

Modeling hybrid tank pilot plant using first principle model and experimental data

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Abstract

The Hybrid tank pilot plant was designed in Process Control Laboratory (PCL) of the University of Alberta at 2009. In this report, we try to construct a model for this process. Using the physical behavior of the plant, it is possible to have the nonlinear first principle dynamic model of that plant. However, the parameters of the model are unknown. Some experiments are designed to estimate the parameters of the model. Then the model is linearized using the relation between the nonlinear and linear model.

Introduction

The Hybrid tank pilot plant was designed in Process Control Laboratory (PCL) of the University of Alberta at 2009. Figure 1 shows a photo of the plant, where its schematic diagram is given in Figure 2. This plant consists of three equal sized process tanks installed in the same level. Each of the right and left tanks are connected to the middle tank through 3 pipes; one in the bottom of the tanks and the others in 2 different levels. All three tanks have another pipe in the bottom to the discharge tank installed in a lower level. All these pipes are equipped with on/off control valve. Also, the tanks are equipped with an overflow discharge pipe installed inside the tanks. The tanks are transparent which are helpful to understand the process behavior. Water is pumped by two pumps to the left and right hand side tanks and can be transferred to the middle tank through the connection pipes. The speeds of the pumps are manipulated and the flows of the water are measured by flow sensor. The levels of all three tanks are measured by differential pressure (DP) sensors. The process instruments are connected through an OPC OPTO22 server to the computer network. A computer is dedicated to operate the process, which can communicate with the process through Matlab. A Simulink, program called "Hybrid_Tank_Simulink.slx", is provided to read all the sensor data and send commands to the 2 pumps and 8 control valves. The program also has the ability to control the process.



Figure 1. Picture of the Hybrid Tank pilot plant

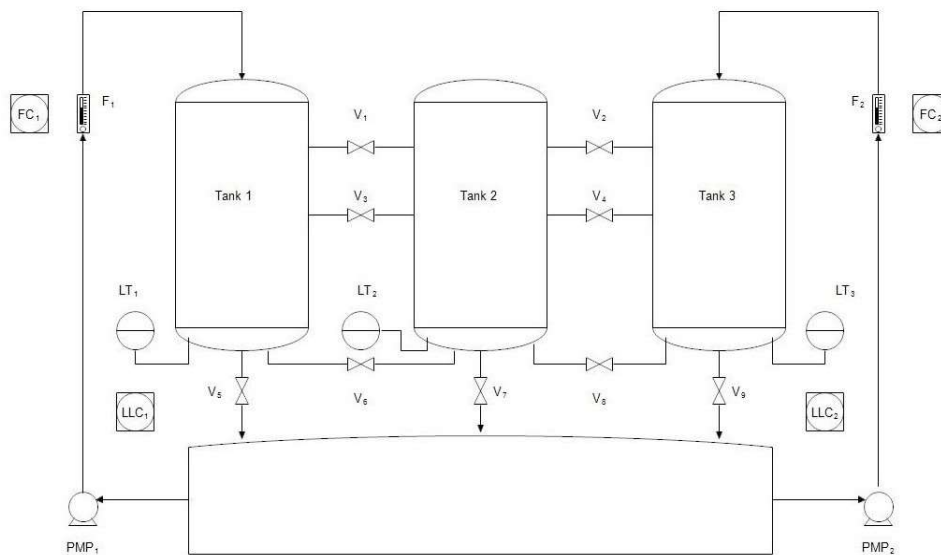


Figure 2. Complete schematic diagram of the hybrid tank plant

In this report, we try to construct a model for this process. Using the physical behavior of the plant, it is possible to have the nonlinear first principle dynamic model of that plant. However, the parameters of the model are unknown. Some experiments are designed to estimate the parameters of the model. Then the model is linearized using the relation between the nonlinear and linear model.

Nonlinear model structure

In this experiment, the input signal to the pump of the left hand side tank is manipulated and the level of both left and middle tanks are measured; right hand side tank does not used in this experiment. The level of the middle tank is also measured using the image of a camera placed in front of the middle tank. Figure 3 depicts the schematic of the process. Depend on the valve positions the dynamic model pf the plant changes. We assume that the valves V_3 , V_6 and V_7 are open and all others are closed. Also we assume that the level of water in the middle tank (Tank 2) is above V_3 , therefore, we can replace V_3 and V_6 with an equivalent valve called V_{36} .

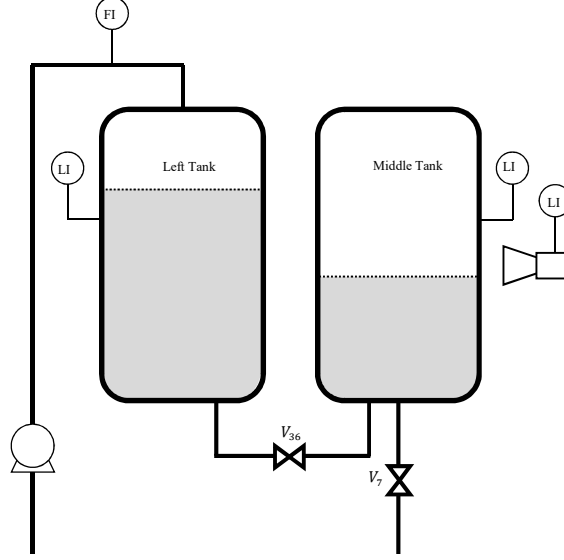


Figure 3. The schematic of the experimental plant

The rate of change of the volume of liquid in a tank equals to the difference between the liquid flows into the tank and the liquid flows out of that:

$$\dot{V} = Q_i - Q_o \quad (1)$$

where V is the volume of liquid inside the tank, Q_i is the sum of all input flow and Q_o is the sum of all output flow. In a cylindrical tank that the cross section of the tank is constant in all levels of the tank, then

$$\dot{H} = \frac{1}{S} (Q_i - Q_o) \quad (2)$$

where S is the cross section area of the tank. On the other hand, the flow passed a pipe is a function of the difference between the pressure in either side of that, the viscosity of the liquid and the conductivity of the pipe:

$$Q = \bar{k} \sqrt{\frac{2(P_i - P_o)}{\rho}} \quad (3)$$

where Q is the flow passed through the pipe, P_i is the pressure in the beginning of the pipe, P_o is the pressure in the end of the pipe, ρ is the density of the liquid and \bar{k} is the coefficient of the pipe which represent the inverse of resistance (conductivity) of the pipe. The value of the parameter \bar{k} depends on the cross section of the pipe and the shape of that. To compute the flow between two tanks, the pressure in either side of the connection pipe is the free air pressure plus the pressure imposed due to the height of the water in them

$$P = P_{air} + 2\rho gH \quad (4)$$

where g is the gravitational acceleration. If both P_i and P_o has the free air pressure, therefore the flow of water between the two tanks is obtained as

$$Q_o = \bar{k} \sqrt{\frac{2\rho g(H_1 - H_2)}{\rho}} = k\sqrt{H_1 - H_2} \quad (5)$$

where H_1 is the higher level and H_2 is the lower height; it is obvious that the flow is from higher level tank to the lower one. Parameter k stands as equivalent to the valve coefficient but relate the flow to the height difference.

Using the above relations, the nonlinear state space model of the hybrid tank plant in Figure 3 is obtained as follows:

$$\begin{aligned} \dot{H}_l &= \frac{-k_1}{S_l} \sqrt{H_l - H_m} + \frac{1}{S_l} Q_i \\ \dot{H}_m &= \frac{k_1}{S_m} \sqrt{H_l - H_m} + \frac{-k_2}{S_m} \sqrt{H_m} \end{aligned} \quad (6)$$

$$\begin{bmatrix} Y_l \\ Y_m \\ Y_c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} H_l \\ H_m \end{bmatrix} + \begin{bmatrix} v_l \\ v_m \\ v_c \end{bmatrix} \quad (7)$$

where H_l and H_m denotes the level of the water in the left and the middle tanks, respectively, and Q_i is the input flow to the left tank. Y_l and Y_m are respectively the level of the left and middle tank measured by the DP level sensors and Y_c is the level of the middle tank measured by the camera image sensor. k_1 , k_2 , S_l and S_m are respectively the coefficients of the valves V_{36} and V_7 and the cross section area of the left and middle tanks.

Parameter estimation

Sensor calibration

The next step is to calibrate the sensor data. Neither of the sensor data read in the Matlab denotes the real value of the variable. This is not anything strange; all sensors should be calibrated either on the sensor transmitter or in the controller side. To calibrate them, we design some experiment to measure the actual value of the variable. Comparing that value with the value read in the Matlab, helps to calibrate the sensor data. Having the value of any variable in two different measurement points is sufficient to construct a simple linear regression calibration formula. However, to improve the accuracy, we measure that in more measurement points.

Level sensors

The level of the water can be measured directly by a tape meter. However, the following points should be taken into account:

- 1) The level measured by the tape meter is supposed to be the real value of the level. So, careful reading of the value is crucial.
- 2) Due to capillary action of water, when we look straight to the water level, we can see two edges for that. The upper one is due to capillary effect. Since the tanks are not too narrow, the lower edge shows the more correct water level.
- 3) The positions of the 3 interactive pipes are different. The bottom one (includes valve V_3) is 18.73 cm (equal to $7\frac{12}{32}$ inch.) lower than the bottom of the tanks (I use an English scale tape meter and then convert that to centimeter. This is the reason the values seems so precise!). According to model (6), on one hand, the difference between tank levels is important for the flow between the tanks. From this point of view this value can be compensated or not for the left tank. On the other hand, the absolute value of the middle tank is important to compute the flow out of middle tank through valve V_7 . So, we add that to the levels.
- 4) The pipe from the middle tank to the storage tank is ended 22.86 cm lower than the bottom of the tank. So this value should be added to the middle tank level to be used in the term $\sqrt{H_m}$ of (6). On the other, as mentioned above, for the other terms including $\sqrt{H_l - H_m}$, the difference between the levels is important instead of their absolute value. So, I add this compensation term to both of them. *Note:* I will show later that this value should again be corrected according to the pipe coefficient.
- 5) The camera data acquisition card is installed in a separate computer dedicated to Multi-Tank pilot plant. Instead of moving it to the hybrid tank computer, we synchronize the data of the image with that of the hybrid tank. A

program is prepared in Matlab to do that. The program is called “HybridTankCameraPlantSynchronization.m”. To use this program, it is necessary to initially synchronize the clock of the two computers. Then, run the plant and get data. The synchronizing program consider the starting time of running each of the programs of camera image and hybrid tank sensor and according to that and the end time, select the portion of the time both run and synchronize the time tag of them. Please notice that you should get the starting time of the programs and save that together with the data file. The file to run the Hybrid Tank is “HybTnkIKFTest2015030501.slx” and the one get the camera data is “tank3_MATLAB_ImageAcquisition_Module.mdl”. Two short Matlab programs are prepared for getting and saving the starting time, called “HybTnkPlantSim2015030501.m” and “HybTnkImage2015030501.m” run in each of the computers.

- 6) If the position of the camera changes, the calibration needs to be corrected. I adjust the camera to be right in front of the middle tank, nearly in the same level as the midpoint between the two top interface valves (valves V_1 and V_3 in Figure 2). Also two legs of the tripod touch the frame of the plant. However, you may need to calibrate it again every time it moves.

To measure the levels for calibration, we filled the tanks when the drain valves to the storage tank (valves V_5 and V_7) are closed but the interface valves are open. Then the pump is stopped and all valves are closed. After a few minutes that the tank level stays in its steady state, the level is measured using tape meter. The DP sensor data value is also available all through the test. This data value is averaged for some (around 100) samples to compensate for the noise and drifts. This gives one data point information. Then, either one of the drain valves or interface valves are opened for a while to get another level. This procedure is repeated to get some measurement data as depicted in Table 1.

Table 1

Data point	Middle Real	Middle DP	Middle Image*		Left real	Left DP
1.	42.00	56.62	-		55.32	92.13
2.	43.26	61.39	-		53.42	87.29
3.	44.69	64.98	-		51.99	83.64
4.	46.83	70.43	-		49.77	78.13
5.	43.66	62.19	-		59.53	102.58
6.	35.72	42.24	-		56.20	94.30
7.	47.86	72.50	24.24		51.67	82.73
8.	44.85	65.24	19.40		48.82	75.35
9.	42.15	58.25	14.63		43.74	62.56
10.	39.05	50.46	9.356		36.20	34.66
11.	35.80	42.24	3.70			-
12.	39.05	50.47	9.356			-
13.	41.04	61.92	16.95			-
14.	46.43	69.25	21.82			-
15.	39.45	51.70	10.16			-
16.	35.88	42.56	4.80			-

* The camera measurement did not saved for some of the early points. Also due to change of the position after calibration and during the rest of the experiment, a constant value of 0.178 is added to the real measurement of the middle tank, when we find the calibration coefficients.

Note: Data of the level sensors are stored in the following files:

Level DP Data: HybTnkPlant20150615Data01 (also 03 & 04)

Image Data: HybTnkImage20150615Data04

For regression, the command “regression” in Matlab is used, where the sensor data is selected as input and the tape measurement as the real output. The regression gives the bias b and coefficient m , which can be sued as following to calibrate the measured data:

$$v_{calib} = m * v_{data} + b \quad (8)$$

Figure 4 shows the real and the calibrated value versus the DP sensor for the middle tank level sensor. It is clear that two of the data points are out of range, which is likely happened due to mistake in reading the tape meter. These are data number 1 and 14 in Table 1. I supposed them to be outlier and remove them. Figure 5.a shows the quality of regression of this variable after removing the outliers. The quality of the regression improves from the error variance of 0.4196 to 0.0036. The same procedure shows that data number 10 is an outlier in the left tank data set and

removed. The quality of the calibration and the calibration coefficients are given in Figure 5 and Table 2. The program File “HybTnk2015061601Calibration.m” perform the whole calibration procedure.

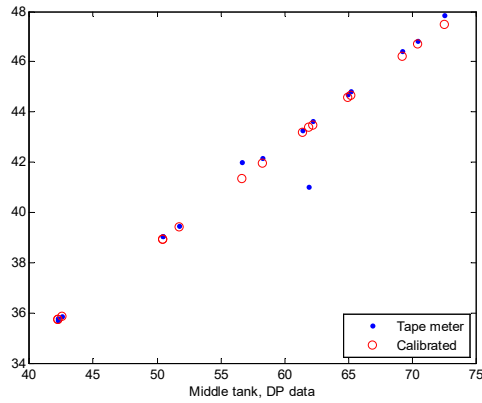


Figure 4. Middle tank calibration performance before removing the outliers.

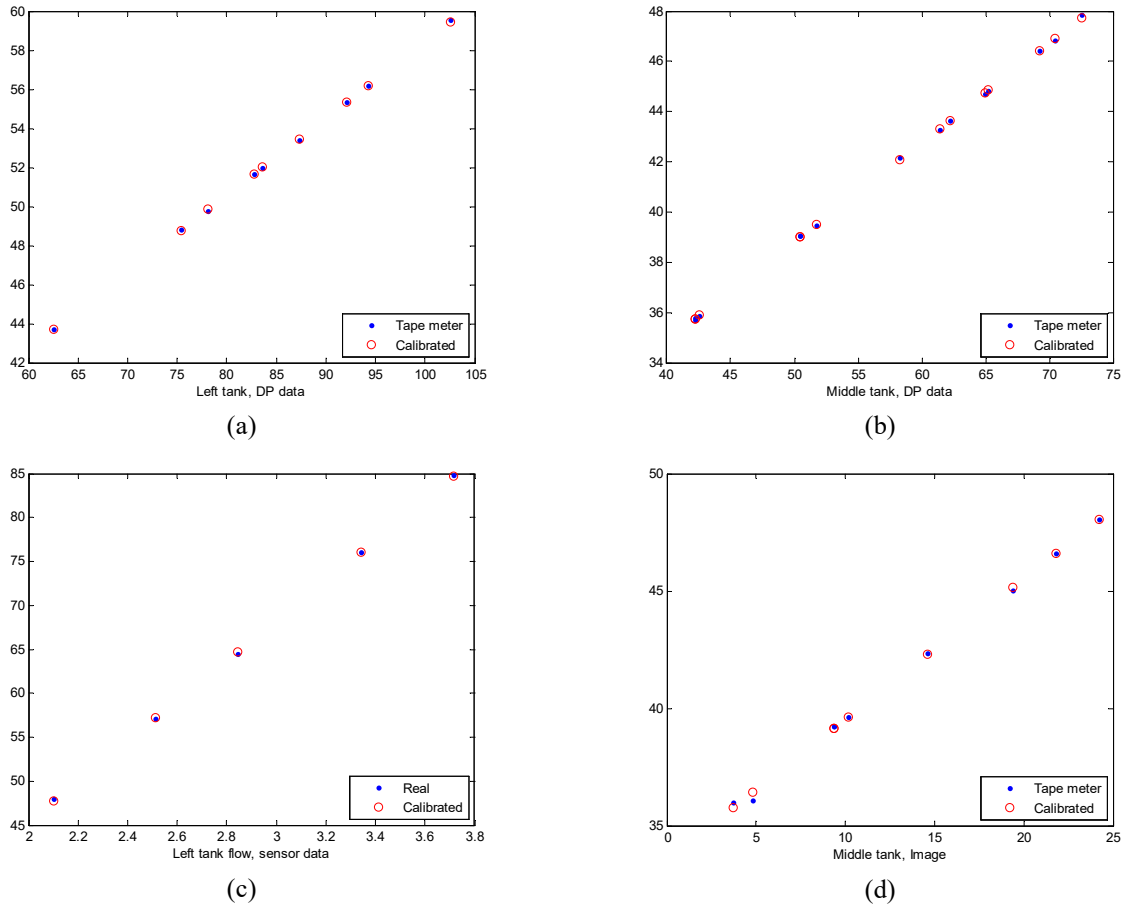


Figure 5. Sensor calibration performance

Table 2. Sensor calibration coefficient and accuracy of calibration

	Coefficient, m	Bias, b	Variance
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Middle Level DP sensor	0.3955	19.0497	0.0036
Middle Level, Image	0.5973	33.5517	0.0262
Left Level, DP sensor	0.3937	19.0796	0.0022
Left Flow	22.9014	-0.3824	0.0293

Note: The coefficient value of the DP sensors shows that the data value read in the Matlab are in inch. Also the level is likely measured from the bottom of the tank.

Flow sensor

To calibrate the flow sensor, we need to measure flow independent of the flow sensor. For this purpose, we apply constant control signal and fill the left side tank for some known time period. The volume of the water can be measured by measuring the height of the water (calibrated value) and considering the cross section area of the tank. Dividing the volume by the time period gives the exact input flow to the tank. This flow value can be used to estimate the calibration coefficient and bias. To perform this test, at the beginning, the level of the tank is very low. Then, while all valves are closed the tank is filled. After filling it to some level, the pump is stopped. We wait for some long period to be sure that the plant is in the steady state and then the level is measured. After that, the drain valve V_5 is opened to drain the tank to some low level. We apply the same procedure in 5 points to improve the estimation accuracy. The whole data set for this test is given in the file “HybTnkPlant20150615Data05.mat”. Figure 6 shows part of the test and Table 3 presents the flow measurement of 5 data points used for calibration. The estimated calibration parameters are given in Table 2. Figure 5.c illustrates the calibration accuracy. It is clear that the calibration has a good performance.

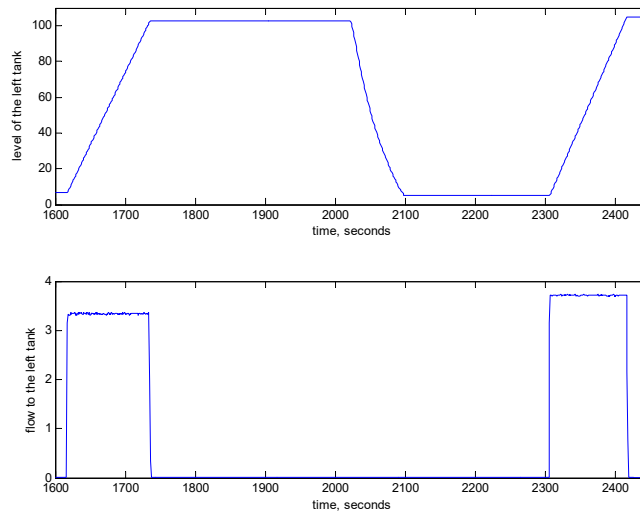


Figure 6. Data of two test point for the flow sensor calibration

Table 3. The real and sensor data of the left tank flow used for sensor calibration

Data point	Real flow	Sensor data
1.	64.48	2.84
2.	57.118	2.51
3.	76.10	3.34
4.	84.84	3.72
5.	47.97	2.10

Measurement noise estimation

DP cell level sensors

To measure the noise, the general rule is to operate the plant in some steady state point and log data for some long period. After removing the mean value, the variance of the noise can be estimated from the data. Figure 7 shows the left tank level measurement. As can be seen even though the pump is stopped and the level is supposed to be

constant for around 25 minutes, but the level sensor is not yet reached to its steady state. To compensate for that, we use only some last few minutes of data, and also use a linear regression to remove the slow rate trend of the data. Figure 8 and Figure 9 presents the data used for the noise variance estimation. This gives the variance of the left tank level sensor equal to 6.33×10^{-6} and variance of middle tank level sensor equal to 2.42×10^{-6} . Since the sensors are form the same type, one possibility is to use the average of this two values as the variance of both. This gives 4.38×10^{-6} as the variance of both DP cell level sensors. The data of this test is stored in the file “HybTnkPlant20150616Data03.mat”.

Figure 8 and Figure 9 also shows that the values are quantized. The quantization value for both of the sensors are 0.002 which give a quantization error variance equals to 3.3×10^{-7} . Although this is a significant value but it is sufficiently small compare to the variance estimation of both sensors. So, the main source of the error in DP cells is not quantization error.

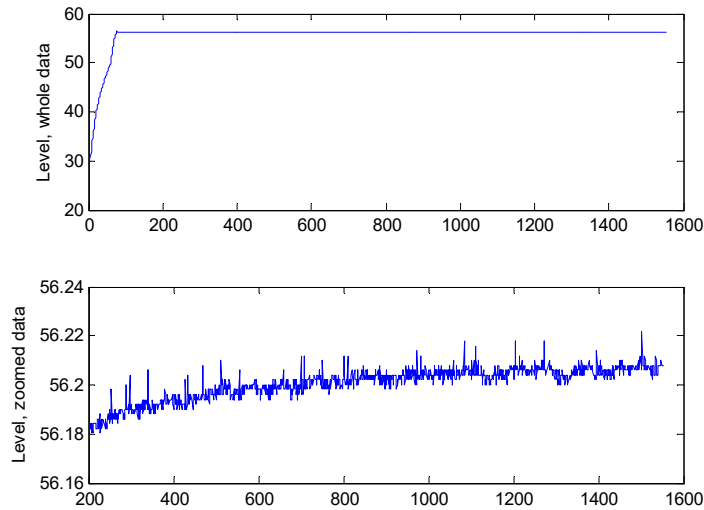


Figure 7. Level sensor output for noise estimation. Top) Whole data. The pump runs for around initial 70 seconds. Bottom) The level zoomed in from time 200 to the end.

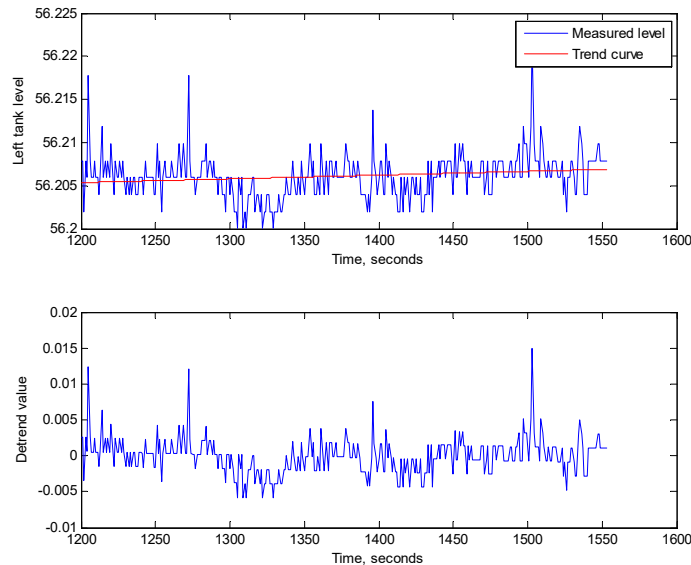


Figure 8. Left tank DP level sensor before and after detrending

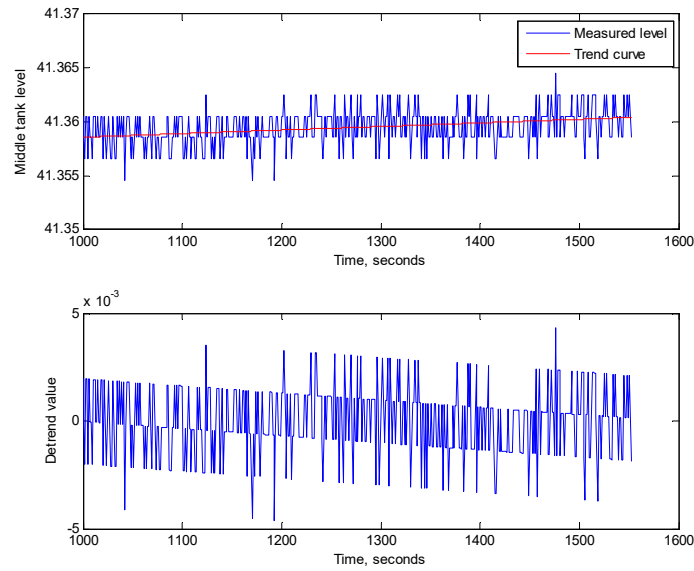


Figure 9. Middle tank DP level sensor before and after detrending

Flow sensor

To measure the flow sensor noise variance, we set the control signal in different value and wait for a long time to reach to steady state. Figure 10 shows the whole test data which is stored in the file “HybTnkPlant20150616Data01.mat”. As can be seen, the noise variance is different depend on the flow value. After estimating the variance in all 8 operating points, the average of that is selected as the estimation of the noise level. Due to the same reason as explained above for the level sensors, to estimate the noise level for each operating point, data in some period of the time is selected and detrended. The final noise variance of the flow sensor is obtained as 0.0719.

Program “HybTank2015061601DPvariance.m” performs the noise level estimation for the DP level sensors and flow meter.

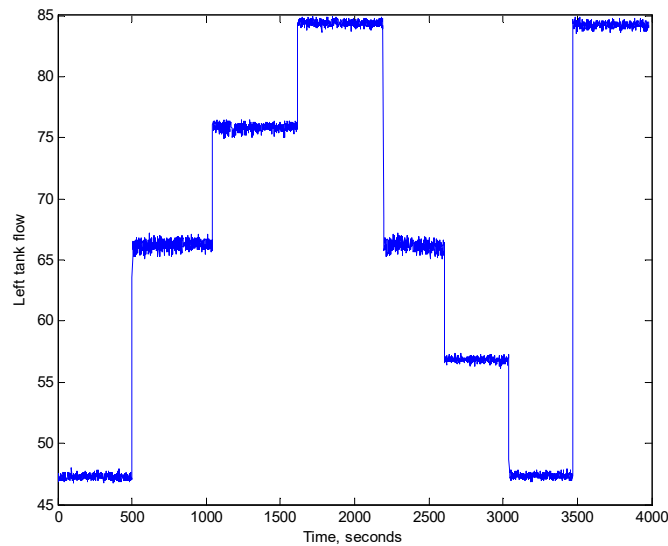


Figure 10. Measured flow to estimate the flow measurement noise variance

Image level sensor

The camera image level sensor has a large quantization error. So, if the level of the tank is kept constant, there will be no change in the output of the image level sensor. The quantization error has a uniform distribution. If we estimate the quantization error it is possible to compute the variance of the error. To measure that, we change the level of the middle tank and measure the quantization error directly from the data. Figure 11 shows the output value of the image level sensor. The step changes are due to quantization value change. It is seen that there are two levels of step. The real quantization step is the small one approximately equal to 0.1227 cm. However, effectively, many steps have larger value approximately equal to 0.2453. We assume that half of the noise is based on either of the steps. As a result, the variance of the noise is obtained as follows

$$var_{image} = 0.5 \times \left(\frac{0.2453^2}{12} + \frac{0.1227^2}{12} \right) = 3.1368 \times 10^{-3} \quad (9)$$

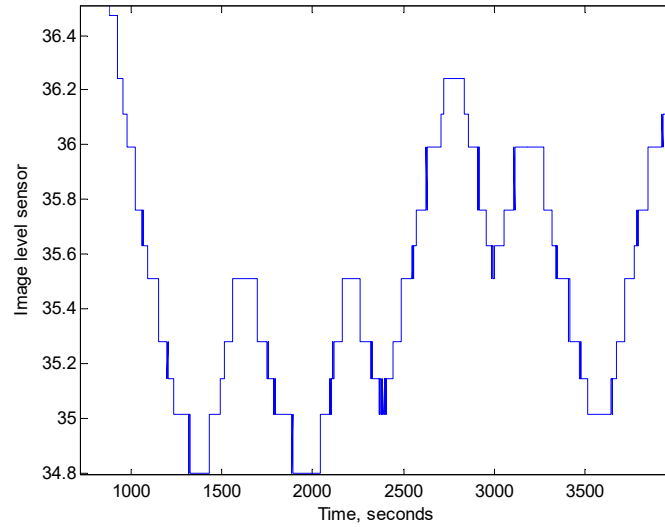


Figure 11. Output of camera level detection

Model parameter estimation

Model (6) has 4 parameters which are cross section area of the tanks and valve coefficients. In this section, these parameters are estimated

Tank cross section area

S_l and S_m are equal in this plant and can be measured directly from the plant. There is an overflow pipe inside the tanks, which its area must be subtracted from the tank tube cross section area. The diameter of the inner side of the tank is $2R = 17.78 \text{ cm}$. The outer side diameter of the overflow pipe is $2r = 2.54 \text{ cm}$. Therefore the cross section area of the tank is

$$S_l = \pi(R^2 - r^2) = 243.22 \text{ cm}^2 \quad (10)$$

$$S_m = S_l \quad (11)$$

Valve coefficient

According to (6), in the steady state when the level of the tanks are constant in the operating point, i.e. $(\dot{H}_l, \dot{H}_m) = (0, 0)$, then the valve coefficients can be computed using the steady state value of the flow and levels using the following equations:

$$k_1 = \frac{Q_i^o}{\sqrt{H_l^o - H_m^o}} \quad (12)$$

$$k_2 = \frac{Q_i^o}{\sqrt{H_m^o}} \quad (13)$$

where Q_i^o , H_l^o and H_m^o are flow and tanks level value in the steady state. So, to estimate the valve coefficient, the plant is run using a constant control signal. After the levels reach the steady state the value of the flow and level variables are measures and used in (12) and (13) to estimate the valve coefficient. To reduce the effect of noise, the mean value of the variables in a couple of hundred samples is used. Program “HybTnk2015061601ValveCoef.m” performs this test. To improve the accuracy, the test is done in multiple operating points and the average of them is used as the final value of valve coefficient. The data are stored in the files “HybTnkPlant2015061701Data01(2,3 & 4).mat”.

Theoretically the valve coefficient is independent of the operating point. So, it is expected that the valve coefficient obtained in all operating points to be very close together. But, as depicted in Table 4, the values of valve coefficient k_2 are meaningfully smaller in the last two operating points. In these operating points, the level of the middle tank is lower than the position of the valve V_6 pipe (so equation (12) is not valid any more to compute k_1). This means k_2 is dependent to the tank level, which cannot be correct. As mentioned in item (4) of the level sensor calibration section, the middle tank is ended 4.13 cm lower than the pipe of valve V_3 . So, in evaluating (12) and (13) this value is added to the level of both tanks. This does not affect the value of k_1 , but will correct k_2 . An assumption for the change in the valve coefficient k_2 is that this compensation is not correct. In fact, also physically the pipe is ended 22.86 cm lower than the bottom of the tank but practically the effective end of the pipe (that is the position that the pressure should be considered equal to the atmospheric pressure) might be in different altitude. So, as illustrated in Figure 12 the end of the pipe is supposed to be unknown H_m^e . Then using two data points we can estimate this unknown extra pipe length, considering that the valve coefficient in both operating point are equal and using the following equation instead of (13):

$$k_2 = \frac{Q_i^o}{\sqrt{(H_m^o + H_m^e)}} \quad (14)$$

where H_m^e is the unknown constant and H_m^o is variable according to the operating point. Using the operating points 4 and 6 according to Table 4, the H_m^e obtains equal to 2.09 cm. This value is added to both middle and left tank levels. The compensated value of k_2 is also shown in this table, which shows very close value in different operating points.

Table 4. Initial computed valve coefficients in various operating points

Operating point	k_1	k_2	
		$H_m^e = 4.13$	$H_m^e = 2.09$
1.	20.3785	13.3439	13.6983
2.	20.6410	13.3248	13.6585
3.	20.4588	13.2852	13.6554
4.	20.6936	13.3700	13.6894
5.	-	13.2340	13.6733
6.	-	13.1611	13.6894
Mean value	20.5430	13.2865	13.6774
Final value	20.3376	-	13.6774

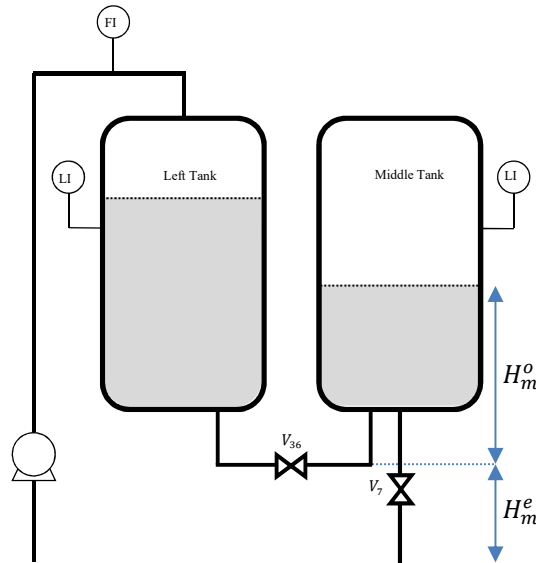


Figure 12. Compensation for the drain pipe line of the middle tank

Figure 13 shows the output of the nonlinear model and the real output. The middle tank output follows the real one but there is a bias in the left tank level. Initially, it seems that the real left tank level is slightly (around 0.18 mm) higher than the calibrated one. But even compensating that has not given a correct model. Since the middle tank level is sufficiently similar the real one, the other possibility is that the coefficient k_1 is slightly less than that obtained from the data, so the model need less value of $H_l - H_m$ to reach the same flow rate. Therefore, we reduce this coefficient by 1%. This results Figure 14, which presents a good coincidence between the model output and the real data. The normalized standard deviation of k_1 is around 0.7% according to the data given in Table 4. So, this one percent error is not very large and can be reasonable. However, I have not yet found any logical reason for this deviation. The valve coefficients used in are given in Table 4 in the final value row.

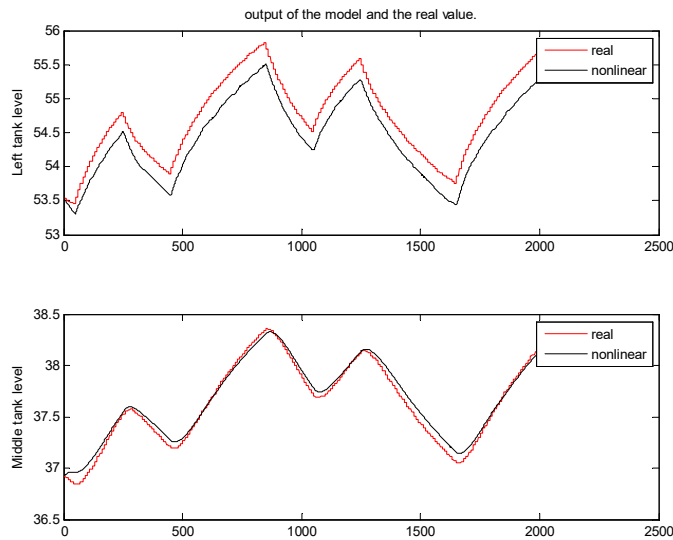


Figure 13. Real value and nonlinear model output using the mean value of valve coefficients in Table 4

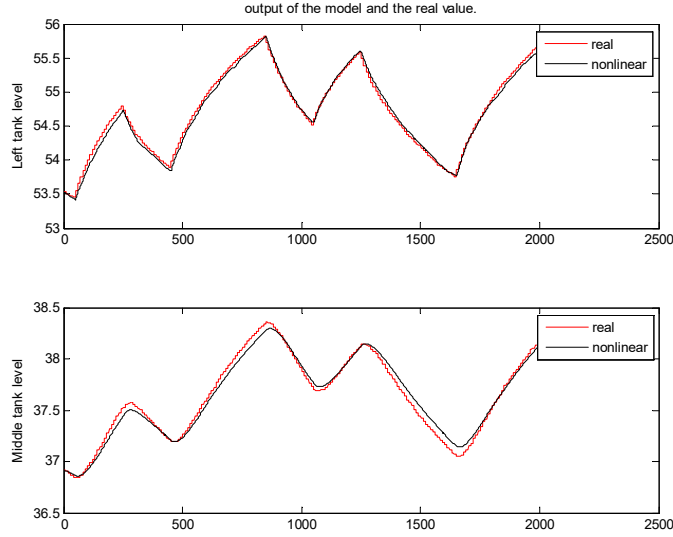


Figure 14. Real value and nonlinear model output using the final value of valve coefficients in Table 4

Constructed Nonlinear Model

Using the above parameter estimations, the final nonlinear model obtained as following:

$$\dot{H}_l = -0.0836\sqrt{H_l - H_m} + 0.0041Q_i \quad (15)$$

$$\dot{H}_m = 0.0836\sqrt{H_l - H_m} + 0.0562\sqrt{H_m}$$

$$\begin{bmatrix} Y_l \\ Y_m \\ Y_c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} H_l \\ H_m \end{bmatrix} + \begin{bmatrix} v_l \\ v_m \\ v_c \end{bmatrix} \quad (16)$$

where the noise variance are $R_l = 6.33 \times 10^{-6}$, $R_m = 3.1 \times 10^{-3}$, $R_c = 2.42 \times 10^{-6}$. Figure 15 illustrates the output of the model and the real output. The model has a good accuracy for the left tank output. However, for the middle tank, although it does not have any bias but it seems that it is slightly slower than the real plant.

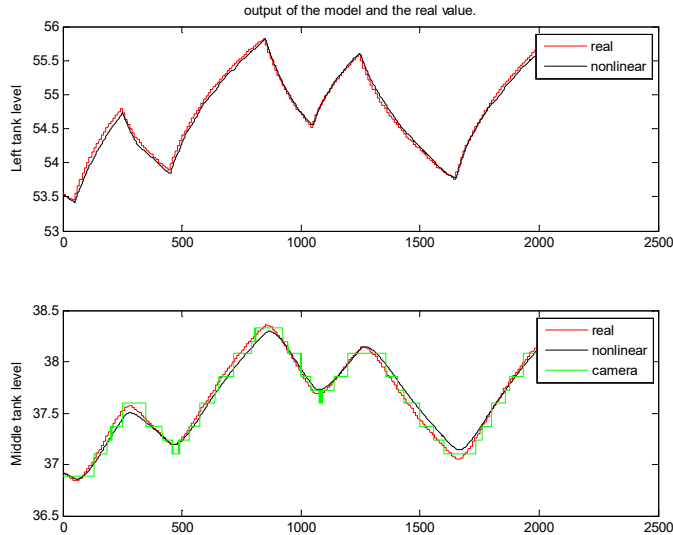


Figure 15. Measured and model level outputs

Linear model construction

The linear model can be obtained using the Taylor expansion of the model (6) (similarly (15)). Let's definition the linearized variables of the plant as follows:

$$\begin{aligned} h_l &= H_l - H_l^o \\ h_m &= H_m - H_m^o \\ q_i &= Q_i - Q_i^o \end{aligned} \quad (17)$$

According to Taylor series expansion, we can write the nonlinear parametric model (6) as

$$\begin{aligned} \dot{h}_l + \dot{H}_l^o &= \left(\frac{-k_1}{s_l} \sqrt{H_l^o - H_m^o} + \frac{1}{s_l} Q_i^o \right) + \left(\frac{-k_1}{s_l} \times \frac{1}{2\sqrt{H_l^o - H_m^o}} (h_l - h_m) + \frac{1}{s_l} q_i \right) + H.O.T. \\ \dot{h}_m + \dot{H}_m^o &= \left(\frac{k_1}{s_m} \sqrt{H_l^o - H_m^o} + \frac{-k_2}{s_m} \sqrt{H_m^o} \right) + \left(\frac{k_1}{s_l} \times \frac{1}{2\sqrt{H_l^o - H_m^o}} (h_l - h_m) - \frac{k_2}{s_m} \times \frac{1}{2\sqrt{H_m^o}} h_m \right) + H.O.T. \end{aligned} \quad (18)$$

The terms \dot{H}_l^o and \dot{H}_m^o in the left hand side are zero by definition (they are the operating point defined in steady state). The first parenthesis in the right hand side are zero, using (6) in the operating point (H_l^o, H_m^o, Q_i^o). The second parenthesis in the right hand side is the right hand side of the linear model. The terms H.T.O. are higher order terms of Taylor series expansion. The effect of them can be included in the process noise which will be shortly identified. As a result the parameterized linear model can be denoted as follows:

$$\begin{bmatrix} \dot{h}_l \\ \dot{h}_m \end{bmatrix} = \begin{bmatrix} \frac{-k_1}{s_l} \times \frac{1}{2\sqrt{H_l^o - H_m^o}} & \frac{k_1}{s_l} \times \frac{1}{2\sqrt{H_l^o - H_m^o}} \\ \frac{k_1}{s_l} \times \frac{1}{2\sqrt{H_l^o - H_m^o}} & \left(\frac{-k_1}{s_l} \times \frac{1}{2\sqrt{H_l^o - H_m^o}} - \frac{k_2}{s_m} \times \frac{1}{2\sqrt{H_m^o}} \right) \end{bmatrix} \begin{bmatrix} h_l \\ h_m \end{bmatrix} + \begin{bmatrix} \frac{1}{s_l} \\ 0 \end{bmatrix} q_i + w(t) \quad (19)$$

To obtain the parameter of the linear model, it is necessary to select the operating point. We assume the mean value of the levels and flow of the experiment given in Figure 14 as the operating point which is $(H_l^o, H_m^o, Q_i^o) = (54.7439, 37.6364, 84.3159,)$ (20)

This ends to the following linearized continuous model:

$$\begin{bmatrix} \dot{h}_l(t) \\ \dot{h}_m(t) \end{bmatrix} = \begin{bmatrix} -0.0101 & 0.0101 \\ 0.0101 & -0.0147 \end{bmatrix} \begin{bmatrix} h_l(t) \\ h_m(t) \end{bmatrix} + \begin{bmatrix} 0.0041 \\ 0 \end{bmatrix} q_i(t) \quad (21)$$

The base sample time for this plant is one second. Using this sample time for discretization the following model is obtained:

$$\begin{bmatrix} h_l(n+1) \\ h_m(n+1) \end{bmatrix} = \begin{bmatrix} 0.99 & 0.01 \\ 0.01 & 0.9855 \end{bmatrix} \begin{bmatrix} h_l(n) \\ h_m(n) \end{bmatrix} + \begin{bmatrix} 0.0041 \\ 2 \times 10^{-5} \end{bmatrix} q_i(n) + w(n) \quad (22)$$

Process noise estimation

The process noise covariance matrix can be estimated by predicting output level using model (22) and subtracting that from the real output, when the real input flow value is introduced to the model, i.e.

$$w(n) = \begin{bmatrix} h_l(n+1) \\ h_m(n+1) \end{bmatrix} - \begin{bmatrix} \hat{h}_l(n+1) \\ \hat{h}_m(n+1) \end{bmatrix} \quad (23)$$

where

$$\begin{bmatrix} \hat{h}_l(n+1) \\ \hat{h}_m(n+1) \end{bmatrix} = \begin{bmatrix} 0.99 & 0.01 \\ 0.01 & 0.9855 \end{bmatrix} \begin{bmatrix} h_l(n) \\ h_m(n) \end{bmatrix} + \begin{bmatrix} 0.0041 \\ 2 \times 10^{-5} \end{bmatrix} q_i(n) \quad (24)$$

Please notice that in the right hand side of (24), h_l and h_m are used which are the real (measured by DP level sensor) are used to predict the levels at step $n+1$.

However In my future research, I am going to use a 10 second sampling time data of the DP cell level sensors and the manipulated variable as a slow rate data. (Please notice that the actual sample time of them is 1 seconds and the data are available in that rate.). So, the slow rate DP level sensors data are used to estimate the process noise covariance matrix. Suppose two consecutive measurement of the slow rate measurements at times $T_i = n$ and $T_{i+1} = n+l$.

$$x(n+l) = A^l x(n) + \sum_{j=1}^l A^{l-j} B u(n-1+j) + \sum_{j=1}^l A^{l-j} w(n-1+j) \quad (25)$$

So the covariance matrix of the error $e(n+l) = x(n+l) - \hat{x}(n+l|x(n), u(n-1+j), j=1, \dots, l)$ is

$$\bar{Q}(l) = \sum_{j=1}^l A^{j-1} Q A^{j-1T} \quad (26)$$

Similar to the equations (25) and (26), the level of the tank at time T_{i+1} given the level measurement at time T_i and the input flow is obtained as follows:

$$h(T_{i+1}) = \hat{h}(T_{i+1}|T_i) + \Omega(l, n) \quad (27)$$

where $h = [h_l \ h_m]^T$ is the level states in (22) **Error! Reference source not found.**, \hat{h} is the $l=10$ sample ahead prediction of level that, similar to (24), is defined as follows and Ω is the slow rate model process noise defined as follows:

$$\hat{h}(T_{i+1}|T_i) = A^l h(T_i) + \sum_{j=1}^l A^{l-j} B q_i(T_i - 1 + j) \quad (28)$$

$$\Omega(l, n) = \sum_{j=1}^l A^{l-j} w(n-1+j) \quad (29)$$

where the matrices A and B are the state space dynamic matrix according to (22). The value of the estimation $\hat{h}(T_{i+1}|T_i)$ can be computed for every slow rate sample time using experimental data. The covariance matrix of the estimation error of this estimation is equal the covariance matrix of $\Omega(l, n)$ denoted as $\bar{Q}(l)$. Using this value, Q can be obtained using the solution of the encapsulating sum matrix equation [1]:

$$\bar{Q}(l) = \sum_{j=1}^l A^{j-1} Q A^{j-1T} \quad (30)$$

According to [1], the encapsulated sum is defined as

$$\sum_l A_l X B_l = C \quad (31)$$

The solution of matrix X can be obtained as follows:

$$vec(X) = (\sum_l B_l^T \otimes A_l)^{-1} vec(C) \quad (32)$$

Where operator vec is defined as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \rightarrow vec(A) = \begin{bmatrix} A_{11} \\ A_{21} \\ A_{12} \\ A_{22} \end{bmatrix} \quad (33)$$

and operator \otimes denotes the Kroneker product, defined for an $m \times n$ matrix A and an $r \times q$ matrix B as an $mr \times nq$ matrix

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix} \quad (34)$$

As a result, using a 10 step ahead prediction error of the model (22), the covariance matrix of process noise of the model (22) is obtained as

$$Q = 10^{-5} \times \begin{bmatrix} 0.6531 & -0.0539 \\ -0.0539 & 0.0458 \end{bmatrix} \quad (35)$$

Linear model evaluation

To evaluate this model we simulate that around the operating point (20). Figure 16 shows the measured level of tanks together with the output of the nonlinear model (15) and the linear model (22). To simulate (22) the input flow is subtracted from the flow in the operating point Q_i^o . This represent the small signal flow q_i . This value is introduced to (22) to compute h_l and h_m . Then, this small signal level h_l and h_m are added with the operating point levels H_l^o and H_m^o to obtain large signal model level H_l and H_m . The result shows that the linear model has around 3 mm bias in average.

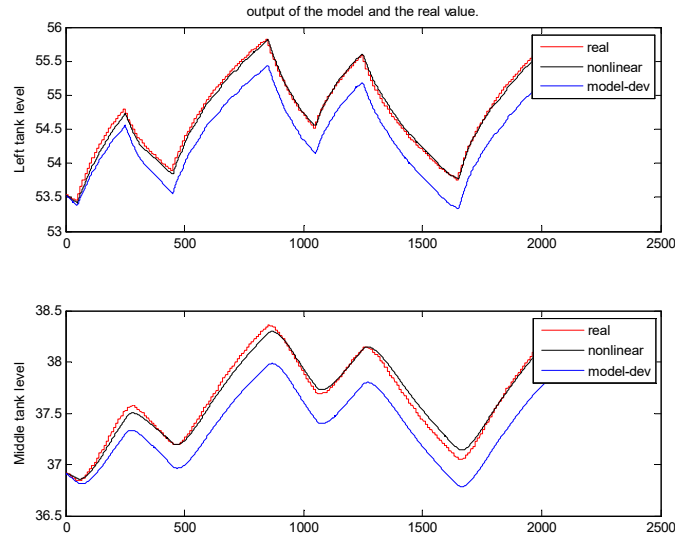


Figure 16. Measured and model outputs

Reference

- [1] K. B. Petersen and M. S. Pedersen, "The matrix cookbook," Available online, March 2015,; http://www.imm.dtu.dk/pubdb/views/edoc_download.php/3274/pdf/imm3274.pdf