RELATIONSHIPS BETWEEN NETWORK CONNECTIVITY AND GLOBAL DYNAMICS OF COMPLEX DYNAMICAL SYSTEMS

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Applied Mathematics

Department of Mathematical and Statistical Sciences

University of Alberta

Abstract

The global dynamics of complex systems is investigated in this thesis, using the framework of coupled dynamical systems. For a coupled dynamical system on an interaction network, we show the impact of the connectivity of the interaction network on its dynamical behavior. We lay particular emphasis on non-strongly connected interaction networks, and clustered behavior of coupled dynamical systems. Two typical kinds of coupled dynamical systems are studied in the thesis: coupled gradient systems and coupled oscillators.

We present a general approach to investigating the dynamical behaviors of coupled gradient systems. The approach is demonstrated through two multi-group epidemic models: one ordinary differential equation model and one functional differential equation model with distributed delay. We show disease either persists in all groups of one strongly connected component or dies out in all groups of one strongly connected component. Moreover, we present a threshold value that determines whether disease persists or dies out in one strongly connected component.

We study both coupled linear and nonlinear oscillators in the thesis. For systems of coupled linear oscillators, we show its dynamical behavior under arbitrary interaction networks. When the interaction network is strongly connected, synchronization occurs; otherwise, clustered behavior may occur. In the case of clustered behavior, we show

the frequency of oscillators in the same strongly connected components are the same. For systems of coupled nonlinear oscillators, synchronization occurs when its interaction network is strongly connected; otherwise, we show synchronization can occur when the coupling strength between any two strongly connected components is sufficiently large.

For coupled gradient systems and coupled oscillators, our analysis shows synchronization occurs under strongly connected interaction networks; while non-strongly connected interaction networks give rise to clustered behavior. In the case of clustered behavior, local systems in one strongly connected components are in the same dynamical cluster.

Acknowledgement

I want to express my deep gratitude to my supervisor, Professor Michael Y. Li, for his guidance and support during my PhD study. His knowledge and insight in applied mathematics leaded me to the start of my research, and his valuable suggestions and comments helped me to complete my thesis.

I would like to thank Professor James Muldowney for many helpful discussions on my research. I would also like to thank Professor Arno Berger, Professor Bin Han, Professor Anthony T-M Lau, Professor Branden Pass, Professor Hao Wang and Professor Xingfu Zou for their help and comments on my thesis.

I would like thank all my friends in University of Alberta. In particular, I would like to thank Professor Zhisheng Shuai, Dr. Chunhua Shan, Ms. Betsy Varughese and Mr. Chuang Xu for the beneficial discussions and collaborations in applied mathematics.

I appreciate the Department of Mathematical and Statistical Sciences, Faculty of Graduate Studies and Research of University of Alberta, for the financial support.

Finally, I am in debt to my parents, and my girl friend Ting Jiang, for their endless love, support and confidence in me.

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Chapter 1

Introduction

A complex dynamical system involves complex coupling among their local agent systems, the state of one agent system can affect other agent systems in a complex way, this results in complex dynamics of the system. It arises in various fields of study, such as consensus problem in multi-agent systems [42, 46], the spread of infectious disease in heterogenous populations [3, 20, 55], synchronization in networks of coupled oscillators [31, 52, 59], the movement of species in ecological systems with complex dispersal networks [26, 41] and the complex behaviors of neurons in neural networks [14]. In the study of complex dynamical systems, one fundamental problem is to understand the relationship between the network connectivity and their global dynamics.

1.1 Coupled dynamical system and its dynamical behaviors

Coupled dynamical systems are used as general mathematical frameworks for the investigation of the dynamics of complex dynamical systems. Let $\mathcal{G}$ be a directed graph with vertices $1, 2, ..., n$. For $i = 1, 2, ..., n$, let $X_i \in \mathbb{R}^{d_i}$ be the variable on vertex i, and

$F_i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_i}$. Assume the local dynamics on vertex i is described by the differential equation

$$\dot{X}_i = F_i(X_i). \tag{1.1}$$

For $i, j = 1, 2, ..., n$, let $h_{ij}(X_i, X_j) : \mathbb{R}^{d_i} \times \mathbb{R}^{d_j} \to \mathbb{R}^{d_i}$ present the influence of X_j to X_i, and $h_{ij} \equiv 0$ if and only if there is no directed edge from j to i in $\mathcal{G}$. The coupled dynamical system on interaction network $\mathcal{G}$ is defined as [35]

$$\dot{X}_i = F_i(X_i) + \sum_{j=1}^{n} h_{ij}(X_i, X_j) \qquad i = 1, 2, ..., n. \tag{1.2}$$

Here the functions F_i, h_{ij} are such that the initial value problems to (1.1), (1.2) have unique solutions. In the thesis, the variables $X_i, i = 1, 2, ..., n$ are called the agents, system (1.1) is called the i-th local agent system, and the directed graph $\mathcal{G}$ is called the interaction network of (1.2).

The dynamics of (1.2) were studied in various disciplines, including ecology [26], mathematical epidemiology [3], neural science [14] and engineering [42, 45]. Generally speaking, there are three typical behaviors for coupled dynamic systems: global synchronization, totally incoherent, and clustered behavior. Global synchronization means all agent systems of (1.2) behave the same, totally incoherent behaviors can be interpreted as no pair of agents behave similarly, and clustered behaviors occur when the agents can be divided into distinct groups such that the dynamical behaviors are similar in each group [52]. These groups of agents with similar dynamical behaviors are called dynamical clusters.

1.2 Literature review

The mathematical research for coupled dynamic systems dates back to the 17th century when Christian Hyugens studied the synchronization of coupled pendulums. There are results on many different types of coupled dynamical systems. In this section we review two kinds of coupled dynamical systems: multi-group epidemic model and coupled oscillators. They are most relevant thesis.

1.2.1 Multi-group epidemic models in mathematical epidemiology

Mathematical epidemiology is a branch of applied mathematics, it concerns the mathematical modelling of the spread of infectious disease in host populations. A common mathematical model in mathematical epidemiology is the compartment model developed by Kermark and Mckendrick [28]. In a compartment model, the total population is divided into distinct compartments, such as susceptible(S), infectious(I) and recovered(R). An example of compartment model is given by

$$\begin{aligned} \dot{S} &= \Lambda - \beta SI - dS, \\ \dot{I} &= \beta SI - (d+\gamma)I, \\ \dot{R} &= \gamma I - dR. \end{aligned} \tag{1.3}$$

Here Λ represents the influx of population, βSI represents the infection, γ is the recover rate, and d is the natural death rate. The flow diagram of system 1.3 is shown in Figure 1.1. From biology considerations, system (1.3) is always considered in the nonnegative octant.

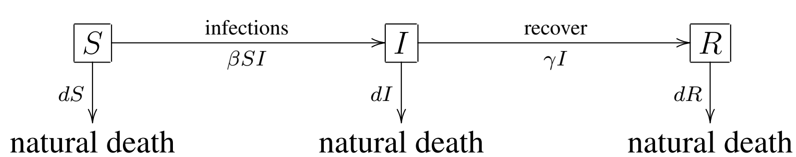


Figure 1.1: Flow diagram of an SIR model.

Let $R_0 = \frac{\Lambda\beta}{d(d+\gamma)}$, in mathematical epidemiology, R_0 is called the *basic reproduction number* of system (1.3). It is the average number of secondary infectious produced when one infectious individual is introduced into a host population [57].

Since variable R does not appear in the first two equations of (1.3), the third equation can be omitted in mathematical analysis. It can be seen that system (1.3) has a boundary equilibrium $P_0 = (\frac{\Lambda}{d}, 0)$, and a positive equilibrium $P^* = (\frac{d+\gamma}{\beta}, \frac{\Lambda}{d+\gamma} - \frac{d}{\beta})$ when $R_0 > 1$. We have the following threshold theorem.

Theorem 1.1. *For system* (1.3)*,*

(i) when $R_0 \leq 1$, the boundary equilibrium P_0 is globally asymptotically stable, the disease will eventually disappear;

(ii) when $R_0 > 1$, the positive equilibrium P^* is globally asymptotically stable with respect to all positive initial conditions, the disease will persist.

When the host population is heterogeneous with respect to the the disease transmission, the spread of infectious disease can be modelled with the multi-group models [55]:

$$\begin{aligned} S_i' &= \Lambda_i - d_i^S S_i - \sum_{j=1}^{n} \beta_{ij} f_{ij}(S_i, I_j), \\ I_i' &= \sum_{j=1}^{n} \beta_{ij} f_{ij}(S_i, I_j) - (d_i^I + \gamma_i) I_i, \end{aligned} \qquad i = 1, \cdots, n, \tag{1.4}$$

where S_i and I_i denote the number of individuals in the susceptible, and infectious compartments in the i-th group of the host population, respectively. For $i \neq j$, the incidence term $\beta_{ij} f_{ij}(S_i, I_j)$ describes the cross-infection from group j to group i, and function $f_{ij} : \mathbb{R}^2 \to \mathbb{R}$, is called the incidence function. For detailed description of system (1.4), please see [15]. Multi-group model (1.4) is an example of coupled dynamical systems, the interaction network $\mathcal{G}$ is obtained from the coupling matrix $(\beta_{ij})_{n\times n}$, it describes cross-transmission among groups.

The mathematical study of multi-group epidemic models lasts for a few decades. Lajmanovich and Yorke proposed and studied the first multi-group model in [32]. The global stability result for multi-group compartment model was firstly established in [21], using a global Lyapunov function, under the assumption that the cross-transmission network is strongly connected. It is shown that the global dynamics are completely determined by basic production number R_0. Specifically, if $R_0 \leq 1$, the disease dies out in all groups; if $R_0 > 1$, the endemic equilibrium is globally asymptotically stable, the disease persists in all groups, this is a threshold result as stated in Theorem 1.1. Using the same method, the threshold result was established in multi-group models with nonlinear incidence functions [30, 35], and multi-group models with distributed delays [13, 36].

Besides multi-group model, there are models in mathematical epidemiology that can be studied within the framework of coupled dynamical systems, such as heterogeneous models structured with infectious stages [20] or spatial dispersal [3, 34].

For heterogeneous epidemic models, we say a group is endemic if the disease eventually persists in the group, a group is disease free if disease eventually dies out in the group. For any two groups, if they are both endemic or disease free, we say these two groups synchronize [15]. The overarching assumption for the previous threshold results is that the underlying network is strongly connected. With this assumption, all groups synchronize.

1.2.2 Coupled oscillators

In a complex system, when each local agent system exhibits periodic behavior, it can be viewed as a system of coupled oscillators. Each local agent systems is called an oscillator, and it is supposed to oscillate under its own natural frequency without coupling. Systems of coupled oscillators are used to model various phenomenons in biology, physics and engineering. Typical examples include networks of pacemaker cells in the heart, arrays of lasers, microwave oscillators and superconducting Josephson junctions [42, 44, 52, 59].

In a system of coupled oscillators, the influence of oscillator X_i to oscillator X_j is usually presented by the term $k_{ij}h(X_i, X_j)$, where k_{ij} is coupling strength, and $h(X_i, X_j)$ specifies the normalized coupling function. A system of coupled oscillators can be written as

$$\dot{X}_i = F_i(X_i) + \sum_{j=1}^{n} k_{ij} h(X_i, X_j) \qquad i = 1, 2, ..., n, \tag{1.5}$$

where $X_i \in \mathbb{R}^d, F_i : \mathbb{R}^d \to \mathbb{R}^d$ and $h(X_i, X_j) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ for $i, j = 1, 2, ..., n$. Let matrix $K = (k_{ij})_{i.j=1,2,...,n}$, K is called the coupling matrix of system (1.5). System (1.5) was studied in various fields of science and engineering, and have been extensively studied [6, 24, 42, 44].

Earlier results on coupled oscillators are based on the assumption that the interaction network is symmetric. Typically, under this assumption, global synchronization occurs when the coupling strength exceeds a critical value, which is often characterized by the eigenvalue of matrix K [6, 24]. Most results on the study of coupled oscillators were estimations of the critical value. In [24], coupled Duffing oscillators under diffusive coupling were investigated, it was shown synchronization will occur when the coupling strength is sufficiently large. In [60], the synchronization problem of coupled oscillators under the symmetric interaction networks was investigated, the estimation of the critical value was made by calculating the second largest eigenvalue of the matrix K. In [6], the authors presented the connection graph approach to calculating the critical coupling strength of coupled oscillators under the symmetric interaction networks.

Recently, models of coupled oscillators were extensively studied under the framework of second order consensus problems [45, 53, 61, 62]. A multi-agent system is formed by local agent systems and the connections among them, it is an example of coupled dynamical system (1.2). Consensus of multi-agent systems means that all agents eventually reach an agreement regarding a certain quantity. The study of second order consensus problem is focused on studying the following equation

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -f_i(x_i) + u_i, \qquad\qquad i = 1, 2, ..., n, \end{aligned} \tag{1.6}$$

where $x_i, y_i \in \mathbb{R}$, $f_i : \mathbb{R} \to \mathbb{R}$ and $u_i = \sum_{j=1}^{n} h(x_i, y_i, x_j, y_j), i = 1, 2, ..., n$. Term u_i specifies the coupling protocol of the i-th oscillator. The consensus of coupled second order oscillators means every oscillator's position and velocity approach the same, it is equivalent with the conventional definition of synchronization. The research of second order consensus problems starts from the study of coupled harmonic oscillators [45], it was shown that global synchronization occurs if the interaction network has a directed spanning tree. Later in [50, 61, 62], synchronization of coupled second order nonlinear oscillators were investigated, under strict conditions on the coupling protocol u_i.

Another approach to the study of coupled oscillators is through the phase models. The phase model approach was developed by Kuramoto [31], he showed that for system of weakly coupled, nearly identical oscillators, the long time dynamics are governed by the following phase equation:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{n} f_{ij}(\theta_j - \theta_i), \qquad i = 1, 2, \ldots, n. \tag{1.7}$$

where $\theta_i \in \mathbb{R}$ represents the phase of oscillator i, and functions $f_{ij} : \mathbb{R} \to \mathbb{R}, i, j = 1, 2, ..., n$ are determined by the original uncoupled oscillator models. Kuramoto model finds applications in a lot of areas, for reviews on Kuramoto models, please refer to [1,52].

1.3 Motivation and objective

The study of complex systems pervades all areas of science [44, 51]. When studying a complex system, it is important to understand the conditions that lead to synchronization or clustered behavior.

Previously, synchronization behavior of coupled dynamical systems has been well studied [35, 42, 43, 46, 52]. In these studies, the interaction network were assumed to be strongly connected, symmetric or more restricted, and synchronization was shown to occur when the coupling strength is greater than a threshold value. On the other hand, clustered behavior of coupled dynamical system was widely reported [2, 3, 16, 18, 26]. In [16, 18, 26], models of coupled ecological system were studied, clustered behavior was observed when the dispersal of species was heterogeneous. In [2, 5, 56], clustered Behavior was reported in systems of coupled oscillators. In [3], clustered behavior was reported in coupled epidemic systems when the interaction network is not strongly connected. Previous observations suggested that clustered behaviors tend to occur when the connectivity of the interaction network is weak. However, mathematical results on clustered behavior were limited [7, 12, 38]. For some models, It was shown that clustered behaviors can occur under very strict conditions [12, 38].

In the thesis, we study both synchronization and clustered behaviors of coupled dynamical systems (1.2) on general interaction networks. Particular emphasis is laid on clustered behaviors, non-strongly connected interaction networks, and the impact of the network connectivity on the dynamical behavior of (1.2). When the interaction network is strongly connected, we show synchronization tends to occur; when the interaction network is not strongly connected, we consider its strongly connected components, and show local agent systems in a same strongly connected component tend to behave the same. The results in the thesis show the non-strongly connectedness of interaction network is a fact that leads to clustered behavior. Specifically, the thesis will answer the following questions for a large proportion of coupled dynamical systems.

(i) What is the global attractor for solutions of the coupled dynamical system (1.2)?

(ii) When does global synchronization occur? When do clustered behaviors occur?

(iii) How does the connectivity of the interaction network determine the dynamical clusters of the coupled dynamical system (1.2)?

1.4 The organization of the thesis

For most coupled dynamical systems, the local agent systems are simple and well studied [35, 42, 45]. In the thesis we focus on two typical local agent systems: gradient system and second order oscillators. A dynamical system is *gradient* if there exists a Lyapunov function that is strictly decreasing in its nonconstant trajectories [23]. Gradient systems are common in models from mathematical epidemiology and mathematical biology [19,29,37]. The multi-group epidemic model mentioned in Section 1.2.1 is an example of coupled gradient system. The second order oscillators are modeled by equation $\ddot{x} + f(x) = 0$, it is very well studied, the orbits of second order oscillators are typically periodic orbits.

In Chapter 2, we present necessary definitions and preliminary results. In particular, we show the convergence results for limiting equations in Section 2.4. The convergence results are needed for the analysis in the following chapters.

In Chapter 3 and Chapter 4 we consider the coupled gradient dynamical systems on networks. For each agent system (4.1), we assume the feasible region is inside the nonnegative quadrant. To characterize the clustered behavior and synchronization, we look at the positiveness of a variable of interest in every local agents. For example, in multi-group epidemic models (1.4), for group i, the infectious population I_i is viewed as the variable of interest, if $\liminf_{t\to\infty} I_i(t), \liminf_{t\to\infty} I_j(t)$ are both positive or zero, we say group i and group j synchronize. The variable of interest is chosen based on biological concern.

In Chapter 3, we present a general approach to studying coupled gradient ordinary differential equations, and demonstrate it with an application to a class of multi-group epidemic ordinary differential equation models; while in Chapter 4, we study coupled gradient functional differential equations, taking a class of multi-group epidemic models with distributed delay as an example. We use a similar approach as stated in Chapter

3. For both multi-group models, we show how the connectivity of interaction networks impact the dynamical behaviors. We are the first to show the global dynamics for multi-group epidemic models on non-strongly connected interaction networks.

In Chapter 5 and Chapter 6 we study coupled second order oscillators. Both synchronization and clustered behavior are investigated. For coupled oscillators, synchronization means that every oscillator move cohesively with each other, rigorous definition of synchronization will be provided in section 5.1. We will show synchronization can occur under weak network connections. If synchronization does not occur, we investigate the frequencies of each oscillator and show the clustered behaviors based on these frequencies. In the end of Chapter 5 and Chapter 6, numerical examples are shown to illustrate the theoretical results. Our results extend various previous results [45, 61, 62] on the this subject.

Chapter 2

Preliminaries

In this chapter we present several definitions and results that are necessary for the thesis. Definitions and results from graph theory are reviewed in Section 2.1. In Section 2.2 we define three auxiliary systems that are related to coupled dynamical system (1.2). In Section 2.3, we review the graph theoretical approach to constructing Lyapunov functions for coupled dynamical system on strongly connected networks. In Section 2.4, we state some convergence results for asymptotically autonomous ordinary, functional differential equations and more general limiting equations.

2.1 Definitions and results from graph theory

A directed graph $\mathcal{G}$, also called a digraph, consists of a nonempty finite set $V(\mathcal{G})$ of elements called vertices and a finite set $E(\mathcal{G})$ of ordered pairs of distinct vertices called arcs, or directed edges [4]. In the thesis, we label all the vertices with positive integers $1, 2, ..., n$, and denote the arc from vertex i to j as (i, j). For the sake of simplicity, we use $i \in \mathcal{G}$ to denote a vertex $i \in V(\mathcal{G})$. A *path* $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with distinct vertices $\{i_1, i_2, ..., i_m\}$ and arcs $\{e_k : k = 1, 2, ..., m-1\}$, such that i_k, i_{k+1} are the end points of e_k. If, moreover, $e_k = (i_k, i_{k+1}), k = 1, 2, ..., m-1$, then $\mathcal{P}$ is a *directed path*. A path

$\{i_1, i_2, ..., i_m\}$ is *closed* if $i_m = i_1$. A closed directed path is called a *directed cycle* [17].

A digraph $\mathcal{G}$ is weighted if each arc (j, i) is assigned a positive weight k_{ij}, if an arc (j_0, i_0) does not exist, then assign $k_{i_0 j_0} = 0$. Matrix $K = (k_{ij})_{n\times n}$ is called the weight matrix of $\mathcal{G}$. For an unweighted digraph, all the arcs have the same weight 1, the weight matrix K is the adjacency matrix of $\mathcal{G}$. In the thesis, we denote a digraph $\mathcal{G}$ with weight matrix K as $\mathcal{G}_K$.

A digraph $\mathcal{G}$ is *connected* if there exists a path between any two vertices of $\mathcal{G}$. A digraph $\mathcal{G}$ is *strongly connected* if exists a directed path between any two vertices of $\mathcal{G}$. A square matrix K is *reducible* if there exists a permutation matrix P, such that

$$PKP^T = \begin{pmatrix} K_1 & 0 \\ K_2 & K_3 \end{pmatrix},$$

where K_1, K_3 are square matrices. A square matrix is *irreducible* if it is not reducible. The next proposition is well known in graph theory [8].

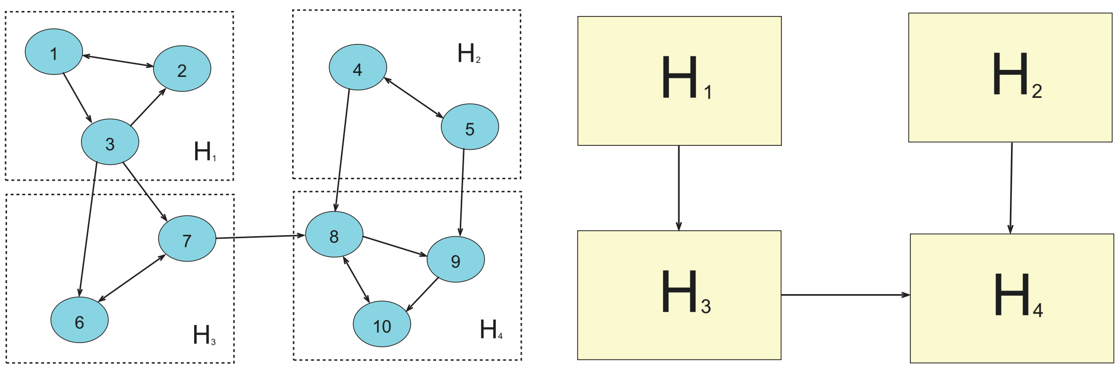
Proposition 2.1. *A directed graph $\mathcal{G}_K$ is strongly connected if and only if matrix K is irreducible.*

A partial order $\preceq$ among vertices of a digraph $\mathcal{G}$ can be defined as follows: for vertices i, j, $i \preceq j$ if there exists a directed path from i to j. We say $i \sim j$ if $i \preceq j$ and $j \preceq i$. It can be verified that relation '$\sim$' is an equivalent relation, each equivalent class is a strongly connected component of $\mathcal{G}$.

Proposition 2.2. *Relation $\sim$ is an equivalent relation.*

Definition 1. *For a directed graph $\mathcal{G}$, its condensed graph $\mathcal{H}$ is a directed graph whose vertices represent the strongly connected components of $\mathcal{G}$. For $H, H' \in V(\mathcal{H})$, a directed edge from H to H' exists if and only if there exist $i \in H$ and $j \in H'$ and a directed edge from i to j in $\mathcal{G}$.*

A canonical partial order $\prec$ can be defined in $\mathcal{H}$ as follows: for $H, H' \in \mathcal{H}$, $H \prec H'$ if there exists a directed path from H to H'. If $H, H' \in \mathcal{H}$ satisfy both $H \prec H'$ and $H' \prec H$, then H and H' are the same strongly connected component. This implies that $\prec$ is a strict partial order. In Figure 2.1, a connected digraph $\mathcal{G}$ and its condensed graph $\mathcal{H}$ are shown. The four strongly connected components H_1, H_2, H_3 and H_4 of $\mathcal{G}$ satisfy the relations $H_1 \prec H_3 \prec H_4$ and $H_2 \prec H_4$.



(a) A connected digraph $\mathcal{G}$ (b) The condensed graph $\mathcal{H}$

Figure 2.1: A digraph $\mathcal{G}$ and its strongly connected components H_i are shown in (a). The corresponding condensed graph $\mathcal{H}$ is shown in (b).

2.2 Auxiliary systems

In this section we describe three auxiliary systems of coupled dynamical system (1.2). Recall that the coupled dynamical system on interaction network $\mathcal{G}$ can be written as

$$\dot{X}_i = F_i(X_i) + \sum_{j=1}^{n} h_{ij}(X_i, X_j) \qquad i = 1, 2, ..., n,$$

and $\mathcal{G} = \{1, 2, ..., n\}$. For a subgraph $G \subseteq \mathcal{G}$, we define

$$X_i' = F_i(X_i) + \sum_{j \in \mathcal{G}} h_{ij}(X_i, X_j), \qquad i \in G. \tag{2.1}$$

as the *G-subsystem of system* (1.2); and

$$X_i' = F_i(X_i) + \sum_{j\in G} h_{ij}(X_i, X_j), \qquad i \in G. \tag{2.2}$$

as the *reduced G-subsystem* of system (1.2);

let $a(t) = (a_1(t), a_2(t), ...a_n(t))$, where $a_i(t) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{d_i}), 1 \le i \le n$, we define

$$X_i' = F_i(X_i) + \sum_{j\in G} h_{ij}(X_i, X_j) + \sum_{l\in \mathcal{G}\setminus G} h_{il}(X_i, a_l), \qquad i \in G \tag{2.3}$$

as the *restricted system* of system (1.2) on G at $a(t)$.

Proposition 2.3.

(a) *If $H \in \mathcal{H}$ is a minimal element, then the H-subsystem is the same as the reduced H-subsystem.*

(b) *If $H \in \mathcal{H}$ is a maximal element, then the $\mathcal{G}\setminus H$-subsystem is the same as the reduced $\mathcal{G}\setminus H$-subsystem.*

The proposition follows from the definitions.

2.3 The graph-theoretical approach to the method of global Lyapunov functions

In 2008, Li et al developed the graph-theoretical approach to the method of global Lyapunov functions for coupled differential equations on networks [21, 35].

Theorem 2.1. *[35] Suppose that the following assumptions are satisfied.*

(1) There exist functions $V_i(t, X_i) : \mathbb{R}\times\mathbb{R}^{d_i} \to \mathbb{R}$, $G_{ij}(t, X_i, X_j) : \mathbb{R}\times\mathbb{R}^{d_i}\times\mathbb{R}^{d_j} \to \mathbb{R}$, and constants $k_{ij} \ge 0$ such that

$$\dot{V}_i(t, X_i) \le \sum_{j=1}^{n} k_{ij} G_{ij}(t, X_i, X_j), \qquad t > 0, i = 1, 2, ...n. \tag{2.4}$$

(2) Let $K = (k_{ij})$, along each directed cycle $\mathcal{C}$ of $\mathcal{G}_K$,

$$\sum_{(s,r)\in E(\mathcal{C})} G_{rs}(t, X_r, X_s) \le 0, \qquad t > 0. \tag{2.5}$$

(3) Digraph $\mathcal{G}_K$ is strongly connected.

Let c_i be the cofactor of diagonal element k_{ii} of the coupling matrix K. Then $c_i > 0$ and function $V(t, X) = \sum_{j=1}^{n} c_i V_i(t, X_i)$ satisfies $\dot{V} \le 0$ for $t > 0$.

In (2.11), $\dot{V}_i(t, X_i) = \frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial X_i}(F_i(X_i) + \sum_{j=1}^{n} h_{ij}(X_i, X_j))$, it is the Lyapunov derivative with respect to system (1.2).

In coupled dynamical system (1.2), if the Lyapunov function V_i for each local agent system is known, and the interaction network $\mathcal{G}$ is strongly connected, Theorem 2.1 shows that the function $V(t, X) = \sum_{j=1}^{n} c_i V_i(t, X_i)$ is a Lyapunov function candidate for the coupled system. When the interaction network $\mathcal{G}$ is not strongly connected, Theorem 2.1 can be applied to prove global stability results for the restricted systems (2.3) on strongly connected components of $\mathcal{G}$.

2.4 Convergence results for nonautonomous differential equations

In this section we review some convergence results for nonautonomous differential equations. In Section 2.4.1, we state the convergence results for asymptotically autonomous semiflows, in Section 2.4.2, we state the convergence results for more general nonautonomous differential equations.

2.4.1 Asymptotic autonomous semiflows

Let (X,d) be a metric space. Consider a mapping $\Phi : \Delta \times X \to X, \Delta = \{(t,s); t_0 \leq s \leq t < \infty\}$, here t_0 is an initial time. Φ is called a nonautonomous semiflow on X if it is continuous and satisfies [54]:

(i) $\Phi(t,s,\Phi(s,r,x)) = \Phi(t,r,x), t \geq s \geq r \geq t_0$,

(ii) $\Phi(s,s,x) = x, s \geq t_0$.

Let $(s,x) \in [t_0,\infty] \times X$, the ω-limit set of (s,x) is defined as

$$\omega_\Phi(s,x) = \bigcap_{\tau \geq s} \overline{\{\Phi(t,s,x); t \geq \tau\}}.$$

It can be verified that $y \in \omega_\Phi(s,x)$ if and only if there exists a sequence $t_k \to \infty$, such that $\Phi(t_k,s,x) \to y$.

Definition 2. *Let Φ be a nonautonomous semiflow on X and Θ be an autonomous semiflow on X. Then Φ is called asymptotically autonomous, with limit semiflow Θ, if*

$$\Phi(t_j + s_j, s_j, x_j) \to \Theta(t,x), \qquad j \to \infty,$$

for any three sequences $t_j \to t, s_j \to \infty, x_j \to x$ when $j \to \infty$, with $x, x_j \in X$, $0 \leq t, t_j < \infty$, and $s_j \geq t_0$.

Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$. Consider ordinary differential equations

$$\dot{x} = f(t,x), \tag{2.6}$$

and

$$\dot{y} = g(y). \tag{2.7}$$

We assume that $f(t,x), g(x)$ are continuous functions and that the initial value problems for each system have unique solutions defined for all future times. Let $x(t,s,x_0)$ be the solution of (2.6) with initial condition $x(s) = x_0$, and let $y(t, y_0)$ be the solution of (2.7) with initial condition $y(t_0) = y_0$.

Lemma 2.1. *[40] Let $\Phi(t, s, x_0) = x(t, s, x_0)$, and $\Theta(t, y_0) = y(t, y_0)$. If $f(t, x) \to g(x), t \to \infty$ on compact subsets of $\mathbb{R}^n$, then Φ is asymptotically autonomous with limit semiflow Θ.*

Suppose $r > 0$ is a given real number, let $C = C([-r, 0], \mathbb{R}^n)$ be the Banach space of continuous functions mapping the interval $[-r, 0]$ to $\mathbb{R}^n$. For $\varphi \in C$, define its norm as $|\varphi| = \sup_{-r \le \theta \le 0} |\varphi(\theta)|$. Define $x_t \in C$ as $x_t(\theta) = x(t + \theta), -r \le \theta \le 0$. Let $f : \mathbb{R} \times C \to \mathbb{R}^n, g : C \to \mathbb{R}^n$ be continuous. Consider the following functional differential equations

$$\dot{x_t} = f(t, x_t), \tag{2.8}$$

$$\dot{y_t} = g(y_t). \tag{2.9}$$

We assume the initial value problems for each system have unique solutions defined for all future times. For $\tilde{x}, \tilde{y} \in C$, let $x(t, s, \tilde{x})$ be the solution of (2.8) with initial condition $x_s = \tilde{x}$, let $y(t, \tilde{y})$ be the solution of (2.9) with initial condition $y_0 = \tilde{y}$. Define a mapping $\tilde{\Phi} : \delta \times C \to C$ as $\tilde{\Phi}(t, s, \tilde{x}) = x_t(\theta, s, \tilde{x})$, and a mapping $\tilde{\Theta} : [0, \infty) \times C \to C$ as $\tilde{\Theta}(t, \tilde{y}) = y_t(\theta, \tilde{y})$.

Lemma 2.2. *[40] If for every compact subset K of C, there is a neighborhood V of K, such that $f(t, \phi) \to g(\phi), t \to \infty$ uniformly for $\phi \in V$, then $\tilde{\Phi}$ is asymptotically autonomous with limit semiflow $\tilde{\Theta}$.*

Note that Lemma 2.1 and Lemma 2.7 can be proved by the continuous dependence of solutions to the initial conditions and the vector fields.

For asymptotic autonomous semiflows, we have the following convergence result.

Theorem 2.2. *[54] Let Φ is asymptotically autonomous on X with limit semiflow Θ. Let e be a locally asymptotically stable equilibrium of Θ and $W_s(e) = \{x \in X; \Theta(t, x) \to$*

$e, t \to \infty\}$ *its basin of attraction. Then every precompact* Φ *orbit whose* ω-Φ-*limit set intersects* $W_s(e)$ *converges to* e.

2.4.2 General limiting equations and the skew product flow

In Section 2.4.1, we considered the asymptotic autonomous differential equations. One question aries, what if the limiting equation itself is not autonomous? For example, in (2.6), $f(t + t_n, x) \to g(t, x), t_n \to \infty$ for some sequence t_n, and $g(t, x)$ is periodic in t. In this subsection we answer this question under the framework of the skew product flow.

Let W be an open set in $\mathbb{R}^n$, consider nonautonomous differential equation

$$\dot{x} = f(t, x), \tag{2.10}$$

where $f : \mathbb{R} \times W \to \mathbb{R}^n$ is continuous and that the solutions of equation (2.10) are unique and can be continued to all t in $\mathbb{R}$. Let $f_\tau(t, x) = f(t + \tau, x)$, $\mathcal{F} = \{f_\tau; \tau \in \mathbb{R}\}$, and $\mathcal{F}^* = \bar{\mathcal{F}}$ be the closure of $\mathcal{F}$ in the compact open topology.

Let X be the product space $W \times \mathcal{F}^*$, a metric d on X is defined by

$$d((x, f), (\hat{x}, \hat{f})) = |x - \hat{x}| + \rho(f, \hat{f}),$$

where ρ is a basic metric on $\mathcal{F}^*$. Define a mapping $\pi : X \times \mathbb{R} \to X$ by

$$\pi((x, f), t) = (u(x, f, t), f_t), \tag{2.11}$$

where $u(x, f, t)$ denotes the solution of (2.10) that satisfies $u(x, f, 0) = x$.

Lemma 2.3. *[47] Let* $f(t, x)$ *be locally Lipschitz in* x *and the Lipschitz constant is independent of* t*. Then the mapping* π *defined by* (2.11) *is a dynamical system on* $X = W \times \mathcal{F}^*$*. The flow defined by mapping* π *is called a skew product flow.*

For $f \in \mathcal{F}^*$, let $\pi^*(f, t) = f_t$, then π^* defines a dynamical system on $\mathcal{F}^*$. Let Ω_f denote the ω-limit set of f in the flow π^*. For a $f^* \in \Omega_f$, we call the equation $\dot{x} = f^*(t, x)$

a limiting equation of (2.10). Note that the limiting equation may not be unique in general.

For $(x, f) \in X$, let $\Omega_{(x,f)}$ denote the ω-limit set of (x, f) in the flow π. If the solution $u(x, f, t)$ to (2.10) is positively compact, $\Omega_{(x,f)}$ is nonempty, compact and invariant [48].

Lemma 2.4. *[48] Let $f(t, x)$ be locally Lipschitz in x and the Lipschitz constant is independent of t. Assume f_t is positively compact in $\mathcal{F}^*$. Let $u(x, f, t)$ be a positively compact solution of* (2.10)*. Then for every point $(x^*, f^*) \in \Omega_{(x,f)}$, the solution $u(x^*, f^*, t)$ is compact. Moreover, there exists a sequence τ_n in R with $\tau_n \to \infty$ and such that $u(x, f, t+\tau_n)$ converges to $u(x^*, f^*, t)$ uniformly on compact set in $\mathbb{R}$.*

To prove the convergence results, we need the following lemma.

Lemma 2.5. *[11] Let M be a compact isolated invariant set of a continuous flow $\mathcal{F}$ on a locally compact metric space. Then for a point x, if $\omega(x) \cap M \neq \phi$ and $\omega(x) \nsubseteq M$, then there exist u and v, such that $u \in \omega(x) \setminus M$, and $\omega(u) \subseteq M$; $v \in \omega(x) \setminus M$, and $\alpha(v) \subseteq M$.*

Theorem 2.3. *If there exists a compact, isolated minimal set γ that asymptotically attracts all the solutions of the limiting equations of* (2.10)*, then γ attracts all the solutions of equation* (2.10)*.*

Proof. For $\forall (x, f) \in X$, Lemma 2.4 shows that $\Omega_{(x,f)}$ is invariant under every limiting flow defined by the limiting equation $x = f^*(t, x)$. Therefore $\gamma \times \Omega_f \subseteq \Omega_{(x,f)}$. If $\gamma \times \Omega_f \neq \Omega_{(x,f)}$, by Lemma 2.8, there exists $(\hat{x}, \hat{f}) \in \Omega_{(x,f)}$, such that the α-limit set of $(\hat{x}, \hat{f})$ is contained in $\gamma \times \Omega_f$, this contradicts with the fact that γ asymptotically attracts all the solutions of the limiting equations of (2.10). Therefore $\Omega_{(x,f)} = \gamma \times \Omega_f$. □

Chapter 3

Coupled gradient ordinary differential equations on networks

In this chapter we study the dynamical behaviors of coupled gradient ordinary differential equations on networks. In Section 3.1, a general approach to studying the coupled gradient ordinary differential equations on networks is presented. In Section 3.2, this approach is applied to a general class of multi-group epidemic models on general interaction networks. In Section 3.3, we show some numerical simulations of the multi-group epidemic models studied in Section 3.2, the simulations illustrate the theoretical results derived in Section 3.2.

3.1 A general approach to studying the coupled gradient ordinary differential equations on networks

Consider the following coupled system on interaction network $\mathcal{G}$,

$$X_i' = F_i(X_i) + \sum_{j=i}^{n} g_{ij}(X_i, PX_j), \qquad i = 1, 2, \ldots, n, \tag{3.1}$$

where $X_i \in \mathbb{R}^d$, $F_i = (F_i^1, F_i^2, ..., F_i^d) : \mathbb{R}^d \to \mathbb{R}^d$, and $g_{ij} = (g_{ij}^1, g_{ij}^2, ..., g_{ij}^d) : \mathbb{R}^d \times \mathbb{R}^r \to \mathbb{R}^d$, where $r \leq n$. The local agent systems $X_i' = F_i(X_i), i = 1, 2, ..., n$ are assumed to be gradient in this chapter. Matrix P is a $r \times d$ matrix with the form $P = (I_r, 0)Q$ for some permutation matrix Q, where I_r is the identity matrix of dimension r. For each $1 \leq i \leq n$, PX_i are referred to as coupling variables. By setting $h_{ij}(X_i, X_j) = g_{ij}(X_i, PX_j)$, system (3.1) becomes the standard form of coupled dynamical system (1.2). The subsystem, reduced subsystem and restricted system of system (3.1) are defined as in Chapter 2. In this chapter, we assume $\mathcal{G}$ is connected, but not necessarily strongly connected.

We make the following general assumptions.

(A_1) For $\forall 1 \leq i \leq n, 1 \leq m \leq d$, $F_i^m(X_i)|_{X_i^m=0} \geq 0$; and $F_i^m(X_i)|_{X_i^m=0} = 0$ only if $PX_i = 0$.

(A_2) For $1 \leq i, j \leq n, X_i, X_j \geq 0$, $Pg_{ij}(X_i, PX_j) \geq 0$; if $Pg_{ij} \not\equiv 0$, then $Pg_{ij}(X_i, PX_j) \neq 0 \Leftrightarrow PX_j \neq 0$.

Assumption (A_1) ensures that the first orthant $\mathbb{R}_+^d$ is positively invariant for each vertex system $\dot{X}_i = F_i(X_i)$. Assumption (A_2) is regarding the non-negativity of the coupling term g_{ij}. We do not require that all entries in the vector g_{ij} to be nonnegative, and only that the coupling entries Pg_{ij} are nonnegative.

In this chapter, we characterize the positiveness of an equilibrium only by the coupling variables. An equilibrium $X^* = (X_1^*, \cdots, X_n^*)$ is said to be *nonnegative* if it belongs to the first orthant $\mathbb{R}_+^d \times \cdots \times \mathbb{R}_+^d$ of the phase space. From Assumption (A_1), we can deduce that at an non-negative equilibrium X^*, for each vertex i, we have either $PX_i^* > 0$ or $PX_i^* = 0$, namely, there is no such i such that the vector PX_i^* has both positive and zero coordinates. Equilibrium X^* is said to be *positive* if and only if $PX_i^* > 0$ for all i. A nonnegative equilibrium X^* is said to be *mixed* if there exists i, j, such that $PX_i^* > 0$ and $PX_j^* = 0$. Generally, if $PX_i^* = 0$ for some i, then X^* is a boundary equilibrium. A mixed equilibrium is necessarily a boundary equilibrium, while a boundary equilibrium may not be mixed since we can have $PX_i^* = 0$ for all $i = 1, 2, ..., n$.

A differential equation $\dot{x} = f(x)$ with $x = (x_1, x_2, ..., x_k) \in \mathbb{R}_+^k$, is said to be uniformly persistent in the nonnegative orthant $\mathbb{R}_+^k$ if

(i) solutions $x(t)$ with $x(0) > 0$ exist for all $t \geq 0$,

(ii) there exists constant $c > 0$ such that $x(0) > 0$ implies $\liminf_{t\to\infty} x_i(t) > c, i = 1, 2, ..., k$.

Let $\mathcal{H}$ be the condensed graph of $\mathcal{G}$. We make the following additional assumptions on system (3.1).

(A_3) For $H \in \mathcal{H}$ and constant vector $a \geq 0$, the restricted system (2.3) on H at a has a nonnegative equilibrium that attracts all positive solutions.

(A_4) For $1 \leq i \leq n$, the local agent system $X_i' = F_i(X_i)$ has at most one boundary equilibrium.

(A_5) For $H \in \mathcal{H}$, if the reduced H-subsystem (2.2) has a positive equilibrium, then system (2.2) is uniformly persistent.

Assumptions (A_1), (A_2), (A_4) and (A_5) are satisfied by many models from mathematical epidemiology and spatial ecology. Assumption (A_3) is a key assumption that needs rigorous verification for each specific model.

Theorem 3.1. *Assume that (A_3) is satisfied. Then there exists a nonnegative equilibrium P^* that attracts all positive solutions of coupled system* (3.1)*.*

Proof. We use induction on the order $|\mathcal{H}|$ of $\mathcal{H}$. When $|\mathcal{H}| = 1$, system (3.1) and the restricted system (2.3) are the same, the theorem holds trivially. Assume that the theorem holds when $|\mathcal{H}| = m$. Then, when $|\mathcal{H}| = m + 1$, let $H \in \mathcal{H}$ be a maximal element with respect to $\prec$. Let $\mathcal{G}_r = \mathcal{G} \backslash H$. Since H is a maximal element, the restricted system of the $\mathcal{G}_r$-subsystem is contained in the set of the restricted system of (3.1). Since $|\mathcal{G}_r/\sim| = m$, by the induction assumption, $\mathcal{G}_r$ subsystem has a nonnegative equilibrium X_r^* that attracts all positive solutions. Now the asymptotic behavior of H-subsystem is the same as the restricted system on H at $a = (X_r^*, 0)$, by Theorem 2.2 in Section 2.4. By assumption (A_3), H-subsystem has a nonnegative equilibrium X_H^* that attracts positive solutions. Therefore system (3.1) has a nonnegative equilibrium $X^* = (X_r^*, u_H^*)$ that attracts all positive solutions of (3.1). □

When $\mathcal{G}$ is not strongly connected, system (3.1) may have multiple mixed equilibria which stay on the boundary of the phase space. It is also possible that system (3.1) does not have any positive equilibrium. In this case, the global attracting equilibrium in Theorem 3.1 could be a mixed type. In the rest of the section, we investigate the properties of equilibria of (3.1) and provide a method for identifying the globally attracting equilibrium P^*.

Let $\mathcal{P}$ denote the set of all nonnegative equilibria of (3.1). The following result holds.

Theorem 3.2. *Assume that assumptions (A_1) and (A_2) are satisfied. For $u^* \in \mathcal{P}$, the following statements hold.*

(a) *If an arc from j to i exists, then $PX_j^* > 0$ implies $PX_i^* > 0$.*

(b) *For $i, j \in \mathcal{G}$ such that $j \preceq i$, $PX_j^* > 0$ implies $PX_i^* > 0$.*

(c) *Let $H \in \mathcal{H}$ be a strongly connected component of $\mathcal{G}$. Then for all $i \in H$, PX_i^* are either all zero or all positive.*

Proof. By assumption (A_2), $PX_j^* > 0$ implies $Pg_{ij}(X_i^*, PX_j^*) \neq 0$. Therefore, there exist a coupling entry m such that $g_{ij}^m(X_i^*, PX_j^*) > 0$. Assume that $PX_i^* = 0$, by assumption (A_1), $PF(t, X_i) \geq 0$. Then $(X_i^m)'|_{X^*} = F_i^m(X_i^*) + \sum_{k=1}^{n} g_{ik}^m(X_i^*, PX_k^*) \geq$

$g_{ij}^m(X_i^*, PX_j^*) > 0$. This contradicts the fact that u^* is an equilibrium. Thus $PX_i^* \neq 0$. By assumption (A_1), for $\forall i$, either $PX_i^* > 0$ or $PX_i^* = 0$. Therefore $PX_i^* > 0$.

For (b), if $j \preceq i$, then there is a directed path from j to i. Applying (1) repeatedly, we get $PX_j^* > 0$ implies $PX_i^* > 0$.

For (c), if vertices i and j are in the same strongly connected components, then $i \preceq j$ and $j \preceq i$. By (2), we get $PX_i^* > 0 \Leftrightarrow PX_j^* > 0$. □

For the restricted system (2.3) on $H \in \mathcal{H}$ at $a \geq 0$, we have the following result.

Proposition 3.1. *Let X^{**} be an equilibrium of the restricted system (2.3). Then PX_i^{**} are either all positive or all zero for all $i \in H$. Furthermore, if there exist $k \in H, l \in \mathcal{G} \backslash H$ such that $g_{kl}(X_i, Pa_l) \neq 0$, then $PX_i^{**} > 0$ for all $i \in H$.*

Proof. Using a similar argument as in the proof of Theorem 3.2, the signs of PX_i^{**} are the same since there is only one strongly connected component in system (2.3). If $\exists k \in H, l \in \mathcal{G} \setminus H$ such that $g_{kl}(X_i, Pa_l) \neq 0$, then $PX_k^{**} > 0$, and thus $PX_i^{**} > 0$ for all $i \in H$. □

We define a mapping $\pi : \mathcal{P} \to (0,1)^{|\mathcal{H}|}$

$$\pi : \ X^* \mapsto \tilde{X}^* = (\tilde{X}_H^*)_{H \in \mathcal{H}}, \tag{3.2}$$

and

$$\tilde{X}_H^* = \begin{cases} 0 & \text{if } PX_i^* = 0, \text{ for } i \in H, \\ 1 & \text{if } PX_i^* > 0, \text{ for } i \in H, \end{cases}$$

where $|\mathcal{H}|$ is the order of set $\mathcal{H}$. From Theorem 3.2, we can see that mapping π is well defined and has the following property.

Proposition 3.2. *For $X^* \in \mathcal{P}$, if $H \prec H'$, then $\tilde{X}_H^* \leq \tilde{X}_{H'}^*$.*

The following result follows from Proposition 3.2.

Proposition 3.3. *An equilibrium $X^* \in \mathcal{P}$ is positive if and only if $\tilde{X}^*_H = 1$ at all minimal elements $H \in \mathcal{H}$.*

Proposition 3.4. *Suppose that assumptions (A_3) and (A_4) are satisfied. Then*

(a) *For $H \in \mathcal{H}$ and $a \geq 0$, the positive or boundary equilibrium of restricted system (2.3) on H at a is unique.*

(b) *The mapping π is one-to-one.*

Proof. When assumption (A_3) is satisfied, the positive equilibrium of system (2.3) is automatically unique, since otherwise the globally stable equilibrium can not attract orbits originated from the other positive equilibria. By Proposition 3.1, if $\exists k \in H, l \in \mathcal{G}\backslash H$ such that $g_{kl}(X_i, PX_l) \neq 0$, then system (2.3) only has positive equilibrium which is automatically unique; if such k, l do not exist, then system (2.3) breaks into independent systems $X_i' = F_i(X_i), i \in H$ when it is restricted on the boundary $\{PX_i = 0, i \in H\}$. Assumption (A_4) guarantees the boundary equilibrium of system (2.3) is unique.

To see that π is one-to-one, suppose that $\pi(P^*) = \pi(P^{**})$ for $P^*, P^{**} \in \mathcal{P}$. We use induction on the order $|\mathcal{H}|$ of $\mathcal{H}$. Suppose that the claim holds for $|\mathcal{H}| = m$. Then, when $|\mathcal{H}| = m + 1$, we can identify a maximal element H in the ordered set $(\mathcal{H}, \prec)$. Let $\mathcal{G}_c = \mathcal{G}\backslash H$, then variables in H will not appear in $\mathcal{G}_c$ subsystem. Let $\mathcal{H}_c = \mathcal{G}_c/\!\sim\, = \mathcal{H}\backslash H$. Then $\mathcal{H}_c$ is a subgraph of $\mathcal{H}$ with $|\mathcal{H}_c| = m$, and $\tilde{P}^*_{H'} = \tilde{P}^{**}_{H'}$, for $H' \in \mathcal{H}_c$. Therefore, by our induction hypothesis,

$$P^*\big|_{\mathcal{G}_c} = P^{**}\big|_{\mathcal{G}_c}.$$

Furthermore, $P^*\big|_H$ and $P^{**}\big|_H$ are the equilibrium of the restricted system on H at $a = (P^*\big|_{\mathcal{G}_c}, 0)$. By (a), $\tilde{P}^*_H = \tilde{P}^{**}_H$ implies $P^* = P^{**}$ on H since the boundary or positive equilibrium is unique for system (2.3). Therefore, $P^* = P^{**}$ over the entire graph $\mathcal{G}$, and the claim holds for $|\mathcal{H}| = m + 1$. □

Define an evaluation function $E: \mathcal{P} \to \mathbb{R}_+$ as $E(X^*) = \sum_{H \in \mathcal{H}} \pi(X^*)_H$ for $u^* \in \mathcal{P}$. The following results identify the global attracting equilibrium.

Theorem 3.3. *Suppose that (A_3) and (A_5) are satisfied. Then the followings hold:*

(a) *all positive solutions of system (3.1) converge to a maximizer of function E;*

(b) *if in addition (A_4) is satisfied, the maximizer of function E is unique.*

Proof. To show (a), we note that since function E takes only integer values and is bounded by $|\mathcal{H}|$, a maximizer exists. Let X^* be a maximizer of function E. For every X^{**} that dose not maximize function E, let $H \in \mathcal{H}$ be the minimal component such that $\tilde{X}^*_H > \tilde{X}^{**}_H$. Then $\tilde{X}^*_H = 1, \tilde{X}^{**}_H = 0$. By Corollary 3.2, for $H' \prec H$, $\tilde{X}^*_{H'} = \tilde{X}^{**}_{H'} = 0$. Therefore the limiting system of H-subsystem is the same as H-reduced subsystem. Since $\tilde{X}^*_H = 1$, the reduced H-subsystem has a positive equilibrium. By Assumption (A_5), the reduced H-subsystem is persistent, and thus the coordinates of a positive solution $X(t)$ in component H will not converge to boundary equilibrium $X^{**}\big|_H$. Therefore solutions in the interior of the nonnegative orthant will not converge to X^{**}.

To show the uniqueness of the maximizer, we assume the opposite. Let X^*, X^{**} be two minimizers and $X^* \neq X^{**}$. By assumption (A_4), the mapping π is one to one, and thus $\pi(X^*) \neq \pi(X^{**})$. There exists a minimal element $H \in \mathcal{H}$ such that $\tilde{X^{**}}_H > \tilde{X^*}_H$. Thus $\tilde{X^{**}}_H = 1, \tilde{X^*}_H = 0$. Using a similar argument as in the proof of (a), we know the solution in the interior of nonnegative orthant will not converge to X^*. This contradicts (a) since X^* is a maximizer of E. Therefore the maximizer of function E is unique. $\square$

Suppose system (3.1) satisfies assumptions $(A_1) - (A_5)$. For every equilibrium P of system (3.1), we can calculate the vector $\pi(P)$, and the number $E(P)$. Then we find the unique maximizer P^* of E. By Theorem 3.3, positive solutions of (3.1) are attracted to P^*.

3.2 A general class of multi-group epidemic models on non-strongly connected networks

In this section we apply the approach described in section 3.1 to a general class of multi-group epidemic models. We consider the following system of multi-group epidemic models of SEIR type:

$$\begin{aligned} S_i' &= \Lambda_i - d_i^S S_i - \sum_{j=1}^{n} \beta_{ij} f_{ij}(S_i, I_j), \\ E_i' &= \sum_{j=1}^{n} \beta_{ij} f_{ij}(S_i, I_j) - (d_i^E + \epsilon_i) E_i, \qquad i = 1, \cdots, n, \\ I_i' &= \epsilon_i E_i - (d_i^I + \gamma_i) I_i, \end{aligned} \tag{3.3}$$

where S_i, E_i, and I_i denote the number of individuals in the susceptible, exposed, and infectious compartments in the i-th group of the host population, respectively. The number of individuals in the recovered compartment of the i-th group is denoted by R_i, and R_i satisfies the following equation:

$$R_i' = \epsilon_i E_i - d_i^R R_i. \tag{3.4}$$

Since equations in (3.3) does not contain variable R_i, we will first establish the global dynamics of system (3.3), and then derive the asymptotic behaviours of R_i from equation (3.4).

All parameters in system (3.3)-(3.4) are assumed to be nonnegative. We further assume that $d_k^S, d_k^E, d_k^I, d_k^R, \Lambda_k > 0$ for all k. For $i \neq j$, the incidence term $\beta_{ij} f_{ij}(S_i, I_j)$ describes the cross-infection from group j to group i. Motivated by biological considerations, we assume that $f_{ij}(0, I_j) = 0$, $f_{ij}(S_i, 0) = 0$, and $f_{ij}(S_i, I_j) > 0$ for $S_i > 0, I_j > 0$. We also assume that $f_{ij}(S_i, I_j)$ are sufficiently smooth. We further make the following assumptions on the incidence function $f_{ij}(S_i, I_j)$:

(F_1) $0 < \lim_{I_j \to 0^+} \frac{f_{ij}(S_i,I_j)}{I_j} = C_{ij}(S_i) \le +\infty, 0 < S_i \le S_i^0$.

(F_2) $f_{ij}(S_i, I_j) \le C_{ij}(S_i) I_j$ for all $I_j > 0$.

(F_3) $C_{ij}(S_i) \le C_{ij}(S_i^0),\ 0 < S_i < S_i^0$.

If a positive equilibrium $P^* = (S_1^*, E_1^*, I_1^*, ..., S_n^*, E_n^*, I_n^*)$ of (3.3) exists, we assume that

(F_4) $(S_i - S_i^*)(f_{ii}(S_i, I_i^*) - f_{ii}(S_i^*, I_i^*)) > 0,\ S_i \ne S_i^*$.

(F_5) $[f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*) - f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)] \left[\frac{f_{ij}(S_i,I_j) f_{ii}(S_i^*,I_i^*)}{I_j} - \frac{f_{ij}(S_i^*,I_j^*) f_{ii}(S_i,I_i^*)}{I_j^*} \right] \le 0,\ S_i, I_j > 0$.

Classes of $f_{ij}(S_i, I_j)$ satisfying (F_1)-(F_3) include common incidence functions such as bilinear incidence $f_{ij}(S_i, I_j) = I_j S_i$, nonlinear incidence $f_{ij}(S_i, I_j) = I_j^{p_j} S_i^{q_i}$, and saturated incidences $f_{ij}(S_i, I_j) = \frac{I_j^{p_j}}{I_j + A_j} \frac{S_i^{q_i}}{S_i + B_i}$. For detailed description of the model, we refer the reader to [21, 35, 55].

For each i, adding the three equations in (3.3) gives

$$(S_i + E_i + I_i)' \le \Lambda_i - d_i^*(S_i + E_i + I_i)$$

with $d_i^* = \min\{d_i^S, d_i^E, d_i^I + \gamma_i\} > 0$. Hence $\limsup_{t \to \infty}(S_i + E_i + I_i) \le \Lambda_i / d_i^*$. Similarly, from the S_i equation we obtain $\limsup_{t \to \infty} S_i \le \Lambda_i / d_i^S$. Therefore, omega limit sets of system (3.3) are contained in the following bounded region in the nonnegative cone of $\mathbb{R}^{3n}$

$$\Gamma = \left\{ (S_1, E_1, I_1, \cdots, S_n, E_n, I_n) \in \mathbb{R}_+^{3n} \mid 0 < S_i \le \frac{\Lambda_i}{d_i^S},\ S_i + E_i + I_i \le \frac{\Lambda_i}{d_i^*}, \text{ for all } i \right\}.$$

It can be verified that region Γ is positively invariant.

System (3.3) always has the *disease-free equilibrium* $P_0 = (S_1^0, 0, 0, \cdots, S_n^0, 0, 0)$, on the boundary of Γ, where $S_i^0 = \Lambda_i / d_i^S$. An equilibrium $P^* = (S_1^*, E_1^*, I_1^*, \cdots, S_n^*, E_n^*, I_n^*)$

is called an *endemic equilibrium* of (3.3) if $S_i^*, E_i^*, I_i^* > 0$ for all $i = 1, \cdots, n$, and P^* is called a *mixed equilibrium* if $E_i^*, I_i^* > 0$ for some $1 \le i \le n$ while $E_j^* = I_j^* = 0$ for some $1 \le j \le n$. It can be seen that mixed equilibria are necessarily on the boundary of Γ, and an endemic equilibrium belongs to the interior of Γ.

The basic reproduction number R_0 for an epidemic model measures the average number of secondary infections caused by a single infectious individual in an entirely susceptible population during its infectious period. Assume that $f_{ij}(S_i, I_j)$ satisfies (F_1), and let

$$M_0 = M(S_1^0, S_2^0, \ldots, S_n^0) = \left(\frac{\beta_{ij}\, \epsilon_i\, C_{ij}(S_i^0)}{(d_i^E + \epsilon_i)(d_i^I + \gamma_i)} \right)_{1 \le i,j \le n}. \tag{3.5}$$

It can be shown using the method of next generation matrix as in [57] that R_0 for system (3.3) is

$$R_0 = \rho(M_0), \tag{3.6}$$

where ρ is the spectral radius of the matrix. If $C_{ij}(S_i^0) = +\infty$ for some i and j, it is understood that $R_0 = +\infty$, see also [35].

System (3.3) can be written as a coupled dynamical system on networks (3.1). Let $X_i = (S_i, E_i, I_i)^T$, the local agent system is a single-group SEIR model with vector field $F_i(X_i) = (\Lambda_i - d_i^S S_i - \beta_{ii} f_{ii}(S_i, I_i), \beta_{ii} f_{ii}(S_i, I_i) - (d_i^E + \epsilon_i)E_i, \epsilon_i E_i - (d_i^I + \gamma_i)I_i)$. Let $B = (\beta_{ij}) \ge 0$, the interaction network is defined by the digraph $\mathcal{G}_B$. A directed arc from vertex j to i exists if and only if $\beta_{ij} > 0$. The general coupling terms are given by $g_{ij} = (-\beta_{ij} f_{ij}, \beta_{ij} f_{ij}, 0)$, and they represents cross-infections among groups. The matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $PX_i = (E_i, I_i)^T$, which shows the coupling variables are E_i and I_i. At an equilibrium $X^* = (X_1^*, \cdots, X_n^*)$ with $X_i^* = (S_i^*, E_i^*, I_i^*)$, the disease is endemic in the group i if $PX_i^* > 0$, and group i is disease free if $PX_i^* = 0$.

Under the assumption that the transmission matrix $B = (\beta_{ij})$ is irreducible, or equivalently, $\mathcal{G}_B$ is strongly connected, the following threshold result was established in [35].

Theorem 3.4 (Li and Shuai)**.** *Assume that $B = (\beta_{ij})$ is irreducible.*

(a) *If $R_0 \leq 1$, then P_0 is the only equilibrium and is globally asymptotically stable in Γ.*

(b) *If $R_0 > 1$, then system (3.3) has a unique endemic equilibrium P^*, and P^* is globally asymptotically stable in the interior of Γ if assumptions (F_4) and (F_5) are satisfied.*

When the irreducibility assumption on the transmission matrix $B = (\beta_{ij})$ is dropped, the digraph $\mathcal{G}_B$ is not necessarily strongly connected. We show in the following that all groups within a strongly connected component H have the same behaviours. From the form of functions $F_i(X_i)$ and $g_{ij}(X_i, X_j)$, we can verify that conditions in (A_1) and (A_2) are satisfied. From Theorem 3.2 we have the following result.

Proposition 3.5. *Let $X^* = (S_i^*, E_i^*, I_i^*)_{i\in\mathcal{G}}$ be an equilibrium and H be a strongly connected component of $\mathcal{G}_B$. Either $PX_i^* = 0$ for all $i \in H$ or $P_i^* > 0$ for all $i \in H$.*

The condensed graph $\mathcal{H}$, sub-systems, reduced sub-systems, restricted systems and the set of equilibria $\mathcal{P}$ are defined in the same way as in Section 3.1. For a strongly connect component $H \in \mathcal{H}$, the reduced H-subsystem is a closed system. Let $R_{0,H}$ denote its basic reproduction number. Since the reduced H-subsystem has an irreducible transmission matrix, it satisfies Theorem 3.4. Therefore, if $R_{0,H} \leq 1$ all solutions of the reduced H-subsystem converge to the disease-free equilibrium, and if $R_{0,H} > 1$, all solutions of reduced H-subsystem converge to an unique endemic equilibrium.

The following result establishes the relation between the basic reproduction number R_0 for the entire system and $R_{0,H}$.

Theorem 3.5. *Let $\mathcal{H}$ be the condensing graph of $(\mathcal{G}, B)$. Then*

$$R_0 = \max\{R_{0,H} \mid H \in \mathcal{H}\}. \tag{3.7}$$

Proof. Theorem 3.5 holds trivially if B is irreducible. Suppose that B is reducible. For a strongly connected component $H \in \mathcal{H}$, let

$$M_{0,H} = \left(\frac{\beta_{ij}\, \epsilon_i\, C_{ij}(S_i^0)}{(d_i^E + \epsilon_i)(d_i^I + \gamma_i)} \right)_{i,j \in V(H)}.$$

Then $R_{0,H} = \rho(M_{0,H})$. If we group the equations in (3.3) according to strongly connected components of $(\mathcal{G}, B)$ and rearrange the components according to the order $\prec$ defined on $\mathcal{H}$, then matrix M_0 in (3.5) can be written in block-triangular form

$$M_0 = \begin{bmatrix} M_{0,H_1} & 0 & \dots & 0 \\ * & M_{0,H_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & M_{0,H_N} \end{bmatrix},$$

where $H_1, \cdots, H_N$ are the vertices of the condensed graph $\mathcal{H}$. Using Cauchy-Bonet formula we see that eigenvalues of M_0 are the ensemble of eigenvalues of M_{0,H_i}, $i = 1, \cdots, N$. This leads to relation (3.7). □

It is clear from Theorem 3.5 that network connectivity has a strong impact on the endemicity of disease transmission. When the interaction network is not strongly connected, $R_0 > 1$ if $R_{0,H} > 1$ on one of the component H. Accordingly, the disease may be endemic in some groups and disappears from other groups. It is possible that an endemic equilibrium may not exist, and that system (3.3) can have multiple mixed equilibria. It is of interest to investigate whether $R_0 > 1$ can imply the existence of a positive endemic equilibrium, and whether global convergence to an equilibrium still holds. Suppose $R_0 > 1$ and no positive equilibrium exists, it is then of interest to investigate which of the many mixed equilibria will be globally stable. The theory developed in Section 3.1 will be applied to investigate the global dynamics of system (3.3) and address these issues.

Each local agent system of (3.3) is a single-group SEIR model with an unique boundary equilibrium, thus assumption (A_4) is satisfied. For every strongly connected component H, global dynamics of the reduced H-subsystem is described by Theorem 3.4. In particular, when a positive equilibrium exists, we know $R_{0,H} > 1$ and this positive equilibrium attracts all the positive solutions, therefore system (3.3) is uniformly persistent, assumption (A_5) is satisfied.

Next we show that system (3.3) satisfies assumption (A_3). Let $H \in \mathcal{H}$ be a strongly connected component and $a_j \geq 0, j \in \mathcal{G} \setminus H$. The restricted subsystem on H at a is:

$$\begin{aligned}
S_i' &= \Lambda_i - d_i^S S_i - \sum_{j \in H} \beta_{ij} f_{ij}(S_i, I_j) - \sum_{j \in \mathcal{G}\setminus H} \beta_{ij} f_{ij}(S_i, a_j) \\
E_i' &= \sum_{j \in H} \beta_{ij} f_{ij}(S_i, I_j) + \sum_{j \in \mathcal{G}\setminus H} \beta_{ij} f_{ij}(S_i, a_j) - (d_i^E + \epsilon_i) E_i, \qquad i \in H. \\
I_i' &= \epsilon_i E_i - (d_i^I + \gamma_i) I_i,
\end{aligned} \tag{3.8}$$

We want to show that system (3.8) has a unique nonegative equilibrium that attracts all positive solutions. We will establish the result for a more general system:

$$\begin{aligned}
S_i' &= \Lambda_i - d_i^S S_i - \sum_{j=1}^{n} \beta_{ij} f(S_i, I_j) - h_i(S_i, I_i), \\
E_i' &= \sum_{j=1}^{n} \beta_{ij} f(S_i, I_j) + p_i h_i(S_i, I_i) - (d_i^E + \epsilon_i) E_i, \qquad i = 1, 2, ...n, \\
I_i' &= \epsilon_i E_i + q_i h_i(S_i, I_i) - (d_i^I + \gamma_i) I_i,
\end{aligned} \tag{3.9}$$

where p_i, q_i satisfy $p_i, q_i \geq 0$ and $0 < p_i + q_i \leq 1$ for all i, $h_i(S_i, I_i) \geq 0$. Other parameters have the same interpretation as in system (3.3). Let $B_1 = (\beta_{ij})$, then $\mathcal{G}_{B_1}$ is the interaction network that associated with system (3.9).

System (3.9) can be regarded as a general multi-group model with vertical transmission: functions $h_i(S_i, I_i)$ can be understood as newborns infected at birth in the i-th group, and enter compartments E_i and I_i by fractions p_i and q_i respectively. It can be verified that the feasible region for system (3.9) is

$$\Delta = \left\{ (S_1, E_1, I_1, ..., S_n, E_n, I_n) \in R_+^{3n} \mid 0 < S_i \leq \frac{\Lambda_i}{d_i^S}, S_i + E_i + I_i \leq \frac{\Lambda_i}{d_i^*}, \text{ for all } i \right\},$$

with $d_i^* = \min\{d_i^S, d_i^E, d_I^I\}$. An equilibrium $P^* = (S_1^*, E_1^*, I_1^*, ..., S_n^*, E_n^* I_n^*)$ of system (3.9) satisfy the following equilibrium equations

$$\begin{aligned} 0 &= \Lambda_i - d_i^S S_i^* - \sum_{j=1}^{n} \beta_{ij} f(S_i^*, I_j^*) - h_i(S_i^*, I_i^*), \\ 0 &= \sum_{j=1}^{n} \beta_{ij} f(S_i^*, I_j^*) + p_i h_i(S_i^*, I_i^*) - (d_i^E + \epsilon_i) E_i^*, \qquad i = 1, 2, ...n. \\ 0 &= \epsilon_i E_i^* + q_i h_i(S_i^*, I_i^*) - (d_i^I + \gamma_i) I_i^*. \end{aligned} \tag{3.10}$$

Proposition 3.6. *Assume matrix B_1 is irreducible and there exists k such that $h_k(S_k, I_k) > 0$ given $S_k > 0$. Then*

(a) *system (3.9) has no equilibria on the boundary of Δ, and*

(b) *system (3.9) has an endemic(positive) equilibrium.*

Proof. By the positive invariance of the compact and convex feasible region Δ and Browder's Fixed Point Theorem [10], we can deduce that system (3.9) has an equilibrium in Δ. Let $P^* = (S_1^*, E_1^*, I_1^*, ..., S_n^*, E_n^*, I_n^*)$ be a equilibrium of system (2.9). From the first equation of (3.10) we get $S_i^* > 0$ for all i.

Suppose $h_k(S_k, I_k) > 0$ for $S_k > 0$. Then from (3.10),

$$(d_k^E + \epsilon_k) E_k^* = \sum_{j=1}^{n} \beta_{kj} f(S_k^*, I_j^*) + p_k h_k(S_k^*, I_k^*),$$

$$(d_k^I + \gamma_) I_k^* = \epsilon_k E_k^* + q_k h_k(S_k^*, I_k^*).$$

Therefore, $h_k(S_k^*, I_k^*) > 0$ implies that $Pu_k^* = (E_k^*, I_k^*)^T > 0$. Since $\mathcal{G}_{B_1}$ is strongly connected, we know $Pu_i^* > 0$ for all $i \in \mathcal{G}_{B_1}$, thus system (3.9) has an endemic equilibrium and no boundary equilibria.

□

Let $P^* = (S_1^*, E_1^*, I_1^*, ..., S_n^*, E_n^*, I_n^*)$ be an endemic equilibrium of (3.9). We make the following assumption on $h_i(S_i, I_i)$:

(F_6) $\left(1 - \frac{I_i f_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)}{I_i^* f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}\right)\left(\frac{f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}{f_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)} - 1\right) \leq 0$, for all $S_i, I_i > 0$.

The next result establishes the global dynamics of system (3.9) when B_1 is irreducible. It generalizes the global stability result in [21, 35].

Theorem 3.6. *Suppose matrix B_1 is irreducible and there exists k such that $h_k(S_k, I_k) > 0$ in Δ. Assume that conditions $(F_1) - (F_6)$ hold. Then system (3.9) has a unique endemic equilibrium and it is globally asymptotically stable in the interior of Δ.*

Proof. Let $P^* = (S_1^*, E_1^*, I_1^*, \cdots, S_n^*, E_n^*, I_n^*)$ be an endemic equilibrium of (3.9). Let

$$V_i = \int_{S_i^*}^{S_i} \frac{f_{ii}(\xi,I_i^*)-f_{ii}(S_i^*,I_i^*)}{f_{ii}(\xi,I_i^*)} d\xi + E_i - E_i^* \log E_i + \frac{d_i^E+\epsilon_i}{\epsilon_i}(I_i - I_i^* \log I_i).$$

Taking the derivative of V_i along solutions to system (3.9), we obtain

$$\begin{aligned}
\dot{V}_i = &\left(1 - \frac{f_{ii}(S_i^*,I_i^*)}{f_{ii}(S_i,I_i^*)}\right)\left(\Lambda_i - d_i^S S_i - \sum_{j=1}^n \beta_{ij} f(S_i, I_j) - h_i(S_i, I_i)\right) + \sum_{j=1}^n \beta_{ij} f(S_i, I_j) \\
&+ p_i h_i(S_i, I_i) - (d_i^E + \epsilon_i)E_i - \frac{E_i^*}{E_i}\sum_{j=1}^n \beta_{ij} f(S_i, I_j) - \frac{E_i^*}{E_i} p_i h_i(S_i, I_i) + (d_i^E + \epsilon_i)E_i^* \\
&+ (d_i^E + \epsilon_i)E_i - \frac{(d_i^E+\epsilon_i)(d_i^I+\gamma_i)}{\epsilon_i} I_i + \frac{(d_i^E+\epsilon_i)}{\epsilon_i} q_i h_i(S_i, I_i) - \frac{I_i^*}{I_i}(d_i^E + \epsilon_i)E_i \\
&+ \frac{(d_i^E+\epsilon_i)(d_i^I+\gamma_i)}{\epsilon_i} I_i^* - \frac{(d_i^E+\epsilon_i)I^*}{\epsilon_i I} q_i h_i(S_i, I_i) \\
\leq & -d_i^S \left[S_i - S_i^*\right)\left(1 - \frac{f_{ii}(S_i^*,I_i^*)}{f_{ii}(S_i,I_i^*)}\right] \\
&+ \sum_{j=1}^n \beta_{ij} f(S_i^*, I_j^*)\left[3 + \frac{f_{ii}(S_i^*,I_i^*)f_{ij}(S_i,I_j)}{f_{ii}(S_i,I_i^*)f_{ij}(S_i^*,I_j^*)} - \frac{f_{ii}(S_i^*,I_i^*)}{f_{ii}(S_i,I_i^*)} - \frac{E_i^* f_{ij}(S_i,I_j)}{E_i f_{ij}(S_i^*,I_j^*)} - \frac{I_i}{I_i^*} - \frac{E_i I_i^*}{E_i^* I_i}\right] \\
&+ p_i h_i(S_i^*, I_i^*)\left[3 + \frac{f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}{f_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)} - \frac{f_{ii}(S_i^*,I_i^*)}{f_{ii}(S_i,I_i^*)} - \frac{E_i^* h_i(S_i,I_i)}{E_i h_i(S_i^*,I_i^*)} - \frac{I_i}{I_i^*} - \frac{E_i I_i^*}{E_i^* I_i}\right] \\
&+ q_i h_i(S_i^*, I_i^*)\left[2 + \frac{f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}{f_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)} - \frac{I_i^* h_i(S_i,I_i)}{I_i h_i(S_i^*,I_i^*)} - \frac{I_i}{I_i^*} - \frac{f_{ii}(S_i^*,I_i^*)}{f_{ii}(S_i,I_i^*)}\right].
\end{aligned}$$

Let

$$Q_i(S_i,E_i,I_i) = 3 + \frac{f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}{f_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)} - \frac{f_{ii}(S_i^*,I_i^*)}{f_{ii}(S_i,I_i^*)} - \frac{E_i^*h_i(S_i,I_i)}{E_ih_i(S_i^*,I_i^*)} - \frac{I_i}{I_i^*} - \frac{E_iI_i^*}{E_i^*I_i},$$

$$R_i(S_i,E_i,I_i) = 2 + \frac{f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}{f_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)} - \frac{I_i^*h_i(S_i,I_i)}{I_ih_i(S_i^*,I_i^*)} - \frac{I_i}{I_i^*} - \frac{f_{ii}(S_i^*,I_i^*)}{f_{ii}(S_i,I_i^*)},$$

and

$$F_{ij}(S_i,E_i,I_i,I_j) = 3 + \frac{f_{ij}(S_i,I_j)f_{ii}(S_i^*,I_i^*)}{f_{ij}(S_i^*,I_j^*)f_{ii}(S_i,I_i^*)} - \frac{f_{ii}(S_i^*,I_i^*)}{f_{ii}(S_i,I_i^*)} - \frac{f_{ij}(S_i,I_j)E_i^*}{f_{ij}(S_i^*,I_j^*)E_i} - \frac{I_i}{I_i^*} - \frac{E_iI_i^*}{E_i^*I_i}.$$

Then

$$\begin{aligned} Q_i(S_i,E_i,I_i) &= \left(4 - \frac{f_{ii}(S_i^*,I_i^*)}{f_{ii}(S_i,I_i^*)} - \frac{E_i^*h_i(S_i,I_i)}{E_ih_i(S_i^*,I_i^*)} - \frac{I_if_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)}{I_i^*f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)} - \frac{E_iI_i^*}{E_i^*I_i}\right) \\ &\quad + \left(1 - \frac{I_if_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)}{I_i^*f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}\right)\left(\frac{f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}{f_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)} - 1\right) \\ &\le \left(1 - \frac{I_if_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)}{I_i^*f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}\right)\left(\frac{f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}{f_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)} - 1\right) \\ &\le 0, \end{aligned}$$

by assumption (F_6). Similarly,

$$\begin{aligned} R_i(S_i,E_i,I_i) &= \left(3 - \frac{f_{ii}(S_i^*,I_i^*)}{f_{ii}(S_i,I_i^*)} - \frac{I_i^*h_i(S_i,I_i)}{I_ih_i(S_i^*,I_i^*)} - \frac{I_if_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)}{I_i^*f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}\right) \\ &\quad + \left(1 - \frac{I_if_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)}{I_i^*f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}\right)\left(\frac{f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}{f_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)} - 1\right) \\ &\le \left(1 - \frac{I_if_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)}{I_i^*f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}\right)\left(\frac{f_{ii}(S_i^*,I_i^*)h_i(S_i,I_i)}{f_{ii}(S_i,I_i^*)h_i(S_i^*,I_i^*)} - 1\right) \\ &\le 0. \end{aligned}$$

Using assumption (F_4) we obtain

$$\dot{V}_i \le \sum_{j=1}^{n} \beta_{ij} f(S_i^*,I_j^*) F_{ij}(S_i,E_i,I_i,I_j).$$

Let $\Phi(a) = 1 - a + \log a$ and $L_i(I_i) = -\frac{I_i}{I_i^*} + \log\frac{I_i}{I_i^*}$. Then $\Phi(a) \le 0$ for $a > 0$ and the equality holds only at $a = 1$. Furthermore,

$$\begin{aligned}
F_{ij} &= L_i(I_i) - L_j(I_j) + \Phi\Big(\tfrac{f_{ii}(S_i^*,I_i^*)}{f_{ii}(S_i,I_i^*)}\Big) + \Phi\Big(\tfrac{E_iI_i^*}{E_i^*I_i}\Big) \\
&\quad + \Phi\Big(\tfrac{I_jf_{ij}(S_i^*,I_j^*)f_{ii}(S_i,I_i^*)}{I_j^*f_{ij}(S_i,I_j)f_{ii}(S_i^*,I_i^*)}\Big) + \Phi\Big(\tfrac{f_{ij}(S_i,I_j)E_i^*}{f_{ij}(S_i^*,I_j^*)E_i}\Big) \\
&\quad + \Big(\tfrac{f_{ij}(S_i,I_j)f_{ii}(S_i^*,I_i^*)}{f_{ij}(S_i^*,I_j^*)f_{ii}(S_i,I_i^*)} - 1\Big)\Big(1 - \tfrac{I_jf_{ij}(S_i^*,I_j^*)f_{ii}(S_i,I_i^*)}{I_j^*f_{ij}(S_i,I_j)f_{ii}(S_i^*,I_i^*)}\Big) \\
&\le L_i(I_i) - L_j(I_j).
\end{aligned}$$

Therefore along each directed cycle $\mathcal{C}$, $\sum_{(i,j)\in E(\mathcal{C})} F_{ij} \le 0$. Let c_i is the cofactor of the i-th diagonal entry of matrix $(\beta_{ij}f(S_i^*, I_j^*))$, and $V = \sum_{i=1}^{n} c_iV_i$. By Theorem 2.1, $\dot{V} \le 0$ for all $(S_1, E_1, I_1, \cdots, S_n, E_n, I_n) \in \Delta$.

In a subset of $\{(S_1, E_1, I_1, ..., S_n, E_n, I_n) \mid \dot{V} = 0\}$ that is invariant for system (3.9), we necessarily have $d_i^S(S_i - S_i^*)(1 - \frac{f_{ii}(S_i^*,I_i^*)}{f_{ii}(S_i,I_i^*)}) = 0$ and $Q_i(S_i, E_i, I_i) = 0$, for all i. It can be verified that these conditions imply $S_i = S_i^*, E_i = E_i^*$ and $I_i = I_i^*$ for all i. Therefore the largest invariant set where $\dot{V} = 0$ is the singleton $\{P^*\}$. LaSalle's Invariance Principle [33] implies that the equilibrium P^* is globally asymptotically stable in Δ. The uniqueness of P^* follows from its global stability. □

It can be seen that the restricted system (3.8) on a strongly connected component H is a special case of the system (3.9), with $p_i = 1$, $q_i = 0$ and

$$h_i(S_i, I_j) = \sum_{j\in\mathcal{G}\setminus H} \beta_{ij}f_{ij}(S_i, c_j). \tag{3.11}$$

The following proposition follows from Theorem 3.6. It shows that system (3.3) satisfies assumption (A_3).

Proposition 3.7. *For every $H \in \mathcal{H}$, let $h_i(S_i, I_j)$ be defined as in (3.11). Assume f_{ij} satisfies assumptions (F_1)-(F_3).*

(a) *Suppose that there exists $k \in H$ such that $h_k(S_k, I_k) > 0$ and that f_{ij} satisfies assumptions (F_4)-(F_6). Then the restricted system (3.8) has an unique endemic* *equilibrium and it is globally asymptotically stable in the interior of the positive octant.*

(b) *Suppose that* $h_i(S_i, I_i) = 0$ *for all* $i \in H$ *and that* f_{ij} *satisfies assumptions* (F_4) *and* (F_5). *If* $R_{0,H} \leq 1$, *then disease free equilibrium is globally asymptotically stable; if* $R_{0,H} > 1$, *then a unique endemic equilibrium exists and is globally asymptotically stable in the interior the positive octant.*

We have shown that assumptions (A_1)-(A_5) are satisfied by system (3.3). Let evaluation map $E : \mathcal{P} \to \mathbb{R}_+$ be defined as in Section 3.1. Applying Theorem 3.3, we have the following result.

Theorem 3.7. *Assume that incidence functions in* (2.6) *satisfy assumptions* (F_1)-(F_6). *Then all positive solutions of system* (3.3) *converge to the unique maximizer of the evaluation map* E.

For a strongly connected component $H \in \mathcal{H}$, define

$$R_{1,H} = \max\{R_{0,H'} | H' \in \mathcal{H}, H' \prec H \text{ or } H' = H\}.$$

R_1 serves as a threshold value that indicate whether disease disappears or persists in each strongly connected component.

Theorem 3.8. *Let* P^* *be the nonnegative globally asymptotically stable equilibrium of system* (3.3). *For* $\forall H \in \mathcal{H}$, *if* $R_{1,H} > 1$, *then* $I_i^* > 0$ *for all* $i \in H$; *if* $R_{1,H} \leq 1$, *then* $I_i^* = 0$ *for all* $i \in H$.

The theorem can be proved using the property of the mapping π.

Corollary 1. *Assume* f_{ij} *satisfies assumptions* (F_1)-(F_6). *For system* (3.3),

(i) if $R_{1,H} \leq 1$ *for every strongly connected component* H, *disease free equilibrium is the only equilibrium, and it is globally asymptotically stable;*

(ii) *if $R_{1,H} > 1$ for every strongly connected component H, there exists an unique positive equilibrium that is globally asymptotically stable;*

(iii) *otherwise, there exist mixed equilibria, and one of them is globally asymptotically stable.*

For the class of multi-group epidemic models whose incidence functions satisfy assumptions (F_1)-(F_6), Theorems 3.4, 3.7, 3.8 and Corollary 1 completely characterize the impact of interaction network on the endemicity of the disease. If the network is strongly connected, then the disease outcomes are entirely determined by the basic reproduction number R_0 (Theorem 3.4): if $R_0 \leq 1$ the disease dies out; if $R_0 > 1$ the disease persists in all groups and all persistent solutions converge to a unique positive endemic equilibrium. The clustered behavior does not occur. If the network is not strongly connected, then R_0 no longer uniquely determines the disease outcomes. While $R_0 \leq 1$ still implies that the disease dies out from all groups, $R_0 > 1$ can not guarantee that the disease is persistent in all groups. A positive endemic equilibrium many not exist in this case, and multiple mixed equilibria may exist depending on the network connectivity. As we show in Theorems 3.7 and 3.8 that it is productive to partition the network $\mathcal{G}$ into strongly connected components and examine the resulting condensed graph $\mathcal{H}$. A natural order $\prec$ and an evaluation map E can be defined on the set of equilibria using the induced network structure on the condensed graph $\mathcal{H}$. The unique maximizer of map E identifies the equilibrium that attracts all positive solutions. The maximizer identifies the groups in which the disease is endemic and groups in which the disease disappears. The order $\prec$ also enables us define the threshold value R_1, its importance in determining whether the disease is endemic in all of the groups is clearly shown in Theorem 3.8 and its corollary.

For multi-group epidemic models, we say two groups are in the same cluster if disease eventually dies out in both groups or persists in both groups. It is shown in Theorem 3.8 that disease either dies out or persists in all groups of a same strongly connected component, this implies that dynamical clusters are composed of strongly connected components. Thus clustered behavior will not occur for system (3.3) on strongly connected networks. The graph connectivity of interaction network $\mathcal{G}_B$ impacts the threshold values $R_{1,H}$ for every strongly connected component, thus impacts the dynamics of system (3.3).

3.3 Numerical simulations

In this section we show several simulation results for the multi-group SEIR model in Section 3.2.

For the multi-group SEIR model (3.3), we choose $n = 3$, with the bilinear incidence function $f_{ij}(S_i, I_j) = S_i I_j$. The coupling matrix $B = (\beta_{ij})$ and the corresponding interaction network are given in Figure 3.1.

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & \beta_{22} & 0 \\ 0 & \beta_{32} & \beta_{33} \end{pmatrix}$$

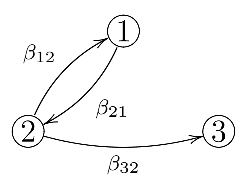


Figure 3.1: Transmission matrix B and the corresponding transmission digraph $\mathcal{G}$.

Here the transmission matrix (β_{ij}) is reducible. The set of strongly connected components $\mathcal{H} = \{H_1, H_2\} = \{\{1, 2\}, 3\}$, and $H_1 \prec H_2$. We can compute the basic reproduction number R_{0,H_i} for H_i reduced subsystem, $i = 1, 2$, using formula (3.6). Then, Theorem 3.5 implies that the basic reproduction number R_0 for the coupled system satisfies

$$R_0 = \max\{R_{0,H_1},\ R_{0,H_2}\}.$$

From Theorem 3.7 in Section 3.2, the global dynamics are determined by these two basic reproduction numbers and the graph structure as summarized in the following three cases:

(1) When $R_{0,H_1} \le 1, R_{0,H_2} \le 1$, we have $R_0 \le 1$, and $\pi(P^*)_{H_1} = \pi(P^*)_{H_2} = 0$.

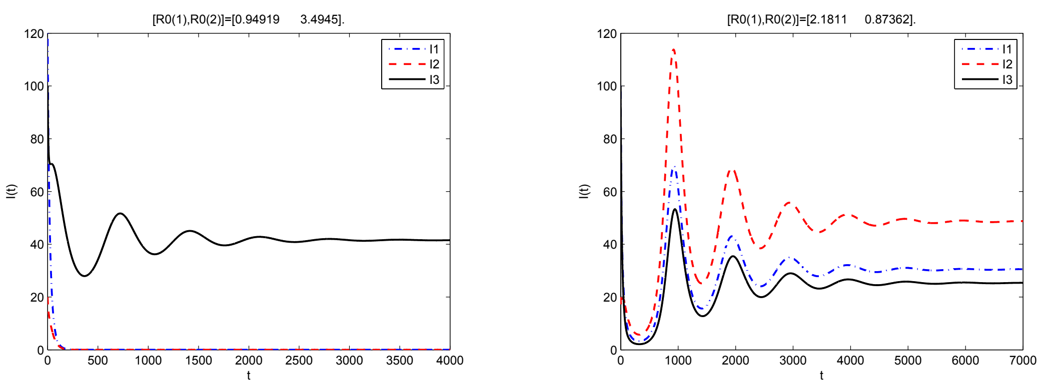
Accordingly, the disease-free equilibrium $P_0 = (S_1^*, 0, 0, S_2^*, 0, 0, S_3^*, 0, 0)$ is the only equilibrium and it is globally asymptotically stable.

(2) When $R_{0,H_1} \leq 1, R_{0,H_2} > 1$, we have $R_0 > 1$, and $\pi(P^*)_{H_1} = 0$, $\pi(P^*)_{H_2} = 1$. There are two equilibria: the disease-free equilibrium P_0 and a mixed equilibrium $P^* = (S_1^*, 0, 0, S_2^*, 0, 0, S_3^*, E_3^*, I_3^*)$, with $\pi(P_0)_{H_1} = 0$, $\pi(P_0)_{H_2} = 1$ and $\pi(P^*)_{H_1} = 0$, $\pi(P^*)_{H_2} = 1$. Thus $E(P_0) = 0, E(P^*) = 1$, the mixed equilibrium P^* is the maximizer of E, all solutions in the interior of Γ are attracted to P^*.

(3) When $R_{0,H_1} > 1$, we have $R_0 > 1$, and $\pi(P^*)_{H_1} = \pi(P^*)_{H_2} = 1$. There are two equilibria: the disease-free equilibrium P_0 and the endemic equilibrium $P^* = (S_1^*, E_1^*, I_1^*, S_2^*, E_2^*, I_2^*, S_3^*, E_3^*, I_3^*)$. It can be seen that $E(P_0) = 0, E(P^*) = 2$. The endemic equilibrium P^* is the maximizer of π and attracts all solutions in the interior of Γ, irrespective of the value of R_{0,H_2}.

Note that the mapping π and E are defined in section 3.1, the feasible Γ region is defined in section 3.2.

We have chosen parameter values to simulate the model and demonstrate the solutions for cases (2) and (3). Our simulation results are shown in Figure 3.2, in which we have plotted $I_i(t)$ for $i = 1, 2, 3$. In Figure 3.2-(a), both $I_1(t)$ and $I_2(t)$ approach zero, and $I_3(t)$ approaches a positive value, so that solutions with positive initial conditions converge to a mixed equilibrium. In Figure 3.2-(b), all $I_i(t)$ approach positive values, thus solutions with positive initial conditions converge to the endemic equilibrium. Note that in Figure 3.2-(b), the parameter values are chosen such that $R_{0,H_2} < 1$, however, $I_3(t)$ approaches a positive value instead of 0.



(a) A positive solution is attracted to a mixed equilibrium in case (2).

(b) A positive solution is attracted to the endemic equilibrium in case (3).

Figure 3.2: Mixed equilibrium and positive equilibrium.

Chapter 4

Coupled gradient functional differential equations on networks

The local agent systems and couplings of a complex system may involve time delays [13, 42, 49]. Coupled functional differential equations on interaction networks are the mathematical framework to study complex systems with time delays. Given a directed graph $\mathcal{G}$ with vertices $1, 2, ..., n$, let $X_i \in \mathbb{R}^d$ be the variable on vertex i, assume the local dynamics on vertex i is described by

$$\dot{X}_i(t) = F_i(X_{it}). \tag{4.1}$$

For some $r > 0$, $X_{it} \in C([-r, 0], \mathbb{R}^d)$ and $X_{it}(s) = X(t+s), -r \leq s \leq 0$, $F_i : C([-r, 0], \mathbb{R}^d) \to \mathbb{R}^d$. Let $h : C([-r, 0], \mathbb{R}^d) \times C([-r, 0], \mathbb{R}^d) \to \mathbb{R}^d$ represent the coupling from vertex j to i, and $h_{ij} \not\equiv 0$ if and only if there is a directed edge from j to i in $\mathcal{G}$. The coupled functional differential equations on $\mathcal{G}$ is

$$\dot{X}_i(t) = F_i(X_{it}) + \sum_{j=1}^{n} h_{ij}(X_{it}, X_{jt}) \qquad i = 1, 2, ..., n. \tag{4.2}$$

In system (4.2), for $i = 1, 2, ...n$, the derivative of X_i at time t is determined by all the information of $X_j(s), j = 1, 2, ..., n, t - r \leq s \leq t$.

Like in Chapter 3, in this chapter we assume the local agent systems (4.1) are gradient, and investigate how the connectivity of interaction network impact the dynamic behaviors of system (4.2). Without stating the general approach for studying coupled functional differential equations as in Section 3.1, we directly investigate a group of multi-group epidemic models with time delays.

Previously, multi-group epidemic models with time delays were only studied under strongly connected networks. In this chapter, our study extends various results in previous literatures. By the theory of asymptotical autonomous semiflows, we firstly establish a global convergence result for the model. Next we prove the global stability result for all restricted systems on strongly connected components, this is the key step. Thirdly, for each group, we define a number R_1 that determines whether disease dies out or persists in the group. R_1 is determined by both basic reproduction number R_0 [57] and the structure of the interaction network $\mathcal{G}$, thus the impact of $\mathcal{G}$ is shown.

The chapter is organized as follows. In Section 4.1, we state the approach to constructing global Lyapunov functions for coupled functional differential equations on networks. In Section 4.2, we present the multi-group model with distributed time delays. In Section 4.3, we prove the global stability result for restricted systems, and the original model. In Section 4.4 we show the impact of the connectivity of interaction network on the disease transmission.

4.1 Approach to constructing global Lyapunov functions for coupled functional differential equations

For an open set $D_i \subset C([-r,0],\mathbb{R}^d)$, $\varphi \in D$, let $X(\varphi)$ be the solution of (4.1) for initial condition φ. For a continuous functional $V_i : D_i \mapsto R$, the derivative $\dot{V}_i$ along the solution of (4.1) is given as [25]

$$\dot{V}_i(\varphi) = \limsup_{h\to 0^+} \frac{V_i(X_h(\varphi)) - V_i(\varphi)}{h}$$

where $X_h(\theta) = X(\theta + h)$.

The following theorem can be derived from [35], it shows the approach to constructing global Lyapunov functions for coupled functional differential equations.

Theorem 4.1. *For open sets $D_i \subset C([-r,0],\mathbb{R}^d), i = 1,2,...,n$, and $D = \prod_{i=1}^{n} D_i$. Assume that the following assumptions are satisfied.*

(1) *There exist functions $V_i(X_{it}) : D_i \mapsto \mathbb{R}$, $F_{ij}(u_{it}, u_{jt}) \in D_i \times D_j \mapsto \mathbb{R}$, and nonnegative constants k_{ij} such that*

$$\dot{V}_i(X_{it}) \leq \sum_{j=1}^{n} k_{ij} F_{ij}(X_i, X_j), \quad u_i \in D_i,\ i = 1,2,\ldots,n.$$

(2) *Let $K = (k_{ij})$, along each directed cycle $\mathcal{C}$ of the weighted digraph $\mathcal{G}_K$,*

$$\sum_{(s,r)\in E(\mathcal{C})} F_{rs}(X_r, X_s) \leq 0, \quad X_r \in D_r,\ X_s \in D_s.$$

(3) *Matrix K is irreducible.*

Let c_i be the cofactor of the i-th diagonal entry of matrix K, and function $V(X_t) = \sum_{i=1}^{n} c_i V_i$. Then $c_i > 0, i = 1,2,...,n$ and $\dot{V}(X_t) \leq 0$ for $X \in D$.

4.2 Model description

We consider a multi-group SIR model with delays. Let S_k, I_k, R_k denote the susceptible, infectious and recovered population in the k-th group, let $l_k(r) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous kernel function that represents the infectivity at infectious age r. Assume the incidence function is of the nonlinear form $Sh(I)$, where $h(I) : \mathbb{R}_+ \mapsto \mathbb{R}_+$. The disease incidence for the k-th group is given by $\sum_{j=1}^{n} \beta_{kj} S_k \int_0^\infty l_j(r) h_j(I_j(t-r))dr$, here β_{kj} is the transmission coefficient from group j to group k. The Multi-group SIR model can be described by the following system of coupled functional differential equations

$$\begin{aligned} S_k' &= \Lambda_k - d_k^S S_k - \sum_{j=1}^{n} \beta_{kj} S_k \int_0^\infty l_j(r) h_j(I_j(t-r))dr, \\ I_k' &= \sum_{j=1}^{n} \beta_{kj} S_k \int_0^\infty l_j(r) h_j(I_j(t-r))dr - (d_k^I + \gamma_k) I_k, \qquad k = 1, 2, ..., n. \\ R_k' &= \gamma_k I_k - d_k^R R_k, \end{aligned} \tag{4.3}$$

Here Λ_k represents the influx of susceptible population in group k, d_k^S, d_k^I, d_k^R represent the death rate of the susceptible, infectious and recovered population in the k-th group, and γ_k represents the rate from infectious compartment to recovered compartment in the k-th group. Note R_k does not appear in the first two equations of (4.3), we can just consider the following system

$$\begin{aligned} S_k' &= \Lambda_k - d_k^S S_k - \sum_{j=1}^{n} \beta_{kj} S_k \int_0^\infty l_j(r) h_j(I_j(t-r))dr, \\ I_k' &= \sum_{j=1}^{n} \beta_{kj} S_k \int_0^\infty l_j(r) h_j(I_j(t-r))dr - (d_k^I + \gamma_k) I_k, \end{aligned} \qquad k = 1, 2, ..., n. \tag{4.4}$$

Let $n \times n$ matrix $B = (\beta_{kj})$, $\mathcal{G}_B$ is the interaction network of coupled functional differential equations (4.4). From biological considerations, we assume that the nonlinear functions $h_j(I_j), j = 1, 2, ..., n$ satisfy the following conditions.

(C_1) $h_j(I) \geq 0, h_j(0) = 0, j = 1, 2, ..., n,$

(C_2) $\lim\limits_{I \to 0} \frac{h_j(I)}{I} = c_j$ and $\frac{h_j(I)}{I} \leq c_j, j = 1, 2, ..., n.$

(C_3) $h_j'(I) \geq 0, h_j''(I) \leq 0$ for $j = 1, 2, ..., n.$

Note that we allow $c_j = \infty$ in condition (C_2).

For $k = 1, 2, ..., n$, we assume that the kernel function $l_k(r)$ has compact support, and satisfies the following property:

$$a_k = \int_0^\infty l_k(r) dr > 0. \tag{4.5}$$

Let N be the smallest number such that $l_k(r) = 0$ when $|r| > N$. Let Banach space $\mathbb{C} = C([-N, 0], \mathbb{R})$ and designate the norm of an element ϕ in $\mathbb{C}$ by $\|\phi\| = \sup_{-N \leq \theta \leq 0} |\phi(s)|$. For $\phi \in \mathbb{C}$, let $\phi_t \in \mathbb{C}$ be defined as $\phi_t(\theta) = \phi(t + \theta), -N \leq \theta \leq 0$. We consider system (4.4) in phase space $X = \prod\limits_{k=1}^{n} \mathbb{R} \times \mathbb{C}$. Standard theory of functional differential equations [25] ensures that system (4.4) with initial conditions

$$(S_1(0), I_{10}, ..., S_n(0), I_{n0}) = (S_{1,0}, \phi_1, ..., S_{n,0}, \phi_n) \in X$$

has a unique solution $(S_1(t), I_{1t}, ..., S_n(t), I_{nt}) \in X$.

By similar argument as in [36], It can be seen that

$$\Theta = \{(S_1, I_1(\cdot), ..., S_n, I_n(\cdot)) \in X | 0 \leq S_k \leq \frac{\Lambda_k}{d_k^S}, S_k + I_k(0) \leq \frac{\Lambda_k}{d_k^*}, I_k \geq 0, k = 1, 2, ...n\}$$

is positively invariant for system (4.4), where $d_k^* = \min\{d_k^S, d_k^I + \gamma_k\}$.

For system (4.4), there exists a disease free equilibrium $P_0 = (S_1^0, 0, S_2^0, 0, ..., S_n^0, 0)$. The basic reproduction number R_0 can be defined as the spectral radius of the next generation matrix [57]

$$M_0 = (\frac{\beta_{kj} S_k^0 a_j c_j}{d_k^I + \gamma_k})_{n \times n}, \tag{4.6}$$

where $a_j, j = 1, 2, ..., n$ are defined in (4.5). When the interaction network $\mathcal{G}_B$ is strongly connected, the following threshold result was established in [49].

Theorem 4.2. *[49] Assume $\mathcal{G}_B$ is strongly connected, and assumptions $(C_1), (C_2)$ are satisfied. Then*

(a) *if $R_0 \leq 1$, P_0 is the only equilibrium of system (4.4) and it is globally asymptotically stable in Θ.*

(b) *if $R_0 > 1$, system (4.4) has a unique endemic equilibrium P^*, and P^* is globally stable in the interior of Θ if assumption (C_3) is satisfied.*

When the interaction network $\mathcal{G}_B$ is not strongly connected, we use similar methods as in Chapter 3 to study system (4.4). Subsystems, reduced subsystems, and restricted systems of system (4.4) are defined as in Chapter 2. Let $G \in \mathcal{G}$ be a subgraph of $\mathcal{G}_B$, the G-subsystem of system (4.4) is

$$\begin{aligned} S_k' &= \Lambda_k - d_k^S S_k - \sum_{j \in \mathcal{G}_B} \beta_{kj} S_k \int_0^\infty l_j(r) h_j(I_j(t-r))dr, \\ I_k' &= \sum_{j \in \mathcal{G}_B} \beta_{kj} S_k \int_0^\infty l_j(r) h_j(I_j(t-r))dr - (d_k^I + \gamma_k) I_k, \end{aligned} \quad k \in G; \tag{4.7}$$

the reduced G-subsystem of system (4.4) is

$$\begin{aligned} S_k' &= \Lambda_k - d_k^S S_k - \sum_{j \in G} \beta_{kj} S_k \int_0^\infty l_j(r) h_j(I_j(t-r))dr, \\ I_k' &= \sum_{j \in G} \beta_{kj} S_k \int_0^\infty l_j(r) h_j(I_j(t-r))dr - (d_k^I + \gamma_k) I_k, \end{aligned} \quad k \in G; \tag{4.8}$$

let $a = (a_1, a_2, ... a_n) \in \mathbb{R}_+^n$, the restricted system on G at a of system (4.4) is

$$\begin{aligned} S_k' &= \Lambda_k - d_k^S S_k - \sum_{j \in G} \beta_{kj} S_k \int_0^\infty l_j(r) h_j(I_j(t-r))dr - \sum_{s \in \mathcal{G}_B \backslash G} \beta_{ks} S_k \int_0^\infty l_s(r) h_s(a_s)dr, \\ I_k' &= \sum_{s \in \mathcal{G}_B \backslash G} \beta_{ks} S_k \int_0^\infty l_s(r) h_s(a_s)dr + \sum_{j \in G} \beta_{kj} S_k \int_0^\infty l_j(r) h_j(I_j(t-r))dr - (d_k^I + \gamma_k) I_k, \\ & k \in G \end{aligned} \tag{4.9}$$

Let $\mathcal{H}$ be the condensed graph of $\mathcal{G}_B$.

Theorem 4.3. *Assume that $\forall H \in \mathcal{H}$ and $\forall a \geq 0$, the restricted system (4.9) on H at a has a nonnegative equilibrium that attracts all positive solutions. Then all positive solutions of coupled system (4.4) converge to a single nonnegative equilibrium P^*.*

Proof. Suppose for $\forall s \in \mathcal{G} \setminus H$, $I_s(t)$ approaches a_s. By Lemma 2.2, the semiflow defined by the H-subsystem (4.2) is asymptotic autonomous with the semiflow defined by the restricted system (4.9) on H at a. By Theorem 2.2, if the restricted system (4.9) has an equilibrium P^* that attracts all positive solutions, then P^* attracts attracts all positive solutions of the H-subsystem (4.2). The rest of the proof is the same as the proof of Theorem 3.1. □

4.3 Analysis of the restricted system

For a restricted system (4.9) on $H \in \mathcal{H}$ at a, let $p_k = \sum\limits_{s \in \mathcal{G}_B \setminus H} \beta_{ks} \int_0^\infty l_s(r) h_s(a_s) dr, k \in H$. Let $m = |H|$, relabel the vertices in H to be $1, 2, ..., m$. Then system (4.9) can be written as

$$\begin{aligned} S_k' &= \Lambda_k - d_k^S S_k - \sum_{j=1}^m \beta_{kj} S_k \int_0^\infty l_k(r) h_j(I_j(t-r)) dr - p_k S_k, \\ I_k' &= p_k S_k + \sum_{j=1}^m \beta_{kj} S_k \int_0^\infty l_k(r) h_j(I_j(t-r)) dr - (d_k^I + \gamma_k) I_k, \end{aligned} \quad k = 1, 2, ..., m. \tag{4.10}$$

Let matrix $B_1 = (\beta_{kj})_{m \times m}$, B_1 is irreducible since H is a strongly connected component. System (4.10) can be viewed as a multi-group model with vertical transmission. Let $X_m = \prod\limits_{k=1}^m \mathbb{R} \times \mathbb{C}$, it can be seen that

$$\Gamma = \{(S_1, I_1(\cdot), ..., S_m, I_m(\cdot)) \in X_m | 0 \leq S_k \leq \frac{\Lambda_k}{d_k^*}, S_k + I_k(0) \leq \frac{\Lambda_k}{d_k^*}, I_k \geq 0, k = 1, 2, ...m\}$$

is the feasible region for system (4.10), where $d_k^* = \min\{d_k^S + p_k, d_k^I + \gamma_k\}$.

When $p_k = 0$ for all k, system (4.10) becomes system (4.4), its dynamical behavior is totally described by Theorem 4.2.

When $p_k > 0$ for some k, the following proposition shows there is no disease free equilibrium in system (4.10).

Proposition 4.1. *Assume $p_{k_0} > 0$ for some $k_0 \in \{1, 2, ..., m\}$. Then system (4.10) has no disease free equilibrium, furthermore, system (4.10) has a positive equilibrium.*

Proof. The set Γ is compact, convex and positively invariant, by Schauder Fixed Point Theorem [10], system (4.10) has an equilibrium in Γ. Let $P^* = (S_1^*, I_1^*, ..., S_m^*, I_m^*)$ be an equilibrium of system (3.1). By the first equation of (4.10), we know $S_k^* > 0$ for all $k = 1, 2, ..., m$.

Consider the second equation of (4.10) with $k = k_0$,

$$0 = \left.\frac{dI_{k_0}}{dt}\right|_{P^*} \geq p_{k_0} S_{k_0}^* - (d_{k_0}^I + \gamma_{k_0}) I_{k_0}^*,$$

thus $I_{k_0}^* > 0$.

Since B_1 is irreducible, there exists $k_1 \in \{1, 2, ..., m\}$, such that $\beta_{k_1 k_0} > 0$. Consider the second equation of (4.10) with $k = k_1$,

$$0 = \left.\frac{dI_{k_1}}{dt}\right|_{P^*} \geq \beta_{k_1 k_0} S_{k_1}^* a_{k_0} h_{k_0}(I_{k_0}^*) - (d_{k_1}^I + \gamma_{k_1}) I_{k_1}^*,$$

thus $I_{k_1}^* > 0$.

Since B_1 is irreducible, we can repeat this process and eventually we get $I_k^* > 0$ for all $k = 1, 2, ..., m$. $\square$

Lemma 4.1. *Assume assumptions $(C_1), (C_2), (C_3)$ are satisfied. For any two positive numbers I, I^*, and $j = 1, 2, ..., m$, $\frac{h_j(I)}{h_j(I^*)} - \ln \frac{h_j(I)}{h_j(I^*)} - \frac{I}{I^*} + \ln \frac{I}{I^*} \leq 0$.*

Proof. Let $x = \frac{I}{I^*}$, $f(x) = \frac{h_j(I^* x)}{h_j(I^*)} = \frac{h_j(I)}{h_j(I^*)}$. Let $g(x) = f(x) - \ln f(x) - x + \ln x$. It can be seen that $g(x) = \frac{h_j(I)}{h_j(I^*)} - \ln \frac{h_j(I)}{h_j(I^*)} - \frac{I}{I^*} + \ln \frac{I}{I^*}$. Then

$$\begin{aligned} g'(x) &= f'(x) - \frac{f'(x)}{f(x)} - 1 + \frac{1}{x} = \frac{f'(x)}{f(x)}(f(x)-1) + \frac{1}{x}(1-x) \\ &= \frac{f'(x)}{f(x)}(f(x)-x) + (\frac{1}{x} - \frac{f'(x)}{f(x)})(1-x). \end{aligned} \tag{4.11}$$

Notice that $h_j(0) = 0$, since $h_j''(I) \leq 0$, for $\forall z \geq 0$,

$$h_j'(z) \leq h_j'(\xi) = \frac{h(z) - 0}{z - 0} = \frac{h(z)}{z} \leq h_j'(0), \tag{4.12}$$

where ξ is a number between 0 and z. We have $f'(x) = \frac{I^* h_j'(I^* x)}{h_j(I^*)}$, by (4.12),

$$0 \leq \frac{f'(x)}{f(x)} = \frac{I^* h_j'(I^* x)}{h_j(I^* x)} \leq \frac{I^* \frac{h_j(I^* x)}{I^* x}}{h_j(I^* x)} \leq \frac{1}{x} \tag{4.13}$$

Let $F(x) = f(x) - x$, then $F(0) = F(1) = 0$, and $F'(x) = f'(x) - 1$, $F''(x) = f''(x) \leq 0$. Then $F(x) \geq 0$ when $x \leq 1$ and $F(x) \leq 0$ when $x \geq 1$. Therefore by (4.11), (4.13), $g'(x) \geq 0$ when $x \leq 1$, $g'(x) \leq 0$ when $x \geq 1$. This implies $g(x) \leq g(1) = 0$. □

With Lemma 4.1, we are able to show the global stability of the positive equilibrium P^* in the following theorem.

Theorem 4.4. *Assume that system (4.10) has an positive equilibrium $P^* = (S_1^*, I_1^*, ..., S_m^*, I_m^*)$. If assumptions $(C_1), (C_2), (C_3)$ are satisfied, then P^* is globally asymptotically stable in Γ and thus unique.*

Proof. First assume $n \geq 2$. Let $\bar{\beta}_{kj} = \beta_{kj} a_j S_k^* h(I_j^*)$, and

$$\bar{B}_1 = \begin{pmatrix} \sum_{j=1}^{m} \bar{\beta}_{1j} & -\bar{\beta}_{12} & \cdots & -\bar{\beta}_{1m} \\ -\bar{\beta}_{21} & \sum_{j=1}^{m} \bar{\beta}_{2j} & \cdots & -\bar{\beta}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{m1} & -\bar{\beta}_{m2} & \cdots & -\sum_{j=1}^{m} \bar{\beta}_{mj} \end{pmatrix}.$$

Let $(c_1, c_2, ..., c_m)$ be the cofactor of the i-th diagonal entry of $\bar{B}_1$. Denote $\alpha_j(r) = \int_r^\infty l_j(r)dr$. Consider a Lyapunov functional $V = \sum\limits_{k=1}^{m} c_k(V_{k1} + V_{k2})$, where

$$V_{k1} = S_k - S_k^* \ln S_k + I_k - I_k^* \ln I_k,$$

$$V_{k2} = \sum_{j=1}^{m} \beta_{kj} S_k^* \frac{h_j(I_j^*)}{I_j^*} \int_0^\infty \alpha_j(r)(I_j(t-r) - I_j^* \ln I_j(t-r))dr.$$

Taking the derivative of V_{k1}, V_{k2} along solutions to system (4.10), and using the equilibrium equation, we obtain

$$\begin{aligned}
\dot{V}_{k1} &= (p_k + d_k^S)S_k^*\Big(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*}\Big) - \sum_{j=1}^{m} \beta_{kj} S_k^* \int_0^\infty l_j(r) h_j(I_j(t-r))dr \\
&\quad + \Big(2 - \frac{S_k^*}{S_k}\Big) \sum_{j=1}^{m} \beta_{kj} S_k^* a_j h_j(I_j^*) + p_k S_k - \frac{I_k}{I_k^*} p_k S_k^* - \frac{I_k}{I_k^*} \sum_{j=1}^{m} \beta_{kj} S_k^* a_j h_j(I_j^*) \\
&\quad - \frac{I_k^*}{I_k} p_k S_k - \frac{I_k^*}{I_k} \sum_{j=1}^{m} \beta_{kj} S_k^* \int_0^\infty l_j(r) h_j(I_j(t-r))dr + p_k S_k^* \\
&= d_k^S S_k^* \Big(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*}\Big) + \sum_{j=1}^{m} \beta_{kj} S_k^* h_j(I_j^*) \int_0^\infty l_j(r) \Big(2 - \frac{I_k}{I_k^*} - \frac{S_k^*}{S_k} \\
&\quad + \frac{h_j(I_j(t-r))}{h_j(I_j^*)} - \frac{I_k^* S_k h_j(I_j(t-r))}{I_k S_k^* h_j(I_j^*)}\Big)dr + p_k S_k^* \Big(3 - \frac{S_k^*}{S_k} - \frac{I_k}{I_k^*} - \frac{I_k^* S_k}{I_k S_k^*}\Big) \\
&\le \sum_{j=1}^{m} \beta_{kj} S_k^* h_j(I_j^*) \int_0^\infty l_j(r) \Big(2 - \frac{I_k}{I_k^*} - \frac{S_k^*}{S_k} + \frac{h_j(I_j(t-r))}{h_j(I_j^*)} - \frac{I_k^* S_k h_j(I_j(t-r)}{I_k S_k^* h_j(I_j^*)}\Big)dr.
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
\dot{V}_{k2} &= \sum_{j=1}^{m} \beta_{kj} S_k^* \frac{h_j(I_j^*)}{I_j^*} \int_0^\infty \alpha_j(r) \Big[-\frac{\partial}{\partial r} \Big(I_j(t-r) - I_j^* \ln I_j(t-r)\Big)\Big]dr \\
&= \sum_{j=1}^{m} \beta_{kj} S_k^* h_j(I_j^*) \int_0^\infty l_j(r) \Big(\frac{I_j}{I_j^*} - \frac{I_j(t-r)}{I_j^*} - \ln \frac{I_j}{I_j(t-r)}\Big)dr
\end{aligned} \tag{4.15}$$

Combining (4.14) and (4.15), using Lemma 4.1, we have

$$\begin{aligned}
\dot{V} &= \sum_{k=1}^{m} c_k(V_{k1}+V_{k2}) \\
&\leq \sum_{k,j=1}^{m} c_k\beta_{kj}S_k^*h_j(I_j^*)\int_0^\infty l_j(r)\Big(2-\frac{I_k}{I_k^*}-\frac{S_k^*}{S_k}+\frac{h_j(I_j(t-r))}{h_j(I_j^*)} \\
&\quad -\frac{I_k^*S_kh_j(I_j(t-r)}{I_kS_k^*h_j(I_j^*)}+\frac{I_j}{I_j^*}-\frac{I_j(t-r)}{I_j^*}-\ln\frac{I_j}{I_j(t-r)}\Big)dr \\
&= \sum_{k,j=1}^{m} c_k\beta_{kj}S_k^*h_j(I_j^*)\int_0^\infty l_j(r)\Big[\Big(1-\frac{S_k^*}{S_k}+\ln\frac{S_k^*}{S_k}\Big)+\Big(-\frac{I_k^*}{I_k}+\frac{I_j^*}{I_j}\Big) \\
&\quad \Big(1-\frac{I_k^*S_kh_j(I_j(t-r)}{I_kS_k^*h_j(I_j^*)}+\ln\frac{I_k^*S_kh_j(I_j(t-r)}{I_kS_k^*h_j(I_j^*)}\Big)+\Big(-\ln\frac{I_k^*}{I_k}+\ln\frac{I_j^*}{I_j}\Big) \\
&\quad +\Big(\frac{h_j(I_j(t-r))}{h_j(I_j^*)}-\ln\frac{h_j(I_j(t-r))}{h_j(I_j^*)}-\frac{I_j(t-r)}{I_j^*}+\ln\frac{I_j(t-r)}{I_j^*}\Big)\Big]dr \\
&\leq \sum_{k,j=1}^{m} c_k\bar{\beta}_{kj}\Big[-\Big(\frac{I_k^*}{I_k}+\ln\frac{I_k^*}{I_k}\Big)+\Big(\frac{I_j^*}{I_j}+\ln\frac{I_j^*}{I_j}\Big)\Big].
\end{aligned}$$

By Theorem 4.1, $\dot{V}\leq 0$.

When $\dot{V}=0$, we have

$$\begin{aligned}
&2-\frac{S_k^*}{S_k}-\frac{S_k}{S_k^*}=0, \\
&1-\frac{I_k^*S_kh_j(I_j(t-r)}{I_kS_k^*h_j(I_j^*)}+\ln\frac{I_k^*S_kh_j(I_j(t-r))}{I_kS_k^*h_j(I_j^*)}=0, \qquad k=1,2,...,m.
\end{aligned} \tag{4.16}$$

Using the same argument as in the proof Theorem 3.1 in [36], one can show the largest invariant subset of the set where $\dot{V}=0$ is the singleton $\{P^*\}$. Therefore P^* is globally asymptotically stable in the interior of Γ, and this implies P^* is the unique positive equilibrium.

At last, for $m=1$, V can still be the Lyapunov function. The argument is similar and much simpler compared to the argument above for $n\geq 2$. □

Note that assumption $(C_1),(C_2),(C_3)$ are satisfied for various incidence functions, like $h(I)=I^q$ for $q\leq 1$ or saturate incidence function $\frac{I^q}{I+pI}$. When $p_k=0$ for all

$k = 1, 2, ..., m$, the basic reproduction number R_0 of system (4.10) is defined as the spectral radius of (4.6). With Theorem 4.2, Theorem 4.4 and Proposition 4.1, we have the following theorem summarizing the dynamics of the restricted system (4.10).

Theorem 4.5. *Assume assumptions $(C_1), (C_2), (C_3)$ are satisfied. For system (4.10),*

(i) when $p_k = 0$ for all k and $R_0 < 1$, the disease free equilibrium is globally asymptotically stable;

(ii) when $p_k = 0$ for all k and $R_0 > 1$, there exist an unique endemic equilibrium and it is globally asymptotically stable in the interior of Γ;

(iii) when $p_k > 0$ for some k, system (4.10) has no disease free equilibrium, there exist an unique endemic equilibrium and it is globally asymptotically stable in Γ.

By Theorem 4.5, we show that the restricted system (4.9) has a nonnegative equilibrium that attracts all positive solutions, by Theorem 4.3, we can establish the global stability result for equation (4.4) on non-strongly connected transmission networks.

Theorem 4.6. *Assume assumptions $(C_1), (C_2), (C_3)$ are satisfied. Then all positive solutions of system (4.4) converge to a nonnegative equilibrium.*

A series of previous results in the study of mathematical epidemiology are extended in this section. Theorem 4.5 generalize previous results [21, 36, 49] by taking vertical transmission term $p_k S_k$ into account. Theorem 4.6 shows the global stability of the nonnegative equilibrium on non-strongly connected connected interaction networks, it generalizes almost all previous results [21, 35, 36, 49] that depended on strongly connected interaction network.

4.4 Impact of the network connectivity

Recall in system (4.4), matrix $B = (\beta_{kj})_{n\times n}$, and the interaction network is $\mathcal{G}_B$. By Theorem 4.2, it is shown that if the interaction network $\mathcal{G}_B$ is strongly connected, then disease either disappears or persists in all groups. If the interaction network is not strongly connected, we can identify the globally asymptotical stable equilibrium by similar methods as stated in Chapter 3.

Proposition 4.2. *Let $P^* = (S_1^*, I_1^*, ..., S_n^*, I_n^*)$ be an equilibrium of system (4.4), the following statements hold.*

(b) *If a path from j to i exists and $I_j^* > 0$, then $I_i^* > 0$.*

(c) *Let $H \in \mathcal{H}$ be a strongly connected component of $\mathcal{G}$. Then for all $i \in H$, I_i^* are either all zero or all positive.*

(d) *Let $H_1, H_2 \in \mathcal{H}$ and $H_1 \prec H_2$, if there exist $j \in H_1$ such that $I_j^* > 0$, then for all $i \in H_2$, $I_i^* > 0$.*

Proposition 4.2 can be proved with the same argument as in the proof of Proposition 4.1. For a strongly connected component H, let $R_{0,H}$ be the basic production number of the reduced H subsystem (4.8).

Proposition 4.3. *Let $P^* = (S_1^*, I_1^*, ..., S_n^*, I_n^*)$ be the globally asymptotically stable non-negative equilibrium of system (4.4).*

(a) *If $R_{0,H} > 1$, then $I_i^* > 0$ for $i \in H$.*

(b) *If $R_{0,H} < 1$ and $R_{0,H'} > 1$ for some $H' \prec H$, then $I_i^* > 0$ for $i \in H$.*

(c) *If $R_{0,H} < 1$, and $R_{0,H'} \le 1$ for all $H' \prec H$, then $I_i^* = 0$ for $i \in H$.*

Proof. For (b), if $R_{0,H'} > 1$ for some $H' \prec H$, then disease persist in all groups of H', then the H-subsystem (4.7) has the same behavior as the restricted system (4.9) on H at some nonzero vector a. By Theorem 4.5, $I_i^* > 0$ for $i \in H$.

For (c), if $R_{0,H'} \le 1$ for all $H' \prec H$, then disease disappear in all groups of H', then H-subsystem (4.7) have the same behavior as the restricted system (4.9) on H at 0, which corresponds to the case $p_k = 0$ for all k in (4.10). By Theorem 4.5, the behavior of the groups on H is determined by the basic reproduction number $R_{0,H}$ of H: if $R_{0,H} < 1$, disease will disappear in all groups of H; if $R_{0,H} > 1$, disease will persist in all groups of H.

Part (a) is implied by the proof of (b) and (c). □

Define

$$R_{1,H} = \max\{R_{0,H'} | H' \in \mathcal{H}, H' \prec H \text{ or } H' = H\}. \tag{4.17}$$

Then $R_{1,H}$ serves as a threshold quantity that indicate whether disease dies out or persists in groups of H.

Theorem 4.7. *Let $P^* = (S_1^*, I_1^*, ..., S_n^*, I_n^*)$ be the globally stable nonnegative equilibrium of system (4.4),*

(i) if $R_{1,H} > 1$, then $I_i^* > 0$ for $i \in H$;

(ii) if $R_{1,H} \le 1$, then $I_i^* = 0$ for $i \in H$.

Theorem 4.7 follows from Proposition 4.3. By Theorem 4.7, we see in a strongly connected component H, disease either dies out in all groups of H, or persists in all groups of H, therefore the dynamical clusters of the multi-group model (4.4) are formed by strongly connected components. For each strongly connected component H, the quantity $R_{1,H}$ determines whether disease dies out or persists in H.

At last we state the connections between R_1 as defined in (4.17) and the basic reproduction number R_0 of the whole system (4.4). It can be verified that $R_0 = \max\{R_{0,H} | H \in$

$\mathcal{H}\}$. By Proposition 4.7, if $R_0 < 1$, disease will disappear in all groups, if $R_0 > 1$, disease will persist in at least one strongly connected component. When the interaction network $\mathcal{G}_B$ is strongly connected, there is only one strongly connected component in $\mathcal{G}_B$, which is $\mathcal{G}_B$ itself. Thus $R_0 = R_{1,\mathcal{G}_B}$. Then Proposition 4.7 reduces to previous global stability result, Theorem 4.2.

Chapter 5

Coupled linear oscillators on networks

In this and next chapter, we study a class of n coupled oscillators

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -f_i(x_i) + \sum_{j=1}^{n} k_{ij}(y_j - y_i), \qquad i = 1, 2, ..., n, \end{aligned} \tag{5.1}$$

where $f_i(x_i) \in C(\mathbb{R})$. The local agent systems of (5.1) are

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -f_i(x_i), \qquad i = 1, 2, ..., n, \end{aligned} \tag{5.2}$$

They are derived from second order oscillators $\ddot{x}_i = -f_i(x_i), i = 1, 2, ..., n$. Let $K = (k_{ij})_{n\times n}$, the interaction network of system (5.1) is $\mathcal{G}_K$. System (5.1) was widely studied in cooperative control problems, classic mechanical problems and electrical problems [24, 42, 45].

In the study of coupled oscillators, frequency plays an important role. Some researchers are interested in the phenomenon of frequency synchronization, in which all agent systems oscillate under a common frequency [31, 59] when time approaches infinity. In the case of clustered behavior, oscillators in the same clusters have the same frequencies, and the dynamical clusters are characterized by their frequencies.

In this chapter, we show the behavior of systems (5.1) with general interaction networks. When the interaction network is strongly connected, synchronization always occurs, regardless of the coupling strength. When the interaction network is not strongly connected, we show clustered behaviors may occur, different agent systems may have different frequencies. Moreover, we show a way to determine the frequencies of each dynamical cluster. The results in this chapter extend various previous results in the study of system (5.1).

This chapter is organized as follows. In Section 5.1, we investigate coupled linear and nonlinear oscillators on strongly connected networks. In Section 5.2, we investigate coupled linear oscillators on networks with a directed spanning tree. In Section 5.3, we investigate coupled linear oscillators on arbitrary networks. Numerical simulations are presented in Section 5.4.

5.1 Coupled oscillators on strongly connected networks

In system (5.1), for $i = 1, 2, ..., n$, let $F_i(x_i)$ be an anti-derivative of $f_i(x_i)$. In this section we assume that

(C_1) $f_i(x_i)x_i > 0, x_i \neq 0, i = 1, 2, ..., n,$

(C_2) $F_i(x_i) \to \infty$ as $|x_i| \to \infty, i = 1, 2, ..., n.$

Definition 3. *System* (5.1) *achieves global synchronization if for every solution* $x(t)$ *of system* (5.1) *and* $\forall i, j \in \{1, 2, ..., n\}$, $\lim_{t\to\infty} \dot{x}_i(t) - \dot{x}_j(t) = 0.$

Let $U = \{(x_1, y_1, ..., x_n, y_n) | y_1 = y_2 = \cdots = y_n\}$. Let W be the largest invariant set contained in U. We observe that $(0, ..., 0) \in W$, hence W is not empty. Then synchronized solutions are all in the set W. In set W, the coupling terms disappear, and system (5.1) becomes n isolated local agent systems (5.2).

Lemma 5.1. *Let $X(t) = (x_1(t), y_1(t), ..., x_n(t), y_n(t))$ be a solution of (5.1) in the set W. Then*

(i) for $i = 1, 2, ..., n$, $(x_i(t), y_i(t))$ is a solution of the local agent system (5.2);

(ii) for $i, j = 1, 2, ..., n$, and $t \in \mathbb{R}$, $y_i(t) - y_j(t) = 0, f_i(x_i(t)) = f_j(x_j(t))$ and $x_i(t) - x_j(t) = C_{ij}$ for some constant C_{ij};

(iii) if assumptions $(C_1), (C_2)$ are satisfied, and $X(t) \not\equiv 0$, the orbit $\{X(t), -\infty < t < \infty\}$ is a periodic orbit.

Proof. The first two statements follow directly from the definition of the set W. For $i = 1, 2, ..., n$, let $F_i(x) = \int_0^x f_i(s)ds$, $V_i(x_i, y_i) = \frac{1}{2}y_i^2(t) + F_i(x_i)$. By assumption (C_1), V_i is positive definite, and $(0, 0)$ is the only equilibrium point of the system

$$\dot{x}_i = y_i,$$
$$\dot{y}_i = -f_i(x_i).$$

By assumption (C_2), all level curves of V_i are closed. It can be seen that along the orbit of $X(t)$, $\frac{d}{dt}V_i = 0$. Then $(x_i(t), y_i(t))$ lies on one level curve of function V_i. Therefore if $(x_i(t), y_i(t)) \not\equiv 0$, the orbit $\{(x_i(t), y_i(t)), -\infty < t < \infty\}$ is a periodic orbit.

By the first and second statement of the theorem, if $X(t) \neq 0$, the orbit $\{X(t), -\infty < t < \infty\}$ is a periodic orbit. □

Theorem 5.1. *In system (5.1), suppose that $\mathcal{G}_K$ is strongly connected, and assumptions $(C_1), (C_2)$ are satisfied. Then all solutions of (5.1) are bounded, and system (5.1) achieves global synchronization.*

Proof. Let c_i be the cofactor of the i-th diagonal element of K. Let $F_i(x) = \int_0^x f_i(s)ds$, $V_i = \frac{1}{2}y_i^2(t) + F_i(x_i)$ and define $V = \sum_{i=1}^{m} c_iV_i$. By assumption (C_1), $V_i, i = 1, 2, ..., n$ and

V are positive definite. Taking derivative of V along a solution of (5.1), we have

$$\begin{aligned}\dot{V}_i &= -f_i(x_i)y_i + \sum_{j=1}^{n} k_{ij}(y_j - y_i)y_i + f_i(x_i)y_i \\ &= \sum_{j=1}^{n} k_{ij}(y_j - y_i)y_i \\ &\leq \frac{1}{2}\sum_{j=1}^{n} k_{ij}(-(y_j - y_i)^2 + y_j^2 - y_i^2) \leq \frac{1}{2}\sum_{j=1}^{n} k_{ij}(y_j^2 - y_i^2).\end{aligned}$$

By Theorem 2.1, $\dot{V} \leq 0$, and $\dot{V} = 0$ only if $y_1 = y_2 = \cdots = y_n$. By Lasalle invariance principle [33], the omega limit set of every bounded orbit of (5.1) is contained in the set W.

By assumption (C_2), V is unbounded only if $|x_i|$ is unbounded or $|y_i|$ is unbounded for some $i = 1, 2, ..., n$. Since $\dot{V} \leq 0$ along every orbit of (5.1), then every orbit of system (5.1) is bounded. Therefore the omega limit set of every orbit of (5.1) is contained in the set W.

For $i, j = 1, 2, ..., n$, let $S_{ij}(t) = y_i(t) - y_j(t)$. Then $\{0\}$ is the only limit point of $S_{ij}(t)$ in ∞. Therefore $\lim_{t\to\infty} y_i(t) - y_j(t) = 0$.

□

Theorem 5.1 shows that synchronization of coupled oscillators can occur when no assumption is made on the coupling strength k_{ij}. Previously, this kind of result was stated in Theorem 3.1 of [45], it only works for systems of coupled linear oscillators. Theorem 5.1 extends the result of Theorem 3.1 in [45], from coupled linear oscillators to coupled nonlinear oscillators. For nonlinear oscillators, previous synchronization results [61, 62] all require the coupling strength to exceed a threshold value.

Functions V and $V_i, i = 1, 2, ..., n$, defined in the proof of Theorem (5.1) can be viewed as the energy functions of the system of coupled dynamical oscillators (5.1) and its local agent systems (5.2). Theorem 5.1 shows that the omega limit set of every orbit of (5.1) is contained in W. The next theorem shows the omega limit set of every orbit of

(5.1) is an orbit in W.

Theorem 5.2. *Suppose that $\mathcal{G}_K$ is strongly connected, and assumptions $(C_1), (C_2)$ are satisfied. Let γ be an orbit of system (5.1), then the omega limit set of an orbit γ of system (5.1) is either $\{0\}$ or a single periodic orbit in W.*

Proof. Let γ_1, γ_2 be two orbits in W, and let $\gamma, \gamma_1, \gamma_2$ be generated by solutions $X(t), X^1(t)$, $X^2(t)$ of system (5.1). Suppose $\gamma_1 \subseteq \omega(\gamma), \gamma_2 \subseteq \omega(\gamma)$. Let $p = 1, 2$, since $\gamma_p \subseteq W$, we know $\frac{d}{dt}V(X^p(t)) = 0$, then there exist constants V_p, such that $V(X^p(t)) = V_p$. Consider function $V(X(t))$, we know $\frac{d}{dt}V(X(t)) \leq 0$, and V is bounded from below, therefore there exists constant V_0, such that $\lim_{t\to\infty} V(X(t)) = V_0$. Since $\gamma_p \subseteq \omega(\gamma)$, $V_1 = V_2 = V_0$. Let $X^p(t) = (x_1^p(t), y_1^p(t), ..., x_n^p(t), y_n^p(t))$. Notice $\gamma_p \subseteq W$, then for $i, j = 1, 2, ..., n$ and $t \in \mathbb{R}$, $y_i^p(t) = y_j^p(t)$, and $-y_i^p(t)f_i(x_i^p(t)) = -y_j^p(t)f_j(x_j^p(t))$. Therefore $(F_i(x_i^p(t)))' = (F_j(x_j^p(t)))'$, by choosing appropriate antiderivatives $F_i, i = 1, 2, ..., n$, we can get $F_i(x_i^p(t)) = F_j(x_j^p(t))$, then $V_i(x_i^p(t), y_i^p(t)) = V_j(x_j^p(t), y_j^p(t))$. Recall $V = \sum_{i=1}^{m} c_i V_i$, we get $V_1 = V_2 \Leftrightarrow V_i(x_i^1(t), y_i^1(t)) = V_i(x_i^2(t), y_i^2(t)), i = 1, 2, ..., n$. By assumption C_1, the level curves $V_i = C, C \in \mathbb{R}, i = 1, 2, ..., n$ of local agent systems (5.2) are isolated, γ_1, γ_2 are disconnected; however, since γ is bounded, $\omega(\gamma)$ is connected, we get a contradiction. Therefore $\omega(\gamma)$ is a single orbit in W. By Lemma 5.1, $\omega(\gamma)$ is either $\{0\}$ or a single periodic orbit in W. □

Note the orbits in W are all synchronized, as described in Lemma 5.1.

Corollary 2. *Suppose that $\mathcal{G}_K$ is strongly connected, $f_i(x_i) = f(x_i), i = 1, 2, ..., n$, $f(x)$ satisfies $f(0) = 0, f'(x) \geq 0$ and $f'(x) = 0$ only on isolated points. Then for every solution $(x_1(t), y_1(t), ..., x_n(t), y_n(t))$ of system (5.1), $\lim_{t\to\infty} x_i(t) - x_j(t) = 0$.*

Proof. It can be seen under the assumption on $f(x)$, assumptions $(C_1), (C_2)$ are satisfied. By Theorem 5.2, the omega limit set of $(x_1(t), y_1(t), ..., x_n(t), y_n(t)), 0 < t < \infty$ is a complete orbit ζ in W. Let $\zeta = \{(\bar{x}_1(t), \bar{y}_1(t), ..., \bar{x}_n(t), \bar{y}_n(t)), -\infty < t < \infty\}$. By

Lemma 5.1, for $i, j = 1, 2, ..., n$ and $t \in \mathbb{R}$, $f(\bar{x}_i(t)) = f(\bar{x}_i(t) + C_{ij})$, since $f'(x) \geq 0$ and $f'(x) = 0$ only in isolated points, $C_{ij} = 0$. Therefore $\bar{x}_i(t) = \bar{x}_j(t)$. Then by the same proof of Theorem 5.1, we get $\lim_{t\to\infty} x_i(t) - x_j(t) = 0$. □

Theorem 5.1 applies to both linear and nonlinear oscillators. Therefore under a strongly connected interaction network, synchronization behavior should be expected. For coupled linear oscillator, $f_i(x_i) = \omega_i^2 x_i$, where ω_i is called the natural frequency of oscillator i.

Theorem 5.3. *Let $(x_1, y_1, ..., x_n, y_n)$ be a solution of system (5.1). Let $f_i(x_i) = \omega_i^2 x_i$ for some ω_i. Suppose the interaction network $\mathcal{G}$ is strongly connected,*

(i) if $\omega_1 = \omega_2 = \cdots = \omega_n = \omega$, for initial condition $(x_1^0, y_1^0, ..., x_n^0, y_n^0)$, there exist constants A and ϕ, such that $\lim_{t\to\infty} x_i(t) = A\cos(\omega t + \phi)$ for $i = 1, 2, ..., n$;

(ii) if there exist i, j such that $\omega_i \neq \omega_j$, then $\lim_{t\to\infty} x_q(t) = 0$ for $q = 1, 2, ..., n$.

Proof. 1. The first statement is a result of Theorem 3.1 in [45].

2. Let $X(t) = (x_1(t), y_1(t), ..., x_n(t), y_n(t))$ be a solution of (5.1), by Theorem 5.2, the omega limit set of $\{X(t), 0 < t < \infty\}$ is an orbit ρ in W. Let $\rho = \{(\bar{x}_1(t), \bar{y}_1(t), ..., \bar{x}_n(t), \bar{y}_n(t)), -\infty < t < \infty\}$. Suppose there exist i, j such that $\omega_i \neq \omega_j$, By Lemma 5.1, $\bar{x}_i = A_i \cos(\omega_i t + \phi_i), \bar{x}_j = A_j \cos(\omega_j t + \phi_j)$, and there exists C_{ij}, such that

$$A_i \cos(\omega_i t + \phi_i) = A_j \cos(\omega_j t + \phi_j) + C_{ij} \tag{5.3}$$

for all t. The left hand side and right hand side of equation (5.3) has different frequencies, for equation (5.3) to hold, we need $A_i = A_j = C_{ij} = 0$. Then $\bar{x}_i(t) = \bar{x}_j(t) = 0$. For $l \neq i, j$, ω_l is different with at least one of ω_i, ω_j, by the same argument as above, $\bar{x}_l(t) = 0$. Thus $\lim_{t\to\infty} x_q(t) = 0$ for $q = 1, 2, ..., n$. □

5.2 Coupled linear oscillators on network with a rooted spanning tree

Definition 4. *A digraph $\mathcal{G}$ has a rooted spanning tree if there exists a vertex i, such that for every vertex j, there exists a directed path from i to j. Vertex i is called a root of $\mathcal{G}$.*

Lemma 5.2. *A digraph $\mathcal{G}$ has a rooted spanning tree if and only if the condensed graph $\mathcal{H}$ of $\mathcal{G}$ has only one minimal element with respect to the partial order $\prec$.*

Proof. If $\mathcal{G}$ has a rooted spanning tree, assume there are two minimal elements H_1 and H_2, By the definition of rooted spanning tree, we have $H_1 \prec H_2$ and $H_2 \prec H_1$, this contradicts the fact that the partial order $\prec$ is a strict partial order. Suppose $\mathcal{H}$ has only one minimal element H_1. Consider the set $\mathcal{H}_1 = \{H \in \mathcal{H}, H \not\succ H_1\}$, if there exists $H_2 \neq H_1$, such that $H_2 \not\succ H_1$, then $\mathcal{H}_1$ is not empty, and there exists a minimal element H_2 in $\mathcal{H}_1$, then H_2 is a minimal element in $\mathcal{H}$, it contradicts the fact that $\mathcal{H}$ has only one minimal element H_1. So for $\forall H \in \mathcal{H}$, if $H \neq H_1, precH_1 \prec H$. Then every vertex of H is a root of $\mathcal{G}$. □

Let $\mathcal{H}$ be the condensed graph of $\mathcal{G}_K$. For a strongly connected components $H \in \mathcal{H}$, consider the restricted system on H at a function $(G(t), g(t), ..., G(t), g(t))$ with $G'(t) = g(t)$.

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -\omega_i^2 x_i + \sum_{j \in H} k_{ij}(y_j - y_i) + \sum_{j \in \mathcal{G}_K \setminus H} k_{ij}(g(t) - y_i), i \in H. \end{aligned} \tag{5.4}$$

Let $m = |H|$ and $p_i = \sum_{j \in \mathcal{G}_K \setminus H} k_{ij}$, then system (5.4) becomes

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -\omega_i^2 x_i + \sum_{j=1}^{m} k_{ij}(y_j - y_i) + p_i(g(t) - y_i), \qquad i = 1, 2, ..., m, \end{aligned} \tag{5.5}$$

System (5.5) can be interpreted as a system of coupled oscillators forced by an external force $g(t)$. Let $(x_1, y_1, ..., x_m, y_m)$, $(\tilde{x}_1, \tilde{y}_1, ..., \tilde{x}_m, \tilde{y}_m)$ be any two solutions of system (5.5), and $u_i = x_i - \tilde{x}_i, v_i = y_i - \tilde{y}_i$. Then u_i, v_i satisfy the following equations.

$$\begin{aligned} \dot{u}_i &= v_i, \\ \dot{v}_i &= -\omega_i^2 u_i + \sum_{j=1}^{m} k_{ij}(v_j - v_i) - p_i v_i, \qquad\qquad i = 1, 2, ..., m. \end{aligned} \tag{5.6}$$

Theorem 5.4. *Suppose there exists j, such that $p_j > 0$. Then all solutions of* (5.6) *approach* 0.

Proof. The interaction network of system (5.6) is H, let its weight matrix be $\tilde{K} = (k_{ij})_{i,j \in H}$. Let c_i be the cofactor of the i-th diagonal element of matrix. Since H is strongly connected, $c_i > 0, i = 1, 2, ..., n$. let $V_i = \frac{1}{2}v_i^2(t) + \frac{u_i^2 \omega_i^2}{2}$, and define $V = \sum_{i=1}^{m} c_i V_i$, we have

$$\begin{aligned} \dot{V}_i &= -\omega_i^2 u_i^2 v_i + \sum_{j=1}^{m} k_{ij}(v_j - v_i)v_i - p_i v_i^2 + \omega_i^2 u_i^2 v_i \\ &= \sum_{j=1}^{m} k_{ij}(v_j - v_i)v_i - p_i v_i^2 \\ &\leq \frac{1}{2} \sum_{j=1}^{m} k_{ij}(-(v_j - v_i)^2 + v_j^2 - v_i^2) - p_i v_i^2 \end{aligned}$$

By Theorem 2.1,

$$\begin{aligned} \dot{V} &= -\sum_{i=1}^{m} c_i p_i v_i^2 - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} c_i k_{ij}(v_j - v_i)^2 \\ &\leq -c_j p_j v_j^2 - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} c_i k_{ij}(v_j - v_i)^2 \leq 0, \end{aligned}$$

and $\dot{V} = 0$ only if $v_j = 0$ and $v_1 = v_2 = \cdots = v_m$, that is $v_i = 0$ for all $i = 1, 2, ..., m$. By a similar argument as in the proof of Theorem 5.1, solutions of system (5.6) are bounded. Let U be the set such that $\dot{V} = 0$, and W be the largest invariant set contained in U. It can be seen that $W = \{0\}$. Therefore all solutions of system (5.6) approach 0. □

Corollary 3. *Suppose* $\omega_1 = \omega_2 = ...\omega_m = \omega$, *and* $G(t)$ *solves the equation* $\ddot{x} = -\omega^2 x$, *then for every solution* $(x_1, y_1, ..., x_m, y_m)$ *of system* (5.5), $\lim_{t\to\infty} x_i(t) - G(t) = 0, i = 1, 2, .., m$.

Proof. If $G(t)$ solves the equation $\ddot{x} = -\omega^2 x$, then $x_i = G(t), i = 1, 2, ..., m$ is a solution of system (5.5). By Theorem 5.4, the difference of any two solutions of system (5.5) approaches 0, therefore $\lim_{t\to\infty} x_i(t) - G(t) = 0, i = 1, 2, .., m$ for every solution $(x_1, y_1, ..., x_m, y_m)$. □

Now we are ready to prove the main theorem of this section.

Theorem 5.5. *Suppose* $f_i(x_i) = \omega^2 x_i$, *and the interaction network* $\mathcal{G}_K$ *has a rooted spanning tree. Then for every solution* $(x_1, y_1, ..., x_n, y_n)$ *of system* (5.1), *and* $i, j = 1, 2, .., n$, $\lim_{t\to\infty} x_i(t) - x_j(t) = 0$.

Proof. Let $\mathcal{H}$ be the condensed graph of $\mathcal{G}_K$. We use induction on the order $|\mathcal{H}|$ of $\mathcal{H}$. When $|\mathcal{H}| = 1$, the interaction network is strongly connected, by Theorem 5.3, the theorem holds. Assume the theorem holds when $|\mathcal{H}| = N$. Then when $|\mathcal{H}| = N + 1$, let H_m be the maximal element with respect to the partial order $\prec$. Let $\mathcal{G}_r = \mathcal{G}_K \setminus H_m$, and let $\mathcal{H}_r$ be the condensed graph of $\mathcal{G}_r$. Then $|\mathcal{H}_r| = N$. By induction assumption, every solution of system (5.1) on interaction network $\mathcal{G}_r$ approaches to a synchronized solution $(\bar{x}(t), \bar{y}(t), ..., \bar{x}(t), \bar{y}(t))$.

Now consider system (5.1) on interaction network $\mathcal{G}_K$. Since H_m is the maximal element with respect to the partial order $\prec$, it does not effect the solution on interaction network $\mathcal{G}_r$. Therefore, for every vertex $i \in \mathcal{G}_r$, $x_i(t)$ approaches to $\bar{x}(t)$. Consider the restricted system on H_m at $(\bar{x}(t), \bar{y}(t), ..., \bar{x}(t), \bar{y}(t))$.

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -\omega^2 x_i + \sum_{j\in H_m} k_{ij}(y_j - y_i) + \sum_{j\in\mathcal{G}_r} k_{ij}(\bar{y}(t) - y_i), i \in H_m. \end{aligned} \tag{5.7}$$

It is clear $\bar{x}(t)$ solves equation $\ddot{x} = -\omega^2 x$. By Corollary 3, $x_i(t)$ approaches $\bar{x}(t)$ for $i \in H_m$. By Theorem 2.3, solutions of H_m subsystem converge to $(\bar{x}(t), \bar{y}(t), ..., \bar{x}(t), \bar{y}(t))$. Therefore the solution of system (5.1) converges to $(\bar{x}(t), \bar{y}(t), ..., \bar{x}(t), \bar{y}(t))$. By induction, the theorem is proved. □

Theorem 5.5 was proved in [45], by explicitly solving the system (5.1). Comparing to the proof in [45], our proof uses Lyapunov functions on each strongly connected component of $\mathcal{G}_K$ in this section. Similar argument can be applied to general cases when the oscillators have different natural frequencies or the interaction network does not have a rooted spanning tree. We will discuss the details in next section.

5.3 Coupled linear oscillators on arbitrary interaction networks

Let $\mathcal{H}$ be the condensed graph of $\mathcal{G}$. For a strongly connected component $H \in \mathcal{H}$, consider the restricted system on H at a function $(G_1(t), g_1(t), ..., G_n(t), g_n(t))$ with $G_i(t) \in C^1(\mathbb{R}), G_i'(t) = g_i(t), i = 1, 2, ..., n$.

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -\omega_i^2 x_i + \sum_{j \in H} k_{ij}(y_j - y_i) + \sum_{j \in \mathcal{G}_K \setminus H} k_{ij}(g_j(t) - y_i), i \in H. \end{aligned} \tag{5.8}$$

Let $m = |H|$, $p_i = \sum\limits_{j \in \mathcal{G}_K \setminus H} k_{ij}$ and $h_i(t) = \sum\limits_{j \in \mathcal{G}_K \setminus H} k_{ij} g_j(t)$. Then system (5.8) becomes

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -\omega_i^2 x_i + \sum_{j=1}^{m} k_{ij}(y_j - y_i) - p_i y_i + h_i(t), \qquad i = 1, 2, ..., m. \end{aligned} \tag{5.9}$$

System (5.9) is similar to system (5.5), it can be interpreted as a system of coupled oscillators driven by an external force $(h_1(t), h_2(t), ..., h_m(t)$. Suppose $h_i(t), i = 1, 2, .., m$

is periodic or quasi-periodic, we are going to show the solutions of system (5.9) approach to functions with the same frequencies of $h_i(t), i = 1, 2, .., m$. To be more specific about frequencies, we introduce the definition of frequency module. Frequency module is associated with almost periodic functions, which are a generalization of periodic functions, for detailed definition and introduction of almost periodic functions, please see [9].

Definition 5. *[22] Suppose $\{\lambda_n\}$ is a set of real numbers. The module $m\{\lambda_n\}$ of the set $\{\lambda_n\}$ is the set consisting of all real numbers which are finite linear combinations of elements of the set $\{\lambda_n\}$ with integer coefficients.*

Definition 6. *[22] For a function $f(t)$ and a real number λ, let $a(\lambda) = \lim\limits_{T\to\infty} \frac{1}{T}\int_0^T f(t)e^{-i\lambda t}dt$. If for a real number λ, $a(\lambda) \neq 0$, then λ is called a Fourier exponent of function $f(t)$. When $f(t)$ is almost periodic, its Fourier exponents are countable, the frequency module of $f(t)$ is defined as $m\{\lambda_n\}$ where the numbers λ_n are the Fourier exponents of function $f(t)$.*

From the definition of frequency modules, we have the following lemma.

Lemma 5.3. *Let $f(t) = \sum\limits_{j=1}^{N} \alpha_j \cos(\omega_j t + \phi_j)$, with $\alpha_j, \omega_j \neq 0$, and $\lim\limits_{t\to\infty} g(t) - f(t) = 0$. Then the frequency modules $m(f) = m(g) = m\{\omega_j, j = 1, 2, ..., N\}$.*

Let $H(t) = (h_1(t), h_2(t), ..., h_m(t))^T$, $x = (x_1, ..., x_m)^T$, system (5.9) can be written in matrix form

$$\ddot{x}(t) + (M + D)\dot{x}(t) + \Omega x(t) = H(t), \tag{5.10}$$

with

$$M = \begin{pmatrix} \sum\limits_{j=1}^{m} k_{1j} & -k_{12} & \cdots & -k_{1m} \\ -k_{21} & \sum\limits_{j=1}^{m} k_{2j} & \ddots & -k_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ -k_{m1} & \cdots & -k_{m(m-1)} & \sum\limits_{j=1}^{m} k_{mj} \end{pmatrix},$$

and

$$D = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & p_m \end{pmatrix}, \Omega = \begin{pmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \omega_m^2 \end{pmatrix}$$

The interaction network of system (5.8) is the strongly connected component H, thus it is strongly connected. By the definition of matrix M, it can be seen that M is irreducible. The homogeneous equation of (5.10) is

$$\ddot{x}(t) + (M + D)\dot{x}(t) + \Omega x(t) = 0. \tag{5.11}$$

Theorem 5.6. *Suppose $p_i > 0$ for some i. Then all solutions of system (5.11) approach to* 0.

Theorem 5.6 can be proved by the same argument as the proof of Theorem 5.4.

Corollary 4. *Let $(x_1, y_1, ..., x_m, y_m), (\tilde{x}_1, \tilde{y}_1, ..., \tilde{x}_m, \tilde{y}_m)$ be any two solutions of system (5.9). Then $\lim_{t\to\infty} x_i(t) - \tilde{x}_i(t) = 0$ for $i = 1, 2, ..., m$.*

Proposition 5.1. *Suppose $p_i > 0$ for some i, and $H(t) = le^{i\mu t}$, with $l \in \mathbb{C}^m, \mu \in \mathbb{R}$. Then one particular solution of system (5.10) is $ve^{i\mu t}$ for some vector $v \in \mathbb{C}^m$.*

Proof. Let $W = (x_1, x_2, ..., x_m, y_1, y_2, ..., y_m)^T, R = \begin{pmatrix} 0, & I \\ -\Omega, & -(M + D) \end{pmatrix}$, and $\tilde{H} = (0, H)^T \in \mathbb{C}^{2m}$, rewrite system (5.10) as

$$\dot{W} = RW + \tilde{H}. \tag{5.12}$$

Let $\tilde{l} = (0, l)^T \in \mathbb{C}^{2m}$, it can be seen that $\tilde{H} = \tilde{l}e^{i\mu t}$. By Theorem 5.6, the eigenvalues of R all have negative real parts, thus $i\mu$ is not an eigenvalue of R, then it can be verified that $(i\mu I - R)^{-1}\tilde{H}e^{i\mu t}$ solves equation (5.12). Then v is the vector formed by the first m entries of $(i\mu I - R)^{-1}\tilde{H}$. □

Proposition 5.2. *Suppose $p_i > 0$ for some i, and $H(t) = \sum_{j=1}^{N} l_j e^{i\mu_j t}$, with $l_j \in \mathbb{C}^m, \mu_j \in \mathbb{R}$. Then there exist $v_j \in \mathbb{C}^m, j = 1, 2, ..., N$, such that all solutions of* (5.10) *converge to* $\sum_{j=1}^{N} v_j e^{i\mu_j t}$.

Proof. By superposition principle and Proposition 5.1, there exists exists $v_j \in \mathbb{C}^m, j = 1, 2, ..., N$, such that $\sum_{j=1}^{N} v_j e^{i\mu_j t}$ is a particular solution of system (5.10). By Corollary 4, all solutions of (5.10) converge to $\sum_{j=1}^{N} v_j e^{i\mu_j t}$. □

In the thesis, we call functions with the form $A\cos(\mu t + \phi)$ *sinusoidal functions*. By Proposition 5.2, in the restricted system (5.8), if $g_i(t), i \in H$ are linear combinations of sinusoidal functions, then the solutions converge to a linear combination of sinusoidal functions that contains all the frequencies of $g_i(t), i \in H$.

Proposition 5.3. *Suppose $g_j(t) = \sum_{i=1}^{N_j} A_{ij}\cos(\mu_{ij}t+\phi_{ij}), j \in \mathcal{G}_K \setminus H$. Let $\mathcal{J} = \{j \in \mathcal{G}_K \setminus H : \exists i \in H, k_{ij} \neq 0\}$. Then solutions of system* (5.8) *converge to a linear combination of sinusoidal functions $x_H(t)$, and $m(x_H(t)) = \sum_{j\in\mathcal{J}} m(g_j)$.*

Proof. Proposition 5.3 is a direct result of Proposition 5.2 as a result of the following three facts.

(i) $Re(Ae^{\phi}e^{i\mu_j t}) = Re(Ae^{i(\mu t+\phi)}) = A\cos(\mu t + \phi)$;

(ii) in system (5.9), functions $h_i(t), i = 1, 2, ..., m$, are linear combinations of $g_i(t)$;

(iii) let f, g be two almost periodic functions, $m(f) + m(g)$ is the smallest module that contains $m(f) \bigcup m(g)$.

□

Theorem 5.7. *In equation* (5.1)*, let $f_i(x_i) = \omega_i^2 x_i$. Let $(x_1, y_1, ..., x_n, y_n)$ be a solution of system* (5.1)*. For every $i \in \{1, 2, ..., n\}$, there exists a finite linear combination of* *sinusoidal functions* $\bar{x}_i(t)$*, such that* $\lim_{t\to\infty} x_i(t) - \bar{x}_i(t) = 0$*. Furthermore, for* $j, l \in \{1, 2, ..., n\}$*, if* i, l *are in the same strongly connected component, then* $m(x_j) = m(x_l)$*.*

Proof. Let $\mathcal{H}$ be the condensed graph of $\mathcal{G}_K$. We use induction on the order $|\mathcal{H}|$ of $\mathcal{H}$. When $|\mathcal{H}| = 1$, the interaction network is strongly connected, by Theorem 5.3, the theorem holds. Assume the theorem holds when $|\mathcal{H}| = N$, when $|\mathcal{H}| = N + 1$, let H_m be the maximal element with respect to the partial order $\prec$. Let $\mathcal{G}_r = \mathcal{G}_K \setminus H_m$, and let $\mathcal{H}_r$ be the condensed graph of $\mathcal{G}_r$. Then $|\mathcal{H}_r| = N$.

Now consider system (5.1) on interaction network $\mathcal{G}$. Since H_m is the maximal element with respect to the partial order $\prec$, it will not effect the solution on interaction network $\mathcal{G}_r$. By induction assumption, for every vertex $i \in \mathcal{G}_r$, $x_i(t)$ approaches a linear combination of sinusoidal functions $\bar{x}_i(t)$, and the frequency module of any two oscillators in the same strongly connected component are the same.

For the case $i \in H_m$, consider the the H_m subsystem of system (5.1)

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -\omega_i^2 x_i + \sum_{j\in H_m} k_{ij}(y_j - y_i) + \sum_{j\in\mathcal{G}_r} k_{ij}(y_j(t) - y_i), i \in H_m. \end{aligned} \tag{5.13}$$

We know that for $j \in \mathcal{G}_r$, $x_j(t)$ approaches to $\bar{x}_j(t)$, $y_j(t)$ approaches to $\dot{\bar{x}}_j(t)$, which is also a linear combination of sinusoidal functions with the same frequency module as $\bar{x}_j(t)$. By theorem 2.3, solutions of (5.13) converges to the global attractor of the restricted system on H_m at $\dot{\bar{x}}(t)$

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -\omega_i^2 x_i + \sum_{j\in H_m} k_{ij}(y - y_i) + \sum_{j\in\mathcal{G}_r} k_{ij}(\dot{\bar{x}}_j(t) - y_i), i \in H_m. \end{aligned} \tag{5.14}$$

By Proposition 5.3, solutions of system (5.14) approach to a linear combination of sinusoidal functions, and the frequency module of this function is $m(x_H(t)) = \sum_{j\in\mathcal{J}} m(\bar{x}_j)$, where $\mathcal{J} = \{j \in \mathcal{G}_K \setminus H_m : \exists i \in H_m, k_{ij} \neq 0\}$. Therefore for $i \in H_m$, $x_i(t)$ approaches to a linear combination of sinusoidal functions $\bar{x}_i(t)$, and the frequency module of any two oscillators in H_m are the same. □

By Theorem 5.7, it is natural to define frequency module for a strongly connected component in system (5.1).

Definition 7. *Let H be a strongly connected component. The frequency module of H is defined as $m(H) = m(x_i)$, $\forall i \in H$.*

The next theorem shows how the interaction network impacts the dynamics of coupled oscillator (2.9).

Theorem 5.8. *Suppose $f_i(x_i) = -\omega_i^2 x_i$. Let $\mathcal{H}$ be the condensed graph of $\mathcal{G}_K$, and $x(t) = (x_1(t), x_2(t), ..., x_n(t))$ be a solution of (5.1). Then*

(i) for every $i \in \{1, 2, ..., n\}$, there exists a finite linear combination of sinusoidal functions $\bar{x}_i(t)$, such that $\lim_{t\to\infty} x_i(t) - \bar{x}_i(t) = 0$;

(ii) for a minimal component H, if $\forall i \in H, \omega_i = \omega$, then $m(H) = m\{\omega\}$; otherwise $m(H) = \{0\}$, and $x_i(t) \to 0, \forall i \in H$;

(iii) for a non-minimal component H, $m(H) = \sum_{H' \prec H} m(H')$.

Proof. Statement 1 directly follows from Theorem 5.7. Statement 2 follows from Theorem 5.3. Statement 3 is implied in the proof of Theorem 5.7. □

When interaction network $\mathcal{G}$ has a rooted spanning tree, Theorem 5.8 generalizes Theorem 5.5 as it applies to the situation when the natural frequencies of the oscillators are not the same.

Corollary 5. *Suppose the interaction network $\mathcal{G}_K$ has a rooted spanning tree, let $\mathcal{H}$ be the condensed graph of $\mathcal{G}_K$. Let $(x_1(t), y_1(t), ..., x_n(t), y_n(t))$ be a solution of system (5.1).*

(i) There is only one minimal element H with respect to the partial order $'\prec'$ in $\mathcal{H}$.

(ii) If the natural frequencies ω_i of each oscillator x_i in H are the same, namely $\omega_i = \omega, i \in H$, then for every $i = 1, 2, ..., n$, $m(x_i) = m\{\omega\}$.

(iii) If the natural frequencies ω_i of each oscillator x_i in H are not the same, then $m(x_i) = \{0\}$, and $x_i(t) \to 0, i = 1, 2, ..., n$.

Theorem 5.8, with Theorem 5.3 and Theorem 5.5 fully describe the dynamics of coupled linear oscillators. If we use the frequencies to differentiate clusters, by Theorem 5.8, we see dynamical cluster of coupled linear oscillators are formed by the strongly connected component of the interaction network. Clustered behavior occurs only if the interaction network does not have a rooted spanning tree, and Theorem 5.8 fully characterize how the connectivity of interaction network determines the clustered behavior of coupled linear oscillator system.

5.4 Numerical simulations

In this section we show several numerical examples to illustrate the results in this chapter. We consider four coupled linear oscillators

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -\omega_i^2 x_i + \sum_{j=1}^{4} k_{ij}(y_j - y_i), \qquad i = 1, 2, 3, 4. \end{aligned} \tag{5.15}$$

Numerical examples are shown under strongly connected interaction network, interaction network with rooted spanning tree, and interaction network without a rooted spanning tree, respectively.

5.4.1 Strongly connected interaction network

In this section, let the interaction network $\mathcal{G}_1$ be described by Figure 5.1. It can be seen

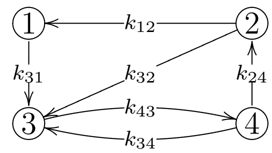


Figure 5.1: Interaction network $\mathcal{G}_1$.

that interaction network $\mathcal{G}_1$ is strongly connected. By Theorem 5.3, we know system (5.15) achieves global synchronization. If all natural frequencies are the same, solutions of system (5.15) synchronize to sinusoidal function; if there exist two different natural frequencies, solutions of system (5.15) synchronize to 0. Numerically, we consider the following two cases corresponding to the conditions stated in Theorem 5.3.

(i) $\omega_1 = \omega_2 = \omega_3 = \omega_4 = 1$.

(ii) $\omega_1 = 1, \omega_2 = 5, \omega_3 = 3, \omega_4 = 4$.

The simulation results are shown in Figure 5.2 and Figure 5.3. In Figure 5.2, x_1, x_2, x_3, x_4 approach to the same sinusoidal function. In Figure 5.3, x_1, x_2, x_3, x_4 approach to 0.

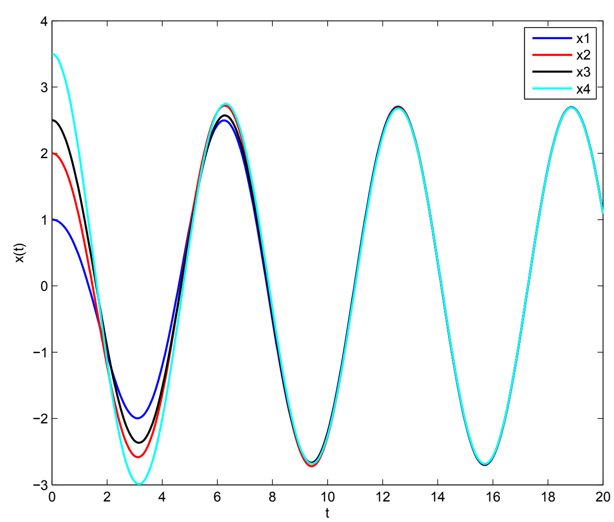


Figure 5.2: Solution approaches to a sinusoidal function when the natural frequencies are the same.

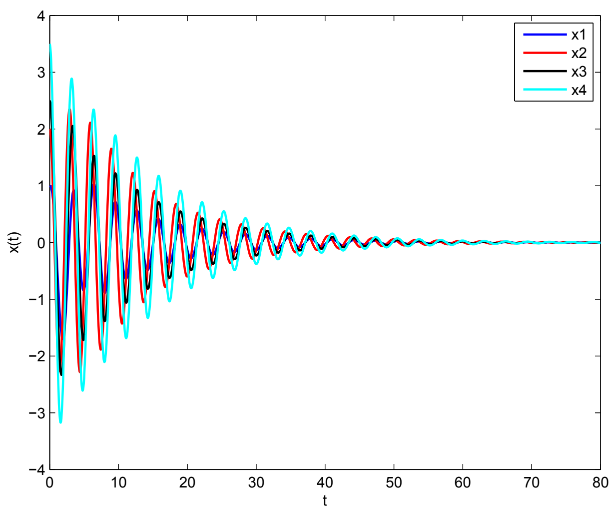


Figure 5.3: Solution approaches to 0 when the natural frequencies are different.

5.4.2 Interaction network with a rooted spanning tree

In this section, let the interaction network $\mathcal{G}_2$ be described by Figure 5.4.

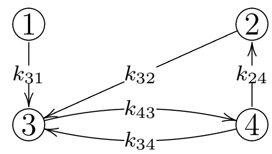


Figure 5.4: Interaction network $\mathcal{G}_2$.

It can be seen that interaction network $\mathcal{G}_2$ has a rooted spanning tree. There are two strongly connected components in interaction network $\mathcal{G}$, $H_1 = \{1\}, H_2 = \{2, 3, 4\}$, and $H_1 \prec H_2$. When all natural frequencies are the same, by Theorem 5.5, solutions of system (5.15) achieves global synchronization. If there exists two different natural frequencies, by Corollary 5, for $i = 2, 3, 4$, $x_i(t)$ converge to functions with the same frequency as $x_1(t)$. Numerically, we still choose ω_i as

(i) $\omega_1 = \omega_2 = \omega_3 = \omega_4 = 1$;

(ii) $\omega_1 = 1, \omega_2 = 5, \omega_3 = 3, \omega_4 = 4$.

The simulation result for the first case is shown in Figure 5.5. It can be seen that x_1, x_2, x_3, x_4 synchronize and approach to the same sinusoidal function. The simulation result for the second case is shown in Figure 5.6. In Figure 5.6-(a), It can be seen that x_1, x_2, x_3, x_4 do not synchronize. In Figure 5.6-(b), We see that the frequencies of x_1, x_2, x_3, x_4 are the same, they all equal to 1, the natural frequency of x_1.

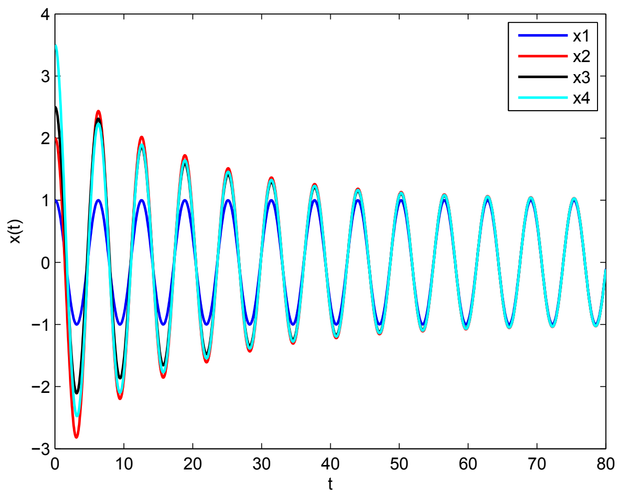
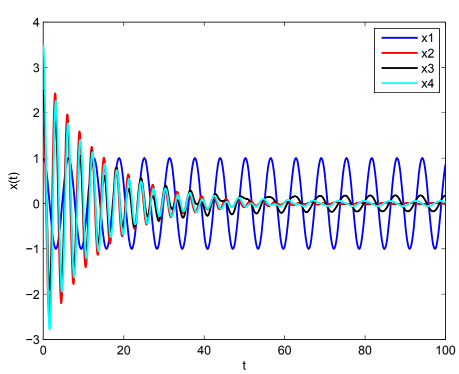
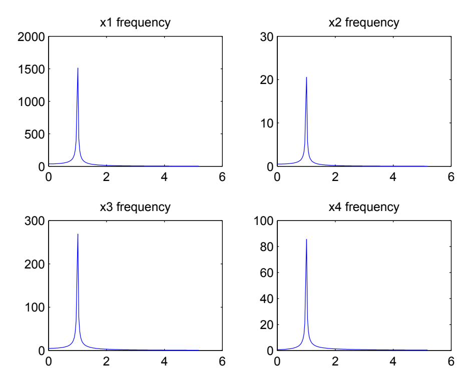


Figure 5.5: Interaction network with a rooted spanning tree and same natural frequency.



(a) Synchronization does not happen.



(b) The frequencies of the four oscillators are the same.

Figure 5.6: Interaction network with a rooted spanning tree and different natural frequency.

5.4.3 Interaction network without a rooted spanning tree

In this section, let the interaction network $\mathcal{G}_3$ be described by Figure 5.7.

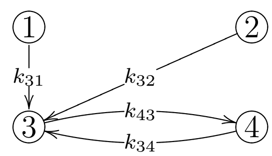
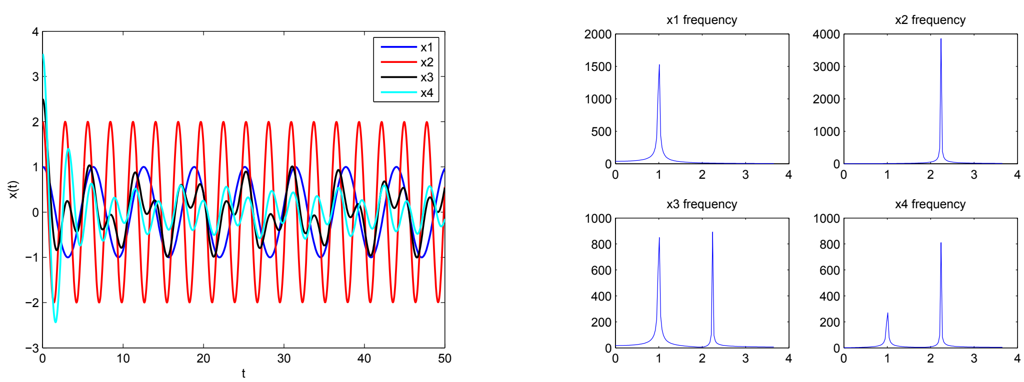


Figure 5.7: Interaction network $\mathcal{G}_3$.

It can be seen that interaction network $\mathcal{G}_2$ does not have a rooted spanning tree. There are three strongly connected components in interaction network $\mathcal{G}$, $H_1 = \{1\}, H_2 = \{2\}, H_3 = \{3, 4\}$, and $H_1 \prec H_3, H_2 \prec H_3$. By Theorem 5.8, x_3, x_4 converge to functions with the frequencies of $x_1(t)$ and $x(t)$. Numerically, we still choose ω_i as $\omega_1 = 1, \omega_2 = 5, \omega_3 = 3, \omega_4 = 4$. The simulation results are shown in Figure 5.8. In Figure 5.8-(a), It can be seen that x_1, x_2, x_3, x_4 do not synchronize. In Figure 5.8-(b), We see that the frequency modules of x_3, x_4 are the same, that is $m(x_3) = m(x_4) = m\{1, \sqrt{5}\}$.



(a) Synchronization does not happen.

(b) x_3, x_4 inherit the frequencies of x_1, x_2.

Figure 5.8: Interaction network without a rooted spanning tree.

Chapter 6

Coupled nonlinear oscillators on networks

In this chapter we investigate system (5.1) when $f_i(x_i), i = 1, 2, ..., n$ are nonlinear and identical.

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -f(x_i) + \sum_{j=1}^{n} k_{ij}(y_j - y_i), \qquad i = 1, 2, ..., n. \end{aligned} \tag{6.1}$$

In this chapter we assume that $f(x) \in C^1(\mathbb{R}), f(0) = 0, f'(x) \geq 0$.

By Theorem 5.1, we know that if the interaction network $\mathcal{G}_K$ is strongly connected, then system (6.1) achieves global synchronization. In this chapter we investigate synchronization problems of system (6.1) when the interaction network has a rooted spanning tree.

In Section 6.1, assuming the interaction network has a rooted spanning tree, we show system (6.1) achieves synchronization when coupling strength between different strongly connected components is sufficiently large. In Section 6.2, we investigate system (6.1) when coupling strength between different strongly connected components is sufficiently small, we show the existence and local stability of a non-synchronized periodic solution with small amplitude. In Section 6.3, numerical examples are given to illustrate the theoretic results in this chapter.

6.1 Synchronization of coupled nonlinear oscillators

In this section, we make the following assumption on $f(x)$ in (6.1).

(A) $f(x) \in C^1(\mathbb{R}), f(0) = 0, f'(x) \geq 0, \lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

Let $\mathcal{H}$ be the condensed graph of the interaction network $\mathcal{G}_K$. For a strongly connected component $H \in \mathcal{H}$, consider the restricted system of (6.1) on H at a function $(G_1(t), g_1(t), ..., G_n(t), g_n(t))$ with $G_i(t) \in C^1(\mathbb{R}), G_i'(t) = g_i(t), i = 1, 2, ..., n$.

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -f(x_i) + \sum_{j\in H} k_{ij}(y_j - y_i) + \sum_{j\in\mathcal{G}_K\setminus H} k_{ij}(g_j(t) - y_i), i \in H. \end{aligned} \tag{6.2}$$

Let $m = |H|$, $p_i = \sum_{j\in\mathcal{G}_K\setminus H} k_{ij}$ and $h_i(t) = \sum_{j\in\mathcal{G}_K\setminus H} k_{ij}g_j(t)$, system (5.8) becomes

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -f(x_i) + \sum_{j=1}^{m} k_{ij}(y_j - y_i) - p_i y_i + h_i(t), \qquad\qquad i = 1, 2, ..., m. \end{aligned} \tag{6.3}$$

Recall $k_{ij} \geq 0$ for $i.j = 1, 2, ..., n$, it can be seen that $p_i \geq 0$ and $h_i(t) = 0$ when $p_i = 0$. The interaction network of system (6.2) and (6.3) are strongly connected component H. Let $\tilde{K} = (k_{ij})_{m\times m}$. Then $H = \mathcal{G}_{\tilde{K}}$.

Proposition 6.1. *Suppose assumption (A) is satisfied, and there exist $i \in \{1, 2, ..., n\}$ and $M > 0$, such that $p_i > 0$ and $|h_i(t)| \leq p_i M$ for sufficiently large t. Then solutions $(x_1(t), y_1(t), ..., x_m(t), y_m(t))$ of system (6.3) are eventually bounded.*

Proof. The proof has three parts.

1. Let set I be the set of index i such that $p_i > 0$. We show $|y_i(t)|, i \in I$ are bounded for sufficiently large t. Let c_i be the cofactor of the i-th diagonal element of matrix $\tilde{K}$. Let $F(x)$ be the anti-derivative of $f(x)$, $V_i = F(x_i) + \frac{y_i^2}{2}$, and $V = \sum_{i=1}^{m} c_i V_i$. By the assumption (A), $V, V_i, i = 1, 2, ..., m$ are positive definite. Along solutions of system (6.3),

$$\dot{V}_i = \sum_{j=1}^{m} k_{ij}(y_j - y_i)y_i + h_i(t)y_i - p_i y_i^2.$$

By Theorem 2.9,

$$\dot{V} \leq \sum_{i \in I}(c_i p_i M|y_i| - c_i p_i y_i^2) \leq \frac{1}{2}\sum_{i \in I} c_i p_i (M^2 - y_i^2).$$

For $i \in I$, let $N_i = \sqrt{\frac{\sum_{j \in I} c_i p_i}{c_i p_i}} M$, and $N = \max_{i \in I} N_i$. By the definition of N, we have $N > M$. Let set $S = \{(x_1, y_1, ..., x_m, y_m) \in \mathbb{R}^{2m} : \exists i \in I,$ such that $|y_i| > N\}$. For $\forall (x_1, y_1, ..., x_m, y_m) \in S$, let $|y_{i_0}| > N$ where $i_0 \in I$, then

$$\begin{aligned}
\dot{V}\big|_{(x_1,y_1,...,x_m,y_m)} &\leq \frac{1}{2}\sum_{i \in I} c_i p_i (M^2 - y_i^2) \\
&\leq \frac{1}{2}[-c_{i_0} p_{i_0} y_{i_0}^2 + \sum_{i \in I} c_i p_i M^2] \\
&< \frac{1}{2}[-c_{i_0} p_{i_0} N^2 + \sum_{i \in I} c_i p_i M^2] \leq 0.
\end{aligned}$$

Therefore solution of system (6.3) will enter and remain in the set $\mathbb{R}^{2m} \backslash S$, that is $|y_i(t)| \leq N$ for $i \in I$ and sufficiently large t.

2. Let $I_0 = \{1, 2, ..., m\} \setminus I$. We show $|y_i(t)|$ is bounded for $i \in I_0$. For $i \in I_0$, we have

$$\dot{V}_i = \sum_{j \in I} k_{ij}(y_j - y_i)y_i + \sum_{j \in I_0} k_{ij}(y_j - y_i)y_i.$$

Let $\tilde{K}_0$ be formed by the rows and columns of $\tilde{K}$ with index in I_0. Let e_i be the cofactor of the i-th diagonal entry of $\tilde{K}_0$. Consider the Lyapunov function $V_0 = \sum_{i \in I_0} e_i V_i$. By

Theorem 2.1,

$$\dot{V}_0 \leq \sum_{i\in I_0}\sum_{j\in I} e_i k_{ij}(y_j - y_i)y_i \leq \frac{1}{2}\sum_{i\in I_0}\sum_{j\in I} e_i k_{ij}(y_j^2 - y_i^2).$$

From part 1 of the proof, $|y_j(t)| \leq N$ for $j \in I$ and sufficiently large t. Then

$$\dot{V}_0 \leq \frac{1}{2}\sum_{i\in I_0}\sum_{j\in I} e_i k_{ij}(N^2 - y_i^2).$$

For $i \in I_0, j \in I$ such that $k_{ij} \neq 0$, let $Q_i = \sqrt{\frac{\sum\limits_{i\in I_0}\sum\limits_{j\in I} e_i k_{ij}}{e_i k_{ij}}} N$, and $Q = \max_{i\in I} Q_i$. Then $Q > N$ By the same argument as in part 1, it can be shown that $|y_i(t)| \leq Q$ for $i \in Q$ and sufficiently large t.

Therefore $|y_i(t)| \leq Q$ for $i = 1, 2, ..., m$ when t is sufficiently large.

3. We show $|x_i(t)|, i = 1, 2, ...m$ are bounded for sufficiently large t. For $i = 1, 2, ..., m$, let $q_i = \sum\limits_{j=1}^{m} k_{ij} + p_i, e_i(t) = \sum\limits_{j=1}^{m} k_{ij} y_j + h_i(t)$. Since $\mathcal{G}_{\tilde{K}}$ is strongly connected, $q_i > 0$. The i-th local agent system of (6.3) is

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -f(x_i) - q_i y_i + e_i(t). \end{aligned} \tag{6.4}$$

Let $z_i = x_i' + q_i x_i$, then system (6.4) is transformed to

$$\begin{aligned} \dot{x}_i &= z_i - q_i x_i, \\ \dot{z}_i &= -f(x_i) + e_i(t). \end{aligned} \tag{6.5}$$

Let $R = \sum\limits_{j=1}^{m} k_{ij} Q + p_i M$. From part 2, for sufficiently large t, $|e_i(t)| \leq R$, and $|z_i(t)| \leq q_i|x_i(t)| + Q$. Consider the Lyapunov function $W_i = \frac{z_i^2}{2} + F(x_i)$, along the solutions of (6.5),

$$\dot{W}_i = -q_i f(x_i)x_i + e(t)z_i \leq -q_i f(x_i)x_i + R(q_i|x_i| + Q)$$

By assumption (A), $f(x_i)x_i > 0$ and approaches ∞ when $x_i \to \infty$ or $x_i \to \infty$. Therefore, there exists $S_i > 0$, such that $\dot{W}_i < 0$ when $|x_i| > S_i$. Therefore $|x_i(t)| \leq S_i$ for sufficiently large t.

$\square$

Let $b = \min_{x\in\mathbb{R}} f'(x)$, and $l(x) = f(x) - bx$, then $b \geq 0$, $l'(x) \geq 0$. Let $(x_1, y_1, ..., x_m, y_m)$, $(\tilde{x}_1, \tilde{y}_1, ..., \tilde{x}_m, \tilde{y}_m)$ be any two solutions of system (6.3), Let $u_i = \tilde{x}_i - x_i, v_i = \tilde{y}_i - y_i$. Then u_i, v_i satisfy the following equation

$$\begin{aligned} &\dot{u}_i = v_i, \\ &\dot{v}_i = -bu_i - l(u_i + x_i) + l(x_i) + \sum_{j=1}^{m} k_{ij}(v_j - v_i) - p_i v_i, \qquad i = 1, 2, ..., m. \end{aligned} \tag{6.6}$$

In system (6.6), for $\forall i$, there exists $\xi_i(t)$ in between $x_i(t)$ and $\tilde{x}_i(t)$, such that $l'(\xi_i(t))u_i = l(u_i + x_i) - l(x_i)$. System (6.6) becomes

$$\begin{aligned} &\dot{u}_i = v_i, \\ &\dot{v}_i = -bu_i - l'(\xi_i(t))u_i + \sum_{j=1}^{m} k_{ij}(v_j - v_i) - p_i v_i, \qquad i = 1, 2, ..., m. \end{aligned} \tag{6.7}$$

Now we introduce the definition detailed balanced digraph [35, 58].

Definition 8. *In a weighted directed graph $\mathcal{G}$, the weight of a cycle is defined as the product of the weights of all the arcs that form the cycle. Let $\mathcal{C}$ be a circle in $\mathcal{G}$, the reversing cycle of $\mathcal{C}$ is a cycle with all arcs of $\mathcal{C}$ reversing direction. A digraph is detailed balanced if the weight of a cycle equals the weight of its reversing cycle.*

Lemma 6.1. *A symmetric digraph is detailed balanced, the converse is not necessarily true. A detailed balanced digraph is strongly connected, the converse is not necessarily true.*

For detailed balanced graph, we have the following result.

Proposition 6.2. *[35] For detailed balanced graph, the conclusions of Theorem 2.2 hold if condition (2.5) is replaced by*

$$\sum_{(s,r)\in E(\mathcal{C})} G_{rs}(t, X_r, X_s) + G_{sr}(t, X_s, X_r) \leq 0, \qquad t > 0. \tag{6.8}$$

Theorem 6.1. *Suppose the interaction network $\mathcal{G}_{\tilde{K}}$ is detailed balanced, and*

$$p_i \geq \frac{l'(\xi_i(t))}{\sqrt{b + l'(\xi_i(t))}} \qquad t > 0, i = 1, 2, ..., m. \tag{6.9}$$

Then every solution of system (6.6) approaches to 0.

Proof. Let c_i be the i-th cofactor of the matrix $\tilde{K}$, and $q_i(t) = l'(\xi_i(t)), i = 1, 2, ..., m$. Then $q_i(t) \geq 0, i = 1, 2, ..., m$. Let $V_i = \frac{1}{2}[(p_i + \frac{2b}{p_i})u_i^2 + 2u_iv_i + \frac{2}{p_i}v_i^2] + \frac{1}{4}\sum_{i=1}^{m} k_{ij}(u_j - u_i)^2$, and $V = \sum_{i=1}^{m} c_iV_i$. Then

$$\begin{aligned} \dot{V}_i \leq & -u_i^2 - \frac{2}{p_i}q_iu_iv_i - (b + q_i)v_i^2 + \sum_{i=1}^{m} k_{ij}(v_j - v_i)v_i \\ & + \sum_{i=1}^{m} k_{ij}(v_j - v_i)u_i + \frac{1}{2}\sum_{i=1}^{m} k_{ij}(v_j - v_i)(u_i - u_j) \end{aligned}$$

If $p_i \geq \frac{l'(\xi_i(t))}{\sqrt{b+l'(\xi_i(t))}}$, it can be verified that the quadratic form $-u_i^2 - \frac{2}{p_i}q_iu_iv_i - (b+q_i)v_i^2 \leq 0$ and equal sign holds only when $u_i, v_i = 0$. Thus

$$\begin{aligned} \dot{V}_i &\leq \sum_{i=1}^{m} k_{ij}(v_j - v_i)v_i + \sum_{i=1}^{m} k_{ij}(v_j - v_i)u_i + \frac{1}{2}\sum_{i=1}^{m} k_{ij}(v_j - v_i)(u_i - u_j) \\ &= \sum_{i=1}^{m} k_{ij}(v_j - v_i)v_i + \frac{1}{2}\sum_{i=1}^{m} k_{ij}(v_j - v_i)(u_i + u_j) \end{aligned} \tag{6.10}$$

By (6.10) and Proposition 6.2, we can verify that $\dot{V} \leq 0$ and $\dot{V} = 0$ only when $u_i, v_i = 0, i = 1, 2, ..., n$. Therefore every solution of system (6.6) approaches to 0. □

Remark 1. *By Proposition 6.1, $x_i, \tilde{x}_i$ are eventually bounded for $i = 1, 2, ..., n$, then $l'(\xi_i(t))$ is eventually bounded, condition 6.9 can be satisfied if constants $p_i, i = 1, 2, ..., m$*

are sufficiently large.

Corollary 6. *Suppose $G_i(t) = G(t)$ for $i \in H$, and $G(t)$ solves the equation $\ddot{x} = -f(x)$, and $\mathcal{G}_{\tilde{K}}$ is detailed balanced. For every solution $(x_1, y_1, ..., x_m, y_m)$ of system (6.2), if condition (6.9) is satisfied, $\lim_{t\to\infty} x_i(t) - G(t) = 0, i = 1, 2, .., m$.*

Proof. If $G(t)$ solves the equation $\ddot{x} = -f(x)$, then $x_i = G(t), i = 1, 2, ..., m$ is a solution of system (6.2). By Theorem 6.1, the difference of any two solutions of system (6.2) approaches 0, therefore $\lim_{t\to\infty} x_i(t) - G(t) = 0, i = 1, 2, .., m$ for every solution $(x_1, y_1, ..., x_m, y_m)$. □

Definition 9. *Consider system of coupled nonlinear oscillators (6.1) on an interaction network $\mathcal{G}$. For $i \in \mathcal{G}$, assume vertex i is in component H. Define the inter-components coupling strength of i as $p_i = \sum_{j\in\mathcal{G}\setminus H} k_{ij}$.*

The following Theorem shows system (6.1) achieves global synchronization when the inter-components coupling strengths are sufficiently large.

Theorem 6.2. *Consider system of coupled nonlinear oscillators (6.1) on an interaction network $\mathcal{G}_K$. Suppose $\mathcal{G}_K$ has a rooted spanning tree. Let $\mathcal{H}$ be the condensed graph of $\mathcal{G}$, let H_0 be the minimal element of $\mathcal{H}$ with respect to the partial order $'\prec'$. Further assume*

(i) condition (A) is satisfied;

(ii) for strongly connected component $H \in \mathcal{H} \setminus \{H_0\}$, H is detail balanced.

Then for $i = 1, 2, ..., n$, there exists constant p_i^0, such that when the inter-components coupling strength $p_i > p_i^0$, system (6.1) achieves global synchronization.

Proof. Firstly we assume $\mathcal{G}$ only have two strongly connected components, H_0 and H_1, and $H_0 \prec H_1$. H_0 is the minimal element, by Corollary 2, solutions of H_0 subsystem synchronize. That is, there exists function $G(t)$, such that $\lim_{t\to\infty} x_i(t) - G(t) = 0$ for $i \in H_0$.

Let $g(t) = G'(t)$, the the H_1 subsystem is

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -f(x_i) + \sum_{j \in H_1} k_{ij}(y_j - y_i) + \sum_{j \in H_0} k_{ij}(y_j - y_i), i \in H_1. \end{aligned} \tag{6.11}$$

By theorem 2.3, solutions of (6.11) converges to the attractor of

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -f(x_i) + \sum_{j \in H_1} k_{ij}(y - y_i) + \sum_{j \in H_0} k_{ij}(g(t) - y_i), i \in H_1. \end{aligned} \tag{6.12}$$

By Proposition 6.1, $x_i(t)$ is eventually bounded for every $i \in H_1$. Thus there exists constant N, such that $l'(x(t)) \leq N$ when t is sufficiently large. Let $p_i^0 = \sqrt{N}$, when $p_i > p_i^0$, condition (6.9) is satisfied. By Corollary 6, $\lim\limits_{t \to \infty} x_i(t) - G(t) = 0$ for $i \in H_1$, system (5.1) achieves global synchronization.

For more general interaction network $\mathcal{G}$, the theorem can be proved by induction, as in the proof of Theorem 5.5. □

System (6.1) is a special form of system (1.6) in Chapter 1. Recall system (1.6) is

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -f_i(x_i) + u_i, \qquad\qquad i = 1, 2, ..., n, \end{aligned}$$

where $u_i = \sum\limits_{j=1}^{n} h(x_i, y_i, x_j, y_j)$ specifies the coupling protocol of the i-th oscillator. Previously, Synchronization behaviors were shown to exist when the coupling protocol u_i was assumed to be $\sum\limits_{j=1}^{n} k_{ij}(y_j - y_i) + k_{ij}(x_j - x_i)$ or $\sum\limits_{j=1}^{n} k_{ij}(x_j - x_i) - p_i y_i$ [50,61,62]. In this chapter, we studied system (1.6) under the coupling protocol $u_i = \sum\limits_{j=1}^{n} k_{ij}(y_j - y_i)$, to the best of our knowledge, this is the first synchronization result under this kind of coupling protocol.

6.2 Non-synchronization when inter-component coupling strength is sufficiently small

By Theorem 6.2, when the inter-component coupling strength is sufficiently large, system (5.1) achieves global synchronization.

In this section we investigate system (6.1) when the inter-component coupling strength is sufficiently small. For better understanding, we firstly investigate the case of two coupled nonlinear oscillators in section 6.2.1. More general coupled nonlinear oscillators are studied in section 6.2.2.

6.2.1 The case of two oscillators

Consider the following coupled system of two oscillators,

$$\begin{aligned} \dot{x}_1 &= y_1, \\ \dot{y}_1 &= -f(x_1), \\ \dot{x}_2 &= y_2, \\ \dot{y}_2 &= -f(x_2) + k_{21}(y_1 - y_2). \end{aligned} \tag{6.13}$$

Assume $k_{12} > 0$, $f \in C^1(\mathbb{R})$, and $f'(x) \geq 0, x \in \mathbb{R}$. The interaction network for system (6.13) is

$$\textcircled{1} \xrightarrow{k_{21}} \textcircled{2},$$

representing that oscillator 1 is a free oscillator, and x_2 is damped driven by x_1.

For every initial condition $(x_1^0, y_1^0, x_2^0, y_2^0)$, we can firstly solve $x_1(t), y_1(t)$. Then plug $y_1(t)$ into the equations of oscillator 2. The equations of oscillator x_2 are

$$\begin{aligned} \dot{x}_2 &= y_2, \\ \dot{y}_2 &= -f(x_2) + k_{21}(y_1 - y_2). \end{aligned} \tag{6.14}$$

Let $x = x_2(t), y(t) = y_2(t), k = k_{21}, g(t) = y_1(t)$, system (6.14) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -f(x) + k(g(t) - y). \end{aligned} \tag{6.15}$$

With the assumption on function $f(x)$, the solutions of $\ddot{x}_1 = f(x_1)$ are periodic. Thus $g(t)$ is a periodic function. Let $b = f'(0)$, $l(x) = f(x) - bx$, then $l(x) \in C^1(\mathbb{R})$ and $l'(0) = 0$. Equation (6.15) can be written as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -bx - l(x) + k(g(t) - y). \end{aligned} \tag{6.16}$$

Let $\mathcal{P}_T$ denote the set of periodic functions of period T, $\mathcal{B}_\delta^d = \{x(t) \in \mathbb{C}(\mathbb{R}, \mathbb{R}^d), \|x(t)\| \leq \delta\}$. For $\delta > 0$, let $\eta(\delta) = max_{(x,y)\in\mathcal{B}_\delta^2} |l'(x)|$. It can be seen that $\eta(\delta) = max_{|x|\leq\delta} |l'(x)|$ and $\lim_{\delta\to 0} \eta(\delta) = 0$. The following theorem is used in this section.

Theorem 6.3. *[22] Let $x \in \mathbb{R}^n$, $f \in \mathcal{P}_T \bigcap C(\mathbb{R})$ and A is an $n \times n$ matrix. Suppose the only solution of $\dot{x} = Ax$ that belongs to $\mathcal{P}_T$ is $x = 0$, then system $\dot{x} = Ax + f(t)$ has an unique solution $x_f \in \mathcal{P}_T$ for every $f(t)$, and there is a constant C such that $|x_f| \leq C|f|$ for all $f \in \mathcal{P}_T \bigcap C(\mathbb{R})$.*

Theorem 6.4. *Assume $g(t) \in \mathcal{P}_T$, and $b \neq \left(\frac{2\pi}{T}\right)^2$. Then there exist sufficiently small $k_0 > 0$, $\delta > 0$ and an unique function $(x^*(t,k), (x^*)'(t,k)) \in \mathcal{B}_\delta^2 \bigcap \mathcal{P}_T$ that solves (6.16), for every $k \in (0, k_0]$.*

Proof. The theorem can be proved with similar argument as in the proof of Theorem 2.1 in [22] Section 4.2. Let $X = (x,y)^T$, $A = \begin{pmatrix} 0 & 1 \\ -b & 0 \end{pmatrix}$, $q(k, X, t) = (0, -l(x) + kg(t) - ky)^T$, write equation (6.16) as

$$\dot{X} = AX + q(t, X, k) \tag{6.17}$$

By Theorem (6.3), for every $Z(t) = (Z_1(t), Z_2(t))^T \in \mathcal{P}_T^2$ and $k \geq 0$, equation

$$\dot{X} = AX + q(t, Z, k) \tag{6.18}$$

has an unique solution $X(t, Z, k) \in \mathcal{P}_T^2$,

$$\|X(t, Z, k)\| \leq C\|q(t, Z, k)\| \leq C(|l(Z_1)| + k|g(t)| + k|Z_2|). \tag{6.19}$$

Since $g(t) \in \mathcal{P}_T$, $g(t)$ is bounded for $t \in \mathbb{R}$, let $M = sup_{t\in\mathbb{R}}|g(t)|$. Let $\mathcal{T} : \mathcal{P}_T \mapsto \mathcal{P}_T$ be a transform defined as $\mathcal{T}(Z) = X(t, Z, k)$. Choose k_0, δ sufficiently small such that

$$C(\eta(\delta)\delta + k_0(M + \delta)) < \delta. \tag{6.20}$$

For $Z, \tilde{Z} \in \mathcal{B}_\delta^2 \bigcap \mathcal{P}_T$, $k \in [0, k_0]$, by (6.20), we have

$$\|\mathcal{T}Z\| \leq C\|q(t, Z, d)\| \leq C(\eta(\delta)\delta + kM + k\delta) \leq C(\eta(\delta)\delta + k_0(M + \delta)) < \delta, \tag{6.21}$$

and

$$\begin{aligned}\|\mathcal{T}Z - \mathcal{T}\tilde{Z}\| = \|\mathcal{T}(Z - \tilde{Z})\| &\leq C\|q(t, Z, k) - q(t, \tilde{Z}, k)\| \\ &= C|l(Z_1) - l(\tilde{Z}_1)| + k_0|Z_2 - \tilde{Z}_2| \\ &\leq C(\eta(\delta) + k_0)\delta \leq \theta < 1.\end{aligned} \tag{6.22}$$

By (6.21) and (6.22), the transform $\mathcal{T}$ is a uniform contraction from $\mathcal{B}_\delta^2 \bigcap \mathcal{P}_T$ to $\mathcal{B}_\delta^2 \bigcap \mathcal{P}_T$ when $k \leq k_0$. Therefore it has a unique fixed point $(x^*(t, k), (x^*)'(t, k)) \in \mathcal{B}_\delta^2 \bigcap \mathcal{P}_T$, and $x^*(t)$ solves equation (6.16). □

Theorem 6.4 establishes the existence of a periodic solution $x^*(t, k)$ for sufficiently small k, its period is the same with the driving force $g(t)$. Next we investigate the stability of $x^*(t, k)$. Let $x(t)$ be a solution of equation (6.16) other than $x^*(t, k)$, let $u(t) = x(t) -$

$x^*(t,k)$, then $u(t)$ satisfies the following differential equation

$$\begin{aligned}\dot{u} &= u,\\ \dot{v} &= -bu - l(x^*(t)+u) + l(x^*(t)) - kv.\end{aligned} \tag{6.23}$$

For $t \in \mathbb{R}$, there exists $\xi(u(t), x^*(t))$, such that $l(x^*(t)+u) - l(x^*(t)) = l'(\xi(u(t), x^*(t)))u$. Let $h(t,u) = l'(\xi(y(t), x^*(t)))$. Equation (6.23) can be written as

$$\begin{aligned}\dot{u} &= u,\\ \dot{v} &= -bu - h(t,u)u - kv.\end{aligned} \tag{6.24}$$

Theorem 6.5. *Assume $b > 0$, $l'(x) \geq 0$ for $x \in \mathbb{R}$, and $\|(x^*(t), (x^*)'(t))\| \leq \mu$. Then $\forall \epsilon > 0$, there exists $k(\epsilon, \mu) > 0$, such that when $k > k(\epsilon, \mu)$, solution $u(t), v(t)$ of system (6.24) approach zero if the initial condition $\|(u(0), v(0))\| < \epsilon$.*

Proof. Consider the Lyapunov function $V(u,v) = \frac{1}{2}[(k + \frac{2b}{k})u^2 + 2uv + \frac{2}{k}v^2]$. It can be verified that $V(u,v)$ is positive definite, and

$$\dot{V} = -u^2 - \frac{2}{k}huv - (b+h)v^2.$$

Therefore $\dot{V} < 0$ when

$$k > \frac{h(t,u)}{\sqrt{b + h(t,u)}}. \tag{6.25}$$

Note that $h(t,u) \geq 0$ since $l'(x) \geq 0$. For a constant $C > 0$, the curve $V(u,v) = C$ is an ellipse E_C. Given $\epsilon > 0$, we can find the minimal ellipse E_{C_1} that encloses $\mathcal{B}^2_\epsilon$, and the minimal disk $\mathcal{B}^2_{\epsilon_1}$ that encloses the ellipse E_{C_1}. Ellipse E_{C_1}, disks $\mathcal{B}^2_\epsilon, \mathcal{B}^2_{\epsilon_1}$ are described in Figure 6.1.

Let $\epsilon_2 = \epsilon_1 + \mu$, then $\|(u(0) + x^*(0), v(0) + (x^*)'(0))\| \leq \epsilon + \mu < \epsilon_1 + \mu = \epsilon_2$. Recall that $\eta(\delta) = max_{|x| \leq \delta}|l'(x)|$, let $k(\epsilon, \mu) = \frac{\eta(\epsilon_2)}{\sqrt{b}}$, then it can be seen for $k > k(\epsilon, \mu)$, (6.25) is satisfied when $\|(u(t), v(t))\| < \epsilon_1$ for all t.

Suppose $\|(u(0), v(0))\| < \epsilon$, let $C(t) = V(u(t), v(t))$, this means that the point

$(u(0), v(0))$ is on the the ellipse $E_{C(0)}$. By the definition of the disk $\mathcal{B}^2_{\epsilon_1}$, the point $(u(0), v(0))$ is inside the disk $\mathcal{B}^2_{\epsilon_1}$. When $k > k(\epsilon, \mu)$, $\dot{V}|_{t=0} < 0$. As time evolves, the point $(u(t), v(t))$ will be on the shrinking ellipse $E_{C(t)}$, then $(u(t), v(t))$ will be always in the interior of the disk $\mathcal{B}^2_{\epsilon_1}$, therefore (6.25) is always satisfied, $\dot{V}(t) < 0$ for all $t > 0$, thus $u(t), v(t)$ approach to 0. □

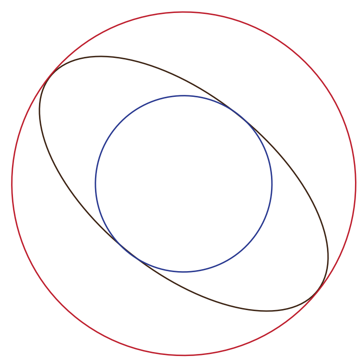


Figure 6.1: The small disk is $\mathcal{B}^2_{\epsilon}$, the ellipse is E_{C_1} and the large disk is $\mathcal{B}^2_{\epsilon_1}$.

Remark 2. *In the proof of Theorem 6.5, the ellipse mentioned has the form $Au^2 + 2uv + Bv^2 = const$, where $A = k + \frac{2b}{k}, B = \frac{2}{k}$. Note that when k is sufficiently small, A and B are both sufficiently large, the rate of the major axis and minor axis approaches to the maximal of $\frac{A}{B}$ and $\frac{B}{A}$, that is the maximal of b and $\frac{1}{b}$. This implies $\frac{\epsilon_1}{\epsilon}$ is bounded.*

Note that in equation (6.23) the solution $x^*(t)$ is an arbitrary solution, if we apply Theorem 6.5 to show the stability of the periodic solution $x^*(t, k)$, the stability of $x^*(t, k)$ can be derived.

Theorem 6.6. *Suppose $g(t) \in \mathcal{P}_T$, $b \neq \left(\frac{2\pi}{T}\right)^2$ and $b > 0$. For function $l(x)$, assume $l'(x) \geq 0$ for $x \in \mathbb{R}$ and $\lim\limits_{x \to 0} \frac{l'(x)}{x} = 0$. Then there exists sufficiently small $k_0 > 0$, such that for every $k \in (0, k_0]$, there exist $\delta > 0$ and a unique function $(x^*(t, k), (x^*)'(t, k)) \in \mathcal{B}^2_\delta \bigcap \mathcal{P}_T$ that solves (6.16), and it is locally asymptotically stable. Let ζ denote the orbit of $(x^*(t, k), (x^*)'(t, k))$, the attraction region is $\{(x, y) \in \mathbb{R}^2, d((x, y), \zeta) < \delta\}$.*

Proof. The key to prove this theorem is to find k, such that both (6.20) and (6.25) are satisfied.

Define the ellipse E_C as in the proof of Theorem 6.6. Given $\delta > 0$, there exists $C_1 > 0$, such that E_{C_1} is the minimal ellipse that encloses the disk $\mathcal{B}^2_\delta$, and there exists $\gamma_1 > 0$, such that the disk $\mathcal{B}^2_{\gamma_1}$ is the minimal disk that encloses the ellipse E_{C_1}. Let $\gamma(\delta) = \gamma_1 + \delta$, and $k(\delta) = \frac{\eta(\gamma(\delta))}{\sqrt{b}}$. It can be seen that $\gamma(\delta) = O(\delta)$.

By our assumption, we have $\lim_{\delta\to 0} \frac{\eta(\gamma)\delta}{\delta} = 0$ and $\lim_{\delta\to 0} \frac{k(\delta)(M+\delta)}{\delta} = \lim_{\delta\to 0} M\frac{k(\delta)}{\delta} + k(\delta) = 0$. Therefore there exists δ_0, such that for $\delta \in (0, \delta_0]$.

$$C(\eta(\delta)\delta + k(\delta)(M + \delta)) < \delta. \tag{6.26}$$

Let $k_0 = k(\delta_0)$, for every $k \in (0, k_0]$, there exists $\delta \in (0, \delta_0]$, such that $k = k(\delta)$. By Theorem 6.4, there exists periodic solution $x^*(t, k)$ and $\|(x^*(t, k), (x^*, k)'(t))\| \leq \delta$. Notice that $k(\delta) > \frac{\eta(\gamma(\delta))}{\sqrt{\eta(\gamma(\delta))+b}}$, by Theorem 6.5, $x^*(t, k)$ is asymptotically stable, and the attraction region is $\{(x, y) \in \mathbb{R}^2, d((x, y), \zeta) < \delta\}$. □

In Theorem 6.6, the assumptions on $f(x)$ are satisfied by various functions, one class of examples is functions $f(x) = a_1x + a_3x^3 + a_5x^5 + \cdots$ with $a_i > 0, i = 1, 3, 5, \ldots$.

6.2.2 The case of multiple oscillators

In this section we consider system (6.1), the general system of coupled nonlinear oscillators. Suppose H is a strongly connected component of the interaction network $\mathcal{G}_K$, with m vertices, the restricted system on H at a function $(G_1(t), g_1(t), \ldots, G_n(t), g_n(t))$ is given by (6.3). Recall $b = f'(0), l(x) = f(x) - bx$. System (6.3) can be written as

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -b - l(x_i) + \sum_{j=1}^{m} k_{ij}(y_j - y_i) - p_i y_i + h_i(t), \qquad i = 1, 2, \ldots, m, \end{aligned} \tag{6.27}$$

where $p_i = \sum\limits_{j \in \mathcal{G}_K \setminus H} k_{ij} \geq 0$ and $h_i(t) = \sum\limits_{j \in \mathcal{G}_K \setminus H} k_{ij} g_j(t)$.

Theorem 6.7. *Assume $h_i(t) \in \mathcal{P}_T, i = 1, 2, ..., n,$ and $b \neq \left(\frac{2\pi}{T}\right)^2$. Then there exist sufficiently small $p_0 > 0$, $\delta > 0$ and a unique function*

$$W^*(t) = (x_1^*(t), (x_1^*)'(t), ..., x_m^*(t), (x_m^*)'(t)) \in \mathcal{B}_\delta^{2m} \bigcap \mathcal{P}_T$$

that solves (6.27), for all $0 < p_i \leq p_0, i = 1, 2, ..., n$.

It can be seen that $\lim\limits_{p_i \to 0} h_i(t) = 0$. Write system (6.27) in vector form, as system (6.17), then Theorem 6.7 can be proved in the same way as Theorem 6.4. Theorem 6.7 shows the existence of the perturbed periodic solution $W^*(t)$ of the restricted system (6.27). With the same arguments as in Theorem 6.5 and Theorem 6.6, we can show the stability of $W^*(t)$.

Theorem 6.8. *For equation (6.27), suppose $h_i(t) \in \mathcal{P}_T, i = 1, 2, ..., n$, $m \neq \left(\frac{2\pi}{T}\right)^2$, $b > 0$ and the strongly connected component H is detailed balanced. For function $l(x)$, assume $l'(x) \geq 0$ for $x \in \mathbb{R}$ and $\lim\limits_{x \to 0} \frac{l'(x)}{x} = 0$. Then there exist sufficiently small p_0, such that for all constants $0 < p_i \leq p_0, i = 1, 2, ..., n$, there exists $\delta > 0$ and a unique function $W^*(t) \in \mathcal{B}_\delta^{2m} \bigcap \mathcal{P}_T$ that solves (6.27), and it is locally asymptotically stable. Let ζ denote the orbit of $W^*(t, P)$, the attraction region is $\{W \in \mathbb{R}^{2m}, d(W, \zeta) < \delta\}$.*

Note that to prove Theorem 6.8, we can use the Lyapunov function given in the proof of Theorem 6.1.

In system (6.1), let $\mathcal{H}$ be the condensed graph of $\mathcal{G}_K$. For $i = 1, 2, ..., n$, suppose $i \in H$, recall $p_i = \sum\limits_{j \in \mathcal{G}_K \setminus H} k_{ij} \geq 0$ is the inter-components coupling strength of vertex i, as defined in Definition 9. For a solution $W(t) = (x_1(t), y_1(t), ..., x_n(t), y_n(t))$ of system (6.1), and a strongly connected component H, let $W_H(t) = (x_i(t), y_i(t))_{i \in H}$. We have the following theorem.

Theorem 6.9. *In (6.1), suppose the interaction network $\mathcal{G}_K$ has a rooted spanning tree, let H_0 be the minimal element of $\mathcal{H}$ with respect to the partial order $' \prec'$. Assume*

(i) $f(x) \in C^1(\mathbb{R}), f'(x) \geq 0, x \in \mathbb{R}$, $f'(0) > 0$ *and* $f''(0) = 0$;

(ii) for strongly connected component $H \in \mathcal{H} \setminus \{H_0\}$, *$H$ is detail balanced;*

Then

(a) *system (6.1) synchronize in component H_0, and $x_i(t), i \in H_0$ approaches to a function in $\mathcal{P}_T$ for some $T > 0$;*

(b) *for every $H \in \mathcal{H} \setminus \{H_0\}$, there exists sufficiently small p_H, such that if the inter-components coupling strength $0 < p_i \leq p_H$ for all $i \in H$, there exist small constants δ_H and a solution $W^*(t) \in \mathcal{P}_T$ of system (6.1), with $|W^*_H(t)| \leq \delta_H$.*

Theorem 6.9 can be proved by applying Theorem 6.8 to the restricted system on $H \neq H_0$, following the partial order $' \prec'$. We briefly describes the process of the proof when the interaction network $\mathcal{G}_K$ has two strongly connected components. Suppose $\mathcal{H} = \{H_0, H_1\}$, and $H_0 \prec H_1$. By Corollary 2, W_{H_0} synchronizes, and $x_i(t), y_i(t), i \in H_0$ approaches to a function $p_0(t)$ in $\mathcal{P}_T$ for some $T > 0$. By Theorem 6.8, there exists a locally asymptotically stable periodic solution $p_1(t)$ of the restricted system on H_1 at $p_0(t)$. Then $W^*_{H_0}(t) = p_0(t)$ and $W^*_{H_1}(t) = p_1(t)$.

Recall in Theorem 6.2, we show that system (6.1) achieves synchronization when the inter-components coupling strength is sufficiently large. Now with Theorem 6.9, we show the behavior of system (6.1) when the inter-components coupling strength is sufficiently small. Then what happens when the the inter-components coupling strength is neither sufficiently large nor sufficiently small? This question is numerically investigated in Section 6.3.3 with a bifurcation diagram.

6.3 Numerical simulations

In this section we provide several numerical examples to illustrate the results in this chapter. We consider four coupled nonlinear oscillators

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -f_i(x_i) + \sum_{j=1}^{4} k_{ij}(y_j - y_i), \qquad i = 1,2,3,4. \end{aligned} \tag{6.28}$$

6.3.1 Strongly connected interaction network

In this section, let the interaction network $\mathcal{G}_1$ be described by Figure 6.2.

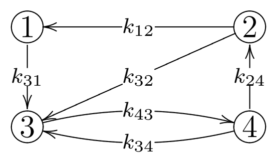


Figure 6.2: Strongly connected interaction network $\mathcal{G}_1$.

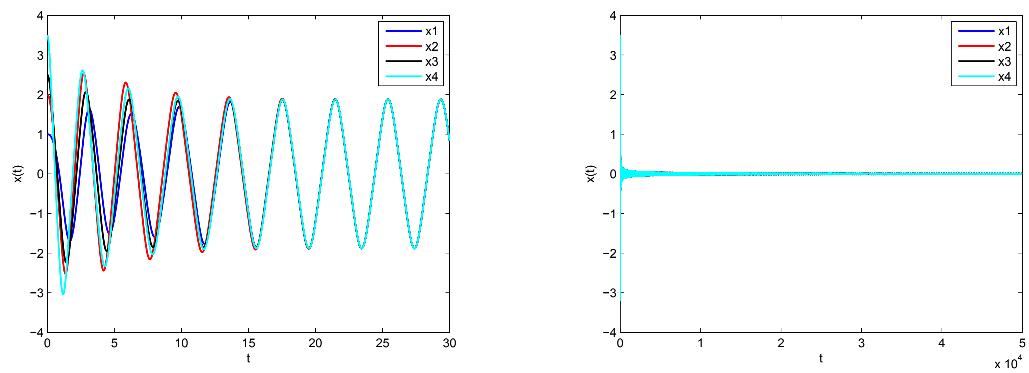
It can be seen that Interaction network $\mathcal{G}_1$ is strongly connected. In system (6.28), we consider the following two cases.

(i) $f_i(x_i) = x_i^3$, for $i = 1,2,3,4$.

(ii) $f_i(x_i) = \alpha_i x_i^3$, where $\alpha_1 = 1, \alpha_2 = 5, \alpha_3 = 3, \alpha_4 = 4$.

Coupled nonlinear oscillators with strongly connected interaction network is studied in section 5.1. By Theorem 5.1, for all the three cases, system (6.28) achieves global synchronization under strongly connected interaction network, that is $\lim\limits_{t\to\infty} \dot{x}_i(t) - \dot{x}_j(t) = 0$ for $i,j = 1,2,3,4$. For the first case, by Corollary 2, $\lim\limits_{t\to\infty} x_i(t) - x_j(t) = 0$ for $i,j = 1,2,3,4$, and they all approach to an orbit of $\ddot{x} + x^3 = 0$. For the second case, for $i,j = 1,2,3,4$, let $C_{ij} = \lim\limits_{t\to\infty} x_i(t) - x_j(t)$, then we have $\lim\limits_{t\to\infty} f_i(x_j(t) + C_{ij}) - f_j(x_j(t)) = 0$, it is clear that $x_j(t)$ approaches 0 for $j = 1,2,3,4$.

The simulation results are shown in Figure 6.3. Case 1 shown in Figure 6.3-(a), x_1, x_2, x_3, x_4 approach to the same periodic function. Case 2 is shown in Figure 6.3-(b), x_1, x_2, x_3, x_4 approach to 0, note here the time it takes for solution to approach 0 is huge, the reason is when x_i is small, $f_i(x_i)$ is the third order infinitesimal of x_i, then $f_i(x_j(t) + 0) - f_j(x_j(t))$ is very small.



(a) case1 (b) case2

Figure 6.3: Strongly connected interaction network case 1 and 2.

6.3.2 Interaction network with a rooted spanning tree

In section, let $f_i(x_i) = x_i + x_i^3, i = 1, 2, 3, 4$. Let the interaction network $\mathcal{G}_2$ be described by Figure 6.4.

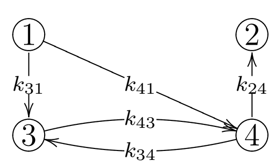


Figure 6.4: Interaction network $\mathcal{G}_2$ with a rooted spanning tree.

It can be seen that Interaction network $\mathcal{G}_2$ has a rooted spanning tree. There are three strongly connected components in interaction network $\mathcal{G}_2$, $H_1 = \{1\}, H_2 = \{3, 4\}, H_3 = \{2\}$, and $H_1 \prec H_2 \prec H_3$. The inter-component coupling strength of vertices in H_2 is $p_3 = k_{31}, p_4 = k_{41}$; the inter-component coupling strength of vertices in H_3 is $p_2 = k_{24}$.

By Theorem 6.2, when the inter-component coupling strength is sufficiently large, system (6.28) achieves global synchronization. By Theorem 6.9, when the inter-component coupling strength is sufficiently small, x_2, x_3, x_4 will converge to periodic orbits with small amplitudes. We consider the following three cases.

(i) $k_{31} = k_{41} = k_{34} = k_{43} = k_{24} = 1$.

(ii) $k_{31} = k_{41} = 0.5, k_{34} = k_{43} = k_{24} = 0.1$.

(iii) $k_{31} = k_{41} = k_{34} = k_{43} = k_{24} = 0.1, k_{41} = 0.01$.

The simulation results are shown in Figure 6.5, Figure 6.6 and Figure 6.7. Case 1 shown in Figure 6.5, oscillators x_1, x_2, x_3, x_4 synchronize, since p_2, p_3, p_4 are all sufficiently large. Case 2 is shown in Figure 6.6, oscillators x_1, x_3, x_4 synchronize while x_2 approaches to a periodic orbit with small amplitude, this is because the inter-component coupling strength p_3, p_4 is large while p_2 is small. Case 3 is shown in Figure 6.7, oscillators x_2, x_3, x_4 all approach to periodic orbits with small amplitudes. The reason is p_2, p_3, p_4 are all sufficiently small.

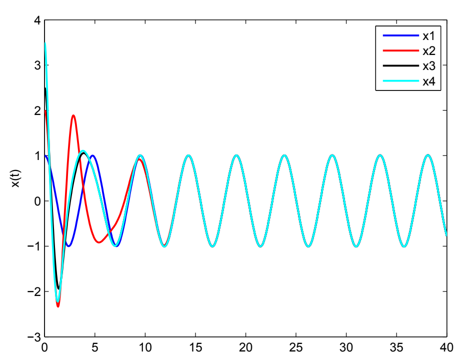


Figure 6.5: Interaction network with a rooted spanning tree, case 1.

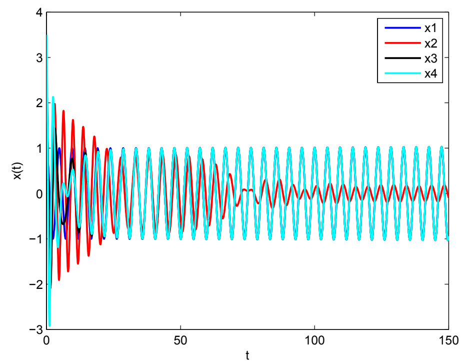


Figure 6.6: Interaction network with a rooted spanning tree, case 2.

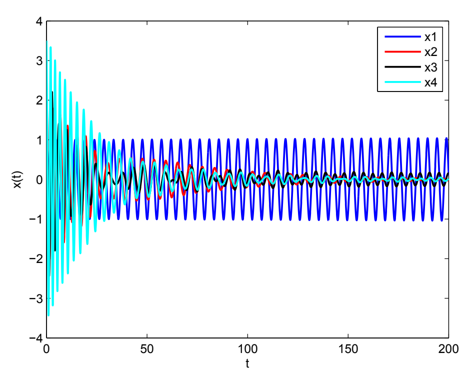


Figure 6.7: Interaction network with a rooted spanning tree, case 3.

6.3.3 Bifurcation diagram of two coupled oscillators

As stated in the end of Section 6.2, in this section we numerically investigate the dynamical behavior of (6.1) when the inter-component coupling strength is neither sufficiently large nor sufficiently small. For simplicity, we consider the following system of two coupled oscillators,

$$\begin{aligned}\dot{x}_1 &= y_1,\\ \dot{y}_1 &= -x_1 - x_1^3,\\ \dot{x}_2 &= y_2,\\ \dot{y}_2 &= -x_2 - x_2^3 + k(y_1 - y_2).\end{aligned} \tag{6.29}$$

Oscillator 1 is a free oscillator, without loss of generality, we assume the initial condition of x_1 oscillator is $(1, 0)$, thus x_1 oscillates with amplitude 1. We focus on the dynamics of oscillator 2.

$$\begin{aligned}\dot{x}_2 &= y_2,\\ \dot{y}_2 &= -x_2 - x_2^3 + k(y_1 - y_2).\end{aligned} \tag{6.30}$$

When k is sufficiently large, by Theorem 6.2, system 6.29 achieves global synchronization, thus the amplitude of oscillator x_2 is also 1. When k is sufficiently small, by Theorem 6.12, a stable periodic solution of (6.30) with a small amplitude exists. Therefore there must exist threshold value $k_0 > 0$, such that for $k \geq k_0$, the small periodic orbit no longer exists. One natural question arises: when k increases continuously from small to large, how do the dynamics of system (6.30) change? Figure 6.8 is the bifurcation diagram of system (6.30), it shed some light on this question.

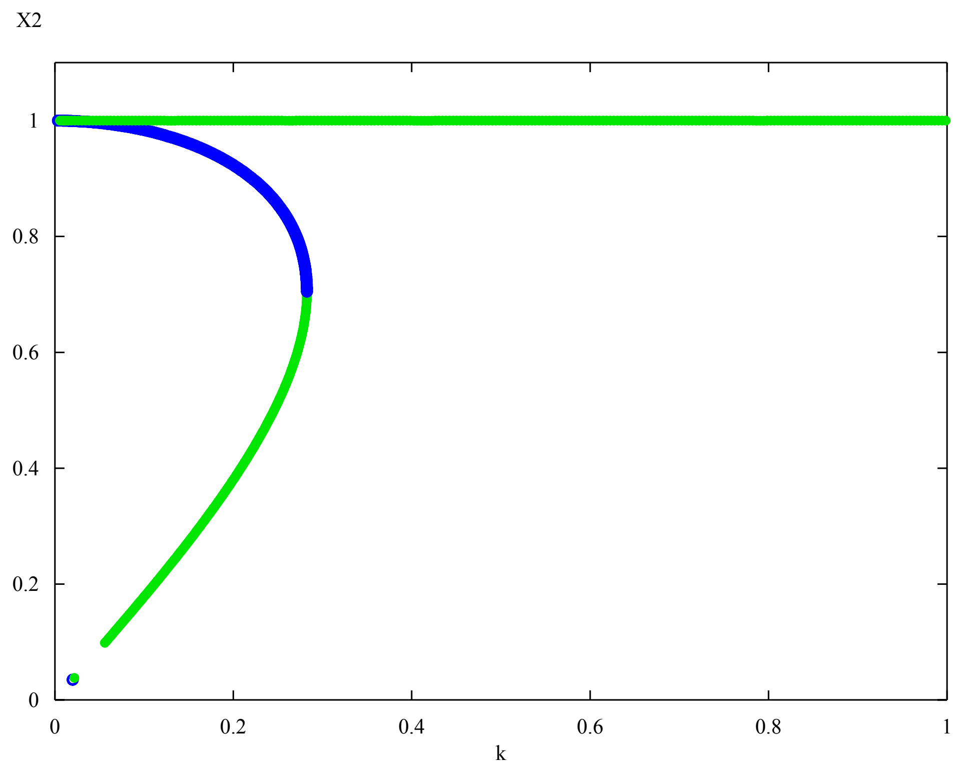


Figure 6.8: The bifurcation diagram of system (6.30).

In Figure 6.8, the horizonal axis describes the coupling strength k, the vertical axis describes the amplitude of periodic orbit of (6.30) for corresponding k. The light green curve represents stable periodic orbit, while dark blue curve represents unstable periodic orbit. It can be seen that there is a turning point at $k_0 \approx 0.3$. When $k < k_0$, there are three periodic solutions of (6.30): the synchronized orbit $x_1(t)$, the stable periodic orbit with small amplitude and an unstable periodic orbit in between of them, when $k > k_0$, system (6.29) achieves global synchronization. $x_2(t)$ approaches to $x_1(t)$, which is periodic function with amplitude 1. The small periodic orbit and the unstable periodic orbit no longer exist.

Chapter 7

Future directions

In the thesis we study coupled gradient systems and coupled oscillators on arbitrary interaction networks, we show the impact of network connectivity on their dynamical behavior. For these two types of systems, we show synchronization occurs when the interaction network is strongly connected, while clustered behavior is expected when the interaction network is not strongly connected. In the case of clustered behavior, the local agent systems in the same strongly connected component are in the same dynamical clusters.

Many mathematical problems regarding coupled dynamical systems can be further investigated. For the future research on this subject, I plan to investigate the following problems.

- Study coupled dynamical systems with multiple interaction networks. For example, in system (1.6), let $u = \sum_{j=1}^{n} k_{ij}(y_j - y_i) + l_{ij}(x_j - x_i)$, $K = (k_{ij}), L = (l_{ij})$. Then there are two independent interaction networks $\mathcal{G}_K$ and $\mathcal{G}_L$ in system (1.6).

- Provide a condition such that core assumption of Chapter 3, assumption (A_3) is satisfied.

- Study coupled nonlinear oscillators on networks without a rooted spanning tree. Unlike the case of linear oscillator, for systems of coupled nonlinear oscillators, there is no superposition principle which is critical for the proof of Theorem 5.8.

- For system (6.29), do bifurcation analysis on system (6.29) on parameter k.

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