UNIVERSITY OF ALBERTA

Normal Functions and their Application to the Hodge Conjecture

by

Jason Colwell ©

A thesis
submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree
of Master of Science

in

Mathematics

DEPARTMENT OF MATHEMATICAL SCIENCES

Edmonton, Alberta
FALL, 1997

0-612-22588-7

University of Alberta

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Title of Thesis: **Normal Functions and their Application to the Hodge Conjecture**

Degree: **Master of Science**

Year this Degree Granted: **1997**

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Date: September 25, 1997

ABSTRACT

The Hodge Conjecture states that for a projective algebraic manifold X, any rational cohomology class of type p,p is the class associated to a rational algebraic cycle of codimension p.

This thesis explains part of the Griffiths program for proving the Hodge Conjecture. The task will amount to proving that the image of the class map

$$[\ ]: C^m(X) \otimes \mathbb{Q} \to H^n(X,\mathbb{Q})$$

contains $\mathrm{Prim}^{m,m}(X,\mathbb{Q})$. We retrict the domain of $[\ ]$ to $\Theta(X/\mathbb{P},\mathbb{Q})$, the preimage of $\mathrm{Prim}^{m,m}(X,\mathbb{Q})$. The Griffiths program then factors the function into the following composition:

$$\begin{array}{ccccc} & & H^0(\mathbb{P},\overline{\mathcal{J}}) & & \\ & \overset{\Phi}{\nearrow} & & \overset{\hat{\Xi}}{\searrow} & \\ \Theta(X/\mathbb{P},\mathbb{Q}) & & \xrightarrow[\ [\ ]\ ]{} & & \mathrm{Prim}^n(X;\mathbb{Z}) \end{array}$$

The approach is to prove the Hodge Conjecture by establishing the surjectivity of Φ and that $\mathrm{im}\hat{\Xi} \supseteq \mathrm{Prim}^{m,m}(X,\mathbb{Q})$.

Our object here will be to show that $\mathrm{Prim}^{m,m}(X;\mathbb{Z}) = \mathrm{im}\hat{\Xi}$ in $\mathrm{Prim}^n(X;\mathbb{Z})$.

ACKNOWLEDGEMENTS

First I would like to thank my supervisor, Dr. James D. Lewis, who has taught me most of the algebraic geometry I know, through courses, his book, and personal instruction. In this last area he proved to be very generous with his time, and willing to discuss my study at unscheduled meetings. As well, I am grateful for the good teaching received in courses I have taken in the Department of Mathematical Sciences. Background in differential geometry and algebra has served me well in learning the material required to write this thesis.

I am thankful to my parents, for their encouragement and support throughout my work on my M.Sc.

Finally, I would like to thank Elizabeth, whose example encouraged me to keep working hard.

NOT TO BE USED AS EXAMPLE OF FORMAT

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- [4] ⇐ [5] & [6] & [7] & [8] & [9] & [10] & [11]
- [5] is true
- [6] is true
- [7] is true
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[12] ⇐ [13]
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INTRODUCTION

Let X be a projective algebraic manifold of dimension n. Of de Rham cohomology $H^i(X)$, there is the Hodge decomposition $H^i(X) = \bigoplus_{p+q=i} H^{p,q}(X)$, where $H^{p,q}(X)$ is the space of closed harmonic forms of type (p,q). The decomposition satifies $\overline{H^{p,q}(X)} = H^{q,p}(X)$, where $\bar{}$ represents complex conjugation. A rational algebraic cycle of codimension p is a sum $z = \sum_{i=1}^{k} r_i z_i$, where $r_i \in \mathbb{Q}$ and the z_i's are irreducible codimension-p subvarieties of X. Integration of a form $\omega \in E^{2n-2p}(X)$ over z is defined

$$\int_z \omega := \sum_{i=1}^{p} r_i \int_{z_i \setminus (z_i)_{sing}} \omega.$$

We know this integral exists because, using a desingularization $p : \tilde{z}_i \to z_i$ of z_i, we have $\int_{z_i\setminus(z_i)_{sing}} \omega = \int_{\tilde{z}} p^*\omega$, which exists. This integration descends to cohomology because $\int_{z_i\setminus(z_i)_{sing}} d\omega = \int_{\tilde{z}} d(p^*\omega) = \int_{\partial\tilde{z}} p^*\omega = 0$. To any such cycle z, we may associate an element of $H^{2p}(X)$. If z is the cycle, the associated class ζ is chosen so that the linear functionals

$$\omega \mapsto \int_z \omega$$

and

$$\omega \mapsto \int_X \omega \wedge \zeta$$

represent the same element of $H^{2n-2p}(X)^*$. Poincaré duality $H^{2n-2p}(X)^* \cong H^{2p}(X)$ guarantees that such a ζ can be found.

The **Hodge Conjecture** (the version considered here is known as Hodge$^{p,p}(X,\mathbb{Q})$) states that for a projective algebraic manifold X, any rational cohomology class of type p,p is the class associated to a rational algebraic cycle of codimension p.

We will show that in proving the Hodge Conjecture an assumption may be made about p and n, and that only part of $H^{p,p}(X,\mathbb{Q}) := H^{p,p}(X) \cap H^{2p}(X,\mathbb{Q})$ need be considered. To the end of justifying these assumptions, which will be stated later, we will state the Lefschetz theorems.

Let X, being a projective algebraic manifold, have Kähler form $\omega \in E^{1,1}(X)$, the space of $(1,1)$-type forms on X. Then

$$L : E^*(X) \to E^*(X)$$

$$\eta \mapsto \omega \wedge \eta$$

descends to cohomology, $H^*(X) \to H^*(X)$. ω may be chosen so that L is dual to the map on homology obtained by intersecting with a smooth hyperplane section.

The *primitive cohomology* is defined to be the kernel of $L^{k+1} : H^{n-k}(X) \to H^{n+k+2}(X)$, written Prim($X$), or Prim$^k(X)$ in the case of the above restriction to $H^k(X)$.

Lefschetz Decompostion

$$H^*(X) = \text{Prim}(X) \dotplus L\text{Prim}(X) \dotplus L^2\text{Prim}(X) \dotplus \ldots$$

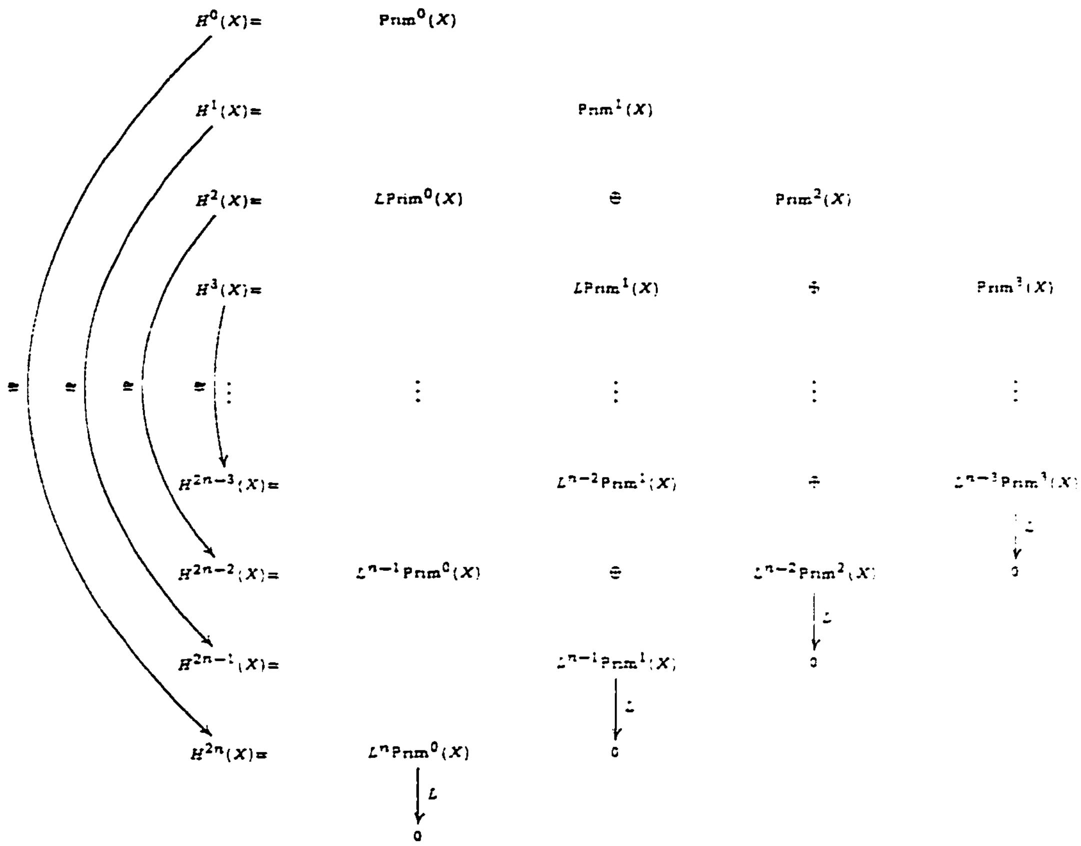
○

Strong Lefschetz Theorem

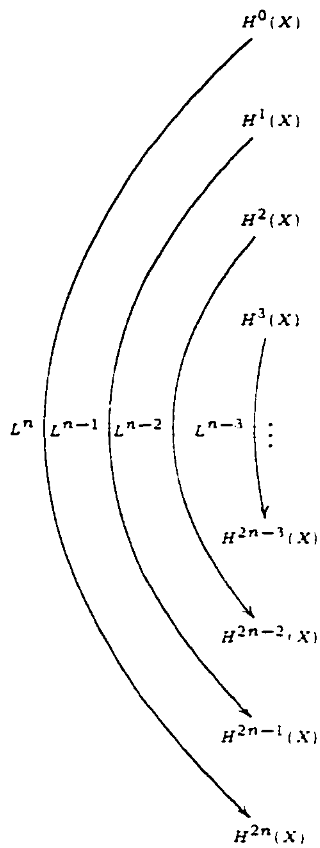
$$L^k : H^{n-k}(X) \to H^{n+k}(X)$$

is an isomorphism. ○

The situation is depicted in the following diagram:



The left side of the above diagram could have been written:



The Lefschetz decomposition is compatible with the Hodge decomposition. Namely, set

$$\mathrm{Prim}^{p,q}(X) = H^{p,q}(X) \cap \mathrm{Prim}^{p+q}(X)$$

$$= \ker\{L^{n-p-q+1} : H^{p,q}(X) \to H^{n-q+1,n-p+1}(X)\}$$

Then $\mathrm{Prim}^k(X) = \bigoplus_{p+q=k} \mathrm{Prim}^{p,q}(X)$. For $p+q < n$.we also have the Strong Lefschetz Theorem. which states that

$$L^{n-p-q} : H^{p,q}(X) \to H^{n-q,n-p}(X)$$

is an isomorphism.

Weak Lefschetz Theorem

Let Y be a smooth hyperplane section of X. with inclusion map $j : Y \hookrightarrow X$. Then $j_* : H_l(Y.\mathbb{Z}) \to H_l(X.\mathbb{Z})$ is surjective for $l = n-1$. and bijective for $l < n-1$. (The correspponding statement for rational homology follows from this.)○

The strong and weak Lefschetz theorems allow us to reduce the Hodge Conjecture to a special case.

We may assume $2p \leq n$.

Suppose $2p > n$. This is equivalent to $2(n-p) < n$.

By the strong Lefschetz theorem we have as an isomorphism

$$L^{2p-n} : H^{2(n-p)}(X.\mathbb{Q}) \to H^{2p}(X.\mathbb{Q})$$

Dual to L^{2p-n} is the map on homology which consists of taking $2p-n$ hyperplane sections:

$$H_{2p}(X.\mathbb{Q}) \to H_{2(n-p)}(X.\mathbb{Q})$$

Now suppose we know $\mathrm{Hodge}^{n-p,n-p}(X.\mathbb{Q})$ and wish to find an algebraic cycle giving a certain class $\eta \in H^{p,p}(X,\mathbb{Q}) \subset H^{2p}(X,\mathbb{Q})$. Find $\xi = (L^{2p-n})^{-1}(\eta) \in H^{n-p,n-p}(X,\mathbb{Q}) \subset H^{2(n-p)}(X.\mathbb{Q})$. By our supposition find $z \in C^{n-p}(X) \otimes \mathbb{Q}$ with $\xi = [z]$. Obtain $y \in C^p(X) \otimes \mathbb{Q}$ from z by taking a smooth hyperplane section $2p-n$ times. This process is dual to L^{2p-n}, so that

$$[y] = L^{2p-n}([z]) = L^{2p-n}(\xi) = \eta$$

and y is a codimension-p algebraic cycle dual to η.

Thus $\mathrm{Hodge}^{p,p}(X,\mathbb{Q})$ follows from $\mathrm{Hodge}^{n-p,n-p}(X,\mathbb{Q})$. Recall $2(n-p) < n$. So we may assume $2p \leq n$. $\square$

We May Assume $n = 2p$.

Suppose $2p < n$. Let F be an $(n-2p)$-dimensional manifold parametrizing an $(n-2p)$-dimensional family of complete intersection subvarieties each obtained by $n-2p$ successive smooth hyperplane sections, covering X. For any $f \in F$, we have the corresponding $(2p)$-dimensional intersection subvariety X_f with inclusion $j_f : X_f \hookrightarrow X$. Now suppose we know $\mathrm{Hodge}^{p,p}(X_f,\mathbb{Q})$ for smooth X_f and wish to find an algebraic cycle giving us a certain class $\eta \in H^{p,p}(X,\mathbb{Q})$. We know that for any general $f \in F$, $j_f^*(\eta) \in H^{p,p}(X_f,\mathbb{Q})$ is algebraic. By $\mathrm{Hodge}^{p,p}(X_f),\mathbb{Q})$ find an algebraic cycle z_f of codimension p. Hilbert scheme arguments show easily that as f varies in F, z_f traces out an algebraic cycle z of codimension n such that $j_f^*(\eta) = [z_f] = j_f^*([z])$. By the weak Lefshetz theorem $j_f^* : H^{p,p}(X,\mathbb{Q} \hookrightarrow H^{2p}(X_f,\mathbb{Q})$ (any $f \in F$ with X_f smooth will do) is an injection and therefore $[z] = \eta$. We have found the desired algebraic cycle z. Our assumption of $\mathrm{Hodge}^{p,p}(X_f,\mathbb{Q})$ was sufficient to obtain $\mathrm{Hodge}^{p,p}(X,\mathbb{Q})$. So we may assume X itself has dimension $2p$. $\square$

A third assumption can be made.

We Need Consider Only $\mathrm{Prim}^{m,m}(X,\mathbb{Q}) \subset H^{m,m}(X,\mathbb{Q})$.

Let Y be a smooth hyperplane section of X with inclusion $j : Y \hookrightarrow X$. By the projection formula, we have the commutative diagram

$$\begin{array}{ccc} H^{n-2}(X,\mathbb{Q}) & \xrightarrow{L} & H^n(X,\mathbb{Q}) \\ (\cong \text{ by weak Lefshetz}) \Big\downarrow j^* & \nearrow_{j_*} & \\ H^{n-2}(Y,\mathbb{Q}) & & \end{array}$$

where j_* is obtained via duality from the pushforward on homology. To see this, let $\langle , \rangle$ be a pairing; then

$$\begin{aligned} j_*j^*(\xi) &= \langle j_*j^*(\xi), X\rangle_X \\ &\overset{\text{projection}}{\underset{\text{formula}}{=}} j_*\langle j^*(\xi), j^*(X)\rangle_X \\ &= j_*\langle j^*(\xi), Y\rangle_Y \\ &\overset{\text{projection}}{\underset{\text{formula}}{=}} j_*\langle \xi, j_*(Y)\rangle_X \end{aligned}$$

$$= \langle \xi, Y \rangle_X$$

$$\stackrel{\text{def}}{=} \text{"}\xi \cap Y\text{"} = L(\xi).$$

Thus $j_* \circ j^* = L$.

From the Lefshetz primitive decomposition we know

$$H^n(X, \mathbb{Q}) = LH^{n-2}(X, \mathbb{Q}) \oplus \text{Prim}^n(X, \mathbb{Q})$$

By the above diagram we obtain

$$H^n(X, \mathbb{Q}) = j_* H^{n-2}(Y, \mathbb{Q}) \oplus \text{Prim}^n(X, \mathbb{Q})$$

Since the Lefshetz and Hodge decompositions are compatible, we have (assuming $n = 2m$)

$$H^{m,m}(X, \mathbb{Q}) = j_* H^{m-1,m-1}(Y, \mathbb{Q}) \oplus \text{Prim}^{m,m}(X, \mathbb{Q}).$$

Now suppose we know $\text{Hodge}^{m-1,m-1}(Y, \mathbb{Q})$ and wish to find an algebraic cycle giving a certain class $\eta \in H^{m,m}(X, \mathbb{Q})$. Let $\eta = j_*\zeta + \xi$ in the above direct sum. By supposition, we can find a $z \in C^{m-1}(Y) \otimes \mathbb{Q}$ with

$$[z] = \zeta.$$

We now know that the success of finding an algebraic cycle giving η depends on our finding an algebraic cycle giving ξ. For if we can find $x \in C^m(X) \otimes \mathbb{Q}$ with

$$[x] = \xi \in \text{Prim}^{m,m}(X, \mathbb{Q}).$$

then, considering z as $\in C^m(X) \otimes \mathbb{Q}$ via $j : Y \to X$, we have our desired algebraic cycle $z + x$:

$$[z + x] = j_*[z] + [x] = j_*\zeta + \xi = \eta$$

By induction, then, it suffices to consider only $\text{Prim}^{m,m}(X, \mathbb{Q}) \subset H^{m,m}(X, \mathbb{Q})$. (The Hodge Conjecture for $\text{Prim}^{m,m}(X, \mathbb{Q})$, in conjunction with $\text{Hodge}^{m-1,m-1}(X, \mathbb{Q})$, implies $\text{Hodge}^{m,m}(X, \mathbb{Q})$.) □

We now outline the Griffiths program for proving the Hodge Conjecture. We have seen so far that our task amounts to proving that the image of

$$[\,] : C^m(X) \otimes \mathbb{Q} \to H^n(X, \mathbb{Q})$$

contains $\text{Prim}^{m,m}(X, \mathbb{Q})$. So we may restrict the domain of [ ] to the pre-image of $\text{Prim}^{m,m}(X, \mathbb{Q})$. Note that $\text{Prim}^{m,m}(X, \mathbb{Q}) = \ker L : H^{m,m}(X, \mathbb{Q}) \to H^{m+1,m+1}(X, \mathbb{Q})$. It follows that the pre-image of $\text{Prim}^{m,m}(X, \mathbb{Q})$ consists of cycles whose generic hyperplane sections are homologically equivalent to 0 (the group of which is denoted $\Theta(X_t)$). We will denote the pre-image of $\text{Prim}^{m,m}(X, \mathbb{Q})$ by $\Theta(X/\mathbb{P}, \mathbb{Q})$. So we are interested in proving that the image of

$$[\ ] : \Theta(X/\mathbb{P}, \mathbb{Q}) \to \text{Prim}^n(X, \mathbb{Q})$$

contains $\text{Prim}^{m,m}(X, \mathbb{Q})$. Most basically, the Griffiths program splits [ ], on $\Theta(X/\mathbb{P}, \mathbb{Q})$, into a composition:

$$\begin{array}{ccc} & H^0(\mathbb{P}, \overline{\mathcal{J}}) & \\ {}^{\Phi}\nearrow & & \searrow^{\hat{\Xi}} \\ \Theta(X/\mathbb{P}, \mathbb{Q}) & \xrightarrow[\ [\,]\ ]{} & \text{Prim}^n(X; \mathbb{Z}) \end{array}$$

Each element of $\Theta(X/\mathbb{P}, \mathbb{Q})$ is fibered by hyperplane sections over $\mathbb{P}$. Φ, the Abel-Jacobi homomorphism, is defined fiberwise, on smooth fibers. So we are actually only interested in those cycles in $\Theta(X/\mathbb{P}, \mathbb{Q})$ for which Φ extends meromorphically across the singular locus of singular hyperplane sections. ($H^0(\mathbb{P}, \overline{\mathcal{J}})$ is the group of normal functions, to be defined later.) The approach is to prove the Hodge Conjecture by establishing the surjectivity of Φ and that $\text{im}\hat{\Xi} \supseteq \text{Prim}^{m,m}$. Its surjectivity is the statement of "Poincaré's existence theorem", which is conjectured to be true in general. What is not true in general is the fiberwise statement "Jacobi inversion", the surjectivity of $C^m(X)_{\text{hom}} \overset{\Phi}{\to} J^m(X_t)$, which would imply Poincaré's existance theorem. We will, in this thesis, demonstrate that $\text{Prim}^{m,m}(X; \mathbb{Z}) \subseteq \text{im}\hat{\Xi}$. The maps Φ and $\hat{\Xi}$ will be defined later.

The object here will be to show that $\text{Prim}^{m,m}(X; \mathbb{Z}) = \text{im}\hat{\Xi}$ in $\text{Prim}^n(X; \mathbb{Z})$.

When X is a surface, then since Jacobi inversion holds for curves, we have the Hodge Conjecture for $p = 1$, known as the Lefschetz (1,1) Theorem. This case is the subject of the Appendix. Jacobi inversion holds for other cases as well, a few of which will be discussed in the section Example Applications. If $\mathbb{Q}$ is replaced by $\mathbb{Z}$ in the statement of the conjecture, this integral version, known as $\text{Hodge}^{p,p}(X, \mathbb{Z})$, in general is false for $p > 1$. A counterexample was found by Atiyah and Hirzbruch.

This thesis is composed of two main sections. The first contains definitions and preliminary results, indexed $\boxed{\text{A}}, \boxed{\text{B}}, \ldots$ The second is the main argument. Its strucure is found in the diagram which precedes it, and its steps are listed in the table of contents. The diagram is interpreted as follows. The statement in a box is implied by the conjunction of the statements in boxes below and attached to it with lines. For example $\boxed{2}$ follows from $\boxed{3}$ and $\boxed{4}$, and $\boxed{3}$ from statements $\boxed{5}$ through $\boxed{10}$. The statements $\boxed{5}$, $\boxed{6}$, $\boxed{7}$, $\boxed{8}$, $\boxed{9}$, $\boxed{10}$, $\boxed{16}$, $\boxed{17}$, $\boxed{19}$, $\boxed{20}$ are the most basic statements, implied by no others in the diagram. The truth of $\boxed{1}$ is the object of the whole argument. Notation found in the diagram will be explained in the first section.

DEFINITIONS AND PRELIMINARY RESULTS

A The setting, and two isomorphic sheaves.

$X \subset \mathbb{P}^N$ is a non-singular projective variety of dimension $n = 2m$. There is a linear pencil of hyperplanes given by $a_0 z_0 + a_1 z_1 = 0$ where $[a_0, a_1] \in \mathbb{P}, (z_0, \ldots z_N) \in \mathbb{P}^N$, and in local coordinates $t = a_1/a_0$. We set X_t as the intersection of X with the hyperplane determined by t. We may choose this *Lefshetz pencil*(the collection $\{X_t : t \in \mathbb{P}\}$) (see [Andreotti & Frankel]) so that X_t is non-singular for all but finitely many points, for all $t \in U := \mathbb{P}\backslash\Sigma$, where Σ is the finite *singular set* $\subset \mathbb{P}$. For $t \in \Sigma$,the singular section X_t has one *ordinary double point*,given in local coordinates by $\left\{(x_1, \ldots, x_n) \in \mathbb{P}^N : \sum_{i=1}^n x_i^2 = 0\right\}$, as its only singularity. We may also arrange the pencil so that X_0, X_∞, and the base locus $D = \bigcap_{t\in\mathbb{P}} X_t = X \cap \{z : z_0 = z_1 = 0\}$ are smooth. Let $\overline{Y}$ be the blowup $B_D(X)$ of X along the *base locus* $D := \bigcap_{t\in\mathbb{P}} X_t = X \cap \{(z_0, \ldots z_N) \in \mathbb{P}^N \mid z_0 = z_1 = 0\}$. Let $\overline{f}$ be the obvious projection $\overline{Y} \to \mathbb{P}$. Let $Y = \overline{f}^{-1}(U)$, and $f = \overline{f}\,|_Y$. We have

$$\begin{array}{ccccc} Y & \longrightarrow & \overline{Y} & \quad & X_t \\ f\downarrow & & \downarrow\overline{f} & & \downarrow\overline{f} \\ U & \underset{j}{\longrightarrow} & \mathbb{P} & & t \end{array}$$

Proposition The two sheaves $\mathcal{O}_U \otimes R^i f_*\mathbb{C}$ and $\mathbf{R}^i f_*\Omega^\bullet_{Y/U}$ over U are isomorphic. ◯

Before giving a proof, we give the definitions of $\mathcal{O}_U \otimes R^i f_*\mathbb{C}$ and $\mathbf{R}^i f_*\Omega^\bullet_{Y/U}$. The latter is derived from the presheaf $V \mapsto \mathbf{H}^i(f^{-1}(V), \Omega^\bullet_{Y/U})$ and $R^i f_*\mathbb{C}$ is derived from the presheaf $V \mapsto H^i(f^{-1}(V), \mathbb{C})$. ($\mathbf{H}$ denotes hypercohomology.)

Briefly we review the definition of hypercohomology [Griffiths & Harris, p.446]. Let M be a complex manifold, and $\mathcal{U}$ be a cover for M. Let $(\mathcal{A}^\bullet, d)$ be a complex of sheaves on M with differential d. $C^p(\mathcal{U}, \mathcal{A}^q)$ is the group of Čech p-cochains with values in $\mathcal{A}^q$. The two operators

$$\delta : C^p(\mathcal{U}, \mathcal{A}^q) \to C^{p+1}(\mathcal{U}, \mathcal{A}^q)$$

$$d : C^p(\mathcal{U}, \mathcal{A}^q) \to C^p(\mathcal{U}, \mathcal{A}^{q+1})$$

satisfy $\delta^2 = d^2 = 0, d\delta + \delta d = 0$. Consequently, we have a double complex

$$\{C^{p,q} = C^p(\mathcal{U}, \mathcal{A}^q) : \delta, d\}$$

and an associated single complex, denoted $(C^\bullet(\mathcal{U}), D)$. (Here $C^i = \bigoplus_{p+q=i} C^{p,q}$ and $D = d + \delta$, so that $D^2 = d^2 + d\delta + \delta d + \delta^2 = d\delta + \delta d = 0$.) If we refine the cover to $\mathcal{U}' < \mathcal{U}$, we get mappings

$$C^p(\mathcal{U}, \mathcal{A}^q) \to C^p(\mathcal{U}', \mathcal{A}^q)$$

$$H^\bullet(C^\bullet(\mathcal{U})) \to H^\bullet(C^\bullet(\mathcal{U}')).$$

So, we may make the definition of hypercohomology:

$$\mathbf{H}^\bullet(M, \mathcal{A}^\bullet) := \lim_{\overrightarrow{\mathcal{U}}} H^\bullet(C^\bullet(\mathcal{U}), D).$$

Proof First, $\Omega^p_{Y/U}$ is the sheaf of relative holomorphic differential p-forms, defined to be the cokernel sheaf in

$$0 \to f^*\Omega^1_U \to \Omega^1_Y \to \Omega^1_{Y/U} \to 0.$$

Consider the following two complexes of sheaves on Y.

$$\Omega^\bullet_{Y/U} : \Omega^0_{Y/U} \to \Omega^1_{Y/U} \to \Omega^2_{Y/U} \to \cdots,$$

with the relative differentials.

$$f^*\mathcal{O}^\bullet_U : f^*\mathcal{O}_U \to 0 \to 0 \to \cdots$$

is a trivial complex with $f^*\mathcal{O}_U$ being obtained from the presheaf

$$V \mapsto \{h \circ f \mid h \text{ holomorphic on } U\}.$$

Now fix, for a time, the open subset V of U. Let $W = f^{-1}(V)$. The natural inclusion $f^*\mathcal{O}^\bullet_U \subset \Omega^\bullet_{Y/U}$ is a quasi-isomorphism by the holomorphic Poincaré lemma. Thus it induces an isomorphism on hypercohomology:

$$\mathbf{H}^\bullet(W, \Omega^\bullet_{Y/U}) \cong \mathbf{H}^\bullet(W, f^*\mathcal{O}^\bullet_U)$$

Examining $\mathbf{H}^\bullet(W, f^*\mathcal{O}^\bullet_U)$ more closely [Griffiths & Harris, p.446], we observe the existence of the spectral sequence $'E_{f^*\mathcal{O}^\bullet_U}$ abutting to $\mathbf{H}^\bullet(W, f^*\mathcal{O}^\bullet_U)$, with

$$('E_{f^*\mathcal{O}^\bullet_U})_2^{p,q} = H^p(W, \mathcal{H}^q(f^*\mathcal{O}^\bullet_U)).$$

the pth Čech cohomology of W with respect to the qth cohomology sheaf of the complex $f^*\mathcal{O}_U^\bullet$ (see Grifiths & Harris, p.446). We have

$$\mathcal{H}^q(f^*\mathcal{O}_U^\bullet) = \begin{cases} f^*\mathcal{O}_U, & q = 0 \\ 0, & q > 0 \end{cases}$$

so that

$$('E_{f^*\mathcal{O}_U^\bullet})_2^{p,q} = H^p(W, \mathcal{H}^q(f^*\mathcal{O}_U^\bullet)) = \begin{cases} H^p(W, f^*\mathcal{O}_U), & q = 0 \\ 0, & q > 0 \end{cases}$$

This shows the spectral sequence $'E_{f^*\mathcal{O}_U^\bullet}$ to be trivial, degenerating at the second term. We conclude

$$\mathbf{H}^*(W, f^*\mathcal{O}_U^\bullet) = H^*(W, f^*\mathcal{O}_U)$$

Combining this with the previous isomorphism $\mathbf{H}^*(W, \Omega^\bullet_{Y/U}) \cong \mathbf{H}^*(W, f^*\mathcal{O}_U^\bullet)$ gives

$$\mathbf{H}^*(W, \Omega^\bullet_{Y/U}) \cong H^*(W, f^*\mathcal{O}_U)$$

We now cease to fix $V \subset U$. The preceding isomorphism indicates an isomorphism between the presheaves given by

$$V \mapsto \mathbf{H}^i(f^{-1}(V), \Omega^\bullet_{Y/U})$$

and

$$V \mapsto H^i(f^{-1}(V), f^*\mathcal{O}_U)$$

and thus between their respective sheaves $\mathbf{R}^i f_* \Omega^\bullet_{Y/U}$ and $R^i f_*(f^*\mathcal{O}_U)$.

Now let us examine the stalks of the sheaves $R^i f_*(f^*\mathcal{O}_U)$ and $\mathcal{O}_U \otimes R^i f_*\mathbb{C}$ over U. The latter has stalk $\mathcal{O}_{U,t} \otimes H^i(X_t, \mathbb{C})$ over t, and the former has stalk $H^i(X_t, f^*\mathcal{O}_U|_{X_t})$ But $f^*\mathcal{O}_U|_{X_t}$ is (from the definition of $f^*\mathcal{O}_U$) a constant sheaf, $\cong \mathcal{O}_{U,t}$. Consequently, we can "factor out" $f^*\mathcal{O}_U|_{X_t}$ to obtain the isomorphism of stalks

$$H^i(X_t, f^*\mathcal{O}_U|_{X_t}) \cong f^*\mathcal{O}_U|_{X_t} \otimes H^i(X_t, \mathbb{C}) \cong \mathcal{O}_{U,t} \otimes H^i(X_t, \mathbb{C})$$

This shows

$$R^i f_*(f^*\mathcal{O}_U) \cong \mathcal{O}_U \otimes R^i f_*\mathbb{C}$$

We already know

$$\mathbf{R}^i f_* \Omega^\bullet_{Y/U} \cong R^i f_*(f^*\mathcal{O}_U)$$

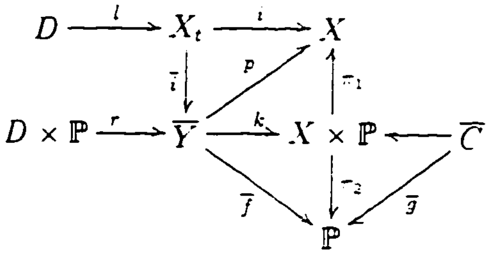
Combining the last two isomorphisms yields the desired result. □

B The complements.

Recall that $\overline{Y}$ was defined to be $B_D(X)$. This is the subset of $X \times \mathbb{P}$ given by $\{(x, s) \in X \times \mathbb{P} \mid x \in X_s\}$, and is also the graph of the linear meromorphic projection

$$X \to \mathbb{P}$$
$$[z] \mapsto [z_0, z_1].$$

We define *the complements* to be the manifold $\overline{C} := X \times \mathbb{P} - \overline{Y} = \{(x, s) \in X \times \mathbb{P} \mid x \notin X_s\}$. We will be working with the following commutative diagram and its cohomology.



Here l, i, r, k are inclusion maps. $\overline{f}$, as previously defined, is $\pi_2 \mid_{\overline{Y}}$, and we define $\overline{g} = \pi_2 \mid_{\overline{C}}, p = \pi_1 \mid_{\overline{Y}}$. The map $\overline{i} : X_t \to \overline{Y}$ is the fiber inclusion, where $k \circ \overline{i}$ is given explcitly by $x \mapsto (x, t)$.

C The splitting of the Gysin map.

As before, let $p : \overline{Y} \to X$ be the projection, being the restriction of $\pi_1 : X \times \mathbb{P} \to X$, and $k : \overline{Y} \to X \times \mathbb{P}$ as above.

We wish to examine the splitting of the Gysin map $k_* : H^n(\overline{Y}) \to H^{n+2}(X \times \mathbb{P})$ under two decompositions. We start by presenting them.

The Künneth decomposition of $H^{n+2}(X \times \mathbb{P})$ is, using $H^1(\mathbb{P}) = 0$ and $H^0(\mathbb{P}) \cong H^2(\mathbb{P}) \cong \mathbb{C}$,

$$H^{n+2}(X \times \mathbb{P})$$
$$\cong H^0(\mathbb{P}) \otimes H^{n+2}(X) \oplus H^1(\mathbb{P}) \otimes H^{n+1}(X) \oplus H^2(\mathbb{P}) \otimes H^n(X)$$
$$\cong H^{n+2}(X) \oplus H^n(X).$$

The projection $\mathrm{pr}_1 = D \times \mathbb{P} \to D$ was omitted from the diagram in **B** to preserve commutivity. (To see that commutivity would not hold were pr_1

included. observe that for $s \neq t$ and any $x \in D$. $\bar{i} \circ l \circ \mathrm{pr}_1((x,s)) = (x,t) \neq (x,s) = r((x,s))$.) From [1],[21], there is a splitting

$$H^n(X) \oplus H^{n-2}(D) \cong H^n(\overline{Y})$$

with the isomorphism given by $p^* + r_*\mathrm{pr}_1^*$. Also, $H^n(X)$ and $H^{n-2}(D)$ are orthogonal under the cup product [21].

Lemma

The Gysin map k_* splits accordingly:

$$\begin{array}{ccccc} H^n(\overline{Y}) & \cong & H^n(X) & \oplus & H^{n-2}(D) \\ \downarrow k_* & & \downarrow \mathrm{id} \quad \searrow L & & \vdots\ i_* l_* \\ H^{n+2}(X \times \mathbb{P}) & \cong & H^n(X) & \oplus & H^{n+2}(X) \end{array}$$

○

Proof

Notice that $k_* \circ p^* = k_* \circ k^* \circ \pi_1^*$, using $p^* = (\pi_1 \circ k)^* = k^* \circ \pi_1^*$.

By the projection formula, $k_* k^*$ is the cup product with the fundamental class of $\overline{Y}$ in $H^2(X \times \mathbb{P})$. This class decomposes into $\omega \otimes [\mathbb{P}] + [X] \otimes \gamma$ under the Künneth decomposition

$$H^2(X \times \mathbb{P}) \cong H^2(X) \otimes H^0(\mathbb{P}) \oplus H^0(X) \otimes H^2(\mathbb{P})$$

where ω is the hyperplane class and γ generates $H^2(\mathbb{P})$.

So k_* is $\cong \mathrm{id} \oplus L$ on $H^n(X)$ (see the preceding diagram), where $H^n(X)$ is identified with its isomorphic image in $H^n(\overline{Y})$.

Now to examine the effect of k_* on the part of $H^n(\overline{Y})$ isomorphic to $H^{n-2}(D)$. We have the *non-commutative* diagram

$$\begin{array}{ccccc} D & \xrightarrow{l} & X_t & \xrightarrow{i} & X \\ \uparrow \mathrm{pr}_1 & & \downarrow \bar{i} & & \uparrow \pi_1 \\ D \times \mathbb{P} & \xrightarrow{r} & \overline{Y} & \xrightarrow{k} & X \times \mathbb{P} \end{array}$$

Non-commutivity was demonstrated in [B]. Though it is not commutative, the cohomology diagram

$$\begin{array}{ccccc} H^{n-2}(D) & \xrightarrow{l_*} & H^n(X_t) & \xrightarrow{i_*} & H^{n+2}(X) \\ \downarrow \mathrm{pr}_1^* & & \downarrow \bar{i}_* & & \uparrow \pi_{1*} \\ H^{n-2}(D \times \mathbb{P}) & \xrightarrow{r_*} & H^n(\overline{Y}) & \xrightarrow{k_*} & H^{n+2}(X \times \mathbb{P}) \end{array}$$

is. because $\bar{i}_* \circ l_* = (\bar{i} \circ l)_*$ does not depend on choice of t. Now the component map $H^{n-2} \to H^n(X) \oplus H^{n-2}(D) \cong H^n(\overline{Y})$ is given by $r_* \circ \mathrm{pr}_1^*$ and the projection $H^{n+2}(X \times \mathbb{P}) \cong H^n(X) \oplus H^{n+2}(X) \to H^{n+2}(X)$ by π_{1*}. So the map we are looking for is

$$\pi_{1*} \circ k_* \circ r_* \circ \mathrm{pr}_1^*$$

which is equal to $i_* \circ l_*$ from the above diagram.

We thus obtain the desired splitting. □

D An isomorphism of sheaves.

Proposition

We also have an isomorphism, for $g = \overline{g}\,|_{\overline{g}^{-1}(U')}$, between $\mathcal{O}_{U'} \otimes R^i g_* \mathbb{C}$ and $\mathbf{R}^i \pi_{2*} \Omega^\bullet_{X \times U'/U'}(\log Y)$.○

Proof

We basically repeat the proof of the previous proposition, except that we begin with the following two complexes of sheaves on $C' := \overline{C} \cap X \times U'$.

$$\Omega^\bullet_{X\times U/U'}(\log Y) : \Omega^0_{X\times U/U'}(\log Y) \to \Omega^1_{X\times U/U'}(\log Y) \to \ldots$$

is the complex of relative differential forms, with the relative differentials.

$$g^* \mathcal{O}^\bullet_{U'} : g^* \mathcal{O}_{U'} \to 0 \to 0 \to \cdots$$

is the trivial complex with $g^* \mathcal{O}_{U'}$ being obtained from the presheaf

$$V \to \{h \circ g \mid h \text{ holomorphic on } U'\}.$$

The exact argument we used before will not work because we do not have a quasi-isomorphism between the complexes. However, this can easily be remedied by restricting the sheaves in both complexes to $X \times U' \backslash Y$. Then we can apply the holomorphic Poincaré lemma and proceed as before. □

E Canonical extensions.

Now we define some bundles (which of course give rise to sheaves) over U'.

Let $\mathcal{H} = \mathbf{R}^{n-1} f_* \Omega^\bullet_{Y/U'} \cong \mathcal{O}_{U'} \otimes R^{n-1} f_* \mathbb{C}$.

Let $\mathcal{K} = \mathbf{R}^n \pi_{2*} \Omega^\bullet_{X\times U'/U'}(\log Y) \cong \mathcal{O}_{U'} \otimes R^n g_* \mathbb{C}$.

The Hodge filtrations on $\Omega^\bullet_{Y/U'}$ and $\Omega^\bullet_{X\times U'/U'}(\log Y)$ are $\{F^p \Omega^\bullet_{Y/U'}\}$ and $\{F^p \Omega^\bullet_{X\times U'/U'}(\log Y)\}$, where $F^p \Omega^\bullet_{Y/U'}$ is the complex

$$\underbrace{0 \to \cdots \to 0}_{p-1} \to \Omega^p_{Y/U'} \to \Omega^{p+1}_{Y/U'} \to \cdots$$

and $F^p \Omega^\bullet_{X\times U'/U'}(\log Y)$ is defined similarly.

Let $\mathcal{F}^p = \mathbf{R}^{n-1} f_* F^p \Omega^\bullet_{Y/U'} \subset \mathcal{H}$, and $\mathcal{F} = \mathcal{F}^m$.

Let $\mathcal{G}^p = \mathbf{R}^n \pi_{2*} F^p \Omega^\bullet_{X\times U'/U'}(\log Y) \subset \mathcal{K}$, and $\mathcal{G} = \mathcal{G}^{m+1}$.

We wish to extend these bundles over all $\mathbb{P}$. For this we need the

Canonical Extension Theorem [Deligne, p.91]

Given a bundle $\mathcal{V}$ on Δ^* with integrable connection and unipotent monodromy, there exists a unique extension $\overline{\mathcal{V}}$ to all of Δ such that

(i) If holomorphic sections of $\overline{\mathcal{V}}$ and $\overline{\mathcal{V}}^*$ are expressed in terms of multi-valued horizontal bases, the coefficients grow like powers of $\log \mid t \mid$.

(ii) The connection matrix of $\mathcal{V}$, expressed in terms of a basis for $\overline{\mathcal{V}}$, has logarithmic poles, and the residue of the connection is nilpotent.

Furthermore, the construction $\mathcal{V} \to \overline{\mathcal{V}}$ is functorial for horizontal maps, is exact, and is compatible with *Hom*, tensor products, and exterior powers.
○

(The connection matrix is the matrix θ of 1-forms $\in \mathcal{E}^1_X(\mathcal{V})$ that in terms of the aforementioned lodal frame $\{\epsilon_1, \dots \epsilon_r\}$ for $\overline{\mathcal{V}}$, satisfies $D\epsilon_i = \sum_j \theta_{ij}\epsilon_j$.)

The integrable *Gauss-Manin connection* for $\mathcal{H}$ and $\mathcal{K}$, and the unipotent *monodromy transformation*, will be discussed in the next section. **F**.

By making the canonical extension at each point of Σ, we obtain extensions $\overline{\mathcal{H}}$ and $\overline{\mathcal{K}}$ of $\mathcal{H}$ and $\mathcal{K}$. Correspondingly, we set

$$\overline{\mathcal{F}} = j_*\mathcal{F} \cap \overline{\mathcal{H}}, \quad \overline{\mathcal{G}} = j_*\mathcal{G} \cap \overline{\mathcal{K}}.$$

F Intermediate Jacobians.

It follows from Hodge Theory that the image of the composite mapping

$$H^{2m-1}(X_s, \mathbb{Z}) \to H^{2m-1}(X_s) \to \frac{H^{2m-1}(X_s)}{F^m H^{2m-1}(X_s)},$$

for smooth X_s, is a lattice. The cokernel J is then a complex torus, called the *intermediate Jacobian* of X_s. Using duality the definition can be rewritten

$$J(X_s) = \frac{F^m H^{2m-1}(X_s)^*}{H_{2m-1}(X_s, \mathbb{Z})},$$

where * denotes the dual space.

Let $\Theta(X_s)$ denote the group of codimension m algebraic cycles on X_s which are homologically equivalent to zero.

There is an *Abel-Jacobi homomorphism*

$$\alpha_s : \Theta(X_s) \to J(X_s)$$

defined by a process of integration. Specifically, for $\xi \in \Theta(X_s)$, by the definition of $\Theta(X_s)$ can find a chain η_ξ of *real* dimension $2m - 1$ in X_s with $\xi = \partial\eta_\xi$. Define

$$\phi_s : \Theta(X_s) \to J(X_s) = \frac{F^m H^{2m-1}(X_s)^*}{H_{2m-1}(X_s, \mathbb{Z})}$$

$$\xi \mapsto \begin{pmatrix} F^m H^{2m-1}(X_s) \to \mathbb{C} \\ \omega \mapsto \int_{\eta_\xi} \omega \end{pmatrix} + H_{2m-1}(X_s, \mathbb{Z}).$$

The intermediate Jacobians fit together to form an analytic fiber space over U, for which the sheaf of germs of holomorphic cross-sections is given by

$$\mathcal{J} := \frac{\mathcal{F}^*}{R^{2m-1} f_* \mathbb{Z}}.$$

We want to extend $\mathcal{J}$ to all $\mathbb{P}$.

An extension $\overline{\mathcal{J}}$ is defined by the short exact sequence

$$0 \to j_* R^{2m-1} f_* \mathbb{Z} \to \overline{\mathcal{F}}^* \to \overline{\mathcal{J}} \to 0.$$

To interpret the sheaf $\overline{\mathcal{J}}$, we examine the local situation around one singular point.

Suppose $\{e_1, \ldots, e_g\}$ is a basis of $\Gamma(\Delta, \overline{\mathcal{F}})$ over $\Gamma(\Delta, \mathcal{O}_\Delta)$. Let $\{u_1, \ldots u_{2g}\}$ be the multi-valued sections of $R^{2m-1} f_* \mathbb{Z}$ determined by generators of $H^{2m-1}(X_t, \mathbb{Z})$ for some $t \in \Delta$, with the first, say k, elements spanning the invariant subspace. Using the map $j_* R^{2m-1} f_* \mathbb{Z} \to \overline{\mathcal{F}}^*$ from the previous exact sequence, $\{u_1, \ldots u_{2g}\}$ can be considered multi-valued functions $\overline{\mathcal{F}} \to \mathbb{C}$. The matrix $[u_j(e_i(t))]$ is, for $t \in \Delta^*$, the *period matrix* of X_t subject to the given choice of bases. The columns of the period matrix determine a lattice in $\mathbb{C}^g$, and $J(X_t)$is the torus obtained by taking the quotient.

For $t = 0$, only the first k columns need be considered, so we divide out only these vectors. We obtain a quotient space $\mathbf{J} \subset \mathbb{C}^g \times \Delta$ with projection $\pi : \mathbf{J} \to \Delta$. The sections of π which are locally liftable to holomorphic mappings $\overline{\mathcal{F}} \to \mathbb{C}^g$. The fiber $\pi^{-1}(0)$ is called the *generalized intermediate Jacobian* of X_0.

There is a *monodromy weight filtration* on $H = H^{2m-1}(X_t)$ (for fixed $t \in \Delta^*$), a sequence of $\mathbb{Q}$-vector spaces $\{H = W_{2n-2} \supset W_{2n-3} \supset \cdots \supset 0\}$.

The monodromy weight filtration provides a *mixed Hodge structure* on H_0 (the fiber of $\overline{\mathcal{H}}$ over 0). A mixed Hodge structure consists of

(i) A finite increasing filtration

$$0 \subset \cdots \subset W_{l-1} \subset W_l \subset W_{l+1} \subset \cdots \subset H_0$$

called a *weight filtration* (in this case the monodromy weight filtration)
(ii) A finite decreasing filtration

$$H_0 \supset \cdots \supset F^{p-1} \supset F^p \supset F^{p+1} \supset \cdots \supset 0$$

called a Hodge filtration (in this case $\{f^p H_0\}_{p \geq 0}$)
Furthermore, the induced filtration on the graded pieces W_l/W_{l-1}:

$$\cdots \supset F^{p-1}W_l/F^{p-1}W_{l-1} \supset F^pW_l/F^pW_{l-1} \supset F^{p+1}W_l/F^{p+1}W_{l+1} \supset \cdots$$

must define a Hodge structure of pure weight l on W_l/W_{l-1}. That the monodromy weight filtration provides a mixed Hodge structure on H_0 is proved in [Schmid].

A few words first about the *monodromy transformation* T on H. Its effect on any $(2m-1)$-cocycle in $H^{2m-1}(X_t)$ is that of translating it through different sections X_s as $s \in \mathbb{P}$ circles once counterclockwise around 0 and returns to its original value t.

$s\bullet$

$\bullet 0$

$\rightarrow \bullet t$

Δ

By the Monodromy Theorem ([Katz],[Landman]) there are integers $p > 0$ and $q \geq 0$ such that

$$(T^p - 1)^{q+1} = 0.$$

We make the further assumption that T is actually unipotent, that $p = 1$. We could always arrange this by passing to the finite covering of Δ^* given by

$$\Delta^* \rightarrow \Delta^*$$

$$z \mapsto z^p$$

Set $N = \log T = -\sum_{k=1}^{q}(I-T)^k/k$. We now present the *monodromy weight filtration*, defined recursively by

$$\left.\begin{array}{ll} W_{2n-i-2} & = \ker N^{n-i} + NW_{2n-i} \\ W_i & = N^{n-i-1}W_{2n-i-2} \end{array}\right\}$$

starting at 0 and -1. (According to [Zucker] we have $N^n = 0$, so $H = W_{2n-2} = W_{2n-1} = \cdots$. Proceed downward from there.)

In the Lefschetz pencil case we can be more particular. On X_t, the subspace of $H^{2m-1}(X_t,\mathbb{Q})$ invariant under T is one-dimensional, generated by the cocycle δ_t. It is called the vanishing cocycle because it approaches 0 as $t \to 0$. We have the

Picard-Lefschetz Formula On $H^{2m-1}(X_t,\mathbb{Q})$, T is given by

$$T(\zeta) = \zeta \pm (\zeta,\delta_t)\delta_t,$$

where the sign $\pm$ depends on m. ○

Therefore $I - T = \pm(\bullet,\delta_t)\delta_t$. Since $(\delta_t,\delta_t) = 0$ (because the dimension of X_t is odd: see [Lewis,14.20]), we infer that $N = I - T$ and that $N^2 = 0$.

We will now compute the weight filtrations on $H^{2m-1}(X_t,\mathbb{Q})$, $H^{2m-1}(X_0,\mathbb{Q})$, and H_0. Applying, we obtain, letting $r = 2m-1$,

$$W_{2r} = \ker N^{r+1} + NW_{2r+2} = H^{2m-1}(X_t,\mathbb{Q}) \text{ (since } N^2 = 0)$$

$$W_{2r} = H^{2m-1}(X_t,\mathbb{Q}) \text{ (for the same reason)}$$

$$\vdots$$

$$W_{2m} = W_{r+1} = H^{2m-1}(X_t,\mathbb{Q})$$

$$W_{2m-1} = W_r = \ker N + NW_{2m+1}$$

$$= ker N = H^{2m-1}(X_t,\mathbb{Q})^T = \ker(\bullet,\delta_t)$$

$$(\text{since} N^2 = 0, \Rightarrow NW_{2m-1} \subset \ker N)$$

$$W_{2m-2} = W_{r-1} = NW_{r+1} = NH^{2m-1}(X_t,\mathbb{Q})$$

$$= \text{im}(\pm(\bullet,\delta_t)\delta_t)$$

$$(\text{where } \delta_t \text{ is the (local) vanishing cocycle (for } t))$$

$$= \mathbb{Q}\delta_t$$

$$W_{2m-3} = W_{r-2} = NW_{r+2} = 0 (\text{since} N^2 = 0)$$

$$\vdots$$

In summary, the monodromy weight filtration is $\{0 \subset W_{2m-2} \subset W_{2m-1} \subset W_{2m}\} = \{0 \subset \mathbb{Q}\delta_t \subset H^{2m-1}(X_t, \mathbb{Q})^T \subset H^{2m-1}(X_t, \mathbb{Q})\}$. Now we can simplify W_{2m-1}. According to a result of H. Clemens, X_0 is a strong deformation retract of $f^{-1}(\Delta)$, so that via the pullback of the inclusion map $j_0^* : X_0 \hookrightarrow f^{-1}(\Delta)$, we have the isomorphism $H^{2m-1}(f^{-1}(\Delta), \mathbb{Q}) \rightleftharpoons H^{2m-1}(X_0, \mathbb{Q})$

But we may also note that because of the

Local Invariant Cycle Property (See [Zucker.4.10] or [Lewis, lec.14].) $j_* R^q f_* \mathbb{Q} \cong R^q \overline{f}_* \mathbb{Q} \quad \forall q \geq 0$

(for $q = 2m - 1$) the pullback of the inclusion map $(j_t : X_t \hookrightarrow f^{-1}(\Delta))$ is an isomorphism

$$j_t^* : H^{2m-1}(f^{-1}(\Delta), \mathbb{Q}) \rightleftharpoons H^{2m-1}(X_t, \mathbb{Q})^T.$$

(By the way $R^{n-1}\overline{f}_*\mathbb{C}$ is defined just as $R^{n-1}f_*\mathbb{C}$ was, with f replaced by $\overline{f}$ and U by $\mathbb{P}$. It is the sheaf associated to the presheaf $V \mapsto H^{n-1}(f^{-1}(V), \mathbb{C})$ over $\mathbb{P}$.) Combining the last two isomorphisms yields

$$H^{2m-1}(X_t, \mathbb{Q})^T \cong H^{2m-1}(X_0, \mathbb{Q})$$

So we may write the weight filtration

$$\{0 \subset W_{2m-2} \subset W_{2m-1} \subset W_{2m}\}$$

$$= \{0 \subset \mathbb{Q} \subset H^{2m-1}(X_0, \mathbb{Q}) \subset H^{2m-1}(X_t, \mathbb{Q})\}$$

This filtration is defined over Δ^* (where the fiber of $\mathcal{H}$ over t is $H^{2m-1}(X_t, \mathbb{Q})$). Taking the limit as $t \to 0$, we obtain a filtration on H_0, the fiber of $\overline{\mathcal{H}}$ over 0:

$$W_{2m} = H_0 \cong \mathbb{Q}^{2g}$$

$$W_{2m-1} = H^{2m-1}(X_0, \mathbb{Q}) \cong \mathbb{Q}^{2g-1}$$

$W_{2m-2} \cong \mathbb{Q}$, and in particular is $\mathbb{Q}\gamma_0$ for some $\gamma_0 \in H^{2m-1}(X_0, \mathbb{Q})$.

We now wish to compute the *period matrix* $M(t)$ for arbitrary $t \in \Delta^*$, a matrix whose columns effectively describe $R^{2m-1}f_*\mathbb{Z}$ as a lattice in $\mathcal{F}^*$. First, let us fix multivalued sections of $R^{2m-1}f_*\mathbb{Z}$, determined (for some $t_0 \in \Delta^*$)

by a (multivalued) basis $\{v_1, \dots, v_{2g}\}$ of $H^{2m-1}(X_{t_0}, \mathbb{Z})$ (the multiple values of v_i deriving from the action of T). We also stipulate that $\{v_1, \dots, v_k\}$ be a basis of the invariant submodule $H^{2m-1}(X_{t_0}, \mathbb{Z})^T$. Since this last subspace is [Lewis. p. 321] $\cong \mathbb{Z}^{2g-1}$, we know $k = 2g - 1$. Let v be the dual cocycle to δ_t (in $H^{2m-1}(X_t, \mathbb{Z})$). Then we claim that $\{v_1, \dots, v_{2g-1}, \tilde{v}\}$, where $\tilde{v} := v - (\pm)\left(\frac{\log t}{2\pi\sqrt{-1}}\right)\delta_t$, determines a frame of $\mathcal{H}$ over Δ^*. To see this, note that $(\delta_t, \delta_t) = 0, \Rightarrow \delta_t \in \ker(\bullet, \delta_t) = H^{2m-1}(X_t, \mathbb{Z})^T = \text{span}\{v_1, \dots, v_{2g-1}\}$, whereas $(v, \delta_t) = 1, \Rightarrow v \notin \ker(\bullet, \delta_t) = \text{span}\{v_1, \dots, v_{2g-1}\}$. This means $v - (\pm)\left(\frac{\log t}{2\pi\sqrt{-1}}\right)\delta_t \notin \text{span}\{v_1, \dots, v_{2g-1}\}$, and consequently $\{v_1, \dots, v_{2g-1}, \tilde{v}\}$ is a basis for $H^{2m-1}(X_t, \mathbb{Z})$, equivalently a frame for $\mathcal{H}$ over Δ^*. We further claim that this frame extends to one for $\overline{\mathcal{H}}$ over Δ. To see this, note that $v_1, \dots, v_{2g-1}$ are by definition invariant under T, thus is single valued on Δ^*. Furthermore $v - (\pm)\left(\frac{\log t}{2\pi\sqrt{-1}}\right)\delta_t$ is single-valued, since

$$T\left(v - (\pm)\left(\frac{\log t}{2\pi\sqrt{-1}}\right)\delta_t\right)$$

$$= (v + (\pm)(v, \delta_t)\delta_t) - (\pm)\left(\frac{\log t + 2\pi\sqrt{-1}}{2\pi\sqrt{-1}}\right)T\delta_t$$

$$= (v + (\pm)\delta_t) - (\pm)\left(1 + \frac{\log t}{2\pi\sqrt{-1}}\right)\delta_t (\text{since } T\delta_t = \delta_t \text{ and } (v, \delta_t) = 1)$$

$$= v - (\pm)\left(\frac{\log t}{2\pi\sqrt{-1}}\right)\delta_t$$

since the choices of sign $(\pm)$ are the same.

Now we want a frame of $\overline{\mathcal{F}}$ over Δ. Note that since $\overline{\mathcal{F}}$ is a holomorphic subbundle of $\overline{\mathcal{H}}$, there is a natural quotient map $\overline{\mathcal{H}} \xrightarrow{q} \overline{\mathcal{H}}/\overline{\mathcal{F}}$, with the cokernel having rank g. Using the fact that

$$F^{m,*}H^{2m-1}(X_t) = \frac{W^{2m-1}H^{2m-1}(X_t) = \text{invariant classes}}{F^m W^{2m-1}H^{2m-1}(X_t)}$$

(see [Lewis, lec. 14]), we may assume without loss of generality that (where $[.]$ denotes image under q) $\{[v_1], \dots, [v_g]\}$ is a basis of $\overline{\mathcal{H}}/\overline{\mathcal{F}}$. Now $v_1, \dots, v_{2g-1}$, being invariant under T, naturally extend over 0, i.e. over all Δ, and are single-valued.

We may write, for $1 \leq i \leq g-1$,

$$[v_{g+i}] = \sum_{j=1}^{g} h_{ij}[v_j],\, h_{ij} \in \mathcal{O}_\Delta.$$

We also have

$$[\tilde{v}] = \sum_{j=1}^{g} h_i[v_j],\, h_{ij} \in \mathcal{O}_\Delta.$$

Examining pre-images under q, we obtain, in $q^{-1}(0) = \overline{\mathcal{F}}$, the set

$$\{v_{g+1} - \sum_{j=1}^{g} h_{1j}v_j, \ldots, v_{2g-1} - \sum_{j=1}^{g} h_{(g-1)j}v_j, \tilde{v} - \sum_{j=1}^{g} h_j v_j\}$$

which, being linearly independant, must be a frame of $\overline{\mathcal{F}}$. Note that since $(v, \delta_t) = 1$ and $\{v_1, \ldots, v_{2g-1}\}$ is a basis of $H^{2m-1}(X_t, \mathbb{Z})^T = \ker(\bullet, \delta_t)$, then $\{v_1, \ldots, v_{2g-1}, v\}$ is a (multivalued) basis of $H^{2m-1}(X_t, \mathbb{Z})$. The period matrix, then, up to a $\mathrm{Gl}(\mathbb{R})$ transformation, is

$$\begin{bmatrix} \int_{X_t} v_1 \wedge (v_{g+1} - \sum_{j=1}^{g} h_{1j}v_j) & \cdots & \int_{X_t} v_{2g-1} \wedge (v_{g+1} - \sum_{j=1}^{g} h_{1j}v_j) & \int_{X_t} v \wedge (v_{g+1} - \sum_{j=1}^{g} h_{1j}v_j) \\ \vdots & & \vdots & \vdots \\ \int_{X_t} v_1 \wedge (v_{2g-1} - \sum_{j=1}^{g} h_{(g-1)j}v_j) & \cdots & \int_{X_t} v_{2g-1} \wedge (v_{2g-1} - \sum_{j=1}^{g} h_{(g-1)j}v_j) & \int_{X_t} v \wedge (v_{2g-1} - \sum_{j=1}^{g} h_{(g-1)j}v_j) \\ \int_{X_t} v_1 \wedge (\tilde{v} - \sum_{j=1}^{g} h_j v_j) & \cdots & \int_{X_t} v_{2g-1} \wedge (\tilde{v} - \sum_{j=1}^{g} h_j v_j) & \int_{X_t} v \wedge (\tilde{v} - \sum_{j=1}^{g} h_j v_j) \end{bmatrix}.$$

We can simplify this notation by setting $n_{ij} = \int_{X_t} v_i \wedge v_j = (v_i, v_j)$, $n_i = \int_{X_t} v_i \wedge v = (v_i, v)$ for $1 \leq i, j \leq 2g-1$. The period matrix becomes

$$\begin{bmatrix} n_{1(g+1)} - \sum_{j=1}^{g} h_{1j}n_{1j} & \cdots & n_{(2g-1)(g+1)} - \sum_{j=1}^{g} h_{1j}n_{(2g-1)j} & (v, v_{g+1} - \sum_{j=1}^{g} h_{1j}v_j) \\ \vdots & & \vdots & \vdots \\ n_{1(2g-1)} - \sum_{j=1}^{g} h_{(g-1)j}n_{1j} & \cdots & n_{(2g-1)(2g-1)} - \sum_{j=1}^{g} h_{(g-1)j}n_{(2g-1)j} & (v, v_{2g-1} - \sum_{j=1}^{g} h_{(g-1)j}v_j) \\ (v_1, \tilde{v} - \sum_{j=1}^{g} h_j v_j) & \cdots & (v_{2g-1}, \tilde{v} - \sum_{j=1}^{g} h_j v_j) & (v, \tilde{v} - \sum_{j=1}^{g} h_j v_j) \end{bmatrix}.$$

The last row has entries (for $1 \leq i \leq 2g-1$):

$$(v_i, \tilde{v} - \sum_{j=1}^{g} h_j v_j)$$

$$= (v_i, \tilde{v}) - \sum_{j=1}^{g} h_j (v_i, v_j)$$

$$= (v_i, v) - (\pm)(\frac{\log t}{2\pi\sqrt{-1}})(v_i, \delta_t) - \sum_{j=1}^{g} h_j n_{ij}$$

$$= n_i - \sum_{j=1}^{g} h_j n_{ij}$$

since $\ker(\bullet, \delta_t) = \text{span}\{v_1, \ldots, v_{2g-1}\}$.

The last column has entries (for $1 \leq i \leq g-1$):

$$(v, v_{g+i} - \sum_{j=1}^{g} h_{ij} v_j)$$

$$= (v_i, v_{g+i}) - \sum_{j=1}^{g} h_{ij}(v, v_j)$$

$$= -n_{g+i} - \sum_{j=1}^{g} h_{ij} n_j.$$

The last, $(g, 2g)$th, entry, is

$$(v, v - (\pm)(\frac{\log t}{2\pi\sqrt{-1}})\delta_t - \sum_{j=1}^{g} h_j v_j)$$

$$= (v, v) - (\pm)(\frac{\log t}{2\pi\sqrt{-1}})(v, \delta_t) - \sum_{j=1}^{g} h_j (v, v_j)$$

$$= 0 - (\pm)(\frac{\log t}{2\pi\sqrt{-1}}) + \sum_{j=1}^{g} h_j n_j).$$

So the whole matrix can be written as

$$\begin{bmatrix} n_{1(g+1)} & \cdots & n_{(2g-1)(g+1)} & -n_{g+1} \\ \vdots & & \vdots & \vdots \\ n_{1(2g-1)} & \cdots & n_{(2g-1)(2g-1)} & -n_{2g-1} \\ n_1 & \cdots & n_{2g-1} & -(\pm)(\frac{\log t}{2\pi\sqrt{-1}}) \end{bmatrix}$$

$$+ \sum_{j=1}^{g} \begin{bmatrix} -h_{1j} n_{1j} & \cdots & -h_{1j} n_{(2g-1)j} & h_{1j} n_j \\ \vdots & & \vdots & \vdots \\ -h_{(g-1)j} n_{1j} & \cdots & -h_{(g-1)j} n_{(2g-1)j} & h_{(g-1)j} n_j \\ -h_j n_{1j} & \cdots & -h_j n_{(2g-1)j} & h_j n_j \end{bmatrix}.$$

The last column of the above sum matrix has unbounded norm as $t \to 0$. The other columns are bounded as $t \to 0$. Consequently, the minimum non-zero norm of the lattice $M(t)\mathbb{Z}^{2g}$ does not approach 0 as $t \to 0$. This means the torus $J(X_t)$ does not degenerate as $t \to 0$. The quotient space $\mathbf{J}$ inherits coordinates from $\mathbb{C}^g \times \Delta$ with the projection $\pi : \mathbf{J} \to \Delta$. $\mathbf{J}$ is therefore a complex manifold with, recall, the generalized intermediate Jacobian $J(X_0) = \pi^{-1}(0)$.

G A connection.

There is a natural integrable connection on $\mathcal{H} = \mathcal{O}_U \otimes R^{n-1}f_*\mathbb{C}$, the Gauss-Manin connection $\nabla = d \otimes 1$. It gives an exact sequence

$$0 \to R^{2m-1}f_*\mathbb{C} \to \mathcal{H} \xrightarrow{\nabla} \Omega^1_U \otimes \mathcal{H} \to 0,$$

which satisfies the Griffiths *infitessimal period relation* $\nabla\mathcal{F} \subset \Omega^1_{\bar{S}} \otimes \mathcal{F}^{p-1}$. For the extended cohomology bundle $\overline{\mathcal{H}}$ we get

$$0 \to j_*R^{2m-1}f_*\mathbb{C} \to \mathcal{H} \xrightarrow{\overline{\nabla}} \Omega^1_{\bar{S}}(\log \Sigma) \otimes \overline{\mathcal{H}}$$

The differentiation $\overline{\nabla}$ is no longer surjective. Let $\mathcal{S}$ denote the image of $\overline{\nabla}$, so that we may write a short exact sequence

$$0 \to j_*R^{2m-1}f_*\mathbb{C} \to \overline{\mathcal{H}} \xrightarrow{\overline{\nabla}} \mathcal{S} \to 0.$$

We have, too, the Gauss-Manin connection $\nabla = \delta \otimes 1$ on $\mathcal{K} = \mathcal{O}_U \otimes R^n g_*\mathbb{C}$. It gives the exact sequence

$$0 \to R^{2m-1}g_*\mathbb{C} \to \mathcal{K} \xrightarrow{\nabla} \Omega^1_{\bar{S}} \otimes \mathcal{K} \to 0$$

For $\overline{\mathcal{K}}$ we get

$$0 \to j_*R^{2m}g_*\mathbb{C} \to \overline{\mathcal{K}} \xrightarrow{\overline{\nabla}} \Omega^1_{\bar{S}}(\log \Sigma) \otimes \overline{\mathcal{K}}$$

Let $\mathcal{T}$ denote the image of $\overline{\nabla}$, so that we may write a short exact sequence

$$0 \to j_*R^{2m}g_*\mathbb{C} \to \overline{\mathcal{K}} \xrightarrow{\overline{\nabla}} \mathcal{T} \to 0.$$

We wish to compute ∇ explicitly.

For a frame of $\mathcal{H}$ we will use, as before

$$\left\{ v_1, \ldots, v_{2g-1}, \tilde{v} = v - (\pm)\left(\frac{\log t}{2\pi\sqrt{-1}}\right)\delta_t \right\}.$$

where $v_1, \ldots, v_{2g-1}$ are invariant under T and v is the dual cocycle to δ_t. Let $a = \sum_{i=1}^{2g-1} a_i v_i + a_{2g}\tilde{v}$ be an arbitrary element of $\mathcal{H}$ (where $a_i \in \mathcal{O}_{\Delta^*}$). We will calculate the image of a in $\Omega^1_{\Delta^*} \otimes \mathcal{H} = \Omega^1_{\Delta^*} \otimes R^{2m-1} f_* \mathbb{C}$.

$$\nabla a = \sum_{i=1}^{2g-1} da_i \otimes v_i + da_{2g} \otimes v - d\left((\pm)\left(\frac{\log t}{2\pi\sqrt{-1}}\right) a_{2g}\right) \otimes \delta_t$$

$$= \sum_{i=1}^{2g-1} a_i' dt \otimes v_i + a_{2g}' dt \otimes v - (\pm)\left(\frac{\log t}{2\pi\sqrt{-1}} a_{2g}' + \frac{a_{2g}}{2\pi\sqrt{-1}t}\right) dt \otimes \delta_t$$

$$= \sum_{i=1}^{2g-1} a_i' dt \otimes v_i + (a_{2g}' dt)\left(1 \otimes v - (\pm)\left(\frac{\log t}{2\pi\sqrt{-1}}\right) \otimes \delta_t\right) - (\pm) a_{2g} \frac{dt}{2\pi\sqrt{-1}t} \otimes \delta_t$$

$$= \sum_{i=1}^{2g-1} a_i'(dt \otimes v_i) - (\pm)\frac{\log t}{2\pi\sqrt{-1}t}(dt \otimes \delta_t) + a_{2g}(dt \otimes \tilde{v})$$

Noting that $\delta_t \in \ker(\bullet . \delta_t) = \operatorname{span}\{v_1, \ldots, v_{2g-1}\}$, and writing $\delta_t = d_1 v_1 + \cdots + d_{2g-1} v_{2g-1}$, with $d_i \in \mathcal{O}_{\Delta^*}$, we have

$$\nabla a = \sum_{i=1}^{2g-1} \left(a_i' - (\pm)\frac{a_{2g} d_i}{2\pi\sqrt{-1}t}\right)(dt \times v_i) + a_{2g}'(dt \otimes \tilde{v}).$$

Now note that $\{dt \otimes v_1, \ldots, dt \otimes v_{2g-1}, dt \otimes \tilde{v}\}$ is a holomorphic frame of $\Omega^1_{\Delta^*} \otimes \mathcal{H}$. We will demonstrate the surjectivity of ∇ on U by finding an element of $\mathcal{H}$ whose image is a given arbitrary element of $\Omega^1_{\Delta} \otimes \mathcal{H}$. Pick from $\Omega^1_{\Delta} \otimes \mathcal{H}$ an arbitrary $b = \sum_{i=1}^{2g-1} b_i(dt \times v_i) + b_{2g}(dt \otimes \tilde{v})$, where $b_i \in \mathcal{O}_{\Delta}$.

Let a_{2g} be an antiderivative of b_{2g}.

For $1 \leq i \leq 2g-1$, let a_i be an antiderivative of

$$b_i + (\pm)\frac{a_{2g} d_i}{2\pi\sqrt{-1}t}.$$

Let $a = \sum_{i=1}^{2g-1} a_i v_i + a_{2g}\tilde{v}$. Then

$$\nabla a = \sum_{i=1}^{2g-1} \left(a_i' - (\pm)\frac{a_{2g} d_i}{2\pi\sqrt{-1}t}\right)(dt \otimes v_i) + a_{2g}'(dt \otimes \tilde{v}).$$

$$= \sum_{i=1}^{2g-1} b_i(dt \otimes v_i) + b_{2g}(dt \otimes \tilde{v})$$

$$= b.$$

We see that ∇ is indeed surjective.

Next, we will compute the extension $\overline{\nabla}$ of ∇ to all $\overline{\mathcal{H}}$. We have an exact sequence,

$$0 \to j_* R^{2m-1} f_* \mathbb{C} \to \mathcal{H} \xrightarrow{\overline{\nabla}} \Omega^1_{\overline{S}}(\log \Sigma) \otimes \overline{\mathcal{H}}.$$

but ∇ is not surjective, and we will see this explicitly. Our (single-valued) frame is again $\{v_1, v_2, \ldots, \tilde{v}\}$. So all the d_i's extend holomorphically over Δ. This time for an arbitrary element of $\overline{\mathcal{H}}$, we pick $a = \sum_{i=1}^{2g-1} a_i v_i + a_{2g}\tilde{v}$, with a_i's holomorphic on all Δ. Using the same computation as before, we obtain

$$\nabla a = \sum_{i=1}^{2g-1} \left(a_i' - (\pm)\frac{a_{2g} d_i}{2\pi\sqrt{-1}t} \right) (dt \otimes v_i) + a_{2g}'(dt \otimes \tilde{v}).$$

$$= \sum_{i=1}^{2g-1} \left(t a_i' - (\pm)\frac{a_{2g} d_i}{2\pi\sqrt{-1}} \right) \left(\frac{dt}{t} \otimes v_i \right) + (t a_{2g}') \left(\frac{dt}{t} \otimes \tilde{v} \right).$$

We have expressed $\overline{\nabla}$ in terms of the holomorphic frame

$$\left\{ \frac{dt}{t} \otimes v_1, \ldots, \frac{dt}{t} \otimes v_{2g-1}, \frac{dt}{t} \otimes \tilde{v} \right\}$$

for $\Omega^1_\Delta(\log 0) \otimes \overline{\mathcal{H}}$. As proof that the holomorphic coefficient of $\frac{dt}{t} \otimes \tilde{v}$ must vanish to order at least 1 at 0.

Recall $\mathcal{S} \subset \Omega^1_\Delta(\log 0) \otimes \mathcal{H}$ is the image of $\overline{\nabla}$. We wrote the short exact sequence:

$$0 \to j_* R^{2m-1} f_* \mathbb{C} \to \overline{\mathcal{H}} \xrightarrow{\overline{\nabla}} \mathcal{S} \to 0.$$

We claim $\Omega^1_\Delta \otimes \overline{\mathcal{H}} \subset \mathcal{S}$. To check this, let $b = \sum_{i=1}^{2g-1} b_i(dt \times v_i) + b_{2g}(dt \otimes \tilde{v})$, where $b_i \in \mathcal{O}_\Delta$, be an arbitrary element of $\Omega^1_\Delta \otimes \overline{\mathcal{H}}$. Then we choose $a = \sum_{i=1}^{2g-1} a_i v_i + a_{2g}\tilde{v}$ exactly as before, so that $\overline{\nabla} a = b$.

$\Omega^1_\Delta \otimes \overline{\mathcal{K}} \subset \mathcal{T}$ is verified in just the same way.

H The Čech cup product.

We recall the general formula for Čech cohomology cup product. Let $\mathcal{A}$ and $\mathcal{B}$ be sheaves on a topological space T, with cover $\mathcal{U}$. The cup product is, first, a binary operation on Čech cochains:

$$\cup : \check{C}^m(\mathcal{U}, \mathcal{A}) \otimes \check{C}^n(\mathcal{U}, \mathcal{B}) \to \check{C}^{m+n}(\mathcal{U}, \mathcal{A} \otimes \mathcal{B})$$

$$(\alpha_{i_0 \dots i_m}) \otimes (\beta_{i_0 \dots i_n}) \mapsto (\gamma_{i_0 \dots i_{m+n}})$$

where

$$\gamma_{i_0 \dots i_{m+n}} = \alpha_{i_0 \dots i_m} \otimes \beta_{i_m \dots i_{m+n}}$$

The cup product descends to cohomology, and to verify this, we must show that $\cup$ takes two cocycles to a cocycle, and takes a cocycle and a coboundary to a coboundary: in other words:

$$\check{Z}^m(\mathcal{U}, \mathcal{A}) \otimes \check{Z}^n(\mathcal{U}, \mathcal{B}) \to \check{Z}^{m+n}(\mathcal{U}, \mathcal{A} \otimes \mathcal{B})$$

$$\check{B}^m(\mathcal{U}, \mathcal{A}) \otimes \check{Z}^n(\mathcal{U}, \mathcal{B}) \to \check{B}^{m+n}(\mathcal{U}, \mathcal{A} \otimes \mathcal{B})$$

$$\check{Z}^m(\mathcal{U}, \mathcal{A}) \otimes \check{B}^n(\mathcal{U}, \mathcal{B}) \to \check{B}^{m+n}(\mathcal{U}, \mathcal{A} \otimes \mathcal{B})$$

These facts will follow quickly from the following observation: If $\alpha \in \check{C}^m(\mathcal{U}, \mathcal{A})$ and $\beta \in \check{C}^n(\mathcal{U}, \mathcal{B})$ Then

$$\begin{aligned}
&(\partial(\alpha \cup \beta))_{i_0 \dots i_{m+n+1}} \\
&= \sum_{l=0}^{m+n+1} (-1)^l (\alpha \cup \beta)_{i_0 \dots \hat{i}_l \dots i_{m+n+1}} \\
&= \sum_{l=0}^{m} (-1)^l (\alpha \cup \beta)_{i_0 \dots \hat{i}_l \dots i_{m+n+1}} + \sum_{l=m+1}^{m+n+1} (-1)^l (\alpha \cup \beta)_{i_0 \dots \hat{i}_l \dots i_{m+n+1}} \\
&= \sum_{l=0}^{m} (-1)^l \alpha_{i_0 \dots \hat{i}_l \dots i_{m+1}} \otimes \beta_{i_{m+1} \dots i_{m+n+1}} + \sum_{l=m+1}^{m+n+1} (-1)^l \alpha_{i_0 \dots i_m} \otimes \beta_{i_m \dots \hat{i}_l \dots i_{m+n+1}} \\
&= \left((\partial\alpha)_{i_0 \dots i_{m+1}} - (-1)^{m+1} \alpha_{i_0 \dots i_m}\right) \otimes \beta_{i_{m+1} \dots i_{m+n+1}} \\
&\quad + \alpha_{i_0 \dots i_m} \otimes (-1)^m \left((\partial\beta)_{i_m \dots i_{m+n+1}} - (-1)^0 \beta_{i_{m+1} \dots i_{m+n+1}}\right) \\
&= (\partial\alpha)_{i_0 \dots i_{m+1}} \otimes \beta_{i_{m+1} \dots i_{m+n+1}} + (-1)^m \alpha_{i_0 \dots i_m} \otimes \beta_{i_{m+1} \dots i_{m+n+1}} \\
&\quad + (-1)^{m+1} \alpha_{i_0 \dots i_m} \otimes \beta_{i_{m+1} \dots i_{m+n+1}} + \alpha_{i_0 \dots i_m} \otimes (-1)^m (\partial\beta)_{i_m \dots i_{m+n+1}} \\
&= (\partial\alpha)_{i_0 \dots i_{m+1}} \otimes \beta_{i_{m+1} \dots i_{m+n+1}} + \alpha_{i_0 \dots i_m} \otimes (-1)^m (\partial\beta)_{i_m \dots i_{m+n+1}}
\end{aligned}$$

$$= (\partial\alpha \cup \beta)_{i_0 \cdots i_{m+n+1}} + (-1)^m(\alpha \cup \partial\beta)_{i_0 \cdots i_{m+n+1}}$$

or equivalently $\partial(\alpha \cup \beta) = \partial\alpha \cup \beta + (-1)^m \alpha \cup \partial\beta$. Now then, if α and β are cocycles, $\partial(\alpha \cup \beta) = \partial\alpha \cup \beta + (-1)^m\alpha \cup \partial\beta = 0. \Rightarrow \alpha \cup \beta$ is a cocycle. Secondly, if α is a cocycle and $\partial\beta$ any coboundary, then

$$\begin{aligned} & \alpha \cup \partial\beta \\ &= (-1)^m((-1)^m\alpha) \cup \partial\beta \\ &= \partial(((-1)^m\alpha) \cup \beta) - (-1)^m\partial((-1)^m\alpha) \cup \beta \\ &= \partial(((-1)^m\alpha) \cup \beta) - \partial\alpha \cup \beta \\ &= \partial(((-1)^m\alpha) \cup \beta) \end{aligned}$$

is a coboundary. Thirdly if $\partial\alpha$ is any coboundary and β is a cocycle, then, similarly, $\partial\alpha \cup \beta$ is a coboundary.

We have verified the three necessary properties of $\cup$. Hence $\cup$ descends to cohomology, giving a map:

$$\cup : \check{H}^m(\mathcal{U}, \mathcal{A}) \otimes \check{H}^n(\mathcal{U}, \mathcal{B}) \to \check{H}^{m+n}(\mathcal{U}, \mathcal{A} \otimes \mathcal{B})$$

Taking the direct limit of $\mathcal{U}$, or alternatively insisting that $\mathcal{U}$ be Leray (i.e. each member of the family $\mathcal{U}$ has trivial higher cohomology groups), we have:

$$\cup : \check{H}^m(T, \mathcal{A}) \otimes \check{H}^n(T, \mathcal{B}) \to \check{H}^{m+n}(T, \mathcal{A} \otimes \mathcal{B})$$

[I] A diagram of pairings.

The following diagram of pairings commutes up to a sign:

$$\begin{array}{ccccc}
H^1(\mathbb{P}, \overline{\mathcal{F}}) & \times & H^0(\mathbb{P}, \Omega^1_{\overline{\mathbb{P}}} \otimes \overline{\mathcal{F}}) & \longrightarrow & H^1(\mathbb{P}, \Omega^1_{\overline{\mathbb{P}}}) \\
\uparrow{\scriptstyle\theta} & & \downarrow & & \downarrow{\scriptstyle\delta'} \\
H^1(\mathbb{P}, j_*R^{2m-1}f_*\mathbb{Z}) & & H^0(\mathbb{P}, \mathcal{S}) & & H^2(\mathbb{P}, \mathbb{C}) \\
\downarrow & & \downarrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\cong} \\
H^1(\mathbb{P}, j_*R^{2m-1}f_*\mathbb{C}) & \times & H^1(\mathbb{P}, j_*R^{2m-1}f_*\mathbb{C}) & \longrightarrow & H^2(\mathbb{P}, j_*R^{4m-2}f_*\mathbb{C})
\end{array}$$

where δ is the connecting homomorphism from

$$0 \to R^{2m-1}f_*\mathbb{C} \to \overline{\mathcal{H}} \to \mathcal{S} \to 0.$$

δ' is the connecting homomorpism from

$$0 \to \mathbb{C} \to \mathcal{O}_{\mathbb{P}} \xrightarrow{d} \Omega^1_{\mathbb{P}} \to 0,$$

and where θ is given by inclusion. The isomorphy of $H^2(\mathbb{P}, \mathbb{C})$ and $H^2(\mathbb{P}, j_* R^{4m-2} f_* \mathbb{C})$ arises because $\overline{Y} \to \mathbb{P}$ has compact fibers X_t, and $\dim X_t = 2m - 1$..

"Commutivity up to a sign" means that, for $\alpha \in H^1(\mathbb{P}, j_* R^{2m-1} f_* \mathbb{Z})$ and $\sigma \in H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}})$, we get $\alpha \cup \delta\sigma = -\delta'(\alpha \cup \sigma)$.

We now give a proof of this:

Choose a covering $\mathcal{U} = \{U_i\}$ that is Leray for all sheaves involved. Let

$$\alpha = (\alpha_{ij}) \in H^1(\mathbb{P}, j_* R^{2m-1} f_* \mathbb{Z})$$

$$\sigma = (\sigma_j) \in H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}})$$

We will first determine $\delta\sigma$. As an element of $\Gamma_{U_j}(\Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}) \subset \Gamma_{U_j}(\mathcal{S})$, we may assume, lifting σ_j to τ_j, that $\sigma_j = \overline{\nabla}\tau_j$ on U_j. As a 0-chain, $\tau := (\tau_i)$ must be passed through the coboundary operator, from $\check{C}^0(\mathcal{U}, \mathcal{S})$ to $\check{C}^1(\mathcal{U}, \mathcal{S})$:

$$(\partial\tau)_{jk} = \tau_k - \tau_j$$

Now we must take the pre-image of this under $j_* R^{2m-1} f_* \mathbb{C} \to \overline{\mathcal{H}}$. But this last map is essentially an inclusion, so $\delta\sigma = \partial\tau \in H^1(\mathcal{U}, j_* R^{2m-1} f_* \mathbb{Z})$, which means

$$(\delta\sigma)_{jk} = \tau_k - \tau_j$$

Using the Čech cup product formula,

$$(\alpha \cup \delta\sigma)_{ijk} = \alpha \wedge (\delta\sigma)_{jk} = \alpha_{ij} \wedge (\tau_k - \tau_j)$$

The image of this under the isomorphism $H^2(\mathbb{P}, j_* R^{4m-2} f_* \mathbb{C}) \xrightarrow{\cong} H^2(\mathbb{P}, \mathbb{C})$ is $\int_{X_t} (\alpha \cup \delta\sigma)_{ijk} = \int_{X_t} \alpha_{ij} \wedge (\tau_k - \tau_j)$

But now consider $\alpha \cup \sigma$, where α is identified with its image under θ.

$$(\alpha \cup \sigma)_{ij} = \alpha_{ij} \cdot \sigma_j = \alpha_{ij} \cdot \overline{\nabla}\tau_j = \alpha_{ij} \cdot \nabla\tau_j$$

Thus we are computing

$$d\left(\int_{X_t} \alpha_{ij} \wedge \tau_j\right)$$

inside $U \subset \mathbb{P}$ (almost everywhere).

Now to calculate $\delta'(\alpha \cup \sigma)$. We just saw that the preimage of $(\alpha \cup \sigma)_{ij}$ under $d : \mathcal{O}_{\mathbb{P}} \to \Omega^1_{\mathbb{P}}$ is $\int_{X_t} \alpha_{ij} \wedge \tau_j$. Passing this through the coboundary operator. from $\check{C}^1(\mathcal{U}.\mathcal{O}_{\mathbb{P}})$ to $\check{C}^2(\mathcal{U}.\mathcal{O}_{\mathbb{P}})$. we get

$$\left(\partial\left(\int_{X_t} \alpha_{ij} \wedge \tau_j\right)\right)_{kln}$$

$$= \int_{X_t} \alpha_{ln} \wedge \tau_n - \int_{X_t} \alpha_{kn} \wedge \tau_n + \int_{X_t} \alpha_{kl} \wedge \tau_l$$

$$= \int_{X_t} (\alpha_{ln} - \alpha_{kn}) \wedge \tau_n + \int_{X_t} \alpha_{kl} \wedge \tau_l$$

$$= \int_{X_t} (\partial(\alpha_{ij})_{kln} - \alpha_{kn}) \wedge \tau_n + \int_{X_t} \alpha_{kl} \wedge \tau_l$$

$$= \int_{X_t} -\alpha_{kl} \wedge \tau_n + \int_{X_t} \alpha_{kl} \wedge \tau_l \text{ (since } \partial(\alpha_{ij}) = 0)$$

$$= \int_{X_t} \alpha_{kl} \wedge (\tau_l - \tau_n).$$

We must take the preimage of this under $\mathbb{C} \to \mathcal{O}_{\mathbb{P}}$. which is essentially an inclusion. So we may write

$$(\delta'(\alpha \cup \sigma))_{ijk} = \int_{X_t} \alpha_{ij} \wedge (\tau_j - \tau_k) = -\int_{X_t} \alpha_{ij} \wedge (\tau_k - \tau_j)$$

and commutivity up to a sign is verified. □

[J] The map $\dot{\Xi}$.

We now construct the map whose image will be our main object of scrutiny.

The Gauss-Manin connection ∇. from [G]. is defined on $\mathcal{H}$.and descends to a function on $\mathcal{F}^* = \mathcal{H}/\mathcal{F}$. Furthermore. ∇ annihilates the image of $R^{n-1}f_*\mathbb{C}$ in $\mathcal{H}$. in particular the image of $R^{n-1}f_*\mathbb{Z}$. Consequently. ∇ descends to $\mathcal{J} = \frac{\mathcal{F}^*}{R^{n-1}f_*\mathbb{Z}}$. The range of the resulting function is determined by the infitessimal period relation (see [G]):

$$\nabla_{\mathcal{J}} : \overline{\mathcal{J}} \to \Omega^1_{\mathbb{P}}(\log \Sigma) \otimes \overline{\mathcal{F}}^{m+1,*}$$

Normal Functions are elements of $H^0(\mathbb{P}.\overline{\mathcal{J}})$. i.e. holomorphic cross-sections of $\sqcup_{\mathbb{P}} J(X_t)$ with "moderate" growth near singular points. and satisfying the horizontality condition $\nabla_{\mathcal{J}}\nu = 0$. In the Lefschetz pencil case.

all normal funtions are horizontal. (This will be presented later, after the calculations in the main argument used to show $\boxed{20}$ is true.) We will be constructing a map taking normal functions to the primitive cohomology of X. This will give the topological invariant associated to a normal function.

First, from the short exact sequence

$$0 \to j_* R^{n-1} f_* \mathbb{Z} \to \overline{\mathcal{F}} \to \overline{\mathcal{J}} \to 0,$$

we get the connecting homomorphism

$$\Xi : H^0(\mathbb{P}, \overline{\mathcal{J}}) \to H^1(\mathbb{P}, j_* R^{n-1} f_* \mathbb{Z})$$

Second, there is (again) the Local Invariant Cycle Property

$$R^q \bar{f}_* \mathbb{C} = j_* R^q f_* \mathbb{C}, \ \forall q \geq 0.$$

Third, there is a splitting

$$H^1(\mathbb{P}, R^{n-1} \bar{f}_* \mathbb{C}) \cong \mathrm{Prim}^n(X) \oplus H^{n-2}(D)_\iota,$$

orthogonal with respect to the cup product (see [Lewis]).

We may now construct the composite map $\hat{\Xi}$.

$$\begin{array}{ccc}
H^0(\mathbb{P}, \overline{\mathcal{J}}) & \xrightarrow{\qquad\qquad\hat{\Xi}\qquad\qquad} & \mathrm{Prim}^n(X) \\
\Xi \downarrow & & \uparrow \mathrm{pr}_1 \\
H^1(\mathbb{P}, R^{n-1} j_* f_* \mathbb{Z}) & \subset H^1(\mathbb{P}, j_* R^{n-1} f_* \mathbb{C}) = H^1(\mathbb{P}, R^{n-1} \bar{f}_* \mathbb{C}) \cong & \mathrm{Prim}^n(X) \oplus H^{n-2}(D)_\iota
\end{array}$$

K An exact sequence.

Here we construct an exact sequence, which will provide an important step in the main argument.

(I) First we restate the theorem of Lefshetz on the cohomology of hyperplane sections.

Theorem

Let $i : X_t \to X$ be the inclusion. Then $i^* : H^q(X) \to H^q(X_t)$ is injective for $q = n-1$ and bijective for $q < n-1$. ○

(II) Second, we define the *vanishing cohomology*. We have the map $i_* : H_{n-1}(X_t) \to H_{n-1}(X)$, which, via Poincaré duality, provides the Gysin map, also called i_*.

$$i_* : H^{n-1}(X_t) \to H^{n+1}(X)$$

The vanishing cohomology is $H^{n-1}(X_t)_v = \ker i_*$. It is called "vanishing" because in homology ($H_{n-1}(X_t)$), $\ker i_*$ is the group of $(n-1)$-classes on X_t which "vanish" (are homologically equivalent to 0) when considered as classes on X. It is generated by the vanishing cocycles $\{\delta_{t_1}, \ldots, \delta_{t_N}\}$ (where $\{t_1, \ldots, t_N\}$ is the singular set $\Sigma \subset \mathbb{P}$).

(III) Third, we have the following, due to the projection formula.

Fact

$i_* i^* : H^{n-1}(X) \to H^{n+1}(X)$ is the cup product with the hyperplane class, by the strong Lefshetz theorem an isomorphism. ○

(IV) We put the two facts together to get

Lemma

$H^{n-1}(X_t) = i^* H^{n-1}(X) \dotplus H^{n-1}(X_t)_v$ for smooth X_t. ○

Proof

We know that the composite

$$H^{n-1}(X) \xrightarrow{i^*} H^{n-1}(X_t) \xrightarrow{i_*} H^{n+1}(X)$$

is an isomorphism (just above), and for this we must have $\operatorname{coker} i^* = \ker i_* = H^{n-1}(X_t)_v$. Therefore

$$H^{n-1} = i^* H^{n-1}(X) \dotplus H^{n-1}(X_t)_v.$$

□

(V) We notice the Gysin sequence for the pair (X, X_t). which in part is

$$H^{n-2}(X_t) \xrightarrow{i_*} H^n(X) \to H^n(X - X_t) \xrightarrow{\text{residue}} H^{n-1}(X_t) \xrightarrow{i_*} H^{n+1}(X_t).$$

The map i_* is constructed just as before. The map $i^* : H^{n-2}(X) \to H^{n-2}(X_t)$ is the pullback. As before we have

Fact

$i_*i^* : H^{n-2}(X) \to H^n(X)$ is the cup product with the hyperplane class (but this time (recall the weak Lefschetz theorem) need not be an isomorphism). Its image is. from the definition and the Lefshetz decompostion. $\mathrm{Prim}^n(X)^{\perp}$. ○

It is said by the previously mentioned theorem of Lefschetz that $i^* : H^{n-2}(X) \to H^n(X_t)$ is an isomorphism. So $\mathrm{Prim}^n(X)^{\perp} = \mathrm{im}\, i_*i^* = \mathrm{im}\, i_*$. and applying the definition of $H^{n-1}(X_t)_v$, we obtain the short exact sequence

$$0 \to \mathrm{Prim}^n(X) \to H^n(X - X_t) \to H^{n-1}(X_t)_v \to 0$$

which sheafifies to give

$$0 \to \mathrm{Prim}^n(X) \to R^n g_* \mathbb{C} \to (R^{n-1} f_* \mathbb{C})_v \to 0$$

where $\mathrm{Prim}^n(X)$ is the constant sheaf on U. Tensoring with $\mathcal{O}_{U^*}$. we obtain the exact sequence of sheaves over U.

$$0 \to \mathrm{Prim}^n(X) \otimes \mathcal{O}_{U^*} \to \mathcal{K} \to \mathcal{H}_v \to 0$$

where to identify $\mathcal{H}_v$ we note that the direct sum

$$H^{n-1}(X_t) = i^* H^{n-1}(X) \dotplus H^{n-1}(X_t)_v$$

passes to $\overline{\mathcal{H}}$. Thus. there is a vanishing part $\mathcal{H}_v \subset \mathcal{H}$ and coresponding $\mathcal{F}_v \subset \mathcal{F}.\overline{\mathcal{H}}_v. \subset \overline{\mathcal{H}}$ and corresponding $\overline{\mathcal{F}}_v \subset \overline{\mathcal{F}}$. Putting Hodge filtration levels into the preceding short exact sequence.

$$0 \to F^{m+1}\mathrm{Prim}^n(X) \otimes \mathcal{O}_{U^*} \to \mathcal{G} \xrightarrow{\text{residue}} \mathcal{F}_v \to 0.$$

(See [E] for a definition of $\mathcal{G}$.)

(VI)

Lemma

There exists an exact sequence

$$0 \to F^{m+1}\mathrm{Prim}^n(X) \otimes \mathcal{O}_{\mathbb{P}} \to \overline{\mathcal{G}} \to \overline{\mathcal{F}}_v \to 0$$

extending the last one.

Proof

From the exact sequence

$$0 \to \mathrm{Prim}^n(X) \to R^n g_* \mathbb{C} \to (R^{n-1} f_* \mathbb{C})_v \to 0$$

we infer $R^n g_* \mathbb{C} \cong \mathrm{Prim}^n(X) \oplus (R^{n-1} f_* \mathbb{C})_v$ over a punctured disc Δ^* around a singular point. In this punctured disc

$$0 \to \mathrm{Prim}^n(X) \otimes \mathcal{O}_{U} \to \mathcal{K} \to \mathcal{H}_v \to 0$$

is horizontally split. By the Canonical Extension Theorem, the splitting passes to the canonical extensions over Δ, so we have

$$\overline{\mathcal{K}} \cong (\mathrm{Prim}^n(X) \otimes \mathcal{O}_\Delta) \oplus \overline{\mathcal{H}}_v$$

On Δ^*, the last exact sequence yields

$$\mathcal{G} \cong (F^{m+1}\mathrm{Prim}^n(X) \otimes \mathcal{O}_\Delta) \oplus \mathcal{F}_v$$

As $\overline{\mathcal{K}}$ modulo $(F^{m+1}\mathrm{Prim}^n(X) \otimes \mathcal{O}_\Delta) \oplus \overline{\mathcal{F}}_v$ is a free $\mathcal{O}_\Delta$-module, we must have

$$\overline{\mathcal{G}} \cong (F^{m+1}\mathrm{Prim}^n(X) \otimes \mathcal{O}_\Delta) \oplus \overline{\mathcal{F}}_v.$$

□

(VII) Take the tensor product of the sequence from the lemma with $\Omega^1_{\mathbb{P}}$ to obtain

$$0 \to \Omega^1_{\mathbb{P}} \otimes F^{m+1}\mathrm{Prim}^n(X) \to \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}} \to \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v \to 0$$

Its cohomology sequence (using $H^0(\mathbb{P}, \Omega^1_{\mathbb{P}}) = 0$) is

$$0 \to H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}) \to H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v) \xrightarrow{\delta} H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes F^{m+1}\mathrm{Prim}^n(X))$$

$$\to H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}) \to H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v) \to \cdots$$

But $F^{m+1}\mathrm{Prim}^n(X))$ is a finite dimensional $\mathbb{C}$-vector space. So $H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes F^{m+1}\mathrm{Prim}^n(X))$ is equal to $H^1(\mathbb{P}, \Omega^1_{\mathbb{P}}) \otimes F^{m+1}\mathrm{Prim}^n(X)$.

The exact sequence becomes

$$0 \to H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}) \to H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_\cdot) \xrightarrow{\delta} H^1(\mathbb{P}, \Omega^1_{\mathbb{P}}) \otimes F^{m+1}\mathrm{Prim}^n(X)$$
$$\to H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}) \to \cdots .$$

L Two residues.

We repeat the computations of the last section, first tensoring the sheaves with $\mathcal{O}(1)$ (pole at ∞). Since $H^1(\mathbb{P}, \Omega^1_{\mathbb{P}}(1)) = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-1)) = 0$ by Serre duality, and $\Omega^1_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-2)$, the last exact sequence becomes

$$0 = H^0(\mathbb{P}, \Omega^1_{\mathbb{P}}(1) \otimes F^{m+1}\mathrm{Prim}^n(X)) \to H^0(\mathbb{P}, \Omega^1_{\mathbb{P}}(1) \otimes \overline{\mathcal{G}})$$
$$\to H^0(\mathbb{P}, \Omega^1_{\mathbb{P}}(1) \otimes \overline{\mathcal{F}}_v) \to H^1(\mathbb{P}, \Omega^1_{\mathbb{P}}(1)) \otimes F^{m+1}\mathrm{Prim}^n(X) = 0,$$

implying an isomorphism

$$H^0(\mathbb{P}, \Omega^1_{\mathbb{P}}(1) \otimes \overline{\mathcal{G}}) \cong H^0(\mathbb{P}, \Omega^1_{\mathbb{P}}(1) \otimes \overline{\mathcal{F}}_v)$$

Given $\sigma \in H^0(\mathbb{P}, \Omega^1_{\mathbb{P}}(1) \otimes \overline{\mathcal{F}}_v)$, its lifting

$$\tilde{\sigma} \in H^0(\mathbb{P}, \Omega^1_{\mathbb{P}}(1) \otimes \overline{\mathcal{G}}) \subset H^0(\mathbb{P}, \Omega^1_{\mathbb{P}}(1) \otimes \overline{\mathcal{K}}) \subset H^0(\mathbb{P}, \mathcal{T})$$

(recall $\mathcal{T} := \mathrm{im}\overline{\nabla}$ (see **G**)) considered (the pole is at infinity) as an element of $H^0(U_0, \mathcal{T})$ gives rise, via the connecting homomorphism from

$$0 \to j_* R^{2m} g_* \mathbb{C} \to \overline{\mathcal{K}} \to \mathcal{T} \to 0,$$

to a class $\xi \in H^1(U_0, R^{2m}\overline{g}_*\mathbb{C}) \subset H^{n+1}(\overline{C} - C_\infty)$ with two residues

$$\alpha = \mathrm{Res}_{\overline{Y} - X_\infty}(\xi) \in H^n(\overline{Y} - X_\infty)$$

and

$$\beta = \mathrm{Res}_{X - X_\infty}(\xi) \in H^n(X - X_\infty).$$

We will see later (in the main argument) that α and β represent inverse classes in $\mathrm{Prim}^n(X)$ (statement **8**).

M Two sheaves to be compared later.

According to [Deligne,p.80] there is an isomorphism on U:

$$\mathcal{G} = \mathbf{R}^n \pi_{2*} G^{m+1} \Omega^\bullet_{X \times U/U}(\bullet Y)$$

where $\{G^p\}$ is the *order-of-pole filtration* on $\Omega^\bullet_{X\times U/U}(\bullet Y)$, with $G^{m+1}\Omega^\bullet_{X\times U/U}(\bullet Y)$ being

$$\underbrace{0 \to \cdots \to 0}_{p-1} \to \Omega^p_{X\times U/U}(Y) \to \Omega^{p+1}_{X\times U/U}(2Y) \to \Omega^{p+2}_{X\times U/U}(3Y) \to \cdots$$

There is an obvious extension of $\mathcal{G}$ (using this expression of it) to all of $\mathbb{P}$:

$$\overline{\mathcal{G}}' = \mathbf{R}^n \pi_{2*} G^{m+1} \Omega^\bullet_{X\times\mathbb{P}/\mathbb{P}}(\bullet \overline{Y})$$

[N] A larger algebraic variety.

Now we express $\overline{Y}$ itself as the fiber of a morphism of a larger algebraic variety to $\mathbb{P}$. To this end construct a suitable degenerate divisor rationally equivalent to $\overline{Y}$

Define

$$Z = \{(x, a, b) \in X \times \mathbb{P} \times \mathbb{P} \subset \mathbb{P}^N \times \mathbb{P} \times \mathbb{P} : a_0 b_0 x_0 + (a_1 b_0 - a_0 b_1) x_1 = 0\}$$

We have the diagram of maps

$$\begin{array}{ccc} Z & \longrightarrow & X \times \mathbb{P} \times \mathbb{P} \\ \downarrow & \swarrow_{\pi} & \\ \mathbb{P}\times\mathbb{P} & & \swarrow_{\chi} \\ \downarrow_{pr_2} & & \\ \mathbb{P} & & \end{array}$$

where $\pi = \mathrm{pr}_{2,3}, \chi = \mathrm{pr}_3$. Define $Z_s = \chi^{-1}(s)$. Note that $Z_0 = \overline{Y}$ and $Z_\infty = X_\infty \times \mathbb{P} \cup X \times \{\infty\}$. Now, using the order-of-pole filtration from [M], we set

$$\mathcal{Z}^\bullet = G^{m+1}\Omega^\bullet_{X\times\mathbb{P}\times\mathbb{P}/\mathbb{P}\times\mathbb{P}}(\bullet Z).$$

The restriction of $\mathcal{Z}^\bullet$ to $\chi^{-1}(s) \cong X \times \mathbb{P}$

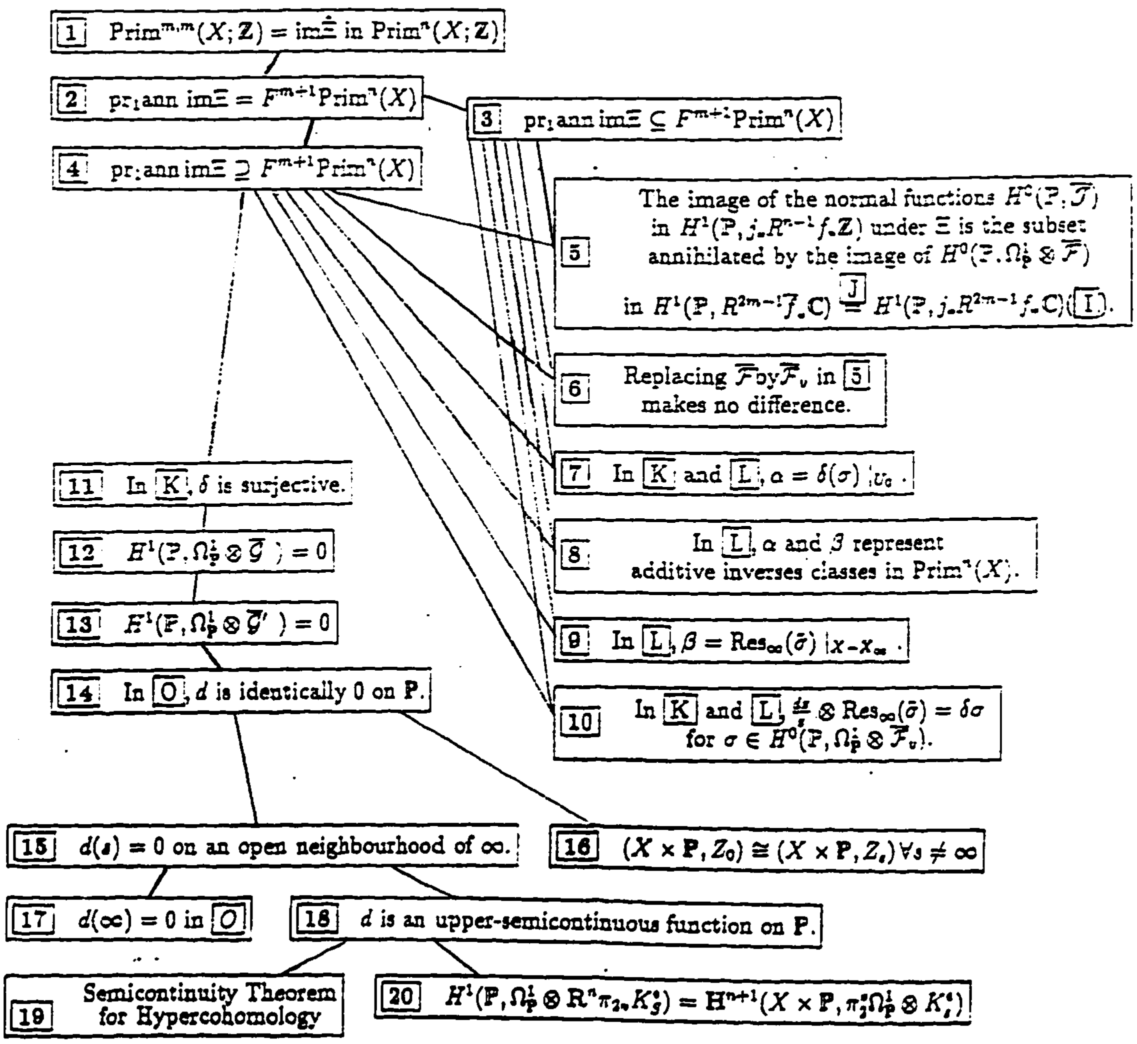
$$\mathcal{Z}^\bullet_s := G^{m+1}\Omega^\bullet_{X\times\mathbb{P}/\mathbb{P}}(\bullet Z_s).$$

$\boxed{\mathbf{O}}$ An upper-semicontinuous function.
Define the function

$$\begin{aligned} d : \mathbb{P} &\to \mathbb{Z}^{\geq 0} \\ s &\mapsto \dim H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n \pi_{2*} \mathcal{Z}^\bullet_s). \end{aligned}$$

Upper-semicontinuity (statement $\boxed{19}$ in the argument) will be proved later.

THE ARGUMENT
Schema



$\boxed{1} \Leftarrow \boxed{2}$

The decomposition $H^1(\mathbb{P}, j_*R^{2m-1}f_*\mathbb{C}) \cong \mathrm{Prim}^n(X) \oplus H^{n-2}(D)_v$ in $\boxed{J}$ is orthogonal, which means that under the cup product pairing in $\boxed{I}$.

$$H^1(\mathbb{P}, j_*R^{2m-1}f_*\mathbb{C}) \otimes H^1(\mathbb{P}, j_*R^{2m-1}f_*\mathbb{C}) \to H^2(\mathbb{P}, j_*R^{4m-2}f_*\mathbb{C}) \cong H^2(\mathbb{P}, \mathbb{C}).$$

and the decomposition, we have, for $\tau \in H^1(\mathbb{P}, j_*R^{2m-1}f_*\mathbb{C})$, $\mathrm{pr}_1 \mathrm{ann}\, \tau = \mathrm{ann}\ \mathrm{pr}_1 \tau \in \mathrm{Prim}(X)$. (By pr_1 we mean projection onto the first term in the decomposition

$$H^1(\mathbb{P}, j_*R^{2m-1}f_*\mathbb{C}) \cong \mathrm{Prim}^n(X) \oplus H^{n-2}(D)_v.$$

And "ann" refers to the annihilator under the cup product pairing.) We conclude

$$\mathrm{Prim}^{m,m}(X:\mathbb{Z}) = \mathrm{im}\hat{\Xi} \text{ in } \mathrm{Prim}^n(X:\mathbb{Z})$$

$$\Uparrow$$

$$\mathrm{ann}\ \mathrm{Prim}^{m,m}(X:\mathbb{Z}) = \mathrm{ann}\ \mathrm{pr}_1(\mathrm{im}\Xi) \text{ in } \mathrm{Prim}^n(X:\mathbb{Z})$$

$$\Uparrow$$

$$F^{m+1}\mathrm{Prim}^n(X:\mathbb{Z}) = \mathrm{pr}_1(\mathrm{ann}\ \mathrm{im}\Xi)$$

□

$\boxed{2} \Leftarrow \boxed{3} \& \boxed{4}$

Trivial. □

$\boxed{3} \Leftarrow \boxed{5} \& \boxed{6} \& \boxed{7} \& \boxed{8} \& \boxed{9} \& \boxed{10}$

By $\boxed{5}$ and $\boxed{6}$, we need only show that the projection (in $\boxed{J}$) of the image (in $\boxed{I}$) of $H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v)$ in $H^1(\mathbb{P}, j_*R^{2m-1}f_*\mathbb{C})$ under δ is contained in $F^{m+1}\mathrm{Prim}(X)$. In other words, we need only show

$$\mathrm{pr}_1 \circ \delta(H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v)) \subseteq F^{m+1}\mathrm{Prim}^n(X).$$

To this end, let $\sigma \in H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v)$. Then form α, β as in $\boxed{L}$. Now $\boxed{7}$, $\boxed{8}$, and $\boxed{9}$ tell us that $\mathrm{pr}_1(\delta(\sigma)\ |_{U_0}) = \mathrm{pr}_1(\alpha) = -\mathrm{pr}_1(\beta) = -\mathrm{Res}_\infty(\tilde{\sigma})\ |_{U_0}$ in $\mathrm{Prim}^n(X)$. But $\boxed{10}$ reveals that $\frac{dz}{z} \otimes \mathrm{Res}_\infty(\tilde{\sigma}) \in H^1(\mathbb{P}, \Omega^1_{\mathbb{P}}) \otimes F^{m+1}\mathrm{Prim}^n(X)$, especially that $\mathrm{Res}_\infty(\tilde{\sigma}) \in F^{m+1}\mathrm{Prim}^n(X)$ This implies

$$\mathrm{pr}_1(\delta(\sigma)) \in F^{m+1}\mathrm{Prim}^n(X).$$

□

[4] ⇐ [5]&[6]&[7]&[8]&[9]&[10]&[11] Again, by [5] and [6], we need only show that $F^{m+1}\mathrm{Prim}^n(X)$ is contained in the projection (in [J]) of the image (in [I]) of $H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v)$ in $H^1(\mathbb{P}, j_*R^{2m-1}f_*\mathbb{C})$ under δ. In other words, we need only show

$$\mathrm{pr}_1 \circ \delta(H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v)) \supseteq F^{m+1}\mathrm{Prim}^n(X).$$

To this end, let $\tau \in F^{m+1}\mathrm{Prim}^n(X)$. Then by [11], $\frac{dz}{z} \otimes \tau \in H^1(\mathbb{P}, \Omega^1_{\mathbb{P}}) \times F^{m+1}\mathrm{Prim}^n(X)$ has a pre-image (call it σ) in $H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}^m_v)$. Form α, β as in [L]. Now [10] says that $\frac{dz}{z} \otimes \mathrm{Res}_\infty(\tilde{\sigma}) = \delta\sigma = \frac{dz}{z} \otimes \tau$, especially that $\mathrm{Res}_\infty(\tilde{\sigma}) = \tau$. Next, [8], [9] reveal that $\mathrm{pr}_1(\alpha) = -\mathrm{pr}_1(\beta) = -\mathrm{Res}_\infty(\tilde{\sigma}) \mid_{X-X_\infty} = -\tau \mid_{X-X_\infty}$ But $\mathrm{pr}_1(\alpha) = \mathrm{pr}_1(\delta(\sigma) \mid_{U_0})$, so that $\tau \mid_{U_0} = -\mathrm{pr}_1(\alpha) = -\mathrm{pr}_1(\delta(\sigma) \mid_{U_0})$.

$$\rightarrow \mathrm{pr}_1(\delta(H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v)))$$

□

[5] is true

We use the diagram from [I]. θ is the from the long exact sequence for

$$0 \rightarrow j_*R^{n-1}f_*\mathbb{Z} \rightarrow \overline{\mathcal{F}}^* \rightarrow \overline{\mathcal{J}} \rightarrow 0,$$

namely

$$\cdots \rightarrow H^0(\mathbb{P}, \overline{\mathcal{J}}) \xrightarrow{\Xi} H^1(\mathbb{P}, j_*R^{n-1}f_*\mathbb{Z}) \xrightarrow{\theta} H^1(\mathbb{P}, \overline{\mathcal{F}}^*) \rightarrow \cdots$$

Thus $\mathrm{im}\Xi = \ker\theta$, $\Rightarrow$ ann $\mathrm{im}\Xi = \mathrm{im}\delta$, since the pairings are dual. This is what we wanted to prove. □

[6] is true

The decomposition $H^{n-1}(X_t) \cong H^n(X) \oplus H^{n-1}(X_t)_v$ passes to $\overline{\mathcal{H}}$. Thus $\overline{\mathcal{H}}$ splits into a constant part and a vanishing part (call it $\overline{\mathcal{H}}_v$):

$$\overline{\mathcal{H}} \cong H^n(X) \oplus \overline{\mathcal{H}}_v.$$

There is a corresponding $\overline{\mathcal{F}}_v \subset \overline{\mathcal{F}}$ (given by $\overline{\mathcal{F}}_v = \overline{\mathcal{F}} \cap \overline{\mathcal{H}}_v$). So $\overline{\mathcal{F}}$ splits thus:

$$\overline{\mathcal{F}} = \overline{\mathcal{F}}_v \oplus C,$$

where C is constant. Then $H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}) = H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v \oplus \Omega^1_{\mathbb{P}} \otimes C) = H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v) \oplus H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes C) = H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v) \oplus 0$. This means we may replace $\overline{\mathcal{F}}$ by $\overline{\mathcal{F}}_v$, equivalently $H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}})$ by $H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v)$, in [I] and [5]. □

[[7] is true]

There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^n\overline{g}_*\mathbb{C} & \longrightarrow & \overline{\mathcal{K}} & \longrightarrow & \mathcal{T} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (R^{n-1}\overline{f}_*\mathbb{C})_v & \longrightarrow & \overline{\mathcal{H}}_v & \longrightarrow & \mathcal{S}_v & \longrightarrow & 0 \end{array}$$

where the first two vertical arrows are residue maps, $\mathcal{S}_v$ is defined as the image of $\overline{\mathcal{H}}_v$ under $\overline{\nabla}$, and the map $\mathcal{T} \to \mathcal{S}_v$ is defined by commutivity via the exactness of the rows.

Use the cover $\{U_0, U_1\} = \{\mathbb{P}-\{\infty\}, \mathbb{P}-\{0\}\}$. Using the previous diagram and the inclusions $\Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{H}} \subset \mathcal{S}$ and $\Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{K}} \subset \mathcal{T}$ from [G], we obtain the other commutative diagram:

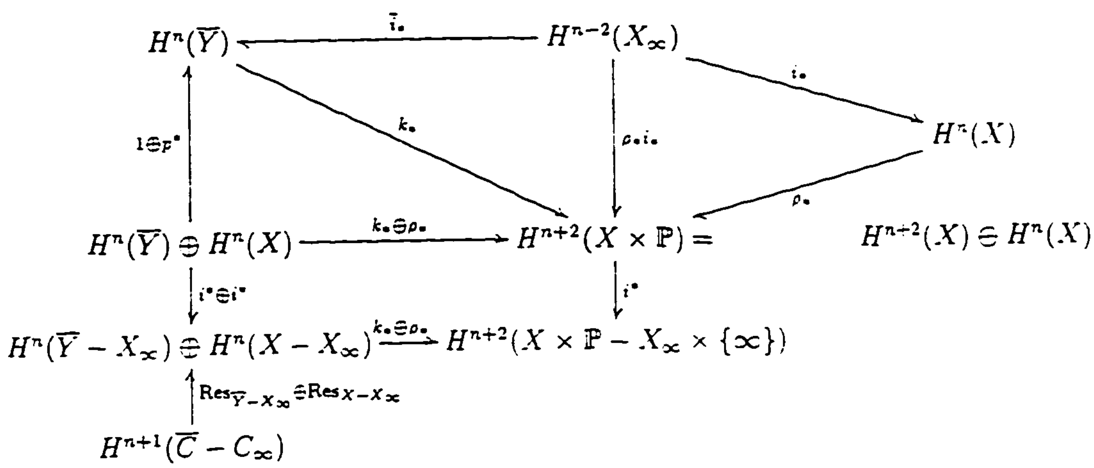
$$\begin{array}{ccccc} H^0(U_0, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}) & \longrightarrow & H^1(U_0, R^n\overline{g}_*\mathbb{C}) & \subset & H^{n+1}(\overline{C} - C_\infty) \\ \downarrow & & \downarrow & & \vdots \\ H^0(U_0, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v) & \xrightarrow{\delta|} & H^1(U_0, R^{n-1}\overline{f}_*\mathbb{C}) & \subset & H^{n+1}(\overline{Y} - X_\infty) \end{array}$$

Note that the map $\delta\,|$ is the same δ, restricted to $U_0 \subset \mathbb{P}$, as the δ in [I], both being the connecting homomorphism from the short exact sequences in [G]. Now, refer to [L] and the diagram above. The image of $\tilde{\sigma} \in H^0(U_0, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}})$ when traced through the diagram, over then down, is $\mathrm{Res}_{\overline{Y}-X_\infty}(\xi) = \alpha$. When traced down then over, it is $\delta\,|_{U_0}(\tilde{\sigma}) = \delta(\tilde{\sigma})\,|_{U_0}$. This shows the result. □

[[8] is true]

What we aim to prove specifically is that for p as in [B] and pr_1 as in [J], $\mathrm{pr}_1(p^*(\beta)) = -\mathrm{pr}_1(\alpha)$. This is equivalent to $p^*(\beta) + \alpha$ lying in the subspace $H^{n-2}(D)_v$ of $H^1(\mathbb{P}, R^{n-1}f_*\mathbb{C})$ under the decomposition $H^1(\mathbb{P}, R^{n-1}f_*\mathbb{C}) \cong \mathrm{Prim}^n(X) \oplus H^{n-2}(D)_v$ from [J].

First construct the diagram:



We specify the maps. $\bar{i}_*$ is the dual map to the pushforward on homology from inclusion $\bar{i} : X_\infty \to \overline{Y}$. i_* is the same for $i : X_\infty \to X$. ρ_* is the pushforward from any of the inclusions $\rho : X \to X \times \mathbb{P}$ as a fiber of π_2. k_* is the Gysin map, the dual to the pushforward on homology from the inclusion $k : \overline{Y} \to X \times \mathbb{P}$. The maps i^* are pullbacks of the obvious inclusions. The remaining map is the sum of residues defined from the unions

$$(\overline{Y} - X_\infty) \cup (\overline{C} - C_\infty) = X \times \mathbb{P} - X_\infty \times \{\infty\}$$

and

$$(X - X_\infty) \cup (\overline{C} - C_\infty) = X \times \mathbb{P} - X_\infty \times \{\infty\}.$$

(As subsets of $X \times \mathbb{P}$, $\overline{Y} - X_\infty$ and $X - X_\infty$ are equal.)

The lower rectangle, intuitively, commutes. Commutivity in the upper rightmost triangle is trivial. The upper middle triangle $\{\bar{i}_*, k_*, \rho_* i_*\}$ commutes, as it is the pushforwards of the maps $\bar{i}, k, \rho \circ i$, which commute:

$$\rho_* i_* = (\rho \circ i)_* = (k \circ \bar{i})_* = k * \bar{i}_*$$

All that is left is the upper left triangle $\{1 \oplus p^*), k_*, k_* \oplus \rho_*\}$. Here, however, we have, from $\boxed{C}$,

$$k_* \circ (1 \oplus p^*) - k_* \oplus \rho_* = k_* \oplus k_* \circ p^* - k_* \oplus \rho_* = 0 \oplus (k_* \circ p^* - \rho_*) = \pi_1^* \circ L.$$

So this triangle is not commutative. But if we replace $H^n(\overline{Y}) \oplus H^n(X)$ by $H^n(\overline{Y}) \oplus \mathrm{Prim}^n(X)$, then since $\mathrm{Prim}^n(X) = \ker L : H^n(X) \to H^{n+2}(X)$,

$$k_* \circ (1 \oplus p^*) - k_* \oplus \rho_* = \pi_1^* \circ L = 0,$$

and so the whole diagram now commutes. □

$\boxed{\boxed{9} \text{ is true}}$

Since we are looking at a residue at ∞, we may consider just a small disc $\Delta \subset \mathbb{P}$ centred at ∞ on which $\overline{f}$ is smooth. The commutative diagram

$$\begin{array}{ccc}
H^0(\Delta, \Omega^1_{\mathbb{P}}(1) \otimes \mathbf{R}^n \pi_{2*} F^{m+1} \Omega^\bullet_{X \times \mathbb{P}/\mathbb{P}}(\log \overline{Y}) & \longrightarrow & H^0(\Delta^*, \Omega^1_{\Delta^*} \otimes \mathcal{G}) \\
 & & \downarrow \\
 & & H^1(\Delta^*, R^n g_* \mathbb{C}) \\
\Big\downarrow \mathrm{Res}_\infty & & \updownarrow \cong \\
 & & H^1(\Delta^*) \otimes_{\mathbb{C}} H^n(C_t) \\
 & & \downarrow \\
\mathbf{H}^n(X, F^{m+1} \Omega^\bullet_X \log X_\infty)) & \xrightarrow{\mathrm{Res}_{X - X_\infty}} & H^n(X - X_\infty)
\end{array}$$

shows $\beta = \mathrm{Res}_{X-X_\infty}(\xi) = \mathrm{Res}_\infty(\tilde{\sigma})$. □

[10] is true

The statement of [10], precisely, is

Lemma Let $\sigma \in H^0(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v)$ and $\tilde{\sigma} \in H^0(\mathbb{P}, \Omega_P \mathbb{P}^1(1) \otimes \overline{\mathcal{G}})$ be its lifting under the isomorphism from [L]. Then

$$\frac{dz}{z} \otimes \mathrm{Res}_\infty(\tilde{\sigma}) = \delta\sigma$$

where $\frac{dz}{z} \in H^1(\mathcal{U}, \Omega^1_{\mathbb{P}})$, and $\mathcal{U}$ is the covering $\{U_0 = \mathbb{P} - \{\infty\}, U_\infty = \mathbb{P} - \{0\}\}$ of $\mathbb{P}$.

Proof Let us calculate $\delta\sigma$. Precisely, σ is the 0-cochain

$$U_0 \mapsto \sigma\,|_{U_0}$$

$$U_\infty \mapsto \sigma\,|_{U_\infty}$$

We must determine the pre-image of $\sigma\,|_{U_0}$ and $\sigma\,|_{U_\infty}$ under $\Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}} \to \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{F}}_v$. We may lift $\sigma\,|_{U_0}$ to $\tilde{\sigma}\,|_{U_0} \in H^0(U_0, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}})$.

By an almost identical construction to that which gave us $\tilde{\sigma}$, we may find $\tau \in H^0(\mathbb{P}, \Omega^1_{\mathbb{P}}(1) \otimes \overline{\mathcal{G}})$ where the pole is at 0 instead of at ∞. Then $\tau\,|_{U_\infty} \in H^0(U_\infty, \Omega^1_{\mathbb{P}}(1) \otimes \overline{\mathcal{G}})$. Now apply the coboundary operator to the cochain

$$U_0 \mapsto \tilde{\sigma}\,|_{U_0}$$

$$U_\infty \mapsto \tau\,|_{U_\infty}$$

We obtain the 1-cochain

$$(U_\infty, U_0) \mapsto (\tilde{\sigma} - \tau)\,|_{U_\infty \cap U_0}\,.$$

Finally, the pre-image of the cochain must be identified, that is, under the map

$$\Omega^1_{\mathbb{P}} \otimes F^{m+1}\mathrm{Prim}^n(X) \to \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}.$$

But this map is essentially an inclusion, so that we finally have

$$\delta\sigma = \tilde{\sigma} - \tau \text{ on } U_\infty \cap U_0$$

For $\lambda = \mathrm{Res}_\infty(\tilde{\sigma})$,

$$\tilde{\sigma} - \tau - \frac{dz}{z} \otimes \lambda \in H^0(U_\infty \cap U_0, \Omega^1_{\mathbb{P}} \otimes F^{m+1}\mathrm{Prim}^n(X))$$

has no pole at ∞ (and only a first order pole at 0), hence is identically 0. □

$\boxed{\boxed{11} \Leftarrow \boxed{12}}$

This follows immediately from the exactness of the sequence in $\boxed{K}$. □

$\boxed{12} \Leftarrow \boxed{13}$

We will prove this by showing that $\overline{\mathcal{G}}' \subseteq \overline{\mathcal{G}}$, for we will then have the exact sequence

$$0 \to \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}' \to \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}} \to \mathcal{Q} \to 0$$

with the quotient sheaf supported on Σ (since $\overline{\mathcal{G}}' |_{U} = \mathcal{G} = \overline{\mathcal{G}} |_{U}$. The long exact cohomology sequence will in part be

$$H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}') \to H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}) \to H^1(\mathbb{P}, \mathcal{Q}) = 0$$

Then $H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}') = 0$ will imply $H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}) = 0$. So again, it suffices to prove $\overline{\mathcal{G}}' \subseteq \overline{\mathcal{G}}$.

We will determine bounds for the growth of the periods of the growth of the periods of local sections of $\mathcal{G}'$ near singular points. It is this property which characterizes the canonical extension in terms of growth.

Let Δ be a local section of $\mathcal{G}'$ is represented by a relatively closed C^∞ n-form ϕ on $X \times \Delta - \overline{Y}$ that has a specified order of pole along $\overline{Y}$ on each local coordinate. In particular, ϕ is a relatively closed form with a pole of order at most $n - (m+1) + 1 = m$.

Let W be a small polydisc in $X \times \mathbb{P}$ centred at the double point of X_0, with $\mathrm{pr}_2(W) = \Delta$. Now the image of $H_n(C_t - W)$ in $H_n(C_t)$ is (for $\neq 0$) a codimension 1 invariant subspace. Chains representing these classes can be kept uniformly away from X_t, so the periods of ϕ are bounded on Δ. A subspace of $H_n(C_t)$ orthogonal to $H_n(C_t - W)$ is generated by the tube on the *dual cycle* (see [12,pp.470,518]). Again, the periods arising from the complement of W is bounded, so we have only a local computation on X_0.

Pick local "double point" coordinates $(x_1, \dots x_n)$ on X and use t for Δ so that $\overline{Y} \times \mathbb{P}$ is given locally by

$$\left\{ (x, t) : \sum_{j=1}^{n} x_j^2 = t \right\}.$$

We may assume $|x_1| < 1$ and $\sum_{j=2}^{n} |x_j|^2 < 1$. On X_t, the relevant portion of the dual cycle is given by [12,p.518]

$$\gamma_t = \{x \in X_t : x_1 \in e^{\frac{1}{2}i\theta}\mathbb{R}, x_{j>1} \in e^{\frac{1}{2}i\theta}\sqrt{-1}\mathbb{R}\}$$

for $t = |t|e^{i\theta}$. Without loss of generality assume t real and positive (i.e. $\theta = 0$).

Since ϕ has a pole of order at most m, we may write, locally,

$$\phi = \left(\sum_{j=1}^{n} x_j^2 - t\right)^{-m} \sum_{\#I+\#J=n} h_{IJ}(x,t)dx_I d\overline{x}_J$$

with the h_{IJ}'s C^∞ functions. Let $\phi_t = \phi \mid_{X\times\{t\}}$, so that $(\mathrm{Res}_{\overline{Y}}(\phi)) \mid_{X_t} = \mathrm{Res}_{X_t}(\phi_t)$. We are interested in the section ϕ integrated over elements of $H_n(C_t)$. But the subspace (image of) $H_n(C_t - W)$ is invariant and the complementary subspace is generatd by tubeγ. Thus the integral of interest is

$$\int_{(\mathrm{tube}\gamma)|X_t} \phi$$

which by duality and since $\gamma_t \in \overline{Y}$, equals

$$\int_{\gamma_t} \mathrm{Res}_{\overline{Y}}(\phi) = \int_{\gamma_t} \mathrm{Res}_{X_t}(\phi_t).$$

We need a result to determine the residue explicitly. It will be proved by reducing the order of pole [Griffiths, p.489]. (Simple-pole forms have well-defined residues (see [Lewis, lec. 9]).)

Let $\phi = y^{-m}\eta$, on the unit polydisc Δ^n, with coordinates $(y, z_2, \ldots, z_n)$, and where η is a C^∞ form.

First we need a lemma, which follows from the similar fact about Taylor series for real variables.

Lemma Let $h(y,z)$ be a C^∞ function on Δ^n. Then there is a partial Taylor expansion

$$h(y,z) = \sum_{l=0}^{m-1} a_l(z)y^l + b_1(y,z)y^m + \overline{y}b_2(y,z)$$

with b_1, b_2 being C^∞, and $a_l = \frac{\partial^l h}{\partial y^l}(0,z)$.○

Applying the lemma to the coefficients of η, we write

$$\phi = y^{-m}\eta_m + \cdots + y^{-1}\eta_1 + \eta_0 + y^{-m}\tilde{\phi}$$

where $\eta_1, \ldots, \eta_m$ are free of $d\overline{y}$ and $\overline{y}$, with coefficients independant of y; η_0 is C^∞, and every term of ϕ involves $\overline{y}$ or $d\overline{y}$. Write $\eta_1 = \mu_1 \wedge dy + \mu_2$, where μ_2 is free of dy.

Proposition $\mathrm{Res}(\phi) = \mu_1$ ○

Proof First, the form o, a closed form with a pole of order m, must have a logarithmic plus a pole of order $m-1$. Applying this expression

$$o = y^{-m}\eta_m + \cdots + y^{-1}\eta_1 + \eta_0 + y^{-m}\tilde{o}$$

enables us to write

$$\eta_m = \tilde{\eta}_m \wedge dy$$
$$\tilde{o} = \upsilon_1 \wedge dy + y\upsilon_2$$

(Every term of υ_1 and υ_2 involves $\overline{y}$ or $d\overline{y}$.)

Subtracting an exact form from o will not change its residue. So for $m > 1$ subtract

$$\frac{d(y^{1-m}(\tilde{\eta}_m + \upsilon_1))}{1-m} = \frac{(d\tilde{\eta}_m + d\upsilon_1)y^{1-m}}{1-m} + y^{-m}(\tilde{\eta}_m + \upsilon_1) \wedge dy$$

$$= \left(\frac{d\tilde{\eta}_m + d\upsilon_1}{1-m}\right) y^{1-m} + (y^{-m}\eta_m + y^{-m}\tilde{o}) - y^{1-m}\upsilon_2$$

The difference contains a pole of order only $m-1$. Using the lemma, we may write the difference as

$$o = y^{1-m}\eta_{m-1} + \cdots + y^{-1}\eta_1 + \eta_0 + y^{1-m}\tilde{o}$$

(thus redefining $\eta_1, \ldots, \eta_{m-1}, \tilde{o}$). Note carefully, however, that $\eta_1, \ldots, \eta_{m-1}$ are the same as before, $\tilde{o}$ has been changed, and to η_{m-1} has been added

$$\frac{d\tilde{\eta}_m}{1-m}$$

which is independant of dy.

We proceed inductively, reducing the order of the pole by one each time, with a process that discards the η_i with the highest i, adds something independent of dy to the η_i with the second-highest i, and changes $\tilde{o}$. We continue the process until the pole is of order 1. It is worth noticing that the process leaves η_1 alone until the final step, in which something is added to $\eta_1 = \mu_1 \wedge dy + \mu_2$ independant of dy. In other words, only μ_2 is affected, not μ_1.

Now we have a form

$$o' = y^{-1}\eta_1 + \eta_0 + y^{-1}\tilde{o}$$

(redefining $\eta, \tilde{o}$) which has the same residue as o. Expressing $\eta_1 = \mu_1 \wedge dy + \mu_2$ we notice that only μ_2 has been changed from the original μ_2, and not μ_1. And the residue of o' is clearly μ_1. □

We may apply the previous proposition to o_t, replacing o with o_t. To make further use of the result, we change variables. Replace $\{x_1, \ldots, x_n\}$ by $\{s = \sum_{j=1}^n x_j^2, x_2, \ldots, x_n\}$. Here $x_1 = \sqrt{s - \sum_{j=2}^n x_j^2}$, which is well-defined since x_1 is positive and real there, gives us a well-defined branch of $\sqrt{}$. We have

$$dx_1 = \left(s - \sum_{j=2}^{n} x_j^2\right)^{-\frac{1}{2}} \left(\frac{1}{2}ds - \sum_{j=2}^{n} x_j dx_j\right)$$

and may simplify the calculation of $\mathrm{Res}_{X_t}(o_t)$ as follows. To use the lemma and proposition, we must compute the $(m-1)$st derivatives of the coefficients of o_t with respect to s and evaluate them at $s = t$, i.e. at $\sqrt{\sum_{j=1}^n x_j^2} = t$, i.e. on X_t. In doing this, only those terms involving ds contribute to the residue. On γ_t, $t \in \mathbb{R}^+$, so that $x_1 \in \mathbb{R}^+, x_2, \ldots, x_n \in \sqrt{-1}\mathbb{R}^+$, whence $dx_1 = d\overline{x}_1, dx_j = -d\overline{x}_j$ for $j > 1$. So any term containing $dx_j \wedge d\overline{x}_j$ will integrate to 0 on γ_t. Thus only some of the terms of

$$o_t = \left(\sum_{j=1}^{n} x_j^2 - t\right)^{-m} \sum_{\#I+\#J=n} h_{IJ}(x,t) dx_I d\overline{x}_J.$$

those in which dx_1, and for $j > 1$, one of dx_j, $d\overline{x}_j$, appears, need be considered. In the substitution

$$dx_1 = \left(s - \sum_{j=2}^{n} x_j^2\right)^{-\frac{1}{2}} \left(\frac{1}{2}ds - \sum_{j=2}^{n} x_j dx_j\right)$$

may ignore the dx terms and pretend that

$$dx_1 = \frac{1}{2}\left(s - \sum_{j=2}^{n} x_j^2\right)^{-\frac{1}{2}} ds$$

We will estimate the growth of

$$\int_{\gamma_t} \mathrm{Res}_{X_t}(o_t)$$

term by term, so we will assume that

$$\omega_t = \left(\sum_{j=1}^{n} x_j^2 - t\right)^{-m} h(x,t)dx_1 \wedge \cdots \wedge dx_n$$

$$= -\left(\sum_{j=1}^{n} x_j^2 - t\right)^{-m} h(x,t)dx_2 \wedge \cdots \wedge dx_n \wedge dx_1$$

$$= -\frac{1}{2}(s-t)^{-m}\left(s - \sum_{j=1}^{n} x_j^2 - t\right)^{-\frac{1}{2}} h(x,t)dx_1 \wedge \cdots \wedge ds.$$

To use the propostion, we must compute

$$\left[\frac{\partial^{m-1}}{\partial s^{m-1}}\left(\frac{1}{2}\left(s - \sum_{j=2}^{n} x_j^2\right)^{-\frac{1}{2}} h(x,t)\right)\right]_{s=t}$$

which by Leibniz's rule becomes

$$\left[\sum_{k=0}^{m-1} c_k \frac{\partial^k h}{\partial s^k}\left(s - \sum_{j=2}^{n} x_j^2\right)^{-\frac{1}{2}-m+k}\right]_{s=t}$$

where the c_k's are constants. The iterated chain rule gives us

$$\frac{\partial^k h}{\partial s^k} = \sum_{\alpha}\left(d_\alpha \frac{\partial^{|\alpha|} h}{\partial x^{|\alpha|}} \prod_{i=1}^{k}\left(s - \sum_{j=2}^{n} x_j^2\right)^{(\frac{1}{2}-i)\alpha_i}\right).$$

where the sum is taken over all non-negative k-tuples $\alpha = (\alpha_1, \ldots, \alpha_k)$ satisfying $\sum_{i=1}^{k} i\alpha_i = k$, $|\alpha| = \sum_{i=1}^{k} \alpha_i$, and the d_i's are constants. The last three expressions are to be evaluated at $s = t$. Using the last two expressions, the third last,

$$\left[\frac{\partial^{m-1}}{\partial s^{m-1}}\left(\frac{1}{2}\left(s - \sum_{j=2}^{n} x_j^2\right)^{-\frac{1}{2}} h(x,t)\right)\right]_{s=t}.$$

is

$$\sum_{k=0}^{m-1} c_k \left(\sum_{\alpha}\left(d_\alpha \frac{\partial^{|\alpha|} h}{\partial x^{|\alpha|}} \prod_{i=1}^{k}\left(s - \sum_{j=2}^{n} x_j^2\right)^{(\frac{1}{2}-i)\alpha_i}\right)\right)\left(t - \sum_{j=2}^{n} x_j^2\right)^{\frac{1}{2}-m+k}. \quad (*)$$

We can now estimate

$$\int_{\gamma_t} \mathrm{Res}_{X_t}(o_t).$$

Use $u_j = \mathrm{Im} x_j$ ($j = 2, \ldots, n$), for coordinates on γ_t. Note

$$\left| t - \sum_{j=2}^{n} u_j^2 \right|$$

$$= \left| t - \sum_{j=2}^{n} x_j^2 \right|$$

$$= \left| - \sum_{j=1}^{n} x_j^2 + t - x_1^2 \right|$$

$$= \left| -x_1^2 \right| < 1.$$

Since the lowest power of $\left(t - \sum_{j=2}^{n} x_j^2\right)$ in $(*)$ is $\frac{1}{2} - m$, occurring only when $k = 0, \alpha = 0$, we may estimate

$$\left| \int_{X_t} \mathrm{Res}(o_t) \right| \leq A \int_{\sum_{j=2}^{n} u_j^2 < 1-t} \left(t + \sum_{j=2}^{n} u_j^2 \right)^{\frac{1}{2} - m} du_2 \ldots du_n$$

for some constant A. This upper bound can be simplified by a change of coordinates. Replace $(u_2, \ldots, u_n)$ by $v_2, \ldots, v_{n-1}, r)$ where

$$u_2 = r v_2$$

$$\vdots$$

$$u_{n-1} = r v_{n-1}$$

$$u_n = r\sqrt{1 - v_2^2 - \cdots - v_{n-1}^2}$$

subject to

$$v_2^2 + \cdots + v_{n-1}^2 < 1$$

$$0 \leq r \leq \sqrt{1 - t}$$

The differentials satisfy

$$du_2 \wedge \cdots \wedge du_n$$

$$= d(rv_2) \wedge \cdots \wedge d(rv_{n-1}) \wedge d\left(r\sqrt{1 - v_2^2 - \cdots - v_{n-1}^2}\right)$$

$$r^{n-2} dv_2 \wedge dv_2 \wedge \cdots \wedge dv_{n-1}$$

So our bound is

$$A \int_{\sum_{j=2}^{n-1} v_j^2 < 1} \int_0^{\sqrt{1-t}} (t + r^2)^{\frac{1}{2} - m} r^{n-2} dr dv_2 \dots dv_{n-1}.$$

Since the v's do not appear in the integrand, the bound becomes

$$A' \int_0^{\sqrt{1-t}} (t + r^2)^{\frac{1}{2} - m} r^{n-2} dr.$$

$$= A' \int_0^{\sqrt{1-t}} (t + r^2)^{\frac{1}{2} - m} (r^2)^{m-1} dr$$

$$\leq A' \int_0^{\sqrt{1-t}} (t + r^2)^{-\frac{1}{2}} dr$$

$$= . \text{ by substituting } w = t^{-\frac{1}{2}} r.$$

$$A' \int_0^{\sqrt{t^{-1} - 1}} (1 + w^2)^{-\frac{1}{2}} dw$$

$$= . \text{ substituting } w = \tan\zeta.$$

$$A' \int_0^{\arctan\sqrt{t^{-1}-1}} (\sec^2\zeta)^{-\frac{1}{2}} \sec^2\zeta d\zeta$$

$$= A' \int_0^{\arctan\sqrt{t^{-1}-1}} \sec\zeta d\zeta$$

$$= A' \left[\frac{1}{2} \log \frac{1 + \sin\zeta}{1 - \sin\zeta}\right]_0^{\arctan\sqrt{t^{-1}-1}}$$

$$= A' \left(\frac{1}{2} \log \frac{1 + \sqrt{\frac{\tan^2\zeta}{\tan^2\zeta + 1}}}{1 - \sqrt{\frac{\tan^2\zeta}{\tan^2\zeta + 1}}}\right)$$

$$= A' \left(\frac{1}{2} \log \frac{1 + \sqrt{\frac{t^{-1}-1}{t^{-1}}}}{1 - \sqrt{\frac{t^{-1}-1}{t^{-1}}}}\right)$$

$$= A'\left(\frac{1}{2}\log\frac{1+\sqrt{1-t}}{1-\sqrt{1-t}}\right)$$

$$= A'\left(\frac{1}{2}\log\left(\frac{\left(1+\sqrt{1-t}\right)^2}{t}\right)\right)$$

$$\leq A'\left(\frac{1}{2}\log\left(\frac{(1+1)^2}{t}\right)\right) \text{ for } t < 1$$

$$B - \frac{1}{2}A'\log t.$$

Thus

$$\left|\int_{\gamma_t} \mathrm{Res}_{X_t}(\omega_t)\right| \leq B - \frac{1}{2}A'\log t$$

for some constants A', B.

What we have proven is that the periods of local sections of $\overline{\mathcal{G}}'$ near the singular points grow at most like powers of $\log t$. Therefore $\overline{\mathcal{G}}' \subseteq \overline{\mathcal{G}}$. □

[13] ⇐ [14]

If d (from [O]) is identically 0 on $\mathbb{P}$, then in particular, we have

$$0 = d(0)$$

$$= \dim H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n\pi_{2*}\mathcal{Z}_0^\bullet)$$

$$= \dim H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n\pi_{2*}G^{m+1}\Omega^\bullet_{X\times\mathbb{P}/\mathbb{P}}(\bullet Z_0))$$

$$= \dim H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n\pi_{2*}G^{m+1}\Omega^\bullet_{X\times\mathbb{P}/\mathbb{P}}(\bullet \overline{Y}))$$

$$= \dim H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}').$$

□

[14] ⇐ [15]&[16]

If the pairs $(X \times \mathbb{P}, Z_0)$ and $(X \times \mathbb{P}, Z_s)$ are isomorphic for all $\neq \infty$, then $\mathcal{Z}_0 = G^{m+1}\Omega^\bullet_{X\times\mathbb{P}/\mathbb{P}}(\bullet Z_0) \cong G^{m+1}\Omega^\bullet_{X\times\mathbb{P}/\mathbb{P}}(\bullet Z_s) = \mathcal{Z}_s$ Hence

$$d(0) = \dim H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n\pi_{2*}\mathcal{Z}_0^\bullet) = \dim H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n\pi_{2*}\mathcal{Z}_s^\bullet) = d(s)$$

for all $s \neq \infty$. But if also $d(s) = 0$ on an open neighboorhood of ∞, then d must be identically 0 on $\mathbb{P}$. □

[15] ⇐ [17]&[18]

This is a direct application of the definition of upper-semicontinuity to a function with integer values. □

[16] is true

Consider the linear automorphism, for $s \in \mathbb{P} - \{\infty\}$:

$$L_s : \mathbb{P} \to \mathbb{P}$$

$$(a_0, a_1) \mapsto (a_0, a_1 - sa_0)$$

Now let us examine the effect of

$$1 \times L_s : X \times \mathbb{P} \to X \times \mathbb{P}$$

on $Z_s = \{(z, a) \in X \times \mathbb{P} : a_0 z_0 + (a_1 - sa_0) z_1 = 0\}$ (from definition in [X]). If $(z, a) \in Z_s$, then

$$(z, L_s(a))$$

satisfies $(z_0, z_1) \cdot L_s(a) = a_0 z_0 + (a_1 - sa_0) z_1 = 0$ which can be rewritten as

$$(L_s(a))_0 z_0 + ((L_s(a))_1 - 0(L_s(a))_0) z_1.$$

This, by definition, means $(z, L_s(a)) \in Z_0$.

We claim that $1 \times L_s$ in fact transforms Z_s isomorphically onto Z_0. To see this consider the inverse transformation

$$(1 \times L_s)^{-1} = 1 \times L_s^{-1},$$

in which $(L_s)^{-1} : \mathbb{P} \to \mathbb{P}$ is given by $(a_0, a_1) \mapsto (a_0, a_1 + sa_0)$. A similar calculation to the above shows that $(z, a) \in Z_0 \Rightarrow (z, L_s^{-1}(a)) \in Z_s$ so that we do indeed have the isomorphism between the pairs $(X \times \mathbb{P}, Z_0)$ and $(X \times \mathbb{P}, Z_s)$.
□

[17] is true

Our task is to show that $d(\infty) = 0$, which means that

$$0 = H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n \pi_{2*} \mathcal{Z}^\bullet_\infty)$$

$$= H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n \pi_{2*} G^{m+1} \Omega^\bullet_{X \times \mathbb{P}/\mathbb{P}}(\bullet Z_\infty))$$

Now $\mathcal{L}^\bullet := G^{m+1} \Omega^\bullet_{X \times \mathbb{P}/\mathbb{P}}(\bullet Z_\infty)$ is the complex

$$\overbrace{0 \to \cdots \to 0}^{m \text{ times}} \to \Omega^m_{X \times \mathbb{P}/\mathbb{P}}(Z_\infty) \to \Omega^{m+1}_{X \times \mathbb{P}/\mathbb{P}}(2Z_\infty) \to \cdots$$

But as $Z_\infty = X_\infty \times \mathbb{P} \cup X \times \{\infty\}$, we may observe

$$\Omega^p_{X\times\mathbb{P}/\mathbb{P}}(lZ_\infty) \cong \pi_1^*\Omega^p_X(lX_\infty) \otimes_{\mathbb{C}} \pi_2^*\mathcal{O}_{\mathbb{P}}(l)$$

with the poles of order l in $\mathcal{O}_{\mathbb{P}}(l)$ being at ∞. So $G^{m+1}\Omega^\bullet_{X\times\mathbb{P}/\mathbb{P}}(\bullet Z_\infty)$ contains the subcomplex $\mathcal{M}^\bullet := G^{m+1}\Omega^\bullet_X(\bullet X_\infty) \otimes_{\mathbb{C}} \pi_2^*\mathcal{O}_{\mathbb{P}}(1)$, which is

$$\overbrace{0 \to \cdots \to 0}^{m \text{ times}} \to \pi_1^*\Omega^m_X(X_\infty) \otimes_{\mathbb{C}} \pi_2^*\mathcal{O}_{\mathbb{P}}(1) \to \cdots,$$

the poles of order 1 in $\mathcal{O}_{\mathbb{P}}(1)$ being again at ∞. The quotient $\mathcal{L}^\bullet/\mathcal{M}^\bullet$ is supported on $\pi_2^{-1}(\infty) = X \times \{\infty\}$. It is clear that $\mathbf{R}^n\pi_{2*}\mathcal{M}^\bullet \cong \mathbf{H}^n(X, G^{m-1}\Omega^\bullet_X(\bullet X_\infty)) \otimes \mathcal{O}_{\mathbb{P}}(1)$, which indicates, via Serre duality, that $H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(1)) \cong H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-1)) = 0. \Rightarrow H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n\pi_{2*}\mathcal{M}^\bullet) = 0$. From the cokernel sequence

$$0 \to \mathcal{M}^\bullet \hookrightarrow \mathcal{L}^\bullet \to \mathcal{L}^\bullet/\mathcal{M}^\bullet \to 0$$

we have, using left exactness,

$$0 = \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n\pi_{2*}\mathcal{M}^\bullet \to \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n\pi_{2*}\mathcal{L}^\bullet\Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n\pi_{2*}(\mathcal{L}^\bullet/\mathcal{L}^\bullet).$$

$$\Rightarrow \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n\pi_{2*}\mathcal{L}^\bullet = 0.$$

This is what we wanted to prove. □

[18] ⇐ [19] & [20]

Apply the Semicontinuity Theorem for Hypercohomology to the morphism $\pi_3 : X \times \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ and the complex of sheaves $\pi_2^*\Omega^1_{\mathbb{P}} \otimes \mathcal{Z}^\bullet$ on $X \times \mathbb{P} \times \mathbb{P}$. Then we have

$$s \mapsto \dim_{k(s)}\mathbf{H}^{n+1}(X \times \mathbb{P}, \pi_2^*\Omega^1_{\mathbb{P}} \otimes \mathcal{Z}^\bullet_s)$$

being upper-semicontinuous on $\mathbb{P}$. But for the scheme $\mathbb{P}$, $k(s) = \mathbb{C}$ for all $s \in \mathbb{P}$. Applying [20], we discover the upper-semicontinuity on $\mathbb{P}$ of

$$s \to \dim\mathbf{H}^{n+1}(X \times \mathbb{P}, \pi_2^*\Omega^1_{\mathbb{P}} \otimes \mathcal{Z}^\bullet_s) = \dim H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n\pi_{2*}\mathcal{Z}^\bullet_s) = d(s)$$

□

[19] is true

This is the proof of the
Semicontinuity Theorem for Hypercohomology

Let $V \to S$ be a projective morphism of Noetherian schemes, and let $\mathcal{N}^\bullet$ be a complex of sheaves on V, with the $\mathcal{N}^i$'s being coherent $\mathcal{O}$-modules flat over S, with differentials linear over $f^{-1}(\mathcal{O}_S)$. For all $p \geq 0$, the function $s \mapsto \dim_k(s)\mathbf{H}^1(V_s, \mathcal{N}_s^\bullet)$ is upper-semicontinuous on S.○.

20 is true

We begin with a

Lemma

$\mathbf{R}^{n+1}\pi_*\mathcal{Z}^\bullet = 0$ ○

Proof

The presheaf associated to $\mathbf{R}^{n+1}\pi_*\mathcal{Z}^\bullet = 0$ is

$$\mathbb{P} \times \mathbb{P} \supset U \to \mathbf{H}^{n+1}(\pi^{-1}(U), \mathcal{Z}^\bullet) = \mathbf{H}^{n+1}(\pi^{-1}(U), G^{m+1}\Omega^\bullet_{X\times\mathbb{P}\times\mathbb{P}/\mathbb{P}\times\mathbb{P}}(\bullet Z)).$$

We know that there exists a spectral sequence $lqE_r\}$ abutting to

$$\mathbf{H}^{n+1}(\pi^{-1}(U), G^{m+1}\Omega^\bullet_{X\times\mathbb{P}\times\mathbb{P}/\mathbb{P}\times\mathbb{P}}(\bullet Z))$$

with

$$'E_2^{p,q} = H_d^p(H^q(\pi^{-1}(U), G^{m+1}\Omega^\bullet_{X\times\mathbb{P}\times\mathbb{P}/\mathbb{P}\times\mathbb{P}}(\bullet Z))).$$

Therefore

$$\begin{aligned} &H^q(\pi^{-1}(U), \Omega^p_{X\times\mathbb{P}\times\mathbb{P}/\mathbb{P}\times\mathbb{P}}((p-m)Z) \\ &= H^q(\pi^{-1}(U), G^{m+1}\Omega^\bullet_{X\times\mathbb{P}\times\mathbb{P}/\mathbb{P}\times\mathbb{P}}(\bullet Z)) \\ &\quad = 0 \forall p \geq m+1, p+q = n+1 \end{aligned}$$

would imply $'E_2^{p,q} = 0$ and thus would imply $\mathbf{H}^{n+1}(\pi^{-1}(U), G^{m+1}\Omega^\bullet_{X\times\mathbb{P}\times\mathbb{P}/\mathbb{P}\times\mathbb{P}}(\bullet Z))$, which would give the desired vanishing of $\mathbf{R}^{n+1}\pi_*\mathcal{Z}^\bullet$.

So it suffices to establish the vanishing of the sheaf associated to the presheaf

$$U \mapsto H^q(\pi^{-1}(U), \Omega^p_{X\times\mathbb{P}\times\mathbb{P}/\mathbb{P}\times\mathbb{P}}((p-m)Z).$$

namely $R^q\pi_*\Omega^p_{X\times\mathbb{P}\times\mathbb{P}/\mathbb{P}\times\mathbb{P}}((p-m)Z)$, for all $p+q = n+1$. For this, we will establish the vanishing of the stalks

$$H^q(X, \Omega^p_X((p-m)X_t), t = \frac{a_1b_0 - a_0b_1}{a_0b_0}$$

over all $(a,b) \in \mathbb{P} \otimes \mathbb{P}$. By Nakano's generalization of the Kodaira Vanishing Theorem [23,p.132] these groups, except for the stalk over $((0,1),(0,1))$ are

0. The entire fiber $\pi_{-1}(((0,1),(0,1)))$ is contained in the polar locus Z, so a special argument is needed. We can replace Z by a linearly equivalent divisor Z' on $X \times \mathbb{P} \times \mathbb{P}$ without changing the cohomology. Take Z' to be given by

$$\frac{a_0 z_0 + a_1 z_1}{a_1 z_0 + a_0 z_1} = \frac{b_1}{b_0}.$$

When $a_0 = b_0$ we get the equation of X_∞. We can now apply the vanishing theorem again. □

This argument is taken, some of it *verbatim*, from [Zucker, 4.46].

To establish $\boxed{20}$ we now use the *Leray spectral sequence for hypercohomology*. We have the sequence $\{\mathbf{E}_r\}$ with

$$\begin{cases} \mathbf{E}_2 & = \mathbf{E}_\infty = \mathbf{H}^{n+1}(X \times \mathbb{P}, \pi_2^*\Omega^1_{\mathbb{P}} \otimes \mathcal{Z}_s^\bullet) \text{ (degeneration at } \mathbf{E}_2) \\ \mathbf{E}_2^{i,j} & = H^i(\mathbb{P}, \mathbf{R}^j\pi_{2*}(\Omega^1_{\mathbb{P}} \otimes \mathcal{Z}_s^\bullet)) \\ & = H^i(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^j\pi_{2*}\mathcal{Z}_s^\bullet) \end{cases}$$

so that

$$\begin{aligned} &\mathbf{H}^{n+1}(X \times \mathbb{P}, \pi_2^*\Omega^1_{\mathbb{P}} \otimes \mathcal{Z}_s^\bullet) \\ &= \bigoplus_{i+j=n+1} H^i(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathrm{R}^j\pi_{2*}\mathcal{Z}_s^\bullet) \end{aligned}$$

But

$$\mathbf{R}^{n+1}\pi_{2*}\mathcal{Z}_s^\bullet = 0$$

and

$$H^{i\geq 2}(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^j\pi_{2*}\mathcal{Z}_s^\bullet) = 0.$$

So the direct sum has a single term; that is,

$$\mathbf{H}^{n+1}(X \times \mathbb{P}, \pi_2^*\Omega^1_{\mathbb{P}} \otimes \mathcal{Z}_s^\bullet) = H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathbf{R}^n\pi_{2*}\mathcal{Z}_s^\bullet).$$

□

From this we may obtain the

Corollary

All normal functions are horizontal ([Zucker]). ○

Proof

We know (statement $\boxed{12}$) that $H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}) = 0$.

From $\boxed{K}$, we have the short exact sequence

$$0 \to \mathrm{F}^{m+1}\mathrm{Prim}^n(X) \otimes \Omega^1_{\mathbb{P}} \to \overline{\mathcal{G}} \otimes \Omega^1_{\mathbb{P}} \to \overline{\mathcal{F}}_v \otimes \Omega^1_{\mathbb{P}} \to 0$$

Since $H^2(\mathbb{P}, \Omega^1_{\mathbb{P}}) = 0$, we have the surjectivity of

$$H^1(\mathbb{P}, \overline{\mathcal{G}} \otimes \Omega^1_{\mathbb{P}}) \to H^1(\mathbb{P}, \overline{\mathcal{F}}_v \otimes \Omega^1_{\mathbb{P}}).$$

So from $H^1(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \overline{\mathcal{G}}) = 0$ we conclude

$$H^1(\mathbb{P}, \overline{\mathcal{F}}_v \otimes \Omega^1_{\mathbb{P}}) = 0.$$

By Serre duality, $H^0(\mathbb{P}, \overline{\mathcal{F}}^{\cdot}_v) = 0$. From the canonical bundle formula, we have the inclusion $\Omega^1_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-2) \hookrightarrow \mathcal{O}_{\mathbb{P}}$, then $\overline{\mathcal{F}}^{\cdot}_v \otimes \Omega^1_{\mathbb{P}} \hookrightarrow \overline{\mathcal{F}}^{\cdot}_v$, which implies

$$H^1(\mathbb{P}, \overline{\mathcal{F}}^{\cdot}_v \otimes \Omega^1_{\mathbb{P}}) = 0.$$

But $H^0(\mathbb{P}, \Omega^1_{\mathbb{P}}) = 0$ and $\overline{\mathcal{F}}^{\cdot} = \overline{\mathcal{F}}^{\cdot}_v \oplus \mathcal{O}_{\mathbb{P}} \otimes (F^{m,*}H^{2m-1}(X))$, which mean that

$$H^1(\mathbb{P}, \overline{\mathcal{F}}^{\cdot} \otimes \Omega^1_{\mathbb{P}}) = 0.$$

Now H^0 is right exact on coherent sheaves because the dimension of $\mathbb{P}$ is 1. Consequently, the surjection $\overline{\mathcal{F}}^{\cdot} \to \overline{\mathcal{F}}^{m+1,*}$ gives a surjection

$$0 = H^0(\mathbb{P}, \overline{\mathcal{F}}^{\cdot} \otimes \Omega^1_{\mathbb{P}}) \to H^0(\mathbb{P}, \overline{\mathcal{F}}^{m+1,*} \otimes \Omega^1_{\mathbb{P}}).$$

Hence

$$H^0(\mathbb{P}, \overline{\mathcal{F}}^{m+1,*} \otimes \Omega^1_{\mathbb{P}}) = 0,$$

which is the vanishing of the middle link in the following factorization of $\overline{\nabla}_{\mathcal{J}}$:

$$\begin{array}{ccc} H^0(\mathbb{P}, \overline{\mathcal{J}}) & \longrightarrow & H^0(\mathbb{P}, \overline{\mathcal{F}}^{m+1,*} \otimes \Omega^1_{\mathbb{P}}(\log \Sigma)) \\ & \searrow \qquad \nearrow & \\ & H^0(\mathbb{P}, \overline{\mathcal{F}}^{m+1,*} \otimes \Omega^1_{\mathbb{P}}) & \end{array}$$

(The reason we have the factorization is ([Lewis.lec.12]) that

$$F^{m,*} = \frac{W_{2m-1}}{F^m W_{2m-1}},$$

W_{2m-1} consisting of invariant classes under monodromy, so that ∇ (on $F^{m,*}$) gives no poles of $\overline{\mathcal{F}}^{m+1,*} \otimes \Omega^1_{\mathbb{P}}$ at Σ.)

Therefore, $\overline{\nabla}_{\mathcal{J}} = 0$. □

EXAMPLE APPLICATIONS

In this section we examine an equivalent condition for Jacobi Inversion, and another special case of the General Hodge Conjecture.

Recall the Abel-Jacobi map

$$\Theta(X_t) \xrightarrow{\Phi} J(X_t).$$

Jacobi Inversion is its surjectivity. Define $J_{\mathrm{a}}(X_t)$ to be the image, in $J(X_t)$, of the subgroup $\Theta(X_t)_{\mathrm{alg}} < \Theta(X_t)$ under Φ. ($\Theta(X_t)_{\mathrm{alg}}$ is the group of codimension-m algebraic cycles on X_t which are algebraically equivalent to 0.) It is known, by an application of Poincaré's complete reducibility theorem, that $J_{\mathrm{a}}(X_t)$ is a subtorus of $J(X_t)$.

Using Hilbert Scheme arguments, it can be shown that the quotient

$$\frac{\Theta(X_t)}{\Theta(X_t)_{\mathrm{alg}}}$$

is at most countable. It follows that

$$\frac{\Phi(\Theta(X_t))}{J_{\mathrm{a}}(X_t)}$$

is also countable.

Jacobi Inversion is equivalent to $J_{\mathrm{a}}(X_t) = J(X_t)$. For if $\Theta(X_t) \xrightarrow{\Phi} J(X_t)$ is surjective, then

$$\frac{J(X_t)}{J_{\mathrm{a}}(X_t)} = \frac{\Phi(\Theta(X_t))}{J_{\mathrm{a}}(X_t)}$$

is both countable, and the quotient of two tori, $J(X_t)$ and $J_{\mathrm{a}}(X_t)$. Hence the tori must be identical, the quotient 0.

What exactly does it mean for a cycle $\xi \in \Theta(X_t)$ to be algebraically equivalent to 0? It means there is a smooth curve Γ, a cycle $z \in C^m(\Gamma \times X_t)$ of codimension m, and points $a, b \in \Gamma$ so that $\xi = z(b) - z(a)$.

In a similar way, given z as above, we have a map on $CH_0(\Gamma)_{\mathrm{alg}}$ (the group of 0-cycles on Γ of degree 0):

$$CH_0(\Gamma)_{\mathrm{alg}} \xrightarrow{z_*} CH^m(X_t)$$

$$(b - a) \mapsto \xi.$$

For any such z we have (see [Lewis.lec.12]) a commutative diagram

$$\begin{array}{ccc} CH_0(\Gamma)_{\text{alg}} & \xrightarrow{z_*} & CH^k(X_t)_{\text{alg}} \\ \cong\downarrow & & \downarrow\Phi \\ J^1(\Gamma) & & J(X_t) \\ \| & & \| \\ \dfrac{H^{0,1}(\Gamma)}{H^1(\Gamma,\mathbb{Z})} & \xrightarrow{[z]_*} & \dfrac{F^{k,*}H^{2k-1}(X_t)}{H^{2k-1}(X_t,\mathbb{Z})} \end{array}$$

We can choose z so that $[z]_*$ is surjective, which if Φ is surjective forces $\text{wt}H^{2k-1}(X_t) = 1$. So the weight of $H^{2k-1}(X_t)$ is necessarily 1 for Jacobi Inversion to hold.

There is another special case of the General Hodge Conjecture (i.e. not $\text{Hodge}^{p,p}(X,\mathbb{Q})$) which has not been discussed so far. Here the dimension of X is odd, say $n = 2k-1$. It is an equality of two filtrations on $H^{2k-1}(X,\mathbb{Q})$:

$$\boxed{N^{k-1}H^{2k-1}(X,\mathbb{Q}) = F_{\mathbb{Q}}^{k-1}H^{2k-1}(X,\mathbb{Q})}$$

The filtration $F_{\mathbb{Q}}^{i}H^{2k-1}(X,\mathbb{Q})$ is defined to be the maximal Hodge Structure in $F^{k-1} \cap H^{2k-1}(X,\mathbb{Q})$. And N is the *filtration by coniveau*, defined by $N^iH^{2k-1}(X,\mathbb{Q}) = \{\text{im}\sigma_* \mid \sigma_* \text{ is the Gysin map } H^{2k-2i-1}(\tilde{Y},\mathbb{Q}) \to H^{2k-1}(X,\mathbb{Q})$ and $\tilde{Y} = \text{desing}(Y)$, where $\text{codim}_X Y = i\}$.

Now suppose the General Hodge Conjecture - and this case in particular - is true. Suppose also that $H^{2k-1}(X,\mathbb{Q})$ has weight 1, and more particularly that $H^{2k-1}(X,\mathbb{Q}) = F_{\mathbb{Q}}^{k-1}H^{2k-1}(X,\mathbb{Q})$. In this case $H^{2k-1}(X,\mathbb{Q}) = F^{k-1} \cap H^{2k-1}(X,\mathbb{Q})$. Then by the finite dimensionality of the cohomology, there exists $\tilde{Y}$ of pure dimension $n-(k-1) = 1+(n-k)$ (with $k \leq n$) so that

$$\sigma_* : H^1(\tilde{Y},\mathbb{Q}) \to H^{k-1}(X,\mathbb{Q})$$

is surjective. By taking $n-k$ general hyperplane sections of $\tilde{Y}$ we get a smooth curve $\Gamma \subset \tilde{Y}$ so that

$$H_1(\Gamma,\mathbb{Q}) \to H_1(\tilde{Y},\mathbb{Q})$$

is surjective by the Weak Lefschetz Theorem.

We also get the inclusion $H^1(\tilde{Y}) \hookrightarrow H^1(\Gamma)$ by duality. Consider a left inverse of it, $H^1(\Gamma) \to H^1(\tilde{Y})$, as a morphism of Hodge structures. This defines an element of $H^1(\Gamma)^* \otimes H^1(\tilde{Y}) \overset{\text{Poincaré duality}}{\cong} H^1(\Gamma) \otimes H^1(\tilde{Y})$ and being a morphism of Hodge strucures it determines a class in

$$(H^1(\Gamma, \mathbb{Q}) \otimes H^1(\tilde{Y}, \mathbb{Q})) \cap F^1.$$

By the Lefschetz (1.1) Theorem, and by taking a suitable integral multiple, this is the class induced by some $z_0 \in CH^1(\Gamma \times \tilde{Y})$. The (surjective) composite

$$H^1(\Gamma, \mathbb{Q}) \longrightarrow H^1(\tilde{Y}, \mathbb{Q}) \xrightarrow{\sigma_*} H^{2k-1}(X, \mathbb{Q})$$

is similarily induced by some

$$[z] \in (H^1(\Gamma) \cong H^1(\Gamma)^*) \otimes H^{2k-1}(X) \subset H^{2k}(\Gamma \times X),$$

where $z \in CH^k(\Gamma \times X)$. We again have the commutative diagram

$$\begin{array}{ccc} CH_0(\Gamma)_{\text{alg}} & \xrightarrow{z_*} & CH^k(X_t)_{\text{alg}} \\ \cong \downarrow & & \downarrow \Phi \\ J^1(\Gamma) & \xrightarrow{[z]_*} & J(X_t) \end{array}$$

And we know $[z]_*$ is surjective, so that Jacobi Inversion follows.

We wish to find some examples of hypersurfaces $X \subset \mathbb{P}^{2k}$ (of dimension $n = 2k - 1$) where $\mathrm{wt}H^n(X) = 1$ and where the General Hodge Conjecture holds.

Let X be such a hypersurface, of degree d. From [Lewis, 9.12], the numerical condition

$$(*)\ \boxed{d \leq 2 + \frac{3}{k-1}}$$

is equivalent to $\mathrm{wt}H^n(X)$ being 1. We want to know whether or not it is sufficient for (the special case of) Hodge Conjecture.

To investigate this, we have the sufficient condition for

$$N^l H^n(X, \mathbb{Q}) = H^n(X, \mathbb{Q}).$$

where $l = \lfloor \frac{n+1}{d} \rfloor$, found in a paper of Lewis:

$$(**)\boxed{l(n+2-l)+1-\binom{d+l}{l} \geq 0}$$

In this case we are assuming $\mathrm{wt} H^n(X) = 1$, so if $l = \lfloor \frac{n+1}{d} \rfloor$ is $k-1$, then $N^l H^n(X, \mathbb{Q}) = H^n(X, \mathbb{Q})$ is $N^{k-1} H^{2k-1}(X, \mathbb{Q}) = F_{\mathbb{Q}}^{k-1} H^{2k-1}(X, \mathbb{Q})$, our special case of the Hodge Conjecture.

Our investigation boils down to the question: "When does condition $(*)$ imply $l = k-1$ and condition $(**)$?"

Assume $(*)$ holds. The cases are:

- $k = 1$ In this case $(*)$ provides no restriction on d, but that hardly matters, for we know the Hodge Conjecture holds for curves (i.e. where $n = 1$). (This is the Lefschetz $(1,1)$ Theorem.)
- $k = 2$ Here, $d \leq 4$.
- $k = 3$ Here, $d \leq 3$.
- $k \geq 4$ Here, $d \leq 2$. We may ignore this case, because the General Hodge Conjecture holds for all cases where $d = 1$ or 2.

We have the two cases $k = 2, 3$ to cover.

If $k = 2$, then $d \leq 4 . \Rightarrow l = 1 = k - 1$. As for $(**)$, it holds:

$$l(n+2-l)+1-\binom{d+l}{l}$$

$$\geq 1(3+2-1)+1-\binom{4+1}{1} = 0.$$

If $k = 3$, then $d \leq 3$. So d is exactly 3, making $l = 2 = k-1$. And $(**)$ again holds:

$$l(n+2-l)+1-\binom{d+l}{l}$$

$$= 2(5+2-2)+1-\binom{3+2}{2} = 1 \geq 0.$$

What we have shown is that condition $(*)$ is equivalent to the version

$$N^{k-1} H^{2k-1}(X, \mathbb{Q}) = F_{\mathbb{Q}}^{k-1} H^{2k-1}(X, \mathbb{Q})$$

of the Hodge Conjecture for odd-dimensional hypersufaces.

INDEX OF NOTATION

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Page	Notation
1	X
1	H^i
1	$H^{p,q}$
1	E^i
1	$\mathrm{Hodge}^{p,p}$
1	L
2	Prim^i
5	$\mathrm{Prim}^{p,q}$
7	$[\,]$
7	C^m
8	Φ
10	X_t
10	Y
10	$\overline{Y}$
10	D
10	$\bar{f}$
10	f
10	$\mathcal{O}$
10	$\mathbf{R}$
10	Ω
10	$\mathbf{H}$
10	$\mathcal{U}$
10	$\mathcal{A}^*$
10	$C^p(\mathcal{U}, \mathcal{A}^q)$
10	$C^{p,q}$
12	$\mathcal{H}^q$
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13	l
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17	$\mathcal{V}$
17	$\underline{\Delta}^*$
17	$\overline{\mathcal{V}}$
17	$\overline{\mathcal{V}}^*$
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17	$\Theta(X_s)$
17	o_s
18	$\mathcal{J}$
18	$\mathbf{J}$
18	H
18	W_l
18	H_0
19	T
19	N
19	δ_t
21	$H^{2m-1}(X_t, \mathbb{Q})^T$
21	$M(t)$
25	∇
25	$\overline{\nabla}$
25	$\mathcal{S}$
25	$\mathcal{T}$
28	$\cup$
28	$\check{C}$
28	∂
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32	$H^{n-2}(D)_v$
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60	$CH_0(\Gamma)_{\mathrm{alg}}$
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61	$N^{k-1}H^{2k-1}(X, \mathbb{Q})$
62	$[z]_*$
64	$\mathcal{S}^k(X_t)$

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APPENDIX

We can explicitly give the Abel-Jacobi map for curves on a surface i.e. for $n = 2$, $m = 1$.

Consider a curve ξ in $\Theta(X/\mathbb{P})$ whose hyperplane sections are homologous to 0 i.e. are 0-cycles of degree 0:

$$\xi_t = \sum_{i=1}^{k}(q_i - p_i)$$

The Abel-Jacobi map takes ξ_t to the functional

$$H^{1,0}(X_t) = F^1 H^1(X_t) \to \mathbb{C}$$

$$\omega \mapsto \sum_{i=1}^{k} \int_{p_i}^{q_i} \omega$$

modulo periods

$$(\omega \mapsto \int_{\eta \in H_1(X_t, \mathbb{Z})} \omega).$$

If we have element ν of $H^0(\mathbb{P}, \mathcal{O}(\mathbb{J}))$, take $\nu(t)$ and invert: get ξ_t with $\alpha_t(\xi_t) = \nu(t)$. Piece the ξ_t's together to get $\xi = \sqcup_{t \in \mathbb{P}} \xi_t$ with $\Phi(\xi) = \nu$.

To show that we do indeed have Jacobi inversion in this case, we can define a map (after choosing p in the connected X_t) on the kth symmetric product of X_t:

$$\mathcal{S}^k(X_t) \to J(X_t)$$

$$q_1 + \cdots + q_k \mapsto \sum_{i=1}^{k} \int_{p}^{q_i} \omega = \alpha_t(\sum_{i=1}^{k} q_i - kp)$$

This map, for $k = g =$ geometric genus of $X_t = \dim H^{1,0}(X_t)$, is analytic, generically 1-1, and surjective. So Φ is surjective since its image contains that of this map.