1(c) = \int_{-\infty}^{c} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du. \tag{2.37}$$

The constants $\tilde{y}_1, \tilde{y}_2$ are defined as

$$\begin{aligned} \tilde{y}_1 &= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \\ \tilde{y}_2 &= \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \end{aligned} \tag{2.38}$$

and

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2. \tag{2.39}$$

Note that Margrabe (1978) and Davis (2002) derive pricing formulas in the case of perfect hedging for contracts with two risky assets, where one of the assets serves as a strike in a call-type payoff.

To calculate the formula for H_0, we rewrite (2.35) as

$$\begin{aligned} H_0 &= E^*\left(\frac{\max\{S_T^1, S_T^2\}}{e^{rT}}\right) \\ &= E^*\left(\frac{S_T^1}{e^{rT}} I_{\{S_T^1 \geq S_T^2\}}\right) + E^*\left(\frac{S_T^2}{e^{rT}} I_{\{S_T^1 < S_T^2\}}\right) \\ &= S_0^1 e^{\frac{-\sigma_1^2}{2}T} E^*\left(e^{\sigma_1 W_T^{1*}} I_{\{\sigma_2 W_T^{2*} - \sigma_1 W_T^{1*} \leq \ln\left(\frac{S_0^1}{S_0^2}\right) + \frac{\sigma_2^2 - \sigma_1^2}{2}T\}}\right) \\ &+ S_0^2 e^{\frac{-\sigma_2^2}{2}T} E^*\left(e^{\sigma_2 W_T^{2*}} I_{\{\sigma_1 W_T^{1*} - \sigma_2 W_T^{2*} < \ln\left(\frac{S_0^2}{S_0^1}\right) + \frac{\sigma_1^2 - \sigma_2^2}{2}T\}}\right). \end{aligned} \tag{2.40}$$

Above, we are using the expressions for evolutions of S^i under P^* given in (2.29).

Next, we simplify the sets of the indicators above by transforming the linear combination of two Wiener processes into a single new Wiener process $\tilde{W}^i = (\tilde{W}_t^i)_{t\in[0,T]}$ under P^*:

$$\begin{aligned} \tilde{W}_t^1 &= \frac{\sigma_2 W_t^{2*} - \sigma_1 W_t^{1*}}{\sigma}, \\ \tilde{W}_t^2 &= \frac{\sigma_1 W_t^{1*} - \sigma_2 W_t^{2*}}{\sigma}, \\ \sigma^2 &= \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2. \end{aligned} \tag{2.41}$$

Note that σ represents the volatility of a risky asset with the underlying risk process $\tilde{W}^i$, and it is the same as the σ in (2.39). Also, since σ^2 must be positive, we need to check that the expression in the definition of σ^2 above is positive:

$$\begin{aligned} \sigma^2 &= \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \\ &> \sigma_1^2\rho^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \quad \text{as} \quad \rho^2 < 1 \\ &= (\sigma_1\rho - \sigma_2)^2 \geq 0 \\ \Rightarrow \sigma^2 &> 0. \end{aligned} \tag{2.42}$$

Using the fact that random variables $\tilde{W}_T^i$ are normally distributed with mean zero and variance T, we can write

$$
\begin{aligned}
\left\{\sigma_2 W_T^{2*} - \sigma_1 W_T^{1*} \le \ln\left(\frac{S_0^1}{S_0^2}\right)\right. \\
\left. + \frac{\sigma_2^2 - \sigma_1^2}{2} T\right\} &= \left\{\sigma \tilde{W}_T^1 \le \ln\left(\frac{S_0^1}{S_0^2}\right) + \frac{\sigma_2^2 - \sigma_1^2}{2} T\right\} \\
&= \left\{ s_1 \le \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \frac{\sigma_2^2 - \sigma_1^2}{2} T}{\sigma\sqrt{T}} \right\}, \\
\left\{\sigma_1 W_T^{1*} - \sigma_2 W_T^{2*} < \ln\left(\frac{S_0^2}{S_0^1}\right)\right. \\
\left. + \frac{\sigma_1^2 - \sigma_2^2}{2} T\right\} &= \left\{\sigma \tilde{W}_T^2 < \ln\left(\frac{S_0^2}{S_0^1}\right) + \frac{\sigma_1^2 - \sigma_2^2}{2} T\right\} \\
&= \left\{ s_2 < \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) + \frac{\sigma_1^2 - \sigma_2^2}{2} T}{\sigma\sqrt{T}} \right\}, \qquad (2.43)
\end{aligned}
$$

where $s_i = \frac{\tilde{W}_T^i}{\sqrt{T}} \sim N(0,1)$ w. r. to P^*.

Now we will utilize a lemma on p. 797 of Shiryaev (1999), referring to it as the *one-asset lemma.* The lemma states that for two normally distributed random variables $\eta \sim N(\mu_\eta, \sigma_\eta^2)$ and $\zeta \sim N(\mu_\zeta, \sigma_\zeta^2)$, and a constant c,

$$
E^*\left(e^{-\eta} \cdot I_{\{\zeta \le c\}}\right) = e^{-\left(\mu_\eta - \frac{\sigma_\eta^2}{2}\right)} \cdot \Psi^1\left(\frac{c - (\mu_\zeta - \operatorname{cov}(\zeta, \eta))}{\sigma_\zeta}\right), \qquad (2.44)
$$

with $\operatorname{cov}(a,b)$ denoting the covariance between a and b, and Ψ^1 the one-dimensional cumulative normal distribution (2.37).

For our case, we take $\eta = -\sigma_i W_T^{i*} \sim N(0, \sigma_i^2 T)$ and $\zeta = s_i \sim N(0,1)$ (w. r. to P^*) and calculate the covariances as required by the lemma:

$$
\operatorname{cov}(\eta, \zeta) = \operatorname{cov}(-\sigma_i W_T^{i*}, s_i) = \frac{(\sigma_i^2 - \rho\sigma_i\sigma_j)T}{\sigma\sqrt{T}}, \qquad (2.45)
$$

where $i, j = 1, 2$. Note that since the underlying random variables W_T^i, $\tilde{W}_T^i$ are continuous (paths of Wiener processes are everywhere continuous), we can use the lemma for sets of type $\{\zeta \le c\}$ as well as $\{\zeta < c\}$.

Following the above considerations, we apply the one-asset lemma (2.44) and obtain

$$
\begin{aligned}
H_0 &= S_0^1 e^{\frac{-\sigma_1^2}{2}T} E^*\left(e^{\sigma_1 W_T^{1*}} I_{\left\{\sigma_2 W_T^{2*}-\sigma_1 W_T^{1*} \leq \ln\left(\frac{S_0^1}{S_0^2}\right)+\frac{\sigma_2^2-\sigma_1^2}{2}T\right\}}\right) \\
&+ S_0^2 e^{\frac{-\sigma_2^2}{2}T} E^*\left(e^{\sigma_2 W_T^{2*}} I_{\left\{\sigma_1 W_T^{1*}-\sigma_2 W_T^{2*} < \ln\left(\frac{S_0^2}{S_0^1}\right)+\frac{\sigma_1^2-\sigma_2^2}{2}T\right\}}\right) \\
&= S_0^1 e^{\frac{-\sigma_1^2}{2}T} e^{\frac{\sigma_1^2}{2}T} \Psi^1\left(\frac{\ln\left(\frac{S_0^1}{S_0^2}\right)+\frac{\sigma_2^2-\sigma_1^2}{2}T}{\sigma\sqrt{T}} + \frac{(\sigma_1^2-\rho\sigma_1\sigma_2)T}{\sigma\sqrt{T}}\right) \\
&+ S_0^2 e^{\frac{-\sigma_2^2}{2}T} e^{\frac{\sigma_2^2}{2}T} \Psi^1\left(\frac{\ln\left(\frac{S_0^2}{S_0^1}\right)+\frac{\sigma_1^2-\sigma_2^2}{2}T}{\sigma\sqrt{T}} + \frac{(\sigma_2^2-\rho\sigma_1\sigma_2)T}{\sigma\sqrt{T}}\right) \\
&= S_0^1 \Psi^1(\tilde{y}_1) + S_0^2 \Psi^1(\tilde{y}_2). \qquad (2.46)
\end{aligned}
$$

This completes the derivation of the formula for the perfect hedge price in (2.36), with $\tilde{y}_i$ given in (2.38).

Since we want to analyze equity-linked life insurance contracts, let us now see how the presence of insurance risk affects the financial payoff H (2.34) and the resulting pricing calculations.

2.3.2 Insurance setting

Let a random variable $\tau(x)$ on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ denote the remaining lifetime of a person of current age x. We can safely assume that the insurance risk arising from clients' mortality and the financial market risk have no (or very minimal) effect on each other, hence the two probability measures P and $\bar{P}$ are independent.

Now, term insurance products are types of policies whose payoff occurs before maturity of the policy. For example, one could buy a 20-year policy paying 10,000 CAD in case the death of the policyholder occurs within 20 years from the date of purchase. On the other hand, the payoff of life insurance products occurs on or after the maturity of the policy, provided some prespecified event did not occur prior to the maturity date. We work with a single-premium equity-linked life insurance contract where the insured receives the payoff H given by (2.34), provided that he/she is alive to collect that payoff. That is, we are interested in the payoff $\bar{H}$

$$
\bar{H} = H \cdot I_{\{\tau(x) > T\}}. \qquad (2.47)
$$

The fair premium U_0 for a contract with such payoff is

$$\begin{aligned} U_0 &= E^* \times \bar{E}\left(He^{-rT} I_{\{\tau(x)>T\}}\right) \\ &= E^*\left(He^{-rT}\right) \bar{P}\{\tau(x) > T\} \\ &= E^*\left(He^{-rT}\right) {}_Tp_x , \end{aligned} \tag{2.48}$$

where ${}_Tp_x = \bar{P}\{\tau(x) > T\}$ denotes the probability of a life aged x surviving T more years.

Notice that mortality of the insured client (as reflected by his/her survival probability ${}_Tp_x$) makes it impossible for the insurance firm writing the contract to hedge its payoff with probability 1:

$$0 < {}_Tp_x < 1 \quad \Rightarrow \quad U_0 < H_0 = E^*\left(He^{-rT}\right). \tag{2.49}$$

As mentioned previously, since mortality is not traded directly, it is not possible to hedge mortality risk, as one hedges the risk associated with trading options by taking positions in the underlying risky asset, for example. Thus the insurance market is incomplete.

2.3.3 Quantile and efficient hedging for life insurance

In the situation when the quantity U_0 (2.48), collected by the firm from the sale of the equity-linked life insurance contract with payoff (2.47), is strictly less than the amount H_0, necessary to hedge the payoff perfectly, the firm faces the risk of default. To reduce this risk, the company must find some appropriate imperfect hedging technique which optimizes the hedging outcomes, given constraints on initial capital available for hedging. Below, we show how quantile or efficient hedging can be applied in this situation.

In this section, we will modify notation from that in Foellmer and Leukert (1999) and Foellmer and Leukert (2000) to reflect the fact that now we are using quantile and efficient hedging results in our setting.

Recall that quantile hedging seeks to find an admissible hedging strategy π^* that maximizes the probability of successful hedging:

$$P\{\omega : V_T^{\pi^*} \geq H\} = \max_{\pi} P\{V_T^{\pi} \geq H\} \quad \text{with} \quad V_0 \leq U_0 < H_0. \tag{2.50}$$

This is the optimization problem described in (2.10), and it is solved by a perfect hedge π^* for the modified contingent claim

$$H^* = HI_{A^*}, \tag{2.51}$$

with

$$A^* = \left\{\frac{1}{Z_T} > a^* e^{-rT} H\right\}. \tag{2.52}$$

Alternatively, efficient hedging aims at obtaining an admissible hedging strategy π^* that minimizes the shortfall risk:

$$E\left(l((H - V_T^{\pi^*})^+)\right) = \min_{\pi} E\left(l((H - V_T^{\pi})^+)\right) \quad \text{with} \quad V_0 \leq U_0 < H_0. \tag{2.53}$$

This problem is described in (2.17); its solution, again, is a perfect hedge π^* for the modified contingent claim

$$H^* = \varphi^* H, \tag{2.54}$$

where

$$\varphi^* = 1 - \left(\frac{I\left(a^* e^{-rT} Z_T\right)}{H} \wedge 1\right), \quad p > 1, \tag{2.55}$$

with $I = (l')^{-1}$ denoting the inverse of the derivative of the loss function l,

$$\varphi^* = I_{\{1 > a^* e^{-rT} H^{1-p} Z_T\}}, \quad 0 < p < 1, \tag{2.56}$$

and finally

$$\varphi^* = I_{\{1 > a^* e^{-rT} Z_T\}}, \quad p = 1. \tag{2.57}$$

The inequalities in (2.50) and (2.53) reflect the fact that the investor is budget-constrained: $U_0 < H_0$, $H_0 = E^*(He^{-rT})$ is the amount needed for a perfect hedge, and the requirement that the initial cost V_0 of the optimal hedging strategy must not be greater than the amount available to the hedger: $V_0 \leq U_0$.

Now, there are several things to note about the adaptation of quantile and efficient hedging results to our setting. First, we use π^* to denote the optimal hedging strategy for both hedging methods; the cost of this strategy is always U_0 (in the notation of previous sections, the optimal strategy and its cost were denoted $(\tilde{\xi}, \tilde{V}_0)$; see discussions following (2.15) and (2.20)). Second, we denote modified contingent claim H^* and use A^* to denote the success set for quantile hedging, and φ^* to denote success ratios for efficient hedging (compare with $\tilde{H}$, $\tilde{A}$ (2.15), and $\tilde{\varphi}$ (2.18), (2.19), (2.20)).

The constraint U_0 on the initial capital available for hedging arises from the insurance risk component (2.48), and is the same for both quantile and efficient hedging; in sections 2.2.3 and 2.2.4, this constraint was denoted $\tilde{V}_0$ (see (2.10), (2.17)). Also, we denote a^* the constant that appears in the explicit forms of A^* and φ^* above; previously, in (2.14), this constant was labeled $\tilde{a}$. Note that in our case, just as in Foellmer and Leukert (1999) and Foellmer and Leukert (2000) for $\tilde{a}$, a^* is calculated from the budget constraint in the setting with nonzero interest rate r. Let us explain this in more detail.

Consider the probability measure Q^* constructed by Foellmer and Leukert when using the Neyman-Pearson lemma to derive quantile and efficient hedging results. The density of this measure is $\frac{dQ^*}{dP^*} = \frac{H}{H_0}$, as given in (2.11), with $H_0 = E^*(H)$. In the notation of sections 2.2.3 and 2.2.4, the cost of the optimal hedge $\tilde{\xi}$ is $\tilde{V}_0$, which, for the case of quantile hedging, becomes

$$\begin{aligned}\tilde{V}_0 &= E^*(HI_{\tilde{A}}) = \int_{\tilde{A}} HdP^* = \int_{\tilde{A}} H\frac{dP^*}{dQ^*}\frac{dQ^*}{dP^*}dP^* \\ &= \int_{\tilde{A}} H\frac{H_0}{H}dQ^* = H_0Q^*(\tilde{A}). \end{aligned} \tag{2.58}$$

This is equivalent to setting the maximal level of Type I error at $Q^*(\tilde{A}) = \alpha = \tilde{V}_0/H_0$ and then minimizing the Type II error according to the Neyman-Pearson lemma. So in the setting of Foellmer and Leukert (1999), $\tilde{a}$ is calculated from the equation $\tilde{V}_0 = E^*(HI_{\tilde{A}})$ (see discussion following (2.15)). Similarly to above considerations, for the case of efficient hedging, $\tilde{a}$ is calculated from $\tilde{V}_0 = E^*(\tilde{\varphi}H)$.

In our case the interest rate must be taken into account when discussing the probability measure Q^* that corresponds to the null hypothesis in the Neyman-Pearson lemma. For our setting, the density of Q^* is defined by

$$\frac{dQ^*}{dP^*}|_{\mathcal{F}_T} = \frac{He^{-rT}}{E^*(He^{-rT})} = \frac{H}{H_0e^{rT}}, \tag{2.59}$$

that is, the density is the ratio of the discounted payoff to the risk-neutral expectation of the discounted payoff. This allows us to express the cost of the optimal hedge π^* for quantile hedging as

$$\begin{aligned}U_0 &= E^*(He^{-rT}I_{A^*}) = \int_{A^*} He^{-rT}dP^* = \int_{A^*} He^{-rT}\frac{dP^*}{dQ^*}\frac{dQ^*}{dP^*}dP^* \\ &= \int_{A^*} He^{-rT}\frac{H_0e^{rT}}{H}dQ^* = H_0Q^*(A^*), \end{aligned} \tag{2.60}$$

just as in Foellmer and Leukert (1999) (see (2.58) above). Note that we dropped the time reference for the densities above with the understanding that all processes are taken on $\mathcal{F}_T$ and A^* is $\mathcal{F}_T$-measurable. The same reasoning and definition of Q^* density (2.59) enable us to calculate a^* for efficient hedging formulas from

$$U_0 = E^*(\varphi^*He^{-rT}) = H_0E^{Q^*}(\varphi^*), \tag{2.61}$$

as is done in Foellmer and Leukert (2000) for $r = 0$.

Now, consider the definition of $\tilde{A} = \{dP_1/dP_0 > \tilde{k}\}$ from the Neyman-Pearson lemma (2.4). Recall that P_0 corresponds to Q^*, the probability measure of the null hypothesis (fail to hedge), and P_1 to P, the real-world measure of the alternate hypothesis (hedge successfully). In Foellmer and Leukert (1999), for $r = 0$, the authors get

$$\begin{aligned}\left\{\frac{dP}{dQ^*} > \tilde{k}\right\} &= \left\{\frac{dP}{dP^*}\frac{dP^*}{dQ^*} > \tilde{k}\right\} \\ &= \left\{\frac{dP}{dP^*}\frac{H_0}{H} > \tilde{k}\right\} = \left\{\frac{dP}{dP^*} > \tilde{a}H\right\},\end{aligned} \tag{2.62}$$

with $\tilde{a} = \tilde{k}/H_0$. For our setting with $r > 0$, we have

$$\begin{aligned}\left\{\frac{dP}{dQ^*} > \tilde{k}\right\} &= \left\{\frac{dP}{dP^*}\frac{dP^*}{dQ^*} > \tilde{k}\right\} \\ &= \left\{\frac{dP}{dP^*}\frac{H_0 e^{rT}}{H} > \tilde{k}\right\} = \left\{\frac{dP}{dP^*} > \frac{a^*}{e^{rT}}H\right\},\end{aligned} \tag{2.63}$$

with $a^* = \tilde{k}/H_0$. This is how we obtain the expression for A^* (2.52) in our setting. The success ratios φ^* for efficient hedging are rewritten as above, taking into account the adjustment for $r > 0$ in the Q^* density, which leads to the modification $a^* e^{-rT}$ from $\tilde{a}$ in Foellmer and Leukert (2000) in the formulas (2.55), (2.56) and (2.57).

It is worthwhile to stress the elegance of the quantile and efficient hedging approaches in the situation with financing constraints. The constraint on the initial hedging capital may arise due to some external factor beyond the hedger's control (such as mortality risk in equity-linked life insurance contracts), or a circumstance within the decision-making power of the hedger (he/she may be unwilling to put up the entire amount required for perfect hedging and be prepared to take some risk as a trade-off for offering the contract at a lower price). In either case, the hedger can solve the problem of insufficient initial capital by maximizing the probability of a successful hedge or minimizing the shortfall risk. For both of these perspectives, the approach is the same: invest into the (optimal) strategy π^*, which perfectly hedges the modified contingent claim H^*, and the desired optimization goal will be achieved. Of course, the structures of H^* and the corresponding strategy π^* differ for quantile hedging and for each risk preference case in efficient hedging. However, conceptually, the above risk management ideas are easy to understand and thus are more likely to be implemented.

In section 2.6, we will illustrate the situation where the investor cannot provide the entire amount H_0 required for a perfect hedge and is ready to accept some default risk when using quantile hedging, or shortfall risk if applying efficient hedging. Default risk is the probability that the hedge fails, we will denote default risk ϵ (in the notation of the Neyman-Pearson lemma, this is the Type II error β; see (2.1) and (2.3)). Shortfall risk, which we will denote δ, is the expected loss from the strategy, defined in (2.16). Note that δ is an amount that could be lost due to imperfect hedging, so it is expressed in dollar terms. We will show the possible risk management strategies based on two different perspectives of the hedger. First, we will calculate the level of default risk (or shortfall risk) if the hedger is willing to invest U_0 taken as a percentage of H_0 into the optimal hedge. Second, we will illustrate how much initial capital U_0 is required if the hedger wants to keep ϵ (or δ) at some acceptable level, specified beforehand. This illustration is based on the payoff (2.47), where the client chooses the larger of the values of two risky assets at maturity of the contract, provided he/she survives to this date.

However, we can extend our study of quantile and efficient hedging far beyond purely financial risk management considerations: the flexibility of the two hedging methods makes them excellent tools for insurance applications, particularly in the case of equity-linked life insurance products. Let us discuss the risk management opportunities presented by quantile and efficient hedging in insurance in more detail. On one hand, we showed in (2.48) that U_0 is the fair premium for an equity-linked life insurance contract with payoff $\bar{H}$ (2.47). On the other hand, the results of quantile and efficient hedging tell us that the budget constraint (U_0 in our case) is also the cost of the optimal strategy π^*, which perfectly hedges the modified contingent claim H^* respectively given by (2.51) and (2.54) for quantile and efficient hedging. Based on this, we obtain the following equalities for the price of the equity-linked life insurance contract in consideration:

$$\begin{aligned} U_0 &= E^* \times \bar{E}(\bar{H}e^{-rT}) = E^*(H^*e^{-rT}) \\ &= E^*(He^{-rT})_T p_x = E^*(H^*e^{-rT}), \end{aligned} \tag{2.64}$$

from which we can express the survival probability of the policyholder as

$$_T p_x = \frac{E^*(H^*)}{E^*(H)}. \tag{2.65}$$

The term $E^*(He^{-rT})$ above is known: it is the perfect hedge price (2.35) for the contract with payoff (2.34). This price is calculated explicitly in (2.36).

Equations (2.64) and (2.65) are essential to the subsequent risk management analysis of quantile and efficient hedging in insurance applications, as they give a quantitative connection between financial and insurance risk components. Such connection, in turn, allows the insurance firm to assess accurately the risks it bears and to implement specific strategies to control these risks according to the preferred risk management approach. That is, the firm can either offer the equity-linked contract in consideration to any client and then, based on the fair price received from this client, maximize the probability of successful hedging $1-\epsilon$, or, for efficient hedging, minimize the shortfall risk δ. Or, the firm can set the acceptable level of financial risk ϵ or the acceptable amount of expected shortfall δ, and then analyze clients for the contract accordingly.

More specifically, in the first approach, the client's survival probability $_T p_x$ can be derived (based on his/her known age x) from some appropriate mortality model. Then, if the firm chooses to apply quantile hedging, it will derive a^* from (2.51) and (2.65), and calculate the maximal probability of successful hedging $1-\epsilon$ from (2.50). When utilizing efficient hedging, the firm will find a^* from (2.54) and (2.65), and then compute the minimal shortfall risk δ using (2.53). Note that the obtained values for the maximal probability of successful hedging or minimal shortfall risk may not fit the company's desired risk profile. Alternatively, the firm can utilize equation (2.65) in reverse: first, choose some acceptable level of default risk ϵ (or shortfall risk δ), calculate a^*, and then find survival probabilities $_Tp_x$ of potential clients. Next, using some particular mortality model, ages of clients paying fair premiums (under the prescribed risk level) can be derived and risk management consequences analyzed in light of the firm's risk preferences.

We will illustrate the application of quantile and efficient hedging in insurance in section 3.6, using the payoff where the client receives the larger of the value of some risky asset or a deterministic guarantee at maturity of the contract, provided the client lives to collect the payoff, of course. The effects of the three mortality models of Gompertz, Makeham and Lee-Carter on the assessment and management of mortality and financial risks will be discussed and illustrated in the context of each of the two risk management approaches described above.

2.4 Theoretical results for two risky assets

In this section we present explicit formulas for the premium of the equity-linked life insurance contract with payoff $\bar{H}$ (2.47) that pays the larger of the values of two risky assets at maturity, conditional upon the policyholder's survival to the maturity date, as well as maximal probability of successful hedging for quantile hedging and minimal shortfall risk for efficient hedging. The results are presented as theorems for each of the imperfect hedging methods, with three cases for efficient hedging based on the three risk preferences of the investor, and are followed by the corresponding proofs.

2.4.1 Quantile hedging

Theorem 1
Suppose that the firm that sells an equity-linked life insurance contract with payoff (2.47) decides to use quantile hedging to maximize the probability of successful hedging.
Part I. The **fair premium** for the contract is

$$\begin{aligned} U_0 &= E^*\left(\frac{\max\{S_T^1, S_T^2\}}{e^{rT}} I_{A^*}\right) \\ &= S_0^1 \cdot \Psi^2(\tilde{x}_1^Q, \tilde{y}_1, \rho_1^Q) + S_0^2 \cdot \Psi^2(\tilde{x}_2^Q, \tilde{y}_2, \rho_2^Q). \end{aligned} \tag{2.66}$$

Part II. The **probability of successful hedging** is given by

$$P(A^*) = \Psi^2(\bar{x}_1^Q, \bar{y}_1^Q, \rho_1^Q) + \Psi^2(\bar{x}_2^Q, \bar{y}_2^Q, \rho_2^Q). \tag{2.67}$$

Above, Ψ^2 denotes two-dimensional cumulative normal distribution of random variables $u_1, u_2 \sim N(0,1)$ with correlation ϱ

$$\Psi^2(c,d,\varrho) = \int_{-\infty}^{c}\int_{-\infty}^{d} \frac{e^{-\frac{1}{2(1-\varrho^2)}(u_1^2+u_2^2-2\varrho u_1 u_2)}}{2\pi\sqrt{1-\varrho^2}} du_1 du_2, \tag{2.68}$$

and

$$\begin{aligned} \rho_1^Q &= \frac{\phi_2(\sigma_2-\sigma_1\rho)-(\sigma_1+\phi_1)(\sigma_1-\sigma_2\rho)}{\sigma_1^Q\sigma}, \\ \rho_2^Q &= \frac{\phi_1(\sigma_1-\sigma_2\rho)-(\sigma_2+\phi_2)(\sigma_2-\sigma_1\rho)}{\sigma_2^Q\sigma}. \end{aligned} \tag{2.69}$$

The constants $\tilde{x}_i^Q$, $\bar{x}_i^Q$, and $\bar{y}_i^Q$ are defined as

$$\begin{aligned} \tilde{x}_1^Q &= \frac{\left[\frac{\sigma_\phi^2-\sigma_1^2}{2}+\phi_1\theta_1+\phi_2\theta_2-\sigma_1(\rho\phi_2+\phi_1)\right]T-\ln\left(a^*S_0^1\right)}{\sigma_1^Q\sqrt{T}}, \\ \tilde{x}_2^Q &= \frac{\left[\frac{\sigma_\phi^2-\sigma_2^2}{2}+\phi_1\theta_1+\phi_2\theta_2-\sigma_2(\rho\phi_1+\phi_2)\right]T-\ln\left(a^*S_0^2\right)}{\sigma_2^Q\sqrt{T}}, \end{aligned} \tag{2.70}$$

$$\begin{aligned} \bar{x}_1^Q &= \frac{\left[r-\mu_1+\frac{\sigma_\phi^2+\sigma_1^2}{2}\right]T-\ln\left(a^*S_0^1\right)}{\sigma_1^Q\sqrt{T}}, \\ \bar{x}_2^Q &= \frac{\left[r-\mu_2+\frac{\sigma_\phi^2+\sigma_2^2}{2}\right]T-\ln\left(a^*S_0^2\right)}{\sigma_2^Q\sqrt{T}}, \end{aligned} \tag{2.71}$$

$$\begin{aligned} \bar{y}_1^Q &= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right)+\left[\mu_1-\mu_2+\frac{\sigma_2^2-\sigma_1^2}{2}\right]T}{\sigma\sqrt{T}}, \\ \bar{y}_2^Q &= \frac{\ln\left(\frac{S_0^2}{S_0^1}\right)+\left[\mu_2-\mu_1+\frac{\sigma_1^2-\sigma_2^2}{2}\right]T}{\sigma\sqrt{T}}, \end{aligned} \tag{2.72}$$

and σ_i^Q are given by

$$\begin{aligned} \sigma_1^{Q^2} &= (\sigma_1+\phi_1)^2+2\rho\phi_2(\sigma_1+\phi_1)+\phi_2^2, \\ \sigma_2^{Q^2} &= (\sigma_2+\phi_2)^2+2\rho\phi_1(\sigma_2+\phi_2)+\phi_1^2. \end{aligned} \tag{2.73}$$

Note that A^* is given in (2.52), $\tilde{y}_i$ in (2.38), ϕ_i in (2.27), θ_i in (2.31), σ in (2.39), and σ_ϕ in (2.28).

Proof.

Part I. To find the fair premium, we need to calculate

$$
\begin{aligned}
U_0 &= E^*\left(\frac{\max\{S_T^1, S_T^2\}}{e^{rT}} I_{A^*}\right) \\
&= E^*\left(\frac{S_T^1}{e^{rT}} I_{\left\{\frac{1}{Z_T} > \frac{a^* S_T^1}{e^{rT}}\right\}} I_{\{S_T^1 \geq S_T^2\}}\right) + E^*\left(\frac{S_T^2}{e^{rT}} I_{\left\{\frac{1}{Z_T} > \frac{a^* S_T^2}{e^{rT}}\right\}} I_{\{S_T^1 < S_T^2\}}\right) \\
&= S_0^1 e^{\frac{-\sigma_1^2}{2}T} E^*\left(e^{\sigma_1 W_T^{1*}} I_{\left\{\frac{1}{Z_T} > \frac{a^* S_T^1}{e^{rT}}\right\}} I_{\{S_T^1 \geq S_T^2\}}\right) \\
&+ S_0^2 e^{\frac{-\sigma_2^2}{2}T} E^*\left(e^{\sigma_2 W_T^{2*}} I_{\left\{\frac{1}{Z_T} > \frac{a^* S_T^2}{e^{rT}}\right\}} I_{\{S_T^1 < S_T^2\}}\right). \qquad (2.74)
\end{aligned}
$$

The indicator sets above can be simplified as follows. In (2.41) and (2.43), we showed that $\{S_T^1 \geq S_T^2\}$ and $\{S_T^1 < S_T^2\}$ can be written as

$$
\begin{aligned}
\{S_T^1 \geq S_T^2\} &= \left\{ s_1 \leq \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \frac{\sigma_2^2 - \sigma_1^2}{2}T}{\sigma\sqrt{T}} \right\} \quad \text{and} \\
\{S_T^1 < S_T^2\} &= \left\{ s_2 < \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) + \frac{\sigma_1^2 - \sigma_2^2}{2}T}{\sigma\sqrt{T}} \right\} \qquad (2.75)
\end{aligned}
$$

with the help of Wiener processes $\tilde{W}^i$ such that $s^i = \frac{\tilde{W}_T^i}{\sqrt{T}} \sim N(0,1)$ under P^*.

Similarly, we want to rewrite $\left\{\frac{1}{Z_T} > \frac{a^* S_T^1}{e^{rT}}\right\}$ and $\left\{\frac{1}{Z_T} > \frac{a^* S_T^2}{e^{rT}}\right\}$. First, consider $\left\{\frac{1}{Z_T} > \frac{a^* S_T^1}{e^{rT}}\right\}$: using the formulas for the density Z_T (2.32) and the evolutions of

S^i (2.29) under P^*, we can write

$$\left\{\frac{1}{Z_T} > \frac{a^* S_T^1}{e^{rT}}\right\} = \left\{e^{-\phi_1 W_T^{1*} - \phi_2 W_T^{2*} + \left(\frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)T} > \frac{a^* S_0^1 e^{rT - \frac{\sigma_1^2}{2}T + \sigma_1 W_T^{1*}}}{e^{rT}}\right\}$$

$$= \left\{(\sigma_1 + \phi_1) W_T^{1*} + \phi_2 W_T^{2*} < \left(\frac{\sigma_\phi^2 + \sigma_1^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right) T - \ln\left(a^* S_0^1\right)\right\}$$

$$= \left\{\sigma_1^Q \tilde{W}_T^{1Q} < \left(\frac{\sigma_\phi^2 + \sigma_1^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right) T - \ln\left(a^* S_0^1\right)\right\}$$

$$= \left\{s_1^Q < \frac{\left(\frac{\sigma_\phi^2+\sigma_1^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right) T - \ln\left(a^* S_0^1\right)}{\sigma_1^Q \sqrt{T}}\right\}. \tag{2.76}$$

To write the last two equalities above, we defined a new Wiener process $\tilde{W}^{1Q} = (\tilde{W}_t^{1Q})_{t\in[0,T]}$ w. r. to P^*:

$$\tilde{W}_t^{1Q} = \frac{(\sigma_1 + \phi_1) W_t^{1*} + \phi_2 W_t^{2*}}{\sigma_1^Q}, \tag{2.77}$$

$$\sigma_1^{Q^2} = (\sigma_1 + \phi_1)^2 + \phi_2^2 + 2\rho(\sigma_1 + \phi_1)\phi_2.$$

Similarly to the requirement $\sigma^2 > 0$ for perfect hedge calculations, we need to check that $\sigma_1^{Q^2}$ is positive:

$$\begin{aligned} \sigma_1^{Q^2} &= (\sigma_1 + \phi_1)^2 + \phi_2^2 + 2\rho(\sigma_1 + \phi_1)\phi_2 \\ &> (\sigma_1 + \phi_1)^2 + \phi_2^2\rho^2 + 2\rho(\sigma_1 + \phi_1)\phi_2 \quad \text{as} \quad \rho^2 < 1 \\ &= ((\sigma_1 + \phi_1) + \phi_2\rho)^2 \geq 0 \\ \Rightarrow \sigma_1^{Q^2} &> 0. \end{aligned} \tag{2.78}$$

Also, since $\tilde{W}_T^{1Q} \sim N(0,T)$, the random variable $s_1^Q = \frac{\tilde{W}_T^{1Q}}{\sqrt{T}} \sim N(0,1)$ w. r. to P^*.

Likewise, we simplify the set $\left\{\frac{1}{Z_T} > \frac{a^* S_T^2}{e^{rT}}\right\}$:

$$\left\{\frac{1}{Z_T} > \frac{a^* S_T^2}{e^{rT}}\right\} = \left\{ e^{-\phi_1 W_T^{1*} - \phi_2 W_T^{2*} + \left(\frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)T} > \frac{a^* S_0^2 e^{rT - \frac{\sigma_2^2}{2}T + \sigma_2 W_T^{2*}}}{e^{rT}} \right\}$$

$$= \left\{ (\sigma_2 + \phi_2) W_T^{2*} + \phi_1 W_T^{1*} < \left(\frac{\sigma_\phi^2 + \sigma_2^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right) T - \ln\left(a^* S_0^2\right) \right\}$$

$$= \left\{ s_2^Q < \frac{\left(\frac{\sigma_\phi^2 + \sigma_2^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right) T - \ln\left(a^* S_0^2\right)}{\sigma_2^Q \sqrt{T}} \right\}, \quad (2.79)$$

with $s_2^Q = \frac{\tilde{W}_T^{2Q}}{\sqrt{T}} \sim N(0,1)$ for the Wiener process $\tilde{W}^{2Q} = (\tilde{W}_t^{2Q})_{t\in[0,T]}$ (w. r. to P^*)

$$\tilde{W}_t^{2Q} = \frac{(\sigma_2 + \phi_2) W_t^{2*} + \phi_1 W_t^{1*}}{\sigma_2^Q}, \quad (2.80)$$

$$\sigma_2^{Q^2} = (\sigma_2 + \phi_2)^2 + \phi_1^2 + 2\rho(\sigma_2 + \phi_2)\phi_1 > 0.$$

Notice that the formulas for σ_i^Q used above with $\tilde{W}^{iQ}$ are the same as the ones in (2.73).

From this point, we require a version of the one-asset lemma (2.44) to calculate explicit expressions for expectations of type

$$E^*\left(e^{-z} I_{\{x<X\}} I_{\{y<Y\}}\right), \quad (2.81)$$

where x, y, z are normally distributed correlated random variables and X, Y are given constants. Note that the approach for deriving the formula for the perfect hedge price of H was similar (see calculations following (2.36)), but we had only one indicator to worry about in that case.

We have derived the result that allows us to calculate expectations in (2.81), it is the multi-asset theorem presented in section 2.5, with proof given in Appendix 2. Here we utilize the theorem for $n = 2$ indicators, referring to it as the *two-asset lemma*:

Two-asset lemma
Let $x \sim N(\mu_x, \sigma_x^2)$, $y \sim N(\mu_y, \sigma_y^2)$ and $z \sim N(\mu_z, \sigma_z^2)$ be three (normally distributed)

random variables with correlations $\rho_{xy}, \rho_{xz}, \rho_{yz}$. Then for given constants X, Y,

$$
\begin{aligned}
E\left(e^{-z} I_{\{x<X\}} I_{\{y<Y\}}\right) &= e^{-\left(\mu_z - \frac{\sigma_z^2}{2}\right)} \cdot \Psi^2(\hat{X}, \hat{Y}, \rho_{xy}), \\
\hat{X} &= \frac{X-\mu_x}{\sigma_x} + \sigma_z \rho_{xz}, \\
\hat{Y} &= \frac{Y-\mu_y}{\sigma_y} + \sigma_z \rho_{yz},
\end{aligned} \tag{2.82}
$$

where Ψ^2 denotes the two-dimensional cumulative normal distribution (2.68).
Proof.
See section 2.5 and Appendix 2.

Now we will use this lemma to finish our calculations of the fair premium for the case of quantile hedging. Based on previous simplifications (see (2.74), (2.75), (2.76) and (2.79)), we have

$$
\begin{aligned}
U_0 &= S_0^1 e^{\frac{-\sigma_1^2}{2}T} E^* \left(e^{\sigma_1 W_T^{1*}} I_{\left\{ s_1^Q < \frac{\left(\frac{\sigma_\phi^2+\sigma_1^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)T - \ln\left(a^* S_0^1\right)}{\sigma_1^Q \sqrt{T}} \right\}} I_{\left\{ s_1 \le \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \frac{\sigma_2^2-\sigma_1^2}{2}T}{\sigma\sqrt{T}} \right\}} \right) \\
&+ S_0^2 e^{\frac{-\sigma_2^2}{2}T} E^* \left(e^{\sigma_2 W_T^{2*}} I_{\left\{ s_2^Q < \frac{\left(\frac{\sigma_\phi^2+\sigma_2^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)T - \ln\left(a^* S_0^2\right)}{\sigma_2^Q \sqrt{T}} \right\}} I_{\left\{ s_2 < \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) + \frac{\sigma_1^2-\sigma_2^2}{2}T}{\sigma\sqrt{T}} \right\}} \right).
\end{aligned} \tag{2.83}
$$

Now we will apply the two-asset lemma with $z = -\sigma_i W_T^{i*} \sim N(0, \sigma_i^2 T)$, $x = s_i^Q \sim N(0,1)$, $y = s_i \sim N(0,1)$, and the respective correlations

$$
\begin{aligned}
\rho_{xz} &= \operatorname{corr}(s_i^Q, -\sigma_i W_T^{i*}) = -\frac{\sigma_i + \phi_i + \rho\phi_j}{\sigma_i^Q}, \\
\rho_{yz} &= \operatorname{corr}(s_i, -\sigma_i W_T^{i*}) = \frac{\sigma_i - \sigma_j \rho}{\sigma}, \\
\rho_{xy} &= \operatorname{corr}(s_i^Q, s_i) = \frac{\phi_j(\sigma_j - \sigma_i\rho) - (\sigma_i + \phi_i)(\sigma_i - \sigma_j\rho)}{\sigma_i^Q \sigma}
\end{aligned} \tag{2.84}
$$

for $i, j = 1, 2$. Note that $\operatorname{corr}(s_i^Q, s_i)$ are exactly the correlations ρ_i^Q given in (2.69).

Let us calculate $\hat{X}, \hat{Y}$: from (2.83), we have

$$\begin{aligned}
\hat{X}_1 &= \frac{\left(\frac{\sigma_\phi^2+\sigma_1^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)T - \ln(a^* S_0^1)}{\sigma_1^Q \sqrt{T}} + \sigma_1\sqrt{T}\left(-\frac{\sigma_1 + \phi_1 + \rho\phi_2}{\sigma_1^Q}\right) \\
&= \frac{\left[\frac{\sigma_\phi^2-\sigma_1^2}{2} + \phi_1\theta_1 + \phi_2\theta_2 - \sigma_1(\rho\phi_2 + \phi_1)\right]T - \ln\left(a^* S_0^1\right)}{\sigma_1^Q\sqrt{T}} = \tilde{x}_1^Q, \\
\hat{X}_2 &= \frac{\left(\frac{\sigma_\phi^2+\sigma_2^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)T - \ln\left(a^* S_0^2\right)}{\sigma_2^Q \sqrt{T}} + \sigma_2\sqrt{T}\left(-\frac{\sigma_2 + \phi_2 + \rho\phi_1}{\sigma_2^Q}\right) \\
&= \frac{\left[\frac{\sigma_\phi^2-\sigma_2^2}{2} + \phi_1\theta_1 + \phi_2\theta_2 - \sigma_2(\rho\phi_1 + \phi_2)\right]T - \ln\left(a^* S_0^2\right)}{\sigma_2^Q\sqrt{T}} = \tilde{x}_2^Q, \qquad (2.85)
\end{aligned}$$

and

$$\begin{aligned}
\hat{Y}_1 &= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \frac{\sigma_2^2-\sigma_1^2}{2}T}{\sigma\sqrt{T}} + \sigma_1\sqrt{T}\left(\frac{\sigma_1 - \sigma_2\rho}{\sigma}\right) \\
&= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} = \tilde{y}_1, \\
\hat{Y}_2 &= \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) + \frac{\sigma_1^2-\sigma_2^2}{2}T}{\sigma\sqrt{T}} + \sigma_2\sqrt{T}\left(\frac{\sigma_2 - \sigma_1\rho}{\sigma}\right) \\
&= \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} = \tilde{y}_2, \qquad (2.86)
\end{aligned}$$

with $\tilde{x}_i^Q$ and $\tilde{y}_i$ introduced in (2.70) and (2.38).

Then for U_0 in (2.83), we obtain

$$\begin{aligned}
U_0 &= S_0^1 e^{\frac{-\sigma_1^2}{2}T} e^{-\left(0-\frac{\sigma_1^2}{2}\right)T} \Psi^2\left(\hat{X}_1, \hat{Y}_1, \mathrm{corr}(s_1^Q, s_1)\right) \\
&+ S_0^2 e^{\frac{-\sigma_2^2}{2}T} e^{-\left(0-\frac{\sigma_2^2}{2}\right)T} \Psi^2\left(\hat{X}_2, \hat{Y}_2, \mathrm{corr}(s_2^Q, s_2)\right) \\
&= S_0^1 \Psi^2(\tilde{x}_1^Q, \tilde{y}_1, \rho_1^Q) + S_0^2 \Psi^2(\tilde{x}_2^Q, \tilde{y}_2, \rho_2^Q), \qquad (2.87)
\end{aligned}$$

which is the formula for the fair premium in the case of quantile hedging.

Part II. Now we derive the formula for the probability of successful hedging $P(A^*)$ in (2.67). For this, we adopt a similar approach to the one above for calculating the fair premium. Note that now we work under the original probability measure P, not the risk-neutral P^*. First, we rewrite A^* as follows:

$$
\begin{aligned}
A^* &= \left\{\frac{1}{Z_T} > \frac{a^* H}{e^{rT}}\right\} \\
&= \left\{\frac{1}{Z_T} > \frac{a^* S_T^1}{e^{rT}}\right\} \cap \{S_T^1 \geq S_T^2\} \\
&+ \left\{\frac{1}{Z_T} > \frac{a^* S_T^2}{e^{rT}}\right\} \cap \{S_T^1 < S_T^2\}. \qquad (2.88)
\end{aligned}
$$

Next, using (2.26) and (2.21), we simplify the sets above by rewriting them in terms of new Wiener processes. For $\{S_T^1 \geq S_T^2\}$ and $\{S_T^1 < S_T^2\}$, we have

$$
\begin{aligned}
\{S_T^1 \geq S_T^2\} &= \left\{\sigma_2 W_T^2 - \sigma_1 W_T^1 \leq \ln\left(\frac{S_0^1}{S_0^2}\right) + \left(\mu_1 - \mu_2 + \frac{\sigma_2^2 - \sigma_1^2}{2}\right) T\right\} \\
&= \left\{ s_1 \leq \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \left(\mu_1 - \mu_2 + \frac{\sigma_2^2 - \sigma_1^2}{2}\right) T}{\sigma\sqrt{T}} \right\} \\
&= \{s_1 \leq \bar{y}_1^Q\}, \\
\{S_T^1 < S_T^2\} &= \left\{\sigma_1 W_T^1 - \sigma_2 W_T^2 < \ln\left(\frac{S_0^2}{S_0^1}\right) + \left(\mu_2 - \mu_1 + \frac{\sigma_1^2 - \sigma_2^2}{2}\right) T\right\} \\
&= \left\{ s_2 < \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) + \left(\mu_2 - \mu_1 + \frac{\sigma_1^2 - \sigma_2^2}{2}\right) T}{\sigma\sqrt{T}} \right\} \\
&= \{s_2 < \bar{y}_2^Q\}, \qquad (2.89)
\end{aligned}
$$

with $\bar{W}^i = (\bar{W}_t^i)_{t\in[0,T]}$ such that

$$
\begin{aligned}
\bar{W}_t^1 &= \frac{\sigma_2 W_t^2 - \sigma_1 W_t^1}{\sigma}, \qquad (2.90) \\
\bar{W}_t^2 &= \frac{\sigma_1 W_t^1 - \sigma_2 W_t^2}{\sigma}, \\
\sigma^2 &= \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 > 0,
\end{aligned}
$$

and $s_i = \frac{\bar{W}_T^i}{\sqrt{T}} \sim N(0,1)$ under P. The constants $\bar{y}_i^Q$ are given in (2.72).

For $\left\{\frac{1}{Z_T} > \frac{a^* S_T^i}{e^{rT}}\right\}$, we get

$$\begin{aligned}
\left\{\frac{1}{Z_T} > \frac{a^* S_T^1}{e^{rT}}\right\} &= \left\{e^{-\phi_1 W_T^1 - \phi_2 W_T^2 + \frac{\sigma_\phi^2}{2}T} > a^* S_0^1 e^{\left(\mu_1 - r - \frac{\sigma_1^2}{2}\right)T + \sigma_1 W_T^1}\right\} \\
&= \left\{(\sigma_1 + \phi_1) W_T^1 + \phi_2 W_T^2 < \left(r - \mu_1 + \frac{\sigma_\phi^2 + \sigma_1^2}{2}\right) T - \ln\left(a^* S_0^1\right)\right\} \\
&= \left\{ s_1^Q < \frac{\left(r - \mu_1 + \frac{\sigma_\phi^2 + \sigma_1^2}{2}\right) T - \ln\left(a^* S_0^1\right)}{\sigma_1^Q \sqrt{T}} \right\} \\
&= \{s_1^Q < \bar{x}_1^Q\}, \\
\left\{\frac{1}{Z_T} > \frac{a^* S_T^2}{e^{rT}}\right\} &= \left\{e^{-\phi_1 W_T^1 - \phi_2 W_T^2 + \frac{\sigma_\phi^2}{2}T} > a^* S_0^2 e^{\left(\mu_2 - r - \frac{\sigma_2^2}{2}\right)T + \sigma_2 W_T^2}\right\} \\
&= \left\{(\sigma_2 + \phi_2) W_T^2 + \phi_1 W_T^1 < \left(r - \mu_2 + \frac{\sigma_\phi^2 + \sigma_2^2}{2}\right) T - \ln\left(a^* S_0^2\right)\right\} \\
&= \left\{ s_2^Q < \frac{\left(r - \mu_2 + \frac{\sigma_\phi^2 + \sigma_2^2}{2}\right) T - \ln\left(a^* S_0^2\right)}{\sigma_2^Q \sqrt{T}} \right\} \\
&= \{s_2^Q < \bar{x}_2^Q\}, \qquad (2.91)
\end{aligned}$$

with $\bar{W}^{iQ} = (\bar{W}_t^{iQ})_{t\in[0,T]}$ such that

$$\begin{aligned}
\bar{W}_t^{1Q} &= \frac{(\sigma_1 + \phi_1) W_t^1 + \phi_2 W_t^2}{\sigma_1^Q}, \qquad (2.92) \\
\bar{W}_t^{2Q} &= \frac{(\sigma_2 + \phi_2) W_t^2 + \phi_1 W_t^1}{\sigma_2^Q},
\end{aligned}$$

and $s_i^Q = \frac{\bar{W}_T^{iQ}}{\sqrt{T}} \sim N(0,1)$ w. r. to P. Also, note that the constants $\bar{x}_i^Q$ and $\sigma_i^{Q^2}$ are given in and (2.71) and (2.73) respectively, and that $\sigma_i^{Q^2} > 0$, as shown in (2.78).

Now (2.88) becomes

$$A^* = \{s_1^Q < \bar{x}_1^Q\} \cap \{s_1 \le \bar{y}_1^Q\} + \{s_2^Q < \bar{x}_2^Q\} \cap \{s_2 \le \bar{y}_2^Q\}, \qquad (2.93)$$

where the respective correlations $\operatorname{corr}(s_i^Q, s_i) = \rho_i^Q$ are given in (2.69) (see also (2.84)). All above considerations allow us to write the set of successful hedging in its finalized form

$$\begin{aligned} P(A^*) &= P\left(\{s_1^Q < \bar{x}_1^Q\} \cap \{s_1 \le \bar{y}_1^Q\} + \{s_2^Q < \bar{x}_2^Q\} \cap \{s_2 \le \bar{y}_2^Q\}\right) \\ &= \Psi^2(\bar{x}_1^Q, \bar{y}_1^Q, \rho_1^Q) + \Psi^2(\bar{x}_2^Q, \bar{y}_2^Q, \rho_2^Q). \end{aligned} \tag{2.94}$$

2.4.2 Efficient hedging: risk-aversion

Theorem 2

Suppose that the firm that sells an equity-linked life insurance contract with payoff (2.47) decides to use efficient hedging to minimize the shortfall risk. Further, suppose that the firm's risk preference is risk-aversion, so that for the loss function $l(x) = x^p$, $p > 1$.

Part I. The **fair premium** for the contract is

$$\begin{aligned} U_0 &= E^*\left(\varphi^* \frac{\max\{S_T^1, S_T^2\}}{e^{rT}}\right) \\ &= S_0^1 \cdot \Psi^2(\tilde{x}_1^A, \tilde{y}_1, \rho_1^A) + S_0^2 \cdot \Psi^2(\tilde{x}_2^A, \tilde{y}_2, \rho_2^A) \\ &- M \cdot \left[\Psi^2(\tilde{c}_1, \tilde{y}_1^c, \rho_1^A) + \Psi^2(\tilde{c}_2, \tilde{y}_2^c, \rho_2^A)\right]. \end{aligned} \tag{2.95}$$

Part II. The **shortfall risk** is given by

$$\begin{aligned} E\left(l((H - V_T)^+)\right) &= N \cdot \left[\Psi^2(\bar{c}_1, \bar{y}_1^c, \rho_1^A) + \Psi^2(\bar{c}_2, \bar{y}_2^c, \rho_2^A)\right] \\ &+ (S_0^1)^p e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)Tp + \frac{\sigma_1^2}{2}Tp^2} \cdot \Psi^2(\bar{k}_1, \bar{y}_1^k, -\rho_1^A) \\ &+ (S_0^2)^p e^{\left(\mu_2 - \frac{\sigma_2^2}{2}\right)Tp + \frac{\sigma_2^2}{2}Tp^2} \cdot \Psi^2(\bar{k}_2, \bar{y}_2^k, -\rho_2^A). \end{aligned} \tag{2.96}$$

Above, Ψ^2 denotes two-dimensional cumulative normal distribution given in (2.68), with

$$\begin{aligned} \rho_1^A &= \frac{\phi_2(\sigma_2 - \sigma_1\rho) - (\phi_1 - \sigma_1(p-1))(\sigma_1 - \sigma_2\rho)}{\sigma_1^E \sigma}, \\ \rho_2^A &= \frac{\phi_1(\sigma_1 - \sigma_2\rho) - (\phi_2 - \sigma_2(p-1))(\sigma_2 - \sigma_1\rho)}{\sigma_2^E \sigma}. \end{aligned} \tag{2.97}$$

The constants M, N, $\tilde{x}_i^A$, $\tilde{c}_i$, $\tilde{y}_i^c$, $\bar{c}_i$, $\bar{y}_i^c$, $\bar{k}_i$, and $\bar{y}_i^k$ are defined as

$$M = \left(\frac{a^*}{p}\right)^{\frac{1}{p-1}} \cdot \frac{e^{\frac{\sigma_\phi^2 T}{2(p-1)^2}}}{e^{rT + \left(r + \frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)\frac{T}{p-1}}}, \tag{2.98}$$

$$N = \left(\frac{a^*}{p}\right)^{\frac{p}{p-1}} \cdot \frac{e^{\frac{\sigma_\phi^2 T p^2}{2(p-1)^2}}}{e^{\left(r+\frac{\sigma_\phi^2}{2}\right)\frac{Tp}{p-1}}}, \tag{2.99}$$

$$\begin{aligned}
\tilde{x}_1^A &= \frac{\ln\left((S_0^1)^{p-1}\frac{p}{a^*}\right) + \left[p\left(r+\frac{\sigma_1^2}{2}\right) + \frac{\sigma_\phi^2-\sigma_1^2}{2} + \phi_1(\theta_1-\sigma_1) + \phi_2(\theta_2-\rho\sigma_1)\right]T}{\sigma_1^E\sqrt{T}}, \\
\tilde{x}_2^A &= \frac{\ln\left((S_0^2)^{p-1}\frac{p}{a^*}\right) + \left[p\left(r+\frac{\sigma_2^2}{2}\right) + \frac{\sigma_\phi^2-\sigma_2^2}{2} + \phi_2(\theta_2-\sigma_2) + \phi_1(\theta_1-\rho\sigma_2)\right]T}{\sigma_2^E\sqrt{T}},
\end{aligned} \tag{2.100}$$

$$\begin{aligned}
\tilde{c}_1 &= \frac{\ln\left((S_0^1)^{p-1}\frac{p}{a^*}\right) + \left[p\left(r-\frac{\sigma_1^2}{2}\right) + \frac{\sigma_\phi^2+\sigma_1^2}{2} - \frac{\sigma_\phi^2}{p-1} + \phi_1(\theta_1+\sigma_1) + \phi_2(\theta_2+\rho\sigma_1)\right]T}{\sigma_1^E\sqrt{T}}, \\
\tilde{c}_2 &= \frac{\ln\left((S_0^2)^{p-1}\frac{p}{a^*}\right) + \left[p\left(r-\frac{\sigma_2^2}{2}\right) + \frac{\sigma_\phi^2+\sigma_2^2}{2} - \frac{\sigma_\phi^2}{p-1} + \phi_2(\theta_2+\sigma_2) + \phi_1(\theta_1+\rho\sigma_2)\right]T}{\sigma_2^E\sqrt{T}},
\end{aligned} \tag{2.101}$$

$$\begin{aligned}
\tilde{y}_1^c &= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \left(\frac{\sigma_2^2-\sigma_1^2}{2}\right)T + \left[\phi_1(\sigma_1-\sigma_2\rho) - \phi_2(\sigma_2-\sigma_1\rho)\right]\frac{T}{p-1}}{\sigma\sqrt{T}}, \\
\tilde{y}_2^c &= \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) + \left(\frac{\sigma_1^2-\sigma_2^2}{2}\right)T + \left[\phi_2(\sigma_2-\sigma_1\rho) - \phi_1(\sigma_1-\sigma_2\rho)\right]\frac{T}{p-1}}{\sigma\sqrt{T}},
\end{aligned} \tag{2.102}$$

$$\begin{aligned}
\bar{c}_1 &= \frac{\ln\left((S_0^1)^{p-1}\frac{p}{a^*}\right) + \left[r + \frac{\sigma_\phi^2}{2} - \frac{\sigma_\phi^2 p}{p-1} + (p-1)\left(\mu_1 - \frac{\sigma_1^2}{2}\right) + p(\phi_1\sigma_1 + \phi_2\sigma_1\rho)\right]T}{\sigma_1^E\sqrt{T}}, \\
\bar{c}_2 &= \frac{\ln\left((S_0^2)^{p-1}\frac{p}{a^*}\right) + \left[r + \frac{\sigma_\phi^2}{2} - \frac{\sigma_\phi^2 p}{p-1} + (p-1)\left(\mu_2 - \frac{\sigma_2^2}{2}\right) + p(\phi_2\sigma_2 + \phi_1\sigma_2\rho)\right]T}{\sigma_2^E\sqrt{T}},
\end{aligned} \tag{2.103}$$

$$\bar{y}_1^c = \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \left[\mu_1 - \mu_2 + \frac{\sigma_2^2-\sigma_1^2}{2} + \frac{p}{p-1}\left(\phi_1(\sigma_1 - \sigma_2\rho) - \phi_2(\sigma_2 - \sigma_1\rho)\right)\right]T}{\sigma\sqrt{T}},$$
$$\bar{y}_2^c = \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) + \left[\mu_2 - \mu_1 + \frac{\sigma_1^2-\sigma_2^2}{2} + \frac{p}{p-1}\left(\phi_2(\sigma_2 - \sigma_1\rho) - \phi_1(\sigma_1 - \sigma_2\rho)\right)\right]T}{\sigma\sqrt{T}}, \tag{2.104}$$

$$\bar{k}_1 = \frac{\ln\left(\frac{a^*}{p}(S_0^1)^{1-p}\right) - \left[r + \frac{\sigma_\phi^2}{2} + (p-1)\left(\mu_1 - \frac{\sigma_1^2}{2}\right) - p(\phi_1\sigma_1 + \phi_2\sigma_1\rho - \sigma_1^2(p-1))\right]T}{\sigma_1^E\sqrt{T}},$$
$$\bar{k}_2 = \frac{\ln\left(\frac{a^*}{p}(S_0^2)^{1-p}\right) - \left[r + \frac{\sigma_\phi^2}{2} + (p-1)\left(\mu_2 - \frac{\sigma_2^2}{2}\right) - p(\phi_2\sigma_2 + \phi_1\sigma_2\rho - \sigma_2^2(p-1))\right]T}{\sigma_2^E\sqrt{T}}, \tag{2.105}$$

$$\bar{y}_1^k = \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \left[\mu_1 - \mu_2 + \frac{\sigma_2^2-\sigma_1^2}{2} + p(\sigma_1^2 - \sigma_1\sigma_2\rho)\right]T}{\sigma\sqrt{T}},$$
$$\bar{y}_2^k = \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) + \left[\mu_2 - \mu_1 + \frac{\sigma_1^2-\sigma_2^2}{2} + p(\sigma_2^2 - \sigma_1\sigma_2\rho)\right]T}{\sigma\sqrt{T}}, \tag{2.106}$$

and σ_i^E are given by

$$(\sigma_1^E)^2 = \sigma_\phi^2 + (1-p)^2\sigma_1^2 - 2\sigma_1(1-p)\theta_1,$$
$$(\sigma_2^E)^2 = \sigma_\phi^2 + (1-p)^2\sigma_2^2 - 2\sigma_2(1-p)\theta_2. \tag{2.107}$$

Note that φ^* is given in (2.55), $\tilde{y}_i$ in (2.38), ϕ_i in (2.27), θ_i in (2.31), σ in (2.39), and σ_ϕ in (2.28). Also, the formulas for the premium and the shortfall risk above hold as long as technical conditions $\rho \neq \frac{\theta_1}{\theta_2}$ and $\rho \neq \frac{\theta_2}{\theta_1}$ are satisfied, as explained in the proof of Theorem 2. The conditions are not restrictive in any way: the likelihood of having two risky assets with correlation of the underlying Wiener processes being exactly equal to the ratio of θ_i is very small. However, in case this does happen, there are ways to deal with the situation (please see the proof for more details).

Proof.

For proofs of pricing and shortfall risk formulas for all risk preference cases of efficient hedging, we first simplify the expression for the modified contingent claim $H^* = \varphi^* H$ (2.54). Second, we rewrite U_0 in terms of indicator sets, which we simplify by introducing new Wiener processes. Third, we evaluate the resulting expectations of type (2.81) by utilizing the two-asset lemma (2.82). Notice that the second and third steps are the same as in the approach of proving the results for quantile hedging. Because of this, we will only show major steps in the derivations, leaving out tedious calculation details.

Part I. We wish to calculate the fair premium for the case of risk-aversion using efficient hedging. The success ratio φ^* for this case is given by (2.55):

$$\varphi^* = 1 - \left(\frac{I\left(a^* e^{-rT} Z_T\right)}{H} \wedge 1 \right), \quad p > 1,$$

with $I = (l')^{-1}$ denoting the inverse of the derivative of the loss function l. Since we are using $l(x) = x^p$, we have

$$l'(x) = px^{p-1} \qquad \Rightarrow \qquad I(x) = x^{\frac{1}{p-1}} \left(\frac{1}{p}\right)^{\frac{1}{p-1}}, \tag{2.108}$$

therefore

$$\begin{aligned} I\left(a^* e^{-rT} Z_T\right) &= k^* \cdot (Z_T)^{\frac{1}{p-1}}, \\ \text{where} \quad k^* &= \left(\frac{a^*}{pe^{rT}}\right)^{\frac{1}{p-1}}. \end{aligned} \tag{2.109}$$

Then H^* simplifies to

$$\begin{aligned} H^* &= \varphi^* H = H - \left(k^* (Z_T)^{\frac{1}{p-1}} \wedge H\right) \\ &= \left(H - k^*(Z_T)^{\frac{1}{p-1}}\right) I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} < H\right\}}, \end{aligned} \tag{2.110}$$

which leads to this expression for the fair premium:

$$\begin{aligned} U_0 &= E^*\left(\frac{\varphi^* H}{e^{rT}}\right) \\ &= E^*\left(\frac{H}{e^{rT}} I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} < H\right\}} - \frac{k^*}{e^{rT}} (Z_T)^{\frac{1}{p-1}} I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} < H\right\}}\right) \\ &= E^*\left(\frac{S_T^1}{e^{rT}} I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^1\right\}} I_{\{S_T^1 \geq S_T^2\}}\right) \\ &- E^*\left(\frac{k^*}{e^{rT}} (Z_T)^{\frac{1}{p-1}} I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^1\right\}} I_{\{S_T^1 \geq S_T^2\}}\right) \\ &+ E^*\left(\frac{S_T^2}{e^{rT}} I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^2\right\}} I_{\{S_T^1 < S_T^2\}}\right) \\ &- E^*\left(\frac{k^*}{e^{rT}} (Z_T)^{\frac{1}{p-1}} I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^2\right\}} I_{\{S_T^1 < S_T^2\}}\right). \end{aligned} \tag{2.111}$$

Now we work with the indicator sets above; we already know that $\{S_T^1 \geq S_T^2\}$ and $\{S_T^1 < S_T^2\}$ simplify as shown in (2.75). So we need to express $\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^1\right\}$ and $\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^2\right\}$ similarly. Since the calculations are symmetric for the formulas involving S^i, from now on we show derivations for S^1 and only state the results for sets involving S^2.

Consider the set $\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^1\right\}$: using (2.32) and (2.29), we can write

$$
\begin{aligned}
&\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^1\right\} \qquad (2.112)\\
&= \left\{k^* e^{\frac{\phi_1}{p-1}W_T^{1*}+\frac{\phi_2}{p-1}W_T^{2*}-\left(\frac{\sigma_\phi^2}{2}+\phi_1\theta_1+\phi_2\theta_2\right)\frac{T}{p-1}} < S_0^1 e^{\left(r-\frac{\sigma_1^2}{2}\right)T+\sigma_1 W_T^{1*}}\right\}\\
&= \left\{\tilde{\sigma}_1^A \tilde{W}_T^{1A} < \ln\left(\frac{S_0^1}{k^*}\right) + \left(r-\frac{\sigma_1^2}{2}\right)T + \left(\frac{\sigma_\phi^2}{2}+\phi_1\theta_1+\phi_2\theta_2\right)\frac{T}{p-1}\right\}\\
&= \left\{s_1^A < \frac{\ln\left(\frac{S_0^1}{k^*}\right) + \left(r-\frac{\sigma_1^2}{2}\right)T + \left(\frac{\sigma_\phi^2}{2}+\phi_1\theta_1+\phi_2\theta_2\right)\frac{T}{p-1}}{\tilde{\sigma}_1^A\sqrt{T}}\right\},
\end{aligned}
$$

where $\tilde{W}^{1A} = (\tilde{W}_t^{1A})_{t\in[0,T]}$ is a Wiener process under P^* such that

$$
\begin{aligned}
\tilde{W}_t^{1A} &= \frac{\left(\frac{\phi_1}{p-1}-\sigma_1\right)W_t^{1*}+\frac{\phi_2}{p-1}W_t^{2*}}{\tilde{\sigma}_1^A}, \qquad (2.113)\\
(\tilde{\sigma}_1^A)^2 &= \left(\frac{\phi_1}{p-1}-\sigma_1\right)^2 + \frac{\phi_2^2}{(p-1)^2} + 2\rho\frac{\phi_2}{p-1}\left(\frac{\phi_1}{p-1}-\sigma_1\right)\\
&= \frac{(\phi_1-\sigma_1(p-1))^2+\phi_2^2+2\rho\phi_2(\phi_1-\sigma_1(p-1))}{(p-1)^2}\\
&= \frac{\sigma_\phi^2+\sigma_1^2(p-1)^2-2\sigma_1(p-1)(\phi_1+\rho\phi_2)}{(p-1)^2},
\end{aligned}
$$

and $s_1^A = \frac{\tilde{W}_T^{1A}}{\sqrt{T}} \sim N(0,1)$ (w. r. to P^*). To simplify the expression for $(\tilde{\sigma}_1^A)^2$ above, we used the definition of σ_ϕ given in (2.28).

We must check that $(\tilde{\sigma}_1^A)^2 > 0$. First, note that ϕ_i (2.27) can be expressed as

$$
\begin{aligned}
\phi_1 = \frac{\rho\theta_2-\theta_1}{1-\rho^2} \quad &\text{and} \quad \phi_2 = \frac{\rho\theta_1-\theta_2}{1-\rho^2} \qquad (2.114)\\
\Rightarrow \quad \phi_1+\rho\phi_2 &= -\theta_1,
\end{aligned}
$$

which allows us to write

$$\begin{aligned}(\tilde{\sigma}_1^A)^2 &= \frac{\sigma_\phi^2 + \sigma_1^2(p-1)^2 + 2\sigma_1\theta_1(p-1)}{(p-1)^2} \\ &= \frac{\sigma_1^2 p^2 + 2\sigma_1(\theta_1 - \sigma_1)p + \sigma_\phi^2 + \sigma_1^2 - 2\sigma_1\theta_1}{(p-1)^2}. \end{aligned} \tag{2.115}$$

Denote

$$Q(p) = \sigma_1^2 p^2 + 2\sigma_1(\theta_1 - \sigma_1)p + \sigma_\phi^2 + \sigma_1^2 - 2\sigma_1\theta_1. \tag{2.116}$$

We examine the discriminant D of the quadratic $Q(p)$ to see when $Q(p) > 0$:

$$\begin{aligned} D &= 4\sigma_1^2(\theta_1 - \sigma_1)^2 - 4\sigma_1^2(\sigma_\phi^2 + \sigma_1^2 - 2\sigma_1\theta_1) \\ &= 4\sigma_1^2(\theta_1^2 - \sigma_\phi^2). \end{aligned} \tag{2.117}$$

Now we need to figure out the sign of $\theta_1^2 - \sigma_\phi^2$. Observe that using (2.114) and (2.28), we can write

$$\sigma_\phi^2 = \frac{\theta_1^2 + \theta_2^2 - 2\rho\theta_1\theta_2}{1-\rho^2}, \tag{2.118}$$

which, in turn, leads to

$$\begin{aligned} \theta_1^2 - \sigma_\phi^2 &= -\frac{\rho^2\theta_1^2 + \theta_2^2 - 2\rho\theta_1\theta_2}{1-\rho^2} \\ &= -\frac{1}{1-\rho^2}(\rho\theta_1 - \theta_2)^2 \le 0, \quad \text{as} \quad \rho^2 < 1. \end{aligned} \tag{2.119}$$

We see that $D < 0$ as long as $\rho\theta_1 - \theta_2 \neq 0$, or, equivalently,

$$\rho \neq \theta_2/\theta_1. \tag{2.120}$$

This implies that $Q(p) > 0$, that is, $(\tilde{\sigma}_1^A)^2 > 0$. For the set with S^2 we get a symmetric condition

$$\rho \neq \theta_1/\theta_2. \tag{2.121}$$

It is precisely (2.120) and (2.121) that give rise to the technical conditions in Theorem 2.

In case that $\rho = \theta_2/\theta_1$, the quadratic $Q(p)$ would have a double root at

$$-(\theta_1 - \sigma_1)/\sigma_1,$$

which means that volatility $\tilde{\sigma}_1^A$ would equal 0 if p happened to be precisely equal to the root of $Q(p)$. This would make the set in (2.112) equal to Ω (or the empty set) and reduce our calculations with sets involving S^1 to those done previously for the case of perfect hedging (see (2.40)). Alternatively, p could be slightly adjusted to not equal the root of $Q(p)$, and we would proceed with computations as shown in the remainder of the proof.

Before we return to the simplification of the set $\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^1\right\}$, consider (2.115):

$$\begin{aligned}(\tilde{\sigma}_1^A)^2 &= \frac{\sigma_\phi^2 + \sigma_1^2(p-1)^2 + 2\sigma_1\theta_1(p-1)}{(p-1)^2} \\ &= \frac{(\sigma_1^E)^2}{(p-1)^2},\end{aligned} \tag{2.122}$$

with σ_1^E defined in (2.107). Now we can write

$$\begin{aligned}&\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^1\right\} \\ &= \left\{ s_1^A < \frac{\ln\left(\frac{S_0^1}{k^*}\right) + \left(r - \frac{\sigma_1^2}{2}\right)T + \left(\frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)\frac{T}{p-1}}{\tilde{\sigma}_1^A\sqrt{T}} \right\} \\ &= \left\{ s_1^A < (p-1)\frac{\ln\left(\frac{S_0^1}{k^*}\right) + \left(r - \frac{\sigma_1^2}{2}\right)T + \left(\frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)\frac{T}{p-1}}{\sigma_1^E\sqrt{T}} \right\}.\end{aligned} \tag{2.123}$$

Consider the expression for U_0 in (2.111): using (2.75) and (2.123), we can write a part of it in the simplified form as

$$\begin{aligned}&E^*\left(\frac{S_T^1}{e^{rT}} I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^1\right\}} I_{\{S_T^1 \geq S_T^2\}}\right) \\ &= S_0^1 e^{-\frac{\sigma_1^2}{2}T} E^*\left(e^{\sigma_1 W_T^{1*}} I_{\{s_1^A < \tilde{k}_1^A\}} I_{\left\{s_1 \leq \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \frac{\sigma_2^2 - \sigma_1^2}{2}T}{\sigma\sqrt{T}}\right\}}\right),\end{aligned} \tag{2.124}$$

where

$$\tilde{k}_1^A = (p-1)\frac{\ln\left(\frac{S_0^1}{k^*}\right) + \left(r - \frac{\sigma_1^2}{2}\right)T + \left(\frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)\frac{T}{p-1}}{\sigma_1^E\sqrt{T}}. \tag{2.125}$$

Now we apply the two-asset lemma (2.82) to evaluate the expectation above. We take $z = -\sigma_1 W_T^{1*} \sim N(0, \sigma_1^2 T)$, $x = s_1^A \sim N(0,1)$ and $y = s_1 \sim N(0,1)$; the necessary correlations are calculated to equal

$$\begin{aligned}\rho_{xz} &= -\frac{p-1}{\sigma_1^E}\left(\frac{\phi_1}{p-1}-\sigma_1+\frac{\phi_2\rho}{p-1}\right),\\ \rho_{yz} &= \frac{\sigma_1-\sigma_2\rho}{\sigma},\\ \rho_{xy} &= \frac{\phi_2(\sigma_2-\sigma_1\rho)-(\phi_1-\sigma_1(p-1))(\sigma_1-\sigma_2\rho)}{\sigma_1^E\sigma}=\rho_1^A, \end{aligned}\tag{2.126}$$

with ρ_1^A defined in (2.97).

Applying the lemma with the above parameters to the expected value in (2.124) and simplifying the resulting constants, we get

$$E^*\left(\frac{S_T^1}{e^{rT}}I_{\left\{k^*(Z_T)^{\frac{1}{p-1}}<S_T^1\right\}}I_{\{S_T^1\geq S_T^2\}}\right)=S_0^1\Psi^2(\tilde{x}_1^A,\tilde{y}_1,\rho_1^A),\tag{2.127}$$

with $\tilde{x}_1^A$ and $\tilde{y}_1$ given in (2.100) and (2.38).

Now let us return to U_0 in (2.111). We will simplify another term in this expression using (2.32), (2.123) and (2.75):

$$\begin{aligned}&E^*\left(\frac{k^*}{e^{rT}}(Z_T)^{\frac{1}{p-1}}I_{\left\{k^*(Z_T)^{\frac{1}{p-1}}<S_T^1\right\}}I_{\{S_T^1\geq S_T^2\}}\right)\\ &=\frac{k^*}{e^{rT}}e^{-\left(\frac{\sigma_\phi^2}{2}+\phi_1\theta_1+\phi_2\theta_2\right)\frac{T}{p-1}}E^*\left(e^{\frac{\phi_1}{p-1}W_T^{1*}+\frac{\phi_2}{p-1}W_T^{2*}}I_{\{s_1^A<\tilde{k}_1^A\}}I_{\left\{s_1\leq\frac{\ln\left(\frac{S_0^1}{S_0^2}\right)+\frac{\sigma_2^2-\sigma_1^2}{2}T}{\sigma\sqrt{T}}\right\}}\right).\end{aligned}\tag{2.128}$$

Define a Wiener process $\tilde{W}^p=(\tilde{W}_t^p)_{t\in[0,T]}$ under P^* as

$$\begin{aligned}\tilde{W}_t^p &= \frac{\frac{\phi_1}{p-1}W_t^{1*}+\frac{\phi_2}{p-1}W_t^{2*}}{\sigma_p},\\ \sigma_p^2 &= \frac{\phi_1^2}{(p-1)^2}+\frac{\phi_2^2}{(p-1)^2}+2\rho\frac{\phi_2\phi_1}{(p-1)^2}\\ &= \frac{\sigma_\phi^2}{(p-1)^2}>0,\end{aligned}\tag{2.129}$$

with σ_ϕ given in (2.28). Then (2.128) becomes

$$\frac{k^*}{e^{rT}} e^{-\left(\frac{\sigma_\phi^2}{2}+\phi_1\theta_1+\phi_2\theta_2\right)\frac{T}{p-1}} E^* \left(e^{\left(\frac{\sigma_\phi}{p-1}\tilde{W}_T^p\right)} I_{\{s_1^A<\tilde{k}_1^A\}} I_{\left\{s_1 \le \frac{\ln\left(\frac{S_0^1}{S_0^2}\right)+\frac{\sigma_2^2-\sigma_1^2}{2}T}{\sigma\sqrt{T}}\right\}} \right). \quad (2.130)$$

Now we apply the two-asset lemma (2.82) with $z = -\frac{\sigma_\phi}{p-1}\tilde{W}_T^p \sim N\left(0, \frac{\sigma_\phi^2}{(p-1)^2}T\right)$, $x, y = s_1^A, s_1 \sim N(0,1)$ under P^*, and correlations

$$\begin{aligned} \rho_{xz} &= \frac{\sigma_1(p-1)(\phi_1+\rho\phi_2)-\sigma_\phi^2}{\sigma_\phi\sigma_1^E}, \\ \rho_{yz} &= \frac{\phi_1(\sigma_1-\sigma_2\rho)-\phi_2(\sigma_2-\sigma_1\rho)}{\sigma_\phi\sigma}, \\ \rho_{xy} &= \rho_1^A. \end{aligned} \quad (2.131)$$

After some simplifications (see (2.125), (2.38) and (2.109)), the expectation in (2.130) becomes

$$E^*\left(\frac{k^*}{e^{rT}}(Z_T)^{\frac{1}{p-1}} I_{\left\{k^*(Z_T)^{\frac{1}{p-1}}<S_T^1\right\}} I_{\{S_T^1\ge S_T^2\}}\right) = M \cdot \Psi^2(\tilde{c}_1, \tilde{y}_1^c, \rho_1^A), \quad (2.132)$$

with M, $\tilde{c}_1$ and $\tilde{y}_1^c$ defined in (2.98), (2.101) and (2.102) respectively.

At this point, to complete the proof, all above calculations would be repeated for expectations involving S^2. But, as mentioned previously, since the results are symmetric, we omit these calculations here and simply state that by putting together (2.127), (2.132) and their respective counterparts for S^2, we obtain the formula for the fair premium for the risk-aversion case of efficient hedging.

Part II. Now let us derive the formula for the shortfall risk for the case of risk-aversion. Based on the discussion after (2.17), the form of success ratio (2.55), and (2.109), the shortfall risk can be expressed as

$$\begin{aligned} E\left(l((H-V_T^{\pi^*})^+)\right) &= E\left(l((1-\varphi^*)H)\right) \\ &= E\left(\left(I(a^*e^{-rT}Z_T)\wedge H\right)^p\right) \\ &= E\left((k^*)^p(Z_T)^{\frac{p}{p-1}} I_{\left\{k^*(Z_T)^{\frac{1}{p-1}}<H\right\}}\right) \\ &+ E\left(H^p I_{\left\{k^*(Z_T)^{\frac{1}{p-1}}\ge H\right\}}\right). \end{aligned} \quad (2.133)$$

Note that the first equality is established on p. 123 in Foellmer and Leukert (2000). This expression further reduces to

$$
\begin{aligned}
E\left(l((H - V_T^{\pi^*})^+)\right) &= E\left((k^*)^p (Z_T)^{\frac{p}{p-1}} I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^1\right\}} I_{\{S_T^1 \geq S_T^2\}}\right) \\
&+ E\left((S_T^1)^p I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} \geq S_T^1\right\}} I_{\{S_T^1 \geq S_T^2\}}\right) \\
&+ E\left((k^*)^p (Z_T)^{\frac{p}{p-1}} I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^2\right\}} I_{\{S_T^1 < S_T^2\}}\right) \\
&+ E\left((S_T^2)^p I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} \geq S_T^2\right\}} I_{\{S_T^1 < S_T^2\}}\right). \qquad (2.134)
\end{aligned}
$$

Now, consider the sets above. We already know how to simplify some of them under P^*, but now we work with the original probability measure P. As in the proof of Part I, we show the derivations for sets involving S^1, as results for sets with S^2 are symmetric. In (2.89), we showed how to reduce the set $\{S_T^1 \geq S_T^2\}$. Now we have to simplify $\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^1\right\}$. Using the same approach as when working with this set in the proof of Part I (see equations (2.112)-(2.123)), we can write

$$
\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^1\right\} = \left\{s_1^A < \bar{k}_1^A\right\}, \qquad (2.135)
$$

where

$$
\bar{k}_1^A = (p-1)\frac{\ln\left(\frac{S_0^1}{k^*}\right) + \left(\mu_1 - \frac{\sigma_1^2}{2}\right)T + \frac{\sigma_\phi^2}{2}\frac{T}{p-1}}{\sigma_1^E\sqrt{T}}, \qquad (2.136)
$$

and $s_1^A = \frac{\bar{W}_T^{1A}}{\sqrt{T}} \sim N(0,1)$ (w. r. to P) with a new Wiener process $\bar{W}^{1A} = (\bar{W}_t^{1A})_{t\in[0,T]}$ defined by

$$
\bar{W}_t^{1A} = \frac{\left(\frac{\phi_1}{p-1} - \sigma_1\right)W_t^1 + \frac{\phi_2}{p-1}W_t^2}{\tilde{\sigma}_1^A}, \qquad (2.137)
$$

for which $(\tilde{\sigma}_1^A)^2 = \frac{(\sigma_1^E)^2}{(p-1)^2} > 0$ as long as $\rho \neq \theta_2/\theta_1$ (see discussion following (2.113)).

We simplify $\left\{k^*(Z_T)^{\frac{1}{p-1}} \geq S_T^1\right\}$ similarly and obtain

$$\begin{aligned}\left\{k^*(Z_T)^{\frac{1}{p-1}} \geq S_T^1\right\} &= \left\{s_1^A \leq (p-1)\frac{\ln\left(\frac{k^*}{S_0^1}\right) - \left(\mu_1 - \frac{\sigma_1^2}{2}\right)T - \frac{\sigma_\phi^2}{2}\frac{T}{p-1}}{\sigma_1^E\sqrt{T}}\right\} \\ &= \{s_1^A \leq -\bar{k}_1^A\}. \end{aligned} \quad (2.138)$$

Now let us simplify the first expectation in (2.134):

$$\begin{aligned} &E\left((k^*)^p(Z_T)^{\frac{p}{p-1}} I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^1\right\}} I_{\{S_T^1 \geq S_T^2\}}\right) \\ &= (k^*)^p e^{-\frac{\sigma_\phi^2 T p}{2(p-1)}} E\left(e^{\left(\frac{\sigma_\phi p}{p-1}\bar{W}_T^p\right)} I_{\{s_1^A < \bar{k}_1^A\}} I_{\{s_1 \leq \bar{y}_1^Q\}}\right), \end{aligned} \quad (2.139)$$

with $\bar{k}_1^A$ and $\bar{y}_1^Q$ given in (2.136) and (2.72). Above, we used a Wiener process $\bar{W}^p = (\bar{W}_t^p)_{t\in[0,T]}$ (w. r. to P) satisfying

$$\bar{W}_t^p = \frac{\frac{\phi_1}{p-1}W_t^1 + \frac{\phi_2}{p-1}W_t^2}{\sigma_p}, \quad (2.140)$$

where $\sigma_p^2 = \frac{\sigma_\phi^2}{(p-1)^2} > 0$ (see (2.129)).

To evaluate the expectation in (2.139), we apply the two-asset lemma (2.82) with $z = -\frac{\sigma_\phi p}{p-1}\bar{W}_T^p \sim N\left(0, \frac{\sigma_\phi^2 p^2}{(p-1)^2}T\right)$, $x, y = s_1^A, s_1 \sim N(0,1)$ under P, and the corresponding correlations given in (2.131). As before, after appropriate simplifications, we obtain

$$E\left((k^*)^p(Z_T)^{\frac{p}{p-1}} I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} < S_T^1\right\}} I_{\{S_T^1 \geq S_T^2\}}\right) = N \cdot \Psi^2(\bar{c}_1, \bar{y}_c^1, \rho_1^A), \quad (2.141)$$

with $\bar{c}_1$, $\bar{y}_1^c$ defined in (2.103), (2.104).

Similarly we simplify the second expectation in (2.134): using (2.21), (2.138) and (2.89), we get

$$\begin{aligned} &E\left((S_T^1)^p I_{\left\{k^*(Z_T)^{\frac{1}{p-1}} \geq S_T^1\right\}} I_{\{S_T^1 \geq S_T^2\}}\right) \\ &= (S_0^1)^p e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)Tp} E\left(e^{\sigma_1 p W_T^1} I_{\{s_1^A < -\bar{k}_1^A\}} I_{\{s_1 \leq \bar{y}_1^Q\}}\right). \end{aligned} \quad (2.142)$$

Now we proceed in the usual manner, applying the two-asset lemma (2.82) with $z = -\sigma_1 p W_T^1 \sim N(0, \sigma_1^2 p^2 T)$, $x, y = s_1^A, s_1 \sim N(0,1)$ under P, and correlations given by

$$\begin{aligned}\rho_{xz} &= \frac{p-1}{\sigma_1^E}\left(\frac{\phi_1}{p-1}-\sigma_1+\frac{\phi_2\rho}{p-1}\right),\\ \rho_{yz} &= \frac{\sigma_1-\sigma_2\rho}{\sigma},\\ \rho_{xy} &= -\frac{\phi_2(\sigma_2-\sigma_1\rho)-(\phi_1-\sigma_1(p-1))(\sigma_1-\sigma_2\rho)}{\sigma_1^E\sigma}=-\rho_1^A. \end{aligned} \tag{2.143}$$

After appropriate simplifications, the expectation in (2.142) becomes

$$\begin{aligned} &E\left((S_T^1)^p I_{\left\{k^*(Z_T)^{\frac{1}{p-1}}\geq S_T^1\right\}} I_{\{S_T^1\geq S_T^2\}}\right)\\ &= (S_0^1)^p e^{\left(\mu_1-\frac{\sigma_1^2}{2}\right)Tp+\frac{\sigma_1^2}{2}Tp^2}\Psi^2(\bar{k}_1,\bar{y}_1^k,-\rho_1^A), \end{aligned} \tag{2.144}$$

with $\bar{k}_1$, $\bar{y}_1^k$ defined in (2.105), (2.106).

Repeating the above steps for the remaining two expectations in (2.134) that involve S^2 and putting together (2.141) with (2.144) enables us to write the final formula for the shortfall risk in Theorem 2.

2.4.3 Efficient hedging: risk-taking

Theorem 3

Suppose that the firm that sells an equity-linked life insurance contract with payoff (2.47) decides to use efficient hedging to minimize the shortfall risk. Further, suppose that the firm's risk preference is risk-taking, so that for the loss function $l(x)=x^p$, $0<p<1$.

Part I. The **fair premium** for the contract is

$$\begin{aligned} U_0 &= E^*\left(\varphi^*\frac{\max\{S_T^1,S_T^2\}}{e^{rT}}\right)\\ &= S_0^1\cdot\Psi^2(\tilde{x}_1^T,\tilde{y}_1,\rho_1^T)+S_0^2\cdot\Psi^2(\tilde{x}_2^T,\tilde{y}_2,\rho_2^T). \end{aligned} \tag{2.145}$$

Part II. The **shortfall risk** is given by

$$\begin{aligned} E\left(l((H-V_T)^+)\right) &= (S_0^1)^p e^{\left(\mu_1-\frac{\sigma_1^2}{2}\right)Tp+\frac{\sigma_1^2}{2}Tp^2}\cdot\left[\Psi^1(\bar{y}_1^T)-\Psi^2(\bar{x}_1^T,\bar{y}_1^T,\rho_1^T)\right]\\ &+ (S_0^2)^p e^{\left(\mu_2-\frac{\sigma_2^2}{2}\right)Tp+\frac{\sigma_2^2}{2}Tp^2}\cdot\left[\Psi^1(\bar{y}_2^T)-\Psi^2(\bar{x}_2^T,\bar{y}_2^T,\rho_2^T)\right]. \end{aligned} \tag{2.146}$$

Above, Ψ^1 and Ψ^2 denote one- and two-dimensional cumulative normal distributions given in (2.37) and (2.68) respectively, with

$$\rho_1^T = \frac{\phi_2(\sigma_2 - \sigma_1\rho) - (\phi_1 + \sigma_1(1-p))(\sigma_1 - \sigma_2\rho)}{\sigma_1^E\sigma},$$

$$\rho_2^T = \frac{\phi_1(\sigma_1 - \sigma_2\rho) - (\phi_2 + \sigma_2(1-p))(\sigma_2 - \sigma_1\rho)}{\sigma_2^E\sigma}. \tag{2.147}$$

The constants $\tilde{x}_i^T$, $\bar{x}_i^T$, and $\bar{y}_i^T$ are defined as

$$\tilde{x}_1^T = \frac{\left[\frac{\sigma_\phi^2-\sigma_1^2}{2} + \phi_1(\theta_1 - \sigma_1) + \phi_2(\theta_2 - \rho\sigma_1) + p\left(r + \frac{\sigma_1^2}{2}\right)\right]T - \ln\left((S_0^1)^{1-p}a^*\right)}{\sigma_1^E\sqrt{T}},$$

$$\tilde{x}_2^T = \frac{\left[\frac{\sigma_\phi^2-\sigma_2^2}{2} + \phi_2(\theta_2 - \sigma_2) + \phi_1(\theta_1 - \rho\sigma_2) + p\left(r + \frac{\sigma_2^2}{2}\right)\right]T - \ln\left((S_0^2)^{1-p}a^*\right)}{\sigma_2^E\sqrt{T}}, \tag{2.148}$$

$$\bar{x}_1^T = \frac{\left[r + \frac{\sigma_\phi^2}{2} - (1-p)\left(\mu_1 - \frac{\sigma_1^2}{2}\right) - p(\sigma_1^2(1-p) + \phi_1\sigma_1 + \phi_2\sigma_1\rho)\right]T - \ln\left((S_0^1)^{1-p}a^*\right)}{\sigma_1^E\sqrt{T}},$$

$$\bar{x}_2^T = \frac{\left[r + \frac{\sigma_\phi^2}{2} - (1-p)\left(\mu_2 - \frac{\sigma_2^2}{2}\right) - p(\sigma_2^2(1-p) + \phi_2\sigma_2 + \phi_1\sigma_2\rho)\right]T - \ln\left((S_0^2)^{1-p}a^*\right)}{\sigma_2^E\sqrt{T}}, \tag{2.149}$$

$$\bar{y}_1^T = \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \left[\mu_1 - \mu_2 + \frac{\sigma_2^2-\sigma_1^2}{2} + p(\sigma_1^2 - \sigma_1\sigma_2\rho)\right]T}{\sigma\sqrt{T}},$$

$$\bar{y}_2^T = \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) + \left[\mu_2 - \mu_1 + \frac{\sigma_1^2-\sigma_2^2}{2} + p(\sigma_2^2 - \sigma_1\sigma_2\rho)\right]T}{\sigma\sqrt{T}}. \tag{2.150}$$

Note that φ^* is given in (2.56), $\tilde{y}_i$ in (2.38), ϕ_i in (2.27), θ_i in (2.31), σ in (2.39), σ_ϕ in (2.28), and σ_i^E in (2.107). Also, as in the case of risk-aversion with $p > 1$, the formulas for the premium and the shortfall risk above hold as long as technical conditions $\rho \neq \frac{\theta_1}{\theta_2}$ and $\rho \neq \frac{\theta_2}{\theta_1}$ are satisfied (see the proof below), but these conditions are not restrictive, as explained in the proof of Theorem 2.

Proof.

Part I. To derive the formula for the fair premium for the case of risk-taking, recall that the success ratio φ^* (2.56) has the form

$$\varphi^* = I_{\{1 > a^* e^{-rT} H^{1-p} Z_T\}}, \quad 0 < p < 1.$$

Using this, we rewrite U_0 as follows:

$$\begin{aligned} U_0 &= E^*\left(\frac{\varphi^* H}{e^{rT}}\right) \\ &= E^*\left(\frac{H}{e^{rT}} I_{\{1>a^* e^{-rT} H^{1-p} Z_T\}}\right) \\ &= E^*\left(\frac{S_T^1}{e^{rT}} I_{\{1>a^* e^{-rT}(S_T^1)^{1-p} Z_T\}} I_{\{S_T^1 \geq S_T^2\}}\right) \\ &+ E^*\left(\frac{S_T^2}{e^{rT}} I_{\{1>a^* e^{-rT}(S_T^2)^{1-p} Z_T\}} I_{\{S_T^1 < S_T^2\}}\right). \end{aligned} \tag{2.151}$$

As in the proof of Theorem 2, we show calculations for S^1; the calculations for S^2 are symmetric. The set $\{1 > a^* e^{-rT}(S_T^1)^{1-p} Z_T\}$ simplifies to

$$\begin{aligned} &\{1 > a^* e^{-rT}(S_T^1)^{1-p} Z_T\} \\ &= \left\{\sigma_1^E \tilde{W}_T^{1T} < \left(r + \frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2 - (1-p)\left(r - \frac{\sigma_1^2}{2}\right)\right) T - \ln\left(a^*(S_0^1)^{1-p}\right)\right\} \\ &= \{s_1^T < \tilde{k}_1^T\}, \end{aligned} \tag{2.152}$$

where

$$\tilde{k}_1^T = \frac{\left(r + \frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2 - (1-p)\left(r - \frac{\sigma_1^2}{2}\right)\right) T - \ln\left(a^*(S_0^1)^{1-p}\right)}{\sigma_1^E \sqrt{T}}, \tag{2.153}$$

the random variable $s_1^T = \frac{\tilde{W}_T^{1T}}{\sqrt{T}} \sim N(0,1)$ under P^*, and the Wiener process $\tilde{W}^{1T} = (\tilde{W}_t^{1T})_{t\in[0,T]}$ is defined by

$$\tilde{W}_t^{1T} = \frac{((1-p)\sigma_1 + \phi_1)W_t^{1*} + \phi_2 W_t^{2*}}{\sigma_1^E}, \tag{2.154}$$

with σ_1^E given in (2.107).

Here we do not have to check again that $(\sigma_1^E)^2 > 0$, since in (2.122) we derived the relation $(\tilde{\sigma}_1^A)^2 = \frac{(\sigma_1^E)^2}{(p-1)^2} > 0$, and (2.115)-(2.119) show that $(\tilde{\sigma}_1^A)^2 > 0$ as long as $\rho \neq \frac{\theta_2}{\theta_1}$ ($\rho \neq \frac{\theta_1}{\theta_2}$) holds. Note that these technical requirements cause the conditions in Theorem 3 and are the same as the ones in Theorem 2 (please refer to the discussion of (2.120) and (2.121)).

Using (2.29), (2.75) and (2.152), the expectation in (2.151) becomes

$$E^*\left(\frac{S_T^1}{e^{rT}} I_{\{1>a^* e^{-rT}(S_T^1)^{1-p} Z_T\}} I_{\{S_T^1 \geq S_T^2\}}\right)$$

$$= S_0^1 e^{\frac{-\sigma_1^2}{2}T} E^* \left(e^{\sigma_1 W_T^{1*}} I_{\{s_1^T < \tilde{k}_1^T\}} I_{\left\{ s_1 \leq \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \frac{\sigma_2^2 - \sigma_1^2}{2} T}{\sigma \sqrt{T}} \right\}} \right). \tag{2.155}$$

Now we apply the two-asset lemma (2.82) to evaluate this expectation. We take $z = -\sigma_1 W_T^{1*} \sim N(0, \sigma_1^2 T)$, $x, y = s_1^T, s_1 \sim N(0,1)$ (w. r. to P^*). The corresponding correlations are

$$\begin{aligned}
\rho_{xz} &= -\frac{(1-p)\sigma_1 + \phi_1 + \rho\phi_2}{\sigma_1^E}, \\
\rho_{yz} &= \frac{\sigma_1 - \sigma_2 \rho}{\sigma}, \\
\rho_{xy} &= \rho_1^T,
\end{aligned} \tag{2.156}$$

with ρ_1^T defined in (2.147).

After appropriate simplifications, we obtain this expression for the expectation in (2.155):

$$E^*\left(\frac{S_T^1}{e^{rT}} I_{\{1>a^* e^{-rT}(S_T^1)^{1-p} Z_T\}} I_{\{S_T^1 \geq S_T^2\}}\right) = S_0^1 \Psi^2(\tilde{x}_1^T, \tilde{y}_1, \rho_1^T). \tag{2.157}$$

The constants $\tilde{x}_1^T$, $\tilde{y}_1$ are given in (2.148), (2.38). Repeating these calculations for the expectation in (2.151) containing S^2 and putting them together with the above result produces the finalized formula for the fair premium for the case of risk-taking.

Part II. To derive the formula for the shortfall risk for the risk-taking case, based on the discussion after (2.17) and arguments on p. 129 of Foellmer and Leukert (2000) that establish the first equality below, we can write

$$\begin{aligned}
E\left(l((H - V_T^{\pi^*})^+)\right) &= E\left(l(H) - \varphi^* l(H)\right) \\
&= E\left(H^p\right) - E\left(\varphi^* H^p\right) \\
&= E\left((S_T^1)^p I_{\{S_T^1 \geq S_T^2\}}\right) + E\left((S_T^2)^p I_{\{S_T^1 < S_T^2\}}\right) \\
&- E\left((S_T^1)^p I_{\{1>a^* e^{-rT}(S_T^1)^{1-p} Z_T\}} I_{\{S_T^1 \geq S_T^2\}}\right) \\
&- E\left((S_T^2)^p I_{\{1>a^* e^{-rT}(S_T^2)^{1-p} Z_T\}} I_{\{S_T^1 < S_T^2\}}\right).
\end{aligned} \tag{2.158}$$

As before, we show calculations for items involving S^1.

Consider the expectation $E\left((S_T^1)^p I_{\{S_T^1 \geq S_T^2\}}\right)$: using (2.89) and (2.72), it can be written as

$$E\left((S_T^1)^p I_{\{S_T^1 \geq S_T^2\}}\right) = (S_0^1)^p e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)Tp} E\left(e^{\sigma_1 p W_T^1} I_{\{s_1 \leq \bar{y}_1^Q\}}\right). \tag{2.159}$$

To evaluate this expression, we use the one-asset lemma (2.44) with $\eta = -\sigma_1 p W_T^1 \sim N(0, \sigma_1^2 p^2 T)$, $\zeta = s_1 \sim N(0,1)$ under P, and

$$\mathrm{cov}(\eta, \zeta) = \frac{(\sigma_1^2 - \sigma_1\sigma_2\rho)Tp}{\sigma\sqrt{T}}. \tag{2.160}$$

Following these considerations, the expectation in (2.159) becomes

$$E\left((S_T^1)^p I_{\{S_T^1 \geq S_T^2\}}\right) = (S_0^1)^p e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)Tp + \frac{\sigma_1^2}{2}Tp^2} \Psi^1(\bar{y}_1^T), \tag{2.161}$$

where Ψ^1 denotes one-dimensional cumulative normal distribution (2.37) and $\bar{y}_1^T$ is defined in (2.150).

Now consider the set $\{1 > a^* e^{-rT}(S_T^1)^{1-p} Z_T\}$: we simplify it under P similarly to what was done in (2.152) under P^*:

$$\begin{aligned}
&\{1 > a^* e^{-rT}(S_T^1)^{1-p} Z_T\} \\
&= \left\{\sigma_1^E \bar{W}_T^{1T} < \left(r + \frac{\sigma_\phi^2}{2} - (1-p)\left(\mu_1 - \frac{\sigma_1^2}{2}\right)\right) T - \ln\left(a^*(S_0^1)^{1-p}\right)\right\} \\
&= \{s_1^T < \bar{k}_1^T\},
\end{aligned} \tag{2.162}$$

where

$$\bar{k}_1^T = \frac{\left(r + \frac{\sigma_\phi^2}{2} - (1-p)\left(\mu_1 - \frac{\sigma_1^2}{2}\right)\right) T - \ln\left(a^*(S_0^1)^{1-p}\right)}{\sigma_1^E \sqrt{T}}, \tag{2.163}$$

the random variable $s_1^T = \frac{\bar{W}_T^{1T}}{\sqrt{T}} \sim N(0,1)$ under P, and the Wiener process $\bar{W}^{1T} = (\bar{W}_t^{1T})_{t \in [0,T]}$ is defined by

$$\bar{W}_t^{1T} = \frac{((1-p)\sigma_1 + \phi_1)W_t^1 + \phi_2 W_t^2}{\sigma_1^E}, \tag{2.164}$$

with σ_1^E given in (2.107).

Let us now return to the other expectation involving S^1 in (2.158):

$$E\left((S_T^1)^p I_{\{1>a^* e^{-rT}(S_T^1)^{1-p} Z_T\}} I_{\{S_T^1 \geq S_T^2\}}\right)$$
$$= (S_0^1)^p e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)Tp} E\left(e^{\sigma_1 p W_T^1} I_{\{s_1^T < \bar{k}_1^T\}} I_{\{s_1 \leq \bar{y}_1^Q\}}\right), \tag{2.165}$$

and, as many times before, we apply the two-asset lemma (2.82) with $z = -\sigma_1 p W_T^1 \sim N(0, \sigma_1^2 p^2 T)$, $x, y = s_1^T, s_1 \sim N(0,1)$ (w. r. to P). The corresponding correlations are the same as those defined in (2.156).

Finally, after some simplifications, we can write

$$E\left((S_T^1)^p I_{\{1>a^* e^{-rT}(S_T^1)^{1-p} Z_T\}} I_{\{S_T^1 \geq S_T^2\}}\right)$$
$$= (S_0^1)^p e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)Tp + \frac{\sigma_1^2}{2}Tp^2} \Psi^2(\bar{x}_1^T, \bar{y}_1^T, \rho_1^T). \tag{2.166}$$

The constants $\bar{x}_1^T$, $\bar{y}_1^T$ are given in (2.149), (2.150). Performing symmetric calculations for expectations with S^2 and putting together (2.161) with (2.166) allows us to write the final result for the shortfall risk for the case of risk-taking.

2.4.4 Efficient hedging: risk-indifference

Theorem 4
Suppose that the firm that sells an equity-linked life insurance contract with payoff (2.47) decides to use efficient hedging to minimize the shortfall risk. Further, suppose that the firm's risk preference is risk-indifference, so that for the loss function $l(x) = x^p$, $p = 1$.
Part I. The **fair premium** for the contract is

$$\begin{aligned} U_0 &= E^*\left(\varphi^* \frac{\max\{S_T^1, S_T^2\}}{e^{rT}}\right) \\ &= S_0^1 \cdot \Psi^2(\tilde{x}_1^I, \tilde{y}_1, \rho_1^I) + S_0^2 \cdot \Psi^2(\tilde{x}_2^I, \tilde{y}_2, \rho_2^I). \end{aligned} \tag{2.167}$$

Part II. The **shortfall risk** is given by

$$\begin{aligned} E\left(l((H - V_T)^+)\right) &= S_0^1 e^{\mu_1 T} \cdot \left[\Psi^1(\bar{y}_1^I) - \Psi^2(\bar{x}_1^I, \bar{y}_1^I, \rho_1^I)\right] \\ &+ S_0^2 e^{\mu_2 T} \cdot \left[\Psi^1(\bar{y}_2^I) - \Psi^2(\bar{x}_2^I, \bar{y}_2^I, \rho_2^I)\right]. \end{aligned} \tag{2.168}$$

Above, Ψ^1 and Ψ^2 denote one- and two-dimensional cumulative normal distributions given in (2.37) and (2.68) respectively, with

$$\begin{aligned} \rho_1^I &= \frac{\phi_2(\sigma_2 - \sigma_1\rho) - \phi_1(\sigma_1 - \sigma_2\rho)}{\sigma_\phi \sigma}, \\ \rho_2^I &= \frac{\phi_1(\sigma_1 - \sigma_2\rho) - \phi_2(\sigma_2 - \sigma_1\rho)}{\sigma_\phi \sigma}. \end{aligned} \tag{2.169}$$

The constants $\tilde{x}_i^I$, $\bar{x}_i^I$, and $\bar{y}_i^I$ are defined as

$$\begin{aligned}\tilde{x}_1^I &= \frac{\left[r + \frac{\sigma_\phi^2}{2} + \phi_1(\theta_1 - \sigma_1) + \phi_2(\theta_2 - \rho\sigma_1)\right] T - \ln(a^*)}{\sigma_\phi \sqrt{T}}, \\ \tilde{x}_2^I &= \frac{\left[r + \frac{\sigma_\phi^2}{2} + \phi_2(\theta_2 - \sigma_2) + \phi_1(\theta_1 - \rho\sigma_2)\right] T - \ln(a^*)}{\sigma_\phi \sqrt{T}}, \end{aligned} \qquad (2.170)$$

$$\begin{aligned}\bar{x}_1^I &= \frac{\left[r + \frac{\sigma_\phi^2}{2} - \phi_1\sigma_1 - \phi_2\sigma_1\rho\right] T - \ln(a^*)}{\sigma_\phi \sqrt{T}}, \\ \bar{x}_2^I &= \frac{\left[r + \frac{\sigma_\phi^2}{2} - \phi_2\sigma_2 - \phi_1\sigma_2\rho\right] T - \ln(a^*)}{\sigma_\phi \sqrt{T}}, \end{aligned} \qquad (2.171)$$

$$\begin{aligned}\bar{y}_1^I &= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \left[\mu_1 - \mu_2 + \frac{\sigma^2}{2}\right] T}{\sigma\sqrt{T}}, \\ \bar{y}_2^I &= \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) + \left[\mu_2 - \mu_1 + \frac{\sigma^2}{2}\right] T}{\sigma\sqrt{T}}. \end{aligned} \qquad (2.172)$$

Note that φ^* is given in (2.57), $\tilde{y}_i$ in (2.38), ϕ_i in (2.27), θ_i in (2.31), σ in (2.39), and σ_ϕ in (2.28).

Proof.

Part I. Let us calculate the fair premium for the case of risk-indifference. Recall that the success ratio φ^* (2.57) has the form

$$\varphi^* = I_{\{1 > Z_T a^* e^{-rT}\}}, \quad p = 1.$$

With this, U_0 can be written as

$$\begin{aligned} U_0 &= E^*\left(\frac{\varphi^* H}{e^{rT}}\right) \\ &= E^*\left(\frac{H}{e^{rT}} I_{\{1 > Z_T a^* e^{-rT}\}}\right) \\ &= E^*\left(\frac{S_T^1}{e^{rT}} I_{\{1 > Z_T a^* e^{-rT}\}} I_{\{S_T^1 \geq S_T^2\}}\right) \\ &+ E^*\left(\frac{S_T^2}{e^{rT}} I_{\{1 > Z_T a^* e^{-rT}\}} I_{\{S_T^1 < S_T^2\}}\right). \end{aligned} \qquad (2.173)$$

Note that now we only need to simplify the set $\{1 > Z_T a^* e^{-rT}\}$ and then apply the two-asset lemma to the expectation with S^1 above. Using the expression for density w. r. to P^* (2.32), we have

$$\begin{aligned} \{1 > Z_T a^* e^{-rT}\} &= \left\{ 1 > e^{\phi_1 W_T^{1*} + \phi_2 W_T^{2*} - \left(\frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right) T} a^* e^{-rT} \right\} \\ &= \{s^I < \tilde{k}^I\}, \end{aligned} \tag{2.174}$$

where

$$\tilde{k}^I = \frac{\left(r + \frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right) T - \ln(a^*)}{\sigma_\phi \sqrt{T}}, \tag{2.175}$$

the random variable $s^I = \frac{\tilde{W}_T^I}{\sqrt{T}} \sim N(0,1)$ for the Wiener process $\tilde{W}^I = (\tilde{W}_t^I)_{t\in[0,T]}$ (w. r. to P^*)

$$\tilde{W}_t^I = \frac{\phi_1 W_t^{1*} + \phi_2 W_t^{2*}}{\sigma_\phi}, \tag{2.176}$$

with σ_ϕ given in (2.28).

Now, using (2.75), we can write

$$\begin{aligned} & E^*\left(\frac{S_T^1}{e^{rT}} I_{\{1 > Z_T a^* e^{-rT}\}} I_{\{S_T^1 \geq S_T^2\}}\right) \\ &= S_0^1 e^{-\frac{\sigma_1^2}{2}T} E\left(e^{\sigma_1 W_T^{1*}} I_{\{s^I < \tilde{k}^I\}} I_{\left\{ s_1 \leq \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \frac{\sigma_2^2 - \sigma_1^2}{2} T}{\sigma\sqrt{T}} \right\}} \right). \end{aligned} \tag{2.177}$$

To evaluate this expectation, we use the two-asset lemma (2.82) with $z = -\sigma_1 W_T^{1*} \sim N(0, \sigma_1^2 T)$, $x, y = s^I, s_1 \sim N(0,1)$ (w. r. to P^*), and correlations

$$\begin{aligned} \rho_{xz} &= -\frac{\phi_1 + \phi_2\rho}{\sigma_\phi}, \\ \rho_{yz} &= \frac{\sigma_1 - \sigma_2\rho}{\sigma}, \\ \rho_{xy} &= \rho_1^I, \end{aligned} \tag{2.178}$$

with ρ_1^I defined in (2.169). After simplifications, the expectation in (2.177) becomes

$$E^*\left(\frac{S_T^1}{e^{rT}} I_{\{1>Z_T a^* e^{-rT}\}} I_{\{S_T^1 \geq S_T^2\}}\right) = S_0^1 \Psi^2(\tilde{x}_1^I, \tilde{y}_1, \rho_1^I). \tag{2.179}$$

Same calculations for the expectation with S^2, together with the ones above, lead to the final formula for the fair premium for the risk-indifference case.

Part II. Now we derive the shortfall risk formula for risk-indifference. Using the expression for φ^* (2.57) and the fact that $l(x) = x$ here, we can write

$$\begin{aligned} E\left(l((H - V_T^{\pi^*})^+)\right) &= E\left((1-\varphi^*)H\right) \\ &= E(H) - E(\varphi^* H) \\ &= E\left(S_T^1 I_{\{S_T^1 \geq S_T^2\}}\right) + E\left(S_T^2 I_{\{S_T^1 < S_T^2\}}\right) \\ &- E\left(S_T^1 I_{\{1>Z_T a^* e^{-rT}\}} I_{\{S_T^1 \geq S_T^2\}}\right) \\ &- E\left(S_T^2 I_{\{1>Z_T a^* e^{-rT}\}} I_{\{S_T^1 < S_T^2\}}\right). \end{aligned} \tag{2.180}$$

For $E\left(S_T^1 I_{\{S_T^1 \geq S_T^2\}}\right)$, we have

$$E\left(S_T^1 I_{\{S_T^1 \geq S_T^2\}}\right) = S_0^1 e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)T} E\left(e^{\sigma_1 W_T^1} I_{\{s_1 \leq \bar{y}_1^Q\}}\right) \tag{2.181}$$

based on (2.89) and (2.72). To this expression we apply the one-asset lemma (2.44) with $\eta = -\sigma_1 W_T^1 \sim N(0,1)$, $\zeta = s_1 \sim N(0,1)$, and

$$\operatorname{cov}(\eta, \zeta) = \frac{(\sigma_1^2 - \sigma_1\sigma_2\rho)T}{\sigma\sqrt{T}}. \tag{2.182}$$

We obtain

$$E\left(S_T^1 I_{\{S_T^1 \geq S_T^2\}}\right) = S_0^1 e^{\mu_1 T} \Psi^1(\bar{y}_1^I), \tag{2.183}$$

with Ψ^1 given in (2.37) and $\bar{y}_1^I$ in (2.172).

Now consider $\{1 > Z_T a^* e^{-rT}\}$: under P, this set simplifies to

$$\begin{aligned} \{1 > Z_T a^* e^{-rT}\} &= \left\{1 > e^{\phi_1 W_T^1 + \phi_2 W_T^2 - \frac{\sigma_\phi^2}{2}T} a^* e^{-rT}\right\} \\ &= \{s^I < \bar{k}^I\}, \end{aligned} \tag{2.184}$$

where

$$\bar{k}^I = \frac{\left(r + \frac{\sigma_\phi^2}{2}\right) T - \ln(a^*)}{\sigma_\phi \sqrt{T}}, \tag{2.185}$$

the random variable $s^I = \frac{\bar{W}_T^I}{\sqrt{T}} \sim N(0,1)$ for the Wiener process $\bar{W}^I = (\bar{W}_t^I)_{t\in[0,T]}$ (w. r. to P), defined as

$$\bar{W}_t^I = \frac{\phi_1 W_t^1 + \phi_2 W_t^2}{\sigma_\phi}, \tag{2.186}$$

with σ_ϕ given in (2.28).

Based on the considerations above, we can write

$$E\left(S_T^1 I_{\{1 > Z_T a^* e^{-rT}\}} I_{\{S_T^1 \geq S_T^2\}}\right) = S_0^1 e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)T} E\left(e^{\sigma_1 W_T^1} I_{\{s^I < \bar{k}^I\}} I_{\{s_1 \leq \bar{y}_1^Q\}}\right) \tag{2.187}$$

and apply the two-asset lemma (2.82) with $z = -\sigma_1 W_T^1 \sim N(0, \sigma_1^2 T)$, $x, y = s^I, s_1 \sim N(0,1)$ (w. r. to P), and the corresponding correlations given in (2.178). After some simplifications, the expectation in (2.187) takes form

$$E\left(S_T^1 I_{\{1 > Z_T a^* e^{-rT}\}} I_{\{S_T^1 \geq S_T^2\}}\right) = S_0^1 e^{\mu_1 T} \Psi^2(\bar{x}_1^I, \bar{y}_1^I, \rho_1^I). \tag{2.188}$$

The constants $\bar{x}_1^I$, $\bar{y}_1^I$ are given in (2.171), (2.172).

Putting together (2.183) with (2.188) and performing similar calculations for expectations involving S^2 enables us to derive the final formula for the shortfall risk for the case of risk-indifference.

2.5 Payoffs with n risky assets

So far, we have focused on discussing the payoff where the client is entitled to receiving the larger of the values of two risky assets at expiration of the contract. A natural question arises: how do we price contracts and manage financial and insurance risks for payoffs involving the larger of n risky assets? Policies with such payoffs are sometimes referred to as "switching-of-funds" contracts. Formally, the payoff is given by

$$\begin{aligned} \bar{H}^n &= H^n \cdot I_{\{\tau(x) > T\}}, \\ \text{where} \quad H^n &= \max\{S_T^1, S_T^2, \ldots, S_T^n\}. \end{aligned} \tag{2.189}$$

As before, $\{\tau(x) > T\}$ refers to the conditioning of the payoff on the policyholder's survival to maturity of the contract.

European-type contracts with payoffs involving several risky assets have been studied previously. For instance, Stulz (1982) derives analytical formulas for prices of European call and put options on the minimum or maximum of two risky assets in the classical Black-Scholes-Merton-type setting. Johnson (1987) generalizes this result to payoffs with n risky assets, using a change of numeraire technique, the characteristics of call/put options, and the lognormal properties of the underlying assets. Boyle and Tse (1990) present a fast and accurate approximation algorithm to value options on the maximum or minimum of several assets. Boyle and Lin (1997) obtain upper bounds for prices of call options on several assets, without making any assumptions about the probability distribution of the underlying assets; thus the bounds depend only on the returns of the assets and their covariances. Laamanen (2000) further extends the result of Johnson (1987) to the payoffs on m best of n risky assets by utilizing a recursive approach in pricing calculations. We derive a more general probabilistic-type result that allows us to value not only payoffs with several assets, but also to calculate directly expectations resulting from such payoffs being contingent upon other events, for example, when using quantile or efficient hedging to price equity-linked life insurance contracts.

If we look carefully through the derivation of pricing formulas in the proofs of Theorems 1, 2, 3 and 4 (section 2.4), we notice that when pricing payoffs with two risky assets, we must evaluate expectations of type

$$E^*\left(e^{-z} I_{\{x<X\}}\right) \quad \text{or} \quad E^*\left(e^{-z} I_{\{x<X\}} I_{\{y<Y\}}\right) \tag{2.190}$$

that contain at most two indicators and three normally distributed correlated random variables. To calculate these expectations, we have used the one-asset and the two-asset lemmas ((2.44) and (2.82) respectively). The same idea applies to the derivation of pricing formulas for payoffs with n assets: we would have to deal with expectations involving n indicators and $n+1$ random variables. For example, for quantile hedging (see (2.66) and (2.74)), we would calculate the fair premium U_0^n as

$$\begin{aligned} U_0^n &= E^*\left(\frac{\max\{S_T^1, S_T^2, \ldots, S_T^n\}}{e^{rT}} I_{A^*}\right) \qquad (2.191) \\ &= E^*\left(\frac{S_T^1}{e^{rT}} I_{\left\{\frac{1}{Z_T} > \frac{a^* S_T^1}{e^{rT}}\right\}} I_{\{S_T^1 \geq S_T^2\}} I_{\{S_T^1 \geq S_T^3\}} \cdots I_{\{S_T^1 \geq S_T^n\}}\right) \\ &+ E^*\left(\frac{S_T^2}{e^{rT}} I_{\left\{\frac{1}{Z_T} > \frac{a^* S_T^2}{e^{rT}}\right\}} I_{\{S_T^2 > S_T^1\}} I_{\{S_T^2 \geq S_T^3\}} \cdots I_{\{S_T^2 \geq S_T^n\}}\right) \\ &+ \cdots + E^*\left(\frac{S_T^n}{e^{rT}} I_{\left\{\frac{1}{Z_T} > \frac{a^* S_T^n}{e^{rT}}\right\}} I_{\{S_T^n > S_T^1\}} I_{\{S_T^n > S_T^2\}} \cdots I_{\{S_T^n > S_T^{n-1}\}}\right), \end{aligned}$$

which, after appropriate simplifications, could be represented similarly to the expectations in (2.190).

As mentioned in the proof of Theorem 2 (section 2.4.2), we have derived a result which allows us to calculate expectations with n indicators. This result, presented below as a theorem, is the key for obtaining explicit formulas for fair premiums for both quantile and efficient hedging (for each risk preference case), as well as explicit expressions for the probability of successful hedging (quantile hedging) or shortfall risk (efficient hedging).

Multi-asset theorem
Let $x_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, \ldots, n$ and $z \sim N(\mu_z, \sigma_z^2)$ be $n+1$ normally distributed random variables with variance-covariance matrix $\mathbf{R}_{n+1}$ given by

$$\mathbf{R}_{n+1} = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_1\sigma_z\rho_{1z} \\ \vdots & \ddots & \vdots \\ \sigma_1\sigma_z\rho_{1z} & \cdots & \sigma_z^2 \end{bmatrix}. \quad (2.192)$$

Then for some given constants X_i,

$$\begin{aligned} E\left(e^{-z} I_{\{x_1 < X_1\}} \cdots I_{\{x_n < X_n\}}\right) &= e^{-\left(\mu_z - \frac{\sigma_z^2}{2}\right)} \cdot \Psi^n(\hat{X}_1, \ldots, \hat{X}_n), \quad (2.193) \\ \hat{X}_i &= \frac{X_i - \mu_i}{\sigma_i} + \sigma_z \rho_{iz}. \end{aligned}$$

In the formulation of the theorem, we refer to x_{n+1} as z, to distinguish the fact that the $n+1$ random variable is in the exponent. Also, Ψ^n, $n > 1$, denotes the n-dimensional cumulative normal distribution (see below) of n random variables with mean 0, variance 1, and correlation matrix

$$\tilde{\mathbf{R}}_n = \begin{bmatrix} 1 & \cdots & \rho_{1n} \\ \vdots & \ddots & \vdots \\ \rho_{1n} & \cdots & 1 \end{bmatrix} \quad (2.194)$$

with the inverse $\tilde{\mathbf{R}}_n^{-1} = \tilde{\mathbf{A}}_n$.

The general formula for the k-dimensional cumulative normal distribution of (k) random variables $y_i \sim N(\mu_i, \sigma_i^2)$ with variance-covariance matrix $\mathbf{D}_k$ is given by

$$\Psi^k_{\text{general}}(c_1, \ldots, c_k) = \quad (2.195)$$
$$\frac{1}{(2\pi)^{k/2}|\mathbf{D}_k|^{1/2}} \int_{-\infty}^{c_1} \cdots \int_{-\infty}^{c_k} e^{-\frac{1}{2}\sum_{i=1}^k \sum_{j=1}^k b_{ij}(y_i-\mu_i)(y_j-\mu_j)} dy_1 \cdots dy_k,$$
$$\mathbf{D}_k = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_1\sigma_k\rho_{1k} \\ \vdots & \ddots & \vdots \\ \sigma_1\sigma_k\rho_{1k} & \cdots & \sigma_k^2 \end{bmatrix}, \quad \mathbf{B}_k = \|b_{ij}\|_k = \mathbf{D}_k^{-1}.$$

Please note that for the remainder of the dissertation (including Appendix 2), Ψ^k,

$k > 1$, without the subscript 'general' will refer to the k-dimensional cumulative normal distribution of correlated random variables with mean 0 and variance 1.

Proof.

Due to the highly technical (and rather messy) derivation details, the proof is provided not in the main text of the thesis, but in Appendix 2.

We believe that this theorem will prove very useful in a number of applications beyond this dissertation: variety of processes in mathematical finance, economics and insurance are modelled as linear or exponential functions of Wiener processes. Frequently, expectations involving several such processes need to be evaluated, and it is likely that some of these expectations can be represented in the form which is suitable for applying the multi-asset theorem to calculate the necessary expressions directly. This would provide higher accuracy and better efficiency in terms of computing power than the use of numerical solutions (such as approximations or simulations).

2.6 Numerical illustration: applying quantile and efficient hedging

In this section, we demonstrate how an investor can use quantile or efficient hedging to deal with insufficient initial capital. We do not deal with insurance risk element yet; this aspect of equity-linked life insurance contracts will be illustrated in section 3.6. For now, we focus on showing what risk management strategies are available to the hedger utilizing quantile or efficient hedging techniques.

2.6.1 Data and parameters

To calculate parameters for our model ($\mu_i, \sigma_i, \rho,\ i = 1, 2$), we used daily stock prices of Russell-2000 (RUT-I) and Dow Jones Industrial Average (DJIA) indices from August 1, 1997 to July 31, 2003. The data was taken from Yahoo! finance (www.finance.yahoo.com). The first index, RUT-I, reflects the performance of 2000 smaller firms in the US, while the second, DJIA, represents 30 large and prestigious US companies. The parameters were calculated using a standard approach in finance (see, for example, Hull (2005)):

$$\ln\left(\frac{S_{t+\Delta t}}{S_t}\right) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t} z, \tag{2.196}$$

where $z \sim N(0,1)$. In our case $\Delta t = \frac{1}{252}$, since we take the business year to have 252 days. We estimate the mean and the standard deviation of $\ln\left(\frac{S_{t+\Delta t}}{S_t}\right)$ in a straightforward manner, and then multiply them by 252 and $\sqrt{252}$ respectively to obtain annualized values. Note that we add to the annualized mean half of the estimated (annualized) variance to obtain μ.

The estimated parameters are given below:

$$\mu_1 = .0482,\ \mu_2 = .0419,\ \sigma_1 = .2234,\ \sigma_2 = .2093,\ \rho = .71.$$

We took the July 31, 2003, values of the indices for the initial prices of the risky assets, correcting for the large difference between the two, so that

$$S_0^1 = (9233.8/476.02) \cdot 476.02, \quad S_0^2 = 9233.8.$$

For the estimate of the interest rate, we looked at 5-year nominal yields on US treasury securities (www.federalreserve.gov) and took $r = .04$, or 4 percent, which is close to the average of the yields in early 2000s.

We consider a 5-year contract with payoff $\max\{S_5^1, S_5^2\}$ and the situation where the seller of the contract is not able to collect (or is not willing to provide) the amount necessary to invest into the perfect hedging strategy. We calculate the perfect hedging price (2.36) and the price of the optimal hedging strategy as prescribed by quantile or efficient hedging (see (2.66), (2.167), (2.145) and (2.95)). We analyze two risk management approaches. First, we find the values for the maximal probability of successful hedging (2.67) for quantile hedging or minimized shortfall risk (for the different risk preferences when using efficient hedging, (2.168), (2.146), (2.96)) based on the available level of initial capital, given as a percentage of the perfect hedging price. Second, we look at what levels of initial capital are required to allow the investor to hedge the payoff with the desired probability of successful hedging (or shortfall risk). We take $p = 1$, $p = .8$ and $p = 1.2$ for the risk-indifference, risk-taking and risk-aversion cases of the investor's risk preference.

2.6.2 Quantile hedging results

The perfect hedging price for the contract is USD $10,587.54$.

Table 1: Probabilities of successful hedging (in percent) based on selected levels of initial hedging capital (given as percentage of the perfect hedging price)

initial capital available	probability of successful hedging
90	95.55
95	98.05
99	99.70

Let us look at the values in Table 1: as expected, we see that by providing larger initial capital for hedging, the investor can expect to hedge with greater probability of success. Or, if the investor chooses to set the acceptable level of the probability of successful hedging, then he/she will need to allot more money for the initial investment into the optimal hedging strategy in order to attain higher probabilities of success, as shown in Table 2. These results agree with our intuition regarding the

Table 2: Initial capital (amount and percentage of the perfect hedging price) required to hedge with given probabilities of success

probability of successful hedging	initial capital needed
90	8,536.23 (80.62)
95	9,422.78 (89.00)
99	10,288.32 (97.17)

relationship between the probability of successful hedging and the capital invested into a hedge. In section 2.7, we explore this relationship further and show that when the initial hedging capital approaches the perfect hedging price, the probability of success goes to 1 (or vice versa). On the other hand, taking smaller and smaller initial capital ($\rightarrow 0$) is equivalent to hedging with increasingly lower probability of success ($\rightarrow 0$).

2.6.3 Efficient hedging results

The perfect hedging price for the contract is still USD $10,587.54$.

Table 3: Expected shortfall (amount and percentage of the perfect hedging price) based on selected levels of initial hedging capital (given as percentage of the perfect hedging price) for risk-indifference, risk-taking and risk-aversion

	initial capital available	expected shortfall
$p = 1.0$	90	1,101.54 (10.40)
	95	533.87 (5.04)
	99	100.51 (0.95)
$p = 0.8$	90	160.06 (1.51)
	95	77.19 (0.07)
	99	14.10 (0.01)
$p = 1.2$	90	5,240.32 (49.50)
	95	2,290.30 (21.63)
	99	326.77 (3.09)

First, we observe some expected patterns across all risk preference cases. The higher the initial capital provided by the investor for the optimal hedging strategy, the smaller is the expected shortfall (Table 3). Equivalently, the lower the shortfall risk acceptable to the investor, the greater is the amount of initial capital required for the optimal hedge (Table 4). Similar to the probability/capital idea in quantile hedging, such results agree with our intuition about the shortfall/capital relationship, which will be examined in more detail in section 2.7.

Next, let us compare the values between the three risk preference cases. Table 3 shows that, for the same given level of initial capital, the amount of expected shortfall

Table 4: Initial capital (amount and percentage of the perfect hedging price) required to maintain selected shortfall risk level (given as percentage of the perfect hedging price) for risk-indifference, risk-taking and risk-aversion

	acceptable shortfall risk	initial capital needed
$p = 1.0$	10	9,568.06 (90.37)
	5	10,062.45 (95.04)
	1	10,476.20 (98.95)
$p = 0.8$	10	4,478.03 (42.30)
	5	7,346.77 (69.40)
	1	9,866.17 (93.19)
$p = 1.2$	10	10,309.31 (97.37)
	5	10,431.13 (98.52)
	1	10,546.32 (99.61)

will be perceived as less by a risk-taker and more by a risk-averse investor than the shortfall expected by a risk-indifferent investor. We take the risk-indifference case as the benchmark since this expected shortfall amount is the actual expected dollar loss. For example, suppose that three investors can provide the initial capital of only 90 percent of the amount required for the perfect hedge. The actual minimized expected loss in this situation is about 1,100 dollars; this is the amount a risk-indifferent investor would see as being lost due to insufficient (for a perfect hedge) initial capital. A risk-taker, providing the same dollar amount for the optimal hedge, would value the expected loss at only 160 dollars; clearly, he/she cares less about losing money than the risk-indifferent person. A risk-averse investor, on the other hand, would 'feel the pain' much more sharply: to him/her, the perceived loss from insufficient initial capital is valued at over 5,000 dollars (Table 3).

Similar pattern is observed in Table 4. To keep the level of acceptable shortfall risk at, say, 5 percent, the risk-indifferent hedger would invest about 10,000 dollars into the optimal hedge. For the same shortfall risk, the risk-taker would give only 7,300, while the risk-averse investor would pay about 400 dollars more than is required by the benchmark case of risk-indifference. Again, this is due to the fact that the risk-taker feels losses less, while the risk-averse investor more than the hedger who values losses based on actual dollar amounts.

Now, let us think about the shortfall/capital relationship a little more. When the level of initial capital available for hedging approaches the perfect hedging price, the expected shortfall approaches zero. Equivalently, the smaller the level of shortfall risk acceptable to the hedger, the larger will be the capital required to invest into the optimal hedging strategy. These intuitive ideas are somewhat illustrated in the two tables above and will be proved in Theorem 5 (section 2.7). But what happens to the expected shortfall in each of the risk preference cases as the capital available for hedging becomes increasingly smaller? Intuitively, it seems correct to think that the expected shortfall would approach the expected (under the subjective probability measure P) full amount of the payoff in the case of risk-indifference: a hedger who values his/her losses 1-for-1 with actual dollar amounts cannot expect to lose more or less than the payoff H which has to be paid to the buyer of the contract at maturity. But then a risk-taking hedger, who cares less about losses, should value the maximal expected shortfall *less* (and a risk-averse investor *more*) than the risk-indifferent person. This is an interesting topic, and we will examine and illustrate it in more detail in the next section.

2.7 How much can you lose?

Following the discussion about maximal expected losses, let us see what happens to the probability of successful hedging (in quantile hedging) or shortfall risk (for efficient hedging) as the initial amount available for hedging
a. approaches the perfect hedging price, or
b. approaches zero.
We already gave the intuition behind the relations capital/probability of success and capital/shortfall. The aim of this section is to justify and quantify the idea that as initial capital approaches the perfect hedging price, probability of success goes to 1 and shortfall risk to 0. And, as initial capital goes to 0, so does the probability of success, while the shortfall risk increases to some boundary which depends on the risk preference of the hedger.

Theorem 5
Part 1: quantile hedging
a. Whenever the initial capital of the optimal hedging strategy (2.66) approaches the perfect hedging price (2.36), the probability of successful hedging (2.67) approaches 1.
b. The probability of successful hedging goes to 0 whenever the price of the optimal hedging strategy goes to 0.
Proof.
Note that all formulas and definitions of constants are given in Theorem 1. Comparing the formulas for the fair premium for quantile and perfect hedging, we see that the quantile price approaches the perfect hedging price whenever

$$\Psi^2(\tilde{x}_i^Q, \tilde{y}_i, \rho_i^Q) \to \Psi^1(\tilde{y}_i) \quad \Leftrightarrow \quad \tilde{x}_i^Q \to \infty \quad \Leftrightarrow \quad a^* \to 0. \tag{2.197}$$

But whenever $a^* \to 0$,

$$\bar{x}_i^Q \to \infty \quad \Leftrightarrow \quad \Psi^2(\bar{x}_i^Q, \bar{y}_i^Q, \rho_1^Q) \to \Psi^1(\bar{y}_i^Q), \tag{2.198}$$

and since $\bar{y}_1^Q = -\bar{y}_2^Q$, $\Psi^1(\bar{y}_1^Q) + \Psi^1(\bar{y}_2^Q) = 1$. Thus part 1a is proved.
To prove part 1b, we note that the quantile price $\to 0$ as $\tilde{x}_i^Q \to -\infty$, or $a^* \to \infty$.

But then $\bar{x}_i^Q$ also $\to -\infty$, so that

$$\Psi^2(\bar{x}_i^Q, \bar{y}_i^Q, \rho_i^Q) \to 0.$$

This concludes the proof of part 1 of Theorem 5.

Part 2: efficient hedging, risk-indifference
a. Whenever the initial capital of the optimal hedging strategy (2.167) approaches the perfect hedging price (2.36), the shortfall risk (2.168) goes to 0.
b. The shortfall risk approaches

$$\text{max shortfall}^I = S_0^1 e^{\mu_1 T} \cdot \Psi^1(\bar{y}_1^I) + S_0^2 e^{\mu_2 T} \cdot \Psi^1(\bar{y}_2^I) \tag{2.199}$$

whenever the price of the optimal hedging strategy goes to 0. Here, Ψ^1 is defined in (2.37), and $\bar{y}_i^I$ in (2.172).
Proof.
All formulas and constants for this case are given in Theorem 4, and the proof is similar to the proofs of parts 1a and 1b. From the formulas for the fair premium for efficient (case $p = 1$) and perfect hedging, we observe that the efficient hedging price approaches the perfect hedging price whenever

$$\Psi^2(\tilde{x}_i^I, \tilde{y}_i, \rho_i^I) \to \Psi^1(\tilde{y}_i) \quad \Leftrightarrow \quad \tilde{x}_i^I \to \infty \quad \Leftrightarrow \quad a^* \to 0. \tag{2.200}$$

And, when $a^* \to 0$,

$$\bar{x}_i^I \to \infty \quad \Leftrightarrow \quad \Psi^1(\bar{y}_i^I) - \Psi^2(\bar{x}_i^I, \bar{y}_i^I, \rho_1^I) \to 0, \tag{2.201}$$

so the shortfall risk goes to 0.

For part 2b, note that the price of the optimal efficient hedging strategy goes to 0 whenever $\tilde{x}_i^I \to -\infty$, or $a^* \to \infty$. But this implies that $\bar{x}_i^I$ also $\to -\infty$, so that

$$\Psi^2(\bar{x}_i^I, \bar{y}_i^I, \rho_i^I) \to 0,$$

which leaves the expression (2.199) for the largest expected shortfall.

Part 3: efficient hedging, risk-taking
a. Whenever the initial capital of the optimal hedging strategy (2.145) approaches the perfect hedging price (2.36), the shortfall risk (2.146) goes to 0.
b. The shortfall risk approaches

$$\begin{aligned} \text{max shortfall}^T &= (S_0^1)^p e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)Tp + \frac{\sigma_1^2}{2}Tp^2} \cdot \Psi^1(\bar{y}_1^T) \\ &+ (S_0^2)^p e^{\left(\mu_2 - \frac{\sigma_2^2}{2}\right)Tp + \frac{\sigma_2^2}{2}Tp^2} \cdot \Psi^1(\bar{y}_2^T) \end{aligned} \tag{2.202}$$

whenever the price of the optimal hedging strategy goes to 0. Here, Ψ^1 is defined in (2.37), and $\bar{y}_i^T$ in (2.150).

Proof.

Note that the formulas and definitions of constants for this case are given in Theorem 3. The price of the optimal efficient hedging strategy goes to the perfect hedging price whenever $\tilde{x}_i^T \to \infty$, or $a^* \to 0$. But in this case $\bar{x}_i^T \to \infty$ and

$$\Psi^2(\bar{x}_i^T, \bar{y}_i^T, \rho_i^T) \to \Psi^1(\bar{y}_i^T),$$

which causes the shortfall risk to approach 0.

On the other hand, when the efficient hedging price approaches 0, $\tilde{x}_i^T \to -\infty$ and $a^* \to \infty$, so that $\bar{x}_i^T \to -\infty$ and $\Psi^2(\bar{x}_i^T, \bar{y}_i^T, \rho_i^T) \to 0$, and the maximal shortfall risk takes the form (2.202). This concludes the proof of part 3 of Theorem 5.

Part 4: efficient hedging, risk-aversion

a. Whenever the initial capital of the optimal hedging strategy (2.95) approaches the perfect hedging price (2.36), the shortfall risk (2.96) goes to 0.

b. The shortfall risk approaches

$$\begin{aligned} \text{max shortfall}^A &= (S_0^1)^p e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)Tp + \frac{\sigma_1^2}{2}Tp^2} \cdot \Psi^1(\bar{y}_1^k) \\ &+ (S_0^2)^p e^{\left(\mu_2 - \frac{\sigma_2^2}{2}\right)Tp + \frac{\sigma_2^2}{2}Tp^2} \cdot \Psi^1(\bar{y}_2^k) \end{aligned} \tag{2.203}$$

whenever the price of the optimal hedging strategy goes to 0. Here, Ψ^1 is defined in (2.37), and $\bar{y}_i^k$ in (2.106).

Proof.

For the risk-aversion case, the formulas and definitions of constants are given in Theorem 2. To prove part 4a, note that

$$\Psi^2(\tilde{x}_i^A, \tilde{y}_i, \rho_i^A) \to \Psi^1(\tilde{y}_i) \quad \Leftrightarrow \quad \tilde{x}_i^A \to \infty \quad \Leftrightarrow \quad a^* \to 0.$$

Then also $M \to 0$ and $\tilde{c}_i \to \infty$ (thus $\Psi^2(\tilde{c}_i, \tilde{y}_i^c, \rho_i^A) \to \Psi^1(\tilde{y}_i^c)$), so that the limit of the product $M \cdot \Psi^2(\tilde{c}_i, \tilde{y}_i^c, \rho_i^A)$ (of type $0 \cdot const$) equals 0 as $a^* \to 0$. But whenever this happens, we also get that $N \to 0$ and $\bar{c}_i \to \infty$, so $\Psi^2(\bar{c}_i, \bar{y}_i^c, \rho_i^A) \to const$, and the product $N \cdot \Psi^2(\bar{c}_i, \bar{y}_i^c, \rho_i^A) \to 0$. At the same time, $a^* \to 0$ implies that $\bar{k}_i \to -\infty$ and $\Psi^2(\bar{k}_i, \bar{y}_i^k, -\rho_i^A) \to 0$. Putting together all of the above, we get that the shortfall risk goes to 0 whenever the price of the optimal efficient hedging strategy for the risk-aversion case approaches the perfect hedging price.

The proof of part 4b requires more work, as we encounter indeterminate forms for some of the limits. First, we establish that the price of the optimal hedging strategy goes to 0 as $a^* \to \infty$. Notice that as $a^* \to \infty$, $\tilde{x}_i^A \to -\infty$ and $\Psi^2(\tilde{x}_i^A, \tilde{y}_i, \rho_i^A) \to 0$. At the same time $\tilde{c}_i \to -\infty$, so $\Psi^2(\tilde{c}_i, \tilde{y}_i^c, \rho_i^A) \to 0$, but $M \to \infty$. Thus we have to show that

$$\lim_{a^* \to \infty} M \cdot \Psi^2(\tilde{c}_i, \tilde{y}_i^c, \rho_i^A) \to 0. \tag{2.204}$$

To deal with this indeterminate form, we refer to the definition of M (2.98) and the expression for the cumulative normal distribution of two correlated random variables (2.68), and apply L'Hospital's rule to evaluate the following limit:

$$\begin{aligned}
\lim_{a^*\to\infty} M\cdot\Psi^2(\tilde{c}_i,\tilde{y}_i^c,\rho_i^A) &= const\cdot\lim_{a^*\to\infty}\frac{\int_{-\infty}^{\tilde{y}_i^c} e^{\frac{-\tilde{c}_i^2+2\rho\tilde{c}_i y-y^2}{2(1-\rho^2)}}\left(\frac{-1}{a^*2\pi\sqrt{1-\rho^2}}\right)dy}{-\frac{1}{p-1}(a^*)^{\frac{-p}{p-1}}} \\
&= const\cdot\lim_{a^*\to\infty}\frac{\int_{-\infty}^{\tilde{y}_i^c} e^{\frac{-(y^2+\rho^2\tilde{c}_i^2-2\rho\tilde{c}_i y)+\rho^2\tilde{c}_i^2-\tilde{c}_i^2}{2(1-\rho^2)}}\frac{dy}{2\pi\sqrt{1-\rho^2}}}{(a^*)^{\frac{-1}{p-1}}} \\
&= const\cdot\lim_{a^*\to\infty}\frac{e^{-\frac{\tilde{c}_i^2}{2}}\int_{-\infty}^{\tilde{y}_i^c} e^{\frac{-(y-\rho\tilde{c}_i)^2}{2(1-\rho^2)}}\frac{dy}{2\pi\sqrt{1-\rho^2}}}{(a^*)^{\frac{-1}{p-1}}}. \qquad (2.205)
\end{aligned}$$

Note that the *const* in the front takes care of all the constants remaining from the definition of $\tilde{c}_i$ and taking of the derivatives.

Now, we can represent $\tilde{c}_i$ as $\tilde{c}_i = \frac{-\ln a^*+k_1}{k_2}$, where k_1, k_2 are constants corresponding to (2.101). Then we rewrite the expression multiplying the integral above as follows:

$$\begin{aligned}
\frac{e^{-\frac{\tilde{c}_i^2}{2}}}{(a^*)^{\frac{-1}{p-1}}} &= \frac{e^{\frac{-1}{2}\left(\frac{-\ln a^*+k_1}{k_2}\right)^2}}{(a^*)^{\frac{-1}{p-1}}} = \frac{e^{-\frac{(\ln a^*)^2}{2k_2^2}-\frac{k_1\ln a^*}{k_2^2}-\frac{k_1^2}{2k_2^2}}}{(a^*)^{\frac{-1}{p-1}}} \\
&= (a^*)^{-\frac{\ln a^*}{2k_2^2}-\frac{k_1}{k_2^2}+\frac{1}{p-1}}\cdot e^{-\frac{k_1^2}{2k_2^2}}. \qquad (2.206)
\end{aligned}$$

Taking the limit of this expression as $a^*\to\infty$, we obtain

$$\lim_{a^*\to\infty}(a^*)^{-\frac{\ln a^*}{2k_2^2}-\frac{k_1}{k_2^2}+\frac{1}{p-1}}\cdot e^{-\frac{k_1^2}{2k_2^2}} = \lim_{a^*\to\infty} const\cdot\frac{1}{(a^*)^{\frac{\ln a^*}{2k_2^2}}} = 0. \qquad (2.207)$$

So far, we showed that the coefficient in front of the integral in (2.205) approaches 0 as $a^*\to\infty$. Now we just need to make sure that the integral does not affect this result. Making the substitution

$$\tilde{z} = \frac{y-\rho\tilde{c}_i}{\sqrt{1-\rho^2}}, \qquad (2.208)$$

we obtain

$$\int_{-\infty}^{\tilde{y}_i^c} e^{\frac{-(y-\rho\tilde{c}_i)^2}{2(1-\rho^2)}}\frac{dy}{2\pi\sqrt{1-\rho^2}} = \frac{1}{\sqrt{2\pi}}\int_{\tilde{l}_1}^{\tilde{l}_2} e^{\frac{-\tilde{z}^2}{2}}\frac{d\tilde{z}}{\sqrt{2\pi}}, \qquad (2.209)$$

which is bounded regardless of what happens to the limits from substitution $\tilde{l}_1$ and $\tilde{l}_2$. Therefore, the product of the (bounded) integral and the coefficient in (2.207) will be of type $0 \cdot const$, so the limit of this product as $a^* \to \infty$ will be 0.

Now that we have established that as $a^* \to \infty$, the price of the optimal hedging strategy for the risk-aversion case of efficient hedging approaches 0, let us see what happens to the shortfall risk. Whenever $a^* \to \infty$, $\bar{k}_i \to \infty$ also, meaning that $\Psi^2(\bar{k}_i, \bar{y}_i^k, -\rho_i^A) \to \Psi^1(\bar{y}_i^k)$. So to show that the maximal expected shortfall is given by (2.203), we need to prove that

$$\lim_{a^* \to \infty} N \cdot \Psi^2(\bar{c}_i, \bar{y}_i^c, \rho_i^A) \to 0. \tag{2.210}$$

We do this in the same way as for the limit (2.204) above. Note that $a^* \to \infty$ means that $N \to \infty$ and $\Psi^2(\bar{c}_i, \bar{y}_i^c, \rho_i^A) \to 0$ (because $\bar{c}_i \to -\infty$).

Based on the definition of N (2.99) and the formula for Ψ^2 (2.68), we have to evaluate

$$\begin{aligned} \lim_{a^* \to \infty} N \cdot \Psi^2(\bar{c}_i, \bar{y}_i^c, \rho_i^A) &= konst \cdot \lim_{a^* \to \infty} \frac{\int_{-\infty}^{\bar{y}_i^c} e^{\frac{-\bar{c}_i^2 + 2\rho\bar{c}_i y - y^2}{2(1-\rho^2)}} \frac{-dy}{a^* 2\pi\sqrt{1-\rho^2}}}{\frac{-p}{p-1}(a^*)^{\frac{-p}{p-1}-1}} \\ &= konst \cdot \lim_{a^* \to \infty} \frac{e^{-\frac{\bar{c}_i^2}{2}} \int_{\bar{l}_1}^{\bar{l}_2} e^{\frac{-\bar{z}^2}{2}} \frac{d\bar{z}}{2\pi}}{(a^*)^{\frac{-p}{p-1}}}, \end{aligned} \tag{2.211}$$

with $konst$ taking care of all the constants resulting from the definition of $\bar{c}_i$, derivatives, and simplifications. Note that the steps to rewrite the expression above are identical to those in (2.205). We substituted

$$\bar{z} = \frac{y - \rho\bar{c}_i}{\sqrt{1-\rho^2}} \tag{2.212}$$

with limits $\bar{l}_1$ and $\bar{l}_2$. Again, as in (2.209), the integral in (2.211) is bounded. Similarly to $\tilde{c}_i$, based on the definition of $\bar{c}_i$ in (2.103) and appropriate constants m_1, m_2, we can write $\bar{c}_i = \frac{-\ln a^* + m_1}{m_2}$. Then, following the same steps as in (2.206), the coefficient multiplying the integral in (2.211) can be written as

$$(a^*)^{-\frac{\ln a^*}{2m_2^2} - \frac{m_1}{m_2^2} + \frac{p}{p-1}} \cdot e^{-\frac{m_1^2}{2m_2^2}}.$$

From this, we see that the coefficient approaches 0 as $a^* \to \infty$. Therefore, the overall product in (2.211) goes to 0 also.

Based on these considerations and the expression for the shortfall risk (2.96), we conclude that as the price of the optimal hedging strategy approaches 0, the shortfall risk approaches its maximal level, given by (2.203), and finish the proof of Theorem 5, part 4b.

2.7.1 Example

Here we provide some numbers to illustrate how the values of the maximal shortfall risk differ based on the risk preference of the hedger. We use the same data, estimates and contract type as in the numerical example in section 2.6. Based on these numbers, we calculate that in the case of risk-indifference ($p = 1$), the largest expected shortfall is USD 13,270.06. Table 5 provides the values for risk-taking ($0 < p < 1$) and risk-aversion ($p > 1$).

Table 5: Values of maximal shortfall risk (dollar amounts) for various risk preferences of the investor

p, risk-taking	max shortfall	p, risk-aversion	max shortfall
0.0001	1.00	1.0001	13,282.81
0.1	2.56	1.1	34,696.96
0.2	6.56	1.2	90,917.44
0.3	16.87	1.3	238.749.10
0.4	43.45	1.4	628,313.24
0.5	112.15	1.5	1,657,112.04
0.6	290.10	1.6	4,379,958.56
0.7	752.02	1.7	11,601,974.26
0.8	1,953.64	1.8	30,799,160.76
0.9	5,086.17	1.9	81,939,309.75
0.9999	13,270.06	2.0	218,470,861.00

First of all, note that as $p \to 1$, the values of maximal expected shortfall for both risk-aversion and risk-taking approach the amount of largest expected shortfall in the risk-indifference case, which is expected. Also, observe that when a risk-taking investor becomes more risk-averse, he/she begins to value potential losses higher and higher. And, when risk aversion grows, the investor becomes increasingly sensitive to shortfall risk. Such results agree with our intuition. It is rather interesting to see that when the level of risk aversion changes by 1 (from $p = 1$ to $p = 2$), the value of the potential loss increases by a factor of about 15,000 (that's a lot!). Or, when risk-taking habits change from 1 to 0, the expected loss decreases by a factor of about 13,000. Such situations seem too extreme; in the real world, risk preferences of majority of the investors probably fall close to $p = 1$.

3 Mortality modelling

Here we discuss the effects of mortality modelling on risk management with equity-linked life insurance contracts.[3]

The notation introduced in this chapter applies only to the current chapter and to Appendix 3 pertaining to the results presented here. Note that some of the names of variables may overlap with those used previously; we choose not to change the names in order to keep them consistent with the notation used in financial and mortality literature. In particular, for the financial setting we use the typical Black-Scholes notation (see, for example, Melnikov et al. (2002) or Hull (2005)), while for the insurance setting we attempt to follow closely the notation of the classical actuarial textbook by Bowers et al. (1997).

3.1 Background

Humans have been trying to understand life and death for as long as they existed. Every culture and nation has legends about the origins and reasons for being born and dying. Scholars would formulate these ideas into questions about whether or not human survival is governed by some law, and if so, what it is and how science can explain it. The obvious source of information on this topic is birth and death data. As early as 1693, English astronomer Edmund Halley constructed a life table from the observed number of deaths in Breslau (now Wroclaw, Poland). Soon after, in 1740, the earliest life tables for males and females were published by Nicholas Struyck (Pitacco (2003)). Around this time, mathematicians became interested in modelling human survival as well. Abraham De Moivre produced the first known analytic model for the probability of survival as a linear function of the person's current age, recognizing, however, that his model failed to represent human survival across all ages accurately.

So the search for a better model continued, and in 1825 Benjamin Gompertz presented his version of the survival probability formula, based on the recognition that human mortality displayed exponential patterns for most ages. His result is believed to be the most influential parametric mortality model in the literature. Some years later, in 1860, Makeham noticed that Gompertz's model was not adequate for higher ages and amended it in an effort to correct this deficiency (Higgins (2003)). Despite further developments after 1860 (including models by Thiele in 1872 and Wittstein in 1883), Gompertz's and Makeham's models remain to this day among the most popular choices for mortality modelling.

In the early 20th century, Italian economist and sociologist Vilfredo Pareto put forth his idea for a model of mortality; Wallodi Weibull's model from the 1940s for predicting time until next failure of a technical system was adapted as a mortality

[3]A version of this chapter has been accepted for publication in *Insurance: Mathematics and Economics* under the title "Evaluating the performance of Gompertz, Makeham and Lee-Carter mortality models for risk management with unit-linked contracts" by A. Melnikov and Yu. Romaniuk.

model, with human organs seen as technical parts that eventually fail. Throughout the last century, there were other contributions to mortality modelling, but most of them were modifications/generalizations of the results of Gompertz and Makeham. In recent decades, the study of mortality has become increasingly more complex. Due to expanding computational capacities, modern parametric models may involve up to ten parameters (for example, the model of Heligman and Pollard (1980) with eight), or rely on processing large amounts of data for parameter estimation (Lee and Carter (1992)). The newest direction in the study of human survival is the idea of modelling and/or forecasting mortality as a stochastic process. For example, Lee and Carter (1992) forecast mortality as a random walk with drift, Dahl (2004) works in a setting where the dynamics of mortality intensity has the form of a diffusion process with drift and volatility dependent on the present state of the process only. Biffis (2005) models asset prices and mortality dynamics by affine jump-diffusion processes, while Luciano and Vigna (2005) use doubly stochastic (Cox) processes to describe mortality dynamics. However, as noted in Higgins (2003), the development of stochastic mortality models is in its infancy stage.

When we speak of mortality models, we should distinguish between static (functions of age only) vs. dynamic (functions of age and current year) and deterministic vs. stochastic models. Of those considered in this paper, the models of Gompertz and Makeham are deterministic and static, while the Lee-Carter method forecasts mortality stochastically and is dynamic. We stress that currently Gompertz- and Makeham-based models are typically used for educational, forecasting and risk valuation purposes (Pitacco (2003)). The particular choice of mortality models discussed in this paper arises from our desire to investigate how the widely used classical models of Gompertz and Makeham compare to one of the most significant recent developments in mortality modelling and forecasting – Lee-Carter's method.

3.2 Mortality models in the thesis

3.2.1 Some actuarial concepts

Before we describe the mortality models of Gompertz, Makeham and Lee-Carter, we need to introduce some actuarial concepts. Working in the insurance setting introduced in section 2.3.2, let l_0 be the number of newborns in a group under observation. Denote $\mathbf{L}(x)$ the number of survivors to age x among the newborns:

$$\mathbf{L}(x) = \sum_{j=1}^{l_0} I_j, \quad I_j = \begin{cases} 1 & \text{if life } j \text{ survives to age } x \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Define *survival function* as the probability that a newborn will attain age x:

$$s(x) = \bar{P}\{\tau(0) > x\}, \tag{3.2}$$

where $\tau(0)$ denotes the remaining lifetime of a newborn. Then from (3.1) and the standard assumption that each newborn's survival function is $s(x)$ (all independent), we have that the expected number l_x of survivors to age x from l_0 newborns is calculated as follows:

$$l_x = \bar{E}(\mathbf{L}(x)) = l_0 s(x). \tag{3.3}$$

Let L_x denote the total expected number of years lived by the survivors between ages x and $x+1$:

$$L_x = \int_0^1 l_{x+t} dt. \tag{3.4}$$

Note that based on (3.2), the following relation holds for the probability that a life aged x survives T more years:

$$_T p_x = \bar{P}\{\tau(x) > T\} = \frac{s(x+T)}{s(x)}. \tag{3.5}$$

Now we can present the classical definitions of the concepts needed to describe the models of Gompertz, Makeham and Lee-Carter. First, the *force of mortality* is given by

$$\mu_x = -\frac{s'(x)}{s(x)}; \tag{3.6}$$

it can be interpreted as the likelihood that a life that survived to age x dies in the next instant of time. The *central death rate* is defined as

$$m_x = \frac{l_x - l_{x+1}}{L_x}; \tag{3.7}$$

we think of this as the expected rate at which survivors in the group of l_0 newborns die between ages x and $x+1$.

3.2.2 Mortality model formulas

Now we are ready to summarize the main contributions of Gompertz, Makeham (as given in Bowers et al. (1997)), and Lee and Carter (1992) to mortality modelling and forecasting:

$$\begin{array}{ll} \textbf{Gompertz:} & \mu_x = B \cdot c^x, \quad c > 1, B > 0; \\ \textbf{Makeham:} & \mu_x = A + B \cdot c^x, \quad c > 1, B > 0, A \geq -B; \\ \textbf{Lee-Carter:} & \ln(m_{x,t}) = a_x + b_x \cdot k_t + \xi_{x,t}. \end{array} \tag{3.8}$$

Above, t refers to historical time period (years), x to the person's age, μ_x is the age-dependent force of mortality, $m_{x,t}$ the central death rate for age x and year t, and $\xi_{x,t}$ the corresponding error. The remaining constants A, B, c (different for

Gompertz and Makeham), and a_x, b_x can be estimated based on historical deaths and death rates data, while the mortality index k_t in the Lee-Carter model can be fitted for past years and forecasted for the future. In section 3.6.1, we describe the estimation of the involved parameters in more detail.

Gompertz's and Makeham's results can be summarized analytically in terms of a single formula for the force of mortality, and parameter estimation is the only potentially challenging task. Lee and Carter's approach requires more explanation. The idea behind the method is to view mortality as a process dependent on age as well as on the time period, so the parameters a_x, b_x are age-specific, while the mortality index k_t reflects the effects of the corresponding year and thus environment on the (past) current and future survival/mortality patterns. The estimate of the mortality index $\hat{k}_t$ is modelled by a random walk with constant drift d and mean-zero random noise ψ:

$$\hat{k}_t = \hat{k}_{t-1} + d + \psi_t. \tag{3.9}$$

Once the mortality index is projected for future years, life table functions (such as survival probabilities and life expectancies) can be extrapolated to use as needed in actuarial/insurance applications.

3.3 Note on financial setting

To focus on the effects of mortality risk in our analysis of pricing and hedging of equity-linked life insurance contracts, we work with a single risky asset, as opposed to the two-asset setting (see section 2.3.1). Note that the study of mortality risk and optimal risk management strategies for equity-linked contracts with two assets is performed in the same way as described in the remainder of this chapter.

We have a typical Black-Scholes-Merton setting: a financial market with interest rate $r > 0$, one riskless money market account $B = (B_t)_{t\in[0,T]}$ and one risky asset $S = (S_t)_{t\in[0,T]}$ satisfying

$$\begin{aligned} dB_t &= rB_t dt \quad \Leftrightarrow \quad B_t = B_0 e^{rt}, \quad B_0 := 1; \\ dS_t &= S_t(\mu dt + \sigma dW_t) \quad \Leftrightarrow \quad S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}, \end{aligned} \tag{3.10}$$

with constants $\mu \in \mathbb{R}$, $\sigma > 0$, and $W = (W_t)_{t\in[0,T]}$ a Wiener process on a standard stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$. All processes are adapted to the filtration $\mathbb{F}$, generated by W. Every predictable process $\pi = (\pi_t)_{t\in[0,T]} = (\beta_t, \gamma_t)_{t\in[0,T]}$ is called a trading strategy (or portfolio) with time t value

$$V_t^\pi = \beta_t B_t + \gamma_t S_t. \tag{3.11}$$

Only self-financing (with no additional inflow/outflow of cash other than the initial premium payment) and admissible (with nonnegative capital) strategies are allowed.

It is well known that in this setting the equivalent martingale measure P^* is unique, and its density Z is given by

$$Z_t = \frac{dP^*}{dP}|_{\mathcal{F}_t} = e^{-\theta W_t - \frac{\theta^2}{2}t}; \quad \theta = \frac{\mu - r}{\sigma}. \tag{3.12}$$

Under P^*, the evolution of S takes form

$$dS_t = S_t(rdt + \sigma dW_t^*) \quad \Leftrightarrow \quad S_t = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t^*}, \tag{3.13}$$

with W^* a Wiener process under P^* (see, for instance, Melnikov et al. (2002)) such that

$$W_t^* = W_t + \theta \cdot t. \tag{3.14}$$

The contract in consideration entitles the client to one unit of some risky asset or a guaranteed amount, whichever is greater, at expiration date T. The payoff H has the form

$$H = \max\{S_T, K_T\} = S_T I_{\{S_T \geq K_T\}} + K_T I_{\{S_T < K_T\}}, \tag{3.15}$$

where K is the deterministic guarantee, calculated as

$$K_T = S_0 e^{gT}. \tag{3.16}$$

Above, g is the rate guaranteed by the contract and S_0 is the initial value of the risky fund. Basically, the client has the right to choose the larger of two funds at maturity of the contract: a risky fund with expected return μ or a risk-free fund earning a rate of g over the duration of the agreement. Clearly, the perfect hedging price of the payoff H can be reduced to a formula similar to that of Black-Scholes for the price of a call option (Black and Scholes (1973)).

3.4 Note on insurance setting

The insurance setting is the same as in the previous chapter ($(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$, see section 2.3.2). Again we run into the problem of the contract premium U_0 being insufficient for a perfect hedge:

$$U_0 = \bar{E} \times E^*(He^{-rT} I_{\{\tau(x) > T\}}) = E^*(He^{-rT})_T p_x < E^*(He^{-rT}). \tag{3.17}$$

Recall that $_T p_x = \bar{P}\{\tau(x) > T\}$ denotes the probability of a life aged x surviving T more years.

3.5 Quantile hedging to the rescue!

In this chapter, we focus on applying quantile hedging to illustrate optimal hedging of financial and insurance risks inherent in equity-linked contracts (alternatively, one could use efficient hedging in a similar manner). Based on the discussion in section 2.3.3, we have that the optimal hedging strategy (the one that maximizes the probability of successful hedging under budget constraints) is the perfect hedge for the modified contingent claim H^*, with cost

$$U_0 = E^*(H^* e^{-rT}) = E^* \left(\frac{\max\{S_T, K_T\} I_{A^*}}{e^{rT}} \right), \tag{3.18}$$

where $A^* = \left\{ \frac{1}{Z_T} > a^* e^{-rT} H \right\}$ (2.52).

From (3.18) and (3.17), we obtain the formula

$$_T p_x = \frac{E^*(\max\{S_T, K_T\} I_A^*)}{E^*(\max\{S_T, K_T\})}, \tag{3.19}$$

which is exactly the equation (2.65) restated here for our particular choice of H. Recall from the discussion of (2.65) that this formula is the key to managing both financial and insurance risk components when hedging contracts with insufficient initial capital. The hedger can either sell the equity-linked contract with payoff H to any client and then determine the maximal probability of successful hedging given that the premium U_0 received from this client was invested into the optimal hedge, or the hedger can set the probability of successful hedging at $1 - \epsilon$ and see what clients are suitable for the policy in consideration (see section 2.3.3 for more details). Before demonstrating the two risk management possibilities in light of the mortality implications of the models of Gompertz, Makeham and Lee-Carter, we derive the formulas for the fair premium U_0 and survival probability $_T p_x$ used in the numerical illustration.

3.5.1 Deriving pricing and survival probability formulas

First, consider equations (3.10), (3.18), (3.12) and the general structure of the maximal set of successful hedging (2.52): they allow us to write

$$\begin{aligned} P(A^*) &= E(I_{A^*}) \\ &= E\left(I_{\left\{\frac{1}{Z_T} > \frac{a^* S_T}{e^{rT}}\right\}} I_{\{S_T \geq K_T\}} \right) + E\left(I_{\left\{\frac{1}{Z_T} > \frac{a^* K_T}{e^{rT}}\right\}} I_{\{S_T < K_T\}} \right) \\ &= E\left(I_{\left\{ e^{\theta W_T + \frac{\theta^2}{2}T} > \frac{a^* S_0}{e^{rT}} e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T} \right\}} I_{\left\{ S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T} \geq S_0 e^{gT} \right\}} \right) \\ &+ E\left(I_{\left\{ e^{\theta W_T + \frac{\theta^2}{2}T} > \frac{a^* S_0 e^{gT}}{e^{rT}} \right\}} I_{\left\{ S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T} < S_0 e^{gT} \right\}} \right). \end{aligned} \tag{3.20}$$

Since S and Z are both functions of W, and $W_T \sim N(0,T)$ (under P), we can rewrite this equation further as

$$\begin{aligned} P(A^*) &= E\left(I_{\{y>J\}} I_{\{y\geq N\}}\right) + E\left(I_{\{y>M\}} I_{\{y<N\}}\right) \quad \text{if} \quad \mu - r - \sigma^2 > 0, \\ P(A^*) &= E\left(I_{\{y<J\}} I_{\{y\geq N\}}\right) + E\left(I_{\{y>M\}} I_{\{y<N\}}\right) \quad \text{if} \quad \mu - r - \sigma^2 < 0, \end{aligned} \tag{3.21}$$

where y is a standard normal random variable (under P), and M, N, J are given by

$$\begin{aligned} M &= \frac{\ln(a^* S_0) + \left(g - r - \frac{\theta^2}{2}\right) T}{\theta \sqrt{T}}, \\ N &= \frac{\left(g - \mu + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}, \\ J &= \frac{\ln(a^* S_0) - \frac{(\theta-\sigma)^2}{2} T}{(\theta - \sigma)\sqrt{T}}. \end{aligned} \tag{3.22}$$

Next, let us see how the constants M, N, J compare to each other: first,

$$\begin{aligned} N - M &= \frac{\left(g - \mu + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} - \frac{\ln(a^* S_0) + \left(g - r - \frac{\theta^2}{2}\right) T}{\theta \sqrt{T}} \\ &= \frac{\frac{T}{\sigma}\left[(\mu - r)\left(g - \mu + \frac{\sigma^2}{2}\right) - \sigma^2 \left(g - r - \frac{\theta^2}{2}\right)\right] - \sigma \ln(a^* S_0)}{\sigma\theta\sqrt{T}} \\ &= \frac{\Lambda}{\sigma\theta\sqrt{T}}, \end{aligned} \tag{3.23}$$

where

$$\Lambda = \frac{T}{\sigma}\left[\left(g - \frac{\mu + r}{2}\right)(\mu - r - \sigma^2)\right] - \sigma \ln(a^* S_0). \tag{3.24}$$

Similarly, we have that

$$\begin{aligned} N - J &= \frac{\left(g - \mu + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} - \frac{\ln(a^* S_0) - \frac{(\theta-\sigma)^2}{2} T}{(\theta - \sigma)\sqrt{T}} \\ &= \frac{\Lambda}{\sigma(\theta - \sigma)\sqrt{T}}, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} M - J &= \frac{\ln(a^* S_0) + \left(g - r - \frac{\theta^2}{2}\right) T}{\theta\sqrt{T}} - \frac{\ln(a^* S_0) - \frac{(\theta-\sigma)^2}{2} T}{(\theta-\sigma)\sqrt{T}} \\ &= \frac{\Lambda}{\theta(\theta-\sigma)\sqrt{T}}. \end{aligned} \tag{3.26}$$

Now, observe that Λ is positive whenever

$$\left(g - \frac{\mu + r}{2}\right)(\mu - r - \sigma^2) > \frac{\sigma^2}{T} \ln(a^* S_0),$$

that is, whenever

$$g > \frac{\frac{\sigma^2}{T}\ln(a^* S_0)}{\mu - r - \sigma^2} + \frac{\mu + r}{2} \text{ and } \mu - r - \sigma^2 > 0$$

or

$$g < \frac{\frac{\sigma^2}{T}\ln(a^* S_0)}{\mu - r - \sigma^2} + \frac{\mu + r}{2} \text{ and } \mu - r - \sigma^2 < 0.$$

Moreover, based on (3.23), (3.25), (3.26), and the fact that the sign of $(\theta - \sigma)$ is determined by the sign of $(\mu - r - \sigma^2)$, if the guaranteed rate g is selected as indicated below, we obtain that

$$\begin{aligned} &J < M < N \quad \text{whenever} \quad \mu - r - \sigma^2 > 0 \quad \text{and} \quad g > G, \\ &M < N < J \quad \text{whenever} \quad \mu - r - \sigma^2 < 0 \quad \text{and} \quad g < G, \\ &G = \frac{\frac{\sigma^2}{T}\ln(a^* S_0)}{\mu - r - \sigma^2} + \frac{\mu + r}{2}. \end{aligned} \tag{3.27}$$

These considerations allow us to simplify $P(A^*)$ (3.21) even further:

$$P(A^*) = P\{y \geq N\} + P\{M < y < N\} = P\{y > M\} \tag{3.28}$$

or

$$P(A^*) = P\{N \leq y < J\} + P\{M < y < N\} = P\{M < y < J\}. \tag{3.29}$$

Therefore, under P, the maximal set of successful hedging A^* takes the form

$$A^* = \{y > M\} \tag{3.30}$$

or

$$A^* = \{M < y < J\}, \tag{3.31}$$

depending on the sign of $(\mu - r - \sigma^2)$ (and the appropriate selection of g).

Using the same reasoning and following the steps done in (3.20) - (3.29) for P, we can simplify the expression for A^* under P^* to

$$\begin{aligned} A^* &= \{y^* > J^*\} \cap \{y^* \geq N^*\} + \{y^* > M^*\} \cap \{y^* < N^*\} \\ &= \{y^* \geq N^*\} + \{M^* < y^* < N^*\} = \{y^* > M^*\} \end{aligned} \quad (3.32)$$

or

$$\begin{aligned} A^* &= \{y^* < J^*\} \cap \{y^* \geq N^*\} + \{y^* > M^*\} \cap \{y^* < N^*\} \\ &= \{N^* \leq y^* < J^*\} + \{M^* < y^* < N^*\} = \{M^* < y^* < J^*\}, \end{aligned} \quad (3.33)$$

where the constants M^*, N^*, J^*

$$\begin{aligned} M^* &= M + \theta\sqrt{T}, \\ N^* &= N + \theta\sqrt{T}, \\ J^* &= J + \theta\sqrt{T} \end{aligned} \quad (3.34)$$

satisfy $J^* < M^* < N^*$ if $\mu - r - \sigma^2 > 0$ and $g > G$, or $M^* < N^* < J^*$ if $\mu - r - \sigma^2 < 0$ and $g < G$ (this is equivalent to (3.27)), and $y^* \sim N(0,1)$ under P^*.

Next, using the evolution of S under P^* (3.13) and the expression for Z_T (3.12), we can write U_0 as

$$\begin{aligned} U_0 &= E^*\left(\frac{S_T}{e^{rT}} I_{\left\{\frac{1}{Z_T} > \frac{a^* S_T}{e^{rT}}\right\}} I_{\{S_T \geq K_T\}}\right) + E^*\left(\frac{K_T}{e^{rT}} I_{\left\{\frac{1}{Z_T} > \frac{a^* K_T}{e^{rT}}\right\}} I_{\{S_T < K_T\}}\right) \\ &= E^*\left(\frac{S_T}{e^{rT}} I_{\{y^* > J^*\}} I_{\{y^* \geq N^*\}}\right) + E^*\left(\frac{K_T}{e^{rT}} I_{\{y^* > M^*\}} I_{\{y^* < N^*\}}\right) \\ &= E^*\left(\frac{S_T}{e^{rT}} I_{\{y^* \geq N^*\}}\right) + E^*\left(\frac{K_T}{e^{rT}} I_{\{M^* < y^* < N^*\}}\right) \end{aligned} \quad (3.35)$$

for A^* in (3.32). For A^* in (3.33), we obtain a similar formula for U_0:

$$\begin{aligned} U_0 &= E^*\left(\frac{S_T}{e^{rT}} I_{\{y^* < J^*\}} I_{\{y^* \geq N^*\}}\right) + E^*\left(\frac{K_T}{e^{rT}} I_{\{y^* > M^*\}} I_{\{y^* < N^*\}}\right) \\ &= E^*\left(\frac{S_T}{e^{rT}} I_{\{N^* \leq y^* < J^*\}}\right) + E^*\left(\frac{K_T}{e^{rT}} I_{\{M^* < y^* < N^*\}}\right). \end{aligned} \quad (3.36)$$

Now we proceed to calculate the explicit formula for the fair premium as follows. The second term in the last line of equation (3.35) (as well as (3.36)) is simply

$$E^*\left(\frac{K_T}{e^{rT}} I_{\{M^* < y^* < N^*\}}\right) = S_0 e^{(g-r)T}(\Psi^1(N^*) - \Psi^1(M^*)), \quad (3.37)$$

with Ψ^1 denoting the one-dimensional cumulative normal distribution (2.37). The first term in the last line of equation (3.35) is calculated directly based on (3.13):

$$\begin{aligned} E^*\left(\frac{S_T}{e^{rT}} I_{\{y^* \geq N^*\}}\right) &= \frac{S_0 e^{-rT}}{\sqrt{2\pi}} \int_{N^*}^{\infty} e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}y^*} e^{\frac{-y^{*2}}{2}} dy^* \\ &= S_0 \Psi^1(\sigma\sqrt{T} - N^*). \end{aligned} \tag{3.38}$$

The first term in the last line of (3.36) is calculated in the same manner; we get that

$$E^*\left(\frac{S_T}{e^{rT}} I_{\{N^* \leq y^* < J^*\}}\right) = S_0(\Psi^1(\sigma\sqrt{T} - N^*) - \Psi^1(\sigma\sqrt{T} - J^*)). \tag{3.39}$$

Finally, putting together (3.37) with (3.38) and (3.39), we obtain these explicit formulas for the fair premium U_0 and survival probability ${}_Tp_x$:

$$\begin{aligned} U_0 &= S_0 \cdot \Psi^1(\sigma\sqrt{T} - N^*) + S_0 e^{(g-r)T} \cdot (\Psi^1(N^*) - \Psi^1(M^*)), \\ {}_Tp_x &= 1 - \frac{e^{(g-r)T} \cdot \Psi^1(M^*)}{\Psi^1(\sigma\sqrt{T} - N^*) + e^{(g-r)T} \cdot \Psi^1(N^*)} \end{aligned} \tag{3.40}$$

for A^* in (3.32). If A^* is given by (3.33), then

$$\begin{aligned} U_0 &= S_0 \cdot (\Psi^1(\sigma\sqrt{T} - N^*) - \Psi^1(\sigma\sqrt{T} - J^*)) + S_0 e^{(g-r)T} \cdot (\Psi^1(N^*) - \Psi^1(M^*)), \\ {}_Tp_x &= 1 - \frac{\Psi^1(\sigma\sqrt{T} - J^*) + e^{(g-r)T} \cdot \Psi^1(M^*)}{\Psi^1(\sigma\sqrt{T} - N^*) + e^{(g-r)T} \cdot \Psi^1(N^*)}. \end{aligned} \tag{3.41}$$

Recall that M^*, N^*, J^* are given in (3.34).

Now let us discuss the conditions (3.27) encountered in the process of calculating the formulas above. These conditions serve as a guide to the firm offering the contract for choosing an appropriate guaranteed rate based on the market situation and the nature of the risky asset (the relation between μ, r and σ). The intuition behind selecting g accordingly is the following. First, we should always have $r < g < \mu$: the guaranteed rate should be higher than the risk-free rate (otherwise clients would find money markets more appealing), and lower than the expected return on the risky asset, since payment of the guarantee involves no risk. But how high or low should g be set between μ and r?

Note that $\mu - r > \sigma^2$ implies that the expected excess return of the risky asset over the risk-free rate is higher than the risk (volatility σ^2) associated with the asset, call it Stock 1. Then Stock 1 is more attractive than Stock 2, whose parameters satisfy $\mu - r < \sigma^2$, since the risk-return relation for Stock 2 is not as appealing as in the previous case. The only restriction is that $\mu - r - \sigma^2 \neq 0$, since this term appears in the denominator of equations in (3.27). However, practically speaking, this should not pose any problems, since finding risky assets satisfying the precise relationship $\mu - r = \sigma^2$ would likely prove difficult.

A simple guideline for a manager deciding how to set guaranteed rates for contracts involving Stocks 1 and 2 is the following. To guarantee the higher expected return of the more appealing Stock 1, the manager should set g to exceed the average of the asset's return and the risk-free rate. That is, for $\mu - r > \sigma^2$, $\frac{\mu+r}{2} < g < \mu$. (Note that this ensures $g > G$, as required by (3.27)).[4] If, on the other hand, the underlying of the contract is the less attractive Stock 2, the manager can set the guarantee below the average of μ and r ($r < g < \frac{\mu+r}{2}$ implies $g < G$ for (3.27)). That is, whenever the contract calls for securing an asset whose expected return is not very high, the guaranteed rate can be lower than in the case of an underlying with high returns.

3.6 Numerical results: effects of the mortality models

3.6.1 Parameter estimation

For the risky asset, we chose the Toronto Stock Exchange/Standard and Poor Composite Index, which mirrors closely the performance of some 300 Canadian and US companies. Note that Canadian Imperial Bank of Commerce (CIBC) offers 3- and 5-year index-linked GICs (Guaranteed Investment Certificates), in which return on an invested amount is linked to the performance of S&P/TSX 60 or 500 fund (over 3 and 5 years respectively). At maturity of the contract, clients receive their investment plus interest earned based on the fund's return, or the originally invested amount if the fund's return is negative over the time of investment. This is essentially the same as the unit-linked contract considered in this paper except for the payoff's dependence on the client's survival to the maturity of the contract; the implied premium for this particular product of CIBC is, of course, the interest the investment would have earned if deposited elsewhere (for example, into treasury bonds) for the same duration.

For TSX/S&P, the annualized return $\mu = 9.11\%$ and volatility $\sigma = 15.73\%$ were estimated using daily data from Jan. 1, 1995 to Jan. 1, 2005 (estimation was performed as outlined in section 2.6.1, see also Hull (2005)). The initial value $S_0 = 9246.7$ was the price of the index as of Dec. 31, 2004 going into 2005. We chose three maturities for the contract, $T = 3$, 10 and 20, to look at the variations in risk management considerations for short-, medium- and long-term agreements. Note that shorter contracts are more likely to be offered under the umbrella of wealth management/investment products, whereas longer contracts belong to the insurance products category. The annual interest rate $r = 5.61\%$ used in calculations was the average annual yield of a 10-year Canada treasury bond from 1997 to 2003, as reported by Bolder et al. (2004). Recall that the contract allows its holder to choose the greater of a risky fund value or some deterministic amount. We used 7 percent

[4]It was shown in section 2.7 that whenever the probability of successful hedging $P(A^*)$ is maximized (or if it is specified beforehand to be close to 1), a^* in (3.27) will be a very small positive number (less than 1, for our purposes). This makes $\ln(a^* S_0)$ and the entire numerator $\frac{\sigma^2}{T}\ln(a^* S_0)$ negative. Thus the sign of the first term of G in equation (3.27) depends on the sign of the denominator, $\mu - r - \sigma^2$.

for the (annualized) guaranteed rate to calculate the corresponding guarantees; for example, for a 3-year contract, $K_3 = 9246.7e^{(.07 \cdot 3)}$.

To estimate the parameters for the three mortality models (see (3.8)), we used 1959 - 1999 mortality data (survival probabilities, deaths and death rates) for the US, Sweden and Japan from the Human Mortality Database (www.mortality.org). To calculate $\hat{A}, \hat{B}, \hat{c}$ for Gompertz and Makeham, we used the standard Least Squares method to minimize the sum (over time and all ages) of the errors squared. Utilizing equations (3.5) and (3.6), we can derive these relations for logs of survival probability values:

$$\begin{aligned} f_{x,t} &= \ln ({}_T p_x) = -\frac{B}{\ln(c)} c^x (c^T - 1) \quad \text{for Gompertz, and} \qquad (3.42) \\ f_{x,t} &= \ln ({}_T p_x) = -AT - \frac{B}{\ln(c)} c^x (c^T - 1) \quad \text{for Makeham.} \end{aligned}$$

Then for the given values of ${}_T p_x$, we seek to minimize

$$\begin{aligned} L &= \sum_t \sum_x \left(f_{x,t} + \frac{\hat{B}}{\ln(\hat{c})} \hat{c}^x (\hat{c}^T - 1) \right)^2 \quad \text{for Gompertz, and} \qquad (3.43) \\ L &= \sum_t \sum_x \left(f_{x,t} + \hat{A}T + \frac{\hat{B}}{\ln(\hat{c})} \hat{c}^x (\hat{c}^T - 1) \right)^2 \quad \text{for Makeham.} \end{aligned}$$

From here, we proceed in a standard manner, calculating partial derivatives of L with respect to $\hat{B}$, $\hat{c}$ for Gompertz and $\hat{A}$, $\hat{B}$, $\hat{c}$ for Makeham, and solving for the corresponding parameter values numerically using MATLAB (the programs are available upon request[5]). The estimated parameters for the models of Gompertz and Makeham are given in Table 6.

Table 6: Estimated parameters for Gompertz (G) and Makeham (M) mortality models for USA (US), Sweden (S), Japan (J)

	A	B	c
G_{US}		$6.148 \cdot 10^{-5}$	1.09159
M_{US}	$9.566 \cdot 10^{-4}$	$5.162 \cdot 10^{-5}$	1.09369
G_S		$1.694 \cdot 10^{-5}$	1.10960
M_S	$4.393 \cdot 10^{-4}$	$1.571 \cdot 10^{-5}$	1.11053
G_J		$2.032 \cdot 10^{-5}$	1.10781
M_J	$5.139 \cdot 10^{-4}$	$1.869 \cdot 10^{-5}$	1.10883

For the Lee-Carter model, there are several methods to estimate $\hat{a}_x$, $\hat{b}_x$ and $\hat{k}_t$, namely, Singular Value Decomposition, Weighted Least Squares, and Maximum

[5]The assistance of M.-C. Koissi and B. Bejanov with parameter estimation and mortality forecasting is gratefully acknowledged.

Likelihood Estimation, also known as Poisson log-bilinear method. For technical details on the methods, see Brouhns et al. (2002), Brouhns et al. (2005), Lee (2000), Lee and Carter (1992), Wilmoth (1993), and for a useful and informative summary and comparison of the estimation methodologies, we refer the reader to Koissi et al. (2006). As pointed out in Koissi et al. (2006) and Wilmoth (1993), all three methods produce similar results, so out of practical considerations, we used Weighted Least Squares approach, thoroughly outlined in the latter article. The weight $w_{x,t}$ is the observed number of deaths in year t at age x. For the estimates of central death rates $m_{x,t}$ (3.7), one can use death rates for the corresponding year and age (Pollard (1973)). Then, similarly to the estimation for Gompertz and Makeham models, we seek to minimize

$$L = \sum_t \sum_x \left(\ln (m_{x,t}) - \hat{a}_x - \hat{b}_x \hat{k}_t \right)^2 \quad \text{for Lee-Carter.} \tag{3.44}$$

Again, we take partial derivatives with respect to $\hat{a}_x$, $\hat{b}_x$, $\hat{k}_t$, set them equal to zero and solve for the required parameters, following the methodology described in Wilmoth (1993).

Once $\hat{a}_x$, $\hat{b}_x$ and $\hat{k}_t$ are estimated, we forecast $\hat{k}_{t+i}$ for $i = 1$ (year 2000) to 26 (year 2025) using (3.9). Note that the drift d is found using Least Squares for the slope of the line $\hat{k}_{1999+t} = \hat{k}_{1999} + d \cdot t$, with the intercept taken to be the estimated value of the mortality index for the last year in the sample (Koissi et al. (2006)). Based on the obtained projections, we calculate forecasts for the central death rate $\hat{m}_{x,t}$ using (3.8), and then obtain survival probabilities ${}_1\hat{p}_{x,t}$ for years 2005 - 2025 based on the relation given in Pollard (1973):

$${}_1p_{x,t} = \frac{2 - m_{x,t}}{2 + m_{x,t}}. \tag{3.45}$$

At this point we find ${}_3\hat{p}_{x,2005}$, for example, as follows:

$${}_3\hat{p}_{x,2005} = {}_1\hat{p}_{x,2005} \cdot {}_1\hat{p}_{x+1,2006} \cdot {}_1\hat{p}_{x+2,2007}. \tag{3.46}$$

For the static models of Gompertz and Makeham, the formulas in (3.42) with the corresponding estimated parameters (Table 6) were used to calculate future survival probabilities ${}_T\hat{p}_x$ for $T = 3$, 10 and 20.

The actual and forecasted values of the mortality index $\hat{k}_t$ for the United States, Sweden and Japan are plotted in Figure 1. Numerical values of $\hat{a}_x$, $\hat{b}_x$ for ages 0 - 100 and $\hat{k}_t$ for years 1959 - 1999 are given in Tables 8 and 9 in Appendix 3. While being aware that there is plenty of room for improvement in the estimation of all parameters and potential for further study of errors in forecasts (not conducted here as the presented analysis is less statistical and more financial in nature), we believe that the numerical results below are reasonable and consistent. They illustrate the importance of selecting an appropriate model to assess and valuate risks inherent in equity-linked contracts in light of contemporary survival patterns, calling to attention the differences in risk management strategies when dealing with clients from varied backgrounds and contracts of short, medium and long duration.

3.6.2 Maximizing the probability of successful hedging

In this setting, the insurance firm selects the first approach of quantile hedging: the firm maximizes the probability of successful hedging given a limited initial capital. Suppose a 60-year old client approaches the firm with the intention of buying a contract that will allow him/her to receive the maximum of TSX/S&P fund value, currently at CAD 9246.70, and a guaranteed amount (based on the guaranteed rate of 7 percent) in 3 (10, 20) years respectively. The client must be alive to collect the payoff.

Since the company knows the client's age, it can estimate his/her survival probability for the corresponding duration based on the selected mortality model. Then the company will use this survival probability ${}_Tp_{60}$ in equation (3.17) to calculate the fair premium U_0 for the contract and quote the price to the client. Upon receipt of amount U_0, the firm will use equation (3.40) to find a^*, which is needed to calculate the maximal probability of successful hedging $1-\epsilon$ based on the form of the maximal set of successful hedging A^* (2.52). Once the company knows how much financial risk (ϵ) it carries, it can decide whether or not such risk profile is acceptable to its managers and shareholders. Table 7 gives sample values of the maximal probability of successful hedging, based on the three mortality models and contracts with 60-year old clients from the US, Sweden and Japan (abbreviated subscripts US, S, J).

Table 7: Probabilities of successful hedging (in percent) based on Gompertz (G), Makeham (M), Lee-Carter (LC) models

T	G_{US}	M_{US}	LC_{US}	G_S	M_S	LC_S	G_J	M_J	LC_J
3	98.2	98.2	98.5	98.7	98.7	99.0	98.6	98.5	99.2
10	94.1	94.1	95.7	95.5	95.5	97.1	95.1	95.0	98.2
20	81.5	81.6	88.9	83.8	83.7	91.7	82.2	82.2	95.6

As we can see from Table 7, Gompertz and Makeham give almost identical results for all countries and all contract durations. Lee-Carter consistently predicts higher probabilities of successful hedging than the other two models, with differences ranging from less than 1 percent for shorter contracts to between 7 (US, Sweden) and 13 (Japan) percent for longer contracts. Since the greatest differences between the models are observed for contracts of longer duration, we can conclude that the choice of a mortality model will affect insurance type products (which are long-term) more so than the short-term investment/wealth management solutions.

Also, insurance firms attracting Swedish and Japanese clients seem to be in a better position, as greater hedging successes are forecasted based on survival trends in these two countries than in the US (Table 7). To analyze this trend, consider

Figure 1: it shows that Japan and Sweden are expected to have lower mortality indices for the next 20 years than the US, meaning that Japanese and Swedish clients are likely to have higher survival probabilities in the next two decades than their US counterparts. Higher survival probabilities imply greater premiums collected from the sale of contracts (see (3.17)) and more initial capital available for hedging, leading to higher probabilities of successful hedging.

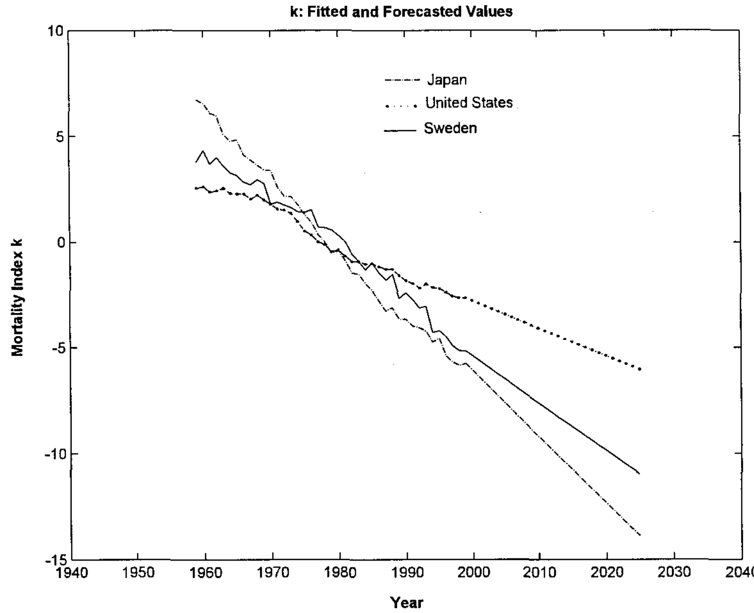

Figure 1: Comparison of mortality indices for USA, Sweden and Japan

Now, although the Lee-Carter model predicts the lowest mortality index values for Japan (Figure 1), Gompertz and Makeham indicate that Sweden should have lower expected mortality rates than Japan, based on the higher probabilities of successful hedging in Table 7. Moreover, the differences between Lee-Carter and Gompertz/Makeham models are observed in all three countries (Table 7). This is a potentially alarming sign in the following sense. As mentioned before, Gompertz- and Makeham-based models are most popular today, so the insurance industry may be relying on models which do not reflect future survival patterns accurately. As shown in Lee and Carter (1992), mortality index values fit past and current mortality patterns and survival expectations in the US fairly well (better than official estimates), so it is reasonable to expect that Lee-Carter approach will forecast future survival patterns successfully at least for the next couple of decades in the US, as well as other developed countries.

Therefore, if the insurance industry is, in fact, relying on Gompertz- and Makeham-type models, it may be overestimating its mortality forecasts, or, vice versa, underestimating future survival tendencies (this is also reflected in Figures 3, 4 and 5, where Lee-Carter survival probability forecasts for most ages are significantly greater than those of Gompertz and Makeham). This, in turn, may result in serious financing problems for insurance firms and significantly undervalued retirement costs in the three countries in consideration. These inferences agree with the concerns about the rising costs of mortality decline and the resulting economic implications of higher tax burdens on the working population, which has to supply tax money to finance pension payments for the retired portion of the population (see, for example, Bongaarts (2004), Koissi et al. (2006), Tuljapurkar et al. (2000), Wong-Fupuy and Haberman (2004)). This concern applies most to Japan. It appears that Sweden and the United States will be somewhat less affected by the choice of mortality model for insurance, economic and demographic considerations (compared to Japan), based on the smaller differences between probabilities of successful hedging in Table 7.

Let us note another obvious and potentially dangerous result for the insurance firm and discuss ways to manage this problem. Consider a 20-year contract offered to a 60-year old client. If the firm charges the client the fair price (as given by (3.40)), the maximal probability of successful hedging, given that the initial capital available for hedging is the premium paid by this client, is between 80 and 95 percent, depending on the country and the mortality model used (see Table 7). It is reasonable to conclude that very few insurance companies would find the level of default risk close to 20 percent acceptable. In this situation, the companies would be better off considering the other direction suggested by quantile hedging: fixing the probability of default first, and then looking at the ages of clients and drawing the corresponding risk management conclusions. This is the approach we examine next.

3.6.3 Fixing the probability of successful hedging

Suppose the insurance firm sells equity-linked life insurance contracts, but requires that the probability of default risk, ϵ, does not exceed some specified value. Based on this chosen risk profile, using equation (3.40) the company calculates the minimal amount of funds needed to hedge the payoff with the prescribed probability and, via one of the three mortality models, determines the ages of clients who would pay a fair price for the contract based on different default levels. The results are presented graphically in Figure 2.

As an example, consider a 10-year contract offered to Swedish clients under the maximum default risk of 1 percent. All models return close to 50 for the critical age, meaning that the premium received from a 50-year old client purchasing the unit-linked contract would guarantee the firm hedging the payoff successfully with probability 99 percent.

A natural question arises: what are the risk implications when clients below or above the age of 50 wish to purchase the same contract? In order to keep the level of default at 1 percent, the firm needs a particular amount of funds (as determined by (3.17)) to invest into a hedging strategy. Therefore, clients of all ages must pay this amount as the contract premium. Clients above 50 will be paying more than

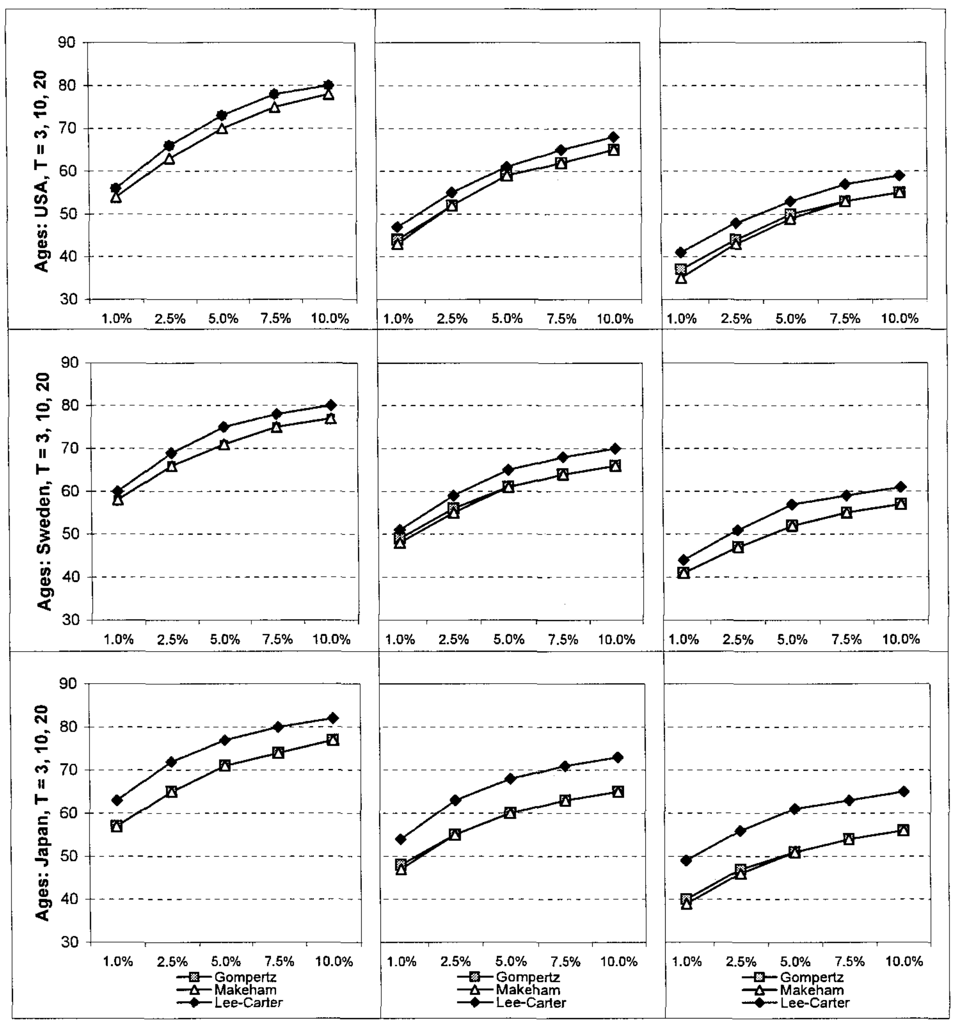

Figure 2: Critical ages of clients in USA, Sweden and Japan based on selected default risk levels

their fair share, as their survival probabilities are lower than those of a 50-year old. However, clients below 50 will enjoy purchasing the contracts at a discount, compared to the premium they should have paid.

From Figure 2, we see that the proportion of people purchasing contracts at a discount decreases with higher maturities (as critical ages decrease). This result is expected: since clients are more likely to die over longer periods of time, to keep its desired risk profile, the company can offer fewer discounts for long-term contracts as compared to short-term ones. Another noticeable pattern is the decrease in critical ages and the proportion of people receiving discounts with lower levels of default risk. This is also logical: the less financial risk the firm is willing to carry, the larger the proportion of people who have to pay higher premiums for the contract. Vice versa, the riskier the firm, the more discounts it can offer to its clients: the proportion of people under the critical age increases with higher ϵ.

Finally, in Figure 2 we also observe some features already discussed in the previous section: the greatest differences between Gompertz, Makeham and Lee-Carter are observed for longer contracts and for Japan. Again, this implies that firms offering long-term contracts are more sensitive to mortality changes, and the insurance industry in Japan should take extra care when selecting mortality models for actuarial and risk management purposes.

3.6.4 Additional observations

Now, let us study the differences between the forecasts of the three models more carefully. In particular, we focus on this question: why does Lee-Carter produce the greatest differences in ages for Japan but closer age predictions in Sweden and the US (Figure 2)? Moreover, why does the curve implied by the critical age values given by Lee-Carter seem to repeat the shapes of Gompertz and Makeham age curves (Figure 2), only higher? We feel that answers to these questions involve general patterns of mortality decline, such as the rectangularization of the survival curve, in developed countries.

Consider Figures 3, 4 and 5: they bring to attention parts of survival curves for the US, Sweden and Japan for the next 3, 10 and 20 years. The plotted survival probabilities were calculated based on the forecasted mortality index $\hat{k}_t$, estimated age-specific parameters $\hat{a}_x$ and $\hat{b}_x$ for Lee-Carter, and model-specific parameters for Gompertz and Makeham. First, we note that survival probability estimates for short future time periods (such as $T = 3$) are closer for all models, with similar values in the US and Sweden. However, for longer time periods ($T = 20$), Lee-Carter predicts significantly higher survival probabilities than Gompertz and Makeham in all countries, with differences being most pronounced in Japan. These trends explain the patterns brought to attention in the first question above.

Furthermore, the shapes of the survival curves in Figures 3, 4 and 5 reveal that the Lee-Carter approach is likely to be much more sensitive to the changes in mortality patterns in the three countries than Gompertz and Makeham. Higher survival probabilities for older ages and lower values for younger ages, as projected by the

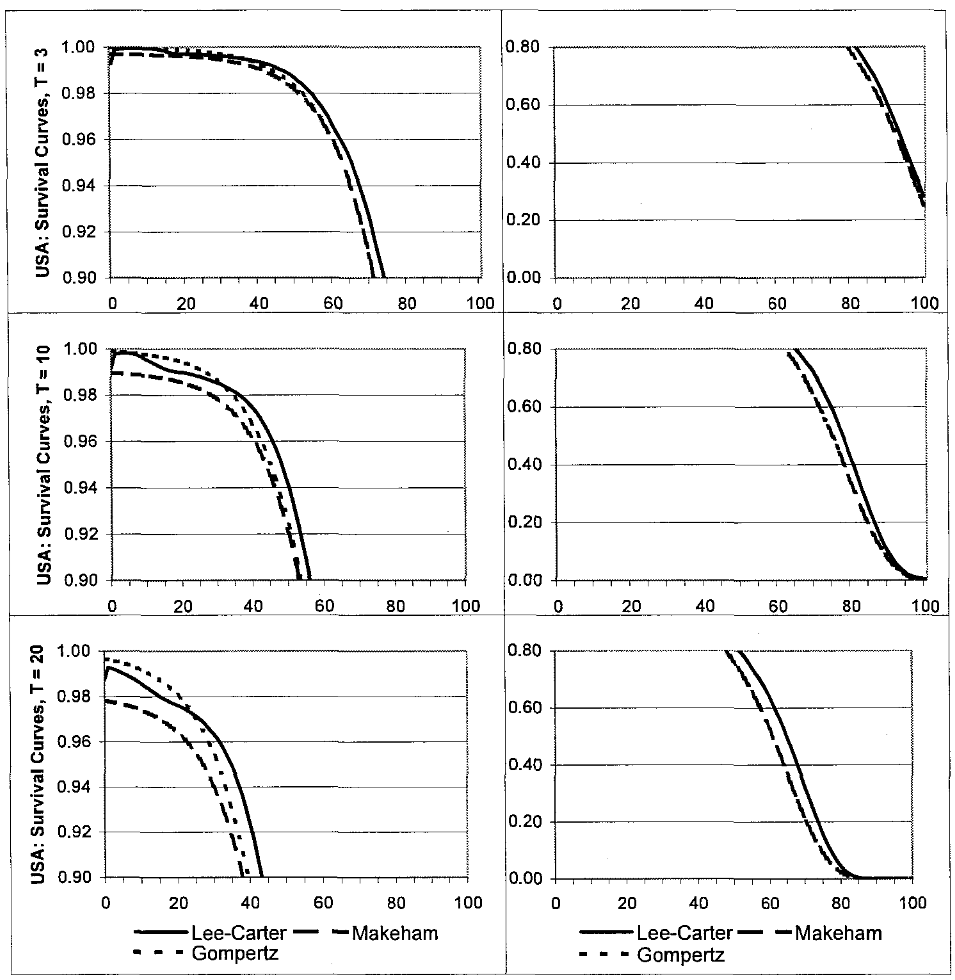

Figure 3: Survival probabilities, USA

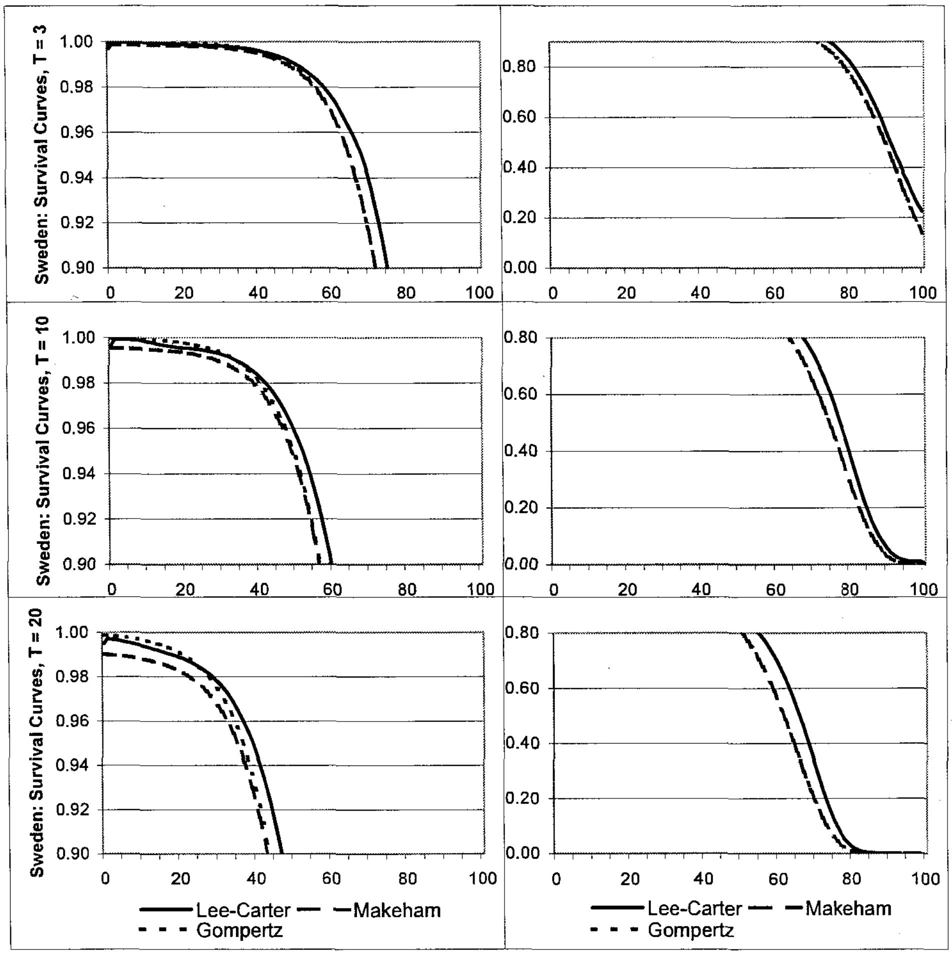

Figure 4: Survival probabilities, Sweden

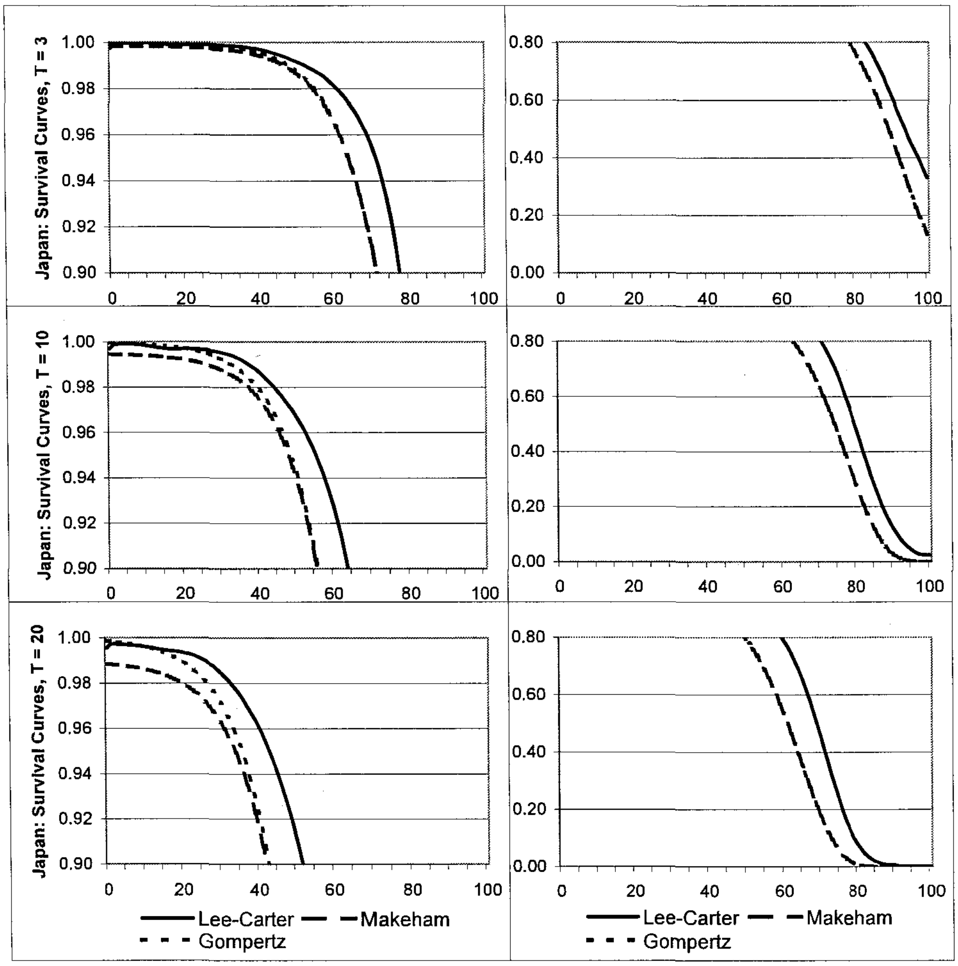

Figure 5: Survival probabilities, Japan

Lee-Carter methodology, reflect the two major trends in the evolution of mortality in these countries over the last century: deceleration of mortality at older ages, and rectangularization of the survival curve. The trends also help to answer the second question concerning the similarity in the shapes of implied critical age curves (Figure 2). The upward "shift" of the Lee-Carter age curve from those of Gompertz and Makeham is a consequence of the rectangularization pattern obvious in Figures 3, 4 and 5. Such observations confirm the conclusions made earlier in the discussion of the two approaches of quantile hedging, emphasizing the importance of choosing the appropriate mortality model that would capture accurately the development of mortality trends within the population in question.

Why do the three survival models produce smaller differences for Sweden and the United States than for Japan? Possibly, such results are explained by patterns of decline in the mortality index in the three countries from 1959 to 1999 (Figure 1). Wilmoth and Horiuchi (1999) point out that drastic changes in the shapes of survival curves in these countries occurred up until the 1950s, with the US characterized by the greatest degree of variability. After this time, Japan's mortality index experienced the most dramatic drop, compared to Sweden and the US. Perhaps the three mortality models are producing greater differences in the survival probability estimates for Japan (compare Figures 3, 4 and 5) in response to the larger change in mortality pattern in this country after the 1950s. Also, the fact that Lee-Carter predicts lower mortality in Japan than the other two countries (Figure 1) is consistent with the generally accepted idea that currently life expectancy in Japan is one of the highest in the world. However, a separate detailed study would be required to make concrete conclusions about the size of differences in the forecasts of the three models for the three countries in question.

3.7 Future direction

For future studies, there are several interesting directions worth exploring. First, a natural extension of the current setting is to consider other types of insurance products and their variations. In particular, one could study term insurance agreements (in which the payoff is paid upon the death of the insured client before maturity of the contract), and also contracts with extra benefits or provisions, such as reversionary or terminal bonuses, paid at maturity of the contract or upon the death of the insured client. This latter type of contracts would then incorporate components of both life and term insurance, with payoff to be received at maturity of the agreement, but bonus paid at or before maturity, at a random time. Another possibility is to study contracts involving several risky funds and payoff variations resulting from the behavior of the underlying asset prices. Such extensions would illustrate the use of our methodology in a more realistic setting, appealing to both researchers and practitioners in the actuarial and insurance fields.

Second, other mortality models should be incorporated into the analysis shown in this thesis. Here, classical mortality models of Gompertz and Makeham were compared with the newer method of Lee and Carter. As a next step, one could compare other recently developed mortality modelling methodologies to the Lee-Carter approach or the classical models. Of particular interest are the increasingly popular stochastic mortality models based on affine processes; typically, they have been used to model the term structure of interest rates (see, for instance, Vasicek (1977) or Cox et al. (1985)). However, today these affine processes are being studied and applied in actuarial context for modelling mortality as a stochastic process (see, for example, Biffis (2005), Luciano and Vigna (2005), and Dahl (2004)); it would be interesting to compare their performance and risk management implications with those of Lee-Carter. The study and the resulting analysis, conducted here for Sweden, Japan and the United States, may prove beneficial to other countries which already have strong markets for insurance products or are developing such markets. The inspection should reveal whether actuarial models used to describe the mortality of the client population in question are adequate for this purpose. Finally, it would be interesting to investigate why the three mortality models agree on forecasts more in some countries than in others; such study could provide insight about possible connections between fluctuations in mortality and the quality and proximity of forecasts given by the different mortality models.

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Appendix 1

We will now calculate the density Z of the risk-neutral measure P^*. We wish to express $Z_t = \frac{dP^*}{dP}|_{\mathcal{F}_t}$ as a stochastic exponent of some process N:

$$Z_t = \mathcal{E}(N_t).$$

Since there are two Wiener processes in our model, N_t has the form $N_t = \phi_1 \cdot W_t^1 + \phi_2 \cdot W_t^2$.

Let us represent B_t, S_t^1 and S_t^2 as stochastic exponents of processes h, H_t^1 and H_t^2 respectively. In our setup,

$$h = rt, \quad H_t^i = \mu_i t + \sigma_i W_t^i.$$

The general methodology for finding martingale measures (Melnikov and Shiryaev (1996), Melnikov et al. (2002)) states that the process

$$\varkappa_t^i(h, H, N) = H_t^i - h_t + N_t + \langle (h - H^i)^c, (h - N)^c \rangle_t$$

should be a martingale w. r. to P, from which the constants ϕ_1 and ϕ_2 are calculated.

For $\varkappa_1$ and $\varkappa_2$, we get the following:

$$\varkappa_t^1 = \mu_1 t + \sigma_1 W_t^1 - rt + \phi_1 W_t^1 + \phi_2 W_t^2 + \sigma_1 \phi_1 t + \sigma_1 \phi_2 \rho t,$$

$$\varkappa_t^2 = \mu_2 t + \sigma_2 W_t^2 - rt + \phi_1 W_t^1 + \phi_2 W_t^2 + \sigma_2 \phi_2 t + \sigma_2 \phi_1 \rho t.$$

To make these martingales, we must have

$$\mu_1 t - rt + \sigma_1 \phi_1 t + \sigma_1 \phi_2 \rho t = 0 \quad \text{and} \quad \mu_2 t - rt + \sigma_2 \phi_2 t + \sigma_2 \phi_1 \rho t = 0,$$

therefore,

$$\phi_1 = \frac{r(\sigma_2 - \sigma_1 \rho) + \rho \mu_2 \sigma_1 - \mu_1 \sigma_2}{\sigma_1 \sigma_2 (1 - \rho^2)} \quad \text{and} \quad \phi_2 = \frac{r(\sigma_1 - \sigma_2 \rho) + \rho \mu_1 \sigma_2 - \mu_2 \sigma_1}{\sigma_1 \sigma_2 (1 - \rho^2)}.$$

Returning to the stochastic exponent form, we get the following expression for Z:

$$\begin{aligned} Z_t &= \frac{dP^*}{dP}|_{\mathcal{F}_t} = \mathcal{E}(N_t) = \mathcal{E}(\phi_1 W_t^1 + \phi_2 W_t^2) \\ &= e^{\left(\phi_1 W_t^1 + \phi_2 W_t^2 - \frac{\sigma_\phi^2}{2} t\right)}, \\ \text{where} \quad \sigma_\phi^2 &= \phi_1^2 + \phi_2^2 + 2\rho\phi_1\phi_2. \end{aligned}$$

This is how we obtain the equations for Z and ϕ_i in (2.26) and (2.27).

Appendix 2

Here we prove the multi-asset theorem (2.193):
for $n+1$ normally distributed correlated random variables $x_i \sim N(\mu_i, \sigma_i^2)$ and $z \sim N(\mu_z, \sigma_z^2)$ with variance-covariance matrix $\mathbf{R}_{n+1}$

$$\mathbf{R}_{n+1} = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_1\sigma_z\rho_{1z} \\ \vdots & \ddots & \vdots \\ \sigma_1\sigma_z\rho_{1z} & \cdots & \sigma_z^2 \end{bmatrix}, \tag{3.47}$$

and given constants X_i, $i = 1, \ldots, n$,

$$\begin{aligned} E\left(e^{-z} I_{\{x_1 < X_1\}} \cdots I_{\{x_n < X_n\}}\right) &= e^{-\left(\mu_z - \frac{\sigma_z^2}{2}\right)} \cdot \Psi^n(\hat{X}_1, \ldots, \hat{X}_n), \\ \hat{X}_i &= \frac{X_i - \mu_i}{\sigma_i} + \sigma_z \rho_{iz}. \end{aligned} \tag{3.48}$$

Above, Ψ^n denotes n-dimensional cumulative normal distribution of correlated random variables with mean 0 and variance 1 (see discussion following (2.195)). Note that we make the standard assumption that all variance-covariance matrices are invertible.

First, let us introduce notation. As mentioned, $\mathbf{R}_{n+1}$ is the variance-covariance matrix for x_i, $i = 1, \ldots, n+1$, where z denotes x_{n+1}, so that $\mu_{n+1} = \mu_z$, $\sigma_{n+1} = \sigma_z$, and whenever used in powers, $i + z = i + n + 1$. The inverse of $\mathbf{R}_{n+1}$ is denoted

$$\mathbf{A}_{n+1} = \|a_{ij}\|_{n+1} = \mathbf{R}_{n+1}^{-1}. \tag{3.49}$$

We denote $\mathbf{R}_n$ the variance-covariance matrix for x_i, $i = 1, \ldots, n$:

$$\mathbf{R}_n = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_1\sigma_n\rho_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_1\sigma_n\rho_{1n} & \cdots & \sigma_n^2 \end{bmatrix}. \tag{3.50}$$

Its inverse is denoted

$$\mathbf{A}_n = \|a_{ij}\|_n = \mathbf{R}_n^{-1}. \tag{3.51}$$

We will also encounter these matrices in the proof:

$$\tilde{\mathbf{R}}_{n+1} = \begin{bmatrix} 1 & \cdots & \rho_{1z} \\ \vdots & \ddots & \vdots \\ \rho_{1z} & \cdots & 1 \end{bmatrix}, \tag{3.52}$$

its inverse

$$\tilde{\mathbf{A}}_{n+1} = \|\tilde{a}_{ij}\|_{n+1} = \tilde{\mathbf{R}}_{n+1}^{-1}; \tag{3.53}$$

as well as

$$\tilde{\mathbf{R}}_n = \begin{bmatrix} 1 & \cdots & \rho_{1n} \\ \vdots & \ddots & \vdots \\ \rho_{1n} & \cdots & 1 \end{bmatrix}, \tag{3.54}$$

and its inverse

$$\tilde{\mathbf{A}}_n = \|\tilde{a}_{ij}\|_n = \tilde{\mathbf{R}}_n^{-1}. \tag{3.55}$$

Next, let us recall some useful facts from linear algebra and apply them to our setting.
1) For any matrix $\mathbf{M}$, we have this relationship between the determinants of $\mathbf{M}$ and its inverse $\mathbf{M}^{-1}$:

$$|\mathbf{M}^{-1}| = \frac{1}{|\mathbf{M}|}. \tag{3.56}$$

2) Constants can be factored from determinants: for us,

$$|\mathbf{R}_{n+1}| = \sigma_1^2\sigma_2^2\cdots\sigma_n^2\sigma_z^2|\tilde{\mathbf{R}}_{n+1}|. \tag{3.57}$$

3) For $\mathbf{M}$, the entries of its inverse $\mathbf{M}^{-1} = \|m_{ij}^{inv}\|$ are given by

$$m_{ij}^{inv} = \frac{(-1)^{i+j}}{|\mathbf{M}|}\cdot|\mathbf{M}^{ji}|, \tag{3.58}$$

where $\mathbf{M}^{ji}$ is the matrix $\mathbf{M}$ with j^{th} row and i^{th} column removed.
4) In our setting, since $\mathbf{R}_{n+1}$ is symmetric, the entries of its (symmetric) inverse $\mathbf{A}_{n+1}$ satisfy

$$a_{ij} = \frac{(-1)^{i+j}}{|\mathbf{R}|}\cdot|\mathbf{R}^{ij}|; \tag{3.59}$$

similar formulas hold for the entries of $\tilde{\mathbf{A}}_{n+1}$, $\mathbf{A}_n$ and $\tilde{\mathbf{A}}_n$.
5) Based on points (2) and (4), we have for $\mathbf{A}_{n+1} = \|a_{ij}\|_{n+1}$

$$\begin{aligned} a_{ij} &= \frac{(-1)^{i+j}\sigma_1^2\cdots\sigma_i\sigma_j\cdots\sigma_n^2\sigma_z^2|\tilde{\mathbf{R}}_{n+1}^{ij}|}{\sigma_1^2\cdots\sigma_n^2\sigma_z^2|\tilde{\mathbf{R}}_{n+1}|} \\ &= \frac{(-1)^{i+j}|\tilde{\mathbf{R}}_{n+1}^{ij}|}{\sigma_i\sigma_j|\tilde{\mathbf{R}}_{n+1}|}; \end{aligned} \tag{3.60}$$

similar relation holds for $\mathbf{A}_n = \|a_{ij}\|_n$.

Now we begin the proof of the multi-asset theorem. Based on the expression for the multi-dimensional cumulative normal distribution (2.195), we see that the following equality has to be established:

$$
\begin{aligned}
&E\left(e^{-z} I_{\{x_1<X_1\}} \cdots I_{\{x_n<X_n\}}\right) = \\
&\frac{1}{(2\pi)^{(n+1)/2}|\mathbf{R}_{n+1}|^{1/2}} \int_{-\infty}^{X_1} \cdots \int_{-\infty}^{X_n} \int_{-\infty}^{\infty} \\
&e^{-z} e^{-\frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_{ij}(x_i-\mu_i)(x_j-\mu_j)} dz\, dx_1 \cdots dx_n = \\
&\frac{e^{-\left(\mu_z - \frac{\sigma_z^2}{2}\right)}}{(2\pi)^{n/2}|\tilde{\mathbf{R}}_n|^{1/2}} \int_{-\infty}^{\frac{X_1-\mu_1}{\sigma_1}+\sigma_z\rho_{1z}} \cdots \int_{-\infty}^{\frac{X_n-\mu_n}{\sigma_n}+\sigma_z\rho_{nz}} e^{-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{a}_{ij}\hat{x}_i\hat{x}_j} d\hat{x}_1 \cdots d\hat{x}_n.
\end{aligned}
\tag{3.61}
$$

Recall that $x_i \sim N(\mu_i, \sigma_i^2)$, $z \sim N(\mu_z, \sigma_z^2)$ with variance-covariance matrix $\mathbf{R}_{n+1}$ and its inverse $\mathbf{A}_{n+1} = \|a_{ij}\|_{n+1}$, where as $\hat{x}_i \sim N(0,1)$ with correlation matrix $\tilde{\mathbf{R}}_n$ and its inverse $\tilde{\mathbf{A}}_n = \|\tilde{a}_{ij}\|_n$.

To save space, let us write

$$
\int_{-\infty}^{\mathbf{a}} f d\mathbf{y} = \int_{-\infty}^{a_1} \cdots \int_{-\infty}^{a_n} f dy_1 \cdots dy_n. \tag{3.62}
$$

Now, let us simplify the expression in the exponent in (3.61) using the substitutions

$$
\begin{aligned}
\tilde{z} &= \frac{z-\mu_z}{\sigma_z} \quad \Rightarrow \quad dz = d\tilde{z}\sigma_z, \\
\tilde{x_i} &= \frac{x_i-\mu_i}{\sigma_i} \quad \Rightarrow \quad dx_i = d\tilde{x}_i\sigma_i,
\end{aligned}
\tag{3.63}
$$

with limits on the integral involving $\tilde{z}$ remaining $\pm\infty$ and the others changing from X_i to $\frac{X_i-\mu_i}{\sigma_i}$, and the simplification of entries of inverses stated earlier in point (5) (see (3.60)):

$$\begin{aligned}
&\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_{ij}(x_i-\mu_i)(x_j-\mu_j)\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_i-\mu_i)(x_j-\mu_j)+2\sum_{i=1}^{n} a_{zi}(z-\mu_z)(x_i-\mu_i)+a_{zz}(z-\mu_z)^2\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\sigma_i\sigma_j\tilde{x}_i\tilde{x}_j+2\sum_{i=1}^{n} a_{zi}\sigma_z\sigma_i\tilde{z}\tilde{x}_i+a_{zz}\sigma_z^2\tilde{z}^2\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(-1)^{i+j}|\tilde{\mathbf{R}}_{n+1}^{ij}|}{\sigma_i\sigma_j|\tilde{\mathbf{R}}_{n+1}|}\sigma_i\sigma_j\tilde{x}_i\tilde{x}_j\\
&\quad+2\sum_{i=1}^{n}\frac{(-1)^{i+n+1}|\tilde{\mathbf{R}}_{n+1}^{iz}|}{\sigma_z\sigma_i|\tilde{\mathbf{R}}_{n+1}|}\sigma_z\sigma_i\tilde{z}\tilde{x}_i+\frac{(-1)^{2(n+1)}|\tilde{\mathbf{R}}_n|}{\sigma_z^2|\tilde{\mathbf{R}}_{n+1}|}\sigma_z^2\tilde{z}^2\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(-1)^{i+j}|\tilde{\mathbf{R}}_{n+1}^{ij}|}{|\tilde{\mathbf{R}}_{n+1}|}\tilde{x}_i\tilde{x}_j\\
&\quad+2\sum_{i=1}^{n}\frac{(-1)^{i+n+1}|\tilde{\mathbf{R}}_{n+1}^{iz}|}{|\tilde{\mathbf{R}}_{n+1}|}\tilde{z}\tilde{x}_i+\frac{|\tilde{\mathbf{R}}_n|}{|\tilde{\mathbf{R}}_{n+1}|}\tilde{z}^2.
\end{aligned}\tag{3.64}$$

Now, with the constant $-\frac{1}{2}$ from the exponent, we complete the square for all terms in the above expression that contain $\tilde{z}$:

$$\begin{aligned}
&-\sigma_z\tilde{z}-\frac{1}{2}\frac{|\tilde{\mathbf{R}}_n|}{|\tilde{\mathbf{R}}_{n+1}|}\tilde{z}^2-\sum_{i=1}^{n}\frac{(-1)^{i+n+1}|\tilde{\mathbf{R}}_{n+1}^{iz}|}{|\tilde{\mathbf{R}}_{n+1}|}\tilde{z}\tilde{x}_i\\
&=-\frac{1}{2|\tilde{\mathbf{R}}_{n+1}|}\left[\tilde{z}|\tilde{\mathbf{R}}_n|^{1/2}+\frac{\sigma_z|\tilde{\mathbf{R}}_{n+1}|+\sum_{i=1}^{n}(-1)^{i+n+1}|\tilde{\mathbf{R}}_{n+1}^{iz}|\tilde{x}_i}{|\tilde{\mathbf{R}}_n|^{1/2}}\right]^2\\
&\quad+\frac{1}{2|\tilde{\mathbf{R}}_{n+1}|}\left[\frac{\sigma_z|\tilde{\mathbf{R}}_{n+1}|+\sum_{i=1}^{n}(-1)^{i+n+1}|\tilde{\mathbf{R}}_{n+1}^{iz}|\tilde{x}_i}{|\tilde{\mathbf{R}}_n|^{1/2}}\right]^2.
\end{aligned}\tag{3.65}$$

Continuing with our calculations, let us now make another substitution:

$$\begin{aligned}
\bar{z} &= \frac{1}{|\tilde{\mathbf{R}}_{n+1}|^{1/2}}\left[\tilde{z}|\tilde{\mathbf{R}}_n|^{1/2}+\frac{\sigma_z|\tilde{\mathbf{R}}_{n+1}|+\sum_{i=1}^{n}(-1)^{i+n+1}|\tilde{\mathbf{R}}_{n+1}^{iz}|\tilde{x}_i}{|\tilde{\mathbf{R}}_n|^{1/2}}\right],\\
d\bar{z} &= d\tilde{z}\frac{|\tilde{\mathbf{R}}_n|^{1/2}}{|\tilde{\mathbf{R}}_{n+1}|^{1/2}}.
\end{aligned}\tag{3.66}$$

Notice that the limits on the integral involving $\bar{z}$ remain $\pm\infty$.

Based on the two substitutions (3.63) and (3.66), point (1) about factoring constants from determinants (3.56), the simplification of the sum (3.64), and the completion of the square (3.65), the expectation in (3.61) takes form

$$\frac{1}{(2\pi)^{(n+1)/2}|\mathbf{R}_{n+1}|^{1/2}}\int_{-\infty}^{\mathbf{X}}\int_{-\infty}^{\infty} e^{-z}e^{-\frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}a_{ij}(x_i-\mu_i)(x_j-\mu_j)}dz\,d\mathbf{x}$$

$$= \frac{e^{-\mu_z}}{(2\pi)^{(n+1)/2}|\tilde{\mathbf{R}}_{n+1}|^{1/2}}\frac{\sqrt{2\pi}|\tilde{\mathbf{R}}_{n+1}|^{1/2}}{|\tilde{\mathbf{R}}_n|^{1/2}}\int_{-\infty}^{\frac{\mathbf{X}-\mu_\mathbf{X}}{\sigma_\mathbf{X}}} e^{-\frac{1}{2|\tilde{\mathbf{R}}_{n+1}|}}$$

$$\cdot e^{\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}|\tilde{\mathbf{R}}_{n+1}^{ij}|\tilde{x}_i\tilde{x}_j-\left(\frac{\sigma_z|\tilde{\mathbf{R}}_{n+1}|+\sum_{i=1}^{n}(-1)^{i+n+1}|\tilde{\mathbf{R}}_{n+1}^{iz}|\tilde{x}_i}{|\tilde{\mathbf{R}}_n|^{1/2}}\right)^2\right]}$$

$$\cdot\int_{-\infty}^{\infty}\frac{e^{-\frac{\tilde{z}^2}{2}}}{\sqrt{2\pi}}d\tilde{z}\,d\tilde{\mathbf{x}}$$

$$= \frac{e^{-\mu_z}}{(2\pi)^{n/2}|\tilde{\mathbf{R}}_n|^{1/2}}\int_{-\infty}^{\frac{\mathbf{X}-\mu_\mathbf{X}}{\sigma_\mathbf{X}}}$$

$$e^{-\frac{1}{2|\tilde{\mathbf{R}}_{n+1}|}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}|\tilde{\mathbf{R}}_{n+1}^{ij}|\tilde{x}_i\tilde{x}_j-\left(\frac{\sigma_z|\tilde{\mathbf{R}}_{n+1}|+\sum_{i=1}^{n}(-1)^{i+n+1}|\tilde{\mathbf{R}}_{n+1}^{iz}|\tilde{x}_i}{|\tilde{\mathbf{R}}_n|^{1/2}}\right)^2\right]}d\tilde{\mathbf{x}}. \tag{3.67}$$

For the next step, let us represent the expression in the exponent as

$$-\frac{1}{2}\cdot J,$$

with J determined as follows:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(-1)^{i+j}|\tilde{\mathbf{R}}_{n+1}^{ij}|}{|\tilde{\mathbf{R}}_{n+1}|}\tilde{x}_i\tilde{x}_j-\frac{1}{|\tilde{\mathbf{R}}_{n+1}|}\left(\frac{\sigma_z|\tilde{\mathbf{R}}_{n+1}|+\sum_{i=1}^{n}(-1)^{i+n+1}|\tilde{\mathbf{R}}_{n+1}^{iz}|\tilde{x}_i}{|\tilde{\mathbf{R}}_n|^{1/2}}\right)^2$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}\left(\frac{|\tilde{\mathbf{R}}_{n+1}^{ij}||\tilde{\mathbf{R}}_n|-|\tilde{\mathbf{R}}_{n+1}^{iz}||\tilde{\mathbf{R}}_{n+1}^{zj}|}{|\tilde{\mathbf{R}}_n||\tilde{\mathbf{R}}_{n+1}|}\right)\tilde{x}_i\tilde{x}_j$$

$$-2\sigma_z\sum_{i=1}^{n}(-1)^{i+n+1}\frac{|\tilde{\mathbf{R}}_{n+1}^{iz}|}{|\tilde{\mathbf{R}}_n|}\tilde{x}_i-\sigma_z^2\frac{|\tilde{\mathbf{R}}_{n+1}|}{|\tilde{\mathbf{R}}_n|}=J. \tag{3.68}$$

Also, let us make notational simplifications

$$r_{ij}=\frac{(-1)^{i+j}}{|\tilde{\mathbf{R}}_n|}\left(\frac{|\tilde{\mathbf{R}}_{n+1}^{ij}||\tilde{\mathbf{R}}_n|-|\tilde{\mathbf{R}}_{n+1}^{iz}||\tilde{\mathbf{R}}_{n+1}^{zj}|}{|\tilde{\mathbf{R}}_{n+1}|}\right) \tag{3.69}$$

and

$$s_i=(-1)^{i+n}\sigma_z\frac{|\tilde{\mathbf{R}}_{n+1}^{iz}|}{|\tilde{\mathbf{R}}_n|}. \tag{3.70}$$

Based on (3.68), (3.69) and (3.70), the expectation in (3.67) becomes

$$\frac{e^{-\mu_z}}{(2\pi)^{n/2}|\mathbf{R}_n|^{1/2}}\int_{-\infty}^{\frac{\mathbf{X}-\mu_{\mathbf{X}}}{\sigma_{\mathbf{X}}}} e^{-\frac{1}{2|\tilde{\mathbf{R}}_{n+1}|}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}|\tilde{\mathbf{R}}_{n+1}^{ij}|\tilde{x}_i\tilde{x}_j-\left(\frac{\sigma_z|\tilde{\mathbf{R}}_{n+1}|+\sum_{i=1}^{n}(-1)^{i+n+1}|\tilde{\mathbf{R}}_{n+1}^{iz}|\tilde{x}_i}{|\tilde{\mathbf{R}}_n|^{1/2}}\right)^2\right]} d\tilde{\mathbf{x}}$$

$$= \frac{e^{-\mu_z}}{(2\pi)^{n/2}|\mathbf{R}_n|^{1/2}}\int_{-\infty}^{\frac{\mathbf{X}-\mu_{\mathbf{X}}}{\sigma_{\mathbf{X}}}} e^{-\frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}\tilde{x}_i\tilde{x}_j+2\sum_{i=1}^{n} s_i\tilde{x}_i-\sigma_z^2\frac{|\tilde{\mathbf{R}}_{n+1}|}{|\tilde{\mathbf{R}}_n|}\right)} d\tilde{\mathbf{x}}. \quad (3.71)$$

Now consider the original equality we are trying to establish, given in (3.61): using this substitution

$$\hat{x}_i = \tilde{x}_i + \sigma_z\rho_{iz}, \quad (3.72)$$

we can rewrite the last expression in the equality as shown:

$$\frac{e^{-\left(\mu_z-\frac{\sigma_z^2}{2}\right)}}{(2\pi)^{n/2}|\tilde{\mathbf{R}}_n|^{1/2}}\int_{-\infty}^{\frac{\mathbf{X}-\mu_{\mathbf{X}}}{\sigma_{\mathbf{X}}}+\sigma_{\mathbf{z}}\rho_{\mathbf{xz}}} e^{-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{a}_{ij}\hat{x}_i\hat{x}_j} d\hat{\mathbf{x}}$$

$$= \frac{e^{-\left(\mu_z-\frac{\sigma_z^2}{2}\right)}}{(2\pi)^{n/2}|\tilde{\mathbf{R}}_n|^{1/2}}\int_{-\infty}^{\frac{\mathbf{X}-\mu_{\mathbf{X}}}{\sigma_{\mathbf{X}}}} e^{-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{a}_{ij}(\tilde{x}_i+\sigma_z\rho_{iz})(\tilde{x}_j+\sigma_z\rho_{jz})} d\tilde{\mathbf{x}}, \quad (3.73)$$

with $\frac{\mathbf{X}-\mu_{\mathbf{X}}}{\sigma_{\mathbf{X}}}+\sigma_{\mathbf{z}}\rho_{\mathbf{xz}}$ in the upper limit referring to each individual upper limit $\frac{X_i-\mu_i}{\sigma_i}+\sigma_z\rho_{iz}$.

At this point, compare (3.71) and (3.73): if we can show that the expressions in the exponents are equal, then we will have completed the proof of the theorem. So, let us proceed with this idea in mind.

First, we expand the exponent of (3.71):

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}\tilde{x}_i\tilde{x}_j + 2\sum_{i=1}^{n} s_i\tilde{x}_i - \sigma_z^2\frac{|\tilde{\mathbf{R}}_{n+1}|}{|\tilde{\mathbf{R}}_n|} \\
&= r_{11}\tilde{x}_1^2 + r_{22}\tilde{x}_2^2 + \cdots + r_{nn}\tilde{x}_n^2 \\
&\qquad +2r_{12}\tilde{x}_1\tilde{x}_2 + 2r_{13}\tilde{x}_1\tilde{x}_3 + \cdots + 2r_{1n}\tilde{x}_1\tilde{x}_n \\
&\qquad\quad +2r_{23}\tilde{x}_2\tilde{x}_3 + \cdots + 2r_{2n}\tilde{x}_2\tilde{x}_n \\
&\qquad\qquad \vdots \\
&\qquad\qquad +2r_{(n-1)n}\tilde{x}_{n-1}\tilde{x}_n \\
&\qquad +2s_1\tilde{x}_1 + 2s_2\tilde{x}_2 + \cdots + 2s_n\tilde{x}_n - \sigma_z^2\frac{|\tilde{\mathbf{R}}_{n+1}|}{|\tilde{\mathbf{R}}_n|}.
\end{aligned} \quad (3.74)$$

Next, let us expand the exponent of (3.73): after some algebraic manipulations, we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{a}_{ij}(\tilde{x}_i+\sigma_z\rho_{iz})(\tilde{x}_j+\sigma_z\rho_{jz})$$
$$=\tilde{a}_{11}\tilde{x}_1^2+\tilde{a}_{22}\tilde{x}_2^2+\cdots+\tilde{a}_{nn}\tilde{x}_n^2$$
$$+2\tilde{a}_{12}\tilde{x}_1\tilde{x}_2+2\tilde{a}_{13}\tilde{x}_1\tilde{x}_3+\cdots+2\tilde{a}_{1n}\tilde{x}_1\tilde{x}_n$$
$$+2\tilde{a}_{23}\tilde{x}_2\tilde{x}_3+\cdots+2\tilde{a}_{2n}\tilde{x}_2\tilde{x}_n$$
$$\vdots$$
$$+2\tilde{a}_{(n-1)n}\tilde{x}_{n-1}\tilde{x}_n$$
$$+2\tilde{x}_1\sum_{j=1}^{n}\tilde{a}_{1j}\sigma_z\rho_{jz}+2\tilde{x}_2\sum_{j=1}^{n}\tilde{a}_{2j}\sigma_z\rho_{jz}+\cdots+2\tilde{x}_n\sum_{j=1}^{n}\tilde{a}_{nj}\sigma_z\rho_{jz}$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{a}_{ij}\sigma_z^2\rho_{iz}\rho_{jz}. \tag{3.75}$$

Comparing the terms in the two expansions, we see that we need to show three things:

1. $r_{ij}=\tilde{a}_{ij}$,
2. $s_i=\sum_{j=1}^{n}\tilde{a}_{ij}\sigma_z\rho_{jz}$, and
3. $\frac{\sigma_z^2}{2}\frac{|\tilde{\mathbf{R}}_{n+1}|}{|\tilde{\mathbf{R}}_n|}=\frac{\sigma_z^2}{2}-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{a}_{ij}\sigma_z^2\rho_{iz}\rho_{jz}$.

Before we prove these, we establish the following relations about determinants:

$$|\tilde{\mathbf{R}}_{n+1}^{zj}|=\sum_{i=1}^{n}(-1)^{i+n}\rho_{iz}|\tilde{\mathbf{R}}_n^{ij}| \tag{3.76}$$

and

$$|\tilde{\mathbf{R}}_n|=|\tilde{\mathbf{R}}_{n+1}|-\sum_{i=1}^{n}(-1)^{i+n+1}\rho_{iz}|\tilde{\mathbf{R}}_{n+1}^{iz}|. \tag{3.77}$$

Now we are ready to prove relations 1, 2 and 3. First, consider 2: using the expression for entries of inverses (3.59), we have

$$\sum_{j=1}^{n}\tilde{a}_{ij}\sigma_z\rho_{jz}=\sum_{j=1}^{n}\frac{(-1)^{i+j}|\tilde{\mathbf{R}}_n^{ij}|}{|\tilde{\mathbf{R}}_n|}\sigma_z\rho_{jz}. \tag{3.78}$$

But from the definition of s_i in (3.70) and relation (3.76), we obtain

$$\begin{aligned} s_i &= (-1)^{i+n}\sigma_z \frac{|\tilde{\mathbf{R}}_{n+1}^{iz}|}{|\tilde{\mathbf{R}}_n|} \\ &= (-1)^{i+n}\sigma_z \sum_{j=1}^{n}(-1)^{j+n}\rho_{jz}\frac{|\tilde{\mathbf{R}}_n^{ij}|}{|\tilde{\mathbf{R}}_n|} \\ &= \sigma_z \sum_{j=1}^{n}(-1)^{i+j}\rho_{jz}\frac{|\tilde{\mathbf{R}}_n^{ij}|}{|\tilde{\mathbf{R}}_n|}. \end{aligned} \tag{3.79}$$

Comparing (3.78) and (3.79), we see that relation 2 is proved.

Next, let us look at 3: we will prove this equation using (3.77) and (3.76). On one hand,

$$\begin{aligned} \frac{\sigma_z^2}{2}\frac{|\tilde{\mathbf{R}}_{n+1}|}{|\tilde{\mathbf{R}}_n|} &= \frac{\sigma_z^2}{2}\left(1+\sum_{i=1}^{n}(-1)^{i+n+1}\rho_{iz}\frac{|\tilde{\mathbf{R}}_{n+1}^{iz}|}{|\tilde{\mathbf{R}}_n|}\right) \\ &= \frac{\sigma_z^2}{2}\left(1+\sum_{i=1}^{n}(-1)^{i+n+1}\rho_{iz}\sum_{j=1}^{n}(-1)^{j+n}\rho_{jz}\frac{|\tilde{\mathbf{R}}_n^{ij}|}{|\tilde{\mathbf{R}}_n|}\right). \end{aligned} \tag{3.80}$$

On the other hand,

$$\frac{\sigma_z^2}{2} - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{a}_{ij}\sigma_z^2\rho_{iz}\rho_{jz} = \frac{\sigma_z^2}{2} + \frac{\sigma_z^2}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(-1)^{i+j+1}|\tilde{\mathbf{R}}_n^{ij}|}{|\tilde{\mathbf{R}}_n|}\rho_{iz}\rho_{jz}. \tag{3.81}$$

Comparing (3.80) and (3.81), we see that relation 3 is also proved.

Finally, note that to prove 1, we need to show that

$$\begin{aligned} \frac{(-1)^{i+j}}{|\tilde{\mathbf{R}}_n|}\left(\frac{|\tilde{\mathbf{R}}_{n+1}^{ij}||\tilde{\mathbf{R}}_n| - |\tilde{\mathbf{R}}_{n+1}^{iz}||\tilde{\mathbf{R}}_{n+1}^{zj}|}{|\tilde{\mathbf{R}}_{n+1}|}\right) &= \frac{(-1)^{i+j}|\tilde{\mathbf{R}}_n^{ij}|}{|\tilde{\mathbf{R}}_n|} \\ \Leftrightarrow \quad |\tilde{\mathbf{R}}_{n+1}^{ij}||\tilde{\mathbf{R}}_n| - |\tilde{\mathbf{R}}_{n+1}^{iz}||\tilde{\mathbf{R}}_{n+1}^{zj}| &= |\tilde{\mathbf{R}}_{n+1}||\tilde{\mathbf{R}}_n^{ij}|. \end{aligned} \tag{3.82}$$

To facilitate the proof of relation 1, we express all matrices in the last identity above in terms of $\tilde{\mathbf{R}}_{n+1}$ with necessary rows/columns removed. Thus we must prove

$$|\tilde{\mathbf{R}}_{n+1}^{ij}||\tilde{\mathbf{R}}_{n+1}^{(n+1)(n+1)}| - |\tilde{\mathbf{R}}_{n+1}^{i(n+1)}||\tilde{\mathbf{R}}_{n+1}^{(n+1)j}| = |\tilde{\mathbf{R}}_{n+1}||\tilde{\mathbf{R}}_{n+1}^{ij,(n+1)(n+1)}|. \tag{3.83}$$

Note that

$$\tilde{\mathbf{R}}_n^{ij} = \tilde{\mathbf{R}}_{n+1}^{ij,(n+1)(n+1)} \tag{3.84}$$

is the matrix $\tilde{\mathbf{R}}_{n+1}$ with rows $i, n+1$ and columns $j, n+1$ removed.

Instead of proving (3.83) directly, we will prove this more general result:

for $n \geq 3$ and any choice of $i, j, k, l \leq n$, the following holds for any $n \times n$ matrix $\mathbf{M}$

$$|\mathbf{M}_n^{ij}||\mathbf{M}_n^{lk}| - |\mathbf{M}_n^{lj}||\mathbf{M}_n^{ik}| = \Gamma_{il}\Gamma_{jk}|\mathbf{M}_n^{ij,lk}||\mathbf{M}_n|. \qquad (3.85)$$

Above, $\mathbf{M}_n^{ij,lk}$ is $\mathbf{M}_n$ with rows i, l and columns j, k removed. Also, the function Γ_{ab} is defined as

$$\Gamma_{ab} = \begin{cases} 1 & \text{if } b > a \\ 0 & \text{if } b = a \\ -1 & \text{if } b < a. \end{cases} \qquad (3.86)$$

We will prove (3.85) by induction on n.

Before we begin the proof, note that if $l = i$ or $k = j$, then (3.85) holds trivially. Next, without loss of generality, we assume that

$$l > i \quad \text{and} \quad k > j. \qquad (3.87)$$

We can make such assumption, because if i were smaller than l (or $k < j$), then we could rename the variables i, l (or j, k), which would introduce a negative sign on both sides of equation (3.85).

Also, we will prove that (3.85) is valid for every matrix $\mathbf{M}_n$ if and only if the following formula holds for every $\mathbf{M}_n$ also:

$$|\mathbf{M}_n^{ij}||\mathbf{M}_n^{nn}| - |\mathbf{M}_n^{nj}||\mathbf{M}_n^{in}| = |\mathbf{M}_n^{ij,nn}||\mathbf{M}_n|. \qquad (3.88)$$

Proof.

$\Rightarrow$ Assume (3.85) holds. Let $k, l = n, n$. Then (3.88) holds as well.

$\Leftarrow$ Assume (3.88) is true. Construct a new matrix $\bar{\mathbf{M}}_n$ by moving row l and column k in matrix $\mathbf{M}_n$ to positions n, n. Then

$$\begin{aligned} |\mathbf{M}_n^{ij}| &= (-1)^{l+k}|\bar{\mathbf{M}}_n^{ij}|, \\ |\mathbf{M}_n^{lk}| &= |\bar{\mathbf{M}}_n^{nn}|, \\ |\mathbf{M}_n^{lj}| &= (-1)^{n-k}|\bar{\mathbf{M}}_n^{nj}|, \\ |\mathbf{M}_n^{ik}| &= (-1)^{n-l}|\bar{\mathbf{M}}_n^{in}|, \\ |\mathbf{M}_n^{ij,lk}| &= |\bar{\mathbf{M}}_n^{ij,nn}|, \\ |\mathbf{M}_n| &= (-1)^{l+k}|\bar{\mathbf{M}}_n|. \end{aligned} \qquad (3.89)$$

Based on the equations above, (3.85) turns into

$$|\bar{\mathbf{M}}_n^{ij}||\bar{\mathbf{M}}_n^{nn}| - |\bar{\mathbf{M}}_n^{nj}||\bar{\mathbf{M}}_n^{in}| = |\bar{\mathbf{M}}_n^{ij,nn}||\bar{\mathbf{M}}_n|. \qquad (3.90)$$

And, since (3.88) holds for any matrix by assumption, in particular, it holds for $\bar{\mathbf{M}}$. Therefore, (3.90) implies that (3.85) is true, which completes the proof.

Now we proceed to prove (3.85) by induction as follows. For the first step, we show that (3.88) holds for $n = 3$, implying that (3.85) also holds for $n = 3$. For the inductive hypothesis, we assume (3.85) holds for some n and show that (3.88) is true for $n+1$, which is equivalent to (3.85) being true for $n+1$.

Step 1. Here we have $n=3$ and $i,j=1,1,\ 1,2$ or $2,2$ (the case for $i,j=2,1$ is equivalent to $i,j=1,2$ by transposition). For each of these possibilities, (3.88) is shown to be true by direct verification (omitted here).

Step 2. Assume that (3.85) holds. That is,

$$|\mathbf{M}_n^{ij}||\mathbf{M}_n^{lk}| - |\mathbf{M}_n^{lj}||\mathbf{M}_n^{ik}| = \Gamma_{il}\Gamma_{jk}|\mathbf{M}_n^{ij,lk}||\mathbf{M}_n| \tag{3.91}$$

is true for every matrix $\mathbf{M}$ and some n, and for all possible values of $i,j,k,l \leq n$. Using this assumption, we will prove that (3.88) holds for $n+1$.

We want to show that for any $i,j < n+1$,

$$|\mathbf{M}_{n+1}^{ij}||\mathbf{M}_{n+1}^{(n+1)(n+1)}| - |\mathbf{M}_{n+1}^{(n+1)j}||\mathbf{M}_{n+1}^{i(n+1)}| = |\mathbf{M}_{n+1}^{ij,(n+1)(n+1)}||\mathbf{M}_{n+1}|. \tag{3.92}$$

First, we expand the determinants above by row/column $n+1$ whenever possible. Using notational simplifications

$$\begin{aligned} \mathbf{M}_n^{ij,lk} &= \mathbf{M}_{n+1}^{ij,l(n+1),(n+1)k}, \\ \mathbf{M}_n^{ij} &= \mathbf{M}_{n+1}^{i(n+1),(n+1)j}, \\ \mathbf{M}_n &= \mathbf{M}_{n+1}^{(n+1)(n+1)}, \end{aligned} \tag{3.93}$$

we can write

$$\begin{aligned} |\mathbf{M}_{n+1}^{ij}| &= \sum_{k=1}^{j-1}\sum_{l=1}^{i-1}(-1)^{l+k+1} m_{l(n+1)} m_{(n+1)k} |\mathbf{M}_n^{ij,lk}| \\ &\quad + \sum_{k=1}^{j-1}\sum_{l=i+1}^{n}(-1)^{l+k} m_{l(n+1)} m_{(n+1)k} |\mathbf{M}_n^{ij,lk}| \\ &\quad + \sum_{k=j+1}^{n}\sum_{l=1}^{i-1}(-1)^{l+k} m_{l(n+1)} m_{(n+1)k} |\mathbf{M}_n^{ij,lk}| \\ &\quad + \sum_{k=j+1}^{n}\sum_{l=i+1}^{n}(-1)^{l+k+1} m_{l(n+1)} m_{(n+1)k} |\mathbf{M}_n^{ij,lk}| + m_{(n+1)(n+1)}|\mathbf{M}_n^{ij}|, \\ |\mathbf{M}_{n+1}^{(n+1)j}| &= \sum_{l=1}^{n}(-1)^{l+n} m_{l(n+1)} |\mathbf{M}_n^{lj}|, \\ |\mathbf{M}_{n+1}^{i(n+1)}| &= \sum_{k=1}^{n}(-1)^{k+n} m_{(n+1)k} |\mathbf{M}_n^{ik}|, \\ |\mathbf{M}_{n+1}| &= m_{(n+1)(n+1)}|\mathbf{M}_n| + \sum_{l=1}^{n}\sum_{k=1}^{n}(-1)^{l+k+1} m_{l(n+1)} m_{(n+1)k} |\mathbf{M}_n^{lk}|. \end{aligned} \tag{3.94}$$

Here, m_{ab} denotes entry in row a and column b in $\mathbf{M}_{n+1}$.

Now we utilize the new notation (3.93) and the expansions for the determinants (3.94) to rewrite (3.88) for $n+1$:

$$\begin{aligned}
&\left[\sum_{k=1}^{j-1}\sum_{l=1}^{i-1}(-1)^{l+k+1}m_{l(n+1)}m_{(n+1)k}|\mathbf{M}_n^{ij,lk}|\right.\\
&+\sum_{k=1}^{j-1}\sum_{l=i+1}^{n}(-1)^{l+k}m_{l(n+1)}m_{(n+1)k}|\mathbf{M}_n^{ij,lk}|\\
&+\sum_{k=j+1}^{n}\sum_{l=1}^{i-1}(-1)^{l+k}m_{l(n+1)}m_{(n+1)k}|\mathbf{M}_n^{ij,lk}|\\
&+\sum_{k=j+1}^{n}\sum_{l=i+1}^{n}(-1)^{l+k+1}m_{l(n+1)}m_{(n+1)k}|\mathbf{M}_n^{ij,lk}|\\
&\left.+m_{(n+1)(n+1)}|\mathbf{M}_n^{ij}|\right]\cdot|\mathbf{M}_n|-\sum_{l=1}^{n}\sum_{k=1}^{n}(-1)^{l+k}m_{l(n+1)}m_{(n+1)k}|\mathbf{M}_n^{lj}||\mathbf{M}_n^{ik}|\\
&=|\mathbf{M}_n^{ij}||\mathbf{M}_n|m_{(n+1)(n+1)}+|\mathbf{M}_n^{ij}|\cdot\sum_{l=1}^{n}\sum_{k=1}^{n}(-1)^{l+k+1}m_{l(n+1)}m_{(n+1)k}|\mathbf{M}_n^{lk}|.
\end{aligned} \tag{3.95}$$

Note that using the Γ function (3.86), we can simplify the sums on the left-hand side of the equation above to get the following:

$$\begin{aligned}
&\left[\sum_{k=1}^{n}\sum_{l=1}^{n}(-1)^{l+k+1}\Gamma_{il}\Gamma_{jk}m_{l(n+1)}m_{(n+1)k}|\mathbf{M}_n^{ij,lk}|\right]\cdot|\mathbf{M}_n| \\
&=\sum_{l=1}^{n}\sum_{k=1}^{n}(-1)^{l+k+1}m_{l(n+1)}m_{(n+1)k}\left(|\mathbf{M}_n^{ij}||\mathbf{M}_n^{lk}|-|\mathbf{M}_n^{lj}||\mathbf{M}_n^{ik}|\right).
\end{aligned} \tag{3.96}$$

Now, comparing the terms inside the sums, we see that they are equal by the induction hypothesis.

Therefore (3.88) holds for $n+1$, thus (3.85), which is equivalent to (3.88) (see (3.90)), holds for $n+1$ also. And, the general formula (3.85) implies that our particular case for $\tilde{\mathbf{R}}_{n+1}$ (3.83) is true as well. This, in turn, shows the equality of the last set of coefficients (relation 1), and completes the proof of the multi-asset theorem.

Appendix 3

Here we include tables with values of the estimated parameters $\hat{a}_x, \hat{b}_x$ and $\hat{k}_t$ for the Lee-Carter model.

Table 8: Estimated parameters $\hat{a}_x$, $\hat{b}_x$ for the Lee-Carter model

	$\hat{a}_x$			$\hat{b}_x$		
Age	USA	Sweden	Japan	USA	Sweden	Japan
0	-4.2672	-4.8263	-4.7669	0.2365	0.1654	0.1875
1	-6.9161	-7.4971	-6.7308	0.1904	0.1480	0.1488
2	-7.3373	-7.9498	-7.2465	0.1732	0.1741	0.1685
3	-7.5993	-8.0820	-7.4993	0.1750	0.1759	0.1712
4	-7.8014	-8.1322	-7.6983	0.1870	0.1706	0.1756
5	-7.9258	-8.2192	-7.8348	0.1909	0.1938	0.1665
6	-8.0263	-8.2954	-7.9452	0.1837	0.1716	0.1507
7	-8.1122	-8.3972	-8.0785	0.1764	0.1740	0.1428
8	-8.1667	-8.4612	-8.2632	0.1544	0.1470	0.1487
9	-8.2357	-8.4903	-8.3794	0.1609	0.1398	0.1446
10	-8.2596	-8.4868	-8.4715	0.1541	0.1278	0.1425
11	-8.2230	-8.5492	-8.4920	0.1449	0.1140	0.1341
12	-8.1152	-8.5226	-8.4405	0.1282	0.1164	0.1257
13	-7.9540	-8.4235	-8.3523	0.1142	0.1117	0.1144
14	-7.7231	-8.2373	-8.2596	0.0923	0.0944	0.1081
15	-7.4651	-7.8444	-8.0411	0.0765	0.0908	0.1005
16	-7.1425	-7.7533	-7.5888	0.0493	0.1011	0.0739
17	-6.9551	-7.6193	-7.4273	0.0532	0.0846	0.0771
18	-6.7716	-7.3407	-7.3279	0.0480	0.0764	0.0754
19	-6.7427	-7.3220	-7.2363	0.0606	0.0733	0.0810
20	-6.7392	-7.2994	-7.2045	0.0596	0.0635	0.0888
21	-6.6911	-7.3081	-7.1989	0.0589	0.0580	0.0984
22	-6.6933	-7.3003	-7.1826	0.0611	0.0706	0.1079
23	-6.7014	-7.2622	-7.1661	0.0533	0.0563	0.1104
24	-6.7166	-7.2991	-7.1618	0.0564	0.0671	0.1173
25	-6.7022	-7.2738	-7.1383	0.0529	0.0521	0.1154
26	-6.7008	-7.2225	-7.1337	0.0454	0.0575	0.1140
27	-6.6778	-7.1836	-7.1232	0.0397	0.0506	0.1164
28	-6.6369	-7.1744	-7.0950	0.0286	0.0548	0.1134
29	-6.6427	-7.1161	-7.0535	0.0425	0.0679	0.1128
30	-6.5857	-7.0597	-7.0289	0.0357	0.0577	0.1104
31	-6.5416	-7.0370	-6.9900	0.0256	0.0545	0.1100
32	-6.4875	-7.0155	-6.9348	0.0314	0.0593	0.1088
33	-6.4353	-6.9309	-6.8737	0.0324	0.0576	0.1075
34	-6.3920	-6.8580	-6.8076	0.0434	0.0543	0.1075
35	-6.3214	-6.7768	-6.7354	0.0471	0.0484	0.1034
36	-6.2570	-6.7273	-6.6713	0.0480	0.0520	0.1006
37	-6.1918	-6.6713	-6.5903	0.0553	0.0554	0.0986
38	-6.0912	-6.5640	-6.5155	0.0529	0.0467	0.0945
39	-6.0612	-6.5351	-6.4258	0.0810	0.0490	0.0951
40	-5.9681	-6.4227	-6.3408	0.0810	0.0512	0.0880
41	-5.8924	-6.3633	-6.2643	0.0762	0.0477	0.0874
42	-5.7905	-6.2548	-6.1773	0.0908	0.0410	0.0814
43	-5.7174	-6.1683	-6.0848	0.0913	0.0519	0.0795
44	-5.6518	-6.0947	-5.9928	0.1014	0.0513	0.0780
45	-5.5530	-5.9968	-5.9053	0.1029	0.0453	0.0744
46	-5.4678	-5.9137	-5.8197	0.0999	0.0506	0.0722
47	-5.3777	-5.8055	-5.7238	0.1063	0.0465	0.0708

	$\hat{a}_x$			$\hat{b}_x$		
Age	USA	Sweden	Japan	USA	Sweden	Japan
48	-5.2757	-5.7289	-5.6374	0.0981	0.0543	0.0700
49	-5.2196	-5.6229	-5.5459	0.1165	0.0517	0.0692
50	-5.1144	-5.5444	-5.4566	0.1164	0.0469	0.0679
51	-5.0246	-5.4492	-5.3685	0.1042	0.0471	0.0683
52	-4.9230	-5.3457	-5.2813	0.1072	0.0493	0.0692
53	-4.8489	-5.2680	-5.1999	0.1065	0.0473	0.0711
54	-4.7685	-5.1724	-5.1122	0.1095	0.0464	0.0711
55	-4.6766	-5.0759	-5.0247	0.1005	0.0422	0.0708
56	-4.5939	-4.9874	-4.9370	0.1003	0.0468	0.0724
57	-4.5157	-4.8860	-4.8487	0.1019	0.0498	0.0748
58	-4.4023	-4.7992	-4.7493	0.0873	0.0478	0.0794
59	-4.3435	-4.7012	-4.6658	0.0976	0.0483	0.0772
60	-4.2480	-4.6006	-4.5780	0.0929	0.0483	0.0767
61	-4.1722	-4.5055	-4.4783	0.0822	0.0489	0.0779
62	-4.0580	-4.4124	-4.3857	0.0876	0.0479	0.0786
63	-3.9960	-4.3005	-4.2898	0.0861	0.0517	0.0803
64	-3.9288	-4.1970	-4.1917	0.0895	0.0504	0.0824
65	-3.8250	-4.1031	-4.0969	0.0962	0.0486	0.0836
66	-3.7651	-4.0053	-3.9989	0.0853	0.0523	0.0842
67	-3.6766	-3.9009	-3.9006	0.0924	0.0543	0.0872
68	-3.5920	-3.7987	-3.7984	0.0844	0.0561	0.0880
69	-3.5250	-3.7016	-3.6976	0.0861	0.0584	0.0885
70	-3.4298	-3.5998	-3.5919	0.0895	0.0603	0.0897
71	-3.3676	-3.4834	-3.4875	0.0782	0.0603	0.0908
72	-3.2586	-3.3866	-3.3789	0.0894	0.0605	0.0912
73	-3.1797	-3.2751	-3.2701	0.0813	0.0614	0.0912
74	-3.1041	-3.1654	-3.1595	0.0863	0.0595	0.0904
75	-3.0148	-3.0615	-3.0482	0.0922	0.0616	0.0902
76	-2.9364	-2.9497	-2.9376	0.0857	0.0614	0.0890
77	-2.8609	-2.8355	-2.8232	0.0838	0.0608	0.0868
78	-2.7720	-2.7333	-2.7120	0.0917	0.0618	0.0855
79	-2.6869	-2.6247	-2.6000	0.0902	0.0589	0.0827
80	-2.5793	-2.5226	-2.4919	0.0857	0.0570	0.0811
81	-2.5102	-2.4092	-2.3825	0.0729	0.0569	0.0787
82	-2.4046	-2.3040	-2.2767	0.0788	0.0568	0.0763
83	-2.3095	-2.2014	-2.1703	0.0818	0.0541	0.0747
84	-2.2148	-2.0972	-2.0699	0.0824	0.0504	0.0722
85	-2.1216	-1.9893	-1.9679	0.0755	0.0499	0.0699
86	-2.0303	-1.8873	-1.8664	0.0740	0.0490	0.0684
87	-1.9426	-1.7942	-1.7691	0.0739	0.0455	0.0656
88	-1.8635	-1.6905	-1.6727	0.0704	0.0430	0.0638
89	-1.7723	-1.6000	-1.5771	0.0689	0.0390	0.0623
90	-1.6779	-1.5066	-1.4858	0.0692	0.0375	0.0598
91	-1.6135	-1.4144	-1.3936	0.0565	0.0316	0.0573
92	-1.5162	-1.3246	-1.3049	0.0561	0.0286	0.0569
93	-1.4303	-1.2352	-1.2298	0.0524	0.0260	0.0538
94	-1.3510	-1.1626	-1.1487	0.0467	0.0260	0.0518
95	-1.2795	-1.0707	-1.0795	0.0398	0.0240	0.0489
96	-1.2050	-0.9910	-0.9964	0.0340	0.0225	0.0511
97	-1.1380	-0.9221	-0.9409	0.0290	0.0145	0.0459
98	-1.0893	-0.8624	-0.8624	0.0186	0.0178	0.0464
99	-1.0474	-0.8037	-0.7921	0.0065	0.0088	0.0495
100	-0.9890	-0.6987	-0.7374	-0.0019	0.0121	0.0456

Table 9: Estimated mortality index values $\hat{k}_t$ for the Lee-Carter model

Year	USA	Sweden	Japan	Year	USA	Sweden	Japan
1959	2.5544	3.7937	6.7274	1980	-0.3782	0.3010	-0.4301
1960	2.6275	4.3281	6.5445	1981	-0.6853	0.0183	-0.8924
1961	2.3719	3.6897	6.0998	1982	-0.9203	-0.5648	-1.4801
1962	2.4350	4.0070	5.9820	1983	-0.9310	-0.9118	-1.5167
1963	2.5415	3.6169	5.1142	1984	-1.0559	-1.3277	-1.9761
1964	2.3038	3.2711	4.7704	1985	-1.0343	-1.0078	-2.2832
1965	2.2729	3.1529	4.8374	1986	-1.1746	-1.4642	-2.7728
1966	2.2731	2.8284	4.1163	1987	-1.3011	-1.8054	-3.2539
1967	2.0242	2.7166	3.8822	1988	-1.2944	-1.5344	-3.1112
1968	2.2170	2.9540	3.6368	1989	-1.5886	-2.6725	-3.6495
1969	1.9851	2.7699	3.4282	1990	-1.8337	-2.4021	-3.6487
1970	1.7825	1.7746	3.3832	1991	-1.9824	-2.7217	-3.9837
1971	1.5421	1.8771	2.5980	1992	-2.1886	-3.1343	-4.0652
1972	1.5065	1.7493	2.1780	1993	-1.9901	-3.0514	-4.2281
1973	1.3587	1.6343	2.1556	1994	-2.1485	-4.2827	-4.7434
1974	0.9483	1.4299	1.7884	1995	-2.2114	-4.2156	-4.5762
1975	0.5051	1.4042	1.2985	1996	-2.3842	-4.4808	-5.4065
1976	0.3230	1.5187	0.9346	1997	-2.5693	-4.8989	-5.6780
1977	0.0017	0.7125	0.3425	1998	-2.6437	-5.1465	-5.8451
1978	-0.1318	0.6676	0.0162	1999	-2.6501	-5.1602	-5.7642
1979	-0.4769	0.5669	-0.5288				