Regularity Criteria for the Energy Equality to the 3D Magneto-Hydrodynamics Equations

by

Mark Pineau

A thesis submitted in partial fulfillment of the requirements for the degree of

Masters of Science

in

Mathematics

Department of Mathematical and Statistical Sciences
University of Alberta

Abstract

We consider the Cauchy problem to the Magneto-Hydrodyanmics Equations (MHD) in $\mathbb{R}^3$, and present specific criteria for which its corresponding energy equality holds. Specifically, we show that very weak solutions to the MHD equations (in the distributional sense) satisfy the energy equality, provided they belong to the space $L^r(0, T; L^s(\mathbb{R}^3))$ with $\frac{2}{r} + \frac{2}{s} = 1$ for $s \geq 4$. Further, we also consider regularity criteria on the gradient of the solution to the MHD Cauchy problem. That is, we show very weak solutions to the MHD equation satisfy the energy equality if $\nabla u, \nabla B \in L^{\frac{8s}{9s-12}}(0, T; L^s(\mathbb{R}^3))$, for $\frac{12}{7} < s \leq \frac{12}{5}$.

Acknowledgements

First and foremost I want to highlight my appreciation for my MSc advisor, Dr. Xinwei Yu at the University of Alberta. He has not only played a crucial part in both my study of general mathematics and theory in PDE, but also has been a great mentor in my introduction to mathematical research. I hope to retain him as a teacher and apply the variety of lessons taught for future research endeavors.

I would also like to thank my brother and fellow mathematician, Ben Pineau, who together with useful discussion, has played a helpful role in mentoring and guiding me through my study of PDE in preparation for my thesis.

Finally, I sincerely give my thanks to the University of Alberta for providing me the opportunity to obtain an MSc degree in mathematics in their department. I hope such a degree, with the tools developed along the way, will prove useful in future career endeavors.

Table of Contents

Chapter 1

Introduction

Fluid dynamics involves the branch of physics relevant to the motion of fluids, specifically to the relevant forces, both internal and external, that may act on a fluid, inducing such a flow. Initially, the forefront of study in this area began in 1757 by the Swiss mathematician Leonhard Euler who derived the so-called incompressible Euler equations, a set of mixed type PDE that describe the flow of an incompressible (density of a fluid particle remains invariant along its flow), inviscid Newtonian fluid under certain initial data. Such equations were then later generalized by French mechanical engineer Claude-Louis Navier and Irish physicist and mathematician George Gabriel Stokes from 1822 (Navier) to 1842-1850 (Stokes), and so named the incompressible Navier-Stokes equations, a set of mixed type PDE that additionally accounted for internal frictional effects (viscosity) on the fluid. In practice, such equations are readily applied in real-world applications (mainly numerically) to optimize velocity flow fields along a variety of geometries (streamline or make more aerodynamic), including car bodies, aircraft hulls, heat exchangers, etc, to attain a desired performance or output/efficiency in engineering. Other areas of application include weather forecasting by predicting future flow/wind patterns, jet/aircraft propulsion applications, as well as fluid flow in pipes for oil transport, among many others. A further generalization of the Navier-Stokes equations, and of main interest in the following thesis, includes certain PDEs that model flows of electrically conducting fluids, or "magneto-fluids", which react differently while in the presence of an electromagnetic field. Such an area (Magnetohydrodynamics) was first developed by Swedish electrical engineer and plasma physicist Hannes

Olof Gosta Alfven, with the derivation of the Magneto-Hydrodynamics Equations (MHD). Interestingly, applications of such fluids (for example plasma and liquid metals), are used in various biomedical areas, including magnetohydrodynamic-based 1 laser beam scanning, and targeted drug delivery, among others [26].

Beyond the physical applications of such equations, basic questions from a purely mathematical viewpoint are often considered next, in an attempt to determine the validity, or well-posedness of such derived PDEs. That is, in particular, it remains famously unsolved whether the incompressible Navier-Stokes equations admit (given smooth initial data) a unique globally defined (in $\mathbb{R}^3$) and smooth solution, as well as other similar equations including the MHD PDE. Interestingly, well-posedness (or global regularity) of solutions to the incompressible Navier-Stokes equations has been shown in the 2-dimensional case. In this direction, it has been shown that by imposing additional integrability conditions (regularity or "Prodi-Serrin" type criteria) on initial data, one can achieve global regularity or well-posedness to the 3D incompressible Navier-Stokes equations. Other pertinent questions regarding PDEs often include transient or long-term time behavior of solutions, numerical algorithms/simulations, as well as regularity criteria, or differentiability conditions of solutions as mentioned above, the latter being considered throughout the remainder of this thesis. Finally, solving these questions for fundamental PDEs of this type is often pertinent, as well as necessary, for understanding properties of more complicated PDE, and thus is a foundational starting point for theoretical work in this area.

The remainder of the chapters of this thesis are presented sequentially as follows: Chapter 2 first presents an overview of the field and review of the literature regarding the Navier-Stokes and MHD equations, as well as underlining the main results and motivation behind the theorems that will be proven in the subsequent chapters. Chapter 3 presents the mathematical theory required for the proofs of each theorem outlined in Chapter 2. Specifically, a more detailed overview of the fluid dynamics equations mentioned above will be presented, outlining required definitions of specific pertinent function spaces, the motivation behind the weak formulation of solutions, such as Leray-Hopf weak solutions, as well as a brief derivation of the

MHD equations. Chapter 4 overviews an essential existence method (Galerkin) of approximate (or regularized) solutions to the MHD equations, which will be necessary for the proofs of the results of this thesis. Chapter 5 presents specific $L^r L^s$ estimates for the non-stationary Stokes system, another necessity for the proofs later in this thesis. Finally, Chapter 6 presents the new main results of the thesis, specifically proving that the energy equality for the MHD equations holds for weak solutions (in the distributional sense) under a variety of regularity criteria (see the abstract or Chapter 2 for details of each theorem).

Chapter 2

Overview and New Results

As an introduction to our discussion, and segue into more complicated topics presented later throughout this thesis, a slightly more in-depth analysis of each fluid equation mentioned in the introduction, as well as historical results will be outlined below, which will serve as an underlying motivation for the following new results. In addition, three new theorems (Theorem 1.1-1.3) are listed near the end of this section, which will serve as the foundational results that the remainder of this thesis will attempt to prove.

2.1 Fluid Equations

2.1.1 Incompressible Euler Equations

As a natural beginning to our discussion, we present chronologically by year derived, the order of fluid PDE mentioned in the introduction, starting with the incompressible Euler equations. Specifically, the remainder of this thesis will assume the fluid in question is Newtonian (viscous stresses on fluid and strain rate are linearly dependent) and incompressible (invariant density along particle flow).

Assuming first an inviscid Newtonian fluid at constant density, the incompressible

Euler equations in the whole space $\mathbb{R}^3$, read

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0, \end{cases} \qquad (t,x) \in \ (0,\infty) \times \mathbb{R}^3 \tag{2.1}$$

with initial data $u(x,0) = u_0(x) \in \mathbb{R}^3$. Here u denotes the velocity flow field, and p the pressure field.

One may note that the first equation of (2.1) is equivalent to Newton's second law of motion, that is, the pressure acting on a fluid particle is linearly dependent on its acceleration, whereas the second divergence-free equation describes the incompressibility assumption. The derivation of (2.1) is elementary and uses a generalization of Leibniz's rule for integrals (Reynold's transport theorem), allowing the interchanging of time derivatives and limits for the material derivative. Since the derivation is rather lengthy, we omit it in our discussion, however see [25] or Tao's blog [47] for a full discussion and proof.

2.1.2 Incompressible Navier-Stokes Equations

A natural generalization of the incompressible Euler equations includes the addition of a viscosity term to account for internal force effects on the fluid. Specifically, the flow of a Newtonian, incompressible fluid may be described by the incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u, \\ \nabla \cdot u = 0, \end{cases} \qquad (t,x) \in \ (0,\infty) \times \mathbb{R}^3 \tag{2.2}$$

with initial data $u(x,0) = u_0(x) \in \mathbb{R}^3$. Here again, u and p denote the velocity flow and pressure fields respectively, whereas ν denotes the viscosity of the fluid. Further, throughout the rest of the thesis, for ease of readability, and without loss of generality, we assume the viscosity $\nu = 1$. The derivation of the incompressible Navier-Stokes equations is done similarly to the incompressible Euler equations; however, more computation is required when dealing with the extra viscosity term. See again Tao's blog for example [47].

Of greatest importance remains the question of the existence and uniqueness of smooth solutions in the whole space, as well as the continuous dependence on smooth initial data (global well-posedness) to the Navier-Stokes equations. Specifically, such a problem was first shown in the 2-dimensional case by Ladyshenkaya (see [40] section 5 for a proof), with the 3-dimensional case remaining unsolved.

A priori/Energy Estimates

One attempt at solving the global well-posedness problem in $\mathbb{R}^3$ starts with a suitable weak formulation of a solution to (2.2) (see section 3.3), where certain "compactness arguments" have been developed that prove the global existence of such solutions (in the distributional sense) to (2.2). Such a method comprises: (1) showing the existence of a sequence of approximate weak solutions to a regularized version of the Navier-Stokes equations, (2) showing the solutions don't blow up in finite time (finding uniform bounds/energy estimates), and (3) proving the limit of the subsequence itself is a weak solution. Importantly however, by construction, such weak solutions lack suitable differentiability criteria, and it remains unknown whether one can construct a smooth weak solution with smooth initial data.

Following step (2) listed above, a priori estimates may be derived which may hint at possible suitable function spaces where weak solutions may lie. This is often the starting point for all analysis regarding existence and smoothness theory for the Navier-Stokes equations, and is the motivation for the construction of the Leray-Hopf weak solution. ([24] and [31]).

Following the discussion above, let $\Omega \subset \mathbb{R}^3$ be open and bounded. We expect regular solutions with $L^2(\Omega)$ initial data of (2.2) to satisfy the equality (2.3) below.

$$\int_\Omega |u(x,t)|^2 \, dx + 2\int_0^t \int_\Omega |\nabla u(\tau,x)|^2 \, dx d\tau = \int_\Omega |u_0(x)|^2 \, dx, \qquad (2.3)$$

where $t \in [0,T]$, $T > 0$. We call this the energy equality (or inequality if we have less than or equal to). Importantly, finding weak solutions that satisfy the energy inequality is a key step in showing in-time global regularity of solutions (this will be rigorously explained in later chapters) We motivate (2.3) below.

Assuming the existence of a solution u to (2.2) of sufficient regularity (differentiable enough so that all operations below are well-defined), dotting the momentum equation of (2.2) by u and rearranging gives

$$\int_{\Omega} (\partial_t u \cdot u - \Delta u \cdot u + (u \cdot \nabla) u \cdot u + \nabla p \cdot u) \, dx = 0, \tag{2.4}$$

Integrating by parts, and using the divergence free condition $\nabla \cdot u = 0$, one notes for each term that

$$\int_{\Omega} (u \cdot \nabla) u \cdot u \, dx = 0, \quad \text{and} \quad \int_{\Omega} \nabla p \cdot u \, dx = 0, \tag{2.5}$$

and

$$-\int_{\Omega} \Delta u \cdot u \, dx = \int_{\Omega} |\nabla u|^2 \, dx, \tag{2.6}$$

whereby combining (2.5) and (2.6) with (2.4) and integrating from 0 to $t \in (0, T)$ with respect to time yields the estimate (2.3). For example, letting $\hat{n}$ denote the outward normal unit vector, the second term of (2.5) is estimated via integration by parts:

$$\int_{\Omega} \nabla p \cdot u \, dx = \int_{\partial\Omega} p \cdot u \cdot \hat{n} \, d\partial\Omega - \int_{\Omega} p \, \text{div}(u) \, dx,$$

Here one often considers solutions u that vanish on the boundary, and thus using as well the divergence free condition, one immediately gets the result. A similar computation shows the other two equality's in (2.5) and (2.6).

Here, one can construct weak solutions with minimal regularity criteria that automatically satisfy the energy inequality and thus attain global in-time existence, however given enough differentiability on the solution, equality can be satisfied instead. This is the main motivation for the following thesis, specifically, what additional regularity criteria is required for a weak solution to satisfy the energy equality.

Historical Results (Navier-Stokes)

One of the first results motivated by the above comes from Leray [31] and Hopf [24] who have shown the existence of global weak solutions to (2.2) for an initial condition in the L^2 sense $u_0 \in L^2_\sigma(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3) : \nabla \cdot u = 0\}$. Moreover, for $T > 0$, such weak solutions u lie in the Leray-Hopf class (2.7)

$$u \in L^2(0,T;H^1(\mathbb{R}^3)) \cap L^\infty(0,T;L^2_\sigma(\mathbb{R}^3)) \tag{2.7}$$

and satisfy the energy inequality (see section (3.3)).

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|\nabla u(\tau)\|^2_{L^2}\, d\tau \leq \|u_0\|^2_{L^2}, \tag{2.8}$$

Despite satisfying the energy inequality, it is still unknown whether Leray-Hopf weak solutions satisfy the corresponding energy equality. Also, despite establishing global in-time existence of Leray-Hopf weak solutions, the question of uniqueness also remains famously unsolved. In this direction, however, uniqueness results have been proven for Leray-Hopf weak solutions that are assumed smooth [32]. Attempts at showing such uniqueness results have proven slow, with only partial progress being made only recently, see e.g. [30, 39, 45].

In this direction, additionally, global regularity results have been shown when extra differentiability criteria are imposed on either the solution u or initial data u_0. In this direction, the first major results were shown by Prodi [37] and Serrin [41] who established global regularity when certain "Prodi-Serrin type criteria" were met. That is, one attains regularity of a solution u past some $T > 0$ if

$$\|u\|^p_{L^p(0,T;L^q(\mathbb{R}^3))} := \int_0^T \|u\|^p_{L^q}\, dt < \infty, \tag{2.9}$$

where p, q are well-defined such that $\frac{2}{p} + \frac{3}{q} < 1$ for $q > 3$

Such results were later made stronger by weakening the differentiability assumption to include values at $\frac{2}{p} + \frac{3}{q} \leq 1$ with $q > 3$, see e.g. [21, 43], as well as for $q = 3$

when the regularity criterion (2.10) holds [15].

$$\text{esssup}_{(0,T)} \|u\|_{L^3} < \infty, \tag{2.10}$$

Extensions and refinements of such Prodi-Serrin type criteria have been studied extensively, with a variety of main results listed, see e.g. [2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 17, 23, 27, 34, 36, 38, 44, 48, 49, 50].

In the direction of satisfying the energy equality, the first main regularity result comes from J.-L. Lions [33], who proved that Leray-Hopf weak solutions that additionally belong to the space $L^4(0,T;L^4(\mathbb{R}^3))$ satisfy the energy equality on $[0,T]$ for $T > 0$. In fact, J.-L. Lions result was later improved by Galdi [19], showing the Leray-Hopf requirement (2.7) was unnecessary and that the regularity criteria $u \in L^4(0,T;L^4(\mathbb{R}^3))$ was sufficient.

Extensions to J.-L. Lions result were later introduced by Shinbrot [42] which generalized the regularity of Leray-Hopf solutions u to a broader array of $L^r L^s$ spaces for $r, s \geq 1$. Specifically, it is proven that if Leray-Hopf weak solutions to (2.2) satisfy

$$u \in L^r(0,T;L^s(\mathbb{R}^3)) \text{ with } \frac{2}{r} + \frac{2}{s} = 1 \text{ for } s \geq 4, \tag{2.11}$$

then the corresponding energy equality (2.3) holds. Similarly to the result by Galdi, the Leray-Hopf condition was again shown redundant and dropped by Berselli and Chiodaroli [5], showing weak solutions to (2.2) satisfy the energy equality if only (2.11) is satisfied.

2.1.3 Magneto-Hydrodynamics Equations

Following the order listed in the introduction, it is also of great interest to study the velocity field of a conducting fluid when under the influence of a magnetic field. Such motion is mathematically described by simultaneously solving both the Navier-Stokes equations for fluid motion and Maxwell's Electromagnetic equations, given certain initial data. (See section 3.2 for a more in-depth discussion of such equations and a brief derivation).

Here the following thesis will be considering the three-dimensional Cauchy problem for the Magneto-Hydrodynamics (MHD) Equations which reads:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu_1\Delta u + \nabla P = (B\cdot\nabla)B, \\ \partial_t B + (u\cdot\nabla)B - \nu_2\Delta B = (B\cdot\nabla)u, \qquad (t,x)\in\ [0,T]\times\mathbb{R}^3 \\ \nabla\cdot u = \nabla\cdot B = 0, \end{cases} \tag{2.12}$$

with initial data

$$u(x,0) = u_0 \in \mathbb{R}^3, \ \ B(x,0) = B_0 \in \mathbb{R}^3. \tag{2.13}$$

for any $T > 0$, where the total pressure P is given by

$$P = p + \frac{1}{2}|B|^2, \tag{2.14}$$

where u and B are the fluid velocity and magnetic field, respectively, p the pressure field, and ν_1, ν_2 the coefficient of viscosity and coefficient of magnetic resistivity, respectively. Here for simplicity and without loss of generality, we assume again that $\nu_1 = \nu_2 = 1$.

Specifically, such equations are derived via coupling the Navier-Stokes equations for fluid flow (2.2), with Maxwell's electromagnetic equations. (see section 3.2 for details, or e.g. Duvaut-Lions [14] and [28]).

Following the discussion regarding the incompressible Navier-Stokes equations, regularity criteria results for weak solutions satisfying the energy equality are listed below. Specifically, a recent paper by Lai and Yang [29] generalized Galdi's [19] result to the 3D MHD Cauchy problem (2.12). Specifically they show that weak solutions (in the distributional sense) to (2.12) that also lie in $L^4(0,T;L^4(\mathbb{R}^3))$ satisfy the energy equality for the MHD equations (2.15)

$$\frac{1}{2}\int_\Omega(\|u(t)\|_2^2+\|B(t)\|_2^2)dx+\int_0^t\int_\Omega(\|\nabla u(\tau)\|_2^2+\|\nabla B(\tau)\|_2^2)d\tau = \frac{1}{2}\int_\Omega(\|u_0\|_2^2+\|B_0\|_2^2)dx, \tag{2.15}$$

for all $0 \le t \le T$ and $\Omega = \mathbb{R}^3$.

Other relevant results regarding the MHD equations are minimal, however see e.g. [51].

2.2 List of New Results

In light of the above discussion, we first seek to generalize the result of Lai and Yang [29] to a format similar to that presented by Berselli and Chiodaroli [5]. That is, one of the main results of the thesis is stated as follows

Theorem 1.1. *Suppose $u, B \in L^2_{\text{loc}}(\mathbb{R}^3 \times (0,T))$ are weak solutions to the MHD Equations (2.12) (in the distributional sense) defined by (3.32), with initial data $u_0, B_0 \in L^2_\sigma(\mathbb{R}^3)$. Then if u, B satisfy the condition (2.16) below, they fall in the Leray-Hopf class (2.7) and satisfy the energy equality (2.15).*

$$u, B \in L^r(0,T; L^s(\mathbb{R}^3)) \text{ with } \frac{2}{r} + \frac{2}{s} = 1 \text{ for } s \ge 4, \tag{2.16}$$

Regularity Criteria on the Gradient of the Solution

Next, mainly as a mathematical curiosity, and natural analogue to Theorem 1 of Berselli and Chiodaroli [5], it is also of interest to study results when regularity criteria are imposed on the gradient of the solution. Specifically, Berselli and Chiodaroli showed that Leray-Hopf weak solutions to the Navier-Stokes Cauchy problem (2.2) imposed with any of the regularity criteria (B1)-(B3) below, satisfy the energy equality (2.3).

(B1) $\nabla u \in L^{\frac{s}{2s-3}}(0,T; L^s(\Omega))$ for $\frac{3}{2} < s < \frac{9}{5}$,

(B2) $\nabla u \in L^{\frac{5s}{5s-6}}(0,T; L^s(\Omega))$ for $\frac{9}{5} \le s \le 3$,

(B3) $\nabla u \in L^{1+\frac{2}{s}}(0,T; L^s(\Omega))$ for $s > 3$, As of current knowledge, this is one of very few, if any result, that deals with regularity criteria on the gradient of the solution so that the energy equality is satisfied.

In this direction, we prove a similar result for Leray-Hopf weak solutions to the MHD Cauchy problem (2.12), as stated by Theorem 1.2.

Theorem 1.2. *Suppose* $u, B \in L^2_{\text{loc}}(\mathbb{R}^3 \times (0,T))$ *are Leray-Hopf weak solutions to the MHD Equations (2.12) defined by (3.32), with initial data* $u_0, B_0 \in L^2_\sigma(\mathbb{R}^3)$*. If in addition* u, B *both satisfy any of the conditions (B1*)-(B3*) above, then the pair satisfy the energy equality (2.15).*

(B1*) $\nabla u, \nabla B \in L^{\frac{s}{2s-3}}(0,T; L^s(\Omega)) \quad \text{for} \quad \frac{3}{2} < s < \frac{9}{5},$

(B2*) $\nabla u, \nabla B \in L^{\frac{5s}{5s-6}}(0,T; L^s(\Omega)) \quad \text{for} \quad \frac{9}{5} \leq s \leq 3,$

(B3*) $\nabla u, \nabla B \in L^{1+\frac{2}{s}}(0,T; L^s(\Omega)) \quad \text{for} \quad s > 3,$

Finally, through a similar method to the proof of Theorem 1.1, we attempt to generalize Theorem 1.2 for very weak solutions, dropping the Leray-Hopf condition. That is, imposing certain regularity criteria on the gradient of the solution, the energy equality (2.15) will hold, making the Leray-Hopf condition redundant. As such we have Theorem 1.3.

Theorem 1.3. *Suppose* $u, B \in L^2_{\text{loc}}(\mathbb{R}^3 \times (0,T))$ *are weak solutions to the MHD Equations (2.12) (in the distributional sense) defined by (3.32), with initial data* $u_0, B_0 \in L^2_\sigma(\mathbb{R}^3)$*. Then if* $\nabla u, \nabla B \in L^{\frac{8s}{9s-12}}(0,T; L^s(\Omega))$*, for* $\frac{12}{7} < s \leq \frac{12}{5}$*, then* u, B *satisfy the energy equality (2.15).*

Chapter 3

Preliminary Setup

In the following chapter, we will introduce pertinent definitions and terminology, as well as useful lemmas that will be crucial later when proving Theorem 1.1-1.3. Specifically, we begin this section by outlining important function spaces that will be used when considering weak solutions to the Navier-Stokes and MHD Cauchy problems. Such function spaces are necessary in our study, which will be deemed very useful when applying certain results from functional analysis. Succeeding this, we provide a more in-depth analysis of the MHD Equations with a brief derivation starting from the incompressible Navier-Stokes and Maxwell's Electromagnetic equations, with at the end, a reformulation of the MHD Cauchy problem in a more condensed form for later ease of use. Nearing the end of this chapter, weak formulations of solutions as well as pertinent L^p estimates are presented (with proof) which will be needed later in Chapter 6.

3.1 Function Spaces

We start by introducing a few important function spaces that will be used throughout this thesis, which will be pertinent when discussing Leray-Hopf weak and weak solutions (in the distributional sense) to (2.12). For a full rigorous discussion, see Evan's textbook on PDE's [16].

Firstly we define an important well known function space that deems useful when approximating solutions (density argument) to PDE's.

Definition 2.1.1. Given an open set Ω, we denote by $C_c^\infty(\Omega)$ the function space of all compactly supported smooth functions on Ω.

Next when discussing sufficient integrability conditions or regularity criteria on weak solutions, so called L^p spaces must be considered.

Definition 2.1.2. Let (Ω, Σ, μ) be a measure space. Denote by $L^p(\Omega)$, $p \geq 1$ the space of all measurable functions $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) whose norm defined by (3.1) or (3.2) is finite.

For $p \in [1, \infty)$:

$$\|f\|_{L^p} := \left(\int_\Omega |f|^p d\mu \right)^{\frac{1}{p}} < \infty, \tag{3.1}$$

For $p = \infty$:

$$\|f\|_\infty := \text{esssup}|f| < \infty, \tag{3.2}$$

Since we will be working with solutions defined on $\Omega \times [0, T)$ for $\Omega \subset \mathbb{R}^3$ and $T > 0$, a natural extension of Definition 2.1.2. is given.

Definition 2.1.3. Let $r, s \geq 1$. Denote by $L^r(0, T; L^s(\Omega))$ the function space of all measurable functions $f : [0, T] \times \Omega \to \mathbb{R}$ (or $\mathbb{C}$) with finite norm

$$\|f\|_{L^r(L^s)} := \left(\int_0^T \|f\|_{L^s(\Omega)}^r dt \right)^{\frac{1}{r}} < \infty, \tag{3.3}$$

Next, in order to deal with regularity on solutions to (2.12), we define the so-called Sobolev function spaces. Theory from functional analysis regarding these spaces will be of great importance when proving Theorem's 1.1-1.3 later.

Specifically, the weak formulation of solutions is necessary when studying the Navier-Stokes or MHD PDE's, hence we define an important weakening of the standard derivative in $\mathbb{R}^n$, which will also prove useful when defining Sobolev function spaces.

Definition 2.1.4. Let $\Omega \subset \mathbb{R}^n$ be an open set. Further let $u, v \in L^1_{loc}(\Omega)$ and α a multi-index. Then v is the α^{th}-weak derivative of u if

$$\int_\Omega u D^\alpha \psi = (-1)^{|\alpha|} \int_\Omega v\psi, \tag{3.4}$$

for all test functions $\psi \in C_c^\infty(\Omega)$.

Here the mixed partial derivative of ψ, $D^\alpha\psi$ is defined by

$$D^\alpha \psi := \frac{\partial^{|\alpha|}\psi}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}, \tag{3.5}$$

Definition 2.1.5. Let $|\alpha| \leq k$ be a multi-index ($\alpha \in \mathbb{N}_0^n$) of order $k \in \mathbb{N}$. We denote the Sobolev space $W^{k,p}(\Omega)$ the space of all measurable functions f on Ω where its mixed partial derivative exists weakly and lies in $L^p(\Omega)$. That is,

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\} \tag{3.6}$$

Lemma 2.1.6. $W^{k,p}(\Omega)$ is a Banach space with norm

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^p(\Omega)} & p \in [1,\infty), \\ \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^\infty(\Omega)} & p = \infty, \end{cases} \tag{3.7}$$

Specifically, when $k = 1$ and $p = 2$, we denote $W^{k,p}(\Omega) := H^1(\Omega)$. Here, as notation would imply, $H^1(\Omega)$ is a Hilbert space.

Also required are various function spaces that are constructed by taking the completions of a restricted class of divergence free functions under a suitable norm. We list a few below for use later.

Definition 2.1.7. We define the list of spaces below as follows

$C^\infty_{c,\sigma}(\Omega) := \{\varphi \in C^\infty_c(\Omega) : \nabla \cdot \varphi = 0\}$

$L^q_\sigma(\Omega) :=$ the completion of $C^\infty_{c,\sigma}(\Omega)$ under the L^q norm.

$H^1_{0,\sigma}(\Omega) :=$ the completion of $C^\infty_{c,\sigma}$ under the $W^{1,2}$norm

$\mathcal{D}_T := \{\varphi \in C^\infty_c(\mathbb{R}^3 \times [0, T)) : \nabla \cdot \varphi = 0\}$

3.2 Overview of the MHD Equations

3.2.1 Derivation of the MHD Equations

In this section we give a brief overview of the MHD partial differential equations, with an additional brief derivation. Specifically, such derivation is achieved via coupling the Navier-Stokes Equations for incompressible flow, with Maxwell's electromagnetic equations, which are again listed below for completeness.

Incompressible Navier-Stokes Equations:

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f, \\ \nabla \cdot u = 0, \end{cases} \qquad (t, x) \in (0, \infty) \times \mathbb{R}^3 \tag{3.8}$$

with initial data $u(x, 0) = u_0(x) \in \mathbb{R}^3$. Here u denotes the velocity flow field, p the pressure field, and f a body force.

Next we list Maxwell's electromagnetic equations, which determine electromagnetic effects on conducting body or fluid.

Maxwell's Electromagnetic Equations:

$$\begin{cases} \partial_t B + \nabla \times E = 0, \\ -\partial_t D + \nabla \times H = J, \\ \nabla \cdot D = \rho_e, \\ \nabla \cdot B = 0, \end{cases} \qquad (t,x) \in (0,\infty) \times \mathbb{R}^3 \tag{3.9}$$

again with some initial data. Here B, E, D, H, J, ρ_e denote the magnetic induction, electric field, electric displacement, magnetic field, electric current density, and electric charge density respectively.

Derivation of the MHD Equations

Here we provide a brief derivation of the MHD Equations, where a full in-depth example can be read in for e.g ([14] and [28]). Specifically we follow the derivation outlined by [28].

It's well known that one can relate the electric current density J to the magnetic induction B and electric field E by

$$J = \rho_e u + \sigma(E + u \times B), \tag{3.10}$$

where $\sigma > 0$ is a constant and denotes the electric conductivity. Or under the assumption of quasi-neutrality ($\frac{\rho_e E}{J \times B} \ll 1$), one finds

$$J = \sigma(E + u \times B), \tag{3.11}$$

Here it is understood that u and B are vector valued functions in $\mathbb{R}^3$, and as such for simplicity in notation, symbols indicating this will not be included throughout the thesis.

Assuming a free space, or non-magnetizable or non-polarizable medium, the magnetic induction B and magnetic field H are equal, scaled by a factor. Namely, we have $B = \mu_0 H$, for some constant μ_0.

Finally assuming no odd phenomena such as high frequencies in the medium, the time change in the displacement current $\partial_t D$ can be neglected in Eq (3.9) resulting in

$$J = \nabla \times H, \tag{3.12}$$

Combining the above assumptions (3.12) with $B = \mu_0 H$ into Eq (3.11) and rearranging yields

$$E = \frac{1}{\sigma\mu_0}\nabla \times B - u \times B, \tag{3.13}$$

whereby substituting (3.13) into (3.9) gives

$$\partial_t B + \frac{1}{\sigma\mu_0}\nabla \times \nabla \times B - \nabla \times (u \times B) = 0, \tag{3.14}$$

Directing our attention to the Navier-Stokes Equations (3.8), we separate the body force f as the sum of the external body force f_{ext} and Lorentz's force f_{em} expressed as

$$f_{\text{em}} = \rho_e E + J \times B \approx J \times B, \tag{3.15}$$

where the RHS is approximated due to the quasi-neutrality assumption, where the term $\rho_e E$ can be neglected. Substituting (3.15) into (3.8) and assuming a zero external body force $f_{\text{ext}} = 0$, one obtains

$$\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = \frac{1}{\mu_0}(\nabla \times B) \times B, \tag{3.16}$$

Finally, using the following general formulas from vector calculus listed below

$$\begin{aligned}
&\nabla \times \nabla \times u = \nabla(\nabla \cdot u) - \Delta u,\\
&(\nabla \times B) \times B = (B \cdot \nabla)B - \nabla\frac{|B|^2}{2},\\
&\nabla \times (v \times B) = v(\nabla \cdot B) - B(\nabla \cdot v) + (B \cdot \nabla)v - (v \cdot \nabla) \times B,
\end{aligned} \tag{3.17}$$

with the equations (3.14) and (3.16) and the divergence free conditions $\nabla \cdot u = \nabla \cdot B = 0$, yield the desired MHD Equations.

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu_1 \Delta u + \nabla P = (B \cdot \nabla)B, \\ \partial_t B + (u \cdot \nabla)B - \nu_2 \Delta B = (B \cdot \nabla)u, & (t,x) \in \; [0,T] \times \mathbb{R}^3 \\ \nabla \cdot u = \nabla \cdot B = 0, \end{cases} \tag{3.18}$$

with initial data

$$u(x,0) = u_0 \in \mathbb{R}^3, \;\; B(x,0) = B_0 \in \mathbb{R}^3. \tag{3.19}$$

3.2.2 Reformulation of the MHD Cauchy Problem

Next, for readability and ease of computation later when proving Theorem's 1.1-1.3, we wish to re-express the above MHD equations into a single condensed PDE that resembles the Navier-Stokes equations. Motivating this, we note that Theorem 1.1 has already been proven by Berselli and Chiodaroli [5] for the Navier-Stokes Cauchy problem (2.2), hence in an attempt to mimic their proof, we will rewrite the MHD equations (2.12) in a similar format to (2.2). Looking at each system, of main difference are the non-linear terms given by $u \cdot \nabla u$ for the Navier-Stokes Cauchy problem, and both $B \cdot \nabla B$ and $B \cdot \nabla u$ for the MHD Cauchy problem. This motivates the following definition.

Notation. Moving forward, we denote the brackets $(\cdot, \cdot)$ by the $L^2_x(\Omega)$ inner product. That is, we have the following definition.

Definition 2.2.1. Let f, g be two real functions on a measure space Ω with measure μ. Then denote by the L^2 inner product of f and g by

$$(f,g)_{L^2} = (f,g) := \int_\Omega fg \, d\mu, \tag{3.20}$$

To condense each system (2.2) and (2.12) into an identical form, a bilinear form $\mathcal{B}$ is defined below for each set of equations. As will be shown soon, such a map $\mathcal{B}$ will help combine both equations of the MHD system (2.12) into a singular coupled equation that looks almost identical to (2.2). We first start by rewriting the

Navier-Stokes PDE (2.2) in a more condensed form by introducing a bilinear form $\mathcal{B}$ through definition 2.2.2. Specifically, such a bilinear form $\mathcal{B}$ is used to combine all non-linear terms of each PDE into a single function.

Definition 2.2.2. Let $u \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}(\mathbb{R}^3))$. Then define when working with the Navier-Stokes Equations (2.2) the bilinear operator $\mathcal{B}_{\text{nav}} : H^1_{0,\sigma}(\Omega) \times H^1_{0,\sigma}(\Omega) \to (H^1_{0,\sigma}(\Omega))^*$ by

$$\mathcal{B}_{\text{nav}}(u,\varphi) := u \cdot \nabla\varphi, \tag{3.21}$$

where $\varphi \in \mathcal{D}_T$.

Assuming u is a solution (or even weak solution (see section 3.3 for a definition)) of sufficient regularity to the Navier-Stokes Cauchy problem, multiplying the Navier-Stokes equations (2.2) by a test function $\varphi \in \mathcal{D}_T$, integrating by parts and using the boundary and divergence free conditions, one easily obtains an equivalent formulation of (2.2). This is done similarly to deriving the energy equality in Chapter 1 and is thus skipped.

$$\begin{cases} \partial_t(u,\varphi) - (u,\partial_t\varphi) + (\nabla u,\nabla\varphi) + \langle\mathcal{B}(u,u),\varphi\rangle = 0, \\ \nabla\cdot u = 0, \end{cases} \qquad (t,x) \in\ (0,\infty)\times\mathbb{R}^3 \tag{3.22}$$

with initial data $u(x,0) = u_0(x) \in \mathbb{R}^3$

Here $\langle\cdot,\cdot\rangle$ denotes the standard dot product (the sum of the products of two vector valued functions coordinates).

Similarly for the MHD Cauchy problem (2.12), one can define a different bilinear form $\mathcal{B}_{\text{MHD}} := \mathcal{B}$ through definition 2.2.3, and with the help of the standard Stokes operator defined in definition 2.2.4, one can condense the MHD equations in a similar fashion done to (3.22).

Definition 2.2.3. Let $\Gamma \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}(\mathbb{R}^3))$ where $\Gamma :=$

(u, B). Then define when working with the MHD Equations (2.12) the bilinear operator $\mathcal{B} : H^1_{0,\sigma}(\Omega) \times H^1_{0,\sigma}(\Omega) \to (H^1_{0,\sigma}(\Omega))^*$ by

$$\mathcal{B}(\Gamma, \Phi) := (u \cdot \nabla\varphi_1 - B \cdot \nabla\varphi_2, u \cdot \nabla\varphi_2 - B \cdot \nabla\varphi_1), \tag{3.23}$$

where $\Phi := (\varphi_1, \varphi_2) \in \mathcal{D}_T$.

Definition 2.2.4. Define by A the standard Stokes operator in Ω by

$$A := -P_\sigma \Delta, \tag{3.24}$$

where P_σ is the Helmholtz-Leray projection given by $(1 < p < \infty)$

$$P_\sigma : L^p(\Omega) \to L^p_\sigma(\Omega), \quad (P_\sigma u)_i = u_i + \partial_i(-\Delta)^{-1}\nabla \cdot u, \tag{3.25}$$

Here, by letting $\Gamma := (u, B)$ be a sufficiently regular solution (or again weak solution) to the MHD equations (2.12) with $\Phi := (\varphi_1, \varphi_2) \in \mathcal{D}_T$, similarily to (3.22), an equivalent formulation for the MHD Cauchy problem is

$$\begin{cases} \partial_t(\Gamma, \Psi) - (\Gamma, \partial_t\Psi) + (\nabla\Gamma, \nabla\Psi) + \langle \mathcal{B}(\Gamma, \Gamma), \Psi\rangle = 0, \\ \nabla \cdot \Gamma_i = 0, \end{cases} \quad (t, x) \in (0, \infty) \times \mathbb{R}^3 \tag{3.26}$$

with initial data $\Gamma(x, 0) = \Gamma_0(x) := (u_0(x), B_0(x)) \in \mathbb{R}^3 \times \mathbb{R}^3$

If the test functions are time independent with $\Psi \in C^\infty_{0,\sigma}(\mathbb{R}^3)$, the first equation of (3.26) reduces to

$$\partial_t(\Gamma, \Psi) + (\nabla\Gamma, \nabla\Psi) + \langle B(\Gamma, \Gamma), \Psi\rangle = 0, \tag{3.27}$$

or equivalently using definition 2.2.3.

$$\partial_t\Gamma + A\Gamma + \mathcal{B}(\Gamma, \Gamma) = 0, \tag{3.28}$$

For details see Lai and Yang [29]

3.3 Leray-Hopf Weak and Weak Solutions

In this section we introduce the notion of weak solutions to both the Navier-Stokes equations (2.2) and MHD equations (2.12), giving a variety of different solution definitions with different 'strengths', some more general than others.

We start by defining the weakest (or most general) formulation of solution to the Navier-Stokes equations, where only satisfaction in the distributional sense is required. That is, construction of strong solutions with sufficient differentiability (or regularity) globally in time is unknown, however one can weaken the regularity requirement and develop theory with a weaker definition of a solution that one normally cannot with strong solutions. This leads to the following definition:

Definition 2.3.1. $u \in L^2_{\mathrm{loc},\sigma}(\mathbb{R}^3 \times (0,T))$ is a weak solution (in the distributional sense) to the Navier-Stokes Cauchy problem (2.2), if

$$\int_0^T \int_{\mathbb{R}^3} (u \cdot \partial_t \varphi + u \cdot \Delta \varphi + u \cdot \nabla \varphi \cdot u) dx dt = -\int_{\mathbb{R}^3} u_0 \cdot \varphi_0 dx, \tag{3.29}$$

$\forall \varphi \in \mathcal{D}_T$ and initial data $u_0 := u(x,0) \in L^2_\sigma(\mathbb{R}^3)$

Strengthening definition 2.3.1. one may then define the notion of a Leray-Hopf weak solution, which has the additional requirement of solving the energy inequality, with other regularity criteria added on top of being a distributional solution. Specifically:

Definition 2.3.2. $u \in L^\infty(0,T; L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T; W^{1,2}(\mathbb{R}^3))$ is a Leray-Hopf weak solution to the Navier-Stokes Cauchy problem (2.2), if it is a weak solution (in the distributional sense), as well as solves the energy inequality

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \le \|u_0\|_{L^2}^2, \tag{3.30}$$

and the initial data strongly converges in the L^2 sense

$$\|u(t) - u_0\|_{L^2} \to 0 \quad \text{as} \quad t \to 0^+, \tag{3.31}$$

Next, as a standard analogue to weak solutions to the Navier-Stokes equations, we give an equivalent formualtion for weak solutions (in the distributional sense) to the MHD equations. Specifically we have

Definition 2.3.3. The pair $(u, B) \in L^2_{\text{loc},\sigma}(\mathbb{R}^3 \times (0,T))$ is a weak solution (in the distributional sense) to the MHD Cauchy problem (2.12), if

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^3} (u \cdot \partial_t \varphi + u \cdot \Delta\varphi + u \cdot \nabla\varphi \cdot u - B \cdot \nabla\varphi \cdot B) dxdt &= -\int_\Omega u_0 \cdot \varphi_0 dx, \\ \int_0^T \int_{\mathbb{R}^3} (B \cdot \partial_t \phi + B \cdot \Delta\phi + u \cdot \nabla\phi \cdot B - B \cdot \nabla\phi \cdot u) dxdt &= -\int_\Omega B_0 \cdot \phi_0 dx,\end{aligned} \tag{3.32}$$

$\forall \varphi, \phi \in \mathcal{D}_T$ and initial data $u_0 := u(x,0), B_0 := B(x,0) \in L^2_\sigma(\mathbb{R}^3)$

Again strengthening the formulation of a weak solution, a Leray-Hopf weak solution to the MHD equations is defined below.

Deifnition 2.3.4. The pair $(u, B) \in L^\infty(0,T; L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T; W^{1,2}(\mathbb{R}^3))$ is a Leray-Hopf weak solution to the MHD Cauchy problem (2.12), if

(1) (u, B) is a distributional solution to (1.1)

$$\begin{aligned}\int_0^T \int_\Omega (u \cdot \partial_t \varphi + u \cdot \varphi - u \cdot \nabla u \cdot \varphi + B \cdot \nabla B \cdot \varphi) dxdt &= -\int_\Omega u_0 \cdot \varphi_0 dx, \\ \int_0^T \int_\Omega (B \cdot \partial_t \phi - \nabla B \cdot \nabla\phi - u \cdot \nabla B \cdot \phi + B \cdot \nabla u \cdot \phi) dxdt &= -\int_\Omega B_0 \cdot \phi_0 dx,\end{aligned} \tag{3.33}$$

$\forall \varphi, \phi \in \mathcal{D}_T$ and initial data $u_0 := u(x,0), B_0 := B(x,0) \in L^2_\sigma(\mathbb{R}^3)$

(2) (u,B) satisfies the energy inequality for the 3D MHD Equations

$$\frac{1}{2}\int_\Omega(\|u(t)\|_2^2+\|B(t)\|_2^2)dx+\int_0^t\int_\Omega(\|\nabla u(s)\|_2^2+\|\nabla B(s)\|_2^2)ds \leq \frac{1}{2}\int_\Omega(\|u_0\|_2^2+\|B_0\|_2^2)dx, \tag{3.34}$$

(3) the initial data strongly converges in the L^2 sense

$$\begin{aligned} \|u(t)-u_0\|_{L^2} \to 0 \quad \text{as} \quad t\to 0^+, \\ \|B(t)-B_0\|_{L^2} \to 0 \quad \text{as} \quad t\to 0^+, \end{aligned} \tag{3.35}$$

Here one notes that smooth solutions to either (2.2) or (2.12) are automatically Leray-Hopf weak solutions, which themselves are weak solutions.

3.4 L^p Estimates

In this section we briefly discuss a few pertinent L^p estimates and interpolation inequalities that will be useful later in our discussion. All estimates presented in this subsection have been developed by other authors and are well-known.

Of first interest is the so called "Young's inequality", which allows one to estimate a product of non-negative real numbers by their sum, scaled and taken to a specific power. Importantly, such an inequality is useful when proving pertinent L^p estimates such as Hölder's inequality, or estimating the non-linear term in the MHD Equations (or in other PDE's).

Theorem 2.4.1. (Young's Inequality for Products). Let $a,b \geq 0$ and $p,q > 1$ be real numbers, such that $\frac{1}{p}+\frac{1}{q}=1$. Then one has

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q}, \tag{3.36}$$

Proof. It's well known that the exponential function $f:\mathbb{R}\to\mathbb{R}_{\geq 0}$, $x\mapsto e^x$ is convex on $\mathbb{R}$. Hence for all $x,y\in\mathbb{R}$ and $t\in[0,1]$, one has

$$e^{tx+(1-t)y} \leq te^x+(1-t)e^y, \tag{3.37}$$

Setting $x = \ln(a^p)$ and $y = \ln(b^q)$ with $t = \frac{1}{p}$ (and thus $1-t = \frac{1}{q}$), one gets through simple arithmetic and (3.37)

$$\begin{aligned} ab &= e^{\ln(ab)} \\ &= e^{\frac{1}{p}p\ln(a)+\frac{1}{q}q\ln(b)} \\ &= e^{\frac{1}{p}\ln(a^p)+\frac{1}{q}\ln(b^q)} \\ &\leq \frac{1}{p}e^{\ln(a^p)} + \frac{1}{q}e^{\ln(b^q)} \\ &= \frac{a^p}{p} + \frac{b^q}{q}, \end{aligned} \tag{3.38}$$

□

Of next importance is a fundamental L^p estimate that generalizes the famous Cauchy Schwartz inequality for L^2 norms. Specifically Hölder's inequality provides L^p bounds for the products of two L^p integrable functions by their individual L^p norms. Such an inequality is integral when studying the convergence of sequences of certain terms in PDE's. Below we provide the statement of the theorem with a brief proof.

Theorem 2.4.2. (Hölder's Inequality) Suppose f, g are measurable functions on Ω. Further let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, with $p, q, r \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then one has

$$\|fg\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}, \tag{3.39}$$

Further when $r = 1$ and $p = q = 2$, Hölder's inequality reduces down to the well-known Cauchy Schwarz inequality.

Proof. Assuming the statement of the theorem, we first prove (3.39) when $r = 1$. Firstly, we assume $p, q < \infty$, since the extremum case $p = \infty$, $q = 1$ (or vice versa) is trivial. Next we may also assume $\|f\|_{L^p(\Omega)}$, $\|g\|_{L^q(\Omega)} \in (0, \infty)$. The limiting case at ∞ is trivial, whereas if either norm is 0 (without loss of generality if $\|f\|_{L^p(\Omega)} = 0$), then $f = 0$ almost everywhere and hence $\|fg\|_{L^1(\Omega)} = 0$ with respect to some measure.

Normalizing f and g, define $F := \frac{f}{\|f\|_p}$ and $G := \frac{g}{\|g\|_q}$. Then one has $\|F\|_p = \|G\|_q = 1$, and thus using Young's inequality one gets

$$\|FG\|_1 \leq \frac{\|F\|_p^p}{p} + \frac{\|G\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1, \tag{3.40}$$

whereby substituting F and G in terms of f and g on the LHS of (3.40) and some simple arithmetic gives the claim for $r = 1$.

For $r \in (1, \infty)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ (the case $r = \infty$ is trivial), one simply notes that if $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, then $|f|^r \in L^{\frac{p}{r}}(\Omega)$ and $|g|^r \in L^{\frac{q}{r}}(\Omega)$ whereby using $\frac{r}{p} + \frac{r}{q} = 1$, one gets through Hölder's inequality for $r = 1$, that

$$\|fg\|_r = \|(fg)^r\|_1^{\frac{1}{r}} \leq \|f^r\|_{\frac{p}{r}}^{\frac{1}{r}} \|g^r\|_{\frac{q}{r}}^{\frac{1}{r}} = \|f\|_p \|g\| q, \tag{3.41}$$

Thus proving the claim. □

Similarly, one can extend Hölder's inequality to products of multiple L^p integrable functions.

Theorem 2.4.3. (Generalized Hölder's Inequality) Let $r, p_1, \ldots, p_m \in [1, \infty]$ with $\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{r}$. Further take $f_j \in L^{p_j}(\Omega)$ for $j = 1, \ldots, m$. Then

$$\left\| \prod_{i=1}^{m} f_i \right\|_{L^r(\Omega)} \leq \prod_{i=k=1}^{m} \|f_i\|_{L^{p_k}(\Omega)}, \tag{3.42}$$

Proof. Follows from an induction argument with Hölder's inequality.

Similar to Hölder's inequality, it is also of interest to find L^p bounds of a single function by a product of L^p norms of itself, with certain weights. That is, an equivalent formulation of Hölder's inequality follows

Theorem 2.4.4. (Convex interpolation inequality) Suppose f, g are measurable functions on Ω. Further let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, with $0 < p < q < \infty$,

$0 < \theta < 1$ and $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$. Then one has

$$\|f\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)}^{1-\theta} \|f\|_{L^q(\Omega)}^{\theta}, \tag{3.43}$$

Proof. Suppose the hypothesis above. Then by Hölder's inequality one gets

$$\|f\|_r = \|f^\theta \cdot f^{1-\theta}\|_r \leq \|f^\theta\|_{\frac{q}{\theta}} \|f^{1-\theta}\|_{\frac{p}{1-\theta}} = \|f\|_q^\theta \|f\|_p^{1-\theta}, \tag{3.44}$$

□

Finally since regularity criteria on the gradient of the solution to the MHD Equations will be considered, interpolation estimates that relate the L^p norms of certain weak derivatives of functions are necessary. Of specific, estimates of the form (3.45) are of interest for functions $u \in C_c^\infty(\mathbb{R}^n)$.

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}, \tag{3.45}$$

where $1 \leq p < n$, $1 \leq q < \infty$ and $C > 0$ is a constant.

It turns out, in fact, q in (3.45) may not be chosen arbitrarily, and instead is related by the dimension of the space $n \in \mathbb{N}$ and the value of p chosen. Motivating this, it should be expected that for arbitrary $u \in C_c^\infty(\mathbb{R}^n)$, the re-scaled mapping $u_\lambda(x) := u(\lambda x)$ (where $\lambda > 0$, $x \in \mathbb{R}^n$), which also lies in $C_c^\infty(\mathbb{R}^n)$, satisfy (3.45). That is we expect a sort of scaling invariance to hold, and thus the following:

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C\|Du_\lambda\|_{L^p(\mathbb{R}^n)}, \tag{3.46}$$

Through a simple change of variables, one notes that

$$\int_{\mathbb{R}^n} |u_\lambda|^q \, dx = \int_{\mathbb{R}^n} |u(\lambda x)|^q \, dx = \lambda^{-n} \int_{\mathbb{R}^n} |u(\tau)|^q \, d\tau, \tag{3.47}$$

$$\int_{\mathbb{R}^n} |Du_\lambda|^p \, dx = \lambda^p \int_{\mathbb{R}^n} |Du(\lambda x)|^p \, dx = \lambda^{p-n} \int_{\mathbb{R}^n} |Du(\tau)|^p \, d\tau, \tag{3.48}$$

whereby substituting (3.47) and (3.48) into (3.46) yields

$$\lambda^{-\frac{n}{q}}\|u\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{1-\frac{n}{p}}\|Du\|_{L^p(\mathbb{R}^n)}, \tag{3.49}$$

or rearranging (3.49) one has

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{1-\frac{n}{p}+\frac{n}{q}}\|Du\|_{L^p(\mathbb{R}^n)}, \tag{3.50}$$

From here one can note that if $\lambda \to 0$ or ∞, the estimate (3.50) breaks down if $1-\frac{n}{p}+\frac{n}{q} \neq 0$. Hence we arrive at a desired relation between p, q and n. That is, we require $1-\frac{n}{p}+\frac{n}{q} = 0$, or rearranging, similarly $q = \frac{np}{n-p}$ for the estimate (3.50) to be true. We make the above rigorous with a statement of the theorem with a proof below.

Theorem 2.4.5. (Sobolev Embedding) Let $1 \leq p < n$ with $n \in \mathbb{N}$. Denote by p^* the Sobolev conjugate of p by

$$p^* := \frac{np}{n-p}, \tag{3.51}$$

then for all functions $u \in C_c^1(\mathbb{R}^n)$, there exists a constant $C > 0$ dependent on p and n such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}, \tag{3.52}$$

Proof. We follow the proof given by Evans [16]. Specifically, we show Theorem 2.4.5. for $p = 1$ first, then generalize to $p \in [1, n)$.

Suppose the hypothesis above, with $p = 1$. Since $u \in C_c^\infty(\mathbb{R}^n)$ has compact support, its components (denote by u_i for $i = 1, \ldots, n$) vanish at $\pm\infty$. Hence we have

$$u(x) = \int_{-\infty}^{x_i} (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)\, dy_i, \tag{3.53}$$

Taking the upper limit of the integral in (3.49) to ∞, one has for each $i = 1, \ldots n$

$$|u(x)| \leq \int_{\mathbb{R}} |Du(x_1, \ldots, y_i, \ldots, x_n)|\, dy_i, \tag{3.54}$$

Or taking the product of (3.54) as $i = 1, \ldots n$ varies, and raising to the power of $\frac{1}{n-1}$, one gets

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}} |Du(x_1, \ldots, y_i, \ldots, x_n)| \, dy_i \right)^{\frac{1}{n-1}}, \tag{3.55}$$

Integrating (3.55) with respect to x_1, we find using the generalized Hölder's inequality (Theorem 2.4.3)

$$\begin{aligned} \int_{\mathbb{R}} |u|^{\frac{n}{n-1}} \, dx_1 &\leq \int_{\mathbb{R}} \prod_{i=1}^{n} \left(\int_{\mathbb{R}} |Du| \, dy_i \right)^{\frac{1}{n-1}} \, dx_1 \\ &= \left(\int_{\mathbb{R}} |Du| \, dy_1 \right)^{\frac{1}{n-1}} \left(\int_{\mathbb{R}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} |Du| \, dy_i \right)^{\frac{1}{n-1}} \, dx_1 \right) \\ &\leq \left(\int_{\mathbb{R}} |Du| \, dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{\mathbb{R}} \int_{\mathbb{R}} |Du| \, dx_1 dy_i \right)^{\frac{1}{n-1}}, \end{aligned} \tag{3.56}$$

Similarly, following verbatim steps done above in (3.56), integrating (3.56) with respect to x_2 and applying the generalized Hölder's inequality again gives

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |u|^{\frac{n}{n-1}} \, dx_1 dx_2 \leq \prod_{i=3}^{n} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |Du| \, dx_1 dx_2 dy_i \right)^{\frac{1}{n-1}}, \tag{3.57}$$

Repeating the above steps n times yields the estimate

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \, dx \leq \left(\int_{\mathbb{R}^n} |Du| \, dx \right)^{\frac{n}{n-1}}, \tag{3.58}$$

proving the result for $p = 1$.

Finally when $p \in (1, n)$, define a scaling of u, $\overline{u}$ by $\overline{u} = |u|^{\alpha}$, where $\alpha = \frac{p(n-1)}{n-p} > 1$.

Then one notes by Hölder's inequality that

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{\frac{\alpha n}{n-1}}\, dx\right)^{\frac{n-1}{n}} &\leq \alpha \int_{\mathbb{R}^n} |u|^{\alpha-1}|Du|\, dx \\ &\leq \alpha \left(\int_{\mathbb{R}^n} |u|^{\frac{(\alpha-1)p}{p-1}}\, dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p\, dx\right)^{\frac{1}{p}},\end{aligned} \tag{3.59}$$

Since by the choice of α, one has $\frac{\alpha n}{n-1} = \frac{(\alpha-1)p}{p-1} = \frac{np}{n-p}$. Hence rearranging (3.59) yields the claim for general $p \in (1, n)$, finishing the proof. $\square$

For added interest, with more machinery, one can extend the Sobolev Embedding Theorem, allowing one to estimate L^p norms of higher order mixed derivatives D^j of L^p integrable functions.

Theorem 2.4.6. (Gagliardo-Nirenberg interpolation inequality) [35] Let $1 \leq q, r \leq +\infty$ and $p \geq 1$. Further let $j, m \in \mathbb{N} \cup \{0\}$ such that $j < m$, with $\theta \in [0, 1]$ satisfying

$$\frac{1}{p} = \frac{j}{n} + \theta\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\theta}{q} \quad \frac{j}{m} \leq \theta \leq 1, \tag{3.60}$$

then for some constant $C > 0$, one has

$$\|D^j u\|_{L^p(\mathbb{R}^n)} \leq C\|D^m u\|^{\theta}_{L^r(\mathbb{R}^n)} \|u\|^{1-\theta}_{L^q(\mathbb{R}^n)}, \tag{3.61}$$

when $u \in L^q(\mathbb{R}^n)$.

Chapter 4

Galerkin Method for Existence of Solutions

One of the main ingredients for the proofs of Theorem's 1.1-1.3 is showing the existence of a solution pair to a so called regularized version of the MHD Cauchy problem (2.12), or Galerkin Method. One may then prove certain properties of the regularized solution and show it approximates the original solution to (2.12), for which both solutions of each system will inherit each others proprieties. As such, before stating the required theorem, we begin by defining a standard space-time mollifying technique for the regularization of the variables in (2.12).

Definition 4.1. Let $g : \mathbb{R}^3 \times (0,T) \to \mathbb{R}^3$ be locally integrable, and $\epsilon > 0$. Then we define by $g^{(\epsilon)}$ and $g_{(\epsilon)}$ the space and space-time mollifiers of g respectively

$$g^{(\epsilon)}(x,\cdot) = \int_{\mathbb{R}^3} k_\epsilon(x-y)g(y,\cdot)dy, \quad g_{(\epsilon)}(x,t) = \int_0^T j_\epsilon(t-s)g^{(\epsilon)}(x,s)ds,$$

where

$$j_\epsilon(\tau) := \epsilon^{-1} j(\tau/\epsilon), \quad k_\epsilon(\xi) := \epsilon^{-3} k(\xi/\epsilon), \quad (\tau,\xi) \in \mathbb{R} \times \mathbb{R}^3,$$

and

$$j \in C_c^\infty(-1,1) \quad \text{and} \quad k \in C_c^\infty(\mathbb{R}^3),$$

Following the mollifying techniques presented above, we consider a regularized version of the 3D Cauchy problem (2.12)

$$\begin{cases} \partial_t u^\epsilon - \Delta u^\epsilon + (u_{(\epsilon)} \cdot \nabla)u^\epsilon - (B_{(\epsilon)} \cdot \nabla)B^\epsilon + \nabla p^\epsilon = f_1, \\ \partial_t B^\epsilon + (u_{(\epsilon)} \cdot \nabla)B^\epsilon - (B_{(\epsilon)} \cdot \nabla)u^\epsilon = \Delta B^\epsilon + f_2, & (t,x) \in (0,T) \times \mathbb{R}^3 \\ \nabla \cdot u^\epsilon = \nabla \cdot B^\epsilon = 0, \end{cases} \tag{4.1}$$

with initial conditions

$$u^\epsilon(\cdot, 0) = u_0^{(\epsilon)}, \quad B^\epsilon(\cdot, 0) = B_0^{(\epsilon)}, \quad \text{on} \quad \mathbb{R}^3, \tag{4.2}$$

where $f_1, f_2 \in C_0^\infty((0,T) \times \mathbb{R}^3)$ with $T > 0$.

Recall from section 3.2 equation (3.26), a condensed form of the MHD Cauchy problem (2.12) is written as

$$\begin{cases} \partial_t(\Gamma, \Psi) - (\Gamma, \partial_t \Psi) + (\nabla\Gamma, \nabla\Psi) + \langle \mathcal{B}(\Gamma, \Gamma), \Psi \rangle = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^3 \\ \nabla \cdot \Gamma_i = 0, \end{cases}$$

with initial data $\Gamma(x,0) = \Gamma_0(x) := (u_0(x), B_0(x)) \in \mathbb{R}^3 \times \mathbb{R}^3$, where $\Gamma := (u, B)$ and $\Psi \in \mathcal{D}_T$.

Mimicking this process, a condensed regularized system to (4.1) is

$$\begin{cases} \partial_t \Gamma^\epsilon + A\Gamma^\epsilon + \mathcal{B}(\Gamma_{(\epsilon)}, \Gamma^\epsilon) = f, & (t,x) \in (0,T) \times \mathbb{R}^3 \\ \nabla \cdot \Gamma^\epsilon = 0, \end{cases} \tag{4.3}$$

with initial conditions

$$\Gamma^\epsilon(\cdot, 0) = (u^\epsilon(\cdot, 0), B^\epsilon(\cdot, 0)) = \Gamma_0^\epsilon, \quad \text{on} \quad \mathbb{R}^3 \tag{4.4}$$

where $\Gamma^\epsilon = (u^\epsilon, B^\epsilon)$, $\Gamma_{(\epsilon)} = (u_{(\epsilon)}, B_{(\epsilon)})$ and $f = (f_1, f_2)$.

Theorem 4.2. Existence of Solutions to the Regularized MHD Cauchy Problem.

Let $\Gamma_0^\epsilon := (u_0^\epsilon, B_0^\epsilon) \in L^2_\sigma(\mathbb{R}^3)$ be initial data to the system (4.3)-(4.4). Then the Cauchy problem (4.3)-(4.4) admits a unique solution $\Gamma^\epsilon = (u^\epsilon, B^\epsilon)$ such that

$$\Gamma^\epsilon \in L^\infty(0,T; L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T; W^{1,2}(\mathbb{R}^3)) \qquad \textit{(is Leray-Hopf)} \tag{4.5}$$

and

$$\begin{aligned} &\max_{t\in[0,T]} (\|u^\epsilon(t)\|_2^2 + \|B^\epsilon(t)\|_2^2) + \int_0^T (\|\nabla u^\epsilon(t)\|_2^2 + \|\nabla B^\epsilon(t)\|_2^2)\, dt \\ &\leq \|u_0^\epsilon\|_2^2 + \|B_0^\epsilon\|_2^2 + C\int_0^T (\|f_1(t)\|^2_{\frac{6}{5}} + \|f_2(t)\|^2_{\frac{6}{5}})\, dt, \end{aligned} \tag{4.6}$$

for some constant $C > 0$.

Proof. Here we follow the proof given by Lai and Yang [29], omitting the proof for uniqueness, since it is not necessary for the thesis. Specifically we consider first the system (4.3)-(4.4) on the restricted domain $\mathbb{B}_R \times (0,T)$ for $T > 0$, where $\mathbb{B}_R$ denotes the ball centered at the origin with radius $R > 0$.

$$\begin{cases} \partial_t \Phi + A\Phi + \mathcal{B}(\Theta, \Phi) = f, \\ \nabla \cdot \Phi = 0, \end{cases} \qquad (t,x) \in (0,T) \times \mathbb{B}_R \tag{4.7}$$

with boundary and initial conditions chosen such that

$$\begin{aligned} &\Phi = 0 && \text{on} \quad \partial\mathbb{B}_R \times (0,T) \\ &\Phi(\cdot, 0) = \Phi_{0_R} && \text{on} \quad \mathbb{B}_R \end{aligned} \tag{4.8}$$

and

$$\lim_{R\to\infty} \|\Phi_0 - \Phi_{0_R}\|_2 = 0, \tag{4.9}$$

Here the existence of such a Φ_{0_R} is shown in Appendix A, Theorem A.2.

4.1 Local in time existence

We first show the existence of local in time solutions to the system (4.7)-(4.9). Specifically, the idea is to construct a sequence of approximate solutions Φ_n that at each time $t > 0$, lay in a finite-dimensional Hilbert space (Banach space that admits an inner product). Exploiting the orthogonality of a specific basis, we may reduce the system (4.7)-(4.9) to a system of ODE's, allowing the use of standard ODE theory for existence and uniqueness, simplifying the problem. Thus before beginning, we provide a well known eigenfunction theorem from PDE theory.

Lemma 4.3. Take $\Omega \subset \mathbb{R}^3$ smooth and bounded. Then there exists functions $\mathcal{N} = \{a_i : i \in \mathbb{N}\}$ such that

$$\begin{aligned}
&(i)\ \ \mathcal{N} \text{ is an orthogonal basis in } L^2_\sigma(\Omega)\\
&(ii)\ \ Aa_i = \lambda_i a_i\ \forall i \in \mathbb{N} \quad \text{such that} \quad 0 < \Lambda_n \leq \lambda_{n+1} \to \infty\ \forall n \in \mathbb{N}\\
&(iii)\ \mathcal{N} \text{ is an orthogonal basis in } H^1_{0,\sigma}(\Omega)
\end{aligned} \tag{4.10}$$

Definition 4.4. We denote the projection operator $P_n : L^2 \to L^2_\sigma$ by

$$P_n\Phi := \sum_{i=1}^{n} \langle \Phi, a_i \rangle a_i, \tag{4.11}$$

where $a_i \in \mathcal{N}$, and $\langle \cdot, \cdot \rangle$ denotes the standard dot/scalar product.

Continuing the proof, we construct approximate solutions Φ_n that belong to the space $P_n L^2_\sigma\ \forall t > 0$, by considering the Cauchy problem

$$\begin{cases} \partial_t \Phi_n + A\Phi_n + P_n\mathcal{B}(\Theta_n, \Phi_n) = P_n f, \\ \nabla \cdot \Phi_n = 0, \end{cases} \quad (t, x) \in (0, T) \times \mathbb{B}_R \tag{4.12}$$

with initial condition

$$\Phi_n(0) = P_n \Phi_{0_R}, \tag{4.13}$$

where we are searching for functions Φ_n and Θ_n of the form

$$\Phi_n(x,t) = \sum_{k=1}^{n} c_k^n(t) a_k(x), \quad \Theta_n(x,t) = \sum_{k=1}^{n} \overline{c}_k^n(t) a_k(x), \tag{4.14}$$

To determine the constants $c_k^n(t)$ and $\overline{c}_k^n(t)$, we exploit the orthogonality of the basis $\mathcal{N}$. Specifically, taking the L^2 inner product of (4.12) with a_k for $k = 1, \ldots, n$, one gets

$$\sum_{j=1}^{n} \left(\frac{d}{dt} c_j^n(t) a_j, a_k \right) + \sum_{j=1}^{n} (c_j^n(t) A a_j, a_k) + \sum_{j=1}^{n} \langle \mathcal{B}(\overline{c}_i^n(t) a_i, c_j^n(t) a_j), a_k \rangle = (f, a_k), \tag{4.15}$$

where we used the well known fact that $(P_n u, a_k) = (u, a_k)\ \forall u \in L^2(\mathbb{R}^3)$.

Since by Lemma 4.3 $Aa_i = \lambda_i a_i\ \forall i \in \mathbb{N}$, and are orthogonal in $L^2(\mathbb{B}_R)$, (3.15) simplifies to a system of linear ODE's

$$\frac{d}{dt} c_k^n(t) + \lambda_k c_k^n(t) + \sum_{i,j=1}^{n} D_{ijk} \overline{c}_i^n(t) c_j^n(t) = f_k, \quad k = 1, \ldots, n \tag{4.16}$$

where

$$D_{ijk} = \langle \mathcal{B}(a_i, a_j), a_k \rangle, \quad f_k(t) = (f(\cdot, t), a_k) \quad \text{and} \quad \overline{c}_i^n(t) = (\Theta_n, a_i), \tag{4.17}$$

where the initial conditions are determined similarly by taking the L^2 inner product of (3.13) with a_k, yielding

$$c_k^n(0) = \langle \Phi_{0_R}, a_k \rangle, \tag{4.18}$$

Here we conclude from classical ODE theory, since we have a countable system of linear ODEs, system (4.16)-(4.18) admits a unique solution tuple $(c_1^n, \ldots, c_n^n)$ on the local time interval $[0, T_n)$ with $0 < T_n \leq T$. Hence we have the existence of a sequence of solutions Φ_n to the system (4.12)-(4.13) of the form (4.14).

4.2 Uniform Estimates on Φ_n

In order to show the existence of a global in-time solution to (4.7)-(4.9), one approach is to extract a strongly convergent subsequence from $\{\Phi_n\}_{n\in\mathbb{N}}$ through the well known Aubin-Lions Lemma (see Appendix A Theorem A.1.) (compactness argument). As such, uniform estimates on Φ_n and $\partial_t\Phi_n$ in $L^2(0,T;(H^1_{0,\sigma}(\mathbb{B}_R)))$ and $L^2(0,T;(H^1_{0,\sigma}(\mathbb{B}_R)^*))$ respectively, are needed.

We first find uniform bounds on Φ_n. By the above argument in section 4.1, we have the existence of a sequence of solutions $\{\Phi_n\}_{n\in\mathbb{N}}$ on $[0,T_n)$ to the system (4.12)-(4.13). Taking the L^2 inner product of $\Phi_n(s)$ with (4.12) yields

$$(\partial_s\Phi_n(s),\Phi_n(s)) + (A\Phi_n(s),\Phi_n(s)) + \langle P_n\mathcal{B}(\Theta_n(s),\Phi_n(s)),\Phi_n(s)\rangle = (f,\Phi_n(s)), \tag{4.19}$$

By the product rule, the first term on the LHS of (4.19) can be rewritten as

$$(\partial_s\Phi_n(s),\Phi_n(s)) = \frac{1}{2}\frac{d}{ds}\|\Phi_n(s)\|^2_{L^2(\mathbb{B}_R)}, \tag{4.20}$$

whereas the second term can be rewritten as (using the definition of the Stokes operator and the projection operator)

$$\begin{aligned}(A\Phi_n(s),\Phi_n(s)) &= (-\mathbb{P}\Delta\Phi_n(s),\Phi_n(s)) = (-\Delta\Phi_n(s),\mathbb{P}\Phi_n(s)) = (-\Delta\Phi_n(s),\Phi_n(s))\\ &= \|\nabla\Phi_n(s)\|^2_{L^2(\mathbb{B}_R)},\end{aligned} \tag{4.21}$$

where in the last step we integrate by parts and use the fact that $\nabla\cdot\Theta_n = 0$.

Next, the third nonlinear term on the LHS of (4.19) vanishes, since

$$\langle P_n\mathcal{B}(\Theta_n,\Phi_n),\Phi_n\rangle = \langle\mathcal{B}(\Theta_n,\Phi_n),P_n\Phi_n\rangle = \langle\mathcal{B}(\Theta_n,\Phi_n),\Phi_n\rangle = 0, \tag{4.22}$$

Hence for $s>0$, combining (4.20)-(4.22) with (4.19), one gets

$$\frac{1}{2}\frac{d}{ds}\|\Phi_n(s)\|_2^2 + \|\nabla\Phi_n(s)\|_2^2 = (f, \Phi_n(s)), \tag{4.23}$$

Integrating (4.23) from 0 to t for all $t \in [0, T_n)$, using (4.9) and applying the Cauchy Schwartz and Young's inequalities, one obtains

$$\|\Phi_n(t)\|_{L^2(\mathbb{B}_R)}^2 + \int_0^t \|\nabla\Phi_n(s)\|_{L^2(\mathbb{B}_R)}^2 \, d\tau \leq \|\Phi_0\|_2^2 + K\int_0^t \|f(s)\|_{\frac{6}{5}}^2 \, d\tau \leq C, \tag{4.24}$$

where $K > 0$ is a constant and $C > 0$ is a constant independent of t and n.

Finally, since the second term on the LHS of (4.24) is non-negative, taking the time supremum over $[0, T]$ of (4.24), one obtains the estimate

$$\int_0^T \|\nabla\Phi_n(s)\|_{L^2(\mathbb{B}_R)}^2 \, d\tau \leq \|\Phi_0\|_2^2 + K\int_0^T \|f(s)\|_{\frac{6}{5}}^2 \, d\tau, \tag{4.25}$$

where $K > 0$ is a constant.

Hence $\Phi_n \in L^\infty(0, T; L^2(\mathbb{B}_R)) \cap L^2(0, T; H_{0,\sigma}^1(\mathbb{B}_R))$ and thus one obtains the uniform bound

$$\|\Phi_n\|_{L^\infty(0,T;L^2(\mathbb{B}_R))\cap L^2(0,T;H_{0,\sigma}^1(\mathbb{B}_R))} \leq C, \tag{4.26}$$

4.3 Uniform Estimates on $\partial_t\Phi_n$

Next we show uniform estimates on $\partial_t\Phi_n$ in $L^2(0, T; H_{0,\sigma}^1(\mathbb{B}_R)^*)$ as mentioned previously. In a similar fashion for the bound on Φ_n, taking the L^2 inner product of (4.12) with an arbitary test function $\varphi \in H_{0,\sigma}^1(\mathbb{R}_R)$, gives

$$\begin{aligned} (\partial_t\Phi_n, \varphi) &= -(A\Phi_n, \varphi) - \langle P_n\mathcal{B}(\Theta_n, \Phi_n), \varphi\rangle + (f, \varphi) \\ &= -(A\Phi_n, \varphi) - \langle \mathcal{B}(\Theta_n, \Phi_n), P_n\varphi\rangle + (f, \varphi) \end{aligned} \tag{4.27}$$

Next we estimate each term on the RHS of (4.27). Specifically, the first term on the RHS of (4.27) estimates as follows

$$|(A\Phi_n, \varphi)| = |(-\mathbb{P}\Delta\Phi_n, \varphi)| = |(-\Delta\Phi_n + \nabla\tilde{\varphi}, \varphi)| = |-\Delta\Phi_n, \varphi| \tag{4.28}$$

whereby integrating by parts and using the divergence free boundary condition, one arrives at

$$|(A\Phi_n, \varphi)| = |(\nabla\Phi_n, \nabla\varphi)| \leq \|\nabla\Phi_n\|_{L^2(\mathbb{B}_R)}\|\varphi\|_{H_0^1(\mathbb{B}_R)} \tag{4.29}$$

Next, for the second term on the RHS of (3.27), using Hölder's inequality with the Sobolev embedding theorem, gives for some constant $c > 0$

$$\begin{aligned} |\langle\mathcal{B}(\Theta_n, \Phi_n), P_n\varphi\rangle| &\leq \|\Theta_n\|_{L^3(\mathbb{B}_R)}\|\nabla\Phi_n\|_{L^2(\mathbb{B}_R)}\|P_n\varphi\|_{L^6(\mathbb{B}_R)} \\ &\leq c\|\nabla\Phi_n\|_{L^2(\mathbb{B}_R)}\|P_n\varphi\|_{H^1_{0,\sigma}(\mathbb{B}_R)} \\ &\leq c\|\nabla\Phi_n\|_{L^2(\mathbb{B}_R)}\|\varphi\|_{H^1_{0,\sigma}(\mathbb{B}_R)} \end{aligned} \tag{4.30}$$

with the last term of (3.27) dealt with similarly

$$|(f, \varphi)| \leq \|f\|_{L^{\frac{6}{5}}(\mathbb{B}_R)}\|\varphi\|_{L^6(\mathbb{B}_R)} \leq c\|f\|_{L^{\frac{6}{5}}(\mathbb{B}_R)}\|\varphi\|_{H^1_{0\sigma}(\mathbb{B}_R)} \tag{4.31}$$

Hence combinging (4.27) with (4.29)-(4.31), one gets

$$\|\partial_t\Phi_n\|_{(H^1_{0,\sigma}(\mathbb{B}_R))^*} \leq c(\|\nabla\Phi_n\|_{L^2(\mathbb{B}_R)} + \|f\|_{L^{\frac{6}{5}}(\mathbb{B}_R)}) \tag{4.32}$$

Squaring both sides of (4.32) yields

$$\begin{aligned} \|\partial_t\Phi_n\|^2_{(H^1_{0,\sigma}(\mathbb{B}_R))^*} &\leq c(\|\nabla\Phi_n\|^2_{L^2(\mathbb{B}_R)} + 2\|\Phi_n\|_{L^2(\mathbb{B}_R)}\|f\|_{L^{\frac{6}{5}}(\mathbb{B}_R)} + \|f\|^2_{L^{\frac{6}{5}}(\mathbb{B}_R)}) \\ &\leq c(\|\nabla\Phi_n\|^2_{L^2(\mathbb{B}_R)} + \|f\|^2_{L^{\frac{6}{5}}(\mathbb{B}_R)}), \end{aligned} \tag{4.33}$$

for which integrating from 0 to T and using (4.24) yields

$$\begin{aligned}\int_0^T \|\partial_t\Phi_n\|^2_{(H^1_{0,\sigma}(\mathbb{B}_R))^*}\,ds &\le c\left(\int_0^T \|\nabla\Phi_n(s)\|^2_{L^2(\mathbb{B}_R)}\,ds + \int_0^T \|f(s)\|^2_{L^{\frac{6}{5}}(\mathbb{B}_R)}\,ds\right)\\ &\le c\left(\|\Phi_0\|_2^2 + \int_0^T \|f(s)\|^2_{\frac{6}{5}}\,ds\right),\end{aligned} \tag{4.34}$$

Hence $\partial_t\Phi_n \in L^2(0,T;(H^1_{0,\sigma}(\mathbb{B}_R))^*)$ and thus one obtains the uniform bound

$$\|\partial_t\Phi_n\|_{L^2(0,T;(H^1_{0,\sigma}(\mathbb{B}_R))^*)} \le C, \tag{4.35}$$

4.4 Global in time Existence

Finally we show the global in time existence of a solution pair $\Gamma^\epsilon = (u^\epsilon, B^\epsilon)$ to the regularized system (4.1), thus in turn proving Theorem 4.2.

The idea is to extract a convergent subsequence with limit Φ_R via the Aubin-Lions Lemma (Appendix A, Theorem A.1.) and prove it's a solution to the system (4.7)-(4.9) for each $R > 0$. Next, one can extend such a sequence of solutions $\{\Phi_R\}_{R>0}$ to the whole domain Ω, defining values of Φ_R outside the ball $\mathbb{B}_R$ to be zero. Applying a standard diagonalization argument then shows such a sequence $\{\Phi_R\}_{R>0}$ strongly converges to a unique solution to (4.1), ending the proof. A summary of the above is outlined below.

Summarizing the above subsections, we have proved the existence of a constant $C > 0$ such that

$$\|\Phi_n\|_{L^\infty(0,T;L^2(\mathbb{B}_R))\cap L^2(0,T;H^1_{0,\sigma}(\mathbb{B}_R))} \le C \quad \text{and} \quad \|\partial_t\Phi_n\|_{L^2(0,T;(H^1_{0,\sigma}(\mathbb{B}_R))^*)} \le C, \tag{4.36}$$

Hence by Aubin-Lions Lemma, there exists $\forall R > 0$ a subsequence (still denote by Φ_n) $\Phi_n \to \Phi_R$ such that

$$\begin{aligned} &\Phi_n \to \Phi_R, \quad \text{strongly in } L^2(0,T; L^2_\sigma(\mathbb{B}_R)) \\ &\Phi_n \to \Phi_R, \quad \text{weakly star in } L^\infty(0,T; L^2_\sigma(\mathbb{B}_R)) \\ &\nabla\Phi_n \to \nabla\Phi_R, \quad \text{weakly in } L^2(0,T; L^2(\mathbb{B}_R)) \end{aligned} \tag{4.37}$$

Dotting $\varphi \in \mathcal{D}_T$ with equation (4.7) and integrating from [0,T) (and integrating by parts), one gets

$$-\int_0^T (\Phi_n, \partial_t\varphi)\, dt + \int_0^T (\nabla\Phi_n, \nabla\varphi)\, dt + \int_0^T \langle \mathcal{B}(\Theta_n, \Phi_n), \varphi\rangle\, dt = (\Phi_{0_R}, \varphi(0)) + \int_0^T (f, \phi)\, dt, \tag{4.38}$$

Here from (4.37), taking the limit of the first and second term of the LHS of (4.38) as $n \to \infty$, one gets

$$\int_0^T (\Phi_n, \partial_t\varphi)\, dt \to \int_0^T (\Phi_R, \partial_t\varphi)\, dt, \tag{4.39}$$

and

$$\int_0^T (\nabla\Phi_n, \nabla\varphi)\, dt \to \int_0^T (\nabla\Phi_R, \nabla\varphi)\, dt, \tag{4.40}$$

where when taking the limit of the third term of the LHS of (4.38), one notes that

$$\mathcal{B}(\Theta_n, \Phi_n) - \mathcal{B}(\Theta, \Phi_R) = \mathcal{B}(\Theta_n - \Theta, \Phi_n) + \mathcal{B}(\Theta, \Phi_n - \Phi_R), \tag{4.41}$$

thus we see by Hölder's inequality that

$$\begin{aligned} \left| \int_0^T \langle \mathcal{B}(\Theta_n - \Theta, \Phi_n), \varphi\rangle\, dt \right| &\leq C \int_0^T \|\Theta_n - \Theta\|_4 \|\nabla\Phi_n\|_2 \|\varphi\|_4\, dt \\ &\leq C_\varphi \left(\int_0^T \|\Theta_n - \Theta\|_4^2\, dt \right)^{\frac{1}{2}} \left(\int_0^T \|\nabla\Phi_n\|_2^2\, dt \right)^{\frac{1}{2}} \to 0, \end{aligned} \tag{4.42}$$

and using (4.37) again we get

$$\int_0^T \langle \mathcal{B}(\Theta, \Phi_n - \Phi_R), \varphi) \rangle \, dt \to 0, \tag{4.43}$$

Hence combining (4.42) and (4.43) with (4.41) we deduce that

$$\int_0^T \langle \mathcal{B}(\Theta_n, \Phi_n), \varphi \rangle \, dt \to \int_0^T \langle \mathcal{B}(\Theta, \Phi_R), \varphi \rangle \, dt, \tag{4.44}$$

Hence combining (4.39), (4.40) and (4.44) with (4.38) after passing to the limit one gets

$$-\int_0^T (\Phi_R, \partial_t \varphi) \, dt + \int_0^T (\nabla \Phi_R, \nabla \varphi) \, dt + \int_0^T \langle \mathcal{B}(\Theta, \Phi_R), \varphi \rangle \, dt = (\Phi_{0_R}, \varphi(0)) + \int_0^T (f, \varphi) \, dt, \tag{4.45}$$

Therefore Φ_R is a weak solution to the system (4.7).

Finally, extending each Φ_R by 0 outside $\mathbb{B}_R$ for each $R > 0$ and denoting these same extensions by Φ_R, we aim to apply the Aubin-Lions Lemma with a standard diagonal argument to show existence.

Firstly it's clear by the estimate (4.24) that such extensions Φ_R satisfy

$$\frac{1}{2}\|\Phi_R(t)\|_2^2 + \int_0^t \|\nabla \Phi_R\|_2^2 \, ds \leq C, \tag{4.46}$$

for some constant $C > 0$ invariant of the radius R.

Hence the above implies $\Phi_R \to \Phi$ weakly in $L^2(0, T; W^{1,2}(\mathbb{R}^3))$.

In addition, using the estimate (4.34), for $R > 1$, one also has

$$\int_0^T \|\partial_t \Phi_R\|^2_{(H^1_{0,\sigma}(\mathbb{R}^3))^*} \, ds \leq c \left(\|\Phi_0\|_2^2 + \int_0^T \|f(s)\|^2_{\frac{6}{5}} \, ds \right), \tag{4.47}$$

where c is another constant invariant of the radius R.

Taking $S \in \mathbb{N}$ such that $S \leq R$, one notes through inclusions of L^p spaces on compact sets (and thus Sobolev spaces) that $(H^1_{0,\sigma})^*(\mathbb{B}_R) \subset (H^1_{0,\sigma})^*(\mathbb{B}_S)$ and thus combining both (4.46) and (4.47), one sees

$$\begin{aligned} \|\partial_t \Phi_R\|_{L^2(0,T;(H^1_{0,\sigma})^*(\mathbb{B}_S))} + \|\Phi_R\|_{L^2(0,T;H^1_0(\mathbb{B}_S))} &\leq \|\partial_t \Phi_R\|_{L^2(0,T;(H^1_{0,\sigma})^*(\mathbb{B}_R))} + \|\Phi_R\|_{L^2(0,T;H^1_0(\mathbb{B}_R))} \\ &\leq C, \end{aligned} \tag{4.48}$$

Applying a diagonalization argument with Aubin-Lions lemma, one can extract a convergent subsequence from $\{\Phi_R\}_R$, (still denote it as $\{\Phi_R\}_R$) such that $\Phi_R \to \Phi$ strongly in $L^2(0,T;L^2(\mathbb{B}_S))$ for all $S \in \mathbb{N}$.

Finally, the limit Φ is indeed a solution to the system (4.1)-(4.2). This is due to the fact that since each Φ_R are weak solutions, using the definiton of a weak solution to (4.1)-(4.2) and taking the limit as $R \to \infty$, one gets the claim, ending the proof. See Lai and Yang for full details [29]. □

Chapter 5

Global in time $L^r L^s$ estimates for the non-stationary Stokes system

Before proceeding with the proofs of Theorems 1.1-1.3, we require further machinery on the so called non-stationary stokes system. Specifically, this section outlines a variety of global in time $L^r L^s$ estimates for the non-stationary stokes system in $\mathbb{R}^3$ (5.1), which will be useful later in the proof of Theorem 1.1. In this section we adapt a portion of Berselli's paper [5] for the MHD Cauchy Problem (2.12).

Theorem 5.1. Solonnikov [46] (see also Giga and Sohr [22]). Let $\Omega \subset \mathbb{R}^3$ be smooth and bounded and $\mathcal{F} \in L^\alpha(0,T;L^\beta(\Omega))$, where $1 < \alpha, \beta < \infty$. Then the initial value non-stationary stokes system

$$\begin{cases} \partial_t \lambda - \Delta\lambda + \nabla\theta = \mathcal{F}, \\ \nabla \cdot \lambda = 0, \end{cases} \qquad (t,x) \in (0,T) \times \Omega \tag{5.1}$$

$$\begin{aligned} &\lambda(t,x) = 0 && (t,x) \in (0,T) \times \partial\Omega, \\ &\lambda(0,x) = 0 && x \in \Omega, \end{aligned}$$

admits a unique solution (γ, θ) such that for some constant $c > 0$ dependent on α, β, T and Ω,

$$\|\partial_t \lambda\|_{L^\alpha(L^\beta)} + \|P_\sigma \Delta\lambda\|_{L^\alpha(L^\beta)} + \|\nabla\theta\|_{L^\alpha(L^\beta)} \leq c\|\mathcal{F}\|_{L^\alpha(L^\beta)}, \tag{5.2}$$

The following is an immediate corollary which will allow us to deal with higher order systems.

Corollary 5.2. Let $\Omega \subset \mathbb{R}^3$ be smooth and bounded and $\mathcal{F} : \mathbb{R}^3 \to \mathbb{R}^6$ with $\mathcal{F} \in L^\alpha(0,T;L^\beta(\Omega))$, where $1 < \alpha, \beta < \infty$. Further, let $\lambda = (\lambda_1, \lambda_2)$ where $\lambda_i : \mathbb{R}^3 \to \mathbb{R}^3$ and $\theta_i : \mathbb{R}^3 \to \mathbb{R}$. Then the Cauchy problem (5.3)

$$\begin{cases} \partial_t \lambda_i - \Delta \lambda_i + \nabla \theta_i = \mathcal{F}_i, \\ \nabla \cdot \lambda_i = 0, \end{cases} \qquad (t,x) \in (0,T) \times \Omega \tag{5.3}$$

$$\lambda(t,x) = 0 \qquad (t,x) \in (0,T) \times \partial\Omega,$$

$$\lambda(0,x) = 0 \qquad x \in \Omega,$$

admits a unique solution (γ, θ) such that for some constant $c > 0$ dependent on α, β, T and Ω,

$$\|\partial_t \lambda\|_{L^\alpha(L^\beta)} + \|P_\sigma \Delta \lambda\|_{L^\alpha(L^\beta)} + \|\nabla \theta\|_{L^\alpha(L^\beta)} \leq c \|\mathcal{F}\|_{L^\alpha(L^\beta)}, \tag{5.4}$$

Proof. The proof follows by applying Theorem 4.1 to each λ_i □

In an attempt to mimic the coupled system (3.26), we set $\mathcal{F} = \mathcal{B}(\gamma, \lambda)$ where $\gamma \in L^r(0,T;L^s(\Omega))$ is a nice enough divergence-free mapping. Here for simplicity, the remainder of this section will assume $\gamma = (\gamma_1, \gamma_2)$ and $\lambda = (\lambda_1, \lambda_2)$ where $\gamma_i, \lambda_i : \mathbb{R}^3 \to \mathbb{R}^3$ and $\theta_i : \mathbb{R}^3 \to \mathbb{R}$.

Adapting Berselli and Chiodaroli [5], we have

Lemma 5.3. Let $\Omega \subset \mathbb{R}^3$ be smooth and bounded, and $\gamma \in L^r(0,T;L^s(\Omega))$ with $\nabla \cdot \gamma_i = 0$ in $\mathcal{D}'_T$ for a.e. $t \in [0,T]$. Then the initial value 3D cauchy problem (5.5)

$$\begin{cases} \partial_t \lambda_i - \Delta \lambda_i + \nabla \theta_i = \mathcal{B}_i(\gamma, \lambda), \\ \nabla \cdot \lambda_i = 0, \end{cases} \qquad (t,x) \in (0,T) \times \Omega \tag{5.5}$$

$$\lambda(t,x) = 0 \qquad\qquad (t,x) \in (0,T) \times \partial\Omega,$$

$$\lambda(0,x) = 0 \qquad\qquad x \in \Omega,$$

admits a unique solution (λ, θ) with $\lambda \in L^2(0,T;H^1(\mathbb{R}^3)) \cap L^\infty(0,T;L^2_\sigma(\mathbb{R}^3))$

In addition if $\nabla\lambda \in L^\alpha(0,TL^\beta(\Omega)) \quad (1 \leq \alpha, \beta \leq \infty)$, then one also has for $c > 0$

$$\|\partial_t \lambda\|_{L^{\frac{r\alpha}{r+\alpha}}(L^{\frac{s\beta}{s+\beta}})} + \|P_\sigma \Delta\lambda\|_{L^{\frac{r\alpha}{r+\alpha}}(L^{\frac{s\beta}{s+\beta}})} + \|\nabla\theta\|_{L^{\frac{r\alpha}{r+\alpha}}(L^{\frac{s\beta}{s+\beta}})} \leq c\|\gamma\|_{L^r(L^s)}\|\nabla\lambda\|_{L^\alpha(L^\beta)}, \tag{5.6}$$

Proof. The existence of a unique solution (γ, θ) in the Leray-Hopf class (2.7) is shown in Section 4 above by Theorem 4.2 (or Lai and Yang [29]). Next, applying the triangle inequality with Hölder's inequality gives

$$\begin{aligned}
\|B(\gamma,\lambda)\|_{L^{\frac{r\alpha}{r+\alpha}}(L^{\frac{s\beta}{s+\beta}})} &= \|(\gamma_1 \cdot \nabla\lambda_1 - \gamma_2 \cdot \nabla\lambda_2, \gamma_1 \cdot \nabla\lambda_2 - \gamma_2 \cdot \nabla\lambda_1)\|_{L^{\frac{r\alpha}{r+\alpha}}(L^{\frac{s\beta}{s+\beta}})} \\
&\leq \sum_{(i,j)\leq 2} \|\gamma_i \cdot \nabla\lambda_j\|_{L^{\frac{r\alpha}{r+\alpha}}(L^{\frac{s\beta}{s+\beta}})} \\
&\leq \sum_{(i,j)\leq 2} \big\| \|\gamma_i\|_{L^s} \|\nabla\lambda_j\|_{L^\beta} \big\|_{L^{\frac{r\alpha}{r+\alpha}}} \\
&\leq \sum_{(i,j)\leq 2} \|\gamma_i\|_{L^r(L^s)} \|\nabla\lambda_j\|_{L^\alpha(L^\beta)} \\
&\leq c\|\gamma\|_{L^r(L^s)}\|\nabla\lambda\|_{L^\alpha(L^\beta)},
\end{aligned} \tag{5.7}$$

in which combining (5.4) with (5.7) gives the result. □

Following the idea presented in Berselli and Chiodaroli [5], we will apply Lemma 5.3 a finite number $n \in \mathbb{N}$ times, using a straightforward induction argument. As such, an interpolation result given by Amann [1] will be needed.

Lemma 5.4. Take $\phi \in W^{1,p}(0,T;L^q(\Omega)) \cap L^p(0,T;W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega))$ with

$\phi(0) = 0$, $(1 < p, q < \infty)$. Then one gets

$$\phi \in L^{p_1}(0,T; W_0^{1,q}(\Omega)) \quad \forall\, p_1 \le p_* \quad \text{where } \frac{1}{p_*} = \frac{1}{p} - \frac{1}{2}, \tag{5.8}$$

Using the interpolation result above, one can show the following result.

Lemma 5.5. Let $\gamma \in L^r(0,T; L^s(\Omega))$ be such that $\frac{1}{r} + \frac{1}{s} = \kappa \le \frac{1}{2}$. If the solution (λ, θ) to (5.3) satisfies $\nabla\lambda \in L^\alpha(0,T; L^\beta(\Omega))$, then

$$\nabla\lambda \in L^{\alpha_1}(0,T; L^{\frac{s\beta}{s+\beta}}(\Omega)) \quad \forall\, \alpha_1 \le \left(\frac{r\alpha}{r+\alpha}\right)_*, \tag{5.9}$$

Proof. Take $\phi = \lambda$ as in Lemma 5.4. Assume $\nabla\lambda \in L^\alpha(0,T; L^\beta(\Omega))$, then we have $\lambda \in L^\alpha(0,T; W_0^{1,\beta})$. We want to show that $\lambda \in L^{\alpha_1}(0,T; W_0^{1,\frac{s\beta}{s+\beta}})$.

By Lemma 5.3, we have for some constant $c > 0$

$$\|P_\sigma \Delta\lambda\|_{L^{\frac{r\alpha}{r+\alpha}}(L^{\frac{s\beta}{s+\beta}})} \le c\|\gamma\|_{L^r(L^s)}\|\nabla\lambda\|_{L^\alpha(L^\beta)}, \tag{5.10}$$

Hence $\lambda \in W^{1,p}(0,T; L^q(\Omega)) \cap L^p(0,T; W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega))$. Further letting $p = \frac{r\alpha}{r+\alpha}$ and $q = \frac{s\beta}{s+\beta}$, applying Lemma 5.4 gives

$$\lambda \in L^{p_1}(0,T; W_0^{1,\frac{s\beta}{s+\beta}}(\Omega)) \quad \forall\, p_1 \le p_*, \text{where } \frac{1}{p_*} := \frac{1}{p} - \frac{1}{2}, \tag{5.11}$$

We note that

$$\frac{1}{\alpha_1} \ge \frac{1}{\left(\frac{r\alpha}{r+\alpha}\right)_*} = \frac{r+\alpha}{r\alpha} - \frac{1}{2} = \frac{1}{p} - \frac{1}{2} = \frac{1}{p_*}, \tag{5.12}$$

for which $p_1 = \alpha_1 \le p_*$ can be chosen, completing the proof. $\square$

Next, we aim to apply a bootstrapping argument to Lemma 5.5 by considering the sequence $(\alpha_n, \beta_n)_{n\in\mathbb{N}}$ defined recursively as (terminating if α_n or β_n reach either 1 or $+\infty$)

$$\alpha_{n+1} := \left(\frac{r\alpha_n}{r+\alpha_n}\right)_* \qquad \beta_{n+1} = \frac{s\beta_n}{s+\beta_n} \qquad n \in \mathbb{N}, \tag{5.13}$$

for which simple computation shows

$$\frac{1}{\alpha_{n+1}} + \frac{1}{\beta_{n+1}} = \kappa - \frac{1}{2} + \frac{1}{\alpha_n} + \frac{1}{\beta_n} \quad n \in \mathbb{N}, \tag{5.14}$$

From here, by the recurrence definition (5.13), it follows that α_n is increasing, while β_n is decreasing. That is, repeated use of Lemma 5.5 results in an increase in regularity of the solution (λ, θ) in the time variable, at the expensive of regularity in the space variable.

Lemma 5.6. Let $\gamma \in L^r(0,T;L^s(\Omega))$ be such that $\frac{1}{r} + \frac{1}{s} = \kappa \leq \frac{1}{2}$. Further define the pair (α_n, β_n) as in Lemma 5.5. Then the solution (λ, θ) to (5.3) satisfies

$$\nabla\lambda \in L^{\tilde{\alpha}_n}(0,T;L^{\beta}_n(\Omega)) \quad \forall \tilde{\alpha}_n \leq \alpha_n, \tag{5.15}$$

with

$$\mathcal{B}(\gamma,\lambda) \in L^{\frac{r\tilde{\alpha}_n}{r+\tilde{\alpha}_n}}(0,T;L^{\frac{s\beta_n}{s+\beta_n}}(\Omega)), \tag{5.16}$$

Proof. (5.15) follows immediately by induction, whereas (5.16) follows similar to the proof of Lemma 5.3, by Hölder's inequality. □

Chapter 6

Main Results

6.1 Initial Outline of Proofs

In this section we begin by providing a broad outline for the proof of Theorem 1.1 (with slight modifications and/or additions to this structure for the proofs of Theorems 1.2 and 1.3). The structure for the proof of Theorem 1.1 can be divided into three crucial parts, specifically, we need to prove:

Step 1. Show weak solutions to (2.12) which also have an additional restricted regularity (in this case are in the space $L^4(0,T;L^4(\mathbb{R}^3))$, the reasoning of which is made clear later in the proof), fall in the Leray-Hopf class 2.7. (The proof of this step is taken directly from Lai and Yang [29]).

Step 2. Show Leray-Hopf weak solutions to (2.12) which are also in $L^4(0,T;L^4(\mathbb{R}^3))$ satisfy the corresponding energy equality.

Step 3. Apply a bootstrapping argument to solutions of Step 1, to extend the allowed regularity of solutions that satisfy the energy equality.

That is, steps 1 and 2 prove Theorem 1.1 for weak solutions in $L^4(0,T;L^4(\mathbb{R}^3))$, with step 3 extending the allowed regularity to the space $L^r(0,T;L^s(\mathbb{R}^3))$ for $\frac{2}{r}+\frac{2}{s}=1$, with $s \geq 4$.

Here Steps 2 and 3 are motivated by the arguments presented by Berselli [5] for the incompressible Navier-Stokes eqations. We adapt such arguments for the MHD equations.

6.2 Proof of Theorem 1.1

6.2.1 Proof of Step 1

Suppose $u, B \in L^2_{\text{loc}}(\mathbb{R}^3 \times (0,T))$ are weak solutions to the MHD Equations (2.12) (in the distributional sense) defined by (3.32), with initial data $u_0, B_0 \in L^2_\sigma(\mathbb{R}^3)$. Then if $u, B \in L^4(0,T; L^4(\mathbb{R}^3))$, they fall in the Leray-Hopf class (2.7) and satisfy the energy equality (2.15).

Proof. Here we follow the arguments presented by Lai and Yang [29]. Let $\Omega := \mathbb{R}^3$. Following the mollifying techniques presented in section 4, we consider a regularized version of the 3D Cauchy problem (2.12)

$$\begin{cases} \partial_t u^\epsilon - \Delta u^\epsilon + (u_{(\epsilon)} \cdot \nabla) u^\epsilon - B_{(\epsilon)} \cdot \nabla B^\epsilon + \nabla p^\epsilon = f_1, \\ \partial_t B^\epsilon + u_{(\epsilon)} \cdot \nabla B^\epsilon - B_{(\epsilon)} \cdot \nabla u^\epsilon = \Delta B^\epsilon + f_2, & (t,x) \in (0,T) \times \mathbb{R}^3 \\ \nabla \cdot u^\epsilon = \nabla \cdot B^\epsilon = 0, \end{cases} \tag{6.1}$$

with initial conditions

$$u^\epsilon(\cdot, 0) = u_0^{(\epsilon)}, \quad B^\epsilon(\cdot, 0) = B_0^{(\epsilon)}, \quad \text{on} \quad \mathbb{R}^3, \tag{6.2}$$

where $f_1, f_2 \in C_0^\infty((0,T) \times \mathbb{R}^3)$ with $T > 0$.

Or following equation (3.26), consider the equivalent coupled system

$$\begin{cases} \partial_t \Gamma^\epsilon + A\Gamma^\epsilon + \mathcal{B}(\Gamma_{(\epsilon)}, \Gamma^\epsilon) = f, \\ \nabla \cdot \Gamma^\epsilon = 0, \end{cases} \quad (t,x) \in (0,T) \times \mathbb{R}^3 \tag{6.3}$$

with initial conditions

$$\Gamma^\epsilon(\cdot,0)=(u^\epsilon(\cdot,0),B^\epsilon(\cdot,0))=\Gamma_0^\epsilon,\quad \text{on}\quad \mathbb{R}^3 \tag{6.4}$$

where $\Gamma^\epsilon=(u^\epsilon,B^\epsilon)$, $\Gamma_{(\epsilon)}=(u_{(\epsilon)},B_{(\epsilon)})$ and $f=(f_1,f_2)$.

Through a standard Galerkin method presented in section 4, it was shown that the system (6.3)-(6.4) with initial data $\Gamma_0^\epsilon\in L^2_\sigma(\mathbb{R}^3)$ admits a unique solution $\Gamma^\epsilon=(u^\epsilon,B^\epsilon)$ in the Leray-Hopf class (2.7) and satisfies the energy estimate (6.5).

$$\begin{aligned}&\max_{t\in[0,T]}(\|u^\epsilon(t))\|_2^2+\|B^\epsilon(t)\|_2^2+\int_0^T(\|\nabla u^\epsilon\|_2^2+\|\nabla B^\epsilon\|_2^2)dt\\&\le\|u_0^\epsilon\|_2^2+\|B_0^\epsilon\|_2^2+c\int_0^T(\|f_1(t)\|^2_{\frac{6}{5}}+\|f_2(t)\|^2_{\frac{6}{5}})dt,\end{aligned} \tag{6.5}$$

for some constant $c>0$.

Notation. Here as an abuse of notation, we use the same variable Γ^ϵ to denote the solution of the system (6.1). This will be done a variety of times to avoid using needlessly many variables.

Here Γ^ϵ is a weak solution to (6.1) and satisfies (following the approach outlined in section 3.2),

$$\partial_t(\Gamma^\epsilon,\Psi)+(\nabla\Gamma^\epsilon,\nabla\Psi)+\langle B(\Gamma^\epsilon,\Psi),\Gamma^\epsilon\rangle=0, \tag{6.6}$$

or equivalently for any $\Psi=(\varphi,\phi)\in\mathcal{D}_T$ integrating from 0 to T gives

$$\int_0^T(\partial_t(\Gamma^\epsilon,\Psi)+(\nabla\Gamma^\epsilon,\nabla\Psi)+\langle B(\Gamma^\epsilon,\Psi),\Gamma^\epsilon\rangle)dt=(\Gamma_0^{(\epsilon)},\Psi(0)), \tag{6.7}$$

Letting Γ be defined by Theorem 1.1, taking the difference of (6.1) with (6.7) and integrating by parts (as to pass all the regularity on Ψ), gives

$$\int_0^T(\Gamma-\Gamma^\epsilon,\partial_t\Psi+\Delta\Psi+\mathcal{B}(\Gamma_{(\epsilon)},\Psi))dt=-\int_0^T(\mathcal{B}(\Gamma-\Gamma_{(\epsilon)},\Psi),\Gamma)dt-(\Gamma_0-\Gamma_0^\epsilon,\Psi(0)), \tag{6.8}$$

From here, the rest of the proof will be dedicated to showing

$$\int_0^T (\Gamma - \Gamma^\epsilon, f)dt \to 0$$

, for which since $f \in C_0^\infty((0,T)\times\mathbb{R}^3)$ is arbitrary, Γ can be identified with $\lim_{\epsilon\to 0}\Gamma^\epsilon$, and hence also Γ is in the Leray-Hopf class (2.7). As such, from the results of step 2, Γ solves the energy equality (2.15) and the proof will be finished.

To work with (6.8), we first wish to condense it by considering the reverse time system of (2.12). Following the argument presented by Lai and Yang [29], for $f \in C_0^\infty((0,T), \times\mathbb{R}^3)$, the 3D Cauchy problem

$$\begin{cases} \partial_t\overline{\Gamma^\epsilon} + A\overline{\Gamma^\epsilon} + \mathcal{B}(\overline{\Gamma_{(\epsilon)}}, \overline{\Gamma^\epsilon}) + \nabla\overline{p^\epsilon} = \overline{f}, \\ \nabla\cdot\overline{\Gamma^\epsilon} = 0, \end{cases} \qquad (t,x) \in \mathbb{R}^3 \times (0,\infty) \tag{6.9}$$

$$\overline{\Gamma_0^{(\epsilon)}} = 0 \quad \text{in } \mathbb{R}^3$$

where $\overline{\Gamma_{(\epsilon)}}(x,t) := -\Gamma_{(\epsilon)}(x, T-t)$ and $\overline{f}(x,t) = -f(x,T-t)$, admits a solution $\overline{\Gamma^\epsilon} \in W^{2,1}_{\frac{4}{3},T} \cap W^{2,1}_{2,T}$.

Letting $\Lambda_\epsilon(t,x) := \overline{\Gamma^\epsilon}(T-t,x)$, with $\Theta^\epsilon(t,x) = p^\epsilon(T-t,x)$, then Λ_ϵ solves the system (4.10)

$$\begin{cases} \partial_t\Lambda_\epsilon + \Delta\Lambda_\epsilon + \mathcal{B}(\Gamma_{(\epsilon)}, \Lambda_\epsilon) - \nabla\Theta^\epsilon = -f, \\ \nabla\cdot\Lambda_\epsilon = 0, \end{cases} \qquad (t,x) \in (0,T)\times\mathbb{R}^3 \tag{6.10}$$

$$\Lambda_\epsilon(x,T) = 0 \quad \text{in } \mathbb{R}^3$$

By using a standard density argument in $\mathcal{D}_T$, taking $\Psi = \Lambda_\epsilon$ as our test function in (6.8), one obtains through (6.10)

$$\int_0^T (\Gamma - \Gamma^\epsilon, f + \nabla\Theta^\epsilon)ds = \int_0^T (\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon), \Gamma)dt - (\Gamma_0 - \Gamma_0^{(\epsilon)}, \Lambda_\epsilon(0)), \tag{6.11}$$

Firstly, since $\nabla \cdot (\Gamma - \Gamma^{\epsilon}) = 0$ in $\mathcal{D}'(T)$ for almost every $t \in (0,T)$, one gets through integrating by parts

$$\int_0^T (\Gamma - \Gamma^{\epsilon}, \nabla \Theta^{\epsilon}) dt = 0, \tag{6.12}$$

Further, by Theorem 4.2, one has for some constant $c > 0$

$$\|\Lambda_{\epsilon}(0)\|_2 \leq c \int_0^T \|f(t)\|_{\frac{6}{5}}^2 dt, \tag{6.13}$$

for which using the Cauchy Schwartz inequality gives

$$\begin{aligned} \lim_{\epsilon \to 0} |\Gamma_0 - \Gamma_0^{(\epsilon)}, \Lambda_{\epsilon}(0)| &\leq \lim_{\epsilon \to 0} \|\Gamma_0 - \Gamma_0^{(\epsilon)}\|_2 \|\Lambda_{\epsilon}(0)\|_2 \\ &\leq c \lim_{\epsilon \to 0} \|\Gamma_0 - \Gamma_0^{(\epsilon)}\|_2 \int_0^T \|f(t)\|_{\frac{6}{5}}^2 dt \\ &\to 0, \end{aligned} \tag{6.14}$$

Finally we aim to show that the nonlinear term in (6.11) goes to zero. Specifically we want to show that

$$\int_0^T (\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_{\epsilon}), \Gamma) dt = 0, \tag{6.15}$$

Indeed, since $\Gamma \in L^4(0,T;L^4(\mathbb{R}^3))$, the nonlinear term is estimated by

$$\begin{aligned} \int_0^T (\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_{\epsilon}), \Gamma) dt &\leq \|\Gamma\|_{L^4(L^4)} \|\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_{\epsilon})\|_{L^{\frac{4}{3}}(L^{\frac{4}{3}})} \\ &\leq \|\Gamma\|_{L^4(L^4)} \sum_{(i,j) \leq 2} \|\Gamma_i - \Gamma_{i(\epsilon)}\|_{L^4(L^4)} \|\nabla \Lambda_j\|_{L^2(L^2)} \\ &\leq c \|\Gamma\|_{L^4(L^4)} \|\nabla \Lambda_{\epsilon}\|_{L^2(L^2)} \|\Gamma - \Gamma_{(\epsilon)}\|_{L^4(L^4)}, \end{aligned} \tag{6.16}$$

Here (6.16) goes to 0 as $\epsilon \to 0$ since $\Gamma \in L^4(0,T,L^4(\mathbb{R}^3))$ and Λ_{ϵ} is Leray-Hopf. The latter is true by Theorem 4.2. for which such solutions lie in $\nabla \Lambda_{\epsilon} \in L^2(0,T;L^2(\mathbb{R}^3))$.

Combining all (6.12)-(6.16) into (6.11) one obtains

$$\int_0^T (\Gamma - \Gamma^\epsilon, f) ds \to 0, \tag{6.17}$$

From here, by Theorem 4.2 (4.6), there exists a convergent subsequence of $\{\Gamma^\epsilon\}_{\epsilon>0}$. We will denote such limit as Γ_{reg}. Hence one gets using (6.17) that

$$\int_0^T (\Gamma - \Gamma_{\text{reg}}, f)\, ds \to \int_0^T (\Gamma - \Gamma^\epsilon, f)\, ds \to 0, \tag{6.18}$$

and since $f \in C_0^\infty(\mathbb{R}^3 \times (0,T))$ is arbitrary, one can identify Γ with Γ_{reg}. Finally by Theorem 4.2 Γ_{reg} is Leray-Hopf, and hence so must Γ be as well, proving Step 1. □

6.2.2 Proof of Step 2:

Suppose $u, B \in L^2_{\text{loc}}(\mathbb{R}^3 \times (0,T))$ is a Leray-Hopf weak solution to the MHD Equations (2.12) (in the distributional sense) defined by (3.32), with initial data $u_0, B_0 \in L^2_\sigma(\mathbb{R}^3)$. Then if $u, B \in L^r(0,T; L^s(\mathbb{R}^3))$ with $\frac{2}{r} + \frac{2}{s} = 1$ for $s \geq 4$, they satisfy the energy equality (2.15).

Proof. The remainder of steps 2 and 3 are modifications of arguments presented by Berselli and Chiodaroli [5]. Let (u, B) be definied as in the above hypothesis, with initial data $u_0, B_0 \in L^2_\sigma(\Omega)$. Fix $T \in (0, \infty)$ with $t_0 \in (0, T]$. Then by a standard density argument, there exists a sequence $\{(u_n, B_n)\}_{n\in\mathbb{N}} \subset C_0^\infty([0,T); C_0^\infty(\Omega))$ that converges to $(u, B) \in L^2(0,T;V) \cap L^r(0,T; W_0^{1,s}(\Omega))$, where V denotes the space of divergent free mappings on $H_0^1(\Omega)$. Letting $\Psi := (\Psi_n)_\epsilon = ((u_n)_\epsilon, (B_n)_\epsilon)$ as our test function, integrating (3.26) from 0 to $t_0 \in (0, T]$ with respect to time gives

$$\begin{aligned}(\Gamma(t_0), (\Psi_n)_\epsilon(t_0)) &= (\Gamma_0, (\Psi_n)_\epsilon(0)) \\ &\quad + \int_0^{t_0} \Big((\Gamma, \partial_t (\Psi_n)_\epsilon) - (\nabla\Gamma, \nabla(\Psi_n)_\epsilon) - \langle \mathcal{B}(\Gamma,\Gamma), (\Psi_n)_\epsilon \rangle \Big) dt,\end{aligned} \tag{6.19}$$

Mollifying and passing to the limit for each term in (6.19) is outlined in Galdi [20], where the two following facts are used

$$\begin{aligned} &\int_0^T (\nabla\Gamma, \nabla\Gamma_\epsilon) = \int_0^T (\nabla\Gamma, (\nabla\Gamma)_\epsilon) dt \xrightarrow[\epsilon\to 0^+]{} \int_0^T \|\nabla\Gamma\|^2 dt, \\ &(\Gamma(t), (\Gamma)_\epsilon(t)) = \frac{\|\Gamma(t)\|^2}{2} + \mathcal{O}(\epsilon), \end{aligned} \tag{6.20}$$

where the first limit requires $\Gamma \in L^2(0, T; V)$.

Next when passing to the limit, the only difficulty arises in the nonlinear term of (6.19). Specifically when comparing to the energy equality, we want to show

$$\int_0^{t_0} \langle \mathcal{B}(\Gamma, \Gamma), (\Psi_n)_\epsilon \rangle \, dt \xrightarrow[\epsilon\to 0^+]{} 0, \tag{6.21}$$

Expanding (6.21) and rearranging into groups of terms we get

$$\begin{aligned} \int_0^{t_0} \langle \mathcal{B}(\Gamma, \Gamma), (\Psi_n)_\epsilon \rangle \, dt &= \int_0^{t_0} (u \cdot \nabla u, (u_n)_\epsilon) \, dt \\ &\quad - \left(\int_0^{t_0} (B \cdot \nabla u, (B_n)_\epsilon) \, dt + \int_0^{t_0} (B \cdot \nabla B, (u_n)_\epsilon) \, dt \right) \\ &\quad + \int_0^{t_0} (u \cdot \nabla B, (B_n)_\epsilon) \, dt \\ &:= L_1 + L_2 + L_3, \end{aligned} \tag{6.22}$$

where the above is organized such that each line goes to 0 if $(\Gamma_m)_\epsilon$ is replaced by Γ via integration by parts. We now show indeed (6.22) goes to 0 as $\epsilon \to 0^+$ and $n \to \infty$.

In an attempt to avoid receptiveness, we estimate L_1 in (6.22) here, and similarily L_2 when looking at the proof of Theorem 1.2. (Both proofs have similar arguments). Estimates for each L_i are accomplished similarily. Specifically, one notes

$$
\begin{aligned}
L_1 &= \int_0^{t_0} (u\cdot\nabla u, (u_n)_\epsilon - (u)_\epsilon)\, dt + \int_0^{t_0} (u\cdot\nabla u, (u)_\epsilon - u)\, dt \\
&\quad + \int_0^{t_0} (u\cdot\nabla u, u)\, dt \\
&= L_1^1 + L_1^2 + L_1^3,
\end{aligned} \tag{6.23}
$$

where each L_1^i denotes each term of L_1 on the RHS of (6.23) for $i = 1, \ldots 3$.

Specifically, estimating L_1^1, one obtains through Hölder's inequality and convex interpolation

$$
\begin{aligned}
\left|\int_0^{t_0} (u\cdot\nabla u, (u_n)_\epsilon - (u)_\epsilon)\, dt\right| &\le \int_0^{t_0} \|u\|_{L^s_x}\|\nabla((u_n)_\epsilon - u_\epsilon)\|_{L^2_x}\|u\|_{L^r_x}\, dt \\
&\le \int_0^{t_0} \|u\|_{L^s_x}\|\nabla((u_n)_\epsilon - u_\epsilon)\|_{L^2_x}\|u\|_{L^2_x}^{2-\frac{r}{2}}\|u\|_{L^s_x}^{\frac{r}{2}-1}\, dt \\
&\le \|u\|_{L^r_T L^s_x}^{\frac{r}{2}}\|(u_n)_\epsilon - u_\epsilon\|_{L^2_T W^{1,2}_x}\|u\|_{L^2_x}^{2-\frac{r}{2}} \\
&\le \|\Gamma\|_{L^r_T L^s_x}^{\frac{r}{2}}\|(\Gamma_n)_\epsilon - \Gamma_\epsilon\|_{L^2_T W^{1,2}_x}\|u\|_{L^2_x}^{2-\frac{r}{2}} \\
&\to 0,
\end{aligned} \tag{6.24}
$$

Similarly for L_1^2, again Hölder's inequality and convex interpolation give

$$
\begin{aligned}
\left|\int_0^{t_0} (u\cdot\nabla u, (u)_\epsilon - u)\, dt\right| &\le \int_0^{t_0} \|u\|_{L^s_x}\|\nabla u\|_{L^2_x}\|u_\epsilon - u\|_{L^r_x}\, dt \\
&\le \int_0^{t_0} \|u\|_{L^s_x}\|\nabla u\|_{L^2_x}\|u_\epsilon - u\|_{L^2_x}^{2-\frac{r}{2}}\|u_\epsilon - u\|_{L^s_x}^{\frac{r}{2}-1}\, dt \\
&\le \|u\|_{L^r_T L^s_x}\|\nabla u\|_{L^2_T L^2_x}\|u_\epsilon - u\|_{L^\infty_T L^2_x}^{2-\frac{r}{2}}\|u_\epsilon - u\|_{L^r_T L^s_x}^{\frac{r}{2}-1} \\
&\le \|\Gamma\|_{L^r_T L^s_x}\|\nabla\Gamma\|_{L^2_T L^2_x}\|\Gamma_\epsilon - \Gamma\|_{L^\infty_T L^2_x}^{2-\frac{r}{2}}\|\Gamma_\epsilon - \Gamma\|_{L^r_T L^s_x}^{\frac{r}{2}-1} \\
&\to 0,
\end{aligned} \tag{6.25}
$$

Finally, for L_1^3, as will be explained similarly later in the proof of Theorem 1.2, one notes that

$$\int_0^{t_0} (u \cdot \nabla u_n, u_n)\, dt \to \int_0^{t_0} (u \cdot \nabla u, u)\, dt, \tag{6.26}$$

hence since the LHS of (6.26) equals zero by integration by parts, we have $L_1^3 \to 0$ if

$$\begin{aligned} &\left| \int_0^{t_0} (u \cdot \nabla u_n, u_n)\, dt - \int_0^{t_0} (u \cdot \nabla u, u)\, dt \right| \\ &\leq \left| \int_0^{t_0} (u \cdot \nabla u_n, u_n - u)\, dt \right| + \left| \int_0^{t_0} (u \cdot \nabla (u_n - u), u)\, dt \right| \\ &\to 0, \end{aligned} \tag{6.27}$$

Estimating the first and second terms on the RHS of (6.27) is done similarly to that shown in (6.24) and (6.25). Hence one gets $L_1^3 \to 0$

Combining the above one gets $L_1 \to 0$, and following a similar procedure for L_2 and L_3, we get

$$\int_0^{t_0} \langle \mathcal{B}(\Gamma, \Gamma), (\Psi_n)_\epsilon \rangle\, dt \to 0, \tag{6.28}$$

Combining (6.28) with (6.19) when passing to the limit (through standard properties of mollifiers), one obtains the energy equality, finishing the proof. □

6.2.3 Proof of Step 3:

Suppose $u, B \in L^2_{\text{loc}}(\mathbb{R}^3 \times (0,T))$ are weak solutions (in the distributional sense) to the MHD Equations (2.12) (in the distributional sense) defined by (3.32), with initial data $u_0, B_0 \in L^2_\sigma(\mathbb{R}^3)$. Then if $u, B \in L^r(0,T; L^s(\mathbb{R}^3))$ with $\frac{2}{r} + \frac{2}{s} = 1$ for $s \geq 4$, they fall in the Leray-Hopf class (2.7) and satisfy the energy equality (2.15).

Proof. Following a verbatim proof of Step 1 up to Equation (6.15), we continue the proof differently by changing the bound on (6.16). Specifically since $\Gamma \in L^r(0,T; L^s(\mathbb{R}^3))$ with $\frac{2}{r} + \frac{2}{s} = 1$ for $s \geq 4$, one has $\Gamma - \Gamma_{(\epsilon)} \to 0$ in $L^r(0,T; L^s(\mathbb{R}^3))$. Letting $\Gamma - \Gamma^\epsilon = (\Gamma_1, \Gamma_2)$ and $\Lambda_\epsilon = (\Lambda_1, \Lambda_2)$, following a similar approach, Hölder's inequality gives

$$\begin{aligned}\int_0^T (\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon), \Gamma) dt &\leq \|\Gamma\|_{L^r(L^s)} \|\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon)\|_{L^{\frac{r}{r-1}}(L^{\frac{s}{s-1}})} \\ &= \|\Gamma\|_{L^r(L^s)} \|\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon)\|_{L^{\frac{2s}{s+2}}(L^{\frac{s}{s-1}})} \\ &\leq \|\Gamma\|_{L^r(L^s)} \sum_{(i,j)\leq 2} \|\Gamma_i - \Gamma_{i(\epsilon)}\|_{L^r(L^s)} \|\nabla \Lambda_j\|_{L^{\frac{s}{2}}(L^{\frac{s}{s-2}})} \\ &\leq c\|\Gamma\|_{L^r(L^s)} \|\nabla \Lambda_\epsilon\|_{L^{\frac{s}{2}}(L^{\frac{s}{s-2}})} \|\Gamma - \Gamma_{(\epsilon)}\|_{L^r(L^s)},\end{aligned} \tag{6.29}$$

Hence the RHS of (6.29) goes to 0, if either

$$\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon) \in L^{\frac{2s}{s+2}}(L^{\frac{s}{s-1}}(\mathbb{R}^3)) \quad \text{or} \quad \nabla \Lambda_\epsilon \in L^{\frac{s}{2}}(L^{\frac{s}{s-2}}(\mathbb{R}^3)), \tag{6.30}$$

Here one notes by setting $r = s = 2$ reduces down to the case proven by Step's 1 and 2. Now we apply a sort of bootstrapping argument iterating Lemma 5.6, starting with our base case $r = s = 2$.

Letting $(\alpha_1, \beta_1) = (2, 2)$ with $\kappa = \frac{1}{2}$, Lemma 4.6 gives for $n \in \mathbb{N}$ (by letting $(\gamma, \lambda, \theta) := (\Gamma_{(\epsilon)}, \Lambda_\epsilon, \Theta^\epsilon)$)

$$\mathcal{B}(\Gamma_{(\epsilon)}, \Lambda_\epsilon) \in L^{\alpha_n}(0, T; L^{\beta_n}(\mathbb{R}^3)) \quad \text{where } \beta_n = \frac{s\beta_{n-1}}{s + \beta_{n-1}} \quad \text{and } \frac{1}{\alpha_n} + \frac{1}{\beta_n} = \frac{3}{2}, \tag{6.31}$$

or by induction, this is equivalent to

$$\mathcal{B}(\Gamma_{(\epsilon)}, \Lambda_\epsilon) \in L^{\frac{s}{s-n}}(0, T; L^{\frac{2s}{2n+s}}) \quad n \in \mathbb{N}, \tag{6.32}$$

where

$$\alpha_n = \frac{s}{s-n} \quad \beta_n = \frac{2s}{2n+s} \quad n \in \mathbb{N}, \tag{6.33}$$

From here we consider two cases, omitting the case when $s = 4$, (since we have already proved it).

Case 1. Suppose $s > 4$ is even. That is, $s = 2k$ with $k > 2$, $k \in \mathbb{N}$. Then it follows that

$$\begin{aligned}\alpha_{k-1} &= \frac{s}{s-k+1} = \frac{2k}{k+1} = \frac{2s}{2+s},\\ \beta_{k-1} &= \frac{2s}{2(k-1)+s} = \frac{2m}{2m-1} = \frac{s}{s-1},\end{aligned} \tag{6.34}$$

giving $\mathcal{B}(\Gamma_{(\epsilon)}, \Lambda_\epsilon) \in L^{\frac{2s}{2+s}}(L^{\frac{s}{s-1}})$ as desired.

Case 2. Suppose $s > 4$ is odd. Here our goal is to interpolate around $L^{\frac{s}{s-1}}(\mathbb{R}^3)$ and $L^{\frac{2s}{2+s}}(\mathbb{R}^3)$. For the former, (6.33) implies β_n is strictly decreasing and hence we wish to find $N \in \mathbb{N}$ such that

$$1 < \beta_{N+1} < \frac{s}{s-1} \leq \beta_N, \tag{6.35}$$

that is

$$1 < \frac{2s}{2(N+1)+s} < \frac{s}{s-1} \leq \frac{2s}{2N+s}, \tag{6.36}$$

Solving (6.36) gives

$$\frac{s}{2} - 2 \leq N < \frac{s}{2} - 1, \tag{6.37}$$

Letting $k \in \mathbb{N}$ denote the integer part of $\frac{s}{2}$, such an $N \in \mathbb{N}$ that satisfies (6.37) is

$$N = k - 1, \tag{6.38}$$

Applying Hölder's inequality, one gets

$$\begin{aligned} &\|\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon)\|_{L^{\frac{2s}{s+2}}(L^{\frac{s}{s-1}})} \\ \leq &\left\| \|\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon)\|^{1-\theta_1}_{L^{\beta_{N+1}}} \right\|_{L^{\frac{2s}{s+2}}} \left\| \|\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon)\|^{\theta_1}_{L^{\beta_N}} \right\|_{L^{\frac{2s}{s+2}}} \\ \leq &\left\| \|\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon)\|^{1-\theta_1}_{L^{\beta_{N+1}}} \right\|^{1-\theta_2}_{L^{\alpha_{N+1}}} \left\| \|\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon)\|^{1-\theta_1}_{L^{\beta_{N+1}}} \right\|^{\theta_2}_{L^{\alpha_N}}, \end{aligned} \tag{6.39}$$

where we have

$$\frac{s}{s-1} = \frac{1-\theta_1}{\beta_{N+1}} + \frac{\theta_1}{\beta_N} \quad \text{and} \quad \frac{s+2}{2s} = \frac{1-\theta_2}{\alpha_{N+1}} + \frac{\theta_2}{\alpha_N}, \tag{6.40}$$

here, both cases of (6.40) gives

$$\theta_1 = \theta_2 = N - \frac{s}{2} + 2, \tag{6.41}$$

where by using (6.37), one gets $\theta_1, \theta_2 \in [0,1)$, allowing the use of Hölder's in (6.39). Finally, the RHS of (6.39) is finite by Lemma 5.6 and one gets $\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon) \in L^{\frac{2s}{s+2}}(L^{\frac{s}{s-1}}(\mathbb{R}^3))$, satisfying the regularity requirement (6.30) and thus finsihing the proof. □

6.3 Proof of Theorem 1.2

We begin this section by proving Theorem 1.2. Specifically we present a modification of the argument given by Berselli and Chiodaroli [5] for the MHD equations. Formally, we show the following.

Theorem 1.2. *Suppose $u, B \in L^2_{\text{loc}}(\mathbb{R}^3 \times (0,T))$ are Leray-Hopf weak solutions to the MHD Equations (2.12) defined by (3.32), with initial data $u_0, B_0 \in L^2_\sigma(\mathbb{R}^3)$. If in addition u, B both satisfy any of the conditions (B1*)-(B3*) below, then the pair satisfy the energy equality (2.15).*

(B1*) $\nabla u, \nabla B \in L^{\frac{s}{2s-3}}(0,T; L^s(\Omega))$ for $\frac{3}{2} < s < \frac{9}{5}$,

(B2*) $\nabla u, \nabla B \in L^{\frac{5s}{5s-6}}(0,T; L^s(\Omega))$ for $\frac{9}{5} \leq s \leq 3$,

(B3*) $\nabla u, \nabla B \in L^{1+\frac{2}{s}}(0,T; L^s(\Omega))$ for $s > 3$,

Before proceeding, the credit of the proof is given to the ideas of Shinbrot [42] and Berselli and Chiodaroli [5], with a few modifications introduced here when dealing with the non-linear term. The main idea's are the same, however Theorem 1.2 is necessary for completeness when dealing with Theorem 1.3, which is an entirely new result.

Proof. We follow a verbtaim approach outlined in step 2 in section 6.2 above.

That is, let (u, B) be definied as in Theorem 1.2, with initial data $u_0, B_0 \in L^2_\sigma(\Omega)$. Fix $T \in (0, \infty)$ with $t_0 \in (0, T]$. By standard density arguments, there exists a sequence $\{(u_n, B_n)\}_{n\in\mathbb{N}} \subset C_0^\infty([0,T); C_0^\infty(\Omega))$ that converges to $(u, B) \in L^2(0,T;V) \cap L^r(0,T; W_0^{1,s}(\Omega))$, where V denotes the space of divergence free mappings on $H_0^1(\Omega)$. Letting $\Psi := (\Psi_n)_\epsilon = ((u_n)_\epsilon, (B_n)_\epsilon)$ as our test function, integrating (3.26) from 0 to $t_0 \in (0, T]$ with respect to time gives

$$\begin{aligned}(\Gamma(t_0), (\Psi_n)_\epsilon(t_0)) &= (\Gamma_0, (\Psi_n)_\epsilon(0)) \\ &\quad + \int_0^{t_0} \Big((\Gamma, \partial_t (\Psi_n)_\epsilon) - (\nabla\Gamma, \nabla(\Psi_n)_\epsilon) - \langle \mathcal{B}(\Gamma,\Gamma), (\Psi_n)_\epsilon\rangle \Big) dt,\end{aligned} \tag{6.42}$$

Expanding the nonlinear term in (6.42) and rearranging into groups of terms we get

$$\begin{aligned}\int_0^{t_0} \langle \mathcal{B}(\Gamma,\Gamma), (\Psi_n)_\epsilon\rangle\, dt &= \int_0^{t_0} (u\cdot\nabla u, (u_n)_\epsilon)\, dt \\ &\quad - \left(\int_0^{t_0} (B\cdot\nabla u, (B_n)_\epsilon)\, dt + \int_0^{t_0} (B\cdot\nabla B, (u_n)_\epsilon)\, dt \right) \\ &\quad + \int_0^{t_0} (u\cdot\nabla B, (B_n)_\epsilon)\, dt \\ &:= L_1 + L_2 + L_3,\end{aligned} \tag{6.43}$$

where the above is organized such that each line goes to 0 if $(\Gamma_n)_\epsilon$ is replaced by Γ via integration by parts. We now show indeed (6.43) goes to 0 as $\epsilon \to 0^+$ and $n \to \infty$. The first line of (6.43) goes to 0 by Berselli and Chiodaroli [5]. Following a similar approach to their proof, the second line can be rewitten as

$$\begin{aligned}
L_2 &= \int_0^{t_0} (B \cdot \nabla u, (B_n)_\epsilon - (B)_\epsilon)\, dt - \int_0^{t_0} (B \cdot \nabla B, (u_n)_\epsilon - (u)_\epsilon)\, dt \\
&\quad + \int_0^{t_0} (B \cdot \nabla u, (B)_\epsilon - B)\, dt - \int_0^{t_0} (B \cdot \nabla B, (u)_\epsilon - u)\, dt \\
&\quad + \int_0^{t_0} (B \cdot \nabla u, B)\, dt - \int_0^{t_0} (B \cdot \nabla B, u)\, dt \\
&\quad = L_2^1 + L_2^2 + L_2^3,
\end{aligned} \tag{6.44}$$

where each L_2^i denotes each line of L_2 in (6.44) for $i = 1, \dots 3$.

We now show that for fixed $\epsilon > 0$, $L_2^1 \to 0$ as $n \to \infty$, $L_2^2 \to 0$ as $\epsilon \to 0^+$, and $L_3^3 = 0$ given the assumptions of Theorem 1.2. Here each term in L_2^1 and L_2^2 can be bounded simlarily. For example the first term in L_2^1 is dealt as follows

Let $(u, B) \in L^r(0, T; L^s(\Omega))$ where either of the conditions from Theorem 1.2 are satisfied. Then if $\frac{2}{\bar{s}} + \frac{1}{s} = 1$, $\frac{1}{\bar{s}} = \frac{1-\theta}{6} + \frac{\theta}{s^*}$ and $\frac{1}{s^*} = \frac{1}{s} - \frac{1}{3}$, Hölder's inequality with Sobolev embedding gives

$$\begin{aligned}
&\left| \int_0^{t_0} ((B \cdot \nabla u, (B_n)_\epsilon - (B)_\epsilon))\, dt \right| \\
&\le \int_0^{t_0} \|B\|_{L_x^{\bar{s}}} \|\nabla u\|_{L_x^s} \|(B_n)_\epsilon - (B)_\epsilon\|_{L_x^{\bar{s}}}\, dt \\
&\le \int_0^{t_0} \|B\|_{L_x^{s^*}}^\theta \|B\|_{L_x^6}^{1-\theta} \|\nabla u\|_{L_x^s} \|(B_n)_\epsilon - (B)_\epsilon\|_{L_x^{s^*}}^\theta \|(B_n)_\epsilon - (B)_\epsilon\|_{L_x^6}^{1-\theta}\, dt \\
&\le C \int_0^{t_0} \|\nabla B\|_{L_x^s}^\theta \|\nabla B\|_{L_x^2}^{1-\theta} \|\nabla u\|_{L_x^s} \|\nabla((B_n)_\epsilon - (B)_\epsilon)\|_{L_x^s}^\theta \|\nabla((B_n)_\epsilon - (B)_\epsilon)\|_{L_x^2}^{1-\theta}\, dt \\
&\le C \|\nabla B\|_{L_T^r L_x^s}^\theta \|\nabla B\|_{L_T^2 L_x^2}^{1-\theta} \|\nabla u\|_{L_T^r L_x^s} \|\nabla((B_n)_\epsilon - (B)_\epsilon)\|_{L_T^r L_x^s}^{1-\theta} \\
&\quad \cdot \|\nabla((B_n)_\epsilon - (B)_\epsilon)\|_{L_T^2 L_x^2}^{1-\theta},
\end{aligned} \tag{6.45}$$

where the final step requires $\frac{2}{\eta_1} + \frac{2}{\eta_2} + \frac{1}{\eta_3} = 1$, where $\eta_1 = \frac{r}{\theta}$, $\eta_2 = \frac{2}{1-\theta}$ and $\eta_3 = r$. Here one can verify that all indicies are well defined if $r, s \ge 1$.

It follows from standard properties of mollifiers that

$$\lim_{n\to\infty} \|\nabla((B_n)_\epsilon - (B)_\epsilon)\|_{L_T^r L_x^s} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\nabla((B_n)_\epsilon - (B)_\epsilon)\|_{L_T^2 L_x^2} = 0, \tag{6.46}$$

and hence

$$\lim_{n\to\infty}\int_0^{t_0}(B\cdot\nabla u,(B_n)_\epsilon-(B)_\epsilon)\,dt=0, \tag{6.47}$$

Applying a similar argument to all the other terms of L_2^1 and L_2^2, shows $L_2^1\to 0$ and $L_2^2\to 0$.

Dealing with the third line in (6.44) requires the specific grouping of terms on the second line in (6.43) for a certain cancellation to occur. We clarify this now by showing $L_2^3=0$. Since $\{(u_n,B_n)\}_{n\in\mathbb{N}}$ is a smooth sequence converging to $(u,B)\in L^2(0,T;V)\cap L^r(0,T;W_0^{1,s}(\Omega))$, integration by parts gives

$$\begin{aligned}&\int_0^{t_0}(B\cdot\nabla u_n,B_n)\,dt+\int_0^{t_0}(B\cdot\nabla B_n,u_n)\,dt\\ &=-\int_0^{t_0}(B\cdot\nabla B_n,u_n)\,dt+\int_0^{t_0}(B\cdot\nabla B_n,u_n)\,dt\\ &\to 0,\end{aligned} \tag{6.48}$$

where the RHS of (6.48) goes to 0 as $n\to\infty$. To finish the proof for L_2, we want to show

$$\int_0^{t_0}(B\cdot\nabla u_n,B_n)\,dt+\int_0^{t_0}(B\cdot\nabla B_n,u_n)\,dt\to\int_0^{t_0}(B\cdot\nabla u,B)\,dt-\int_0^{t_0}(B\cdot\nabla B,u)\,dt, \tag{6.49}$$

in which the proof will be concluded for L_2. Taking the absolute value of the difference of (6.49), applying the Triangle Inequality, we see

$$\left|\int_0^{t_0}(B\cdot\nabla u_n,B_n)\,dt+\int_0^{t_0}(B\cdot\nabla B_n,u_n)\,dt-\int_0^{t_0}(B\cdot\nabla u,B)\,dt-\int_0^{t_0}(B\cdot\nabla B,u)\,dt\right|$$

$$\begin{aligned}&\le\left|\int_0^{t_0}(B\cdot\nabla u_n,B_n-B)\,dt\right|+\left|\int_0^{t_0}(B\cdot\nabla(u_n-u),B)\,dt\right|\\ &+\left|\int_0^{t_0}(B\cdot\nabla B_n,u_n-u)\,dt\right|+\left|\int_0^{t_0}(B\cdot\nabla(B_n-B),u)\,dt\right|,\end{aligned} \tag{6.50}$$

Here we show each term of (6.50) goes to 0 by considering each of the 3 cases of Theorem 1.2. Specifically, we will deal only with the first two terms on the RHS of (6.50), since the last two terms are dealt with similarly.

Case 1: Suppose $\nabla\Gamma_i \in L^{\frac{s}{2s-3}}(0,T;L^s(\Omega)), \quad \text{for} \quad \frac{3}{2} < s < \frac{9}{5}$.

The first term on the RHS of (6.50) is shown first. Specifically let $\frac{2}{\tilde{s}} + \frac{1}{s} = 1$, $\frac{1}{\tilde{s}} = \frac{\theta}{s^*} + \frac{1-\theta}{6}$ and $\frac{1}{s^*} = \frac{1}{s} - \frac{1}{3}$. Then for some constant $C > 0$, Hölder's inequality with Sobolev embedding gives

$$\begin{aligned}
&\left|\int_0^{t_0} ((B\cdot\nabla u_n, B_n - B)\, dt\right| \\
&\le \left|\int_0^{t_0} \|B\|_{L_x^{\tilde{s}}}\|\nabla(u_n)\|_{L_x^s}\|B_n - B\|_{L_x^{\tilde{s}}}\, dt\right| \\
&\le \left|\int_0^{t_0} \|B\|^{\theta}_{L_x^{s^*}}\|B\|^{1-\theta}_{L_x^6}\|\nabla u_n\|_{L_x^s}\|B_n - B\|^{\theta}_{L_x^{s^*}}\|B_n - B\|^{1-\theta}_{L_x^6}\, dt\right| \\
&\le C\left|\int_0^{t_0} \|\nabla B\|^{\theta}_{L_x^{s^*}}\|\nabla B\|^{1-\theta}_{L_x^2}\|\nabla u_n\|_{L_x^s}\|\nabla(B_n - B)\|^{\theta}_{L_x^{s^*}}\|\nabla(B_n - B)\|^{1-\theta}_{L_x^2}\, dt\right| \\
&\le C\|\nabla B\|^{\theta}_{L_T^rL_x^s}\|\nabla B\|^{1-\theta}_{L_T^2L_x^2}\|\nabla u_n\|_{L_T^rL_x^s}\|\nabla(B_n - B)\|^{\theta}_{L_T^rL_x^s}\|\nabla(B_n - B)\|^{1-\theta}_{L_T^2L_x^2} \\
&\le C\|\nabla\Gamma\|^{\theta}_{L_T^rL_x^s}\|\nabla\Gamma\|^{1-\theta}_{L_T^2L_x^2}\|\nabla\Gamma_n\|_{L_T^rL_x^s}\|\nabla(\Gamma_n - \Gamma)\|^{\theta}_{L_T^rL_x^s}\|\nabla(\Gamma_n - \Gamma)\|^{1-\theta}_{L_T^2L_x^2} \\
&\to 0,
\end{aligned} \tag{6.51}$$

where the final step requires $\frac{2}{\eta_1} + \frac{2}{\eta_2} + \frac{1}{\eta_3} = 1$, where $\eta_1 = \frac{r}{\theta}$, $\eta_2 = \frac{2}{1-\theta}$ and $\eta_3 = r$. Here one can verify that all indicies are well defined if $\frac{3}{2} < s < \frac{9}{5}$.

Likewise, let $\frac{2}{\tilde{s}} + \frac{1}{s} = 1$, $\frac{1}{\tilde{s}} = \frac{\theta}{s^*} + \frac{1-\theta}{6}$ and $\frac{1}{s^*} = \frac{1}{s} - \frac{1}{3}$. Then for $C > 0$, Hölder's inequality with Sobolev embedding gives

$$\begin{aligned}
\left|\int_0^{t_0} ((B\cdot\nabla(u_n - u), B)\, dt\right| &\le \left|\int_0^{t_0} \|B\|_{L_x^{\tilde{s}}}\|\nabla(u_n - u)\|_{L_x^s}\|B\|_{L_x^{\tilde{s}}}\, dt\right| \\
&\le \left|\int_0^{t_0} \|B\|^{2\theta}_{L_x^{s^*}}\|B\|^{2(1-\theta)}_{L_x^6}\|\nabla(u_n - u)\|_{L_x^s}\, dt\right| \\
&\le C\left|\int_0^{t_0} \|\nabla B\|^{2\theta}_{L_x^s}\|\nabla B\|^{2(1-\theta)}_{L_x^2}\|\nabla(u_n - u)\|_{L_x^s}\, dt\right| \\
&\le C\|\nabla B\|^{2\theta}_{L_T^rL_x^s}\|\nabla B\|^{2(1-\theta)}_{L_T^2L_x^2}\|\nabla(u_n - u)\|_{L_T^rL_x^s} \\
&\le C\|\nabla\Gamma\|^{2\theta}_{L_T^rL_x^s}\|\nabla\Gamma\|^{2(1-\theta)}_{L_T^2L_x^2}\|\nabla(\Gamma_n - \Gamma)\|_{L_T^rL_x^s} \\
&\to 0,
\end{aligned} \tag{6.52}$$

where the final step requires $\frac{1}{\eta_1} + \frac{1}{\eta_2} + \frac{1}{\eta_3} = 1$, where $\eta_1 = \frac{r}{2\theta}$, $\eta_2 = \frac{1}{1-\theta}$ and $\eta_3 = r$.

Here one can verify that all indices are well defined if $\frac{3}{2} < s < \frac{9}{5}$.

The third and fourth terms on the RHS of (6.50) can be bounded similarly to (6.51) and (6.52) respectively.

Case 2.a: Suppose $\nabla\Gamma_i \in L^{\frac{5s}{5s-6}}(0,T;L^s(\Omega)), \quad \text{for} \quad \frac{9}{5} \le s < \frac{12}{5}$.

We provide estimates for the first term of (6.50), for which the other terms are bounded similarly. Specifically, let $\frac{2}{\tilde{s}} + \frac{1}{s} = 1$, $\frac{1}{\tilde{s}} + \frac{\theta}{2} + \frac{1-\theta}{s^*}$ and $\frac{1}{s^*} = \frac{1}{s} - \frac{1}{3}$. Then Hölder's inequality with Sobolev embedding gives

$$\begin{aligned}
&\left|\int_0^{t_0} ((B\cdot\nabla u_n, B_n - B)\,dt\right| \\
&\le \left|\int_0^{t_0} \|B\|_{L_x^{\tilde{s}}}\|\nabla(u_n)\|_{L_x^s}\|B_n - B\|_{L_x^{\tilde{s}}}\,dt\right| \\
&\le \left|\int_0^{t_0} \|B\|_{L_x^2}^{\theta}\|B\|_{L_x^{s^*}}^{1-\theta}\|\nabla u_n\|_{L_x^s}\|B_n - B\|_{L_x^2}^{\theta}\|B_n - B\|_{L_x^{s^*}}^{1-\theta}\,dt\right| \\
&\le C\left|\int_0^{t_0} \|B\|_{L_x^2}^{\theta}\|\nabla B\|_{L_x^s}^{1-\theta}\|\nabla u_n\|_{L_x^s}\|B_n - B\|_{L_x^2}^{\theta}\|\nabla(B_n - B)\|_{L_x^s}^{1-\theta}\,dt\right| \\
&\le C\|B\|_{L_T^\infty L_x^2}^{\theta}\|\nabla B\|_{L_T^r L_x^s}^{1-\theta}\|\nabla u_n\|_{L_T^r L_x^s}\|B_n - B\|_{L_T^\infty L_x^2}^{\theta}\|\nabla(B_n - B)\|_{L_T^r L_x^s}^{1-\theta} \\
&\le C\|\Gamma\|_{L_T^\infty L_x^2}^{\theta}\|\nabla\Gamma\|_{L_T^r L_x^s}^{1-\theta}\|\nabla\Gamma_n\|_{L_T^r L_x^s}\|\Gamma_n - \Gamma\|_{L_T^\infty L_x^2}^{\theta}\|\nabla(\Gamma_n - \Gamma)\|_{L_T^r L_x^s}^{1-\theta} \\
&\to 0,
\end{aligned} \tag{6.53}$$

where the final step requires $\frac{2}{\eta_1} + \frac{1}{\eta_2} = 1$, where $\eta_1 = \frac{r}{1-\theta}$ and $\eta_2 = r$.

Case 2.b: Suppose $\nabla\Gamma_i \in L^{\frac{5s}{5s-6}}(0,T;L^s(\Omega)), \quad \text{for} \quad \frac{12}{5} \le s \le 3$.

Again for the first term of (6.50), let $\frac{1}{s} + \frac{1}{\tilde{s}} = \frac{1}{2}$, $\frac{1}{\tilde{s}} = \frac{\theta}{2} + \frac{1-\theta}{s^*}$ and $\frac{1}{s^*} = \frac{1}{s} - \frac{1}{3}$. Then

Hölder's inequality with Sobolev embedding gives

$$
\begin{aligned}
&\left|\int_0^{t_0} ((B \cdot \nabla u_n, B_n - B)\, dt\right| \\
&\leq \left|\int_0^{t_0} \|B\|_{L_x^2} \|\nabla(u_n)\|_{L_x^s} \|B_n - B\|_{L_x^{\tilde{s}}}\, dt\right| \\
&\leq \left|\int_0^{t_0} \|B\|_{L_x^2} \|\nabla u_n\|_{L_x^s} \|B_n - B\|_{L_x^2}^{\theta} \|B_n - B\|_{L_x^{s^*}}^{1-\theta}\, dt\right| \\
&\leq C\left|\int_0^{t_0} \|B\|_{L_x^2} \|\nabla u_n\|_{L_x^s} \|B_n - B\|_{L_x^2}^{\theta} \|\nabla(B_n - B)\|_{L_x^s}^{1-\theta}\, dt\right| \\
&\leq C\|B\|_{L_T^\infty L_x^2} \|\nabla u_n\|_{L_T^r L_x^s} \|B_n - B\|_{L_T^\infty L_x^2}^{\theta} \|\nabla(B_n - B)\|_{L_T^r L_x^s}^{1-\theta} \\
&\leq C\|\Gamma\|_{L_T^\infty L_x^2} \|\nabla \Gamma_n\|_{L_T^r L_x^s} \|\Gamma_n - \Gamma\|_{L_T^\infty L_x^2}^{\theta} \|\nabla(\Gamma_n - \Gamma)\|_{L_T^r L_x^s}^{1-\theta} \\
&\to 0,
\end{aligned}
\tag{6.54}
$$

where the final step requires $\frac{1}{\eta_1} + \frac{1}{\eta_2} = 1$, where $\eta_1 = \frac{r}{1-\theta}$ and $\eta_2 = r$. Again it is easily checked that all indicies are well defined.

Case 3: Suppose $\nabla\Gamma_i \in L^{1+\frac{2}{s}}(0, T; L^s(\Omega)), \quad \text{for} \quad s > 3.$

Again we only provide estimates for the first term of (5.50), since the others are done similarly. Specifically let $\frac{2}{\tilde{s}} + \frac{1}{s} = 1$ and $\frac{1}{\tilde{s}} = \frac{\theta}{2}$. Then for some constant $C > 0$, Hölder's inequality with Sobolev embedding gives

$$
\begin{aligned}
&\left|\int_0^{t_0} ((B \cdot \nabla u_n, B_n - B)\, dt\right| \\
&\leq \left|\int_0^{t_0} \|B\|_{L_x^{\tilde{s}}} \|\nabla(u_n)\|_{L_x^s} \|B_n - B\|_{L_x^{\tilde{s}}}\, dt\right| \\
&\leq \left|\int_0^{t_0} \|B\|_{L_x^2}^{\theta} \|B\|_{L_x^\infty}^{1-\theta} \|\nabla u_n\|_{L_x^s} \|B_n - B\|_{L_x^2}^{\theta} \|B_n - B\|_{L_x^\infty}^{1-\theta}\, dt\right| \\
&\leq C\left|\int_0^{t_0} \|B\|_{L_x^2}^{\theta} \|\nabla B\|_{L_x^s}^{1-\theta} \|\nabla u_n\|_{L_x^s} \|B_n - B\|_{L_x^2}^{\theta} \|\nabla(B_n - B)\|_{L_x^s}^{1-\theta}\, dt\right| \\
&\leq C\|B\|_{L_T^\infty L_x^2}^{\theta} \|\nabla B\|_{L_T^r L_x^s}^{1-\theta} \|\nabla u_n\|_{L_T^r L_x^s} \|B_n - B\|_{L_T^\infty L_x^2}^{\theta} \|\nabla(B_n - B)\|_{L_T^r L_x^s}^{1-\theta} \\
&\leq C\|\Gamma\|_{L_T^\infty L_x^2}^{\theta} \|\nabla \Gamma\|_{L_T^r L_x^s}^{1-\theta} \|\nabla \Gamma_n\|_{L_T^r L_x^s} \|\Gamma_n - \Gamma\|_{L_T^\infty L_x^2}^{\theta} \|\nabla(\Gamma_n - \Gamma)\|_{L_T^r L_x^s}^{1-\theta} \\
&\to 0,
\end{aligned}
\tag{6.55}
$$

where the final step requires $\frac{2}{\eta_1} + \frac{1}{\eta_2} = 1$, where $\eta_1 = \frac{r}{1-\theta}$ and $\eta_2 = r$. Here one can verify that all indices are well defined if $s > 3$.

Finally, following a verbatim approach outlined above, the last line L_3 of (6.43) can be shown to go to 0, completing the proof of Theorem 1.2. $\square$

6.4 Proof of Theorem 1.3

To finish the thesis, we prove a new result for very weak solutions to the MHD Cauchy problem (2.12), dropping the Leray-Hopf condition. That is, we show

Theorem 1.3. *Suppose $u, B \in L^2_{\text{loc}}(\mathbb{R}^3 \times (0,T))$ are weak solutions to the MHD Equations (2.12) (in the distributional sense) defined by (3.32), with initial data $u_0, B_0 \in L^2_\sigma(\mathbb{R}^3)$. Then if $\nabla u, \nabla B \in L^{\frac{8s}{9s-12}}(0,T; L^s(\Omega))$, for $\frac{12}{7} < s \leq \frac{12}{5}$, then u, B satisfy the energy equality (2.15).*

We follow a similar approach to the proof of Theorem 1.1 for regularity criteria on Γ instead of $\nabla\Gamma$. Specifically the goal is to show that assuming the conditions of Theorem 1.3, then the solution is automatically Leray-Hopf, and thus the problem reduces down to the simpler case of Theorem 1.2.

Following a verbatim argument to the proof of Theorem 1.1, the proof remains unchanged, with the exception of bounding the nonlinear term in (6.11).

Proof. By definition, we write $\mathcal{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon)$ in its components where subscripts $(\cdot)_i$ denote mappings from $\mathbb{R}^3 \to \mathbb{R}^3$. Letting superscripts $(\cdot)_i^k$ denote the individual components from $\mathbb{R} \to \mathbb{R}$ of each $(\cdot)_i$ component, rewriting the nonlinear term one gets

$$\begin{aligned}\int_0^T (\mathcal{B}(\Gamma-\Gamma_{(\epsilon)},\Lambda_\epsilon),\Gamma)dt &= \int_0^T\int_\Omega \mathcal{B}(\Gamma-\Gamma_{(\epsilon)},\Lambda_\epsilon)\cdot\Gamma dt\\ &= \int_0^T\int_\Omega \sum_{i=1}^{2}(-1)^{i+1}(\Gamma-\Gamma_{(\epsilon)})_i\cdot\nabla\Lambda_{\epsilon_i}\cdot\Gamma_1\,dxdt\\ &+ \int_0^T\int_\Omega \sum_{\substack{(i,j)\le 2\\ i\neq j}}(-1)^{i+1}(\Gamma-\Gamma_{(\epsilon)})_i\cdot\nabla\Lambda_{\epsilon_j}\cdot\Gamma_2\,dxdt\\ &:= I_1+I_2,\end{aligned} \tag{6.56}$$

Next, writing I_1 and I_2 in its components (where summing over indices $m,n=1,\ldots,3$ is carried out but not written), and integrating by parts, one gets

$$\begin{aligned} I_1 &= \int_0^T\int_\Omega \sum_{i=1}^{2}(-1)^{i+1}(\Gamma-\Gamma_{(\epsilon)})_i^m\cdot\partial_m\Lambda_{\epsilon_i}^n\cdot\Gamma_1^n\,dxdt\\ &= \int_0^T\int_\Omega \sum_{i=1}^{2}(-1)^{i}(\Gamma-\Gamma_{(\epsilon)})_i^m\cdot\Lambda_{\epsilon_i}^n\cdot\partial_m\Gamma_1^n\,dxdt\\ &= \int_0^T\int_\Omega \sum_{i=1}^{2}(-1)^{i}(\Gamma-\Gamma_{(\epsilon)})_i\cdot\Lambda_{\epsilon_i}\cdot\nabla\Gamma_1\,dxdt\\ &\le \int_0^T\int_\Omega \sum_{i=1}^{2}\left|(\Gamma-\Gamma_{(\epsilon)})_i\cdot\Lambda_{\epsilon_i}\cdot\nabla\Gamma_1\right|\,dxdt,\end{aligned} \tag{6.57}$$

Likewise, doing the same for I_2 and combining everything we get

$$\begin{aligned}\int_0^T (\mathcal{B}(\Gamma-\Gamma_{(\epsilon)},\Lambda_\epsilon),\Gamma)dt &\le \int_0^T\int_\Omega \sum_{i=1}^{2}\left|(\Gamma-\Gamma_{(\epsilon)})_i\cdot\Lambda_{\epsilon_i}\cdot\nabla\Gamma_1\right|\,dxdt\\ &+ \int_0^T\int_\Omega \sum_{\substack{(i,j)\le 2\\ i\neq j}}\left|(\Gamma-\Gamma_{(\epsilon)})_i\cdot\Lambda_{\epsilon_j}\cdot\nabla\Gamma_2\right|\,dxdt \quad (6.58)\\ &:= J_1+J_2,\end{aligned}$$

Next we find suitable L^rL^s estimates for J_1 and J_2. Specifically, since Λ_ϵ is Leray-Hopf, we interpolate between $L^\infty_T L^2_x$ and $L^2_T L^6_x$, using the fact that $\nabla\Lambda_\epsilon \in L^2(0,T;L^2(\Omega))$.

Indeed, let $\frac{1}{a}+\frac{1}{q}+\frac{1}{s}=1$, $\frac{1}{a}=\frac{1}{s}-\frac{1}{3}$ and $1=\frac{1}{p}+\frac{2}{r}$. Then for some constant $C>0$, Hölder's inequality with Sobolev embedding gives (checking if each norm is well-defined later)

$$\begin{aligned} J_1 &\le \left| \int_0^T \sum_{i=1}^{2} \|(\Gamma-\Gamma_{(\epsilon)})_i\|_{L^a_x} \|\Lambda_{\epsilon_i}\|_{L^q_x} \|\nabla\Gamma_1\|_{L^b_x}\, dt \right| \\ &\le C \left| \int_0^T \sum_{i=1}^{2} \|\nabla(\Gamma-\Gamma_{(\epsilon)})_i\|_{L^s_x} \|\Lambda_{\epsilon_i}\|_{L^q_x} \|\nabla\Gamma_1\|_{L^s_x}\, dt \right| \\ &\le C \sum_{i=1}^{2} \|\nabla(\Gamma-\Gamma_{(\epsilon)})_i\|_{L^r_T L^s_x} \|\Lambda_{\epsilon_i}\|_{L^p_T L^q_x} \|\nabla\Gamma_1\|_{L^r_T L^s_x}, \end{aligned} \tag{6.59}$$

Continuing the estimate for J_1 (6.59), for $\frac{3}{2}=\frac{2}{p}+\frac{3}{q}$ with $2\le q\le 6$, convex interpolation with sobolev embedding gives

$$\begin{aligned} J_1 &\le C \sum_{i=1}^{2} \|\nabla(\Gamma-\Gamma_{(\epsilon)})_i\|_{L^r_T L^s_x} \|\nabla\Gamma_1\|_{L^r_T L^s_x} \left\| \|\Lambda_{\epsilon_i}\|_{L^2_x}^{\frac{3}{q}-\frac{1}{2}} \|\Lambda_{\epsilon_i}\|_{L^6_x}^{\frac{3}{2}-\frac{3}{q}} \right\|_{L^p_T} \\ &\le C \sum_{i=1}^{2} \|\nabla(\Gamma-\Gamma_{(\epsilon)})_i\|_{L^r_T L^s_x} \|\nabla\Gamma_1\|_{L^r_T L^s_x} \|\Lambda_{\epsilon_i}\|_{L^\infty_T L^2_x}^{\frac{3}{q}-\frac{1}{2}} \|\nabla\Lambda_{\epsilon_i}\|_{L^2_T L^2_x}^{\frac{3}{2}-\frac{3}{q}} \\ &\le C \|\nabla\Gamma-\Gamma_{(\epsilon)}\|_{L^r_T L^s_x} \|\nabla\Gamma\|_{L^r_T L^s_x} \|\Lambda_\epsilon\|_{L^\infty_T L^2_x}^{\frac{3}{q}-\frac{1}{2}} \|\nabla\Lambda_\epsilon\|_{L^2_T L^2_x}^{\frac{3}{2}-\frac{3}{q}}, \\ &\to 0, \end{aligned} \tag{6.60}$$

where the above norms are well-defined when $\frac{12}{7}<s\le\frac{12}{5}$.

Hence J_1 (and by similar argument J_2) go to 0 as $\epsilon\to 0^+$. Hence by reasoning similar to Theorem 1.1, Γ is also Leray-Hopf. Finally, it's clear that the space $L^{\frac{8s}{9s-12}}(0,T;L^s(\Omega))$ for $\frac{12}{7}<s\le\frac{12}{5}$ is covered by the cases listed in Theorem 1.2, since by the chain of inclusions of L^p spaces on compact sets, we note

$$
\begin{aligned}
&1 \leq \frac{s}{2s-3} \leq \frac{8s}{9s-12} \leq \infty \quad \text{for} \quad \frac{12}{7} < s \leq \frac{9}{5} \\
&\implies L^{\frac{8s}{9s-12}}(0,T;L^s(\Omega)) \subset L^{\frac{s}{2s-3}}(0,T;L^s(\Omega)),
\end{aligned} \tag{6.61}
$$

and

$$
\begin{aligned}
&1 \leq \frac{5s}{5s-6} \leq \frac{8s}{9s-12} \leq \infty \quad \text{for} \quad \frac{9}{5} \leq s \leq \frac{12}{5} \\
&\implies L^{\frac{8s}{9s-12}}(0,T;L^s(\Omega)) \subset L^{\frac{5s}{5s-6}}(0,T;L^s(\Omega)),
\end{aligned} \tag{6.62}
$$

where (6.61) and (6.62) reduces down to conditions (B1*) and (B2*) of Theorem 1.2, finishing the proof. □

Here one also notes that the same result of Theorem 1.3 holds for the incompressible Navier-Stokes system (2.2), where the operator $\mathcal{B}$ changes to $\mathcal{B}(u) := \int_\Omega u \cdot \nabla u \, dx$. We omit the proof since it is almost identical to the above.

Chapter 7

Conclusion and Future Work

In the preceding thesis, we have shown a variety of regularity results on weak solutions to the incompressible MHD Cauchy problem, generalizing previous results only known for the incompressible Navier-Stokes equations. Of specific, we first presented by Theorem 1.1, a natural analogue of Berselli and Chiodaroli's result for the Navier-Stokes equations [5] to the MHD equations. That is, it was shown that weak solutions in the distributional sense to the MHD Cauchy problem (2.12) with additional regularity in $L^r(0,T;L^s(\mathbb{R}^3))$ with $\frac{2}{r}+\frac{2}{s}=1$ for $s\geq 4$, satisfies the corresponding energy equality (2.15). This was proven using three crucial steps, beginning by showing MHD weak solutions in $L^4(0,T;L^4(\mathbb{R}^3))$ are Leray-Hopf, then that MHD Leray-Hopf weak solutions in the same space satisfy the energy equality, and finally by applying a bootstrapping argument on weak solutions to extend the allowed space of integrability where the equality is satisfied (with other pertinent lemma's used including an existence or Galerkin method for solutions). Of next results presented was Theorem 1.2, which in contrast to Theorem 1.1 where regularity criteria was imposed on the solution itself, out of sheer curiosity, and an analogue to Berselli and Chiodaroli's result for the Navier-Stokes Cauchy problem, integrability conditions were imposed on the gradient of the solution to the MHD equations instead. In particular, it was first shown (Theorem 1.2) that Leray-Hopf weak solutions to the MHD Cauchy problem with additional integrability conditions imposed on its gradient, satisfy the corresponding energy equality (2.15). With Theorem 1.1 in mind, we finished the thesis with Theorem 1.3, attempting to drop the Leray-Hopf condition, such that only sufficient regularity criteria on the gradient of a weak solution is sufficient to satisfy the energy equality.

A variety of future work on the above theorem's can be considered, first and foremost with extending the integrability condition of Theorem 1.3 to match that of Theorem 1.2. Specifically, since Theorem 1.1 generalized Berselli and Chiodaroli's result of the Navier-Stokes equations to the MHD equations without weakening the prescribed regularity condition, it seems expected (however unknown) that the same should hold true for regularity criteria on the gradient of the solution when dropping the Leray-Hopf condition. Of additional interest is extending the space of regularity from Theorem 1.1 of weak solutions (to both the Navier-Stokes and MHD equations) outside $L^r(0,T;L^s(\mathbb{R}^3))$ with $\frac{2}{r}+\frac{2}{s}=1$ for $s\geq 4$. Specifically, it was shown by Beirao da Veiga and Yang [13] that Leray-Hopf weak solutions to the incompressible Navier-Stokes equations that also lie in $L^r(0,T;L^s(\mathbb{R}^3))$ with $\frac{1}{r}+\frac{3}{s}=1$ for $3\leq s\leq 4$, satisfy the energy equality. It is currently unknown if this is true for both the Navier-Stokes and MHD equations if we drop the Leray-Hopf condition.

Bibliography

[1] Herbert Amann. Linear and quasilinear parabolic problems. vol. i, volume 89 of monographs in mathematics, 1995.

[2] Hugo Beirao da Veiga et al. A new regularity class for the Navier-Stokes equations in $\mathbb{R}^n$. *Chinese Annals of Mathematics Series B*, 16:1–6, 1995.

[3] Luigi Berselli and Giovanni Galdi. Regularity criteria involving the pressure for the weak solutions to the Navier-Stokes equations. *Proceedings of the American Mathematical Society*, 130(12):3585–3595, 2002.

[4] Luigi C Berselli. Some criteria concerning the vorticity and the problem of global regularity for the 3D Navier-Stokes equations. *Annali Dell'Universita'Di Ferrara*, 55(2):209–224, 2009.

[5] Luigi C Berselli and Elisabetta Chiodaroli. On the energy equality for the 3D Navier-Stokes equations. *Nonlinear Analysis*, 192:111704, 2020.

[6] Clayton Bjorland and Alexis Vasseur. Weak in space, log in time improvement of the Ladyženskaja–Prodi–Serrin criteria. *Journal of Mathematical Fluid Mechanics*, 13:259–269, 2011.

[7] Stefano Bosia, Monica Conti, and Vittorino Pata. A regularity criterion for the Navier-Stokes equations in terms of the pressure gradient. *Open Mathematics*, 12(7):1015–1025, 2014.

[8] Chongsheng Cao and Edriss S Titi. Regularity criteria for the three-dimensional Navier-Stokes equations. *Indiana University Mathematics Journal*, pages 2643–2661, 2008.

[9] Chongsheng Cao and Edriss S Titi. Global regularity criterion for the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor. *Archive for rational mechanics and analysis*, 202:919–932, 2011.

[10] Dongho Chae and Jihoon Lee. Regularity criterion in terms of pressure for the Navier-Stokes equations. 1999.

[11] Chi Hin Chan. Smoothness criterion for Navier-Stokes equations in terms of regularity along the streamlines. 2010.

[12] Chi Hin Chan and Alexis Vasseur. Log improvement of the Prodi-Serrin criteria for Navier-Stokes equations. *arXiv preprint arXiv:0705.3659*, 2007.

[13] Hugo Beirao da Veiga and Jiaqi Yang. On the Shinbrot's criteria for energy equality to newtonian fluids: a simplified proof, and an extension of the range of application. *Nonlinear Analysis*, 196:111809, 2020.

[14] Georges Duvaut and Jacques-Louis Lions. Inéquations en thermoélasticité et magnétohydrodynamique. *Archive for Rational Mechanics and Analysis*, 46:241–279, 1972.

[15] Luis Escauriaza, Gregory Serëgin, and Vladimir Sverak. On L^3, Infinity-solutions to the Navier-Stokes equations and backward uniqueness. 2002.

[16] Lawrence C Evans. *Partial differential equations*, volume 19. American Mathematical Society, 2022.

[17] Jishan Fan and Tohru Ozawa. Regularity criterion for weak solutions to the Navier-Stokes equations in terms of the gradient of the pressure. *Journal of Inequalities and Applications*, 2008:1–6, 2008.

[18] Giovanni Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems*. Springer Science & Business Media, 2011.

[19] Giovanni Galdi. On the energy equality for distributional solutions to Navier-Stokes equations. *Proceedings of the American Mathematical Society*, 147(2):785–792, 2019.

[20] Giovanni P Galdi. An introduction to the Navier-Stokes initial-boundary value problem. In *Fundamental directions in mathematical fluid mechanics*, pages 1–70. Springer, 2000.

[21] Yoshikazu Giga. Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. *Journal of differential equations*, 62(2):186–212, 1986.

[22] Yoshikazu Giga and Hermann Sohr. Abstract lp estimates for the cauchy problem with applications to the navier-stokes equations in exterior domains (evolution equations and applications to nonlinear problems). , 730:120–127, 1990.

[23] Bin Han, Zhen Lei, Dong Li, and Na Zhao. Sharp one component regularity for Navier-Stokes. *Archive for Rational Mechanics and Analysis*, 231:939–970, 2019.

[24] Eberhard Hopf. Über die anfangswertaufgabe für die hydrodynamischen grundgleichungen. erhard schmidt zu seinem 75. geburtstag gewidmet. *Mathematische Nachrichten*, 4(1-6):213–231, 1950.

[25] John K Hunter. An introduction to the incompressible Euler equations. *Notes, Univ. of California, Davis*, 1:19, 2006.

[26] Mohsen Sheikholeslami Kandelousi, Sadia Ameen, M. Shaheer Akhtar, and Hyung-Shik Shin. *Nanofluid Flow in Porous Media*. IntechOpen, Rijeka, Aug 2020.

[27] Hideo Kozono and Yasushi Taniuchi. Bilinear estimates in BMO and the Navier-Stokes equations. *Mathematische Zeitschrift*, 235:173–194, 2000.

[28] Baishun Lai and Ge Tang. The regularity of very weak solutions to magneto-hydrodynamics equations. *Journal of Mathematical Fluid Mechanics*, 25(3):57, 2023.

[29] Baishun Lai and Yifan Yang. On the energy equality for very weak solutions to 3D MHD equations. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 152(6):1565–1588, 2022.

[30] Pierre Gilles Lemarié-Rieusset. Recent developments in the Navier-Stokes problem. 2002.

[31] Jean Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta mathematica*, 63:193–248, 1934.

[32] Jean Leray and Robert Terrell. On the motion of a viscous liquid filling space (a translation to english of the 1934 paper of leray, sur le mouvement d'un liquide visqueux emplissant l'espace.). *arXiv preprint arXiv:1604.02484*, 2016.

[33] JL Lions. Sur la régularité et l'unicité des solutions turbulentes des équations de Navier Stokes. *Rendiconti del Seminario Matematico della Universita di Padova*, 30:16–23, 1960.

[34] Jiří Neustupa and Patrick Penel. Regularity of a suitable weak solution to the Navier-Stokes equations as a consequence of regularity of one velocity component. *Applied nonlinear analysis*, pages 391–402, 2002.

[35] Louis Nirenberg. On elliptic partial differential equations. *Annali della Scuola Normale Superiore di Pisa-Scienze Fisiche e Matematiche*, 13(2):115–162, 1959.

[36] Manuel Núñez. Regularity criteria for the Navier-Stokes equations involving the ratio pressure-gradient of velocity. *Mathematical methods in the applied sciences*, 33(3):323–331, 2010.

[37] Giovanni Prodi. Un teorema di unicita per le equazioni di Navier-Stokes. *Annali di Matematica pura ed applicata*, 48:173–182, 1959.

[38] Chen Qionglei and Zhang Zhifei. Regularity criterion via the pressure on weak solutions to the 3D Navier-Stokes equations. *Proceedings of the American Mathematical Society*, 135(6):1829–1837, 2007.

[39] James C Robinson, José L Rodrigo, and Witold Sadowski. *The three-dimensional Navier–Stokes equations: Classical theory*, volume 157. Cambridge university press, 2016.

[40] Gregory Seregin. *Lecture notes on regularity theory for the Navier-Stokes equations*. World Scientific, 2014.

[41] James Serrin. *On the interior regularity of weak solutions of the Navier-Stokes equations*. Mathematics Division, Air Force Office of Scientific Research, 1961.

[42] Marvin Shinbrot. The energy equation for the Navier-Stokes system. *SIAM Journal on Mathematical Analysis*, 5(6):948–954, 1974.

[43] Hermann Sohr. Zur regularitätstheorie der instationären gleichungen von Navier-Stokes. *Mathematische Zeitschrift*, 184(3):359–375, 1983.

[44] Hermann Sohr. A regularity class for the Navier-Stokes equations in lorentz spaces. *Journal of Evolution Equations*, 1:441–467, 2001.

[45] Hermann Sohr. *The Navier-Stokes equations: An elementary functional analytic approach*. Springer Science & Business Media, 2012.

[46] Vsevolod Alekseevich Solonnikov. Estimates of solutions of an initial-and boundary-value problem for the linear nonstationary Navier-Stokes system. *Zapiski Nauchnykh Seminarov POMI*, 59:178–254, 1976.

[47] Terrance Tao. 254a, notes 0: Physical derivation of the incompressible Euler and Navier-Stokes equations, Jan 2019.

[48] Chuong V Tran and Xinwei Yu. Note on Prodi-Serrin-Ladyzhenskaya type regularity criteria for the Navier-Stokes equations. *Journal of Mathematical Physics*, 58(1):011501, 2017.

[49] Chuong V Tran and Xinwei Yu. Regularity of Navier-Stokes flows with bounds for the pressure. *Applied Mathematics Letters*, 67:21–27, 2017.

[50] Alexis Vasseur. Regularity criterion for 3d Navier-Stokes equations in terms of the direction of the velocity. *arXiv preprint arXiv:0705.2446*, 2007.

[51] Yong Zeng. Note on energy equality of MHD system. *Zeitschrift für angewandte Mathematik und Physik*, 73:1–11, 2022.

Appendix A (Auxiliary Lemmas)

Theorem A.1. (Aubin-Lions Lemma in L^2) [39] Let $\Omega \subset \mathbb{R}^3$ be smooth and bounded and $H(\Omega)$ a Hilbert Space. Then for $R > 1$ such that

$$\|u_n\|_{L^2(0,T;H^1_{0,\sigma}(\mathbb{B}_R))} + \|\partial_t u_n\|_{L^2(0,T;(H^1_{0,\sigma}(\mathbb{B}_R))^*)} \leq C \quad \text{for all } n \in \mathbb{N}, \tag{7.1}$$

There exists $u \in L^2(0,T;H(\Omega))$ and a subsequence $\{u_{n_i}\}_{i\in\mathbb{N}}$ of u_n such that

$$u_{n_i} \to u \quad \text{strongly in} \quad L^2(0,T;H). \tag{7.2}$$

Here we prove a necessary approximation result required for the proof of Theorem 4.2. Specifically equation (4.9). The proof is given by Lai and Yang [29] and we present it here for completeness.

Theorem A.2. Let $\Phi \in H^1(\mathbb{R}^3) \cap \mathrm{L}^2_\sigma(\mathbb{R}^3)$. Then there exists a sequence $\{\Phi_R\}_{R>0} \subset H^1_0(\mathbb{B}_R) \cap L^2_\sigma(\mathbb{B}_R)$ such that

$$\lim_{R\to\infty} \|\Phi - \Phi_R\|_{H^1(\mathbb{R}^3)} = 0, \tag{7.3}$$

Proof. Let $\Phi \in H^1(\mathbb{R}^3) \cap \mathrm{L}^2_\sigma(\mathbb{R}^3)$ and $\phi \in C^1(\mathbb{R})$ be a cut off function defined such that

$$\phi = \begin{cases} 1 & \text{if} \quad |\phi(x)| \leq 1, \\ 0 & \text{if} \quad |\phi(x)| \geq 2, \end{cases} \tag{7.4}$$

with $\phi^R(x) = \phi(\frac{|x|}{R})$.

Denote by $B := \{x : R < |x| < 2R\}$ the open annulus bounded radially by R and $2R$.

We consider the system

$$\nabla w_R = -v \cdot \nabla \psi^R, \tag{7.5}$$

By Galdi [18] there exists a unique solution $w_R \in H^1_0(B)$ with suitable initial data to (7.5) satisfying the estimate (7.6)

$$\|\nabla w_R\|_{L^2(B)} \leq c \|v \cdot \nabla \psi^R\|_{L^2(B)}, \tag{7.6}$$

for some constant $c > 0$ independent of the radius R.

Next by Poincare's inequality and the estimate (7.6), one gets

$$\|\nabla w_R\|_{L^2(\mathbb{B}_{R,2R})} \leq C_1 R \|\nabla w_R\|_{L^2(\mathbb{B}_{R,2R})} \leq C_2 \|v\|_{L^2(\mathbb{B}_{R,2R})}, \tag{7.7}$$

where $C_1, C_2 > 0$ are constants.

Extending w_R to $\mathbb{R}^3$ such that $w_R := 0$ in $\mathbb{R}^3 \backslash B$ and defining

$$\overline{v} := \psi^R v + w_R, \tag{7.8}$$

one can show through a standard density argument that $\overline{v} \in H^1_{0,\sigma}(\mathbb{B}_{2R})$, and thus for $\epsilon \geq 0$ arbitrary, there exists a sequence $\{\overline{v}_\epsilon\}_{\epsilon \geq 0} \subset C^\infty_{0,\sigma}(\mathbb{B}_{2R})$ such that

$$\|\overline{v} - \overline{v}_\epsilon\|_{H^1_0(\mathbb{B}_{2R})} < \epsilon, \tag{7.9}$$

Choosing $v_R := \overline{v}_\epsilon$ we estimate the following

$$\|v - v_R\|_{H^1(\mathbb{R}^3)} \leq \|v - \overline{v}_\epsilon\|_{L^2(\mathbb{R}^3)} + \|\nabla v - \nabla \overline{v}_\epsilon\|_{L^2(\mathbb{R}^3)}, \tag{7.10}$$

Evaluating the first term on the RHS of (7.10) one notes by the triangle inequality and the definition of $\overline{v}$ that

$$\begin{aligned} \|v - \overline{v}_\epsilon\|_{L^2(\mathbb{R}^3)} &\leq \|\overline{v}_\epsilon - \overline{v}\|_{L^2(\mathbb{R}^3)} + \|v - \overline{v}\|_{L^2(\mathbb{R}^3)} \\ &< \epsilon + \|(1 - \psi^R) v\|_{L^2(\mathbb{R}^3)} + \|w_R\|_{L^2(B)}, \end{aligned} \tag{7.11}$$

where by the construction of ψ^R, taking $R \to \infty$ and $\epsilon \to 0$, one obtains

$$\|v - \overline{v}_\epsilon\|_{L^2(\mathbb{R}^3)} \to 0, \tag{7.12}$$

In a similar fashion, the second term on the RHS of (7.10) is estimated by (using the product rule when evaluating $\nabla(\psi^R v)$)

$$\begin{aligned}\|\nabla v - \nabla \overline{v}_\epsilon\|_{L^2(\mathbb{R}^3)} &\leq \|\nabla \overline{v}_\epsilon - \nabla \overline{v}\|_{L^2(\mathbb{R}^3)} + \|\nabla v - \nabla \overline{v}\|_{L^2(\mathbb{R}^3)} \\ &< \epsilon + \|(1-\psi^R)\nabla v\|_{L^2(\mathbb{R}^3)} + \|\nabla \psi^R v\|_{L^2(\mathbb{R}^3)} + \|\nabla w_R\|_{L^2(B)},\end{aligned} \tag{7.13}$$

giving similarly as $R \to \infty$ and $\epsilon \to 0$

$$\|\nabla v - \nabla \overline{v}_\epsilon\|_{L^2(\mathbb{R}^3)} \to 0, \tag{7.14}$$

Hence combining (7.11) and (7.13) with (7.10) one achieves the desired result. □