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Connectedness of the Invertible Group of a Nest Algebra

by

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Abstract

We will study whether or not the invertible group of a nest algebra is connected. This is an open problem in general for nest algebras. In this thesis, we will survey the known results. The invertible group of a nest algebra is connected when there is a finite bound on the number of consecutive finite rank atoms in the nest. As particular cases, this shows that the invertible group is connected for finite dimensional nest algebras, continuous nest algebras, and nest algebras of infinite multiplicity.

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CHAPTER 1

Introduction and Historical Notes

Whether or not the group of invertibles in a unital Banach algebra is connected is an interesting question in general. Let G be the group of invertible elements of a unital Banach algebra $\mathbb{A}$ and let G_0 denote the connected component of G which contains the identity. Then, G_0 is an open, closed, and normal subgroup of G. We may then define the **abstract index group** $\Lambda_{\mathbb{A}}$ to be the quotient group G/G_0. We see that the invertible elements are connected in a unital Banach algebra if and only if $\Lambda_{\mathbb{A}}$ is the trivial group.

The invertibles are connected in the finite dimensional matrix algebra $M_n(\mathbb{C})$. It is also known that the invertible group is connected in $\mathcal{B}(\mathcal{H})$, the set of bounded linear operators on a separable, infinite dimensional, complex Hilbert space, $\mathcal{H}$. If we are dealing with a commutative unital Banach algebra $\mathbb{A}$, there is a nice description of G_0, the connected component of the identity. It turns out that $G_0 = \exp(\mathbb{A})$, the range of the exponential function. (The exponential of $F \in \mathbb{A}$ is defined by the convergent series $\exp(F) := \sum_{n=0}^{\infty} \frac{F^n}{n!}$.)

An example of a Banach algebra where the invertibles are not connected is the **Calkin algebra**, which is the quotient algebra $\mathcal{A}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Once again, $\mathcal{H}$ is a separable infinite dimensional Hilbert space, and $\mathcal{K}(\mathcal{H})$ denotes the set of compact operators on $\mathcal{H}$. The abstract index group for the Calkin algebra is isomorphic to $\mathbb{Z}$. We can obtain a description of the connected components of the invertible group of $\mathcal{A}(\mathcal{H})$ in terms of the Fredholm index. An operator $T \in \mathcal{B}(\mathcal{H})$ is called **Fredholm** if $\text{ran}(T)$ is closed, and both $\text{nul}(T)$ and $\text{nul}(T^*)$ are finite. For a Fredholm operator T, we define the (Fredholm) **index** of T to be $\text{ind}(T) = \text{nul}(T) - \text{nul}(T^*)$. It turns out that $T \in \mathcal{B}(\mathcal{H})$ is Fredholm if and only if $\pi(T)$ is invertible in the Calkin algebra, where π denotes the canonical quotient map.

Also, $\pi(T_1)$ and $\pi(T_2)$ lie within the same connected component of the group of invertibles if and only if T_1 and T_2 are Fredholm with $\text{ind}(T_1) = \text{ind}(T_2)$. Each connected component is non-empty, since $\text{ind}(I) = 0$, $\text{ind}(S^n) = n$ if $n > 0$. and $\text{ind}((S^*)^{-n}) = n$ if $n < 0$, where S denotes the backward unilateral shift in $\mathcal{B}(\mathcal{H})$. For further details, see [4].

The connectedness question has been studied recently in the case of nest algebras. Though a complete answer to the question has not yet been found, it has been shown that the invertible elements are connected for a large class of nest algebras. These results will be examined in this thesis.

Nests and nest algebras were first defined precisely by Ringrose in 1965 [**11**], though these objects were also studied in a more general setting by Kadison and Singer as early as 1960 [**5**]. One of the most important results for nest algebras is the Similarity Theorem, which characterizes when two nest algebras are isomorphic. D. Larson proved the first version of the theorem in 1982 [**6**], showing that any two continuous nest algebras are similar, and hence, their nest algebras are isomorphic. The Similarity Theorem was proved in its most general form by K. R. Davidson in 1983 [**1**]. He showed that two nests are similar precisely when there is an order preserving isomorphism between the nests which also preserves the rank of the atoms.

The first result on the invertibility question in nest algebras came in 1993, when K. R. Davidson and J. Orr showed that the invertible group is connected in nest algebras of infinite multiplicity. These are the nest algebras in which all of the atoms have infinite rank; and in particular, includes the nest algebras of continuous nests. The proof of this result uses results of D. Larson and D. Pitts on idempotents in nest algebras [**7**], and the Interpolation Theroem of J. Orr [**8**].

Soon after, in 1994, the invertibility question was again answered in the affirmative for a more general class of nest algebras. In [**3**], K. R. Davidson, J. Orr and D. Pitts showed that the invertibles are connected in nest algebras for which there is a finite bound on the number of consecutive finite rank atoms in the nest.

The invertibility question remains an open problem in general. It is unknown whether or not the invertibles are connected in the nest algebra consisting of upper-triangular (bounded) matrices on an infinite dimensional, separable Hilbert space. This is somewhat surprising, since it was originally thought that this was one of the most "natural" nest algebras to deal with.

CHAPTER 2

Introduction to Nests and Nest Algebras

In this chapter, we provide a brief introduction to nests and nest algebras. We will give the definitions we will need, some examples of nests and nest algebras, and a few basic results on connectedness. We also discuss the useful Similarity Theorem for nest algebras. For a more complete treatment of nest algebras, including the Similarity Theorem, see [**2**].

1. Preliminaries

A Hilbert space, $\mathcal{H}$, is a vector space equipped with an inner product, $\langle\cdot,\cdot\rangle$, such that $\mathcal{H}$ is complete under the norm $\|x\| := \langle x, x\rangle^{\frac{1}{2}}$. We denote by $\mathcal{B}(\mathcal{H})$ the set of linear, bounded (or equivalently, continuous) functions from $\mathcal{H}$ into $\mathcal{H}$. We will always assume that the field of scalars for our Hilbert spaces is $\mathbb{C}$. We will also assume throughout that our Hilbert spaces are separable; that is to say, that each Hilbert space has a countable orthonormal basis.

To each operator $T \in \mathcal{B}(\mathcal{H})$ we associate the **adjoint** operator T^* which is the unique operator in $\mathcal{B}(\mathcal{H})$ which satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathcal{H}$. An **idempotent** in an algebra is an element E which satisfies $E^2 = E$. A **projection** is an operator $P \in \mathcal{B}(\mathcal{H})$ which is a self-adjoint (that is, $P = P^*$) idempotent.

We define a norm on $\mathcal{B}(\mathcal{H})$ by setting $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$. This norm makes $\mathcal{B}(\mathcal{H})$ into a metric space, via the metric $d(S,T) = \|S - T\|$. We define the **rank** of an operator to be the dimension of its range (possibly ∞), and write $\mathrm{rank}(T)$. The set of finite rank operators in $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{F}(\mathcal{H})$. The set of **compact** operators, $\mathcal{K}(\mathcal{H})$, is the (norm) closure of $\mathcal{F}(\mathcal{H})$.

Along with the metric topology defined by the norm as above, there are various other topologies we can put on $\mathcal{B}(\mathcal{H})$. We will define each by describing which nets converge. The **strong operator topology (SOT)** is defined by the convergence $T_\alpha \xrightarrow{\text{SOT}} T$ if and only if $T_\alpha x \to Tx$ for all $x \in \mathcal{H}$. The **weak operator topology (WOT)** is defined by $T_\alpha \xrightarrow{\text{WOT}} T$ if and only if $\langle T_\alpha x, y\rangle \to \langle Tx, y\rangle$ for all $x, y \in \mathcal{H}$.

We also have a **weak-$*$ topology** on $\mathcal{B}(\mathcal{H})$ because it is the dual space of the space of trace class operators, $\mathcal{C}_1(\mathcal{H})$, which we will define presently. We will say that an operator $T \in \mathcal{B}(\mathcal{H})$ is in $\mathcal{C}_1(\mathcal{H})$ if it is compact, and if the spectrum of $|K| = (K^*K)^{\frac{1}{2}}$, $(s_n)_{n=1}^{\infty}$, is a sequence in $l^1(\mathbb{N})$. The sequence $(s_n)_{n=1}^{\infty}$ is usually listed in decreasing order, and these s_n are called the **s-numbers** of K. For $K \in \mathcal{C}_1(\mathcal{H})$ we may define its **trace** by $\text{tr}(K) := \sum_k \langle Ke_k, e_k\rangle$ where (e_k) is some orthonormal basis for $\mathcal{H}$. It can be shown that the trace is independent of the choice of orthonormal basis. If $\mathcal{H}$ is finite dimensional, this agrees with the usual notion of trace of a matrix. Now, weak-$*$ convergence in $\mathcal{B}(\mathcal{H})$ can be described by $T_\alpha \xrightarrow{\text{weak}-*} T$ if and only if $\text{tr}(T_\alpha K) \to \text{tr}(TK)$ for all $K \in \mathcal{C}_1(\mathcal{H})$.

2. Nests and Nest Algebras: Definitions and Examples

2.1. Definition. Let $\mathcal{H}$ be a Hilbert space. A **nest** on $\mathcal{H}$ is a linearly ordered subset β of projections in $\mathcal{B}(\mathcal{H})$ which is closed in the strong operator topology, and contains 0 and I.

2.2. Definition. For a nest β, the **nest algebra**, $\text{Alg}(\beta)$, consists of all operators which leave the range of each projection in β invariant. That is,

$$\text{Alg}(\beta) = \{\text{T} \in \mathcal{B}(\mathcal{H}) : \text{TP} = \text{PTP for all P} \in \beta\}.$$

We will check that $\text{Alg}(\beta)$ is in fact an algebra. Let $A, B \in \text{Alg}(\beta)$, $\lambda \in \mathbb{C}$. It is easy to see that $A + B$ and λA are in $\text{Alg}(\beta)$. Let $P \in \beta$ and let $\mathcal{M} = \text{ran}(P)$. Then $AB\mathcal{M} \subseteq A\mathcal{M} \subseteq \mathcal{M}$ and hence $AB \in \text{Alg}(\beta)$.

Let us also show that $\text{Alg}(\beta)$ is WOT closed. Let $P \in \beta$ and let $\mathcal{M} = \text{ran}(P)$. Suppose $(T_\alpha)_{\alpha \in A}$ is a net in $\text{Alg}(\beta)$ which converges (WOT) to $T \in \mathcal{B}(\mathcal{H})$. Let $x \in \mathcal{M}$ and $y \in \mathcal{M}^\perp$. Since $T_\alpha \in \text{Alg}(\beta)$, we have $\langle T_\alpha x, y \rangle = 0$. Then,

$$\langle Tx,\ y \rangle = \lim_\alpha \langle T_\alpha x, y \rangle = 0,$$

which implies that $Tx \in \mathcal{M}$ as $y \in \mathcal{M}^\perp$ was arbitrary. Since $x \in \mathcal{M}$ was arbitrary, $T\mathcal{M} \subseteq \mathcal{M}$ and thus $T \in \text{Alg}(\beta)$.

Here are some examples of nests and nest algebras:

2.3. Example. Let $\mathcal{H}$ be a finite dimensional Hilbert space, with basis $\{e_1, \ldots, e_n\}$. Let P_k be the orthogonal projection onto the span of $\{e_1, \ldots, e_k\}$, and let $P_0 = 0$. Then,

$$\beta = \{P_k : 0 \leq k \leq n\}$$

is a nest. In this case, the nest algebra is just $\mathcal{T}_n(\mathbb{C})$, the algebra of upper triangular $n \times n$ matrices (with respect to the basis $\{e_1, \ldots, e_n\}$).

2.4. Example. Let $\mathcal{H}$ be an infinite dimensional (separable) Hilbert space with basis $\{e_1, e_2, \ldots\}$. Let P_k be the projection onto the span of $\{e_1, \ldots, e_k\}$. Then

$$\{0,\ P_k, I : k = 1, 2, \ldots\}$$

is a nest. The nest algebra consists of all bounded operators which have an upper triangular matrix with respect to the basis $\{e_1,\ e_2, \ldots\}$.

2.5. Example. Let $\mathcal{H} = L^2([0,1], m)$, where m denotes Lebesgue measure. The multiplication operator $P_t := M_{\chi_{[0,t]}}$ is a projection in $\mathcal{B}(\mathcal{H})$. The set

$$\beta = \{P_t : 0 \leq t \leq 1\}$$

is a nest, known as the **Volterra nest.**

2.6. Example. Let $\mathcal{H} = l^2(\mathbb{Q}) = L^2(\mathbb{Q}, \mu)$ where μ denotes counting measure:

$$\mu(E) = \begin{cases} \text{card}(E) & \text{if } E \text{ is a finite set} \\ \infty & \text{otherwise.} \end{cases}$$

For any $t \in \mathbb{R}$ define

$$N_t^- = M_{\chi_{(-\infty,t)\cap\mathbb{Q}}} \text{ and } N_t^+ = M_{\chi_{(-\infty,t]\cap\mathbb{Q}}}$$

where χ_Y denotes the characteristic function of the set Y. The multiplication operators above are projections in $\mathcal{B}(\mathcal{H})$. We define the nest

$$\beta = \{0, N_t^-, N_t^+, I : t \in \mathbb{R}\}$$

which is known as the **Cantor nest.**

Given $P \in \beta$, define $P_- = \sup\{Q \in \beta : Q < P\}$ and $P_+ = \inf\{Q \in \beta : Q > P\}$. Notice that these are both in β since nests are SOT closed. If $P_- \neq P$, then P_- is the immediate predecessor of P; likewise, when $P_+ \neq P$, P_+ is the immediate successor of P. If it happens that $P_- \neq P$, then we call $P - (P_-)$ an **atom** of β. If the atoms of β sum (in the strong operator topology) to the identity, we say that β is **atomic.** If a nest has no atoms at all, we call the nest **continuous.**

The nests in Examples 2.3 and 2.4 are atomic, whereas the Volterra nest of Example 2.5 is continuous. We claim that the Cantor nest (Example 2.6) is atomic. If t is irrational, then $N_t^+ = N_t^-$ and both are equal to $(N_t^-)_-$. If $t \in \mathbb{Q}$, then $N_t^+ - N_t^- = M_{\chi_{\{t\}}}$ is an atom of β. Also,

$$\sum_{t \in \mathbb{Q}} N_t^+ - N_t^- = \sum_{t \in \mathbb{Q}} M_{\chi_{\{t\}}} = M_{\chi_{\mathbb{Q}}} = I$$

where I is the identity in $\mathcal{B}(l^2(\mathbb{Q}))$. Hence, the Cantor nest is atomic.

More generally, a projection of the form $P - Q$ where $P, Q \in \beta$ is called an **interval** of β, also written β-interval. Notice that atoms are minimal intervals. Each β-interval. $N - M$, is **semi-invariant** for $T \in \mathrm{Alg}(\beta)$, meaning that $T(N - M) = NT(N - M)$. and $(N - M)T = (N - M)T(I - M)$.

2.7. Definition. Let $\mathcal{H}$ be a Hilbert space. A nest β on $\mathcal{H}$ is called **maximal** if whenever γ is a nest on $\mathcal{H}$ with $\beta \subseteq \gamma$ then $\gamma = \beta$.

Here is a nice characterization of when a nest is maximal.

2.8. Proposition. *A nest is maximal if and only if all of its atoms have rank one.*

Proof. Let β be a maximal nest on a Hilbert space $\mathcal{H}$ and suppose that $E = N - M$ is an atom of β with $\mathrm{rank}(E) \geq 2$. Choose some non-zero vector $x \in \mathrm{ran}(E)$. The orthogonal projection P onto $(\mathrm{ran}(M) + \mathrm{span}\{x\})$ satisfies $M < P < N$. Hence $\gamma = \beta \cup \{P\}$ is a nest which is strictly larger than β, which is a contradiction.

Conversely, let β be a nest on $\mathcal{H}$ such that all of its atoms have rank one. Let $P \in \mathcal{B}(\mathcal{H})$ be a projection such that $P < N$ or $P \geq N$ for all $N \in \beta$. These are the only possible projections with which we might hope to enlarge our nest β. Define

$$N_0 = \sup\{N \in \beta : N \leq P\} \text{ and } N_1 = \inf\{N \in \beta : N \geq P\}.$$

Note that $N_0, N_1 \in \beta$, and $N_0 \leq P \leq N_1$. For any $N \in \beta$, $N \leq N_0$ or $N \geq N_1$. Hence $(N_1)_- = N_0$. By hypothesis, $\mathrm{rank}(N_1 - N_0) \leq 1$. Thus, $P = N_0$ or $P = N_1$, and hence our nest β is maximal.

$\square$

Notice that in particular, continuous nests are maximal (because there are no atoms to check). Each one of the nests in the Examples above is maximal. Here is an example of a nest which is not maximal.

2.9. Example. Let $\mathcal{H}$ be an infinite dimensional Hilbert space with orthonormal basis $(e_n)_{n=1}^{\infty}$. Let P_k be the orthogonal projection onto the span of $\{e_1, e_2, \ldots, e_{2k}\}$. The nest $\beta = \{0, P_k, I : k \geq 1\}$ is not maximal, because each atom of β has rank 2. In fact, this nest is contained in the maximal nest from Example 2.4.

We define the **diagonal** of a nest algebra to be

$$\mathcal{D}(\beta) = \text{Alg}(\beta) \cap (\text{Alg}(\beta))^*.$$

If we are in the maximal finite dimensional case (Example 2.3), $\text{Alg}(\beta) = \mathcal{T}_n(\mathbb{C})$, then the diagonal of $\text{Alg}(\beta)$ is precisely the set of diagonal matrices.

The **commutant** of a subset $\mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ is the set

$$\mathbb{A}' = \{T \in \mathcal{B}(\mathcal{H}) : TA = AT \text{ for all } A \in \mathbb{A}\}.$$

We will check that $\mathcal{D}(\beta) = \beta'$. If $T \in \mathcal{D}(\beta)$, then for $P \in \beta$,

$$TP = PTP = (PT^*P)^* = (T^*P)^* = PT.$$

On the other hand, if $T \in \beta'$, then $TP = (TP)P = PTP$ and $T^*P = (PT)^* = (PTP)^* = PT^*P$. Hence $\mathcal{D}(\beta) = \beta'$. The double commutant $\beta'' = (\beta')'$ is known as the **core** of β. By von Neumann's Double Commutant Theorem, the core of β is the SOT closure of the algebra generated by β.

Here is another way to think about nests. The map sending $P \mapsto \text{ran}(P)$ is an order preserving homeomorphism which carries the strong operator topology to the order topology (the order on subspaces of $\mathcal{H}$ is just containment). So we could have also defined a nest as a linearly ordered chain of subspaces of $\mathcal{H}$ which contains $\{0\}$ and $\mathcal{H}$, and is closed under taking intersections and closed spans.

For any nest β on a separable Hilbert space, there is an order preserving homeomorphism from (β, SOT) onto a compact subset $\omega \subseteq [0, 1]$ [**2**, Theorem 2.13]. The homeomorphism is given by the map $P \mapsto \langle Px, x\rangle$ where x is a unit separating vector for β. (A **separating vector** for β is a vector $x \in \mathcal{H}$ such that $Px = 0$ implies that $P = 0$ for all $P \in \beta$.) In particular, notice that nests are (SOT) compact. We call ω the **order type** of β.

Observe that the order type depends on the choice of unit separating vector, x, but is well defined up to homeomorphism.

2.10. Example. The order types of the nests in the first three examples above are $\{1 - \frac{1}{k} : k = 1, \dots n\} \cup \{1\}$, $\{1 - \frac{1}{k} : k = 1, 2, \dots\} \cup \{1\}$, and $[0, 1]$ respectively. If β is the Volterra nest (Example 2.5), an example of a separating vector for β is the constant function 1. Thus, the map $\Phi : P \mapsto \langle P1, 1\rangle$ gives rise to the natural parameterization $P_t \leftrightarrow t$ of the nest. The order type of the Cantor nest turns out to be the Cantor set. Perhaps surprisingly, the order type of the nest in Example 2.9 is also $\{1 - \frac{1}{k} : k = 1, 2, \dots\} \cup \{1\}$, the same order type as in Example 2.4.

Each nest β has an associated spectral measure, $\mathcal{E}$. Let h be an order preserving homeomorphism of β onto a compact subset $\omega \subseteq [0, 1]$. The spectral measure $\mathcal{E}$ is defined on the Borel subsets of ω or equivalently (via the homeomorphism h) on the Borel subsets of β. For each Borel set, Σ, $\mathcal{E}(\Sigma)$ is a projection in the core, β''. The spectral measure satisfies countable additivity; that is, if $(B_n)_{n=1}^{\infty}$ is a countable family of disjoint Borel sets of β, then $\mathcal{E}(\bigcup_{n=1}^{\infty} B_n) = SOT - \sum_{n=1}^{\infty} \mathcal{E}(B_n)$. Also, for any vectors $x, y \in \mathcal{H}$, the map

$B \mapsto \langle \mathcal{E}(B)x, y\rangle$ is a complex-valued measure on the Borel sets of β. The spectral measure has the property that $\mathcal{E}([0,t]) = h^{-1}(t)$ for $t \in \omega$.

3. Connectedness

For a general topological space, connectedness and path connectedness are different. However, we will see that for the invertible group of a nest algbebra, $(\mathrm{Alg}(\beta))^{-1}$, connectedness and path connectedness are the same. We will use some standard topological results; the proofs here are based on [**12**].

3.1. Definition. We say that an invertible element $T \in \mathrm{Alg}(\beta)$ is **connected to the identity** if there is a (norm) continuous function $f : [0,1] \to (\mathrm{Alg}(\beta))^{-1}$ such that $f(0) = T$ and $f(1) = I$.

It is worth pointing out that the range of f being in $(\mathrm{Alg}(\beta))^{-1}$ really requires three things: for every $t \in [0,1]$, $f(t)$ must be invertible, and $f(t)$ and $f(t)^{-1}$ must be in the nest algebra, $\mathrm{Alg}(\beta)$.

3.2. Definition. A topological space X is called **connected** if there do not exist disjoint non-empty open sets A, B such that $X = A \cup B$. We call X **path (or pathwise) connected** if for every $x, y \in X$, there exists a continuous function $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. Such an f will be called a **path** from x to y. A space X is called **locally path connected** if each point $x \in X$ has a neighbourhood base consisting of path connected sets.

The next proposition compares connectedness to path connectness.

3.3. Proposition. *Let X be a topological space. If X is path connected, then it is connected. If X is connected and locally path connected, then it is path connected.*

Proof. Let X be path connected, and suppose A, B are disjoint non-empty open sets such that $X = A \cup B$. Choose $a \in A$ and $b \in B$ and let $f : [0, 1] \to X$ be a path from a to b. Then, the sets $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty open sets whose union is $[0, 1]$. This is a contradiction, because $[0, 1]$ is connected. Hence, X is connected.

Next, suppose that X is connected and locally path connected. Let $x \in X$ and define

$$A = \{y \in X : \text{there exists a path in } X \text{ from } x \text{ to } y\}.$$

Notice that $A \neq \emptyset$ because $x \in A$. Let $a \in A$ and choose a path connected neighbourhood U of a. Any point $u \in U$ is also in A, because there is a path from x to a within X, and a path from a to u within $A \subseteq X$. So $U \subseteq A$ and hence A is open.

We will show that A is also closed. Let $b \in \bar{A}$ and let U be a path connected neighbourhood of b. Then, $U \cap A$ is non-empty, say $z \in U \cap A$. Then, we have a path from x to z (within X), and a path from z to b (within U). Adjoining these two paths shows that $b \in A$, and hence A is closed.

Since X is connected, we must have $A = X$, for otherwise A and $(X \setminus A)$ would be disjoint non-empty open sets whose union is X. This means that X is path connected.

□

3.4. Proposition. *Let β be a nest. Then the group of invertibles,* $(\text{Alg}(\beta))^{-1}$ *is locally path connected.*

Proof. First, notice that $\text{Alg}(\beta)$ is a Banach algebra because it is a norm closed subalgebra of $\mathcal{B}(\mathcal{H})$. Indeed, we have seen that $\text{Alg}(\beta)$ is even WOT closed. Fix $T \in (\text{Alg}(\beta))^{-1}$ and for $n \in \mathbb{N}$ with $n > ||T^{-1}||$, define

$$U_n = \{X \in (\text{Alg}(\beta))^{-1} : ||X - T|| < \frac{1}{n}\}$$

Then, $\{U_n : n \in \mathbb{N}, n > \|T^{-1}\|\}$ is a neighbourhood base at T. Let $X \in U_n$ and define $f(t) = (1-t)T + tX$ for $t \in [0,1]$. Then, for each t,

$$\|f(t) - T\| = t\|X - T\| < \frac{1}{n} < \frac{1}{\|T^{-1}\|}$$

which implies that $f(t)$ is invertible, and also that $f(t) \in U_n$. Also, f is continuous because $\|f(t_1) - f(t_2)\| = |t_1 - t_2|\|X - T\|$ for any $t_1, t_2 \in [0, 1]$. This shows that $(\text{Alg}(\beta))^{-1}$ is locally path connected.

□

As an immediate corollary to the two results above, we see that path connectedness and connectedness are the same for $(\text{Alg}(\beta))^{-1}$.

3.5. Corollary. *Let β be a nest. The invertibles are connected in* Alg(β) *if and only if the invertibles are path connected in* Alg(β).

So when we consider the question of whether or not the invertibles are connected a nest algebra, the Corollary above shows that we may equally mean "connected" or "path connected".

Whether or not the invertibles are connected is a nest algebra remains an open problem in general. However, it has been shown that the invertibles are connected in all the cases where the answer is known. The purpose of this thesis is to examine the connectedness question for nest algebras. We will first look at a simple case to give an idea of the flavour of this problem.

3.6. Proposition. *Let $\mathcal{H}$ be a finite dimensional Hilbert space with orthonormal basis $\{e_1, \ldots, e_n\}$ and let $\beta = \{0, P_k : k = 1, \ldots, n\}$ where P_k is the orthogonal projection onto* span$\{e_1, \ldots, e_k\}$. *Then, the invertibles are connected in* Alg(β).

Proof. Recall that the nest algebra $\text{Alg}(\beta)$ for this nest is simply $\mathcal{T}_n(\mathbb{C})$, the algebra of upper-triangular $n \times n$ matrices. A matrix in $\text{Alg}(\beta)$ is invertible if and only if all of its diagonal entries are non-zero.

Let $T \in \text{Alg}(\beta)$ be invertible, and write $T = D + N$ where $D = \text{diag}\{d_k\}_{k=1}^n$ is diagonal, and N is strictly upper triangular. Note that each d_k is non-zero.

We will first define the function $f : [0,1] \to (\text{Alg}(\beta))^{-1}$ by

$$f(t) = D + (1-t)N.$$

Note that $f(0) = T$ and $f(1) = D$. For each t, $f(t) \in (\text{Alg}(\beta))^{-1}$ because it is an upper triangular matrix with non-zero diagonal entries. The function f is continuous since $\|f(t_1) - f(t_2)\| = |t_1 - t_2| \, \|N\|$ for any $t_1, t_2 \in [0,1]$.

For each $k, 1 \le k \le n$, write $d_k = r_k e^{i\theta_k}$ and define functions g_k on $[0,1]$ by

$$g_k(t) = \begin{cases} r_k e^{i(1-2t)\theta_k} & \text{if } 0 \le t \le \frac{1}{2} \\ (2-2t)r_k + (2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Set $g(t) = \text{diag}\{g_k(t)\}_{k=1}^n$. We have $g(0) = D$ and $g(1) = I$ (the $n \times n$ identity matrix). For every $t \in [0,1]$, $g(t)$ is a diagonal matrix with non-zero entries, and is thus in $(\text{Alg}(\beta))^{-1}$. Also g is continuous since each function g_k is continuous, and since the norm in this case is given by $\|g(t)\| = \max\{|g_k(t)| : k = 1, \ldots, n\}$.

So, the function

$$h(t) = \begin{cases} f(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

is a continuous function into $(\text{Alg}(\beta))^{-1}$ with $h(0) = T$ and $h(1) = I$. Hence, the invertibles are connected in $\text{Alg}(\beta)$.

□

It is important here that our scalar field for $\mathcal{H}$ was $\mathbb{C}$. If we take only real scalars, it is false that the invertibles are connected, even in the nest algebra $\mathcal{T}_1(\mathbb{R}) \simeq \mathbb{R}$. The set of invertible elements of the nest algebra is $\mathbb{R} \setminus \{0\}$ which is not connected.

4. Similarity

4.1. Definition. We say that two nests β, γ on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ are **similar** if there is an invertible operator $S : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\gamma = S\beta := \{[SN] : N \in \beta\}$$

where $[SN]$ denotes the orthogonal projection onto the range of SN.

One of the most important results for nest algebras is the Similarity Theorem (Theorem 4.3 below). The first version of the Similarity Theorem, which showed that all continuous nest algebras are similar, was proved by D. Larson [**6**]. This paper also showed that a similarity between nests can fail to preserve multiplicity, which answered a long standing question of J. R. Ringrose. We will see an example of this phenomenon below. The Similarity Theorem in its full generality was proved by K. R. Davidson [**1**].

Similarity of nests gives us useful information about the corresponding nest algebras; namely, if β and γ are similar nests, then $\mathrm{Alg}(\beta)$ and $\mathrm{Alg}(\gamma)$ are isomorphic. In fact, if $S : \beta \to \gamma$ is the similarity, then $\mathrm{Alg}(\beta) = S^{-1}\mathrm{Alg}(\gamma)S$. To see this, let $T \in \mathrm{Alg}(\beta)$, and let $L \in \gamma$. Choose $N \in \beta$ so that $L = [SN]$. Let $\mathcal{N} = \mathrm{ran}(N)$ and $\mathcal{L} = \mathrm{ran}(L)$. We have

$$STS^{-1}\mathcal{L} = STS^{-1}S\mathcal{N} = ST\mathcal{N} \subseteq S\mathcal{N} = \mathcal{L},$$

and hence, $STS^{-1} \in \mathrm{Alg}(\gamma)$. Thus, $\mathrm{Alg}(\beta) \subseteq S^{-1}\mathrm{Alg}(\gamma)S$. For the converse, let $R \in \mathrm{Alg}(\gamma)$ and $N \in \beta$. Choose $L \in \gamma$ with $L = [SN]$ and set $\mathcal{N} = \mathrm{ran}(N)$, $\mathcal{L} = \mathrm{ran}(L)$ as before. Then, $S^{-1}RS\mathcal{N} = S^{-1}R\mathcal{L} \subseteq S^{-1}\mathcal{L} = \mathcal{N}$, and so, $S^{-1}RS \in \mathrm{Alg}(\beta)$. This shows that $\mathrm{Alg}(\beta) = S^{-1}\mathrm{Alg}(\gamma)S$.

4.2. Definition. We call a map $\theta : \beta \to \gamma$ between two nests **rank preserving** if for every atom $P - Q$ in β,

$$\operatorname{rank}(P - Q) = \operatorname{rank}(\theta(P) - \theta(Q)).$$

We wish to gain some information on when nests will be similar. First, suppose we are given two nests which are similar, say $S : \beta \to \gamma$ is a similarity. We define the map $\theta_S(N) = [SN]$. By definition of similarity of nests, this map is onto. If $N_1 < N_2$ are in β, then $\theta_S(N_1) = [SN_1] < [SN_2] = \theta_S(N_2)$; thus, θ_S is one-to-one and order preserving. Since S is an invertible operator, θ_S also preserves the rank of the atoms of β. To summarize, θ_S is an order and rank preserving isomorphism between the nests. The Similarity Theorem shows that the converse holds; that is, if there exists an order and rank preserving isomorphism between two nests, then they are similar. If this is the case, we may even take the similarity to be a small compact perturbation of a unitary.

4.3. Theorem. (Similarity Theorem [1]) *Let β_1, β_2 be nests on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ respectively. There exists an order and rank preserving isomorphism $\theta : \beta_1 \to \beta_2$ if and only if there is an invertible operator $S \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $S\beta_1 = \beta_2$. Given such a θ and $\varepsilon > 0$, we can find a unitary U and a compact K with $\|K\| < \varepsilon$ such that $S = U + K$ is invertible, $S\beta_1 = \beta_2$ and $\theta = \theta_S$, where $\theta_S(N) = [SN]$ for $N \in \beta_1$.*

We will end with an example which shows that two nests can be similar, and yet not unitarily equivalent. We will say that nests β, γ on Hilbert spances $\mathcal{H}_1$, $\mathcal{H}_2$ respectively are **unitarily equivalent** if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\gamma = U\beta := \{[UN] : N \in \beta\}$.

4.4. Example. Let $\beta = \{N_t : t \in [0,1]\}$ be the Volterra nest from Example 2.5. Define the nest

$$\gamma = \beta \oplus \beta := \{N_t \oplus N_t : t \in [0,1]\}$$

on the Hilbert space $L^2([0,1], m) \oplus L^2([0,1], m)$, where m denotes Lebesgue measure. The map $N_t \mapsto N_t \oplus N_t$ is an order preserving isomorphism, which also (vacuously) preserves the rank of atoms because β and γ are continuous nests. By the Similarity Theorem, the nests are similar. However, the nests are not unitarily equivalent, since $\beta' = L^\infty([0,1], m)$ is commutative, whereas $\gamma' = M_2(L^\infty([0,1], m))$ is not.

CHAPTER 3

Idempotents in a Nest Algebra

In this chapter, we will collect some results on idempotents in a nest algebra. We will define algebraic equivalence and similarity of idempotents, and obtain a description of when idempotents in a nest algebra are algebraically equivalent or similar in terms of the ranks of their compressions to intervals of the nest. These results will be of use to us in the next chapter when we prove the Interpolation Theorem, and again when we study the connectedness problem for nest algebras in Chapter 6.

1. The Idempotent Theorem

1.1. Lemma. *Let β be a nest and $E \in \mathrm{Alg}(\beta)$ be an idempotent. Let $S \in \mathcal{B}(\mathcal{H})$ be invertible. For any $N, M \in \beta$ with $N > M$,*

$$\mathrm{rank}((N-M)E(N-M)) = \mathrm{rank}(([SN]-[SM])SES^{-1}([SN]-[SM]))$$

where $[T]$ denotes the projection onto $\mathrm{ran}(T)$.

Proof. Fix $N, M \in \beta$ with $N > M$. Consider the finite subnest $\gamma = \{0, M, N, I\}$. We may factor $S = U_1 A$ where U_1 is unitary, and $A \in \mathrm{Alg}(\gamma)$ [**2**, Lemma 14.4]. Since γ is a finite nest, we may further factor $A = U_2 B$ where U_2 is unitary, $B \in \mathrm{Alg}(\gamma)$ and B is invertible with $B^{-1} \in \mathrm{Alg}(\gamma)$ [**2**, Corollary 14.3]. Hence, $S = UB$ where $U = U_1U_2$ is unitary, and $B \in \mathrm{Alg}(\gamma)^{-1}$.

Now, we have $SN = UBN = UNBN = UNU^*SN$. Thus, $\mathrm{ran}(SN) \subseteq \mathrm{ran}(UNU^*)$.

On the other hand, $UNU^* = SB^{-1}NBS^{-1} = SNB^{-1}NBS^{-1}$ because B^{-1} is also in $\mathrm{Alg}(\gamma)$. Hence, we have shown that $\mathrm{ran}(SN) = \mathrm{ran}(UNU^*)$. This implies that $[SN] = UNU^*$ since both are projections with the same range.

The same proof shows that $[SM] = UMU^*$. So, $[SN] - [SM] = U(N - M)U^*$. Using the fact that $E, B, B^{-1} \in \mathrm{Alg}(\gamma)$, and the semi-invariance of $N - M$, we get

$$\begin{aligned}
&\mathrm{rank}(([SN]-[SM])SES^{-1}([SN]-[SM])) \\
&= \mathrm{rank}(U(N-M)U^*SES^{-1}U(N-M)U^*) \\
&= \mathrm{rank}(U(N-M)BEB^{-1}(N-M)U^*) \\
&= \mathrm{rank}(U(N-M)B(N-M)E(N-M)B^{-1}(N-M)U^*) \\
&= \mathrm{rank}((N-M)B(N-M)E(N-M)B^{-1}(N-M)) \\
&= \mathrm{rank}((N-M)E(N-M)),
\end{aligned}$$

because $(N-M)B(N-M)$ is invertible in $\mathcal{B}((N-M)\mathcal{H})$: its inverse is $(N-M)B^{-1}(N-M)$. To verify this last comment, consider the matrices

$$B = \begin{pmatrix} * & * & * \\ 0 & B_4 & * \\ 0 & 0 & * \end{pmatrix} \begin{matrix} M\mathcal{H} \\ (N-M)\mathcal{H} \\ N^\perp\mathcal{H} \end{matrix} \quad \text{and } B^{-1} = \begin{pmatrix} * & * & * \\ 0 & C_4 & * \\ 0 & 0 & * \end{pmatrix} \begin{matrix} M\mathcal{H} \\ (N-M)\mathcal{H} \\ N^\perp\mathcal{H} \end{matrix},$$

both of which have this block upper-triangular form because $B, B^{-1} \in \mathrm{Alg}(\gamma)$. So multiplying these matrices shows that $B_4C_4 = I = C_4B_4$, as advertised.

□

If $\mathbb{A}$ is a algebra, we say that two idempotents $A, B \in \mathbb{A}$ are **algebraically equivalent** if there exist $X, Y \in \mathbb{A}$ (not necessarily idempotents) such that $A = XY$ and $B = YX$, and we write $A \sim B$. Algebraic equivalence is an equivalence relation on the set of idempotents in $\mathbb{A}$. We will check that $\sim$ is transitive; the reflexivity and symmetry are easy to see.

Let $A, B, C \in \mathbb{A}$ be idempotents with $A \sim B$ and $B \sim C$. Let $X, Y \in \mathbb{A}$ be such that $A = XY$ and $B = YX$; and choose $Z, W \in \mathbb{A}$ so that $B = ZW$ and $C = WZ$. Now, $(XZ)(WY) = XBY = X(YX)Y = A$ and $(WY)(XZ) = WBZ = W(ZW)Z = C$. Hence $A \sim C$.

Algebraic equivalence is a useful notion in other settings as well. For example, it is used when defining the K-theory of rings.

If $\mathbb{A}$ is a unital algebra, we call $A, B \in \mathbb{A}$ **similar** if there exists an invertible element $S \in \mathbb{A}$ with $A = SBS^{-1}$. The next theorem, due to D. Larson and D. Pitts gives us useful information on when idempotents in a nest algebra are algebraically equivalent or similar.

1.2. Theorem. (Idempotent Theorem [7]) *Let β be a nest on $\mathcal{H}$ and $E, F \in \mathrm{Alg}(\beta)$ be idempotents.*

1. E, F are algebraically equivalent within $\mathrm{Alg}(\beta)$ if and only if

$$\mathrm{rank}((N - M)E(N - M)) = \mathrm{rank}((N - M)F(N - M))$$

for all $N, M \in \beta, N > M$.

2. E, F are similar within $\mathrm{Alg}(\beta)$ if and only if

$$\mathrm{rank}((N - M)E(N - M)) = \mathrm{rank}((N - M)F(N - M))$$

and

$$\mathrm{rank}((N - M)(I - E)(N - M)) = \mathrm{rank}((N - M)(I - F)(N - M))$$

for all $N, M \in \beta, N > M$.

Proof. We first prove that E is similar to F if and only if $E \sim F$ and $I - E \sim I - F$.

Suppose that E is similar to F. Let $A \in \mathrm{Alg}(\beta)^{-1}$ be such that $E = AFA^{-1}$. Then, $E = (AF)A^{-1}$, $F = A^{-1}(AF)$ and $I - E = (A(I - F))A^{-1}$, $I - F = A^{-1}(A(I - F))$. So, $E \sim F$ and $I - E \sim I - F$.

Conversely, choose $A', B', C', D' \in \text{Alg}(\beta)$ with $E = A'B', F = B'A'$ and $I - E = C'D', I - F = D'C'$. Next, put

$$A = EA'F, \ B = FB'E, \ C = (I-E)C'(I-F), \ D = (I-F)D'(I-E).$$

Notice that $AB = EA'FB'E = EA'(B'A')B'E = E$; and similarly, $F = BA$, $I - E = CD$ and $I - F = DC$.

Now, we have $(B+D)E = BE = BAB = FB = F(B+D)$. Finally,

$$(A+C)(B+D) = AB + CB + AD + CD = E + 0 + 0 + (I-E) = I$$

and similarly, $(B+D)(A+C) = I$. So, E is similar to F. Having established this, it now suffices to prove only the first part of our Theorem.

Firstly, suppose that $E \sim F$. Choose $A, B \in \text{Alg}(\beta)$ such that $E = AB$ and $F = BA$. Let $N, M \in \beta$ with $N > M$. Then,

$$\begin{aligned}(N-M)E(N-M) &= (N-M)A(N-M)B(N-M) \\ &= ((N-M)A(N-M))((N-M)B(N-M)).\end{aligned}$$

Similarly, $(N-M)F(N-M) = ((N-M)B(N-M))((N-M)A(N-M))$ so we have $(N-M)E(N-M) \sim (N-M)F(N-M)$.

Notice that any compression of the idempotent $E \in \text{Alg}(\beta)$ to an interval of β is again an idempotent. To see this, let $G - L$ be the interval. Since $E^2 = E$, the decomposition

$$E = \begin{pmatrix} * & * & * \\ 0 & E_4 & * \\ 0 & 0 & * \end{pmatrix} \begin{matrix} L\mathcal{H} \\ (G-L)\mathcal{H} \\ G^{\perp}\mathcal{H} \end{matrix}$$

shows that $E_4^2 = E_4$; however, E_4 is just the compression of E to $G - L$, namely $E_4 = (G-L)E(G-L)$.

We claim that if X, Y are algebraically equivalent *idempotents*, then they have the same rank. For consider the algebraic equivalence $X = CD$, $Y = DC$. Then,

$$\begin{aligned} \text{rank}(X) &= \text{rank}(X^2) \\ &= \text{rank}(CDCD) \\ &= \text{rank}(CYD) \\ &\leq \text{rank}(Y). \end{aligned}$$

The same argument gives the reverse inequality upon switching the roles of X and Y. Hence $\text{rank}(X) = \text{rank}(Y)$ and the first half our result follows.

For the converse, suppose that $\text{rank}((N-M)E(N-M)) = \text{rank}((N-M)F(N-M))$ for all $N > M$ in β.

First, we claim that every idempotent is similar (within $\mathcal{B}(\mathcal{H})$) to a projection. To see this let $X \in \mathcal{B}(\mathcal{H})$ be an idempotent. We have a matrix decomposition

$$X = \begin{pmatrix} I & X_2 \\ 0 & 0 \end{pmatrix} \begin{matrix} X\mathcal{H} \\ (I-X)\mathcal{H} \end{matrix}$$

so that

$$\begin{pmatrix} I & X_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & X_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -X_2 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

and the operator on the right hand side is a projection.

Choose $S, T \in \mathcal{B}(\mathcal{H})$ so that $P = SES^{-1}$ and $Q = TFT^{-1}$ are projections. Notice that $P \in S\text{Alg}(\beta)S^{-1} = \text{Alg}(S\beta)$ and is self adjoint. Hence, $P \in \text{Alg}(S\beta) \cap \text{Alg}(S\beta)^* = (S\beta)'$. In other words, P commutes with $[SL]$ for all $L \in \beta$. We have similar results for Q.

If γ is a nest on a Hilbert space $\mathcal{H}'$ and R is a projection in γ', we will use the notation γ_R for the nest $\{(RN)|_{\text{ran}(R)} : N \in \gamma\}$ on the Hilbert space $\text{ran}(R)$.

Let $N, M \in \beta$ with $N > M$. By Lemma 1.1 (and by hypothesis), we have

$$\text{rank}(P([SN] - [SM])) = \text{rank}(Q([TN] - [TM]))$$

In particular, $P[SN] = P[SM]$ if and only if $Q[TN] = Q[TM]$.

This means that the map $\theta : (S\beta)_P \to (T\beta)_Q$ given by

$$\theta((P[SL])\,|_{\text{ran}(P)}) = (Q[TL])\,|_{\text{ran}(Q)}$$

is well-defined and one-to-one. Also notice that θ is onto, and is order preserving.

By applying Lemma 1.1 again, we see that θ is dimension preserving; namely

$$\begin{aligned}
&\text{rank}((P[SN])\,|_{\text{ran}(P)}) - (P[SM])\,|_{\text{ran}(P)})) \\
&= \text{rank}(P([SN] - [SM])) \\
&= \text{rank}(Q([TN] - [TM])) \\
&= \text{rank}(\theta((P[SN])\,|_{\text{ran}(P)}) - \theta((P[SM])\,|_{\text{ran}(P)})
\end{aligned}$$

So, by the Similarity Theorem (Theorem 2.4.3), there is an invertible operator $J_0 : \text{ran}(P) \to \text{ran}(Q)$ which implements θ. That is,

$$J_0(\text{ran}(P[SL])) = \text{ran}(Q[TL]) \quad \text{for all } L \in \beta.$$

Put $K_0 = J_0^{-1}$. Extend J_0, K_0 to operators J, K defined on all of $\mathcal{B}(\mathcal{H})$ by setting them equal to zero on $(\text{ran}(P))^\perp$ and $(\text{ran}(Q))^\perp$ respectively. Then, we have $KJ = P$, $JK = Q$. Also note that $K = PKQ$ and $J = QJP$.

Let $A = S^{-1}KT$ and $B = T^{-1}JS$. Then,

$$AB = S^{-1}KTT^{-1}JS = S^{-1}KJS = S^{-1}PS = E$$

and similarly, $BA = F$.

It only remains to check that $A, B \in \text{Alg}(\beta)$. Let $L \in \beta$. Since P commutes with $[SL]$,

$$\begin{aligned} AL\mathcal{H} &= S^{-1}KTL\mathcal{H} = S^{-1}K[TL]\mathcal{H} \\ &= S^{-1}KQ[TL]\mathcal{H} = S^{-1}P[SL]\mathcal{H} \\ &\subseteq S^{-1}[SL]\mathcal{H} \\ &= L\mathcal{H}. \end{aligned}$$

Similarly, $BL\mathcal{H} \subseteq L\mathcal{H}$. Hence, $A, B \in \text{Alg}(\beta)$ and they show that $E \sim F$.

□

CHAPTER 4

Interpolation

We will introduce the Ringrose lower diagonal seminorm function and Larson's Ideal in this chapter. These tools, along with several other preliminary results, will lead us to the Interpolation Theorem of J. Orr (Theorem 3.6). We will use the Interpolation Theorem to show that the so called interpolating operators (see Definition 3.7) are dense in a nest algebra of infinite multiplicity. Most of the results in this chapter are based on [**8**].

1. The Diagonal Seminorm Function

1.1. Definition. Let β be a nest, $N \in \beta$, $X \in \mathcal{B}(\mathcal{H})$. We define

$$i_N(X) = \inf\{\|(N-M)X(N-M)\| : M \in \beta, M < N\}$$

if $N > 0$ and set $i_0(X) = 1$. We define the **support** of X to be the set

$$\mathrm{supp}_\beta(X) = \{N \in \beta : i_N(X) \neq 0\}.$$

The function $i_N(X)$ was first defined by Ringrose in [**11**] to study ideals in nest algebras, and is called the **(lower) diagonal seminorm function**. Ringrose also defined the upper diagonal seminorm function, $i_N^+(X) = \sup\{\|(M-N)X(M-N)\| : M \in \beta, M > N\}$, but we will not make use of this function here. The (lower) diagonal seminorm function will be studied in the next few results, which will lead up to the Interpolation Theorem of J. Orr (Theorem 3.6). Some elementary properties of $i_N(X)$ are given in the next Lemma.

1.2. Lemma. *Let β be a nest on $\mathcal{H}$ and $X \in \mathcal{B}(\mathcal{H})$. Then, $i_{(\cdot)}(X)$ is a positive valued, bounded function on β. If $N \in \beta$ is not the upper endpoint of an atom of β, then $\limsup_{M\uparrow N} i_M(X) \leq i_N(X)$. The function $i_N(\cdot)$ is a seminorm, and is submultiplicative on* $\mathrm{Alg}(\beta)$. *The map $Y \mapsto i_{(\cdot)}(Y)$ is continuous from $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$ into $(B(\beta), \|\cdot\|_\infty)$ where $B(\beta)$ denotes the space of bounded functions on β.*

Proof. Clearly $i_{(\cdot)}(X)$ is positive valued. For $N \in \beta$,

$$\begin{aligned} i_N(X) &= \inf\{\|(N-M)X(N-M)\| : M < N\} \\ &\leq \inf\{\|N-M\|\,\|X\|\,\|N-M\| : M < N\|\} \leq \|X\|, \end{aligned}$$

and so $i_{(\cdot)}(X)$ is bounded. Let $N \in \beta$, and $\varepsilon > 0$. Suppose N is not the upper endpoint of an atom of β. Choose $L \in \beta$, $L < N$ such that $\|(N-L)X(N-L)\| < i_N(X) + \varepsilon$. For all $M \in (L, N)$,

$$i_M(X) \leq \|(M-L)X(M-L)\| \leq \|(N-L)X(N-L)\| < i_N(X) + \varepsilon.$$

Hence, $\limsup_{M\uparrow N} i_M(X) \leq i_N(X)$. Let $X, Y \in \mathcal{B}(\mathcal{H})$ and fix $N \in \beta$.

$$\begin{aligned} & i_N(X+Y) \\ &= \inf\{\|(N-M)(X+Y)(N-M)\| : M < N\} \\ &\leq \inf\{\|(N-M)X(N-M)\| + \|(N-M)Y(N-M)\| : M < N\}. \end{aligned}$$

Let $\varepsilon > 0$. Choose $M_1, M_2 \in \beta$ such that $\|(N-M_1)X(N-M_1)\| < i_N(X) + \frac{\varepsilon}{2}$ and $\|(N-M_2)Y(N-M_2)\| < i_Y(X) + \frac{\varepsilon}{2}$ and let $M_0 = \max\{M_1, M_2\}$. By the above inequality, we have $i_N(X+Y) \leq \|(N-M_0)X(N-M_0)\| + \|(N-M_0)Y(N-M_0)\| \leq i_N(X) + i_N(Y) + \varepsilon$. Hence $i_N(X+Y) \leq i_N(X) + i_N(Y)$.

If $S, T \in \text{Alg}(\beta)$, then using the semi-invariance of the interval $N - M$ we have

$$\begin{aligned} & i_N(ST) \\ &= \inf\{\|(N-M)ST(N-M)\| : M < N\} \\ &\leq \inf\{\|(N-M)S(N-M)\|\,\|(N-M)T(N-M)\| : M < N\}. \end{aligned}$$

Let $\varepsilon > 0$, and set $\delta = \min\{\sqrt{\frac{\varepsilon}{2}}, \frac{\varepsilon}{2(i_N(S)+i_N(T))}\}$. As above, choose M_0 such that $\|(N-M_0)S(N-M_0)\| < i_N(S) + \delta$ and $\|(N-M_0)T(N-M_0)\| < i_N(T) + \delta$. We have $i_N(ST) \leq (i_N(S)+\delta)(i_N(T)+\delta) \leq i_N(S)\, i_N(T) + \delta(i_N(S)+i_N(T)) + \delta^2 \leq i_N(S)\, i_N(T) + \varepsilon$. So, $i_N(ST) \leq i_N(S)\, i_N(T)$.

Let $(X_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{B}(\mathcal{H})$ with $X_n \xrightarrow{\|\cdot\|} 0$. For any $N \in \beta$,

$$\|i_{(\cdot)}(X_n)\|_{\infty} = \sup\{\|i_N(X_n)\| : N \in \beta\} \leq \|X_n\|$$

and hence, $i_{(\cdot)}(X_n) \xrightarrow{\|\cdot\|_{\infty}} 0$ as $n \to \infty$.

□

1.3. Proposition. *Let β be a continuous nest and $E \in \text{Alg}(\beta)$ be an idempotent. For all $N \in \beta$, either $i_N(E) = 0$ or $i_N(E) \geq 1$.*

Proof. First recall that for any $M \in \beta$ with $M < N$, $(N-M)E(N-M)$ is an idempotent. If T is any non-zero idempotent in $\mathcal{B}(\mathcal{H})$, then

$$\|T\| = \|T^2\| \leq \|T\|^2$$

which implies that $\|T\| \geq 1$. Hence, it is clear from the definition of i_N that either $i_N(E) = 0$ or $i_N(E) \geq 1$.

□

The next result gives us a condition for algebraic equivalence of idempotents in a continuous nest algebra based on the lower diagonal seminorm function.

1.4. Proposition. *Let β be a continuous nest and let $E, F \in \operatorname{Alg}(\beta)$ be idempotents. Then E, F are algebraically equivalent if and only if*

$$\{N \in \beta : i_N(E) = 0\} = \{N \in \beta : i_N(F) = 0\}.$$

Proof. Let $N, M \in \beta$, $M < N$. Let $T \in \operatorname{Alg}(\beta)$ be an idempotent. We will first show that

$$\operatorname{rank}((N - M)T(N - M)) = \begin{cases} \infty & \text{if } i_L(T) \neq 0 \text{ for some } L \in (M, N] \\ 0 & \text{otherwise.} \end{cases}$$

First, suppose we have some $L \in (M, N]$ such that $i_L(T) \neq 0$. Since β is a continuous nest, we can find an increasing sequence $(M_n)_{n=1}^{\infty} \subseteq (M, L)$ such that $M_n \to L$. (The convergence here is in the strong operator topology, the usual topology on a nest β.)

Suppose that $\operatorname{rank}((L - M_1)T(L - M_1))$ is finite. Then, $\operatorname{rank}((L - M_n)T(L - M_n))$ is finite for all n, and hence the convergence $M_n \to L$ is also in norm. However, this contradicts the fact that $i_L(T) \neq 0$. So,

$$\operatorname{rank}((N - M)T(N - M)) \geq \operatorname{rank}((L - M_1)T(L - M_1)) = \infty.$$

Conversely, suppose that $i_L(T) = 0$ for all $L \in (M, N]$. In particular, $i_N(T) = 0$, and so there exists $N_0 \in (M, N)$ such that

$$\|(N - N_0)T(N - N_0)\| < 1.$$

Since it is an idempotent, we must have $(N - N_0)T(N - N_0) = 0$. Define

$$N_1 = \inf\{P \in (M, N] : (N - P)T(N - P) = 0\}.$$

Notice that $(N - N_1)T(N - N_1) = 0$, since multiplication is jointly SOT-continuous in the ball of radius $\|T\|$ in $\mathcal{B}(\mathcal{H})$. We claim that $N_1 = M$; suppose it does not. Since $i_{N_1}(T) = 0$, we can find some $N_2 \in (M, N_1)$ so that

$$(N_1 - N_2)T(N_1 - N_2) = 0$$

by the same argument as above. Now,

$$\begin{aligned}
&(N - N_2)T(N - N_2) \\
&= (N - N_1 + N_1 - N_2)T(N - N_1 + N_1 - N_2) \\
&= 0 + (N - N_1)T(N_1 - N_2) + (N_1 - N_2)T(N - N_1) + 0 \\
&= (N_1 - N_2)T(N - N_1)
\end{aligned}$$

because $TN_1 = N_1TN_1$. Since $(N - N_2)T(N - N_2)$ is an idempotent, we have

$$(N - N_2)T(N - N_2) = ((N_1 - N_2)T(N - N_1))^2 = 0.$$

However, this contradicts our definition of N_1, and so $N_1 = M$ and $(N - M)T(N - M) = 0$. Our result now follows immediately from the Idempotent Theorem (Theorem 3.1.2).

□

2. Larson's Ideal

2.1. Definition. Let β be a nest, and let $N - M$ be a β-invterval. A **partition** of $N - M$ is a (not necessarily finite) collection $(E_\alpha)_{\alpha \in A}$ of pairwise orthogonal β-intervals such that $\sum_{\alpha \in A} E_\alpha = N - M$. Let $X \in \mathcal{B}(\mathcal{H})$. We will call a partition $(E_\alpha)_{\alpha \in A}$ of a β-interval an **ε-partition** for X if $\|E_\alpha X E_\alpha\| < \varepsilon$ for all $\alpha \in A$.

For $N, M \in \beta$, we will use the notation

$$(N, M) = \{L \in \beta : M < L < N\},$$

with analogous definitions for $(N, M]$, $[N, M)$, and $[N, M]$. Notice that when we use this notation we are always dealing with subsets of β; that is, if we say $L \in (N, M)$, we are including the fact that $L \in \beta$.

2.2. Lemma. *Let β be a nest, $X \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. Suppose there exist $G, L \in \beta$ such that $i_N(X) < \varepsilon$ for all $N \in \beta$ with $G < N \leq L$. Then, there exists a partition (E_α) of $L - G$ such that*

$$\|E_\alpha X E_\alpha\| < \varepsilon$$

for all α. In other words, there exists an ε-partition of $L - G$ for X.

Proof. Since $i_L(X) < \varepsilon$, we can choose $N_0 \in [G, L)$ such that $\|(L - N_0)X(L - N_0)\| < \varepsilon$. Hence, the class

$$\mathcal{C} \doteq \{(E_\alpha)_{\alpha \in A} : (E_\alpha)_{\alpha \in A} \text{ is an } \varepsilon\text{-partition of } L - N \text{ for some } N \in [G, L)\}$$

is non-empty. We define a partial order on $\mathcal{C}$ by setting $(E_\alpha)_{\alpha \in A} \preceq (F_\delta)_{\delta \in D}$ if (and only if) for all $\alpha \in A$, there exists $\delta \in D$ such that $E_\alpha = F_\delta$. That is, if each interval in $(E_\alpha)_{\alpha \in A}$ is also in $(F_\delta)_{\delta \in D}$.

Suppose $(\mathbf{C}_\lambda)_{\lambda \in \Lambda}$ is an increasing linearly ordered subset of $\mathcal{C}$. Then, the union $\bigcup_{\lambda \in \Lambda} \mathbf{C}_\lambda$ is again in $\mathcal{C}$ and is an upper bound for $(\mathbf{C}_\lambda)_{\lambda \in \Lambda}$. Hence, by Zorn's Lemma, there exists a maximal element in $\mathcal{C}$, say $(E_\alpha)_{\alpha \in A}$ which is an ε-partition of $L - N_1$ for some $N_1 \in [G, L)$.

We claim that $N_1 = G$. Suppose to the contrary that $N_1 > G$. By hypothesis, $i_{N_1}(X) < \varepsilon$ and thus, we may choose $N_2 \in [G, N_1)$ such that $\|(N_1 - N_2)X(N_1 - N_2)\| < \varepsilon$. But then $(E_\alpha)_{\alpha \in A}$ together with the interval $N_1 - N_2$ is a partition which lies in $\mathcal{C}$ and is strictly greater than $(E_\alpha)_{\alpha \in A}$. This contradicts the maximality of $(E_\alpha)_{\alpha \in A}$. Hence, $N_1 = G$ and we have a partition of $L - G$ with the required property.

□

Next, we will examine Larson's Ideal for a nest algebra (defined below). This ideal is meant to generalize the notion of strict upper triangularity for finite-dimensional matrices. D. Larson defines this ideal in [**6**], and also shows that for an uncountable nest, $\mathcal{R}_\beta^\infty$ contains non-zero idempotents. This may be somewhat surprising, given that the only strictly upper-triangular matrix which is an idempotent is 0 in finite dimensions.

2.3. Definition. Let β be a nest. We define **Larson's ideal** to be the set

$$\mathcal{R}_\beta^\infty = \{T \in \text{Alg}(\beta) : \text{for all } \varepsilon > 0, \text{ there exists an } \varepsilon\text{-partition of } I \text{ for T}\}.$$

Another generalization of strict upper-triangularity for nest algebras is the Jacobson radical. One way to define the Jacobson radical of a nest algebra is via the so called Ringrose condition (see [**2**, Theorem 6.7]). We will say that $T \in \text{Alg}(\beta)$ is in the **Jacobson radical** if and only if for all $\varepsilon > 0$, there exists a *finite* ε-partition of I for T. It is obvious from this definition that $\mathcal{R}_\beta^\infty$ contains the Jacobson radical. In [**8**], it is shown that $\mathcal{R}_\beta^\infty$ is actually the largest diagonal-disjoint ideal in $\text{Alg}(\beta)$.

The next Theorem gives an equivalent definition of $\mathcal{R}_\beta^\infty$ in terms of the Ringrose diagonal seminorm function. It will follow as an easy Corollary to the Theorem, that Larson's ideal is in fact a two-sided ideal in $\text{Alg}(\beta)$.

2.4. Theorem. *Let β be a nest. Then,*

$$\mathcal{R}_\beta^\infty = \{T \in \text{Alg}(\beta) : i_N(T) = 0 \text{ for almost all } N \in \beta\}.$$

Proof. For each $a > 0$ and $X \in \mathcal{B}(\mathcal{H})$, the set $Y_a = \{N \in \beta : i_N(X) \geq a\}$ is a Borel set. This follows from the fact that the map $N \mapsto i_N(X)$ is left lower semi-continuous, and hence (Borel) measurable. Thus, the set $Y = \{N \in \beta : i_N(X) = 0\}$ is also a Borel set. So, the set on the right hand side in the theorem is well-defined.

Let $T \in \mathcal{R}_\beta^\infty$ and let $\varepsilon > 0$. Choose a partition $(E_\alpha)_{\alpha\in A}$ of I such that $\|E_\alpha X E_\alpha\| < \varepsilon$ for all $\alpha \in A$. Write $E_\alpha = L_\alpha - G_\alpha$ (where $L_\alpha, G_\alpha \in \beta$). Then, $i_N(X) < \varepsilon$ for all $N \in (G_\alpha, L_\alpha]$. Let

$$Y = \{N \in \beta : i_N(T) \geq \varepsilon\}.$$

Then,

$$\begin{aligned} \mathcal{E}(Y) &\leq \mathcal{E}(\beta \setminus \bigcup_{\alpha\in A}(G_\alpha, L_\alpha]) \\ &= I - \sum_{\alpha\in A}(L_\alpha - G_\alpha) = I - I = 0. \end{aligned}$$

So, $i_N(T) = 0$ for almost all $N \in \beta$.

Next, suppose $i_N(T) = 0$ for almost all $N \in \beta$. Let $\varepsilon > 0$. The set $Y = \{N \in \beta : i_N(T) > 0\}$ has measure zero. Also, if $(M_\alpha)_{\alpha\in A}$ as an increasing net in Y which converges to $M \in \beta$, then $M \in Y$. We may write the complement $\beta \setminus Y = \bigcup_{\alpha\in A}(G_\alpha, L_\alpha]$, where the intervals $(G_\alpha, L_\alpha]$ are pairwise disjoint. By Lemma 2.2, for each (fixed) $\alpha \in A$ there exists a partition $(E_{\alpha,\delta})_{\delta\in D}$ of $L_\alpha - G_\alpha$ such that $\|E_{\alpha,\delta} T E_{\alpha,\delta}\| < \varepsilon$ for all $\delta \in D$. Since the collection $(E_{\alpha,\delta})_{\alpha\in A,\delta\in D}$ is a partition of I, we have $T \in \mathcal{R}_\beta^\infty$ and we are done.

□

2.5. Corollary. *Larson's ideal, $\mathcal{R}_\beta^\infty$, is a two-sided ideal in* Alg(β).

Proof. Let $T \in \mathcal{R}_\beta^\infty$ and $X, Y \in \mathrm{Alg}(\beta)$. By submultiplicativity of i_N on Alg(β), we have $i_N(XTY) \leq i_N(X)\, i_N(T)\, i_N(Y)$. By the previous Theorem, the right hand side is equal to zero for almost all $N \in \beta$. Using the Theorem again, $XTY \in R_\beta^\infty$.

□

3. The Interpolation Theorem

We now require several technical Lemmas on our way to the proof of the Interpolation Theorem (Theorem 3.6).

3.1. Lemma. *Let β_1, β_2 be similar nests, with similarity $S : \mathcal{H}_1 \to \mathcal{H}_2$. Let $i_N^{(1)}(X)$, $i_{[SN]}^{(2)}(X)$ denote the respective diagonal seminorm functions. For $T \in \operatorname{Alg}(\beta_1)$ and $N \in \beta_1$.*

$$\frac{1}{k}\, i_{[SN]}^{(2)}(STS^{-1}) \leq i_N^{(1)}(T) \leq k\, i_{[SN]}^{(2)}(STS^{-1})$$

where $k = \|S\|\|S^{-1}\|$ is the condition number for S.

Proof. Let $N, M \in \beta_1$ with $M < N$. Let $z \in \mathcal{H}$ and write $z = y_1 + y_2$, where $y_1 \in \operatorname{ran}(SN)$, $y_2 \in (\operatorname{ran}(SN))^{\perp}$. We may write $y = SNx$ for some $x \in \mathcal{H}$. Then,

$$\begin{aligned} S^{-1}[SN]y_1 &= S^{-1}y_1 = S^{-1}SNx \\ &= Nx = NS^{-1}SNx \\ &= NS^{-1}y_1 = NS^{-1}[SN]y_1. \end{aligned}$$

Since $S^{-1}[SN]y_2 = 0 = NS^{-1}[SN]y_2$, we have $S^{-1}[SN] = NS^{-1}[SN]$. For $z \in \mathcal{H}$ we also have

$$\begin{aligned} [SM]^{\perp}SM^{\perp}z &= [SM]^{\perp}S(I - M)z \\ &= [SM]^{\perp}Sz - [SM]^{\perp}SMz \\ &= [SM]^{\perp}Sz, \end{aligned}$$

and so $[SM]^{\perp}S = [SM]^{\perp}SM^{\perp}$.

Write $E = [SN] - [SM]$. Using the equalities above, we have

$$\begin{aligned} \|ESTS^{-1}E\| &= \|E([SM]^{\perp}SM^{\perp})T(NS^{-1}[SN])E\| \\ &\leq \|S\|\,\|M^{\perp}TN\|\,\|S^{-1}\| \\ &= k\|(N-M)T(N-M)\| \end{aligned}$$

because $T \in \mathrm{Alg}(\beta_1)$. Hence, $\frac{1}{k}\, i^{(2)}_{[SN]}(STS^{-1}) \leq i^{(1)}_{N}(T)$. To get the other required inequality, we repeat the same argument considering S^{-1} as a similarity from β_2 to β_1.

□

3.2. Lemma. *Let β be a continuous nest. There exists a sequence of pairwise orthogonal projections $(F_n)_{n=1}^{\infty}$ in the core, β'', such that*

$$\mathrm{supp}_{\beta}(F_n) = \beta$$

for all n.

Proof. We will construct a sequence $(J_n)_{n=1}^{\infty}$ of Lebesgue measurable subsets of $[0,1]$ which are pairwise disjoint, and are such that $m(J_n \cap G) > 0$ for every open interval $G \subseteq [0,1]$ and for all n.

Let $(G_m)_{m=1}^{\infty}$ be a countable (topological) basis of open intervals for $[0,1]$. Fix a well-ordering of $\mathbb{N} \times \mathbb{N}$. We will define $J_{n,m} \subseteq G_m$ inductively according to this well-ordering as follows.

First, select an open interval contained in

$$G_m \setminus \bigcup_{(k,l)<(m,n)} J_{k,l}$$

say $(a_{m,n}, b_{m,n})$. Let $C \subseteq [0,1]$ be a non-null Cantor set. Put

$$J_{m,n} = \frac{(b_{m,n} - a_{m,n})}{2} C + \frac{5a_{m,n}}{4}$$

so that $J_{m,n}$ is a set of non-zero measure which lies within G_m. Also, by construction, the $J_{m,n}$ are pairwise disjoint. Set $J_n = \bigcup_{m=1}^{\infty} J_{m,n}$. The sequence $(J_n)_{n=1}^{\infty}$ consists of pairwise disjoint subsets. Since each J_n meets every basic open interval G_m in a set of non-zero measure, the same holds for an arbitrary open interval $G \subseteq [0,1]$.

The sequence of spectral projections $F_n = \mathcal{E}(J_n)$ $n = 1, 2, 3, \ldots$, are the required projections for our result. Indeed, the F_n are pairwise orthogonal; and for $N \in \beta$,

$$i_N(F_n) = \inf_{M<N} \|(N-M)F_n(N-M)\| = 1$$

because $\mathcal{E}^{-1}((M, N))$ is an open interval in $[0,1]$ whenever $0 < M < N$.

□

3.3. Lemma. *Let β be a continuous nest. There is a bounded sequence $(Q_n)_{n=1}^{\infty}$ of idempotents in $\mathcal{R}_\beta^\infty$ and a sequence of pairwise orthogonal projections $(F_n)_{n=1}^{\infty}$ in β'' such that* $\mathrm{supp}_\beta(F_n) = \beta$, $Q_n = F_n Q_n F_n$ *(for all n) and* $\mathrm{supp}_\beta(Q) = \beta$ *where* $Q =$ SOT$-\sum_{n=1}^{\infty} Q_n$.

Proof. We will first work with the Volterra nest (Example 2.2.5). Let γ be the Volterra nest. We may parameterize γ as $\gamma = \{N_t : t \in [0,1]\}$, where $N_t = M_{\chi_{[0,t]}}$.

Let $K \subseteq [0,1]$ be a non-null version of the Cantor set which is constructed by deleting semi-open intervals of the form $(a, b]$ from $[0,1]$. Let $P = M_{\chi_K}$. Notice that $\mathrm{supp}_\gamma(P) = \{N_t : t \in K\}$.

Now, let β be the given continuous nest, and let $N \in \beta$. Let $f : [0,1] \to [0,1]$ be an order preserving homeomorphism which maps K onto a version of the Cantor set, C, which does have measure zero. Let Φ be an order homeomorphism of $[0,1]$ onto β such that $N \in \Phi(C)$. Composing these maps (and using of the parameterization of γ) we obtain an order preserving, and (vacuously) rank preserving isomorphism, θ from the Volterra nest γ onto β. By the Similarity Theorem (Theorem 2.4.3), we can choose a similarity of the form

$S = U + K$ where U is unitary, K is compact, and $\|K\| < \frac{1}{6}$. Hence, the condition number $\|S\|\|S^{-1}\| < (1+\frac{1}{6})(1-\frac{1}{6})^{-1} < 2$.

Set $Q = SPS^{-1}$. By Lemma 3.1,

$$i_N(Q) \geq \frac{1}{2} i_{[S^{-1}N]}(P) > 0,$$

and thus, $N \in \text{supp}_\beta(Q)$. Again by Lemma 3.1, $i_M(Q) = 0$ for almost all $M \in \beta$ because C has measure zero. Hence, $Q \in \mathcal{R}_\beta^\infty$ (Theorem 2.4). Also notice that Q is an idempotent with $\|Q\| < 2$.

Since (β, SOT) is separable (it is homeomorphic to $[0,1]$), we can choose a countable (SOT) dense subset $(N_i)_{i=1}^\infty \subseteq \beta$. For each i, use the construction above to obtain P_i, where P_i is an idempotent in $\mathcal{R}_\beta^\infty$ such that $\|P_i\| < 2$ and $N_i \in \text{supp}_{N_i}(P_i)$.

Let $(F_i)_{i=1}^\infty$ be the sequence of core projections obtained in Lemma 3.2. Fix $i \in \mathbb{N}$. The map $N \mapsto F_i N|_{\text{ran}(F_i)}$ is an order and rank preserving isomorphism of β onto the nest $\beta_{F_i} := \{F_i N|_{\text{ran}(F_i)} : N \in \beta\}$. By the Similarity Theorem, we can find a similarity $S_i : \mathcal{H} \to \text{ran}(F_i)$ with $\|S\|\|S^{-1}\| < 2$. We can extend the domain and range of S_i and S_i^{-1} if necessary by defining them to be zero where they are not already defined. This way, we may think of each as an operator in $\mathcal{B}(\mathcal{H})$. Also, $S_i, S_i^{-1} \in \text{Alg}(\beta)$. Since $\mathcal{R}_\beta^\infty$ is a two-sided ideal in $\text{Alg}(\beta)$, $Q_i := S_i P_i S_i^{-1}$ lies inside $\mathcal{R}_\beta^\infty$. Hence, $(Q_i)_{i=1}^\infty$ is a sequence of idempotents with $\sup_{i\geq 1} \|Q_i\| \leq 2$.

We claim that for each i, $N_i \in \text{supp}_{N_i}(Q_i)$. Suppose $M \in \beta$, $M < N_i$. Then, $MF_i < N_i$. By Lemma 3.1, we have

$$\|(N_i - M)Q_i(N_i - M)\| \geq \frac{1}{2}\|(N_i - M)P_i(N_i - M)\| > 0.$$

Let $Q = \text{SOT} - \sum_{i=1}^\infty Q_i$. Since each $N \in \beta$, $N > 0$ is the limit of some increasing subsequence of $(N_i)_{i=1}^\infty$, we have $\text{supp}_\beta(Q) = \beta$.

□

3.4. Definition. Let x, y be vectors in $\mathcal{H}$ We define the rank one operator $x \otimes y^* \in \mathcal{B}(\mathcal{H})$ by the action

$$x \otimes y^*(z) = \langle z, y \rangle x$$

for all $z \in \mathcal{H}$.

3.5. Lemma. *Let β be a continuous nest and $X \in \mathcal{B}(\mathcal{H})$. Let $(E_n)_{n=1}^\infty$ be a sequence of intervals of β such that for all n, there exists $N \in \beta$ such that $i_N(E_n X E_n) > 1$. Let $(x_n), (y_n)$ be orthonormal sequences in $\mathcal{H}$. There exist orthogonal sequences $(a_n), (b_n)$ in $\mathcal{H}$ such that $a_n, b_n \in \operatorname{ran}(E_n)$, $\|a_n\|, \|b_n\| \leq 1$, and*

$$AXB = \sum_{n=1}^\infty x_n \otimes y_n^*$$

where $A = \sum_{n=1}^\infty x_n \otimes a_n^*$, $B = \sum_{n=1}^\infty b_n \otimes y_n^*$.

Proof. We will choose our orthogonal sequences $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty$ inductively. Select a_1, b_1 to be any unit vectors in $\operatorname{ran}(E_1)$. Let $n \in \mathbb{N}$ be fixed, and suppose that a_k, b_k have already been chosen for $1 \leq k \leq n-1$. Put

$$\mathcal{M} = \operatorname{span}\{b_k, X^* a_k, X E_n a_k, X^* E_n X b_k : k = 1, 2, \ldots, n-1\}$$

Choose $N \in \beta$ with $i_N(E_n X E_n) > 1$. There exists an infinite dimensional subspace $\mathcal{Q} \subseteq \operatorname{ran}(E_n)$ such that $\|E_n X z\| \geq \|z\|$ for all $z \in \mathcal{Q}$.

Because $\mathcal{M}$ is finite dimensional, there exists a unit vector in $\mathcal{Q} \cap \mathcal{M}^\perp$, call it b_n. Let

$$a_n = \frac{E_n X b_n}{\|E_n X b_n\|^2}$$

Note that $\|E_n X b_n\| \geq \|b_n\| = 1$, and so $\|a_n\| \leq 1$. The sequence $(b_n)_{n=1}^\infty$ is orthogonal by construction of $\mathcal{M}$. If $1 \leq k \leq n-1$,

$$\langle a_n, a_k \rangle = \frac{\langle E_n X b_n, a_k \rangle}{\|E_n X b_n\|^2} = \frac{\langle b_n, X^* E_n a_k \rangle}{\|E_n X b_n\|^2} = 0$$

since $b_n \in \mathcal{M}^\perp$ and $X^* E_n a_k \in \mathcal{M}$. We have

$$\langle X b_n, a_k \rangle = \langle b_n, X^* a_k \rangle = 0$$

and similarly, $\langle X b_k, a_n \rangle = 0$. Lastly,

$$\langle X b_n, a_n \rangle = \frac{\langle X b_n, E_n X b_n \rangle}{\|E_n X b_n\|^2} = \frac{\langle E_n X b_n, E_n X b_n \rangle}{\|E_n X b_n\|^2} = 1.$$

So, we have shown that for $i, j \in \mathbb{N}$,

$$\langle X b_i, a_j \rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Set $A = \sum_{m=1}^{\infty} x_m \otimes a_m^*$ and $B = \sum_{m=1}^{\infty} b_n \otimes y_n^*$. If $z \in \mathcal{H}$, we have

$$\begin{aligned} AXBz &= (\sum_{m=1}^{\infty} x_m \otimes a_m^*) X (\sum_{n=1}^{\infty} b_n \otimes y_n^*) z \\ &= (\sum_{m=1}^{\infty} x_m \otimes a_m^*) X (\sum_{n=1}^{\infty} \langle z, y_n \rangle b_n) \\ &= (\sum_{m=1}^{\infty} x_m \otimes a_m^*)(\sum_{n=1}^{\infty} X b_n \langle z, y_n \rangle) \\ &= \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \langle X b_n, a_m \rangle x_m \langle z, y_n \rangle \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \delta_{mn} x_m \langle z, y_n \rangle \\ &= \sum_{m=1}^{\infty} \langle z, y_m \rangle x_m = (\sum_{m=1}^{\infty} x_m \otimes y_m^*) z \end{aligned}$$

□

3.6. Theorem. (Interpolation Theorem [**8**, Theorem 3.1]) *Let β be a continuous nest. Let $X \in \mathcal{B}(\mathcal{H})$ and $a > 0$. Let $\Sigma = \{N \in \beta : i_N(X) \geq a\}$. There exist $A, B \in \mathrm{Alg}(\beta)$ such that $AXB = \mathcal{E}(\Sigma)$ where $\mathcal{E}$ is the spectral measure for β.*

Proof. Let $\gamma = \mathcal{E}(\Sigma)\beta$. Notice that γ is also a continuous nest. We apply Lemma 3.3 to γ to obtain sequences $(F_n)_{n=1}^{\infty}$ and $(Q_n)_{n=1}^{\infty}$ as in the Lemma. Set $Q = \sum_{n=1}^{\infty} Q_n$. For fixed

$n \in \mathbb{N}$, Q_n is an idempotent in $\mathcal{R}_\gamma^\infty$, so we may choose a partition $(E_m^{(n)})_{m=1}^\infty$ of γ such that $\|E_m^{(n)} Q_n E_m^{(n)}\| < 1$. As it is an idempotent, $E_m^{(n)} Q_n E_m^{(n)} = 0$.

Next, fix $m, n \in \mathbb{N}$. Choose a strictly increasing sequence $(M_k)_{k \in \mathbb{Z}}$ in γ so that the intervals $M_k - M_{k-1}$ form a partition of $E_m^{(n)}$. Define $E_{m,k}^{(n)} = M_k - M_{k-1}$.

For each $m, n \in \mathbb{N}$ and $k \in \mathbb{Z}$, the projection $E_{m,k}^{(n)} F_n$ is non-zero, and hence has infinite rank (because β is a continuous nest). Let $U_{m,n,k}$ be a partial isometry which sends $\text{ran}(E_{m,k+2}^{(n)} F_n)$ onto $\text{ran}(E_{m,k}^{(n)} F_n)$; and put $U_n = \sum_{m=1}^\infty \sum_{k=-\infty}^\infty U_{m,n,k}$.

Notice that U_n maps F_n onto F_n, $U_n \in \text{Alg}(\gamma)$ and U_n commutes with each $E_m^{(n)}$. Also $U := \sum_{n=1}^\infty U_n$ is a unitary in γ.

We will now show that $U^*Q \in \text{Alg}(\gamma)$. Write $E_m^{(n)} = G_m^{(n)} - L_m^{(n)}$, where $G_m^{(n)}, L_m^{(n)} \in \text{Alg}(\gamma)$. Then,

$$\begin{aligned} E_m^{(n)} Q_n &= E_m^{(n)} (I - L_m^{(n)}) Q_n \\ &= E_m^{(n)} Q_n (I - G_m^{(n)}) \end{aligned}$$

because $E_m^{(n)} Q_n E_m^{(n)} = 0$ and $Q_n \in \text{Alg}(\gamma)$. If $N \in \text{Alg}(\gamma)$, then

$$\begin{aligned} U_n^* Q_n N &= \sum_{m=1}^\infty E_m^{(n)} U_n^* Q_n N \\ &= \sum_{m=1}^\infty E_m^{(n)} U_n^* E_m^{(n)} Q_n (I - G_m^{(n)}) N. \end{aligned}$$

Notice that $(I - G_m^{(n)})N$ is non-zero only when $G_m^{(n)} < N$; and thus, when $E_m^{(n)} < N$. Therefore, $U_n^* Q_n N = N U_n^* Q_n N$. Since $N \in \gamma$ was arbitrary, $U_n^* Q_n \in \text{Alg}(\gamma)$ and then so is $U^* Q = \sum_{n=1}^\infty U_n^* Q_n$.

We think of $\text{Alg}(\gamma)$ as a subalgebra of $\text{Alg}(\beta)$ by identifying $T \in \text{Alg}(\gamma)$ with $\mathcal{E}(\Sigma) T \mathcal{E}(\Sigma)$. Write B_1 for the element which corresponds to U^*Q within $\text{Alg}(\beta)$.

We claim that $\text{supp}_\beta(Q) = \text{supp}_\beta(\mathcal{E}(\Sigma))$. For $N \in \beta$,

$$i_N(Q) = i_N(\mathcal{E}(\Sigma)Q) \leq \|Q\| i_N(\mathcal{E}(\Sigma)).$$

So, $\text{supp}_\beta(Q) \subseteq \text{supp}_\beta(\mathcal{E}(\Sigma))$. On the other hand, suppose that $i_N(\mathcal{E}(\Sigma)) > 0$. Thus, for $M \in \beta$ with $M < N$, we have $M\mathcal{E}(\Sigma) < N\mathcal{E}(\Sigma)$. Since $\text{supp}_\gamma(Q) = \gamma$, $(N-M)Q(N-M) \neq 0$, and hence $i_N(Q) \geq 1$. Thus $\text{supp}_\beta(\mathcal{E}(\Sigma)) \subseteq \text{supp}_\beta(Q)$ which establishes the claim.

By Proposition 1.4, Q is algebraically equivalent (within $\text{Alg}(\beta)$) to $\mathcal{E}(\Sigma)$. Choose $A_2, B_2 \in \text{Alg}(\beta)$ with $Q = B_2A_2$ and $\mathcal{E}(\Sigma) = A_2B_2$. Hence, $A_2QB_2 = (A_2B_2)^2 = \mathcal{E}(\Sigma)$.

For m, n, k we will redefine our $E^{(n)}_{m,k}$ to be the β-interval

$$\inf\{F : F \text{ is a } \beta\text{-interval, and } E^{(n)}_{m,k} \leq F\}$$

Since $E^{(n)}_{m,k}\mathcal{E}(\Sigma) \neq 0$, the open interval $(L^{(n)}_{m,k}, G^{(n)}_{m,k})$ (where $E^{(n)}_{m,k} = G^{(n)}_{m,k} - L^{(n)}_{m,k}$) must contain some point from Σ. So, $i_N(E^{(n)}_{m,k}XE^{(n)}_{m,k}) \geq a$ for some $N \in \beta$. (Indeed, this holds for some $N \in \Sigma$.)

Relabel the intervals $E^{(n)}_{m,k}$ as $(E_i)_{i=1}^\infty$ in such a way that each $E^{(n)}_{m,k}$ occurs infinitely many times in the sequence $(E_i)_{i=1}^\infty$. For each i, choose a vector y_i so that the set $Y_{m,n,k} = \{y_i : i \text{ is such that } E_i = E^{(n)}_{m,k}\}$ is a basis for $\text{ran}(E^{(n)}_{m,k+1}F_i)$.

Let $x_n = Uy_n$. Note that $x_n \in \text{ran}(E^{(n)}_{m,k-1}F_n)$. By Lemma 3.5 we can find $A_0, B_0 \in \text{Alg}(\beta)$ such that $A_0XB_0 = U$. Finally,

$$(A_2A_0)X(B_0B_1B_2) = A_2(UB_1)B_2 = A_2QB_2 = \mathcal{E}(\Sigma).$$

□

3.7. Definition. We say that an element X in a unital algebra $\mathbb{A}$ is **interpolating** if there exist $A, B \in \mathbb{A}$ such that $AXB = I$.

As an immediate Corollary to the Interpolation Theorem, we get a nice description of when an element of a nest algebra is interpolating in terms of the Ringrose diagonal seminorm funtion.

3.8. Corollary. *Let β be a continuous nest. If $X \in \mathcal{B}(\mathcal{H})$ is such that $i_N(X) \geq a$ for some $a > 0$ and for all $N \in \beta$, then there exist $A, B \in \mathrm{Alg}(\beta)$ such that $AXB = I$. That is, X is interpolating in $\mathrm{Alg}(\beta)$.*

Proof. We apply the Interpolation Theorem. By hypothesis, $\Sigma = \beta$, and hence $\mathcal{E}(\Sigma) = I$.

□

CHAPTER 5

Nest Algebras of Infinite Multiplicity

1. Infinite Multiplicity

1.1. Definition. We say a Banach subalgebra $\mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ has **infinite multiplicity** if it is isomorphic to $\mathbb{A} \otimes \mathcal{B}(\mathcal{H}')$ for some infinite dimensional Hilbert space $\mathcal{H}'$. For the tensor product, we are taking the completion of the algebraic tensor product in the weak operator topology on $\mathcal{B}(\mathcal{H} \otimes \mathcal{H}')$.

We are now interested in the nest algebras of infinite multiplicity. Let β be a nest such that all of the atoms of β are infinite dimensional. We will show that the nest algebra, $\text{Alg}(\beta)$, has infinite multiplicity.

Let $\mathcal{H}'$ be an infinite dimensional Hilbert space, and let I be the identity for $\mathcal{B}(\mathcal{H}')$. Consider the nest

$$\beta \otimes I := \{P \otimes I : P \in \beta\}.$$

The map $P \mapsto P \otimes I$ is an order preserving isomorphism of β onto $\beta \otimes I$. Also note that P is an atom of β precisely when $P \otimes I$ is an atom of $\beta \otimes I$, and that the dimension of the atom is infinite in each nest. Hence, by the Similarity Theorem, the nests β and $\beta \otimes I$ are similar. So, the corresponding nest algebras are isomorphic. Since $\text{Alg}(\beta \otimes I) \simeq \text{Alg}(\beta) \otimes \mathcal{B}(\mathcal{H}')$, $\text{Alg}(\beta)$ has infinite multiplicity.

Conversely, suppose β is a nest such that $\text{Alg}(\beta)$ is of infinite multiplicity. Then $\text{Alg}(\beta) \simeq \text{Alg}(\beta) \otimes \mathcal{B}(\mathcal{H})$, where $\mathcal{H}'$ is some infinite dimensional Hilbert space. Notice that $\text{Alg}(\beta) \otimes \mathcal{B}(\mathcal{H}')$ is a nest algebra, with $\gamma := \text{Lat}(\text{Alg}(\beta) \otimes \mathcal{B}(\mathcal{H}'))$ as its nest.

Suppose that T is in the diagonal, $\mathcal{D}(\gamma) = \text{Alg}(\gamma) \cap (\text{Alg}(\gamma))^*$. Write T in matrix form as $(T_{ij})_{i,j \in \mathbb{N}}$ where $T_{ij} \in \text{Alg}(\beta)$. Since T^* has the matrix $T^* = (T_{ji}^*)_{i,j \in \mathbb{N}}$, we have $T_{ij} \in \mathcal{D}(\beta)$ for each i, j. On the other hand, suppose that $T \in \text{Alg}(\beta) \otimes \mathcal{B}(\mathcal{H}')$ has the matrix $(T_{ij})_{i,j \in \mathbb{N}}$ and $T_{ij} \in \mathcal{D}(\beta)$ for each i, j. Then, $T_{ji}^* \in \text{Alg}(\beta)$ for all $i, j \in \mathbb{N}$. So, $T^* \in \text{Alg}(\beta) \otimes \mathcal{B}(\mathcal{H}')$. Hence,

$$\mathcal{D}(\gamma) = \{(T_{ij}) \in \text{Alg}(\beta) \otimes \mathcal{B}(\mathcal{H}') : T_{ij} \in \mathcal{D}(\beta) \text{ for all } i, j\}.$$

Next, let $P \in \gamma$ and write $P = (P_{ij})$. Since $\mathcal{D}(\gamma) = \gamma'$, P commutes with each $T \in \mathcal{D}(\gamma)$. Let $A_k \in \mathcal{D}(\gamma)$ be defined to be the matrix with the identity in the (k, k) entry, and zero elsewhere. By examining the matrix multiplication, $PA_k = A_k P$ implies that $P_{mk} = P_{km} = 0$ whenever $m \neq k$. Since this holds for all k, P is diagonal, say $P = \text{diag}\{P_{ii}\}$. Also, P commutes with each $A_{k(k+1)}$, where $A_{k(k+1)}$ has the identity in the $(k, k+1)$ entry, and zero elsewhere. The matrix multiplication this time shows that $P_{11} = P_{22} = P_{33} = \cdots$. Hence $P = P_{11} \otimes I$, where I denotes the identity in $\mathcal{B}(\mathcal{H}')$.

Let $T_{11} \in \text{Alg}(\beta)$. Then, $T := \text{diag}\{T_{11}, 0, 0, \ldots\}$ is in $\text{Alg}(\gamma) = \text{Alg}(\beta) \otimes \mathcal{B}(\mathcal{H}')$. Since $PTP = TP$, we also have (by matrix multiplication) $P_{11}T_{11}P_{11} = T_{11}P_{11}$. Thus, $P_{11} \in \text{Lat}(\text{Alg}(\beta)) = \beta$.

If $P_1 < P_2$ are in β, then

$$\text{rank}(P_2 - P_1) = \text{rank}(P_2 \otimes I - P_1 \otimes I) = \infty$$

and hence, β cannot have any finite rank atoms.

2. Approximation by Interpolating Operators

In this section, we will show that the interpolating operators are dense in a nest algebra of infinite multiplicity.

2.1. Proposition. *Let β be a continuous nest. The interpolating operators are dense in* Alg(β).

Proof. We may parameterize our continuous nest as $\beta = \{N_t : 0 \leq t \leq 1\}$. Write i_t for i_{N_t}.

Let $Y \in \text{Alg}(\beta)$ and let $\varepsilon > 0$. Define

$$U = \{t \in (0, 1] : i_t(Y) \leq \frac{\varepsilon}{2}\}$$

Notice that U is measurable because the map $t \mapsto i_t(X)$ is left lower semicontinuous, and hence measurable.

Let $X = Y + \varepsilon\mathcal{E}(U)$ where $\mathcal{E}$ is the spectral measure for β. Notice that $X \in \text{Alg}(\beta)$.

Suppose that t is an interior point of U. Then, $i_t(\mathcal{E}(U)) = 1$. On the other hand, if t is an interior point of the complement U^c, then $i_t(\mathcal{E}(U)) = 0$.

Because i_t is a seminorm,

$$\begin{aligned} i_t(X) &\geq |i_t(Y) - \varepsilon\, i_t(\mathcal{E}(U))| \\ &\geq \frac{\varepsilon}{2} \end{aligned}$$

whenever t is in $\text{Int}(U) \cup \text{Int}(U^c)$. Since $t \mapsto i_t(X)$ is left lower semi-continuous, the above inequality holds for all $t \in (0, 1]$.

Hence, $||Y - X|| \leq \varepsilon$ and $i_t(X) \geq \frac{\varepsilon}{2}$ for all t. By Corollary 4.3.8, X is interpolating, and so the interpolating operators are dense in Alg(β).

□

Next, we will show that the interpolating operators are dense in a nest algebra of infinite multiplicity.

2.2. Definition. An operator $T \in \mathcal{B}(\mathcal{H})$ is called **semi-Fredholm** if $\text{ran}(T)$ is closed and one of $\text{nul}(T) := \dim(\ker(T))$ and $\text{nul}(T^*) := \dim(\ker(T^*))$ is finite. If T is semi-Fredholm, we define its (semi-Fredholm) **index** to be $\text{ind}\,(T) = \text{nul}(T) - \text{nul}(T^*)$. The index can take either integer values, or the values $\pm\infty$.

2.3. Lemma. *Let $\mathcal{H}$ be an infinite dimensional (separable) Hilbert space. Then, the interpolating operators are dense in $\mathcal{B}(\mathcal{H})$.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ and let $\varepsilon > 0$. We will split the proof into two cases. For the first case, assume T is semi-Fredholm.

Suppose that $\text{nul}(T) \leq \text{nul}(T^*)$. Then, $n = \text{nul}(T)$ is finite. Let $(e_k)_{k=1}^n$ be an orthonormal basis for $\ker(T)$, and let $(f_k)_{k=1}^n$ be an orthonormal set in $(\text{ran}(T))^\perp = \ker(T^*)$. Let $F = \varepsilon(\sum_{k=1}^n f_k \otimes e_k^*)$. (If n was 0, T is already left invertible, so just take $F = 0$.)

Let $x \in \mathcal{H}$ be a vector, and write $x = x_1 + x_2$ where $x_1 \in \ker(T)$ and $x_2 \in (\ker(T))^\perp$. If $x_1 \neq 0$, then $(T+F)x = Tx + Fx = Tx_2 + Fx_1 \neq 0$ since $Fx_1 \neq 0$ and $Fx_1 \in (\text{ran}(T))^\perp$. If $x_2 \neq 0$, then $(T+F)x = Tx_2 + Fx_1 \neq 0$ since $x_2 \notin \ker(T)$. Thus, $\ker(T+F) = \{0\}$ so that $T+F$ is one-to-one. Hence, $T+F$ is left invertible.

On the other hand, suppose that $n = \text{nul}(T^*) \leq \text{nul}(T)$. Let $(e_k)_{k=1}^n$ be an orthonormal set in $\ker(T)$, and let $(f_k)_{k=1}^n$ be an orthonormal basis for $(\text{ran}(T))^\perp$. Let $F = \varepsilon(\sum_{k=1}^n f_k \otimes e_k^*)$. (Again, if $n = 0$ take $F = 0$.)

For $y \in \mathcal{H}$, write $y = y_1 + y_2$ where $y_1 \in \text{ran}(T)$ and $y_2 \in (\text{ran}(T))^\perp$. Choose $x_1 \in (\ker(T))^\perp$ with $Tx_1 = y_1$, and $x_2 \in \ker(T)$ with $Fx_2 = y_2$. Then, $(T+F)(x_1 + x_2) = Tx_1 + Fx_2 = y_1 + y_2 = y$. So, T+F is onto, and hence right invertible.

Notice that left invertible and right invertible operators are interpolating. Also, $\|T - (T+F)\| = \|F\| \leq \varepsilon$.

Next, suppose that T is not semi-Fredholm. It is easy to see from the definition that T^* is also not semi-Fredholm. Let $|T| = (T^*T)^{\frac{1}{2}}$. Since $|T|$ and $|T^*|$ are self-adjoint (in fact, positive), we have the associated spectral projections $E_{|T|}$ and $E_{|T^*|}$. Then, $E_{|T|}([0, \frac{\varepsilon}{4}])$ and $E_{|T^*|}([0, \frac{\varepsilon}{4}])$ are infinte rank projections. Also, $\|TE_{|T|}([0, \frac{\varepsilon}{4}])\| \leq \frac{\varepsilon}{4}$ and $\|T^*E_{|T^*|}([0, \frac{\varepsilon}{4}])\| \leq \frac{\varepsilon}{4}$. Define

$$T_0 = (I - E_{|T^*|}([0, \frac{\varepsilon}{4}]))T(I - E_{|T|}([0, \frac{\varepsilon}{4}])).$$

Then, since T and T^* commute with their spectral projections, we have $\|T - T_0\| \leq \frac{3\varepsilon}{4}$ by the triangle inequality.

Define U_0 to be a partial isometry which sends $\ker(T_0)$ onto $\ker(T_0^*)$. This is possible since both kernels are infinite dimensional. Then, the operator $T_0 + \frac{\varepsilon}{4}U_0$ is bounded below, and hence invertible. Since $\|T - (T_0 + \frac{\varepsilon}{4}U_0)\| \leq \varepsilon$, we have approximated T by an interpolating (it was even invertible) operator to within ε. Hence, the interpolating operators are dense in $\mathcal{B}(\mathcal{H})$.

□

2.4. Proposition. *Let β be a nest whose nest algebra has infinite multiplicity. Then, the interpolating operators are dense in* $\mathrm{Alg}(\beta)$.

Proof. By hypothesis, every atom of β is infinite dimensional. Let (E_n) be the set of atoms of β, and put $\mathcal{H}_n = \mathrm{ran}(E_n)$ for each n. Let $T \in \mathrm{Alg}(\beta)$ and $0 < \varepsilon < 1$.

By the Lemma above, the interpolating operators are dense in $\mathcal{B}(\mathcal{H}_n)$. For each n, choose $A_n, B_n, T_n' \in \mathcal{B}(\mathcal{H}_n)$ so that $A_nT_n'B_n$ is the identity for $\mathcal{H}_n$, and $\|E_nTE_n - T_n'\| < \frac{\varepsilon}{4}$. If $\|T_n'\| \leq \frac{\varepsilon}{4}$ for some n, replace T_n' with $T_n' + \frac{\varepsilon}{2}E_n$. If we make this replacement, also choose new A_n, B_n in $\mathcal{B}(\mathcal{H}')$ such that $A_nT_n'B_n$ is the identity for $\mathcal{H}_n$, and we have $\|E_nTE_n - T_n'\| < \frac{3\varepsilon}{4}$. Now, we have the additional condition that $\|T_n'\| \geq \frac{\varepsilon}{4}$ for all n, since $\|T' + \frac{\varepsilon}{2}E_n\| \geq |\,\|T_n\| - \frac{\varepsilon}{2}\|E_n\|\,| \geq \frac{\varepsilon}{4}$ if we had to redefine our T_n.

Define operators $A, B \in \mathcal{B}(\mathcal{H})$ by $A = I - \sum_n(E_n - E_nA_nE_n)$ and $B = I - \sum_n(E_n - E_nB_nE_n)$. Let $T' = T - \sum_n(E_nT_n'E_n - E_nTE_n)$. By this construction, we have $AT'B$

equal to the identity on each atom, and $||T - T'|| \leq \frac{3\varepsilon}{4}$. Notice that A, B, and T' are all in $\text{Alg}(\beta)$.

Let h be an order preserving homeomorphism of β onto a compact subset $\omega \subseteq [0, 1]$. For each $t \in \omega$, let $N_t = h^{-1}(t)$; this paramaterizes our nest, β. Define the Borel set

$$U = \{t \in \omega : i_{N_t}(T') \leq \frac{\varepsilon}{8}\}.$$

Let $\mathcal{E}$ be the spectral projection for β. This gives us the projection $\mathcal{E}(U) \in \beta''$. Notice that for each atom E_n of β, $\mathcal{E}(U)E_n = 0$, since $i_{G_n}(T') = ||T_n|| \geq \frac{\varepsilon}{4}$, where G_n is the upper endpoint for E_n. Define $T'' = T' + \frac{\varepsilon}{4}\mathcal{E}(U)$. Notice that $AT''B$ is still equal to the identity on each atom E_n of β. Also, $T'' \in \text{Alg}(\beta)$.

Form a new nest, γ, by replacing each infinite rank atom E_n with a continuous nest on $\mathcal{H}_n$, which is possible because each $\mathcal{H}_n$ is infinite dimensional. So, γ is a continuous nest. Notice that $AT''B$ is also in $\text{Alg}(\gamma)$, because replacing each atom E_n of β with a continuous nest will not change the identity operator that was on that atom. Let i_M^γ be the lower diagonal seminorm function for γ. We have $i_M^\gamma(AT''B) \geq \frac{\varepsilon}{8}$ for all $M \in \gamma$. Hence $AT''B$ is interpolating in $\text{Alg}(\gamma)$, by (the Corollary to) the Interpolation Theorem (Corollary 4.3.8). Choose $A', B' \in \text{Alg}(\gamma)$ such that $A'(AT''B)B' = I$. Since A', B' are also in $\text{Alg}(\beta)$, T'' is interpolating in $\text{Alg}(\beta)$. Since we also have

$$||T - T''|| \leq ||T - T'|| + \frac{\varepsilon}{4}||\mathcal{E}(U)|| \leq \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon,$$

the interpolating operators are dense in $\text{Alg}(\beta)$.

□

3. Connectedness in Infinite Multiplicity Algebras

Let $\mathbb{A}$ be an algebra of infinite multiplicity. Notice that $\mathbb{A}$ is (continuously) isomorphic to $M_2(\mathbb{A})$ because

$$\mathbb{A} \simeq \mathbb{A} \otimes \mathcal{B}(\mathcal{H}') \simeq \mathbb{A} \otimes \mathcal{B}(\mathcal{H}' \oplus \mathcal{H}') \simeq \mathbb{A} \otimes \mathcal{B}(\mathcal{H}') \otimes M_2(\mathbb{C}) \simeq \mathbb{A} \otimes M_2(\mathbb{C}) \simeq M_2(\mathbb{A}).$$

Because of this isomorphism, the results of this section will focus on the connectedness of invertibles within $M_2(\mathbb{A})$.

3.1. Lemma. *Let $\mathbb{A}$ be an infinite multiplicity algebra. An invertible operator of the form*

$$\begin{pmatrix} 1 & A \\ B & C \end{pmatrix}$$

in $M_2(\mathbb{A})$ is connected within the invertibles to I_2.

Proof. We can factor

$$\begin{pmatrix} 1 & A \\ B & C \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C - BA \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}.$$

For any $t \in [0, 1]$, the matrix $\begin{pmatrix} 1 & 0 \\ tB & 1 \end{pmatrix}$ is invertible; the inverse is simply $\begin{pmatrix} 1 & 0 \\ -tB & 1 \end{pmatrix}$. Hence, the outer two factors are connected to I_2. The middle factor of the right hand side is invertible, so it only remains to show that any invertible operator of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}$$

is connected to I_2 within $M_2(\mathbb{A})$.

Let $\mathbb{Y}$ be a unital Banach algebra and suppose that $Y \in \mathbb{Y}$ is invertible. Then, we claim that the element $\begin{pmatrix} Y^{-1} & 0 \\ 0 & Y \end{pmatrix} \in M_2(\mathbb{Y})$ is connected to I_2. Consider the path

$$f(t) = \begin{pmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix} \begin{pmatrix} Y^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix}.$$

Notice that $f(t)$ is invertible for each t. Since $\|f(t)\| \leq 2(|\cos(\frac{\pi}{2}t)| + |\sin(\frac{\pi}{2}t)|)$, we see that $f(t)$ is a continuous path. Finally, notice that $f(0)$ is the given operator, and $f(1) = I_2$.

Now, let $X \in \mathbb{A}$ and let $\tilde{X}$ be the image of X under the isomorphism $\mathbb{A} \to \mathbb{A} \otimes \mathcal{B}(\mathcal{H}')$. Let I be the identity in $\mathcal{B}(\mathcal{H}')$. Since $\mathcal{H}'$ is infinite dimensional, $\tilde{X} \oplus I$ is unitarily equivalent to $\tilde{X} \oplus I \oplus I \oplus I \oplus \cdots$; and hence the latter is also unitarily equivalent to $I \oplus \tilde{X}$.

As we saw above, $I \oplus I$ is connected to $\tilde{X}^{-1} \oplus \tilde{X}$ in $M_2(\mathbb{A} \otimes \mathcal{B}(\mathcal{H}'))$. So, $\tilde{X} \oplus I \oplus I \oplus I \oplus \cdots$ is connected to

$$\tilde{X} \oplus \tilde{X}^{-1} \oplus \tilde{X} \oplus \tilde{X}^{-1} \oplus \cdots$$

which (by the same argument) is connected to

$$I \oplus I \oplus I \oplus I \oplus \cdots.$$

Using the continuous isomorphism between $\mathbb{A}$ and $\mathbb{A} \otimes \mathcal{B}(\mathcal{H}')$, we obtain a continuous path of invertibles which connects $\begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $M_2(\mathbb{A})$ as required.

□

3.2. Corollary. *Let $\mathbb{A}$ be an infinite multiplicity algebra. An invertible operator of the form* $\begin{pmatrix} 0 & A \\ B & C \end{pmatrix} \in M_2(\mathbb{A})$ *is connected to* I_2.

Proof. Let $T = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$. If $0 \leq r < \frac{1}{\|A^{-1}\|}$, then the operator

$$T_r := \begin{pmatrix} rI & A \\ B & C \end{pmatrix}$$

is also invertible.

Thus, T is connected (by a straight line segment) to T_ε where $\varepsilon = \frac{1}{2\|T^{-1}\|}$. At each point in the path, the operator differs from T in norm by at most ε, and hence is invertible.

Next, we can connect T_ε to

$$\begin{pmatrix} 1 & \frac{1}{\varepsilon}A \\ \frac{1}{\varepsilon}B & \frac{1}{\varepsilon}C \end{pmatrix}$$

via a path of scalar multiples of T_ε. The last matrix is connected to I_2 by Lemma 3.1.

□

Similar results to Lemma 3.1 and Corollary 3.2 hold even when the 'special' entry is somewhere other than the $(1,1)$ corner. For example, in the proof of Lemma 3.1 we could change our factorization to

$$\begin{pmatrix} A & B \\ 1 & C \end{pmatrix} = \begin{pmatrix} A & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ B - AC & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & C \end{pmatrix}$$

to obtain the same results for the $(2,1)$ entry.

3.3. Theorem. *Let $\mathbb{A}$ be an infinite multiplicity algebra. Let $X \in \mathbb{A}$ be interpolating. If the operator $\begin{pmatrix} X & A \\ B & C \end{pmatrix} \in M_2(\mathbb{A})$ is invertible, then it is connected to I_2.*

Proof. Suppose that $UV = I$. Then, the operators

$$D = \begin{pmatrix} U & 0 \\ 1 - VU & V \end{pmatrix} \text{ and } E = \begin{pmatrix} V & 1 - VU \\ 0 & U \end{pmatrix}$$

are invertible; they are inverses of each other. By Corollary 3.2, each is connected to I_2. Let $S, T \in \mathbb{A}$ be such that $SXT = I$.

Substitute $U = S$, $V = XT$ in A, and $U = SX$, $V = T$ in B. We have that $\begin{pmatrix} X & A \\ B & C \end{pmatrix}$ is connected to

$$D\begin{pmatrix} X & A \\ B & C \end{pmatrix}E = \begin{pmatrix} S & 0 \\ 1 - XTS & XT \end{pmatrix}\begin{pmatrix} X & A \\ B & C \end{pmatrix}\begin{pmatrix} T & 1 - TSX \\ 0 & SX \end{pmatrix}$$

$$= \begin{pmatrix} I & * \\ * & * \end{pmatrix}.$$

Since the last matrix is invertible, it is connected to the identity by Lemma 3.1.

□

CHAPTER 6

Connectedness of the Invertibles

We will now show that the invertibles are connected in a nest algebra where there is a finite bound on the number of consecutive finite rank atoms in the nest. In particular, this shows that the invertibles are connected in any continuous nest algebra (where there are no atoms), and any nest algebra of infinite multiplicity (where there are no finite rank atoms). The results here are based on [**3**].

1. Nests with no Isolated Finite Rank Atoms

1.1. Definition. Let β be a nest, and let (E_n) be the set of finite rank atoms of β. The projection $P_f = \sum_n E_n$ is called the **projection onto the finite part** of β. The **projection onto the infinite part** of β is $P_\infty = I - P_f$. For $T \in \mathrm{Alg}(\beta)$, we can define the corresponding compressions $\Delta_f(T) = P_f T P_f$ and $\Delta_\infty(T) = P_\infty T P_\infty$.

1.2. Definition. Let β be a nest. A β-interval $E = N - M$ is called **isolated** if either of its endpoints (N or M) is an isolated point of the topological space (β, SOT).

1.3. Theorem. *Let β be a nest with no isolated finite rank atoms and assume that β has maximal non-atomic part. Let $T \in \mathrm{Alg}(\beta)$ be such that its compression to the finite rank atoms, $\Delta_f(T)$, is zero. Then, there exist operators $A, B \in \mathrm{Alg}(\beta)$ such that $T = AP_\infty B$.*

Proof. If E is a β-interval which is not a finite rank atom, then $P_\infty E$ has infinite rank. Hence we may write

$$P_\infty = P_0 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{m,n}$$

where all of the projections P_0 and $P_{m,n}$ are pairwise orthogonal, and also satisfy

$$\operatorname{rank}(P_0 E) = \operatorname{rank}(P_{m,n} E) = \infty$$

whenever E is not a β-interval which is not a finite rank atom. In particular, it follows from Theorem 3.1.2 that P_∞ is algebraically equivalent to P_0.

Let $E_n = G_n - L_n$ be a finite rank atom for some (fixed) $n \in N$. There are at most countably many such atoms, because we are always assuming our Hilbert spaces are separable; this allows us to use $n \in \mathbb{N}$ to index the finite rank atoms. Since $E_n T = E_n T(I - G_n)$ has finite rank, it is compact.

Select a strictly descreasing sequence $(N_{m,n})_{m=1}^{\infty} \subseteq \beta$ such that

$$G_n = \inf_{m \geq 1} \{N_{m,n}\} \text{ and } \|E_n T N_{m,n}\| < \frac{1}{m2^n}.$$

This sequence exists, for suppose to the contrary that we could not find such a sequence. Then we would have $\varepsilon > 0$, and an infinite dimensional subspace $\mathcal{Z}$ on which $\|E_n T(I - G_n)z\| \geq \varepsilon$ for $z \in \mathcal{Z}$, which contradicts compactness. (Recall that the finite rank atom E_n is isolated by hypothesis.)

As a result,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m\|E_n T N_{m,n}\| \leq 1.$$

For each $m, n \geq 1$ choose a partial isometry $V_{m,n}$ which maps $\operatorname{ran}(E_n)$ into $\operatorname{ran}((N_{m,n} - G_n)P_{m,n})$. This is possible since the latter space is infinite dimensional. Notice that we may write

$$V_{m,n} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & 0 \end{pmatrix} \begin{matrix} L_n\mathcal{H} \\ E_n\mathcal{H} \\ (I - G_n)\mathcal{H} \end{matrix}$$

and hence, $V_{m,n}^*$ is in the nest algebra, $\operatorname{Alg}(\beta)$.

Set $N_{0,n} = I$ and $J_{m,n} = N_{m-1,n} - N_{m,n}$, and define

$$A = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m} V_{m,n}^{*} \text{ and } B = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m V_{m,n} E_n T J_{m,n}$$

Note that A is bounded because of the scalars $\frac{1}{m}$. We claim that B is also bounded. Define

$$B_0 = \sum_{m=2}^{\infty}\sum_{n=1}^{\infty} m V_{m,n} E_n T J_{m,n}.$$

Then, B_0 is compact. Also,

$$B - B_0 = (\sum_{n=1}^{\infty} V_{1,n})(P_f - \sum_{n=1}^{\infty} E_n T N_{1,n})$$

where $P_f = I - P_\infty$ is the projection onto the finite part of β. Since the last term on the right hand side is compact, $B - B_0$ is bounded, and hence so is $B = B_0 + (B - B_0)$.

Set $P_1 = \sum_{n=1}^{\infty} P_{1,n}$. We have $AB = AP_1B = P_fT$. Since P_∞ is algebraically equivalent to P_0 we can choose operators $A_1, B_1 \in \mathrm{Alg}(\beta)$ such that $P_\infty = A_1B_1$ and $P_0 = B_1A_1$. Set $A_2 = A_1P_0$ and $B_2 = P_0B_1$. Then, $A_2P_0B_2 = A_1P_0B_1 = A_1(B_1A_1)B_1 = P_\infty$. Also, $A_2P_0 = A_2$ and $P_0B_2 = B_2$. So, we calculate

$$(A + A_2)P_\infty(B + B_2P_\infty T) = AB + A_2B_2P_\infty T = P_fT + P_\infty T = T.$$

□

1.4. Theorem. *Let β be a nest with no isolated finite rank atoms. Then, the invertibles in* $\mathrm{Alg}(\beta)$ *are connected.*

Proof. We may assume that β has maximal non-atomic part, since β is similar to such a nest; hence, the corresponding nest algebras are isomorphic. So, the subspace $P_\infty\mathcal{H}$ can be decomposed as $P_\infty\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are unitarily equivalent. Let P_1 and P_2 be the orthogonal projections onto $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.

Let $T \in \mathcal{B}(\mathcal{H})$ be invertible. We decompose T as

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{matrix} P_f\mathcal{H} \\ P_\infty\mathcal{H}, \end{matrix}$$

and decompose D further as

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2. \end{matrix}$$

Since $P_2\beta$ has infinite multiplicity, there exist $X, Y, D'_4 \in \mathrm{Alg}(\beta)$ such that $XD_4Y = I$ and $\|D_4 - D'_4\| < \frac{1}{2\|T^{-1}\|}$ (Proposition 5.2.4). Consider the path

$$f(T) = \begin{pmatrix} A & B \\ C & D_t \end{pmatrix} \begin{matrix} P_f\mathcal{H} \\ P_\infty\mathcal{H} \end{matrix} \quad \text{where } D_t := \begin{pmatrix} D_1 & D_2 \\ D_3 & (1-t)D_4 + tD_4 \end{pmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$

for $t \in [0,1]$. Then, f is continuous, for $\|f(t_1) - f(t_2)\| = |t_1 - t_2|\|D_4 - D'_4\|$ for any $t_1, t_2 \in [0,1]$. Since $\|f(T) - T\| = \|(1-t)D_4 + tD'_4 - D_4\| = |t|\|D_4 - D'_4\| \leq \frac{1}{2\|T^{-1}\|}$, $f(t)$ is invertible for $t \in [0,1]$. Hence, T is connected to the matrix

$$\begin{pmatrix} A & B \\ C & D' \end{pmatrix} \begin{matrix} P_f\mathcal{H} \\ P_\infty\mathcal{H} \end{matrix} \quad \text{where } D' := \begin{pmatrix} D_1 & D_2 \\ D_3 & D'_4 \end{pmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2. \end{matrix}$$

For convenience of notation, we will replace D_4 with D'_4.

As in the proof of Theorem 5.3.3, there exist invertible operators X' and Y' of the form

$$X' = \begin{pmatrix} * & * \\ 0 & X \end{pmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix} \quad \text{and } Y' = \begin{pmatrix} * & 0 \\ * & Y \end{pmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2, \end{matrix}$$

both of which are connected to I_2 by Corollary 5.3.2. Let

$$T' = \begin{pmatrix} I & 0 \\ 0 & X' \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Y' \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{matrix} P_f\mathcal{H} \\ P_\infty\mathcal{H}. \end{matrix}$$

It now suffices to show that T' is connected to the identity. Notice that

$$D' = X' \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} Y' = \begin{pmatrix} * & * \\ * & I \end{pmatrix}.$$

Using the same factorization as in the proof of Lemma 5.3.1, we may connect T' to in invertible operator T'' of the form

$$T'' = \begin{pmatrix} A'' & B'' & 0 \\ C'' & D'' & 0 \\ 0 & 0 & I \end{pmatrix} \begin{matrix} P_f\mathcal{H} \\ \mathcal{H}_1 \\ \mathcal{H}_2. \end{matrix}$$

Set

$$E = \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix}$$

and $E_0 = E - \Delta_f(E) - P_1$. Then,

$$E_0 = \begin{pmatrix} A'' - \Delta_f(A'') & B'' \\ C'' & D'' - I \end{pmatrix} \begin{matrix} P_f\mathcal{H} \\ \mathcal{H}_1, \end{matrix}$$

because (with respect to the same decomposition) $\Delta_f(E) = \begin{pmatrix} A'' & 0 \\ 0 & 0 \end{pmatrix}$ and $P_1 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$. Also notice that $\Delta_f(E_0) = 0$, since $\Delta_f(\Delta_f(A'')) = \Delta_f(A'')$.

By Theorem 1.3 we can factor $E_0 = FP_1G = \begin{pmatrix} 0 & F_1 \\ 0 & F_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ G_1 & G_2 \end{pmatrix}$, where $F, G \in \mathrm{Alg}((P_f + P_1)\beta)$. Now,

$$\begin{pmatrix} \Delta_f(A'') & 0 & F_1 \\ 0 & I & F_2 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ G_1 & G_2 & I \end{pmatrix} = \begin{pmatrix} A'' & B'' & F_1 \\ C'' & D'' & F_2 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} E & F'' \\ G'' & I \end{pmatrix}$$

is connected to the identity, where $F'' = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ and $G'' = \begin{pmatrix} G_1 & G_2 \end{pmatrix}$. Since E is invertible, we have

$$\begin{pmatrix} I & 0 \\ -G''E^{-1} & I \end{pmatrix}\begin{pmatrix} E & F'' \\ G'' & I \end{pmatrix}\begin{pmatrix} I & -E^{-1}F'' \\ 0 & I \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & I - G''E^{-1}F'' \end{pmatrix}.$$

The right hand side is connected to $\begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix} = T''$, and so we have connected our original T to the identity.

□

2. Nests with a Finite Bound on the Number of Consecutive Finite Rank Atoms

2.1. Corollary. *Let β be a nest such that there is a finite bound on the number of consecutive finite rank atoms of β. Then, the invertibles are connected in* $\mathrm{Alg}(\beta)$.

Proof. Let $N \in \mathbb{N}$ be an upper bound for the number of consecutive finite rank atoms of β. Let (J_n) be the set of maximal β-intervals which are formed by summing consecutive finite rank atoms. Let $T \in \mathrm{Alg}(\beta)$ be invertible.

Let $P_a = \sum_i E_i$, where E_i runs over the set of all atoms of β. Then, P_a is the projection onto the atomic part of β. Let Δ be the compression onto the atomic part; namely, $\Delta(X) := P_a X P_a$. The projection onto the continuous part of β is $P_c = I - P_a$.

Set $D = \Delta(T) + P_c$. Notice that D is invertible, with $D^{-1} = \Delta(T^{-1}) + P_c$. Also, D is connected to the identity within the diagonal $\beta' = \mathrm{Alg}(\beta) \cap (\mathrm{Alg}(\beta))^*$.

Next, let $A = \sum_n J_n D^{-1} T J_n + P_\infty$. For each n, β_{J_n} is a nest with at most $N+1$ elements. So, A is of the form $I + B$, where B is nilpotent of order at most $N+1$. For any

$t \in [0,1]$, the inverse of $I + tB$ exists. Indeed, it is given by the finite sum $I + \sum_{k=1}^{N+1}(-tB)^k$. Hence, A is connected to I by a path of invertibles in $\mathrm{Alg}(\beta)$.

Finally, let $C = A^{-1}D^{-1}T$. We claim that C is also connected to the identity. On showing this, we will have connected the product $T = DAC$ to the identity, which is what we want.

We form a new nest γ from β by replacing each J_n with a single atom (which is finite rank); and by replacing each infinite rank atom, say G_k, with a continuous nest on the Hilbert space $G_k\mathcal{H}$. Then, our nest γ is such that no finite rank atom of γ is isolated. Hence, by Theorem 1.4, we may connect C to the identity in $(\mathrm{Alg}(\gamma))^{-1}$ by some path, say $f(t)$. Then, define a new path

$$g(t) = \Delta_\gamma(f(t))f(t)^{-1}$$

where Δ_γ denotes the compression onto the atomic part of γ. The path $g(t)$ also connects C to the identity within $(\mathrm{Alg}(\gamma))^{-1}$. Hovever, the compression of $g(t)$ to the atomic part of γ is always the identity, since

$$\Delta_\gamma(g(t)) = \Delta_\gamma^2(f(t))\Delta_\gamma(f(t)^{-1}) = \Delta_\gamma(f(t)f(t)^{-1}) = \Delta_\gamma(I).$$

So, $g(t)$ is a path which also lies within $(\mathrm{Alg}(\beta))^{-1}$. Hence, $T = DAC$ is connected to the identity within $\mathrm{Alg}(\beta)$.

□

CHAPTER 7

Conclusions

Let us apply the connectedness results we have studied to some of the examples of nest algebras given in the Introduction.

We already have seen a direct proof that the invertibles are connected in the nest algebra of a maximal nest on a finite dimensional Hilbert space. More abstractly, any nest on a finite dimensional Hilbert space certainly has a finite bound on the number of consecutive finite rank atoms in the nest: we could take the dimension of the Hilbert space itself as our bound. Hence, the invertible group is connected for a nest algebra on a finite dimensional Hilbert space by Theorem 6.2.1.

Since continuous nests have no atoms at all, $N = 0$ is a finite bound on the number of consecutive finite atoms in the nest. Again by Theorem 6.2.1, the invertibles are connected in any continuous nest algebra. In particular, the invertibles are connected in the Volterra nest algebra.

More generally, a nest with an infinite multiplicity nest algebra has no finite rank atoms. In particular then, the nest has no isolated finite rank atoms. Hence, Theorem 6.1.4 shows that the invertibles are connected in any infinite multiplicity nest algebra. Let us consider a special case. Let $\mathcal{H}$ is an infinite dimensional (separable) Hilbert space. The nest algebra for the trivial nest $\{0, I\}$ is $\mathcal{B}(\mathcal{H})$. Since $\mathcal{B}(\mathcal{H})$ is an infinite multiplicty nest algebra (the nest has no finite rank atoms), the invertible group of $\mathcal{B}(\mathcal{H})$ is connected.

Finally, let β be the Cantor nest. The atoms of β are of the form $N_t^+ - N_t^- = M_{\chi_{\{t\}}}$ for $t \in \mathbb{Q}$. Each atom has rank one, since the range is just $\{c\delta_t : c \in \mathbb{C}\}$ where δ_t is the point mass function at t. For any $t \in \mathbb{Q}$, we can choose a strictly increasing sequence of rationals,

$(s_n)_{n=1}^{\infty}$, with $s_n \to t$ as $n \to \infty$. Then,

$$\text{SOT} - \lim_{n\to\infty} N_{s_n}^- = N_{(\lim_{n\to\infty} s_n)}^- = N_t^-.$$

So, the atom $N_t^+ - N_t^- = M_{\chi_{\{t\}}}$ is not isolated. Since t was arbitrary (in $\mathbb{Q}$), there are no isolated finite rank atoms in β. By Theorem 6.1.4, the invertibles are connected in $\mathrm{Alg}(\beta)$.

It is still unknown whether or not the invertibles are connected in the nest algebra of upper-triangular (bounded) matrices on an infinite dimensional Hilbert space (Example 2.2.4). It is thought by many that the invertible group of this nest algebra is not connected, because this is the case in the commutative analogue, $H^\infty(\mathbb{D})$; this is the space of analytic functions on the closed unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$.

Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{Z}}$. Let γ be the nest $\{0, P_k, I : k \in \mathbb{Z}\}$, where P_k is the projection onto the closed span of $\{e_n : n \leq k\}$. In [**10**], D. Pitts shows that a specific invertible operator $T \in \mathrm{Alg}(\gamma)$ is actually connected to the identity, though it was previously thought that this T would be a good candidate to be an operator which would lie outside the connected component of I.

Another approach to the invertibility question for this nest algebra might be through K-theory. D. Pitts has calculated the K_0 group of an arbitrary nest algebra [**9**]. The K_1 group of a nest algebra is still unknown in general, and, if found, could provide information about the connected components of the invertibles of the nest algbera.

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