University of Alberta

Generalised Optimal Stopping and Financial Markets

by

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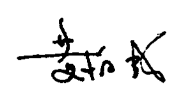

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Abstract

The concept of optimal stopping is fundamental in financial markets. It models the best time to buy or sell assets, as well as the valuation of options. Due to time constraints in markets, for example, off-hours and contracts, classical optimal stopping theories are inadequate. A generalisation of the optimal stopping problem is introduced here, in which stopping can take place only in subsets of the time interval. Under certain conditions most results in the classical theory are extended. This leads to the valuation of Bermudan option which is exercisable only on certain specified days of its life. In particular, European and American options are special cases of Bermudan option.

In addition, an economic agent may have constraints in his trading portfolios. Cvitanić and Karatzas [22] employed a stochastic control approach to study the valuation of such options. We also include a brief summary of this topic.

Preface

During the past decade there has been an accelerating interest in the development of mathematical finance. Furthermore, the sophistication of the mathematics used to model financial phenomena has also increased dramatically. It is surpising that, although there are many research papers in mathematical finance, there are not many standard texts available in the English-speaking world.

The aim of this book is, therefore, to provide an introduction to the mathematics used to describe financial markets. In addition it provides a unifying approach to the valuation of different kinds of options. Also, this book is arranged so it presents a natural transition from the theory of optimal stopping problems to the valuation of options.

There is much material in mathematical finance in journal papers which awaits a clear, self-contained treatment that can be easily mastered by students without considerable preparation, or extra reading. It is hoped this research note serves this purpose and provides an accessible introduction to the theory of option pricing.

Another purpose of this book is to introduce generalised optimal stopping problems, in which stopping can take place only in subsets of the time interval. Under certain conditions most results in the classical optimal stopping problems, due to Fakeev, Bismut and Skalli, can be extended to this setting. This generalisation is necessary for the valuation of options with constrained exercise times. In financial markets these are often called Bermudan options.

The content of this book consists of six chapters. Chapter 1 provides a survey of definitions and results from probability theory and stochastic calculus. By the end of Chapter 1 the reader will have seen all the probability theory that is needed in the rest of the book. Chapter 2 discusses the generalised optimal stopping problem. This chapter requires no prior knowledge of classical optimal stopping problem. Chapter 3 provides the mathematical settings in financial markets and is based on the works of El Karoui, Karatzas and Cvitanić. Chapter 4 summarizes options into three categories: constrained exercise time, constrained portfolios and path dependent options. Chapters 5 and 6 discuss options with constrained exercise times and portfolios in detail.

It is anticipated that this work will be useful to graduate students with a reasonably strong mathematical background and to professionals who may be interested in acquiring at least a working knowledge of option pricing.

Dennis Wong

Acknowledgements

Part of the contents of this book rely heavily on the pioneering developments of many mathematicans cited in the pages that follow. I would like to take this opportunity to acknowledge my intellectual debt to all authors mentioned in the sequel.

I would also like to thank my supervisor, Dr. Robert Elliott. In the preparation of this book he generously supplied advice and encouragement, and at the completion of the manuscript he offered further insight, corrections and suggestions.

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Last but not least I would like to thank my parents and my fiancée, Helen. They have been unfailingly supportive and understanding during the preparation of this work.

Dennis Wong

Contents

Chapter 1

Preliminary

In this chapter we present the definitions and concepts needed in later section. Further details can be found in [13], [17], [22], [27] and [28].

1.1 Probability Space

A probability space is a triple $(\Omega, \mathcal{F}, P)$, where Ω is a set, $\mathcal{F}$ is a σ-field of subsets of Ω and P is a non-negative measure with total mass 1. The system of sets $\mathcal{F}^P$ is called the completion of $\mathcal{F}$ if $\mathcal{F}^P$ contains all those sets $A \subset \Omega$ for which there exist $A_1, A_2 \in \mathcal{F}$ such that $A_1 \subset A \subset A_2$ and $P(A_2 - A_1) = 0$. The system of sets $\mathcal{F}^P$ is a σ-algebra and the probability P can be uniquely extended to sets from $\mathcal{F}^P$. The probability space $(\Omega, \mathcal{F}, P)$ is said to be complete if $\mathcal{F}^P$ coincides with $\mathcal{F}$.

An $\mathcal{F}$-measurable function $X : \Omega \to R^d$ is called a random variable if $d = 1$ or a random vector if $d \geq 2$. For an arbitary index set J and arbitary functions X_α from Ω to R^d or $\overline{R} = R \cup \{\pm\infty\}$, the σ-algebra $\sigma\{X_\alpha, \alpha \in J\}$ is the smallest σ-algebra on Ω such that each X_α is measurable. This is called the σ-algebra generated by the collection $\{X_\alpha, \alpha \in J\}$. If $\mathcal{G}$ is a sub σ-algebra of $\mathcal{F}$, the augmentation of $\mathcal{G}$ is the smallest σ-algebra containing $\mathcal{G}$ and all the P-null sets in $\mathcal{F}$.

1.2 Stochastic Processes and Filtration

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ be a measurable space. A mapping $X : T \times \Omega \to E$, where T is a subset of the extended positive real line $\overline{R}^+$, is called a *stochastic process* if for every $t \in T$, X_t is an E-valued measurable function. $(\Omega, \mathcal{F}, P)$ is called the base of the process X, $(E, \mathcal{E})$ is called the state space of X, and T is called the time index. For fixed $\omega \in \Omega$ the map $t \to X_t(\omega)$ is called a *sample path* of the process X.

If E is a topological space, then X is called a *continuous process*, if P-a.s. all paths are continuous for the induced topology on T. Similarly we define *right continuous* and *left continuous* processes, and the property that a process has *left hand* or *right hand limits* respectively. In particular, a process with $RCLL$ paths means almost all paths of it are right continuous with left limits.

A stochastic process X is called *measurable* if the map X is measurable with respect to the σ-algebra $\mathcal{B} \otimes \mathcal{F}$, where $\mathcal{B}$ is the Borel σ-algebra on T and $\mathcal{B} \otimes \mathcal{F}$ is the product σ-algebra of $\mathcal{B}$ and $\mathcal{F}$.

Two processes X and Y are called *indistinguishable* if P-a.s. all paths of X and Y coincide. Two processes X and Y are *modifications* if $X_t =$

Y_t P-a.s. for each $t \in T$.

A stochastic process X, such that P-a.s. all paths are real functions, is called a *real process.* A real stochastic process X will be called *evanescent* if X is indistinguishable from the process which is identically equal to zero, and a set $A \subset T \times \Omega$ is *evanescent* if the process 1_A is evanescent.

A stochastic process X is said to be separable if there exists a denumerable set D in T such that for almost surely every ω, given $t \in T$, there exists a sequence $(t_n) \subset D$ converging to t and $X_{t_n}(\omega) \to X_t(\omega)$.

If X_t is an R^d-valued process, it is possible to find an R^d-valued modification of X_t which is separable. In general, we shall work with the separable modification of a process.

Kolmogorov's continuity criterion [32] states that, if X_t is a separable process defined on a compact subset of R^+ such that

$$E|X_t - X_s|^\alpha \leq C|t-s|^{1+\beta}, \quad \alpha, \beta, C > 0,$$

then X is a continuous process.

A *filtration* with time index T is a family $(\mathcal{F}_t, t \in T)$ of sub σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s < t$ in T. We define the sub σ-algebras $\mathcal{F}_{t+}$ and $\mathcal{F}_{t-}$ of $\mathcal{F}_t$ by $\mathcal{F}_{t+} \triangleq \bigwedge_{s>t} \mathcal{F}_s$ and $\mathcal{F}_{t-} \triangleq \bigvee_{s<t} \mathcal{F}_s$. The filtration $(\mathcal{F}_t, t \in T)$ is called *right continuous* if $\mathcal{F}_{t+} = \mathcal{F}_t$ holds for every $t \in T$ satisfying $\inf\{s \mid s > t, s \in T\} = t$. *Left continuity* of $(\mathcal{F}_t, t \in T)$ is defined similarly. Finally, a filtration $(\mathcal{F}_t, t \in T)$ is *complete* if $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$.

1.3 Adapted Processes and Stopping Times

A stochastic process $(X_t)_{t\in T}$ is called $\mathcal{F}_t$-adapted if for every $t \in T$, the mapping $\omega \to X_t(\omega)$ is $\mathcal{F}_t$-measurable. It is said to be progressively measurable, if for every $t \in T$ the mapping $(t,\omega) \to X_t(\omega)$, defined on $[0,t] \times \Omega$, is measurable for the product $\mathcal{B}_t \otimes \mathcal{F}_t$. Here $\mathcal{B}_t$ denotes the Borel σ-algebra on $[0,t]$.

It is easy to see that every progressively measurable process is adapted. The converse is true provided that the process is right or left continuous.

A T-valued random variable τ is a stopping time with repect to $\mathcal{F}_t$ if for every $t \in T$ the set $\{\tau \leq t\}$ belongs to $\mathcal{F}_t$. We denote the set of all stopping times as $\mathcal{T}$.

The motivation for this definition is that the process knows whether τ has happened by time t from information available in $\mathcal{F}_t$. Suppose a student is asked to hand in his examination paper one minute after the bell rings. At any moment he knows whether he could continue writing, or not. This is a stopping time. However, if he is asked to hand in his paper one minute before the bell rings, and he does not know in advance when the bell will ring, then he will not know when to stop; such a time is not a stopping time.

We define the σ-algebra up to a stopping time τ, the σ-algebra $\mathcal{F}_\tau$ by

$$A \in \mathcal{F}_\tau \iff A \cap \{\tau \leq t\} \in \mathcal{F}_t \quad \forall\, t \in T.$$

The following properties of stopping times will be useful later:

(T1) Let σ, τ be two stopping times; then the random variables $\tau \wedge \sigma$, $\tau \vee \sigma$, $\tau + \sigma$ are stopping times.

(T2) Let τ be a stopping time; then τ is $\mathcal{F}_\tau$ measurable.

(T3) Let τ be a stopping time and σ be a $\mathcal{F}_\tau$-measurable random variable such that $\sigma \geq \tau$ P-a.s. ; then σ is a stopping time.

(T4) Let σ, τ be two stopping times and $A \in \mathcal{F}_\sigma$; then we have $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_\tau$.

(T5) Let σ, τ be two stopping times such that $\sigma \leq \tau$ P-a.s. ; then we have $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$.

(T6) Let σ, τ be two stopping times; then the sets $\{\sigma < \tau\}$, $\{\sigma = \tau\}$ and $\{\sigma > \tau\}$ belong to both $\mathcal{F}_\tau$ and $\mathcal{F}_\sigma$.

(T7) Let τ_n be a sequence of stopping times such that $\tau_n \uparrow \tau$; then τ is a stopping time.

Let B be a Borel set in R^d. Let X_t be an R^d-valued process which is right continuous and adapted to $\mathcal{F}_t$. Furthermore, assume $(\mathcal{F}_t, t \in T)$ is complete. Then the entry time of the process X into B,

$$\tau(B) \triangleq \inf\{t \mid X_t \in B\},$$

is a stopping time.

If $(X_t)_{t\in T}$ is a progressively measurable process and τ is a stopping time, then the mapping $\omega \to X_{\tau(\omega)}(\omega)$ is $\mathcal{F}_\tau$-measurable.

A stochastic process $(X_t)_{t\in T}$ is said to be in class D if the set of random variables $\{X_\tau \mid \tau \in \mathcal{T}\}$ is uniformly integrable. That is

$$\lim_{c\to\infty} \sup_{\tau\in\mathcal{T}} \int_{|X_\tau|\geq c} |X_\tau(\omega)|\, dP(\omega) = 0.$$

1.4 Concepts Relating to Martingales

A real valued stochastic process $(X_t)_{t \in T}$, is said to be a *supermartingale*, (respectively a *submartingale*), on $(\Omega, \mathcal{F}, P)$ with respect to the filtration $(\mathcal{F}_t, t \in T)$ if each X_t is integrable, $\mathcal{F}_t$-measurable and

$$E\left[X_t \mid \mathcal{F}_s\right] \leq (\geq)\, X_s \quad P\text{-a.s.} \qquad \text{for } s \leq t \text{ in } T$$

If the process X is both a supermartingale and a submartingale then it is said to be a martingale. Note that X is a submartingale if and only if $-X$ is a supermartingale. Also a supermartingale or submartingale with constant expectation is a martingale.

Suppose $(X_t)_{t \in T}$ is a right continuous supermartingale and $J = [\, u, v\,] \subset T$. Then for any $\lambda > 0$ we have

$$\lambda P\left(\sup_{t \in J} X_t \geq \lambda\right) \leq E\left[X_u\right] + E\left[X_v^-\right].$$

Also, almost surely, X_t has left-hand limits and almost every path is bounded on every compact interval.

Let $(X_t)_{t \in T}$ be an arbitrary supermartingale and Q be a separable subset of T. Then the restriction to Q of the map $s \to X_s(\omega)$ has a left and right limit at every point $t \in T$, for almost every $\omega \in \Omega$. Assume the filtration is right continuous, then the supermartingale X has a right continuous modification if and only if the function $t \to E\left[X_t\right]$ is right continuous on T.

Suppose $\sigma \leq \tau$ are two stopping times and X is a right continuous supermartingale, then the random variables X_σ and X_τ are integrable and $X_\sigma \geq E\left[\, X_\tau \mid \mathcal{F}_\sigma\,\right]$. This result is called the *optional stopping theorem.*

If $\mathcal{C}$ is some family of processes, then $\mathcal{C}_{loc}$ will denote the family of processes which are locally in $\mathcal{C}$. That is, $(Y_t)_{t\in T} \in \mathcal{C}_{loc}$ if there is an increasing sequence of stopping times (τ_n) such that $\lim_n \tau_n = \infty$ P-a.s. and each stopped process $(Y_{t\wedge\tau_n})_{t\in T} \in \mathcal{C}$. The sequence (τ_n) is called a *localizing sequence* for $\mathcal{C}$. For example, suppose $\mathcal{M}$ denotes the set of uniformly integrable martingales on $(\Omega, \mathcal{F}, P)$ with respect to the filtration $(\mathcal{F}_t, t \in T)$, then $\mathcal{M}_{loc}$ will denote the set of processes which are locally in $\mathcal{M}$, or the set of local martingales.

Obviously, every martingale is a local martingale. The converse is true provided that it is of class D. Similarly, one can show a positive local martingale M satisfying $E[M_0] < \infty$, is a supermartingale.

1.5 Lévy Processes

The Lévy processes, which include the Poisson process and Brownian motion as special cases, are a special class of stochastic processes studied first by the French mathematician Paul Lévy. In this section, we are assuming given a probability space with a complete, right continuous filtration.

An adapted process $(X_t)_{0\le t<\infty}$ with $X_0 = 0$ P-a.s. is a *Lévy process* if

i) X has *independent increments*: that is, $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \le s < t < \infty$;

ii) X has *stationary increments*: that is, $X_t - X_s$ has the same distribution as X_{t-s}, $0 \le s < t < \infty$;

iii) X is *continuous in probability*: that is, $\lim_{t \to s} X_t = X_s$ where the limit is taken in probability.

If we take the Fourier transform of each X_t we get a function

$$f(t,u) = E\left[e^{iu^* X_t}\right],$$

where $f(0,u) = 1$ and $f(t+s,u) = f(t,u)f(s,u)$, and $f(t,u) \neq 0$ for every (t,u). Using the continuity in probability, there exists a continuous function $\phi(u)$, with $\phi(0) = 0$, such that $f(t,u) = e^{-t\phi(u)}$. In particular, there is a one-to-one correspondence between the set of Lévy processes and the set of infinitely divisible distributions. Also for each fixed u, the process $M_t^u \triangleq \frac{e^{iu^* X_t}}{f(t,u)}$ is a complex-valued martingale with respect to $(\mathcal{F}_t, t \in T)$.

Every Lévy process has a unique modification which is right continuous and has left limits. The augmentation of the filtration generated by an arbitrary Lévy process is right continuous

Let X be a Lévy process and τ be a stopping time. On the set $\{\tau < \infty\}$ the process Y defined by $Y_t \triangleq X_{\tau+t} - X_\tau$ is a Lévy process adapted to $\mathcal{F}_{\tau+t}$. Furthermore, Y is independent of $\mathcal{F}_\tau$ and has the same distribution as X.

Suppose X_t has a Poisson distribution with parameter λt, then the Lévy process X is said to be a *Poisson process*. Almost surely, the paths of a Poisson process are right continuous and constant except for upward jumps of size one, of which there are finitely many in each bounded time interval, but infinitely many in $[0, \infty)$.

Suppose X_t has a Gaussian distribution with mean zero and variance matrix $t \cdot \Sigma$, for a given non random matrix Σ. Then the Lévy process X is said to be a *Brownian motion* starting at the origin. Moreover if $\Sigma = I$, we call it a standard Brownian Motion. By Kolmogorov's continuity criterion, instead of a right continuous version, we even have a continuous version of the Brownian motion. However, almost all sample paths of a Brownian motion are of unbounded variation and nowhere differentiable.

1.6 Elementary Stochastic Calculus

Suppose W_t is a d-dimensional Brownian motion with respect to a complete, right continuous filtration $(\mathcal{F}_t, t \in T)$ on the probability space $(\Omega, \mathcal{F}, P)$. We wish to define the Itô integral

$$\int_0^{T_0} X^*(s,\omega)\, dW(s,\omega), \quad [0, T_0] \subset T,$$

for some d-dimensional process X which satisfies certain measurability and integrability assumptions.

A first attempt at such definition might be a path-by-path Riemann-Stieltjes Integral. That is, by fixing $\omega \in \Omega$ we try to define the integral in the usual sense for each sample path. Unfortunately this approach does not work because almost all paths of a Brownian motion are of unbounded variation.

Let $M_w^2(0, T_0; R^d)$ be the set of all measurable, adapted and R^d-valued stochastic processes X such that

$$\int_\Omega \int_0^{T_0} ||X(t,\omega)||^2 dt\ dP(\omega) < \infty.$$

Notice $M_w^2(0, T_0; R^d)$ is a Hilbert subspace of $L^2\left((0, T_0) \times \Omega, dt \otimes dP; R^d\right)$.

Suppose ϕ is a piecewise constant function in $M_w^2(0, T_0; R^d)$. That is, there exists a partition of $[0, T_0]$:

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = T_0$$

such that

$$\phi(t, \omega) = \sum_{j=0}^{n-1} e(j, \omega)\, 1_{[t_n, t_{n+1})}(t).$$

Here $e(j, \cdot) \in L^2(\Omega, \mathcal{F}_{t_j}, P)$ for $j = 0, \cdots, n-1$. We denote the set of all such piecewise constant functions in $M_w^2(0, T_0; R^d)$ by $\Phi_w^2(0, T_0; R^d)$.

Define a linear operator $\mathcal{I} : \Phi_w^2(0, T_0; R^d) \to L^2(\Omega, \mathcal{F}, P)$ as follows:

$$\mathcal{I}(\phi)(\omega) \triangleq \sum_{j=0}^{n-1} e^*(j, \omega)\, (W(t_{n+1}, \omega) - W(t_n, \omega)).$$

One can show the linear operator $\mathcal{I}$ is an isometry, that is

$$||\mathcal{I}(\phi)||_{L^2} = ||\phi||_{\Phi_w^2}.$$

Suppose $X \in M_w^2(0, T_0; R^d)$; for any positive integer j define

$$X_j(t, \omega) \triangleq \begin{cases} \frac{j}{T_0} \int_{\frac{nT_0}{j}}^{\frac{(n+1)T_0}{j}} X(s, \omega)\, ds, & \frac{nT_0}{j} \le t < \frac{(n+1)T_0}{j},\ n \ge 1 \\ 0, & 0 \le t < \frac{T_0}{j} \end{cases}.$$

The sequence $(X_j) \subset \Phi_w^2(0, T_0; R^d)$ and converges to X in norm, so we have $\overline{\Phi_w^2}^{||\cdot||}(0, T_0; R^d) = M_w^2(0, T_0; R^d)$. We thus can extend the linear map $\mathcal{I}$ to $M_w^2(0, T_0; R^d)$.

Now for any $X \in M_w^2(0, T_0; R^d)$, we define the stochastic integral:

$$\int_0^{T_0} X^*(s, \omega)\, dW(s, \omega) \triangleq \mathcal{I}(X)(\omega).$$

The stochastic linear operator $\mathcal{I}$ has the following properties: For any $X, Y \in M_w^2(0, T_0; R^d)$,

i) $E[\mathcal{I}(X)] = 0.$

ii) $E[\mathcal{I}(X)^2] = ||X||^2_{M^2_w}.$

iii) $E[\mathcal{I}(X)\mathcal{I}(Y)] = E\left[\int_0^{T_0} X^*(s)Y(s)\,dt\right].$

We can also define the stochastic integral for any measurable, adapted and R^d-valued process X such that

$$\int_0^{T_0} ||X(s,\cdot)||^2\,dW(s,\cdot) < \infty \ P\text{-a.s.}$$

We denote this class of process by $L^2_w(0,T_0;R^d)$. Suppose $X \in L^2_w(0,T_0;R^d)$; for any positive integer j put,

$$X_j(t,\omega) \triangleq X(t,\omega)\,1_{\int_0^t ||X(s,\omega)||^2\,ds<j}(t).$$

Then $X_j \in M^2_w(0,T_0;R^d)$. Also we see the sequence $\mathcal{I}(X_j)$ converges almost surely and we define

$$\int_0^{T_0} X^*(s,\omega)\,dW(s,\omega) \triangleq \lim_{j\to\infty} \mathcal{I}(X_j)(\omega).$$

Suppose $X \in L^2_w(0,T_0;R^d)$. We can define for all $t \in [0,T_0]$,

$$I(t,\omega) \triangleq \int_0^{T_0} 1_{[0,t)}(s)X^*(s,\omega)\,dW(s,\omega).$$

Here $I(\cdot,\omega)$ is a martingale with respect to $(\mathcal{F}_t, t \in T)$ and we can find a continuous modification of I.

Remark 1.1 *Let $\sigma(t,\omega)$ be a matrix. Then $\int_0^t \sigma(s,\omega)dW(s,\omega)$ is the vector with components*

$$\int_0^t \sigma_i(s,\omega)dW(s,\omega),$$

where σ_i is the i-th row of σ.

Let $L^1_w(0, T_0; R^d)$ and $L^2_w(0, T_0; R^{d\times d})$ be defined in a manner similar to $L^2_w(0, T_0; R^d)$. Let $\alpha \in L^1_w(0, T_0; R^d)$ and $\beta \in L^2_w(0, T_0; R^{d\times d})$. Consider a family $X(t)$ of adapted continuous processes with values in R^d, which depends on an $\mathcal{F}_0$-measurable random variable x, and is defined by

$$(1.1) \qquad X(t, \omega) = x(\omega) + \int_0^t \alpha(s, \omega)\, ds + \int_0^t \beta(s, \omega)\, dW(s, \omega).$$

The stochastic integral equation (1.1) is called a solution of the following stochastic differential equation:

$$(1.2) \qquad dX(t) = \alpha(t)\ dt + \beta(t)\, dW(t),$$

with initial value $X(0) = x$.

Suppose X solves (1.2) and $u(t, x)$ is a functional on $[0, T_0] \times R^d$, belonging to $C^{1,2}\left([0, T_0] \times R^d\right)$. Then the Itô's Lemma [23] states that

$$\begin{aligned} u(t, X(t)) &= u(0, X(0)) + \int_0^t \left(u_t + u_x^* \alpha + \frac{1}{2}\, trace\, (\beta\beta^* u_{xx})\right)(s, X(s))\, ds \\ &\quad + \int_0^t (\beta^* u_x)(s, X(s))\, dW(s), \quad \forall\, t \in [0, T_0]. \end{aligned}$$

Let W_t be a d-dimensional Brownian motion and $\mathcal{F}_t$ be the augmentation of the filtration generated by W. Suppose M is a local martingale with $M_0 = 0$, which is right continuous and has left limits paths almost surely. Then there exists a d-dimensional progressively measurable processes π such that

$$\int_0^{T_0} ||\pi(s)||^2\, ds < \infty; \quad 0 \le T_0 < \infty,$$

and

$$M_t = \int_0^t \pi^*(s)\, dW(s); \quad 0 \le t < \infty.$$

This result is known as the *Martingale Representation Theorem.*

Suppose θ is a d-dimensional, measurable and adapted process satisfying

$$\int_0^{T_0} ||\theta(s)||^2 ds < \infty \ P\text{-a.s.} ; \quad 0 \le T_0 < \infty.$$

Set

$$Y_t \triangleq \exp\left[\int_0^t \theta^*(s)\, dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2\, ds\right]; \quad 0 \le t \le T_0.$$

Then the process Y satisfies the stochastic integro-differential equation

$$Y_t = 1 + \int_0^t Y_s\, \theta^*(s) dW(s).$$

If this continuous local martingale Y is a martingale we can define a family of probability measures $\tilde{P}_t$ by:

$$\tilde{P}_t(A) \triangleq E\left[1_A Y_t\right]; \quad A \in \mathcal{F}_t,\ 0 \le t \le T_0.$$

The martingale property shows that the family of probability measures satisfies the consistency condition

$$\tilde{P}_{T_0}(A) = \tilde{P}_t(A); \quad A \in \mathcal{F}_t,\ 0 \le t \le T_0.$$

Now define a process $\tilde{W}$ by:

$$\tilde{W}_t \triangleq W_t - \int_0^t \theta(s)\, ds; \quad 0 \le t \le T_0.$$

Then the process $\tilde{W}$ is a d-dimensional Brownian motion on $(\Omega, \mathcal{F}_{T_0}, \tilde{P}_{T_0})$. This result is known as the *Girsanov's Theorem* [22].

1.7 Classical Optimal Stopping Problems

At what time do we decide to stop playing roulette? Optimal stopping techniques are mathematical methods for discussing this type of problem, in which the decision taken at each instant can be based only on past experiments.

Let $(X_n)_{n\geq 0}$ be a sequence of successive fortunes of a gambler. These random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$. At time n, the events which the gambler knows, belong to a σ-algebra $\mathcal{F}_n$. We assume the fortune process is $\mathcal{F}_n$-adapted. The gambler must choose a stopping time τ, that is a random number of times that he plays before stopping. The problem of optimal stopping is then the following:

i) Determine the *optimal average gain* $V = \sup_\tau E[X_\tau]$.

ii) Determine if possible, the *optimal stopping time* τ^* such that $V = E[X_{\tau^*}]$.

In addition to the gambling situation above, there are many optimal stopping problems in real life situations. We describe a few more examples.

Example. A participant in a contest is given a sequence of questions. Every time he answers a question correctly, he gets a reward and the right to proceed, or withdraw, from the contest. If he cannot answer a question correctly he looses all the money he has received and he is out of the contest. Suppose that he can answer the i-th question correctly with probability p_i. What is the best time for him to stop the game? What is the expected gain when stopping at such an optimal time?

Example. (Marriage or Secretary Problem)
Assume we are going to investigate n objects in succession: they are of different quality and can be ranked accordingly. Suppose at time k, the object x_k passes by, in random order, for inspection. Each object can be evaluated to be either inferior or superior to the objects examined so far. Every time an object passes by we have to decide whether the object is accepted or not. If we choose to accept the object, then the process stops. However, if an object is rejected we can no longer choose it and the process proceeds. What selection scheme should we use to maximize the probability of obtaining the best object?

These two scenarios are some of the classical examples of optimal stopping problems. In fact, we can extend these problems to the continuous-time setting.

Example. Let X_t be a progressively measurable process which denotes the price of a stock at time t. Suppose there is a fixed transaction cost $\alpha > 0$ for the sale of such stock with a discount rate of $r > 0$. We would like to decide when to sell the stock so that the expected, discounted net sale price is maximized. In this case, the *reward* that one gets for selling the stock at time t would be $e^{-rt}(X_t - \alpha)$. Hence, we would like to seek the optimal stopping time $\tau^* \in \mathcal{T}$ such that

$$E\left[e^{-r\tau^*}(X_{\tau^*} - \alpha)\right] = \sup_{\tau \in \mathcal{T}} E\left[e^{-r\tau}(X_\tau - \alpha)\right]. \quad \square$$

Optimal stopping problems are ubiquitous in decision making situations. However, sometimes the setting is too general for real life applications. For instance, in the last example, we cannot sell the stock at any time, twenty four hours a day, seven days a week. We wish to model the situation where there are some restrictions on the time that we can stop, or take action. This leads to a more general class of optimal stopping problems, where details are discussed in the following chapters.

Chapter 2

Optimal Stopping Problems

2.1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and consider a filtration $(\mathcal{F}_t, t \in T)$ where the index set T is a subset of the set $\overline{R^+}$ of non-negative extended real numbers. Also, for simplicity, we assume T contains a maximal element $\hat{\infty}$. Now consider a stochastic process $X = (X_t, t \in T)$ adapted to $(\mathcal{F}_t, t \in T)$. Let $\mathcal{T}$ be the collection of all P-a.s. T-valued stopping times with respect to $(\mathcal{F}_t, t \in T)$, and let $\mathcal{T}(X)$ be the collection of those $\tau \in \mathcal{T}$ for which X_τ is a random variable and $E[X_\tau]$ is defined.

An optimal stopping time for the process X is a stopping time $\tau_0 \in \mathcal{T}(X)$ for which $E[X_{\tau_0}] = \sup_{\tau \in \mathcal{T}(X)} E[X_\tau]$.

The results on optimal stopping were first developed for the case where $T = \overline{N}$, the set of extended positive integers. Fundamental results for optimal stopping in the discrete case were published by Snell [31] , and later by

Dynkin [14], Chow and Robbins [8], Neveu [26] and Siegmund [30]. With the foundations in the discrete parameter case, Fakeev [18], Bismut and Skalli [4] studied the case where $T = \overline{R^+}$.

In the first section we are going to investigate the optimal stopping problem for a collection of stopping times whose ranges are restricted to a particular subset in T. With this generalisation, we can study a wider class of optimization problems and obtain a unified approach to the optimal stopping problem in both the discrete and continuous - parameter cases.

2.2 Basic Properties and Definitions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a complete right continuous filtration $(\mathcal{F}_t, t \in T)$ where $T = \overline{R^+}$ or $[0, T_0]$ for some $T_0 \in R^+$. In the first case $\hat{\infty} = \infty$, in the latter case we have $\hat{\infty} = T_0$.

Let $(X_t, t \in T)$ be a real-valued stochastic process satisfying:

(2.1) $$X_t \geq 0 \quad P \text{ a.s. for all } t \in T,$$

(2.2) X_t has right continuous paths and is adapted to $(\mathcal{F}_t, t \in T)$,

(2.3) $$E\left[\sup_{t\in T} X_t\right] < \infty.$$

Remark 2.1 *Condition 2.1 can be removed if we strengthen condition 2.3 to* $E\left[\sup_{t\in T}|X_t|\right] < \infty$.

A measurable subset S of T is called a stopping region if $\hat{\infty} \in S$. A stopping region S is called *feasible* if $(s_n) \subset S$ and $s_n \downarrow s$ implies $s \in S$.

Finally, a stopping region S is said to be *right continuous* if for each $s \in S - \{\hat{\infty}\}$, there exists a $(s_n) \subset S$ such that $s_n \downarrow\downarrow s$, which means (s_n) is strictly decreasing and $s_n \to s$. The symbol $\mathcal{S}$ is used to denote the collection of all P-a.s. S-valued stopping times with respect to $(\mathcal{F}_t, t \in T)$. Obviously, $\mathcal{S}$ is non-empty and $\mathcal{S} \subset \mathcal{T}$.

Example. $T = \overline{R^+}$, $S = \overline{N}$ is a feasible stopping region. □

Example. $T = [0, T_0]$, $S = [S_0, T_0]$ is a feasible right continous stopping region for $S_0 \in T - \{T_0\}$. □

Let $\gamma, \sigma \in \mathcal{T}$ and $\gamma \leq \sigma$, we denote

$$
\begin{aligned}
\mathcal{S}_{\gamma}^{\sigma} &\triangleq \{\tau \in \mathcal{S} \mid P(\gamma \leq \tau \leq \sigma) = 1\}, \\
\mathcal{S}_{\gamma+}^{\sigma} &\triangleq \{\tau \in \mathcal{S} \mid P(\gamma < \tau \leq \sigma) = 1\}.
\end{aligned}
$$

If $\sigma = \hat{\infty}$, then we shall omit the upper index.

2.3 Essential Supremum

Let $\mathcal{G}$ be a sub-sigma field of $\mathcal{F}$. For an arbitrary family $\mathcal{H} = \{Y_\alpha, \alpha \in A\}$ of $\mathcal{G}$-measurable extended real-valued functions defined on $(\Omega, \mathcal{F}, P)$, let

$$Z \triangleq \operatorname*{ess\,sup}_{\alpha \in A} Y_\alpha$$

denote the $\mathcal{G}$-measurable function that satisfies:

i) $Z \geq Y_\alpha$ P-a.s. for all $\alpha \in A$

ii) For any $\mathcal{G}$-measurable function $Z' \geq Y_\alpha$ P-a.s. for all $\alpha \in A$, we have $Z' \geq Z$ P-a.s.

The function Z is called the *essential supremum* of the system $\mathcal{H}$. By definition, we assume the essential supremum of an empty family of measurable functions to be the measurable function idential to zero P-a.s.

THEOREM 2.3.1 (Neveu) *Suppose $\mathcal{G}$ is a sub-sigma field of $\mathcal{F}$. For every non-empty family $\mathcal{H}$ of $\mathcal{G}$-measurable extended real-valued functions defined on $(\Omega, \mathcal{F}, P)$, the essential supremum Z of $\mathcal{H}$ always exists and is unique P-a.s. Further, there exists at least one sequence $(Y_n, n \in N) \subset \mathcal{H}$ such that $Z = \sup_{n \in N} Y_n$. If the family $\mathcal{H}$ is directed upwards, the sequence $(Y_n, n \in N)$ can be chosen to be increasing P-a.s. and then $Y_n \uparrow Z$.*

Proof:

Since only the order structure of $\overline{R}$ is involved in this theorem, we can restrict ourselves to the case where the functions in $\mathcal{H}$ take their values in [0,1] by mapping $\overline{R}$ onto [0,1] by an increasing bijection.

Let $\tilde{\mathcal{J}}$ be the class of all countable sub-families of $\mathcal{H}$. For every $\mathcal{J} \in \tilde{\mathcal{J}}$, introduce the measurable function $Y_{\mathcal{J}}$ defined as the countable supremum $Y_{\mathcal{J}} = \sup_{Y \in \mathcal{J}} Y$. Next let us consider the supremum $\alpha = \sup_{\mathcal{J} \in \tilde{\mathcal{J}}} E[Y_{\mathcal{J}}]$. This supremum is attained because if $(\mathcal{J}_n)$ is a sequence in $\tilde{\mathcal{J}}$ such that $E[Y_{\mathcal{J}_n}] \to \alpha$, then $\mathcal{J}^* = \bigcup_n \mathcal{J}_n \in \tilde{\mathcal{J}}$ and $E[Y_{\mathcal{J}^*}] = \alpha$. Now denote $Z' = Y_{\mathcal{J}^*}$, Z' is $\mathcal{G}$-measurable, and we are going to show Z' satisfies property i) and ii) in the definition of essential supremum.

For every $Y' \in \mathcal{H}$, we have $Z' \vee Y' = \sup_{Y \in \mathcal{J}^* \cup \{Y'\}} Y$. We thus have $\alpha = E[Z'] \leq E[Z' \vee Y'] \leq \alpha$ which, since α is finite, is only possible if $Y' \leq Z'$ P-a.s. Property ii) is followed from the definition of Z'. Hence

$Z = Z'$. Also, the uniqueness of the essential supremum follows directly from property ii).

Arrange $\mathcal{J}^*$ in a sequence (Y_n); then $Z = Z' = \sup_n Y_n$. Now if the family $\mathcal{H}$ is directed upward (that is, if for any pair Y_1, Y_2 of functions in $\mathcal{H}$, there exists a $Y_3 \in \mathcal{H}$ such that $Y_3 \geq Y_i$ P-a.s. for $i = 1, 2$), set $Y'_1 = Y_1$ and take Y'_{n+1} to be a function in $\mathcal{H}$ which dominates Y'_n and Y_{n+1}. Then $(Y'_n) \subset \mathcal{H}$ and $Y'_n \uparrow Z$. □

2.4 'Supremum' Measure

Given a stochastic process $(X_t, t \in T)$ satisfying (2.1), (2.2) and (2.3). Let $\gamma, \sigma \in \mathcal{T}$ such that $\gamma \leq \sigma$ P-a.s. and S be a stopping region in T. We define set functions Γ_γ^σ and $\Gamma_{\gamma+}^\sigma$ on $\mathcal{F}_\gamma$ by,

$$\begin{aligned} \Gamma_\gamma^\sigma(A) &\triangleq \sup_{\tau \in \mathcal{S}_\gamma^\sigma} E[X_\tau 1_A], \\ \Gamma_{\gamma+}^\sigma(A) &\triangleq \sup_{\tau \in \mathcal{S}_{\gamma+}^\sigma} E[X_\tau 1_A] \quad \text{for any } A \in \mathcal{F}_\gamma. \end{aligned}$$

For consistency we define the supremum over an empty set to be zero. Note that the above definitions are well defined since $(X_t, t \in T)$ is adapted and right continuous, we have $(X_t, t \in T)$ progressively measurable and X_τ is $\mathcal{F}_\tau$-measurable.

Remark 2.2 *A similar set function of this kind was introduced by Fakeev in [18], but was not widely used in the theory of optimal stopping problems. Here, we are going to investigate these set functions in detail.*

THEOREM 2.4.1 *Suppose $\gamma, \sigma \in \mathcal{T}$ are such that $\gamma \leq \sigma$ P-a.s. and S is a stopping region in T. Then Γ_γ^σ and $\Gamma_{\gamma+}^\sigma$ are finite measures on $\mathcal{F}_\gamma$.*

Proof:

We only need to show Γ_γ^σ is a finite measure on $\mathcal{F}_\gamma$. The proof for $\Gamma_{\gamma+}^\sigma$ follows similarly. Also we assume $\mathcal{S}_\gamma^\sigma \neq \emptyset$, otherwise the result is trival.

Suppose $A = \bigcup_{i=1}^\infty A_i$, $A_i \bigcap A_j = \emptyset$ for $i \neq j$ and $A_i \in \mathcal{F}_\gamma$. Obviously, we have $\sum_{i=1}^\infty \Gamma_\gamma^\sigma(A_i) = \Gamma_\gamma^\sigma(A) + \varepsilon$ for some $\varepsilon \geq 0$. We only need to show $\varepsilon = 0$.

Suppose not, then for every $i \in N$, there exists $\tau_i^* \in \mathcal{S}_\gamma^\sigma$ such that $\Gamma_\gamma^\sigma(A_i) - \varepsilon/2^{i+1} \leq E\left[X_{\tau_i^*} 1_{A_i}\right]$. Take $\tau^* = \sum_{i=1}^\infty \tau_i^* 1_{A_i} + \gamma^* 1_{A^c}$ for any $\gamma^* \in \mathcal{S}_\gamma^\sigma$. It is routine to check $\tau^* \in \mathcal{S}_\gamma^\sigma$ and $\sum_{i=1}^\infty \Gamma_\gamma^\sigma(A_i) - \varepsilon/2 \leq E\left[X_{\tau^*} 1_A\right]$. Hence $\Gamma_\gamma^\sigma(A) + \varepsilon/2 \leq E\left[X_{\tau^*} 1_A\right]$. This is a contradiction. We, therefore, have $\varepsilon = 0$.

Finally, because of (2.3), we have Γ_γ^σ is a finite measure. $\square$

THEOREM 2.4.2 *Suppose $\gamma, \sigma \in \mathcal{T}$ are such that $\gamma \leq \sigma$ P-a.s. Then,*

$$\frac{d\Gamma_\gamma^\sigma}{dP} = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_\gamma^\sigma} E\left[X_\tau \mid \mathcal{F}_\gamma\right], \tag{2.4}$$

$$\frac{d\Gamma_{\gamma+}^\sigma}{dP} = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{\gamma+}^\sigma} E\left[X_\tau \mid \mathcal{F}_\gamma\right]. \tag{2.5}$$

Proof:

We shall only prove the case for (2.5). (2.4) can be proved similarly. Also, we assume $\mathcal{S}_{\gamma+}^\sigma \neq \emptyset$, otherwise the result would be trivial.

First of all, $\frac{d\Gamma_{\gamma+}^\sigma}{dP}$ is $\mathcal{F}_\gamma$-measurable. Also, given any $\tau \in \mathcal{S}_{\gamma+}^\sigma$, we have

$$E\left[\frac{d\Gamma_{\gamma+}^\sigma}{dP} 1_A\right] = \Gamma_{\gamma+}^\sigma(A) \geq E\left[E\left[X_\tau \mid \mathcal{F}_\gamma\right] 1_A\right] \quad \text{for all } A \in \mathcal{F}_\gamma.$$

Hence $\frac{d\Gamma^{\sigma}_{\gamma+}}{dP} \geq E[X_\tau \mid \mathcal{F}_\gamma]$ for all $\tau \in \mathcal{S}^{\sigma}_{\gamma+}$.

Now, let Y be any $\mathcal{F}_\gamma$-measurable function such that $Y \geq E[X_\tau \mid \mathcal{F}_\gamma]$ for all $\tau \in \mathcal{S}^{\sigma}_{\gamma+}$. Then $E[Y 1_A] \geq \Gamma^{\sigma}_{\gamma+}(A)$ for all $A \in \mathcal{S}^{\sigma}_{\gamma+}$.

Also, we have $Y \geq \frac{d\Gamma^{\sigma}_{\gamma+}}{dP}$. Hence the result follows. □

Remark 2.3 *Note that the collection $\{E[X_\tau \mid \mathcal{F}_\gamma], \tau \in \mathcal{S}^{\sigma}_{\gamma}\}$ of $\mathcal{F}_\gamma$ - measurable functions is directed upwards.*

THEOREM 2.4.3 *Let $\gamma, \lambda, \sigma \in \mathcal{T}$ be such that $\gamma \leq \lambda \leq \sigma$ P-a.s. If $B \in \mathcal{F}_\gamma$ satisfies the condition,*

$$\Gamma^{\sigma}_{\lambda}(A \cap B) \geq \Gamma^{\lambda}_{\gamma}(A \cap B) \quad \textit{for all} \quad A \in \mathcal{F}_\gamma,$$

then we have

$$\Gamma^{\sigma}_{\gamma}(A \cap B) = \Gamma^{\sigma}_{\lambda}(A \cap B) \quad \textit{for all} \quad A \in \mathcal{F}_\gamma.$$

Proof:

Let A be an arbitary set in $\mathcal{F}_\gamma$; since $\Gamma^{\sigma}_{\gamma}(A \cap B) \geq \Gamma^{\sigma}_{\lambda}(A \cap B)$, we only need to show $\Gamma^{\sigma}_{\gamma}(A \cap B) \leq \Gamma^{\sigma}_{\lambda}(A \cap B)$.

1. If $\mathcal{S}^{\sigma}_{\gamma} = \emptyset$, then $\mathcal{S}^{\sigma}_{\lambda} = \emptyset$. Hence the result.

2. If $\mathcal{S}^{\sigma}_{\gamma} \neq \emptyset$ and $\mathcal{S}^{\lambda}_{\gamma} = \emptyset$, then $\mathcal{S}^{\sigma}_{\gamma} = \mathcal{S}^{\sigma}_{\lambda}$. Hence the result.

3. If $\mathcal{S}^{\sigma}_{\gamma} \neq \emptyset$ and $\mathcal{S}^{\sigma}_{\lambda} = \emptyset$, then $\mathcal{S}^{\sigma}_{\gamma} = \mathcal{S}^{\lambda}_{\gamma}$, and $\Gamma^{\lambda}_{\gamma}(A \cap B) = \Gamma^{\sigma}_{\gamma}(A \cap B) \geq \Gamma^{\sigma}_{\lambda}(A \cap B) \geq \Gamma^{\lambda}_{\gamma}(A \cap B)$. Hence the result. Note the last inequality follows from the given hypothesis on B.

4. If $\mathcal{S}_\gamma^\lambda \neq \emptyset$ and $\mathcal{S}_\lambda^\sigma \neq \emptyset$, then obviously $\mathcal{S}_\gamma^\sigma \neq \emptyset$. For any $\tau \in \mathcal{S}_\gamma^\sigma$, we can write $\tau = 1_{\tau\leq\lambda}\tau_1 + 1_{\tau>\lambda}\tau_2$ for some $\tau_1 \in \mathcal{S}_\gamma^\lambda$ and $\tau_2 \in \mathcal{S}_\lambda^\sigma$. Hence

$$\begin{aligned}
E\left[X_\tau 1_{A\cap B}\right] &= E\left[X_{\tau_1} 1_{A\cap B\cap\{\tau\leq\lambda\}}\right] + E\left[X_{\tau_2} 1_{A\cap B\cap\{\tau>\lambda\}}\right] \\
&\leq \Gamma_\gamma^\lambda(A\cap B\cap\{\tau\leq\lambda\}) + \Gamma_\lambda^\sigma(A\cap B\cap\{\tau>\lambda\}) \\
&\leq \Gamma_\lambda^\sigma(A\cap B\cap\{\tau\leq\lambda\}) + \Gamma_\lambda^\sigma(A\cap B\cap\{\tau>\lambda\}) \\
&= \Gamma_\lambda^\sigma(A\cap B).
\end{aligned}$$

Therefore, $\Gamma_\gamma^\sigma(A\cap B) \leq \Gamma_\lambda^\sigma(A\cap B)$.

Hence, it follows that $\Gamma_\gamma^\sigma(A\cap B) = \Gamma_\lambda^\sigma(A\cap B)$. □

THEOREM 2.4.4 *Let $\gamma, \lambda, \sigma \in \mathcal{T}$ be such that $\gamma \leq \lambda \leq \sigma$ P-a.s. If $B \in \mathcal{F}_\gamma$ satisfies the condition,*

$$\Gamma_\lambda^\sigma(A\cap B) \leq \Gamma_\gamma^\lambda(A\cap B) \quad \textit{for all} \quad A \in \mathcal{F}_\gamma,$$

then we have

$$\Gamma_\gamma^\sigma(A\cap B) = \Gamma_\gamma^\lambda(A\cap B) \quad \textit{for all} \quad A \in \mathcal{F}_\gamma.$$

Proof:

Mimic the proof of Theorem 2.4.3. □

Corollary 2.4.5 *Let $\gamma, \lambda, \sigma \in \mathcal{T}$ be such that $\gamma \leq \lambda \leq \sigma$ P-a.s. If $B \in \mathcal{F}_\gamma$ satisfies the conditions,*

$$\begin{aligned}
\Gamma_\lambda^\sigma(A\cap B) &\geq \Gamma_\gamma^\lambda(A\cap B) \\
\Gamma_\lambda^\sigma(A\cap B^c) &\leq \Gamma_\gamma^\lambda(A\cap B^c) \quad \textit{for all} \quad A \in \mathcal{F}_\gamma,
\end{aligned}$$

then we have

$$\Gamma^{\sigma}_{\gamma}(A) = \Gamma^{\lambda}_{\gamma}(A \cap B^c) + \Gamma^{\sigma}_{\lambda}(A \cap B) \quad \textit{for all} \;\; A \in \mathcal{F}_{\gamma}.$$

Proof:

Apply Theorem 2.4.3 and Theorem 2.4.4. □

THEOREM 2.4.6 *Suppose that $\gamma, \lambda \in \mathcal{T}$, then we have*

$$\Gamma_{\gamma}(A \cap \{\gamma = \lambda\}) = \Gamma_{\lambda}(A \cap \{\gamma = \lambda\}) \quad \textit{for any} \;\; A \in \mathcal{F}_{\gamma} \cup \mathcal{F}_{\lambda}.$$

Proof:

Note that $\mathcal{S}_{\gamma}$ and $\mathcal{S}_{\lambda}$ are non-empty sets. Now given $A \in \mathcal{F}_{\gamma} \cup \mathcal{F}_{\lambda}$, without loss of generality, we can assume

$$\Gamma_{\gamma}(A \cap \{\gamma = \lambda\}) = \Gamma_{\lambda}(A \cap \{\gamma = \lambda\}) + \varepsilon \quad \text{for some } \varepsilon \geq 0$$

Assume $\varepsilon > 0$, then there exists $\tau' \in \mathcal{S}_{\gamma}$ such that

$$\Gamma_{\gamma}(A \cap \{\gamma = \lambda\}) \leq E\left[X_{\tau'} 1_{A \cap \{\gamma = \lambda\}}\right] + \varepsilon/2.$$

Take $\tau^* = 1_{\tau' \geq \lambda}\tau' + 1_{\tau' < \lambda}\lambda'$ for any $\lambda' \in \mathcal{S}_{\lambda}$. It is routine to check $\tau^* \in \mathcal{S}_{\lambda}$. Then we have,

$$\begin{aligned}
\Gamma_{\lambda}(A \cap \{\gamma = \lambda\}) &\geq E\left[X_{\tau^*} 1_{A \cap \{\gamma = \lambda\}}\right] \\
&= E\left[X_{\tau'} 1_{A \cap \{\gamma = \lambda\} \cap \{\tau' \geq \lambda\}}\right] + E\left[X_{\lambda'} 1_{A \cap \{\gamma = \lambda\} \cap \{\tau' < \lambda\}}\right] \\
&= E\left[X_{\tau'} 1_{A \cap \{\gamma = \lambda\} \cap \{\tau' \geq \gamma\}}\right] + E\left[X_{\lambda'} 1_{A \cap \{\gamma = \lambda\} \cap \{\tau' < \gamma\}}\right] \\
&= E\left[X_{\tau'} 1_{A \cap \{\gamma = \lambda\}}\right] \\
&\geq \Gamma_{\gamma}(A \cap \{\gamma = \lambda\}) - \varepsilon/2 \\
&= \Gamma_{\lambda}(A \cap \{\gamma = \lambda\}) + \varepsilon/2.
\end{aligned}$$

This is a contradiction. Hence the result follows. □

2.5 Generalised Envelopes

Given a stochastic process $(X_t, t \in T)$ satisfying (2.1), (2.2) and (2.3), let S be a stopping region and $\gamma, \sigma \in \mathcal{T}$ be such that $\gamma \leq \sigma$ P-a.s. We define

$$Y_\gamma^\sigma(S) \triangleq \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_\gamma^\sigma} E\,[\, X_\tau \mid \mathcal{F}_\gamma \,]\,.$$

If $\sigma = \hat{\infty}$ P-a.s. , then we shall omit the upper index. The collection of random variables $\{Y_\gamma(S) \mid \gamma \in \mathcal{T}\}$ is called the *Generalised Dirichlet Envelope (GDE)* of $(X_t, t \in T)$ with stopping region S.

Similarly, for any $t \leq s \in T$, we define

$$Z_t^s(S) \triangleq Y_t^s(S) = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_t^s} E\,[\, X_\tau \mid \mathcal{F}_t \,]\,.$$

Again, if $s = \hat{\infty}$, then we shall omit the upper index. The adapted process $(Z_t(S) \mid t \in T)$ is called the *Generalised Snell Envelope, or process, (GSE)* of $(X_t, t \in T)$ with stopping region S.

Without loss of ambiguity, we shall omit the stopping region S in the above notations.

When $S = T$, these Generalised envelopes coincide with the definitions in the classical theory as in [4], [7] and [18].

Remark 2.4 *Obviously, the GSE is a subset of the GDE. We would like to investigate under what conditions the Generalised Snell Envelope is sufficient. That is, when does $Z_\gamma(S) = Y_\gamma(S)$ for any $\gamma \in \mathcal{T}$? In that case one could construct the GDE from the GSE.*

THEOREM 2.5.1 *Suppose $\gamma \in \mathcal{T}$ and $\mathcal{G}$ is a sub-sigma field of $\mathcal{F}_\gamma$. Then,*

$$E\left[Y_\gamma \mid \mathcal{G}\right] = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_\gamma} E\left[X_\tau \mid \mathcal{G}\right].$$

In particular, suppose $\mathcal{G}$ is a sub-sigma field of $\mathcal{F}_t$. Then,

$$E\left[Z_t \mid \mathcal{G}\right] = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_t} E\left[X_\tau \mid \mathcal{G}\right].$$

Proof:

We prove only the first assertion; the other follows directly. First of all, $E[Y_\gamma \mid \mathcal{G}]$ is a $\mathcal{G}$-measurable function such that for any $\tau \in \mathcal{S}_\gamma$,

$$\begin{aligned} E\left[\,E[Y_\gamma \mid \mathcal{G}]1_A\,\right] &= E\left[Y_\gamma 1_A\right] \\ &= \Gamma_\gamma(A) \qquad\qquad \text{by } (2.4) \\ &\geq E\left[X_\tau 1_A\right] \\ &= E\left[\,E[X_\tau \mid \mathcal{G}]1_A\,\right] \quad \text{for any } A \in \mathcal{G}. \end{aligned}$$

Hence $E[Y_\gamma \mid \mathcal{G}] \geq E[X_\tau \mid \mathcal{G}]$ for any $\tau \in \mathcal{S}_\gamma$.

Next, let Y be any $\mathcal{G}$ measurable random variable such that $Y \geq E[X_\tau \mid \mathcal{G}]$ for all $\tau \in \mathcal{S}_\gamma$. Then

$$\begin{aligned} E[Y 1_A] &\geq \sup_{\tau \in \mathcal{S}_\gamma} E[X_\tau 1_A] \\ &= \Gamma_\gamma(A) \\ &= E\left[\,E[Y_\gamma \mid \mathcal{G}]1_A\,\right] \quad \text{for any } A \in \mathcal{G}. \end{aligned}$$

Hence $Y \geq E[Y_\gamma \mid \mathcal{G}]$, and we can conclude that

$$E[Y_\gamma \mid \mathcal{G}] = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_\gamma} E[X_\tau \mid \mathcal{G}]. \qquad \square$$

Corollary 2.5.2 *Suppose $\gamma \in \mathcal{T}$, then we have*

$$E[Y_\gamma] = \sup_{\tau \in \mathcal{S}_\gamma} E[X_\tau].$$

In particular, we have

$$E[Z_t] = \sup_{\tau \in \mathcal{S}_t} E[X_\tau] \quad \textit{for any } t \in T.$$

Proof:

Use the previous theorem and take $\mathcal{G}$ as the null sigma field, $\{\emptyset, \Omega\}$. □

THEOREM 2.5.3 *If $\gamma, \lambda \in \mathcal{T}$ and $\gamma \leq \lambda$ P-a.s. , then we have*

$$E[Y_\lambda \mid \mathcal{F}_\gamma] \leq Y_\gamma.$$

In particular, the Generalised Snell process $(Z_t, t \in T)$ of X is a supermartingale with respect to $(\mathcal{F}_t, t \in T)$.

Proof:

$$\begin{aligned} E[E[Y_\lambda \mid \mathcal{F}_\gamma] 1_A] &= \Gamma_\lambda(A) \\ &\leq \Gamma_\gamma(A) = E[Y_\gamma 1_A] \qquad \text{for all } A \in \mathcal{F}_\gamma \end{aligned}$$

Hence the result follows. Again, the second assertion is a direct consequence from the first assertion. □

LEMMA 2.5.4 *Suppose X satisfies (2.1) - (2.3), then the Generalised Dirichlet Envelope of X is uniformly integrable.*

Proof:

By (2.1), let $M = \sup_{\tau \in \mathcal{S}_0} E[X_\tau] < \infty$; then by corollary 2.5.2 and Theorem 2.5.3, we have $E[Y_\gamma] \leq M$ and hence $P(Y_\gamma \geq c) \leq M/c$. Then by (2.4),

$$E\left[Y_\gamma 1_{Y_\gamma \geq c}\right] = \sup_{\tau \in \mathcal{S}_\gamma} E\left[X_\tau 1_{Y_\gamma \geq c}\right].$$

As $P(Y_\gamma \geq c) \leq M/c$ is independent of γ and X is uniformly integrable, the right hand side tends to 0 as $c \uparrow \infty$, uniformly in γ and τ. Hence the result follows. □

Given a stopping time $\gamma \in \mathcal{T}$, one can always express γ as a limit of a decreasing sequence of stopping times which take on discrete values. Define for $n \geq 1$;

$$\lceil \gamma \rceil_n \triangleq \begin{cases} \frac{\lceil 2^n \gamma \rceil}{2^n} & \text{if } \gamma < \hat{\infty} \\ \hat{\infty} & \text{if } \gamma = \hat{\infty} \end{cases}$$

where $\lceil r \rceil$ is the smallest integer greater than or equal to the real number r. This is called the dyadic approximation of γ from above. Obviously, each $\lceil \gamma \rceil_n$ belongs to $\mathcal{T}$ and takes on discrete values. Also we have $\lceil \gamma \rceil_n \downarrow \gamma$ P-a.s.

Now given a Generalised Dirichlet Envelope $\{Y_\gamma \mid \gamma \in \mathcal{T}\}$ of X satisfying conditions (2.1), (2.2) and (2.3), we can define

$$Y_{\gamma+} \triangleq \lim_{n \to \infty} Y_{\lceil \gamma \rceil_n} \qquad \text{for any } \gamma \in \mathcal{T}.$$

By Theorem 2.5.3, $\left(Y_{\lceil \gamma \rceil_n}, \mathcal{F}_{\lceil \gamma \rceil_n}\right)$ for $n \in N$ is a reversed supermartingale. With lemma 2.5.4, we can establish the existence of the above limit. From the next lemma, one would also determine the measurability of $Y_{\gamma+}$ given $\gamma \in \mathcal{T}$.

LEMMA 2.5.5 *If γ is a stopping time in $\mathcal{T}$, then $Y_{\gamma+}$ is $\mathcal{F}_{\gamma+}$-measurable.*

Proof:

Refer to Section 1.3 of Chung [9]. □

Given a Generalised Snell Envelope $(Z_t, t \in T)$ of the process X satisfying conditions (2.1), (2.2) and (2.3), one can define

$$Z_{t+} \triangleq \begin{cases} \lim_{\substack{s \in D \\ s \downarrow\downarrow t}} Z_s & \text{if } t < \hat{\infty} \\ Z_{\hat{\infty}} & \text{if } t = \hat{\infty} \end{cases}$$

where D is a countable dense subset in T. Because the Snell Envelope is a supermartingale, we have for P-a.s. ω, $Z_{t+}(\omega)$ exists for all $t \in T$. Notice the definition of Z_{t+} is independent of the countable dense set D. One can see this by considering the union of two different countable dense sets and taking the limits of suitable subsequences.

Now for $\gamma \in \mathcal{T}$, we can define

$$Z_{\gamma+}(\omega) \triangleq Z_{\gamma(\omega)+}(\omega)$$

THEOREM 2.5.6 (Bellman equation) *Suppose $\gamma, \sigma \in \mathcal{T}$ are such that $\gamma \leq \sigma$ P-a.s. , then we have*

$$Y_\gamma = E\,[\,Y_\sigma \mid \mathcal{F}_\gamma\,] \vee Y_\gamma^\sigma.$$

In particular, for $t \leq s$ we have

$$Z_t = E\,[\,Z_s \mid \mathcal{F}_t\,] \vee Z_t^s.$$

Proof:

We prove only the first assertion; the second assertion would follow immediately.

Let $B = \left\{ E\left[Y_\sigma \mid \mathcal{F}_\gamma \right] \geq Y_\gamma^\sigma \right\} \in \mathcal{F}_\gamma$. For any $A \in \mathcal{F}_\gamma$, we have

$$\begin{aligned}
\Gamma_\sigma(A \cap B) &= E\left[Y_\sigma 1_{A\cap B} \right] \\
&= E\left[E\left[Y_\sigma \mid \mathcal{F}_\gamma \right] 1_{A\cap B} \right] \\
&\geq E\left[Y_\gamma^\sigma 1_{A\cap B} \right] \\
&= \Gamma_\gamma^\sigma(A \cap B).
\end{aligned}$$

Similarly, for any $A \in \mathcal{F}_\gamma$, we also have

$$\begin{aligned}
\Gamma_\sigma(A \cap B^c) &= E\left[Y_\sigma 1_{A\cap B^c} \right] \\
&= E\left[E\left[Y_\sigma \mid \mathcal{F}_\gamma \right] 1_{A\cap B^c} \right] \\
&\leq E\left[Y_\gamma^\sigma 1_{A\cap B^c} \right] \\
&= \Gamma_\gamma^\sigma(A \cap B^c).
\end{aligned}$$

Therefore, by corollary 2.4.5, we have

$$\Gamma_\gamma(A) = \Gamma_\sigma(A \cap B) + \Gamma_\gamma^\sigma(A \cap B^c) \qquad \text{for any} \quad A \in \mathcal{F}_\gamma$$

That is,

$$E\left[Y_\gamma 1_A \right] = E\left[\left(E\left[Y_\sigma \mid \mathcal{F}_\gamma \right] \vee Y_\gamma^\sigma \right) 1_A \right] \qquad \text{for any} \quad A \in \mathcal{F}_\gamma.$$

Hence the result. □

Corollary 2.5.7 *Suppose γ is a stopping time in T, then we have*

$$Y_\gamma(S) = \begin{cases} Y_{\gamma+}(S) \vee X_\gamma & \text{if } \gamma \in \mathcal{S} \\ Y_{\gamma+}(S) & \text{if } \gamma \notin \mathcal{S}. \end{cases}$$

In particular, we also have

$$Z_t(S) = \begin{cases} Z_{t+}(S) \vee X_t & \text{if } t \in S \\ Z_{t+}(S) & \text{if } t \notin S. \end{cases}$$

where S is the stopping region corresponding to the GDE and GSE.

Proof:

Again, we prove only first assertion. By the Bellman Equation, we can write

$$Y_\gamma(S) = E\left[Y_{\lceil\gamma\rceil_n}(S) \mid \mathcal{F}_\gamma \right] \vee \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_\gamma^{\lceil\gamma\rceil_n}} E\left[X_\tau \mid \mathcal{F}_\gamma \right].$$

Using uniform integrability of the GDE and right continuity of the filtration, we have

$$Y_\gamma(S) = Y_{\gamma+}(S) \vee \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_\gamma^\gamma} E\left[X_\tau \mid \mathcal{F}_\gamma \right].$$

Because

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{S}_\gamma^\gamma} E\left[X_\tau \mid \mathcal{F}_\gamma \right] = \begin{cases} X_\gamma & \text{if } \gamma \in \mathcal{S} \\ 0 & \text{if } \gamma \notin \mathcal{S}. \end{cases}$$

Hence the result follows. □

2.6 Regularity of the Envelopes

In this section, we are going to look at the properties of the Envelopes when the stopping region S is specified. Under some regular conditions on S, the

Generalised Snell Envelope is right continuous, regular and characterized P-a.s. uniquely by some of its properties. With $S = T$, these properties coincide with those in the literature. Again, we adopt the same notations as in the last section.

LEMMA 2.6.1 *For any $\gamma \in \mathcal{S}$, we have $Y_\gamma(S) \geq X_\gamma$ P-a.s. In particular, for any $t \in S$, we have $Z_t(S) \geq X_t$ P-a.s.*

Proof:

If $\gamma \in \mathcal{S}$, then $\gamma \in \mathcal{S}_\gamma$. Hence, we have

$$Y_\gamma(S) = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_\gamma} E[X_\tau \mid \mathcal{F}_\gamma] \geq X_\gamma \quad P\text{-a.s.} \qquad \square$$

LEMMA 2.6.2 *Suppose $\gamma \in \mathcal{T}$ takes on discrete values, then*

$$Z_\gamma = Y_\gamma \; P\text{-a.s.}$$

Proof:

Let $\gamma = \sum_{i=1}^{\infty} t_i 1_{\gamma=t_i}$ P-a.s. For $A \in \mathcal{F}_\gamma$, we have

$$\begin{aligned}
E[Z_\gamma 1_A] &= \sum_{i=1}^{\infty} E\left[Z_{t_i} 1_{A \cap \{\gamma = t_i\}}\right] \\
&= \sum_{i=1}^{\infty} \Gamma_{t_i}(A \cap \{\gamma = t_i\}) \\
&= \sum_{i=1}^{\infty} \Gamma_\gamma(A \cap \{\gamma = t_i\}) \qquad \text{by Theorem 2.4.6} \\
&= \Gamma_\gamma(A) \\
&= E[Y_\gamma 1_A].
\end{aligned}$$

Hence $Z_\gamma = Y_\gamma$. $\square$

LEMMA 2.6.3 *If γ is a stopping time in $\mathcal{T}$, then we have*

$$Y_{\gamma+} = Z_{\gamma+} \quad P\text{-a.s.}$$

Proof:

Suppose $\Omega_0 \subset \Omega$ is such that $Z_{\gamma+}$ and $Y_{\gamma+}$ exist on Ω_0, and $P(\Omega \backslash \Omega_0) = 0$. Then for $\omega \in \Omega_0$,

$$\begin{aligned} Y_{\gamma+}(\omega) &= \lim_{n\to\infty} Y_{\lceil\gamma\rceil_n(\omega)}(\omega) \\ &= \lim_{n\to\infty} Z_{\lceil\gamma\rceil_n(\omega)}(\omega) \\ &= Z_{\gamma+}(\omega). \quad \Box \end{aligned}$$

THEOREM 2.6.4 *Suppose S is right continuous and $\gamma \in \mathcal{T}$. Then we have*

$$Y_\gamma(S) = Y_{\gamma+}(S).$$

In particular, the Generalised Snell Envelope $(Z_t(S) \mid t \in T)$ is a right continuous supermartingale with respect to $(\mathcal{F}_t, t \in T)$.

Proof:

By corollary 2.5.7, one must show $Y_{\gamma+}(S) \geq X_\gamma$ P-a.s. for $\gamma \in \mathcal{S}$. In fact, because of lemma 2.6.3, we only have to show $Z_{\gamma+}(S) \geq X_\gamma$ P-a.s. for $\gamma \in \mathcal{S}$.

Without loss of generality, we assume $\gamma < \hat{\infty}$ P-a.s. For P-a.s. ω, $\gamma(\omega) \in S$ and there exists a distinct sequence $\{s_n(\gamma_\omega)\} \subset S$ such that $s_n \downarrow\downarrow \gamma(\omega)$. Then we have,

$$Z_{\gamma+}(S)_\omega = \lim_{n\to\infty} Z_{s_n}(S)_\omega$$

$$
\begin{aligned}
&\geq \limsup_{n\to\infty} X_{s_n}(\omega) \qquad \text{by lemma 2.6.1} \\
&= X_{\gamma+}(\omega) \\
&= X_{\gamma}(\omega).
\end{aligned}
$$

The last equality holds because of (2.2).

By Theorem 2.5.3, it follows that $(Z_t(S) \mid t \in T)$ is a right continuous supermartingale with respect to $(\mathcal{F}_t, t \in T)$. □

THEOREM 2.6.5 *If S is a right continuous stopping region, then the Generalised Snell Envelope is sufficient. That is*

$$Z_{\gamma}(S) = Y_{\gamma}(S) \quad P\text{-a.s.} \quad \textit{for any} \ \ \gamma \in \mathcal{T}.$$

Proof:

By lemma 2.6.3, we have

$$Z_{\gamma+}(S) = Y_{\gamma+}(S) \quad P\text{-a.s.}$$

Since S is right continuous, following from Theorem 2.6.4

$$Z_{\gamma}(S) = Y_{\gamma}(S) \quad P\text{-a.s.} \qquad \square$$

THEOREM 2.6.6 *Suppose S is a discrete stopping region, then $Z(S)$ is the minimal supermartingale dominating X on S.*

Proof:

By Theorem 2.5.3 and lemma 2.6.1, $Z(S)$ is a supermartingale dominating X on S. Suppose G is a supermartingale such that $G_s \geq X_s$ P-a.s. for all $s \in S$.

Let $t \in T$; by considering the restriction of G to the time set $S \cup \{t\}$, and the discrete optional stopping rule, we have for all $\tau \in \mathcal{S}_t$

$$\begin{aligned} G_t &\geq E[G_\tau \mid \mathcal{F}_t] \quad P\text{-a.s.} \\ &\geq E[X_\tau \mid \mathcal{F}_t] \quad P\text{-a.s.} \end{aligned}$$

Hence, $G_t \geq Z_t(S)$ P-a.s. for all $t \in T$. □

THEOREM 2.6.7 *Suppose S is a right continuous stopping region, then $Z(S)$ is the minimal right continuous supermartingale dominating X on S.*

Proof:

Mimic the proof in Theorem 2.6.6 and use the continuous time optional stopping rule. □

Remark 2.5 *Suppose S is a right continuous stopping region and H is a minimal right continuous supermartingale dominating X on S. By the minimality of H and Theorem 2.6.7, H is a modification of $Z(S)$. Because H and $Z(S)$ are right continuous, then H and $Z(S)$ are indistinguishable. Hence, Theorem 2.6.7 is a characterization of the Generalised Snell Envelope with respect to $(\mathcal{F}_t, t \in T)$.*

LEMMA 2.6.8 *Let γ be a stopping time in $\mathcal{T}$ and G be a supermartingale. Suppose either γ takes on discrete values, or G has right continuous paths, then $\{G_{\gamma \wedge t} \mid t \in T\}$ is a supermartingale.*

Proof:

We need to prove $G_{\gamma\wedge s} \geq E[G_{\gamma\wedge r} \mid \mathcal{F}_s]$ for $r \geq s$. Suppose $\gamma > s$ P-a.s. . Then using the discrete or continuous version of optional stopping rules, we have

$$G_s \geq E[G_{\gamma\wedge r} \mid \mathcal{F}_s].$$

Now suppose $\gamma \leq s$ P-a.s. , we have

$$G_\gamma = E[G_\gamma \mid \mathcal{F}_s] = E[G_{\gamma\wedge r} \mid \mathcal{F}_s]$$

Putting these together, we have

$$\begin{aligned} G_{\gamma\wedge s} &= G_s 1_{\gamma>s} + G_\gamma 1_{\gamma\leq s} \\ &\geq E[G_{\gamma\wedge r} 1_{\gamma>s} \mid \mathcal{F}_s] + E[G_{\gamma\wedge r} 1_{\gamma\leq s} \mid \mathcal{F}_s] \\ &= E[G_{\gamma\wedge r} \mid \mathcal{F}_s]. \qquad \square \end{aligned}$$

Given a stochastic process X satisfying conditions (2.1) - (2.3) and a stopping region $S \subset T$, the process X in the optimal stopping problem with respect to S is called the reward process. A stopping rule $\tau_0^* \in \mathcal{S} \subset \mathcal{T}$ is called an optimal stopping rule if

$$E\left[X_{\tau_0^*}\right] = \sup_{\tau\in\mathcal{S}} E[X_\tau].$$

The optimal stopping problem with respect to S for the reward process X is the problem of determining whether or not an optimal stopping rule exists, and of characterizing it if it does exist. If $\varepsilon > 0$, a stopping rule $\tau_0' \in \mathcal{S}$ is called an ε-optimal stopping rule if

$$E\left[X_{\tau_0'}\right] \geq \sup_{\tau\in\mathcal{S}} E[X_\tau] - \varepsilon.$$

Generally, if we want to maximize our expected reward after a random or deterministic time $\gamma \in \mathcal{T}$, we can generalise the above setting by considering the optimal and ε-optimal stopping rules $\tau_\gamma^*, \tau_\gamma' \in \mathcal{S} \subset \mathcal{T}$ after time γ such that

$$\begin{aligned} E\left[X_{\tau_\gamma^*}\right] &= \sup_{\tau \in \mathcal{S}_\gamma} E\left[X_\tau\right], \\ E\left[X_{\tau_\gamma'}\right] &\geq \sup_{\tau \in \mathcal{S}_\gamma} E\left[X_\tau\right] - \varepsilon. \end{aligned}$$

Notice that because $\sup_{\tau \in \mathcal{S}} E\left[X_\tau\right] = \Gamma_0(\Omega)$ and $\sup_{\tau \in \mathcal{S}_\gamma} E\left[X_\tau\right] = \Gamma_\gamma(\Omega)$, it is reasonable to believe the optimal stopping problem is closely related to the 'Supremum' measure as well as the Generalised Snell Envelope.

THEOREM 2.6.9 *Let Z be the Generalised Snell Envelope of the reward process X satisfying conditions (2.1) - (2.3) with respect to a discrete or right continuous stopping region S. Suppose $\tau_0^* \in \mathcal{S} \subset \mathcal{T}$. Then the following are equivalent,*

(1) τ_0^* *is an optimal stopping rule.*

(2) $E\left[Z_0\right] = E\left[Z_{\tau_0^*}\right] = E\left[X_{\tau_0^*}\right]$.

(3) $Z_{\tau_0^*} = X_{\tau_0^*}$ *P-a.s. and* $\left(Z_{\tau_0^* \wedge t} \mid t \in T\right)$ *is a martingale.*

Proof:

(1) $\Rightarrow$ (2) : Because S is a discrete or right continuous stopping region, then either τ_0^* takes on discrete values or Z is right continuous. By Theorem 2.5.3, Z is a supermartingale dominating X on S, so similarly to the proofs of

Theorem 2.6.6 and Theorem 2.6.7, we have

$$E[Z_0] \geq E\left[Z_{\tau_0^*}\right] \geq E\left[X_{\tau_0^*}\right].$$

On the other hand, τ_0^* is optimal so we have

$$E[Z_0] = \Gamma_0(\Omega) = E\left[X_{\tau_0^*}\right].$$

Therefore, $E[Z_0] = E\left[Z_{\tau_0^*}\right] = E\left[X_{\tau_0^*}\right]$.

(2) $\Rightarrow$ (3) : Since $\tau_0^* \in \mathcal{S}$, we have $Z_{\tau_0^*} \geq X_{\tau_0^*}$ P-a.s. If $P\left(Z_{\tau_0^*} > X_{\tau_0^*}\right) > 0$, then $E\left[Z_{\tau_0^*}\right] > E\left[X_{\tau_0^*}\right]$. This is a contradiction. Therefore, $Z_{\tau_0^*} = X_{\tau_0^*}$ P-a.s.

Again, by a similar argument to that above, we have

$$E[Z_0] \geq E\left[Z_{\tau_0^* \wedge t}\right] \geq E\left[Z_{\tau_0^*}\right].$$

Since $E[Z_0] = E\left[Z_{\tau_0^*}\right]$, we see $E\left[Z_{\tau_0^* \wedge t}\right] = E[Z_0]$. Now $\left(Z_{\tau_0^* \wedge t} \mid t \in T\right)$ is a supermartingale with constant expectation, hence it is a martingale.

(3) $\Rightarrow$ (1) : As $\left(Z_{\tau_0^* \wedge t} \mid t \in T\right)$ is a martingale and $Z_{\tau_0^*} = X_{\tau_0^*}$, we have

$$E\left[X_{\tau_0^*}\right] = E\left[Z_{\tau_0^*}\right] = E\left[Z_{\tau_0^* \wedge \tau_0^*}\right] = E\left[Z_{\tau_0^* \wedge t}\right] = E[Z_0].$$

Hence, τ is an optimal stopping rule. □

Let S be a feasible stopping region and $\gamma \in \mathcal{T}$. Define

$$\sigma_\gamma^*(\omega) \triangleq \inf\{t \in S \mid t \geq \gamma(\omega), Z_t(\omega) = X_t(\omega)\}.$$

Notice the infimum is always taken over a non-empty set as $Z_\infty = X_\infty$ P-a.s. Also, as S is feasible, we have $\sigma_\gamma^* \in \mathcal{S}_\gamma$. Further, if S is discrete or right continuous, we have $Z_{\sigma_\gamma^*} = X_{\sigma_\gamma^*}$ P-a.s.

LEMMA 2.6.10 *Suppose S is a feasible and right continuous (discrete) stopping region. Let $\gamma \in \mathcal{T}$ and $\tau \in \mathcal{S}_\gamma$. Then the reward process X satisfies the following inequality,*

$$E\left[X_{\tau\wedge\sigma_\gamma^*} \mid \mathcal{F}_\gamma\right] \geq E\left[X_\tau \mid \mathcal{F}_\gamma\right].$$

Proof:

$$\begin{aligned} X_{\tau\wedge\sigma_\gamma^*} &= 1_{\tau\geq\sigma_\gamma^*} X_{\sigma_\gamma^*} + 1_{\tau<\sigma_\gamma^*} X_\tau \\ &= 1_{\tau\geq\sigma_\gamma^*} Z_{\sigma_\gamma^*} + 1_{\tau<\sigma_\gamma^*} X_\tau \\ &\geq E\left[1_{\tau\geq\sigma_\gamma^*} Z_\tau \mid \mathcal{F}_{\sigma_\gamma^*}\right] + 1_{\tau<\sigma_\gamma^*} X_\tau \\ &\geq E\left[1_{\tau\geq\sigma_\gamma^*} X_\tau \mid \mathcal{F}_{\sigma_\gamma^*}\right] + 1_{\tau<\sigma_\gamma^*} X_\tau. \end{aligned}$$

Now for any $A \in \mathcal{F}_\gamma$, we have

$$\begin{aligned} E\left[X_{\tau\wedge\sigma_\gamma^*} 1_A\right] &\geq E\left[1_{\tau\geq\sigma_\gamma^*} X_\tau 1_A\right] + E\left[1_{\tau<\sigma_\gamma^*} X_\tau 1_A\right] \\ &= E\left[X_\tau 1_A\right]. \quad \square \end{aligned}$$

THEOREM 2.6.11 *Let S be a right continuous, or discrete, stopping region. Moreover, suppose S is feasible. If $\gamma \in \mathcal{T}$ either takes on discrete values or S is right continuous, then we have*

$$\Gamma_\gamma(\Omega) = \Gamma_{\sigma_\gamma^*}(\Omega).$$

That is,

$$E\left[Z_\gamma\right] = E\left[Z_{\sigma_\gamma^*}\right].$$

Proof:

Assume the equality fails, so we have $\Gamma_\gamma(\Omega) = \Gamma_{\sigma_\gamma^*}(\Omega) + \varepsilon$ for some $\varepsilon > 0$. Define

$$\mathcal{E} \triangleq \left\{\tau \in \mathcal{S}_\gamma^{\sigma_\gamma^*} \mid E[X_\tau] \geq \Gamma_{\sigma_\gamma^*}(\Omega) + \varepsilon/2\right\}.$$

We shall see $\mathcal{E}$ is non-empty.

First of all, because of the 'supremum' property of $\Gamma_\gamma(\Omega)$, there exists $\tau' \in \mathcal{S}_\gamma$ such that

$$\begin{aligned} E[X_{\tau'}] &\geq \Gamma_\gamma(\Omega) - \varepsilon/2 \\ &= \Gamma_{\sigma_\gamma^*}(\Omega) + \varepsilon/2. \end{aligned}$$

Next, since $\tau' \wedge \sigma_\gamma^* \in \mathcal{S}_\gamma^{\sigma_\gamma^*}$ and by the previous lemma, we have

$$E\left[X_{\tau' \wedge \sigma_\gamma^*}\right] \geq E[X_{\tau'}] \geq \Gamma_{\sigma_\gamma^*}(\Omega) + \varepsilon/2.$$

Hence, $\tau' \wedge \sigma_\gamma^* \in \mathcal{E}$.

Our next step is to show $\mathcal{E}$ has a unique maximal element. Suppose $\mathcal{L}$ is a linear order set in $\mathcal{E}$ in which the order relation is $\geq$ P-a.s. Define

$$\tau^* \triangleq \operatorname*{ess\,sup}_{\tau \in \mathcal{L}} \tau.$$

It is known that τ^* is a stopping time and there exists a sequence $(\tau_n) \subset \mathcal{L}$ such that $\tau_n \uparrow \tau^*$ P-a.s. Because of the uniform integrability of X, we have $E[X_{\tau_n}] \to E[X_{\tau^*}]$. Also, as $E[X_{\tau_n}] \geq \Gamma_{\sigma_\gamma^*}(\Omega) + \varepsilon/2$, we then have

$$E[X_{\tau^*}] \geq \Gamma_{\sigma_\gamma^*}(\Omega) + \varepsilon/2.$$

At the same time we have $\gamma \leq \tau_n \leq \sigma_\gamma^*$ P-a.s. implying $\tau^* \in \mathcal{S}_\gamma^{\sigma_\gamma^*}$. Therefore, $\tau^* \in \mathcal{E}$. Since $\mathcal{L}$ has an upper bound in $\mathcal{E}$, by Zorn's Lemma, there exists an unique maximal element λ in $\mathcal{E}$.

Obviously $\sigma^*_\gamma \notin \mathcal{E}$, and we have $\lambda \leq \sigma^*_\gamma$, with strict inequality holding on some non-null subset of Ω. By the definition of σ^*_γ one can deduce

$$\Gamma_\lambda(\Omega) = E[Z_\lambda] > E[X_\lambda].$$

Therefore, there exists $\lambda' \in \mathcal{S}_\lambda$ such that $E[X_{\lambda'}] > E[X_\lambda]$. Similarly to the above argument, we have $\lambda' \wedge \sigma^*_\gamma \in \mathcal{S}_\gamma^{\sigma^*_\gamma}$ and

$$E\left[X_{\lambda' \wedge \sigma^*_\gamma}\right] \geq E[X_{\lambda'}] > E[X_\lambda] \geq \Gamma_{\sigma^*_\gamma}(\Omega) + \varepsilon/2.$$

Therefore, $\lambda' \wedge \sigma^*_\gamma \in \mathcal{E}$. Now $\lambda' \wedge \sigma^*_\gamma \geq \lambda$ implies $\lambda' \wedge \sigma^*_\gamma = \lambda$. This would contradict the fact that $E\left[X_{\lambda' \wedge \sigma^*_\gamma}\right] > E[X_\lambda]$. Hence, we have $\varepsilon = 0$. □

Corollary 2.6.12 *Let S be a discrete or right continuous stopping region. Moreover, suppose S is feasible. If $\gamma \in \mathcal{T}$ takes on discrete values, or S is right continuous, then we have that σ^*_γ is an optimal stopping rule after time γ. In particular, σ^*_0 is an optimal stopping time and satisfies all equivalent statements in Theorem 2.6.9.*

Proof:

$$E\left[X_{\sigma^*_\gamma}\right] = E\left[Z_{\sigma^*_\gamma}\right] = E[Z_\gamma] = \sup_{\tau \in \mathcal{S}_\gamma} E[X_\tau]. \qquad \square$$

So far, we have shown that if S is right continuous, the Generalised Snell Envelope $Z(S)$ with respect to S is sufficient and it can be characterised as the minimal supermartingale dominating the reward process X on the stopping region S. Moreover, $Z(S)$ has P-a.s. right continuous paths if S is right continuous. In addition to these regularity conditions on S, when we have the feasibility condition, we can find an optimal stopping rule for the optimal stopping problem.

A supermartingale G is called *regular* if for any monotone sequence $\{\gamma_n\} \subset \mathcal{T}$ converging P-a.s. to a stopping time $\gamma \in \mathcal{T}$, we have

$$\lim_{n\to\infty} E[G_{\gamma_n}] = E[G_\gamma].$$

LEMMA 2.6.13 *Suppose X satisfies (2.1) - (2.3). If S is right continuous and feasible, then $Z(S)$ is regular.*

Proof:

We only need to prove $\gamma_n \uparrow \gamma$ implies $E[Z_{\gamma_n}] \downarrow E[Z_\gamma]$ as the monotone decreasing case is trivial.

Notice $\sigma^*_{\gamma_n} \to \gamma'$ for some $\gamma' \in \mathcal{S}$, and $\gamma' \geq \gamma$. Then

$$\begin{aligned} E[X_{\gamma'}] &= \lim_{n\to\infty} E\left[X_{\sigma^*_{\gamma_n}}\right] \\ &= \lim_{n\to\infty} E\left[Z_{\sigma^*_{\gamma_n}}\right] \\ &= \lim_{n\to\infty} E[Z_{\gamma_n}]. \end{aligned}$$

However, $E[Z_\gamma] \geq E[Z_{\gamma'}] \geq E[X_{\gamma'}] = \lim_{n\to\infty} E[Z_{\gamma_n}]$. □

LEMMA 2.6.14 *Suppose X satisfies (2.1) - (2.3). If S is right continuous, then $Z(S)$ is in class D.*

Proof:

Z is said to be of class D if the set of random variables $\{Z_\tau \mid \tau \in \mathcal{T}\}$ is uniformly integrable. However, if S is right continuous, then the GSE is sufficient for the GDE. That is $Z_\tau = Y_\tau$ P-a.s. Hence, the result follows from lemma 2.5.4.

THEOREM 2.6.15 *Suppose Z is the Generalised Snell Envelope of the reward process X satisfying conditions (2.1) - (2.3) with respect to a feasible, right continuous stopping region S. Then Z has a unique decomposition of the form*

$$Z_t = M_t - A_t.$$

Here A is a continuous, non-decreasing predictable process such that $A_0 = 0$ P-a.s. and M is a martingale.

Proof:

It follows from Theorem 2.6.7, lemma 2.6.13, lemma 2.6.14 and the Doob Meyer Decomposition [13]. □

From Theorem 2.6.9, one can see σ_0^* is the smallest optimal stopping rule. One can also ask for the largest optimal stopping rule. The idea then is to discover when $Z_{\lambda\wedge t}$ ceases to be a martingale, where $\lambda \in \mathcal{T}$ is given. Suppose S is a right continuous stopping region, so we can decompose $Z = M - A$ into a martingale M and a predictable non-decreasing process A with $A_0 = 0$. Since A is predictable,

$$\lambda_0^* \triangleq \sup\{t \in S \mid A_t = 0\},$$

is a stopping rule. This exit time of A from 0 thus gives the last time when $E\left[A_{\lambda_0^*}\right] = E[A_0] = 0$.

LEMMA 2.6.16 *Suppose $\tau_0^* \in \mathcal{S}$ is an optimal stopping rule. Then we have*

$$\tau_0^* \leq \lambda_0^* \quad P\text{-}a.s.$$

Proof:

Since $E[Z_0] = E\left[Z_{\tau_0^*}\right]$, by writing $Z = M - A$ for the Doob Meyer Decomposition of Z, we have $E\left[M_{\tau_0^*}\right] = E[M_0]$. So $E\left[A_{\tau_0^*}\right] = E[A_0]$, and hence $A_{\tau_0^*} = 0$ P-a.s. This implies $\tau_0^* \leq \lambda_0^*$. □

THEOREM 2.6.17 *λ_0^* is the largest optimal stopping rule.*

Proof:

Since $E[Z_0] = E\left[Z_{\lambda_0^*}\right]$, by Theorem 2.6.9 we have λ_0^* is an optimal stopping rule. Using the previous lemma, we see λ_0^* is the largest optimal stopping rule. □

Corollary 2.6.18 *A stopping rule $\tau \in \mathcal{S}$ is an optimal stopping rule if and only if $X_\tau = Z_\tau$ P-a.s. and $\tau \leq \lambda_0^*$.*

Proof:

This follows from Theorem 2.6.9 and the definition of λ_0^*. □

2.7 Lagrange Multipliers

In [12], Davis and Karatzas proposed a deterministic approach to the (ordinary) optimal stopping problem. They proved that

$$\sup_{\tau\in\mathcal{T}} E\left[X_\tau\right] = E\left[\sup_{t\in\mathcal{T}}\left(X_t + M_\infty - M_t\right)\right],$$

where M is the martingale component in the Doob Meyer Decomposition of the ordinary Snell Envelope Z of X. Thus, the process $\alpha_t = M_\infty - M_t$ is the Lagrange multiplier corresponding to the "non-anticipativity constraint" that τ be a stopping rule, (rather than a general non-adapted random time). Davis and Karatzas used many properties of the optimal and ε-optimal stopping rules to prove their proposition.

A natural question is : Can we obtain the "Lagrange multiplier" for the generalised optimal stopping problem with respect to a stopping region S? Similarly to the properties of the *GSE*, this "Lagrange process" (α_t) should depend on the stopping region S. Putting $S = \mathcal{T}$ as a special case, one would expect that we would require S to be right continuous and feasible. At the same time, as Davis and Karatzas obtained their result by using the properties of optimal and ε-optimal rules, feasibility seems to be an essential prerequisite. However, it turns out that feasibility is not required in the proposition.

In this section, we are going to investigate the above problem. However, we cannot imitate Davis and Karazatas's approach. The problem is that S may not be connected and piecewise estimations become tedious. Here, we are going to prove the generalised version of the above proposition by using a completely different but elementary approach.

LEMMA 2.7.1 *Let X be a reward process satisfying (2.1) - (2.3). Suppose S is a right continuous stopping region. Then we have*

$$E\left[\sup_{s\in S_t}(X_s - M_s) \mid \mathcal{F}_t\right] = \operatorname*{ess\,sup}_{\tau\in S_t} E\left[X_\tau - M_\tau \mid \mathcal{F}_t\right].$$

where M is the martingale part in the Doob Meyer Decomposition of $Z(S)$ and $S_t \triangleq [t, \hat{\infty}] \cap S$.

Proof:

Notice we do not require feasibility of S for Z to have the Doob Meyer Decomposition. The only loss is the continuity of the predictable part in the Decomposition. For simplicity, we define

$$V_t \triangleq E\left[\sup_{s\in S_t}(X_t - M_t) \mid \mathcal{F}_t\right].$$

If $r_1 \leq r_2$, then we have

$$\begin{aligned} E[V_{r_2} \mid \mathcal{F}_{r_1}] &= E\left[\sup_{s\in S_{r_2}}(X_s - M_s) \mid \mathcal{F}_{r_1}\right] \\ &\leq E\left[\sup_{s\in S_{r_1}}(X_s - M_s) \mid \mathcal{F}_{r_1}\right] \\ &= V_{r_1}. \end{aligned}$$

Hence V is a supermartingale. Furthermore, using monotone convergence theorem and the right continuity of $X - M$, we can show $E[V_t]$ is right continuous in t, and thus V is a right continuous supermartingale.

Next, we are going to show V dominates $X - M$ on S. Given $t \in S$, we have

$$\begin{aligned} X_t - M_t &\leq \sup_{s \in S_t} (X_s - M_s) \\ X_t - M_t &\leq E\left[\sup_{s \in S_t} (X_s - M_s) \mid \mathcal{F}_t\right] \\ &= V_t. \end{aligned}$$

Then, last but not least, we show V is the minimal right continuous supermartingale dominating $X - M$ on S. The lemma then follows directly from Theorem 2.6.7.

Suppose G is a right continuous supermartingale dominating $X - M$ on S. Hence we have

$$G_t + M_t \geq X_t \qquad \text{for } t \in S.$$

Hence $G + M$ is a right continuous supermartingale dominating X on S. By Theorem 2.6.7, we have

$$G_t + M_t \geq Z_t(S) \geq X_t \qquad \text{for } t \in S.$$

Subtracting M_t from the above equation, we have

$$G_t \geq -A_t \geq X_t - M_t \qquad \text{for } t \in S.$$

Since A is the non-decreasing predictable process in the Doob Meyer Decomposition, we have for $t \in S$,

$$G_t \geq -A_t \geq -\inf_{s \in S_t} A_s \geq \sup_{s \in S_t} (X_s - M_s)$$

Hence, for any $t \in S$, we have

$$G_t \geq E\left[\sup_{s \in S_t} (X_s - M_s) \mid \mathcal{F}_t\right] = V_t \qquad \square$$

THEOREM 2.7.2 *Suppose that X satisfies (2.1) - (2.3), S is right continuous and define $\alpha_t \triangleq M_{\hat{\infty}} - M_t$. Then*

$$Z_t(S) = E\left[\sup_{s \in S_t} (X_s + \alpha_s) \mid \mathcal{F}_t\right].$$

Proof:

$$\begin{aligned}
Z_t(S) &= \operatorname*{ess\,sup}_{\tau \in S_t} E\left[X_\tau \mid \mathcal{F}_t\right] \\
&= \operatorname*{ess\,sup}_{\tau \in S_t} E\left[X_\tau - M_\tau \mid \mathcal{F}_t\right] + M_t \\
&= \operatorname*{ess\,sup}_{\tau \in S_t} E\left[X_\tau - M_\tau \mid \mathcal{F}_t\right] + E\left[M_{\hat{\infty}} \mid \mathcal{F}_t\right] \\
&= E\left[\sup_{s \in S_t} (X_s - M_s) \mid \mathcal{F}_t\right] + E\left[M_{\hat{\infty}} \mid \mathcal{F}_t\right] \\
&= E\left[\sup_{s \in S_t} (X_s + \alpha_s) \mid \mathcal{F}_t\right]. \qquad \square
\end{aligned}$$

Corollary 2.7.3 *Suppose that X satisfies (2.1) - (2.3) and S is right continuous. Then*

$$\sup_{\tau \in S_t} E\left[X_\tau\right] = E\left[\sup_{s \in S_t} (X_s + \alpha_s)\right].$$

Proof:

This is a direct consequence of Theorem 2.7.2 by taking expectations on both sides. $\square$

Corollary 2.7.4 (Davis and Karatzas) *Suppose that X satisfies (2.1) - (2.3) and S is right continuous. Then*

$$\sup_{\tau \in S} E[X_\tau] = E\left[\sup_{t \in S}(X_t + \alpha_t)\right].$$

Proof:

This is a direct consequence of Corollary 2.7.3. □

Remark 2.6 *By taking $S = T$ in corollary 2.7.4, we have the original version of Davis and Karatzas's proposition.*

Corollary 2.7.4 has a very interesting intrepretation. Because $E[\alpha_\tau] = 0$ for any $\tau \in \mathcal{T}$, we have

$$\sup_{\tau \in S} E[X_\tau + \alpha_\tau] = E\left[\sup_{t \in S}(X_t + \alpha_t)\right].$$

The left hand side of the equation represents the best a "gambler" can achieve and the other side represents the best a "prophet" can achieve. Hence, given a process X, we can find a "penalty α" so that a prophet can only do as well as a gambler in the game corresponding to $X + \alpha$.

THEOREM 2.7.5 *Suppose X satisfies (2.1) - (2.3) and S is a right continuous such that $Z(S)$ is a non-increasing predictable process. Then*

$$\sup_{\tau \in S} E[X_\tau] = E\left[\sup_{t \in S} X_t\right].$$

Proof:

We have $M_t = 0$ and hence the result. □

2.8 Special Cases of the Stopping Region

In this section, we are going to summarize the chapter by looking at some special cases of the stopping region.

In the case where $S = T$ is a right continuous and feasible stopping region, we have most of the results from the previous sections. The Snell Envelope Z of process X is the minimal regular, right continuous supermartingale dominating X over T. For stopping times $\sigma \geq \gamma$ P-a.s. , we have

$$\begin{aligned} Z_\gamma &= \operatorname*{ess\,sup}_{\tau \geq \gamma} E\left[X_\tau \mid \mathcal{F}_\gamma \right], \\ Z_\gamma &= E\left[Z_\sigma \mid \mathcal{F}_\gamma \right] \vee Z_\gamma^\sigma, \\ Z_\gamma &= Z_{\gamma+} \vee X_\gamma. \end{aligned}$$

Also, the stopping time $\sigma_0^* \triangleq \inf\{t \geq 0 \mid Z_t = X_t\}$ is the optimal stopping rule.

It is quite interesting to see how we pass from the continuous time results to the discrete case in which the time index is $\overline{N}$. Suppose $\left(X_n \mid n \in \overline{N}\right)$ is a discrete time stochastic process and $\mathcal{N}$ is the collection of all stopping times taking values on $\overline{N}$. We define

$$X'_t \triangleq X_{\lfloor t \rfloor} \text{ and } \mathcal{F}'_t \triangleq \mathcal{F}_{\lfloor t \rfloor} \quad \text{for any } t \in \overline{R^+}$$

where $\lfloor t \rfloor$ is the largest integer smaller than or equal to t and $\lfloor \infty \rfloor = \infty$. Then X'_t satisfies (2.1) and (2.2) with respect to the right continuous filtration $\mathcal{F}'_t$. Denote $\mathcal{R}$ as the collection of all stopping times taking values on $\overline{R^+}$. With the discrete stopping region $\overline{N} \subset \overline{R^+}$, we can express the discrete optimal stopping problem in term of the generalised optimal stopping problem that we have proposed, because for $m \in \overline{N}$ we have,

$$Z_m \triangleq \operatorname*{ess\,sup}_{\tau \in \mathcal{N}_m} E\,[\,X_\tau \mid \mathcal{F}_m\,] = \operatorname*{ess\,sup}_{\tau \in \mathcal{N}_m} E\,[\,X'_\tau \mid \mathcal{F}'_m\,] \triangleq Z'_m(N).$$

Again we can characterise Z_n as the minimal supermartingale dominating X_n over all integers. For stopping times $\sigma \geq \gamma$ P-a.s. in $\mathcal{N}$, we have

$$\begin{aligned} Z_\gamma &= \operatorname*{ess\,sup}_{\tau \geq \gamma} E\,[\,X_\tau \mid \mathcal{F}_\gamma\,]\,, \\ Z_\gamma &= E\,[\,Z_\sigma \mid \mathcal{F}_\gamma\,] \vee Z^\sigma_\gamma, \\ Z_m &= E\,[\,Z_{m+1} \mid \mathcal{F}_m\,] \vee X_m. \end{aligned}$$

Also, the stopping time $\sigma_0^* \triangleq \inf\{n \geq 0 \mid Z_n = X_n\}$ is the optimal stopping rule.

Chapter 3

Financial Markets

3.1 Introduction

The aim of this chapter is to model a financial market in the context of stochastic calculus. We shall deal with a financial market $\mathcal{M}$ in which $d+1$ assets (or securities) can be traded continuously. These assets include a risk free asset, called a bond, and d risky assets called stocks. The prices of these stocks are driven by the same number of independent Brownian motions, which model the exogenous forces of uncertainty that influence the market. The interest rate of the bond, the appreciation rates of the stocks, their dividend paying rates, as well as their volatilities, constitute the coefficients of the market model. We impose some reasonable constraints on these coefficients so that the financial market behaves regularly. At the same time, we are going to define wealth, investment strategies or portfolios and consumption processes in terms of stochastic calculus. This language assists us in describing continuous time trading in terms of mathematics.

Instrumental in the approach that we adopt are two fundamental results of stochastic calculus: the *Girsanov's change of probability measure* and the *representation of Brownian martingales as stochastic integrals*. The former constructs processes that are independent Brownian motions under a new, equivalent probability measure, which equates the rates of return of all stocks and interest rate of the bond. The latter result provides the right investment portfolio for an investor who wants to achieve a certain level of wealth at any particular time. We assume that the reader is familiar with both these results. They are discussed in several monographs and texts dealing with stochastic calculus, such as [6] and [22].

This chapter is quite basic in its mathematical content. However, it serves as an important prerequisite for the theory of option pricing. Most of the results here are discussed in the the paper of Karatzas [21].

3.2 The Financial Market Model

Let $\mathcal{M}$ be a financial market with $d+1$ assets which can be traded continuously. One of them is a non-risky asset, called the bond, (also frequently called a savings account), with a price $P_0(t)$ given by

$$dP_0(t) = P_0(t)r(t)dt, \quad P_0(0) = 1. \tag{3.1}$$

and $P_0(t)$ determines the discount factor

$$\beta(t) \triangleq (P_0(t))^{-1}. \tag{3.2}$$

The remaining d assets are risky; we shall refer to them as stocks, and assume that the price $P_i(t)$ per share of the i-th stock, is governed by the linear stochastic differential equation

$$dP_i(t) = P_i(t)\left[b_i(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t)\right], \tag{3.3}$$

$$P_i(0) = p_i, \quad i = 1, 2, \cdots, d.$$

In this model, $W(t) = (W_1(t), \cdots, W_d(t))^*$ is a standard Brownian motion in R^d, whose components represent the external, independent sources of uncertainty in the market $\mathcal{M}$; with this interpretation, the *volatility coefficient* $\sigma_{ij}(\cdot)$ in (3.3) models the instantaneous intensity with which the j-th source of uncertainty influences the price of the i-th stock.

As is standard in the literature, our market $\mathcal{M}$ is assumed to be an *ideal market*. In other words, we have infinitely divisible assets, no constraints on consumption, no transaction costs or taxes. However, we shall allow for constraints on portfolio choice, such as limitations on borrowing from a savings account, or on short-selling of stocks, and so on.

3.3 Probabilistic Setting

We fix, from now on, a finite time interval $T = [\,0, T_0\,]$ in which we shall treat all of our problems.

The Brownian motion W in (3.3) will be defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and we shall denote by $(\mathcal{F}_t, t \in T)$ the P-augmentation of the natural filtraton $\mathcal{F}_t^W = \sigma\,(W(s);\ 0 \le s \le t)$ for $t \in T$. The coefficients of

$\mathcal{M}$, that is, the *interest rate* process $r(t)$, the *appreciation rate* vector process $b(t) = (b_1(t), \cdots, b_d(t))^*$ of the stocks, and the *volatility* matrix-valued process $\sigma(t) = \{\sigma_{ij}(t)\}_{1 \le i,j \le d}$, will all assumed to be progressively measurable with respect to $(\mathcal{F}_t, t \in T)$, and bounded unıformly in $(t, \omega) \in T \times \Omega$. We shall also assume that the *covariance matrix* process $a(t) \triangleq \sigma(t)\sigma(t)^*$ is strongly nondegenerate; that is there exists a number $\varepsilon > 0$ such that

$$(3.4) \qquad \xi^* a(t)\xi \ge \varepsilon ||\xi||^2, \qquad \forall\, \xi \in R^d,\ (t, \omega) \in T \times \Omega.$$

The assumption (3.4) amounts to what is called the *completeness* of the market model in the finance literature, and will enable our analysis to go through without serious technical difficulties. It is a straightforward consquence of the strong non-degeneracy condition that the matrices $\sigma(t), \sigma^*(t)$ are invertible and that the norms of these inverses are bounded above and below by δ and $1/\delta$ for some $\delta > 1$.

3.4 Portfolio, Consumption and Wealth

Let us consider now an economic agent, who invests in the various securities and whose decisions cannot affect the prices in the market (a small investor). At any time $t \in T$, he can decide,

i) how many shares of bond $\phi_0(t)$ and how many shares of stocks, $(\phi_1(t), \cdots, \phi_d(t))$ to hold, and

ii) what amount of money $C(t + h) - C(t) \ge 0$ to withdraw for consumption during the interval $(t, t + h]$, $h > 0$.

Of course, all these decisions can only be based on the current information $\mathcal{F}_t$, without anticipation of the future. More precisely, we have

Definition 3.4.1 *A trading strategy in the market $\mathcal{M}$ is a progressively measurable vector process $(\phi_0(t), \cdots, \phi_d(t))$ such that $\int_0^{T_0} \phi_i^2(t)dt < \infty$ almost surely for $0 \leq i \leq d$.*

The processes ϕ_0 and ϕ_i represent the number of shares of the bond and the i-th stock respectively, which are held or shortened at any given time t. A short position in the bond (respectively, the i-th stock), that is $\phi_0 < 0$ (repectively, $\phi_i < 0$), should be thought of as a loan.

Definition 3.4.2 *A consumption process is a progressively measurable process $C(t)$ with non-decreasing RCLL paths such that $C(0) = 0$, $C(T_0) < \infty$, almost surely.*

A basic assumption in the market $\mathcal{M}$, is that trading and consumption strategies should satisfy the so-called *self financing* condition, that is for $t \in T$, we have

$$\sum_{i=0}^{d} \phi_i(t)P_i(t) = \sum_{i=0}^{d} \phi_i(0)P_i(0) + \sum_{i=0}^{d} \int_0^t \phi_i(t)dP_i(t) - C(t). \tag{3.5}$$

The meaning of the equation is that, starting with an initial amount $x = \phi_0(0) + \sum_{i=1}^{d} \phi_i(0)p_i$ of wealth, all changes in wealth are due to capital gains (appreciation of stocks, and interest from bond), minus the amount consumed.

Definition 3.4.3 *The wealth process $X(t)$ is defined to be*

$$X(t) \triangleq \sum_{i=0}^{d} \phi_i(t)P_i(t), \quad t \in T. \tag{3.6}$$

The process can clearly take both positive and negative values, since the same is true for $\phi_i(\cdot)$, $i = 0, 1, \cdots, d$.

Definition 3.4.4 *The vector process $\pi = (\pi_1, \cdots, \pi_d)^*$ defined by*

$$\pi_i(t) \triangleq \begin{cases} \phi_i(t)P_i(t)/X(t) & \text{if } X(t) \neq 0 \\ 0 & \text{otherwise} \end{cases}, \tag{3.7}$$

for $1 \leq i \leq d$, $t \in T$, is called the portfolio process. The portfolio process, $\pi_i(t)$, simply represents the proportion of the wealth to invest in the i-th stock at time $t \in T$.

The adaptivity condition in the definition of consumption and portfolio process means, of course, that the investor cannot anticipate the future market prices; thus "insider trading" is excluded.

THEOREM 3.4.5 *With the above interpretation and notation, the wealth process $X(t)$ satifies the following stochastic differential equation:*

$$\begin{aligned} dX(t) &= r(t)X(t)dt + \pi^*(t)\left(b(t) - r(t)\vec{1}\right)X(t)dt \\ &\quad + \pi^*(t)\sigma(t)X(t)dW(t) - dC(t), \end{aligned} \tag{3.8}$$

where $\vec{1}$ is the vector in R^d with all components 1.

Proof:

$$\begin{aligned} dX(t) &= \sum_{i=0}^{d} \phi_i(t)dP_i(t) - dC(t) \\ &= \sum_{i=1}^{d} \phi_i(t)\left[b_i(t)P_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}dW_j(t)\right] \end{aligned}$$

$$
\begin{aligned}
& + \phi_0(t)r(t)P_0(t)dt - dC(t) \\
= & \sum_{j=1}^{d} \pi_j(t)X(t)\left[b_j(t)dt + \sum_{k=1}^{d}\sigma_{jk}(t)dW_k(t)\right] \\
& + \left(1 - \sum_{j=1}^{d}\pi_j(t)\right) r(t)X(t)dt - dC(t) \\
= & \; r(t)X(t)dt + \pi^*(t)\left(b(t) - r(t)\vec{1}\right)X(t)dt \\
& + \pi^*(t)\sigma(t)X(t)dW(t) - dC(t). \quad \square
\end{aligned}
$$

Now (3.8) is a simple linear stochastic differential equation for X. One can remove the drift term $\pi^*(t)\left(b(t) - r(t)\vec{1}\right)$ by Girsanov's theorem (1960). With a newly constructed Brownian motion $\tilde{W}$ under a different measure $\tilde{P}$, all the appreciation rates b_j are effectively equal to the interest rate r in the stochastic differential equation. We will discuss this further in the next section.

3.5 Auxiliary Probability Measure

Let us introduce the process

$$
\theta(t) \triangleq (\sigma(t))^{-1}\left[b(t) - r(t)\vec{1}\right],
$$

and observe that from the boundedness condition (3.4) on the coefficients in the market $\mathcal{M}$, it follows that the exponential supermartingale

$$
Y_t \triangleq \exp\left\{-\int_0^t \theta^*(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds\right\}, \quad t \in T
$$

is actually a martingale with respect to $(\mathcal{F}_t, t \in T)$. Denote by K an upper bound on both $||\theta(t,\omega)||$ and $\beta(t,\omega)$ for $(t,\omega) \in T \times \Omega$. For every finite $\alpha > 1$

we obviously have

$$
\begin{aligned}
Y_{T_0}^{\alpha} &= \exp\left(-\int_0^{T_0} \alpha\theta^*(s)dW(s) - \frac{1}{2}\int_0^{T_0} ||\alpha\theta(s)||^2 ds\right) \\
&\quad \cdot \exp\left(\frac{\alpha(\alpha-1)}{2}\int_0^{T_0} ||\theta(s)||^2 ds\right),
\end{aligned}
$$

and thus

$$
E\left[Y_{T_0}^{\alpha}\right] \leq \exp\left(\frac{\alpha(\alpha-1)}{2} T_0 K^2\right) < \infty. \tag{3.9}
$$

We shall define a new probability measure by setting

$$
\tilde{P}(A) \triangleq E\left[Y_{T_0} 1_A\right], \quad \forall A \in \mathcal{F}_{T_0}
$$

on $(\Omega, \mathcal{F}_{T_0})$. Then by the Girsanov's theorem we have that:

i) P and $\tilde{P}$ are mutually absolutely continuous on $\mathcal{F}_{T_0}$.

ii) The process

$$
\tilde{W}(t) \triangleq W(t) + \int_0^t \theta(s)ds; \quad t \in T,
$$

is an R^d-valued Brownian motion on $(\Omega, \mathcal{F}_{T_0}, \tilde{P})$.

In terms of this new process $\tilde{W}$, the equations (3.3), (3.8) can be written

$$
dP_j(t) = P_j(t)\left[r(t)dt + \sum_{k=1}^{d} \sigma_{jk}(t)d\tilde{W}(t)\right], \tag{3.10}
$$

$$
dX(t) = r(t)X(t)dt + X(t)\pi^*(t)\sigma(t)d\tilde{W}(t) - dC(t). \tag{3.11}
$$

Their solutions are given by

$$
\begin{aligned}
\beta(t)P_j(t) &= P_j(0)\exp\left\{\int_0^t \sigma_j^*(s)d\tilde{W}(s) - \frac{1}{2}\int_0^t ||\sigma_j(s)||^2 ds\right\}, \\
\beta(t)X(t) &= x + \int_0^t \beta(s)X(s)\pi^*(s)\sigma(s)d\tilde{W}(s) - \int_0^t \beta(s)dC(s),
\end{aligned}
$$

where $t \in T = [0, T_0]$. Here $X(0) = x > 0$ is the investor's initial wealth and $\sigma_j(t) \triangleq (\sigma_{j1}(t), \cdots, \sigma_{jd}(t))^*$.

In particular, we conclude the discounted stock price processes $\beta(t)P_j(t)$ are martingales under $\tilde{P}$ and the process

$$\tilde{M}(t) \triangleq \beta(t)X(t) + \int_0^t \beta(s)dC(s) = x + \int_0^t \beta(s)X(s)\pi^*(s)\sigma(s)d\tilde{W}(s),$$

which consists of discounted wealth at t, plus total discounted consumption on $[0, t]$, is a continuous local martingale under $\tilde{P}$.

3.6 Admissible Strategies

Definition 3.6.1 *A portfolio/consumption process pair (π, C) is called admissible for the initial capital $x \in R$, and we write $(\pi, C) \in \mathcal{A}(x)$, if*

i) $\pi(\cdot)$ is a progressively measurable, R^d-valued process that satisfies $\int_0^{T_0} ||\pi(t)||^2 dt < \infty$, almost surely.

ii) $C(\cdot)$ is a consumption process.

iii) The weath process corresponding to (π, C), that is, the solution $X^{x,\pi,C}(t) \equiv X(t)$ of equation (3.11) satisfies, almost surely:

$$X^{x,\pi,C}(t) \geq 0, \quad \forall t \in T.$$

The admissibility requirement is imposed in order to prevent pathologies like *doubling strategies* (c.f. Harrison & Pliska [19]); these achieve arbitrarily large levels of wealth at $t = T_0$, but require $X(\cdot)$ to be unbounded from below on $[0, T_0]$.

Remark 3.1 *For $x \in R$ and $(\pi, C) \in \mathcal{A}(x)$, let $X(\cdot)$ be the corresponding wealth process, and define*

$$\phi_i(t) = \begin{cases} X(t)\pi_i(t)/P_i(t), & i = 1, \cdots, d \\ X(t)\left(1 - \sum_{j=1}^d \pi_j(t)\right)/P_0(t), & i = 0 \end{cases} ; \quad \forall t \in T.$$

Then $\phi(\cdot) = (\phi_0(\cdot), \cdots, \phi_d(\cdot))^*$ constitutes a trading strategy corresponding to (x, π, C).

Remark 3.2 *For any $x \in R$, $(\pi, C) \in \mathcal{A}(x)$ and for any $a \neq 0$, we have $X^{ax,\pi,aC}(\cdot) = aX^{x,\pi,C}(\cdot)$.*

Remark 3.3 *If $(\pi, C) \in \mathcal{A}(x)$, the continuous $\tilde{P}$-local martingale $\tilde{M}(t)$ is bounded uniformly from below, and is thus a continuous $\tilde{P}$-supermartingale. Consequently, for any $\tau \in \mathcal{T}$, we have*

$$\tilde{E}\left[\beta(\tau)X^{x,\pi,C}(\tau) + \int_0^\tau \beta(t)dC(t)\right] \leq x, \quad \forall (\pi, C) \in \mathcal{A}(x) \tag{3.12}$$

THEOREM 3.6.2 *Given $x \in R$, and C a consumption process such that*

$$\tilde{E}\left[\int_0^{T_0} \beta(t)dC(t)\right] \leq x,$$

there exists a portfolio process π such that $(\pi, C) \in \mathcal{A}(x)$.

Proof:

Denote $D \triangleq \int_0^{T_0} \beta(t)dC(t)$, and consider the non-negative process

$$\tilde{X}(t) \triangleq (\beta(t))^{-1}\left(\tilde{E}\left[\int_t^{T_0} \beta(s)dC(s) \mid \mathcal{F}_t\right] + (x - \tilde{E}[D])\right).$$

Obviously $\tilde{X}(0) = x$ and, with $m(t) \triangleq \tilde{E}[D \mid \mathcal{F}_t] - \tilde{E}[D]$, we can write

$$\beta(t)\tilde{X}(t) = x + m(t) - \int_0^t \beta(s)dC(s).$$

Now using the fundamental martingale representation theorem [22], it can be shown that there exists an $\mathcal{F}_t$-progressively measurable process $\psi(\cdot)$ with values in R^d and $\int_0^{T_0} ||\psi(s)||^2 ds < \infty$, for which the martingale $m(t)$ takes the form

$$m(t) = \int_0^t \psi^*(s)d\tilde{W}(s).$$

It sufficies then to define

$$\pi(t) \triangleq \left(\beta(t)\tilde{X}(t)\sigma^*(t)\right)^{-1} \psi(t).$$

Thanks to the assumptions on the coefficients in $\mathcal{M}$, we see that $\pi(\cdot)$ is a portfolio process. Also, it is clear that $\tilde{X}(t) = X^{x,\pi,C}(t)$. Hence the result follows. □

THEOREM 3.6.3 *Given $x \in R$, and a non-negative $\mathcal{F}_{T_0}$ random variable B which satisfies the condition*

$$\tilde{E}[\beta(T_0)B] \leq x,$$

there exists $(\pi, C) \in \mathcal{A}(x)$ such that $X^{x,\pi,C}(T_0) = B$.

Proof:

Define the non-negative process $\tilde{X}$ by

$$\begin{aligned} \beta(t)\tilde{X}(t) &\triangleq \tilde{E}[Q \mid \mathcal{F}_t] + (x - \tilde{E}[Q])\left(1 - \frac{t}{T}\right) \\ &= x + m(t) - \rho t, \end{aligned}$$

where $Q \triangleq \beta(T_0)B$, $\rho \triangleq \frac{x-\tilde{E}[Q]}{T}$ and $m(t) \triangleq \tilde{E}[Q \mid \mathcal{F}_t] - \tilde{E}[Q]$. The desired consumption rate process is then $C(t) = \rho \int_0^t (\beta(s))^{-1} ds$. The rest of the argument follows the proof of Theorem 3.6.2.

Remark 3.4 *Theorem 3.6.3 still holds if T is replaced by an arbitary stopping time $\tau \in \mathcal{T}$. One would have to replace the definition of $\tilde{X}(\cdot)$ by*

$$\beta(t)\tilde{X}(t) \triangleq \tilde{E}[Q \mid \mathcal{F}_t] + (x - \tilde{E}[Q])\left(1 - \frac{t \wedge \tau}{\tau}\right),$$

and take $C(t) \equiv C(\tau)$, $\pi(t) \equiv 0$ for $\tau \leq t \leq T_0$. The rest of the argument goes through without change.

THEOREM 3.6.4 *Suppose $B(t)$ is a process such that $(\beta(t)B(t) \mid t \in T)$ is a regular, right continuous supermartingale in the class $D(T)$ under $\tilde{P}$. Then there exists $(\pi, C) \in \mathcal{A}(B(0))$ such that $X^{B(0),\pi,C}(t) = B(t)$ $\tilde{P}$-almost surely.*

Proof:

By the Doob Meyer decomposition and the fundamental martingale representation theorem, there exists a non-decreasing continuous predictable process $A(\cdot)$ and a progressively measurable process $\psi(\cdot)$ with values in R^d and $\int_0^{T_0} ||\psi(s)||^2 ds < \infty$ such that

$$d(\beta(t)B(t)) = \psi^*(t)d\tilde{W}(t) - dA(t).$$

Applying Itô's Lemma, we obtain

$$dB(t) = r(t)B(t)dt + (\beta(t))^{-1}\psi^*(t)d\tilde{W}(t) - (\beta(t))^{-1}dA(t).$$

Set $\pi(t) \triangleq [\beta(t)B(t)\sigma^*(t)]^{-1}\psi(t)$ and $C(t) \triangleq \int_0^t (\beta(s))^{-1}\,dA(s)$. Using a similar argument to that of Theorem 3.6.2, we have $(\pi, C) \in \mathcal{A}(B(0))$ and see that $B(t)$ is the solution of the wealth process with respect to (π, C). Hence the result follows. $\square$

Chapter 4

Options - A General View

4.1 Introduction

The valuation of options is a central problem in the modern theory of finance. Options on stocks were first traded on an organized exchange in 1973. Since then there has been a dramatic growth in options markets. Options are now traded on many different exchanges throughout the world. Huge volumes of options are also traded over the counter by banks and other financial institutions. The underlying assets include stocks, stocks indices, foreign currencies, debt instruments, commodities and futures contracts.

There are two basic types of options. A *call (put) option* is a right but not an obligation to buy (sell) a certain asset at a specified price until or at a future date. The situation can be colorfully imagined as a game where the reward is the payoff of the option and the option holder pays a fee (the option price) for playing the game.

In addition to these two basic types of options, there are many kind of options which differ in their payoff methods. In an excellent series of articles that appeared in *RISK* magazine in 1991 and 1992, Mark Rubinstein discusses quite a number of exotic options. We are not going to study them one by one. However, we are going to categorize these options briefly.

4.2 Options: Constrained Exercise Times

Suppose the investor is interested in the market $\mathcal{M}$ on the time horizon $T = [0, T_0] \subset R$. Let S be a stopping region in T, and $B(\cdot)$ be a non-negative progressively measurable process. An ***option with constrained exercise times*** is a financial instrument (T, B, S) consisting of

i) A payoff method $B(t)$ at time t.

ii) A selection of an exercise time $\tau \in \mathcal{S}$.

An investor who holds an option of this type can exercise his right only at time in S within the time horizon T. At the exercise time $\tau \in \mathcal{S}$, he would receive an income $B(\tau)$. In [20], Hull terms this category as *Bermudan options*. The Bermudan options consist of the most usual types of option in the market. We have the following examples.

4.2.1 European Options

An European option is an option which specifies that the holder can exercise his right only at a specified future date. This is a special case when $S = \{T_0\}$.

The pricing of European puts and calls on stocks has an interesting history, beginning with Bachelier [1] in 1900. The theory only reached a satisfactory level with the celebrated paper by Black and Scholes [5] using certain notions of *hedging* and *arbitrage free pricing.* These ideas were formalized and extended in Harrison and Pliska [19] by applying the fundamental concepts of stochastic integrals and the Girsanov's theorem.

Example. With $B(t) = \sum_{i=1}^{d} (P_i(t) - c_i)^+$, we have the *classical European call option* with strike price c_i on the i-th asset.

Example. With $B(t) = \sum_{i=1}^{d} (c_i - P_i(t))^+$, we have the *classical European put option* with strike price c_i on the i-th asset.

Example. With $d = 1$ in market $\mathcal{M}$ and $B(t) = (P_1(t) - c)^+ \vee (c - P_1(t))^+$, we have the *"As you like it" European option* with strike price c.

Example. With $d = 2$ in market $\mathcal{M}$ and $B(t) = (c_1 - P_1(t))^+ \vee (c_2 - P_2(t))^+$, we have the *Two rainbow European put option.*

4.2.2 American Options

Another common option is one with exercise possible at any instant until a given future time. These options are termed *American.* This is a special case in which $S = T = [0, T_0]$.

The earliest, and still one of the most penetrating analysis on the pricing of the American option is that of McKean [25]. There, the problem of pricing the American option is transformed into a *Stefan problem.* Although the American option problem was treated as an optimal stopping problem by

McKean, a financial justification using hedging arguments was given only later by Bensoussan [2] and Karatzas [21].

Example. Again with $d = 1$ in $\mathcal{M}$ and $B(t) = (P_1(t) - c)^+$ or $B(t) = (c - P_1(t))^+$, we have the classical *American call* and *American put* respectively.

4.3 Options: Constrained Portfolios in $\mathcal{M}$

We can consider options with constrained portfolios in $\mathcal{M}$. Suppose an investor agent is not allowed to short sell his stock, to borrow, and so on. Then the set of admissible strategies is reduced.

Generally we shall fix a non-empty, closed, convex set $K \subset R^d$. Then the set of admissible strategies with initial capital x would be

$$\mathcal{A}(x, K) \triangleq \{(\pi, C) \in \mathcal{A}(x) \mid \pi \in K\}.$$

In particular, we can consider either the European or American type of option with constrained portfolios in $\mathcal{M}$ such that the portfolio π is restricted to lie in the non-empty, closed convex set K.

Example. *Unconstrained case:* $K = R^d$.

Example. *Prohibition of short selling:* $K = [0, \infty)^d$.

Example. *Constraints on short selling:* $K = [-k, \infty)^d$, $k \geq 0$.

Example. *Prohibition of borrowing:* $K = \left\{\pi \in R^d \mid \sum_{i=1}^d \pi_i \leq 1\right\}$.

Example. *Incomplete Market, only the first m stocks can be traded:* $K = \left\{x \in R^d \mid x_i = 0, \forall\, i = m+1, \cdots, d\right\}$.

4.4 Options: Path Dependent

The payoff methods for this category of options depend not only on $P(t)$ at time t, but may depend on all the history of $P(\cdot)$ before t.

Example. *Lookback put option:* With $d = 1$ in the market $\mathcal{M}$, the payoff method at time t is $\left(\sup_{0\le s\le t} P_1(s) - P_1(t)\right)$.

Example. *Lookback call option:* With $d = 1$ in the market $\mathcal{M}$, the payoff method at time t is $(P_1(t) - \inf_{0\le s\le t} P_1(s))$.

Example. *Asian put option:* With $d = 1$ in the market $\mathcal{M}$, the payoff method at time t is $\left(c - \frac{\int_0^t P_1(s)ds}{t}\right)^+$ with strike price c.

Example. *Asian call option:* With $d = 1$ in the market $\mathcal{M}$, the payoff method at time t is $\left(\frac{\int_0^t P_1(s)ds}{t} - c\right)^+$ with strike price c.

In general, as in the previous section, we can consider European or American types of path dependent options.

Chapter 5

Options - Constrained Exercise Times

5.1 Basic Setting

Suppose S is a feasible, right continuous stopping region in $T = [0, T_0] \subset R$. An *option with constrained exercise times* or *Bermudan option*, is a financial instrument (T, B, S) consisting of

i) A payoff method $B(t)$ at time t.

ii) A selection of an exercise time $\tau \in \mathcal{S}$.

Here $\{B(t) \mid t \in T\}$ is a continuous, non-negative progressively measurable process which satisfies

$$E\left[\sup_{t\in T} B(t)\right]^{\mu} < \infty, \qquad \text{for some } \mu > 1.$$

We are interested in the following ***pricing problem*** for the option with constrained exercise time : What is a fair price to pay at $t = 0$ for this instrument? How much is it worth at any later time $t \in T$?

Let us suppose for a moment that the selection of $\tau \in \mathcal{S}$ has been made and we have a discounted income of $\beta(\tau)B(\tau)$ when we exercise the right of the option; suppose we have an initial capital

$$x = \tilde{E}\,[\,\beta(\tau)B(\tau)\,]\,.$$

Then from Remark 3.4, we can invest and consume in some way so that the corresponding discounted wealth process is

$$\beta(t)X^{(\tau)}(t) = \tilde{E}\,[\,\beta(\tau)B(\tau)\mid \mathcal{F}_t\,]\,, \quad \text{for} \;\; t \in [\,0,\tau\,],$$

with $X^{(\tau)}(\tau) = B(\tau)$. This suggests to us that we pay at most a price x for the option, or else we should ***work on our own*** to invest and consume. If the price of the option is less than x, then we would be better off to buy the option.

In particular, we should expect the fair price at $t = 0$ to be given by

$$\sup_{\tau \in \mathcal{S}} \tilde{E}\,[\,\beta(\tau)B(\tau)\,]\,,$$

and the value of this option at any time $t \in T$ should be

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{S}_t} \tilde{E}\left[\,-\exp\left(\int_t^\tau r(s)ds\right) B(\tau)\mid \mathcal{F}_t\right].$$

This is because an investor would select the optimal stopping time to exercise the option, if he has the choice to do so. The question is whether the above process is the wealth corresponding to an admissible portfolio/consumption process pair, that again ***duplicates*** the payoff from the option and does so with the minimal initial capital.

5.2 Hedging Strategy and Fair Price

In this section, we are going to define the fair price for the option with constrained exercise times. It turns out that this price is *arbitage free*. In short, arbitage free means nobody can obtain positive wealth from negative initial capital by any admissable strategy.

Definition 5.2.1 *Given a level of initial wealth $x \geq 0$, consider a pair $(\pi, C) \in \mathcal{A}(x)$. Let $X^{x,\pi,C}(\cdot)$ denote the corresponding wealth process. We say that (π, C) is a hedging strategy against the financial instrument (T, B, S), and write $(\pi, C) \in \mathcal{H}(x)$, if the following requirements hold, $\tilde{P}$ almost surely:*

i) $X^{x,\pi,C}(t) \geq B(t); \qquad \forall\, t \in S,$

ii) $X^{x,\pi,C}(T_0) = B(T_0)$.

Suppose that an investor buys the financial instrument (T, B, S) at $t = 0$ for the price $x \geq 0$, and there exists a pair $(\pi, C) \in \mathcal{H}(x)$. Then it makes no sense for the investor to exercise the option at t if $X^{x,\pi,C}(t) > B(t)$, because he could have done strictly better in terms of terminal wealth, and at least as well in terms of consumption, by investing instead in the market and consuming his wealth according to (π, C).

Remark 5.1 *In the case of European option, i.e.* $S = \{T_0\}$, *we have the hedging criterions degenerate to*

$$X^{x,\pi,C}(T_0) = B(T_0),$$

which coincides with the classical definition.

Definition 5.2.2 *The fair price at* $t = 0$ *for financial instrument* (T, B, S), *is the number*

$$\hat{V} \triangleq \inf\{x > 0 \mid \exists(\pi, C) \in \mathcal{H}(x)\}.$$

It is not difficult to see the *fair price* defined as above is an arbitage free price. Suppose the price of (T, B, S) is greater than $\hat{V}$; one could follow a suitable admissible strategy with a smaller initial capital to achieve the same terminal wealth. If the price of (T, B, S) is less than $\hat{V}$, the investor would be better off to hold an option instead. Roughly speaking, an investor with zero initial capital would not be better off borrowing money to buy the financial instrument (T, B, S) with price $\hat{V}$.

THEOREM 5.2.3 *The fair price at* $t = 0$ *for* (T, B, S) *is given by*

$$\hat{V} = v(0) \triangleq \sup_{\tau \in S} \tilde{E}[\beta(\tau)B(\tau)].$$

Moreover, there exists a strategy $(\pi^*, C^*) \in \mathcal{A}(u(0))$ *with wealth process,*

$$X^{u(0),\pi^*,C^*}(t) = \operatorname*{ess\,sup}_{\tau \in S_t} \tilde{E}\left[\exp\left(-\int_t^\tau r(s)ds\right) B(\tau) \mid \mathcal{F}_t\right] \quad \tilde{P}\text{-a.s.}$$

Proof:

Let $Q(\cdot) \triangleq \beta(\cdot)B(\cdot)$ and Z be the Generalised Snell Envelope of Q with respect to the feasible and right continuous stopping region S under $\tilde{P}$. We obtain from the Hölder inequality and (3.9), with $p \triangleq \sqrt{\mu} > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ that,

$$\tilde{E}\left[\sup_{t\in T} Q(t)\right] \leq \left(K^{\mu} E\left[\sup_{t\in T} B(t)\right]^{\mu}\right)^{\frac{1}{p}} \cdot \left(E\left[Y_{T_0}^{q}\right]\right)^{\frac{1}{q}} < \infty.$$

Thus we see that the process Q under $\tilde{P}$ satisfies (2.1) - (2.3). We define the process $v(\cdot)$ as

$$v(t) \triangleq [\beta(t)]^{-1} Z(t).$$

Suppose $x > 0$ and $(\pi, C) \in \mathcal{H}(x) \subset \mathcal{A}(x)$. From equation (3.12) in conjunction with the definition of hedging strategy, we have, for every stopping rule $\tau \in \mathcal{S}$,

$$\begin{aligned} \tilde{E}[Q(\tau)] &= \tilde{E}[\beta(\tau)B(\tau)] \\ &\leq \tilde{E}\left[\beta(\tau)X^{x,\pi,C}(\tau)\right] \\ &\leq x. \end{aligned}$$

Then

$$v(0) = [\beta(0)]^{-1} Z(0) = \sup_{\tau\in\mathcal{S}} \tilde{E}[Q_{\tau}] \leq x.$$

Therefore, $v(0) \leq \hat{V}$.

Next, $\beta(t)v(t) = Z(t)$ is a regular, right continuous supermartingale in class $D(T)$. By theorem 3.6.4, there exist $(\pi^*, C^*) \in \mathcal{A}(v(0))$ such that

$$\begin{aligned} X^{x,\pi^*,C^*}(t) &= v(t) \\ &= [\beta(t)]^{-1} Z(t) \\ &= \tilde{E}\left[-\exp\left(\int_t^{\tau} r(s)ds\right) B(\tau) \mid \mathcal{F}_t\right]. \end{aligned}$$

Since Z dominates Q on the stopping region S and $Z(T_0) = Q(T_0)$, we have v dominates B on the stopping region S and $v(T_0) = B(T_0)$. Thus, $(\pi^*, C^*) \in \mathcal{H}(v(0))$ and we have $v(0) \geq \hat{V}$. □

Remark 5.2 *The process $v(\cdot)$ is called the valuation process of (T, B, S) and the fair price of this option at time t is defined to be $v(t)$.*

In the first section of this chapter, we have seen that if the exercise time $\tau \in \mathcal{S}$ is selected by the economic agent in advance, then the corresponding price for this option should be $\tilde{E}\left[\beta(\tau)B(\tau)\right]$. Suppose the economic agent selects the optimal time

$$\sigma_0^* \triangleq \inf\left\{t \in S \mid \beta(t)B(t) = Z(t)\right\},$$

then the corresponding price for this option is the fair price $\hat{V}$.

THEOREM 5.2.4 *Consider a classical financial market $\mathcal{M}$ with constant interest rate $r(t) \equiv r \geq 0$ and volatility $\sigma(t) \equiv \sigma$. Let us consider an option $(T, \varphi(P(\cdot)), S)$ where $P(\cdot) \triangleq (P_1(\cdot), \cdots, P_d(\cdot))^*$ and $\varphi : R_+^d \to [0, \infty)$ is a bounded continuous function. Suppose*

$$u(t,x) \triangleq \sup_{\tau \in \mathcal{S}_t} \tilde{E}\left[\, e^{-r(\tau - t)}\varphi(P(\tau)) \mid P(t) = x\,\right], \quad \forall (t,x) \in T \times R_+^d$$

and u belongs to the Sobolev space $W^{1,2}(T \times R_+^d)$. Then u satisifies the following con[illegible]ns:

$$\begin{aligned} \mathcal{L}\left[e^{-rt}u\right](t,x) &\leq 0 && \forall (t,x) \in T \times R_+^d \\ \left(\mathcal{L}\left[e^{-rt}u\right](t,x)\right) \cdot (u(t,x) - \varphi(x)) &= 0 && \forall (t,x) \in S \times R_+^d \\ u(T_0, x) &= \varphi(x) && \forall x \in R_+^d \\ u(t,x) &\geq \varphi(x) && \forall t \in S \end{aligned}$$

in the sense of Schwartz distributions. Here

$$\mathcal{L} \triangleq \frac{\partial}{\partial t} + \sum_{i=1}^{d} r x_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j}.$$

Proof:

By the Markov property, we have the discounted valuation process

$$e^{-rt} v(t) = e^{-rt} u(t, P(t)).$$

This is a regular, right continuous supermartingale in class $D(T)$ under $\tilde{P}$. By the Doob Meyer decomposition and the fundamental martingale representation theorem, there exist a non-decreasing continuous predictable process $A(\cdot)$ and a progressively measurable process $\psi(\cdot)$, with values in R^d and satisfting $\int_0^{T_0} ||\psi(s)||^2 ds < \infty$, such that

$$d\left(e^{-rt} u(t, P(t))\right) = \psi^*(t) d\tilde{W}(t) - dA(t).$$

On the other hand, applying the generalized Itô's Lemma [23], we have

$$d\left(e^{-rt} u(t, P(t))\right) = \mathcal{L}\left[e^{-rt} u\right](t, P(t))dt + e^{-rt} diag(P(t)) \nabla u(t, P(t)) \cdot \sigma^* d\tilde{W}(t).$$

By comparing the coefficients, we have $\mathcal{L}\left[e^{-rt} u\right](t, x) \leq 0$.

We introduce the following two regions which partition $S \times R_+^d$,

$$\begin{aligned} \mathcal{C} &\triangleq \left\{(t,x) \in S \times R_+^d \mid u(t,x) > \varphi(x)\right\}, \\ \mathcal{W} &\triangleq \left\{(t,x) \in S \times R_+^d \mid u(t,x) = \varphi(x)\right\}. \end{aligned}$$

We shall call $\mathcal{C}$ the *continuation region* and $\mathcal{W}$ the *stopping region.*

By Theorem 2.6.11, we have $\left\{e^{-r(s\wedge\sigma_t^*)}u(s\wedge\sigma_t^*, P(s\wedge\sigma_t^*))\right\}_{t\leq s\leq T_0}$ is a martingale, thus

$$\mathcal{L}\left[e^{-rt}u\right](t,x)=0, \qquad \forall\,(t,x)\in\mathcal{C}.$$

The rest of the proof follows easily. □

Example. (Merton) In [24], Merton states the result that: *An American call option with payoff* $B(t)=(P_1(t)-c_1)^+$ *and positive exercise price* c_1*, written on a stock in the classical market with* $d=1$*, should not be exercised before the expiration date* T_0. In fact, a similiar statement holds for the financial instrument (T,B,S) with $B(t)=(P_1(t)-c_1)^+$. In that case, the process $Q(\cdot)=\beta(\cdot)B(\cdot)$ is a submartingale under $\tilde{P}$. Therefore, the optimal stopping time would be T_0.

Example. (Zero exercise price problem)
Consider the same setting as the last example. If the process Q is a supermartingale under $\tilde{P}$, then we have $v(t)=B(t)$. Suppose we have a zero exercise price for (T,B,S), that is $B(t)=P_1(t)$. Then Q is a supermartingale. Hence, the fair price for (T,P_1,S) is $\hat{V}=P_1(0)$. In short, (T,P_1,S) is equivalent to P_1.

5.2.1 Pricing an European Option

We have already seen that European option is just a special case when the stopping region $S=\{T_0\}$. Hence the pricing of European option is just an easy consequence of the general case.

THEOREM 5.2.5 (Karatzas [21]) *The fair price for the European option* $(T, B, \{T_0\})$ *at time* $t = 0$ *is given by*

$$\hat{V} = \tilde{E}\,[\,\beta(T_0)B(T_0)\,]\,,$$

and its valuation process is given by

$$v(t) = P_0(t)\tilde{E}\,[\,\beta(T_0)B(T_0) \mid \mathcal{F}_t\,]\,.$$

Proof:

This is a simple corollary of Theorem 5.2.3. □

Example. (Black and Scholes)

Consider the *classical financial market model* $\mathcal{M}$ with constant interest rate $r(t) \equiv r \geq 0$, volatility matrix $\sigma(t) \equiv \sigma$, and the payoff method $B(t) = \varphi(P(t))$. Here, $\varphi : R^d_+ \to [\,0, \infty\,)$ is a continuous function and $P(t) \triangleq (P_1(t), \cdots, P_d(t))^*$. The solution of P is given by,

$$P(t) = D(t, \sigma\tilde{W}(t)) \cdot P(0)$$

where $D : [\,0, \infty\,) \times R^d \to M_{d\times d}$ is such that

$$D(t, z) = \begin{bmatrix} e^{(r-a_{11})t+z_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{(r-a_{dd})t+z_d} \end{bmatrix}.$$

From the previous Theorem 5.2.5, we have the valuation process is given by

$$v(t) \;=\; \tilde{E}\left[e^{-r(T_0-t)}\varphi(P(T_0)) \mid \mathcal{F}_t\right]$$

$$
\begin{aligned}
&= e^{-r(T_0-t)}\tilde{E}\left[\varphi\left(D\left(T_0,\sigma\tilde{W}_{T_0}\right)P(0)\right) \mid \mathcal{F}_t\right] \\
&= e^{-r(T_0-t)}\tilde{E}\left[\varphi\left(D\left(T_0-t,\sigma\left[\tilde{W}_{T_0}-\tilde{W}_t\right]\right)P(t)\right) \mid \mathcal{F}_t\right] \\
&= e^{-r(T_0-t)}\int_{R^d}\varphi\left(D(T_0-t,\sigma z)P(t)\right)\phi(z,T_0-t)dz
\end{aligned}
$$

where $\phi(z,t) \triangleq (2\pi t)^{-d/2}\exp\left\{-\frac{\|z\|^2}{2t}\right\} \quad \forall z \in R^d,\ t>0.$

Define

$$
u(t,x) \triangleq \begin{cases} e^{-r(T_0-t)}\int_{R^d}\varphi\left(D(T_0-t,\sigma z)x\right)\phi(z,T_0-t)dz, & (t,x)\in[0,T_0)\times R^d_+ \\ \varphi(x), & (t,x)\in\{T_0\}\times R^d_+ \end{cases}
$$

With this notation, we have

$$
v(t) = u(t,P(t)) \quad \forall t \in T. \tag{5.1}
$$

Suppose $\varphi(x) = \sum_{i=1}^d (x_i - c_i)^+$, we can deduce a special result in Colwell, Elliott, Kopp [6]. In that case, we have

$$
\hat{V} = u(0,p) = e^{-rT_0}\sum_{i=1}^d \left(x_i\Delta_{i+}(T_0,p_i) - c_i\Delta_{i-}(T_0,p_i)\right), \qquad p = P(0)
$$

where

$$
\Delta_{i\pm}(s,y) \triangleq \Phi\left(\frac{1}{\sqrt{a_{ii}s}}\left[\ln\left(\frac{y}{c_i}\right) + \left(r \pm \frac{a_{ii}}{2}\right)s\right]\right)
$$

and $\Phi(\cdot)$ is the standard normal distribution. When $d=1$ the above result reduces to the well-known *Black-Scholes formula.*

Remark 5.3 By Theorem 3.6.4, it is possible to compute the portfolio π through the martingale representation theorem. This is the essence of the paper [6] by Colwell, Elliott and Kopp. However we can look at the problem in another way. Indeed, under appropriate growth conditions on φ in the above Black and Scholes Example, the function $u(t,x)$ of (3.12) is the unique solution of the Cauchy problem

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}a_{ij}x_ix_j\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{d}rx_i\frac{\partial u}{\partial x_i}-ru=0 \qquad (t,x)\in[0,T_0)\times R_+^d,$$

$$u(T_0,x)=\varphi(x), \qquad \forall x\in R_+^d,$$

by the Feynman-Kac theorem. Applying Itô's rule to the valuation process $v(\cdot)$ in (5.1), we have

$$dv(t)=rv(t)dt+\sum_{i=1}^{d}\sum_{j=1}^{d}\sigma_{ij}P_i(t)\frac{\partial}{\partial x_i}u(t,P(t))d\tilde{W}(t).$$

Also, since $v(\cdot)=X^{\hat{v},\pi^*,C^*}$, comparing the above with equation (3.11), we see that

$$\pi_i(t)=\frac{P_i(t)}{u(t,P(t))}\cdot\frac{\partial}{\partial x_i}u(t,P(t)), \qquad \forall t\in T,\ i=1,\cdots,d$$

is the portfolio process.

5.2.2 Pricing an American Option

We have also seen that the American option is another special case of (T,B,S). Therefore the pricing of an American option (T,B,T) is just an easy consequence of the general case. We have the following theorem.

THEOREM 5.2.6 (Karatzas [21]) *The fair price for the American option (T,B,T) at time $t=0$ is given by*

$$\hat{V}=\sup_{\tau\in\mathcal{T}}\tilde{E}\left[\beta(\tau)B(\tau)\right],$$

and its valuation process is given by

$$v(t)=\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t}\tilde{E}\left[\exp\left(-\int_t^{\tau}r(s)ds\right)B(\tau)\mid\mathcal{F}_t\right].$$

Proof:

This is a simple corollary of Theorem 5.2.3. □

Example. (American put)

Consider a classical financial market $\mathcal{M}$ with constant interest rate $r(t) \equiv r \geq 0$ and volatility $\sigma(t) \equiv \sigma$ with $d = 1$ and $\varphi(x) = (c_1 - x)^+$.

Let $H^{m,\lambda}$ be the set of measurable, real functions f on R whose distributional derivatives of order up to and including m belong to $L^2(R, e^{-\lambda|x|}dx)$ for some positive λ. This space is given the norm

$$||f|| \triangleq \left[\sum_{i \leq m} \int_R |\partial^i f(x)|^2 e^{-\lambda|x|} dx\right]^{\frac{1}{2}}.$$

The space $L^2\left(T; H^{m,\lambda}\right)$ is the set of measurable functions $g : [0, T_0] \to H^{m,\lambda}$ such that $\int_{[0,T_0]} ||g(t)||^2 dt < \infty$.

Suppose a continuous function $(t, x) \to w(t, x)$, defined on $T \times R_+$, such that $w(e^x, t) \in L^2\left(T; H^{2,\lambda}\right)$ and $\frac{\partial w}{\partial t} \in L^2\left(t; H^{0,\lambda}\right)$, satisfies the following system on $T \times R_+$:

$$\begin{aligned} \mathcal{L}\left[e^{-rt}w(t,x)\right] &\leq 0, \\ \left(\mathcal{L}\left[e^{-rt}w(t,x)\right]\right)\left(w(t,x) - (K-x)^+\right) &= 0, \\ w(t,x) &\geq (K-x)^+, \\ w(T_0,x) &= (K-x)^+. \end{aligned}$$

in the sense of Schwartz distributions where

$$\mathcal{L} \triangleq \frac{\partial}{\partial t} + \sum_{i=1}^{d} r x_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j}$$

Then w is unique and $w(t,x) = u(t,x)$.

The existence and uniqueness of the solution is proved in chapter 3 of Bensoussan and Lions [3], but, of course, it is not known explicity. Despite this, the system provides a useful characterization of the option value from which one can derive many properties of the value function. Note that the above conditions on w are exactly the same conditions for u in Theorem 5.2.4 when $S = T$.

5.3 Early Exercise Premium

The financial instrument (T, B, S) is more flexible than $(T, B, \{T_0\})$ in terms of its exercise date. Therefore, one would expect to pay more for (T, B, S). Let $v(t)$ be the valuation of (T, B, S) and $p(t)$ be the valuation of $(T, B, \{T_0\})$. We [illegible] the *Early Exercise Premium* $e(t)$ as the extra amount one should to pa[illegible]der to have such privilege. That is

$$e(t) \triangleq v(t) - p(t).$$

In the classical literature, the Early Exercise Premium is defined to be the difference in prices between the American and European types of option. That is the special case of the above definition when $S = T$.

THEOREM 5.3.1 *Suppose* $(\hat{\pi}, \hat{C}) \in \mathcal{H}(v(0))$ *is a strategy such that its wealth process coincides with* $v(\cdot)$. *We have*

$$e(t) = P_0(t) \cdot \tilde{E}\left[\int_t^{T_0} \beta(u) d\hat{C}(t) \mid \mathcal{F}_t\right].$$

Proof:

We adopt all the notations as in Theorem 5.2.3 and Theorem 3.6.4.

$$\begin{aligned} e(t) &= P_0(t) \cdot \left(\operatorname*{ess\,sup}_{\tau \in \mathcal{S}_t} \tilde{E}\,[\,Q(\tau) \mid \mathcal{F}_t\,] - \tilde{E}\,[\,Q(T_0) \mid \mathcal{F}_t\,] \right) \\ &= P_0(t) \cdot \left(Z(t) - \tilde{E}\,[\,Z(T_0) \mid \mathcal{F}_t\,] \right) \\ &= P_0(t) \cdot \tilde{E}\,[\,A(t) - A(T_0) \mid \mathcal{F}_t\,], \end{aligned}$$

where A is the non-decreasing predictable part of the Doob Meyer decomposition of Z. Since $d\hat{C}(t) = -P_0(t)dA(t)$, we have

$$e(t) = P_0(t) \cdot \tilde{E}\left[\int_t^{T_0} \beta(t) d\hat{C}(t) \mid \mathcal{F}_t \right]. \qquad \square$$

5.4 Gittens Index Processes

Consider a financial instrument (T, B, S) with $B(t) = (c - B'(t))^+$ in which $B'(\cdot)$ is a non-negative, adapted continuous process. In this section we are going to look at the valuation of (T, B, S) as a family of optimal stopping problems

$$v(t; c) \triangleq P_0(t) \cdot \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_t} \tilde{E}\left[\beta(\tau)(c - B'(\tau))^+ \mid \mathcal{F}_t \right]$$

parametrized by $c \in [\,0, \infty\,)$.

Similarly, we also look at the family of European valuation processes of $(T, B, \{T_0\})$

$$p(t; c) \triangleq P_0(t) \cdot \tilde{E}\left[\beta(T_0)(c - B'(T_0))^+ \mid \mathcal{F}_t \right]$$

parametrized by $c \in [\,0, \infty\,)$.

With $d = 1$, $B'(t) = P_1(t)$ and $S = T$, the valuation processes $v(t;c)$ is just a family of classical American put options with exercise price c. With $d = 1$, $B'(t) = P_1(t)$ and $S = \{T_0\}$, we have a family of European put options $p(t;c)$ with exercise price c.

We shall assume throughout that the process $p(\cdot;c)$ is strictly positive on $[0, T_0)$. It is also obvious that

$$v(t;c) \geq p(t;c) \vee (c - B'(t))^+ \qquad (t,c) \in S \times R_+$$

From Theorem 5.2.3, we know that, with fixed c and

$$Q(t;c) \triangleq \beta(t)(K - B'(t))^+ \qquad t \in T,$$

the process

$$Z(t;c) \triangleq \beta(t)v(t;c) = \operatorname*{ess\,sup}_{\tau \in S_t} \tilde{E}\left[Q(\tau;c) \mid \mathcal{F}_t \right] \qquad t \in T,$$

is the generalised Snell Envelope of $Q(\cdot, c)$ with respect to the feasible right continuous stopping region S.

Clearly, $Z(T_0;c) = Q(T_0, c)$ and the stopping time

$$\begin{aligned} \sigma_t^*(c) &\triangleq \inf\{s \in S_t \mid Z(s;c) = Q(s,c)\} \\ &= \inf\left\{s \in S_t \mid v(s;c) = (c - B'(s))^+\right\} \end{aligned}$$

is the optimal stopping rule after time t, and hence

$$\{Z(s \wedge \sigma_t^*(c);c)\}_{t \leq s \leq T_0}$$

is a martingale with respect to the continuous augmented filtration $(\mathcal{F}_t, t \in T)$.

Furthermore, since $Z(\cdot, c)$ is a regular, right continuous supermartingle of class $D(T)$, and hence quasi-left continuous, we have

$$\lim_{n\to\infty} Z(\tau_n, c) = Z(\tau, c) \qquad \forall \tau_n, \tau \in \mathcal{T}, \tau_n \uparrow \tau.$$

LEMMA 5.4.1 *For each $t \in [0, T_0)$, the mapping $v(t; \cdot)$ is convex, increasing, strictly positive except $c = 0$.*

Proof:

The convexity and monotonicity follow from the facts that the mapping $c \to (c - x)^+$ has these properties, and that we are taking supremum over the class $\mathcal{S}_t$ of stopping times. □

LEMMA 5.4.2 *For each $t \in [0, T_0)$, the mapping $c \to c - v(t; c)$ is concave, increasing, null at $c = 0$. Furthermore, we have $c - v(t; c) \leq c \wedge B'(t)$ for $t \in S - \{T_0\}$.*

Proof:

$$\begin{aligned}
c - v(t;c) &= c - P_0(t) \cdot \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_t} \tilde{E}\left[\beta(\tau)(c - B'(\tau))^+ \mid \mathcal{F}_t\right] \\
&= c - P_0(t) \cdot \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_t} \tilde{E}\left[\beta(\tau)(c - c \wedge B'(\tau)) \mid \mathcal{F}_t\right] \\
&= \operatorname*{ess\,inf}_{\tau \in \mathcal{S}_t} \tilde{E}\left[c(1 - e^{-\int_t^\tau r(s)ds}) + e^{-\int_t^\tau r(s)ds}(c \wedge B'(\tau)) \mid \mathcal{F}_t\right] \\
&\leq c \wedge B'(t) \quad \text{if } t \in S - \{T_0\}.
\end{aligned}$$

The two functions of c inside the expectation are linear and concave, respectively, and both are increasing. Since we are taking an infimum over $\mathcal{S}_t$, these properties persist. □

LEMMA 5.4.3 *For each $t \in [0, T_0)$, the mapping $c \to \sigma_t^*(c)$ is decreasing and right continuous.*

Proof:

Using the previous lemma, we introduce the random field

$$\varphi(t;c) \triangleq v(t;c) - (c - B'(t))^+ = (c \wedge B'(t)) - (c - v(t;c))$$

which is continuous, decreasing in c and non-negative for all $t \in S - \{T_0\}$. We can rewrite the optimal stopping time $\sigma_t^*(c)$ as $\inf\{u \in S_t \mid \varphi(u;c) = 0\}$.

If $c_1 \geq c_2$, then we have

$$0 \leq \varphi(\sigma_t^*(c_2); c_1) \leq \varphi(\sigma_t^*(c_2); c_2) = 0.$$

Hence $\varphi(\sigma_t^*(c_2); c_1) = 0$ and we have $\sigma_t^*(c_1) \leq \sigma_t^*(c_2)$ by the property of the infimum. Thus, $\sigma_t^*(\cdot)$ is decreasing.

Next, suppose $c_n \downarrow\downarrow c$. We need to show $\sigma_t^*(c_n) \uparrow \sigma_t^*(c)$. Since $\sigma_t^*(\cdot)$ is decreasing, $\sigma_t^* \triangleq \lim_{n\to\infty} \sigma_t^*(c_n)$ exists and $\sigma_t^* \leq \sigma_t^*(c)$. Since $\varphi(\sigma_t^*(c_l); c_m) = 0$ for $l > m$, we use quasi-left continuity of the process $\varphi(\cdot, c_m)$ and take $l, m \to \infty$, so that we have $\varphi(\sigma_t^*, c) = 0$. Thus $\sigma_t^*(c) \leq \sigma_t^*$. □

THEOREM 5.4.4 *For every $t \in [0, T_0)$, the convex mapping $v(t;\cdot)$ has a right hand derivative given by*

$$\begin{aligned} \beta(t) \cdot \frac{\partial^+}{\partial c} v(t;c) &= \tilde{E}\left[\beta(\sigma_t^*(c)) 1_{\sigma_t^*(c) < T_0} + \beta(T_0) 1_{B'(T_0) \leq c, \sigma_t^*(c) = T_0} \mid \mathcal{F}_t\right] \\ &= \tilde{E}\left[\beta(\sigma_t^*(c)) - \beta(T_0) 1_{B'(T_0) \leq c, \sigma_t^*(c) = T_0} \mid \mathcal{F}_t\right]. \end{aligned}$$

Proof:

Fix $(t,c) \in [0,T_0) \times (0,\infty)$, and for any $\varepsilon > 0$, denote $\sigma^\varepsilon \triangleq \sigma_t^*(c+\varepsilon)$ and $\sigma^0 \triangleq \sigma_t^*(c)$. Since $\sigma^\varepsilon \le \sigma^0$, we have $Z(\cdot \wedge \sigma^\varepsilon; c)$ is a martingale and thus, $\beta(t)\, v(t;c) = \tilde{E}\,[\beta(\sigma^\varepsilon) v(\sigma^\varepsilon;c) \mid \mathcal{F}_t]$. On the other hand, from the optimality of σ^ε at $(t, c+\varepsilon)$, we have $\beta(t)\, v(t;c+\varepsilon) = \tilde{E}\left[\beta(\sigma^\varepsilon)\,(c+\varepsilon - B'(\sigma^\varepsilon))^+ \mid \mathcal{F}_t\right]$. Thus, we have the following

$$\beta(t)\ (v(t;c+\varepsilon) - v(t;c)) = \tilde{E}\left[\beta(\sigma^\varepsilon)\left((c+\varepsilon - B'(\sigma^\varepsilon))^+ - v(\sigma^\varepsilon;c)\right) \mid \mathcal{F}_t\right]$$

On the event $\{\sigma^\varepsilon < T_0\}$, we have $c + \varepsilon - B'(\sigma^\varepsilon) = v(\sigma^\varepsilon; c+\varepsilon) > 0$ and $v(\sigma^\varepsilon;c) > c - B'(\sigma^\varepsilon)$, and thus $(c+\varepsilon - B'(\sigma^\varepsilon))^+ - v(\sigma^\varepsilon;c) \le \varepsilon$.

On the event $\{\sigma^\varepsilon = T_0\}$, we have $v(\sigma^\varepsilon;c) = (c - B'(T_0))^+$, and thus $(c+\varepsilon - B'(\sigma^\varepsilon))^+ - v(\sigma^\varepsilon;c) \le \varepsilon 1_{B'(T_0)\le c+\varepsilon}$.

These observations lead to the upper bound,

$$\beta(t)\,\frac{v(t;c+\varepsilon) - v(t;c)}{\varepsilon} \le \tilde{E}\left[\beta(\sigma^\varepsilon) 1_{\sigma^\varepsilon < T_0} + \beta(T_0) 1_{\sigma^\varepsilon = T_0\,,\,B'(T_0)\le c+\varepsilon} \mid \mathcal{F}_t\right].$$

Now $\sigma^\varepsilon \uparrow \sigma^0$ as $\varepsilon \downarrow 0$, so we have

$$\beta(t)\ \limsup_{\varepsilon\downarrow 0} \frac{v(t;c+\varepsilon) - v(t;c)}{\varepsilon} \le \tilde{E}\left[\beta(\sigma^0) 1_{\sigma^0 < T_0} + \beta(T_0) 1_{\sigma^0 = T_0\,,\,B'(T_0)\le c} \mid \mathcal{F}_t\right]$$

To obtain a lower bound, recall the supermartingale properties of $Z(\cdot; c+\varepsilon)$. These give $\beta(t)\, v(t;c+\varepsilon) \ge \tilde{E}\,[\beta(\sigma^0) v(\sigma^0; c+\varepsilon) \mid \mathcal{F}_t]$, and in conjunction with $\beta(t)\, v(t;c) = \tilde{E}\,[\beta(\sigma^0)(K - B'(\sigma^0))^+ \mid \mathcal{F}_t]$, we have

$$\begin{aligned}\beta(t)\,(v(t;c+\varepsilon) - v(t;c)) \ &\ge\ \tilde{E}\left[\beta(\sigma^0)\left(v(\sigma^0;c+\varepsilon) - (c - B'(\sigma^0))^+\right) \mid \mathcal{F}_t\right] \\ &\ge\ \tilde{E}\left[\beta(\sigma^0)\left((c+\varepsilon - B'(\sigma^0))^+ - (c - B'(\sigma^0))^+\right) \mid \mathcal{F}_t\right].\end{aligned}$$

On the event $\{\sigma^0 \le T_0\}$, we have $c - B'(\sigma^0) = v(\sigma^0;c) > 0$, and hence $(c+\varepsilon - B'(\sigma^0))^+ - (c - B'(\sigma^0))^+ = \varepsilon$.

On the event $\{\sigma^0 = T_0\}$, we have $(c + \varepsilon - B'(\sigma^0))^+ - (c - B'(\sigma^0))^+ = (c + \varepsilon - B'(T_0))^+ - (c - B'(T_0))^+ \geq \varepsilon 1_{B'(T_0) \leq c}$.

These observations lead to the upper bound,

$$\beta(t)\, \frac{v(t; c+\varepsilon) - v(t;c)}{\varepsilon} \geq \tilde{E}\left[\beta(\sigma^0) 1_{\sigma^0 < T_0} + \beta(T_0) 1_{\sigma^0 = T_0, B'(T_0) \leq c} \mid \mathcal{F}_t\right]$$

Thus we have,

$$\beta(t) \liminf_{\varepsilon \downarrow 0} \frac{v(t; c+\varepsilon) - v(t;c)}{\varepsilon} \geq \tilde{E}\left[\beta(\sigma^0) 1_{\sigma^0 < T_0} + \beta(T_0) 1_{\sigma^0 = T_0, B'(T_0) \leq c} \mid \mathcal{F}_t\right]$$

Hence the result follows. □

For any $t \in T$, we define the *Gittens Index Process* for the financial instrument (T, B, S) where $B(t) = (c - B'(t))^+$ as

$$M(t) \triangleq \begin{cases} \inf\left\{c > 0 \mid v(t;c) = (c - B'(t))^+\right\} & 0 \leq t < T_0 \\ B'(T_0) & t = T_0 \end{cases}$$

For each $t \in [0, T_0)$ and $t' \geq t$, we define the lower envelope of $M(\cdot)$ on $[t, t']$ as

$$\underline{M}(t, t') \triangleq \inf_{u \in S \cap [t, t']} M(u)$$

Again, as in lemma 5.4.3, we can introduce the random field $\varphi(t;c) \triangleq v(t;c) - (c - B'(t))^+$ which is continuous, decreasing in c and non-negative for all $t \in S - \{T_0\}$. We can rewrite the Gittens Index and optimal stopping time after t as

$$\begin{aligned} M(t) &= \begin{cases} \inf\{c > 0 \mid \varphi(t;c) = 0\} & t < T_0 \\ B'(T_0) & t = T_0 \end{cases} \\ \sigma_t^*(c) &= \inf\{u \in S_t \mid \varphi(u;c) = 0\}. \end{aligned}$$

Then, for $t' < T_0$ we have

$$\begin{aligned}
\underline{M}(t,t') > c &\Leftrightarrow M(u) > c \qquad \forall u \in [t,t'] \cap S \\
&\Leftrightarrow \varphi(u;c) > 0 \qquad \forall u \in [t,t'] \cap S \\
&\Leftrightarrow \sigma_t^*(c) > t'.
\end{aligned} \tag{5.2}$$

At the same time, we have

$$\begin{aligned}
\{\sigma_t^*(c) < T_0\} &= \cup_{\alpha>0} \{\sigma_t^*(c) \leq T_0 - \alpha\} \\
&= \cup_{\alpha>0} \{\underline{M}(t, T_0 - \alpha) \leq c\} \\
&= \{\underline{M}(t, T_0-) \leq c\}.
\end{aligned} \tag{5.3}$$

THEOREM 5.4.5 *In terms of the lower envelope $\underline{M}(t,\cdot)$ of the Gittens Index $M(\cdot)$, we have for every $t \in [0, T_0)$ the following representation of the early exercise premium for the financial instrument (T, B, S) with $B(t) = (c - B'(t))^+$:*

$$\begin{aligned}
v(t;c) - p(t;c) &= P_0(t) \cdot \tilde{E}\,[\int_t^{T_0} \beta(u) r(u)\,(c - \underline{M}(t,u))^+\,du \\
&\qquad + \beta(T_0)\,(c \wedge B'(T_0) - \underline{M}(t, T_0-))^+ \mid \mathcal{F}_t\,].
\end{aligned}$$

Proof:

It is not difficult to see $\beta(t) \cdot \frac{\partial^+}{\partial c} p(t;c) = \tilde{E}\left[\beta(T_0) 1_{B'(T_0) \leq c} \mid \mathcal{F}_t\right]$. Then, from Theorem 5.4.4 we have,

$$\begin{aligned}
\beta(t)\,\frac{\partial^+}{\partial c}\,[v(t;c) - p(t;c)] &= \tilde{E}\left[1_{\sigma_t^*(c) < T_0}\left(\beta(\sigma_t^*(c)) - \beta(T_0) 1_{B'(T_0) \leq c}\right) \mid \mathcal{F}_t\right] \\
&= \tilde{E}\,[(\beta(\sigma_t^*(c)) - \beta(T_0)) \mid \mathcal{F}_t] \\
&\qquad + \tilde{E}\left[\beta(T_0) 1_{B'(T_0) > c,\, \sigma_t^*(c) < T_0} \mid \mathcal{F}_t\right].
\end{aligned}$$

Using (5.2), we have

$$\begin{aligned}\beta(\sigma_t^*(c)) - \beta(T_0) &= \int_t^{T_0} \beta(u)r(u)1_{\sigma_t^*(c)\leq u}\, du \\ &= \int_t^{T_0} \beta(u)r(u)1_{\underline{M}(t,u)\leq c}\, du.\end{aligned}$$

From (5.3), we also have

$$1_{B'(T_0)>c\,,\sigma_t^*(c)<T_0} = 1_{\underline{M}(t,T_0-)\leq c<B'(T_0)}.$$

Hence

$$\begin{aligned}\beta(t)\,\frac{\partial^+}{\partial c}\left[v(t;c)-p(t;c)\right] &= \tilde{E}\left[\int_t^{T_0} \beta(u)r(u)1_{\underline{M}(t,u)\leq c}\, du \mid \mathcal{F}_t\right] \\ &\quad + \tilde{E}\left[\beta(T_0)1_{\underline{M}(t,T_0-)\leq c<B'(T_0)} \mid \mathcal{F}_t\right]\end{aligned}$$

Now, integrate both sides over $[0,c]$ and use $v(t;c) = p(t;0) = 0$, we have proved the theorem from the conditional Fubini theorem. □

Remark 5.4 *El Karoui and Karatzas had derived the above results in discrete setting [15] and continuous setting [16] with $S = T$.*

Chapter 6

Options - Constrained Portfolios in $\mathcal{M}$

6.1 Introduction

In this chapter, we are going to evaluate the price of options with constrained portfolios in the market $\mathcal{M}$. For simplicity, we only consider the European type of these options. We shall adopt the same setting as in chapter 4 and fix a non-empty, closed, convex set $K \subset R^d$. Denote by

$$\delta(x) \equiv \delta(x|K) \triangleq \sup_{\pi \in K}(-\pi^* x) : R^d \to R \cup \{+\infty\},$$

the support function of the convex set $-K$. This is a closed, positively homogeneous, proper convex function [29] on R^d. It is finite on its *effective domain*

$$\tilde{K} \triangleq \left\{x \in R^d \mid \delta(x|K) < \infty\right\}$$

$$= \left\{x \in R^d \mid \exists \beta \in R \ s.t. -\pi^* x \le \beta, \forall \pi \in K\right\},$$

which is a convex cone (called the barrier cone of $-K$). It will be assumed throughout the chapter that the support function δ is continuous on $\tilde{K}$ and bounded below on R^d. These two assumptions are satisfied if K contains the origin and locally simplicial [29].

From now on, we consider only portfolios that take values in this given non-empty, closed, convex set $K \subset R^d$. Then we shall replace the set of admissible strategies $\mathcal{A}(x)$ with

$$\mathcal{A}(x; K) \triangleq \{(\pi, C) \in \mathcal{A}(x) \mid \pi(t,\omega) \in K \text{ for } \ell \otimes P \ a.s.\}.$$

Let us consider the class $\mathcal{H}$ of $\tilde{K}$-valued progressively measurable process $\nu(\cdot)$ that satisfies

$$E\left[\int_0^{T_0} ||\nu(t)||^2 + \delta(\nu(t))dt\right] < \infty,$$

and introduce for every $\nu \in \mathcal{H}$ the analogues,

$$\theta_\nu(t) \triangleq \sigma^{-1}(t)\left[b(t) - r(t)\vec{1} + \nu(t)\right], \tag{6.1}$$

$$\beta_\nu(t) \triangleq \exp\left[-\int_0^t r(s) + \delta(\nu(s))ds\right], \tag{6.2}$$

$$Y_\nu(t) \triangleq \exp\left[-\int_0^t \theta_\nu^*(s)dW(s) - \frac{1}{2}\int_0^t ||\theta_\nu(s)||^2 ds\right], \tag{6.3}$$

$$\tilde{W}_\nu(t) \triangleq W(t) + \int_0^t \theta_\nu(s)ds, \tag{6.4}$$

$$\tilde{P}_\nu(A) \triangleq E\left[\tilde{Y}_\nu(T_0)1_A\right], \quad A \in \mathcal{F}_{T_0}, \tag{6.5}$$

of the settings in chapter 2. Finally, denote by $\mathcal{D}$ the subset consisting of the processes $\nu \in \mathcal{H}$ for which the exponential local martingale $Y_\nu(\cdot)$ is actually a martingale. Thus, for every $\nu \in \mathcal{D}$, the measure $\tilde{P}_\nu$ is a probability measure and the process $\tilde{W}_\nu(\cdot)$ is a Brownian motion under $\tilde{P}_\nu$ by the Girsanov theorem. In terms of this new Brownian motion $\tilde{W}_\nu(\cdot)$, the stock price equations (3.3) can be rewritten as

$$(6.6) \qquad dP_i(t) = P_i(t)\left[(r(t) - \nu_i(t))dt + \sum_{j=1}^{d} \sigma_{ij}(t) d\tilde{W}_\nu^{(j)}(t)\right], \ i = 1, \cdots, d.$$

As we saw in Theorem 5.2.5, in the case $K = R^d$, the number

$$u^{(0)} = \tilde{E}_0\left[\beta_0(T_0)B(T_0)\right] = \tilde{E}\left[\beta(T_0)B(T_0)\right]$$

is the unconstrained hedging price for the financial instrument $(T, B, \{T_0\})$. In the framework of [10], the number

$$u^\nu \triangleq \tilde{E}_\nu\left[\beta_\nu(T_0)B(T_0)\right]$$

is the unconstrained hedging price for $(T, B, \{T_0\})$ in an auxiliary market $\mathcal{M}_\nu$; this consists of a bond with interest rate $r^{(\nu)}(t) \triangleq r(t) + \delta(\nu(t))$ and d stocks, with the same volatility matrix σ as before and appreciation rates $b^{(\nu)}(t) \triangleq b_i(t) + \nu_i(t) + \delta(\nu(t))$, $i = 1, \cdots, d$, for any given $\nu \in \mathcal{D}$. Thus, in the market $\mathcal{M}_\nu$, the price of the bond in $\mathcal{M}_\nu$ is given as

$$dP_0^\nu(t) = P_0^\nu(t)\left[r(t) + \delta(\nu(t))\right]dt,$$

and the prices of the stocks are given as

$$dP_i^\nu(t) = P_i^\nu(t)\left[\{b_i(t) + \nu_i(t) + \delta(\nu(t))\}\, dt + \sum_{j=1}^{d} \sigma_{ij}(t) dW^{(j)}(t)\right], \ i = 1, \cdots, d.$$

We shall show that the price for hedging $(T, B, \{T_0\})$ with a constrained portfolio in the market $\mathcal{M}$ is given by the supremum of the unconstrained prices u_ν in these auxiliary markets $\mathcal{M}_\nu$, $\nu \in \mathcal{D}$.

6.2 K-Constrained Portfolio Option

This section is a brief summary of the paper [11] by Cvitanić and Karatzas. We shall again investigate the pricing problem. For details and examples we refer the reader to [11].

Definition 6.2.1 *A financial instrument* $(T, B, \{T_0\})$ *is called* K*-hedgeable if it satisfies*

$$v(0) \triangleq \sup_{\nu \in \mathcal{D}} \tilde{E}_\nu \left[\beta_\nu(T_0) B(T_0)\right] = \sup_{\nu \in \mathcal{D}} u_\nu < \infty.$$

This definition will be justified later. Actually, it can be shown that for any K-hedgeable financial instrument $(T, B, \{T_0\})$, there exists a trading strategy $(\pi, C) \in \mathcal{A}(v(0); K)$ such that $X^{v(0),\pi,C}(T_0) = B(T_0)$, and that $v(0)$ is the minimal initial wealth for which this can be achieved.

Definition 6.2.2 *An European type of* K*-constrained portfolios option* $(T, B, \{T_0\}, K)$ *is a* K*-hedgeable financial instrument* $(T, B, \{T_0)\})$ *with admissable trading strategies lying in a non-empty, closed, convex set* $K \subset R^d$*. A fair price of* $(T, B, \{T_0\}, K)$ *is defined by*

$$\hat{V} \triangleq \inf\left\{x > 0 \mid \exists\, (\pi, C) \in \mathcal{A}(x; K) \ s.t. \ X^{x,\pi,C}(T_0) \geq B(T_0) \ a.s.\right\}.$$

For any stopping times $\gamma, \sigma \in \mathcal{T}$ such that $\gamma \leq \sigma$, denote by $\mathcal{D}_\gamma^\sigma$ the restriction of $\mathcal{D}$ to the stochastic interval $[[\gamma, \sigma]]$. For every $\tau \in \mathcal{T}$, consider also the $\mathcal{F}_\tau$-measurable random variable

$$v(\tau) \triangleq \operatorname*{ess\,sup}_{\nu \in \mathcal{D}} \tilde{E}_\nu \left[\beta(T_0) B(T_0) \exp\left\{-\int_\tau^{T_0} \delta(\nu(s))ds\right\} \mid \mathcal{F}_\tau\right].$$

LEMMA 6.2.3 *For any K-hedgeable financial instrument $(T, B, \{T_0\})$, the family of random variables $\{v(\tau)\}_{\tau \in \mathcal{T}}$ satisfies the equation of Dynamic Programming*

$$v(\tau) = \operatorname*{ess\,sup}_{\nu \in \mathcal{D}_\tau^\theta} \tilde{E}_\nu \left[v(\theta) \exp\left\{ -\int_\tau^\theta \delta(\nu(u))du \right\} \mid \mathcal{F}_\tau \right], \quad \forall\, \theta \in \mathcal{T}_\tau^{T_0}.$$

Proof:

See the appendix in Cvitanić and Karatzas [11]. □

THEOREM 6.2.4 *The process $v(\cdot)$ can be considered in its RCLL modification and, for every $\nu \in \mathcal{D}$,*

$$(6.7) \quad \left\{ \begin{array}{c} Q_\nu(t) \triangleq v(t) \exp\left\{ -\int_0^t \delta(\nu(u))du \right\}, \quad t \in T \\ \textit{is a RCLL } \tilde{P}_\nu\textit{-supermartingale with respect to } (\mathcal{F}_t, t \in T) \end{array} \right\}.$$

Furthermore, $v(\cdot)$ is the smallest adapted, RCLL process that satisfies (6.7) together with the equality

$$(6.8) \qquad v(T_0) = \beta_0(T_0)B(T_0).$$

Proof:

See the appendix in Cvitanić and Karatzas [11]. □

The following theorem can be regarded as the main result of this chapter; it justifies Definition 6.2.1.

THEOREM 6.2.5 *For a financial instrument $(T, B, \{T_0\}, K)$ such that K is a non-empty, closed, convex subset in R^d, we have $\hat{V} = v(0)$.*

Proof:

We first want to show $\hat{V} \leq v(0)$. From (6.7), the martingale representation theorem and the Doob-Meyer decomposition, we have for every $\nu \in \mathcal{D}$,

$$dQ_\nu(t) = \psi_\nu^*(t)d\tilde{W}_\nu(t) - dA_\nu(t),$$

where $\psi_\nu(\cdot)$ is a progressively measurable process with values in R^d and $\int_0^{T_0} ||\psi(s)||^2 ds < \infty$, and $A_\nu(\cdot)$ is a non-decreasing, continuous and predictable process such that $A_\nu(0) = 0$. Let us introduce a positive, adapted $RCLL$ process

$$\hat{X}(t) \triangleq \frac{v(t)}{\beta_0(t)} = \frac{Q_\nu(t)}{\beta_\nu(t)} \quad t \in T; \quad \forall \nu \in \mathcal{D}$$

with $\hat{X}(0) = v(0)$, $\hat{X}(T_0) = B(T_0)$. Now we need to find a trading strategy $(\hat{\pi}, \hat{C}) \in \mathcal{A}(v(0); K)$ such that $\hat{X}(\cdot) = X^{v(0),\hat{\pi},\hat{C}}(\cdot)$. This will prove $\hat{V} \leq v(0)$.

From (6.7), we have for any ν, $\mu \in \mathcal{D}$, we have from (6.7) that,

$$Q_\mu(t) = Q_\nu(t) \cdot \exp\left[\int_0^t \{\delta(\nu(s)) - \delta(\mu(s))\} ds\right].$$

Applying Itô's rule, we obtain

$$\begin{aligned} dQ_\mu(t) &= \exp\left[\textstyle\int_0^t \{\delta(\nu(s)) - \delta(\mu(s))\} ds\right] \\ &\quad \times \left[Q_\nu(t)\{\delta(\nu(t)) - \delta(\mu(t))\} dt - dA_\nu(t) + \psi_\nu^*(t)d\tilde{W}_\nu(t)\right] \\ &= \exp\left[\textstyle\int_0^t \{\delta(\nu(s)) - \delta(\mu(s))\} ds\right] \\ &\quad \times \left[\hat{X}(t)\beta_\nu(t)\{\delta(\nu(t)) - \delta(\mu(t))\} dt - dA_\nu(t)\right. \\ &\quad \left. + \psi_\nu^*(t)\sigma^{-1}(t)(\nu(t) - \mu(t))\, dt + \psi_\nu^*(t)d\tilde{W}_\mu(t).\right]. \end{aligned} \tag{6.9}$$

Comparing this with the Doob-Meyer decomposition

$$dQ_\mu(t) = \pi_\mu^*(t)d\tilde{W}_\mu(t) - dA_\mu(t), \tag{6.10}$$

we conclude from the uniqueness of the decomposition

$$\psi_\nu^*(t)\exp\left(\int_0^t \delta(\nu(s))ds\right) = \psi_\mu^*(t)\exp\left(\int_0^t \delta(\mu(s))ds\right)$$

and hence this expression is independent of $\nu \in \mathcal{D}$. We can define a portfolio process $\hat{\pi}$ via

$$\text{(6.11)} \quad \psi_\nu^*(t)\exp\left(\int_0^t \delta(\nu(s))ds\right) = \hat{X}(t)\beta_0(t)\hat{\pi}^*(t)\sigma(t), \ t \in T; \ \nu \in \mathcal{D}.$$

Similarly, by comparing (6.9) and (6.10), and substituting (6.11) into the expression, we conclude that

$$\begin{aligned} & \exp\left(\int_0^t \delta(\nu(s))ds\right) dA_\nu(t) - \beta_0(t)\hat{X}(t)\left[\delta(\nu(t)) + \hat{\pi}^*(t)\nu(t)\right] dt \\ = \ & \exp\left(\int_0^t \delta(\mu(s))ds\right) dA_\mu(t) - \beta_0(t)\hat{X}(t)\left[\delta(\mu(t)) + \hat{\pi}^*(t)\mu(t)\right] dt, \end{aligned}$$

and hence this expression is also independent of $\nu \in \mathcal{D}$. Now define

$$\text{(6.12)} \quad \hat{C}(t) \triangleq \int_0^t \beta_\nu^{-1}(s)dA_\nu(s) - \int_0^t \hat{X}(s)\left[\delta(\nu(s)) + \nu^*(s)\hat{\pi}(s)\right] ds,$$

for every $t \in T$, $\nu \in \mathcal{D}$. Take $\nu \equiv 0$ we obtain $\hat{C}(t) = \int_0^t \beta_0^{-1}(s)dA_0(s)$, and hence $\hat{C}(\cdot)$ is a consumption process.

Finally, we need to show $\hat{\pi}$ takes values in K. By the arguments of [10], Theorem 9.1, we only need to show that

$$\text{(6.13)} \qquad \delta(\nu(t,\omega)) + \nu^*(t,\omega)\hat{\pi}(t,\omega) \geq 0, \quad \ell \otimes P \ a.s.$$

holds for every $\nu \in \mathcal{D}$. (*These arguments need the continuity condition of* $\delta(\cdot|K)$ *and the assumption that the set* K *is closed.*) Notice from (6.11), we obtain

$$\begin{aligned} A_\nu(t) &= \int_0^t \beta_\nu(s)\left[\{\delta(\nu(s)) + \nu^*(s)\hat{\pi}(s)\}\hat{X}(s)ds + d\hat{C}(s)\right] \\ &\leq k\left[\int_0^t \{\delta(\nu(s)) + \nu^*(s)\hat{\pi}(s)\}\hat{X}(s)ds + \hat{C}(t)\right], \quad t \in T, \ \nu \in \mathcal{D}, \end{aligned}$$

for some $k > 0$. Fix $\nu \in \mathcal{D}$ and define the set

$$F_t \triangleq \{\omega \in \Omega \mid \delta(\nu(t,\omega)) + \nu^*(t,\omega)\hat{\pi}(t,\omega) < 0\}, \quad t \in T.$$

Let

$$\mu \triangleq \frac{\nu(t)1_{F_t^c} + n\nu(t)1_{F_t}}{1 + ||\nu(t)||}, \quad n \in N.$$

Then $\mu \in \mathcal{D}$ and by assuming (6.13) does not hold, we get

$$\begin{aligned} E\left[A_\mu(T_0)\right] &\leq E\left[k\int_0^{T_0} \frac{\hat{X}(t)1_{F_t^c}\{\delta(\nu(t)) + \nu^*(t)\hat{\pi}(t)\}}{1 + ||\nu(t)||}dt + k\hat{C}(t)\right] \\ &\quad + nE\left[k\int_0^{T_0} \frac{\hat{X}(t)1_{F_t}\{\delta(\nu(t)) + \nu^*(t)\hat{\pi}(t)\}}{1 + ||\nu(t)||}dt\right] \\ &< 0. \end{aligned}$$

This is a contradiction.

Now we have $\hat{X}(\cdot)$ is the wealth process corresponding to $(\hat{\pi}, \hat{C})$ since

$$\begin{aligned} d(\beta_0(t)\hat{X}(t)) &= dQ_0(t) = \psi_0(t)d\tilde{W}_0(t) - dA_0(t) \\ &= \beta_0(t)\hat{X}(t)\hat{\pi}^*(t)\sigma(t)d\tilde{W}_0(t) - \beta_0^{-1}(t)d\hat{C}(t), \ \hat{X}(0) = v(0). \end{aligned}$$

Therefore, $(\hat{\pi}, \hat{C}) \in \mathcal{A}(v(0); K)$ and $\hat{V} \leq v(0)$ follows.

To complete the proof, we need to show $v(0) \leq \hat{V}$. Since $\hat{V} < \infty$, there exists $x \in R^+$ such that $X^{x,\pi,C}(T_0) \geq B(T_0)$ for some $(\pi, C) \in \mathcal{A}(x; K)$. Using Itô's rule on $\beta_\nu(t)X^{x,\pi,C}(t)$ we have that

$$\begin{aligned} M_\nu(t) &\triangleq \beta_\nu(t)X^{x,\pi,C}(t) + \int_0^t \beta_\nu(s)dC(s) \\ &\quad + \int_0^t \beta_\nu(s)X^{x,\pi,C}(s)\left[\delta(\nu(s)) + \nu^*(s)\pi(s)\right]ds \\ &= x + \int_0^t \beta_\nu(s)X^{x,\pi,C}(s)\pi^*(s)\sigma(s)d\tilde{W}_\nu(s). \end{aligned}$$

Consequently, $M_\nu(\cdot)$ is a non-negative $\tilde{P}_\nu$ local martingale and hence a supermartingale. Therefore,

$$x \geq \tilde{E}_\nu \left[M_\nu(T_0) \right] \geq \tilde{E}_\nu \left[\beta_\nu(T_0) B(T_0) \right],$$

for any given $\nu \in \mathcal{D}$. Therefore, $x \geq v(0)$ and thus $\hat{V} \geq v(0)$. □

Bibliography

[1] Bachelier, L., *Théorie de la spéculation*, Ann. Ecole Norm. Sup. 17, 21-86 (1900).

[2] Bensoussan, A., *On the theory of option pricing*, Acta Appl. Math. 2, 139-158 (1984).

[3] Bensoussan, A. and Lions, J.L., *Applications of Variational Inequalities in Stochastic Inequalities in Stochastic Control*, North Holland, Amsterdam (1982).

[4] Bismut, J.M. and Skalli, B., *Temps d'arret optimal, théorie générale des processus et processus de Markov*, Z. Wahr. Verw. Geb. 39, 301-303 (1977).

[5] Black, F. and Scholes, M., *The pricing of options and corporate liabilities*, J. Political Economy 81, 637-659 (1973).

[6] Colwell, D., Elliott, R. and Kopp, E., *Martingale representation and hedging polices*, Stochastic Processes and their Applications 38, 335-345 (1991).

[7] Chow, Y.S., Robbins, H.E. and Siegmund, D., *Great Expectations, The Theory of Optimal Stopping*, Houghton Mifflin Company, Boston.

[8] Chow, Y.S. and Robbins, H.E., *On optimal stopping rules*, Z. Wahr. Verw. Geb. 2, 33-49 (1963).

[9] Chung, K.L., *Lectures from Markov Process to Brownian Motion*, Grundlehren der mathematischen Wissenschaften 249, Spinger-Verlag.

[10] Cvitanić, J. and Karatzas, I., *Convex duality in constrained portfolio optimization*, Ann. Appl. Probab. 2, 767-818 (1992).

[11] Cvitanić, J. and Karatzas, I., *Hedging Contingent Claims*, Ann. Appl. Probab. 3, 625-681 (1993).

[12] Davis, M., and Karatzas, I., *A deterministic approach to optimal stopping with applications to a prophet inequality*. To appear in P. Whittle Festschrift. (1993)

[13] Dellacherie, C. and Meyer, P., *Probabilities et Potentiels*, Hermanu (1980).

[14] Dynkin, E.B., *Optimal choice of the stopping moment of a Markov process*, Dokl. Akad. Nauk SSSR 150, 238-240 (1963).

[15] El Karoui, N. and Karatzas, I., *General Gittens index processes in discrete time*, Proc. Natl. Acad. Sci. USA. 90, 1232-1236 (1992).

[16] El Karoui, N. and Karatzas, I., *The optimal stopping problem for a general American put option*, Proc. IMA Workshop, June 1993.

[17] Elliott, R.J., *Stochastic Calculus and Applications*, Springer-Verlag (1982).

[18] Fakeev, A.G., *Optimal stopping rules for stochastic process with continous parameter*, Theory of Prob. and Appl. 15, 324-331 (1970).

[19] Harrison, J.M. and Pliska, S., *Martingales and stochastic integrals in the theory of continuous trading*, Stochastic Process Appl. 11, 215-260 (1981).

[20] Hull, John C., *Options, Futures and Other Derivative Securities*, Prentice Hall (1993).

[21] Karatzas, I., *Optimization problems in the theory of continuous trading*, SIAM J. Control Optim. 27, 1221-1259 (1989).

[22] Karatzas, I. and Shreve, S., *Brownian Motion and Stochastic Calculus*, Springer, New York (1988).

[23] Krylov, N.V., *Controlled Diffusion Processes*, Springer Verlag

[24] Merton, R.C., *Theory of rational option pricing*, Bell J. Econ. Managmt. Sci. 4, 141-183. (1973).

[25] McKean, H.P., *Appendix: A free boundary problem for the heat equation arising from a problem in mathematical economics*, Indust. Manage. Rev. 6, 32-39 (1965).

[26] Neveu, J., *Discrete-Parameter Martingales*, North-Holland (1975).

[27] Oksendal, B., *Stochastic Differential Equations*, Springer-Verlag (1991).

[28] Protter, P., *Stochastic Integration and Differential Equations*, Springer-Verlag (1990).

[29] Rockafellar, R.T. *Convex Analysis* Princeton University Press (1970).

[30] Siegmund, D., *Some problems in the theory of optimal stopping rules*, Ann. Math. Stat. 38, 1627-1640 (1967).

[31] Snell, J.L., *Application of martingale system theorems*, Trans. Amer. Math. Soc. 73, 293-312 (1952).

[32] Stroock D.W. and Varadhan, S.R.S., *Multidimensional Diffusion Processes*, Springer-Verlag (1979).