An overview of soil heterogeneity: quantification and implications on geotechnical field problems

Tamer Elkateb, Rick Chalaturnyk, and Peter Robertson

Abstract: Engineering judgment and reliance on factors of safety have been the conventional tools for dealing with soil heterogeneity in geotechnical practice. This paper presents a review of recent advances in treating soil variability. It presents the implications of geostatistical techniques and up-scaling methods used for quantifying the heterogeneous permeability of soil as addressed in the petroleum industry. Moreover, the interest of geotechnical practice to incorporate the statistical properties of soil in a probabilistic design framework is also discussed. This ranges from conventional Monte Carlo simulation based design and stochastic finite element analysis to the recent techniques that take into account the effect of spatial correlation of soil properties. Example applications of these techniques to different types of field problems, such as foundation settlement, seepage flow, and liquefaction assessment, are discussed with emphasis on the limitations of the current practice and trends for future research. In addition, different decision making algorithms are addressed with examples of their applications to geotechnical field problems.

Key words: heterogeneity, spatial variability, geostatistics, stochastic analysis, decision making.

Résumé :

Mots clés :

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Introduction

Almost all natural soils are highly variable in their properties and rarely homogeneous. Soil heterogeneity can be classified into two main categories. The first is lithological heterogeneity, which can be manifested in the form of thin soft/stiff layers embedded in a stiffer/softer media or by the inclusion of pockets of different lithology within a more uniform soil mass. The second source of heterogeneity can be attributed to inherent spatial soil variability, which is the variation of soil properties from one point to another in space due to different deposition conditions and different loading histories.

Early attention to the problem of soil nonhomogeneity emerged from the field of petroleum engineering where efforts were devoted towards assessing the effect of heterogeneity on the production of oil fields. Geostatistical theories and up-scaling techniques were implemented to estimate equivalent permeabilities for the fields of interest that honored detailed reservoir heterogeneity.

The conventional tools for dealing with ground heterogeneity in the field of geotechnical engineering have been the reliance upon high safety factors and local experience. Morgenstern (2000) introduced case histories for different geotechnical applications where relying solely on engineering judgment resulted in poor to bad predictions in up to 70% of the cases considered. As a result, it has been readily accepted that there is a need to develop more reliable tools to incorporate ground heterogeneity in a rather quantitative scheme amenable to engineering design. Early attempts to rationally deal with the variability of soil properties in geotechnical engineering involved the introduction of reliability-based design methods that combined limit equilibrium analysis with Monte Carlo simulation techniques. In addition, the stochastic finite element method was introduced as an effective way to incorporate soil variability into a numerical analysis framework. Recently, attempts have been made to incorporate spatial correlation between soil properties into a statistical design scheme using either of the above approaches or by implementing the outcome of a

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Monte Carlo simulation into deterministic numerical analysis schemes. It is worth noting that almost no attention has been given to assess the effect of lithological heterogeneity on the macro (overall) behavior of heterogeneous soil media. The treatment of such heterogeneity has been exclusively left to local experience and engineering judgment.

The main objective of this paper is to discuss the different techniques developed to deal with soil heterogeneity in both petroleum and geotechnical engineering and their applicability to geotechnical field problems. In addition, attempts will be made to identify the difficulties associated with obtaining representative parameters that honor detailed ground heterogeneity. In the following sections, techniques developed in the petroleum literature to deal with lithological heterogeneity are discussed. Then, different elements of inherent soil variability will be presented along with their implications on geotechnical field problems, such as settlement of shallow foundation, liquefaction susceptibility, and seepage flow. Limitations of the current practice will be addressed, and potential trends for future studies will be suggested. Finally, different decision algorithms will be discussed together with examples of their applications in geotechnical analyses.

Lithological heterogeneity

The impact of lithological heterogeneity of the ground on the production of oil and gas reservoirs has been a major area of study in petroleum engineering practice. Several up-scaling techniques were developed to deal with complex ground profiles, such as the sand shale sequence shown in Fig. 1. The main aim of these techniques was to scale-up fine scale permeability to coarser scales amenable to flow simulation and engineering calculations. These averaging techniques can be classified into the following methods:

(1) Empirical techniques, such as the power averaging technique (Deutsch 1989). These are the simplest forms of up-scaling laws.

(2) Semi-empirical methods, such as the renormalization (King 1989) and the representative elementary volume (REV) – renormalization (Norris et al. 1991). They are more sophisticated than the previous type but have limited theoretical basis.

(3) Analytical techniques, such as that proposed by Warren and Price (1961). These methods are rather cumbersome to implement in practice.

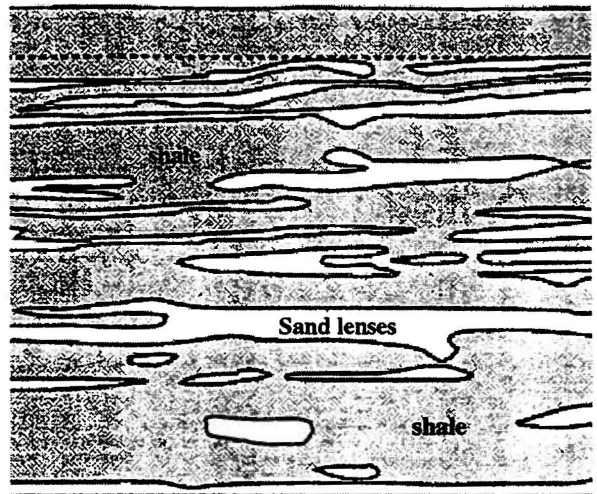
The power averaging method was obtained through nonlinear regression of the results obtained from a three-dimensional numerical simulation of flow through sandstone–shale formations. The analysis was carried out under different target shale volumes, and the equivalent permeability was regarded as that of a homogeneous soil mass producing similar flow under the same head difference and boundary conditions. This equivalent permeability, k_e, was found to satisfy the relation

$$[1] \qquad k_e = [V_{sh} k_{sh}^{\omega} + (1 - V_{sh}) k_{ss}^{\omega}]^{1/\omega}$$

where k_{sh} and k_{ss} are the permeabilities of the shale and sandstone, respectively; V_{sh} is the volume fraction of shale; and ω is an averaging power.

The value of ω was suggested to range from –1 to 1 depending on the direction of flow and the geometrical anisotropy of the shale, i.e., the ratio between the vertical to the lateral extent of shale. The major advantage of this method is its simplicity, while the main drawback is that the shale blocks were assumed to be uncorrelated to each other.

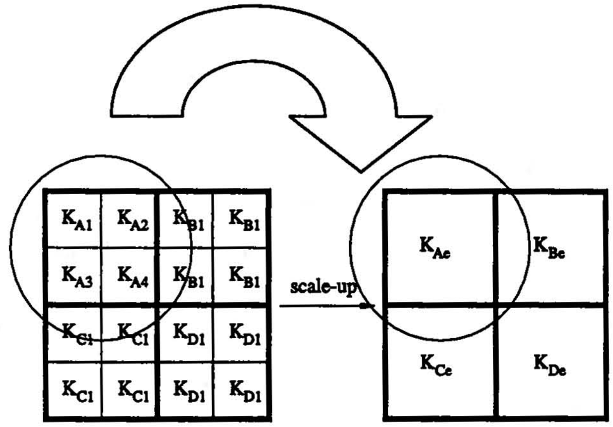
Fig. 1. Sand–shale sequence in a petroleum field, WY, U.S.A. (modified from Norris et al. 1991).



In the renormalization technique (King 1989), a simulation grid is generated across the analysis domain and a constant value of soil permeability is assigned to each element of the simulation grid. Then, these elements are grouped into blocks of four and assigned an effective (equivalent) permeability value, k_e, as illustrated in Fig. 2. This effective permeability was obtained based on the analogy between water flow through soils of different permeabilities and electric current flow through a network of resistors. The above procedure can be applied to the new grid and repeated several times depending on the scale of interest. This method was originally developed for uncorrelated permeability fields, but it is also valid for correlated fields. It is worth noting that only isotropic media of equal permeabilities in vertical and horizontal directions were considered during the development of this technique. However, the method can be extended to anisotropic media by applying the up-scaling procedure to both the vertical and horizontal directions. In spite of the theoretical basis implemented in this technique, it can be regarded as a relatively complicated method compared with the empirical formula presented in eq. [1].

The REV-renormalization approach (Norris et al. 1991) to up-scale sand–shale formations for flow simulation combined the representative elementary volume (REV) theory with the renormalization technique. The REV theory defines a specific averaging volume at which all microscopic variations are averaged out producing a representative single macroscopic value, which is usually referred to as the representative elementary property (REP) (Norris et al. 1991). The REV technique was originally developed to assess the representative property of porous materials, where the representative property at the smallest scale represents the property of either a void or a solid. By gradually increasing the averaging volume, more voids and solids are included in the averaging volume, resulting in fluctuation in the representative property, as shown in Fig. 3. The REV can be defined as

Fig. 2. Schematic diagram of up-scaling using the renormalization technique (modified from King 1989).



$$k_e = \frac{4(k_1+k_3)(k_2+k_4)[k_2k_4(k_1+k_3)+k_1k_3(k_2+k_4)]}{[k_2k_4(k_1+k_3)+k_1k_3(k_2+k_4)][k_1+k_2+k_3+k_4]+3(k_1+k_2)(k_3+k_4)(k_1+k_3)(k_2+k_4)}$$

a critical averaging volume beyond which there is no significant fluctuation in the representative property as the addition of extra voids or solids has a minor effect on the averaged property. In the REV-renormalization technique, subsurface soil is discretized into exclusive geological units characterized by a specific type of sedimentary structure. Within each unit, the spatial distribution of sand and shale is translated into binary maps. The renormalization technique is then employed to determine soil permeabilities at different averaging volume scales to determine the REV and the associated equivalent permeability, which is regarded in this case as the REP. It was concluded from the results obtained using this approach that effective permeability was mainly dependant on the relative volume and connectivity of different lithologies rather than their inherent spatial variability.

The first rational attempt to provide an analytical solution to the problem of soil lithological heterogeneity and its effect on flow was proposed by Warren and Price (1961). They combined the results of physical modeling with that of numerical simulation and suggested the geometric mean (Deutsch 2002) as an estimate of the effective permeability of heterogeneous media. Afterwards, several studies were carried out to develop enhanced measures of effective permeability. Two main approaches were adopted in these studies; the effective medium theory and the perturbation expansion (King 1989). In either case, the effective permeability estimates were considered accurate only for small ranges of permeability fluctuations.

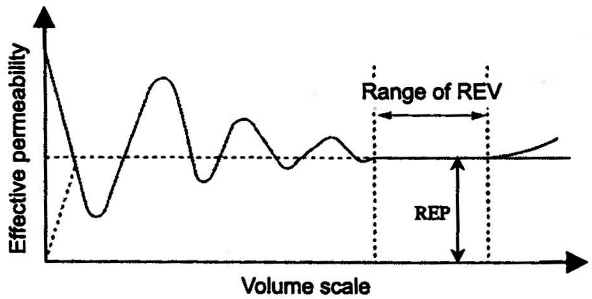
In the field of geotechnical engineering, almost no attempt has been made to assess the effect of lithological heterogeneity on the macro (overall) behavior of soil mass in spite of the need to develop such algorithms for certain geotechnical applications. An example of these applications is codepositional mixed fine tailings embankments, as shown in Fig. 4. The basic idea of this tailings disposal system is to mix fine tailings, which behave as very soft clay, with sand to obtain relatively steeper embankments, compared with conventional thickened tailings embankments (Robinsky 1999). The heterogeneous nature of these embankments requires an estimation of equivalent engineering parameters that take into consideration the effect of the spatial distribution of fine tailings pockets on the overall behavior of these embankments.

Inherent spatial variability of soil properties

Most geotechnical analyses are deterministic in the sense that average soil parameters are given to each distinct layer. The uncertainties in these properties and their variation from one point to another in space have been accounted for, qualitatively, by the use of safety factors and by implementing local experience and engineering judgment. The selection of these design parameters, however, has contained some degree of uncertainty and consequently a degree of unavoidable risk. These uncertainties can be attributed to the following factors (Phoon and Kulhawy 1999):
(1) soil inherent spatial variability due to variation in deposition conditions and stress history from one point to another in space;
(2) measurement errors due to insufficient control of testing procedure and equipment;
(3) deterministic trends in soil properties, such as the increase in soil strength with depth due to the increase in confining pressure; and
(4) the collection of field data over long time periods.

This paper will focus primarily on inherent soil variability, where stochastic analyses can be employed to assess the effect of this type of variability on engineering design. To proceed with a stochastic analysis, the main elements of soil spatial variability have to be identified, such as
(1) classical statistical characteristics, such as the mean, coefficient of variation (COV), and probability distribution of the soil data;
(2) the spatial correlation structure that describes the variation of soil properties from one point to another in space;

Fig. 3. Representative elementary volume concept (modified from Norris et al. 1991).



(3) the limit of spatial continuity, beyond which no or small correlation between soil data exists; and
(4) the volume–variance relationships, which help assess the reduction in the variance of field data upon averaging over a certain volume of interest.

Details of the above elements are discussed in the following sections.

Classical statistical characteristics of soil properties

Several attempts have been made to obtain the classical statistical properties of soil, such as the mean value, COV, and probability distribution, throughout geotechnical engineering practice. These statistical characteristics have been discussed by several authors, such as Lumb (1970), Schultz (1975), and Griffiths and Fenton (1993). Phoon and Kulhawy (1999) provided an excellent summary of different statistical characteristics for different soil types and field tests. Generally, it was found that high variability, expressed in terms of a high COV, was usually associated with strength parameters, and that undrained shear strength was usually highly variable compared to the drained friction angle. It is worth noting that different probability distribution models such as normal, lognormal, and beta distributions have been implemented by different authors to curve fit the results of field data. This implies that these distributions are probably site and parameter specific and that there is no generic distribution pattern for soil properties.

Spatial correlation between soil properties

Soil properties do not vary randomly in space; rather such variation is gradual and follows a pattern that can be quantified using spatial correlation structures, where soil properties are treated as random variables. The spatial correlation structure is often expressed in terms of the variogram (Deutsch 2002) or the covariance function (Vanmarcke 1977).

The variogram is a measure of dis-similarity between two points in space separated by a distance h, according to the relation

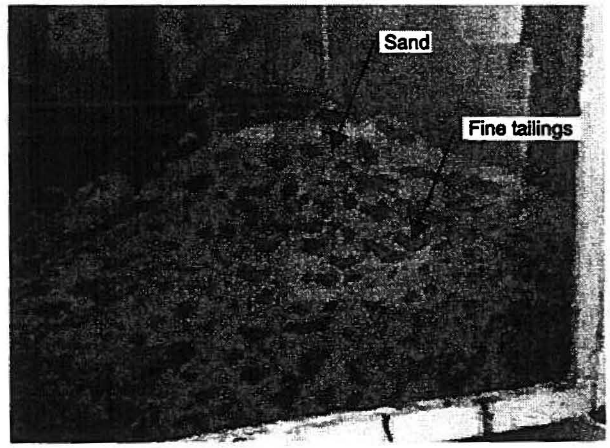
$$[2] \qquad 2\gamma(h) = \mathrm{Var}[Z(u+h) - Z(u)]$$

where $2\gamma(h)$ is the variogram value at a separation distance h; $Z(u)$ is the value of the random variable at location u; $Z(u+h)$ is the value of the random variable at distance h from $Z(u)$; and Var[] is the variance operator.

On the other hand, the covariance is a measure of similarity between the above two points and can be obtained through

$$[3] \qquad C(h) = E[Z(u)\, Z(u+h)] - m^2$$

Fig. 4. Fine tailings – sand mixture in a laboratory model of co-depositional mixed fine tailings embankment (modified from Hutcheson 2000).



where $C(h)$ is the value of the covariance function at a separation distance h; m is the mean value of Z; and $E[\ ]$ is the mean operator.

Variogram and covariance functions are correlated through the variance of field data, σ^2, in the form

$$[4] \qquad \gamma(h) = \sigma^2 - C(h)$$

It should be emphasized that the above variogram and covariance relations are only valid for stationary random fields where both the mean and standard deviation are constants across the domain of interest. Most soil mechanical properties, however, are expected to exhibit spatial trends especially in the vertical direction due to their sensitivity to change in confining pressure. An example of these vertical trends is shown in Fig. 5*a* where the tip resistance, q_c, of cone penetration tests tends to increase with depth. To satisfy the stationarity condition, these trends must be removed (detrended) in a process often referred to as detrending of field data. The detrending process is usually carried out by identifying deterministic trends in field data implementing regression analysis (Deutsch 2002), as shown in Fig. 5*a*. It should be realized that the linear variation of cone tip resistance with depth, shown in Fig. 5, is a simplifying assumption for practical application; as such variation can take other forms especially for sandy soils. Spatial trends in field data, however, should be kept as simple as possible to minimize the uncertainty associated with the assessment of these trends (Baecher 1987). This uncertainty in spatial trends may have a significant influence on the outcomes of stochastic geotechnical analyses especially in the presence of limited field data. Neter et al. (1996) and El-Ramly (2001) provided an excellent discussion on the assessment of this uncertainty and its implications on statistical analyses. The detrending process results in generating detrended field data, as shown in Fig. 5*b* that can be considered as stationary random variables using the relation

$$[5] \qquad q = q_c - q_o(z)$$

where q is the detrended cone tip resistance and $q_o(z)$ is the deterministic vertical trend.

Fig. 5. Detrending of CPT tip resistance data (*a*) identifying linear vertical trend, and (*b*) detrended data.

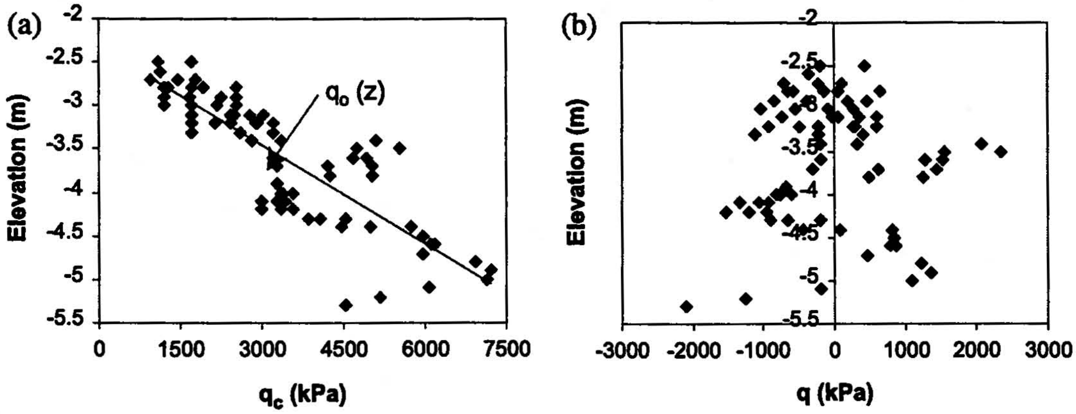
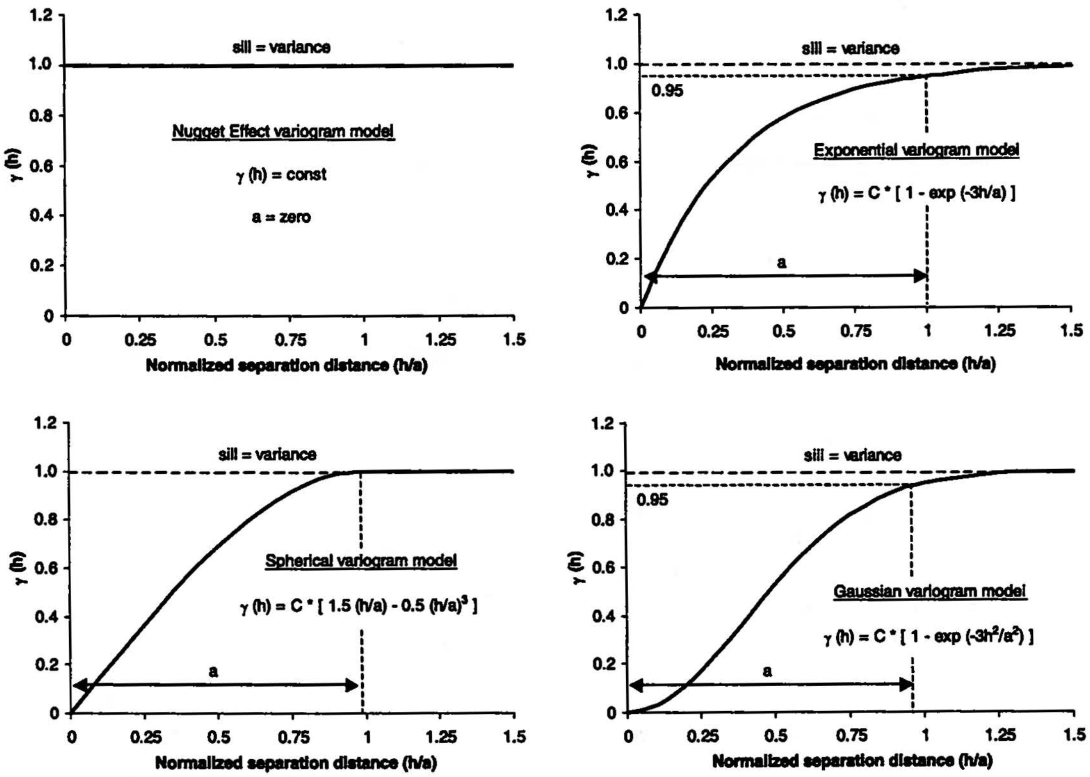


Fig. 6. Examples of variogram models commonly used in practice.



Spatial correlation structures are usually characterized by their model types and the limit of spatial correlation between field data. Spatial correlation models are parametric relationships used to curve fit the experimental variograms, or covariance functions, obtained from analysis of the field data. Deutsch (2002) has provided an excellent summary of common variogram models used in practice. Examples of these models (i.e., spherical, exponential, and Gaussian) are shown in Fig. 6. These models help to determine the spatial correlation between field data at any separation distance and in different directions. In addition, they can incorporate other geological information such as the direction of maximum continuity and maintain numerical stability of stochastic simulation (Deutsch 2002). The limit of spatial continuity is discussed in more detail in the following section.

Limit of spatial continuity between field data

The limit of spatial continuity is defined as the separation distance between field data at which there is no, or insignificant, spatial correlation. This limit can be expressed in terms of the spatial range (Deutsch 2002), the scale of fluctuation (Vanmarcke 1977), or the autocorrelation distance (DeGroot and Baecher 1993). The spatial range, a, can be defined as the separation distance at which the semivariogram reaches the sill (variance) and correlation between data no longer exists, as shown in Fig. 6. For variogram models where the variogram is asymptotic to the sill (σ^2), as in the case of exponential and Gaussian models, an effective range can be considered as the separation distance at which the variogram reaches a value equal to 0.95 times that of the sill. The scale of fluctuation, θ, estimates the distance within which soil properties show relatively strong correlation and data become either above or below the mean value. Vanmarcke (1977) developed a simplified procedure to estimate the scale of fluctuation for different spatial correlation structure models. The autocorrelation distance, R, is the separation distance at which the covariance function decays to a value of σ/e, where e is the base of the natural logarithm, and correlation between data can be considered relatively weak. A relationship between these different measures was developed by Elkateb (2002), as shown in Table 1.

Volume-variance relationships

The volume-variance relationships are analytical expressions used to obtain the variance of spatial averages of field data over certain volumes of interest. These spatial averages usually have a narrower probability distribution function than those associated with field data (Vanmarcke 1977) and consequently a smaller variance. The variance of these spatial averages can be correlated to the point variance using the variance reduction factor, Γ_v^2, as discussed by Vanmarcke (1984) through

$$[6] \qquad \sigma_\Gamma = \Gamma_v \sigma$$

where σ is the standard deviation of field data (point statistics); σ_Γ is the standard deviation of the spatial average of the data over volume v; and Γ_v^2 is the variance reduction factor.

The variance reduction factor depends on the averaging volume, type of correlation structure, and the limit of spatial correlation between field data. Several analytical expressions for the variance reduction factor were introduced by Vanmarcke (1984), in the form

$$[7a] \qquad \Gamma_T^2 = 2\left(\frac{R}{T}\right)^2\left(\frac{T}{R} - 1 + e^{-T/R}\right) \qquad \text{for exponential}$$

correlation structures

and

$$[7b] \qquad \Gamma_T^2 = 2\left(\frac{R}{T}\right)^2\left[\sqrt{\pi}\frac{T}{R}\xi\left(\frac{T}{R}\right) - 1 + e^{-T/R}\right] \qquad \text{for}$$

Gaussian correlation structures

where Γ_T^2 is the one-dimensional variance reduction factor; R is the autocorrelation distance; T represents the size of the average volume; and $\xi(T/R)$ is the error function, which varies from 0 to 1 as T increases from 0 to ∞.

Table 1. Comparison between different measures of the limit of spatial continuity between field data.

Type of correlation structure model	Spatial range	Scale of fluctuation	Autocorrelation distance
Exponential	a	$2a/3$	$a/3$
Gaussian	a	a	$0.58a$
Spherical	a	$0.55a$	a

The above expressions are based on the assumption that the averaging process occurs in one direction only. These expressions can be easily extended to the three-dimensional case by assuming separable correlation structures (Vanmarcke 1984). Such an assumption implies that the three-dimensional variance reduction factor could be expressed as the product of its one-dimensional components in the form

$$[8] \qquad \Gamma_v = \Gamma_{Tx}\Gamma_{Ty}\Gamma_{Tz}$$

where Γ_{Tx}, Γ_{Ty}, and Γ_{Tz} are the one-dimensional variance reduction factors in the x, y, and z directions, respectively.

The variance reduction factor approaches 1 when the parameter T is small compared to the limit of spatial continuity. For many geotechnical applications, the size of the averaging volume in the horizontal direction is usually small compared to the spatial range of the horizontal correlation structure. As a result, it has been a common practice in many geotechnical implementations of the variance reduction factor to assume that its value will be affected only by the size of the averaging volume in the vertical direction, i.e., layer thickness. This is because the variance reduction factor in the horizontal direction can be reasonably assumed to be equal to one.

In a similar fashion, the variance reduction factor for spherical correlation structures was developed in this study, as explained in Appendix A, and can be expressed in the form

$$[9] \qquad \Gamma_T^2 = 1 - \frac{T}{2a} + \frac{T^3}{20a^3}$$

Spatial averages of random variables are spatially correlated in a way similar to point (field data) statistics. This correlation can be quantified in a pair-wise manner by assessing the coefficient of correlation between any couple of one-dimensional spatial averages, as shown in Fig. 7, through the relationship (Vanmarcke 1977)

$$[10] \qquad \rho_{12} = \frac{D_0^2\Gamma^2(D_0) - D_{01}^2\Gamma^2(D_{01}) - D_{02}^2\Gamma^2(D_{02}) + D_{012}^2\Gamma^2(D_{012})}{2D_1\Gamma(D_1)D_2\Gamma(D_2)}$$

where ρ_{12} is the correlation coefficient between the spatial averages over the depths D_1 and D_2; $\Gamma^2(D_0)$, $\Gamma^2(D_{01})$, $\Gamma^2(D_{02})$, and $\Gamma^2(D_{012})$ are the variance reduction factors over averaging thickness equal to D_0, D_{01}, D_{02}, and D_{012}, respectively; and $\Gamma(D_1)$ and $\Gamma(D_2)$, are the square roots of the variance reduction factors over averaging thickness equal to D_1 and D_2, respectively.

Stochastic analysis techniques in geotechnical engineering

Stochastic analysis provides an excellent tool to account for the variability of soil properties and to develop rational algorithms to estimate soil design parameters on a probabilistic basis where the associated risk level can be quantified.

Several approaches have been adopted by geotechnical practitioners to implement stochastic analyses in geotechnical field problems, such as liquefaction assessment, slope stability analysis, and foundation settlement. Examples of these approaches are (*i*) application of reliability principles to limit equilibrium analyses; (*ii*) stochastic finite element analysis; and (*iii*) application of stochastic input soil parameters into deterministic numerical analysis.

A detailed discussion of the above approaches is provided in the following sections together with examples of their applications to geotechnical field problems.

Application of reliability principles to limit equilibrium analysis

Statistical analysis of limit equilibrium problems was primarily developed to perform probabilistic slope stability analysis using different techniques, such as analytical approaches, approximate solutions, and Monte Carlo simulation. Analytical approaches, such as those proposed by Tobutt and Richards (1979), were primarily concerned with obtaining closed form solutions for the statistical properties of earth slopes factors of safety. These solutions do not provide information about the output probability distribution and become cumbersome when considering different sources of uncertainty.

Approximate solutions of probabilistic slope stability analysis, such as the first order second moment (FOSM) and the point estimate method (PEM), have been advocated by several authors, such as Christian et al. (1994). The basic idea of the FOSM method (Harr 1987) is to express the factor of safety as a function of different random variables considered in the statistical analysis. This function is then expanded about the mean values of these random variables using the Taylor expansion, retaining only linear (first order) terms, where the mean and variance of the safety factor can be assessed through

$$[11a]\quad E[\mathrm{F.S}] = F(E[x_1], E[x_2], \ldots\ldots\ldots E[x_n])$$

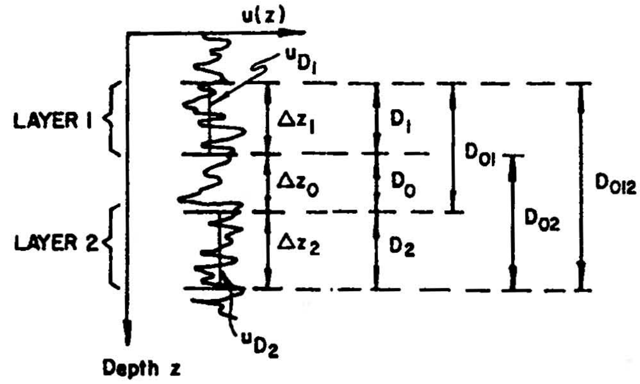
$$[11b]\quad \mathrm{Var}[\mathrm{F.S}] = \sum_{i=1}^{i=n}\left(\frac{\partial F}{\partial x_i}\sigma_{x_i}\right)^2 + 2\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}\left(\frac{\partial F}{\partial x_i}\frac{\partial F}{\partial x_j}\right)C[x_i, x_j]$$

where E[F.S] and Var[F.S] are the mean and variance of the factor of safety, respectively; $C[x_i, x_j]$ is the covariance between the random variables x_i and x_j; and n is the number of random variables.

The major advantage of this technique is its simplicity, especially when considering different sources of uncertainty, as it provides a direct estimation of the mean and variance for the factors of safety. However, the accuracy of this technique is questionable, especially when dealing with highly nonlinear relations and large soil variability, due to the truncation of high order terms in the Taylor expansion.

In the PEM (Rosenblueth 1975 and 1981), the probability distribution of each of the random variables is represented by two points estimates x_+ and x_- with probability densities of p_+ and p_-, respectively. This is based on the analogy between probability distributions and distributed vertical loads on a horizontal rigid beam resting on two pin supports (Harr (1987). For symmetrical probability distributions, p_+ and p_- are taken as equal to 0.5; while x_+ and x_- are taken as one standard deviation above and below the mean value of the random variable, respectively. The mathematical details of this method are complicated enough to be beyond the scope of this paper and readers interested in these details can refer to Harr (1987). This technique is quite useful when it is difficult or even impossible to apply the FOSM method as a result of difficulties in obtaining derivatives of the factor of safety with respect to different random variables. The main limitation of this technique is the complexity in calculations when considering multiple random variables in the assessment of safety factors. An excellent summary of the accuracy and limitations of this method has been provided by Christian and Baecher (1999).

Fig. 7. Schematic diagram illustrating different terms used to obtain correlation between spatial averages of random variables (modified from Vanmarcke 1977).

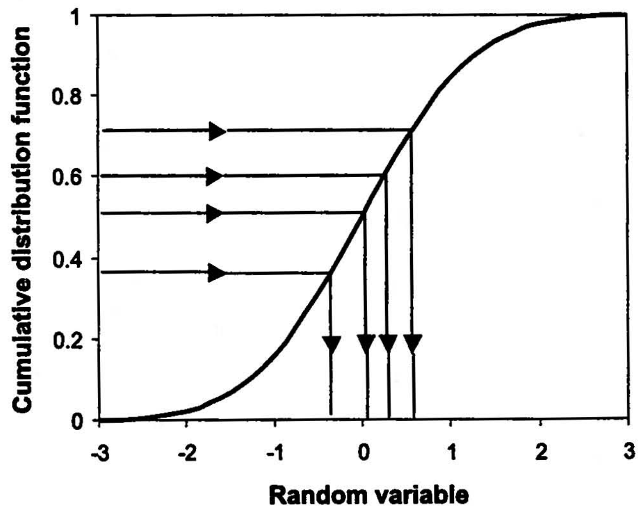


The early implementation of Monte Carlo simulation to limit equilibrium analyses considered soil or rock properties as uncorrelated random variables (Kim and Major 1978). Several realizations of soil design parameters were obtained and used to develop a histogram for the factor of safety of earth slopes. Recently, the effect of spatial correlation between soil properties has been accounted for through the application of geostatistics principles and volume-variance relationships (El-Ramly 2001). A complete probability distribution of output variables, such as the factor of safety, can be obtained and the failure probability can be reasonably assessed. This is a major advantage over other analysis techniques where some assumptions have to be made about the probability distribution of output variables. It should be noted, however, that Monte Carlo simulation has its own limitations, which can be summarized as follows:

(1) The need to define a reliable input reference distribution, which requires a considerable number of field data. In addition, older versions of Monte Carlo simulation algorithms used to deal only with parametric probability distribution functions, i.e., probability distributions that can be defined through mathematical relationships such as normal and lognormal distribution. Field data, however, do not necessarily fit into any of these parametric distributions. This problem has been overcome by recent versions of Monte Carlo simulations, such as that of Deutsch and Journel (1998) that are

capable of dealing with nonparametric distribution functions directly inferred from field data;
(2) Clustering of the simulation outcome into a limited zone of the input probability distribution, as the drawn samples are more likely to be in areas of higher probability, as shown in Fig. 8. This problem mainly arises in cases where an insufficient number of realizations (number of iterations in Monte Carlo algorithm) are used in the simulation process (Palisade Corporation 1996). This may result in sampling values of the random variable away from the tails of the input probability distribution, which can be on the unsafe (nonconservative) side. This problem, however, can be overcome by using a number of realizations large enough to reproduce the input distribution; and
(3) Depending on the number of variables involved in the simulation process, Monte Carlo simulation may require a significantly large number of iterations and consequently a considerable computational effort. However, the authors believe that this problem has been overcome by the new generation of fast computers.

Fig. 8. Clustering of the outcome of Monte Carlo simulations resulting from an insufficient number of realizations (modified from Palisade Corporation 1996).



Stochastic finite element method

The stochastic finite element method (SFEM) is a modification of the traditional finite element method to capture the effect of soil spatial variability on numerical analysis. This is carried out by using finite element discretization to obtain a direct assessment of the mean and variance of nodal displacements together with the covariance between displacements at different nodes of the numerical analysis mesh (Baecher and Ingra 1981). This assessment is usually accomplished by calculating a covariance matrix whose value depends on the characteristics of spatial correlation between soil properties, such as variogram model and spatial range. These characteristics are captured into the finite element scheme by introducing the matrix of differentials thereby assessing the effect of the variation of mechanical soil properties from one element to another on the global stiffness matrix. For more details about SFEM, the reader can refer to Baecher and Ingra (1981) and Auvinet et al. (1996). Different modifications of SFEM have been developed by introducing different numerical techniques to capture soil spatial variability. Examples of these modifications are the probabilistic finite element method (Righetti and Harrop-Williams 1988) and stochastic integral formulations (Zeitoun and Baker 1992).

The major advantage of the SFEM is the direct assessment of statistical characteristics of output variables, such as the mean and variance. This helps avoid long computational time associated with incorporating several realizations of spatially variable soil parameters into deterministic analysis scheme, as discussed in the following section. On the other hand, different limitations of the SFEM have been discussed by several authors, such as Baecher and Ingra (1981) and Auvinet et al. (1996), and they can be summarized as follows:
(1) The analysis results are not affected by the probability distribution of the input random variables. Furthermore, a distribution has to be assumed for output variables as SFEM provides only an assessment of the mean and standard deviation;
(2) Element variance and covariance matrices are functions of element shape and geometry and their determination becomes quite tedious for irregular element shape and complicated boundary conditions;
(3) It is limited to small variability due to the error associated with the truncation of higher order terms in the Taylor expansion (which is used for the determination of mean values of the response (output) variables, such as nodal settlements);
(4) Integration of the random variable field over each element may result in a change in the anisotropy ratio of the spatial correlation structure of soil properties; and
(5) It is usually limited to the linear elastic behavior of soil to avoid extreme complexity in the computation process and does not adequately capture the behavior of soil properties with skewed probability distributions.

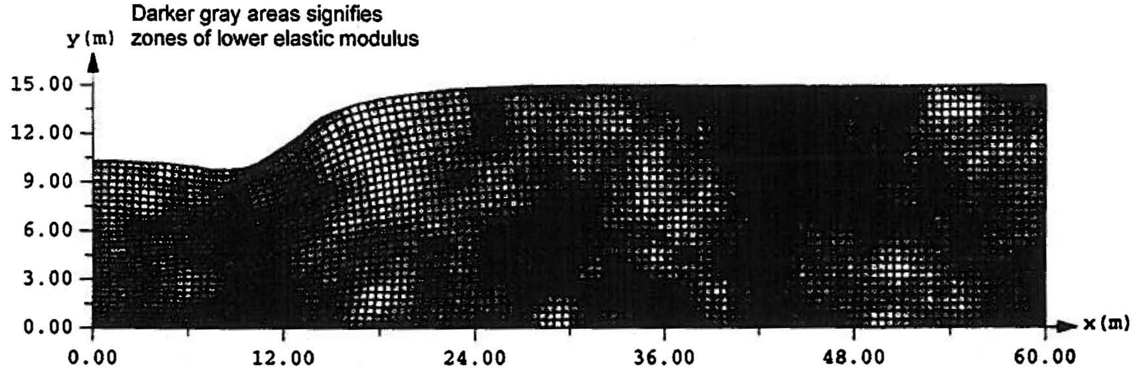
Due to the above limitations, the use of stochastic finite element analysis has received limited attention from geotechnical practitioners and researchers.

Application of stochastic input parameters into deterministic numerical analysis

Deterministic numerical analysis with stochastic input soil parameters has been recently adopted by many researchers, such as Paice et al. (1994) and Popescu et al. (1998), as a technique to incorporate soil spatial variability in geotechnical design. Monte Carlo based simulation techniques have been used to generate several realizations of soil properties that vary from one point to another across the domain of interest, as shown in Fig. 9. This spatial variation is usually employed into the numerical analysis scheme by assessing soil properties at the center of each element of the numerical simulation grid and assuming them to be constant within that element. By analyzing several realizations of the spatially variable soil medium, histograms of response (output) variables can be obtained. Examples of the simulation algorithms used in practice are the sequential Gaussian, the sequential indicator simulations (Deutsch 2002), and the local average subdivision technique (Fenton and Vanmarcke 1990).

The sequential Gaussian simulation (SGS) is the most commonly used technique, especially in the field of petro-

Fig. 9. Deformed mesh with spatially variable elastic modulus below flexible strip footing (modified from Paice et al. 1994).



leum engineering. The basic idea of this technique is illustrated in Fig. 10. Input random variables are transformed into standardized normally distributed random variables with zero means and unit variances for which different variogram characteristics are assessed. Simulated values of a standardized variable, Z, can be determined at any node of the simulation grid according to the relationship

$$[12] \quad Z_s(u) = Z^*(u) + R(u)$$

where $Z_s(u)$ is the simulated value of the variable Z at location u; $Z^*(u)$ is the krigged estimate of the variable Z at location u; and $R(u)$ is a random residual.

The krigged estimate is a linear estimator of the variable Z at location u in space, where the value of Z is unknown, using the kriging interpolation techniques (Journel and Huijbregts 1978). This estimate depends on different characteristics of spatial correlation structure (variogram), does not vary from one realization to another, and can be assessed through

$$[13] \quad Z^*(u) = \sum_{i=1}^{n} \lambda_i Z(u_i)$$

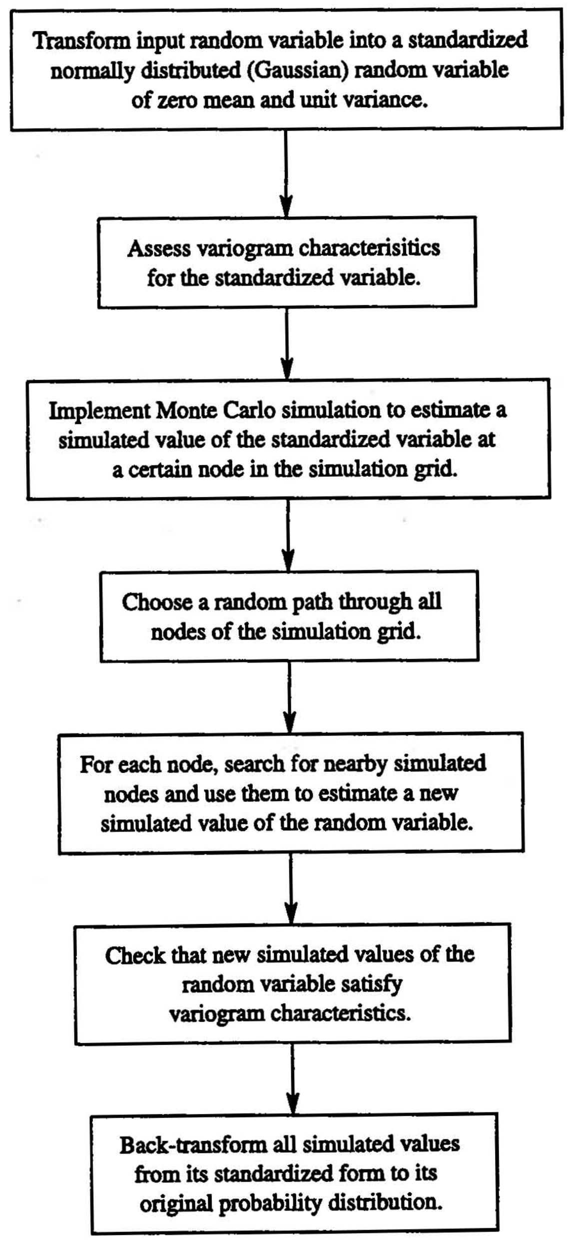
where $Z(u_i)$ is a known value of Z at location u_i in space, either from field data at that location or previously simulated nodes; and λ_i is a weight given to field data at location u_i that depends on the characteristics of the spatial correlation structure.

The random residual $R(u)$ follows a normal distribution with zero mean and a variance equal to the kriging variance (Deutsch 2002). A different value of $R(u)$ is obtained in each realization using Monte Carlo simulation resulting in a variation of the simulated value of the random variable, $Z(u)$, from one realization to another. A random path is followed to assess the value of the standardized random variable at each node of the numerical simulation grid. The simulated values across the analysis domain are then back-transformed to their original probability distribution. By repeating the above procedure, several realizations of soil spatial variation across the analysis domain can be obtained.

Application of stochastic analysis to geotechnical field problems

The stochastic analysis techniques discussed in the previous sections have been implemented in several applications

Fig. 10. The basic idea of the sequential Gaussian simulation.



throughout the history of geotechnical engineering practice to assess the impact of ground variability on geotechnical field problems. In the following sections, an attempt is made to address the current state of practice in some of these applications and its limitations together with potential trends for future research.

Stochastic analysis of shallow foundation settlement

Early attempts to perform probabilistic analysis of foundation settlement started in the late 1960s. Wu and Kraft (1967) estimated the uncertainty in soil bearing capacity and foundation settlement through assessing the uncertainty in applied load, soil strength, and deformation parameters. The uncertainty in soil strength was estimated by assessing the variability of laboratory undrained shear strength for clayey soils and that of SPT data for sandy soils. Resendiz and Herrera (1969) carried out a probabilistic analysis of settlement and rotation of flexible and rigid footings over randomly variable compressible soils. A one-dimensional settlement model was adopted in which the coefficient of volume change was characterized as a normally distributed random variable. The analysis results were used to obtain design parameters that satisfied tolerable settlements and rotations criteria together with the minimum expected monetary loss. These studies can be considered as a good start towards addressing such a complex problem. However, they were fairly primitive as some elements of soil inherent variability, such as spatial correlation between soil properties, were not adequately considered.

The modern approach to deal with ground variability and its implications on foundation settlement started in the early 1980s with the pioneer work of Baecher and Ingra (1981). In their study, two-dimensional stochastic finite element analyses were carried out to assess the uncertainty in total and differential settlement. Soil elastic modulus was treated as a random variable, whereas Poisson's ratio was assumed to be constant across the soil mass. Two spatial correlation models, the exponential and the squared exponential (Gaussian), were considered and the response variables (total and differential settlement) were assumed to be normally distributed. A limiting assumption of the study was the linear elastic soil behavior, which implies, together with the use of SFEM, a small variation in soil properties to avoid the development of plastic zones and the onset of nonlinear constitutive behavior. In addition, the effect of different probability distributions of soil properties on the expected uncertainty was not assessed.

In a similar fashion, Zeitoun and Baker (1992) proposed a stochastic approach for settlement prediction of shallow foundations using the stochastic integral formulation (SIF) technique, which is a modification of the SFEM. It was assumed that soil would exhibit linear elastic behavior under both axisymmetrical and plane strain conditions. Soil shear modulus was treated as a random variable and the Gaussian model was chosen to represent the spatial variation of shear modulus across the problem domain. The technique used in the study had serious limitations as unrealistic spatial correlations were assumed either through the use of a very high horizontal autocorrelation distance or by considering the soil medium to be in the form of concentric rings of constant elastic modulus. Furthermore, no information was provided about the probability distribution of output variables, and the effect of different spatial correlation structures was not accounted for. Finally, the use of SFEM implied some restrictions on the range of soil variability used in the analysis to prevent development of plastic zones as discussed in the previous paragraph.

The effect of random soil stiffness on foundation settlement was reinvestigated by Paice et al. (1994) through the use of deterministic finite element analysis with stochastic input soil parameters. Soil elastic modulus was regarded as a spatially random variable resulting in a ground profile as previously shown in Fig. 9. The elastic modulus was assumed to follow a lognormal probability distribution and exponential correlation structure. This study has some limiting assumptions such as the linear elastic soil behavior, the isotropic correlation structure, and the symmetry of spatial distribution of elastic modulus around the footing centerline. In addition, the effect of different types of correlation structures and the sensitivity of the results to the number of realizations were not considered.

The effect of random fluctuations of the interface between soil layers on the uncertainty in foundation settlements has been accounted for by Brzakala and Pula (1996). This uncertainty in soil geometry was converted into a new random field expressed in terms of the interface fluctuation and was incorporated into the stochastic finite element analysis. The main limitations of the study lie in the linear elastic soil behavior and the neglecting of the inherent soil variability within layers. In addition, the effect of different types of probability distributions and correlation structures on the stochastic analyses outcomes was not accounted for.

It is worth noting that the above studies have not considered the effect of changing the state of stresses in soil mass on the outcome of statistical analyses. In other words, the sensitivity of the statistical characteristics of output variables to wide ranges of applied vertical and horizontal stresses was not adequately addressed. In addition, no attempt has been made to provide risk-based representative soil parameters that can be implemented in deterministic analyses while continuing to honor detailed ground heterogeneity.

Stochastic analysis of liquefaction problems

The early attempts to quantify the stochastic nature of liquefaction problems were focused on developing analytical expressions to estimate the uncertainty in liquefaction potential assessment. Yegian and Whitman (1978) conducted a pioneer study to provide a statistical evaluation of the annual probability of failure for potentially liquefiable sites. This was carried out by combining the annual probability of given earthquakes with the probability of ground failure under these earthquakes. In addition, an analytical expression was developed to assess the uncertainty in a limit state parameter, proposed to estimate the maximum shear resistance of the ground. A major limitation of that study was that the effect of spatial correlation between soil properties was not accounted for and that the uncertainty in the results were assumed to be insensitive to the probability distribution of the input random variables. Furthermore, the derivation of the expression for the limit state parameter was based on the assumption that the soil shear resistance and vertical effective stress were two independent random variables.

Recently, the use of deterministic finite element analysis with stochastic input soil parameters has gained much popularity in the field of probabilistic liquefaction analysis. Several attempts have been made to apply this technique to study case histories that involved potentially liquefiable ground conditions.

Fenton and Vanmarcke (1991) performed one-dimensional finite element analyses to assess the effect of inherent variability of soil properties on liquefaction potential at the Wildlife site, California. Soil properties, such as porosity, Poisson's ratio, elastic modulus, permeability, and the dilation angle, were considered as random variables. The first two properties were considered to be normally distributed, whereas the rest were assumed to follow a lognormal distribution. The effect of correlation structure was taken into consideration through the application of the variance reduction factor proposed by Vanmarcke (1977). Several realizations of soil properties were generated using the local average subdivision technique, and were excited by various earthquake motions applied at the base of the soil columns using DYNA1D software. A main limitation of the study is that the results sensitivity to the number of realizations was not taken into consideration. Moreover, the effect of different probability distributions of soil properties was not accounted for together with the use of one-dimensional analysis, in which the analysis domain was divided into soil columns neglecting the coupling between soil elements.

Popescu et al. (1996) carried out one of the pioneer investigations on the effect of soil spatial variability on liquefaction assessment using the results of cone penetration tests, where the cone tip resistance, q_c, and the cone index, I_c, were treated as random variables. A simulation algorithm was developed in the study for the simulation of non-Gaussian multivariate random fields. A nonlinear regression algorithm was adopted to determine the probability distribution and the correlation structure of the random variables. The problem was analysed using the DYNAFLOW software implementing stochastic input soil parameters obtained using the correlations between soil properties and CPT data. For comparison, a deterministic numerical analysis was carried out using the mean values of soil parameters. A characteristic percentile of cone tip resistance was proposed for use in deterministic analyses to predict the same maximum pore pressure obtained from stochastic analysis. The major limitation of the study lies in the use of only four realizations to quantify the effect of soil variability, which may not be sufficient to sample the expected range of response, as discussed earlier. In addition, the effect of spatial correlation range on the study outcomes was not accounted for. The authors believe, however, that this spatial range may have a profound effect on liquefaction susceptibility and needs to be considered in future stochastic liquefaction studies. In addition, the strength percentile proposed for use in deterministic analysis to capture the effect of soil spatial variability was subjectively assessed.

Popescu et al. (1998) extended the previous study to provide risk-based liquefaction potential measures through two-dimensional stochastic analysis where cone tip resistance was treated as a spatially random variable. The effect of inherent variability was assessed through 25 realizations of the spatial distribution of CPT data across the site. A series of deterministic finite element analysis was carried out, using different percentiles of the recorded CPT data, to estimate equivalent (representative) soil parameters that can capture the implications of soil spatial variability. These parameters were considered to be associated with the upper limit of the Monte Carlo simulation response range. It was concluded that the initiation of soil liquefaction could not be accurately predicted by deterministic models, which could not account for the presence of loose pockets within soil masses. This study, as in the case of the previous one, did not take into consideration the sensitivity of the analysis to the number of realizations of soil properties. In addition, the proposed equivalent parameters could be considered overconservative, as they were associated with the most critical response of the stochastic analysis. Moreover, the effect of the type of correlation structure and its spatial limit was not accounted for.

Once again, the effect of changing the state of stresses in soil mass on the outcome of stochastic analyses has not been adequately considered in any of the above studies. For example, more investigation is needed to ascertain whether or not the same values of the representative cone tip resistance percentiles of Popescu et al. (1998) would be obtained if potentially liquefiable layers were at different depths below ground surface.

Stochastic analyses of seepage flow

The problem of water flow through heterogeneous porous media has been studied thoroughly along the history of petroleum engineering and water resources research. One of the pioneer attempts to apply the principles of geostatistics into the geotechnical engineering practice to study the effect of soil spatial variability on seepage flow was made by Griffiths and Fenton (1993). In their study, the local average subdivision simulation technique was used to generate 1000 realizations of spatially variable hydraulic conductivity below a water retaining structure. The resulting field was then mapped onto a finite element mesh to perform numerical analysis of the problem under deterministic boundary conditions. The hydraulic conductivity, k, was assumed to follow lognormal probability distribution, and the effect of spatial correlation structure was accounted for through quantifying the influence of the scale of fluctuation on different response (output) variables. The limitations of the study lie in the use of an isotropic correlation structure where vertical and horizontal ranges were assumed equal, and the effect of different probability distributions and correlation structure models were not accounted for. Furthermore, the sensitivity of the analysis to the number of realizations of the random variable, hydraulic conductivity, was not considered.

Decision making in geotechnical engineering

One of the major challenges that faces geotechnical engineers is the need to make decisions regarding the soil parameter to be used in engineering analysis. These decisions have to be based on information that invariably has a certain degree of uncertainty. Consequently, the decision making process is considered to be governed by two factors, the uncertainty in the decision variables and the risk level of the project. Several decision making algorithms have been used throughout the history of geotechnical engineering practice, such as the worst case and quasi worst case approaches, reliability-based techniques, confidence interval approach, and Bayesian decision analyses. Details of these algorithms will be discussed in the following paragraphs.

The worst case approach aims at achieving the absolute safety of the project and relies on the notion of maximum loss and maximum expected hazards, often referred to as the maxi-max criterion (Ang and Tang 1984). For example, if the range of the measured friction angle of a sandy deposit at a certain site ranges from 30–40°, the design value will be assessed as 30°. This approach is overly conservative and rarely used in practice. On the other hand, the quasi worst case approach (Pate-Cornell 1987) tries to apply some kind of engineering judgment into the above approach to provide an upper bound for the risk level. Revisiting the above example, the sandy soil at the site is classified (say medium dense sand) and the minimum value associated with such classification (say 33°) will be used as the design value. A common problem of the two approaches is that no information can be obtained about the risk level associated with the design value.

The reliability-based approach relies on selecting design parameters that satisfy a desired degree of reliability or a certain probability of failure. This approach has been commonly used in slope stability analysis. Wolff (1996) proposed soil design parameters to be associated with a reliability index, β, of 3 for routine slopes and 4 for critical slopes such as dams. The reliability index can be obtained through

$$[14] \quad \beta = \frac{m_{FS} - L}{\sigma_{FS}}$$

where m_{FS} is the mean factor of safety; L is a limit state value usually equal to 1; and σ_{FS} is the standard deviation of the factor of safety.

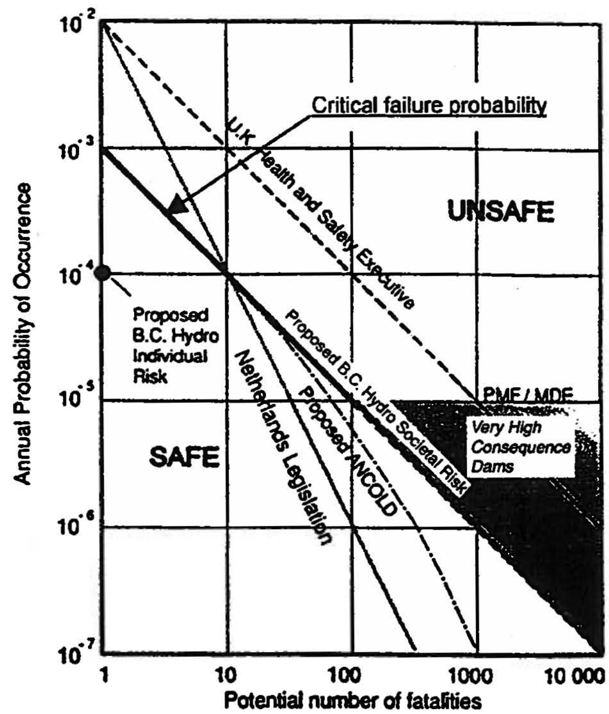
In a similar fashion, the US Army Corps of Engineers (1995) proposed an assessment of the performance level of embankments depending on the target reliability index and the corresponding failure probability, as shown in Table 2. Comparing the recommendation presented in Table 2 with the suggested values of reliability index of Wolff (1999) implies that the selection of design parameters for earth slopes should be associated with critical failure probabilities no more than 0.1%. British Columbia (BC) Hydro developed a similar approach for dam design based on a thorough review of different potential hazards (Whitman 2000). In their criterion, critical failure probabilities were assessed as a function of potential number of fatalities, as shown in Fig. 11. On the other hand, El-Ramly (2001) concluded that critical failure probabilities developed in geotechnical literature were overly conservative and that a critical failure probability of 2% could be regarded as an upper bound for the satisfactory performance of earth slopes. This critical value was assessed based on extensive probabilistic slope stability analyses of several case histories in North America and Hong Kong.

In the confidence interval approach (Harr 1987), soil parameters associated with the upper and lower limits of a certain degree of confidence are proposed as design parameters.

Table 2. Assessment of performance of earth slopes and the associated failure probability as proposed by the US Army Corps of Engineers (1995).

Expected level of performance	β	Probability of failure (%)
High	5	3×10^{-5}
Good	4	0.003
Above average	3	0.1
Below average	2.5	0.6
Poor	2	2.3
Unsatisfactory	1.5	7
Hazardous	1	16

Fig. 11. Critical probabilities of failure for dam design in terms of expected number of fatalities (modified from Whitman 2000). ANCOLD, Australian National Committee on Large Dams; PME/MDE, XXXXXXXXXXXX.



The selection of design parameters associated with a 90% level of confidence, commonly used in practice, is illustrated in Fig. 12. Using these design parameters provides a range for output (response) variables, such as factor of safety, with only a 5% chance that the actual value of these variables will be either larger than the upper limit or smaller than the lower limit of this range.

The most robust decision making algorithm is the Bayesian decision analysis (Benjamin and Cornell 1970; Deutsch 2002), where the impact of making mistakes in estimating design parameters can be expressed in terms of monetary values. This approach utilizes loss functions and histograms of soil design parameters to obtain optimal estimates of these parameters associated with minimum expected monetary loss. The loss functions are mathematical relations used to quantify the effect of making mistakes in

Fig. 12. Selection of design parameters associated with a 90% confidence level.

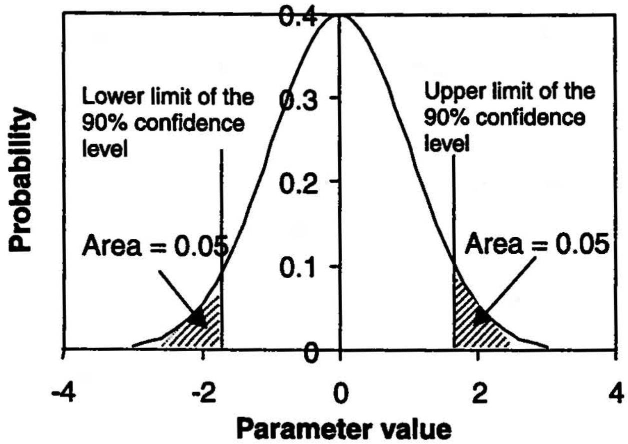
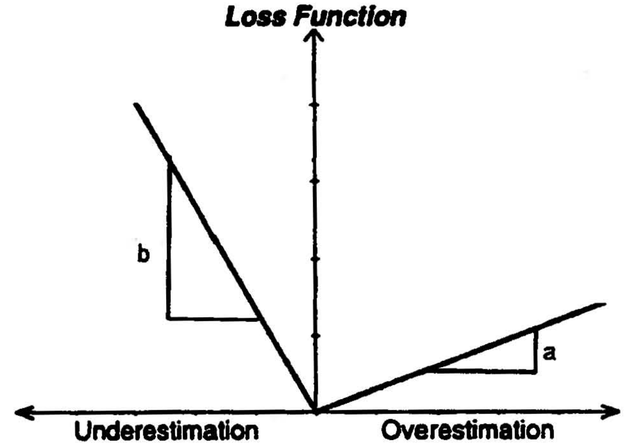


Fig. 13. Linear loss functions to quantify the effect of making mistakes in estimating soil design parameters.



selecting design parameters. These functions can take different forms, such as linear, quadratic, and exponential, as shown in Fig. 13 for linear loss functions. For more details of the application of this approach, the reader can refer to Deutsch (2002). The main limitation of this approach is the difficulty associated with its application in cases where loss of human lives may be expected.

Throughout the history of geotechnical engineering few attempts have been made to implement the above approaches into field problems. One of the pioneer works in this area was that by Folayan et al. (1970), where the Bayesian decision analysis was applied to settlement prediction analysis. In their study, the compression index, C_c, was treated as a random variable and the results of 27 one-dimensional consolidation tests were used to obtain a histogram for C_c. In addition, an exponential loss function was adopted to assess a value of C_c that produced the minimum expected loss. A main limitation of the study was that it ignored the effect of the spatial correlation characteristics of C_c on the analysis results.

Conclusions

A thorough review of the different techniques developed to deal with soil heterogeneity has been presented in this study. Different approaches developed to handle the lithological heterogeneity of the ground in the petroleum engineering field were briefly addressed. Different elements of soil inherent spatial variability, such as mean, variance, spatial correlation characteristics, and volume-variance relationships were thoroughly discussed together with their implications in geotechnical design. Different approaches adopted throughout the history of geotechnical engineering to perform stochastic analyses of different geotechnical applications were thoroughly reviewed and criticized. In addition, an expression for the variance reduction factor of spherical spatial correlation structures (variograms) was developed in this study. Examples of the applications of stochastic analysis to field problems such as shallow foundation settlement, liquefaction assessment, and seepage flow were presented with emphasis on the limitations of the current practice. Finally, different decision making algorithms were discussed together with comments regarding their applicability in geotechnical engineering.

From this study it can be concluded that there is a need for a comprehensive study of soil heterogeneity that takes into consideration different sources of nonhomogeneity and their implications on different geotechnical applications. Furthermore, there is a need to ascertain whether or not the outcomes of stochastic geotechnical analyses are sensitive to changes in the state of stresses in subsurface layers. Finally, the risk level of geotechnical projects should be incorporated in a decision-making framework to provide estimates of representative soil parameters that honor the detailed ground heterogeneity.

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Appendix A

Derivation of variance reduction factor for spherical correlation structure

The variance reduction factor, Γ^2, can be determined according to the following relation proposed by Vanmarcke (1984):

$$[A1] \quad \Gamma_T^2 = \frac{2}{T}\int_0^T \left(1 - \frac{h}{T}\right) C(h)\mathrm{d}h$$

where h is the separation distance and $C(h)$ is the standard covariance, i.e., covariance with a unit variance.

The standard covariance of spherical correlation structure can be expressed in the form

$$[A2] \quad C(h) = 1 + 0.5\left(\frac{h}{a}\right)^3 - 1.5\left(\frac{h}{a}\right)$$

Substituting in eq. [A1] provides

$$[A3] \quad \Gamma_T^2 = \frac{2}{T}\int_0^T \left(1 - \frac{h}{T}\right)\left[1 + 0.5\left(\frac{h}{a}\right)^3 - 1.5\left(\frac{h}{a}\right)\right]\mathrm{d}h$$

Integrating and rearranging results provides the following expression for the variance reduction factor for spherical correlation structures:

$$[A4] \quad \Gamma_T^2 = 1 - \frac{T}{2a} + \frac{T^3}{20a^3}$$