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COMPLEX STRUCTURES AND UNCONDITIONAL DECOMPOSITIONS IN SUBSPACES OF BANACH SPACES

by

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Abstract

The aim of the first part of this thesis is to propose a method of constructing explicit complex Banach spaces not isomorphic to their complex conjugates as subspaces of a natural large class of Banach spaces. As a consequence, such constructions provide examples of real Banach spaces which admit at least two non-isomorphic complex structures. In particular, it is shown that L_p, for $1 \leq p < 2$, and $(\sum_n \oplus l_{r_n})_{l_2}$, for some $r_n \nearrow 2$, contain this type of subspaces.

The second part of the thesis establishes the following new characterization of a Hilbert space in terms of unconditionality: a Banach space X is isomorphic to a Hilbert space if and only if for every subspace Y of $l_2(X)$ there is $k \geq 1$ such that Y can be decomposed as an unconditional sum of k-dimensional subspaces. A consequence of our construction is that $l_2(X)$ contains at least countably many mutually non-isomorphic infinite dimensional subspaces, when X is a non-hilbertian Banach space with a finite cotype.

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Chapter 1

Introduction

This is a dissertation in Geometric Functional Analysis devoted to the study of structural properties of infinite dimensional Banach spaces.

Among all Banach spaces, the Hilbert space l_2 is the"nicest" and most "regular". It has lots of symmetries and, in particular, all of its infinite dimensional subspaces are isomorphic to the entire space. This is not true anymore even for such classical spaces as l_p, L_p $(p \neq 2)$, whose subspaces admit much more diversity.

In general terms, we concentrate on constructing Banach spaces which have "few" symmetries while they also have a very decent structure. We are looking for arguments which allow us to obtain these constructions as subspaces of arbitrary Banach spaces or at least inside Banach spaces from certain large classes of spaces. This would support the idea that phenomena of this type are not merely accidental but that they reflect a common behavior.

We describe now the results of the thesis and we comment on their place in the already existing literature.

In Chapter 2 we discuss the motivation for the problems considered in this dissertation and we present some fundamental facts in the Banach space theory.

The first topic of this dissertation, described in Chapter 3, is devoted to

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constructing Banach spaces on which no good comparison between its linear structure over real numbers and over complex numbers can be made.

In considering isomorphism of complex Banach spaces, a natural question is whether real isomorphic spaces are complex isomorphic. The constructions we exhibit in Chapter 3 not only yield a negative answer to this question but also provide examples of real Banach spaces which admit at least two non-isomorphic complex structures.

There are two types of known examples of real Banach spaces with more than one complex structure, one constructed by J. Bourgain [Bo] (with a variant by S. Szarek [S1]) and the other by N. Kalton [Ka].

In Bourgain's example, the space X is an l_2-direct sum $X = (\sum_k \oplus X_k)_{l_2}$, where X_k are suitable finite dimensional spaces obtained by considering certain random norms on $\mathbf{C}^N$. Szarek's variant of this example has the finite dimensional spaces X_k obtained (again by random methods) as proportional dimensional subspaces of $l_{q_k}^{n_k}$, for certain $q_k \searrow 2$ and $n_k \nearrow \infty$. It should be noted that by this method it is not possible to obtain an example of the same type with $X_k \subset l_{q_k}^{n_k}$, for $q_k < 2$.

The space that Kalton constructed is a *twisted sum of Hilbert spaces* i.e., X has a closed subspace E so that E and X/E are Hilbertian, while X itself is not isomorphic to a Hilbert space. His example is a variant of the Kalton-Peck space [Ka-P] and is constructed with a complex twisting function.

The purpose of Chapter 3 is to propose a method of constructing real Banach spaces with at least two non-isomorphic complex structures (in fact we can easily get a continuum of such structures) as subspaces of a natural large class of Banach spaces, thus showing that the phenomena from [Bo], [S1], [Ka] can be found in a more general situation. In particular, we prove that L_p, for $1 \leq p < 2$, and $(\sum_n \oplus l_{r_n})_{l_2}$, for some $r_n \nearrow 2$, contain this type of subspaces. This latter example complements the results by Bourgain and Szarek.

As we mentioned before, the Bourgain-Szarek argument does not yield ex-

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plicit examples, it relies on probabilistic methods. Our method allows us also to exhibit a constructive version of their example.

The heart of our argument is based on a comparison of different convergence behavior of certain series in the chosen space. The proofs are based on successive perturbations and appropriate restrictions of the operators involved in order to simplify their representations.

Similarly as for twisted sums, the spaces obtained here admit unconditional decompositions into 2-dimensional subspaces, while they do not have an unconditional basis.

This latter remark leads us to the second topic of this dissertation, discussed in Chapter 4, which is connected to the existence of unconditional basis or unconditional finite dimensional decompositions in Banach spaces.

The first known example of a Banach space without an unconditional basis which still has an unconditional decomposition into 2-dimensional subspaces is the already mentioned Kalton-Peck space [Ka-P]. This fact was observed by Johnson, Lindenstrauss and Schechtman in [J-L-S]. Their technique was further refined by Ketonen [Ke] and Borzyszkowski [B], who used it for subspaces of L_p, and subsequently generalized in the work of Komorowski and Tomczak-Jaegermann [K], [K-T1], [K-T2], [K-T3], where a general method of constructing subspaces without unconditional basis (or even without local unconditional structure) was developed. Among other results, in [K-T1] it is proved that every Banach space either contains l_2 or a subspace without an unconditional basis. This theorem was later used by Gowers in the solution to the homogeneous space problem [G]: an infinite dimensional Banach space which is isomorphic to all its infinite dimensional closed subspaces must be isomorphic to a Hilbert space.

Chapter 4 of this dissertation is concerned with a higher-dimensional generalization of the classical notion of unconditional basis and its relation with some subspaces of $l_2(X)$, for X a non-hilbertian Banach space. This work continues

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and, in part, generalizes the series of constructions that were done before in the framework of arbitrary Banach spaces ([K-T1], [K-T2], [K-T3]).

The main result of this chapter provides the following new characterization of a Hilbert space in terms of unconditionality: a Banach space X is isomorphic to a Hilbert space if and only if for every subspace Y of $l_2(X)$ there is $k \geq 1$ such that Y can be decomposed as an unconditional sum of k-dimensional subspaces.

The main step consists of constructing, for X a non-hilbertian Banach space with a finite cotype, and for all integers $k \geq 2$, a subspace of $l_2(X)$ which has a k-dimensional unconditional decomposition and for which k is the minimal number with such property. Similar constructions were previously done for subspaces of L_p, $1 \leq p < 2$ (in [B]).

Another consequence of this construction is that $l_2(X)$ contains at least countably many infinite dimensional subspaces, when X is a non-hilbertian Banach space with a finite cotype.

Our argument has roots in the techniques introduced in [J-L-S] and developed later in the literature, as we mentioned above. In particular, we employ many ideas from the tensor product presentation in [K-T3]. The essential idea is summarized in Proposition 4.2.2, which is a version of Proposition A in [B] and generalizes Proposition 1.1 in [K-T1]. This result will be our main criterion for recognizing that a space with a special structure does not have higher-order local unconditional structure. The proof of Proposition 4.2.2 is slightly different and shorter than the one presented in [B] and, in addition, gives an important estimate.

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Chapter 2

Preliminaries in the Banach space theory

2.1 Motivation and fundamental notions

We will start with some basic definitions in functional analysis.

A *normed space* is a pair $(X, \|\cdot\|)$ where X is a vector space over $\mathbf{R}$ or $\mathbf{C}$ and $\|\cdot\| : X \to \{r \in \mathbf{R} \mid r \geq 0\}$ satisfies

(i) $\|x\| = 0$ iff $x = 0$

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and scalars λ

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

A normed space is called a *Banach space* if every Cauchy sequence is convergent: if $(x_n)_{n\geq 1} \subset X$ is such that $\|x_n - x_m\| \to 0$ as $\min\{n, m\} \to \infty$ then $(x_n)_{n\geq 1}$ converges to some point x_0 in X (i.e., $\|x_n - x_0\| \to 0$).

If X and Y are two normed spaces over the same field we define a *linear operator* from X to Y to be a map $T : X \to Y$ such that $T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T x_1 + \lambda_2 T x_2$, for all $x_1, x_2 \in X$ and scalars λ_1, λ_2. A linear operator $T : X \to$

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Y is *bounded* if there exists $M > 0$ such that

$$\|Tx\| \leq M\|x\|$$

for all $x \in X$. The smallest constant M satisfying the above inequality is denoted by $\|T\|$.

Two normed spaces X and Y are said to be *isomorphic* if there is a one-to-one operator from X to Y such that T and T^{-1} are both bounded.

If X and Y are two isomorphic normed spaces, the *Banach-Mazur distance* between X and Y is

$$d(X,Y) = \inf\{\|T\|\|T^{-1}\| \mid T : X \to Y \text{ isomorphism}\}.$$

The norm of an operator $T : X \to Y$ depends on the linear structure considered on X and Y; it is important whether they are real Banach spaces or complex Banach spaces. A very natural question is the following : if X and Y are complex Banach spaces which are real isomorphic, does this imply that they are complex isomorphic? The content of Chapter 3 of this dissertation revolves around this topic.

For a real normed space $(X, \|\cdot\|)$ it is not always the case that X admits a complex structure, that is there exists a multiplication of the elements of X by complex scalars which is compatible with the norm

$$\|\lambda x\| = |\lambda|\|x\| \quad ,\forall x \in X, \quad \forall \lambda \in \mathbf{C}$$

(or compatible with a norm $|||\cdot|||$ equivalent to $\|\cdot\|$). Consider, for example, the trivial case $X = (\mathbf{R}^{2n+1}, \|\cdot\|_2)$, for any $n = 1, 2, \ldots$. If X admits a complex structure then, denoting by $\{e_j\}_j$ a basis for X, treated as a complex space, we get that $\{e_j, ie_j\}_j$ is a basis for the real space X, contradicting with the fact that X has odd dimension.

However, if X is a real normed space then the cartesian square $X \oplus X$, endowed with the norm $\|(x,y)\| = \|x\|_X + \|y\|_X$ or any other equivalent norm,

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always has a structure of a complex normed space, with respect to the multiplication

$$(a+ib)(x,y) = (ax - by, ay + bx)$$

and the norm (equivalent to $\|\cdot\|$)

$$|||(x,y)||| = \sup_{\theta\in[0,2\pi]} \|(x\cos\theta, y\sin\theta)\|.$$

If $(X, \|\cdot\|)$ is a real normed space, the complex structures on X correspond (in the one-to-one correspondence) to the $\mathbf{R}$-linear isometries A on X such that $A^2 = -I$. For one implication take $Ax = ix$; conversely, if such an isometry exists, define $(a+ib)x = ax + bAx$ and consider on X the equivalent norm

$$|||x||| = 1/2\pi \int_0^{2\pi} \|(\cos\theta)x + (\sin\theta)Ax\| d\theta.$$

By using probabilistic methods, S. Szarek was able to construct in [S1] an infinite dimensional space which does not admit a complex structure. As a consequence of the previous discussion, his space is not isomorphic to the cartesian square $Z \oplus Z$ of any Banach space Z.

All real Banach spaces discussed throughout Chapter 3 will admit a structure of complex Banach space. The question we will concentrate on is whether we can obtain at least two non-isomorphic (of course we mean non-complex isomorphic) complex structures for such a space.

In the theory of Banach spaces there are results which are true for the real Banach spaces only. The following theorem will not only exemplify this but also will allow us to introduce in a natural way the notion of the *complex conjugate* of a Banach space, which will play a central role in the sequel.

Theorem 2.1.1 (Mazur-Ulam) *Every isometry F (i.e. a mapping preserving the distance) from a real normed space X* onto *a real normed space Y, with $F(0) = 0$, is linear.*

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For a complex Banach space X define $\overline{X}$, the *complex conjugate* of X, to be the Banach space with the same elements and norm, the same addition of vectors, while the multiplication by scalars is given by $\lambda \odot x = \overline{\lambda} x$, for $\lambda \in \mathbf{C}$ and $x \in X$.

Obviously X and $\overline{X}$ are identical as real spaces. However, the identity map $I : X \longrightarrow \overline{X}$ is an isometry which is clearly not complex linear, hence such a naive extension of the Mazur-Ulam theorem is not true.

As we have already mentioned, X and $\overline{X}$ are identical as real spaces and X and $\overline{X}$, treated as complex spaces, provide two complex structures for this real space. In many cases the spaces X and $\overline{X}$ are (complex) isomorphic. For example, when X has an unconditional basis $\{e_j\}_j$, the natural map $J : X \longrightarrow \overline{X}$ given by $J(\sum_j t_j e_j) = \sum_j t_j \odot e_j$ is an isomorphism between X and $\overline{X}$.

The purpose of Chapter 3 is to propose a method of constructing complex Banach spaces not isomorphic to their complex conjugates, and hence having at least two non-isomorphic complex structures. Also, such constructions provide examples of complex Banach spaces which are isomorphic as real spaces and non-isomorphic treated as complex spaces.

We will now pass to some more specific definitions and notations from the Banach space theory, that can be found e.g., in [L-T1] and [T], together with some other terminology not explained here.

A sequence $\{e_j\}_{j\geq 1}$ in a Banach space X is called a *(Schauder) basis* if every vector $x \in X$ has a unique representation $x = \sum_{j\geq 1} a_j e_j$ as a sum of a convergent series. A Schauder basis represents a sort of a "coordinate system". We say that $\{u_l\}_{l\geq 1} \subset X$ are *successive blocks* of $\{e_j\}_{j\geq 1}$ if each vector u_l is of the form $u_l = \sum_{j=p_l+1}^{p_{l+1}} a_j e_j$, with $\{a_j\}_{j\geq 1}$ scalars and $0 \leq p_1 < p_2 < \ldots$ an increasing sequence of integers.

For a basis $\{e_j\}_{j\geq 1}$ in a Banach space X let $P_n : X \to X$ be the projection defined by $P_n\left(\sum_{j=1}^{\infty} a_j e_j\right) = \sum_{j=1}^{n} a_j e_j$, for all $n \geq 1$. It can be easily shown that $\sup_n \|P_n\| < \infty$. The number $\sup_n \|P_n\|$ is called the *basis constant* of

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$\{e_j\}_{j\geq 1}$.

The existence of a basis in a Banach space does not give much information on the structure of the space. In order to study in more detail the structural properties of a Banach space one needs to consider bases with certain additional properties, among the most important being the unconditional basis.

A basis $\{e_j\}_{j\geq 1}$ in a Banach space X is *unconditional* if there exists a constant $C > 0$ such that

$$\|\sum_j \mu_j a_j e_j\| \leq C\|\sum_j a_j e_j\|,$$

for all $x = \sum_j a_j e_j \in X$ and all signs $\mu_j = \pm 1$ $(j = 1, 2, \ldots)$. The infimum of such constants C is called the *unconditional constant* of $\{e_j\}_{j\geq 1}$. As an example, the Haar basis is unconditional in $L_p[0,1]$, for $1 < p < \infty$ and it is not unconditional for $L_1[0,1]$.

Comparing to the case of $\{e_j\}_{j\geq 1}$ being only a basis of X, in the case of unconditional basis we have $\sup_{\sigma\subset\{1,2,\ldots\}} \|P_\sigma\| < \infty$, where $\{P_\sigma\}_{\sigma\subset\{1,2,\ldots\}}$ are the natural projections associated to the unconditional basis $\{e_j\}_{j\geq 1}$, defined by $P_\sigma\left(\sum_{j=1}^{\infty} a_j e_j\right) = \sum_{\sigma\subset\{1,2,\ldots\}} a_j e_j$.

Another often used observation concerning unconditional bases is the following: if $\{e_j\}_{j\geq 1}$ is an unconditional basis with the unconditional constant C then, for every $x = \sum_{j\geq 1} a_j e_j$ and every choice of bounded scalars $\{\lambda_j\}_{j\geq 1}$, we have

$$\|\sum_{j\geq 1} \lambda_j a_j e_j\| \leq 2C \sup_n |\lambda_n| \, \|\sum_{j\geq 1} a_j e_j\|$$

(in the case of a real Banach space we can take C instead of $2C$).

A Banach space X with a Schauder basis can be viewed as a sum of one-dimensional spaces. It is sometimes useful to consider coarser decompositions of X, with the components into which we decompose being subspaces of dimension larger than 1.

Let X be a Banach space. A sequence $\{Z_k\}_{k\geq 1}$ of closed subspaces of X is called a *Schauder decomposition* of X if every vector $x \in X$ has a unique

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representation $x = \sum_k z_k$ as a sum of a convergent series such that $z_k \in Z_k$ for every $k = 1, 2, \ldots$. In this case we define the *support* of x with respect to the decomposition $\{Z_k\}_k$ to be $\operatorname{supp} x = \{k \mid z_k \neq 0\}$. If $\dim Z_k < \infty$, for all $k \geq 1$, we say that X has a *finite dimensional decomposition.*

A decomposition $\{Z_k\}_k$ is called *C-unconditional* for some constant $C > 0$, if for all $x = \sum_k z_k \in X$ and $\mu_k = \pm 1$ $(k = 1, 2, \ldots)$ one has $\|\sum_k \mu_k z_k\| \leq C\|\sum_k z_k\|$. The infimum of such constants C is denoted by $\operatorname{unc}\{Z_k\}$.

It is well-known that even if a Banach space has an unconditional decomposition into 2-dimensional subspaces it is still possible that X may fail to have an unconditional basis. The first example of such phenomenon is the Kalton-Peck space ([Ka-P]).

In the context of arbitrary Banach spaces, it was proved by Komorowski and Tomczak-Jaegermann ([K-T1], [K-T2]) that every Banach space either contains l_2 or a subspace without an unconditional basis which still has an unconditional decomposition into 2-dimensional subspaces.

Motivated by these results, it is natural to investigate properties related to unconditionality in Banach spaces which admit an unconditional finite dimensional decomposition.

In connection with the problem of constructing spaces with an unconditional basis, some new parameters have been introduced in the literature. An important example is the *local unconditional structure* of a Banach space. This is a localization of the notion of unconditional basis.

A Banach space X has *local unconditional structure* if there is $C \geq 1$ such that for every finite dimensional subspace $E \subset X$ there exists a Banach space F with a 1-unconditional basis and operators $u : E \to F$ and $\omega : F \to X$ such that the natural embedding $i_E : E \to X$ admits a factorization $i_E = \omega u$ and $\|u\|\|\omega\| \leq C$.

As it turns out, the arguments which appear in the literature regarding the construction of spaces without unconditional basis can be modified to obtain

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that such spaces have a stronger property, namely they don't have local unconditional structure.

Regarding the unconditional decomposition into k-dimensional subspaces, the natural analog of the local unconditional structure can be also defined for a Banach space. The purpose of Chapter 4 of this dissertation is to study this type of property in general Banach spaces.

2.2 More specific concepts and facts

Let us discuss another type of Schauder basis, which is used mainly in connection with duality problems, namely shrinking basis.

Let $\{x_n\}_n$ be a Schauder basis in a Banach space X. We say that $\{x_n\}_n$ is *shrinking* if, for every $x^* \in X^*$, the norm of $x^*_{|\overline{\text{span}}\,\{x_i\}_{i=n}^{\infty}}$ tends to 0 as $n \to \infty$. An example of a shrinking basis consists of the unit vector basis in l_p, for all $1 < p < \infty$. On the other hand, for $X = l_1$ or $X = C(0,1)$ there is no basis which is shrinking.

It is a known fact ([L-T1]) that $\{x_n\}_n$ is a shrinking basis if and only if the *biorthogonal functionals* $\{x_n^*\}_n$ associated to the basis $\{x_n\}_n$, defined by the relation $x_n^*(x_m) = \delta_n^m$ for all $n, m \geq 1$, form a Schauder basis of X^*.

A sequence $\{x_n\}_n$ in a Banach space X is called *w-null* if it converges to 0 in the weak topology, that is $x^*(x_n) \to 0$ as $n \to \infty$, for all $x^* \in X^*$. Clearly a shrinking basis is w-null. Indeed, for any $x^* \in X^*$, $|x^*(x_n)| \leq \|x_n\|\, \|x^*_{|\overline{\text{span}}\,\{x_i\}_{i=n}^{\infty}}\| \to 0$ as $n \to 0$.

In order to verify whether a basis in a Banach space is shrinking, an useful result is the following proposition, due to R. C. James [J]. We present the proof for the sake of completeness, and also for illustrating the use of the classical "gliding-hump" argument, which will appear often in the sequel.

In general terms, the "gliding hump" argument is typically used in situations in which we have vectors $(x_j)_{j\geq 1}$ in a Banach space X whose expansions with

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respect to the basis (or with respect to certain Schauder decompositions of X) start arbitrarily far. In this situation we are able to approximate an infinite subset of the initial vectors with some other vectors $(y_s)_{s\geq 1}$ whose corresponding expansions are now disjoint.

Proposition 2.2.1 *Let X be a Banach space with a Schauder basis $\{x_n\}_{n\geq 1}$. Suppose that there exist $0 = i_1 < i_2 < \ldots < i_k < \ldots$ such that, denoting by $Z_k = \operatorname{span}\{x_n : i_k + 1 \leq n \leq i_{k+1}\}$, $\{Z_k\}_{k\geq 1}$ forms an unconditional finite dimensional decomposition for X.*

If $\{x_n\}_{n\geq 1}$ is not a shrinking basis then we can find $\delta > 0$ and normalized vectors $\{w_l\}_{l\geq 1}$ in X, successive blocks with respect to the decomposition $\{Z_k\}_{k\geq 1}$, such that

$$\|\sum_l a_l w_l\| \geq \delta \sum_l |a_l|,$$

for all finite sequences of scalars $\{a_l\}_l$.

Remark. It immediately follows from the triangle inequality that

$$\delta \sum_l |a_l| \leq \|\sum_l a_l w_l\| \leq \sum_l |a_l|$$

for all $\{a_l\}_{l\geq 1} \in l_1$. In particular, the subspace $\overline{\operatorname{span}}\{w_i\}$ is isomorphic to l_1.

Proof. For all $s \geq 1$, denote by $P_s : X \to X$ the natural projection onto $\operatorname{span}\{Z_k\}_{k\leq s}$.

Since $\{x_n\}_{n\geq 1}$ is not a shrinking basis there exist $x^* \in X^*$, with $\|x^*\| = 1$, an $\epsilon \in (0,1)$ and, for $s = 1, 2, \ldots$, a normalized vector $u_s \in \operatorname{span}\{x_n\}_{n\geq s}$ so that

$$|x^*(u_s)| > \epsilon, \ \forall\, s = 1, 2, \ldots.$$

The following inductive argument is based on the "gliding-hump" procedure.

Let $s_1 = 1$ and let k_1 be such that $\|u_{s_1} - P_{k_1} u_{s_1}\| < \epsilon/2$. Let $w_1 = P_{k_1} u_{s_1}$ and then $|x^*(w_1)| > \epsilon/2$.

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Choose now s_2 large enough such that $P_{k_1}u_{s_2} = 0$ (it is sufficient to take $s_2 > k_1 + 1$) and take k_2 such that $\|u_{s_2} - P_{k_2}u_{s_2}\| < \epsilon/2$. Letting $w_2 = P_{k_2}u_{s_2}$ we obtain $|x^*(w_2)| > \epsilon/2$.

Inductively, we can construct $\{w_l\}_{l\geq 1}$ successive blocks with respect to the decomposition $\{Z_k\}_{k\geq 1}$ and satisfying

$$|x^*(w_l)| > \frac{\epsilon}{2}, \ \forall\, l = 1, 2, \ldots.$$

Fix an arbitrary finite sequence of scalars $\{a_l\}_l$. Choose $\{\theta_l\}_l$, with $|\theta_l| = 1$, such that $|\sum_l \theta_l a_l x^*(w_l)| = \sum_l |a_l x^*(w_l)|$. Then

$$\begin{aligned} 2K\|\sum_l a_l w_l\| &\geq \|\sum_l \theta_l a_l w_l\| \geq |x^*\left(\sum_l \theta_l a_l w_l\right)| \\ &= \sum_l |a_l||x^*(w_l)| \geq \frac{\epsilon}{2}\sum_{l\geq 1}|a_l|, \end{aligned}$$

where K is the unconditional constant of the decomposition $\{Z_k\}_{k\geq 1}$.

The vectors $\{w_l\}_{l\geq 1}$ satisfy the conclusion, except that they are not normalized (we have $1 + \epsilon/2 \geq \|w_l\| \geq 1 - \epsilon/2$, for all $l = 1, 2, \ldots$). To this end substitute them with $\{w_l/\|w_l\|\}_{l\geq 1}$. □

Let us mention now some few more notations that will be used in the sequel.

Let (X_n) be a sequence of Banach spaces and let $1 \leq p < \infty$. We denote by $(\sum_n \oplus X_n)_{l_p}$ the space of all sequences (x_n) in ΠX_n such that the following expression representing the norm is finite

$$\|(x_n)\| = \left(\sum_n \|x_n\|^p_{X_n}\right)^{1/p} < \infty.$$

For simplicity, when $p = 1$ instead of $(\sum_n \oplus X_n)_{l_1}$ we use the notation $\sum_n \oplus X_n$. Also, if $X_n = X$ for all n we will write $l_p(X)$ instead of $(\sum_n \oplus X)_{l_p}$. For a natural number n, by $l_p^n(X)$ we denote the space of all n-tuples $(x_j)_{j=1}^n$ endowed with the corresponding norm.

For a Banach space X denote by $L_2(X)$ the set of all measurable functions $f : [0, 1] \to X$ with the property that $\|f\|^2$ is integrable with respect to the

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Lebesgue measure on $[0,1]$. The Rademacher functions $\{r_i\}_i$ on $[0,1]$ are defined by $r_i(t) = \operatorname{sgn} \sin 2^i \pi t$, for $i = 1, 2, \ldots$. Denote by $\mathrm{Rad}_n(X)$ the subspace of $L_2(X)$ consisting of the functions of the form $f(t) = \sum_{i \le n} r_i(t) x_i$, with $x_i \in X$ for $i \le n$.

We will require in the sequel several well known facts about the above spaces. Namely, it is clear that $\mathrm{Rad}_n(X)$ contains X as a subspace and it is isometric to a subspace of $l_2^{2^n}(X)$ via the map

$$\begin{array}{ccc} \mathrm{Rad}_n(X) & \longleftrightarrow & l_2^{2^n}(X) \\ f(t) = \sum_{i \le n} r_i(t) x_i & \longleftrightarrow & x = \left(\frac{1}{2^{n/2}} \sum_{i \le n} \epsilon_i x_i \right)_{(\epsilon_i)} \end{array},$$

where above we considered all sequences of signs $(\epsilon_i)_{i \le n}$.

Finally, we will turn our attention to the tensor product of normed spaces, which proves to be an important technical tool in functional analysis.

If X and Y are vector spaces then we can define the algebraic tensor product of X with Y, $X \otimes Y$ to be the space whose elements have a representation as finite sums of elementary tensors $\sum_{k=1}^n x_k \otimes y_k$, with $x_k \in X$, $y_k \in Y$, where the elementary tensors satisfy

$$(x + x') \otimes y = x \otimes y + x' \otimes y$$

$$x \otimes (y + y') = x \otimes y + x \otimes y'$$

$$\lambda(x \otimes y) = \lambda x \otimes y = x \otimes \lambda y,$$

for all $x, x' \in X$, $y, y' \in Y$ and scalars λ.

Such an object can be obtained by using a quotient construction. An important feature of this construction is the universality result stating that bilinear maps defined on $X \times Y$ (that is maps which are linear in both the first and second variable) can be uniquely extended to linear maps on $X \otimes Y$.

The situation considered in this dissertation is the tensor product $X_1 \otimes \ldots \otimes X_m$, where X_i are Banach spaces ($i = 1, \ldots, m$). We say that a norm $\| \cdot \|$

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on the tensor product $X_1 \otimes \ldots \otimes X_m$ is a *cross-norm* if $\|x_1 \otimes \ldots \otimes x_m\| = \|x_1\|_{X_1} \cdot \ldots \cdot \|x_m\|_{X_m}$, for all $x_i \in X_i$, $i = 1, \ldots, m$. If each of the spaces X_i is finite dimensional with algebraic basis $\{f_j^{(i)}\}_j$, then $\{f_{j_1}^{(1)} \otimes \ldots \otimes f_{j_m}^{(m)}\}_{j_1,\ldots,j_m}$ is an algebraic basis in $X_1 \otimes \ldots \otimes X_m$, which will be called the *natural tensor basis*.

There are many examples of tensor products spaces used in the theory of Banach spaces, including the injective tensor product and the projective tensor product, but in this dissertation we will use only certain simple tensor product spaces. A characteristic example consists of $l_2^n \otimes X$ endowed with the cross-norm induced by the space $l_2^n(X)$ via the map

$$\begin{array}{ccc} l_2^n(X) & \longleftrightarrow & l_2^n \otimes X \\ (x_1, \ldots, x_n) & \longleftrightarrow & e_1 \otimes x_1 + \ldots + e_n \otimes x_n, \end{array}$$

where $e_1, \ldots, e_n$ form the standard unit vector basis in l_2^n.

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Chapter 3

Subspaces of L_p with more than one complex structure

3.1 Introduction

We will first present a construction of infinite dimensional subspaces of $l_{p_1} \oplus l_{p_2} \oplus l_{p_3} \oplus l_{p_4} \oplus l_{p_5}$ $(1 < p_5 < \ldots < p_1 < \infty)$ whose Banach-Mazur distance to their complex conjugates is arbitrarily large (Proposition 3.2.2 and Corollary 3.2.3).

Then, by "glueing" together such spaces (i.e., we consider their l_2-direct sums) we get the desired constructions of spaces non-isomorphic to their complex conjugates (Theorem 3.3.1, Corollary 3.3.2 and Theorem 3.4.2). Moreover, as we will see later, these examples provide the existence of real Banach spaces which admit not only two, but a continuum of non-isomorphic complex structures.

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3.2 Subspaces of $l_{p_1} \oplus l_{p_2} \oplus l_{p_3} \oplus l_{p_4} \oplus l_{p_5}$ not-well isomorphic to their complex conjugates

Let W, V be Banach spaces having finite dimensional decompositions $\{W_k\}_k$ and $\{V_j\}_j$ respectively. Let $T : W \longrightarrow V$ be a bounded linear operator. We say that T is *block-diagonal* with respect to $\{W_k\}_k$ and $\{V_j\}_j$ if for every k there exists a finite set $B_k \subset \{1, 2, \ldots\}$ such that

$$\begin{cases} \max B_k < \min B_l \quad \forall k, l \in \{1, 2, \ldots\} \text{ with } k < l, \\ \operatorname{supp} Tw_k \subset B_k \quad \forall w_k \in W_k, \ \forall k \in \{1, 2, \ldots\} \end{cases}$$

where $\operatorname{supp} Tw_k$ is taken with respect to the decomposition $\{V_j\}_{j \in J}$.

The following general observation will be often used in the sequel.

Proposition 3.2.1 *Let W, V be Banach spaces having decompositions into 2-dimensional spaces $\{W_k\}_k$ and $\{V_j\}_j$ respectively. Let $W_k = \operatorname{span}\{w_{1,k}, w_{2,k}\}$, for $k = 1, 2, \ldots$, and $V_j = \operatorname{span}\{v_{1,j}, v_{2,j}\}$, for $j = 1, 2, \ldots$, and suppose that $\{w_{1,k}, w_{2,k}\}_k$ is a w-null normalized basis in W and $\{v_{1,j}, v_{2,j}\}_j$ is a normalized basis in V. Let $T : W \to V$ be a bounded linear operator.*

Then, for every $\epsilon > 0$, there exist a subsequence $I_0 \subset \{1, 2, \ldots\}$ and $T_0 : W^0 = \overline{\operatorname{span}}\{W_k\}_{k \in I_0} \to V$ a block-diagonal operator with respect to $\{W_k\}_{k \in I_0}$ and $\{V_j\}_j$ such that

$$\|T_{|W^0} - T_0\| \leq \epsilon.$$

Proof. Denote by $\{w_{1,k}^*, w_{2,k}^*\}_k$ and $\{v_{1,j}^*, v_{2,j}^*\}_j$ the biorthogonal functionals in W^* and V^* associated to $\{w_{1,k}, w_{2,k}\}_k$ and $\{v_{1,j}, v_{2,j}\}_j$ respectively. Then, since $\{v_{1,j}, v_{2,j}\}_j$ is a normalized basis in V, we have

$$\lim_j v_{1,j}^*(Tw_{1,k}) = \lim_j v_{1,j}^*(Tw_{2,k}) = 0, \quad \text{for all } k = 1, 2, \ldots$$

and similarly for $v_{2,j}^*$.

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On the other hand, since $\{w_{1,k}, w_{2,k}\}_k$ is w-null

$$\lim_k v_{1,j}^*(Tw_{1,k}) = \lim_k v_{1,j}^*(Tw_{2,k}) = 0, \quad \text{for all } j = 1, 2, \ldots$$

and similarly for $v_{2,j}^*$.

Let $\epsilon > 0$. Let C denote the basis constant of the basis $\{w_{1,k},\ w_{2,k}\}_k$.

By a classical gliding-hump argument we can find a subsequence I_0 of $\{1, 2, \ldots\}$ and a block-diagonal operator $T_0 : W^0 = \overline{\text{span}}\{W_k\}_{k\in I_0} \to V$ such that the columns of T_0 are approximated by the correspondent columns of T. We can then write $T_{|W^0} - T_0 = \sum_{k\in I_0}(S_k + U_k)$ with $S_k : W^0 \to V$ and $U_k : W^0 \to V$ satisfying, for every $k \in I_0$,

$$\begin{cases} S_k(w_{1,l}) = 0, \text{ if } l \neq k, \quad \text{and} \quad \|S_k(w_{1,k})\| < (1/2C)\,\epsilon/2^{k+1} \\ S_k(w_{2,l}) = 0, \text{ for all } l \in I_0 \end{cases}$$

and similarly

$$\begin{cases} U_k(w_{2,l}) = 0, \text{ if } l \neq k, \quad \text{and} \quad \|U_k(w_{2,k})\| < (1/2C)\,\epsilon/2^{k+1} \\ U_k(w_{1,l}) = 0, \text{ for all } l \in I_0. \end{cases}$$

Observe that $\|S_k\| \le \epsilon/2^{k+1}$ for $k \in I_0$. Indeed, for all $x = \sum_{t\in I_0}(a_t w_{1,t} + b_t w_{2,t}) \in W^0$

$$\begin{aligned} \|S_k x\| &= \|S_k(a_k w_{1,k})\| \le \frac{1}{2C}\frac{\epsilon}{2^{k+1}}|a_k| \\ &\le \frac{1}{2C}\frac{\epsilon}{2^{k+1}}2C\|x\| = \frac{\epsilon}{2^{k+1}}\|x\|. \end{aligned}$$

Similarly $\|U_k\| \le \epsilon/2^{k+1}$ for $k \in I_0$. Thus

$$\|T_{|W^0} - T_0\| \le \sum_{k\in I_0}(\epsilon/2^{k+1} + \epsilon/2^{k+1}) \le \epsilon.$$

□

We will now pass to the main construction of this section, which will play a central role throughout the present chapter.

Let $\wp = \{(p_1, p_2, ..., p_5) \mid 1 < p_5 < \ldots < p_1 < \infty\}$.

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For every $\eta = (p_1, ..., p_5) \in \wp$ and $N \in \mathbf{N}$ we will construct a Banach space $X_{N,\eta}$ as follows: we will define 2-dimensional subspaces Z_k of $l_{p_1} \oplus \ldots \oplus l_{p_5}$ (depending on N and η) which will form an unconditional decomposition for $X_{N,\eta} = \overline{\operatorname{span}}\, \{Z_k\}_{k \geq 1}$.

Fix $\eta \in \wp$ and $N \in \mathbf{N}$. For $i = 2, \ldots, 5$ set $\alpha_i = 1/p_i - 1/p_{i-1}$, and let $\alpha = \min\{\alpha_2, \ldots, \alpha_5\}$. Fix a positive integer $\lambda > 2\,\alpha_3/\alpha_4 + 5$.

Denote by $\{f_{j,k}\}_k$ the natural basis of l_{p_j} $(j = 1, \ldots, 5)$. Define the vectors x_k and y_k spanning Z_k $(k = 1, 2, \ldots)$ by the formulas

$$\begin{aligned} x_k &= f_{1,k} \qquad\quad +\gamma_1 f_{3,k} \; +\gamma_2 f_{4,k} \; +\gamma_3 f_{5,k} \\ y_k &= \qquad f_{2,k} \qquad\qquad\quad +\gamma_2 f_{4,k} \; +i\gamma_3 f_{5,k} \end{aligned}$$

where $\gamma_1 = N^{-2\alpha_3}$, $\gamma_2 = N^{-4(\alpha_3+\alpha_4)}$ and $\gamma_3 = N^{-\lambda(\alpha_3+\alpha_4+\alpha_5)}$.

It is easy to see that for any scalars s and t we have

$$\max(|s|, |t|) \leq \|sx_k + ty_k\| \leq 4(|s| + |t|).$$

It follows that the decomposition $\{Z_k\}_{k\geq 1}$ is 1-unconditional and $x_1, y_1, x_2, y_2, \ldots$ form a Schauder basis in $X_{N,\eta}$ (and also in $\overline{X}_{N,\eta}$). This is a shrinking basis (and hence w-null), since otherwise we can find (by Proposition 2.2.1) $\delta > 0$ and successive normalized blocks $\{w_l\}_l$ (with respect to the decomposition $\{Z_k\}_{k\geq 1}$) such that for every finite sequence of scalars $\{a_l\}_l$

$$\|\sum_l a_l w_l\| \geq \delta \sum_l |a_l|.$$

The contradiction occurs when we observe that $\{w_l\}_{l\geq 1}$ satisfy an upper p_5-estimate. Indeed, if we denote by $Q_s : l_{p_1} \oplus \ldots \oplus l_{p_5} \to l_{p_s}$ the canonical projection $(s = 1, \ldots, 5)$, the fact that $\{w_l\}_{l\geq 1}$ are normalized implies

$$\|Q_s w_l\| \leq 1 \text{ , for } s = 1, \ldots, 5 \text{ and } l = 1, 2, \ldots.$$

Since $\{w_l\}_{l\geq 1}$ are successive blocks with respect to the decomposition $\{Z_k\}_{k\geq 1}$ it follows that, for each $1 \leq l \leq 5$, $\{Q_s w_l\}_l$ are successive blocks in l_{p_s}. Thus,

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for every finite sequence of scalars $\{a_l\}_l$

$$\begin{aligned}
\|\sum_{l\geq 1} a_l w_l\| &= \|Q_1(\sum_{l\geq 1} a_l w_l)\| + \ldots + \|Q_5(\sum_{l\geq 1} a_l w_l)\| \\
&= \|\sum_{l\geq 1} a_l Q_1 w_l\| + \ldots + \|\sum_{l\geq 1} a_l Q_5 w_l\| \\
&\leq \left(\sum_{l\geq 1} |a_l|^{p_1}\right)^{1/p_1} + \ldots + \left(\sum_{l\geq 1} |a_l|^{p_5}\right)^{1/p_5} \\
&\leq 5\left(\sum_{l\geq 1} |a_l|^{p_5}\right)^{1/p_5}.
\end{aligned}$$

The next result, concerning the behavior of linear operators acting from $X_{N,\eta}$ to $\overline{X}_{N,\eta}$, will be essential for the proof of Theorem 3.3.1.

Proposition 3.2.2 *Let $\eta \in \wp$ and N be a positive integer.*
Let $I \subset \{1, 2, \ldots\}$ be an infinite set and let Y be the subspace of $X_{N,\eta}$ defined by $Y = \overline{\text{span}}\{Z_k\}_{k\in I}$. Consider $T : Y \longrightarrow \overline{X}_{N,\eta}$ a block-diagonal operator (with respect to $\{Z_k\}_{k\in I}$ and $\{\overline{Z_k}\}_{k\geq 1}$) with $\|T\| \leq 1$. Then

(i) There exists a finite set $J \subset I$ such that

$$\max\{\|Tx_k\|, \|Ty_k\|\} \leq 24N^{-\alpha}, \quad \text{for all } k \in I \setminus J.$$

(ii) Let $\{I_l\}_{l\geq 1}$ be a family of disjoint subsets of I with the property that $|I_l| = N$, for all $l \geq 1$. Let $\widetilde{x}_l = \sum_{k\in I_l} a_l(k)x_k$, $\widetilde{y}_l = \sum_{k\in I_l} a_l(k)y_k$ satisfy $\sum_{k\in I_l} |a_l(k)|^{p_2} = 1$, for $l = 1, 2, \ldots$. Then there exists a finite subset $J_0 \subset \{1, 2, ...\}$ such that

$$\max\{\|T\widetilde{x}_l\|, \|T\widetilde{y}_l\|\} \leq 57N^{-\alpha}, \quad \text{for all } l \in \{1, 2, ...\} \setminus J_0.$$

Remark. The proof Proposition 3.2.2 will still work if we consider $X_{N,\eta}$ as a subspace of $l_{p_1} \oplus_q l_{p_2} \oplus_q \ldots \oplus_q l_{p_5}$, for some $q \geq 1$ (note that, in this case, $X_{N,\eta}$ is the same vector space as before endowed with an equivalent norm).

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Corollary 3.2.3 *Let $\eta \in \wp$ and N be a positive integer. Then*

$$d(X_{N,\eta}, \overline{X}_{N,\eta}) \geq \frac{1}{100} N^{\alpha}.$$

Proof of Corollary 3.2.3 Let $T : X_{N,\eta} \longrightarrow \overline{X}_{N,\eta}$ be an isomorphism satisfying $\|T\| = 1/2$. By Proposition 3.2.1 there exist an infinite dimensional subspace $Y = \overline{\text{span}}\ \{Z_k\}_{k \in I}$ of $X_{N,\eta}$ and $T_0 : Y \longrightarrow \overline{X}_{N,\eta}$ a block-diagonal operator with respect to $\{Z_k\}_{k\in I}$ and $\{\overline{Z_k}\}_{k \geq 1}$ such that

$$\frac{1}{2\|T^{-1}\|}\|x\| \leq \|T_0 x\| \leq \|x\|, \qquad \text{for all } x \in Y. \tag{3.1}$$

By Proposition 3.2.2 (i) we can find $k_0 \in I$ such that $\|T_0 x_{k_0}\| \leq 24N^{-\alpha}$, which, combined with (3.1), concludes the corollary. □

Remark. In the same circle of problems, we should mention the result of Szarek [S2] showing that in the finite dimensional case there is an n-dimensional complex space Y such that $d(Y, \overline{Y}) \geq cn$, with c an absolute constant.

Let us comment more on this result. In the local theory the set $\mathcal{B} = \mathcal{B}_n = \{X \mid X \text{ normed space },\ \dim X = n\}$, endowed with the Banach-Mazur distance, is usually called *the Minkowski compactum.* By a result of F. John [Jo], $d(X, l_2^n) \leq n^{1/2}$ for every $X \in \mathcal{B}_n$ and thus, if $X, Y \in \mathcal{B}_n$ we have $d(X,Y) \leq n$. We should also recall that, by the result of Gluskin [Gl], the diameter of $\mathcal{B}_n$, $\sup_{X,Y \in \mathcal{B}_n} d(X,Y)$, is asymptotically of order n. This remarkable fact can be also seen as a consequence of the Szarek's result.

Proof of Proposition 3.2.2. Because T is block-diagonal with respect to $\{Z_k\}_{k\in I}$ and $\{\overline{Z_k}\}_{k\geq 1}$, for every $k \in I$ there exist a finite set $B_k \subset \{1, 2, \ldots\}$ and sequences of scalars $u_k = \left(\overline{u_k(j)}\right)_j$, $v_k = \left(\overline{v_k(j)}\right)_j$, $w_k = \left(\overline{w_k(j)}\right)_j$, $s_k = \left(\overline{s_k(j)}\right)_j$ such that

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$$\begin{cases} \max B_k < \min B_l, \quad \forall k,l \in I \text{ with } k<l \\ Tx_k = \sum_{j\in B_k} \left(\overline{u_k(j)} \odot x_j + \overline{v_k(j)} \odot y_j\right) = \sum_{j\in B_k} \left(u_k(j)x_j + v_k(j)y_j\right) \\ Ty_k = \sum_{j\in B_k} \left(\overline{w_k(j)} \odot x_j + \overline{s_k(j)} \odot y_j\right) = \sum_{j\in B_k} \left(w_k(j)x_j + s_k(j)y_j\right). \end{cases}$$

We start off with the complex conjugate sequences in the definitions of each u_k, v_k, w_k, s_k for convenience only, since this will produce later a simplification of writing.

Taking into account the definitions of x_j and y_j we can write, for all $k \in I$

$$\begin{aligned} Ty_k &= \sum_{j\in B_k} w_k(j) f_{1,j} + \sum_{j\in B_k} s_k(j) f_{2,j} + \sum_{j\in B_k} \gamma_1 w_k(j) f_{3,j} \\ &+ \sum_{j\in B_k} \gamma_2 \left(w_k(j) + s_k(j)\right) f_{4,j} + \sum_{j\in B_k} \gamma_3 \left(w_k(j) + i s_k(j)\right) f_{5,j}. \end{aligned} \tag{3.2}$$

We will only prove the estimates in **(i)** and **(ii)** involving y_k's (the others can be obtained similarly). The proof of **(i)** is presented in a few steps.

Let $Q_t : l_{p_1} \oplus \ldots \oplus l_{p_5} \longrightarrow l_{p_t}$ be the canonical projection ($t = 1, \ldots, 5$).

Step 1. We show first that there exists a set $A_1 \subset I$, $|A_1| < N$ such that

$$\gamma_1 \| \sum_{j\in B_k} w_k(j) f_{3,j} \| \leq 3N^{-\alpha}, \quad \text{for all } k \in I \setminus A_1. \tag{3.3}$$

Indeed, let A_1 be the set of all $k \in I$ such that $\gamma_1 \| \sum_{j\in B_k} w_k(j) f_{3,j} \| > 3N^{-\alpha}$, and assume that $|A_1| \geq N$. Then choose a subset $\tilde{A}$ of A_1 of cardinality N and consider the vector $y = \sum_{k\in\tilde{A}} y_k$. We have

$$\|y\| = N^{1/p_2} + N^{1/p_4 - 4(\alpha_3+\alpha_4)} + N^{1/p_5 - \lambda(\alpha_3+\alpha_4+\alpha_5)} \leq 3N^{1/p_2}$$

and

$$\|Ty\| \geq \|Q_3 Ty\| = \| \sum_{k\in\tilde{A}} \sum_{j\in B_k} \gamma_1 w_k(j) f_{3,j} \| > 3N^{-\alpha} N^{1/p_3}.$$

Since $\|Ty\| \leq \|T\| \, \|y\| \leq \|y\|$, we get the contradiction.

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In a similar manner as above we can obtain $A_2, A_3 \subset I$, $|A_2|, |A_3| < N$ satisfying

$$\gamma_2 \| \sum_{j \in B_k} (w_k(j) + s_k(j)) f_{4,j} \| \leq 3N^{-\alpha}, \quad \text{for all } k \in I \setminus A_2 \tag{3.4}$$

$$\gamma_3 \| \sum_{j \in B_k} (w_k(j) + i s_k(j)) f_{5,j} \| \leq 3N^{-\alpha}, \quad \text{for all } k \in I \setminus A_3. \tag{3.5}$$

Combining (3.2), (3.3), (3.4) and (3.5) we find a set $A \subset I$, with $|A| < 3N$ such that

$$\|Ty_k\| \leq \| \sum_{j \in B_k} w_k(j) f_{1,j} \| + \| \sum_{j \in B_k} s_k(j) f_{2,j} \| + 9N^{-\alpha}, \quad \forall k \in I \setminus A. \tag{3.6}$$

Step 2. By considering elements of the form $x = \sum_{k \in \tilde{A}} x_k$, with $\tilde{A} \subset I$, $|\tilde{A}| = N$, we can obtain in an analogous way as (3.3) a set $A_4 \subset I$, with $|A_4| < N$ such that

$$\| \sum_{j \in B_k} v_k(j) f_{2,j} \| \leq 4N^{-\alpha}, \quad \text{for all } k \in I \setminus A_4. \tag{3.7}$$

Step 3. This is a stronger estimate than (3.3) (and could have been proved directly instead of (3.3)). We show that there is a subset $A_6 \subset I$, with $|A_6| < N^3$ such that

$$\| \sum_{j \in B_k} w_k(j) f_{3,j} \| \leq 3N^{-\alpha_3}, \quad \text{for all } k \in I \setminus A_6. \tag{3.8}$$

For the proof take a vector $y = \sum_{k \in \tilde{A}} y_k$, with $\tilde{A}$ a subset of cardinality N^3 of the set of all $k \in I$ such that $\| \sum_{j \in B_k} w_k(j) f_{3,j} \| > 3N^{-\alpha_3}$. Then

$$\|y\| = (N^3)^{1/p_2} + N^{-4(\alpha_3+\alpha_4)} (N^3)^{1/p_4} + N^{-\lambda(\alpha_3+\alpha_4+\alpha_5)} (N^3)^{1/p_5} \leq 3(N^3)^{1/p_2}$$

and

$$\|Ty\| \geq \|Q_3 Ty\| = \gamma_1 \| \sum_{k \in \tilde{A}} \sum_{j \in B_k} w_k(j) f_{3,j} \| \geq N^{-2\alpha_3} 3N^{-\alpha_3} (N^3)^{1/p_3} = 3(N^3)^{1/p_2},$$

contradiction.

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As a remark, note that since $\{f_{1,j}\}_j$ is dominated by $\{f_{3,j}\}_j$ (as the unit vector bases in l_{p_1} and l_{p_3}, respectively) we also have the estimate analogous to (3.8)

$$\|\sum_{j\in B_k} w_k(j) f_{1,j}\| \leq 3N^{-\alpha_3}, \quad \text{for all } k \in I \setminus A_6. \tag{3.9}$$

Step 4. We show that there exists a set $A_5 \subset I$, $|A_5| < N^\lambda$ such that

$$\|\sum_{j\in B_k} (u_k(j) + v_k(j) - w_k(j) - s_k(j))\, f_{4,j}\| \leq 5N^{-\alpha_4}, \quad \forall k \in I \setminus A_5. \tag{3.10}$$

Indeed, let A_5 be the set of all $k \in I$ such that

$$\|\sum_{j\in B_k} (u_k(j) + v_k(j) - w_k(j) - s_k(j))\, f_{4,j}\| > 5N^{-\alpha_4}$$

and assume that $|A_5| \geq N^\lambda$. Then pick a subset $\tilde{A}$ of A_5 of cardinality $K := N^\lambda$ and consider the vector $z = \sum_{k\in\tilde{A}} (x_k - y_k)$. We have

$$\|z\| = K^{1/p_1} + K^{1/p_2} + N^{-2\alpha_3} K^{1/p_2+\alpha_3} + N^{-\lambda(\alpha_3+\alpha_4+\alpha_5)} \sqrt{2} K^{1/p_2+\alpha_3+\alpha_4+\alpha_5}$$

while

$$\begin{aligned} \|Tz\| &\geq \|Q_4 Tz\| = \gamma_2 \|\sum_{k\in\tilde{A}} \sum_{j\in B_k} (u_k(j) + v_k(j) - w_k(j) - s_k(j))\, f_{4,j}\| \\ &> N^{-4(\alpha_3+\alpha_4)}\, 5N^{-\alpha_4} K^{1/p_2+\alpha_3+\alpha_4}. \end{aligned}$$

But this contradicts $\|Tz\| \leq \|z\|$ since, by the choice of λ,

$$N^{-4(\alpha_3+\alpha_4)} N^{-\alpha_4} K^{1/p_2+\alpha_3+\alpha_4} \geq$$

$$\geq \max\{K^{1/p_2}, N^{-2\alpha_3} K^{1/p_2+\alpha_3}, N^{-\lambda(\alpha_3+\alpha_4+\alpha_5)} K^{1/p_2+\alpha_3+\alpha_4+\alpha_5}\}.$$

Notice that the above inequality is equivalent to

$$\max\{N^{4+\alpha_4/(\alpha_3+\alpha_4)}, N^{5+2\alpha_3/\alpha_4}\} \leq K \leq N^{\lambda+(\lambda-4)\alpha_3/\alpha_5+(\lambda-5)\alpha_4/\alpha_5}$$

which is satisfied since $\lambda > 5 + 2\alpha_3/\alpha_4$.

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Step 5. By considering elements of the form $z = \sum_{k\in\tilde{A}}(x_k + iy_k)$, with $|\tilde{A}|$ large enough, and taking into account that $Tz = \sum_{k\in\tilde{A}}(Tx_k + i\odot Ty_k) = \sum_{k\in\tilde{A}}(Tx_k - i\,Ty_k)$, one can find a finite set $A_7 \subset I$ such that

$$\|\sum_{j\in B_k}(u_k(j) + s_k(j) + iv_k(j) - iw_k(j))\, f_{5,j}\| \le 5N^{-\alpha}, \ \forall k \in I\setminus A_7. \tag{3.11}$$

Since $p_1 > p_2 > p_3 > p_4 > p_5$ it follows that $\{f_{2,j}\}_j$ is dominated by $\{f_{3,j}\}_j$, $\{f_{4,j}\}_j$, $\{f_{5,j}\}_j$. Combining this with (3.7), (3.10), (3.8), (3.11) we obtain a finite set $A'(= A_4 \cup A_5 \cup A_6 \cup A_7)$ such that

$$\|\sum_{j\in B_k} s_k(j)\ f_{2,j}\| \le 12N^{-\alpha}, \quad \text{for all } k \in I\setminus A'. \tag{3.12}$$

Using (3.9) and (3.12) in (3.6) we get a finite set $J(= A\cup A')$ satisfying

$$\|Ty_k\| \le 24N^{-\alpha}, \quad \text{for all } k \in I\setminus J.$$

(ii) Let $J \subset I$ be the subset constructed in (i). By ignoring a finite number of sets from the family $\{I_l\}_{l\ge 1}$ we can suppose that $I_l \subset I\setminus J$ for all $l \ge 1$. In particular, for each $l \in \{1, 2, ...\}$ we have

$$\begin{cases} \|Q_1Ty_k\| \le 24N^{-\alpha} \\ \|Q_2Ty_k\| \le 24N^{-\alpha}, \end{cases}$$

for all $k \in I_l$.

Looking at $T\widetilde{y}_l$ we can write, for each $l \ge 1$,

$$\begin{aligned}
\|T\widetilde{y}_l\| &= \|\sum_{k\in I_l} a_l(k)Q_1Ty_k\| + \|\sum_{k\in I_l} a_l(k)Q_2Ty_k\| + \|(Q_3+Q_4+Q_5)T\widetilde{y}_l\| \\
&\le 24N^{-\alpha}\left((\sum_{k\in I_l}|a_l(k)|^{p_1})^{1/p_1} + (\sum_{k\in I_l}|a_l(k)|^{p_2})^{1/p_2}\right) + \\
&\qquad\qquad + \|(Q_3+Q_4+Q_5)T\widetilde{y}_l\| \\
&\le 48N^{-\alpha} + \|Q_3T\widetilde{y}_l\| + \|Q_4T\widetilde{y}_l\| + \|Q_5T\widetilde{y}_l\|.
\end{aligned}$$

We show that there exists a subset $A \subset \{1, 2, ...\}$, $|A| < N$ such that

$$\|Q_3T\widetilde{y}_l\| \le 3N^{-\alpha}, \quad \forall l \in \{1, 2, ...\}\setminus A.$$

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Indeed, let A be the set of all $l \in \{1, 2, ...\}$ satisfying $\|Q_3 T \widetilde{y}_l\| > 3N^{-\alpha}$, and assume that $|A| \geq N$. Choose a subset A_0 of A of cardinality N and consider the vector $y = \sum_{l \in A_0} \widetilde{y}_l$. We have

$$
\begin{aligned}
\|y\| &= \|\sum_{l \in A_0} \sum_{k \in I_l} a_l(k)\ y_k\| \\
&= (\sum_{l \in A_0} \sum_{k \in I_l} |a_l(k)|^{p_2})^{1/p_2} + \gamma_2 (\sum_{l \in A_0} \sum_{k \in I_l} |a_l(k)|^{p_4})^{1/p_4} + \gamma_3 (\sum_{l \in A_0} \sum_{k \in I_l} |a_l(k)|^{p_5})^{1/p_5} \\
&\leq N^{1/p_2} + \left(\gamma_2 \, (N^2)^{1/p_4 - 1/p_2} + \gamma_3 \, (N^2)^{1/p_5 - 1/p_2}\right) (\sum_{l \in A_0} \sum_{k \in I_l} |a_l(k)|^{p_2})^{1/p_2} \\
&= N^{1/p_2} + N^{-2(\alpha_3 + \alpha_4)} N^{1/p_2} + N^{-(\lambda-2)(\alpha_3 + \alpha_4 + \alpha_5)} N^{1/p_2}.
\end{aligned}
$$

This contradicts $\|T\| \leq 1$ since

$$
\|Ty\| \geq \|Q_3 T y\| = \|\sum_{l \in A_0} Q_3 T \widetilde{y}_l\| > 3N^{-\alpha} N^{1/p_3}.
$$

Arguing similarly for $Q_4 T \widetilde{y}_l$ and $Q_5 T \widetilde{y}_l$ we obtain the conclusion. □

Remark. The proof of Proposition 3.2.2 doesn't require I to be infinite, just to have a certain (large) cardinality. However, the fact that $X_{N,\eta}$ is infinite dimensional is crucial when we consider arbitrary operators defined on $X_{N,\eta}$, since this will allow us to approximate them by block-diagonal operators.

3.3 Subspaces of L_p, $1 \leq p < 2$, with at least two non-isomorphic complex structures

Theorem 3.3.1 *Let $(r_n)_{n \geq 1}$ be a strictly decreasing sequence of real numbers, with $r_n > 1$ for all n, and let $q \in [1, \lim_{n \to \infty} r_n]$. There exists a subspace X of $(\sum_{n \geq 1} \oplus\, l_{r_n})_{l_q}$ which is not isomorphic to its complex conjugate. Furthermore, we can construct the subspace X such that, as a real space, it has a continuum of non-isomorphic complex structures.*

Proof. For each $m = 1, 2, \ldots$ we will define X_m as one of the spaces $X_{N,\eta}$ discussed before for the following choice of the parameters involved. Let $\eta_m =$

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$(r_{5m+1}, r_{5m+2}, \ldots, r_{5m+5}) \in \wp$. Set

$$\alpha_m = \min\left\{\frac{1}{r_{5m+2}} - \frac{1}{r_{5m+1}}, \ldots, \frac{1}{r_{5m+5}} - \frac{1}{r_{5m+4}}\right\}.$$

(This definition of α_m corresponds to α from the main construction in Section 3.2, and it hopefully will not get confused with the notation $\alpha_2, \ldots, \alpha_5$ used there). Finally, fix a natural number $N_m \geq (456 \cdot m)^{2/\alpha_m}$. Now, let $X_m = X_{N_m,\eta_m}$ be the space defined in Section 3.2, treated as a subspace of $l_{r_{5m+1}} \oplus_q l_{r_{5m+2}} \oplus_q \cdots \oplus_q l_{r_{5m+5}}$ (see remark after Proposition 3.2.2).

Similarly as in Section 3.2, for every $m = 1, 2, \ldots$, let $Q_{t,m} : l_{r_{5m+1}} \oplus_q l_{r_{5m+2}} \oplus_q \cdots \oplus_q l_{r_{5m+5}} \longrightarrow l_{r_{5m+t}}$ be the canonical projection, where $t = 1, \ldots, 5$.

We will show that the space $X = (\sum_{m\geq 1} \oplus X_m)_{l_q}$ is not isomorphic to its complex conjugate $\overline{X} = (\sum_{m\geq 1} \oplus \overline{X}_m)_{l_q}$.

Suppose that $T : X \longrightarrow \overline{X}$ is an isomorphism with $\|T\| \leq 1/4$. Denote by $a = \|T^{-1}\|$ and by $P_j : \overline{X} \longrightarrow \overline{X}_j$ the projection of $\overline{X}$ onto its j-th term.

The proof is based on successive passing to appropriate subspaces in order to simplify the representation of the isomorphism T.

Fix an arbitrary $m \geq 1$. Recall that $X_m = \overline{\text{span}}\,\{Z_k\}_{k\geq 1}$.

Let $s > m$. We will show that

$$\begin{gathered} \forall L \subset \{1, 2, \ldots\} \text{ infinite set } \forall \epsilon_s > 0\ \exists k \in L \text{ such that} \\ \|P_s T z_k\| < \epsilon_s \|z_k\|, \qquad \forall z_k \in Z_k. \end{gathered} \tag{3.13}$$

If not we can find $\epsilon_s > 0$, an infinite set $\{k_j\}_{j\geq 1}$ and, for each $j \geq 1$, normalized elements $z_j \in Z_{k_j}$ satisfying

$$\epsilon_s \leq \|P_s T z_j\| \left(= (\|Q_{1,s} P_s T z_j\|^q + \ldots + \|Q_{5,s} P_s T z_j\|^q)^{1/q}\right).$$

By passing to a subsequence of j's (apply Proposition 3.2.1 to the operator $P_s T_{|\overline{\text{span}}\,\{Z_{k_j}\}_{j\geq 1}}$) we may assume that $(P_s T z_j)_{j\geq 1}$ are successive blocks in $\overline{X}_s$. Also, after taking a further subsequence, we get $t \in \{1, \ldots, 5\}$ such that

$$\|Q_{t,s} P_s T z_j\| \geq \frac{\epsilon_s}{10}, \quad \text{for all } j \geq 1.$$

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After relabeling we may consider $t = 1$. To obtain the contradiction observe that, for all positive integers M

$$5M^{1/r_{5m+5}} \geq \|\sum_{j=1}^{M} z_j\|_{X_m} \geq \|P_sT(\sum_{j=1}^{M} z_j)\|_{\overline{X}_s} \geq \|Q_{1,s}(\sum_{j=1}^{M} P_sTz_j)\| \geq \frac{\epsilon_s}{10}M^{1/r_{5s+1}}.$$

Now a standard argument easily shows that for $s > m$

$$\forall L \subset \{1,2,\ldots\} \text{ infinite set } \forall \epsilon_s > 0\ \exists L_s \subseteq L \text{ infinite set such that}$$
$$\|P_sTx\| < \epsilon_s\|x\|, \qquad \forall x \in \overline{\text{span}}\,\{Z_k\}_{k\in L_s}. \tag{3.14}$$

Indeed, fix L and ϵ_s. By successive applications of (3.13) we can construct by induction infinite subsets $L = L'_1 \supset L'_2 \supset L'_3 \supset \ldots$ and a sequence of integers $k'_1 < k'_2 < k'_3 < \ldots$ such that $k'_j \in L'_j \setminus L'_{j+1}$ for $j = 1, 2, \ldots$ and

$$\|P_sTz_j\| \leq \epsilon_s/2^j\,\|z_j\|, \quad \text{for all } z_j \in Z_{k'_j}.$$

Then let $L_s = \{k'_1, k'_2, \ldots\}$. Since the decomposition $\{Z_k\}_{k\in L_s}$ is 1-unconditional in X_m, it is easy to see that

$$\|P_sTx\| \leq \epsilon_s\|x\|, \quad \text{for all } x \in \overline{\text{span}}\,\{Z_k\}_{k\in L_s}.$$

Applying (3.14) inductively for $s = m+1, m+2, \ldots$ we obtain that for every sequence $\{\epsilon_s\}_{s>m}$, $\epsilon_s \searrow 0$ there exist infinite sets of positive integers $L_{m+1} \supset L_{m+2} \supset \ldots \supset L_s \supset \ldots$ such that

$$\|P_sT_{|\,\overline{\text{span}}\,\{Z_k\}_{k\in L_s}}\| < \epsilon_s, \quad \text{for all } s > m.$$

Letting $I = \{k_j\}_{j>m}$ to be a diagonal sequence, so that $k_j \in L_j$ for $j > m$, we have

$$\|P_sT_{|\,\overline{\text{span}}\,\{Z_{k_j}\}_{j\geq s}}\| < \epsilon_s, \quad \text{for all } s > m.$$

Let $Y_m = \overline{\text{span}}\,\{Z_k\}_{k\in I} \subset X_m$. Perturbing the operator $T_{|Y_m}$ we obtain an operator (denoted again by) $T : Y_m \longrightarrow \overline{X}$ satisfying

$$\begin{cases} P_sT_{|\,\overline{\text{span}}\,\{Z_{k_j}\}_{j\geq s}} = 0, & \text{for all } s > m \\ \frac{1}{2a}\|x\| \leq \|Tx\| \leq \frac{1}{2}\|x\|, & \text{for all } x \in Y_m. \end{cases} \tag{3.15}$$

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Let denote by $R_m : \overline{X} \longrightarrow (\sum_{s>m} \oplus \overline{X}_s)_{l_q}$ the natural projection. We will show that there exists an infinite subset $\tilde{I} \subset I$ so that, after further perturbations of T, we have

$$\begin{cases} R_m T_{\mid \overline{\operatorname{span}}\{Z_k\}_{k\in\tilde{I}}} = 0 \\ \frac{1}{4a}\|x\| \leq \|Tx\| \leq \|x\|, \quad \text{for all } x \in \overline{\operatorname{span}}\{Z_k\}_{k\in\tilde{I}} \end{cases} \tag{3.16}$$

To this end, it is enough to prove that for all $\delta > 0$ and every infinite set $L \subset I$ there is $k \in L$ so that

$$\|R_m T z_l\| \leq \delta \|z_l\|, \quad \text{for all } z_l \in Z_l.$$

Suppose that the above statement is not true and hence we can find $\delta > 0$, an infinite set $L \subset I = \{k_j\}_{j>m}$ and, for each $l \in L$, normalized elements $z_l \in Z_l$ such that

$$\|R_m T z_l\| > \delta.$$

If $L = \{l_1, l_2, \ldots, l_t, \ldots\}$ with $l_1 < l_2 < \ldots < l_t < \ldots$ then, by (3.15), we have $\operatorname{supp} R_m T z_{l_1} \supset \operatorname{supp} R_m T z_{l_2} \supset \ldots \supset \operatorname{supp} R_m T z_{l_t} \supset \ldots$, where the support is considered with respect to the decomposition $\{\overline{X}_s\}_{s>m}$. After a gliding hump argument we may assume that $(R_m T z_l)_{l\in L}$ are successive blocks in $(\sum_{s>m} \oplus \overline{X}_s)_{l_q}$ with respect to the decomposition $\{\overline{X}_s\}_{s>m}$. Since $r_{5m+1} > \ldots > r_{5m+5} > q$ it is now clear that we can find real scalars $\{a_l\}_{l\in L}$ such that $z = \sum_{l\in L} a_l z_l$ is convergent in Y_m while $R_m T z = \sum_{l\in L} a_l R_m T z_l$ is divergent in $\overline{X}$, showing that the above assumption is false. Therefore we have (3.16).

By applying successively Proposition 3.2.1 we may also assume that the operator $T : \overline{\operatorname{span}}\{Z_k\}_{k\in\tilde{I}} \to \overline{X}$ satisfies, besides (3.16),

$$\begin{cases} P_1 T : \overline{\operatorname{span}}\{Z_k\}_{k\in\tilde{I}} \longrightarrow \overline{X}_1 \text{ is block} - \text{diagonal} \\ \qquad\ldots \\ P_m T : \overline{\operatorname{span}}\{Z_k\}_{k\in\tilde{I}} \longrightarrow \overline{X}_m \text{ is block} - \text{diagonal.} \end{cases} \tag{3.17}$$

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Applying Proposition 3.2.2 (ii) to $P_mT : \overline{\text{span}}\,\{Z_k\}_{k\in\tilde{I}} \longrightarrow \overline{X}_m$ we find $I_0 \subset \tilde{I}$, $|I_0| = N_m$ with the property that, considering $y = \sum_{k\in I_0} y_k$,

$$\|P_mTy\| \leq 57N_m^{-\alpha_m}{N_m}^{1/r_{5m+2}} \leq 57N_m^{-\alpha_m/2}{N_m}^{1/r_{5m+2}}.$$

Thus we can write

$$\|(P_1 + ... + P_{m-1})Ty\| \geq \|Ty\| - \|P_mTy\| \geq (\frac{1}{4a} - 57N_m^{-\alpha_m/2}){N_m}^{1/r_{5m+2}}.$$

Assume $1/(8a) - 57{N_m}^{-\alpha_m/2} \geq 0$. There exists $s \in \{1, ..., m-1\}$ such that

$$\|P_sTy\| \geq \frac{1}{m-1}\|(P_1 + ... + P_{m-1})Ty\| \geq \frac{1}{m-1}\,\frac{1}{8a}{N_m}^{1/r_{5m+2}}.$$

Since $1/(m-1) \geq {N_m}^{-\alpha_m/2}$ and, by our assumption, $1/(8a) \geq 57{N_m}^{-\alpha_m/2}$, the last quantity from above is larger than or equal to $57{N_m}^{1/r_{5m+1}}$. This is a contradiction since

$$\begin{aligned}\|P_sTy\| &= \|\sum_{k\in I_0} P_sTy_k\| \leq \|\sum_{k\in I_0} Q_{1,s}P_sTy_k\| + \ldots + \|\sum_{k\in I_0} Q_{5,s}P_sTy_k\| \\ &\leq 2{N_m}^{1/r_{5s+1}} + \ldots + 2{N_m}^{1/r_{5s+5}}\end{aligned}$$

where, at the last inequality, we used (3.17) and

$$\|Q_{t,s}P_sTy_k\| \leq \|P_sTy_k\| \leq \|y_k\| \leq 2, \quad \forall k \in I_0, \quad \forall t = 1, \ldots, 5.$$

Hence we must have $a \geq 1/456\ N_m^{\alpha_m/2} \geq m$, for all $m \geq 1$, proving that X is not isomorphic to its complex conjugate.

We will indicate how we can obtain continuum non-isomorphic complex structures on X. For a set $A \subset \{1, 2, \ldots\}$ denote by $X^{(A)}$ the Banach space defined by $X^{(A)} = (\sum_{m\geq 1} \oplus \widetilde{X}_m)_{l_q}$ where

$$\widetilde{X}_m = \begin{cases} X_m, & \text{if } m \notin A \\ \overline{X}_m, & \text{if } m \in A. \end{cases}$$

It is well known that there exists a family of cardinality continuum of infinite subsets of positive integers $\{A_t\}_{t\in\mathbf{R}}$ such that $|A_t \cap A_s| < \infty$, for $t \neq s$. Indeed,

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identifying $\mathbf{N}$ with the set of all rational numbers, we let A_t to be an arbitrarily fixed infinite sequence of rational numbers converging to t, for every $t \in \mathbf{R}$.

Now notice that any two Banach spaces from the family $\{X^{(A_t)}\}_{t\in\mathbf{R}}$ are not isomorphic. Indeed, let $A, B \in \{A_t\}_{t\in\mathbf{R}}$ and let T be an isomorphism between $X^{(A)}$ and $X^{(B)}$. Denoting $A^c \cap B = \{n_1, n_2, ..., n_l, ...\}$ we can repeat the whole argument for $T_{|X_{n_l}}$ and get $\|T^{-1}\| \geq n_l$, for all $l \geq 1$. □

Remark. The proof of Theorem 3.3.1 yields, for the case $q = 2$, a constructive version of the Bourgain-Szarek example.

Corollary 3.3.2 *For $1 \leq p < 2$, the space L_p contains a real subspace having a continuum of non-isomorphic complex structures.*

Proof. Let $\{r_n\}_{n\geq 1}$ be a strictly decreasing sequence of real numbers such that $p < r_n < 2$, for all n. It is well known that if $1 \leq p < q < 2$ then L_p contains an isomorphic copy of l_q. Also, L_p is isomorphic to an l_p sum of infinitely many copies of L_p, $(\sum \oplus L_p)_{l_p}$. Then L_p has a subspace isomorphic to $(\sum_{n\geq 1} \oplus\, l_{r_n})_{l_p}$. The conclusion follows now from Theorem 3.3.1. □

3.4 Another Banach space with at least two non-isomorphic complex structures

The following fact is well known.

Lemma 3.4.1 *Let $\{q_n\}_{n\geq 1}$ be a sequence of real numbers with $q_n \geq 1$, for all n, and let E be a M-dimensional subspace of $(\sum_{n\geq 1} \oplus l_{q_n})_{l_2}$. Then*

$$d(E, l_2^M) \leq M^\beta,$$

where $\beta = \sup_n |1/q_n - 1/2|$.

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Proof. Denoting by P_j the natural projection of $(\sum_{n\geq 1} \oplus l_{q_n})_{l_2}$ onto its j-th term, we have $E \subset (\sum_{n\geq 1} \oplus P_n E)_{l_2}$ and, by the result of Lewis [Le],

$$d(P_n E, l_2^{\dim P_n E}) \leq (\dim P_n E)^{|1/q_n - 1/2|} \leq M^{|1/q_n - 1/2|}, \quad \text{for all } n \geq 1.$$

Thus

$$d(E, l_2^M) \leq d((\sum_{n\geq 1} \oplus P_n E)_{l_2}, l_2) \leq M^{\beta}.$$

□

We can now prove the main result of this section.

Theorem 3.4.2 *There exists a sequence $r_n \nearrow 2$ such that $\left(\sum_{n\geq 0} \oplus l_{r_n}\right)_{l_2}$ contains a real subspace with a continuum of non-isomorphic complex structures.*

Proof. The sequence $\{r_n\}_{n\geq 0}$ will be defined inductively. We will also construct inductively a sequence of positive integers $\{N_m\}_{m\geq 1}$. Denoting by $\eta_m = (r_{5m-1}, r_{5m-2}, \ldots, r_{5m-5})$ for all $m \geq 1$, we will then define X_{N_m, η_m} as one of the spaces discussed in Section 3.2. Set $\alpha_m = \min\{1/r_{5m-2} - 1/r_{5m-1}, \ldots, 1/r_{5m-5} - 1/r_{5m-4}\}$ (this definition of α_m corresponds to α from the construction in Section 3.2). We start the inductive construction with $\eta_1 = (r_4, r_3, \ldots, r_0)$ such that $2 > r_4 > \ldots > r_0 \geq 1$. Having defined $\eta_1, N_1, \ldots, \eta_{m-1}, N_{m-1}$ and $\eta_m = (r_{5m-1}, r_{5m-2}, \ldots, r_{5m-5})$ we take $N_m \in \{1, 2, \ldots\}$ such that

$$[N_m^{\alpha_m}]^{1/r_{5m-2} - 1/2} \geq 100m. \tag{3.18}$$

Setting $M_m = [N_m^{\alpha_m}]$ we can then choose $\eta_{m+1} = (r_{5m+4}, \ldots, r_{5m})$ such that $2 > r_{5m+4} > \ldots > r_{5m} > r_{5m-1} > \ldots > r_{5m-5}$ and

$$M_m^{1/r_{5m} - 1/2} \leq 2. \tag{3.19}$$

Notice that the sequence $\{r_n\}_{n\geq 0}$ converges to 2, since by (3.18) and (3.19) we have

$$0 \leq \frac{1}{r_{5m}} - \frac{1}{2} \leq \frac{\ln 2}{\ln M_m} \leq \frac{\ln 2}{\ln 100m}, \quad \text{for all } m \geq 1.$$

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Let $X_m = X_{N_m,\eta_m}$ be the space defined in Section 3.2, treated as a subspace of $l_{r_{5m-1}} \oplus_2 l_{r_{5m-2}} \oplus_2 \ldots \oplus_2 l_{r_{5m-5}}$. We will show that the space $X = (\sum_{m\geq 1} \oplus X_m)_{l_2}$ is not isomorphic to its complex conjugate.

Suppose that $T : X \longrightarrow \overline{X}$ is an isomorphism with $\|T\| \leq 1/2$. Denote by $a = \|T^{-1}\|$ and by $P_j : \overline{X} \longrightarrow \overline{X}_j$ the projection of $\overline{X}$ onto its j-th term.

Let $m \geq 1$ be arbitrarily fixed. Recall that $X_m = \overline{\text{span}}\,\{Z_k\}_{k\geq 1}$. A similar argument as in Theorem 3.3.1 (see (3.14)) shows that we can find an infinite set of positive integers $I(= I_m)$ and an operator (denoted again by) $T : Y_m = \overline{\text{span}}\,\{Z_k\}_{k\in I} \longrightarrow \overline{X}$ such that

$$\begin{cases} P_s T = 0, \quad \forall s = 1, \ldots, m-1 \\ \frac{1}{2a}\|x\| \leq \|Tx\| \leq \|x\|, \quad \forall x \in Y_m. \end{cases}$$

We may also assume that $P_m T : \overline{\text{span}}\,\{Z_k\}_{k\in I} \rightarrow \overline{X}_m$ is a block-diagonal operator (apply Proposition 3.2.1). By Proposition 3.2.2 (i) we may extract a subset $K \subset I$, with $|K| = M_m$ such that

$$\|P_m T y_k\| \leq 24 N_m^{-\alpha_m}, \quad \text{for all } k \in K,$$

where $y_k \in Z_k$ is one of the basis vectors defining Z_k, for $k \in K$. In particular $\{y_k\}_{k\in K}$ are 1-unconditional. Let $R_m : \overline{X} \longrightarrow (\sum_{s>m} \oplus \overline{X}_s)_{l_2}$ be the natural projection. Since $T = P_m T + R_m T$, for every choice of signs $\{\epsilon_k\}_{k\in K}$ we can write

$$\begin{aligned} \|R_m T(\sum_{k\in K} \epsilon_k y_k)\| &\geq \frac{1}{2a}\|\sum_{k\in K} \epsilon_k y_k\| - \|P_m T(\sum_{k\in K} \epsilon_k y_k)\| \\ &\geq \frac{1}{2a}\|\sum_{k\in K} y_k\| - 24 M_m N_m^{-\alpha_m}. \end{aligned} \tag{3.20}$$

We have two cases. Assume first that $1/4a\,\|\sum_{k\in K} y_k\| \geq 24 M_m N_m^{-\alpha_m}$. Since by Lemma 3.4.1 any M_m -dimensional subspace E of $(\sum_{s>m} \oplus \overline{X}_s)_{l_2} \subset (\sum_{s\geq 5m} \oplus \bar{l}_{r_s})_{l_2}$ satisfies

$$d(E, l_2^{M_m}) = d(E, \bar{l}_2^{M_m}) \leq M_m{}^{1/r_{5m}-1/2} \leq 2,$$

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using the parallelogram identity, estimate (3.20) and our hypothesis we obtain

$$\sum_{k\in K}\|R_mTy_k\|^2 \geq \frac{1}{4}\frac{1}{2^{M_m}}\sum_{(\epsilon_k)_{k\in K}}\|\sum_{k\in K}\epsilon_k R_m T y_k\|^2 \geq \frac{1}{64a^2}\|\sum_{k\in K} y_k\|^2.$$

Since $\|R_mTy_k\| \leq \|Ty_k\| \leq \|y_k\| \leq 2$ for all $k \in K$, we get

$$4M_m \geq \frac{1}{64a^2}\|\sum_{k\in K} y_k\|^2 \geq \frac{1}{64a^2}M_m^{2/r_{5m-2}}.$$

Thus, by (3.18),

$$a \geq \frac{1}{16}M_m^{1/r_{5m-2}-1/2} \geq m.$$

The second case is $1/4a\|\sum_{k\in K} y_k\| < 24M_mN_m^{-\alpha_m} \leq 24M_mN_m^{-\alpha_m/2}$. Then we have

$$a > \frac{1}{96}M_m^{-1}N_m^{\alpha_m/2}\|\sum_{k\in K} y_k\| \geq \frac{1}{96}M_m^{-1}N_m^{\alpha_m/2}M_m^{1/r_{5m-2}} \geq \frac{1}{96}M_m^{1/r_{5m-2}-1/2} \geq m.$$

Thus $a \geq m$ in this case as well. Since m is arbitrary, it means that spaces X and $\overline{X}$ are not isomorphic. The fact that X, as a real space, has continuum non-isomorphic complex structures follows in the same manner as in Theorem 3.3.1. $\square$

3.5 Final remarks

As we already mentioned in Chapter 2, a space which is non-isomorphic to its complex conjugate cannot admit an unconditional basis. As a consequence of our construction, the spaces obtained here have a stronger property than being without unconditional basis.

Comparing to constructions of subspaces without unconditional basis, like the ones discussed in Chapter 4 of this dissertation, in the present case of isomorphisms of the complex conjugates many fewer linear operators are available, and no criterion of a similar type as Proposition 4.2.2 is known. Thus it is not clear how to construct spaces with at least two non-isomorphic complex structures as subspaces of arbitrary Banach spaces X, or at least inside $l_2(X)$.

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If we allow ourselves to look for *quotients of subspaces* (which is essentially different than *subspaces*) of $l_2(X)$ then there is the following characterization of a real Hilbert space in terms of the cardinality of its complex structures.

Theorem 3.5.1 ([M-T]) *A real Banach space X is isomorphic to a Hilbert space if and only if every infinite dimensional quotient of every subspace of $l_2(X)$ admits a unique, up to an isomorphism, complex structure.*

It is clear that constructions of spaces non-isomorphic to their complex conjugates are more difficult if we require the space to be close, in a sense, to a Hilbert space. By refining the arguments of this chapter it seems possible to obtain spaces which are very close indeed to l_2, namely *weak Hilbert spaces* (see [P]), and still not isomorphic to their complex conjugates.

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Chapter 4

Unconditional Decompositions in Subspaces of $l_2(X)$

4.1 Introduction

As we already mentioned, the essential idea of the construction is summarized in Proposition 4.2.2, which is our main criterion for recognizing that a space with a special structure does not have higher-order local unconditional structure.

Using this criterion we will then describe an abstract setting in which it is possible to construct subspaces of tensor product spaces without the higher-order local unconditional structure (Theorem 4.3.1). This will enable us to obtain Theorem 4.4.1 and to provide the characterization of a Hilbert space from Corollary 4.4.2.

Comparing to the situation from Chapter 3, in this chapter there are no major differences between the cases of real Banach spaces and complex Banach spaces. A choice of a particular field of scalars (real or complex) may affect only the absolute constants which appear in the estimates. To fix our attention we will assume that all the Banach spaces involved are real Banach spaces.

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4.2 Local unconditional structure of order k

Definition 4.2.1 *A Banach space X has local unconditional structure of order $\leq k$ if there is $C \geq 1$ such that for every finite dimensional subspace $E \subset X$ there exist a Banach space V and operators $u : E \to V$, $w : V \to X$ such that $i_E = wu$, $\|u\| \, \|w\| \leq C$ and V has a 1-unconditional decomposition $\{V_j\}_{j \leq N}$, for some positive integer N, with $\dim V_j \leq k$, for $j \leq N$. Here i_E is the natural embedding $i_E : E \to X$. The infimum of such constants C is denoted by $\mathcal{U}_k(X)$.*

This definition generalizes the local unconditional structure (or l.u.s.t.) defined in Chapter 2. Clearly, if a space X has a k-dimensional unconditional decomposition then $\mathcal{U}_k(X) < \infty$. For an arbitrary Banach space X these different types of local unconditional structure are related via the following inequalities

$$\mathcal{U}_1(X) \geq \mathcal{U}_2(X) \geq \ldots \geq \mathcal{U}_k(X) \geq \ldots.$$

The following result generalizes [K-T1], Proposition 1.1 (see also [K-T2]) and is a version of a criterion due to Ketonen [Ke] and Borzyszkowski [B].

Proposition 4.2.2 *Let $k \geq 2$. Let Y be a Banach space of cotype r, for some $r < \infty$, with the cotype constant $C_r(Y)$. Suppose that there exists $s \leq k-1$ such that $\mathcal{U}_s(Y) < \infty$ and that Y has a λ-unconditional decomposition $\{Z_i\}_i$, with $\dim Z_i = k$ for all i, for some $\lambda \geq 1$. Then there exists an operator $T : Y \to Y$ such that*

(i) *$T(Z_i) \subset Z_i$, for $i = 1, 2, \ldots,$*

(ii) *$\|T\| \leq \lambda^2 M \, \mathcal{U}_s(Y)$, where M depends on s, r and $C_r(Y)$ only,*

(iii) *$\inf_\mu \|T_{|Z_i} - \mu I_{Z_i}\| \geq \frac{1}{2k^2}$, for $i = 1, 2, \ldots,$ where the infimum is taken over all real scalars μ.*

Proof Assume $\mathcal{U}_s(Y) < \infty$. It is enough to construct a sequence of operators $T_n : Y \to Y$ such that, for all n, the operator T_n satisfies (i), (ii) and

$$\inf_\mu \|T_{n|Z_i} - \mu I_{Z_i}\| \geq \frac{1}{2k^2}, \text{ for } i = 1, \ldots, n.$$

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The existence of the operator T will then follow by a diagonal construction.

Namely, pass to an infinite subsequence $L_1 \subset \mathbf{N}$ such that the limit over n, $\lim\{T_{n\,|\,Z_1}\,|\,n \in L_1\}$ exists (since $\dim Z_1 = k$, we may take the limit in an arbitrary norm on the space of operators, for example in the operator norm). By induction, for $i \geq 2$, pick an infinite subsequence $L_i \subset L_{i-1}$ such that the limit $\lim\{T_{n\,|\,Z_i}\,|\,n \in L_i\}$ exists. Let then $L = \{l_1, l_2, \ldots\}$ be an infinite (increasing) subsequence of $\mathbf{N}$ such that $l_i \in L_i$ for all i. Clearly, for every i, the limit over n, $\lim\{T_{n\,|\,Z_i}\,|\,n \in L\}$ exists. Define the operator $T : \operatorname{span}\{Z_i\}_i \to \operatorname{span}\{Z_i\}_i$ by $T_{|\,Z_i} = \lim\{T_{n\,|\,Z_i}\,|\,n \in L\}$, for all i. It is easy to see that T satisfies all the required conditions.

Fix n and $\epsilon > 0$. Let $Y^n = \operatorname{span}\{Z_i\}_{i \leq n}$. Since $\mathcal{U}_s(Y) < \infty$, as it was proved in [B], Proposition 3.1 (although the actual formulation was slightly weaker), there exist a space V with a finite 1-unconditional decomposition $\{V_l\}_{l \leq N}$ and operators $u : Y^n \to V$, $w : V \to Y$ such that $j = wu$, $\|u\|\|w\| \leq (1+\epsilon)\mathcal{U}_s(Y)$, $\dim V_l \leq s$ for all $l \leq N$ and, for every positive integer m, the decomposition $\{r_i V_l\}_{i \leq m, l \leq N}$ is M-unconditional in $\operatorname{span}\{r_i V_l\}_{i \leq m, l \leq N} = \mathrm{Rad}_m(V) \subset L_2(V)$, with M depending on s, r and $C_r(Y)$ only. Above $j : Y^n \to Y$ stands for the canonical inclusion map.

Let P_i be the natural projection from Y onto Z_i, for $i = 1, 2, \ldots$. Also, by $Q_l : V \to V$ denote the natural projection from V onto V_l ($l \leq N$).

For a sequence of signs $\Theta = \{\theta_l\}_{l \leq N}$, with $\theta_l = \pm 1$ if $l \leq N$, define an operator $\Lambda_\Theta : V \to V$ by $\Lambda_\Theta = \sum_{l \leq N} \theta_l Q_l$.

For $i = 1, 2, \ldots$ choose a sequence of signs $\Theta_i = \{\theta_l(i)\}_{l \leq N}$ such that

$$\sup\nolimits_\Theta \inf\nolimits_\mu \|P_i w \Lambda_\Theta u P_i - \mu I_{Z_i}\| \leq \frac{k^2}{k^2 - 1} \inf\nolimits_\mu \|P_i w \Lambda_{\Theta_i} u P_i - \mu I_{Z_i}\|. \tag{4.1}$$

Define now $T_n : Y \to Y$ by $T_n = \sum_{i=1}^n P_i w \Lambda_{\Theta_i} u P_i$.

Fix an arbitrary $y \in Y$. Since $\operatorname{unc}\{Z_i\}_i \leq \lambda$ we get

$$\|T_n y\| \;=\; \|\sum_{i=1}^n P_i w \Lambda_{\Theta_i} u P_i(y)\|$$

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$$
\begin{aligned}
&= \|\int_0^1 \left(\sum_{i=1}^n r_i(t)P_i\right)\left(\sum_{i=1}^n r_i(t)w\Lambda_{\Theta_i}uP_i(y)\right) dt\| \\
&\leq \sup_{0\leq t\leq 1}\|\sum_{i=1}^n r_i(t)P_i\|\|w\| \int_0^1 \|\sum_{i=1}^n r_i(t)\Lambda_{\Theta_i}uP_i(y)\|dt \\
&\leq \lambda\|w\|(\int_0^1 \|\sum_{i=1}^n r_i(t)\Lambda_{\Theta_i}uP_i(y)\|^2 dt)^{1/2}.
\end{aligned}
$$

For each $i = 1, \ldots, n$ write $uP_i(y) \in V$ as $uP_i(y) = \sum_{l\leq N} v_l(i)$, with $v_l(i) \in V_l$ for $l = 1, \ldots, N$. Then, since $\text{unc}\{r_iV_l\}_{i\leq n, l\leq N} \leq M$ in $L_2(V)$,

$$
\begin{aligned}
\|T_ny\| &\leq \lambda\|w\|(\int_0^1 \|\sum_{i=1}^n r_i(t)\,(\sum_{l\leq N}\theta_l(i)v_l(i))\|^2 dt)^{1/2} \\
&= \lambda\|w\|(\int_0^1 \|\sum_{i=1}^n\sum_{l\leq N}\theta_l(i)r_i(t)v_l(i)\|^2 dt)^{1/2} \\
&\leq \lambda M\|w\|(\int_0^1 \|\sum_{i=1}^n\sum_{l\leq N} r_i(t)v_l(i)\|^2 dt)^{1/2} \\
&= \lambda M\|w\|(\int_0^1 \|\sum_{i=1}^n r_i(t)uP_i(y)\|^2 dt)^{1/2} \\
&\leq \lambda M\|w\|\|u\|(\int_0^1 \|\sum_{i=1}^n r_i(t)P_i(y)\|^2 dt)^{1/2} \\
&\leq \lambda^2 M(1+\epsilon)\mathcal{U}_s(Y)\|y\|.
\end{aligned}
$$

Hence we have (ii) satisfied.

Before we prove (iii'), recall that for any $\{x_j\}_j$ in an m-dimensional Banach space we have (as a consequence of Auerbach lemma)

$$\sup_{\epsilon_j=\pm 1}\|\sum \epsilon_j x_j\| \geq \frac{1}{m}\sum \|x_j\|. \tag{4.2}$$

Returning to our proof, fix an arbitrary $i \in \{1, \ldots, n\}$. Consider the k^2-dimensional space H of all linear operators on Z_i with the operator norm and consider the quotient space H/H_0, with $H_0 = \text{span}\,[I_{Z_i}]$. For each $R \in H$, let $\tilde{R}$ be the canonical image of R in H/H_0.

Define $R_l : Z_i \to Z_i$ by $R_l = P_iwQ_luP_i$. Since $\dim R_l(Z_i) \leq s < k$

$$\|\tilde{R}_l\| = \inf_\mu \|R_l - \mu I_{Z_i}\| \geq \frac{1}{2}\|R_l\|, \quad \text{for } l = 1, \ldots, n. \tag{4.3}$$

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This is clearly true for $|\mu| < \|R_l\|/2$, and in the case $|\mu| \geq \|R_l\|/2$ it is sufficient to notice that, since $R_l(Z_i)$ is s-dimensional, there is an $x \in \ker R_l$ such that $\|x\| = 1$.

By (4.1), (4.2) and (4.3) we now have

$$\inf_\mu \|T_{n\,|\,Z_i} - \mu I_{Z_i}\| \geq \frac{k^2-1}{k^2} \sup_\Theta \inf_\mu \|P_i w \Lambda_\Theta u P_i - \mu I_{Z_i}\|$$

$$= \frac{k^2-1}{k^2} \sup_{\theta_l = \pm 1} \|\sum_{l \leq N} \theta_l \tilde{R}_l\| \geq \frac{k^2-1}{k^2} \frac{1}{k^2-1} \sum_{l \leq N} \|\tilde{R}_l\|$$

$$\geq \frac{1}{2k^2} \sum_{l \leq N} \|R_l\| \geq \frac{1}{2k^2} \|\sum_{l \leq N} R_l\| = \frac{1}{2k^2} \|I_{Z_i}\| = \frac{1}{2k^2}.$$

□

Let us also state the following lemma which can be obtained in a similar way as [K-T3], Lemma 3.4.

Lemma 4.2.3 *Let $k \geq 2$ be an integer, $\alpha \in [\frac{k^2-1}{k^2}, 1)$, $n \geq k^2(k^2-1)$ and let $I = \{1, \ldots, n\}$.*

(i) *For all $i_1, \ldots, i_{k+1} \in I$ and $\delta = 1, \ldots, k-1$, let $A^{(\delta)}_{0,i_2,\ldots,i_{k+1}}, A^{(\delta)}_{i_1,0,i_3,\ldots,i_{k+1}}, \ldots,$ $A^{(\delta)}_{i_1,\ldots,i_k,0}$ be subsets of I, each of cardinality at least $\lceil \alpha n \rceil$. Then there exist $j_1, \ldots, j_{k+1} \in I$ such that*

$$j_1 \in A^{(\delta)}_{0,j_2,\ldots,j_{k+1}}, \ldots, j_{k+1} \in A^{(\delta)}_{j_1,\ldots,j_k,0}\,, \quad \text{for all } \delta = 1, \ldots, k-1.$$

(ii) *For all $i_1, \ldots, i_k \in I$ and $\delta = 1, \ldots, k-1$, let $A^{(\delta)}_{0,i_2,\ldots,i_k}, A^{(\delta)}_{i_1,0,i_3,\ldots,i_k}, \ldots,$ $A^{(\delta)}_{i_1,\ldots,i_{k-1},0}$ be subsets of I, each of cardinality at least $\lceil \alpha n \rceil$. Also, for all $i_2, \ldots, i_k \in I$, let $C_{0,i_3,\ldots,i_k}, C_{i_2,0,i_4,\ldots,i_k}, \ldots, C_{i_2,\ldots,i_{k-1},0}$ be subsets of $I \times I$, each of cardinality at least $\lceil \alpha n^2 \rceil$. Then there exists $j_1, \ldots, j_k \in I$ such that*

$$j_1 \in A^{(\delta)}_{0,j_2,\ldots,j_k}, \ldots, j_k \in A^{(\delta)}_{j_1,\ldots,j_{k-1},0} \quad \text{for all } \delta = 1, \ldots, k-1$$

and

$$(j_1, j_2) \in C_{0,j_3,\ldots,j_k}, \ldots, (j_1, j_k) \in C_{j_2,\ldots,j_{k-1},0}.$$

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Proof (i) Consider, for all $\delta = 1, \ldots, k-1$, the following subsets of $\underbrace{I \times \ldots \times I}_{k+1}$

$$\begin{aligned}
I_{1,\delta} &= \bigcup_{i_2,\ldots,i_{k+1}=1}^{n} A^{(\delta)}_{0,i_2,\ldots,i_{k+1}} \times \{i_2\} \times \ldots \{i_{k+1}\} \\
I_{2,\delta} &= \bigcup_{i_1,i_3,\ldots,i_{k+1}=1}^{n} \{i_1\} \times A^{(\delta)}_{i_1,0,i_3,\ldots,i_{k+1}} \times \{i_3\} \times \ldots \{i_{k+1}\} \\
&\vdots \\
I_{k+1,\delta} &= \bigcup_{i_1,\ldots,i_{k+1}=1}^{n} \{i_1\} \times \ldots \times \{i_k\} \times A^{(\delta)}_{i_1,\ldots,i_k,0}.
\end{aligned}$$

Each set from the collection of k^2-1 sets $\{I_{l,\delta}\}_{l=1,\ldots,k+1;\ \delta=1,\ldots,k-1}$ has cardinality larger than or equal to $[\alpha n] \cdot n^k$. Since $\alpha \geq (k^2-1)/k^2$ and $n \geq k^2(k^2-1)$ we have

$$\begin{aligned}
[\alpha n] - \frac{k^2-2}{k^2-1} n &> (\alpha n - 1) - \frac{k^2-2}{k^2-1} n \\
&= (\alpha - \frac{k^2-2}{k^2-1}) n - 1 \\
&\geq (\frac{k^2-1}{k^2} - \frac{k^2-2}{k^2-1}) n - 1 \\
&= \frac{n}{k^2(k^2-1)} - 1 \geq 0.
\end{aligned}$$

Hence $[\alpha n] n^k > (k^2-2)/(k^2-1)\, n^{k+1}$, which implies

$$|I_{l,\delta}| > \frac{k^2-2}{k^2-1} |\underbrace{I \times \ldots \times I}_{k+1}|, \quad \forall l = 1, \ldots, k+1;\ \delta = 1, \ldots, k-1.$$

It follows that

$$\bigcap_{l=1,\ldots,k+1;\ \delta=1,\ldots,k-1} I_{l,\delta} \neq \emptyset.$$

Taking $j_1, \ldots, j_{k+1} \in I$ so that $(j_1, \ldots, j_{k+1}) \in \bigcap_{l=1,\ldots,k+1;\ \delta=1,\ldots,k-1} I_{l,\delta}$ we obtain the conclusion.

(ii) Consider the following k^2-k subsets of $\underbrace{I \times \ldots \times I}_{k}$

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$$
\begin{aligned}
I_{1,\delta} &= \bigcup_{i_2,\ldots,i_k=1}^{n} A^{(\delta)}_{0,i_2,\ldots,i_k} \times \{i_2\} \times \ldots \{i_k\} \\
&\vdots \\
I_{k,\delta} &= \bigcup_{i_1,\ldots,i_{k-1}=1}^{n} \{i_1\} \times \ldots \times \{i_{k-1}\} \times A^{(\delta)}_{i_1,\ldots,i_{k-1},0},
\end{aligned}
$$

for all $\delta = 1, \ldots, k-1$. Also let

$$
\begin{aligned}
J_{12} &= \{(i_1, \ldots, i_k) \in \underbrace{I \times \ldots \times I}_{k} \mid (i_1, i_2) \in C_{0,i_3,\ldots,i_k}\} \\
J_{13} &= \{(i_1, \ldots, i_k) \in \underbrace{I \times \ldots \times I}_{k} \mid (i_1, i_3) \in C_{i_2,0,i_4,\ldots,i_k}\} \\
&\vdots \\
J_{1k} &= \{(i_1, \ldots, i_k) \in \underbrace{I \times \ldots \times I}_{k} \mid (i_1, i_k) \in C_{i_2,\ldots,i_{k-1},0}\}.
\end{aligned}
$$

Each of the $k^2 - 1$ sets from above has cardinality at least $\frac{k^2-2}{k^2-1} |\underbrace{I \times \ldots \times I}_{k}|$. The conclusion is satisfied for $j_1, \ldots, j_k \in I$ such that

$$
(j_1, \ldots, j_k) \in \left(\bigcap_{l=1,\ldots,k;\ \delta=1,\ldots,k-1} I_{l,\delta} \right) \cap \left(\bigcap_{l=1,\ldots,k-1} J_{1l} \right).
$$

$\square$

4.3 Main construction

The next result describes a method of constructing subspaces of tensor product spaces without local unconditional structure of order $\leq k$.

Theorem 4.3.1 *Let $\lambda \geq 1$ and $D \geq \lambda k \sqrt{k^2 - 1}$. Let F be an n-dimensional Banach space with a λ-unconditional normalized basis $\{f_i\}_{i=1}^n$.*

(i) *Suppose that $\| \sum_{i=1}^n f_i \| \geq n^{1/2} D$. Consider $k+1$ tensor product spaces $X_1 = F \otimes \underbrace{l_2^n \otimes \ldots \otimes l_2^n}_{k}$, $X_2 = l_2^n \otimes F \otimes \underbrace{l_2^n \otimes \ldots \otimes l_2^n}_{k-1}$, $\ldots$, $X_{k+1} = \underbrace{l_2^n \otimes \ldots \otimes l_2^n}_{k} \otimes F$,*

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each endowed with a cross-norm. Suppose that the natural tensor basis in each X_i *is* C_λ*-unconditional, for some* $C_\lambda \geq 1$. *Set* $X = X_1 \oplus \ldots \oplus X_{k+1}$ *and let* $C_r(X)$ *be the cotype* r *constant of* X, *for some* $2 \leq r < \infty$. *Then there exists a subspace* $Y \subset X$, *which admits a* k*-dimensional* C_λ*-unconditional decomposition, such that* $\mathcal{U}_{k-1}(Y) \geq a\,\lambda^{-1} C_\lambda^{-2} D$.

(ii) *Suppose that* $\|\sum_{i=1}^n f_i\| \leq n^{1/2}/D$. *Consider* k *tensor product spaces* $X_1 = F \otimes \underbrace{l_2^n \otimes \ldots \otimes l_2^n}_{k-1}$, $X_2 = l_2^n \otimes F \otimes \underbrace{l_2^n \otimes \ldots \otimes l_2^n}_{k-2}$, $\ldots$, $X_k = \underbrace{l_2^n \otimes \ldots \otimes l_2^n}_{k-1} \otimes F$, *each endowed with a cross-norm. Suppose that the natural tensor basis in each* X_i *is* C_λ*-unconditional, for some* $C_\lambda \geq 1$. *Set* $X = X_1 \oplus \ldots \oplus X_k \oplus l_2^{n^k}$ *and let* $C_r(X)$ *be the cotype* r *constant of* X, *for some* $2 \leq r < \infty$. *Then there exists a subspace* $Y \subset X$, *which admits a* k*-dimensional* C_λ*-unconditional decomposition, such that* $\mathcal{U}_{k-1}(Y) \geq a\,C_\lambda^{-2} D^{1/2}$.

Here $a > 0$ *depends on* k, r *and* $C_r(X)$ *only.*

Proof **(i)** Set $\alpha = (k^2-1)/k^2$. By [K-T3], Lemma 3.3 (i) there exists $D^2 \leq n_0 \leq n$ and a subset $I \subset \{1, \ldots, n\}$ with $|I| = n_0$ such that

$$\|\sum_{i \in I} f_i\| \geq n_0^{1/2} D$$

and for any real scalars $c_1, \ldots, c_{n_0}$ there exists a subset $S \subset I$, with $|S| \geq [\alpha n_0]$ such that

$$\|\sum_{i \in I} c_i f_i\| \geq \max_{i \in S} |c_i| (1 - \alpha^{1/2}) n_0^{1/2} \lambda^{-1} D. \tag{4.4}$$

Without loss of generality we may assume that $F' := \operatorname{span}\{f_i\}_{i \in I}$ is the original space F (and in particular $n = n_0$ and $I = \{1, \ldots, n\}$).

Let $\{e_i\}_{i=1}^n$ be the standard unit basis in l_2^n.

For $i_1, \ldots, i_{k+1} = 1, \ldots, n$ let $Z_{i_1,\ldots,i_{k+1}}$ be the k-dimensional subspace of $X = X_1 \oplus \ldots \oplus X_{k+1}$ spanned by the vectors $x^{(1)}_{i_1,\ldots,i_{k+1}}, \ldots, x^{(k)}_{i_1,\ldots,i_{k+1}}$ defined as follows:

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$$\begin{bmatrix} x^{(1)}_{i_1,\dots,i_{k+1}} \\ x^{(2)}_{i_1,\dots,i_{k+1}} \\ \dots \\ x^{(k)}_{i_1,\dots,i_{k+1}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 1 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix} \begin{bmatrix} f_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_{k+1}} \\ e_{i_1} \otimes f_{i_2} \otimes \dots \otimes e_{i_{k+1}} \\ \dots \\ \dots \\ e_{i_1} \otimes e_{i_2} \otimes \dots \otimes f_{i_{k+1}} \end{bmatrix}$$

Consider the space $Y = \operatorname{span}\{Z_{i_1,\dots,i_{k+1}}\}^n_{i_1,\dots,i_{k+1}} \subset X$. The k-dimensional decomposition $\{Z_{i_1,\dots,i_{k+1}}\}^n_{i_1,\dots,i_{k+1}}$ is C_λ unconditional since, by our assumptions, $\{f_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_{k+1}}\}_{i_1,\dots,i_{k+1}}, \dots, \{e_{i_1} \otimes \dots \otimes e_{i_k} \otimes f_{i_{k+1}}\}_{i_1,\dots,i_{k+1}}$ are C_λ-unconditional in $X_1, \dots, X_{k+1}$, respectively.

Also, for all $i_1, \dots, i_{k+1} = 1, \dots, n$ and scalars $s_1, \dots, s_k$ we have

$$\max(|s_1|, \dots, |s_k|) \le \|s_1 x^{(1)}_{i_1,\dots,i_{k+1}} + \dots + s_k x^{(k)}_{i_1,\dots,i_{k+1}}\| \le 2(|s_1| + \dots + |s_k|). \quad (4.5)$$

Let $T : Y \to Y$ be an operator obtained in Proposition 4.2.2, for the case $s = k-1$. On each $Z_{i_1,\dots,i_{k+1}}$ write the operator $T_{|Z_{i_1,\dots,i_{k+1}}}$ in the matrix form with respect to the basis $\{x^{(\delta)}_{i_1,\dots,i_{k+1}}\}_{\delta=1,\dots,k}$, $T_{|Z_{i_1,\dots,i_{k+1}}} = \left[a^{i_1,\dots,i_{k+1}}_{ij}\right]^k_{i,j=1}$. That is, for all $i_1, \dots, i_{k+1} = 1, \dots, n$ and $\delta = 1, \dots, k$

$$T(x^{(\delta)}_{i_1,\dots,i_{k+1}}) = a^{i_1,\dots,i_{k+1}}_{1\delta} x^{(1)}_{i_1,\dots,i_{k+1}} + \dots + a^{i_1,\dots,i_{k+1}}_{k\delta} x^{(k)}_{i_1,\dots,i_{k+1}}.$$

Notice that for all $i_1, \dots, i_{k+1} = 1, \dots, n$ we have

$$\max\left(\{|a^{i_1,\dots,i_{k+1}}_{ij}| : i \ne j\} \cup \{|a^{i_1,\dots,i_{k+1}}_{11} - a^{i_1,\dots,i_{k+1}}_{\delta\delta}| : 2 \le \delta \le k\}\right) \ge \frac{1}{4k^4}. \quad (4.6)$$

Indeed, fix arbitrary $i_1, \dots, i_{k+1}$, and let

$$\widetilde{a}_{ij} = \begin{cases} a^{i_1,\dots,i_{k+1}}_{ij} - a^{i_1,\dots,i_{k+1}}_{11}, & \text{if } i = j \\ a^{i_1,\dots,i_{k+1}}_{ij}, & \text{if } i \ne j. \end{cases}$$

Observe that then $[\widetilde{a}_{ij}]_{i,j}$ is the matrix of the operator $T_{|Z_{i_1,\dots,i_{k+1}}} - a^{i_1,\dots,i_{k+1}}_{11} I_{Z_i}$ with respect to the basis $\{x^{(\delta)}_{i_1,\dots,i_{k+1}}\}_{\delta=1,\dots,k}$. Then (4.6) follows from (4.5) and Proposition 4.2.2 (iii).

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Fix arbitrary $i_2, \ldots, i_{k+1} = 1, \ldots, n$ and $\delta = 2, \ldots, k$.

We have, by the cross-norm property,

$$\begin{aligned}
\|\sum_{i_1=1}^{n} x^{(\delta)}_{i_1,\ldots,i_{k+1}}\| &\leq \|(\sum_{i_1=1}^{n} e_{i_1}) \otimes \ldots \otimes f_{i_\delta} \otimes \ldots \otimes e_{i_{k+1}}\| + \\
&\qquad + \|(\sum_{i_1=1}^{n} e_{i_1}) \otimes e_{i_2} \otimes \ldots \otimes e_{i_k} \otimes f_{i_{k+1}}\| \\
&= 2\,\|\sum_{i_1=1}^{n} e_{i_1}\| = 2n^{1/2}.
\end{aligned}$$

Thus

$$\begin{aligned}
2n^{1/2}\,\|T\| &\geq \|T(\sum_{i_1=1}^{n} x^{(\delta)}_{i_1,\ldots,i_{k+1}})\| \\
&= \|\sum_{i_1=1}^{n} (a_{1\delta}^{i_1,\ldots,i_{k+1}} x^{(1)}_{i_1,\ldots,i_{k+1}} + \ldots + a_{k\delta}^{i_1,\ldots,i_{k+1}} x^{(k)}_{i_1,\ldots,i_{k+1}})\| \\
&\geq \|\sum_{i_1=1}^{n} a_{1\delta}^{i_1,\ldots,i_{k+1}} f_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_{k+1}}\|
\end{aligned}$$

and so

$$\|\sum_{i_1=1}^{n} a_{1\delta}^{i_1,\ldots,i_{k+1}} f_{i_1}\| \leq 2n^{1/2}\,\|T\|. \tag{4.7}$$

By (4.4) there exists a subset $A^{(\delta)}_{0,i_2,\ldots,i_{k+1}} \subset \{1, \ldots, n\}$ of cardinality at least $[\alpha n]$ such that

$$\max_{i_1 \in A^{(\delta)}_{0,i_2,\ldots,i_{k+1}}} |a_{1\delta}^{i_1,\ldots,i_{k+1}}| \cdot (1 - \alpha^{1/2}) n^{1/2} \lambda^{-1} D \leq 2n^{1/2}\,\|T\|.$$

Thus, for every $i_2, \ldots, i_{k+1} = 1, \ldots n$ and $\delta = 2, \ldots, k$ we have

$$\max_{i_1 \in A^{(\delta)}_{0,i_2,\ldots,i_{k+1}}} |a_{1\delta}^{i_1,\ldots,i_{k+1}}| \leq 2D^{-1}(1 - \alpha^{1/2})^{-1} \lambda \|T\|.$$

A similar argument shows that for arbitrary $i_1, i_3, \ldots, i_{k+1} = 1, \ldots, n$ and $\delta \in \{1, 3, \ldots, k\}$ we obtain a set $A^{(\delta)}_{i_1,0,i_3,\ldots,i_{k+1}} \subset \{1, \ldots, n\}$ of cardinality at least $[\alpha n]$ such that

$$\max_{i_2 \in A^{(\delta)}_{i_1,0,i_3,\ldots,i_{k+1}}} |a_{2\delta}^{i_1,\ldots,i_{k+1}}| \leq 2D^{-1}(1 - \alpha^{1/2})^{-1} \lambda \|T\|.$$

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After k steps, for $i_1, \dots, i_{k-1}, i_{k+1} = 1, \dots, n$ and $\delta = 1, \dots, k-1$ we get $A^{(\delta)}_{i_1,\dots,i_{k-1},0,i_{k+1}} \subset \{1, \dots, n\}$, with $|A^{(\delta)}_{i_1,\dots,i_{k-1},0,i_{k+1}}| \geq [\alpha n]$, such that

$$\max_{i_k \in A^{(\delta)}_{i_1,\dots,i_{k-1},0,i_{k+1}}} |a_{k\delta}^{i_1,\dots,i_{k+1}}| \leq 2D^{-1}(1-\alpha^{1/2})^{-1}\lambda\|T\|.$$

We can repeat once more the procedure of obtaining (4.7) (starting this time from $\| \sum_{i_{k+1}=1}^{n} (x^{(1)}_{i_1,\dots,i_{k+1}} - x^{(\delta)}_{i_1,\dots,i_{k+1}})\| \leq 2n^{1/2}$, for $\delta = 2, \dots, k$) to get, for all $i_1, \dots, i_k = 1, \dots, n$ and $\delta = 2, \dots, k$, a set $A^{(\delta)}_{i_1,\dots,i_k,0} \subset \{1, \dots, n\}$ of cardinality at least $[\alpha n]$ such that

$$\max_{i_{k+1} \in A^{(\delta)}_{i_1,\dots,i_k,0}} |\gamma_{1\delta}^{i_1,\dots,i_{k+1}}| \leq 2D^{-1}(1-\alpha^{1/2})^{-1}\lambda\|T\|,$$

where $\gamma_{1\delta}^{i_1,\dots,i_{k+1}} := a_{11}^{i_1,\dots,i_{k+1}} + \dots + a_{k1}^{i_1,\dots,i_{k+1}} - a_{1\delta}^{i_1,\dots,i_{k+1}} - \dots - a_{k\delta}^{i_1,\dots,i_{k+1}}$.

By Lemma 4.2.3 (i) (note that $n \geq D^2 \geq k^2(k^2-1)$) we can find a $k+1$-tuple $(j_1, \dots, j_{k+1})$ such that

$$\begin{cases} j_1 \in A^{(\delta)}_{0,j_2,\dots,j_{k+1}} & \text{for all } \delta = 2, \dots, k \\ \vdots & \\ j_k \in A^{(\delta)}_{j_1,\dots,j_{k-1},0,j_{k+1}} & \text{for all } \delta = 1, \dots, k-1 \\ j_{k+1} \in A^{(\delta)}_{j_1,\dots,j_k,0} & \text{for all } \delta = 2, \dots, k. \end{cases}$$

From the inequalities defining $A^{(\delta)}_{0,j_2,\dots,j_{k+1}}, \dots, A^{(\delta)}_{j_1,\dots,j_k,0}$ this yields

$$\max\{|a_{ij}^{j_1,\dots,j_{k+1}}| : i \neq j\} \leq 2D^{-1}(1-\alpha^{1/2})^{-1}\lambda\|T\|$$

and, for all $\delta = 2, \dots, k$,

$$\begin{aligned} |a_{11}^{j_1,\dots,j_{k+1}} - a_{\delta\delta}^{j_1,\dots,j_{k+1}}| &\leq |\gamma_{1\delta}^{j_1,\dots,j_{k+1}}| + \sum_{i\neq 1} |a_{i1}^{j_1,\dots,j_{k+1}}| + \sum_{i \neq \delta} |a_{i\delta}^{j_1,\dots,j_{k+1}}| \\ &\leq 2(2k-1)D^{-1}(1-\alpha^{1/2})^{-1}\lambda\|T\|. \end{aligned}$$

Thus, by (4.6) we obtain $\|T\| \geq cD\lambda^{-1}$, where $c = (1-\alpha^{1/2})/8k^4(2k-1)$ depends on k only. In the same time, by Proposition 4.2.2 (ii) we have $\|T\| \leq C_\lambda^2 M\,\mathcal{U}_{k-1}(Y)$. This finally yields $\mathcal{U}_{k-1}(Y) \geq cD\lambda^{-1}C_\lambda^{-2}M^{-1}$.

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(ii) Set again $\alpha = (k^2-1)/k^2$.

For this case, the role of the relation (4.4) will be played by Lemma 3.3 (ii) from [K-T3], which states that for every sequence of real numbers $c_1, \dots, c_m$ there exists a subset $S \subset \{1, \dots, m\}$, with $|S| \geq [\alpha m]$, such that

$$(\sum_{i=1}^{m} c_i^2)^{1/2} \geq (1-\alpha)^{-1/2} n^{1/2} \max_{i \in S} |c_i|. \tag{4.8}$$

Consider, for $i_1, \dots, i_k = 1, \dots, n$, the k-dimensional subspace $Z_{i_1,\dots,i_k}$ of $X = X_1 \oplus \dots \oplus X_k \oplus l_2^{n^k}$ spanned by the vectors $x^{(1)}_{i_1,\dots,i_k}, \dots, x^{(k)}_{i_1,\dots,i_k}$ defined as follows:

$$\begin{bmatrix} x^{(1)}_{i_1,\dots,i_k} \\ x^{(2)}_{i_1,\dots,i_k} \\ \dots \\ x^{(k)}_{i_1,\dots,i_k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & D^{-1/2} \\ 0 & 1 & 0 & \dots & 0 & D^{-1/2} \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & D^{-1/2} \end{bmatrix} \begin{bmatrix} f_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k} \\ \dots \\ \dots \\ e_{i_1} \otimes e_{i_2} \otimes \dots \otimes f_{i_k} \\ e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k} \end{bmatrix}$$

Let $Y = \text{span}\,\{Z_{i_1,\dots,i_k}\}^n_{i_1,\dots,i_k} \subset X$. By our assumptions $\{Z_{i_1,\dots,i_k}\}^n_{i_1,\dots,i_k}$ forms a C_λ-unconditional decomposition of Y.

If $T : Y \to Y$ is an operator from Proposition 4.2.2 write, on each $Z_{i_1,\dots,i_k}$, $T_{|Z_{i_1,\dots,i_k}} = \left[a^{i_1,\dots,i_k}_{ij}\right]^k_{i,j=1}$. That is, for all $i_1, \dots, i_k = 1, \dots, n$ and $\delta = 1, \dots, k$

$$T(x^{(\delta)}_{i_1,\dots,i_k}) = a^{i_1,\dots,i_k}_{1\delta} x^{(1)}_{i_1,\dots,i_k} + \dots + a^{i_1,\dots,i_k}_{k\delta} x^{(k)}_{i_1,\dots,i_k}.$$

Similarly as in (4.6) we get

$$\max \left(\{|a^{i_1,\dots,i_k}_{11} - a^{i_1,\dots,i_k}_{\delta\delta}| : 2 \leq \delta \leq k\} \cup \{|a^{i_1,\dots,i_k}_{ij}| : i \neq j\} \right) \geq \frac{1}{4k^4}. \tag{4.9}$$

Fix arbitrary $i_2, \dots, i_k = 1, \dots, n$. We have

$$\begin{aligned} \|\sum_{i_1=1}^{n} x^{(1)}_{i_1,\dots,i_k}\| &\leq \|\sum_{i_1=1}^{n} f_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k}\| + D^{-1/2} \|\sum_{i_1=1}^{n} e_{i_1} \otimes \dots \otimes e_{i_k}\| \\ &= \|\sum_{i_1=1}^{n} f_{i_1}\| + D^{-1/2} n^{1/2} \leq 2n^{1/2} D^{-1/2}. \end{aligned} \tag{4.10}$$

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Hence

$$\begin{aligned} 2n^{1/2}D^{-1/2}\,\|T\| &\geq \|\sum_{i_1=1}^{n} Tx^{(1)}_{i_1,\ldots,i_k}\| \\ &= \|\sum_{i_1=1}^{n} (\, a_{11}^{i_1,\ldots,i_k} x^{(1)}_{i_1,\ldots,i_k} + \ldots + a_{k1}^{i_1,\ldots,i_k} x^{(k)}_{i_1,\ldots,i_k}\,)\,\|. \end{aligned}$$

By (4.8) we get, for all $\delta = 2, \ldots, k$

$$\begin{aligned} 2n^{1/2}D^{-1/2}\,\|T\| &\geq \|\sum_{i_1=1}^{n} a_{\delta 1}^{i_1,\ldots,i_k}\, e_{i_1} \otimes \ldots \otimes f_{i_\delta} \otimes \ldots \otimes e_{i_k}\| \\ &= (\sum_{i_1=1}^{n} |a_{\delta 1}^{i_1,\ldots,i_k}|^2\,)^{1/2} \\ &\geq (1-\alpha)^{1/2} n^{1/2} \max_{i_1 \in A^{(\delta)}_{0,i_2,\ldots,i_k}} |a_{\delta 1}^{i_1,\ldots,i_k}| \end{aligned} \tag{4.11}$$

and thus, for a certain subset $A^{(\delta)}_{0,i_2,\ldots,i_k} \subset \{1, \ldots, n\}$ with $|A^{(\delta)}_{0,i_2,\ldots,i_k}| \geq [\alpha n]$,

$$\max_{i_1 \in A^{(\delta)}_{0,i_2,\ldots,i_k}} |a_{\delta 1}^{i_1,\ldots,i_k}| \leq 2D^{-1/2}(1-\alpha)^{-1/2}\,\|T\|.$$

Similarly, for all $i_1, i_3, \ldots, i_k = 1, \ldots, n$ and $\delta \in \{1, 3, \ldots, k\}$ we can obtain $A^{(\delta)}_{i_1,0,i_3,\ldots,i_k} \subset \{1, \ldots, n\}$ a set of cardinality at least $[\alpha n]$ such that

$$\max_{i_2 \in A^{(\delta)}_{i_1,0,i_3,\ldots,i_k}} |a_{\delta 2}^{i_1,\ldots,i_k}| \leq 2D^{-1/2}(1-\alpha)^{-1/2}\,\|T\|$$

and, after k steps, for all $i_1, \ldots, i_{k-1} = 1, \ldots, n$ and $\delta = 1, \ldots, k-1$ one can choose $A^{(\delta)}_{i_1,\ldots,i_{k-1},0} \subset \{1, \ldots, n\}$, with $|A^{(\delta)}_{i_1,\ldots,i_{k-1},0}| \geq [\alpha n]$, such that

$$\max_{i_k \in A^{(\delta)}_{i_1,\ldots,i_{k-1},0}} |a_{\delta k}^{i_1,\ldots,i_k}| \leq 2D^{-1/2}(1-\alpha)^{-1/2}\,\|T\|.$$

In an analogous way to (4.10) and (4.11) (estimating $\|\sum_{i_1,i_2=1}^{n}(x^{(1)}_{i_1,\ldots,i_k} - x^{(2)}_{i_1,\ldots,i_k})\|$ from above, then using the boundeness of T and (4.8)) we get, for all $i_3, \ldots, i_k = 1, \ldots, n$, a certain subset $C_{0,i_3,\ldots,i_k} \subset \{1, \ldots, n\} \times \{1, \ldots, n\}$ such that $|C_{0,i_3,\ldots,i_k}| \geq [\alpha n^2]$ and

$$\max_{(i_1,i_2) \in C_{0,i_3,\ldots,i_k}} |\gamma_{12}^{i_1,\ldots,i_k}| \leq 2D^{-1/2}(1-\alpha)^{-1/2}\,\|T\|,$$

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where we denoted, for all the choices of the parameters involved,

$$\gamma_{1\delta}^{i_1,\ldots,i_k} := a_{11}^{i_1,\ldots,i_k} + \ldots + a_{k1}^{i_1,\ldots,i_k} - a_{1\delta}^{i_1,\ldots,i_k} - \ldots - a_{k\delta}^{i_1,\ldots,i_k}.$$

Indeed, we have for all $i_3, \ldots, i_k = 1, \ldots, n$

$$\begin{aligned} \|\sum_{i_1,i_2=1}^{n} (x^{(1)}_{i_1,\ldots,i_k} - x^{(2)}_{i_1,\ldots,i_k})\| &\leq \|\sum_{i_1=1}^{n} f_{i_1}\| \, \|\sum_{i_2=1}^{n} e_{i_2}\| + \|\sum_{i_1=1}^{n} e_{i_1}\| \, \|\sum_{i_2=1}^{n} f_{i_2}\| \\ &\leq 2nD^{-1} \end{aligned}$$

and then

$$\begin{aligned} 2nD^{-1}\,\|T\| &\geq \|\sum_{i_1,i_2=1}^{n} T(x^{(1)}_{i_1,\ldots,i_k} - x^{(2)}_{i_1,\ldots,i_k})\| \\ &\geq D^{-1/2}\|\sum_{i_1,i_2=1}^{n} \gamma_{12}^{i_1,\ldots,i_k}\, e_{i_1} \otimes \ldots \otimes e_{i_k}\| \\ &\geq D^{-1/2}(1-\alpha)^{1/2} n \max\nolimits_{(i_1,i_2)\in C_{0,i_3,\ldots,i_k}} |\gamma_{12}^{i_1,\ldots,i_k}|. \end{aligned}$$

After $k-1$ similar steps we obtain, for all $i_2, \ldots, i_{k-1} = 1, \ldots, n$, a subset $C_{i_2,\ldots,i_{k-1},0} \subset \{1, \ldots, n\} \times \{1, \ldots, n\}$ of cardinality at least $\lceil \alpha n^2 \rceil$ such that

$$\max\nolimits_{(i_1,i_k)\in C_{i_2,\ldots,i_{k-1},0}} |\gamma_{1k}^{i_1,\ldots,i_k}| \leq 2D^{-1/2}(1-\alpha)^{-1/2}\,\|T\|.$$

By Lemma 4.2.3 (ii) (note that $n^{1/2}D^{-1} \geq \|\sum_{i=1}^{n} f_i\| \geq 1/\lambda\,\|f_1\|$ and hence $n \geq D^2/\lambda^2 \geq k^2(k^2-1)$) we find a k-tuple $(j_1, \ldots, j_k)$ such that

$$\begin{cases} j_1 \in A^{(\delta)}_{0,j_2,\ldots,j_k} & \text{for all } \delta = 2, \ldots, k \\ \vdots & \\ j_k \in A^{(\delta)}_{j_i,\ldots,j_{k-1},0} & \text{for all } \delta = 1, \ldots, k-1 \\ (j_1, j_2) \in C_{0,j_3,\ldots,j_k}, \ldots, (j_1, j_k) \in C_{j_2,\ldots,j_{k-1},0}. & \end{cases}$$

The conclusion of the theorem will follow from Proposition 4.2.2 (ii) and (4.9) (similarly as in (i)). □

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4.4 Local unconditional structures in subspaces of $l_2(X)$

Let $\{e_i\}_i$ be the standard unit vector basis in l_2. For all positive integers n and k, the space $l_2^{n^k}(X)$ can be (algebraically) identified to $X \otimes \underbrace{l_2^n \otimes \ldots \otimes l_2^n}_{k}$, via

$$(x_{i_1,\ldots,i_k})_{i_1,\ldots,i_k}^n \quad \longleftrightarrow \quad \sum_{i_1,\ldots,i_k=1}^{n} x_{i_1,\ldots,i_k} \otimes e_{i_1} \otimes \ldots \otimes e_{i_k}$$

We will consider on $X \otimes \underbrace{l_2^n \otimes \ldots \otimes l_2^n}_{k}$ the norm induced by $l_2^{n^k}(X)$. This is a cross-norm, since if $x \in X$ and $u_1 = \sum_{i_1=1}^n a_{i_1}^{(1)} e_{i_1} \in l_2^n, \ldots, u_k = \sum_{i_k=1}^n a_{i_k}^{(k)} e_{i_k} \in l_2^n$ then

$$\begin{aligned} x \otimes u_1 \otimes \ldots \otimes u_k &= x \otimes (\sum_{i_1=1}^n a_{i_1}^{(1)} e_{i_1}) \otimes u_2 \ldots \otimes u_k \\ &= \sum_{i_1=1}^n (x \otimes a_{i_1}^{(1)} e_{i_1} \otimes u_2 \otimes \ldots u_k) \\ &= \sum_{i_1=1}^n (a_{i_1}^{(1)} x \otimes e_{i_1} \otimes u_2 \otimes \ldots u_k) \\ &\;\vdots \\ &= \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n (a_{i_1}^{(1)} a_{i_2}^{(2)} \ldots a_{i_k}^{(k)} x \otimes e_{i_1} \otimes \ldots \otimes e_{i_k}) \end{aligned}$$

and hence

$$\begin{aligned} \|x \otimes u_1 \otimes \ldots \otimes u_k\| &= \|x\| \left(\sum_{i_1=1}^n \cdots \sum_{i_k=1}^n |a_{i_1}^{(1)} a_{i_2}^{(2)} \ldots a_{i_k}^{(k)}|^2 \right)^{1/2} \\ &= \|x\| \left(\sum_{i_1=1}^n |a_{i_1}^{(1)}|^2 \right)^{1/2} \cdots \left(\sum_{i_k=1}^n |a_{i_k}^{(k)}|^2 \right)^{1/2} \\ &= \|x\| \|u_1\| \ldots \|u_k\|. \end{aligned}$$

Also, if $\{f_1, \ldots, f_m\}$ is a 1-unconditional sequence in X then $\{f_j \otimes e_{i_1} \otimes \ldots \otimes e_{i_k}\}_{j=1,\ldots,m;\, i_1,\ldots,i_k=1,\ldots,n}$ is 1-unconditional in $X \otimes \underbrace{l_2^n \otimes \ldots \otimes l_2^n}_{k}$.

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Indeed, if $\{\epsilon_{j,i_1,\ldots,i_k}\}_{j,i_1,\ldots,i_k}$ is a sequence of signs, then

$$\begin{aligned}
&\| \sum_{j=1}^{m} \sum_{i_1,\ldots,i_k=1}^{m} \epsilon_{j,i_1,\ldots,i_k}(f_j \otimes e_{i_1} \otimes \ldots \otimes e_{i_k})\| = \\
&= \| \sum_{i_1,\ldots,i_k=1}^{n} (\sum_{j=1}^{m} \epsilon_{j,i_1,\ldots,i_k} f_j) \otimes e_{i_1} \otimes \ldots \otimes e_{i_k}\| = \\
&= \left(\sum_{i_1,\ldots,i_k=1}^{n} \|\sum_{j=1}^{m} \epsilon_{j,i_1,\ldots,i_k} f_j\|^2 \right)^{1/2} \leq \left(\sum_{i_1,\ldots,i_k=1}^{n} \|\sum_{j=1}^{m} f_j\|^2 \right)^{1/2} = \\
&= \| \sum_{i_1,\ldots,i_k=1}^{n} (\sum_{j=1}^{m} f_j) \otimes e_{i_1} \otimes \ldots \otimes e_{i_k}\| = \\
&= \|\sum_{j=1}^{m} \sum_{i_1,\ldots,i_k=1}^{n} f_j \otimes e_{i_1} \otimes \ldots \otimes e_{i_k}\|.
\end{aligned}$$

We have analogous identifications for each of the tensor spaces $l_2^n \otimes X \otimes \underbrace{l_2^n \otimes \ldots \otimes l_2^n}_{k-1}, \ldots, \underbrace{l_2^n \otimes \ldots \otimes l_2^n}_{k} \otimes X$, resulting in the same type of properties.

Before we pass to the main result of this section, we recall some necessary information.

Let X be a Banach space. For any positive integer l, let $K_l(X) \geq 1$ be the smallest constant K such that for every 1-unconditional normalized sequence of vectors $\{x_i\}_{i=1}^m \in X$, with $1 \leq m \leq l$, one has

$$K^{-1} m^{1/2} \leq \|\sum_{i=1}^{m} x_i\| \leq K m^{1/2}.$$

We say that X has property (H) if $K(X) := \sup_l K_l(X) < \infty$. This notion was introduced by Pisier in [P] and studied by Nielsen and Tomczack-Jaegermann in [N-T].

For an n-dimensional Banach space X we have, as it was proved in [N-T], Proposition 1.2 (see also [K-T3], Proposition 4.3),

$$d_X \leq c K_n \left(\mathrm{Rad}_n(X)\right)^4, \tag{4.12}$$

where c is an universal constant.

We can now prove the main result of this section.

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Theorem 4.4.1 *Let X be a Banach space not isomorphic to a Hilbert space.*

(i) *If X has cotype r, for some $2 \leq r < \infty$, then, for all $k \geq 2$, there exists a subspace Y in $l_2(X)$ which admits a k-dimensional 1-unconditional decomposition and $\mathcal{U}_{k-1}(Y) = \infty$.*

(ii) *There exists a subspace Z in $l_2(X)$ such that Z has unconditional finite dimensional decomposition but $\mathcal{U}_k(Z) = \infty$ for all $k \geq 1$.*

(iii) *If X has cotype r, for some $2 \leq r < \infty$, then $l_2(X)$ has at least countably many mutually non-isomorphic subspaces.*

Proof **(i)** Let $k \geq 2$ arbitrarily fixed.

Since X is not isomorphic to a Hilbert space we can find finite dimensional subspaces of X, $\{X_n\}_{n\geq 1}$, such that the euclidean distances $d_{X_n} \nearrow \infty$.

In order to get the result it is enough to show that if Z is a finite dimensional Banach space, with the cotype r constant $C_r(Z)$, then, if d_Z is sufficiently large we can obtain a subspace $Y \subset l_2(Z)$ having a k-dimensional 1-unconditional decomposition and satisfying $\mathcal{U}_{k-1}(Y) \geq a\, d_Z^{1/4}$, with a depending on k, r and $C_r(Z)$ only. Having proved this finite dimensional statement we can conclude as follows: for all $n \geq 1$, denote by Y_n a subspace of $l_2(X_n)$ which has a k-dimensional 1-unconditional decomposition and satisfies $\mathcal{U}_{k-1}(Y_n) \geq a\, d_{X_n}^{1/4}$, with a depending on k, r and $C_r(X)$ only and set $Y = (\sum_{n\geq 1} \oplus Y_n)_{l_2} \subset l_2(X)$. Then Y_n is 1-complemented in Y, for all $n \geq 1$, while $\sup_{n\geq 1} \mathcal{U}_{k-1}(Y_n) = \infty$. Since having local unconditional structure of order $\leq k-1$ passes to complemented subspaces, we will obtain $\mathcal{U}_{k-1}(Y) = \infty$.

To prove the quantitative estimate above, let $\dim Z = n$. By (4.12) there exist a universal constant $c > 0$ and 1-unconditional normalized vectors $f_1, \ldots, f_m$ in $\mathrm{Rad}_n(Z)$, with $1 \leq m \leq n$, such that either

$$\|\sum_{i=1}^{m} f_i\| \geq c d_Z^{1/4} m^{1/2}, \tag{4.13}$$

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or

$$\|\sum_{i=1}^{m} f_i\| \leq c^{-1} d_Z^{-1/4} m^{1/2}. \tag{4.14}$$

If $F = \operatorname{span}\{f_1, \ldots, f_m\} \subset \operatorname{Rad}_n(Z)$, then F is a m-dimensional space with a 1-unconditional normalized basis.

In the case that (4.13) is satisfied, set $W_1 = (F \otimes \underbrace{l_2^m \otimes \ldots \otimes l_2^m}_{k}) \oplus \ldots \oplus (\underbrace{l_2^m \otimes \ldots \otimes l_2^m}_{k} \otimes F)$ where on each tensor space entering in the definition of W_1 we consider the cross-norm induced by $l_2^{m^k}(F)$. Thus W_1 is isometric to a subspace of $l_2(F)$. Since $F \subset \operatorname{Rad}_n(Z)$ and $\operatorname{Rad}_n(Z)$ can be identified to a subspace of $l_2^{2^n}(Z)$, we obtain that W_1 is isometric to a subspace of $l_2(Z)$.

By an earlier remark, each of the tensor spaces entering in the definition of W_1 has its natural tensor basis 1-unconditional. Also, looking at the cotype r constant of W_1, we have (see [T]) $C_r(W_1) \leq C_r(l_2(Z)) = C_r(Z)$.

Thus Theorem 4.3.1 (i) (we assume that d_Z is large enough, since this is the case for which we will use the result, and hence $c d_Z^{1/4} \geq k\sqrt{k^2-1}$) yields the existence of a subspace Y of W_1 satisfying

$$\mathcal{U}_{k-1}(Y) \geq a\, d_Z^{1/4},$$

with a depending on k, r and $C_r(Z)$ only.

In the second case, that is (4.14) is true, the proof is similar, by considering the subspace W_2 of $l_2(Z)$ defined by $W_2 = (F \otimes \underbrace{l_2^m \otimes \ldots \otimes l_2^m}_{k-1}) \oplus \ldots \oplus (\underbrace{l_2^m \otimes \ldots \otimes l_2^m}_{k-1} \otimes F) \oplus l_2^{m^k}$. Then we use Theorem 4.3.1 (ii).

(ii) In the case X has cotype r, for some $2 \leq r < \infty$, consider, for each $k \geq 2$, the subspace Y_k of $l_2(X)$ obtained in (i).

Let $Z = (\sum_{k\geq 2} \oplus Y_k)_{l_2} \subset l_2(X)$. For every $k \geq 2$, Y_k is 1-complemented in Z and $\mathcal{U}_{k-1}(Y_k) = \infty$. It follows that $\mathcal{U}_{k-1}(Z) = \infty$ for all $k \geq 2$. Since each of the spaces Y_k has a k-dimensional 1-unconditional decomposition we infer that Z has a 1-unconditional finite dimensional decomposition.

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Suppose now that X does not have finite cotype, which is equivalent to X containing l_∞^n's uniformly.

Let $\{E_n\}_n$ be a sequence of subspaces of X such that $\dim E_n = n$ and $d(E_n, l_\infty^n) \leq 2$. Since a "random" $[n/2]$-dimensional subspace of l_∞^n has the Gordon-Lewis constant of maximal order (see [F-J]), there exists, for all n, a $[n/2]$- dimensional subspace Y_n of E_n satisfying $GL(Y_n) \geq c\sqrt{n}$, with $c > 0$ an absolute constant. By [B], Proposition 1.3, we have, for all $k \geq 1$,

$$\sqrt{k}\,\mathcal{U}_k(Y_n) \geq GL(Y_n) \geq c\sqrt{n}, \quad \text{for all } n \geq 1.$$

If we let $Z = (\sum_{n\geq 1} \oplus Y_n)_{l_2} \subset l_2(X)$, then clearly Z has a 1-unconditional finite dimensional decomposition. For every $n \geq 1$, Y_n is 1-complemented in Z, while $\sup_n \mathcal{U}_k(Y_n) = \infty$, for $k = 1, 2 \ldots$. This shows that $\mathcal{U}_k(Z) = \infty$, for all $k \geq 1$.

(iii) Let X be a Banach space of finite cotype not isomorphic to a Hilbert space. For each $k \geq 2$, let Y_k be the subspace of $l_2(X)$ obtained in **(i)**. For $s > t \geq 2$ we have $\mathcal{U}_{s-1}(Y_s) = \infty$, while $\mathcal{U}_{s-1}(Y_t) \leq \mathcal{U}_t(Y_t) < \infty$. Therefore $l_2(X)$ contains infinitely many mutually non-isomorphic subspaces. □

Theorem 4.4.1 provides now the following characterization of a Hilbert space.

Corollary 4.4.2 *For a Banach space X the following are equivalent:*

(i) *X is isomorphic to a Hilbert space.*

(ii) *For every subspace Y of $l_2(X)$ there exists $k \geq 1$ such that $\mathcal{U}_k(Y) < \infty$.*

(iii) *For every subspace Y of $l_2(X)$ admitting an unconditional finite dimensional decomposition there exists $k \geq 1$ such that $\mathcal{U}_k(Y) < \infty$.*

(iv) *For every subspace Y of $l_2(X)$ admitting an unconditional finite dimensional decomposition there exists $k \geq 1$ such that Y admits an unconditional decomposition into k-dimensional subspaces.*

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4.5 Final remarks

We can continue the study of local unconditional structure of higher order for Banach spaces which admit an unconditional finite dimensional decomposition, in the direction of some recent results of Casazza and Kalton [C-Ka].

Their main result states that, for a space X with an unconditional decomposition $\{Z_k\}_k$ such that $\sup_k \dim Z_k < \infty$, X has local unconditional structure if and only if there is an unconditional basis $\{f_{kj}\}_{j=1}^{\dim Z_k}$ on each Z_k so that $(f_{kj})_{k,j}$ is an unconditional basis for X.

By the use of Proposition 4.2.2 it seems possible to obtain a generalization of this result in the following form: under the same assumptions as above, the condition $\mathcal{U}_s(X) < \infty$ is actually equivalent to X having an unconditional decomposition into s-dimensional subspaces, where $s \geq 1$ is a positive integer.

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