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THE AESTHETICS OF MATHEMATICAL PROOFS

by

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Abstract

Mathematical theories and entities may be praised for many reasons: for their truth, certainty, depth, applicability, fruitfulness, and beauty. Of these virtues, some (truth, certainty, and to a lesser degree applicability) have received a great deal of philosophical scrutiny and analysis while the others have remained vague and mysterious. This thesis attempts to dispel some of the fog around one member of the latter category, namely *beauty*. While many objects of mathematics are of aesthetic interest, this thesis focuses specifically on the aesthetics of proofs because of their central position in mathematical practice. It is argued that proofs are conceptual artefacts, and that our perception of them is framed by how we use them. Contemporary theories of functional beauty are then adapted in order to understand and analyse aesthetic claims made about proofs.

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Chapter 1

Aesthetic Talk in Mathematics

1.1 Introduction

References to aesthetic factors are a recurring theme in discussions by mathematicians of mathematics, however they have very rarely been taken up with any philosophical seriousness. My aim in this chapter is to tease out some common threads in these discussions which illuminate the phenomenon of aesthetics within mathematics as well as help isolate why it is a recurring topic, and therefore why its study is important for philosophy of mathematics. In doing this, I hope to also shed light on the reasons philosophers have until quite recently remained relatively mum on the topic.

In many respects, it is self evident that the aesthetic (or at least, the appearance of the aesthetic) plays some role in mathematics. Sinclair (2004) identifies three distinctive ways in which the aesthetics is relevant to mathematical practice: it may be evaluative, leading to the aesthetic judgements of various mathematical objects, generative insofar as it aids inquiry and discovery, and motivational in that leads mathematicians to study particular subjects and problems (264). This breakdown is insightful, although we must still ask the degree to which such factors impact the field, especially in comparison to others. The evaluative role also remains mysterious as it is unclear why such judgements should be possible to make in the first place, and what precisely they entail.

To motivate and clarify these problems, I will focus on three notable and influential attempts by 20th century mathematicians to describe, explain, and sometimes define the mathematical aesthetic. The work under consideration is by mathematicians themselves who are, in many cases, writing so as to give the layman a better sense of what they as mathematicians do, and to explain the value and importance of mathematics as they see it. As a result, these accounts are in places semi-autobiographical and largely focused on conveying personal experience and insight rather than providing a proper philosophical analysis. Nonetheless, these texts have helped set the stage for the ongoing discussion of these issues.

The first work I consider is that of Poincaré (1910), which addresses the psychology underlying mathematical creativity. Poincaré provides an extended description and analysis of his personal experiences while working on proofs. Poincare views aesthetics as playing an important although subconscious role in the process of discovery. The second text is G.H. Hardy's 1940 semi-autobiographical text *A Mathematician's Apology.* The aim of this short book is not to give a rigorous account of aesthetics in mathematics (in fact Hardy acknowledges that he has "no qualifications for any serious discussion in aesthetics"(90)) but rather to justify the practice of mathematics, and discuss in what ways it is valuable.

Finally, I consider Jerry King who follows in Hardy's footsteps with his

1992 book *The Art of Mathematics*. Here King aims to explain what it is that mathematicians do, and how it is that the aesthetic is part of mathematical experience. Through this project, he hopes both to help the non-mathematician appreciate the appeal of mathematics, and show what kinds of cultural and institutional changes might be needed in the academic world to enhance teaching such that non-mathematicians are more aware of and receptive to genuine mathematics. Notably, King's book contains perhaps the most sustained attempt at a detailed aesthetic theory for mathematics.

I focus primarily on the reasons these authors have for taking up the topic in a semi-theoretical capacity: why they have come to believe and examine the fact that the aesthetic dimension is valuable to their work and experiences as mathematicians. Thus, to facilitate the discussion, I identify two major recurring themes which drive the authors in question to raise aesthetic issues with regard to mathematics. The first theme, of particular interest to Hardy and King, is a concern with *why* mathematics (especially pure mathematics) is practised at all, and why it should be considered a worthwhile human endeavour in spite of its highly theoretical nature. There is a normative shade to this theme; these mathematicians wish to defend their work and demonstrate not just *that* it is done, but that it *should* be done. Practical and moral justifications being hard to establish, the hope on the part of these authors is that mathematics can be shown to have independent aesthetic value, perhaps putting it on a par with art. The aesthetic becomes an important source of value for mathematics as a discipline.

The second theme is a concern with *how* mathematics is practiced by (human) mathematicians, in particular how problems are posed and solved. A focus on these elements is exemplified by Poincaré and those who pursue a more distinctively cognitive approach to the question of aesthetic appreciation in mathematics. Here, Sinclair's generative and motivational roles come into the spotlight, with the aesthetic seeming to occupy important, and perhaps essential, places in mathematical reasoning processes. Where the former theme was normative, the theme here is more descriptive in nature with the question of what is necessary to have mathematics at all being at issue.

1.2 Arguing from Value

A recurring motivation for identifying aesthetic features of mathematical objects involves the need to make value judgements. The claim is that mathematicians must make such distinctions in order to direct themselves toward new research projects, or improve upon old ones, and so the question becomes *what makes mathematics valuable?* There are three obvious potential answers:

1. Mathematics is valuable because it provides us with a body of a-priori, necessary *truths*.

2. Mathematics is valuable because it is *useful*. In particular, it is of enormous importance to the natural sciences.

3. Mathematics is valuable because it is *beautiful.*.

Although most would agree that all three of these elements contribute to the overall value of mathematics (with the third being the most controversial), the relative importance mathematicians place on each can dramatically alter their view of the subject. This harkens back to Sinclair's motivational role, and indeed it has been argued that "the basic viewpoint people adopt toward mathematics can influence the direction of research"(Krull, 1987, 50) .

Those interested primarily in the first element gravitate toward foundationalist projects such as axiomatizing known theories, those interested in the second choose projects in applied areas at the cross section of mathematics and the sciences, and those motivated primarily by the aesthetic will work on theoretical projects with special concern for finding and presenting aesthetically satisfying results.

In the cases where (3) is treated as a value of primary importance to mathematical work, it is generally argued that (1) and (2) are each, in some sense, lacking. In any endeavour, it is necessary to make assessments of recently completed, newly adopted, and potential future work, as well as long standing results and products. These judgements are clearly made by the mathematician choosing where to direct their energies, but might also be made by outsiders to the field (who might be in politically or financially relevant positions). Part of the aim of some authors, such as Hardy, is to reveal to outsiders some of the value judgements that they as mathematicians make.

Judging the quality of a proof or theorem based on validity alone might be considered inadequate for these purposes because such a measure is binary and as such does not provide any way to assess relative quality: we cannot make assessments about one proof of a given theorem being better or worse than another proof of the same theorem if this is all we look to in our evaluations. It fails to distinguish run of the mill results from those that are innovative, and it does not seem to provide value assessments that could be considered goal directing for mathematicians themselves.

Utility seems like a reasonable value to assert for mathematics, especially if you are concerned with making a large scale judgement of the discipline as a whole. There can be no denying the prevalence of applications of mathematical concepts to domains from the scientific, to the technological. Moreover, such considerations certainly have played a role in some of the development of the field, especially when considering decisions about what theories to pursue. Consider, for instance, the history of the quaternions which, after an initial burst of activity was briefly abandoned when it appeared matrices could do the same sort of applied work, and has come back into popularity with the discovery of new uses.

Yet despite its promising nature, this sort of criterion still fails to account for all aspects of mathematical development. It does not allow us to adequately assess mathematical works, because in many cases we require decades or even centuries to make the transition from theory to application (and even the case of the quaternions illustrates how unpredictable the process can be, with promising ideas being dropped and readopted). This fact has recently been pointed out by Rowlett (2011) who challenges political and institutional pressure "for researchers to predict the impact of their work before it is undertaken"(166) by cautioning that "there is no way to guarantee in advance what pure mathematics will later find application"(167) and cataloguing examples of applications with gestation periods so long that they could not reasonably have been anticipated when the work was initially carried out.

We are left to look for features (such as a streamlined presentation, and intra–mathematical utility) of the result or proof which allow us to judge its relative value without depending entirely on either the fact of its validity, or on non-mathematical applications. This leads naturally into consideration of the aesthetic as potentially filling that hole. For instance, King holds that aesthetic valuations are used to distinguish correct and 'good' mathematics from correct but nonetheless 'bad' mathematics. That is, he considers a good piece of mathematics to be a "beautiful mathematical artifact"(King, 1992, 179). Moreover, King feels that "it is hard to overestimate the degree of unanimity which exists among mathematicians regarding the beauty of certain mathematical results"(174). In his view, there is general agreement among mathematicians about what is and is not beautiful and this strongly suggests "some set of commonly accepted aesthetic principles."(179) [1]

Hardy strongly and quite self-consciously separates the aesthetic dimension from that of utility. In fact, he is somewhat unusual insofar as he views non-mathematical applicability as being not merely aesthetically neutral but almost incompatible with beauty. This is most apparent in the distinction he makes between "real mathematics" and "trivial mathematics"(Hardy, 1992, 139). Trivial mathematics, he claims, is largely elementary and makes up the bulk of what the lay person believes to be mathematics. It can be (and usually is) justified on the basis of its abundant applications to industry and the sciences. Real mathematics, by contrast, is the cutting edge practised by professional mathematicians themselves and of which non-mathematicians,

[1] King is not unusual in this regard. Many others, have stated similar views about aesthetic universalism in mathematics. For instance, Rota (1997) claims that "mathematical beauty and mathematical truth, like other objective characteristics of mathematics, are subject to the eidetic laws of the real world, on a par with the laws of physics" (176) while at the same time noting that beauty (like truth, for him) is non-arbitrary but still context-dependant. Thus, conclusions about what is and is not beautiful may change over time, but will still be agreed upon within a particular mathematical community. Assumptions that there is an objective and universally recognizable fact of the matter are also implicit in attempts (e.g. Aigner and Ziegler (2009)) to identify 'the best' proofs.

This assumption has been challenged to varying degrees by others such as Krull (1987), Le Lionnais (1971), and Wells (1990). Wells performed an informal survey of mathematicians, asking them to rank various theorems for their beauty and give explanations for their responses, from which he concluded that "the idea that mathematicians largely agree in their aesthetic judgements is at best grossly oversimplified."(40). Krull and Le Lionnais expressed doubts about the claim by arguing that there can be, and in fact have been, significant differences in taste and style among mathematicians (Le Lionnais in particular distinguishes the classical from the romantic). I will take up the problem of aesthetic taste in the last chapter.

even those who use the tools emerging from trivial mathematics, tend to be quite ignorant. At the same time, Hardy considers many *theoretical* physicists (namely Maxwell, Einstein, Eddington and Dirac whose work on quantum mechanics and relativity theory he considers "almost as 'useless' as the theory of numbers"(Hardy, 1992, 131-132)) to be real mathematicians. "Real mathematics" is thus the term Hardy uses to isolate and refer to the body of mathematical work which lacks practical applicability or which only produces practical results after it has been around for long periods of time, in the manner discussed above.

In large part because of the wedge he has drawn between usefulness and aesthetic merit, the aesthetic is a core element of Hardy's apology. He argues that "such mathematics is useful as is wanted by a superior engineer or a moderate physicist; and that is roughly the same thing as to say, such mathematics has no particular aesthetic merit"(Hardy, 1992, 134), and therefore finds that real mathematics "must be justified as an art if it can be justified at all"(139). Hardy is, through this distinction, able to divorce real mathematics from ethical concerns which arise in applications of trivial mathematics (in particularly to warfare), finding it to be "harmless". Pure mathematics, on this view, is the domain of aesthetics, while applied mathematics is the domain of ethics.

While King does not separate usefulness from beauty in the way Hardy does, he finds the fact that there are applications for mathematics which was originally explored (he claims) for aesthetic motives to be a "miracle of second order of magnitude"(King, 1992, 121). King recalls the famous problem raised by Wigner (1960) of the "unreasonable effectiveness" of mathematics in applied domains. This phenomenon has an uncanny and "miraculous" quality largely because while "the concepts of elementary mathematics and particularly elementary geometry were formulated to describe entities which are directly suggested by the actual world, the same does not seem to be true of the more advanced concepts, in particular the concepts which play such an important role in physics."(Wigner, 1960, 2) Instead, these concepts "were so devised that they are apt subjects on which the mathematician can demonstrate his ingenuity and sense of formal beauty."(3) It is precisely because the applications come later, and frequently not by initial design, that their pervasiveness is so striking. For King, Wigner's observation combined with a presumed dominantly aesthetic motivation behind mathematical work create "the *paradox of the utility of beauty*"(King, 1992, 121). There is no reason to suppose that beautiful mathematics should ever produce useful results, and so the fact that utility emerges from the process appears all the more mysterious.

Like Hardy, and in a manner that naturally grows from the value theoretic approach discussed here, King holds that mathematics has much in common with the fine arts. He motivates this view by describing an analogy between Dickie's "artworld" and the cultural institution around mathematics, which he refers to as the "mathworld". For Dickie

1. An artist is a person who participates with understanding in the making of a work of art.
2. A work of art is an artifact of a kind created to be presented to an artworld public.
3. A public is a set of person the members of which are prepared in some degree to understand an object which is presented to them.
4. The artworld is the totality of all artworld systems.

5. An artworld system is a framework for the presentation of a work of art by an artist to the artworld public. (Dickie, 1984, 80-82)

King uses the mathematical metaphor of an *isomorphism* (that is "a one-to-one correspondence between two sets which preserves particular operations on these sets." (King, 1992, 166)) to describe the relation between Dickie's theory of an artworld and his own suggested theory of a mathworld. In particular, he adapts Dickie's definitions by replacing the words "artist" with "mathematician", "art" with "mathematics", and "artworld" with "mathworld". This transformation yields the mathworld. King then claims that the various practices described by words such as "participates", "making work", "presented", "created", and "presentation" are the operations preserved under such a transformation, being common to both the mathworld and the artworld. Thus, according to King, the artworld and the mathworld are structurally similar.

The only significant differences between the mathworld and the artworld are in how they actually manifest in our culture. In particular, while the artworld contains art critics, the math world does not, according to King, contain critics of the same sort. This could reasonably be taken as evidence against mathematics being considered an art, being an importantly different feature between the two fields; however King suggests that this lack is instead a problem that should be addressed. He argues that "the mathworld *allows* for the possibility of mathematical critics of all types" but nonetheless "no critics live in the math world system"(King, 1992, 172) other than mathematicians themselves.[2] As a result, mathematicians are not held accountable to the

[2] The extent to which mathematical criticism is really absent can be disputed. There are sometimes discussions embedded in journal articles which might be considered criticisms,

public, and most nonmathematicians are denied the preparation necessary to allow them to understand and appreciate works of mathematics, thereby entering into the mathworld public. Without the critics to act as a guide, the mathematical community remains highly isolated. To change this fact, King suggests that numerous alterations must be made within mathematical culture and education systems. He would like to "embed" the mathworld in the artworld, making it a "subset" thereof (173).

King is not alone in viewing a strong kinship between mathematics and art. Hardy, John Sullivan[3], and others have voiced similar claims. Views of this form are derived through a combination of focusing on the aesthetic value of mathematics and consciously separating mathematics from other "real world" concerns, as is exemplified by Hardy's real/trivial distinction. The trend reflects a general desire to justify mathematics on its own terms. This may be in part related to historical and political trends, especially in the 20th century, being a time when "the field of mathematics made its final separation from the sciences"(Sinclair and Pimm, 2007, 16) to which it had been tightly bound in the past. Because of this sort of change, questions of how to categorize math (should we think of it as an art, a science, both, or

although these do not play the same role as art criticism in communicating with the public. There are also numerous sources of "recreational" mathematics, as well as collections of proofs and results aimed at varying levels of mathematical skill. These sometimes contain brief discussions and analyses of the various merits of a particular mathematical artefact which could be considered elementary criticism. Nonetheless there are not very many sustained commentaries on the aesthetic values of particular theorems, proofs, or texts, although the analyses of ancient mathematical texts in Netz (2005, 2009) could certainly be considered examples of such projects undertaken recently.

[3] "...it is certain that the real function of art is to increase our self-consciousness; to make us more aware of what we are, and therefore of what the universe in which we live really is. And since mathematics, in its own way, also performs this function, it is not only aesthetically charming but profoundly significant. It is an art, and a great art. It is on this, besides its usefulness in practical life, that its claim to esteem must be based."(Sullivan, 1956, 2021)

something entirely different?) arose. Exploration of aesthetic themes, and the justification of mathematical work on grounds unrelated to its scientific uses thus represents, alongside foundational projects, a part of the introspection that the discipline continues to undergo. Authors such as Hardy and King are particularly concerned with how mathematics can be defended as an essentially autonomous and intrinsically valuable field of study, and the aesthetic domain plays an important role in this project. The resulting rather romantic idea that mathematics may be pursued on aesthetic grounds "for the greatest good of all and for the greatest glory of the human adventure, by men who expect no material profit for themselves"(Le Lionnais, 1971, 147) makes it tempting for many to place mathematics with the fine arts rather than with the natural sciences.

1.3 Arguing from Practice

Those interested in the topic of aesthetics with respect to mathematical practice consider the manner in which it may come to bear on the actual development of mathematics as a field, and seek concrete examples that can help shed light on how and why this occurs. The suggestion is that aesthetic considerations can be seen in many places, ranging from the projects mathematicians choose, to the methods they employ. Aesthetic evaluations both of a mathematician's personal work and that of their peers may also impact the dissemination of information. An example of this is given by Krull (1987) who points out how Dedekind withheld a proof for years because "it did not satisfy his aesthetic demands"(57).

In their most basic form, projects investigating practice lack the normative undertones of their value based brethren (although, the two themes are often mixed). The claim is that, whether or not it is beneficial or harmful, the aesthetic *does* play a role in mathematics. As a result "beauty in mathematics must be incorporated into any adequate epistemology of mathematics. Philosophies of mathematics that ignore beauty will be inherently defective and incapable of effectively interpreting the activities of mathematicians"(Wells, 1990, 41). Since it appears that there is some sort of aesthetic presence in mathematical practice, it is incumbent upon philosophers to take the topic seriously if they are to develop a rich and adequate understanding of mathematics as a whole.

While Hardy obviously treats the aesthetic as being an important feature in mathematical practice, he largely uses this in support of his value theoretic theme. By suggesting that this is the value that mathematicians themselves appeal to in their work, he is able to both provide an explanation for why the subject is studied and give support to the idea that such a valuation is plausible. Issues of mathematical practice are of more central importance to Poincaré (1910) who is an early example of a mathematician discussing the aesthetic in order to explain aspects of reasoning and thought.

Poincaré places the primary place of aesthetics at the beginning of the creative process. A mathematician, on his view, must have an intuitive faculty attuned to aesthetic features such as "order" and "harmony" in order to make the judgements necessary in research. This "delicate feeling"(Poincaré, 1910, 324) is rare, but nonetheless known to "all real mathematicians" (331) (here Poincaré seems to make a similar distinction as Hardy between "real" serious mathematicians and those doing more trivial work). Mathematical creation, he claims, consists in combining already known entities in new ways. This cannot be done haphazardly because there are infinitely many potential combinations, most of which are trivial or uninteresting. The role of Poincaré's aesthetic intuition is thus to help filter potential research directions towards those which are worthwhile. That is, the mathematician must direct his attention toward only those projects which may be "useful" insofar as they help us better understand "mathematical law" (325), and reveal connections between what appear to be disparate areas of mathematics (like Hardy and King, Poincaré is here concerned with internal uses of mathematics only, a feature with will be discussed below). The manner in which a mathematician identifies these projects is subconscious, difficult (or impossible) to define, and emotional rather than mechanical.

Poincaré's theory of mathematical intuition and creativity was taken up more extensively in Hadamard (1996) with the help of an informal survey of professional mathematicians asking them to describe their creative experiences. Hadamard agrees with Poincaré that an aesthetic sensibility is the main driving force behind mathematical discovery. He also views the sense of beauty as fulfilling an indispensable role, allowing mathematicians to make decisions about their research direction for which they are otherwise in the dark. The mathematician needs his aesthetic sensibility because "concerning the fruitfulness of the future result, about which, strictly speaking, we most often do not know anything in advance, that sense of beauty can inform us and I cannot see anything else allowing us to foresee"(Hadamard, 1996, 127). Significantly, the direction taken is very often a good one; the mathematician is moved to pursue problems of genuine interest and such problems are likely to be fruitful insofar as they result in solutions "of some value for science, whether it permits further applications or not" (Hadamard, 1996, 127). Using beauty as a guide is not an unfortunate necessity with which we are saddled, but actually a reasonably reliable method.

For Hardy the role of the aesthetic is largely motivational with the mathematician aiming to produce beautiful works. The aesthetic separates good mathematics from bad, although it is not what part it plays in the actual process of mathematical reasoning. Poincaré, by contrast, highlights the role of the aesthetic in creation, treating the achievement of aesthetic harmony as being both of aesthetic and epistemic merit. In addition to giving pleasure to the mathematician, it has heuristic value as a fallible but nonetheless indispensable "aid to the mind"(Poincaré, 1910, 331), leading the mathematician down fruitful paths of inquiry and creation by guiding his choices. Indeed, this heuristic is of such importance that to Poincaré an individual who entirely lacks aesthetic intuition cannot understand (let alone contribute to) higher mathematics. Success as a mathematician is correlated not only with the presence of the intuitive faculty (by no means a universal human attribute) but also with how well developed this faculty is: those who have heightened sensitivity can both understand and create mathematics, while those who have only minimal sensitivity might be able to understand and even apply mathematical concepts, but can never be "real" (i.e. creative) mathematicians. The research mathematician is by necessity an aesthete.

1.4 The Shades of Theories

Poincaré attempts not merely to assert the existence of an aesthetic sense, but also to provide a sketch of what features that sense may be picking up, and why. He subscribes to the view that beautiful mathematical entities are "those whose elements are harmoniously disposed so that the mind without effort can embrace their totality while realizing the details"(Poincaré, 1910, 331). A feeling of mathematical beauty arises when we as human beings are able to clearly grasp a concept or chain of reasoning.

Despite the fact that over a century has passed since Poincaré's writing, his cognitive approach to describing mathematical beauty retains some appeal. One recent proposal suggests that mathematical beauty should be defined as "cognisability" such that "A beautiful proof is one which the mind can play its way through with a natural grace"(Blåsjö, 2012). Echoing Poincaré further, the author suggests that "We grasp a beautiful proof as a whole, yet see the role of every detail; it is vivid and transparent..."(1) According to this kind of theory, the aesthetic develops out of a sense of pleasure we experience when a piece of abstract reasoning is conveyed with particular clarity such that we may think through it without encountering mental snares.[4]

In developing his own rudimentary theory regarding the mathematical aesthetic, Hardy attempts to align mathematicians with artists. He draws a con-

[4]The thought that a feeling of beauty should arise when we somehow perceive complexity and detail as brought together harmoniously echoes the slogan "unity in variety", and gives the theories of Poincaré's kind a noticeably 18th century character. It may be that such definitions of beauty remain appealing to some mathematicians because the core idea can, to a certain extent, be interpreted in mathematical terms or using mathematical metaphors. As an example, George Birkhoff builds a formal theory of aesthetics around the idea that "aesthetic feelings arise primarily because of an unusual degree of harmonious interrelationship within the object". He suggests that the feeling of value we get from an object can be captured by a mathematical formula for *aesthetic measure*: $M = \frac{O}{C}$ With O being a measure of the objects *order*, and C its *complexity*. Birkhoff explicitly acknowledging the theory's lineage:

> The well known aesthetic demand for "unity in variety" is evidently closely connected with this formula. The definition of the beautiful as that which exhibits the greatest number of ideas in the shortest time, given by the Dutchman Hemsterhuis in the eighteenth century, is also of an analogous nature.(Birkhoff, 1931, 134)

nection between math and art not only in terms of aesthetic value of the objects created but also in terms of the methods employed by their creators. To this end, he directly compares mathematics to poetry and painting. For Hardy, the three disciplines have in common a concern with constructing patterns. They can be distinguished largely in terms of what materials their patterns are built from. The poet's distinctive medium in this enterprise is words, the painter's is colour, and the mathematician's is *ideas* (what he means by idea is not precisely defined, but presumably Hardy wants them to be abstractions of some sort). Having described mathematics in these terms, he goes on to make his famous claim that:

> The mathematician's patterns, like the painter's or poet's, must be *beautiful*; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. (Hardy, 1992, 85)

Thus Hardy too seems to view the aesthetic as being a guide. Implicit in his claim, however, is that while "beautiful" mathematics is the actual aim, and what is ultimately immortalized, "ugly" mathematics may still be useful as a tool, or stepping stone toward other results. Further, it is *pattern* that is the source of the aesthetic element in mathematics. It is unsurprising, then, that Hardy places a significant emphasis on the structural features of mathematical objects in his analysis of what is of aesthetic import, considerations which accord in many respects with those Poincaré also singles out.

Yet Hardy also admits factors that are more readily viewed as contextual. He suggests, for instance, that real mathematics (having no practical applications) derives its worth not just from being beautiful but also from being "serious". While superficially plausible, the relation between the concept of beauty and the concept of seriousness is somewhat foggy (not in the least because of the difficulty in defining each of these concepts independently). We are told that "the best mathematics is *serious* as well as beautiful"(Hardy, 1992, 89), suggesting that the two concepts are distinct, and that each is to be considered a virtue separate from the other. On the other hand, Hardy also tells us that "the beauty of a mathematical theorem *depends* a great deal on its seriousness"(Hardy, 1992, 90), which suggests, in seeming contradiction, that seriousness is actually a prerequisite for beauty, or an element thereof. I will assume that this ambiguity arises because Hardy, speaking loosely, is using the term beauty in two different respects: in the first place to identify what may be more or less considered formal or structural aesthetic features (the "purely aesthetic" in his words (Hardy, 1992, 113)), and in the second place to talk about overall aesthetic merit in which both factors such as seriousness and structural beauty may both be relevant–the greater aesthetic effect being mixed.

Hardy considers not just seriousness, but numerous other qualities as being of aesthetic relevance to mathematics. It is worth briefly examining the features he identifies in order to get a richer sense of what makes up the full aesthetic package on his view.

Seriousness is meant to allow us to distinguish between the important mathematics and essentially recreational mathematics. A lack of seriousness is what keeps problems in the theory of chess, for instance, from being on an aesthetic par with the theorems more central to higher mathematics. In an attempt to add some precision to the notion, Hardy suggests that seriousness is itself a matter of *generality* and *depth*, both of which have to do with the relation of mathematical concepts to one another.

To be *general* means to be "a constituent in many mathematical constructs, which is used in the proof of theorems of many kinds" where "the relations revealed by the proof should be such as connect many different mathematical ideas" (Hardy, 1992, 104). Generality is thus a matter of striking a balance: a theorem should not be so specific that it admits of no generalizations, but not so general that it is overly obvious and uninteresting.

Depth can be thought of as the inverse of generality. It too is concerned with the relation of mathematical ideas to one another, but instead of describing how a given theorem is applied to other mathematical constructions, it describes how other mathematical constructions are applied to it. That is, the depth of a theorem depends on how many other fundamental concepts and results must be appealed to in its proof.

What this seems to reveal is that for Hardy, generality and depth are features that can only be identified within the context of a larger mathematical body of knowledge. Seriousness is thus a relational property between a given mathematical object (theorem, proof, etc), and mathematics as a whole. The aesthetic is in this way dependent not only on the aesthetic object, but also on its intellectual context.

Hardy's "purely aesthetic" qualities recall in many respects Poincaré's view of beauty, although Hardy takes a more externalist approach than Poincaré by focusing on the object itself rather than the psychological effect it has on the mathematician. Hardy doesn't give a precise description of what constitutes a (structurally) beautiful mathematical pattern, but he does outline a number of qualities which he considers to be of "purely aesthetic" significance: *unexpectedness, inevitability* and *economy* (Hardy, 1992, 113). His explanation of these features is minimal, but the suggestion is that he has a preference for streamlined proofs which produce surprising results by simple means (relative to their effect). He also suggests that they should avoid too many "variations" as occurs when you have a proof containing many cases. While generality and depth describe the relation between a construct and mathematics as a whole, inevitability and economy are features of the construct in isolation.

King echoes Hardy on many notes, but streamlines his account further by arguing that there are only two aesthetic principles active with respect to mathematics: the principle of minimum completeness, and the principle of maximum applicability.

The *principle of minimum completeness* states that:

> A mathematical notion N satisfies the principle of minimal completeness provided that N contains within itself all properties necessary to fulfill its mathematical mission, but N contains no extraneous properties. (King, 1992, 181)

King wishes this principle to be a somewhat more general version of Occam's Razor. He stipulates that a proof (or other artefact) should appeal to all and only the "properties" necessary for success, that it should be "in some sense, complete or self-contained yet contains nothing extraneous"(King, 1992, 182). There are obvious difficulties with this principle, in large part because of the imprecision of terms involved. King wishes his concept of *notion* to be as general as possible, embracing any sort of mathematical object, but this results in an overly general statement which one suspects may even be impossible to actually satisfy in the case of many mathematical objects. It is particularly difficult to assess this criterion because what constitutes a mathematical property, what it means for N to *contain* such a property (as opposed to N *having* the property), and in what a "mathematical mission" consists are all left vague. Nonetheless, we can imagine that a proof satisfying this principle should not take any logical detours, and make minimal appeal to outside theorems or axioms. This principle represents a simplicity desideratum.

King's second principle, *the principle of maximum applicability* states:

> A mathematical notion N satisfies the principle of maximum applicability provided that N contains properties which are widely applicable to mathematical notions other than N. (King, 1992, 181)

While this suffers from the same defects as the minimality principle, it is certainly more clear. King wishes for the target mathematical object to have applications to other problems in mathematics. It is easy to see the parallels between King's two principles and Hardy's criteria which inspired them. The principle of minimum completeness translates most plausibly into ideals of simplicity and economy, while that of maximum applicability is very similar to generalizability. The mapping is not perfect, however; Hardy's depth (the property of appealing to, and so bringing together, many other aspects of mathematics) is captured by neither of these principles, nor do they seem to allow for "surprise". Through his two principles, King attempts greater precision and posits fewer sources for the aesthetic than did Hardy.

It worth is commenting upon the way in which utility, earlier identified as a feature many authors wished to distance themselves from, actually occupies a central place in the aesthetic theories presented. Hardy's seriousness and King's principle of mathematical applicability both treat utility as being of aesthetic importance while Poincaré even more strongly aligns the two, telling us that “the useful combinations are precisely the most beautiful”(Poincaré, 1910, 332). These elements seem to be at odds with the attempts to isolate the value of mathematics from practical applications – to drive a wedge between beauty and utility. Yet King clarifies this issue somewhat, noting that “the word ‘applicability’ does not refer to the notion’s utility in explaining or describing real world phenomena, but rather to the relevance of N to mathematics itself”(King, 1992, 182). Similarly, Poincaré takes usefulness to mean “capable of leading us to the knowledge of a mathematical law” or the capacity to “reveal to us unsuspected kinship between other facts”(Poincaré, 1910, 325). The apparent tension may in this way be resolved through a consideration of the various forms of application that exist in mathematics, as in (Hacking, 2011). As Hacking suggests, the interest, from the aesthetic perspective, is not in problems from ordinary life, nor physics, nor industry, but only in “maths applied to maths”(Hacking, 2011, 11). Nonetheless, while the attempt at detachment from external applications motivates the view that mathematics might be considered a kind of art, the preoccupation with this kind of internal mathematical utility suggests that there may still be an important *difference* between mathematics and the fine arts.

1.5 Concluding Remarks

The topic of aesthetics arises both when mathematicians attempt to describe their experiences, and when they try to explain why what they do is of any value. In both cases, the concern is largely with internal aspects of mathematics – mathematics for the sake of mathematics. In addition to this, there appear to be two very different places in which the authors locate the aesthetic element. The first is in the mathematical construction itself and its own internal features. That is, the aesthetic may be identified within particular objects by identifying it with certain formal and structural elements such as symmetry and simplicity. In a similar vein, features may be considered aesthetic because of the manner in which they impact mathematicians psychologically. The second major source of the aesthetic is, roughly speaking, the construction's place, especially its *usefulness*, within the larger context of mathematics. It is here that mathematical usefulness emerges as an important aesthetic virtue.

The recurring interest in the internal elements of mathematics as a discipline, as well as its value as an autonomous field sheds some light on why the topic of the aesthetic has not been extensively discussed by philosophers. Many of the features considered of aesthetic importance may be missed when mathematics is viewed from the outside (as by philosophy). We may, for instance, overlook the aesthetic if we adopt an overly instrumentalist stance. Many disciplines have a vested interest in mathematics because they make extensive use of its products, and so the questions that naturally arise from the uncontested importance of mathematics to other areas enquiry have come to dominate much of the discussion regarding its purpose and role. For philosophers, the interest in mathematics has historically centred on the conundrums abstract mathematical entities pose for metaphysics, the remarkable reliability of the mathematical method, and the resulting special epistemic status of its products. These projects foreground foundational questions, and treat *truth* as the distinctive feature giving mathematics its value.

That mathematics can be considered from an aesthetic perspective at all is not immediately obvious to those lacking personal hands-on experience with the subject matter. While certain objects we encounter (art objects, for instance) are clearly aesthetic, recognition of aesthetic experiences with respect to objects of mathematics seems, on the whole, limited to practitioners. This fact has not gone unnoticed by those interested in the phenomenon: Poincaré builds such a specialized aesthetic sensitivity into the very nature of a mathematician and King goes further, offering a fairly complex explanation for why mathematicians seem privy to the aesthetic features of mathematical artefacts, but non-mathematicians (even those who use mathematics on a daily basis, such as scientists or engineers) generally are not. To do this, he borrows Bullough's notion of aesthetic distance, suggesting that scientists are "too close" to mathematics making them view it on purely practical terms, while those in the humanities are "too far" resulting in their having no emotional response to the material. Mathematicians alone lie on the boundary (King, 1992, 207–208).

It seems more likely, however, that the paucity of discussion of mathematical aesthetics outside mathematical circles is due not to an inherent inability by non-mathematicians to appreciate mathematical artefacts on an aesthetic level, or recognize the possibility of such appreciation *in principle*, but rather with a lack of opportunity to see the phenomenon at work, or to experience it themselves. Despite this, it is not hard to see that questions about the role and nature of the aesthetic are of importance to *any* situation where we are interested in the relationship between people and the world. Since mathematics is a way to study the world (both the physical world, and the mathematical world, whatever that may be), it is unsurprising that aesthetics has emerged as a topic of interest among mathematicians themselves.

Chapter 2

Functional Beauty and the Aesthetic Induction

2.1 Introduction

Although there have been few extended attempts at aesthetic theories targeting the topic of mathematical proofs, some theories have been advanced which may be adapted in a manner suitable to such a purpose. This chapter examines and criticises two aesthetic theories which are plausible candidates for shedding light on the mathematical case. The first of these theories is the aesthetic induction of James McAllister, and the second is Parsons and Carlson's theory of functional beauty.

The basic idea behind the aesthetic induction is that a community develops preferences for aesthetic properties which have been associated in the past with empirical success. McAllister first uses this theory to describe the role of the aesthetic in scientific theory choice, but speculates that the aesthetic phenomenon in mathematics, in particular regarding proofs, might be explained by appeal to the same principles. Parsons and Carlson's theory takes a different approach by focusing on the way in which our awareness of an object's function can affect our aesthetic perception of it. This theory is not directly aimed at intellectual domains, although it attempts to be as general as possible across domains in which functionality is important. As such, if it can be shown that proofs are in some way functional artefacts, then we can reasonably expect them to be subject to similar aesthetic responses, and so explicable by appeal to similar principles.

Both aesthetic theories presented here focus primarily on non-art objects which receive aesthetic attention, and both fit within the broad category of functionalist theories insofar as they suggest that there is a connection between an object's apparent ability to perform its function and our attribution of aesthetic terms such as "beauty" to it. Neither of these theories is a perfect fit with the task at hand; however each offers valuable insight which can be applied to create a more robust theory suitable for explaining the case of mathematical proofs. The last part of the chapter will be devoted to this task.

2.2 What do we need in an aesthetic theory?

The first chapter described the manner in which aesthetics has been a recurring theme in mathematics. In particular, it has arisen in discussions of the value and justification of mathematics, especially pure mathematics, and in discussions of the way in which mathematicians as human beings practice mathematics. An aesthetic theory should present us with the tools with which we can properly evaluate the claims made with respect to these issues. For that reason, the theory should have three major explanatory goals:

Accurately depict the phenomenon An aesthetic theory for mathematics should be able to accurately describe and generate predictions about aesthetic evaluations in mathematics. In particular, we would like to be able to explain and critically evaluate aesthetic claims about particular proofs, as well as the recurring appeal to specific properties as being either aesthetically appealing or unappealing. The theory should thus provide a framework in which we are able to describe what aesthetic effects are possible within the domain, how various properties of proofs can create these aesthetic effects, and to what extent these evaluations can change over time or differ depending on context.

Answer questions of practice The theory should allow us to explore the role of aesthetic criteria in mathematical reasoning, and to critically evaluate intuitions to the effect that a proof which is aesthetically pleasing is more likely to be successful or true than one which is displeasing. Importantly, the theory need not justify or excuse the use of aesthetic guides as heuristic, but it should be able to explain why some might be inclined in such a direction. In order to provide such an explanation, there must be a proper examination of what we mean by success in the mathematical context, as well as what we mean by beauty. This should elucidate any relation that there may be between the two concepts.

Answer questions of value Since the aesthetic is sometimes treated as a source of value for mathematics, it should be possible to explain in what way judgements of aesthetic value differ from other sorts of judgements of value (such as judgements of epistemic value). To this end, it is desirable to explain how the aesthetic in the mathematical context relates to the aesthetic in other contexts, and how it is that aesthetic predicates come to be applied to such different sorts of objects.

2.3 The Aesthetic Induction

The theory of aesthetic induction, given detailed presentation in McAllister (1999), remains dominant when discussing the case of aesthetics in the sciences. Although aesthetic induction was developed specifically to explain theory choice in the natural sciences, McAllister (2005) suggests that aspects of mathematics, in particular the development and evolution of mathematical proofs, may be modelled using the same theory. In this section I will explain McAllister's theory as it applies to proofs. I argue that the aesthetic induction does have a large degree of explanatory power, but suffers from significant deficiencies especially when imported into the mathematical case. The theory I will later defend attempts to retain the successful aspects of McAllister's theory, while not falling prey to the same criticisms.

McAllister's theory deals specifically with attributions of beauty made by observers of certain mathematical entities. Thus "a mathematical entity has certain aesthetic properties, such as simplicity and symmetry. On the strength of perceiving[1] these properties in a mathematical entity, and by virtue of holding to aesthetic criteria that attach value to these properties, an observer is moved to project beauty into the entity"(McAllister, 2005, 16). Beauty, on this account, is not itself a property of an object. Rather it is an aesthetic

[1]Since the entities in question are conceptual or abstract in nature, "perception" in this instance cannot be thought of straightforwardly as sense perception. Instead it must consist in a kind of recognition or understanding that the entity has a particular aesthetic property. This assumption is not unproblematic and points to a manner in which the mathematical case may be importantly different from more standard cases in aesthetics.

value which we project onto objects when they satisfy certain aesthetic criteria. The aesthetic induction is meant to be the process by which such criteria develop and change. The evolutionary aspect of the induction is important to McAllister because he applies his theory in part to explain the history of science, and Kuhn style paradigm shifts.

The major aspects of McAllister's theory can be summarized as follows:[2]

1. Aesthetic properties are those intrinsic properties of object O that evoke an aesthetic response.

2. An aesthetic response is a projection of value (in particular beauty) into O in accordance with aesthetic criteria.

3. Aesthetic criteria take the following form: if O has property p then "attach more aesthetic value to it than if, other circumstances being equal, it did not" (McAllister, 1999, 34). Various properties will have different weightings which will determine just how much aesthetic value they contribute to the overall assessment of O.

4. The set of all aesthetic criteria held either by a community or an individual is referred to as the *aesthetic canon*. The canon can thus be thought of as a set of ordered pairs $(p_1, W_{p_1}), (p_2, W_{p_2}), \ldots$ where the p_i are all the (potentially) aesthetic properties, and W_{p_i} is the weighting of p_i. McAllister emphasizes the fact that many of these properties will have a weighting of zero, rendering them aesthetically indifferent.

5. The overall aesthetic value of O, V(O) can be thought as the weighted sum of the values of all O's aesthetic properties: $V(O) = W_{p_1} + W_{p_2} + \ldots$

[2] I have added some formalism to render aspects of McAllister's theory more explicit.

6. At any given time, the weight of a property p is determined by the degree of success objects with that property have enjoyed in the past. That is, in the scientific case,"scientists at a given time attach value to an aesthetic property roughly in proportion to the degree of empirical success scored up to that time by the set of all past theories that exhibit that property"(McAllister, 2005, 28) while in the mathematical case, McAllister suggests that these valuations "evolve in response to the perceived practical utility of mathematical constructs"(29), although he does little to explain what"practical utility" might be for mathematics. The process by which weightings are adjusted over time in response to empirical/practical success is referred to as the *aesthetic induction*.

2.3.1 Strengths and Weaknesses

The aesthetic induction is meant to explain both how aesthetic preferences can change over time, and also how such preferences can be considered in some way rational. That is, the aesthetic evaluation, according to this theory, can act as a (highly fallible) indicator of potential success. Scientists are vindicated to some degree in applying aesthetic criteria to theory choice because the criteria are themselves grounded in an inductive process which will over time be expected to assign greater value to genuinely useful properties, and lower value to those properties which are associated with failure.

The aesthetic induction is thus well suited to satisfying the second desideratum since it can explain why the aesthetic has sometimes been viewed as having heuristic value, as Poincaré suggested. In particular, a mathematician might, when seeking a proof, be thought of as doing so under the guiding prescription to maximize V(O). This would lead them to favour properties and styles that had been successful in the past, and therefore had acquired a high aesthetic value.

Nonetheless, McAllister's theory suffers from some weaknesses when it comes to satisfying the other two desiderata described above. With respect to the third desideratum, McAllister's theory treats beauty as a direct reflection of aesthetic value itself, rather than an aesthetic property. For this reason, it is not clear how to interpret other sorts of aesthetic predicates. With respect to the first desideratum, it is not clear that McAllister's theory accurately describes the case of mathematical proofs, since it has been developed in order to explain the pattern of evolution of aesthetic preferences in scientific theories. McAllister argues that proofs obey similar trends, however we must be careful not to generalize too swiftly from one case to the other.

It is also not clear that McAllister's model accurately captures the manner in which properties generate aesthetic effects. In part, this is because certain aspects of the theory need to be fleshed out such as the question of how new properties fare within the aesthetic canon. Other issues arise because the aesthetic induction assumes that preferences for each property develop independently from one another, and that the overall aesthetic merit of an object is found by simply adding the preferences for all its properties together. This simple calculation ignores contextual elements as well as the fact that properties may be related to one another. These could easily be confounding factors in the aesthetic evaluation.

The Focus on Beauty

McAllister's program, in particular the form of projectivism which he adopts, is focused on assigning aesthetic value to various objects. Specifically, he is interested in how some factors privilege one object over another aesthetically. This provides a means of measuring and comparing rival scientific theories on aesthetic grounds. The aspect of aesthetic experience that is explored is thus preference, and the induction is meant to explain preference formation and change.

Because of this, the theory is limited in that it only addresses one aesthetic "value" (namely beauty), a fact which McAllister acknowledges. He suggests that there may be other sorts of aesthetic values to consider, though does not devote much time to them. In particular, he suggests that there may be an analogue to "artistic merit" having to do not only with perceptible properties internal to the object, but also to"relational properties that may not be manifest in it, such as the property of having had a particular influence on the development of science"(McAllister, 1999, 31). Thus, to McAllister, beauty is to be understood as the value of the internal properties of the object, while other values may exist having to do with specifically relational properties between the object and other elements in the context such as the history of the field.

This raises questions about whether the division between internal and relational measures can be made quite so cleanly, whether other aesthetic values (especially pertaining to internal properties) are possible with respect to mathematics (or scientific theories), and how such values, should they exist, relate to one another. Because McAllister views beauty as a value projected into an object, and not as an aesthetic property of that object, it is also unclear how to interpret other aesthetic terms such as “pretty”, “ugly”, “elegant”, “cute”, and “graceful”. It seems unlikely that these are all separate values, but suggesting that they are all different ways of expressing degrees of one value (beauty) oversimplifies a great deal of aesthetic talk. The most natural way in which to extend the theory would be to allow for many distinct aesthetic values, and to have an inductive process for each such value, but this severs them from one another in a manner which makes it unclear where the values are coming from in the first place.

Responding to Novelty

It is not clear, under the aesthetic induction, how to handle novel developments in the field. McAllister suggests at times that new properties will be met with aesthetic indifference, and at other times that we should expect them to be welcomed with displeasure. This is not a problem of the theory being unable to provide an explanation, since it might be adapted to be consistent with either of these possibilities, but rather with it being under-developed. More importantly, however, I believe there is good reason to suppose that novel properties can have an *active* aesthetic effect, and that this effect is neither necessarily one of displeasure nor necessarily one of pleasure.

The problem of novelty involves the mechanism by which new properties get added to a canon. Clearly, there are three basic ways in which McAllister could do this:

1. the new property p enters the canon with weighting $W_p = 0$

2. the new property p enters the canon with weighting $W_p < 0$

3. the new property p enters the canon with weighting $W_p > 0$

McAllister claims that "if a property has no association with empirical success(...) scientists attach no aesthetic value to it and thus feel no aesthetic attraction for theories that exhibit it" (McAllister, 2005) and "in the canon of any actual scientist or community, the weightings of most of the properties will be zero, indicating that no aesthetic value is attached" (McAllister, 1999, 78). Based on this, McAllister seems to be saying that we add new properties to the canon in the first way, with a weighting of zero. If, as this suggests, mathematicians attach no aesthetic value to a property, then they should be neither attracted nor repelled by such theories, hence the attitude of mathematicians to new features should be one of neutrality, and any explanation of aesthetic displeasure must be made using the inductive process itself. That is, we arrive at a negative assessment of a property only after it has accumulated a track record of failures.

Yet McAllister also suggests that novelty is often met initially with active displeasure. This tendency is described in his discussion of the response of the mathematical community to the introduction of new classes of numbers:

> Each of these classes of numbers had to undergo a gradual process of acceptance: whereas initially each new class of numbers was regarded with aesthetic revulsion, in due course–as it demonstrated its empirical applicability in mathematical theorizing–it came to be attributed growing aesthetic merit. (McAllister, 2005, 29)

So McAllister seems to say here that novelty is met, in general, with aesthetic displeasure (sometimes even with "revulsion"), and, moreover that this tendency is evidence in favour of his theory. The aesthetic induction is required not simply so that a property can be viewed as beautiful, but so that it can cease to be viewed as ugly. This suggests that the default position toward a property is actually a negative weighting, case 2.

While McAllister's theory can easily accommodate this solution, any instance in which an object presenting novel features is met with a largely positive initial response presents some difficulty. Indeed, if it is ever the case that novel features themselves are identified as meritorious then this directly contradicts McAllister's predictions since the response cannot be attributed to the compensating effects of other features. Cases such as that of Cantor's highly original work on set theory and transfinite arithmetic suggest that the introduction of new concepts and methods may receive a mixed reaction within the community, rather than a uniformly positive or negative one.[3] The idea that novelty might in fact increase aesthetic admiration in some cases lies at the heart of what Le Lionnais considers "romantic beauty". Far from mathematicians always greeting new methods with revulsion, he suggests that "seeing a certain method succeed regularly every time it is employed makes it more disconcerting to discover its inefficacy in a new instance, and increases all the more the enthusiasm with which we greet an entirely original and successful procedure"(Le Lionnais, 1971, 143). When tried and true methods have failed at a particular problem, new methods which do succeed at this task can be especially impressive and pleasing.

A way to look at novelty that accounts for this is to take new properties as

[3]In his discussion of this period of mathematical history, Kline (1972) points out that some prominent figures, such as Kronecker, Klein, and Poincaré, responded to the new material with open hostility and suspicion, while others, such as Hilbert and Russell, admired and praised it. He notes that Hilbert was particularly effusive, famously referring to Cantor's work as "one of the most beautiful realizations of human activity in the domain of the purely intelligible"(Hilbert, 1926, 167).

aesthetically neutral, as McAllister initially suggests, but also to treat the second order property of *being novel* or perhaps (similarly) being *rare or unusual* as itself having an aesthetic impact. In particular, if the object as a whole would get a negative assessment, then the novel and rare make it worse. The ugly becomes repulsive, or even monstrous. In a parallel manner, however, if the proof would get a positive assessment, then the presence of novel features will increase its merit. It may be praised as "clever," "ingenious," or "daring". This points again to the need for a more textured account of the mathematical aesthetic than McAllister, by focusing almost entirely on beauty, provides.

Acceptance and Preference

The problems with novelty signal broader issues with whether McAllister's theory is able to accurately describe the mathematical case. The aesthetic induction is designed to explain how preferences of both communities and individuals can evolve over time in response to the success of past theories or proofs. As a result, it generates predictions about how, historically, the aesthetic canon should have developed and how the community will respond in terms of their aesthetic preferences to other new developments. As an example, McAllister cites attitudes toward computer assisted proofs as following the template predicted by the aesthetic induction. Early examples of such proofs (the proof of the 4-colour theorem being the most famous) were met, at first, with a degree of suspicion as to their definitiveness[4]. Despite this, they have

[4] A detailed exposition of the development of and reaction to the proof of the 4-colour theorem can be found in Mackenzie (1999). The major complaint levied against the proof was that the extensive use of computer, and overall structure of the proof inhibit our capacity to understand the proof. Stewart (1981, 34) is cited as an example of a mathematician who found the proof valid but unsatisfying for these reasons, while Bonsall (1982, 13–14) "went as far as to deny that the Appel-Haken solution should be considered a proof at all" (Mackenzie, 1999, 41).

come to be accepted by the community as valid methods. Thus, according to McAllister, "we can forecast that the computer-assisted proof will remold the concept of understanding and aesthetic criteria in mathematics and will gain an acceptance similar to that of quantum theories in physics" (McAllister, 2005, 28). As these forms of proof continue to be successful, the aesthetic criteria will adjust so that mathematicians find them more pleasing.

The history of the development of proof styles and strategies, and of scientific theory choice do not seem entirely parallel, however. In particular, the scientific case involves periods of upheaval and dramatic change (Kuhn's revolutions) which are not so clearly seen in the mathematical case. McAllister allows for this by holding that the aesthetic canon tends to lead to a preference for theories illustrating familiar and already successful properties. The empirical success of new but aesthetically displeasing theories leads to a scientific revolution in which aesthetic and empirical commitments clash. This eventually results in the community abandoning its old aesthetic canon, after which, he claims, the canon is gradually rebuilt centred on aesthetic criteria emerging from the new paradigm. The shifts and transitions in the case of mathematical proofs have been, arguably, substantially more modest. As a result, it is worth inquiring whether the predictions made by the theory of aesthetic induction truly manifest in the mathematical case.

One of the significant differences between modelling theory choice and modelling the evolution of mathematical proofs is that the question underlying the former is, at its core, one of what we should accept when presented with mutually incompatible options, whereas this is not what is at stake with the latter. Rival theories are incompatible in that at most one may be correct, while when presented with two 'rival' proofs, we may view one as being superior to the other, and yet still accept both as valid. Except perhaps when we use aesthetic elements as a heuristic guide in the process of proof discovery, we do not choose one proof at the expense another (nor one proof kind over another). We may prove the same theorem in many ways, and doing so can actually increase our understanding of the result. Hence, different proofs (even of the same result) simply are not rivals in the same way that different theories of the same phenomenon are rivals. As a result, while acceptance of a scientific theory will require us to prefer it over others, this may not be the case for mathematical proofs. We may accept a proof without preferring it; we may even accept a proof which we think is inferior to another of the same result.

There are several ways in which developments in mathematics can contradict McAllister's theory: a property might be associated more often with unsuccessful proofs than successful ones, and yet retain a high degree of preference. In this case, the aesthetic canon has failed to adjust preferences negatively. Similarly, a property might be associated with success more often than failure and retain a low degree of preference. Montano (2013) refers to these sorts of anomalies as positive and negative "historical constants" (6) and argues convincingly that the property of complexity in mathematical proofs is an example of a negative historical constant. Complexity can mean any number of things, so Montano focuses specifically on proofs by cases, claiming that these "have a very long history of success but mathematicians' preference for them has not increased"(Montano, 2013, 7). Heavy appeal to varied cases remains on the whole a property that is considered aesthetically undesirable, a position that has not much changed over time despite the increased prevalence and general respectability of proofs with this feature. The acceptance of the community, and the repeated exposure to the proof style does not seem to have translated into the aesthetic preference for such proofs that induction would predict, making them an important counter-example.[5]

These problems arise in part because McAllister often treats aesthetic preference as merely tracking *acceptance* of a property. That said, since he also claims that "aesthetic preferences evolve in response to the perceived practical utility of mathematical constructs"(McAllister, 2005, 29), there is room in his theory to explain the apparent resilience of certain preferences by better defining what it means for a proof to be of "practical utility", and thereby giving more nuanced criteria for success in the mathematical case. Should proofs by cases be found to have less "practical utility", in spite of being correct, then the failure of aesthetic induction to increase our preference with regard to this property would be explained. A precisification of "practical utility" could allow for the required separation between the notions of acceptance and preference: the community might accept proofs which are largely impractical, but not prefer them.

An issue that lends force to Montano's objection stems from the way in which properties are explicated in McAllister's theory. McAllister has us use

[5]Montano's proposed solution to this and other problems with McAllister's theory is to suggest that certain preferences and aversions are biologically determined, or at least biologically constrained. As a result, the aesthetic induction does not lead to dramatic changes in preferences toward these "robust" properties. Although Montano's naturalistic modification of the aesthetic induction provides a manner of modelling the behaviour of such properties, I think there are good reasons to seek other explanations. In particular while Montano motivates his theory by appeal to the fact that "our innate preference for sweetness and aversion to bitterness show that some preferences are not formed by a history of experiences with stimuli"(Montano, 2013, 15), it is not clear how or why we would have evolved analogous constraints with respect to the aesthetic properties of largely abstract objects, such as proofs. As a result of this lacking in background theory, it is virtually impossible to predict how robust a given property is prior to actually gathering observations on its performance. This makes the theory able only to provide an essentially ad-hoc explanation of empirical findings. Any property which violates the basic model of aesthetic induction could be postulated as being robust, effectively making the new theory immune to counter-example.

terms like “complex” and “simple” as though they refer to distinct properties, however it is more precise to consider them short-hand for evaluations relative to a single gradeable property (hence, “complex” should be read as “of high complexity” and “simple” as “of low complexity”). Such a move complicates the inductive model because it requires us measure properties on a scale. That is, it is not merely the presence or absence of a property which we mark, but the degree to which it has been present in successful or unsuccessful past proofs. The pay-off is that this approach allows for a way out of Montano’s objection. Complex proofs may score many successes, but if simple proofs continue to score substantially more, then we should expect the complex proofs to remain aesthetically displeasing. This is because, when no longer treated as distinct properties, the successes of simple proofs count against complex proofs (and vice versa). Since they are predicates referencing the same property, complexity is incompatible with simplicity: by being complex, a proof is trivially precluded from also being simple. We must, in this way, take into account the logical relations between properties and mark when we are dealing with genuinely distinct properties, or merely names applied to graded properties.

McAllister also might be aided by considering the other (non-logical) ways in which properties are related to one another. The aesthetic induction treats the valuation of an object as a simple additive function summing the weights of its properties. This ignores several possibilities having to do with the manner in which properties can be combined to create different effects. A property might be weighted higher in one context than another, or it might get a higher weighting in virtue of other properties being present in the same object. This may, in part, be due to the fact that certain properties are complementary or non-complementary with one another: we can expect to see certain properties together while others typically stand apart. In some cases the presence of one property might even make it easier for another to appear. We can expect, for instance, that exploiting effects in argumentative symmetry will produce a more unified proof with fewer cases. We can also expect that proofs in specific sub-fields will make appeal to particular strategies and techniques – proofs in many areas of combinatorics and discrete mathematics rely on mathematical induction, for instance. The use of other argument forms, in these cases, might strike us as being more pleasing and illuminating, or at least, surprising. In general, properties might accentuate or draw attention to one another. This suggests that we should have a theory in which properties are not treated wholly independently from one another.

2.4 Parsons and Carlson

By connecting the aesthetic to practical success, McAllister presents a kind of functionalist theory for scientific (and mathematical) beauty.[6] Despite its shortcomings, McAllister's theory has strong appeal and remains the best candidate available for explaining the appearance of the aesthetic in scientific and mathematical domains. Much of the success of McAllister's theory is owing to the intuition that the aesthetic objects he considers are the sorts

[6]McAllister himself holds that his theory is "not quite an expression of aesthetic functionalism" on account of the fact that "I endorse the functionalist conviction that fitness to a purpose is an important factor determining aesthetic valuations. However, the aesthetic induction does not consist in attributing beauty to each scientific theory to the degree to which it shows fitness to its purpose."(McAllister, 1999, 80–81) That is, the attributions of beauty are based on the empirical track-record of the properties of a theory, rather than the empirical track-record of the theory itself.

In fact, not all forms of functionalism require that beauty be identified directly with fitness. One may deal instead with the more indirect appearance of fitness, just as McAllister does. At the most basic level, functionalist theories assert that the function of an object has some sort of bearing on our aesthetic appreciation of it. Precisely what bearing it has can be fleshed out in many different ways.

of things for which function really does have aesthetic import, in large part because most of our interactions with such objects involve us actively using them. Assuming that this intuition has some substance to it, we should expect these cases to be similar in some respects to other cases where function plays an active part. It is therefore worth considering a recent attempt at fleshing out the functionalist position with respect to other domains if we are to come to a better understanding of how it works in the mathematical case.

A detailed and general functionalist theory of aesthetics is presented in Parsons and Carlson (2008). According to Parsons and Carlson, many objects fall within functional categories and knowledge of these categories structures and informs our aesthetic perceptions of these objects. As a result, we do not, in general, view functional objects with aesthetic disinterest. Instead we use our understanding of an object's function, as well as any knowledge we have about how it performs this function, to determine if it looks functional, dysfunctional, efficient, clumsy and so on. The same features may be viewed positively with respect to one functional category, and negatively with respect to another.

This theory has several major elements:

1. Artefacts are attributed proper functions according to *selected effects*. That is, proper functions are selected for: performance of such functions ensures that the artefact continues to be manufactured or maintained.

2. We perceive artefacts as falling into functional categories. Functional categories are a way by which we understand proper functions.

3. The functional category of an artefact determines whether its properties appear standard, contra-standard, or variable. Following Walton (1970),

the standard features are those which are generally considered essential to category membership, the contra-standard features are those which are considered essentially absent from category members, and the variable features are those which have no effect on category membership.[7]

4. The number and type of properties an artefact has determines our aesthetic response. Three kinds of functional beauty are then described based on the number and kind of properties it has.

Parsons and Carlson recognize that providing a complete functionalist aesthetic theory requires developing a theory of function. We must have a way of determining what the function of an object is (referred to by them as the problem of indeterminacy), as well as a way of accounting for how an observer can come to know an object's function and how this knowledge can impact their aesthetic perception of the object (the problem of translation).

In tackling the problem of how to attribute function, Parsons and Carlson consider and reject an intentionalist theory for function according to which the function of an object would be determined by the intentions of an agent (whether it be a designer, maker, or user). In particular, they criticise intentionalism's inability to distinguish between proper functions and other sorts of functions. The proper functions are those that in some way are central to an object, or which "belong to the object itself"(Parsons and Carlson, 2008, 66), rather than the idiosyncratic functions which may be acquired accidentally. They argue that "the intentions of users of an artefact, for example,

[7]It is worth observing that, as defined, the members of the set of contra-standard properties will be the negations of those in the set of standard properties. Thus, Parsons and Carlson sometimes use phrases such as "lacks standard properties" and "possesses contra-standard properties" interchangeably. Given that how we perceive absence and presence of certain types of properties may differ, and also that many of these properties are likely fuzzy, this is not an entirely insubstantial assumption.

may bestow a multiplicity of functions on an object (...) but these functions do not thereby become proper functions of the object."(Parsons and Carlson, 2008, 66) As an example, they suggest that despite the fact that frying pans can have numerous functions, such as being used as a weapon should the need arise, their proper function remains that of being a hot cooking surface. While an intentionalism that allowed only for designer intent could account for this, such a view is unable to handle the fact that certain improvised or discovered functions do, over time, become the proper function of an object. Functions intended by users sometimes (but not always) seem to override functions intended by designers, and it is unclear why this should only sometimes be the case. Many agents are involved in the design, production, and use of functional objects, and there is no guarantee that their intentions will agree with one another, nor is it clear how to break ties. This, according to Parsons and Carlson, undermines the capacity of such theories to provide a satisfactory accounts of proper function.

Instead of relying on the intent of some agent in fixing function, Parsons and Carlson borrow from theories of function put forward in philosophy of science in which function is tied to evolutionary success[8]. For Parsons and Carlson, economic success is treated as analogous to this kind of evolutionary success. Artefacts which do well in the marketplace are "reproduced" in the sense that tokens of their type continue to be manufactured, distributed, or (in the case of one-off artefacts such as individual buildings) preserved for ongoing

[8]Specifically, they adapt the definition:

> Trait X has a proper function F if and only if X currently exists because, in the recent past, ancestors of X were successful in enhancing fitness because they performed F, leading to reproduction of the genotype for trait X.(Parsons and Carlson, 2008, 72)

use. Based on this, they suggest that a better account of functionality for artefacts is a selected effects view according to which:

> X has a proper function F if and only if Xs currently exist because, in the recent past, ancestors of X were successful in meeting some need or want in the marketplace because they performed F, leading to manufacture and distribution, or preservation, of Xs. (Parsons and Carlson, 2008, 148)

Here an ancestor of X is an object similar to X "whose success in the marketplace was a causal factor in the production of X" (Parsons and Carlson, 2008, 76). Thus, rather than having proper function fixed by human intentions, Parsons and Carlson tie it into the object's causal history. In the case of the frying pan, its function as a weapon is not a proper function because the existence of frying pans is due not to their success as bludgeons, but rather to their success as cooking devices. If the use of frying pans as weapons were to come to be sufficiently wide spread that more frying pans were produced in order to satisfy this need, then weaponry would be a proper function of the frying pan.

A peculiarity of this theory is that it is unable to attribute proper functions to genuinely novel artefacts. This is a conclusion Parsons and Carlson accept, arguing that it is rare that an artefact truly is novel. Most artefacts we encounter have ancestors with which we are familiar. It would, they argue, be impossible to assign a proper function to a truly novel artefact. They provide a thought experiment involving new kind of can-opener bearing no resemblance to those that came before it. They suggest that in these cases, we fall back on our knowledge of the designer's intent to determine an object's function, but that this does not yield a proper function.

The second major part of Parsons and Carlson's theory considers how an object's function affects our aesthetic appreciation of it. They argue that we perceive artefacts as belonging to functional categories determined by their proper functions. Functional categories are meant to be the analogue of categories of art proposed by Walton, which include "media, genre, styles, forms and so forth"(Walton, 1970, 339). According to Walton's theory, understanding an artwork as belonging to a certain category causes us to perceive its non-aesthetic properties as being standard, contra-standard, or variable relative to that category, and this in turn impacts the aesthetic properties we perceive the object as having. Parsons and Carlson suggest that our functional categories work in a similar manner as these categories of art. These categories are "forms of understanding of function" (Parsons and Carlson, 2008, 92) meaning not just an understanding of what an object's function is, but also of how it performs its function. They suggest that knowledge of how the function is fulfilled is particularly important for our recognizing properties as being of a particular type. It is therefore not enough to know what an object is for; you must, to a certain degree, understand how it works.

In Walton's original theory, the standard/variable/contra-standard distinction helps us structure our expectations by laying out what features are almost always present in a given category (the standard), rarely present (contra-standard), and sometimes present (variable). The standard features are perceived as necessary for category membership, while the contra-standard are perceived as necessarily absent such that an object having many such properties will no longer strike us as belonging to that particular category at all. Since the variable properties are neither necessary, nor necessarily absent, they are those properties in which most of the artistic play occurs, and by virtue of which much artistic expression is made possible. In a sense, the appraiser who is familiar with the category in which a particular work falls will learn to bracket the standard features, focus on the variable, and might potentially be shocked by the presence of contra-standard features (so long as they are not so dominant as to jettison the work completely from the category). Walton points out that this perceptual framing explains some otherwise perplexing artistic phenomena. For instance, we are able to perceive strong (even perfect) resemblances between a portrait and the subject of the portrait despite the fact that there are radical and obvious differences between the two, such as one being three-dimensional, moving, and composed of flesh and blood while the other is flat, static, and composed of canvas and paint. The appropriate judgement is possible due to the fact that being on a canvas, rendered using paints, still, and two-dimensional are all features standard to the category *painting* and so do not factor into our perception of a resemblance. Instead, the variable, contingent features of the painting are foregrounded (such as colours and shapes) and these are precisely the features in which a resemblance to the subject of the portrait may be made out. Thus, while standard features are not unimportant to appreciation of works of art relative to their category, the viewer of an art-work uses their knowledge of the category in large part to identify which features ought to be ignored and which should receive their attention.

Walton does not hold that standard features are entirely without aesthetic effect. In fact, he suggests that they behave a bit like "'rules' for producing works in the given category" (Walton, 1970, 351) which when followed produce a sense of "rightness" or perfection. The aesthetic effect can be particularly pronounced if such "rules" come into conflict because an adept artist can navigate such a situation in a manner that resolves the tension, producing a sense of relief. Walton even suggests that this phenomenon is at work with respect to aesthetic experiences in mathematics.[9] The sense of rightness evoked by standard properties is what Parsons and Carlson focus on in adapting Walton's approach to functional categories. The presence of standard features directly contributes to the sense of aptness in the case of functional categories, just as it suggests rightness with respect to categories of art. Similarly, Parsons and Carlson suggest that the presence of contra-standard features makes the object appear less apt, or in some way "discordant"(Parsons and Carlson, 2008, 94).

Embracing Walton's theory of category-relative features allows Parsons and Carlson to present an enriched phenomenology with regard to the functional aesthetic in which there is not one but three distinct forms of beauty:

looks fit: the object has no contra-standard features and many variable features which are "indicative of functionality".

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[9]The passage in full:

> Suppose that the first movement of a sonata in G major modulates to C-sharp major by the end of the development section. A rule of sonata form decrees that it must return to G for the recapitulation. But the keys of G and C-sharp are as unrelated as any two keys can be; it is difficult to modulate smoothly and quickly from one to the other. Suppose also that while the sonata is in C-sharp there are signs that, given other rules of sonata form, indicate that the recapitulation is imminent (for example, motivic hints of the return, an emotional climax, or a cadenza). Listeners who hear it as a work in sonata form are likely to have a distinct feeling of unease, tension, uncertainty as the time for the recapitulation approaches. If the composer with a stroke of ingenuity accomplishes the necessary modulation quickly, efficiently, and naturally, this will give them a feeling of relief-one might say of deliverance. The movement to C-sharp (which may have seemed alien and brashly adventurous) will have proven to be quite appropriate, and the entire sequence will in retrospect have a sense of correctness and perfection about it. *Our impression of it is likely, I think, to be very much like our impression of a "beautiful" or "elegant" proof in mathematics. (Indeed the composer's task in this example is not unlike that of producing such a proof.)*(Walton, 1970, 351, emphasis added)

looks elegant: the object has no (or few) contra-standard or variable features, and thus only (or mostly) standard features. Every feature appears essential.

displays visual tension: the object has some contra-standard features, but still appears able to perform its function. (Parsons and Carlson, 2008, 96–97)

In this phenomenology, the most widely recognized form of functional beauty is looking fit. Looking fit is "an aesthetic quality that things possess when the functional category employed to perceive them causes us to see those things as having no contra-standard features at all, and as having, to a high degree, certain variable features that are indicative of functionality" (Parsons and Carlson, 2008, 95-96). Elegance, simplicity, and grace[10], on the other hand, is achieved by objects which "have few contra-standard or variable features, but only standard ones"(97)[11]; that is, their properties are all (or are almost all) standard relative to their functional category. This is meant to

[10]While these are distinct predicates, they are treated as all being the same 'kind' of aesthetic quality, and the terms are used interchangeably by Parsons and Carlson.

[11]This definition is ambiguous. Parsons and Carlson could mean either *have few contra-standard or variable features, and mainly standard features* or *have neither variable nor contra-standard features, but only standard features.* The former is the more forgiving definition and includes objects satisfying the strict interpretation as a limiting case. It is also consistent with the claim that "appearing economical and efficient is a matter of degree, and so are the aesthetic qualities produced by this appearance."(Parsons and Carlson, 2008, 98). The latter definition, however, is more clear cut and makes for a better theoretical model. I therefore assume this definition for simplicity, but recognize that it represents a potentially unattainable ideal.

De Clercq (2013) argues that "type 2 functional beauty is not a genuine possibility. The functional objects that we are familiar with always seem to have perceptual features that one could regard as variable for the kind in question."(39) I share the concern that the presence of some variable properties is inevitable, and wonder as well if (given this fact) those objects under the heading of type 1 beauty might actually appear more elegant than those under the heading of type 2, since their variable features are actively suggestive of functionality. This worry is an indication that Parsons and Carlson's treatment of property types requires some adjustment.

capture what it means to appear efficient and streamlined, lacking in superfluous elements. Finally, an object is perceived as having a sort of "visual tension" when it has many contra-standard properties for its function, and yet manages to perform that function nonetheless. It should be noted that while the tension arises from the object appearing capable of performing in spite of its contra-standard features, this knowledge is itself dependent on us somehow being aware that it *does* work. Although Parsons and Carlson specifically dub this "visual tension" (and give entirely visual examples) the concept is more generally perceptual, having to do with sensory "dissonance" (which is perhaps a better name). The artefacts are clearly functional but "possess perceptual features that are at odds with this manifest functionality"(Parsons and Carlson, 2008, 99). The pleasure derived when a simple 'childish' sounding instrument such as the Ukelele is used to play highly sophisticated music might be considered an auditory example of this kind of pleasing dissonance.

2.4.1 Strengths and Weaknesses

By considering how our perceptual categories affect our aesthetic experiences, Parsons and Carlson's theory introduces much depth into discussions of functionalism. Applying this theory to the case of proofs can potentially yield explanations for how a wide variety of aesthetic effects can be generated, and thus offer an overall more accurate and nuanced theory. The fact that Parsons and Carlson's theory can be applied to a wide variety of objects is also beneficial. If conceptual artefacts such as proofs can be brought into this fold, then their aesthetic dimension can be seen as one aspect of a larger phenomenon. Appeals to value judgements in this area can be compared to those in other areas which Parsons and Carlson examine, such as architecture.

Nonetheless, there are certain difficulties involved in adapting a theory of art to the functional case. One of these issues is that the manner in which Parsons and Carlson assign proper function leaves open the possibility of more than one proper function, and hence of an object being in more than one category. This is something the authors seem aware of, but do not take up in much detail. Nonetheless, it requires some attention because it can be expected to complicate the phenomenology when considering objects such as proofs which demonstrate precisely this sort of pluralism. Another more substantial set of issues for Parsons and Carlson arises from the observation that functional categories may not be entirely analogous to categories of art. This has certain down-stream effects when evaluating how standard, contra-standard, and variable properties interact to generate aesthetic qualities. Further detail and modifications may be necessary to best describe the functional case.

Differences Between Categories of Art and Functional Categories

There are differences between the roles of standard, contra-standard, and variable features as described in Walton's original theory and as described by Parsons and Carlson. In particular, variable features no longer behave in the same manner when generating an aesthetic effect. In both the second and third forms of functional beauty, it is the presence of standard and contra-standard features which does the work, with variable features being either absent or playing a minor role. Only with respect to the first form of functional beauty do variable features play an active role. In Walton's theory, by contrast, it is the variable features which are the more aesthetically salient features for evaluating art. This suggests that the analogy between how functional categories affect our aesthetic perceptions and how categories of art affect our aesthetic perceptions may not be as close as it first appears.[12] Indeed, I believe that there are some important points of dissimilarity which necessitate further refinements of the theory if it is to be robust enough to handle complex cases, such as that of proofs.

It is not surprising that differences between our perception of functional objects and art objects should arise. They stem from the fact that in the case of aesthetic appreciation of objects which fall within functional categories, the eye is trained to pick out those features which it perceives as directly contributing to functional success. How we view features is, by the very nature of the categories involved, goal directed. Standard features will be perceptually foregrounded because of their contribution to the appearance of fitness. This makes how we perceive functional categories fundamentally different (indeed inverted!) from the case described by Walton where standard features were often placed in the background.

How we determine what are standard, contra-standard, and variable properties in the functional case will also differ from how we determine them in the case of categories of art. In particular, we cannot in the same way conceive of standard properties as those that seem necessary for category membership, because there are multiple paths to fulfilling a particular function. That is, we have cases where we can say that it is necessary to have either feature A, or feature B in order to perform function F, but the object need not have both

[12] Parsons and Carlson argue that certain categories of art (such as commemorative art, or tragedy) ought to be thought of as functional categories (Parsons and Carlson, 2008, 223). This raises interesting questions about whether our aesthetic appreciation of objects within such a category shifts when we view it as a category of art or a functional category. Although I don't take those problems up here, I believe we can expect such considerations to muddy the situation somewhat and so should keep them in mind as possible confounding factors.

since neither A nor B is independently necessary. The functional category of *flight* is an example of this: it is essential that flying objects must either flap wings, be lighter than air, glide, or have some other means of achieving lift-off. Similarly, a printer must use toner, or ink, or one of many other means of marking a page. It is not necessary that it use any particular one of these, but it is necessary that it use at least one. We could assert that it is a standard property for printers to have a mechanism for marking the page, but that is most of the way to the trivial statement that it is necessary for a printer to be able to print. In these cases, we should instead claim that it is the disjunction of these simple properties which we perceive as necessary for category membership.

There is, however, a danger in allowing disjunctive properties to be standard: it is too permissive if left unrestricted. In particular, it is necessary that an object must either have or lack any given property P, so the disjunctive property (P or not-P) would be standard, where P is any property whatsoever. Should this be allowed, then every object will have an over-abundance of standard (and contra-standard, for parallel reasons) properties, completely skewing the expected phenomenology of the object. The easiest way to make the required restriction is to not allow any disjunctions where one of the disjuncts is, on its own, standard. Thus we must classify the individual disjuncts themselves as variable since they are not (on their own) perceived as necessary to an object's membership in a particular functional category, although they are individually considered helpful if present (they might even guarantee success). Indeed, despite the fact that together they may be considered necessary, certain such features may appear substantially more 'normal' relative to a particular category, and similarly some such features may simply seem better suited to the task than others. This suggests that many variable properties are actually active contributors to the appearance of fitness, playing part of the aesthetic role previously attributed only to standard features.

Variable properties are therefore relevant when considering all aspects of functional objects, although not in the same manner as with Walton's original discussion of variable properties of art. Some of these variable properties even directly contribute to an object's appearing able to perform its function. Close inspection reveals that variable properties of this sort are central to Parsons and Carlson's theory. Consider, for instance the following complaint raised by Ross (2009):

> If the absence of contra-standard properties generates type 2 functional beauty, while the presence of such properties is a hallmark of type 3 functional beauty, then these two claims together with the Law of the Excluded Middle threaten to establish that we're always in the presence of functional beauty.

Ross' argument in detail is as follows:

premise 1 Contra-standard properties must be either absent or present

premise 2 If they are absent, then the object looks elegant (type 2 beauty)

premise 3 While if they are present then they display visual tension (type 3 beauty).

conclusion As a result, she claims, all artefacts are perceived to be in some way beautiful given Parsons and Carlson's theory.

Parsons and Carlson escape this sort of worry by blocking the third premise by saying more specifically: if contra-standard properties are present *and* the object appears to be functioning or capable of functioning[13], then it displays visual tension. Should the object have contra-standard properties but not appear functional, it would be expected to receive a more straightforwardly negative assessment such as being unfit, dysfunctional, or ugly. That is, the phenomenon targeted by Parsons and Carlson is not simply when the object displays contra-standard features, but when the presence of contra-standard features inexplicably does not seem to hinder the object's functionality as we might expect it to. This underscores how important it is to the description of visual tension that we have some way of perceiving whether the given object appears actually functional and that this be independent of its standard and contra-standard features. It may even be important to say that the object appears able to perform its function *well* or *properly* to distinguish the cases of visual tension from those in which an object appears dysfunctional but still chugs along (as in the case of a car with flat tires), which we still consider ugly on account of the object in question being broken or malfunctioning.

We must then pin-point what gives the object the appearance of *looking able to function*, or *looking functional*. In order for it to be useful for Parsons and Carlson, this quality must be distinguished from that of *looking fit*. More precisely, we perceive an object as looking fit in virtue of it lacking contra-standard features, and so looking fit is incompatible with the presence of contra-standard features. At the same time, since visual tension requires both that object look functional and that it display contra-standard features, these things must be compatible (or there are no objects with this kind of

[13]In their words, such objects are *manifestly functional*: "although they are manifestly functional, appearing capable of performing their function, such objects also possess perceptual features that are 'at odds with' this manifest functionality."(Parsons and Carlson, 2008, 99)

beauty). Since the quality of looking functional must be compatible with the presence of some contra-standard features, and the quality of looking fit is incompatible with contra-standard features, looking fit and looking functional cannot be the same.

One way an object may look functional is that it may have other features that suggest to us that it is able to perform its task. A means of defining looking functional along these lines is suggested by Parsons and Carlson's reference to variable features "indicative of functionality"[14]. Based on this, we can say that *an object looks functional if it displays variable features that are indicative of functionality.* Such a definition is consistent with the object also displaying either standard or contra-standard features, but requires further detail.

A (rather uninteresting) way in which an object can look functional is that we might catch it in the act. In this case, we are perceiving an empirical, non-aesthetic property (the property "works!") of the object. Although it is perhaps intuitive to consider this a standard property, it is better thought of as variable since we do not need to perceive an object work, nor not-work, before placing it in a functional category. In fact, as mentioned above, an object which doesn't work appears broken or malfunctioning, a perception that only makes sense if the object retains its category membership regardless of whether it works or not. "Works!" may be considered the simplest sort of variable property which is actively indicative of functionality. For the concept to be of greater use, we should be able to identify other less trivial variable properties indicative of functionality. Wear and tear, for instance, suggests that the object has been used in the recent past and therefore allows us to

[14]These were originally mentioned in the definition of looking fit.

plausibly infer its functionality. An artefact which is well maintained, polished, oiled, etc. looks ready to use and functional. If it is rusting or gathering dust, on the other hand, we might reasonably doubt its functionality.

Perceiving that the object looks functional is one way in which variable properties can contribute to our aesthetic perceptions in a non-neutral manner. Another, however, might be if we think they increase the object's probability of success. This too could be considered an interpretation of the phrase "indicative of functionality" but should be distinguished from the previous case where the features established *that* the object functions. Properties which make an object seem more likely to succeed or to perform particularly well are perhaps better thought of as *indicators of effectiveness*. Parsons and Carlson emphasize that knowledge of *how* an object performs its function plays an active role in our aesthetic perceptions. This is perhaps where such knowledge can be effectively applied: with it, we may be able to discern features that contribute positively to performance (even if they are non-essential). The shape of most vehicles is certainly a variable property, but we know an aerodynamic form makes it move more swiftly and efficiently. Having such a shape therefore contributes to the object looking fit (though perhaps not its looking functional) because it is an indicator of effectiveness.

Given these observations, the phenomenology can be clarified as follows. An object:

looks fit whenever all properties are either standard, indicative of functionality, or indicative of effectiveness.

looks elegant whenever all properties are standard.

looks visually tense/dissonant whenever many properties are indicative of functionality and some properties are contra-standard.

The fact that Parsons and Carlson must appeal heavily to sub-kinds of variable properties in order to capture the aesthetic effects they are after suggests that that it may be better to focus on these than to so literally generalize Walton's original taxonomy. Among artefacts, the role that standard and contra-standard properties played for Walton is, in many respects, taken up by these different varieties of variable properties. Given their central role, these would be well served by a heavier focus and further elaboration. The division between properties as laid out by Walton and adapted by Parsons and Carlson simply fails to distinguish between properties in a manner salient to many aspects of functionalism.

Multi-Functionality

A final complication worth briefly considering is that Parsons and Carlson's definition does not in and of itself exclude the possibility that there might be multiple proper functions for an artefact and hence that it might fall into more than one functional category. The potential for multiple proper functions arises from the potential for multiple marketplace needs. Should there be different wants and needs satisfied by an object performing different functions, and should these each lead to the manufacture and distribution of that object, then they would each qualify as a proper function. We can consider certain kinds of food as an example of this phenomenon. Junk food has the proper function of tasting good, with little to no regard for nutrition. It is difficult to assert that such food has the proper function of nourishing us, since its success (or lack thereof) at this function has had little bearing on its recent marketplace performance. Health foods (or perhaps an IV drip, or soylent green) are the opposite, selected for their nutritional value regardless of the pleasantness of their consumption. Yet there also exist foods such as smoothies and energy bars which have both proper functions: they owe their ongoing manufacture to satisfying both sets of needs. Marketplaces consist of many individuals with many wants and needs.

One way in which to handle these situations is perhaps also adaptable from Walton who suggests that in addition to considering properties as standard, contra-standard or variable relative to their category, we also consider properties of an object as being standard (etc.) for a particular individual at a particular time based on all of the categories in which that individual perceives a work of art as falling:

> A feature of a work is standard for a particular person on a particular occasion when, and only when, it is standard relative to some category in which he perceives it, and is not contra-standard relative to any category in which he perceives it. A feature is variable for a person on an occasion just when it is variable relative to all of the categories in which he perceives it. And a feature is contra-standard for a person on an occasion just when it is contra-standard relative to any of the categories in which he perceives it. (Walton, 1970, 342)

Effectively, Walton advocates the creation of hybrid categories of art indexed to an individual, an approach which we can also adopt in cases where more than one functional category is perceived as active. Using Walton's definitions of standard/variable/and contra-standard properties for these hybrid categories, we can consider some of the ways the interactions of such features can affect Parsons and Carlson's phenomenology when dealing with multi-functional objects.

We may note, for instance, that according to this way of adapting the theory, an object looks fit as a hybrid whenever it looks fit relative to all its functional sub-categories, and has some features which are variable for all of them. This is because to look fit relative to a category, an object must lack contra-standard properties, and a property which is contra-standard in any sub-category (preventing it from being either fit or elegant relative to that category) will also be contra-standard for the hybrid. Surprisingly, on the other hand, it is possible for an object to look elegant for a hybrid category while not looking elegant for any of its sub-categories; it may also look fit for some (but not all). This is because the appearance of elegance requires not only a lack of contra-standard features, but also a lack of variable features. A property which is variable for one functional category may be standard (or contra-standard) for another, and thus present as standard (or contra-standard) in the hybrid category. Finally, the appearance of visual tension requires that the object have contra-standard properties but still appear functional. We may conclude that an object displays visual tension in a hybrid category whenever it displays the tension relative to at least one sub-category, and also appears functional relative to its sub-categories.

A typical example of a multi-functional object is a sofa-bed. Here, looking fit requires that the sofa-bed look fit as both a bed and a sofa.[15] It is elegant if all of its features are standard relative to one of these categories, even if they

[15] This is the same as the "absolute aesthetic judgement about a multifunctional object" found in De Clercq (2013, 46) according to which "a sofa bed is functionally beautiful [period] if and only if it is beautiful both as a bed and as a sofa"

are variable relative to the other (so for instance, the sofa-bed having arms might be considered standard for the sofa category but not the bed category[16]). Finally, it displays tension if it displays contra-standard features of either bed or sofa, but appears able to perform both functions regardless.

One of the assumptions behind functionalism is that the categories in which we perceive an object as falling are important to our aesthetic perception of that object. Although Parsons and Carlson largely bracket consideration of multi-functional objects, it is necessary to bear in mind the manner in which these interactions occur, since some proofs are examples of multi-functional objects.

2.5 Functional Beauty Refined

The greatest strength of McAllister's aesthetic induction is its capacity to explain how aesthetic preferences can develop and evolve in response to the performance of an object which has a particular function. This is particularly well suited to discussing cases involving objects of intellectual discourse such as theories (and perhaps proofs) because it makes sense of the manner in which both individual experience with the subject matter and historical evolution of the field as a whole can impact aesthetic assessments. The interplay pictured between success and preference can also help explain why aesthetic factors are sometimes treated as being indicative of correctness, and provides tools with which to assess the degree to which such evaluations are rational. As such, this theory provides a strong framework from which to evaluate the role of aesthetic judgements in mathematical practice. Since references to beauty being a test,

[16] Again, we might wonder if this is possible since, for instance, having a specific colour is clearly variable relative to both categories.

indicator, or guide are pervasive in informal discussions of the aesthetic in mathematics, an explanation of this form is highly desirable. The theory suffers from the fact that the mathematical case may differ somewhat from the case of scientific theory choice both with respect to historical developments as well as the notion of functional success. It also generates an overly simplistic view of the aesthetic in which objects can receive aesthetic evaluation and comparison based solely on how high they score on a single scale (their beauty valuation). This makes it harder to explain different kinds of aesthetic effect which might come up in discussion, and to take on certain questions of value since the theory makes it somewhat unclear what about the value judgements are specifically aesthetic, and how the aesthetic in mathematics relates to the aesthetic as it is encountered in other aspects of human life.

Parsons and Carlson do not provide the same tools as McAllister for evaluating claims about the aesthetic in mathematical practice, but their theory, if adapted for the mathematical case, gives a means to situate it within the wider context of aesthetic experience. They focus on the manner in which an understanding of functionality can structure our perceptions of an object, thereby creating aesthetic effects, and they provide a nuanced treatment of properties and a sophisticated development of how these properties can interact to generate different sorts of phenomenological effects. Parsons and Carlson are not restricted to a single form of beauty, as McAllister is, but are able to describe three different experiences emerging out of an object's properties and their perceived relations to functional success. Unfortunately, some of the ways in which Parsons and Carlson adapt concepts from the theory of art do not appear to be an ideal fit with the functional case. Moreover, the theory does not adequately explain how we come to perceive an object as belonging to a particular functional category, nor why certain properties are perceived as standard, contra-standard, or variable relative to their functional category. Indeed, an aspect of non-aesthetic properties other than whether they are standard (etc.) takes centre stage: that of whether the property is an indicator of functionality, and this aspect of the theory remains under-explored.

In light of these observation, I present a theory adapted from those of McAllister and Parsons and Carlson. By combining elements of both, I aim to address the issues raised and arrive at a theory better suited to the mathematical case.

2.5.1 Functional Categories

I am sympathetic to the view that how we categorize objects, be they tools or artworks, impacts our aesthetic perception of them. In many cases, multiple systems of classification might be available to us. In the case of proofs, I will focus on the manner in which we perceive the core argument in terms of its epistemic functions, however I do not deny that issues of genre and style also play an operative role in aesthetic perceptions of proofs, perhaps in a manner which quite directly echoes Walton's theory. Indeed, the question of genre and style is not, I think, wholly distinct from that of function in mathematics. Certain styles developed because they are better suited to performing certain functions, or are imposed by the subject matter of a particular sub-discipline. As will be explained in more detail in chapter 3, I hold that what makes an aesthetic consideration stylistic has to do with the sorts of non-aesthetic properties it is responding to. As such, style may also be subject to functionalist considerations. Understanding functional categories is therefore an important first step toward understanding the other category systems relative to which perceptions of mathematical proofs may take place.

As discussed above, Parsons and Carlson suggest using a selected effects theory to determine the proper functions of various objects. It is reasonable to adopt this strategy for determining proper functions of mathematical proofs also, thereby allowing proofs to fall under their fairly unified account of functionality. This strategy divorces function from the intents of individual mathematicians, making it depend instead on the actual use and success of proofs in the community.[17] As a result, it provides a coherent explanation for how proofs could change function, and how they might acquire new functions over time. That said, proofs are not the same sorts of artefacts as physical artefacts or buildings, and so the selected effects theory may require further amendment to be suitable.

While Parsons and Carlson use the marketplace to establish functionality for everyday artefacts, something analogous must be appealed to in order to establish functionality of conceptual artefacts like proofs. It should be recalled that the marketplace was itself put forward as an analogy to a genetic theory, suggesting that the development of technology has an evolutionary quality to it. Just as genotypes survive and are reproduced because of their ability to perform certain functions in an organism, artefacts are manufactured, maintained, and distributed because of their ability to perform certain functions for agents. Thus, the existence of a particular artefact token is the result of its

[17] This is a way of attributing function to proof that focuses not on the proofs ideal logical structure, but its ongoing development in practice (even when still informal). In this way it is consistent with the depiction of proof seen in Lakatos (1976) where proofs may be of use even if they are falsified because, for instance, they may point the way toward more refined and interesting conjectures. In this sense, Lakatos also does not trust original intentions of mathematicians to determine a proof's function, treating the view "that a proof necessarily proves what it has set out to prove" as a "dogmatist bias" (23).

ancestors performing well. The marketplace represents the (economic) system in which all of this occurs. Proofs also survive and reproduce, in the sense that they get preserved, copied, shared, translated, taught, and referenced in other works. The "marketplace" for this sort of artefact can therefore be thought of as the epistemic community in which it is embedded, complete with that community's conferences, journals, and classrooms. The proper functions of these artefacts are, as a result, those tasks at which success has led to ongoing exposure within the community.

There are some aspects of this analogy which might be problematic and should be considered briefly. We might worry that success at certain functions lead to substantially more copies of particular proofs than success at others, and that this might not always reflect our intuitions about the relative importance of these proofs. That is, we might arrive at a view of proper function which leads us to believe that certain functions are more important than others because they have met with greater success in the intellectual "marketplace". For instance, elementary proofs are widely distributed, while longer proofs are rarely reproduced (and in some case not even read in full by most people), and many more standard but perhaps unexciting proofs go unmarked. It might be feared that while the functions of trivial proofs would be well explained, more sophisticated and mathematically significant proofs might fail to receive any attribution of function at all because of their lower visibility. To avoid this, we must accept more than simply copies and restatements of a proof as analogous to the manufacturing and distribution referenced by Parsons and Carlson, but also the references to such proofs which are found elsewhere in the literature. Citation is one of the most potent devices we have for, in some sense, "preserving" intellectual artefacts.[18]

We might also worry about the fact that proofs which are known to be incorrect are sometimes proliferated anyway. This sort of failed proof is no longer serving any of the functions that we normally associate with a proof in good standing, but its reproduction can still be attributed to its performing *some* function. For instance, it might be referred to as an example of common or subtle errors (as when we consider the various mistaken proofs leading up to the proof of the 4-colour theorem), or it might simply be amusing (as in the case of fallacious proofs that 1=0[19]). This illustrates the manner in which proper functions can change, and how they differ from intended function. It does not mean that proofs can never be considered to have malfunctioned or failed on account of errors being found in them, only that we must consider the proof's history. When an error is first detected in a proof that is circulating, then that proof ought to be considered either *defective* because it is unable to perform its proper function, or even *broken* if that error was introduced in a sloppy reproduction (as when a good proof is taught incorrectly, or there is a typo in its presentation). If instances of a defective proof continue to be referenced and reproduced *because it is is invalid*, then that proof's proper function has shifted. It no longer has the proper function of establishing a result, but now functions in another capacity, such as vividly illustrating what

[18]It is interesting to note that success at performing different functions might be reflected in different ways. We would expect explanatory success to manifest in the classroom and textbook, while proofs which connect or tool-build might be more heavily referenced in the journals and higher-level literature (these functions of proof are discussed in chapter 3). This suggests that performing more than one function will increase a proof's overall success, and points to how simply establishing a result may, in a sense, be insufficient for its overall success. A proof which falls out of use or memory no longer serves any function within the community, even if it is correct.

[19] See Aberdein (2013) for a discussion of the role humour can play in mathematical reasoning and practice.

not to do.

The question of what an object's function is differs somewhat from the question of how we perceive its function, or rather what functional category it forms a part of. Functional categories as treated here are cognitive categories, rather than metaphysical. They are conceptual groupings of objects with similar functionality. They order and structure our perceptions, as well as frame our expectations, but should be considered distinct from artefact kinds. We may say that an object belongs to a functional category F, whenever we perceive that it has F as a proper function. We may learn that an object is F by being told that it is, or by being familiar with others of its kind. We may surmise that an unfamiliar object is F by seeing it perform F (or attempt to perform F), or by its bearing a strong resemblance to other F objects (such as its ancestors). We could, of course, be wrong, since on the one hand we could witness improvised or accidental functionality, and on the other hand we may encounter an object designed to mimic an F object without being an F object itself (for instance we might mistakenly categorize a replica gun as a weapon rather than a prop). A more reliable source for our categories might be taken from the selected effects theory itself. Since proper function depends on the ongoing production of that particular object based on its successfully performing a particular function, perception of a proper function should depend similarly on familiarity with the particular "marketplace" in which it is situated, and that marketplace's demands.

2.5.2 Non-Aesthetic Properties and Induction

I argued above that an account which better explained the role of standard and variable properties and their relation to the appearance of fitness was required than that provided by Parsons and Carlson. To this end, rather than focusing on the perceived necessary conditions (the perceived essential features) for an object, I focus on the manner in which particular features are correlated with the performance of particular functions. Thus, instead of standard, variable, and contra-standard properties, I suggest that we speak of positive indicators, non-indicators, and negative indicators of functionality as follows:

1. Property P *positively indicates* functionality F whenever P-objects tend to succeed at F (more than objects lacking P).

2. Property P *negatively indicates* functionality F whenever P-objects tend to fail at F (more than objects lacking P).

3. Property P is a *non-indicator* of functionality F whenever it is neither a positive indicator nor a negative indicator of F. This may be either due to P-objects both succeeding and failing at F in equal measure, or perhaps to P being a new property with which we have little experience.

For Walton, standard and contra-standard features were seen as the negations of one another, allowing us to refer to the presence of contra-standard features and the absence of standard features synonymously. That is not necessarily the case for positive and negative indicators: having a feature may positively indicate success and increase the apparent fitness of the object, while lacking that same feature might not actively decrease the appearance of fitness. We should also note that the sets of positive and negative indicators are internally consistent. That is, the properties P and not-P cannot both be positive (or negative) indicators because if P is a positive indicator then the presence of P must indicate a greater chance of success than its absence, not-P (and similarly with negative indication). The "more than objects lacking P" clause is meant to capture how when a P object and a not-P object are compared, the viewer is likely to judge the P object more apt relative to functionality F than the not-P object. The same viewer will not necessarily find the not-P object actively dysfunctional. For that to be the case, not-P must itself be a negative indicator, but in fact it could be a non-indicator. Thus, when discussing mathematics, we can (and often do) have cases such as the following: brevity is an indicator of success at certain functions (such as convincing), while prolixity is a negative indicator thereof. A proof cannot be both brief and prolix–these properties are incompatible. Nonetheless, we may have a proof which is neither brief nor prolix. We therefore have a property (the property have being of a middling length) which can be neither a positive indicator nor a negative indicator of explanatory success; instead, it is a non-indicator.

This behaviour fits with the observation that many properties should be considered fuzzy or vague, admitting to degrees. Thus, when discussing indicators, we may occasionally be referring to ranges of values for fuzzy properties, and cannot assume that an object having a property which falls outside the positive indicating range also falls within the negative indicating range. For these reasons it is somewhat deceptive to speak of an object simply having or lacking non-aesthetic properties. This observation is particularly relevant in the case of proofs, where such things as complexity and length have aesthetic bearing. While it is desirable in the long run to provide a model which more precisely reflects these kinds of features, we may for now consider the properties under discussion as being short form for a more complex and precise description. Thus, in the above example, we mean by "brevity" and "prolixity" a particular ranges of values: a proof may be said to be "long", "short" or "average" based on how its length compares to that of proofs in general. The more below average its length, the shorter it will seem; the more above average, the longer. This is obviously a simplification, but does reflect the manner in which we actually speak of these topics.

As we did in the context of Parsons and Carlson's original theory, we may note that there is a difference between properties which indicate functionality in the sense discussed above, and properties which indicate to us that the object is functioning. The former contribute to the perceived likelihood of future success, whereas the latter suggest current or past success, and perhaps provide clues which allow us to perceive it in the correct functional category, but may not contribute in any way to the perceived likelihood of future success.

We must have a way of determining what properties are positive or negative indicators, and this is where an inductive mechanism such as that of McAllister is of importance. A property P gets weighting W_P *relative to* functional category F, beginning at a neutral zero. This gives us the weighting $W_{F,P}$ which reflects the degree to which we associate possessing P with succeeding or failing at F. When an object that possesses P is successful at a particular function F, that property's weighting relative to F increases. When it is unsuccessful, the weighting decreases. Highly negatively weighted properties are considered negative indicators of F while positively weighted properties are considered positive indicators of F, and those with a weighting close to 0 are perceived by us as non-indicators. Based on this, we may use the weightings of properties in category F to determine how F-fit an objects appears.[20]

Where beauty is identified with this appearance of fitness, this is essentially equivalent to the aesthetic induction but with some useful distinctions introduced: an explicit relativization of the weightings to functional categories allows for the potential for the same property to receive different aesthetic assessments in different contexts, and clearly differentiating between positive and negative indicators opens up the space to bring in the more sophisticated phenomenological machinery of Parsons and Carlson. There are more complex aesthetic responses available based upon not simply how fit an object looks for a function, but also what sorts of properties it has and in what proportions. While the aesthetic induction focuses largely on the cumulative effect of an object's properties together, Parsons and Carlson have given us tools we can use to consider a more detailed breakdown of the overall effect. For instance, we can use their approach to distinguish between an object which scores highly because all of its features are positively indicated, and an object which scores highly because only a few features are positively indicated but the weightings of these features dominate in magnitude those of the negative indicators. The aesthetic induction alone cannot do this.

2.5.3 Aesthetic Effects

I agree with Parsons and Carlson that our perception of aesthetic properties such as functional beauty depends on the presence of non-aesthetic properties, and the relations of these properties to the functional categories in which

[20]This process essentially involves subtracting the probability that we assign to a P object of failure at F, from the probability that we assign to it of success. One straightforward way this can be modelled is by calculating $\frac{\text{observed successes}-\text{observed failures}}{\text{total observed cases}}$. This is obviously quite simplistic and runs into difficulties, especially with small sample sizes. The theory could be better fleshed out by making use of heavier (and likely Bayesian) machinery.

we perceive the aesthetic object. Thus, echoing their approach but with the required modifications, we may describe the phenomenology as follows:

Appearance of Fitness When an object presents many positive indicators of functionality F, and few negative indicators, then it may be said to look fit for F (or F-fit).

Elegance When an object presents only positive indicators of functionality F, and neither negative indicators nor many non-indicators, then it looks especially elegant with respect to F. Every feature of the object seems to contribute to its potential success at F.[21]

Dissonance When an object presents negative indicators of functionality F, but still appears to be actively functional, then there may be the same kind of tension or dissonance that Parsons and Carlson refer to.

It is helpful to remark here that this breakdown of Parsons and Carlsons theory has certain things in common with the functionalism proposed by Sauchelli (2013) according to which our *expectations* frame our aesthetic experiences. More precisely, for Sauchelli, our past experiences with functional objects produce expectations (beliefs) about how other objects with similar functions will look, and this in turn alters our perceptual experiences (Sauchelli, 2013, 46). Sauchelli's version of Parsons and Carlson's phenomenology states that looking fit is a result of our expectations being satisfied "possibly in a clever way", elegance is a result of such conformity combined with the additional constraint that it "avoid any unnecessary element in the arrangement of the parts of the object", and tension is a matter of the object violating

[21]Elegance may perhaps be thought of as a kind of super-fitness.

expectations while still being able to function (Sauchelli, 2013, 49). This is essentially consistent with my view if we take positive and negative indicators as a more precise way to describe what our expectations for an object are. That is, we can say that we expect objects to present many positive indicators of their functionality, and that we expect them to to lack most negative indicators. The induction borrowed from McAllister is a concrete mechanism for modelling how such expectations might be formed and altered over time, and allows us to see clearly the manner in which "these beliefs are the result of our previous experiences with objects appearing to have the same (or similar) functions." (Sauchelli, 2013, 46)

Sauchelli also points out that satisfying expectations is not always sufficient in itself, but "rather, the composition of its parts should be perceived as clever and ingenious to stimulate aesthetic interest" (Sauchelli, 2013, 49). In the case of proofs, as with other more mundane functional objects to which it seems that we do not *always* have an aesthetic response, an aesthetic theory should tell us why we take aesthetic interest in some objects of a particular kind and do not take interest in others. A plausible explanation for this is that we require something to grab our attention first. For Sauchelli this is being "clever or ingenious", but we might instead speak more simply of novelty.[22] In his discussion of aesthetic themes to which surveyed mathematicians seemed to respond, Wells describes the effect of novelty in a similar manner:

[22] I think there is an implicit assumption of novelty built into many references of ingenuity and cleverness. How clever can something be if it is expected? We might worry that this could produce some problems for Sauchelli's theory since it is built upon the concept of expectation. In this regard, one of the things that an appeal to positive and negative indicators allows us to do is distinguish between the unexpectedness of an object with *new non-indicators*, and the unexpectedness of and object with *familiar negative indicators*. The first is arises when the object has features which we have never encountered before, while the second arises when it has features we have come to think will undermine its success at a given function.

> Surprise and novelty are expected to provoke emotion, often pleasant, but also often negative. New styles in popular and high culture have a novelty value, albeit temporary. As usual there is a psychological connection. Human beings do not respond to just any stimulus: they do tend to respond to novelty, surprisingness, incongruity, and complexity. (Wells, 1990, 39)

A proof may look fit or even elegant if we consciously analyse it, but yet not elicit a spontaneous aesthetic response. Such a response will usually only be elicited if there is some manner in which the proof is exceptional. The effect of novelty is to transform the object into the sort of exceptional thing which provokes such a response where it might not otherwise occur. In this way, a novel feature draws and directs our attention. This idea may easily be incorporated into the framework I propose:

Banality An object dominated by familiar non-indicators is banal and unlikely to elicit a strong response of any sort.

Novelty When an object presents non-indicators which we have not encountered much in the past, then it appears novel. Despite not yet having an association with fitness, these properties are attention grabbing, and as such may have an amplifying effect with respect to our overall perception of the object. In the case of proofs, the novel features may be regarded as clever or suspicious depending on their resemblance with other more familiar positive or negative features.

As a final note, in the case of mathematics, novelty is not the only thing that may draw our attention and amplify our aesthetic response; historical and sociological factors such as the fame of the theorem proven may have similar effects. Mathematics is littered with holy-grail problems (such as Hilbert's problems, the Clay institute problems, Erdős' puzzles, and so on). When proofs of such famous conjectures are offered, they are subject to far greater scrutiny than other proofs of mathematics. The aesthetic reaction is also, in these cases, amplified beyond what it might otherwise be (some of the intensely negative aesthetic responses to the 4-colour theorem might in part be a result of this phenomenon).

2.6 Concluding Remarks

In some respects, the functionalism I've proposed is a result of connecting dots left by other theorists. It retains the spirit and structure of McAllister's aesthetic induction, but also approaches more closely the theory of Parsons and Carlson, which has the advantage of being much more general. The result is a refined theory which is better able to satisfy the desiderata identified at the start of this chapter than either of the two major theories on which it is based. The richness of Parsons and Carlson's phenomenology can be used to describe in detail how particular proofs evoke particular responses in us, thus helping the theory satisfy the first desideratum that it should accurately describe the phenomenon. The appeal to an inductive mechanism allows us to explain the manner in which the aesthetic can be in mathematical practice, and allows us to consider the degree to which this is justified, thus satisfying the second desideratum. Finally, by relating aesthetic responses to mathematical objects to those of other objects, the theory may help us understand and interpret value judgements made about mathematics by appeal to aesthetics. The next chapter will be devoted to more of the particulars involved in applying this theory to proofs.

Chapter 3

Functional Beauty and the Mathematical Proof

3.1 Introduction

This chapter concerns the application of a functionalist aesthetic to the case of mathematical proofs. Since proofs are not the only objects in mathematics which may draw our aesthetic eye, however, it is worth first considering why they are of particular importance to the more general problem of applying aesthetic theory to mathematics.

There are numerous aspects of mathematics that may be subject to aesthetic evaluation, and it is important not to conflate them. Rota, for instance points out "theorems, proofs, entire mathematical theories, a short step in the proof of some theorem, and definitions are at various times thought to be beautiful or ugly by mathematicians."(Rota, 1997, 171). It is by no means certain, or even likely, that each of these manifestations of the aesthetic will bear the same explanation. At the same time, it does seem likely that the aesthetic judgements at one level of mathematical discourse may be related to those at others, such that we cannot have a complete understanding of the overall aesthetic phenomenon within mathematics without some discussion of the entire field. Similarly, it is unlikely that the aesthetic in the mathematical case should remain entirely insulated from other aesthetic trends active within our culture at large. The problem therefore calls for examination of the various individual objects of mathematics, while still bearing in mind their place within a larger mathematical context.

A twofold distinction is often treated as aesthetically salient when discussing mathematics. It can be expressed as being between facts and methods (Le Lionnais, 1971, 122-123), or as products and processes (McAllister, 2005, 17)[1]. In the first group we have various entities produced by or found within mathematics such as theorems, equations, formulae, numbers, and particular geometric constructs. In the second grouping are found the various tools of the craft such as proofs, algorithms, heuristics, and perhaps even notational styles (for instance, Leibniz and Newton's notations for calculus).

The distinction is by no means perfect. It is neither clear that it is exhaustive nor disjoint and as such may not be able to easily accommodate every relevant category of object. The mathematical theory, for instance, is a product of mathematics but also a means of collating results and methods, as well as structuring a research program. It does not so neatly fit into these groupings, yet we still see aesthetic assertions such as that "Some beautiful theories may never be given a presentation which matches their beauty" and that an

[1] I favour McAllister's terminology, not only because it is nicely alliterative but also because some may be uncomfortable with the implications of the term "facts". Indeed, I wish to allow for incorrect products of mathematics to be something the discipline on occasion outputs (and then, hopefully, corrects).

"instance of a beautiful theory which has never been matched in beauty of presentation is Gentzen's natural deduction"(Rota, 1997, 173-174), or more simply that (e.g.) group theory is a particularly beautiful branch of mathematics.

We might also wonder where important counter-examples such as the Weierstrass function[2] lie. Here we have an entity that is clearly a product, yet which also acts as a disproof for the once widely believed conjecture that a function continuous on an interval should also be differentiable somewhere on that interval. Counterexamples such as this occupy an important place in mathematical practice and act as both products and processes. Granting the assumption that products and processes are ruled by slightly different aesthetic principles, we would expect mixed objects such as counter-examples and theories to be impacted by both, with potentially quite complex results.

Despite its limitations, the intuition that this might represent an important difference for aesthetic analysis strikes me as quite plausible. Objects of the first group are more obviously analogous to works of art than those in the second group. We frequently produce visualizations and physical representations of mathematical objects of this sort which can be appreciated aesthetically without necessarily referencing the particularly mathematical features of the object. Triangles, spirals, fractals such as the Mandelbrot set, and even graphical representations of functions like Weierstrass', can all be viewed aesthetically without any mathematical aptitude or awareness whatsoever. In these cases, there does not appear to be a specifically mathematical aesthetic at work. They can reasonably be subsumed under other aesthetic theories and

[2] A continuous but nowhere differentiable function found by Weierstrass in the mid 19th century. This demonstrated that differentiability does not follow from continuity as many had believed.

do not particularly strain our existing frameworks.

Theorems (and some theoretical entities such as equations) present more of a challenge for aesthetics, as they are more conceptually loaded. Similarly, most objects of the second group (processes) require a reasonable degree of mathematical training before their aesthetic elements are in any way appreciable. I conjecture that the aesthetic for processes is more tightly integrated into the mathematical context itself than is the aesthetic for products. It cannot be properly understood without an analysis that takes into consideration the mathematical content and purpose. Moreover, the best way to see the kind of aesthetic interaction we may have with this sort of object is by recognizing their role as tool.

I have chosen to focus on one specific member of the second group: the proof. I treat the proof here as a particularly important nexus of aesthetic evaluation in part because of its central role in mathematics as a discipline, and because discussion of the aesthetic properties of proofs are quite prevalent. Of all the tools available to mathematicians, it represents the most quintessentially mathematical. The proof is intimately bound up with the theorem, however, so a complete aesthetic theory of proof awaits the study of theorems as well (and vice versa).

I assume that at the level with which we interact with them, mathematical proofs are artefacts: humanly created objects with particular functions. More specifically, I hold that proofs are artefacts with functions which are largely epistemic in nature. They are peculiar artefacts in certain ways because they are in large part abstract, but this is not entirely without precedent.[3] As with other sorts of objects discussed by Parsons and Carlson, proofs can be

[3]See for instance Turner (2012), Knuuttila (2005)

understood as falling within functional categories. Because of the fact that it is through their use that we primarily interact with them, our functional understanding of proofs is key to our aesthetic perception of them. I adopt the functionalist aesthetic theory outlined in chapter 2 based on the assumption that, as with other artefacts, our interactions with and perceptions of proofs are structured in part by our understanding of their functions. The functional categories inform our aesthetic perceptions. In particular, a proof may have the aesthetic property of appearing fit or unfit for various functions on the basis of its non-aesthetic properties.

In order to more precisely explain how some of the aesthetic assessments which we have seen can be explained using the provided theory, I will identify in this chapter many of the central functions proofs have, and have previously had, within mathematics. Based on the functionalism I've adopted, an individual's views and priorities with respect to the various functions of proof can be expected to affect their aesthetic sensibilities and taste. Moreover, the long-standing tendency to reprove known theorems can in part be explained by a desire to accomplish all the functions of proof, even though these functions may sometimes be in tension with one another. To aid in the analysis, a number of examples are provided in Appendix A and referred to throughout this chapter.

3.2 An Aside on Style

In the case of considerations of all the potential aesthetic objects described above, be they products or processes, proofs or theorems, or indeed any other sort of abstract or conceptual object, we are faced with some unique problems having to do with distinguishing the object of evaluation from its mode of representation. The presentation of the proof is the means by which we gain our epistemic access to the proof in the first place, but the same proof may be rendered in multiple ways. That is, there is a difference between the proof itself and any given presentation of it. When faced with a proof in some form (written, spoken, perhaps thought), we make two aesthetic assessments because we may respond both to the presentation and to the underlying argument. This kind of dual vision can be considered analogous to that involved in viewing representational forms of art where we may speak sensibly of, for instance, a beautiful photograph of an ugly object, or an ugly photograph of a beautiful object. That there is such a difference is precisely why we are able to make sense of Rota's above-mentioned claim that there may be cases where a proof or theory does not receive a presentation that matches it in beauty.

The distinction I make between the presentation of the proof and the proof itself also echoes that made by Netz (2005, 254) who claims that beauty appears in the activity or experience of doing mathematics where it is "a property of states of mind", in the products of the pursuit where is "a property of texts", and in the various entities of mathematics itself where it is "a property of the ontological realm of mathematics". I largely agree with Netz on this division, but where he focuses his study on the second dimension of the aesthetic, I am concerned mainly with the third.

I refer to the textual dimension of aesthetic experience as the realm of mathematical style, whereas when we speak of the proof itself being beautiful, I say that we are dealing with the aesthetics of the actual mathematical content. Both of these are valid topics of aesthetic analysis, although our aesthetic responses to them are not necessarily governed by all the same principles. The functionalist theory I outline pertains mainly to assessments of mathematical content. Certain aspects of mathematical style may also be explained by appeal to functional beauty, however I believe that style likely has a great deal in common with various forms of artistic expression as well.

The difference between style and content is well demonstrated by expositions of *the marriage algorithm* which are steeped in metaphor and narrative (see A.3). Embedding such a proof in a fictional scenario such as this can make for a humorous read, and even highlight potential applications for the result, but it is an entirely stylistic choice. Other aspects of the proof, such as its being direct and constructive, are of a different character being integral to the mathematics itself. In general, I say that we are able to distinguish between style and content in the same proof because we consider different groups of properties in each case. Properties related to content are those which pertain to the logical structure and argument form of the proof. Although it is very difficult to precisely define what makes one proof identical or different from another, we may generally consider these properties the sort that when altered result in something we would consider a different proof. In a sense, then, these are the properties that we perceive as essential to a proof's identity; the logically salient features.[4] Examples of properties of this sort include:

- Structural properties such as symmetry of cases, number of cases, proof form used (for instance, inductive).
- Length of proof (when presented as efficiently as possible).
- Overall complexity.

[4]This bears some resemblance to the distinction between standard and variable by Walton, although it is not relative to a category at all but simply has to do with a specific proof's identity.

- Appeals to axioms as well as other theorems.
- Appeals to visualizations which are either essential or literal representations of the subject matter.

Stylistic properties, on the other hand, are those that relate directly to the presentation, and which may be chosen and varied by the writer. They are non-essential to a proof's identity in that we may change them without affecting the underlying argument. These properties include such things as:

- The language in which the proof is written.
- Notational choices (for instance, Leibniz's or Newton's notation for the calculus, dots or parentheses in formal logic).
- The use of literary devices such as narrative, metaphor, and analogy in the exposition.
- The use of signposting.
- The order in which lemmas and sub-proofs are presented, whenever this doesn't have an effect on the logic of the proof itself.
- The length of the exposition of the proof, when this involves non-essential text.
- The use of diagrams which are aids (as opposed to those which are integrated into the argument itself).

Style can involve similar features as content, such as the length of the actual text. This is stylistic when pertaining not to the length of argument in full, but rather to the granularity and detail provided in a particular explanation, as well as to the amount of background and contextual exposition provided. A mathematician may provide a long exposition for one audience, such as a popular audience or class, and a shorter one for another, such as fellow experts in the field. Similarly, visualization can be either integral to the proof or simply be a means of helping the reader understand the abstract material presented. Visualizations central to the material are used in many elementary geometry proofs such as the famous "windmill" proof of the Pythagorean theorem, or Kelly's proof of the Sylvester-Gallai theorem (see A.4).[5]

But visualization may also be used idiosyncratically or as a supplement, as is illustrated by Hadamard's mental pictures for Euclid's proof that there are infinitely many primes (see A.2). While Hadamard's imagery may be helpful to him (and possibly others) it does not play any sort of direct mathematical role in the proof.

3.3 Properties of Proofs

In addition to separating out specifically stylistic properties, it is useful to group the mathematical properties as structural, cognitive, or methodological[6]. While all of these have parallel groupings in the stylistic domain, the focus here remains on features of content.

Structural properties are what we would consider the more formal properties of the proof. Important structural properties include: the *length* of the proof, *the number of assumptions* made (such as axioms or prior results re-

[5]While diagrammatic proofs have been considered by some to be a-logical, there is ongoing work to present formal diagrammatic logical systems. For instance, Mumma (2010) presents a formalization of Euclid's *Elements*, thereby claiming to provide "a formal model where Euclid's diagrams are part of the rigorous proof" (256).

[6]These groupings are fairly rough around the edges and are presented only as a means to facilitate discussion.

quired), the *logical form* of the proof, whether the proof *appeals to different cases* (and how many there are), and whether there is *symmetry* in arguments and cases.

Cognitive properties are those which impact how humans read and understand proofs. Clearly, certain cognitive properties are affected by structural properties, for instance *surveyability* and *complexity* are both affected by the *length* of the proof in question. As such, it may be possible to entirely reduce these properties to combinations of structural properties. In the cases where this is possible, it is still the presence or absence of such a combination which acts as a whole to indicate a particular function. These may still be discussed separately from the individual structural properties, even when understood as property clusters of this kind. Important cognitive properties include: *surveyability* (that is, the capacity for the proof to be read through and understood by a person), *graspability* (the capacity for a proof to (in some sense) be understood all at once, or in a single thought), *visualizability* (the capacity for the proof to be depicted visually), *complexity* (how difficult it is to understand the proof) and *intuitiveness* (how well the proof, or theorem being proven, fits with our expectations).

Methodological properties are those having to do with the standards and methods of mathematical practice. As with cognitive properties, some of these will be combinations of structural properties. Important methodological properties include: the *argument type* (such as inductive, constructive), what sorts of *external aids* are required for the proof (for instance, a straight-edge and compass, or a computer), how well the proof stands up to current *standards of rigour*, and whether or not it illustrates *purity of methods* (that is, whether the proof keeps to the same concepts as are appealed to in the theorem itself).

3.4 Functions of Proofs

I have suggested that there is an interplay between the functionality and aesthetics, and in particular that certain properties of proofs may positively or negatively indicate particular kinds of functionality. We come to perceive these properties as falling under such headings based on our experiences with proofs which manifest them. In some (though not all) cases, there may be a causal connection that we can uncover which explains the correlation which we perceive. Identifying properties which might help or inhibit various functions, as well as properties to which we have a great deal of exposure, can allow us to explain or predict aesthetic responses. To that end, this section discusses a number of important functions of proofs and identifies properties which can be expected to positively or negatively indicate fitness for that function.

I focus on functions which there is evidence to suggest many proofs perform or have in the past performed. In some cases, the importance of that function in their own practice has been highlighted by mathematicians, while in other cases we can see that functionality at work by examining the history of the field. It is important to remember that, given the selected effects theory of function, functions can and often will change over time, so this taxonomy should not be considered in any way settled, permanent, or normative. Indeed, there have been a number of other breakdowns of what proofs do or are for[7]. One may thus find more fine-grained taxonomy than that provided here, although I believe that most of the functions listed in these cases fall into the broader functional categories I outline. Finally, I note that these functions are in some ways related to one another such that the successful performance of one

[7]See for instance Dawson (2006) who offers around 30.

function can sometimes improve a proof's ability to perform another.

3.4.1 Verify and Certify

As its name indicates, the proof is meant to establish the truth of a given proposition given certain starting assumptions. It is characterized in the mathematical case by a sense of certainty and even inevitability which accompanies it. Thus, the proof plays a justificatory role: it is *conclusive* evidence for a proposition. Once a proof has been provided, no other is required. This is the manner in which the proof is able to act as a certificate, a kind of rubber stamp permitting the application of the proven theorem to other problems in mathematics (and elsewhere).

The emphasis on the proof as justification rightly remains central to most serious attempts at exploring the topic. It is the most important and commonly thought of function of proof. In their certifying role, proofs are being used in an evidential capacity. Having a proof allows us to trust a theorem and apply it as we see fit. It is the manner by which new theorems are added to the existing storehouse. Mathematical proofs so certified are further honoured with adjectives such as "permanent", "certain", and "infallible" (although in reality, they are not always error free). The proof is thus an unusual form of evidence because of the high degree of certainty it confers.

This function of proof can only be fulfilled when it has been properly verified. Positive indicators will therefore play to our presuppositions and expectations. These are the properties which make the proof 'look right' even prior to a thorough verification, while the negative indicators are those that might make the reader suspicious. Obvious positive indicators include methodological properties such as the use of what we know to be logically valid methods, as well as (more generally) techniques and argument patterns which are well-established and uncontroversial in the community. Factors having to do with the 'fit' between theorem and proof might also come into play. To that end, purity of methods might be seen as a positive indicator, while appeals to theorems in unexpected and seemingly unrelated areas might seem to be negative indicators. Such things as length and complexity are likely to be negative indicators since they can easily conceal subtle and hard to find errors, and indeed often do. Proofs with these features may thus evoke a greater sense of suspicion when encountered.

3.4.2 Unify and Connect

One of the central uses of proof is not just to show the correctness of a given theorem, but also to connect various theorems and concepts to one another in a manner which helps unify or shed light on the field. Proofs can do this by demonstrating equivalences and dependencies between apparently distinct structures, appealing to results and methods from other areas than those in which the theorem originated, revealing what axioms the theorem depends on, and by just generally exposing the structural underbelly of mathematics conceived of as a larger system. In this way, proofs may serve to place a theorem within the larger mathematical context. They can help generate a picture of the overall mathematical system, and to locate a particular theorem or entity within it.

Though we may sometimes think of mathematics as being formally unified in ZFC set theory, it historically has tended to develop around particular problems, with any axiomatization coming quite late in a field's development. As a result, it is not always obvious in what ways one area relates to another, and proofs which expose the relations between fields can be highly informative. The proof of Brouwer's fixed point theorem by way of Sperner's lemma (and vice versa) derives much of its importance from the fact that it connects topology and combinatorics.

By making these sorts of connections, a proof can also be used as (not necessarily conclusive) evidence for or against adopting various primitive assumptions, axioms, and conjectures that it makes use of. For instance, when the foundations of set theory were still under debate, the proof of the Banach-Tarski theorem, a highly paradoxical result according to which we can decompose a sphere and then reassemble the pieces to create two spheres both as large as the first, was appealed to as evidence against the Axiom of Choice, of which it makes vital use[8]. Here, the argument was not that the Banach-Tarski theorem should or should not get the rubber-stamp, nor that the proof was in itself flawed, but rather that one of our basic assumptions should be abandoned because of its essential role in generating strange results.

A conjecture may also gain plausibility when it is proven to follow from another unproven conjecture which is considered even more plausible. That is, proofs of the form "Y follows from X" may be taken as reasonable evidence

[8]As explained by Suppes:

> Banach and Tarski show that by using the axiom of choice a sphere of fixed radius may be decomposed into a finite number of parts and put together again in such a way as to form two spheres with the given radius.(...) As the matter is often put, here is a clear case of mathematics violating physical reality in the sense that we certainly do not believe that anything like these decompositions can actually be carried out in real space. The very existence of these paradoxical results has often been used in the past to argue against the admissibility of the axiom of choice. (Suppes, 1988, 89)

for Y itself if X is generally thought to be correct. There are many proofs of theorems from outstanding conjectures such as the Riemann Hypothesis, and similarly many algorithms (in cryptography especially) which depend on the unproven hypothesis $P \neq NP$[9]. These results await a proof of the active assumptions in order to be elevated to theorem-hood themselves, but they still stand above their lesser brethren which have not been tied to such plausible assumptions. In the meanwhile, they increase confidence in their parent conjectures.

Certain proofs of this sort function largely to simplify other outstanding problems. This can be done in a number of ways, for instance the proof could be of a particular case but suggest a generalization which might be of use to a larger problem. Similarly, the proof may form part of a larger process: if we can show that one theorem follows from another (or better, that they are equivalent), and the second seems easier to prove, then we have potentially made good progress on proving the first. Polya discusses the pervasiveness and usefulness of the such methods in his heuristic guide *How to Solve It*:

> Chains of equivalent auxiliary problems are frequent in mathematical reasoning. We are required to solve a problem A; we cannot see the solution, but we may find that A is equivalent to another problem B. Considering B we may run into a third problem C equivalent to B. Proceeding in the same way, we reduce C to D, and so on, until we come upon a last problem L whose solution is known or immediate. (Polya, 1988, 54)

[9] Computer scientist Stephen Cook points out that "one negative consequence of a feasible proof that $P = NP$ is that complexity based cryptography would become impossible" but that most complexity theorists (including himself) believe $P \neq NP$.(Cook, 2003, 28)

The work leading up to the proof of Fermat's Last Theorem by way of the Taniyama-Shimura conjecture is but one example of such an incremental use of proof.

Positive indicators of proofs with this sort of function include transparency, explicitness, and attention to detail. All of these properties contribute to a sense that the proof is revealing its underlying structure and assumptions. Mixed methods is a positive indicator that the proof is making unexpected connections, while purity of methods is a negative indicator since it suggests that we are remaining within the boundaries standard for the sub-field. Finally, generality is a positive indicator of this sort of proof, for it is suggestive of the potential to export the result or methods to other contexts.

3.4.3 Explain

Explanation is considered by many one of the most central roles proofs play and also one of the most complex. It is tied to the informative element of proving, the manner in which the proof gives more than the theorem alone could. Consider, for instance, a maximally non-explanatory proof: a black-box proof, a proof by God. This kind of proof would be acquired by asking an omnipotent and infallible divine authority whether a conjecture holds. An answer of "yes" would clearly be able to act in the certifying capacity required of proof, yet for the most part it would (I think) be of very limited mathematical value.

This role of proof has been explored in great detail, and remains under active discussion.[10] Providing a full account of explanation is well beyond the scope of this project, however explanation is something for which we use proofs (even if it is not always effective), and some proofs clearly provide

[10] For instance in Hafner and Mancosu (2005), Mancosu (2001), Steiner (1978)

more satisfying explanations than others. Moreover, while providing a positive account of necessary and sufficient conditions for explanation is extremely difficult, in part because explanation seems to take many forms in mathematical discourse, there do seem to be very many properties that we expect to *inhibit* explanation, and such properties are likely to be perceived as negative indicators. This is a manner in which explanation can be a rather peculiar function: it is much easier to identify examples that fail to be explanatory, and even give some reason for this failure, than to identify precisely what constitutes a proper mathematical explanation. The divine black-box proof method described above would lack any explanatory value because of the complete absence of information. It is *too simple*. A proof so long, complex, or arcane that no person in the mathematical community is able to read or understand it also lacks effective explanatory power, even if it does add information. At the same time, apparently ad-hoc steps, even when they are short and easily followed, can inhibit explanation. Such steps may appear both ingenious and clever, but are also likely to arose curiosity about where they came from.

Certain proof methods, such as reductio ad absurdum and mathematical induction are sometimes viewed as being less explanatory than others, or in the least as negative indicators of explanatory power. This may have to do with methods themselves tending to display certain of the negative indicators described above. In particular, aspects of these proofs, such as the source of the formula or assumption itself, may be murky and unclear and for this reason seem to be ad-hoc. More precisely, one of the elements that can make direct (so non-reductio) and non-inductive proofs seem more explanatory in many cases is that they show a strategy for generating the given theorem, rather than making what may seem to be a "magical" assumption at first and then verifying or falsifying its correctness (see A.5 for a popular example of this).[11]

A proof functions in an explanatory capacity by somehow conveying a sense of *why* the given result holds, and perhaps even allowing the audience to retrace the process of discovery themselves. As a result, features which positively indicate explanatory potential tend to be cognitive properties which we consider to be psychological aids, in particular being relatively simple, surveyable, and visualizable. Hadamard, for instance, suggests that "any mathematical argument, however complicated, must appear to me as a unique thing. I do not feel that I have understood it as long as I do not succeed in grasping it in one global idea..."(Hadamard, 1996, 65). At the same time, explanatory proofs should be sufficiently detailed as to not make the result seem mysterious or magical. Symmetry and analogy may help with understanding because they exploit similarities between cases to reduce the number of individual arguments that must be considered.

3.4.4 Convince

Reuben Hersh suggests that "in mathematical practice, in the real life of living mathematicians, proof is *convincing argument*, as judged by qualified judges."(Hersh, 1993, 389) Indeed, there is a rhetorical role that proofs must fulfil; convincing other mathematicians of the correctness of the proof being provided is a necessary prerequisite for its use in any evidential capacity. It is the first function temporally, even if the function of certification is considered

[11]Similarly, Mark Lange has recently argued that there might be a connection between explanatory proofs and mathematical coincidences in that "we may come to recognize a given proof as explanatory by first appreciating that it makes some result non-coincidental" (Lange, 2010, 332), and that proofs by cases suffer because they "cannot show the theorem to be no coincidence."(337-338) Lange also argues that inductive proofs are in general non-explanatory because any explanation they give would be circular in nature (Lange, 2009).

first in importance.

Conviction can operate on multiple levels: we may convince the reader that the result being proven is correct, or we may convince them that the proof itself is correct. These are distinct but related functions of proofs. Ideally a proof should succeed at both, however this is by no means always the case. Some proofs can convince of the result while still leaving doubts as to the proof itself. This may be because there is a weakness in the proof that makes the reader suspicious, but they nonetheless consider any problems arising from that vulnerability to be repairable. On the other hand, some proofs can (paradoxically) convince the reader of the correctness of the reasoning itself, but not the result. This situation can arise if it is thought that what the proof proves is not what it was claimed to prove (or in other words, the stated theorem is incorrect). In such a case the theorem statement itself might be adjusted as required in the manner described in Lakatos (1976). Proofs of highly counter-intuitive theorems might also encounter this difficulty if the conviction generated by the argument does not outweigh the initial doubt about the theorem. Sometimes such situations may result in a questioning of methods or of primitives (as discussed above).

There are a number of factors that might indicate a proof's being convincing, and unsurprisingly some of these overlap with the indicators for certification, and others with explanation. For instance, following familiar logical structure (using tried and true methods), and its appearing *in principle* formalizable are likely associated with generating conviction because they produce a sense of trust. The cognitive features such as visualizability and surveyability also contribute to generating conviction, since they make it easier for the reader to follow and understand the proof in full.

3.4.5 Tool-Build

Finally, we note that proofs may function to introduce new or adapt existing mathematical theories, methods, and reasoning patterns. In this way, they are tool-builders. Proofs that perform this function are as important or sometimes even more important for *how* they prove than they are for *what* they prove. Sometimes they explicitly describe algorithms for finding a particular solution to a general problem (as in A.3), while other times they introduce constructions or concepts which may be useful for solving other problems, have non-mathematical applications, or are a topic of study in and of themselves. By their very nature, tool-building proofs are usually novel in some way, and as a result, the aesthetic reaction to such proofs is often quite strong. Positive and negative indicators for tool-building will be largely methodological: positive indicators will involve the introduction of new techniques, the extension of old ones, or the seeming exportability of a result or method to other problems. Negative indicators will, by contrast, involve familiarity. Tool-building can in this way come into aesthetic conflict with functions such as convincing.

Cantor's proof for the uncountability of the real numbers (see A.6) is an excellent example of a conceptual tool-builder. It introduced the diagonalization technique, a proof strategy which was novel at the time but would form the basic scaffolding and inspiration for many other proofs.

3.5 Mathematical Taste

Having considered some of the functions and properties of proofs, we are in a better place to re-examine questions having to do with particular aesthetic evaluations and preferences. The functionalist theory provided does allow us to account for the historical constants mentioned in the previous chapter. Preferences which remain on the whole stable may be explained in one of two ways: either the given property is a positive/negative indicator of a functionality which has retained its importance to the community over time (the positive indicators of the verification function are such properties), or it is a positive/negative indicator of multiple functions, and so is relatively invulnerable to shifts or differences in the view of proof (simplicity may be such a property, being a positive indicator for almost every major function).

While the stability of certain preferences is explained by the theory, it nonetheless does not predict any sort of universal aesthetic taste. Differences in preferences regarding the objects of mathematics may, and likely will, emerge. This is because the more an individual looks to proofs to fulfil a particular function, the more their taste will favour proofs which perform that function well, and the more they will identify as aesthetically pleasing those properties which are common to their preferred proofs, and thus positive indicators of that same function. The mathematical aesthetic will, in this way, track certain philosophical views about proof, as well as standards of practice and rigour within both the community at large and various sub-disciplines.

Where there are differences in taste among individuals, we should usually be able to also find differences in their views on proof. For instance, interests in verifying/certifying and unifying/connecting will generate preferences for particular bundles of methodological and structural properties, while interests in convincing and explaining will generate preferences for more of the cognitive properties. Differences in taste between pure and applied mathematicians may be explained by the latter putting greater stock in proofs which tool-build by generating external applications, while the former focus more on functions such as unification and explanation.

As an example, recall that Hardy identified *economy, inevitability, unexpectedness, depth,* and *seriousness* as aesthetically salient properties. *Unexpectedness*, being the effect of novelty, is predicted within the theory to enhance the pleasure of aesthetically pleasing proofs. The other ideals may be tested against Hardy's own views on proof as discussed in Hardy (1929). Here he argues that, on the one hand, proofs generate conviction (a role he associates largely with Hilbert's metamathematical proofs). On the other hand, certain proofs (in particular properly logical proofs such as those in *Principia Mathematica*) do not have this function at all. He argues that the theorems under consideration in such cases are too obvious for the proof to be needed in order for us to be convinced of their truth[12], so our interest in these proofs must derive instead from"their formal and aesthetic properties"(Hardy, 1929, 18). Ultimately, says Hardy, "Our object is *both* to exhibit the pattern and to obtain assent. We cannot exhibit the pattern completely, since it is far too elaborate; and we cannot be content with mere assent from a hearer blind to its beauty"(Hardy, 1929, 18). Hardy thus views convincing and unifying as the main functions of proof. The former is consistent with his stated ideals of economy, inevitability, while the latter explains his concern for depth and seriousness.

Poincaré displays an interest in unifying and connecting, insofar as he considers the most fruitful areas of study "those which reveal to us unsuspected kinship between other facts, long known, but wrongly believed to be strangers to one another"(Poincaré, 1910, 325). That said, he is even more deeply con-

[12] "I believe the Prime Number Theorem because of de la Vallee-Poussin's proof of it, but I do not believe that 2 + 2 = 4 because of the proof in *Principia Mathematica*."(Hardy, 1929, 17)

cerned with proof's explanatory capacity – the manner in which it can allow him to "perceive at a glance the reasoning as a whole", and "re-invent it in so far as I repeat it"(324). An emphasis on the explanatory function is unsurprising given Poicaré's general interest in the psychology of mathematical creativity. This viewpoint leads naturally to preferences for cognitive properties such as surveyability.

Rota and Krull also have aesthetic views formed by an interest in explanation. Rota goes so far as to suggest that "enlightenment, not truth, is what the mathematician seeks when he or she asks the question 'What is this good for?' "(Rota, 1997, 181) and argues that the concept of mathematical beauty is actually a means of elliptically discussing how some proofs can be "enlightening". Explanation acts as Rota's aesthetic filter. Indeed, for him, proofs cannot be seen as beautiful *at all* if they are poorly understood or divorced from their explanatory context:

> A great many beautiful arguments require an extensive buildup, and are long-winded. Familiarity with a huge amount of background material is the condition for understanding any piece of mathematics, and beauty can be appreciated only in the light of previous experiences of other presentations of similar statements or variants of an argument. Very frequently, a proof is viewed as beautiful only after one is made aware of previous clumsy and longer proofs. (Rota, 1997, 179)

Beauty, for Rota, comes rarely with new proofs but generally only at the end of a long process of developing, understanding, and refining existing proofs. As a result, a beautiful proof carries with it a sense of completeness and "definitiveness"(Rota, 1997, 178).

Krull echoes Rota's fascination with enlightenment, proclaiming that he "simply cannot stop playing around with the material, hoping that one day enlightenment will come"(Krull, 1987, 52). He further acknowledges that his focus on "simplicity, clarity, and great 'line' as characteristics of beauty" is rooted in his own particularly "abstract" taste whereas, he claims, "if I had a more 'concrete' taste, I would instead have spoken of diversity, variegation, and the like"(52). Krull's abstract taste is much like that of Poincaré; it revolves around the explanatory and connective functions of mathematical work, precisely those which make theorems appear "evident and compelling"(Krull, 1987, 49), and which "illustrate the underlying mathematical structure in an extremely simple and transparent way."(50)

Krull's discussion of "concrete" and "abstract" taste recalls somewhat the "romantic" and "classical" sensibility which Le Lionnais (1971) speaks of. The romantic viewpoint is supposedly motivated not by anything particularly concrete, but rather by a "glorification of violent emotion, non-conformism and eccentricity"(Le Lionnais, 1971, 130). Proofs of this sort are often indirect and surprising: they are convincing, but at the same time not necessarily *illuminating* because they appear to use tricks and artifices. Such proofs are, as a result, ill-suited to explanation, but can still make deep connections and suggest powerful new methods.

As a final note, although Krull considers the viewpoint of the foundationalist in *contrast* to the aesthetic viewpoint, the theory presented here suggests that this too can understood as representing a kind of aesthetic taste. Mathematicians (mainly logicians) with such taste will favour precisely the sort of fiddly detail involved in what Krull describes as "painstaking and often necessarily complicated and unattractive investigations"(Krull, 1987, 50). They can be expected to admire formal gap-less proofs where the underlying logical assumptions are transparent, and to look for simplicity in the system underlying a proof even if this comes at the cost of length and complexity in the proof itself. All this stems from a view of the proof which emphasises its functions of verification and unification.

In general, where there is aesthetic agreement, it is likely owing either to a fundamental accord as to the functional categories in which a proof is perceived, or to the presence of properties which are positive or negative indicator across multiple different categories. The functionalism I've presented thus predicts a diversity of aesthetic sensibilities stemming from differences of practice, experience, and even philosophical viewpoints about the subject matter. The theory thus explicitly denies aesthetic universalism among mathematicians: there is space for personal disagreements and well as the development of subcommunities of individuals with similar taste.[13]

3.6 Concluding Remarks

In the first chapter, I claimed that, by and large, arguments to the effect that the aesthetic is of importance to mathematics were based either on considerations of practice, and considerations of value. The aesthetic analysis provided was aimed not just at describing how aesthetic experiences can be had in mathematics, but also at critically examining the arguments from value and practice. We are now in a position to do so.

[13] Of course, some of these communities may be more vocal about their aesthetic viewpoint than others. Other cultural factors can come into play here, such as whether the subcommunity itself is open to or wary of aesthetic appeals in their work.

The Hardy style arguments appealing to the aesthetic value of mathematics as a justification for the field are only modestly supported by my theory. The theory connects the aesthetic in mathematics with the aesthetic in other areas of human life, and as a result it does not allow mathematics to make any sort of special claim to value on the basis of a unique aesthetic. Mathematics represents an interesting and somewhat idiosyncratic case for aesthetics, but under this theory it cannot be seen as aesthetically privileged over other conceptual non-art domains in which function plays a perceptually active role. By the same lights, if we accept that there is aesthetic value in these other areas, then we should be willing to allot value to mathematics on similar ground. A potential limitation of this argument is that such aesthetic value emerges from our perception of proofs in relation to their functional roles, and so in some ways may be seen as a derivative value. Perceiving aesthetic value may depend on our appreciating the functions of mathematical proofs within the field, and thus on our already understanding their value in this light.

Arguments from practice, on the other hand are largely redeemed (or at least accounted for) by this theory, since it suggests that the manner in which we use proofs within mathematical practice is precisely that which leads to our aesthetic perception of them. The aesthetic simply pops out of mathematical practice in the same way as it pops out of our interactions with other objects in our environment. Further, by adopting an inductive mechanism for at least some of the process, we are able to explain why some mathematicians have put stock in aesthetic evaluations as heuristic guides in their work. The aesthetic plays a role in generating intuitive judgements because our aesthetic sensibilities are keyed to look for features which we associate with success. As with McAllister's original theory, aesthetic appeals when engaged in mathematical reasoning are shown to be not entirely unfounded. Certainly, we can understand why some might view beauty as somehow being tied to truth. That said, trust in such appeals should be tempered by a recognition that they are highly fallible.

Finally, it is worth remarking that this theory suggests we cannot have any aesthetic sense for the objects of mathematics such as proofs until we are familiar with how those objects are used, and their place within the larger mathematical context. That is, proper aesthetic appreciation requires more than understanding of the mathematical content of the proof itself; it requires that the proof be properly situated. Some familiarity with the history and standards of the field, with past proofs and proof strategies, as well as active interaction with the proof as a tool all constitute essential prerequisites for any aesthetic appreciation. Greater background knowledge should be expected to dramatically increase the depth of the sophistication with which the object is perceived and the level of appreciation the observer has for its aesthetic qualities, since such knowledge actively fuels the inductive mechanisms underlying much of the aesthetic perception. This may in part be why aesthetic talk about mathematics has mainly occurred by and among experts, and why students and non-mathematicians may have so much difficulty in recognizing the source of this admiration, let alone sharing in it. There is no substitute for experience.

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Appendix A

Examples and Discussion

A.1 Pythagoras on Irrationality

A popular historically important example, and one of those Hardy considered particularly pleasing, is Pythagoras' proof for the irrationality of $\sqrt{2}$:

Theorem 1: $\sqrt{2}$ *is irrational.*

Proof. Suppose $\sqrt{2}$ is rational

Then it can be expressed as a fraction $\frac{p}{q}$ which is in its lowest terms (that is, the greatest common divisor of p and q is 1). Thus:

$$\sqrt{2} = \frac{p}{q}$$

$$2q^2 = p^2$$

So p^2 is divisible by 2, and hence p is also divisible by 2. $(\star)$

Since p is divisible by 2, there is some integer k such that $p = 2k$.

By substitution we get:

$$2q^2 = (2k)^2$$
$$q^2 = 2k^2$$

So q^2 is divisible by 2, and hence q is also divisible by 2.
But now we have found that 2 is a common divisor of both p and q, and this is a contradiction. □

The proof is short, simple, and also nicely illustrates symmetry of argument since the steps establishing that p is divisible by 2 mirrors those establishing the same of q. Despite its use of a reductio form, the argument overall seems fairly explanatory, which might be because of the manner in which it straightforwardly states and tests the definition of rational number against a particular case. In fact, Dreyfus and Eisenberg (1986) found that when presented with five different proofs of this theorem, mathematicians consistently considered this to be one of the two most elegant and preferable for teaching, because of "their simplicity and the fact that they can be completely understood with a minimal amount of mathematical background"(8).

The only wrinkle inhibiting, to a degree, the elegance of the proof is the apparently arbitrary choice to analyse a specific integer. Attempting to eliminate the arbitrary step immediately reveals that it is 2 being prime that makes this proof work for 2 (this fact is used on line ($\star$)). This observation gives way to a more elegant proof and more general theorem:

Theorem 2: *For any prime number r, $\sqrt{r}$ is irrational.*

A proof may be given here which is precisely the same as that for the original theorem, but with r substituted for 2 as needed. Our first theorem is now a special case of a much more general result, and the elimination of an arbitrary assumption has made the proof all the more pleasing (and explanatory). Further generalizations may be similarly made; as Hardy himself points out "Pythagoras's argument is capable of far-reaching extension, and can be applied, with little change of principle, to very wide classes of 'irrationals'"(Hardy, 1992, 100). This process illustrates how aesthetic improvements can sometimes go hand in hand with non-trivial mathematical improvements.

A.2 Euclid on Primes

Another historically important and elementary proof, also identified as aesthetically pleasing by Hardy, is Euclid's proof for the infinitude of primes. I give Euclid's own presentation followed by a modern presentation. It is worth remarking that while Euclid relies on a somewhat foreign (to modern sensibilities) geometric approach, the structure of the argument is clearly unchanged.

Theorem 3: *Prime numbers are more than any assigned multitude of prime numbers. (there are infinitely many prime numbers)*

Euclid's proof. Let A, B, C be the assigned prime numbers;
I say that there are more prime numbers than A, B, C.

For let the least number measured by A, B, C be taken,
and let it be DE;
let the unit DF be added to DE.

Then EF is either prime or not.

Figure 1: A visual aid for Euclid's proof with $A = 2, B = 3, C = 5, ED = 30, DF = 1, G = 7$

First, let it be prime;

then the prime numbers A, B, C, EF have been found which are more than A, B, C.

Next, let EF not be prime;

therefore it is measured by some prime number.

Let it be measured by the prime number G.

I say that G is not the same with any of the numbers A, B, C.

For, if possible, let it be so.

Now A, B, C measure DE;

therefore G also will measure DE.

But it also measures EF.

Therefore G, being a number, will measure the remainder, the unit DF:

which is absurd

Therefore G is not the same with any one of the numbers A, B, C.

And by hypothesis it is prime.

Therefore the prime numbers A, B, C, G have been found which are more than the assigned multitude of A, B, C.[1] □

Modern rendition. Consider any set of finitely many prime numbers: $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$.

[1]Euclid's elements Book IX, proposition 20, reprinted in (Hawking, 2007, 101)

We will show that there is a prime number not in $\mathcal{P}$.

Consider the integer $q = p_1 \cdot p_2 \cdots p_n + 1$.

q must either be prime or composite.

If q is prime, then it is an example of a prime not in $\mathcal{P}$

and so $\mathcal{P}$ does not contain all prime numbers.

If q is composite, then it has some prime divisor p_i

If p_i is in $\mathcal{P}$ then it is a divisor of both q and $q - 1$

and thus p_i must divide their difference, 1, which is impossible.

So p_i is an example of a prime number not in $\mathcal{P}$.

Thus, there is no finite set containing all prime numbers. □

Euclid's proof opens Aigner and Ziegler (2009), and, along with the above proof from Pythagoras, received Hardy's praise. For Hardy, both these proofs exemplify his ideals in that:

> ...there is a very high degree of *unexpectedness*, combined with *inevitability* and *economy*. The arguments take so odd and surprising a form; the weapons used seem so childishly simple when compared with the far reaching results; but there is no escape from the conclusions. There are no complications of detail-one line of attack is enough in each case...(Hardy, 1992, 113)

Hadamard uses this proof (with $\mathcal{P}$ containing the first 11 primes) to illustrate the manner in which he goes about most mathematical investigations with a heavy reliance on imagery or "concrete representations" rather than symbols, even when the material appears quite abstract. Although this bears a passing resemblance to the visualization in figure 1, it is substantially less precise.

Steps in the proof	My Mental Pictures
I consider all primes from 2 to 11, say 2,3,5,7,11	I see a confused mass.
I form their product $2 \times 3 \times 5 \times 7 \times 11 = N$	N being a rather large number, I imagine a point rather remote from the confused mass.
I increase that product by 1, say N plus 1.	I see a second point a little beyond the first.
That number, if not a prime, must admit of a prime divisor, which is the required number.	I see a place somewhere between the confused mass and the first point.

(Hadamard, 1996, 76-77)

A.3 Gale and Shapley on Marriage

A proof which typically receives a highly stylized presentation is the marriage algorithm provided by Gale and Shapley.

Definition: We consider two finite sets of equal size: one of men and one of women. Each man lists all the women in order of preference, with the women he would most like to be married to ranked highest. The women do the same with the men. A set of marriages is called a *stable matching* if no unmatched couple prefer each other over their actual spouses.

Theorem 4: *There is always a stable matching.*

Proof. We apply the following algorithm to find a matching:

First, each man proposes to his top ranked woman. The women then choose their preferred man from those who have proposed to them so far and "keeps him on a string".

In the next round, every man who was rejected previously proposes to his next highest ranked woman. She compares all the new suitors to each other as well as to whomever she has on a string from the previous round, and again chooses her favourite.

This process continues until every woman has been proposed to, at which point they accept their preferred proposal.

A short argument establishes that the matching found is indeed stable:

> suppose John and Mary are not married to each other but John prefers Mary to his own wife. Then John must have proposed to Mary at some stage and subsequently been rejected in favor of someone that Mary liked better. It is now clear that Mary must prefer her husband to John and there is no instability. (Gale and Shapley, 1962, 13)

□

A.4 Kelly on Collinear Points

An interesting example from geometry is L.M. Kelly's proof of the Sylvester-Gallai theorem:

Theorem 5: *For any arrangement of finitely many points, where the points are not all on a line, there is a line passing through exactly two of these points.*

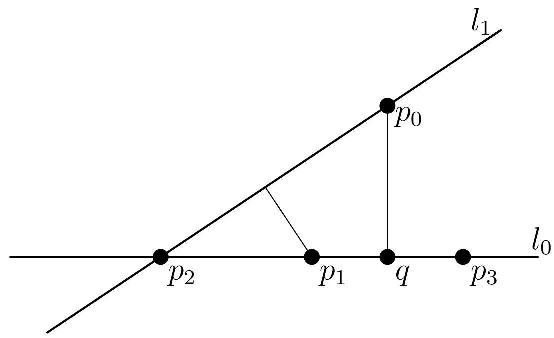

Figure 2: The construction used in Kelly's proof of the Sylvester-Gallai theorem. Note that it is possible for $p_1 = q$, and that p_3 may be located anywhere on the line l_0.

Proof. We will find a line satisfying the requirements given.

Let $\mathcal{P}$ be the set of points, let $\mathcal{L}$ be the set of all lines defined by two points of $\mathcal{P}$.

Consider all the pairs consisting of a line l_i from $\mathcal{L}$, and a point p_i from $\mathcal{P}$ which is not on l_i.

Since both $\mathcal{P}$ and $\mathcal{L}$ are finite, there must be a pair having the smallest perpendicular distance between the point and the line.

Let (p_0, l_0) be such a pair.

Let q be the point on l_0 closest to p_0 (q is not necessarily in $\mathcal{P}$).

Suppose l_0 passes through three points of $\mathcal{P}$: p_1, p_2, p_3.

At least two of these points, say p_1 and p_2, must be on the same side of q (inclusive of point q itself).

Without loss of generality, let the point p_2 be further from q than p_1.

Consider the line l_1 passing through p_2 and p_0.

Then the distance between p_1 and l_1 is shorter than that between p_0 and l_0, a contradiction.

So l_0 is a line passing through only two points of $\mathcal{P}$, as required. $\square$

This proof is short and provides a nice construction illustrating why a particular pattern must be satisfied. It also connects this to another pattern: we can locate a line going through exactly two points somewhere among those lines which are closest to other points in the $\mathcal{P}$. For some, the proof has certain important defects, however. In the paper where he first drew attention to it, Coxeter (1948) indicates that Kelly's proof is deficient because it is not "synthetic", by which he means something akin to pure of methods:

> Synthetic geometry is ultimately based on certain primitive concepts and axioms, appropriate to the particular kind of geometry under consideration (e.g.,projective or affine, real, complex or finite). Each problem belongs to one kind (or to a few kinds), and I would call a solution synthetic if it remains in that kind, analytic if it goes outside. (Coxeter, 1948, 26)

> It seems to me that parallelism and distance are essentially foreign to this problem, which is concerned only with incidence and order. Thus I would not regard the proofs by Grünwald and Kelly as strictly synthetic. (Coxeter, 1948, 27)

Coxeter's complaint is that Kelly's proof is Euclidean when the problem itself is not. More specifically, it is the appeal to the shortest perpendicular distance which limits Kelly's proof to the Euclidean case, and violates purity of methods. Coxeter then suggests that an existing projective proof by Steinberg is preferable overall, although it is more complex, precisely because it does obey purity of methods in this way. Pambuccian (2009) sheds light on this debate by performing a reverse analysis on Kelly's proof, Steinberg's proof, and one other popular proof of the theorem. He points out how in this case "the two concerns, for minimal and for pure axiom systems, lead to different, incompatible results"(Pambuccian, 2009, 246). In this way, this example illustrates how ideals of purity and minimality (a form of simplicity) can come into tension with one another, and how such considerations can lead to the development and proliferation of many proofs for the same theorem.

A.5 Gauss on Tedious Sums

Here I present two well known proofs of the same result: an inductive proof, and a proof from Gauss made by pairing off terms, which may even be rendered pictorially. According to folklore, Gauss as a child vexed his teacher by coming up with this short-cut for rapidly summing the digits from 1 to 100 rather than doing the sum the long way. Gauss' proof is usually considered aesthetically superior to the inductive proof, especially by those interested in mathematical explanation.

Theorem 6: *for all natural numbers i,* $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Inductive proof. Base case: for $n = 1, 1 = \frac{1(1+1)}{2}$ is true.
Inductive step: Suppose the formula holds for some $n = k$

Then it also holds for $n = k + 1$, since:

$$\begin{aligned}\sum_{i=1}^{k+1} i &= 1 + 2 + \cdots + k + (k+1) \\ &= \sum_{i=1}^{k} i + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

□

Gauss' proof.

$$\begin{aligned}2\sum_{i=1}^{n} i &= (1 + 2 + \cdots + n) + (n + (n-1) + \cdots + 1) \\ &= (1+n) + (2 + (n-1)) + \cdots + (n+1) \\ &= n(n+1)\end{aligned}$$

□

This method is amenable to easy and fairly direct visualization, as described by Polya:

> Take a rectangle with sides n and $n+1$, and divide it in two halves by a zigzag line(...) Each of the halves is "staircase-shaped" and its area has the expression $1 + 2 + \cdots + n$(...) Now, the whole area of the rectangle is $n(n+1)$ of which the staircase-shaped area is one half. This proves the formula.(Polya, 1988, 118)

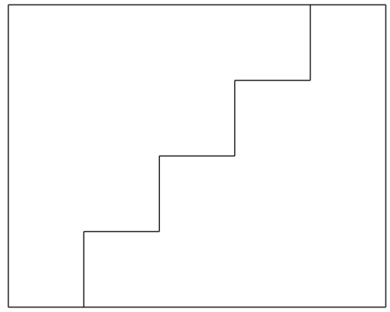

Figure 3: Polya's illustration of Gauss' proof for $n = 4$.

A.6 Cantor on Uncountability

Georg Cantor had an uncanny ability to prove uncanny results in slick ways. One in particular stands out as being of aesthetic interest:

Theorem 7: *The set of Real numbers* $\mathbb{R}$ *is uncountable.*

Proof. Suppose $\mathbb{R}$ is countable, then the unit interval $(0, 1]$ is also countable. We assume this is the case.

We can assign each number on this interval a unique infinite decimal expansion. To ensure uniqueness, we only assign expansions without infinitely many zeroes at the end (so, for instance, we assign $\frac{1}{2} = 0.499999\ldots$ rather than $\frac{1}{2} = 0.50000\ldots$)

In general, for $a_i \in (0, 1]$, write the expansion as $a_i = 0.a_{i,1}a_{i,2}a_{i,3}\ldots$
If $(0, 1]$ is countable, then there is a bijection between it and the natural numbers $\mathbb{N}$.
We can represent this bijection as a list exhausting our interval:

$$
\begin{array}{l}
a_1 = \boldsymbol{a_{1,1}}\ a_{1,2}\ a_{1,3} \dots a_{1,n} \dots \\
a_2 = a_{2,1}\ \boldsymbol{a_{2,2}}\ a_{2,3} \dots a_{2,n} \dots \\
a_3 = a_{3,1}\ a_{3,2}\ \boldsymbol{a_{3,3}} \dots a_{3,n} \dots \\
\vdots \qquad\qquad\qquad \ddots \\
a_n = a_{n,1}\ a_{n,2}\ a_{n,3} \dots \boldsymbol{a_{n,n}} \dots
\end{array} \tag{1}
$$

Consider the bold diagonal number: $\alpha = 0.a_{1,1}a_{2,2}a_{3,3} \dots a_{n,n} \dots$

Using α, we now define a new number, $\beta = 0.b_1b_2 \dots$ as follows:

for each k, if $a_{k,k} = 1$ then let $b_k = 2$, else let $b_k = 1$.

By assumption, β must appear on the list (1).

But it cannot, for it it did then it would intersect α at some point m, giving us $b_m = a_{m,m}$, which is impossible. □

Cantor's proof is remarkable for several reasons. First of all, it presents the method of diagonalization, a technique which enjoys an ongoing and important place in modern mathematics, logic, and computer science. As such, Cantor's proof is a significant tool-builder. Although it has much ancestry in other reductio proofs (such as our first example), the diagonalization Cantor performs also represents one of the clearest instances of novelty, since this proof is the first place in which this method is encountered, both for the mathematical community historically and for most students of mathematics. Novelty accentuates the already striking aspects of the proof, making it appear extremely clever.

While the method applied is fairly easy to understand, even with little background, the result being shown is highly counter-intuitive. This loads our expectations against the proof's validity in the first place, and results in a kind of incredulity when it works out. Indeed, many people resist the conclusion when the proof is first put to them. This tendency is examined in a fair amount of detail in Hodges (1998). Hodges analyses various failed attempts to refute Cantor's proof which had been submitted to him for peer review. He speculates that the proof is targeted for refutation by these individuals because "this argument is often the first mathematical argument that people meet in which the conclusion bears no relation to anything in their practical experience or their visual imagination"(Hodges, 1998, 3). This failing applies to more than just students and amateurs, for he adds on a more historical note that "until Cantor first proved his theorem ([6], by a much longer argument, as it happens[2]), nothing like its conclusion was in anybody's mind's eye. And even now we accept it because it is proved, not for any other reason"(Hodges, 1998, 3).

The proof fails to achieve complete elegance only because we must get around the possibility of non-unique decimal expansions, and decide on a manner to define β such that it cannot be in our list. Neither task is difficult and both can be done in any number of ways such that in any given presentation of this proof, different techniques might be applied (and sometimes are simply gestured at). These steps are needed to ensure the validity of the proof but their particulars might seem to be a somewhat ad-hoc measure, or a fairly uninteresting hoop which we must jump through before the true content of the proof can be presented.

[2]Here Hodges references Cantor's initial 1884 proof. This proof is translated in Stedall (2008) and given a modern presentation in Franks (2010). It is not *that* much longer.