University of Alberta

Orbital Influence of the Oscillations Excited Inside a Rotating Star in a Binary System

by

Claudine Couture ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Master of Science

Department of Physics

Edmonton, Alberta

Fall 2002

0-612-81385-1

University of Alberta

Library Release Form

Name of author:	Claudine Couture
Title of thesis:	Orbital Influence of the Oscillations Excited Inside a Rotating Star in a Binary System
Degree:	Master of Science
Year this Degree Granted:	2002

Claudine Couture

Claudine Couture
Department of Physics
University of Alberta
Edmonton, Alberta
Canada

August 28th, 2002

University of Alberta

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Orbital Influence of the Oscillations Excited Inside a Rotating Star in a Binary System** submitted by **Claudine Couture** in partial fulfillment of the requirements for the degree of Master of Science.

Sharon Morsink (Supervisor)

Eric Woolgar (External)

Richard Sydora

Moritz Heimpel

DATE: Aug 26, 2002

... le présent est indéfini, le futur n'a de réalité qu'en tant qu'espoir présent, le passé n'a de réalité qu'en tant que souvenir présent...

Jorge Luis Borges

Abstract

Tidal interactions can modify the orbital evolution of a binary system of neutron stars. The equation of motion for the amplitudes of the oscillations is easily obtained for non-rotating stars. For rotating stars, there is a complication in the obtention of this equation. Previous authors have used a configuration space expansion of the modes, but the amplitudes are then coupled together. In this work, I use a phase space expansion to get the equation describing the evolution of the amplitudes, and the amplitudes are no longer coupled. I show that the use of the configuration space expansion introduces non-negligible errors in the orbital evolution of a binary system of neutron stars.

Acknowledgements

I would like to thank all the people who helped me to complete this work, either directly or indirectly. A special thank goes to my supervisor, Sharon Morsink, for her precious advice, her help in demystifying the aspects of this field with which I was not familiar, and for her support. I have to thank my friend Stéphane for the endless hours of discussion about some of the mathematical subjects involved in this project. I also want to thank NSERC for their financial support.

Table of Contents

List of Figures

Chapter 1

Introduction

The possibility to detect gravitational waves in the near future with a new generation of detectors, like the interferometer LIGO, opens up a whole new branch of research in astrophysics. Since the signal-to-noise ratio for these detectors is weak, it is very important to have the best knowledge possible of the waveform emitted by the sources we want to study in order to be able to extract the signal from the noise. The needed accuracy on the phase of the theoretical template is $< 0.01\%$ [31]. We have to pay particular attention to the phase of the waves while constructing the templates.

The design of LIGO, which is presently scheduled to take its first runs of data late in 2002, is optimized to detect the gravitational waves emitted by an inspiraling binary system of neutron stars. It has its peak sensitivity for frequencies between 10Hz and 1000Hz [32], a frequency range that is swept by binary neutron stars during the final stages of their coalescence. We then have to consider all of what could possibly alter the evolution of this type of system.

One of the effects that could modify the orbital evolution of a binary system of neutron stars is the internal structure of the stars. The gravitational waves emitted by the internal structure itself are negligible ([6],[21], [22], and [33]).

However, the tidal interactions between the stars can produce a detectable phase difference between the waveform from a binary model of stars with an internal structure and the one from a binary system of point masses [17].

Many authors have worked on the oscillations excited inside non-rotating neutron stars. Kochanek gives in [17] a complete review of the behavior of coalescing neutron stars just before merger. He studies the internal structure of the neutron stars, the impact of viscosity and the effect on the emitted gravitational waves. Excitation of modes in a non-rotating neutron star during its inspiral have been studied in [27]. Resonant tidal effects in coalescing binaries have been studied for non-rotating stars by many authors (see [30], [2], and [18]).

However, the stars are not always non-rotating. Pulsars are believed to be rotating neutron stars. The study of tidal interaction is more complicated because a particular attention has to be paid to the modes decomposition. Witte and Savonije [36] have studied the tides in a rotating main sequence star. For rotating neutron stars, I have to mention the work of Lai and Ho, [19] and [15]. In these articles, they study the excitation of oscillations inside rotating neutron stars and their resonances and look at their impact on the orbital evolution of the system. However, they use an approximation in their formalism that would be valid only for slowly rotating stars.

What I propose here is to use a formalism proposed by Schenk et al. [28] to study the impact of tidal interactions on the orbital motion of a coalescing binary system of neutron stars and compare the results to those obtained with the formalism used by Ho and Lai.

I model the neutron star as a fluid with no viscosity and constant density throughout the star. Even though this is not a realistic model, it makes the calculations easier and allows us to compare the results obtained with the two formalisms. I consider a binary system on a circular orbit since the emission of gravitational waves circularizes the orbit [29].

The presentation of the material is as follows.

In chapter 2, I present the description of non-radial oscillations inside a rotating star, starting with the basic fluid equations. The oscillations are introduced as perturbations of the equilibrium solution. Then, the equations describing the evolution of their amplitudes is given for the two formalisms, as well as the eigenvectors.

Chapter 3 gives all the background material necessary to describe the orbital motion of the binary system. This includes the tidal potential, the Lagrangian equations of motion, the loss of energy by the emission of gravitational radiation, and the possibility to have resonances.

The numerical methods are presented in chapter 4, along with the description of the assumptions made and the parameters used. It is followed by the presentation of the results.

Finally, in chapter 5, a summary of the work is given.

Chapter 2

Non-Radial Oscillations

In this chapter, I introduce the equation of motion for the oscillations inside an incompressible star. I first present the basic equations describing the fluid motion. Then, I give a summary of the properties of Lagrangian perturbations and use them to derive the equation of motion for driven and free oscillations. Following this, I present the expansion on the eigenmodes that should be used in the case of non-rotating stars, and the one appropriate for rotating stars.

I assume that the star can be described by Newtonian mechanics, is incompressible and is made of a one component fluid. The equations I present are valid in the inertial frame.

2.1 General Equations

2.1.1 Basic Fluid Equations

I summarize here the basic fluid equations needed later in this chapter.

The continuity equation expresses the conservation of mass locally [14]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{2.1}$$

where ρ is the mass density and $\mathbf{v}$ is the fluid velocity while the Navier-Stokes equation, describing the motion of a Newtonian fluid, is, if we neglect the viscosity, the electrostatic force, and the magnetic force [14]

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho}\nabla P - \nabla\Phi. \tag{2.2}$$

In this case, it is called *Euler's equation.* Here, P is the fluid pressure, Φ is the gravitational potential, and $\frac{d}{dt}$ is the total time derivative following a fluid element [29]

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla. \tag{2.3}$$

In Euler's equation (2.2), if we put $\frac{\partial\vec{v}}{\partial t} = 0$, we get the hydrostatic equilibrium equation

$$\frac{1}{2}\rho\nabla\left(\vec{v}\cdot\vec{v}\right) = -\nabla P - \rho\nabla\Phi. \tag{2.4}$$

The gravitational potential Φ is given by the well known Poisson equation :

$$\nabla^2\Phi = 4\pi G\rho \tag{2.5}$$

where G is the gravitational constant.

2.1.2 Solution for a Non-Rotating Constant Density Spherical Star

We can easily find a solution to equations (2.4) and (2.5) for a non-rotating constant density spherical star. In this case, we have $v = 0$, $P = P(r)$, and $\Phi = \Phi(r)$. This leads to

$$\nabla\Phi = \frac{4\pi r\rho G}{3} \tag{2.6}$$

and

$$\nabla P = -\rho\nabla\Phi = -\frac{4\pi r\rho^2 G}{3}. \tag{2.7}$$

Since the star is spherical and has a constant density, we can define the interior mass, M_r, as the mass contained inside a sphere of radius r, with r smaller than the radius of the star (R),

$$M_r = \frac{4\pi r^3 \rho}{3}. \tag{2.8}$$

Using M_r, we have the following equation

$$\frac{dP}{dr} = -\frac{GM_r \rho}{r^2}. \tag{2.9}$$

The surface of the star is defined as the area where the gas pressure becomes zero. The boundary condition for this problem is then

$$P \xrightarrow{r \to R} 0. \tag{2.10}$$

Using the boundary condition (2.10), equation(2.9) has the solution

$$P(r) = \frac{4\pi G \rho^2}{6} \left(R^2 - r^2\right). \tag{2.11}$$

2.1.3 Lagrangian and Eulerian Perturbations

To derive the equation of motion for the oscillations inside a star that I am using, one needs to use the linear perturbation theory. This means that the considered perturbations are small enough to neglect the terms in which they appear at the second order or at a higher order. There are two types of perturbations that are generally used in the linear theory. The Lagrangian perturbations and the Eulerian perturbations. The material I present in the section can be found in [10] and in [29].

The Eulerian perturbations represent a macroscopic point of view. It consists of looking at one point of the flow in the unperturbed solution, and then looking at the same point in the perturbed solution. If we consider the physical quantity Q, its Eulerian perturbation is then

$$\delta Q \equiv Q(\vec{x}, t) - Q_0(\vec{x}, t) \tag{2.12}$$

where the 0 subscript indicates the unperturbed solution.

On the other hand, the Lagrangian perturbations represent a more microscopic point of view. Instead of always looking at the same point, we look at the same fluid element in the unperturbed and perturbed solutions. To do so, we need to know the new position of this element. We define the Lagrangian displacement which gives the difference in the position of the element between the two solutions as

$$\vec{\xi}\,(\vec{x},t) = \vec{x}\,(t) - \vec{x}_0\,(t)\,. \tag{2.13}$$

The Lagrangian perturbation of Q is then given by

$$\Delta Q \equiv Q\left[\vec{x} + \vec{\xi}\,(\vec{x},t)\,,t\right] - Q_0\,(\vec{x},t)\,. \tag{2.14}$$

By comparing equations (2.12) and (2.14), we can see the following first order relation between the two types of perturbations

$$\Delta = \delta + \vec{\xi}\cdot\nabla. \tag{2.15}$$

Also note that the Lagrangian perturbation of the fluid velocity is a simple function of ξ

$$\Delta\vec{v} = \frac{d\vec{\xi}}{dt}. \tag{2.16}$$

There are different commutation relations between the two types of perturbations. Two of them are necessary for the derivations presented later in this chapter. They are

$$\Delta\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}\Delta - \frac{\partial \xi^j}{\partial x^i}\nabla_j \tag{2.17}$$

and

$$\Delta\frac{d}{dt} = \frac{d}{dt}\Delta. \tag{2.18}$$

Considering the perturbation of the integral giving the mass in an arbitrary volume of fluid, and knowing that this mass is conserved, it is possible to derive the following equations for the Lagrangian and Eulerian perturbations of the mass density ρ :

$$\Delta\rho = -\rho\nabla\cdot\vec{\xi} \tag{2.19}$$

$$\delta\rho = -\nabla\cdot\left(\rho\vec{\xi}\right). \tag{2.20}$$

I am going to consider only adiabatic perturbations. This means that ΔP and $\Delta\rho$ are related through the equation

$$\frac{\Delta P}{P} = \Gamma_1\frac{\Delta\rho}{\rho} \tag{2.21}$$

where Γ_1 is called the adiabatic exponent

$$\Gamma_1 \equiv \left(\frac{\partial\ln P}{\partial\ln\rho}\right)_s, \tag{2.22}$$

which is evaluated at constant entropy.

Using the previous equations, it is possible to derive a relation describing the fluid inside an incompressible star. Since it is incompressible, it should have a constant density. Then from equation (2.19),

$$\Delta\rho = 0 \quad\Rightarrow\quad \nabla\cdot\vec{\xi} = 0 \tag{2.23}$$

inside the star. Substituting equation (2.19) in equation (2.21), and considering that $\Delta P = 0$ at the surface of the star, we get that

$$\nabla\cdot\vec{\xi} = 0 \tag{2.24}$$

everywhere inside and outside the star.

2.1.4 Perturbation About Equilibrium

We are now ready to derive the equation of motion for the oscillations inside the star. It is done by considering the perturbation of the star about its equilibrium configuration. I am considering a star of constant density that is rotating about the z-axis. For this part, I mainly follow what is done in [29]. We begin with the perturbation of Euler's equation (2.2), which is

$$\Delta\left(\frac{dv^i}{dt}+\frac{1}{\rho}\nabla_i P+\nabla_i\Phi\right)=0. \tag{2.25}$$

Using the previous commutation relations (2.17) and (2.18), the hydrostatic equilibrium equation (2.4) and (2.16), we get

$$\rho_0\frac{d^2\xi^i}{dt^2}-\frac{\Delta\rho}{\rho_0}\nabla_i P+\nabla_i\Delta P_0+\rho_0\nabla_i\Delta\Phi=0. \tag{2.26}$$

Then, using (2.3), (2.4), (2.15), (2.19), and (2.21) we get

$$\rho_0\frac{\partial^2\xi_i}{\partial t^2}+2\rho_0\left(\mathbf{v_0}\cdot\nabla\right)\frac{\partial\xi_i}{\partial t}+\rho_0 v_0^k\nabla_k\left(v_0^j\nabla_j\xi_i\right)-L_{ij}\xi^j=0 \tag{2.27}$$

where

$$L_{ij}\xi^j\equiv\nabla_i\left(\Gamma_1 P_0\nabla_j\xi^j\right)-\left(\nabla_j\xi^j\right)\nabla_i P_0+\left(\nabla_i\xi^j\right)\nabla_j P_0-\rho_0\xi^j\nabla_j\nabla_i\Phi_0-\rho_0\nabla_i\delta\Phi \tag{2.28}$$

with

$$\nabla_i\delta\Phi=-4\pi G\rho_0\xi^i. \tag{2.29}$$

We can also write the last equation in the following form using operators

$$\frac{\partial^2\vec{\xi}}{\partial t^2}+\mathbf{B}\cdot\frac{\partial\vec{\xi}}{\partial t}+\mathbf{C}\cdot\vec{\xi}=0 \tag{2.30}$$

if we make the identifications

$$\mathbf{B}=2\left(\vec{v}_0\cdot\nabla\right) \tag{2.31}$$

$$\left(\mathbf{C}\cdot\vec{\xi}\right)_i=\left(\vec{v}_0\cdot\nabla\right)^2\xi_i-\frac{L_{ij}\xi^j}{\rho_0}. \tag{2.32}$$

It is possible to prove that **B** is an anti-hermitian operator while **C** is hermitian with respect to the following inner product between two eigenvectors [28]

$$\langle\vec{\xi},\vec{\xi}'\rangle=\int d^3x\rho\left(\vec{x}\right)\vec{\xi}^*\left(\vec{x}\right)\cdot\vec{\xi}'\left(\vec{x}\right). \tag{2.33}$$

Equation (2.30) describes the motion of free oscillations inside the star and has many different solutions, called the different eigenmodes of oscillation, identified by the subscript α. In fact, it represents a set of indices, $\alpha = \{njk\}$ where n gives the number of nodes in the r-direction, j the number of nodes in the θ-direction and k is the azimuthal index. We can separate the time part of $\vec{\xi_\alpha}(\vec{r}, t)$ in the following way

$$\vec{\xi_\alpha}(\vec{x}, t) \equiv \vec{\xi_\alpha}(\vec{x})\, e^{i\omega_\alpha t} \propto e^{i\omega_\alpha t + ik\phi}. \tag{2.34}$$

The equation of motion describing the eigenmodes then becomes

$$-\omega_\alpha^2 \vec{\xi_\alpha} + 2i\omega_\alpha (\vec{v_0} \cdot \nabla)\, \vec{\xi_\alpha} + \mathbf{C} \cdot \vec{\xi_\alpha} = 0. \tag{2.35}$$

Oscillations inside the star can also be driven by an external force. We now have a general displacement vector $\vec{\xi}$. The equation of motion for $\vec{\xi}(\vec{r}, t)$ is then

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} + \mathbf{B} \cdot \frac{\partial \vec{\xi}}{\partial t} + \mathbf{C} \cdot \vec{\xi} = \vec{a}_{ext} \tag{2.36}$$

with $\vec{a}_{ext}$ being the acceleration caused by the external force. This acceleration can be given by a potential U as

$$\vec{a}_{ext} = -\nabla U. \tag{2.37}$$

2.2 Non-Rotating Stars

The equations of motion for the mode amplitudes are different for rotating and non-rotating stars, because the orthogonality condition for the eigenvectors is not the same in both cases. In this section, I present the case of non-rotating stars. The equations for rotating stars are studied in the next section.

When the star is not rotating, the fluid is not moving in the unperturbed configuration. This means $\mathbf{v}_0 = 0$. Equation (2.36) becomes simpler since $\mathbf{B}$ is then zero. This means that, when there is no external acceleration,

$$\mathbf{C} \cdot \vec{\xi}_{\alpha} = \omega_{\alpha}^{2} \vec{\xi}_{\alpha}. \tag{2.38}$$

Since **C** is a hermitian operator, two eigenmodes corresponding to two different eigenvalues are orthogonal [7]:

$$\langle \vec{\xi}_{\alpha}, \vec{\xi}_{\beta} \rangle = A \delta_{\alpha\beta} \tag{2.39}$$

where A is a constant. It is also always possible to form a basis with the eigenvectors of a hermitian operator. We can then express a general displacement vector as a linear superposition of eigenmodes, a **configuration space expansion**

$$\vec{\xi}(\vec{x}, t) = \sum_{\alpha} a_{\alpha}(t) \vec{\xi}_{\alpha}(\vec{x}). \tag{2.40}$$

I chose the proportionality constant A in the orthogonality relation to be $A = MR^2$, with M the mass of the star and R its radius, so that the eigenmodes have dimension of length and the mode amplitudes a_α are dimensionless. With this normalization constant, $a_\alpha = 1$ corresponds to a displacement of the order of the size of the star. The equation of motion for the mode amplitudes can be found by substituting the expansion (2.40) into (2.36) and taking the inner product with an eigenvector $\vec{\xi}_\beta$. The final result is a second order differential equation for the amplitude, [28]

$$\ddot{a}_{\alpha}(t) + \omega_{\alpha}^{2} a_{\alpha}(t) = \langle \vec{\xi}_{\alpha}, \vec{a}_{ext}(t) \rangle. \tag{2.41}$$

2.3 Rotating Stars

The case of rotating stars is not as simple as the previous one. When the star is rotating, it is incorrect to assume that the eigenvectors satisfy a simple orthogonality condition like (2.39). In this section, I am going to derive the equations using the correct orthogonality condition and show the difference with what is generally done.

Here, I am going to suppose that the frequencies are real (the system is stable) and that the modes are not degenerate.

2.3.1 Slow Rotation Approximation

Since the star is rotating, $\mathbf{v}_0 \neq 0$. We now have an equation of the form

$$\mathbf{H} \cdot \vec{\xi}_\alpha = 0 \tag{2.42}$$

with

$$\mathbf{H} = -\omega_\alpha^2 + i\omega_\alpha \mathbf{B} + \mathbf{C}. \tag{2.43}$$

Even though $\mathbf{H}$ is hermitian, two eigenvectors $\vec{\xi}_\alpha$ with different eigenvalues ω_α are eigenvectors of different operators since $\mathbf{H}$ depends on ω_α. This means that eigenvectors do not need to be orthogonal [28]. It would still be possible to find a set of eigenvectors forming a basis [28], but another problem arises if we try to expand a general displacement vector $\vec{\xi}$ as in (2.40), what we call an expansion in the configuration space. If we tried to do so, the modes would generally be coupled to each other in the following way:

$$\sum_\alpha \Big(\left(\ddot{a}_\alpha + \omega_\alpha^2 a_\alpha\right) \langle \vec{\xi}_\beta, \vec{\xi}_\alpha \rangle - i \left(\dot{a}_\alpha - i\omega_\alpha a_\alpha\right) \langle \vec{\xi}_\beta, i\mathbf{B} \cdot \vec{\xi}_\alpha \rangle \Big) = \langle \vec{\xi}_\beta, \vec{a}_{ext} \rangle. \tag{2.44}$$

To circumvent this obstacle, previous authors (see for example [15] and [19]) have generally assumed that the previous orthogonality condition (2.39) holds (which would be a good approximation for a slowly rotating star) and that

$$\langle \vec{\xi}_\beta, i\mathbf{B}\vec{\xi}_\alpha \rangle = \langle \vec{\xi}_\alpha, i\mathbf{B}\vec{\xi}_\alpha \rangle \delta_{\alpha\beta} \tag{2.45}$$

so that they get the following equation of motion (using $A = MR^2$):

$$\left(\ddot{a}_\alpha + \omega_\alpha^2 a_\alpha\right) MR^2 - i \left(\dot{a}_\alpha - i\omega_\alpha a_\alpha\right) \langle \vec{\xi}_\alpha, i\mathbf{B}\vec{\xi}_\alpha \rangle = \langle \vec{\xi}_\alpha, \vec{a}_{ext} \rangle. \tag{2.46}$$

Now, defining

$$T_\alpha = i \int d^3x \rho \vec{\xi}^*_\alpha \cdot (\vec{v}_0 \cdot \nabla) \vec{\xi}_\alpha \tag{2.47}$$

we can write this equation as

$$\ddot{a}_\alpha MR^2 - 2iT_\alpha \dot{a}_\alpha + \left(\omega_\alpha^2 MR^2 - 2T_\alpha \omega_\alpha\right) a_\alpha = \langle \vec{\xi}_\alpha, \vec{a}_{ext} \rangle. \tag{2.48}$$

Remember that this equation is a good approximation only for slowly rotating stars, which means $\Omega_s^2 \ll \frac{GM}{R^3}$, even though previous authors have used it for general rotating stars. Here, Ω_s is the spin frequency of the star, M is its total mass, and R its radius.

2.3.2 General Solution

A more rigorous solution to this problem is to do an expansion in the phase space and to use the correct orthogonality condition for this case.

A **phase space expansion** is of the form

$$\begin{bmatrix} \vec{\xi}(\vec{x}, t) \\ \dot{\vec{\xi}}(\vec{x}, t) \end{bmatrix} = \sum_A c_A(t) \begin{bmatrix} \vec{\xi}_A(\vec{x}) \\ i\omega_A \vec{\xi}_A(\vec{x}) \end{bmatrix} \tag{2.49}$$

where I use the subscript A instead of α because the vector space is twice as big as the previous one. The orthogonality condition that these eigenvectors obey is derived in the following way. First define

$$W\left(\vec{\xi}_A, \vec{\xi}_B\right) = \langle \vec{\xi}_A, \dot{\vec{\xi}}_B + \frac{1}{2}\mathbf{B} \cdot \vec{\xi}_B \rangle - \langle \dot{\vec{\xi}}_A + \frac{1}{2}\mathbf{B} \cdot \vec{\xi}_A, \vec{\xi}_B \rangle. \tag{2.50}$$

If $\vec{\xi}_A$ and $\vec{\xi}_B$ are solutions of (2.30), we have [11]

$$\frac{\partial W\left(\vec{\xi}_A, \vec{\xi}_B\right)}{\partial t} = 0. \tag{2.51}$$

In our case, this leads to the following orthogonality condition [12],

$$\langle \vec{\xi}_A, i\mathbf{B} \cdot \vec{\xi}_B \rangle - (\omega_A + \omega_B) \langle \vec{\xi}_A, \vec{\xi}_B \rangle = Z_A \delta_{AB} \tag{2.52}$$

which gives us an equation for Z_A when A = B, using (2.47), and using the same normalization as in the previous case ($\langle\vec{\xi}_\alpha, \vec{\xi}_\alpha\rangle = MR^2$)

$$2T_A - 2\omega_A MR^2 = Z_A. \tag{2.53}$$

We can invert equations (2.49) and (2.52). Using the expansion in equation (2.49), we can identify

$$i\dot{\vec{\xi}} = -\sum_A c_A \omega_A \vec{\xi}_A, \tag{2.54}$$

$$i\mathbf{B}\cdot\vec{\xi} = i\sum_A c_A \mathbf{B}\cdot\vec{\xi}_A, \tag{2.55}$$

$$-\omega_B\vec{\xi} = -\omega_B\sum_A c_A\vec{\xi}_A. \tag{2.56}$$

Adding these and using the orthogonality condition (2.52), we find [28]

$$c_A(t) = \frac{1}{Z_A}\langle\vec{\xi}_A, i\dot{\vec{\xi}} + i\mathbf{B}\cdot\vec{\xi} - \omega_A\vec{\xi}\rangle. \tag{2.57}$$

Using the previous results, the equation of motion for the coefficients c_A is

$$\dot{c}_A - i\omega_A c_A = \frac{1}{Z_A}\langle\vec{\xi}_A, i\vec{a}_{ext}\rangle. \tag{2.58}$$

With this equation, the modes are naturally uncoupled from each other and no approximation has been used. Moreover, this equation is simpler than (2.48) and is only a first order differential equation rather than a second order one.

So far, the equations presented were valid for a general $\vec{\xi}$, which could be complex. If we consider that $\vec{\xi}$ has to be real, like in any physical case, we can reduce the number of eigenvectors needed in the expansion by 2, getting the same number as in the non-rotating star case [28].

First, by looking at equation (2.35), we can see that, if ($\vec{\xi}_A$,ω_A) is a solution of the eigenvalue equation, then ($\vec{\xi}^*_A$,$-\omega_A$)has to be another solution. All the modes come in pairs. We can use this to rewrite the summation over A as a summation over α. Equation (2.49) becomes

$$\begin{bmatrix} \vec{\xi}(\vec{x}, t) \\ \dot{\vec{\xi}}(\vec{x}, t) \end{bmatrix} = \sum_{\alpha} \left(c_{\alpha}(t) \begin{bmatrix} \vec{\xi}_{\alpha} \\ i\omega_{\alpha}\vec{\xi}_{\alpha} \end{bmatrix} + c'_{\alpha}(t) \begin{bmatrix} \vec{\xi}^{*}_{\alpha} \\ -i\omega_{\alpha}\vec{\xi}^{*}_{\alpha} \end{bmatrix} \right). \tag{2.59}$$

To have $\vec{\xi}$ element of $\mathbb{R}$, we need $c'_{\alpha}(t) = c^{*}_{\alpha}(t)$. We can get the equation of motion in the same way we got the one for a general displacement vector. It is

$$\dot{c}_{\alpha} - i\omega_{\alpha}c_{\alpha} = \frac{i}{Z_{\alpha}}\langle\vec{\xi}_{\alpha}, \vec{a}_{ext}\rangle. \tag{2.60}$$

This equation uses the same eigenvectors as does equation (2.48) for the slow rotation approximation since we need the same number of basis vectors. It makes it possible to compare their results.

2.4 Eigenvectors

In this section, I am going to present the expressions for the eigenvectors for non-rotating stars as well as for rotating stars. We will need the one for rotating stars in the next chapter.

2.4.1 Non-Rotating Star

For a non-rotating star, the unperturbed fluid is not moving, $v_0 = 0$. Equation (2.27) is

$$\rho_0 \frac{d^2\xi_i}{dt^2} = L_{ij}\xi^j \tag{2.61}$$

Using equations (2.4), (2.15), and (2.19), it is possible to rewrite $L_{ij}\xi^j$ as

$$L_{ij}\xi^j = -\left(\nabla\cdot\vec{\xi}\right)\nabla_i P\left(1 - \Gamma_1\frac{P}{\rho}\frac{\partial\rho}{\partial P}\right) - \rho\nabla_i\left(\frac{\delta P}{\rho} + \delta\Phi\right). \tag{2.62}$$

Now, if we consider that the star is made of an ideal gas of noninteracting neutrons at a temperature of 0K, it is described by a polytropic equation of state [29] of the form

$$P = K\rho^{\gamma} \tag{2.63}$$

where K is only a proportionality constant and γ is called the polytropic index. For example, when the neutrons are nonrelativistic, $\gamma = 5/3$, and $\gamma = 4/3$ for extremely relativistic neutrons. Using this equation of state, we now have

$$L_{ij}\xi^j = -\left(\nabla \cdot \vec{\xi}\right) \nabla_i P \left(1 - \frac{\Gamma_1}{\gamma}\right) - \rho \nabla_i \left(\frac{\delta P}{\rho} + \delta\Phi\right). \tag{2.64}$$

Putting equation (2.63) into the definition of Γ_1 (2.21), we find $\Gamma_1 = \gamma$. If we define

$$\delta\psi \equiv \frac{\delta P}{\rho} + \delta\Phi, \tag{2.65}$$

equation (2.61) reduces to

$$-\frac{d^2\vec{\xi}}{dt^2} = \nabla \delta\psi. \tag{2.66}$$

Taking the divergence of equation (2.66), we get

$$-\frac{d^2}{dt^2}\left(\nabla \cdot \vec{\xi}\right) = \nabla^2 \delta\psi. \tag{2.67}$$

Since I am considering an incompressible star, $\nabla \cdot \vec{\xi} = 0$ from equation (2.24), so that we have Laplace's equation,

$$\nabla^2 \delta\psi = 0. \tag{2.68}$$

It is known that the solution of Laplace's equation is of the form

$$\delta\psi = \sum_{j=0}^{\infty} \sum_{k=-j}^{j} \left[A_{jk} r^j + B_{jk} r^{-(j+1)}\right] Y_{jk}(\theta, \phi) \tag{2.69}$$

where A_{jk} and B_{jk} are constants. Since $\delta\psi$ has to be finite inside the star, we have to impose $B_{jk} = 0$. This then reduces $\delta\psi$ to

$$\delta\psi = \sum_{jk} A_{jk} r^j Y_{jk}(\theta, \phi). \tag{2.70}$$

On the other hand, it is possible to work on $\delta\psi$ to get a more convenient expression. Using equations (2.2), (2.15), and (2.21), and remembering that we neglect terms of second order or higher in the perturbations, we get

$$\frac{1}{\rho}\delta P = \frac{GM}{R^3}\xi^r r, \tag{2.71}$$

with ξ^r the r-component of $\vec{\xi}$. But since

$$\frac{1}{\rho}\delta P \propto \sum_{jk} A_{jk} r^j Y_{jk}(\theta, \phi), \tag{2.72}$$

we have

$$\xi^r = \chi \sum_{jk} A_{jk} r^{j-1} Y_{jk}(\theta, \phi) \tag{2.73}$$

with χ a proportionality constant.

It is also possible to get a similar expression for $\delta\Phi$. First, the general expression is

$$\delta\Phi(\vec{x}) = -G \int \frac{\delta\rho(r', \theta', \phi')}{|\vec{x} - \vec{x}\,'|} d^3x'. \tag{2.74}$$

Using equations (2.19) and (2.24), we obtain

$$\delta\rho = -\vec{\xi} \cdot \nabla\rho. \tag{2.75}$$

We could be tempted to say that $\nabla\rho = 0$ because the star is incompressible, but we have to consider the surface of the star. Instead, we have

$$\rho = \bar{\rho}\Theta(R - r) \tag{2.76}$$

with

$$\bar{\rho} = \frac{3M}{4\pi R^3} \tag{2.77}$$

and $\Theta(R - r)$ is the the Heaviside step function. We then have

$$\delta\rho = \bar{\rho}\xi^r \delta(R - r) \tag{2.78}$$

$$= \bar{\rho} \sum_{jk} \xi^r_{jk}(r) Y_{jk}(\theta, \phi) \delta(R - r). \tag{2.79}$$

It is possible to expand $\frac{1}{|\vec{x} - \vec{x}\,'|}$ in the spherical harmonics as follows [1]

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = 4\pi \sum_{j=0}^{\infty} \sum_{k=-j}^{j} \frac{1}{2j+1} \frac{r_<^j}{r_>^{j+1}} Y_{jk}(\theta_1, \phi_1) Y^*_{jk}(\theta_2, \phi_2), \tag{2.80}$$

where the $<$ and $>$ mean the smallest and the biggest of r_1 and r_2. Using the previous expansion and the orthogonality condition between spherical harmonics [1],

$$\int_{\phi=0}^{\pi}\int_{\theta=0}^{\pi} Y^*_{j_1k_1}(\theta,\phi)\, Y_{j_2k_2}(\theta,\phi)\sin(\theta)\, d\theta d\phi = \delta_{j_1,j_2}\delta_{k_1,k_2}, \tag{2.81}$$

the fact that $r < R$, and

$$\xi^r(R) = \chi R^{j-1}, \tag{2.82}$$

we get

$$\delta\Phi = -\frac{3GM}{R^3}\chi\sum_{jk} A_{jk}\frac{r^j}{2j+1}Y_{jk}(\theta,\phi). \tag{2.83}$$

We can now join the two parts of $\delta\psi$,

$$\delta\psi = \frac{1}{\rho}\delta P + \delta\Phi \tag{2.84}$$

$$= \frac{GM}{R^3}\chi\sum_{jk} A_{jk}\frac{2(j-1)}{2j+1}r^j Y_{jk}(\theta,\phi). \tag{2.85}$$

In the case of eigenvectors, there is no driving force and

$$\frac{d^2\vec{\xi}_\alpha}{dt^2} = -\omega_\alpha^2\vec{\xi}_\alpha, \tag{2.86}$$

so that

$$\frac{d^2\vec{\xi}}{dt^2} = -\sum_\alpha \omega_\alpha^2\vec{\xi}_\alpha. \tag{2.87}$$

With $\alpha = \{j,k\}$,

$$\sum_\alpha \omega_\alpha^2\vec{\xi}_\alpha = \frac{GM}{R^3}\chi\sum_{jk} A_{jk}\frac{2(j-1)}{2j+1}r^j Y_{jk}(\theta,\phi), \tag{2.88}$$

and this leads us to

$$\omega_\alpha^2 = \frac{GM}{R^3}\frac{2j(j-1)}{2j+1} \tag{2.89}$$

and

$$\vec{\xi}_\alpha = \frac{\chi_{jk}}{j}\nabla\left(r^j Y_{jk}(\theta,\phi)\right) \tag{2.90}$$

where

$$\chi_{jk} = \chi A_{jk}. \tag{2.91}$$

These modes are called the f-modes, for fundamental modes, because there is no node in the r-direction. So, from now on, I am going to consider $\alpha = \{jk\}$.

Recall that these equations are valid for non-rotating stars only.

2.4.2 Rotating Star

The derivation for the eigenvectors of a rotating star is somewhat similar to the one for non-rotating stars.

We begin with Euler's equation in the inertial frame (2.2). Since we are going to work in the co-rotating frame, we need the expression for the Coriolis force to make the transformation. It is given by [13]

$$\left(\frac{d\vec{v}_i}{dt}\right)_i = \left(\frac{d\vec{v}_r}{dt}\right)_r + 2\left(\vec{\Omega}_s \times \vec{v}_r\right) + \vec{\Omega}_s \times \left(\vec{\Omega}_s \times \vec{r}_r\right), \tag{2.92}$$

where the subscript i refers to the inertial frame, the subscript r refers to the co-rotating frame, and $\vec{\Omega}_s$is the spin frequency of the star as seen from the inertial frame. Applying equation (2.92) to equation (2.2), we get the Euler equation in the co-rotating frame,

$$\left(\frac{d\vec{v}_r}{dt}\right)_r + 2\left(\vec{\Omega}_s \times \vec{v}_r\right) + \vec{\Omega}_s \times \left(\vec{\Omega}_s \times \vec{r}_r\right) = -\frac{1}{\rho}\nabla P - \nabla\Phi. \tag{2.93}$$

In the co-rotating frame, the equilibrium velocity is zero. So, $\vec{v}_r = \delta\vec{v}_r$. In the first order in the perturbations, $\delta\vec{v}_r = \Delta\vec{v}_r = \frac{\partial\vec{\xi}}{\partial t}$.

Again to first order in the perturbations, and remembering that $\frac{\partial\vec{\xi}}{\partial t} = i\omega_\alpha\vec{\xi}$, equation (2.93) becomes

$$-\omega^2_{\alpha,\,r}\vec{\xi}_r + 2i\omega_{\alpha,\,r}\Omega_s\left(-\xi_{r,\,y}\hat{x} + \xi_{r,\,x}\hat{y}\right) = -\nabla\Psi, \tag{2.94}$$

where

$$\Psi = \frac{1}{\rho}P + \Phi - \frac{1}{2}\Omega_s^2\left(x^2 + y^2\right). \tag{2.95}$$

Combining this with the equation for an incompressible fluid, equation (2.24), we find

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \left(1 - \frac{4\Omega_s}{\omega_{\alpha,\,r}^2}\right)\frac{\partial^2\Psi}{\partial z^2} = 0. \tag{2.96}$$

A specific set of solutions for this equation is given by [8]

$$\Psi = A_k(x+iy)^k, \tag{2.97}$$

where A_k is a constant. Substituting equation (2.97) into (2.94), we find

$$\vec{\xi}_\alpha\,(x,y,z) = \frac{kA_k}{\omega_{\alpha,\,r}\left(\omega_{\alpha,\,r} - 2\Omega_s\right)}\,(x+iy)^{k-1}\,(\hat{x}+i\hat{y})\,. \tag{2.98}$$

This is the form in the cartesian coordinates. In cylindrical coordinates, it becomes

$$\vec{\xi}_\alpha\,(r_\perp,\phi,z) = \frac{kA_k}{\omega_{\alpha,\,r}\left(\omega_{\alpha,\,r} - 2\Omega_s\right)}\,r_\perp^{k-1}e^{ik\phi}\left(\hat{r}_\perp + i\hat{\phi}\right) \tag{2.99}$$

and in spherical coordinates

$$\vec{\xi}_\alpha\,(r,\theta,\phi) = \frac{kA_k}{\omega_{\alpha,\,r}\left(\omega_{\alpha,\,r} - 2\Omega_s\right)}\,r^{k-1}\,(\sin\theta)^{k-1}\,e^{ik\phi}\left(\sin\theta\hat{r} + \cos\theta\hat{\theta} + i\hat{\phi}\right). \tag{2.100}$$

Since

$$Y_{kk}\,(\theta,\phi) = \frac{(-1)^k}{2^k k!}\sqrt{\frac{(2k+1)!}{4\pi}}\,(\sin\theta)^k\,e^{ik\phi}, \tag{2.101}$$

the eigenvectors in spherical coordinates can also be given by, neglecting the factor $(-1)^k$,

$$\vec{\xi}_\alpha\,(r,\theta,\phi) = \frac{kA_k}{\omega_{\alpha,\,r}\left(\omega_{\alpha,\,r} - 2\Omega_s\right)}\,\frac{2^k k!}{\sqrt{(2k+1)!/4\pi}} \times r^{k-1}\left(\hat{r} + \frac{1}{k}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{k\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}\right)Y_{kk}\,(\theta,\phi)\,. \tag{2.102}$$

By comparing (2.102) with (2.90), you can see that equation (2.98) represents the solutions for the modes with $j = k$. An expression for $j \neq k$ is very difficult to derive. In fact, it has been derived [3], but it is given in spheroidal coordinates.

Recall that these equations have been derived in the co-rotating frame. However, equation (2.98) can be changed to be valid in the inertial frame by using the following relation,

$$\phi = \phi^{(r)} + \Omega_S t. \tag{2.103}$$

This relation can be used to derive a similar equation for ω_α. Since,

$$\omega_\alpha^{(r)} t + k\phi^{(r)} = \omega_\alpha^{(r)} t + k\left(\phi - \Omega_s t\right) \tag{2.104}$$

$$= \left(\omega_\alpha^{(r)} - k\Omega_s\right) t + k\phi, \tag{2.105}$$

we have [15]

$$\omega_\alpha = \omega_\alpha^{(r)} - k\Omega_S, \tag{2.106}$$

where the superscript (r) means that the quantity is taken in the co-rotating frame, and Ω_S is the rotation angular frequency of the star. r and θ are unaffected by the change of frame. $\omega_\alpha^{(r)}$ is given in (4.8) for the case j=k.

Chapter 3

Binary System

In this chapter, I discuss further aspects of stellar oscillations for binary stellar systems. I first introduce the tidal interactions between the stars. After that, I present the equations describing the orbital motion, including the driving of fluid oscillations by the orbital motion, the emission of gravitational radiation, and the possibility of resonances.

3.1 Tidal Potential

I consider two stars in a binary system. These two stars have a gravitational influence on each other, provoking displacements of the fluid they are made of. Even though it is only an approximation, I treat one of the two as a point mass and I look at the oscillations it triggers in its companion and the effects these oscillations are going to have on the orbital motion of the stars.

First of all, the point star of mass M' induces oscillations inside its companion of mass M by the gravitational interaction with the fluid elements through the potential

$$U(\vec{r}, t) = -\frac{GM'}{\left|\vec{r} - \vec{D}(t)\right|} \tag{3.1}$$

where the coordinate system used is centered on the star M, with the z-axis pointing along the spin-axis of the star. $\vec{r} = (r, \theta, \phi)$ is the position vector of a fluid element of the star M and $\vec{D}(t)$ is the position vector of the point mass star M'. In spherical coordinates, with respect to the z-axis, $\vec{D}(t)$ is given by $(D(t), \frac{\pi}{2}, \Phi(t))$. This potential can be expanded on the spherical harmonics as , supposing that the spin axis of the star M is aligned with the orbital spin axis, [26]

$$U(\vec{r}, t) = -GM' \sum_{lm} \frac{W_{lm} r^l}{[D(t)]^{l+1}} e^{-im\Phi(t)} Y_{lm}(\theta, \phi) \tag{3.2}$$

with W_{lm} being defined as

$$W_{lm} = (-1)^{(l+m)/2} \left[\frac{4\pi}{2l+1}(l+m)!\,(l-m)!\right]^{\frac{1}{2}} \left[2^l \left(\frac{l+m}{2}\right)! \left(\frac{l-m}{2}\right)\right]^{-1}. \tag{3.3}$$

The symbol $(-1)^n \neq 0$ only if $n \,\epsilon\, \mathbb{Z}$. This gives us the restriction

$$l + m \;\epsilon\; 2\mathbb{Z}. \tag{3.4}$$

We can now use this potential to calculate the external acceleration in the equations of motion for the mode amplitudes in the previous chapter. If we say

$$\vec{a}_{ext} = -\nabla U, \tag{3.5}$$

we get

$$\langle \vec{\xi}_\alpha, \vec{a}_{ext} \rangle = \sum_{lm} \frac{GM'}{[D(t)]^{l+1}} W_{lm} Q_{\alpha,lm} e^{-im\Phi} \tag{3.6}$$

where $Q_{\alpha,lm}$ is the overlap integral and is given by

$$Q_{\alpha,lm} = \int d^3x \rho \vec{\xi}_\alpha^* \cdot \nabla \left(r^l Y_{lm}(\theta, \phi)\right) = \int d^3x \delta\rho_\alpha^* \left(r^l Y_{lm}(\theta, \phi)\right). \tag{3.7}$$

For $Q_{\alpha,lm}$ to be nonzero, we need

$$m = k. \tag{3.8}$$

Recall $\alpha = \{j, k\}$

If we put this in the equations for rotating stars, we get, for the slow rotation approximation

$$\ddot{a}_\alpha MR^2 - 2iT_\alpha\dot{a}_\alpha + \left(\omega_\alpha^2 MR^2 - 2T_\alpha\omega_\alpha\right)a_\alpha = \sum_{lm}\frac{GM'}{\left[D\left(t\right)\right]^{l+1}}W_{lm}Q_{\alpha,lm}e^{-im\Phi} \quad (3.9)$$

and in the general case

$$\dot{c}_\alpha - i\omega_\alpha c_\alpha = \frac{i}{Z_\alpha}\sum_{lm}\frac{GM'}{\left[D\left(t\right)\right]^{l+1}}W_{lm}Q_{\alpha,lm}e^{-im\Phi}. \quad (3.10)$$

3.2 Orbital Motion

It is now time to describe the motion of two stars around each other. The problem of two stars in motion around their centre of mass can be reduced to a one body problem. An initial problem consisting of a star of mass m_1 orbiting at a distance r_1 from the centre of mass and a star of mass m_2 at a distance r_2 is equivalent to the problem of a star of mass μ

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} \quad (3.11)$$

on an orbit of radius $D = |\vec{r}_1 - \vec{r}_2|$.

3.2.1 Gravitational Potential

The stars are under the influence of their mutual gravitational attraction. The gravitational potential is given by

$$U_{grav} = -GM'\int d^3x\frac{\rho}{\left|\vec{D} - \vec{r}\right|} \quad (3.12)$$

where the integration is made on the volume of the star M, $\vec{D}(t) = (D(t), \pi/2, \Phi(t))$ is the vector distance between the centres of the two stars and $\vec{r}$ is the position of a fluid element inside the studied star measured from the star's centre. Remember that $\rho = \rho_0 + \delta\rho$. The integral then becomes

$$U_{grav} = -GM' \int d^3x \frac{\rho_0}{\left|\vec{D} - \vec{r}\right|} - GM' \int d^3x \frac{\delta\rho}{\left|\vec{D} - \vec{r}\right|} \tag{3.13}$$

$$= U_0 + U_{tide}. \tag{3.14}$$

Let's first look at U_0. It represents the gravitational potential of a ball of constant density at a point outside of its surface. It is proven in many textbooks, in [20] for example, that this potential is equal to the one caused by a point-like object of the same mass at the centre of the star,

$$U_0 = -\frac{GMM'}{D}. \tag{3.15}$$

U_{tide} can be expanded using the fact that $\delta\rho = -\nabla \cdot (\rho\vec{\xi})$ and

$$\frac{1}{\left|\vec{D} - \vec{r}\right|} = \sum_{lm} \frac{W_{lm} r^l}{[D(t)]^{l+1}} e^{-im\Phi} Y_{lm}(\theta, \phi) \tag{3.16}$$

$$= \sum_{lm} \frac{W_{lm} r^l}{[D(t)]^{l+1}} e^{im\Phi} Y^*_{lm}(\theta, \phi) \tag{3.17}$$

where W_{lm} is given in equation (3.3). The two expressions are equal because m runs from $-l$ to l. This makes the left-hand side real.

The final result depends on the expansion of the displacement vector used. Let's first look at the slow rotation approximation. In this case, remember that

$$\vec{\xi} = \sum_{\alpha} a_\alpha(t)\, \vec{\xi}_\alpha(\vec{x}). \tag{3.18}$$

This gives

$$\delta\rho = \sum_{\alpha} a_\alpha(t)\, \delta\rho_\alpha \tag{3.19}$$

with $\delta\rho_\alpha = -\nabla \cdot (\rho\vec{\xi}_\alpha)$. Putting these results in the equation (3.13) for U_{tide}, we get [19]

$$U_{tide} = -GM' \sum_{\alpha,lm} \frac{W_{lm} Q_{\alpha,lm}}{[D(t)]^{l+1}} a_\alpha(t)\, e^{im\Phi(t)} \tag{3.20}$$

considering that, since $Q_{\alpha,lm}$ is real, it is equal to its complex conjugate,

$$Q_{\alpha,lm} = \int d^3x \delta\rho_\alpha \left[r^l Y^*_{lm}(\theta,\phi)\right]. \tag{3.21}$$

For the general solution, we proceed the same way but we use equation (2.49) for the expansion of the displacement vector. $\delta\rho$ is expanded as

$$\delta\rho = \sum_\alpha \left(c_\alpha(t)\,\delta\rho_\alpha + c^*_\alpha(t)\,\delta\rho^*_\alpha\right) \tag{3.22}$$

so that we have for U_{tide}

$$U_{tide} = -GM' \sum_{\alpha,lm} \frac{W_{lm} Q_{\alpha,lm}}{[D(t)]^{l+1}} \left(c_\alpha(t)\, e^{im\Phi(t)} + c^*_\alpha(t)\, e^{-im\Phi(t)}\right). \tag{3.23}$$

3.2.2 Lagrange's Equations of Motion

Once we have an expression for the potential, we can find Lagrange's equations of motion for the orbit. First, the Lagrangian is given by

$$L \equiv T - U \tag{3.24}$$

$$= \frac{1}{2}\mu \left|\dot{\vec{D}}\right|^2 - U_{grav} \tag{3.25}$$

where T is the kinetic energy and U the potential energy associated to a conservative force. Lagrange's equations are

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0. \tag{3.26}$$

q_j represents the generalized coordinates and $\dot{q}_j$ the generalized velocities. In our case, we have

$$\left|\dot{\vec{D}}\right|^2 = \dot{D}^2 + D^2\dot{\Phi}^2. \tag{3.27}$$

So,

$$L = \frac{1}{2}\mu\left(\dot{D}^2 + D^2\dot{\Phi}^2\right) - U_{grav}. \tag{3.28}$$

Since U_{grav} is independent of the velocities, the two equations of motion are

$$\mu\ddot{D} - \mu D\dot{\Phi}^2 + \frac{\partial U_{grav}}{\partial D} = 0 \tag{3.29}$$

and

$$\frac{d\left(\mu D^2\dot{\Phi}\right)}{dt} + \frac{\partial U_{grav}}{\partial \Phi} = 0. \tag{3.30}$$

Now, let's look at what these equations are in the two expansions used for rotating stars. For the slow rotation approximation, equations (3.29) and (3.30) lead to

$$\ddot{D} = D\dot{\Phi}^2 - \frac{GMM'}{\mu D^2} - \frac{GM'}{\mu}\sum_{\alpha lm} \frac{(l+1)\, W_{lm}Q_{\alpha,lm}}{D^{l+2}} a_\alpha e^{im\Phi} \tag{3.31}$$

and

$$\ddot{\Phi} = -\frac{2\dot{D}\dot{\Phi}}{D} + \frac{GM'}{\mu D^2}\sum_{\alpha lm} \frac{imW_{lm}Q_{\alpha,lm}}{D^{l+1}} a_\alpha e^{im\Phi}. \tag{3.32}$$

For the general solution expansion, we get instead

$$\ddot{D} = D\dot{\Phi}^2 - \frac{GMM'}{\mu D^2} - \frac{GM'}{\mu}\sum_{\alpha lm} \frac{(l+1)\, W_{lm}Q_{\alpha,lm}}{D^{l+2}} \left(c_\alpha e^{im\Phi} + c_\alpha^* e^{-im\Phi}\right) \tag{3.33}$$

and

$$\ddot{\Phi} = -\frac{2\dot{D}\dot{\Phi}}{D} + \frac{GM'}{\mu D^2}\sum_{\alpha lm} \frac{imW_{lm}Q_{\alpha,lm}}{D^{l+1}} \left(c_\alpha e^{im\Phi} - c_\alpha^* e^{-im\Phi}\right). \tag{3.34}$$

3.3 Gravitational Radiation

When there is a non-spherically symmetric movement of masses, gravitational radiation is emitted, carrying energy away. This is the case for a binary system, and the loss of energy affects the orbital motion. In this section, I want to introduce the basic equation describing this phenomenon.

3.3.1 Metric And Gravitational Field

Once the waves are away from their source, we can use the weak field approximation because they are generally in an area where the gravitation is weak. In this approximation, the metric is given by [23]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.35}$$

with

$$|h_{\mu\nu}| \ll 1. \tag{3.36}$$

In our solar system, $h_{\mu\nu}$ would include the quasistatic contributions of the sun, the planets, etc., and gravitational waves from astrophysical sources. Here, we are going to ignore the quasistatic contributions.

The gravitational field $\bar{h}_{\mu\nu}$ is related to the metric in the following way [23]

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \tag{3.37}$$

where

$$h \equiv h_{\alpha}{}^{\alpha}. \tag{3.38}$$

The equations satisfied by that field are

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0 \tag{3.39}$$

and the Einstein field equations

$$\Box \bar{h}_{\mu\nu} = 0. \tag{3.40}$$

The simplest solution is the plane-wave solution [23], of the form

$$\mathbb{R}\left[A_{\mu\nu} e^{(ik_\alpha x^\alpha)}\right] \tag{3.41}$$

where $A_{\mu\nu}$ are the amplitudes and k_α the wave vectors. They satisfy

$$k_\alpha k^\alpha = 0 \tag{3.42}$$

$$A_{\mu\nu} k^\alpha = 0. \tag{3.43}$$

By imposing the following constraints, where the spatial indices are summed even if they are both down,

$$h_{\mu 0} = 0 \tag{3.44}$$

$$h_{kj,j} = 0 \tag{3.45}$$

$$h_{kk} = 0 \tag{3.46}$$

we choose a gauge. The new gauge is called the transverse-traceless gauge (transverse because it is transverse to its direction of propagation and traceless because $h_{kk} = 0$) and is identified by the superscript TT. Note that in this gauge, since $h = h_{\mu}{}^{\mu} = h_{kk} = 0$, there is no distinction between $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$.

In the TT gauge, the only non-vanishing components of $h^{TT}_{\mu\nu}$ are

$$h^{TT}_{xx} = -h^{TT}_{yy} \tag{3.47}$$

$$h^{TT}_{xy} = h^{TT}_{yx} \tag{3.48}$$

for the two possible polarizations.

3.3.2 Stress-Energy Tensor

Gravitational waves can exert forces and do work, so they must carry energy. We cannot define the the stress-energy tensor in a point because the gravitational force can be transformed to take any value, e.g. zero, at a point. The stress-energy tensor is given by [23]

$$T^{GW}_{\mu\nu} = \langle h^{TT}_{jk,\mu} \; h^{TT}_{jk,\nu} \rangle \tag{3.49}$$

where $\langle \quad \rangle$ means an average over many wavelengths and GW stands for gravitational wave.

3.3.3 Quadrupole Nature Of Gravitational Radiation

To get an order-of-magnitude estimate of the energy carried away from a source by gravitational radiation, we can use the electromagnetic radiation formulas, substituting e^2 by $-m^2$ [23]. This doesn't give the exact numerical factors, but can be used to find the leading order of radiation.

In electromagnetism, the radiation is dominated by the electric dipole. The electric dipole moment is given by [16]

$$\vec{d}_{el} = \sum_i e_i \vec{x}_i \tag{3.50}$$

where the summation is over all the particles involved. e_i is the charge of the i^{th} particle and $\vec{x}_i$ its position. The gravitational analog is

$$\vec{d}_{GW} = \sum_i m_i \vec{x}_i. \tag{3.51}$$

The power output $L \propto \ddot{d}$. But $\dot{d}_{GW}$ represents the momentum of the system,

$$\dot{\vec{d}}_{GW} = \sum_i m_i \dot{\vec{x}}_i = \vec{p}_{total}. \tag{3.52}$$

Since the momentum is conserved, the is no gravitational radiation similar to the electric dipole radiation.

In electromagnetism, the next leading-order radiation is the magnetic dipole. The magnetic dipole moment is given by [16]

$$\vec{\mu}_{el} = \sum_i \vec{x}_i \times (e_i \vec{v}_i) \tag{3.53}$$

where $\vec{v}_i$ is the velocity of the i^{th} particle. The gravitational analog is equal to the angular momentum,

$$\vec{\mu}_{GW} = \sum_i \vec{x}_i \times (m_i \vec{v}_i) = \vec{J}. \tag{3.54}$$

Here again, since the angular momentum is a conserved quantity, there is no gravitational analog to the magnetic dipole radiation.

We have to look at the next component, the electric quadrupole, to find the first term to emit gravitational radiation. In electromagnetism, the electric quadrupole moment is given by [16]

$$Q_{ij} = \sum_i e_i \left(x_{ij} x_{ik} - \frac{1}{3} \delta_{jk} \left(x_i \right)^2 \right). \tag{3.55}$$

For gravitational radiation, this becomes

$$I_{jk} = \sum_i m_i \left(x_{ij} x_{ik} - \frac{1}{3} \delta_{jk} \left(x_i \right)^2 \right). \tag{3.56}$$

The power is, with the proper numerical factor, [23]

$$L_{GW} = \frac{dE}{dt} \tag{3.57}$$

$$= \frac{1}{5} \frac{G}{c^5} \langle \dddot{I}_{jk} \dddot{I}_{jk} \rangle. \tag{3.58}$$

3.3.4 Binary System

For a binary system of stars of masses m_1 and m_2 on a circular orbit with d_1 and d_2 as the respective distances from the center of mass,

$$m_1 d_1 = m_2 d_2 = \mu D, \tag{3.59}$$

it is easy to prove that the power is [25]

$$L_{GW} = \frac{32}{5} \frac{G^4}{c^5} \frac{M_+^3 \mu^2}{D^5} \tag{3.60}$$

with

$$M_+ = m_1 + m_2. \tag{3.61}$$

Recall Kepler's third law,

$$\Omega_{orb}^2 = \frac{G M_+}{D^3}. \tag{3.62}$$

Ω_{orb} is the orbital frequency and is related to the orbital period P_{orb} by $\Omega_{orb} = \frac{2\pi}{P_{orb}}$. The energy of the system, $E = K + U$, is

$$E = \frac{1}{2}\left(m_1 d_1^2 + m_2 d_2^2\right)\Omega_{orb}^2 - \frac{G m_1 m_2}{D} \tag{3.63}$$
$$= -\frac{1}{2}\frac{G\mu M_+}{D}. \tag{3.64}$$

From (3.60) and (3.63), we find the time derivative of the distance between the two stars,

$$\frac{dD}{dt} = -\frac{64}{5}\frac{G^3}{c^5}\frac{\mu M_+^2}{D^3}. \tag{3.65}$$

This equation can be solved to find the function D(t),

$$D(t) = \left(D_0^4 - 4\beta t\right)^{1/4} \tag{3.66}$$

where

$$\beta = \frac{64}{5}\frac{G^3 \mu M_+^2}{c^5} \tag{3.67}$$

and D_0 is the initial separation.

Therefore, the time the binary system takes to collapse is given by

$$T(D_0) = \frac{D_0^4}{4\beta}. \tag{3.68}$$

For a binary system of two stars of mass about 2.785×10^{30} kg and initial orbital frequency of 20Hz, which is the case I will study later, this time is approximatively 25 seconds. For the pulsar PSR 1913+16, which is the observed pulsar with the smallest distance between the stars, the time remaining before the binary system collapses is approximately 300 million years [4].

It is also possible to get an expression for the time derivative of the angular momentum, $J = \mu\dot{\Phi}D^2$, [24]

$$\frac{dJ}{dt} = -\frac{32}{5}\frac{G^{7/2}}{c^5}\frac{\mu^2 M_+^{5/2}}{D^{7/2}}. \tag{3.69}$$

This gives us the term that should be added to the lagrangian equation (3.30).

3.4 Resonances

The different modes of oscillation can enter resonance. They then interact more strongly with the orbital motion, taking more energy away.

To find the condition for resonance, first consider equations (3.9) or (3.10). Since $\vec{\xi} \propto e^{i\omega_\alpha t}$ when there is no external force, it would be natural to replace respectively a_α and c_α by $q_\alpha e^{i\omega_\alpha t}$. Also, notice that, for a particular m, the main term in the summation

$$\sum_{lm} \frac{GM'}{\left[D\left(t\right)\right]^{l+1}} W_{lm} Q_{\alpha,lm} e^{-im\Phi} \tag{3.70}$$

is the one with l = m. If we make the substitution for q_α, we get the following equations. For the slow rotation approximation it is

$$\ddot{q}_\alpha + 2i\left(\omega_\alpha - T_\alpha\right)\dot{q}_\alpha = \sum_{lm} \frac{GM'}{\left[D\left(t\right)\right]^{l+1}} W_{lm} Q_{\alpha,lm} e^{-im\Phi - i\omega_\alpha t} \tag{3.71}$$

and in the general case,

$$\dot{q}_\alpha = \frac{i}{Z_\alpha} \sum_{lm} \frac{GM'}{\left[D\left(t\right)\right]^{l+1}} W_{lm} Q_{\alpha,lm} e^{-im\Phi - i\omega_\alpha t}. \tag{3.72}$$

A mode enters resonance when the argument of the exponential is zero, which is when

$$\omega_\alpha = -m\Omega_{orb}. \tag{3.73}$$

Chapter 4

Numerical Methods and Results

In this chapter, I to present the equations I need in a form suitable for numerical calculations. I also present the assumptions made, the numerical methods used and the results.

4.1 Modification of the Equations

In this section, I use the equations presented in the last chapters for rotating stars and put them in a format that can be used for numerical integrations.

4.1.1 Cases Studied

As stated chapter 2, equations (2.102) describes the modes with $j = k$ and a more general expression is given in spheroidal coordinates. So, I am going to make calculations for the modes $j = k$ for simplicity, and also because it is what has been done in previous articles on the subject (see [15]).

Also, by looking at equations (3.9) and (3.10), we can see that the main contribution for a given m to the modes excitation by the tidal potential is the one with $l = m$. So, I am going to study these contributions only.

Since we already had $m = k$, all the indices of the different contributions are going to be equal,

$$j = k = l = m. \tag{4.1}$$

4.1.2 Maclaurin Spheroid

Since the star is rotating, its surface does not have the shape of a sphere, but of an ellipsoid, described by a Maclaurin spheroid. The star has an equatorial radius a_1 and a polar radius a_3. The eccentricity of the star, e, is defined as

$$e = \sqrt{1 - \frac{a_3^2}{a_1^2}}. \tag{4.2}$$

where we assume that $a_3 < a_1$. We can also define a mean radius, R, as

$$R = \left(a_1^2 a_3\right)^{1/3}. \tag{4.3}$$

The shape of the spheroid is described by the following equations. The first one,

$$r_{max}\left(\theta\right) = \left(\frac{\sin^2\theta}{a_1^2} + \frac{\cos^2\theta}{a_3^2}\right)^{-1/2} \tag{4.4}$$

gives the maximum value of r for a given θ. The next one,

$$\frac{x^2 + y^2}{a_1^2} + \frac{z^2}{a_3^2} = 1, \tag{4.5}$$

describes the surface of the spheroid in cartesian coordinates.

The spin frequency of the star Ω_s and the eccentricity are related by the following expression [5],

$$\Omega_s^2 = \frac{2\pi G\rho}{e^3}\left[\sqrt{1-e^2}\left(3 - 2e^2\right)\arcsin e - 3e\left(1 - e^2\right)\right] \tag{4.6}$$

where ρ is the density of the star, which we assume to be constant,

$$\rho = \frac{3M}{4\pi R^3}. \tag{4.7}$$

Figure (4.1) shows $\frac{\Omega_s}{\sqrt{G\rho}}$ as a function of eccentricity.

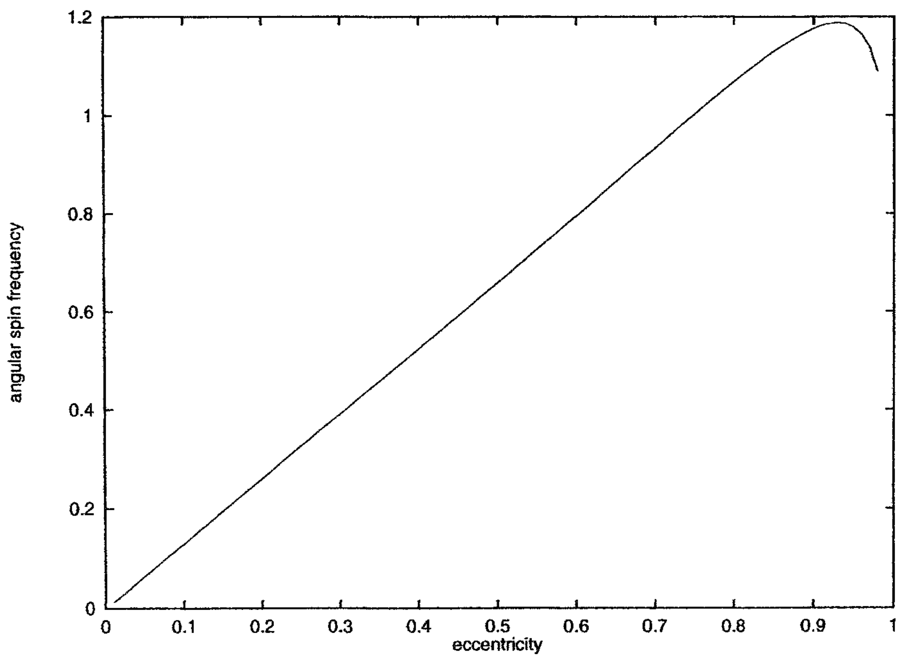

Figure 4.1: $\frac{\Omega_s}{\sqrt{G\rho}}$ as a function of eccentricity.

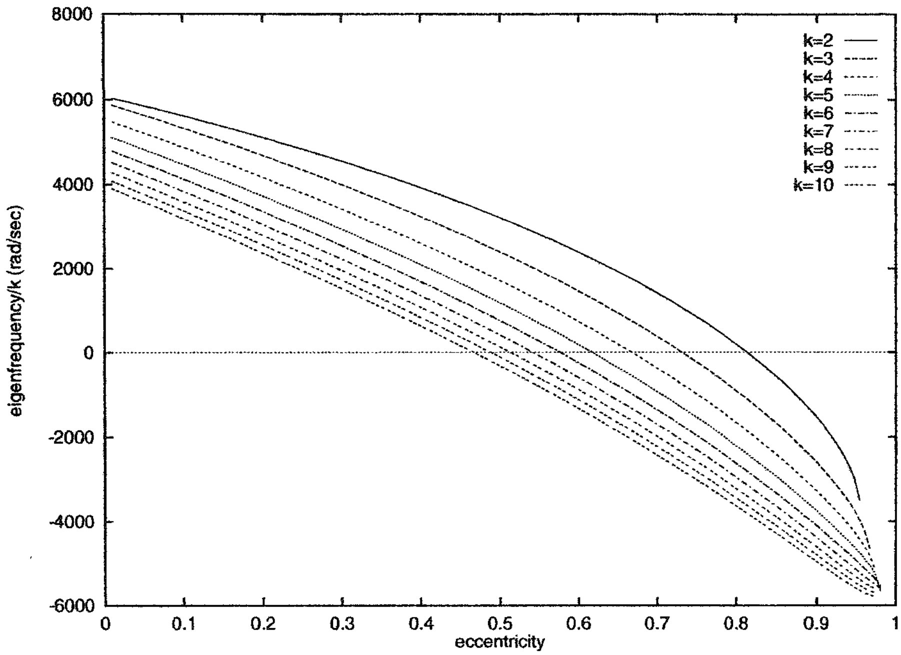

Figure 4.2: Positive solution of $\frac{\omega_\alpha}{k}$ as a function of eccentricity for a star of density 6.6×10^{17} kg/m^3.

4.1.3 Eigenmodes

We need an expression for ω_α. In the case $j = k$, it has been derived in [9] and is given by

$$\frac{\omega_{\alpha,r}}{\Omega_s} = 1 \pm \left[1 - \frac{2ke^2R_k}{(3-2e^2)\sin^{-1}e - 3e\sqrt{1-e^2}}\right] \tag{4.8}$$

where R_k is

$$R_k = \frac{\sqrt{1-e^2}}{e}\frac{(2k-1)!!}{(2k)!!}\sum_{p=k+1}^{\infty}\frac{(2p-2)!!}{(2p-1)!!}\,e^{2(p-k)} + \frac{(1-e^2)}{e^2}\left[\frac{1}{\sin e} - \frac{e}{\sqrt{1-e^2}}\right]. \tag{4.9}$$

For a star with a density of 6.6×10^{17} kg/m^3, in the case $j = k = m$ and $k > 0$, figures (4.2) and (4.3) show the two solutions for $\frac{\omega_\alpha}{k}$.

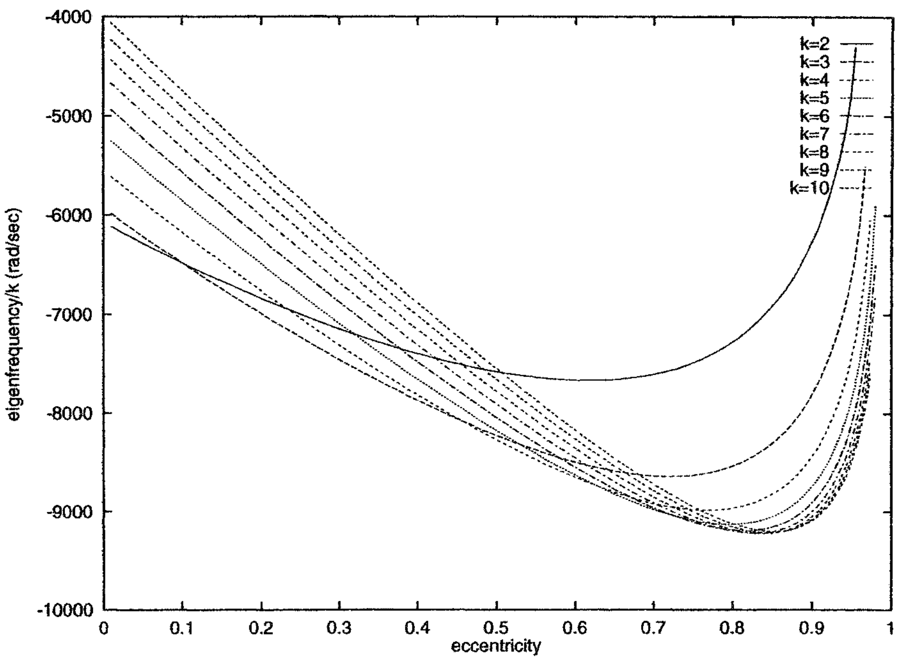

Figure 4.3: Negative solution of $\frac{\omega_\alpha}{k}$ as a function of eccentricity for a star of density 6.6×10^{17} kg/m^3.

The expressions for the eigenvectors in the co-rotating frame in three coordinate systems were given in chapter 2 (equations (2.98) to (2.102)). Now, we need to determine their normalization constant. Recall that

$$\langle \vec{\xi}_\alpha \, , \, \vec{\xi}_\alpha \rangle = MR^2. \tag{4.10}$$

To find the normalization constant of the eigenvectors, I am going to use the expression given in the cylindrical coordinate system (2.99). Defining

$$U_k \equiv \frac{kA_k}{\omega_{\alpha,\, r}\left(\omega_{\alpha,\, r} - 2\Omega_s\right)}, \tag{4.11}$$

we have

$$\langle \vec{\xi}_\alpha \, , \, \vec{\xi}_\alpha \rangle = 2U_k^2 \rho \int d^3x r_\perp^{2(k-1)}. \tag{4.12}$$

Equation (4.5) gives us

$$r_{\perp max} = a_1 \sqrt{1 - \frac{z^2}{a_3}}. \tag{4.13}$$

$\langle \vec{\xi}_\alpha \, , \, \vec{\xi}_\alpha \rangle$ then becomes

$$\langle \vec{\xi}_\alpha \, , \, \vec{\xi}_\alpha \rangle = \frac{2\pi\rho U_k^2}{k} \left(\frac{a_1}{a_3}\right)^k \int_{-a_3}^{a_3} dz \left(a_3^2 - z^2\right)^k. \tag{4.14}$$

This integral can be performed using Mathematica or Maple. The result is

$$\int_{-a_3}^{a_3} \left(a_3^2 - z^2\right)^k dz = \frac{a_3^{2k+1}\sqrt{\pi}\,\Gamma\left(k+1\right)}{\Gamma\left(k+\frac{3}{2}\right)} \tag{4.15}$$

for $a_3 > 0$ and $Re(k) \geq 0$, and where Γ is the gamma function related to the factorial function by

$$\Gamma\left(k+1\right) = k!. \tag{4.16}$$

Another useful relation is [35]

$$\Gamma\left(k + \frac{3}{2}\right) = \frac{(2k+1)!!}{2^{k+1}}\sqrt{\pi}. \tag{4.17}$$

Using these, we finally get

$$U_k = \frac{1}{2^k k!}\sqrt{\frac{MR^2}{4\pi\rho a_1^{2k} a_3} k\left(2k+1\right)!}. \tag{4.18}$$

The normalized eigenvectors in the three coordinate systems are

$$\vec{\xi}_\alpha(x,y,z) = \frac{1}{2^k k!}\sqrt{\frac{MR^2}{4\pi\rho a_1^{2k} a_3} k\,(2k+1)!}\,(x+iy)^{k-1}(\hat{x}+i\hat{y}) \tag{4.19}$$

$$\vec{\xi}_\alpha(r_\perp,\phi,z) = \frac{1}{2^k k!}\sqrt{\frac{MR^2}{4\pi\rho a_1^{2k} a_3} k\,(2k+1)!}\, r_\perp^{k-1} e^{ik\phi}\left(\hat{r}_\perp + i\hat{\phi}\right) \tag{4.20}$$

$$\vec{\xi}_\alpha(r,\theta,\phi) = \frac{1}{2^k k!}\sqrt{\frac{MR^2}{4\pi\rho a_1^{2k} a_3} k\,(2k+1)!}\, r^{k-1}(\sin\theta)^{k-1} e^{ik\phi} \left(\sin\theta\hat{r} + \cos\theta\hat{\theta} + i\hat{\phi}\right). \tag{4.21}$$

4.1.4 T_α

I will now derive an expression for T_α more suitable for numerical calculations. First, recall equation (2.47),

$$T_\alpha = i\int d^3x\rho\vec{\xi}_\alpha^* \cdot (\vec{v}_0\cdot\nabla)\,\vec{\xi}_\alpha. \tag{4.22}$$

To work on the part $(\vec{v}_0\cdot\nabla)\,\vec{\xi}_\alpha$, I am going to work in the cartesian coordinates. First, since we are considering rigid rotation,

$$\vec{v}_0\cdot\nabla = \Omega_s\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right). \tag{4.23}$$

Applying this to (4.19), we find

$$(\vec{v}_0\cdot\nabla)\,\vec{\xi}_\alpha = \Omega_s(1-k)\left(-\xi_{\alpha y}\hat{x} + \xi_{\alpha x}\hat{y}\right) \tag{4.24}$$

$$= (1-k)\,\vec{\Omega}_s\times\vec{\xi}_\alpha \tag{4.25}$$

and

$$\vec{\xi}_\alpha^*\cdot(\vec{v}_0\cdot\nabla)\,\vec{\xi}_\alpha = -2i\Omega_s U_k^2\left(r_\perp^2\right). \tag{4.26}$$

Making use of (4.10) and (4.12), we finally find

$$T_\alpha = \Omega_s(1-k)\,MR^2. \tag{4.27}$$

4.1.5 $Q_{\alpha,\,kk}$

From equation (3.7), $Q_{\alpha,\,lm}$ for the case l=m=k is given by

$$Q_{\alpha,kk} = \int d^3x\rho\vec{\xi}^*_\alpha \cdot \nabla \left(r^k Y_{kk}(\theta,\phi)\right). \tag{4.28}$$

First notice that

$$r^k Y_{kk}(\theta,\phi) = \frac{(-1)^k}{2^k k!}\sqrt{\frac{(2k+1)!}{4\pi}} r^k (\sin\theta)^k e^{ik\phi} \tag{4.29}$$

$$= S_k \left(r \sin\theta e^{i\phi}\right)^k \tag{4.30}$$

$$= S_k (x+iy)^k \tag{4.31}$$

with

$$S_k \equiv \frac{(-1)^k}{2^k k!}\sqrt{\frac{(2k+1)!}{4\pi}}, \tag{4.32}$$

so that

$$\nabla\left(r^k Y_{kk}\right) = S_k k (x+iy)^{k-1}(\hat{x}+i\hat{y}). \tag{4.33}$$

Substituting into (4.28), we get

$$Q_{\alpha,\,kk} = 2U_k S_k k\rho \int d^3x r_\perp^{2(k-1)}. \tag{4.34}$$

Using (4.12), we finally find

$$Q_{\alpha,\,kk} = (-1)^k a_1^k a_3^{1/2}\sqrt{k\rho M R^2}. \tag{4.35}$$

4.1.6 Gravitational Radiation

To introduce the contribution of gravitational radiation into the equations of motion, I use different methods for the D and the Φ components.

As stated in an earlier chapter, equation (3.69) gives a term that has to be added to the equation for Φ. This gives, for the configuration space expansion

$$\ddot{\Phi} = -\frac{2\dot{D}\dot{\Phi}}{D} + \frac{GM'}{\mu D^2}\sum_k \frac{ikW_{kk}Q_{\alpha,kk}}{D^{k+1}} a_\alpha e^{ik\Phi} - \frac{32}{5}\frac{G^{7/2}}{c^5}\frac{\mu^2 M_+^{5/2}}{D^{7/2}} \tag{4.36}$$

and, for the phase space expansion, it is

$$\ddot{\Phi} = -\frac{2\dot{D}\dot{\Phi}}{D} + \frac{GM'}{\mu D^2}\sum_k \frac{ikW_{kk}Q_{\alpha,kk}}{D^{k+1}}\left(c_\alpha e^{ik\Phi} - c_\alpha^* e^{-ik\Phi}\right) - \frac{32}{5}\frac{G^{7/2}}{c^5}\frac{\mu^2 M_+^{5/2}}{D^{7/2}}. \quad (4.37)$$

For the D equation, since we already have an analytical solution giving $D(t)$ for a Keplerian orbit with the contribution of gravitational radiation (3.66), we can use this solution and add to it the contribution of the tidal interaction with the oscillations

$$D = D_{gw} + D_{tide}, \quad (4.38)$$

where (recall (3.66))

$$D_{gw} = \left(D_0^4 - 4\beta t\right)^{1/4} \quad (4.39)$$

and we have, for the slow rotation approximation

$$\ddot{D}_{tide} = -\frac{GM'}{\mu}\sum_k \frac{(k+1)\,W_{kk}Q_{\alpha,kk}}{D^{k+2}} a_\alpha e^{ik\Phi}, \quad (4.40)$$

and in the general case

$$\ddot{D}_{tide} = -\frac{GM'}{\mu}\sum_k \frac{(k+1)\,W_{kk}Q_{\alpha,kk}}{D^{k+2}}\left(c_\alpha e^{ik\Phi} + c_\alpha^* e^{-ik\Phi}\right). \quad (4.41)$$

This split is not strictly correct, since the equations are nonlinear in D. However, as long as the contribution from the tidal interactions is small compared to the contribution from the gravitational radiation part, it will be a good approximation, and it is the case here.

4.1.7 Real and Imaginary Parts

To numerically integrate the set of equations, I have to separate the real and imaginary parts of the equations. In the case of the slow rotation approximation, if I replace

$$a_\alpha = u + iv, \quad (4.42)$$

where I have omitted the subscript α on u and v for simplicity, the equations to be solved for the amplitude of the modes become

$$\ddot{u} = \kappa \cos(k\Phi) - 2T_\alpha \dot{v} - \left(\omega_\alpha^2 - 2T_\alpha\omega_\alpha\right) u \tag{4.43}$$

$$\ddot{v} = -\kappa \sin(k\Phi) + 2T_\alpha \dot{u} - \left(\omega_\alpha^2 - 2T_\alpha\omega_\alpha\right) v \tag{4.44}$$

where

$$\kappa = \sum_k \frac{GM' W_{kk} Q_{\alpha,\, kk}}{D^{k+1}} \tag{4.45}$$

and those for the orbital motion,

$$\ddot{D}_{tide} = \frac{-GM'}{\mu} \sum_{kk} \frac{(k+1)\, W_{kk} Q_{\alpha,\, kk}}{D^{k+2}} \left(u\cos(k\Phi) - v\sin(k\Phi)\right) \tag{4.46}$$

and

$$\ddot{\Phi} = \frac{2\dot{D}\dot{\Phi}}{D} - \frac{GM'}{\mu D^2} \sum_k \frac{k W_{kk} Q_{\alpha,\, kk}}{D^{k+1}} \left(v\cos(k\Phi) + u\sin(k\Phi)\right). \tag{4.47}$$

For the general solution, with

$$c_\alpha = a + ib, \tag{4.48}$$

the equations for the amplitudes become

$$\dot{a} = \frac{1}{z_\alpha} \kappa \sin(k\Phi) - \omega_\alpha b \tag{4.49}$$

and

$$\dot{b} = \frac{1}{z_\alpha} \kappa \cos(k\Phi) + \omega_\alpha a. \tag{4.50}$$

The equations describing the orbital motion are

$$\ddot{D}_{tide} = \frac{-2GM'}{\mu} \sum_k \frac{(k+1)\, W_{kk} Q_{\alpha,\, kk}}{D^{k+2}} \left(a\cos(k\Phi) - b\sin(k\Phi)\right) \tag{4.51}$$

and

$$\ddot{\Phi} = \frac{2\dot{D}\dot{\Phi}}{D} + \frac{2GM'}{\mu D^2} \sum_k \frac{k W_{kk} Q_{\alpha,\, kk}}{D^{k+1}} \left(b\cos(k\Phi) + a\sin(k\Phi)\right). \tag{4.52}$$

4.2 Simulations

For my simulations, I used a fourth order Runge-Kutta algorithm. To be able to compare the results using the two different formalisms, I first integrated simultaneously equations (4.43) to (4.47), substituting the results of (4.43) and (4.44) into (4.46) and (4.47) for every loop of the algorithm. I did the same thing for equations (4.49) to (4.52). Recall that I am using the split (4.38).

4.2.1 Parameters

To perform the calculations, I needed to choose the numerical parameters to be used. First recall that one of the stars, the companion, is point-like. It is thus described uniquely by its mass. Since a neutron star is born when the core of an aging supergiant star reaches the Chandrasekhar limit (the maximum mass the electron degeneracy pressure can support) and collapses, I use the Chandrasekhar mass as the mass of the two neutron stars forming my binary system, which is

$$M_{Ch} = 1.4 M_{\odot}, \tag{4.53}$$

where $M_{\odot}$ is the solar mass and is approximately 2.0×10^{30} kg.

Calculations considering the weight of a neutron star which is supported by neutron degeneracy pressure can be made (see [4] for example) and it is found that the radius of a $1.4M_{\odot}$ neutron star lies roughly between 10km and 15km. I am using a radius of 10km, so that the star has a higher density and is able to rotate more rapidly.

I have performed the calculations with M and a_3 constant instead of keeping ρ constant. In this situation, figure (4.1) does not give me the eccentricity at which the star has the fastest spin frequency. Keeping M $= 1.4M_{\odot}$ and $a_3 =$ 10km, the representation of the angular spin frequency is given in figure (4.4).

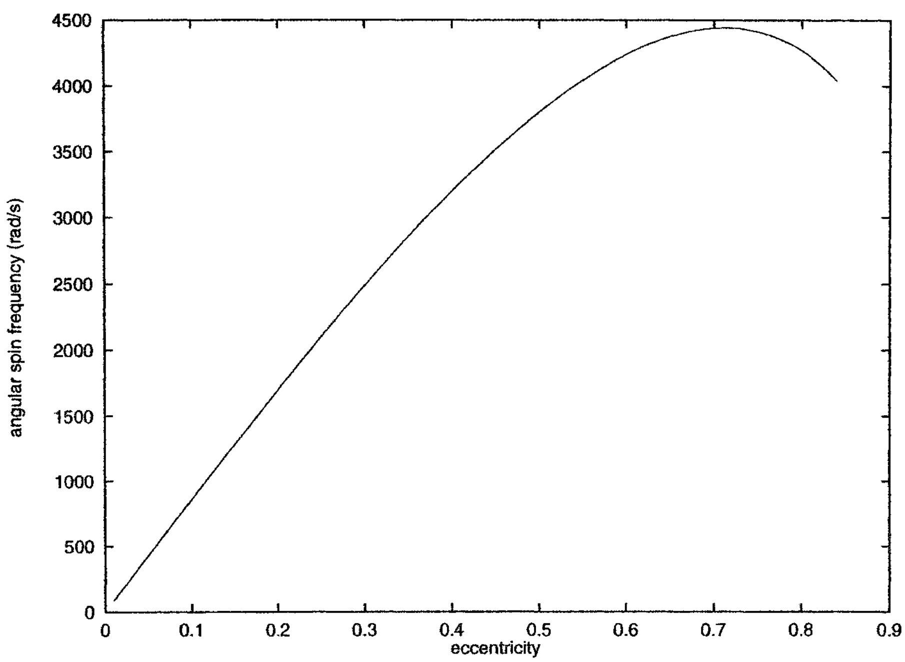

Figure 4.4: Angular spin frequency as a function of eccentricity for a $1.4M_{\odot}$ with a polar radius of 10km.

I have performed simulations using stars of two different eccentricities. As can be seen on figure (4.4), the maximum spin frequency is for a star of eccentricity about 0.71. Since I want to compare an approximation valid only for slowly rotating stars with the solution valid for a general case, I will use this eccentricity. It corresponds to an angular frequency of approximately 4440 rad/sec.

Pulsars are believed to be rotating neutron stars. The fastest pulsar observed (see [4] for example) has an angular frequency of approximately 4000 rad/sec. A neutron star with a frequency of 4440 rad/sec does not seem to be completely unrealistic. However, it is thought [2] that the stars will probably not be rotating that fast, so I will also show results for another case, that of a star rotating with a frequency of about 850 rad/s, which means an eccentricity of 0.1.

The binary system is going to emit gravitational waves strong enough to be possibly detected on Earth only when it has an orbital frequency between 100-1000 Hz [32], which is during the few last seconds of the final in-spiral. I am then going to study the end of their coalescence. Concretely, in the code, this means choosing an initial orbital period of 0.05 s. The equations derived in the earlier chapters are not valid when the stars are very close to each other since they do not consider the tidal deformations in the companion. Thus, I stop the calculations when the stars are getting too close. This occurs when there is a divergence in the behavior or when the distance between the stars is twice the radius of the studied star, since they would be merging.

The term with l = 0 (in equations (3.31) to (3.34) for example) represents a global expansion or contraction of the star, with no angular dependence. It is then not relevant for tidal interactions. For the term with l = 1 (and m = 1),

$$rY_{11} = const \times r \sin\theta e^{i\phi} \tag{4.54}$$

$$= const \times (x + iy) \,. \tag{4.55}$$

Since we are working in the center of mass system of coordinates, $Q_{\alpha,\,11}$ gives zero.

This term doesn't have to be considered either. We then begin with $k = l = 2$.

In equations (4.42) to (4.52), there is a summation over all possible values of k. However, the main contribution comes from the term with $k = 2$. The contribution for $k = 3$ is 100 times smaller, it is even smaller for $k = 4$ and so on. I could use only $k = 2$, but to add some precision, the modes for $k = 2, 3, 4$ are also included.

The effects of tidal interactions are more obvious at resonance. The condition for resonance was given in equation (3.73). But during the final in-spiral of the binary system, the orbital frequency changes so quickly that the passage through resonances is almost instantaneous and no effect can be seen. It is then not relevant to try to adjust the parameters of the simulation to study them. I am using the positive solution of ω_α and positive values of Ω_{orb} because this situation is more likely to occur.

Finally, the initial amplitudes of the oscillations are set to zero. They are excited by the tidal interaction with the companion.

4.2.2 Results

It is finally the time to show the results. Since the main point of this project is to see if the fact of using a slow rotation approximation introduces noticeable error in the results of the calculation, first look at figure (4.5). It shows the distance between the two stars as a function of time for a star with eccentricity 0.71 and an initial orbital period of 0.05 second. It is obvious that the slow rotation approximation introduces a big error in the result during the final in-spiral of a rapidly rotating star. The stars fall toward each other faster when using the phase space expansion. This would involve a difference in the gravitational waves emitted.

As stated earlier, $e = 0.71$ is not the case the most likely to occur. Look

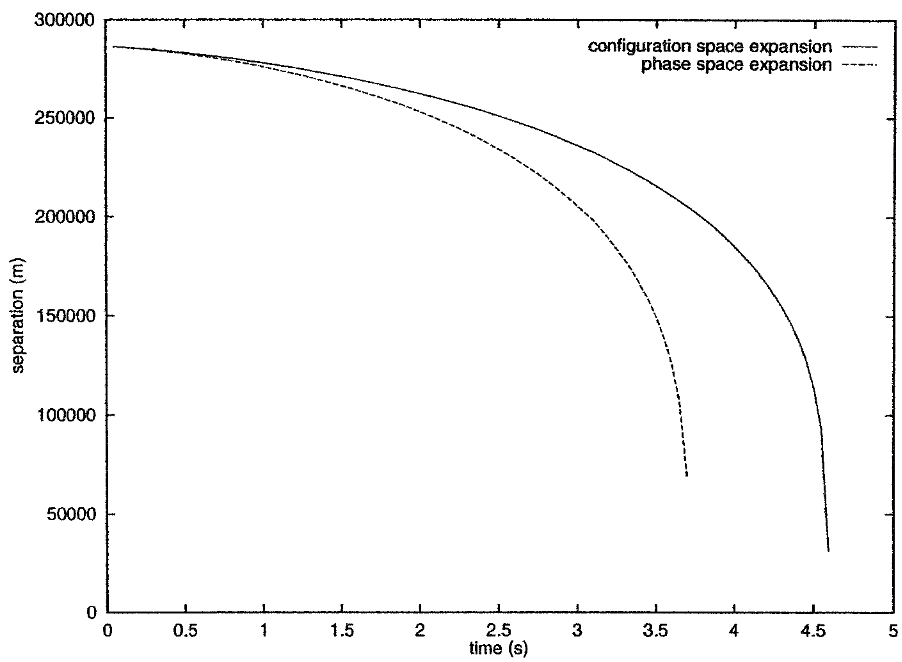

Figure 4.5: Separation of the stars for $e = 0.71$ and an initial orbital period of 0.05 second.

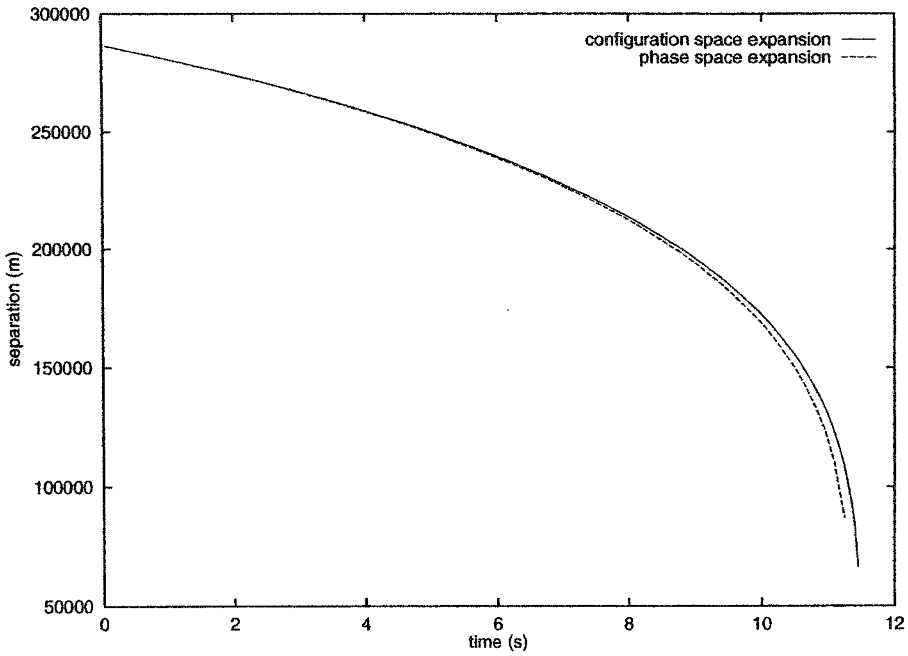

Figure 4.6: Separation of the stars for $e = 0.1$ and an initial orbital period of 0.05 second.

at figure (4.6). It shows the separation between the stars as a function of time, for a star of eccentricity $e = 0.1$. They start at the same initial orbital period as in the previous case, which means their initial separation is the same. The stars still move towards each other faster with the phase space expansion, but the difference between the results using the two expansions is smaller then for a more rapidly rotating star. The effect of tidal interactions is not as strong as in the previous case and the acceleration of the two stars is smaller. Nevertheless, this would still make a difference in the emitted gravitational waves.

If we define ΔD as the difference between the separation of the stars for the phase space expansion and the configuration space expansion,

$$\Delta D = D_{phase} - D_{conf}, \tag{4.56}$$

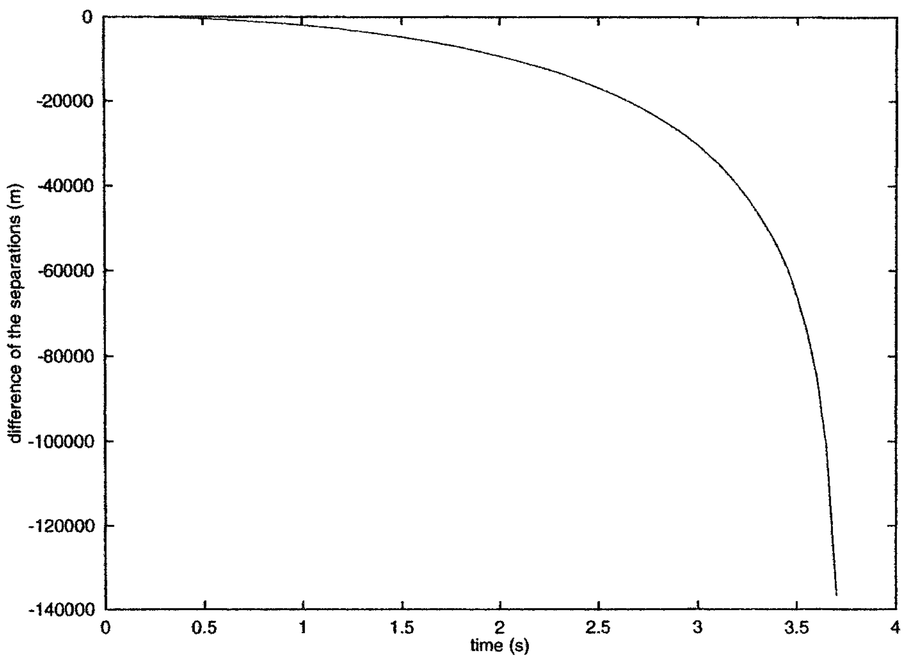

Figure 4.7: ΔD for a star with $e = 0.71$ and an initial orbital period of 0.05 second.

it is possible to have a better idea of the errors introduced by the slow rotation approximation in the two previous cases. It is shown on figures (4.7) and (4.8).

As you can see, even for a star with $e = 0.1$, the use of a slow rotation approximation introduces an error of over 20 km in the distance between the stars in less then 12 seconds.

These errors affect the number of orbits that the stars perform during a determined period of time, and consequently the phase. Let's define ΔN as the difference between the number of revolutions for the two expansions,

$$\Delta N = \frac{\Phi_{phase} - \Phi_{conf}}{2\pi}. \tag{4.57}$$

Figures (4.9) and (4.10) gives ΔN as a function of time for the two studied cases. For the case of a star with $e = 0.1$, the total number of orbits lies

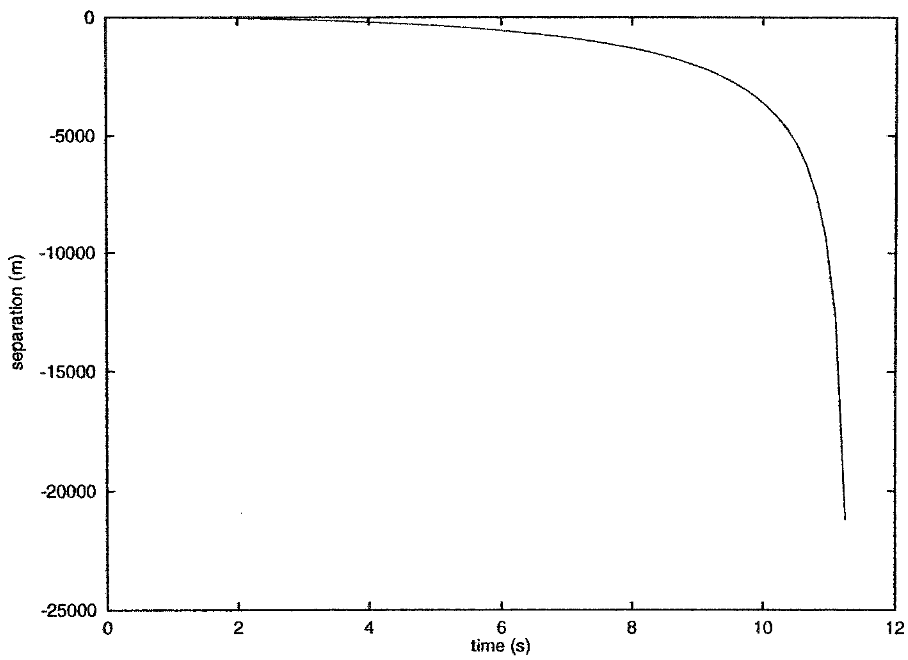

Figure 4.8: ΔD for a star with $e = 0.1$ and an initial orbital period of 0.05 second.

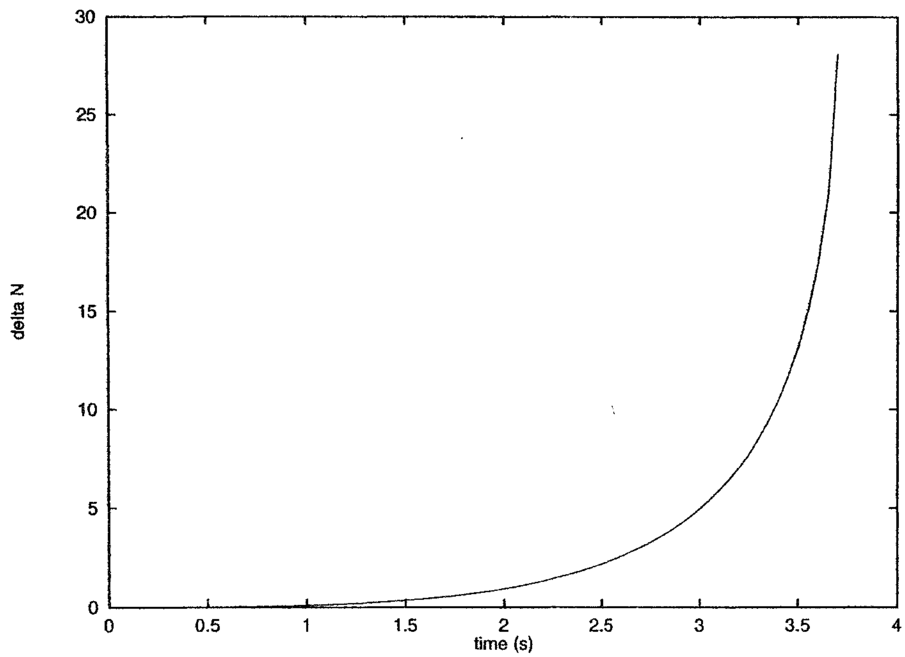

Figure 4.9: ΔN for a star with e = 0.71 and an initial orbital period of 0.05 second.

between 310 and 320, depending of the formalism used, while the coalescence lasts between 115 and 145 orbits in the case of a star with $e = 0.71$. As you can see in the figures, $\Delta N(e = 0.1)$ is approximately 10 orbits, and $\Delta N(e = 0.71)$ is approximately 30. We then have a relative error $\frac{\Delta N}{N}$ of 3% for e = 0.1 and 23% for $e = 0.71$. Recall that the accuracy has to be $< 0.01\%$ for detection.

The main conclusion that can be drawn from these results is that the approximation involved in using the configuration space expansion is not precise when it comes to determining the phase of the waveform of the gravitational radiation emitted by a coalescing binary system of neutron stars. Even though the errors involved are bigger for a faster rotating star, using a configuration space expansion as an approximation reduces the efficiency of the extraction of the signal from the noise even for a star with a spin frequency of approximately

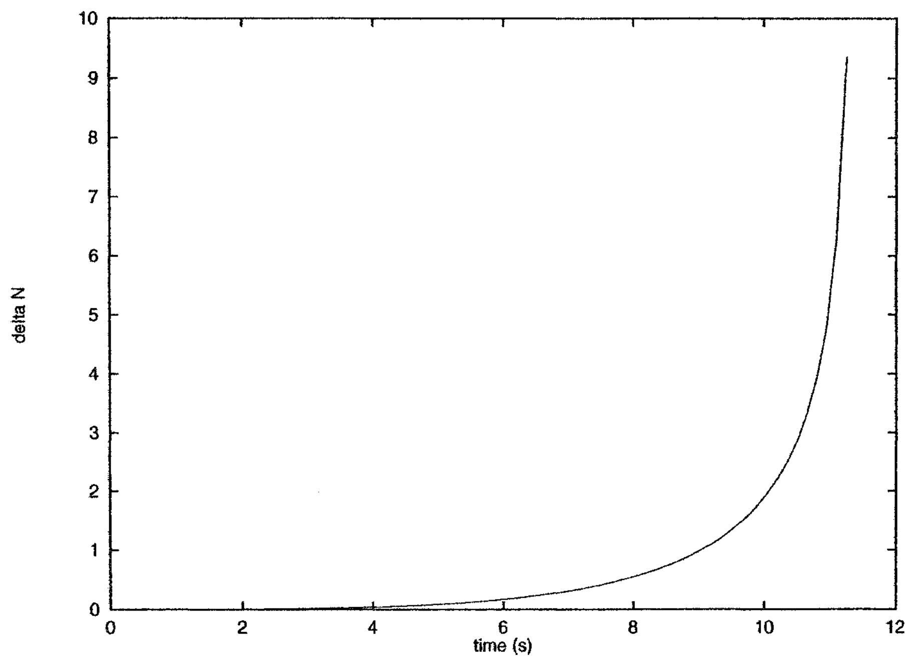

Figure 4.10: ΔN for a star with e = 0.1 and an initial orbital period of 0.05 second.

136Hz ($e = 0.1$).

Chapter 5

Conclusion

The main goal of this project was to study the orbital evolution of a binary system of rotating neutron stars using two different formalisms. One of the stars was supposed to be point-like, and it excited f-mode oscillations inside the other star. The star was supposed to be made of fluid and to have a constant density.

The first formalism involves an expansion of the eigenmodes of oscillation in the configuration space and the resulting equations for the amplitudes of the modes are coupled. In order to be able to solve these equations, one has to assume that the coupling terms are negligible. It is done by assuming that the orthogonality condition valid for the non-rotating stars holds for rotating stars. This is likely to introduce errors in the results.

The second formalism, developed by Schenk et al. in [28] involves an expansion in the phase space. The resulting equations describing the amplitudes of the modes are this time uncoupled from each other. There is no need to cut out terms in order to solve the equation.

By comparing the results using the two formalisms, I could determine the error on the predicted orbital evolution of the system when the configuration space expansion is used. This is important to know because a difference in the phase of the orbital motion creates a difference in the phase of the waveform of the emitted gravitational radiation.

I studied the case of a $1.4M_{\odot}$ star of polar radius of 10km with two eccentricities, $e = 0.1$ and $e = 0.71$. A star with $e = 0.1$ has an angular spin frequency of approximatively 850 rad/sec, and a star with $e = 0.71$ has a spin frequency of 4440 rad/sec.

In both cases, there was a non-negligible difference between the the results from the two formalisms. In the case of the e = 0.1 star, the relative error in the number of orbits during coalescence, $\frac{\Delta N}{N}$ was about 3% while it was about 23% in the case of a e = 0.71 star. Even though the error is a lot bigger for a rapidly rotating star, in both cases the error is big enough to reduce the ability to extract the signal from the noise for the data that are eventually going to be provided by gravitational wave detectors. In order to build an accurate theoretical template, the phase space expansion formalism should be used.

These calculations were made using the simplest model of a neutron star. To be more realistic and to get more accurate results, these calculations should be done using corrections for general relativistic effects. They should also involve a star with a density higher in its core than in its outer layers, and they should take into consideration the viscosity of the neutron star material.

Bibliography

[1] Arfken, G.B.; *Mathematical Methods for Physicists*, Third Edition; Academic Press, New York; (1985).

[2] Bildsten, L. & Cutler C.; *Tidal Interactions of Inspiraling Compact Binaries*; The Astrophysical Journal, **400**:175-180; (1992).

[3] Bryan, G.H.; *The waves on a Rotating Liquid Spheroid of Finite Ellipticity*; Philos. Trans. R. Soc. London A **180**:187; (1889).

[4] Carroll, B.W. & Ostlie, D.A.; *Modern Astrophysics*; Addison-Wesley, (1996).

[5] Chandrasekhar, S.; *Ellipsoidal Figures of Equilibrium*; Dover Publications, Inc.; (1987).

[6] Clark, J.P.A.; *Limits on the finite size effect in gravitational radiation*; Astrophysical Letters, **18**:73-74; (1977).

[7] Cohen-Tannoudji, C., Diu, B., & Laloë, F.; *Mécanique quantique I*; Hermann, éditeur des sciences et des arts, Paris; (1996).

[8] Comins, N.; *On secular instabilities of rigidly rotating stars in general relativity - I. Theoretical formalism*; Mon. Not. R. Astron. Soc. **189**:233-253; (1979).

[9] Comins, N.; *On secular instabilities of rigidly rotating stars in general relativity - II. Numerical results*; Mon. Not. R. Astron. Soc. **189**:255-272; (1979).

[10] Cox, J.P.; *Theory of Stellar Pulsation*; Princeton University Press, Princeton, New Jersey; (1980).

[11] Friedman, J.L., & Schutz, B.F.; *Lagrangian Perturbation Theory of Nonrelativistic Fluids*; The Astrophysical Journal, **221**:937-957; (1978).

[12] Friedman, J.L., & Schutz, B.F.; *Secular Instability of Rotating Newtonian Stars*; The Astrophysical Journal, **222**:281-296 (1978).

[13] Goldstein, H.; *Classical Mechanics second edition*; Addison-Wesley; (1980).

[14] Guyon, É.,& Hulin, J.P.,& Petit, L.; *Hydrodynamique physique*; InterEditions/Editions du CNRS, Paris; (1991).

[15] Ho, W. C. G., & Lai, D.; *Resonant Tidal Excitations of Rotating Neutron Stars in Coalescing Binaries*; Mon. Not. R. Astron. Soc. **308**:153-166, (1999).

[16] Jackson, J.D.; *Classical Electrodynamics* third edition; John Wiley & Sons, Inc.; (1999).

[17] Kochanek, C.S.; *Coalescing Binary Neutron Stars*; The Astrophysical Journal, **398**:234-247, (1992).

[18] Kokkotas, K.D. & Schäfer,G.; *Tidal and tidal-resonant effects in coalescing binaries*; Mon. Not. R. Astron. Soc., **275**:301-308, (1995).

[19] Lai, D.; *Dynamical Tides in Rotating Binary Stars*; The Astrophysical Journal, **490**:847-862, (1997).

[20] Marion, J.B., & Thornton, S.T.; *Classical Dynamics of Particles & Systems, Third Edition*; Harcourt Brace Jovanovich, Publishers; (1988).

[21] Mashhoon, B.; *Tidal Gravitational Radiation*; The Astrophysical Journal, **185**:83-86; (1973).

[22] Mashhoon, B.; *Tidal radiation*; The Astrophysical Journal, **216**:591-609; (1977).

[23] Misner, C.W., Thorne, K.S., & Wheeler, J.A.; *Gravitation*; W. H. Freeman And Company, New York; (1973).

[24] Peters, P.C.; *Gravitational Radiation and the Motion of Two Point Masses*; Phys. Rev. **136**:1224-1232, (1963).

[25] Peters, P.C. & Mathews, J.; *Gravitational Radiation from Point Masses in a Keplerian Orbit*; Phys. Rev. **131**:435-440, (1964).

[26] Press, W.H. & Teukolsky, S.A.; *On Formation of Close Binaries by Two-Body Tidal Capture*; The Astrophysical Journal, **213**:183-192, (1977).

[27] Reisenegger, A. & Goldreich, P.; *Excitation of Neutron Star Normal Modes During Binary Inspiral*; The Astrophysical Journal, **426**:688-691; (1994).

[28] Schenk, A. K., Arras, P., Flanagan, É. É., Teukolsky, S. A., & Wasserman, I.; *Nonlinear mode coupling in rotating stars and the r-mode instability in neutron stars*; Phys. Rev. D 65 (2002) 024001.

[29] Shapiro, S.L. & Teukolsky, S.A.; *Black Holes, White Dwarfs, and Neutron Stars*; John Wiley & Sons, Inc.; (1983).

[30] Shibata, M.; *Effects of Tidal Resonances in Coalescing Binary Systems*; Progress of Theoretical Physics, **91**, No. 5:pp. 871-883; (1994).

[31] Thorne, K.S.; in 300 Years of Gravitation; editions S.W. Hawking and W. Israel; Cambridge University Press; (1987).

[32] Thorne, K.S.; *Probing Black Holes and Relativistic Stars with Gravitational Waves*; in Black Holes and Relativistic Stars, Edited by Robert M. Wald; The University of Chicago Press; (1998).

[33] Turner, M.; *Tidal Generation of Gravitational Waves From Orbiting Newtonians Stars I. General Formalism*; The Astrophysical Journal, **216**:914-929; (1977).

[34] Weinberg, S.; *Gravitation and Cosmology*; John Wiley & Sons; (1972).

[35] Weisstein, E.; *Eric Weisstein's World of Mathematics*; http://mathworld.wolfram.com/ ; (2002).

[36] Witte, M.G. & Savonije, G.J.; *The dynamical tide in a rotating 10* $M_\odot$ *main sequence star*; Astron. Astrophys., **341**:842-852; (1999).