# Economic Model Predictive Control with Extended Horizon

by

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# Abstract

In this thesis, we propose a computationally efficient economic model predictive control (EMPC) design which is based on a well-known methodology — the separation of the control and prediction horizon. The extension of the prediction horizon of EMPC is realized by employing an auxiliary control law which asymptotically stabilizes the optimal steady state. The contributions of this thesis are to systematically analyze the stability and performance of the general EMPC scheme with extended horizon, and to explore its extensions and applications to several specific scenarios.

Specifically, we establish stabilizing conditions of the proposed EMPC in a progressive manner. First, we establish practical stability of the EMPC with respect to the extended horizon for strictly dissipative systems satisfying mild assumptions. Then, under stronger conditions involving Lipschitz continuities and exponential stability of the auxiliary controller, the shrinkage of the practical stability region is shown to be exponential. Further, we characterize a general condition on the storage function under which exponential stability of the optimal steady state can be established. Conventional set-point tracking MPC with quadratic cost falls into the latter category. The achievable performance of the proposed EMPC design is also discussed in a similar manner under different stabilizing conditions. It is revealed that the performance of the proposed EMPC is approximately upper-bounded by the auxiliary controller. Our theoretical results provide valuable insights into the intrinsic properties of EMPC as the discussions are laid out in a very general setting and the results are compatible with the analysis of existing MPC / EMPC designs.

With a deepened theoretical understanding of EMPC with extended horizon, we further explored the extension of the proposed EMPC design in several scenarios: (i) We consider the case where a locally optimal LQR control law can be found. A terminal cost is constructed as the value function of the LQR controller plus a linear term characterized by the Lagrange multiplier associated with the steady-state constraint. This design results in an EMPC this is locally optimal. (ii) We take advantage of EMPC with extended horizon to handle systems with scheduled switching operations. The EMPC operations are divided into two phases — an infinite-time operation phase and a mode transition phase. The proposed EMPC schemes are much more efficient and achieves improved mode transition performance than existing EMPC designs. (iii) We consider systems with zone tracking objectives which can be viewed as a special case of economic objective. The proposed zone MPC penalizes the distance of the predicted state and input trajectories to a desired target zone which is not necessarily positive invariant. We resort to LaSalle's invariance principle and develop invariance-like theorem which is suitable for stability analysis of zone control. Auxiliary controllers which asymptotically stabilize an invariant subset of the target zone are employed to enlarge the region of attraction. (iv) We apply the proposed EMPC algorithm to the control of oilsand primary separation vessel (PSV) to maximize the bitumen recovery rate.

# Preface

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Chapter 2 of this thesis is a revised version of Su Liu and Jinfeng Liu, Economic model predictive control with extended horizon. *Automatica*, 73:180-192, 2016.

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Chapter 4 of this thesis is a revised version of Su Liu and Jinfeng Liu. Economic model predictive control for scheduled switching operations. In *Proceedings of the American Control Conference*, pages 1784-1789, Boston, MA, 2016.

Chapter 5 of this thesis is a revised version of Su Liu, Yawen Mao, and Jinfeng Liu, Nonlinear model predictive control for zone tracking. *IEEE Transactions on Automatic Control*, submitted.

Chapter 6 of this thesis is a revised version of Su Liu, Jing Zhang and Jinfeng Liu. Economic MPC with terminal cost and application to an oilsand primary separation vessel. *Chemical Engineering Science*, 136:27-37, 2015.

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# Chapter 1

# Introduction

## 1.1 Motivation

Model predictive control (MPC) is a rich fruit in the branch of model control theory which is rooted in computer science and informatics. Ever since its origin in chemical plants in the late 1970s, both the industrial applications and theory development of MPC have been booming. A survey [4] conducted in mid-1999 showed more than 4500 industrial MPC applications prior to the new millennium and indicated a rapid annual growth of approximately 20%. Presently, not only has MPC become the standard advanced control technique in chemical engineering, it is also being applied to various areas in mechanical engineering, electrical engineering, automotive engineering, aerospace engineering and so on. The landscape of MPC is shaped by three major aspects: theory, computation and application. To a large extent, early development of MPC theory was pushed by its successful industrial applications. The progresses in MPC theory in return stimulated more applications which forms a positive feedback loop. Since the implementation of MPC requires online or real-time optimization, all MPC designs and applications have to reckon with the computational burden. This entails either the use of more powerful optimization algorithms or the design of computationally efficient MPC.

Traditionally, in the process industry, MPC is designed for tracking set-point or reducing variations in crucial process variables. The huge success of MPC in industrial applications owes much to its ability to optimally handle process constraints and interactions. However, in many applications, especially at a higher level of decision-making, the objectives are often economic-oriented and may be different from the classical control objective of set-point tracking. For example, some typical objectives include the maximization of operation profit, the minimization of energy consumption, or the maximization of a certain product. These economic objectives may not be seamlessly translated into set-point tracking objectives. It is possible that non steady-state operation, such as periodic operation, yields superior performance. These considerations motivate the recent development of a more general form of MPC called economic MPC or EMPC. EMPC optimizes general cost functions that are directly linked to the economic metrics (profit, efficiency, sustainability) of the plant. This promising approach, which has great potential to improve the dynamic performance during transient processes, is still a relatively new research area. New theories, algorithms and tools are being developed for EMPC.

In this thesis, we propose a computationally efficient EMPC design which is based on a well-known methodology in conventional MPC — separation of the control and prediction horizon. Our objectives are to systematically analyze the stability and performance of the general EMPC scheme with extended horizon, and to explore its extensions and applications to several specific scenarios.

## 1.2 Research overview

### 1.2.1 A brief review of MPC

Technically, MPC refers to a control methodology that repeatedly solves online a finite-horizon open-loop optimal control problem in a receding horizon fashion. Figure 1.1 shows a conceptual picture of the MPC scheme. At a sampling time $k$, the MPC algorithm computes a future input trajectory by optimizing the predicted trajectories over a finite prediction horizon with the initial state $x(k)$. The first element of the input sequence is then injected to the plant. Upon availability of an updated state measurement (or estimation) at the subsequent sampling time $k+1$, the MPC algorithm is repeated with the prediction horizon shifted one step further.

MPC differs from the classical optimal control mainly in the way the optimal control problem is solved. In classical optimal control such as LQR control, an optimal feedback control law is obtained offline whereas in MPC, an input sequence is calculated online based on the current system state. The former requires the solution of the

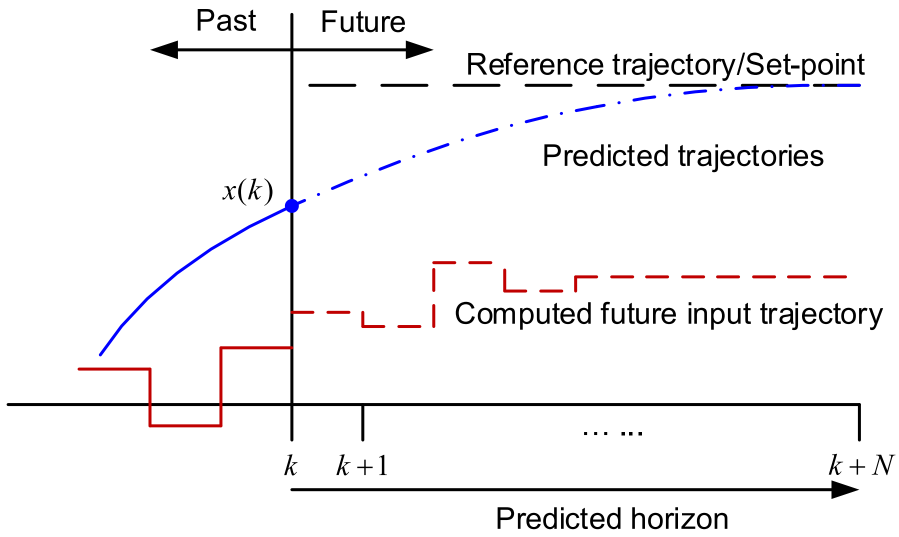


**Figure 1.1:** A conceptual picture of the MPC scheme

Hamilton-Jacobi-Bellman equation which can be very difficult or merely impossible to obtain for generic nonlinear systems. The latter amounts to solving a finite dimensional mathematical programming problem in real-time which is becoming more and more viable thanks to the development of computer hardware/software and the advances in optimization algorithms.

An essential feature that shapes the landscape of MPC theory and design is that the open-loop optimal control problem be solvable in a short period of time (as compared to the dynamics of the process). For one thing, online optimization necessitates the use of a finite horizon which creates a discrepancy between MPC theory and classical infinite-horizon optimal control. This discrepancy has spawned considerable research interests in the design and analysis of MPC with nominal stability guarantee. Consensus was reached in the milestone paper [5] in which the use of the value function as Lyapunov function and several 'ingredients' for nominal stability of MPC are formalized. Since then, more insights into the inherent stability of MPC have been revealed by leveraging the dynamic programming principle [6], [7], [8], [9], [10].

For another thing, computational efficiency of MPC will remain a constant theme throughout the development of MPC. This is a twofold mission. On the one hand, development of fast optimization algorithms serve as the most straightforward impetus for the computational efficiency of MPC. The majority of industrial MPC optimization problems are formulated as quadratic programming (QP) problems which can be efficiently solved using standard algorithms such as the interior-point methods [11, 12], the active-set method [13], and fast gradient methods [14]. Recent advances in nonlinear programming has also made online implementation of MPC to nonlinear, large-scale systems possible, promoting an integrated MPC scheme for plant-wide real-time optimization [15]. On the other hand, significant research efforts have been made to reduce the computational load of MPC. These include heated research areas such as (i) distributed MPC (see [16] and the references therein), where a centralized MPC is decomposed into several subsystem MPCs which communicate and cooperate; and (ii) explicit MPC (e.g., [17], [18]), where an explicit feedback control law is obtained offline by solving a parametric programming problem or obtaining its approximate solution.

### 1.2.2 Separation of control and prediction horizon

One of the most commonly adopted approach to improve the computational efficiency of MPC is to separate the control horizon and the prediction horizon. The idea is to extend the prediction horizon beyond the control horizon by employing an auxiliary control law which asymptotically stabilizes the desired set-point. In this way, the optimization horizon of MPC can be increased without increasing the number of decision variables (i.e. the number of free inputs). As a matter of fact, separation between the control horizon and the prediction horizon arises along early versions of MPC and is well embraced in industrial MPC [4]. With the development of MPC theory, however, separation between the control horizon and the prediction horizon is gradually being phased out in the context of linear MPC. It was shown in [19] that the constrained infinite-horizon LQR can be formulated as a finite-dimensional QP. Specifically, the infinite-horizon tail of the LQR objective function can be lumped together as a terminal cost function which is the value function of an unconstrained infinite-horizon LQR. Under this design, if the control horizon is sufficiently large such that the predicted state at the end of the control horizon can be steered to the origin by the unconstrained infinite-horizon LQR without violating state and input constraints, then the MPC is equivalent to the optimal constrained infinite-horizon LQR controller. The availability of the locally optimal terminal cost function as well as the efficient QP algorithms mentioned above make the separation between control and prediction horizon less appealing for linear MPC designs.

However, for nonlinear systems, the infinite-horizon cost-to-go is in general not available because the solution to the corresponding HJB equation may not exist. A practical remedy is to use a control Lyapunov function (CLF) which provides an incremental upper bound on the stage cost [20, 21, 7] as the terminal cost function. NMPC designed this way has guaranteed stability but can be overly conservative, especially when the terminal cost is much larger than the actual infinite-horizon cost-to-go. Theoretically, one could use a large control horizon to achieve asymptotic stability as well as near-optimal performance [6, 8, 22]. But the use of a large control horizon poses serious challenge to the computational efficiency of MPC. An ideal trade-off is proposed in [23], where a locally optimal control law is utilized to extend the prediction horizon of the NMPC design. The NMPC in [23] is capable of achieving enlarged stability region and locally optimal performance without relying on a large control horizon. These desired features of NMPC design are only possible via the separation of control and prediction horizons. It is therefore safe to say that the methodology to separate control and prediction horizon bears its indispensable merits in the context of NMPC design.

### 1.2.3 Economic MPC

With the maturing of MPC theory, recent development of MPC has seen an exploration into economic-oriented model predictive control. This line of research originates from [24] where unreachable set-points are considered in the conventional tracking MPC scheme. The authors showed that tracking unreachable set-point results in improved transient performance as compared to the conventional MPC that tracks reachable set-point. These results simulated research efforts into a novel form of MPC called economic MPC (EMPC). In EMPC, the quadratic-type cost functions used in conventional MPC are replaced with general economic cost functions that are not necessarily positive-definite with respect to the optimal steady-state operation. Consequently, standard stability analysis techniques to use the value function of conventional MPC as a Lyapunov function is no longer viable. In fact, steady-state operation may not even be the economically optimal operation for EMPC. It has been realized that dissipativity plays an important role in characterizing the optimality of steady-state operation as well as establishing the stability of EMPC [25, 26, 27].

Different EMPC designs have been proposed which stem from the conventional NMPC designs. For example, EMPC with point-wise terminal constraint [28], EMPC with terminal cost [2, 29], EMPC with Lyapunov-based constraint [30, 1]. These EMPC designs also suffer the problems encountered in the conventional NMPC design — they could be overly conservative or computationally demanding. In another line of research [10, 31], EMPC without terminal conditions is studied. This line of research reveals some intrinsic properties of EMPC. It is shown that under certain controllability and dissipativity conditions, near-optimal performance can be achieved if a sufficiently large control horizon is used. However, a large control horizon could make online implementations of the EMPC design computationally impractical. It is thus natural to also resort to the separation of control and prediction horizon in the context of EMPC to improve its computational efficiency.

## 1.3 Contributions and thesis outline

The rest of the thesis is organized as follows:

In Chapter 2, we propose the basic EMPC formulation with extended prediction horizon based on an auxiliary controller. The extension of the prediction horizon is realized by employing a terminal cost which characterizes the economic performance of the auxiliary controller over a finite prediction horizon. The proposed EMPC design is easy to construct and computationally efficient. We analyze the stability and performance of the proposed EMPC design with special attention paid to the impact of the extended horizon. It is shown that for strictly dissipative systems satisfying mild assumptions, a finite terminal horizon is sufficient to guarantee the convergence and performance of the EMPC to be approximately upper-bounded by that of the auxiliary controller.

In Chapter 3, we design a terminal cost for economic model predictive control (EMPC) which preserves local optimality. From the results in Chapter 2, the performance of EMPC is upper-bounded by the auxiliary controller. A very natural question to ask is whether it is possible to find a locally optimal control law and whether the corresponding infinite-horizon cost to go or its approximation can be found. We first show, based on the strong duality and second order sufficient condition (SOSC) of the steady-state optimization problem, that the optimal operation of the system is locally equivalent to an infinite-horizon LQR controller. The proposed terminal cost is constructed as the value function of the LQR controller plus a linear term characterized by the Lagrange multiplier associated with the steady state constraint. EMPC with the proposed terminal cost is stabilizing with an appropriately chosen control horizon, and preserves the local optimality of the LQR controller. Simulation results of an isothermal CSTR verify our analysis.

In Chapter 4, we extend the proposed EMPC design to control systems with scheduled switching operations. The proposed EMPC scheme takes advantage of a set of auxiliary controllers that locally stabilizes the optimal steady state of each operating mode. In the proposed approach, EMPC operations are divided into two phases — an infinite-time operation phase and a mode transition phase, depending on the current sampling time and the scheduled mode switching time. Sufficient conditions to ensure recursive feasibility of the proposed EMPC design are established. The proposed EMPC design is computationally efficient and enjoys enlarged feasibility regions than the auxiliary controllers. Simulation results of a chemical process example demonstrate the superiority of our design over existing MPC designs for switched scheduling operations.

In Chapter 5, we propose a general framework for the design and analysis of nonlinear model predictive control for zone tracking. Zone tracking objective can be viewed as a special case of economic objective. The proposed zone MPC penalizes the distance of the predicted state and input trajectories to a desired target zone which is not necessarily positive invariant. We resort to the invariance principle and develop invariance-like theorem which is suitable for stability analysis of zone control. It is proved that under the zone MPC design, the system converges to the largest control invariant subset of the target zone. Further discussions are made on enlargement of the region of attraction by employing an auxiliary controller as well as handling a secondary economic objective via a second-step economic optimization. Two numerical examples are used to demonstrate the superiority of zone control over set-point control and the efficacy of the zone MPC design.

In Chapter 6, we apply the proposed EMPC algorithm to an oilsand primary separation vessel (PSV). We show how previously developed EMPC design and analysis results in the context of discrete-time system can be extended to continuous-time systems where the issue of sampling needs to be addressed.

Chapter 7 summaries the contributions of this work and discusses future research directions.

# Chapter 2

# Economic MPC with extended horizon

In this chapter, we propose the basic EMPC formulation with extended prediction horizon based on an auxiliary controller. The system and EMPC formulation are set up in Section 2.1. Section 2.2 addresses the stability and convergence of the EMPC design. Practical stability of the proposed EMPC design is established for strictly dissipative systems satisfying mild assumptions. Under stronger conditions, the shrinkage of the practical stability region is shown to be exponential with respect to the increase of the terminal horizon. For a special type of systems which satisfy a further condition on the storage function (including conventional MPC with quadratic cost), exponential stability can be achieved. Interestingly, the same result for this type of systems may not be achieved by an EMPC without terminal condition. Section 2.3 discusses the asymptotic and transient performance of the EMPC design. Results on the asymptotic performance of the proposed EMPC design for general nonlinear systems are provided first. Stronger results on the transient performance of the EMPC design for strictly dissipative systems are derived subsequently, based on different stability conditions from Section 2.2. Two numerical examples are used to verify our results in Section 2.4. Finally, we conclude our results in Section 2.5.

## 2.1 Problem setup

### 2.1.1 Notation

Throughout this work, the operator $|\cdot|$ denotes the Euclidean norm of a scalar or a vector. The symbol '$\setminus$' denotes set substraction such that $\mathbb{A} \setminus \mathbb{B} := \{x \in \mathbb{A}, x \notin \mathbb{B}\}$. The symbol $\mathcal{B}_r(x_s)$ denotes the open ball centered at $x_s$ with radius $r$ such that $\mathcal{B}_r(x_s) := \{x : |x - x_s| < r\}$. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and satisfies $\alpha(0) = 0$. A class $\mathcal{K}$ function $\alpha$ is called a class $\mathcal{K}_\infty$ function if $\alpha$ is unbounded. A continuous function $\sigma : [0, \infty) \to [0, a)$ is said to belong to class $\mathcal{L}$ if it is strictly decreasing and satisfies $\lim_{x\to\infty} \sigma(x) = 0$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if for each fixed $r$, $\beta(r, s)$ belongs to class $\mathcal{L}$, and for each fixed $s$, $\beta(r, s)$ belongs to class $\mathcal{K}$.

### 2.1.2 System description

We consider a class of nonlinear systems which can be described by the following discrete state-space model:

$$x(k + 1) = f(x(k), u(k)) \tag{2.1}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state vector and $u \in \mathbb{R}^{n_u}$ denotes the control input vector. The system state and input are subject to the constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$ respectively, where $\mathbb{X} \subset \mathbb{R}^{n_x}$ and $\mathbb{U} \subset \mathbb{R}^{n_u}$ are compact sets. We assume that there exists an optimal steady state $(x_s, u_s)$ that uniquely solves the following steady-state optimization problem:

$$\begin{aligned} (x_s, u_s) = \quad & \arg\min_{x,u} \; l(x, u) \\ \text{s.t.} \quad & x = f(x, u) \\ & x \in \mathbb{X} \\ & u \in \mathbb{U} \end{aligned} \tag{2.2}$$

where $l(x, u) : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is the economic stage cost function.

### 2.1.3 EMPC based on an auxiliary controller

It is assumed that there exists an auxiliary explicit controller $u = h(x)$ which renders $x_s$ asymptotically stable with $u_s = h(x_s)$ while satisfying the input constraint for all $x \in \mathbb{X}_f$, where $\mathbb{X}_f \subseteq \mathbb{X}$ is a compact set containing $x_s$ in its interior. It is also assumed that the region $\mathbb{X}_f$ is forward invariant under the controller $u = h(x)$. Namely, $f(x, h(x)) \in \mathbb{X}_f$ holds for all $x \in \mathbb{X}_f$. We use $x_h(k, x)$ to denote the closed-loop state trajectory under the controller $h$ at time instant $k$ with the initial state $x_h(0, x) = x$. The above assumptions imply that there exists a class $\mathcal{KL}$ function $\beta_x$ such that:

$$\begin{aligned} |x_h(k,x) - x_s| &\leq \beta_x(|x - x_s|, k) \\ x_h(k,x) &\in \mathbb{X}_f \\ h(x_h(k,x)) &\in \mathbb{U} \end{aligned} \tag{2.3}$$

for all $k \geq 0$ and $x \in \mathbb{X}_f$.

Our EMPC design takes advantage of the auxiliary controller $h(x)$ to extend the prediction horizon. Specifically, this is implemented by employing the following terminal cost $V_f(x, N_h)$, which characterizes the economic performance of the controller $h(x)$ for $N_h$ steps with the initial state $x \in \mathbb{X}_f$:

$$V_f(x, N_h) = \sum_{k=0}^{N_h-1} l(x_h(k,x), h(x_h(k,x)))$$

At a time instant $n$, our EMPC design is formulated as the following optimization problem $\mathcal{P}(n)$:

$$\min_{u(0),u(1),\ldots,u(N-1)} \sum_{k=0}^{N-1} l(\tilde{x}(k), u(k)) + V_f(\tilde{x}(N), N_h) \tag{2.4a}$$

$$\text{s.t. } \tilde{x}(k+1) = f(\tilde{x}(k), u(k)), \quad k = 0, ..., N-1 \tag{2.4b}$$

$$\tilde{x}(0) = x(n) \tag{2.4c}$$

$$\tilde{x}(k) \in \mathbb{X}, \quad k = 0, ..., N-1 \tag{2.4d}$$

$$u(k) \in \mathbb{U}, \quad k = 0, ..., N-1 \tag{2.4e}$$

$$\tilde{x}(N) \in \mathbb{X}_f \tag{2.4f}$$

where $\tilde{x}(k)$ denotes the predicted state trajectory, $x(n)$ is the state measurement at time instant $n$. The optimal solution to the above optimization problem is denoted as $u^*(k|n)$, $k = 0, ..., N-1$. The corresponding optimal state trajectory is $x^*(k|n)$, $k = 0, ..., N$. The manipulated input of the closed-loop system under the EMPC at a time instant $n$ is: $u(n) = u^*(0|n)$. At the next sampling time $n+1$, the optimization of Eq. (2.4) is re-evaluated. The feasibility region of the optimization problem of

Eq. (2.4) is denoted by $\mathbb{X}_N := \{x(n) : \mathcal{P}(n) \text{ is feasible}\}$. $\mathbb{X}_N$ is forward invariant under the EMPC design of Eq. (2.4) due to the forward-invariance of the terminal region $\mathbb{X}_f$. In other words, the EMPC design is recursively feasible.

It is well understood that conventional MPC with an infinite-horizon terminal cost (i.e., $V_f(x, N_h)$ with $N_h \to \infty$) is stabilizing [21]. Similar result has been established in the framework of EMPC for systems satisfying a strong duality condition [28]. However, the framework of Eq. (2.4) with an infinite terminal horizon is not implementable unless an analytical form of $V_f(x, \infty)$ is available, which is difficult to construct for generic nonlinear systems, if it exists at all. Thus it is natural to ask whether the MPC/EMPC of Eq. (2.4) with a finite terminal horizon $N_h$ will stabilize the system or provide satisfactory performance. The issue has been partly addressed for conventional MPC in [23]. It is shown in [23] that for any stabilizing linear controller $h(x)$, there is always a finite prediction horizon such that the MPC with extended prediction horizon is stabilizing. Most impressively, if $h(x)$ is chosen as the locally optimal linear quadratic (LQ) controller, then the MPC behaves like the LQ controller when $x(n)$ is close to $x_s$ for sufficiently large $N_h$, regardless of $N$. This locally near optimal behaviour cannot be achieved by other terminal cost designs where nonlinearity is approximately handled to make the terminal cost compatible with the stage cost (e.g., [20, 2]). To our knowledge, so far there are no results addressing the finite terminal horizon in the framework of EMPC. This work fills this gap. Our analysis is carried out in a general setting where we do not assume continuous differentiability of the system — a condition under which a linear stabilizing auxiliary controller and the corresponding forward invariant set can be readily constructed (see e.g., [32] (pp.136-137) ). This allows our analysis to be applicable to a broader class of nonlinear systems. Interested readers may refer to [33, 34, 35] and references therein for some existing nonlinear controller design techniques. In the following we will proceed with our discussions assuming that the auxiliary controller $h(x)$ is known while we focus on the impact of a finite terminal horizon $N_h$.

## 2.2 Stability and convergence

We restrict our attentions to systems that are strictly dissipative with respect to the economic cost functions. Systems of this type are optimally operated at steady state [25, 26, 27]. Since the optimal steady state is not necessarily a minimizer of the economic stage cost, EMPC with a finite horizon in general cannot stabilize the optimal steady state. We will first establish practical stability of the EMPC design with respect to the terminal horizon $N_h$ in a general setting. Then under a set of stronger conditions, we show that the shrinkage of the practical stability region can be exponential. Finally, we show that for a special type of systems satisfying a further condition on the storage function (including conventional MPC with quadratic cost), exponential stability can be achieved.

### 2.2.1 Practical stability

**Definition 1** *(Strictly dissipative systems) The system of Eq. (2.1) is strictly dissipative with respect to the supply rate* $s:\ \mathbb{X}\times\mathbb{U}\to\mathbb{R}$ *if there exists a storage function* $\lambda:\ \mathbb{X}\to\mathbb{R}$ *and a class* $\mathcal{K}$ *function* $\alpha_l$ *such that the following holds for all* $x\in\mathbb{X}$ *and* $u\in\mathbb{U}$*:*

$$\lambda(f(x,u)) - \lambda(x) \leq s(x,u) - \alpha_l(|x-x_s|)$$

**Assumption 1** *(Strict dissipativity) The system of Eq. (2.1) is strictly dissipative with respect to the supply rate* $s(x,u) = l(x,u) - l(x_s,u_s)$

**Assumption 2** *(Continuity) The functions* $f$ *and* $l$ *are continuous on* $\mathbb{X}\times\mathbb{U}$*,* $h$ *is continuous on* $\mathbb{X}_f$*,* $\lambda$ *is continuous on* $\mathbb{X}$*.*

**Assumption 3** *(Bounded supply under* $h(x)$*) There exists a class* $\mathcal{K}_\infty$ *function* $\alpha_h$ *such that the accumulated supply rate* $s(x,u) = l(x,u) - l(x_s,u_s)$ *under the auxiliary controller* $h(x)$ *is bounded such that*

$$\sum_{k=0}^{N_h-1} l(x_h(k,x), h(x_h(k,x)) - N_h l(x_s,u_s) \leq \alpha_h(|x-x_s|)$$

*for all* $x\in\mathbb{X}_f$ *and* $N_h\geq 1$*.*

**Lemma 1 (c.f [2, 36])** *Let $V(x)$ be a bounded function defined on a closed set $\mathbb{X} \subset \mathbb{R}^{n_x}$ containing $x_s$. If $V(x)$ is continuous at $x_s$ with $V(x_s) = 0$, then there exists a class $\mathcal{K}_\infty$ function $\alpha$ such that:*

$$V(x) \leq \alpha(|x - x_s|), \quad \forall x \in \mathbb{X} \tag{2.5}$$

**Remark 1** *In the light of Lemma 1 and based on Assumption 2, the definition of strictly dissipative systems in Definition 1 which employs a class $\mathcal{K}$ function is equivalent to the definition made in [25] where a positive-definite function is employed. Similarly, if the continuity Assumption 2 holds, then Assumption 3 is equivalent to assuming that the accumulated supply rate under the auxiliary controller $h(x)$ is bounded from above.*

To proceed, let us define the rotated cost function $\bar{l}(x, u)$ as follows:

$$\bar{l}(x, u) = l(x, u) - l(x_s, u_s) + \lambda(x) - \lambda(f(x, u)) \tag{2.6}$$

Based on Assumption 1, the rotated economic cost function $\bar{l}(x, u)$ is bounded from below by:

$$\bar{l}(x, u) \geq \alpha_l(|x - x_s|) \tag{2.7}$$

for all $x \in \mathbb{X}$, $u \in \mathbb{U}$. The rotated terminal cost is defined accordingly as:

$$\bar{V}_f(x, N_h) = \sum_{k=0}^{N_h - 1} \bar{l}(x_h(k, x), h(x_h(k, x)))$$

And we define the rotated optimization problem $\bar{\mathcal{P}}(n)$ as follows:

$$\begin{aligned} \min_{u(0),u(1),\ldots,u(N-1)} \quad & \sum_{k=0}^{N-1} \bar{l}(\tilde{x}(k), u(k)) + \bar{V}_f(\tilde{x}(N), N_h) + \lambda(x_h(N_h, \tilde{x}(N)) - \lambda(x_s) \\ \text{s.t.} \quad & \text{(2.4b)-(2.4f)} \end{aligned} \tag{2.8}$$

**Lemma 2** *The optimal solutions to the optimization problems $\mathcal{P}(n)$ of Eq. (2.4) and $\bar{\mathcal{P}}(n)$ of Eq. (2.8) are identical.*

**Proof.** Let us use $\bar{V}_{N,N_h}(x(n), u)$ to denote the objective function of the rotated optimization problem $\bar{\mathcal{P}}(n)$. It can be shown that:

$$V_{N,N_h}(x(n), u) = \bar{V}_{N,N_h}(x(n), u) + (N + N_h) l(x_s, u_s) - \lambda(x(n)) + \lambda(x_s) \tag{2.9}$$

Since $\mathcal{P}(n)$ and $\bar{\mathcal{P}}(n)$ have the same constraint set and their objective functions only differ by constant terms $(N+N_h)l(x_s, u_s)$, $\lambda(x(n))$ and $\lambda(x_s)$ which are all independent on the decision variables $u(0), ..., u(N-1)$, they have the same optimal solution. ■

The equivalence of solutions between the original problem $\mathcal{P}(n)$ and the rotated problem $\bar{\mathcal{P}}(n)$ allows us to carry out stability analysis of the closed-loop system based on $\bar{\mathcal{P}}(n)$. In the following, we will first define a so-called relaxed practical Lyapunov function as an analysis tool for practical stability and then show that the optimal objective function value of $\bar{\mathcal{P}}(n)$ is a relaxed practical Lyapunov function.

**Definition 2** *(Relaxed practical Lyapunov function) A function $V(x) : \mathbb{S} \to \mathbb{R}$ defined on a forward-invariant set $\mathbb{S}$ is called a relaxed practical Lyapunov function with respect to positive scalars $\delta_1,\ \delta_2,\ \delta_3$, if there exist class $\mathcal{K}_\infty$ functions $\alpha_1,\ \alpha_2,\ \alpha_3$ such that the closed-loop state trajectory $x(k)$, $k \geq 0$ satisfies:*

$$\alpha_1(|x(k) - x_s|) - \delta_1 \leq V(x(k)) \leq \alpha_2(|x(k) - x_s|) + \delta_2 \tag{2.10a}$$

$$V(x(k+1)) \leq V(x(k)) - \alpha_3(|x(k) - x_s|) + \delta_3 \tag{2.10b}$$

**Theorem 1** *If there exists a relaxed practical Lyapunov function $V(x)$ on a forward-invariant set $\mathbb{S}$ with positive scalars $\delta_1,\ \delta_2,\ \delta_3$ and $\mathcal{K}_\infty$ functions $\alpha_1,\ \alpha_2,\ \alpha_3$ as defined in Eq. (2.10) for the closed-loop system of Eq. (2.1), and if $\mathcal{B}_r(x_s) \subset \mathbb{S}$ where $r = \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_1 + \delta_2 + \delta_3)$, then there exists a $\mathcal{KL}$ function $\beta$ such that the following holds for all $x(0) \in \mathbb{S}$, $k \geq 0$:*

$$|x(k) - x_s| \leq \max\{\beta(|x(0) - x_s|, k),\ r\}$$

**Proof.** In this proof, we first show that if $V(x) \geq \alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_2 + \delta_3$, it keeps decreasing until it reaches $V(x) < \alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_2 + \delta_3$. Then we show that the $\alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_2 + \delta_3$ level set of $V(x)$ is forward invariant. Finally, we construct $\beta$ and $r$ based on these results.

First, if $V(x(k)) \geq \alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_2 + \delta_3$, from Eq. (2.10a) we have: $\alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_3 \leq \alpha_2(|x(k) - x_s|)$, substituting the above into Eq. (2.10b), the following can be obtained:

$$V(x(k+1)) \leq V(x(k)) - \alpha_3(\alpha_2^{-1}(\alpha_2(\alpha_3^{-1}(\delta_3))) + \delta_3) + \delta_3$$

Since $\delta_3 > 0$, there exists a positive scalar $\epsilon > 0$ such that

$$\alpha_3(\alpha_2^{-1}(\alpha_2(\alpha_3^{-1}(\delta_3))) + \delta_3) = \alpha_3(\alpha_2^{-1}(\alpha_2(\alpha_3^{-1}(\delta_3)))) + \epsilon = \delta_3 + \epsilon$$

which gives: $V(x(k+1)) - V(x(k)) \leq -\epsilon$. Thus, the following holds for all $V(x(0))$ and $V(x(k)) \geq \alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_2 + \delta_3$:

$$V(x(k)) \leq V(x(0)) - k\epsilon$$

Taking into account that $V(x)$ is bounded on $\mathbb{S}$, the above implies that $V(x)$ decreases to $\alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_2 + \delta_3$ in finite time.

Second, we show that the $\alpha_2(\alpha_3^{-1}(\delta_3))+\delta_2+\delta_3$ level set of $V(x)$ is forward invariant. That is, $V(x(k+1)) < \alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_2 + \delta_3$ if $V(x(k)) < \alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_2 + \delta_3$. We consider two cases:

(1) $|x(k) - x_s| \geq \alpha_3^{-1}(\delta_3)$. In this case, from Eq. (2.10b):

$$V(x(k+1)) \leq V(x(k)) - \alpha_3(\alpha_3^{-1}(\delta_3)) + \delta_3 = V(x(k)) < \alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_2 + \delta_3$$

(2) $|x(k) - x_s| < \alpha_3^{-1}(\delta_3)$. In this case, from Eq. (2.10a):

$$V(x(k)) \leq \alpha_2(|x(k) - x_s|) + \delta_2 < \alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_2$$

Substituting the above into Eq. (2.10b):

$$V(x(k+1)) < \alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_2 - \alpha_3(|x(k) - x_s|) + \delta_3 < \alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_2 + \delta_3$$

From the above results, the following holds for all $x(0) \in \mathbb{S}$:

$$V(x(k)) \leq \max\{V(x(0)) - k\epsilon,\ \alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_2 + \delta_3\}$$

Using Eq. (2.10a) and the above, the following can be obtained:

$$\alpha_1(|x(k)-x_s|) \leq \max\{\alpha_2(|x(0)-x_s|)+\delta_1+\delta_2-k\epsilon,\ \alpha_2(\alpha_3^{-1}(\delta_3))+\delta_1+\delta_2+\delta_3\} \quad (2.11)$$

Let us define $\hat{\beta}$:

$$\hat{\beta}(|x(0) - x_s|, k) = \frac{|x(0) - x_s|}{\alpha_3^{-1}(\delta_3)} \max\{\alpha_2(|x(0) - x_s|) + \delta_1 + \delta_2 - k\epsilon, 0\}$$

It can be checked that $\hat{\beta}$ belongs to class $\mathcal{KL}$ by definition, and that Eq. (2.11) still holds with the replacement:

$$\alpha_1(|x(k) - x_s|) \leq \max\{\hat{\beta}(|x(0) - x_s|, k),\ \alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_1 + \delta_2 + \delta_3\}$$

To verify the above, note that when $|x(0) - x_s| < \alpha_3^{-1}(\delta_3)$, the first term on the right-hand-side of Eq. (2.11) is smaller than the second term; And when $|x(0) - x_s| \geq \alpha_3^{-1}(\delta_3)$, $\hat{\beta}(|x(0) - x_s|, k) \geq \alpha_2(|x(0) - x_s|) + \delta_1 + \delta_2 - k\epsilon$.

Theorem 1 is thus proved with $\beta(|x(0) - x_s|, k) = \alpha_1^{-1}(\hat{\beta}(|x(0) - x_s|, k))$ and $r = \alpha_1^{-1}\alpha_2(\alpha_3^{-1}(\delta_3)) + \delta_1 + \delta_2 + \delta_3)$. ■

Theorem 1 characterizes the practical stability of the system on a forward-invariant set $\mathbb{S}$. Specifically, the system state is driven into an open ball $\mathcal{B}_r(x_s)$ in finite time and maintained in it thereafter. Let us use $\bar{V}^*_{N,N_h}(x(n))$ to denote the optimal objective function value of $\bar{\mathcal{P}}(n)$. In the following, we show that under the EMPC design, $\bar{V}^*_{N,N_h}(x(n))$ is a relaxed practical Lyapunov function on $\mathbb{X}_N$, with the corresponding scalars $\delta_1,\ \delta_2,\ \delta_3$ all being class $\mathcal{L}$ functions of $N_h$.

**Lemma 3** *If Assumption 2 holds, then there exists a class $\mathcal{KL}$ function $\beta_l$ such that:*

$$|l(x_h(k, x), h(x_h(k, x)) - l(x_s, u_s)| \leq \beta_l(|x - x_s|, k) \tag{2.12}$$

*for all $x \in \mathbb{X}_f$.*

**Proof.** Based on the continuity of $l(\cdot)$ and $h(\cdot)$ and the fact that $u_s = h(x_s)$, $|l(x, h(x)) - l(x_s, u_s)|$ is bounded and continuous at 0. Applying Lemma 1, there exists a class $\mathcal{K}_\infty$ function $\alpha$ such that

$$|l(x, h(x)) - l(x_s, u_s)| \leq \alpha(|x - x_s|)$$

for all $x \in \mathbb{X}_f$. Using $x_h(k, x)$ to replace $x$ and from Eq. (2.3), the following holds for all $x \in \mathbb{X}_f$:

$$|l(x_h(k, x), h(x_h(k, x))) - l(x_s, u_s)| \leq \alpha(|x_h(k, x) - x_s|) \leq \alpha(\beta_x(|x - x_s|, k))$$

By definition, $\alpha(\beta_x(|x - x_s|, k))$ belongs to class $\mathcal{KL}$ with respect to $|x - x_s|$ and $k$. Let $\beta_l(|x - x_s|, k) = \alpha(\beta_x(|x - x_s|, k))$, Eq. (2.12) is obtained. ■

**Lemma 4** *If Assumption 2 holds, then there exists a class $\mathcal{KL}$ function $\beta_\lambda$ such that:*

$$|\lambda(x_h(k,x)) - \lambda(x_s)| \leq \beta_\lambda(|x - x_s|, k) \tag{2.13}$$

*for all $x \in \mathbb{X}_f$.*

**Proof.** The proof is similar to the proof of Lemma 3 and is omitted here. ■

Before presenting the final result in Theorem 2, we still need to establish an upper bound for the auxiliary optimization problem in Lemma 5 below, which will then be used to bound $\bar{V}^*_{N,N_h}(x(n))$.

**Lemma 5** *If Assumptions 1-3 hold, then there exists a class $\mathcal{K}_\infty$ function $\alpha_v$ such that:*

$$\bar{V}^r_{N,N_h}(x(n)) \leq \alpha_v(|x(n) - x_s|) \tag{2.14}$$

*for all $x(n) \in \mathbb{X}_N$, where the function $\bar{V}^r_{N,N_h}(x(n))$ is the optimal objective function value of the following optimization problem $\mathcal{R}(n)$:*

$$\begin{aligned} \bar{V}^r_{N,N_h}(x(n)) = & \min_{u(0),u(1),\ldots,u(N-1)} \sum_{k=0}^{N-1} \bar{l}(\tilde{x}(k), u(k)) + \bar{c}(\tilde{x}(N), N_h) \\ & s.t.\ (2.4b) - (2.4f) \end{aligned} \tag{2.15}$$

**Proof.** In this proof, we show that Lemma 1 can be applied to $\bar{V}^r_{N,N_h}(x(n))$. It can be verified that $\bar{V}^r_{N,N_h}(x_s) = 0$. Also, $\bar{V}^r_{N,N_h}(x(n))$ is continuous at $x(n) = x_s$ because of Assumption 2.

We show next that $\bar{V}^r_{N,N_h}(x(n))$ is bounded on $\mathbb{X}_N \subseteq \mathbb{X}$. Since $\bar{l}(x,u)$ is bounded on $\mathbb{X} \times \mathbb{U}$, the first term in the objective function of Eq. (2.15) is bounded for a given control horizon $N$. Based on the definition of $\bar{l}$, the rotated terminal cost $\bar{c}(\tilde{x}(N), N_h)$ can be equivalently written as:

$$\bar{c}(\tilde{x}(N), N_h) = \sum_{k=0}^{N_h-1} l(x_h(k,\tilde{x}(N)), h(x_h(k,\tilde{x}(N)))) + \lambda(\tilde{x}(N)) - \lambda(x_h(N_h, \tilde{x}(N))$$

Based on Assumption 3, $\sum_{k=0}^{N_h-1} l(x_h(k,\tilde{x}(N)), h(x_h(k,\tilde{x}(N))))$ is bounded by $\alpha_h(\tilde{x}(N))$, the terms $\lambda(\tilde{x}(N))$ and $\lambda(x_h(N_h, \tilde{x}(N))$ are all bounded on $\mathbb{X}_f$. The rotated terminal cost is thus bounded on $\mathbb{X}_f$, which further implies that $\bar{V}^r_{N,N_h}(x(n))$ is bounded on $\mathbb{X}_N$. Applying Lemma 1, there exists a class $\mathcal{K}_\infty$ function $\alpha_v$ such that Eq. (2.14) holds. ■

Let us define the size of the terminal set:

$$d_{\max} := \max\{|x - x_s| : x \in \mathbb{X}_f\} \tag{2.16}$$

The practical stability of the EMPC design is summarized in the following theorem.

**Theorem 2** *Consider the system of Eq. (2.1) in closed-loop under the EMPC of Eq. (2.4), if Assumptions 1-3 hold, then there exist class $\mathcal{KL}$ functions $\beta_n$ and $\beta_r$ such that the following holds for all $x(0) \in \mathbb{X}_N$ and $n \geq 0$:*

$$|x(n) - x_s| \leq \max\{\beta_n(|x(0) - x_s|, n),\ \beta_r(d_{\max}, N_h)\}$$

**Proof.** We will first establish $\bar{V}^*_{N,N_h}(x(n))$ as a relaxed practical Lyapunov function on $\mathbb{X}_N$ such that:

$$\alpha_l(|x(n) - x_s|) - \beta_\lambda(d_{\max}, N_h) \leq \bar{V}^*_{N,N_h}(x(n)) \leq \alpha_v(|x(n) - x_s|) + \beta_\lambda(d_{\max}, N_h) \tag{2.17a}$$

$$\bar{V}^*_{N,N_h}(x(n+1)) \leq \bar{V}^*_{N,N_h}(x(n)) - \alpha_l(|x(n) - x_s|) + \beta_l(d_{max}, N_h) \tag{2.17b}$$

Once the above is established, applying Theorem 1, there exist $\beta_n$ and $r$ such that

$$|x(n) - x_s| \leq \max\{\beta_n(|x(0) - x_s|, n),\ r\}$$

where $r = \alpha_l^{-1}\Big(\alpha_V(\alpha_l^{-1}(\beta_l(d_{max}, N_h))) + 2\beta_\lambda(d_{\max}, N_h) + \beta_l(d_{max}, N_h)\Big)$ is the class $\mathcal{KL}$ function $\beta_r(d_{\max}, N_h)$.

**Left half of Eq. (2.17a)**. From the definition, $\bar{V}^*_{N,N_h}(x(n))$ can be written in terms of the corresponding optimal state and input trajectories $x^*(k|n)$ and $u^*(k|n)$ as follows:

$$\bar{V}^*_{N,N_h}(x(n)) = \sum_{k=0}^{N-1} \bar{l}(x^*(k|n), u^*(k|n)) + \bar{c}(x^*(N|n), N_h) + \lambda(x_h(N_h, x^*(N|n))) - \lambda(x_s) \tag{2.18}$$

Based on Eq. (2.7), we have:

$$\sum_{k=0}^{N-1} \bar{l}(x^*(k|n), u^*(k|n)) + \bar{c}(x^*(N|n), N_h) \geq \bar{l}(x(n), u^*(0|n)) \geq \alpha_l(|x(n) - x_s|) \tag{2.19}$$

Using Lemma 4, and taking into account the definition of $d_{max}$, the following holds:

$$|\lambda(x_h(N_h, x^*(N|n))) - \lambda(x_s)| \leq \beta_\lambda(|x^*(N|n) - x_s|, N_h) \leq \beta_\lambda(d_{max}, N_h) \tag{2.20}$$

Combining Eqs. (2.18), (2.19) and (2.20), the left half of Eq. (2.17a) is obtained.

**Right half of Eq. (2.17a)**. We denote the the optimal solution to the optimization problem $\mathcal{R}(n)$ of Eq. (2.15) as $u_r^*(k|n)$ and the corresponding optimal state trajectory as $x_r^*(k|n)$, $k = 0, ..., N-1$. Since $\bar{\mathcal{P}}(n)$ and $\mathcal{R}(n)$ have the same constraint set, $u_r^*(k|n)$ is also feasible for $\bar{\mathcal{P}}(n)$. The following inequality holds:

$$\bar{V}_{N,N_h}^*(x(n)) \leq \bar{V}_{N,N_h}(x(n), u_r^*) = \bar{V}_{N,N_h}^r(x(n)) + \lambda(x_h(N_h, x_r^*(N|n))) - \lambda(x_s) \quad (2.21)$$

Using Lemma 5 and following Eq. (2.20), the right half of Eq. (2.17a) is obtained.

**Eq. (2.17b)**. Since the input sequence:

$$\tilde{u}(n+1) = [u^*(1|n), ..., u^*(N-1|n), h(x^*(N|n))]$$

provides a feasible solution for $\bar{\mathcal{P}}(n+1)$, the following holds:

$$\begin{aligned} \bar{V}_{N,N_h}^*(x(n+1)) \leq\ & \bar{V}_{N,N_h}(x(n+1), \tilde{u}(n+1)) \\ =\ & \bar{V}_{N,N_h}^*(x(n)) - \bar{l}(x(n), u(n)) + \lambda(x_h(N_h+1, x^*(N|n))) \\ & -\lambda(x_h(N_h, x^*(N|n))) + \bar{l}(x_h(N_h, x^*(N|n)), h(x_h(N_h, x^*(N|n))) \\ =\ & \bar{V}_{N,N_h}^*(x(n)) - \bar{l}(x(n), u(n)) - l(x_s, u_s) \\ & +l(x_h(N_h, x^*(N|n)), h(x_h(N_h, x^*(N|n))) \end{aligned} \quad (2.22)$$

Using Lemma 3, the following holds:

$$\begin{aligned} & l(x_h(N_h, x^*(N|n)), h(x_h(N_h, x^*(N|n)))) - l(x_s, u_s) \\ \leq\ & \beta_l(|x^*(N|n) - x_s|, N_h) \leq \beta_l(d_{max}, N_h) \end{aligned} \quad (2.23)$$

Substituting Eqs. (2.23) and (2.7) into Eq. (2.22), Eq. (2.17b) is obtained.

From the above analysis, $\bar{V}_{N,N_h}^*(x(n))$ is a relaxed practical Lyapunov function of the closed-loop system on $\mathbb{X}_N$. Applying Theorem 1, Theorem 2 is proved. ■

Theorem 2 shows that the system state will be driven into an open ball $\mathcal{B}_r(x_s)$ in finite time. The radius of the ball is a class $\mathcal{KL}$ function $r = \beta_r(d_{\max}, N_h)$, which implies that for a sufficiently large but finite terminal horizon $N_h$, the system state will converge into a small region containing the optimal steady state. In addition, if the terminal region is small, then the need to use a large terminal horizon $N_h$ to ensure convergence of the system state is mitigated. In the extreme case where $\mathbb{X}_f$ shrinks to a point $\{x_s\}$ so that $d_{\max} = 0$, Theorem 2 reduces to $|x(n) - x_s| \leq \beta_n(|x(0) - x_s|, n)$. This is consistent with the asymptotic stability of EMPC with a point-wise terminal constraint [28].

### 2.2.2 Exponential shrinkage

In Theorem 2, the decreasing rate of $\beta_r(d_{\max}, N_h)$ with respect to $N_h$ is not characterized. In other words, under Assumptions 1-3, the magnitude of $N_h$ to ensure a reasonably small size of the practical stability region is "uncontrolled". In this subsection, we show that under a set of stronger conditions, the size of the ball $r = \beta_r(d_{\max}, N_h)$ shrinks exponentially as $N_h$ increases. Moreover, we show that for a special case which satisfies a further condition on the storage function, exponential stability can be achieved under the EMPC design with sufficiently large $N_h$.

**Assumption 4** *(Exponential stability of h) There exist positive constants $a \geq 1$ and $0 < s < 1$ such that the following holds for all $k \geq 0$ and $x \in \mathbb{X}_f$ :*

$$|x_h(k,x) - x_s| \leq as^k|x - x_s|$$

**Assumption 5** *(Polynomial bounds) There exist positive constants $c_1$, $c_2$, and $p$ such that the following holds for all $x \in \mathbb{X}_N$ and $u \in \mathbb{U}$:*

$$c_1|x - x_s|^p \leq \bar{l}(x,u) \leq c_2(|x - x_s|^p + |u - u_s|^p)$$

**Assumption 6** *(Lipschitz continuity) The functions $f$ and $l$ are Lipschitz continuous on $\mathbb{X} \times \mathbb{U}$, $h$ is Lipschitz continuous on $\mathbb{X}_f$, $\lambda$ is Lipschitz continuous on $\mathbb{X}$.*

**Lemma 6** *If Assumptions 1, 4, 5, 6 hold, then there exists a positive constant $a_f$ such that the following holds for all $x \in \mathbb{X}_f$ and $N_h \geq 1$:*

$$\bar{c}(x, N_h) \leq a_f|x - x_s|^p$$

**Proof.** Let $L_h$ denotes the Lipschitz constant of $h$ on $\mathbb{X}_f$. From Assumptions 5 and 6, the following holds for all $x \in \mathbb{X}_f$:

$$\bar{l}(x, h(x)) \leq c_2(1 + {L_h}^p)|x - x_s|^p$$

Thus,

$$\bar{c}(x, N_h) = \sum_{k=0}^{N_h-1} \bar{l}(x_h(k,x), h(x_h(k,x))) \leq \sum_{k=0}^{N_h-1} c_2(1 + {L_h}^p)|x_h(k,x) - x_s|^p$$

Using Assumption 4:

$$\begin{aligned}\bar{c}(x, N_h) &\leq \sum_{k=0}^{N_h-1} c_2(1 + {L_h}^p) a^p s^{pk} |x - x_s|^p \\ &\leq \sum_{k=0}^{\infty} c_2(1 + {L_h}^p) a^p s^{pk} |x - x_s|^p \\ &= c_2(1 + {L_h}^p) a^p \frac{1}{1 - s^p} |x - x_s|^p\end{aligned}$$

Lemma 6 is thus established with $a_f = c_2(1 + {L_h}^p) a^p \frac{1}{1 - s^p}$. ■

**Lemma 7** *If Assumptions 1, 4, 5, 6 hold, then there exists a positive constant* $a_v$ *such that the optimal objective function of the auxiliary optimization problem* $\mathcal{R}(n)$ *is bounded by*

$$\bar{V}^r_{N,N_h}(x(n)) \leq a_v |x(n) - x_s|^p$$

*for all* $x(n) \in \mathbb{X}_N$ *and* $N_h \geq 1$.

**Proof.** If $x(n) \in \mathbb{X}_f$, the input sequence generated by the auxiliary controller $h(x)$:

$$U_h = [h(x_h(0, x(n))), h(x_h(1, x(n)))..., h(x_h(N - 1, x(n)))] \tag{2.24}$$

provides a feasible solution to the optimization problem $\bar{\mathcal{R}}(n)$, with the corresponding objective function $\bar{c}(x, N + N_h)$. From Lemma 6, $\bar{V}^r_{N,N_h}(x(n)) \leq \bar{c}(x, N + N_h) \leq a_f |x(n) - x_s|^p$.

Next, we extend the polynomial upper-bound of $\bar{V}^r_{N,N_h}(x(n))$ on $\mathbb{X}_f$ to $\mathbb{X}_N$. We first note that $\bar{V}^r_{N,N_h}(x(n))$ is bounded on $\mathbb{X}_N$ because all the terms in the objective function of $\mathcal{R}(n)$ are bounded. Specifically, $\bar{c}(\tilde{x}(N), N_h)$ is bounded because of Lemma 6; $\sum_{k=0}^{N-1} \bar{l}(\tilde{x}(k), u(k))$ is bounded for a given finite prediction horizon $N$ due to Assumption 5. Let us define the maximum possible value of $\bar{V}^r_{N,N_h}(x(n))$ as $\bar{V}^r_{\max} =: \max\{\bar{V}^r_{N,N_h}(x(n)) : x(n) \in \mathbb{X}_N\}$. Let us further define $d_{\min} := \inf\{|x - x_s| : x \in \mathbb{X}_N \setminus \mathbb{X}_f\}$. Since the terminal region $\mathbb{X}_f$ is a compact set containing $x_s$ in its interior, $d_{\min} > 0$. From the definition of $\bar{V}^r_{\max}$ and $d_{\min}$, the following holds for all $x(n) \in \mathbb{X}_N \setminus \mathbb{X}_f$:

$$\bar{V}^r_{N,N_h}(x(n)) \leq \frac{\bar{V}^r_{\max}}{{d_{\min}}^p} |x(n) - x_s|^p$$

Let

$$a_v = \max\{a_f, \frac{\bar{V}^r_{\max}}{{d_{\min}}^p}\} \tag{2.25}$$

then $\bar{V}^r_{N,N_h}(x(n)) \leq a_v|x(n) - x_s|^p$ holds for all $x(n) \in \mathbb{X}_N$. ■

**Lemma 8** *If Assumptions 4, 6 hold, then there exists a positive constant $a_l$ such that the following holds for all $x \in \mathbb{X}_f$:*

$$|l(x_h(k, x), h(x_h(k, x)) - l(x_s, u_s)| \leq a_l s^k |x - x_s|$$

**Proof.** Let $L_l$ and $L_h$ be the Lipschitz constants of $l$ and $h$, the following can be obtained for all $x \in \mathbb{X}_f$:

$$|l(x_h(k, x), h(x_h(k, x))) - l(x_s, u_s)| \leq L_l(1 + L_h)|x_h(k, x) - x_s|$$

Using Assumption 4 yields

$$l(x_h(k, x), h(x_h(k, x))) - l(x_s, u_s) \leq L_l(1 + L_h)as^k|x - x_s|$$

Lemma 8 is thus established with $a_l = L_l(1 + L_h)a$. ■

**Lemma 9** *If Assumptions 1, 4, 6 hold, then there exists a positive constant $a_\lambda$ such that the following holds for all $x \in \mathbb{X}_f$:*

$$|\lambda(x_h(k, x)) - \lambda(x_s)| \leq a_\lambda s^k |x - x_s|$$

**Proof.** Let $L_\lambda$ be the Lipschitz constant of $\lambda(x)$ on $\mathbb{X}_f$, from Assumption 4

$$|\lambda(x_h(k, x)) - \lambda(x_s)| \leq L_\lambda|x_h(k, x) - x_s| \leq L_\lambda as^k|x - x_s|$$

Lemma 9 is thus satisfied with $a_\lambda = L_\lambda a$. ■

**Theorem 3** *Consider the system of Eq. (2.1) in closed-loop under the EMPC of Eq. (2.4). If Assumptions 1, 4, 5, 6 hold, then there exist class $\mathcal{KL}$ functions $\beta_p$ and $\beta^e_r$ such that the following holds for all $x(0) \in \mathbb{X}_N$ and $n \geq 0$:*

$$|x(n) - x_s| \leq \max\{\beta_n(|x(0) - x_s|, n),\ \beta^e_r(d_{\max}, N_h)\}$$

*where $\beta^e_r$ decreases to 0 exponentially fast as $N_h$ increases.*

**Proof.** Following the similar arguments as in the proof of Theorem 2, it can be shown that $\bar{V}^*_{N,N_h}(x(n))$ is a relaxed practical Lyapunov function on $\mathbb{X}_N$ satisfying:

$$\begin{aligned} & c_1|x(n)-x_s|^p - a_\lambda d_{\max} s^{N_h} \le \bar{V}^*_{N,N_h}(x(n)) \le a_v|x(n)-x_s|^p + a_\lambda d_{\max} s^{N_h} \\ & \bar{V}^*_{N,N_h}(x(n+1)) \le \bar{V}^*_{N,N_h}(x(n)) - c_1|x(n)-x_s|^p + a_l d_{\max} s^{N_h} \end{aligned} \tag{2.26}$$

Specifically, the above can be established following the similar arguments as used in Theorem 2 with $\alpha_l(|x-x_s|)$ replaced by $c_1|x-x_s|^p$ due to Assumption 5, $\beta_\lambda(d_{\max}, k)$ replaced by $a_\lambda d_{\max} s^k$ due to Lemma 9, $\alpha_v|x(n)-x_s|$ replaced by $a_v|x(n)-x_s|^p$ because of Lemma 7, and $\beta_l(d_{\max}, k)$ replaced by $a_l d_{\max} s^k$ because of Lemma 8. The details are omitted here for brevity. Applying Theorem 1, Eq. (2.26) implies the existence of $\beta_n$ and $\beta^e_r$ with

$$\beta^e_r(d_{\max}, N_h) = \Big(\frac{d_{\max}}{c_1}[\frac{a_v}{c_1}a_l + 2a_\lambda + a_l]\Big)^{\frac{1}{p}} \Big(s^{\frac{1}{p}}\Big)^{N_h}$$

which decreases to 0 exponentially fast as $N_h$ increases. ■

Note that in Theorem 2 or Theorem 3, the exponential decreasing of $\beta_n(|x(0) - x_s|, n)$ with respect to $n$ is trivial. This is because for any given $N_h$, there is always a finite time instant $n^*$ such that $\beta_n(|x(0)-x_s|, n) \le \beta^e_r(d_{\max}, N_h)$ for all $n \ge n^*$, which means that the effect of the term $\beta_n(|x(0)-x_s|, n)$ disappears in finite time. Therefore, we can always consider the decreasing rate of $\beta_n(|x(0)-x_s|, n)$ with respect to $n$ as exponential. In the following, we show that for a special case where the following condition on the storage function is satisfied, exponential stability instead of practical stability can be achieved. That is, $\beta_n(|x(0)-x_s|, n)$ decreases to 0 exponentially fast as $n$ increases and $\beta^e_r(d_{\max}, N_h) \equiv 0$.

**Assumption 7** *(Polynomial bound on $\lambda$) There exists a positive constant $c_\lambda$, such that the following holds*

$$|\lambda(x) - \lambda(x_s)| \le c_\lambda |x - x_s|^p$$

*for all $x \in \mathbb{X}_f$, where $p$ is defined in Assumption 5.*

**Theorem 4** *Consider the system of Eq. (2.1) in closed-loop under the EMPC of Eq. (2.4) with $x(0) \in \mathbb{X}_N$. if Assumptions 1, 4, 5, 6, 7 hold, then there exists a finite $N^*_h$ such that for all $N_h \ge N^*_h$, $x_s$ is exponentially stable.*

**Proof.** We prove that for some sufficiently large $N_h$ , $\bar{V}^*_{N,N_h}(x(n))$ is a Lyapunov function satisfying

$$c_1|x(n) - x_s|^p \leq \bar{V}^*_{N,N_h}(x(n)) \leq c_v|x(n) - x_s|^p \tag{2.27a}$$

$$\bar{V}^*_{N,N_h}(x(n+1)) - \bar{V}^*_{N,N_h}(x(n)) \leq -c_3|x(n) - x_s|^p \tag{2.27b}$$

where $c_1$ is defined in Assumption 5, $c_v$ and $c_3$ are positive constants.

**Left half of Eq. (2.27a)** . We show that the left half of Eq. (2.27a) holds for all $N_h \geq N_1$ where

$$N_1 = \min\{N_h \in \mathbb{Z}^+ : N_h \geq \frac{1}{p \ln s} \ln(\frac{c_1}{c_\lambda a^p})\} \tag{2.28}$$

where $\mathbb{Z}^+$ denotes the set of positive integers. Based on Assumption 4 and Assumption 7,

$$\lambda(x_h(N_h, x^*(N|n))) - \lambda(x_s) \geq -c_\lambda|x_h(N_h, x^*(N|n)) - x_s|^p \geq -c_\lambda a^p s^{N_h p}|x^*(N|n) - x_s|^p$$

Taking into Assumption 5, it can be verified that the following holds for all $N_h \geq N_1$:

$$\begin{aligned} & \bar{l}(x^*(N|n), h(x^*(N|n))) + \lambda(x_h(N_h, x^*(N|n))) - \lambda(x_s) \\ \geq \ & (c_1 - c_\lambda a^p s^{N_h p})|x^*(N|n) - x_s|^p \geq 0 \end{aligned} \tag{2.29}$$

Substituting the above into Eq. (2.18) and using Assumption 5, the following holds for all $x(n) \in \mathbb{X}_N$ and $N_h \geq N_1$:

$$\bar{V}^*_{N,N_h}(x(n)) \geq \bar{l}(x^*(0|n), u^*(0|n)) \geq c_1|x(n) - x_s|^p$$

**Right half of Eq. (2.27a)** The proof is similar to that of Lemma 7. We first show that there exists a positive constant $c_f$ such that $\bar{V}^*_{N,N_h}(x(n)) \leq c_f|x(n) - x_s|^p$ for all $x(n) \in \mathbb{X}_f$. Since the input sequence $U_h$ of Eq. (2.24) provides a feasible solution for $\bar{\mathcal{P}}(n)$, using Lemma 6 and Assumptions 4 and 7, the following holds for all $x(n) \in \mathbb{X}_f$:

$$\begin{aligned} \bar{V}^*_{N,N_h}(x(n)) \ & \leq \bar{V}_{N,N_h}(x(n), U_h) \\ & = \bar{c}(x(n), N + N_h) + \lambda(x_h(x(n), N + N_h)) - \lambda(x_s) \\ & \leq a_f|x - x_s|^p + c_\lambda a^p s^{(N+N_h)p}|x(n) - x_s|^p \\ & \leq a_f|x - x_s|^p + c_\lambda a^p|x(n) - x_s|^p \end{aligned}$$

Thus, we have found a $c_f = a_f + c_\lambda a^p$. Following the similar arguments as in the proof of Lemma 7, the polynomial upper bound can be extended from $\mathbb{X}_f$ to $\mathbb{X}_N$, which means that there exists a positive constant $c_v$ such that the right half of Eq. (2.27a) holds for all $x(n) \in \mathbb{X}_N$ and $N_h \geq 0$.

**Eq. (2.27b)** . Recall from Eq. (2.22) that

$$\begin{aligned}&\bar{V}^*_{N,N_h}(x(n+1)) - \bar{V}^*_{N,N_h}(x(n)) \leq \lambda(x_h(N_h+1, x^*(N|n))) - \lambda(x_h(N_h, x^*(N|n))) \\ &+ \bar{l}(x_h(N_h, x^*(N|n)), h(x_h(N_h, x^*(N|n))) - \bar{l}(x(n), u(n))\end{aligned} \tag{2.30}$$

For the first two terms on the right-hand-side of Eq. (2.30), the following can be obtained using Assumption 7 and Assumption 4:

$$\begin{aligned} & \lambda(x_h(N_h+1, x^*(N|n))) - \lambda(x_h(N_h, x^*(N|n))) \\ \leq\; & |\lambda(x_h(N_h+1, x^*(N|n))) - \lambda(x_s)| + |\lambda(x_h(N_h, x^*(N|n))) - \lambda(x_s)| \\ \leq\; & c_\lambda |x_h(N_h+1, x^*(N|n)) - x_s|^p + c_\lambda |x_h(N_h, x^*(N|n)) - x_s|^p \\ \leq\; & c_\lambda (1+s^p) a^p s^{N_h p} |x^*(N|n) - x_s|^p \end{aligned} \tag{2.31}$$

To replace $x^*(N|n)$ with $x(n)$ in the above, we note that Eq. (2.29) can be shifted one step forward if $N_h \geq N_1 + 1$:

$$\bar{l}(x_h(1, x^*(N|n)), h(x_h(1, x^*(N|n)))) + \lambda(x_h(N_h, x^*(N|n))) - \lambda(x_s) \geq 0$$

substituting the above into Eq. (2.18) and using Assumption 5 and the right half of Eq. (2.27a), the following holds:

$$c_1 |(x^*(N|n) - x_s|^p \leq \bar{l}(x^*(N|n), h(x^*(N|n))) \leq \bar{V}^*_{N,N_h}(x(n)) \leq c_V |x(n) - x_s|^p \tag{2.32}$$

Combining Eqs. (2.31) and (2.32) yields:

$$\lambda(x_h(N_h+1, x^*(N|n))) - \lambda(x_h(N_h, x^*(N|n))) \leq c_\lambda (1+s^p) a^p s^{N_h p} \frac{c_V}{c_1} |x(n) - x_s|^p \tag{2.33}$$

For the third term on the right-hand-side of Eq. (2.30), from Assumptions 4, 5, 6 and using Eq. (2.32):

$$\begin{aligned} \bar{l}(x_h(N_h, x^*(N|n)), h(x_h(N_h, x^*(N|n))) &\leq c_2(1 + {L_h}^p) |x_h(N_h, x^*(N|n)) - x_s|^p \\ &\leq c_2(1 + {L_h}^p) a^p s^{N_h p} |(x^*(N|n) - x_s)|^p \\ &\leq c_2(1 + {L_h}^p) a^p s^{N_h p} \frac{c_V}{c_1} |x(n) - x_s|^p \end{aligned} \tag{2.34}$$

For the last term on the right-hand-side of Eq. (2.30), from Assumption 5:

$$-\bar{l}(x(n), u(n)) \leq -c_1|x(n) - x_s|^p \tag{2.35}$$

Substituting Eqs. (2.33), (2.34) and (2.35) into Eq. (2.30):

$$\begin{aligned} & \bar{V}^*_{N,N_h}(x(n+1)) - \bar{V}^*_{N,N_h}(x(n)) \\ \leq \ & \Big(c_\lambda(1+s^p)a^p s^{N_h p}\frac{c_V}{c_1} + c_2(1+{L_h}^p)a^p s^{N_h p}\frac{c_V}{c_1} - c_1\Big)|x(n) - x_s|^p \end{aligned} \tag{2.36}$$

Let $c_3 = -c_\lambda(1+s^p)a^p s^{N_h p}\frac{c_V}{c_1} - c_2(1+{L_h}^p)a^p s^{N_h p}\frac{c_V}{c_1} + c_1$, it can be verified that $c_3 > 0$ for all $N_h > N_3$ (not necessarily positive) where

$$N_3 = -\frac{1}{p \ln s} \ln\big((c_\lambda(1+s^p) + c_2(1+L_h^p))\frac{c_V}{c_1^2}\big)$$

From the above analysis, let

$$N_h^* = \min\{N_h \in \mathbb{Z}^+ : N_h \geq N_1, N_h > N_3\} \tag{2.37}$$

Then $\bar{V}^*_{N,N_h}(x(n))$ satisfies Eq. (2.27) for all $N_h \geq N_h^*$ and $x(n) \in \mathbb{X}_N$, which implies that $x_s$ is exponentially stable on $\mathbb{X}_N$. ■

Note that Assumption 7 does not necessarily hold if the conditions for Theorem 3 are satisfied. If $0 < p \leq 1$ in Assumption 5, then Assumption 6 (Lipschitz continuity of $\lambda$) is a sufficient condition for Assumption 7. If $p > 1$, then Assumption 7 is stronger than Assumption 6. We will show in the numerical examples in Section 5 that Assumption 7 is satisfied for only one of the examples. Note also that Assumption 7 is satisfied automatically in conventional MPC where the stage cost $l(x, u)$ is positive-definite with respect to $(x_s, u_s)$. In this case, we can always define the rotated stage cost as the original stage cost $\bar{l}(x, u) = l(x, u)$, with the storage function $\lambda(x) = 0$. Thus Theorem 4 applies to conventional MPC with quadratic stage cost. It is consistent with the existing results for conventional MPC that a finite terminal horizon is sufficient to guarantee stability [37, 23].

## 2.3 Asymptotic and transient performance

In this section, we characterize upper bounds on the performance of the EMPC design. First, we consider the asymptotic performance of the EMPC for systems that are not necessarily dissipative. Then we show that for strictly dissipative systems satisfying different stability conditions as discussed in Section 3, stronger results on the transient performance can be obtained. In our analysis, the transient performance of the EMPC is compared against a benchmark system $h^*$ which is an augmented version of the auxiliary controller $h(x)$ with the same feasibility region as EMPC. It is shown that if the terminal horizon $N_h$ is sufficiently large, then transient performance of the EMPC is approximately upper-bounded by that of the benchmark system $h^*$. This implies that the EMPC either extends the feasibility region of the auxiliary controller or improves its performance. Moreover, if $h(x)$ is locally optimal on the terminal region $\mathbb{X}_f$ or around the optimal steady state $x_s$, then the local optimality is approximately preserved by the EMPC if a sufficiently large terminal horizon $N_h$ is used.

### 2.3.1 Asymptotic performance

A fundamental consideration when it comes to the design of EMPC is whether it achieves satisfactory asymptotic performance which is defined as follows:

$$\bar{J}_{asy} := \lim_{K\to\infty} \sup \frac{1}{K} \sum_{k=0}^{K-1} l(x(k), u(k))$$

Ideally, the asymptotic performance of the EMPC design should be no worse than the optimal steady state operation, i.e., $\bar{J}_{asy}^{EMPC} \leq l(x_s, u_s)$. In general, this goal may not be achieved by the proposed EMPC design with a finite $N_h$. The following Theorem shows that near-optimal asymptotic performance can be achieved if $N_h$ is sufficiently large.

**Theorem 5** *Consider the system of Eq. (2.1) in closed-loop under the EMPC of Eq. (2.4) with $x(0) \in \mathbb{X}_N$. If $f$, $l$, $h$ are continuous, then the asymptotic performance of the system is bounded by:*

$$\bar{J}_{asy}^{EMPC} \leq l(x_s, u_s) + \beta_l(d_{\max}, N_h) \tag{2.38}$$

*where $\beta_l$ is defined in Lemma 3 and $d_{\max}$ in Eq. (2.16). Moreover, if Assumptions 4, 6 hold, then there exists a class $\mathcal{KL}$ function $\beta_l^e(d_{\max}, N_h)$ which decreases exponentially to 0 with respect to $N_h$ such that:*

$$\bar{J}_{asy}^{EMPC} \leq l(x_s, u_s) + \beta_l^e(d_{\max}, N_h) \tag{2.39}$$

**Proof.** First we prove Eq. (2.38) under the continuity assumptions of the system. The proof follows the standard approach to construct a feasible solution for $\mathcal{P}(n+1)$ by discarding the first entry of the solution of $\mathcal{P}(n)$ and implementing the auxiliary controller $h(x)$ one more step at the end. Specifically, if $\mathcal{P}(n)$ is feasible, the input sequence $\tilde{U}(n+1) = [u^*(1|n), ..., u^*(N-1|n), h(x^*(N|n))]$ provides a feasible solution for $\mathcal{P}(n+1)$. Based on the principal of optimality $V^*_{N,N_h}(x(n+1)) \leq V_{N,N_h}(x(n+1), \tilde{U}(n+1))$ and the construction of $\tilde{U}(n+1)$, the following can be obtained:

$$l(x(n), u(n)) \leq V^*_{N,N_h}(x(n)) - V^*_{N,N_h}(x(n+1)) + l(x_h(N_h, x^*(N|n)), h(x_h(N_h, x^*(N|n)))$$

Using Lemma 3 and taking into account the definition of $d_{\max}$, the following can be obtained:

$$l(x(n), u(n)) \leq V^*_{N,N_h}(x(n)) - V^*_{N,N_h}(x(n+1)) + l(x_s, u_s) + \beta_l(d_{\max}, N_h) \quad (2.40)$$

Summing up both sides of the above inequality from $n = 0$ to $K - 1$:

$$\sum_{n=0}^{K-1} l(x(n), u(n)) \leq V^*_{N,N_h}(x(0)) - V^*_{N,N_h}(x(K)) + K(l(x_s, u_s) + \beta_l(d_{\max}, N_h)) \quad (2.41)$$

Dividing both sides by $K$ with $K$ approaching infinity:

$$\begin{aligned} &\lim_{K\to\infty} \sup \frac{1}{K} \sum_{n=0}^{K-1} l(x(n), u(n)) \\ \leq \; &\lim_{K\to\infty} \sup \frac{1}{K} \big(V^*_{N,N_h}(x(0)) - V^*_{N,N_h}(x(K))\big) + l(x_s, u_s) + \beta_l(d_{\max}, N_h) \end{aligned}$$

The left-hand-side of the above is $\bar{J}^{EMPC}_{asy}$.Since $V^*_{N,N_h}(x(n))$ is bounded on $\mathbb{X}_N$ for any finite $N$ and $N_h$, we have $\lim_{K\to\infty} \sup \frac{1}{K}\big(V^*_{N,N_h}(x(0)) - V^*_{N,N_h}(x(K))\big) = 0$. This proves Eq. (2.38).

Eq. (2.39) under Assumptions 4 and 6 can be proved following the similar arguments with $\beta_l(d_{\max}, N_h)$ in Eq. (2.40) replaced with $a_l s^k d_{\max}$ (because of Lemma 8) which is a class $\mathcal{KL}$ function $\beta^e_l(d_{\max}, N_h)$ which decreases exponentially to 0 with respect to $N_h$. ■

Note that the result of Theorem 5 is established for general systems that are not necessarily dissipative. In other words, the closed-loop system state does not necessarily converge to the optimal steady state. For strictly dissipative systems, the near optimal asymptotic performance of Theorem 5 will be a direct result of practical stability. We show in the following that stronger results on the transient performance can be obtained for strictly dissipative systems.

### 2.3.2 Transient performance

We employ the solution to $\bar{\mathcal{P}}(0)$ with the objective function $\bar{V}^*_{N,\infty}(x(0))$ as the benchmark system which is denoted as $h^*$. Speficically, $\bar{V}^*_{N,\infty}(x(0)) = \sum_{k=0}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k))$ with $x_{h^*}(0) = x(0)$. The benchmark system $h^*$ can be viewed as an improved version of the auxiliary controller $h$ with enlarged feasiblity region or improved performance. The transient performance of the system over K time steps is denoted as

$$J_K := \sum_{k=0}^{K-1} \Big( l(x(k), u(k)) - l(x_s, u_s) \Big) \tag{2.42}$$

In the following, we compare $J_K^{EMPC}$ with $J_K^{h^*}$ given the same initial state $x(0)$.

**Remark 2** *Note that $\bar{V}^*_{N,N_h}(\cdot)$ is bounded and well-defined for $N_h \to \infty$ if the conditions in Section 3 hold. The boundedness of $\bar{V}_{N,N_h}(\cdot)$ can be seen from Eq. (2.17a) or Eq. (2.26). Essentially, $\bar{V}_{N,N_h}(\cdot)$ is bounded from below because of the strict dissipativity Assumption 1, and is bounded from above because of Assumption 3. In the case of exponential shrinkage, Assumptions 4 and 5 are sufficient conditions for Assumption 3. Our analysis based on the rotated problem $\bar{\mathcal{P}}(n)$ remains valid even though the original objective function $V_{N,N_h}(\cdot)$ may grow unbounded as $N_h \to \infty$. In this case, the result of Lemma 2 can be recovered by subtracting $l(x_s, u_s)$ from the original stage cost $l(x, u)$.*

**Lemma 10** *If the conditions of Theorem 2 are satisfied, then there exists a class $\mathcal{KL}$ function $\beta_{x^*}$ such that:*

$$|x_{h^*}(k) - x_s| \leq \beta_{x^*}(|x(0) - x_s|, k) \tag{2.43}$$

*Moreover, if the conditions of Theorem 3 are satisfied, then the following holds:*

$$|x_{h^*}(k) - x_s| \leq a s^{k-N} (\frac{a_v}{c_1})^{\frac{1}{p}} |x(0) - x_s| \tag{2.44}$$

*where $a$ and $s$ are defined in Assumption 4, $c_1$ and $p$ are defined in Assumption 5, $a_v$ is defined in Lemma 7.*

**Proof.** First, we construct $\beta_{x^*}$ of Eq. (2.43) under the conditions of Theorem 2. From Eq. (2.3), the following holds for all $k \geq N$:

$$|x_{h^*}(k) - x_s| \leq \beta_x(|x_{h^*}(N) - x_s|, k - N) \tag{2.45}$$

Using Lemma 5 and taking into account that $\bar{V}^r_{N,\infty}(x(n)) = \bar{V}^*_{N,\infty}(x(n))$ (because $\lim_{N_h \to \infty} \lambda(x_h(N_h, \tilde{x}(N)) = \lambda(x_s))$, $\bar{V}^*_{N,\infty}(x(0))$ is also upper-bounded by $\bar{V}^*_{N,\infty}(x(n)) \leq \alpha_v(|x(0) - x_s|)$. Combining this with Assumption 1, the following can be obtained

$$\alpha_l(|x_{h^*}(N) - x_s|) \leq \bar{l}(x_{h^*}(N), u_{h^*}(N)) \leq \bar{V}^*_{N,\infty}(x(n)) \leq \alpha_v(|x(0) - x_s|) \qquad (2.46)$$

Substituting the above into Eq. (2.45), the following holds for all $k \geq N$:

$$|x_{h^*}(k) - x_s| \leq \beta_x(\alpha_l^{-1}(\alpha_v(|x(0) - x_s|)),\ k - N)$$

The right-hand-side of the above is a class $\mathcal{KL}$ function with respect to $|x(0) - x_s|$ and $k$ for $k \geq N$. We can extend the above to account for $k = 0, ..., N-1$ by noting that $|x_{h^*}(k) - x_s| \leq \alpha_l^{-1}(\alpha_v(|x(0) - x_s|))$ for all $k = 0, ..., N-1$. $\beta_{x^*}$ can be constructed as follows

$$\beta_{x^*}(|x(0) - x_s|, k) = \begin{cases} \max\{\alpha_l^{-1}(\alpha_v(|x(0) - x_s|)),\quad \beta_x(\alpha_l^{-1}(\alpha_v(|x(0) - x_s|)), 0)\} \\ k = 0, ..., N-1. \\ \beta_x(\alpha_l^{-1}(\alpha_v(|x(0) - x_s|)),\ k - N), \quad k \geq N \end{cases}$$

It can be verified that $\beta_{x^*}$ defined above is a class $\mathcal{KL}$ function that satisfies Eq. (2.43).

Under the conditions of Theorem 3 , Eq. (2.44) can be proved following the similar procedure. We provide a sketch of the proof. From Assumption 4, the following holds for all $k \geq N$

$$|x_{h^*}(k) - x_s| \leq a s^{k-N} |x_{h^*}(N) - x_s| \qquad (2.47)$$

Based on Assumption 5 and Lemma 7, Eq. (2.46) can be further written as

$$c_1 |x_{h^*}(N) - x_s|^p \leq \bar{l}(x_{h^*}(N), u_{h^*}(N)) \leq \bar{V}^*_{N,\infty}(x(n)) \leq a_v |x(0) - x_s|^p \qquad (2.48)$$

Combining Eqs. (2.47) and (2.48) it can be shown that Eq. (2.44) holds for all $k \geq N$. It can be also verified that Eq. (2.44) holds for all $0 \leq k < N$ taking into account that $a s^{k-N} > 1$ and that Eq. (2.48) holds if $x_{h^*}(N)$ is replaced with $x_{h^*}(k)$ for all $0 \leq k < N$. ■

**Lemma 11** *If the conditions of Theorem 2 are satisfied, then the following holds:*

$$\bar{V}^*_{N,N_h}(x(0)) \leq \bar{V}^*_{N,\infty}(x(0)) + \beta_\lambda(d_{\max}, N_h) \qquad (2.49)$$

*where $\beta_\lambda$ is defined in Lemma 4. Moreover, if the conditions of Theorem 3 are satisfied, then the following holds:*

$$\bar{V}^*_{N,N_h}(x(0)) \leq \bar{V}^*_{N,\infty}(x(0)) + a_\lambda d_{\max} s^{N_h} \tag{2.50}$$

*where $a_\lambda$ is defined in Lemma 9.*

**Proof.** First we prove Eq. (2.49) under the conditions of Theorem 2. Recall from Eq. (2.21):

$$\bar{V}^*_{N,N_h}(x(0)) \leq \bar{V}^r_{N,N_h}(x(0)) + \lambda(x_h(N_h, x^*_r(N|0))) - \lambda(x_s) \tag{2.51}$$

Due to the positive-finiteness of the rotated cost,

$$\bar{V}^r_{N,N_h}(x(0)) \leq \bar{V}^r_{N,\infty}(x(0)) = \bar{V}^*_{N,\infty}(x(0)) \tag{2.52}$$

Using Lemma 4,

$$\lambda(x_h(N_h, x^*_r(N|0))) - \lambda(x_s) \leq \beta_\lambda(|x_h(N_h, x^*_r(N|0))) - x_s|, k) \leq \beta_\lambda(d_{\max}, N_h) \tag{2.53}$$

Substituting Eqs. (2.52) and (2.53) into Eq. (2.51), Eq. (2.49) is obtained.

Following the similar arguments, Eq. (2.50) under the conditions of Theorem 3 can be obtained with Eq. (2.53) further written as:

$$\lambda(x_h(N_h, x^*_r(N|0))) - \lambda(x_s) \leq a_\lambda |x^*_r(N|0) - x_s| s^{N_h} \leq a_\lambda d_{\max} s^{N_h} \tag{2.54}$$

because of Lemma 9. The details are omitted here for brevity. ■

**Lemma 12** *If Assumptions 1-3 hold, then there exists a class $\mathcal{KL}$ function $\beta_{h^*}$ such that the following holds for all $x(0) \in \mathbb{X}_N$:*

$$\sum_{k=K}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k)) \leq \beta_{h^*}(|x(0) - x_s|, K) \tag{2.55}$$

*Moreover, if Assumptions 1, 4, 5, 6 hold, then there exists a class $\mathcal{KL}$ function $\beta^e_{h^*}(|x(0) - x_s|, K)$ which decreases to 0 exponentially fast as $K$ increases such that the following holds for all $x(0) \in \mathbb{X}_N$:*

$$\sum_{k=K}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k)) \leq \beta^e_{h^*}(|x(0) - x_s|, K) \tag{2.56}$$

**Proof.** First we prove Eq. (2.55) under Assumptions 1-3. To construct $\beta_{h^*}$, we first note that there exists a class $\mathcal{K}_\infty$ function $\alpha_{h^*}$ such that

$$\sum_{k=K}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k)) \leq \alpha_{h^*}(|x_{h^*}(K) - x_s|) \tag{2.57}$$

The above can be shown based on Lemma 1 by treating $\sum_{k=K}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k))$ as a function of $x_{h^*}(K)$. Specifically, it can be verified that (i): if $x_{h^*}(K) = 0$, then $\sum_{k=K}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k)) = 0$ because of the positive-definiteness of the rotated cost; (ii): $\sum_{k=K}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k))$ is continuous at $x_{h^*}(K) = 0$ because of the continuity Assumption 2; and (iii): $\sum_{k=K}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k))$ is bounded because $\bar{V}^*_{N,\infty}(x) = \sum_{k=0}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k))$ is bounded. Using Eq. (2.43) of Lemma 10 to replace $x_{h^*}(K)$ on the right-hand-side of Eq. (2.57).

$$\sum_{k=K}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k)) \leq \alpha_{h^*}(\beta_{x^*}(|x(0) - x_s|, K))$$

Eq. (2.55) is thus established with $\beta_{h^*} = \alpha_{h^*}(\beta_{x^*}(|x(0) - x_s|, K))$.

Next we prove Eq. (2.56) under Assumptions 1, 4, 5, 6. It can be verified that the following holds for all $x(0) \in \mathbb{X}_N$ and $K \geq 0$:

$$\sum_{k=K}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k)) \leq a_v |x_{h^*}(K) - x_s|^p \tag{2.58}$$

where the positive constant $a_v$ is defined in Eq. (2.25). The proof is similar to that of Lemma 7 and is omitted here for brevity. Using Eq. (2.44) of Lemma 10 to replace $x_{h^*}(K)$ on the right-hand-side of Eq. (2.58):

$$\sum_{k=K}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k)) \leq \frac{{a_v}^2 a^p}{c_1} s^{(k-N)p} |x(0) - x_s|^p$$

It can be verified that the right-hand-side of the above is a class $\mathcal{KL}$ function $\beta^e_{h^*}(|x(0) - x_s|, K)$ which decreases to 0 exponentially fast as $K$ increases. ■

Now we are ready to state the main results of this subsection:

**Theorem 6** *If the conditions of Theorem 2 are satisfied, then there exist class $\mathcal{KL}$ functions $\beta_K$, $\beta_{N_h}$ and $\beta_l$ such that the following holds:*

$$J_K^{EMPC} \leq J_K^{h^*} + \beta_K(|x(0) - x_s|, K) + \beta_{N_h}(d_{\max}, N_h) + K\beta_l(d_{\max}, N_h) \tag{2.59}$$

*Moreover, if the conditions of Theorem 3 are satisfied, then the following holds*

$$J_K^{EMPC} \le J_K^{h^*} + \beta_K(|x(0) - x_s|, K) + \beta_{N_h}^e(d_{\max}, N_h) + K\beta_l^e(d_{\max}, N_h) \tag{2.60}$$

*where* $\beta_{N_h}^e$ *and* $\beta_l^e$ *decreases to 0 exponentially fast as* $N_h$ *increases.*

**Proof.** First we prove Eq. (2.59) under the conditions of Theorem 2. Recall from Eq. (2.41)

$$J_K^{EMPC} = \sum_{n=0}^{K-1} l(x(n), u(n)) \le V_{N,N_h}^*(x(0)) - V_{N,N_h}^*(x(K)) + K\beta_l(d_{\max}, N_h)$$

Using Eq. (2.9) to replace the original objective function with the rotated objective function:

$$J_K^{EMPC} \le \bar{V}_{N,N_h}^*(x(0)) - \bar{V}_{N,N_h}^*(x(K)) - \lambda(x(0)) + \lambda(x(K)) + K\beta_l(d_{\max}, N_h) \tag{2.61}$$

Based on Lemma 11,

$$\begin{aligned} \bar{V}_{N,N_h}^*(x(0)) &\le \bar{V}_{N,\infty}^*(x(0)) + \beta_\lambda(d_{\max}, N_h) \\ &= \sum_{k=0}^{K-1} \bar{l}(x_{h^*}(k), u_{h^*}(k)) + \sum_{k=K}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k)) + \beta_\lambda(d_{\max}, N_h) \end{aligned} \tag{2.62}$$

For the first term on the right-hand-side of Eq. (2.62):

$$\sum_{k=0}^{K-1} \bar{l}(x_{h^*}(k), u_{h^*}(k)) = J_K^h + \lambda(x(0)) - \lambda(x_{h^*}(K)) \tag{2.63}$$

Substituting Eqs. (2.63) and (2.55) of Lemma 12 into Eq. (2.62):

$$V_{N,N_h}^*(x(0)) \le J_K^h + \lambda(x(0)) - \lambda(x_{h^*}(K)) + \beta_{h^*}(|x(0) - x_s|, K) + \beta_\lambda(d_{\max}, N_h) \tag{2.64}$$

From Lemma 4 and the positive-definiteness of the rotated cost, the following can be obtained

$$\bar{V}_{N,N_h}^*(x(K)) \ge -\beta_\lambda(d_{\max}, N_h) \tag{2.65}$$

Substituting Eqs. (2.64) and (2.65) into Eq. (2.61):

$$J_K^{EMPC} \le J_K^h + \lambda(x(K)) - \lambda(x_{h^*}(K)) + \beta_{h^*}(|x(0) - x_s|, K) + 2\beta_\lambda(d_{\max}, N_h) + K\beta_l(d_{\max}, N_h) \tag{2.66}$$

Based on Lemma 1, there exists a class $\mathcal{K}_\infty$ function $\alpha_\lambda$ such that

$$|\lambda(x) - \lambda(x_s)| \leq \alpha_\lambda(|x - x_s|) \tag{2.67}$$

for all $x \in \mathbb{X}_N$. From Theorem 2, the following holds:

$$\lambda(x(K)) - \lambda(x_s) \leq \alpha_\lambda(|x(K) - x_s|) \leq \alpha_\lambda(\max\{\beta_n(|x(0) - x_s|, K),\ \beta_r(d_{\max}, N_h)\})$$

which further yields:

$$\lambda(x(K)) - \lambda(x_s) \leq \alpha_\lambda(\beta_n(|x(0) - x_s|, K)) + \alpha_\lambda(\beta_r(d_{\max}, N_h)) \tag{2.68}$$

Using Lemma 10,

$$\lambda(x_s) - \lambda(x_{h^*}(K)) \leq \alpha_\lambda(|x_{h^*}(K) - x_s|) \leq \alpha_\lambda(\beta_{x^*}(|x(0) - x_s|, K)) \tag{2.69}$$

Substituting Eqs. (2.68) and (2.69) into Eq. (2.66) yields:

$$\begin{aligned} J_K^{EMPC} \leq\ & J_K^{h^*} + \alpha_\lambda(\beta_n(|x(0) - x_s|, K)) + \alpha_\lambda(\beta_r(d_{\max}, N_h)) + \alpha_\lambda(\beta_{x^*}(|x(0) - x_s|, K)) \\ & + \beta_{h^*}(|x(0) - x_s|, K) + 2\beta_\lambda(d_{\max}, N_h) + K\beta_l(d_{\max}, N_h) \end{aligned}$$

Eq. (2.59) is thus proved with

$$\beta_K(|x(0)-x_s|, K) = \alpha_\lambda(\beta_p(|x(0)-x_s|, K)) + \alpha_\lambda(\beta_{x^*}(|x(0)-x_s|, K)) + \beta_{h^*}(|x(0)-x_s|, K)$$

$$\beta_{N_h}(d_{\max}, N_h) = \alpha_\lambda(\beta_r(d_{\max}, N_h)) + 2\beta_\lambda(d_{\max}, N_h)$$

Under the conditions of Theorem 3, Eq. (2.60) can be proved following the similar arguments with $\alpha_\lambda(|x - x_s|)$ in Eq. (2.67) replaced with $L_\lambda|x - x_s|$ where $L_\lambda$ is the Lipschitz constant of $\lambda$ on $\mathbb{X}_N$; $\beta_r$ replaced with $\beta_r^e$ as in Theorem 3; $\beta_\lambda(d_{\max}, N_h)$ replaced with $a_\lambda d_{\max} s^{N_h}$ because of Lemma 11. The resulting $\beta_{N_h}^e(d_{\max}, N_h)$ is $\beta_{N_h}^e(d_{\max}, N_h) = L_\lambda \beta_r^e(d_{\max}, N_h) + 2a_\lambda d_{\max} s^{N_h}$. $\beta_l$ can be replaced with $\beta_l^e$ because of Theorem 5. ■

**Remark 3** *Comparing the results of Theorem 5 and Theorem 6, it can be seen that the results of Theorem 6 are stronger. Theorem 5 can be recovered from Theorem 6 by dividing $K$ on both sides of Eq. (2.59) or Eq. (2.60) with $K \to \infty$ (the asymptotic performance of $h^*$ is $l(x_s, u_s)$). Note that the terms $\beta_K(|x(0) - x_s|, K)$ and $\beta_{N_h}(d_{\max}, N_h)$ or $\beta_{N_h}^e(d_{\max}, N_h)$ do not affect the asymptotic performance. The vanishing of these terms are due to the practical stability of the EMPC design as well as the stability of* *the auxiliary controller h. The above results also call for the auxiliary controller to be not only stabilizing but also to provide satisfying transient performance in the terminal region. For strictly dissipative systems, stability implies asymptotic performance, which is a weaker index than transient performance.*

For the special case when exponential stability as in Theorem 4 is achieved, we have the following results on the transient performance of the EMPC design:

**Theorem 7** *If the conditions of Theorem 4 are satisfied, then there exists a finite $N_h^*$ such that the following holds for all $N_h \geq N_h^*$:*

$$J_K^{EMPC} \leq J_K^{h^*} + \beta_K^e(|x(0) - x_s|, n) + \beta_{N_h}^e(|x(0) - x_s|, N_h)$$

*where $\beta_K^e(|x(0) - x_s|, n)$ decreases exponentially to 0 as $n$ increases, $\beta_{N_h}^e(|x(0) - x_s|, N_h)$ decreases exponentially to 0 as $N_h$ increases.*

**Proof.** We prove Theorem 7 with $N_h^*$ defined in Eq. (2.37). From Eq. (2.36) and Assumption 5, the following can be obtained:

$$\begin{aligned} & \bar{V}_{N,N_h}^*(x(n+1)) - \bar{V}_{N,N_h}^*(x(n)) \\ \leq \; & \Big(c_\lambda(1+s^p)a^p s^{N_h p}\frac{c_V}{{c_1}^2} + c_2(1+{L_h}^p)a^p s^{N_h p}\frac{c_V}{{c_1}^2} - 1\Big)\bar{l}(x(n), u(n)) \end{aligned}$$

Let $L_e(N_h) = c_\lambda(1+s^p)a^p s^{N_h p}\frac{c_V}{{c_1}^2} + c_2(1+{L_h}^p)a^p s^{N_h p}\frac{c_V}{{c_1}^2}$. It can be verified that $L_e$ is a class $\mathcal{L}$ function of $N_h$ that decreases to 0 exponentially fast with $L_e(N_h^*) < 1$. Summing the above from $n = 0$ to $K - 1$ and taking into account that $\bar{V}_{N,N_h}^*(x(n))$ is nonnegative because Theorem 4 is satisfied, the following can be obtained:

$$(1 - L_e(N_h))\sum_{n=0}^{K-1}\bar{l}(x(n), u(n)) \leq \bar{V}_{N,N_h}^*(x(0)) - \bar{V}_{N,N_h}^*(x(K)) \leq \bar{V}_{N,N_h}^*(x(0)) \quad (2.70)$$

From Theorem 4 and taking into account that $N_h \geq N_h^*$ and $L_e(N_h^*) < 1$, the following holds from Eq. (2.70):

$$\sum_{n=0}^{K-1}\bar{l}(x(n), u(n)) \leq \frac{c_v|x(0) - x_s|^p}{1 - L_e(N_h^*)}$$

Substituting the above back into Eq. (2.70), the following can be obtained:

$$\sum_{n=0}^{K-1}\bar{l}(x(n), u(n)) \leq \bar{V}_{N,N_h}^*(x(0)) + L_e(N_h)\frac{c_v|x(0) - x_s|^p}{1 - L_e(N_h^*)} \quad (2.71)$$

From Lemma 7 and Assumption 5, and taking into account the positive-definiteness of $\bar{l}$, the following holds:

$$c_1|x_r^*(N|0) - x_s|^p \leq \bar{l}(x_r^*(N|0), u_r^*(N|0)) \leq \bar{V}_{N,N_h}^r(x(0)) \leq a_v|x(0) - x_s|^p$$

Sustituting the above into Eq. (2.54), Eq. (2.50) of Lemma 11 can be futher written as

$$\bar{V}_{N,N_h}^*(x(0)) \leq \bar{V}_{N,\infty}^*(x(0)) + a_\lambda s^{N_h} \frac{a_v}{c_1}|x(0) - x_s|^p$$

Sustituting the above into Eq. (2.71):

$$\sum_{n=0}^{K-1} \bar{l}(x(n), u(n)) \leq \bar{V}_{N,\infty}^*(x(0)) + a_\lambda s^{N_h} \frac{a_v}{c_1}|x(0) - x_s|^p + L_e(N_h)\frac{c_v|x(0) - x_s|^p}{1 - L_e(N_h^*)} \quad (2.72)$$

$$\bar{V}_{N,\infty}^*(x(0)) = \sum_{k=0}^{K-1} \bar{l}(x_{h^*}(k), u_{h^*}(k)) + \sum_{k=K}^{\infty} \bar{l}(x_{h^*}(k), u_{h^*}(k))$$

Using Eq. (2.63) and Eq. (2.56) of Lemma 12 to replace the two parts on the right-hand-side of the above

$$\bar{V}_{N,\infty}^*(x(0)) \leq J_K^h + \lambda(x(0)) - \lambda(x_{h^*}(K)) + \beta_{h^*}^e(|x(0) - x_s|, K) \quad (2.73)$$

The left-hand-side of Eq. (2.71) can be written as:

$$\sum_{n=0}^{K-1} \bar{l}(x(n), u(n)) = J_K^{EMPC} + \lambda(x(0)) - \lambda(x(K)) \quad (2.74)$$

Substituting Eqs (2.73) and (2.74) into Eq. (2.71):

$$\begin{aligned} J_K^{EMPC} \leq\ & J_K^h + \lambda(x(K)) - \lambda(x_{h^*}(K)) + \beta_{h^*}^e(|x(0) - x_s|, K) \\ & + a_\lambda s^{N_h} \frac{a_v}{c_1}|x(0) - x_s|^p + L_e(N_h)\frac{c_v|x(0) - x_s|^p}{1 - L_e(N_h^*)} \end{aligned} \quad (2.75)$$

From Theorem 4, $x_s$ is exponentially stable under the proposed EMPC which implies that there exists a class $\mathcal{KL}$ function $\beta_n^e(|x(0) - x_s|, n)$ which decreases to 0 exponentially as $n$ increases such that:

$$|x(n) - x_s| \leq \beta_n^e(|x(0) - x_s|, n)$$

Using the above and Eq. (2.44) of Lemma 10, and taking into account the Lipschitz continuity of $\lambda$, the following holds:

$$\begin{aligned} \lambda(x(K)) - \lambda(x_{h^*}(K)) &= \lambda(x(K)) - \lambda(x_s) + \lambda(x_s) - \lambda(x_{h^*}(K)) \\ &\leq L_\lambda \beta_n^e(|x(0) - x_s|, K) + L_\lambda \beta_{x^*}^e(|x(0) - x_s|, K) \end{aligned} \quad (2.76)$$

Substituting Eq. (2.76) into Eq. (2.75):

$$\begin{aligned} J_K^{EMPC} \leq \quad & J_K^h + L_\lambda \beta_n^e(|x(0) - x_s|, K) + L_\lambda \beta_{x^*}^e(|x(0) - x_s|, K) + \beta_{h^*}^e(|x(0) - x_s|, K) \\ & + a_\lambda s^{N_h} \frac{a_v}{c_1} |x(0) - x_s|^p + L_e(N_h) \frac{c_v |x(0) - x_s|^p}{1 - L_e(N_h^*)} \end{aligned} \tag{2.77}$$

Taking

$$\beta_K^e(|x(0) - x_s|, n) = L_\lambda \beta_n^e(|x(0) - x_s|, K) + L_\lambda \beta_{x^*}^e(|x(0) - x_s|, K) + \beta_{h^*}^e(|x(0) - x_s|, K)$$

and

$$\beta_{N_h}^e(|x(0) - x_s|, N_h) = a_\lambda s^{N_h} \frac{a_v}{c_1} |x(0) - x_s|^p + L_e(N_h) \frac{c_v |x(0) - x_s|^p}{1 - L_e(N_h^*)}$$

Theorem 7 is proved. ■

**Remark 4** *The results of Theorems 6 and 7 indicate that for sufficiently large $N_h$, the transient or asymptotic performance of the EMPC design is approximately upper-bounded by the performance of the auxiliary controller $h$ or its improved version $h^*$. This implies that the EMPC either extends the feasibility region of the auxiliary controller or improves its performance. Moreover, if $h(x)$ is (near) optimal on the terminal region $\mathbb{X}_f$ or around the optimal steady state $x_s$, then the local optimality is approximately preserved by the EMPC if a sufficiently large terminal horizon $N_h$ is used. Since our analysis is carried out for an arbitrary control horizon $N$, these results also explain the computational efficiency of the EMPC design.*

## 2.4 Numerical examples

**Example 5.1** This example is a linearized continuously stirred tank reactor model taken from [10, 28]. Consider the control system:

$$x(k+1) = \begin{pmatrix} 0.8353 & 0 \\ 0.1065 & 0.9418 \end{pmatrix} x(k) + \begin{pmatrix} 0.00457 \\ -0.00457 \end{pmatrix} u(k) + \begin{pmatrix} 0.5559 \\ 0.5033 \end{pmatrix} \tag{2.78}$$

with the stage cost $l(x, u) = |x|^2 + 0.05u^2$. The state and input constraints are: $\mathbb{X} = [-100, 100]^2$ and $\mathbb{U} = [-10, 10]$. The optimal steady state that solves the steady-state optimization problem of Eq. (2.2) is $x_s \approx [3.5463, 14.6531]^T$ and $u_s \approx 6.1637$. We choose the auxiliary controller to be the open-loop optimal steady-state input

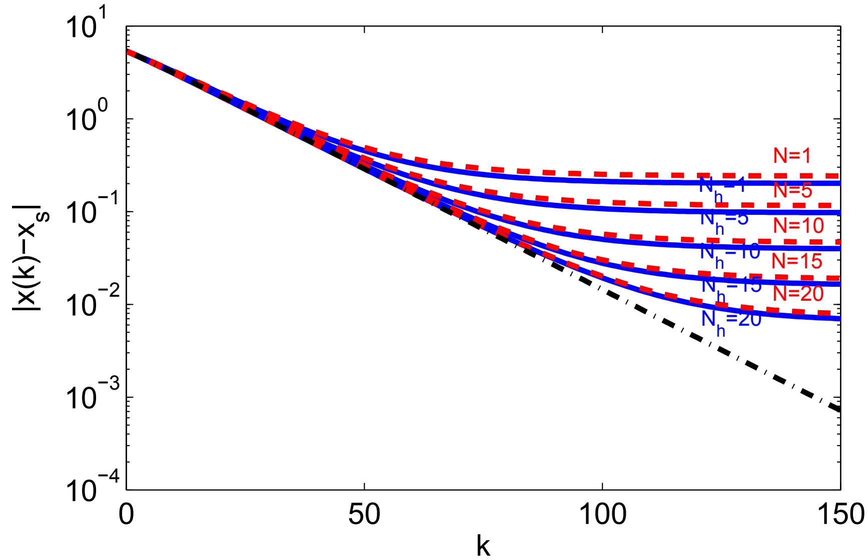


**Figure 2.1:** Closed-loop system state trajectories under the proposed EMPC with $N = 1$ and $N_h = 1,\ 5,\ 10,\ 15,\ 20$ (solid lines), EMPC without terminal cost with $N = 1,\ 5,\ 10,\ 15,\ 20$ (dashed lines), and the auxiliary controller $h$ (dash-dotted line).

$h(x) = u_s$ with the terminal region $\mathbb{X}_f = \{x : |x - x_s| \le 85\}$. Under these settings, it can be verified that all the conditions for Theorem 3 are satisfied. In particular, Assumption 1 is satisfied with a linear storage function $\lambda(x) = c^T x$ where $c = [-368.6684, -503.5415]^T$. The corresponding rotated stage cost is a quadratic function that achieves its minimum at $(x_s, u_s)$.

In the simulations, we make comparisons between the following configurations: (1) the proposed EMPC with control horizon $N = 1$ and different terminal horizons $N_h = 1,\ 5,\ 10,\ 15,\ 20$; (2) An EMPC without terminal cost and with different control horizons $N = 1,\ 5,\ 10,\ 15,\ 20$; (3) the auxiliary controller $h(x) = u_s$. The initial state is $x(0) = [4, 20]^T$.

Figure 2.1 shows the state trajectories under the different configurations. The system state converges to the optimal steady state under the proposed EMPC design. Moreover, the practical stability region shrinks exponentially fast as $N_h$ increases. These results agree with the results in Theorem 3. Figure 2.2 shows the transient performance defined in Eq. (2.42) under different configurations from $K = 50$ to 150. The performance of the proposed EMPC design approaches that of the auxiliary controller as $N_h$ increases, agreeing with the results of Theorem 6. Note that the slope

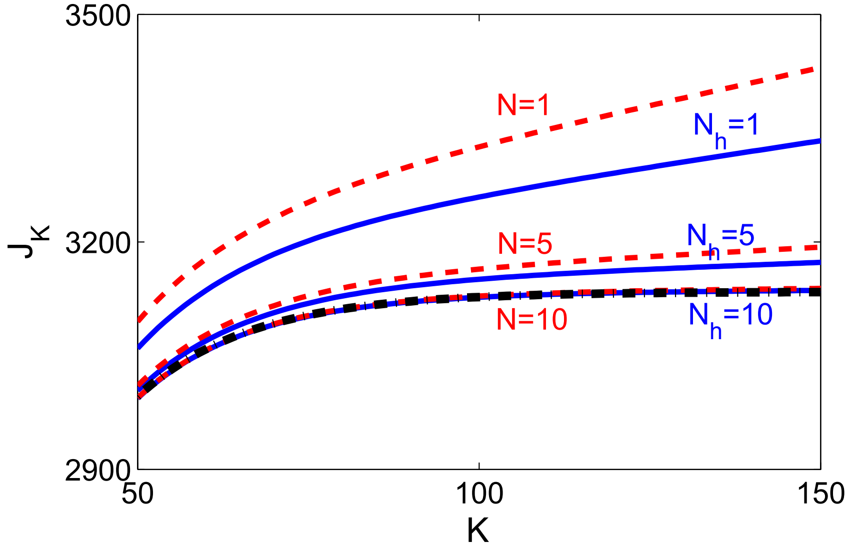


**Figure 2.2:** Transient performance under the proposed EMPC with $N = 1$ and $N_h = 1,\ 5,\ 10$ (solid lines), EMPC without terminal cost with $N = 1,\ 5,\ 10$ (dashed lines), and the auxiliary controller $h$ (dash-dotted line).

**Table 2.1:** Shifted asymptotic performance ($\bar{J}_{\text{asy}} - l(x_s, u_s)$) and average optimization evaluation time (CPU) of the proposed EMPC with $N = 1$

| | $N_h = 1$ | $N_h = 5$ | $N_h = 10$ | $N_h = 15$ | $N_h = 20$ |
|---|---|---|---|---|---|
| $\bar{J}_{\text{asy}} - l(x_s, u_s)$ | 1.3524 | 0.3149 | 0.0518 | 0.0086 | 0.0014 |
| CPU(s) | 0.0132 | 0.0160 | 0.0214 | 0.0234 | 0.0255 |

of $J_K$ decreases as $K$ increases which corresponds to the vanishing term $\beta_K(|x(0) - x_s|, K)$ of Eq. (2.60).

The asymptotic performance and average optimization problem evaluation time under the proposed EMPC and EMPC without terminal cost are shown in Table 2.1 and Table 2.2, respectively. The asymptotic performance converges to the optimal steady state performance exponentially as $N_h$ increases, which verifies our analysis in Theorem 5. For roughly the same prediction horizon, the proposed EMPC design achieves approximately the same level of convergence and performance as that of the EMPC without terminal conditions, but with much less computational efforts. These results demonstrate the computational efficiency of the proposed EMPC design.

**Table 2.2:** Shifted asymptotic performance ($\bar{J}_{\text{asy}} - l(x_s, u_s)$) and average optimization evaluation time (CPU) of the EMPC without terminal cost.

| | $N = 1$ | $N = 5$ | $N = 10$ | $N = 15$ | $N = 20$ |
|---|---|---|---|---|---|
| $\bar{J}_{\text{asy}} - l(x_s, u_s)$ | 1.9581 | 0.4487 | 0.0720 | 0.0116 | 0.0018 |
| CPU(s) | 0.0120 | 0.0351 | 0.0709 | 0.1192 | 0.1921 |

**Example 5.2** This example is also taken from [10]. Consider the control system:

$$x(k+1) = 2x(k) + u(k) \tag{2.79}$$

with state and input constraints: $\mathbb{X} = [-0.5,\ 0.5]$, $\mathbb{U} = [-2, 2]$. The stage cost to be minimized is $l(x, u) = u^2$. Thus, the objective of the control problem is to maintain the system state $x$ inside the region $\mathbb{X}$ with the minimum control effort. It can be easily checked that the optimal steady state is $(x_s, u_s) = (0, 0)$. The system of Eq. (2.79) satisfies the strict dissipativity Assumption 1 with a quadratic storage function $\lambda(x) = -x^2/2$. The corresponding rotated stage cost of Eq. (2.6) is $\bar{l}(x, u) = 2u^2 + 1.5x^2 + 2xu$, which is subject to polynomial bounds: $x^2 \le \bar{l}(x, u) \le 3(x^2 + u^2)$. Assumption 5 is thus satisfied with $c_1 = 1$, $c_2 = 3$ and $p = 2$. Assumption 7 is satisfied with $c_\lambda = 0.5$. We design the auxiliary controller $h(x) = -1.5x$, which satisfies Assumption 4 with $a = 1$ and $s = 0.5$ on $\mathbb{X}_f = \mathbb{X}$. Assumption 6 is also satisfied. Thus, all the conditions for Theorem 4 are satisfied.

In the simulations, we compare (1) the proposed EMPC with $N = 1$ and $N_h = 1,\ 2,\ 5$; (2) EMPC without terminal cost (but with terminal region constraint) with $N = 1,\ 5,\ 10,\ 15,\ 20$; And (3) the auxiliary controller. The initial state is $x(0) = 0.5$. The state and stage cost under different scenarios are shown in Figure 2.3 and Figure 2.4 , respectively. The transient performance $J_K$ with $K = 30$ under different scenarios and the corresponding average optimization evaluation times are shown in Table 2.3 and Table 2.4. From Figure 2.3, it can be seen that EMPC with the proposed terminal cost exponentially stabilizes the optimal steady state (note that the y-axis is in logarithmic coordinate), which agrees with Theorem 4. Figure 2.3 and Table 2.4 verify our analysis in Theorem 7 that the transient performance of the proposed EMPC design approaches that of the auxiliary controller exponentially fast as $N_h$ increases. In this example, employing the extended terminal horizon not

**Table 2.3:** Transient performance ($J_{30}$) and optimization evaluation time (CPU) of the proposed EMPC with terminal cost with $N = 1$.

| | $N_h = 1$ | $N_h = 2$ | $N_h = 5$ |
|---|---|---|---|
| $J_{30}$ | 0.7714 | 0.7508 | 0.7500 |
| CPU(s) | 0.0208 | 0.0210 | 0.0215 |

only improves the computational efficiency for EMPC, but also leads to exponential stability which cannot be achieved by EMPC without terminal condition.

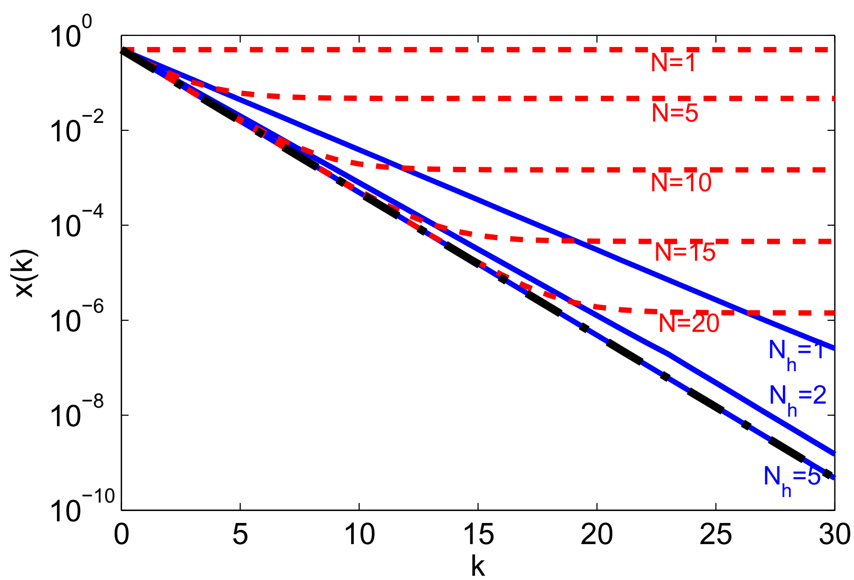


**Figure 2.3:** State trajectories of the closed-loop system under the proposed EMPC with $N = 1$ and $N_h = 1,\ 2,\ 5$ (solid lines), EMPC without terminal cost with $N = 1,\ 5,\ 10,\ 15,\ 20$ (dashed lines), and the auxiliary controller $h$ (dash-dotted line).

## 2.5 Summary

In this chapter, we provided a general framework to analyze the stability and performance of EMPC with extended terminal horizon. While a finite terminal horizon is in general not sufficient to ensure stability of the optimal steady state, it is sufficient to achieve practical stability for strictly dissipative systems under mild assumptions. Further conditions to ensure the exponential shrinkage of the practical stability region are provided. For a special case including conventional MPC with positive-definite

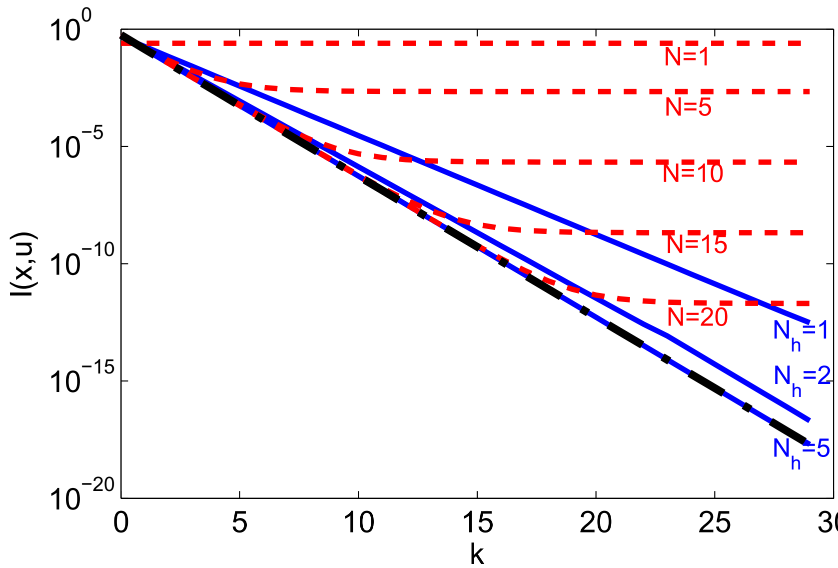


**Figure 2.4:** Stage costs of the closed-loop system under the proposed EMPC with $N = 1$ and $N_h = 1,\ 2,\ 5$ (solid lines), EMPC without terminal cost with $N = 1,\ 5,\ 10,\ 15,\ 20$ (dashed lines), and the auxiliary controller $h$ (dash-dotted line).

**Table 2.4:** Transient performance ($J_{30}$) and optimization evaluation time (CPU) of the EMPC without terminal cost.

| | $N = 1$ | $N = 5$ | $N = 10$ | $N = 15$ | $N = 20$ |
|---|---|---|---|---|---|
| $J_{30}$ | 7.500 | 0.8088 | 0.7501 | 0.7500 | 0.7500 |
| CPU(s) | 0.0212 | 0.0356 | 0.0460 | 0.0712 | 0.0899 |

stage costs, exponential stability can be achieved. Performance of the EMPC is also shown to be approximately upper-bounded by that of the auxiliary controller if a large terminal horizon is used. These results provide insights into the intrinsic properties of EMPC and also explain the computational efficiency of the EMPC design.

# Chapter 3

# Economic MPC with local optimality

In this chapter, we design a terminal cost which preserves the local optimality for EMPC. We first show, based on the strong duality and second order sufficient condition (SOSC) of the steady-state optimization problem, that the optimal operation of the system is locally equivalent to an LQR controller. The proposed terminal cost is constructed as the value function of the LQR controller plus a linear term characterized by the Lagrange multiplier associated with the steady-state constraint. From the perspective of dissipative systems, the linear term corresponds to a linear storage function for the system. EMPC with the proposed terminal cost is stabilizing with an appropriately chosen control horizon, and preserves the local optimality of the LQR controller. Simulation results of an isothermal CSTR verify our analysis.

## 3.1 Preliminaries

### 3.1.1 Notation

Throughout this chapter, the operator $|\cdot|$ denotes the Euclidean norm of a scalar or a vector. The notation $int(\mathbb{X})$ denotes the interior of the set $\mathbb{X}$. The symbol $\mathcal{B}_r(x_s)$ denotes the open ball centered at $x_s$ with radius $r$ such that $\mathcal{B}_r(x_s) := \{x : |x - x_s| < r\}$. The operator $\lambda_{\min(\max)}(H)$ denotes the minimum (maximum) eigenvalue of matrix $H$. The symbol '$\setminus$' denotes set substraction such that $\mathbb{A} \setminus \mathbb{B} := \{x \in \mathbb{A}, x \notin \mathbb{B}\}$.

### 3.1.2 System description

We consider the nonlinear discrete state-space model:

$$x(k+1) = f(x(k), u(k)) \tag{3.1}$$

where $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$ denote the system state and input respectively, $k \geq 0$ is the discrete time index. The performance of the system is measured by an economic cost function $l(x, u) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$. It is assumed that both $f(x, u)$ and $l(x, u)$ are twice continuously differentiable. The system state and input are subject to constraint $g(x) \leq 0$ and $h(u) \leq 0$ respectively, where the sets $\mathbb{X} := \{x : g(x) \leq 0\}$ and $\mathbb{U} := \{u : h(u) \leq 0\}$ are compact. For simplicity of discussion, we assume that the set $\mathbb{X}$ is forward-invariant. That is, for any $x \in \mathbb{X}$, there exists $u \in \mathbb{U}$ such that $f(x, u) \in \mathbb{X}$. The forward invariance of the constraint set ensures recursive feasibility of the moving horizon optimization problems. This condition can be relaxed, see e.g., [6] and [9] (Sections 8.2-8.3) for results in the context of MPC with positive definite stage cost.

We use $(x_s, u_s)$ to denote the optimal steady state and input of the system, which is the solution to the following steady-state optimization problem:

$$\begin{aligned} (x_s, u_s) = \quad & \arg\min_{x,u} \; l(x, u) \\ \text{s.t.} \quad & x = f(x, u) \\ & g(x) \leq 0 \\ & h(u) \leq 0 \end{aligned} \tag{3.2}$$

Note that the pair $(x_s, u_s)$ is not necessarily a minimizer of the economic cost function $l(x, u)$.

## 3.2 Optimal operation of the system

A key difference between conventional MPC with positive-definite stage cost and economic MPC is that in EMPC, steady-state operation is not necessarily the optimal operation. For generic nonlinear systems, it is possible that a dynamic trajectory (e.g., a periodic orbit) outperforms the average performance of steady-state operation. The optimality of steady-state operation is shown to be closely connected to the notion of dissipativity or the so-called turnpike property [25], [26], [27]. Verifying dissipativity of the system amounts to finding a storage function, which is in general a difficult task. In this chapter, we consider systems satisfying the strong duality condition (see e.g., [38] Section 5) where a linear storage function can be found. Similar approach was used in [28] to establish stability of EMPC with point-wise terminal constraint. We make the following assumption on the system:

**Assumption 8** *The steady-state optimization problem of Eq. (3.2) satisfies strong duality and second order sufficient condition (SOSC). Moreover, the state and input constraints are not active at the optimal steady state* $(x_s, u_s)$. *That is,* $g(x_s) < 0$, $h(u_s) < 0$.

**Lemma 13** *If Assumption 8 holds, then there exists a positive scalar* $c_l > 0$ *such that the following holds for all* $x \in \mathbb{X}$ *and* $u \in \mathbb{U}$*:*

$$l(x,u) + \lambda_s^T(f(x,u) - x) - l(x_s,u_s) \geq c_l(|x - x_s|^2 + |u - u_s|^2) \tag{3.3}$$

*where* $\lambda_s$ *is the KKT multiplier associated with the equality constraint* $f(x,u) - x = 0$.

**Proof.** Consider the Lagrangian:

$$L(w, \lambda_s, \mu_s, \nu_s) = l(x,u) + \lambda_s^T(f(x,u) - x) + \mu_s^T g(x) + \nu_s^T h(x) \tag{3.4}$$

where $w = [x^T, u^T]^T$ with $w_s = [x_s^T, u_s^T]^T$, $\lambda_s$, $\mu_s$ and $\nu_s$ are the KKT multipliers. Since strong duality holds, $L(w_s, \lambda_s, \mu_s, \nu_s) = l(x_s, u_s)$. Also taking into account the SOSC: $\frac{\partial L}{\partial w}|_{w=w_s} = 0$, and $H = \frac{\partial^2 L}{\partial w^2}|_{w=w_s} > 0$, the Taylor series of $L(w, \lambda_s, \mu_s, \nu_s)$ at $w = w_s$ can be written as follows:

$$L(w, \lambda_s, \mu_s, \nu_s) = l(x_s, u_s) + (w - w_s)^T H (w - w_s) + O(|w|^2)$$

Due to the positive definiteness of $H$, for any positive scalar $c_s$ such that $0 < c_s < \lambda_{\min}(H)$, there exists $\epsilon > 0$ such that the following holds for all $w \in \mathcal{B}_\epsilon(w_s)$:

$$L(w, \lambda_s, \mu_s, \nu_s) - l(x_s, u_s) \geq c_s(|x - x_s|^2 + |u - u_s|^2)$$

Substituting Eq. (3.4) into the above and taking into account that $\lambda_s = 0$ and $\nu_s = 0$ (because of the strict complimentary condition and the assumption that $g(x_s) < 0$, $h(u_s) < 0$), the following holds for all $w \in \mathcal{B}_\epsilon(w_s)$:

$$l(x,u) + \lambda_s^T(f(x,u) - x) - l(x_s,u_s) \geq c_s(|x - x_s|^2 + |u - u_s|^2)$$

The quadratic lower bound on $\mathcal{B}_\epsilon(w_s)$ can be extended to the compact set $\mathbb{Z} := \{w : x \in \mathbb{X}, u \in \mathbb{U}\}$. Specifically, let $\bar{l}_{\max} = \sup\{l(x,u)+\lambda_s^T(f(x,u)-x)-l(x_s,u_s) : w \in \mathbb{Z}\}$ and $\bar{l}_{\min} = \inf\{|x-x_s|^2+|u-u_s|^2 : w \in \mathbb{Z} \setminus \mathcal{B}_\epsilon(w_s)\}$, then $c_l = \max\{c_s, \frac{\bar{l}_{\max}}{\bar{l}_{\min}}\}$ satisfies Eq. (3.3) for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$. ■

Assumption 8 is slightly stronger than the assumptions made in [28]. In [28], the left-hand-side of Eq. (3.3) (the rotated cost) is lower bounded by a $\mathcal{K}_\infty$ function of $|x - x_s|$. In this chapter, a quadratic lower bound with respect to $(x_s, u_s)$ is assumed. This will allow us to find the locally optimal controller and to approximate the corresponding economic cost using linear quadratic regulator (LQR) approach.

**Remark 5** *The strong duality condition is satisfied for convex optimization problems with affine equality constraints satisfying the Slater condition [38]. In the context of control systems, it is satisfied for the steady-state optimization of linear systems with convex cost functions and constraints (also satisfying the Slater condition). It is important to note that Assumption 8 is rather strong for systems with optimal steady-state operation. A more general condition which implies optimal steady-state operation is dissipativity [2]. The merit of Assumption 8 is that it can be verified numerically and provides us with a linear storage function that can be used to design the terminal cost. Specifically, once $(x_s, u_s)$ and the corresponding KKT multipliers are obtained, one can verify Eq. (3.3) by carrying out the following optimization:*

$$\begin{aligned} \min_{x,u} \quad & l(x,u) + \lambda_s^T(f(x,u) - x) - l(x_s,u_s) \\ s.t. \quad & g(x) < 0 \\ & h(u) < 0 \end{aligned}$$

*If $(x_s, u_s)$ is the optimal solution to the above optimization problem, and if the hessian matrix of the objective function is positive definite at $(x_s, u_s)$, then Assumption 8 holds.*

Under Assumption 8, the system is strictly dissipative with respect to the stage cost $l(x,u)-l(x_s,u_s)$, with a linear storage function $-\lambda_s x$ for all $x \in \mathbb{X}$, $u \in \mathbb{U}$. Under further controllability conditions (which we will make in Assumption 9), Assumption 8 implies that the system is suboptimally operated off steady state [25], [26], [27]. Specifically, this means that the asymptotic performance, or the infinite-time average performance, is lower bounded by the steady-state economic performance:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} l(x(k), u(k)) \geq l(x_s, u_s)$$

with the equality attained only if $\lim_{k\to\infty} x(k) = x_s$. The asymptotic performance however, is a relatively weak performance index which does not reflect transient performance of the system. To this end, we consider the following infinite-time transient performance:

$$J_\infty(x) = \inf \sum_{k=0}^{\infty} \Big( l(x(k), u(k)) - l(x_s, u_s) \Big) \tag{3.5a}$$

$$\text{s.t. } x(k+1) = f(x(k), u(k)) \tag{3.5b}$$

$$g(x(k)) \leq 0 \tag{3.5c}$$

$$h(u(k)) \leq 0 \tag{3.5d}$$

$$x(0) = x \tag{3.5e}$$

$$\lim_{k\to\infty} x(k) = x_s \tag{3.5f}$$

The goal of EMPC is to approximate the above infinite-horizon optimization problem in a receding horizon fashion. We make the following controllability assumption:

**Assumption 9** *$J_\infty(x) < \infty$ for all $x \in \mathbb{X}$. Moreover, the linearized system at $(x_s, u_s)$ with $[A, B] = [\frac{\partial f}{\partial x}|_{(x_s,u_s)}, \frac{\partial f}{\partial u}|_{(x_s,u_s)}]$ is controllable.*

The assumption that $J_\infty(x) < \infty$ for all $x \in \mathbb{X}$ is equivalent to assuming stabilizability of the system on $\{(x, u) : x \in \mathbb{X}, u \in \mathbb{U}\}$ and that the system can be driven to the steady state with a finite cost. Note that $J_\infty(x) > -\infty$ holds automatically due to strong duality. Now let us investigate the property of $J_\infty(x)$ for $x$ near $x_s$.

**Lemma 14** *If Assumption 8 and Assumption 9 hold, then there exists $\epsilon > 0$ such that for all $x \in \mathcal{B}_\epsilon(x_s)$, $J_\infty(x)$ can be locally approximated as:*

$$J_\infty(x) = \lambda_s(x - x_s) + \frac{1}{2}(x - x_s)^T P(x - x_s) + O(|x - x_s|^2)$$

*where $P$ is the solution to the following discrete time algebraic Riccati equation:*

$$P = A^T PA - (A^T PB + N)(R + B^T PB)^{-1}(B^T PA + N^T) + Q \tag{3.6}$$

*where*

$$H = \frac{\partial^2 L}{\partial w^2}|_{w=w_s} = \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} > 0$$

*is the Hessian matrix of the Lagrangian evaluated at* $(x_s, u_s)$*. The corresponding locally optimal controller is*

$$u = K_{LQ}(x - x_s) + u_s \tag{3.7}$$

*where* $K_{LQ} = (R + B^T PB)^{-1} B^T PA$*.*

**Proof.** Let us define the rotated stage cost

$$\bar{l}(x, u) = l(x, u) + \lambda_s^T (f(x, u) - x) - l(x_s, u_s) \tag{3.8}$$

which is positive definite with respect to $(x_s, u_s)$ on $x \in \mathbb{X}$, $u \in \mathbb{U}$, as established in Lemma 13. From the definition of $\bar{l}(x, u)$ and taking into account the constraint of Eq. (3.5f), the objective function of Eq. (3.5a) can be rewritten in the following rotated form:

$$\begin{aligned} & \textstyle\sum_{k=0}^{\infty} \Big( l(x(k), u(k)) - l(x_s, u_s) \Big) \\ = & \textstyle\sum_{k=0}^{\infty} \bar{l}(x(k), u(k)) - \lambda_s x(0) + \lambda_s \lim_{k \to \infty} x(k) \\ = & \textstyle\sum_{k=0}^{\infty} \bar{l}(x(k), u(k)) - \lambda_s (x - x_s) \end{aligned}$$

Consider the following optimization problem:

$$\begin{aligned} \bar{J}_\infty = \inf \quad & \textstyle\sum_{k=0}^{\infty} \bar{l}(x(k), u(k)) \\ \text{s.t.} \quad & (3.5b) - (3.5e) \end{aligned} \tag{3.9}$$

The solutions to the optimizations of Eq. (3.5) and Eq. (3.9) are identical, because the objective functions of the two problems differ by a bounded constant term $\lambda_s(x - x_s)$, and the constraint sets are the same except for Eq. (3.5f) which is automatically satisfied in Eq. (3.9) because of Assumption 9 and the positive-definiteness of $\bar{l}(x, u)$. Eq. (3.9) is essentially a conventional MPC with positive-definite stage cost which is quadratically lower bounded. It can be shown by geometric methods (see e.g., [39] [7] for results of continuous systems which can be readily extended to discrete systems) that there exists $\epsilon > 0$ such that for all $x \in \mathcal{B}_\epsilon(x_s)$, the optimal trajectory of Eq. (3.9) does not violate the state and input constraints, and that $J_\infty(x)$ can be approximated by:

$$\bar{J}_\infty = \frac{1}{2}(x - x_s)^T P (x - x_s) + O(|x - x_s|^2)$$

where $P$ is defined in Eq. (3.6). Thus, for all $x \in \mathcal{B}_\epsilon(x_s)$:

$$\begin{aligned} J_\infty = & \ \bar{J}_\infty + \lambda_s(x - x_s) \\ = & \ \lambda_s(x - x_s) + (x - x_s)^T P(x - x_s) + O(|x - x_s|^2) \end{aligned}$$

This proves Lemma 14. ■

Lemma 14 shows that the control problem of Eq. (3.5) can be locally approximated as a linear quadratic regulator design problem. The optimal objective function can be locally approximated as the corresponding infinite-horizon LQR cost plus the difference of the storage function at the steady state and the initial state. In the following, we will use this approximation as the terminal cost for EMPC and discuss the stability and performance of the EMPC design.

## 3.3 EMPC with the proposed terminal cost

At a time instant $n$, our EMPC design is formulated as the following optimization problem:

$$\min_{u(0),...,u(N-1)} \sum_{k=0}^{N-1} l(\tilde{x}(k), u(k)) + V_f(\tilde{x}(N)) \tag{3.10a}$$

$$\text{s.t.} \quad \tilde{x}(k+1) = f(\tilde{x}(k), u(k)), \quad k = 0, ..., N-1 \tag{3.10b}$$

$$\tilde{x}(0) = x(n) \tag{3.10c}$$

$$g(\tilde{x}(k)) \leq 0, \quad k = 0, ..., N-1 \tag{3.10d}$$

$$h(u(k)) \leq 0, \quad k = 0, ..., N-1 \tag{3.10e}$$

where $\tilde{x}(k)$ denotes the predicted state trajectory, $x(n)$ is the state measurement at time instant $n$. The terminal cost $V_f(x)$ is the local second order approximation of $J_\infty(x)$ characterized in Lemma 14. That is,

$$V_f(x) = \lambda_s(x - x_s) + \frac{1}{2}(x - x_s)^T P(x - x_s)$$

The optimal solution to the above optimization problem is denoted as $u^*(k|n)$, $k = 0, ..., N-1$. The corresponding optimal state trajectory is $x^*(k|n)$, $k = 0, ..., N$. The manipulated input of the closed-loop system under the EMPC at a time instant $n$ is: $u(n) = u^*(0|n)$. At the next sampling time $n+1$, the optimization of Eq. (3.10) is re-evaluated.

In the following, we consider the stability and performance of the EMPC design of Eq. (3.10). To proceed, let us consider the following MPC design

$$\begin{aligned} \min_{u(0),\ldots,u(N-1)} \quad & \textstyle\sum_{k=0}^{N-1} \bar{l}(\tilde{x}(k), u(k)) + \bar{V}_f(\tilde{x}(N)) \\ \text{s.t.} \quad & (3.10b) - (3.10e) \end{aligned} \tag{3.11}$$

where $\bar{V}_f(x) = \frac{1}{2}(x - x_s)^T P (x - x_s)$, $\bar{l}(x, u)$ is the rotated stage cost defined in Eq. (3.8). It can be easily checked that the objective functions of Eq. (3.10) and Eq. (3.11) only differ by a constant term $\lambda_s \tilde{x}(0) = \lambda_s x(n)$. Thus, the two MPCs deliver the same solution. Proving stability of the proposed EMPC design is equivalent to proving stability of the MPC of Eq. (3.11), which has positive definite stage cost and terminal cost. This task has been dealt with in [7] and [8] where the discussions are made under more general settings. In the following, we provide a tailored proof for the task at hand. Let us use $\bar{V}_N^*(x)$ to denote the optimal objective function value of Eq. (3.11) with control horizon $N$.

**Lemma 15** *There exists a positive scalar $c_{V_\infty}$ such that the following holds for all $x \in \mathbb{X}$:*

$$\bar{V}_\infty^*(x) \leq c_{V_\infty} |x - x_s|^2 \tag{3.12}$$

**Proof.** From the proof of Lemma 14, $\bar{V}_f(\tilde{x}(N))$ is a local quadratic approximation of $\bar{V}_\infty^*(x)$ (note that $\lim_{N \to \infty} \tilde{x}(N) = x_s$ and the terminal cost $\bar{V}_f(\tilde{x}(N))$ can be dropped for $N \to \infty$). Thus, for any positive scalar $c_s > \lambda_{\max}(P)$, there exists $\epsilon > 0$ such that the following holds for all $x \in \mathcal{B}_\epsilon(x_s)$: $\bar{V}_\infty^*(x) \leq c_s |x - x_s|^2$. Following the similar arguments as in the proof of Lemma 13, and based on the controllability assumption that $\bar{V}_\infty^*(x)$ is bounded on $\mathbb{X}$, the quadratic upper bound of $\bar{V}_\infty^*(x)$ can be extended from the local region $\mathcal{B}_\epsilon(x_s)$ to $\mathbb{X}$. That is, there exists a positive scalar $c_{V_\infty}$ such that Eq. (3.12) holds for all $x \in \mathbb{X}$. ■

Lemma 15 shows that the infinite-horizon cost of the MPC design of Eq. (3.11) is quadratically upper bounded. Our final proof requires a stronger condition ( Lemma 16) that $\bar{V}_N^*(x)$ is quadratically upper bounded for all $N \geq 1$ and $x \in \mathbb{X}$. This condition is stronger than Lemma 15 because $\bar{V}_N^*(x)$ is not necessarily an increasing sequence with respect to $N$.

**Lemma 16** *There exists a positive scalar* $c_{V_f}$ *such that the following holds for all* $x \in \mathbb{X}$ *and* $N \geq 1$*:*

$$\bar{V}_N^*(x) \leq c_{V_f}|x - x_s|^2 \tag{3.13}$$

**Proof.** Let us use $\bar{x}_\infty(k, x)$ and $\bar{u}_\infty(k, x)$, $k \geq 0$ to denote the state and input trajectories of $\bar{V}_\infty^*(x)$ respectively. The following holds:

$$\bar{V}_N^*(x) \leq \sum_{k=0}^{N-1} \bar{l}(\bar{x}_\infty(k, x), \bar{u}_\infty(k, x)) + \bar{V}_f(\bar{x}_\infty(N, x)) \tag{3.14}$$

Since $\bar{l}$ is positive definite, and from Lemma 15:

$$\sum_{k=0}^{N-1} \bar{l}(\bar{x}_\infty(k, x), \bar{u}_\infty(k, x)) \leq \bar{V}_\infty^*(x) \leq c_{V_\infty}|x - x_s|^2 \tag{3.15}$$

So we only need to establish a quadratic upper bound of $\bar{V}_f(\bar{x}_\infty(N, x))$. From Lemma 13 and Lemma 15:

$$c_l|\bar{x}_\infty(N, x) - x_s|^2 \leq \bar{l}((\bar{x}_\infty(N, x), \bar{u}_\infty(N, x))) \leq c_{V_\infty}|x - x_s|^2$$

Thus,

$$\begin{aligned} \bar{V}_f(\bar{x}_\infty(N, x)) &= |\bar{x}_\infty(N, x) - x_s|_P^2 \\ &\leq \lambda_{\max}(P)|\bar{x}_\infty(N, x) - x_s|^2 \\ &\leq \lambda_{\max}(P)\frac{c_{V_\infty}}{c_l}|x - x_s|^2 \end{aligned} \tag{3.16}$$

Substituting Eqs. (3.15) and (3.16) into Eq. (3.14), Lemma 16 is proved with $c_{V_f} = c_{V_\infty} + \lambda_{\max}(P)\frac{c_{V_\infty}}{c_l}$ ■

**Theorem 8** *Consider the EMPC of Eq. (3.10) with* $x(0) \in \mathbb{X}$*. If Assumption 8 and Assumption 9 hold, then there exists a finite* $N^*$ *such that the EMPC is stabilizing for all* $N \geq N^*$*.*

**Proof.** Let us use $\bar{x}_N(k, x)$, $k = 0, ..., N$ and $\bar{u}_N(k, x)$, $k = 0, ..., N - 1$ to denote the state and input trajectories of $\bar{V}_N^*(x)$ respectively. From Lemma 16, $\bar{V}_N^*(x) = \sum_{k=0}^{N-1} \bar{l}(\bar{x}_N(k, x), \bar{u}_N(k, x)) + \bar{V}_f(\bar{x}_N(N, x)) \leq c_{V_f}|x - x_s|^2$. Since $\bar{l}$ and $\bar{V}_f$ are all positive definite, the above implies that there exists a $p \in \{1, ..., N - 1\}$ (we assume $N \geq 2$) such that $\bar{l}(\bar{x}_N(p, x), \bar{u}_N(p, x)) \leq \frac{c_{V_f}}{N-1}|x - x_s|^2$. Using Lemma 13, this further implies that

$$|\bar{x}_N(p, x) - x_s|^2 \leq \frac{c_{V_f}}{(N-1)c_l}|x - x_s|^2 \tag{3.17}$$

Now let us consider $\bar{V}_N^*(\bar{x}_N(1,x))$, which is the optimal value of the subsequent optimization problem. A candidate solution to the subsequent optimization is to adopt the state and input trajectories of $\bar{x}_N(k,x)$ and $\bar{u}_N(k,x)$ for $k = 1, ...p-1$, and then evaluate $\bar{V}_{N-p}^*(\bar{x}_N(p,x))$ to fill up the rest state and input trajectories. The objective function of this candidate solution, which is no smaller than $\bar{V}_N^*(\bar{x}_N(1,x))$, is:

$$\bar{V}_N^*(\bar{x}_N(1,x)) \leq \sum_{k=1}^{p-1} \bar{l}(\bar{x}_N(k,x), \bar{u}_N(k,x)) + \bar{V}_{N-p}^*(\bar{x}_N(p,x)) \tag{3.18}$$

From Lemma 13 and the positive definiteness of the rotated stage and terminal cost, the first term of the right-hand-side of Eq. (3.18) is bounded by:

$$\begin{aligned} \sum_{k=1}^{p-1} \bar{l}(\bar{x}_N(k,x), \bar{u}_N(k,x)) \leq & \; \bar{V}_N^*(x) - \bar{l}(x, \bar{u}_N(0,x)) \\ \leq & \; \bar{V}_N^*(x) - c_l |x - x_s|^2 \end{aligned} \tag{3.19}$$

From Lemma 16 and Eq. (3.17), the second term of the right-hand-side of Eq. (3.18) is bounded by:

$$\bar{V}_{N-p}^*(\bar{x}_N(p,x)) \leq \frac{{c_{V_f}}^2}{(N-1)c_l} |x - x_s|^2 \tag{3.20}$$

Substituting Eqs. (3.19) and (3.20) into Eq. (3.18), we have:

$$\bar{V}_N^*(\bar{x}_N(1,x)) \leq \bar{V}_N^*(x) + (\frac{{c_{V_f}}^2}{(N-1)c_l} - c_l)|x - x_s|^2$$

Thus, it is clear that the value function $V_N^*(x)$ is a Lyapunov function of the closed-loop system if $\frac{{c_{V_f}}^2}{(N-1)c_l} - c_l < 0$ which is satisfied if $N > (\frac{c_{V_f}}{c_l})^2 + 1$. This proves Theorem 13. ■

Theorem 13 shows that EMPC with the proposed terminal cost is stabilizing if a sufficiently large $N$ is used. The minimum value of $N$ required to ensure stability could be much smaller than what can be inferred from the above proof. For more discussions on the size of $N$ to ensure stability, interested readers may refer to [9], Chapter 6.

**Theorem 9** *The EMPC of Eq. (3.10) is locally equivalent to the infinite-horizon LQR controller of Eq. (3.7)*

**Proof.** Let $\bar{x}(k) = x(k) - x_s$ and $\bar{u}(k) = u(k) - u_s$, consider the following quadratic programming (QP):

$$\begin{aligned} \min \; & \textstyle\sum_{k=0}^{N-1} \; \bar{x}(k)^T H \bar{x}(k) + \bar{x}(N)^T P \bar{x}(N) \\ \text{s.t.} \; & \bar{x}(k+1) = A\bar{x}(k) + B\bar{u}(k) \end{aligned} \tag{3.21}$$

where $H$, $P$, $A$ and $B$ are defined in Lemma 14. The above QP delivers the LQR control law characterized in Eq. (3.7). If the system state is sufficiently close to $x_s$ such that no state or input constraints are active, the MPC of Eq. (3.11) which is an equivalent transformation of our EMPC design, is locally equivalent to the QP of Eq. (3.21). Thus the two optimization problems have locally first order equivalent solutions, as a result of Theorem 16 in [40]. ■

## 3.4 Case study

This example is taken from [2], [28]. Consider a single first-order irreversible chemical reaction in an isothermal CSTR: $A \to B$, where the reactant $A$ is converted into product $B$. The material balances of the process are:

$$\begin{aligned} \frac{dc_A}{dt} &= \frac{Q}{V_R}(c_{Af} - c_A) - k_r c_A \\ \frac{dc_B}{dt} &= \frac{Q}{V_R}(c_{Bf} - c_B) + k_r c_A \end{aligned} \tag{3.22}$$

where $c_A$ and $c_B$ are the molar concentrations of $A$ and $B$ respectively, $c_{Af}$ and $c_{Bf}$ are feed concentrations of $A$ and $B$, $Q$ is the volumetric flow through the reactor, $V_R$ is the volume of the reactor, $k_r$ is the reaction rate constant. The corresponding process parameters are shown in Table 3.1

**Table 3.1:** Process parameters

| $c_{Af}$ | $c_{Bf}$ | $V_R$ | $k_r$ |
|---|---|---|---|
| $1\ mol/L$ | $0\ L$ | $10\ L$ | $1.2\ L/(mol\ min)$ |

In the controller design, the flow rate $0 \leq Q \leq 20L/min$ is the manipulated control input. The economic objective is $l_e(c_A, x_B, Q) = -2Qc_B + 0.5Q$, which corresponds to maximizing the product $B$ and meanwhile minimizing the operational cost. The optimal steady state of the system is $c_A = c_B = 0.5mol/L$, $Q = 12L/min$. Quadratic penalty terms are added to the economic cost function to make the steady-state optimization strongly dual: $l(c_A, x_B, Q) = -2Qc_B + 0.5Q + |c_A - 0.5|^2_{Q_A} + |c_B - 0.5|^2_{Q_B} + |Q - 12|^2_R$, where $Q_A = Q_B = R = 0.505$. The Lagrangian of the the steady-state optimization with the regularized cost $l(c_A, x_B, Q)$ is: $L(c_A, x_B, Q) = l(c_A, x_B, Q) + \lambda^T f(c_A, x_B, Q) + [-Q,\ Q - 20]$, where $f(c_A, x_B, Q)$ denotes the right-hand-side of Eq. (3.22). The optimal KKT multipliers are $\lambda = [-10, -20]^T$ and $\nu = [0, 0]^T$. The corresponding Hessian matrix is:

$$H = \begin{bmatrix} 1.01 & 0 & 1 \\ 0 & 1.01 & 0 \\ 1 & 0 & 1.01 \end{bmatrix}$$

which is positive definite with eigenvalues 0.01, 1.01 and 2.01. Thus, Assumption 8 is satisfied. The sampling time is 0.2 minutes. The discrete linearized model at the optimal steady state is

$$A = \begin{bmatrix} 0.6188 & 0 \\ 0.1678 & 0.7866 \end{bmatrix} \qquad B = \begin{bmatrix} 0.0079 \\ -0.0079 \end{bmatrix}$$

Our proposed terminal cost for EMPC is

$$V_f(x) = \lambda^T(x - x_s) + \frac{1}{2}(x - x_s)^T P (x - x_s)$$

where

$$P = \begin{bmatrix} 0.4036 & 0.7048 \\ 0.7048 & 2.6490 \end{bmatrix}$$

In the simulation, we compare three EMPC configurations: (1) EMPC with the proposed terminal cost, (2) EMPC without terminal cost, and (3) EMPC with terminal cost for stability [2] where the candidate terminal cost is

$$\lambda^T(x - x_s) + \frac{1}{2}(x - x_s)^T P_s (x - x_s)$$

with $P_s = \begin{bmatrix} 117.1 & 34.1 \\ 34.1 & 132.5 \end{bmatrix}$. The initial state is $[c_A, c_B] = [0.3,\ 2.2]$. For comparison purposes, we use a short horizon $N = 2$. The closed-loop system states, inputs and stage costs are shown in Fig. 4.2, Fig. 4.3 and Fig. 3.3, respectively. The steady states of the two EMPCs with terminal cost are the optimal steady state $x_s = [0.5,\ 0.5]^T$ and $u_s = 12$, whereas the steady state of EMPC without terminal cost is slightly off the optimal with $x_s = [0.5049,\ 0.4961]^T$ and $u_s = 12.2371$. The overall performance $\bar{J} = \sum l(x, u) - l(x_s, u_s)$ of each configurations shown in Fig. 3.3 are summarized in Table 3.2. It is seen from these results that EMPC with terminal cost for stability [2] yields the fastest convergence but with relatively poor performance. EMPC without

**Table 3.2:** Process parameters

| | Proposed | No terminal cost | Terminal cost for stability |
|---|---|---|---|
| $\bar{J}$ | -177.94 | -174.13 | -142.20 |

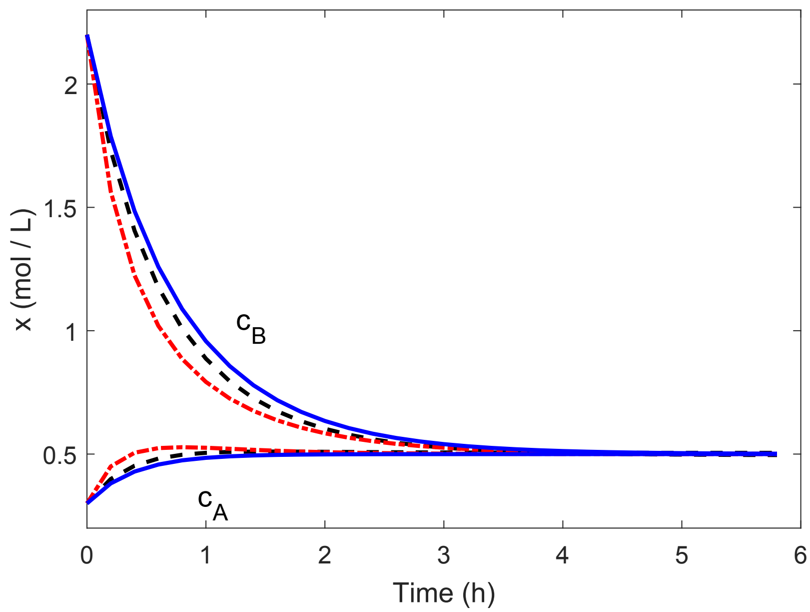


**Figure 3.1:** Closed-loop state trajectories under EMPC with the proposed terminal cost (solid lines), EMPC without terminal cost (dashed lines), and EMPC with terminal cost for stability [2] (dash-dotted line).

terminal cost provides near-optimal transient performance but with the steady state slightly off the optimal (due to the indefinite cost). EMPC with the proposed terminal cost achieves both the near-optimal transient performance and stability of the optimal steady state.

## 3.5 Summary

In this chapter, we propose a terminal cost for EMPC which preserves local optimality. The EMPC design provides a balanced solution for EMPC without terminal condition and EMPC with terminal cost for stability. Our results are derived based on the strong duality and second order sufficient condition of the steady-state optimization problem. Our future work will consider more general conditions under which suitable

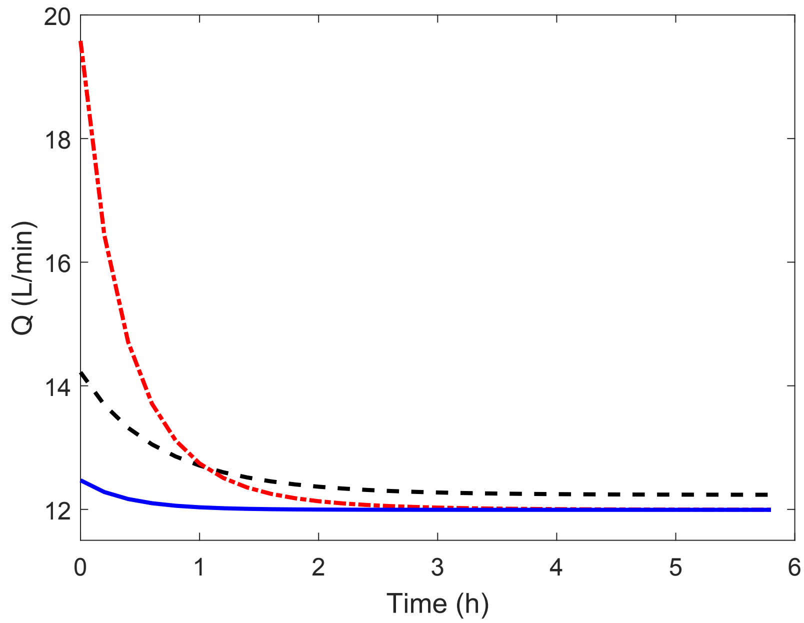


**Figure 3.2:** Closed-loop input trajectories under EMPC with the proposed terminal cost (solid lines), EMPC without terminal cost (dashed lines), and EMPC with terminal cost for stability [2] (dash-dotted line).

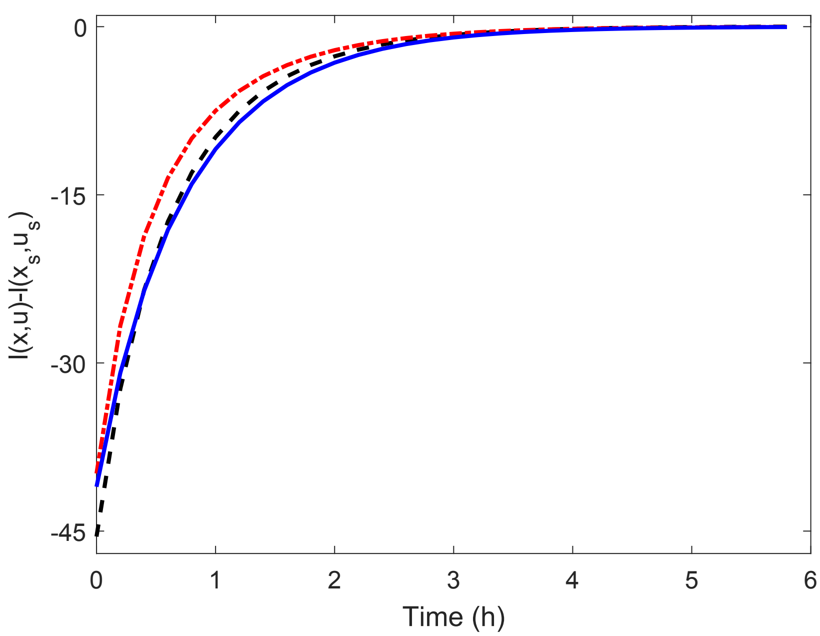


**Figure 3.3:** Closed-loop stage costs under EMPC with the proposed terminal cost (solid lines), EMPC without terminal cost (dashed lines), and EMPC with terminal cost for stability [2] (dash-dotted line).

storage functions of dissipative systems can be constructed and employed for terminal cost design.

# Chapter 4

# Economic MPC for scheduled switching operations

Switched systems consist of a set of subsystems or operating modes and a switching schedule that specifies the switching between each operating modes. In the operation of chemical processes, switching operations may arise under various situations. For instance, a change in the raw material, the hybrid nature of the system stemming from the discrete components such as switches, or the switching between different controllers necessitated by different performance criteria. Stability and design of switched systems have been a heated research area in the past decades (e.g., [41, 42, 43, 44]). Problems of interest include the controller design and stability analysis for switched systems under arbitrary switching schedule, or identifying a class of stabilizing switching schedules.

In this chapter, we extend the EMPC design in Chapter 2 for scheduled switching operations. In the proposed approach, EMPC operations are divided into two phases. If the current time is far away from the next scheduled mode switching time, EMPC takes infinite-time operation under the current operating mode. If the scheduled mode switching time is within the prediction horizon of the infinite-time EMPC design, EMPC is operated in a mode transition phase. The proposed EMPC design enjoys enlarged feasibility regions and practically improved performance than the auxiliary controllers. Sufficient conditions to ensure recursive feasibility of the proposed EMPC design are established. The simulation example of a chemical process demonstrates the applicability and effectiveness of our approach.

## 4.1 Preliminaries

### 4.1.1 Notation

Throughout this chapter, the operator $|\cdot|$ denotes the Euclidean norm of a scalar or a vector. The notation $int(\mathbb{X})$ denotes the interior of the set $\mathbb{X}$. The symbol $\mathcal{B}_r(x_s)$ denotes the open ball centered at $x_s$ with radius $r$ such that $\mathcal{B}_r(x_s) := \{x : |x - x_s| < r\}$. A continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and satisfies $\alpha(0) = 0$. A continuous function $\sigma : [0, \infty) \rightarrow [0, a)$ is said to belong to class $\mathcal{L}$ if it is strictly decreasing and satisfies $\lim_{x \to \infty} \sigma(x) = 0$. A continuous function $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{KL}$ if for each fixed $r$, $\beta(r, s)$ belongs to class $\mathcal{L}$, and for each fixed $s$, $\beta(r, s)$ belongs to class $\mathcal{K}$.

### 4.1.2 System description

We consider a class of switched nonlinear systems composed of $p$ operating modes described by the following discrete state-space model:

$$x(k+1) = f_{M(k)}(x(k), u(k)) \tag{4.1}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state vector and $u \in \mathbb{R}^{n_u}$ denotes the input vector. The function $M(k) : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{I}$ characterizes the prescribed switching schedule with $\mathbb{I} := \{1, 2, ..., p\}$. We will use $m_r^{in}$ and $m_r^{out}$ to denote the time instant when, for the $r$-th time, the system of Eq. (4.1) is switched in and out of mode $m \in \mathbb{I}$, respectively. Specifically, for $k \in [m_r^{in}, m_r^{out}]$, $M(k) = m$ and the system (4.1) is represented by $x(k+1) = f_m(x(k), u(k))$.

Under an operating mode $m \in \mathbb{I}$, the system state and input are subject to constraint $x \in \mathbb{X}_m$ and $u \in \mathbb{U}_m$ where $\mathbb{X}_m \subset \mathbb{R}^{n_x}$ and $\mathbb{U}_m \subset \mathbb{R}^{n_u}$ are compact sets. It is assumed that under each operating mode $m$, there exists an optimal steady state $(x_m^{ss}, u_m^{ss})$ that uniquely solves the following steady-state optimization problem:

$$\begin{aligned} (x_m^{ss}, u_m^{ss}) = \quad & \arg\min_{x,u} \; l_m(x, u) \\ \text{s.t.} \quad & x = f_m(x, u) \\ & x \in \mathbb{X}_m \\ & u \in \mathbb{U}_m \end{aligned} \tag{4.2}$$

where $l_m(x,u) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}$ is the economic stage cost function under the operating mode $m$.

### 4.1.3 A set of auxiliary controllers

It is assumed that under each operating mode $m \in \mathbb{I}$, there exists an explicit controller $u = h_m(x)$ which renders the optimal steady state $x_m^{ss}$ asymptotically stable with $u_m^{ss} = h_m(x_m^{ss})$ while satisfying the state and input constraints for all $x \in \mathbb{D}_m$, where $\mathbb{D}_m \subseteq \mathbb{X}_m$ is a compact set containing $x_m^{ss}$ in its interior. In the remainder, the region $\mathbb{D}_m$ will be referred to as the stability region of the controller $h_m(x)$ under mode $m$. We use $x_{h_m}(k,x)$ to denote the closed-loop state trajectory under the controller $h_m$ at time instant $k$ with the initial state $x_{h_m}(0,x) = x$. The above assumptions imply that there exists a class $\mathcal{KL}$ function $\beta_{x_m}$ such that:

$$\begin{aligned} |x_{h_m}(k,x) - x_m^{ss}| &\leq \beta_{x_m}(|x - x_m^{ss}|, k) \\ x_{h_m}(k,x) &\in \mathbb{D}_m \\ h_m(x_{h_m}(k,x)) &\in \mathbb{U}_m \end{aligned} \tag{4.3}$$

for all $k \geq 0$ and $x \in \mathbb{D}_m$. In addition, we assume that the optimal steady state of an operating mode $m$, $x_m^{ss}$, lies in the interior of the stability region of the auxiliary controller in the subsequent operating mode. This assumption is made explicit as follows:

**Assumption 10** *For any two consecutive operating modes $m, l \in \mathbb{I}$ as prescribed by $M(k)$ such that $m_r^{out} + 1 = l_q^{in}$, the following holds: $x_m^{ss} \in int(\mathbb{D}_l)$.*

Assumption 10 allows us to conclude the following result:

**Proposition 1** *If Assumption 10 holds, then there exists a finite $N_{ml}$ such that $x_{h_m}(k,x) \in \mathbb{D}_l$ for all $k \geq N_{ml}$ and $x \in \mathbb{D}_m$.*

**Proof** Since $x_m^{ss} \in int(\mathbb{D}_l)$, there exists $\epsilon > 0$ such that $\mathcal{B}_\epsilon(x_m^{ss}) \subset \mathbb{D}_l$. Taking into account the asymptotic stability of the controller $h_m(x)$ characterized in Eq. (4.3), there exists a finite $N_{ml}$ such that $\beta_{x_m}(|x - x_m^{ss}|, k) \leq \epsilon$ for all $k \geq N_{ml}$ and $x \in \mathbb{D}_m$, which implies that $x_{h_m}(k,x) \in \mathbb{D}_l$ for all $k \geq N_{ml}$ and $x \in \mathbb{D}_m$. ■

Proposition 1 shows that there is always a finite time in which the system state under the operating mode $m$ and controller $h_m(x)$ will be driven into the stability region of its subsequent operating mode $l$. Let us define the minimal time interval satisfying Proposition 1 as follows:

$$N^*_{ml} := \{\min N_{ml} : \ x_{h_m}(k, x) \in \mathbb{D}_l, \ \forall k \geq N_{ml} \geq 0, \forall x \in \mathbb{D}_m\} \tag{4.4}$$

Note that the minimal time interval $N^*_{ml}$ essentially depends on the convergence rate of the controller $h_m(x)$ as well as the overlapping of the stability regions $\mathbb{D}_m$ and $\mathbb{D}_l$. For instance, if $\mathbb{D}_m = \mathbb{D}_l$, then $N^*_{ml} = 0$. Based on $N^*_{ml}$, we make the following assumption on the lengths of the operating modes according to the prescribed schedule $M(k)$:

**Assumption 11** *For any two consecutive operating modes $m, l \in \mathbb{I}$ as prescribed by $M(k)$ such that $m^{out}_r + 1 = l^{in}_q$ , the length of the operating mode $m$ is no less than $N^*_{ml}$. That is, $m^{out}_r - m^{in}_r \geq N^*_{ml}$, where $N^*_{ml}$ is defined in Eq. (4.4).*

Under Assumption 10 and Assumption 11, the set of controllers $h_m(x)$, $m \in \mathbb{I}$ forms a feasible set of controllers for the scheduled switching operations:

**Proposition 2** *Consider the system of Eq. (4.1) in closed-loop under the scheduled switching operations $M(k)$, $k \geq 0$ where under each operating mode $m \in \mathbb{I}$, the controller $h_m(x)$ is implemented. If Assumption 10 and Assumption 11 hold, and if $x(0) \in \mathbb{D}_{M(0)}$, then the set of controllers $h_m(x)$, $m \in \mathbb{I}$ is feasible for all $k \geq 0$.*

In this chapter, we design EMPC for the scheduled switching operations based on the set of auxiliary controllers $h_m(x)$, $m \in \mathbb{I}$, that provides enlarged feasibility region and practically improved performance over the set of auxiliary controllers.

**Remark 6** *Note that Assumption 10 and Assumption 11 essentially implies the stabilizability of the optimal steady states of each operating mode of the switched system of Eq. (4.1), and that the stabilizability of the optimal steady states is not lost between mode transitions.*

## 4.2 Proposed empc for scheduled switching operations

In this section, we present the proposed EMPC design for scheduled switching operations and discuss the recursive feasibility and performance of the proposed EMPC design. In Chapter 2, a terminal cost for the EMPC design for single-mode infinite-time operation was proposed. The terminal cost is designed such that it characterizes the economic performance of an auxiliary controller $h(x)$ for $N_h$ steps. Adopting such a terminal cost extends the prediction horizon of the EMPC design by $N_h$ steps beyond the control horizon $N$. In this chapter, we extend the previous results in [45] to account for scheduled switching operations. In the proposed EMPC design, we adopt the same control horizon $N$ while a different terminal cost horizon $N_{h_m}$ is assigned for each operating mode $m \in \mathbb{I}$ based on the auxiliary controller $h_m(x)$. Specifically, the terminal cost function for the infinite-time EMPC design under mode $m$, $c_m(x, N_{h_m})$, is defined as follows:

$$c_m(x, N_{h_m}) = \sum_{k=0}^{N_{h_m}-1} l(x_{h_m}(k, x), h_m(x_{h_m}(k, x))) \tag{4.5}$$

where $x \in \mathbb{D}_m$. The cost function $c_m(x, N_{h_m})$ above characterizes the economic performance of the closed-loop system under mode $m \in \mathbb{I}$ under the controller $h_m(x)$ for $N_{h_m}$ steps with the initial state $x \in \mathbb{D}_m$.

### 4.2.1 Implementation strategy

The implementation of the proposed EMPC design is divided into two phases. In one phase when the next scheduled mode switching time is beyond the prediction horizon (i.e., control horizon plus the terminal cost horizon), EMPC takes infinite-time operation under the current mode; when the next scheduled mode switching time is within the prediction horizon, EMPC is operated under a mode-switching phase. The implementation strategy of the proposed EMPC between two consecutive operating modes $m \in \mathbb{I}$ and $l \in \mathbb{I}$ such that $m_r^{out} + 1 = l_q^{in}$ is outlined in Algorithm 1 below:

**Algorithm 1**

1. *Infinite-time operation under mode $m$.* At a sampling time $k \in [m_r^{in},\ m_r^{out} - N - N_{h_m}]$ when the scheduled mode switching time $l_q^{in}$ is beyond the prediction horizon $k + N + N_{h_m}$, EMPC takes the infinite-time operation of Eq. (4.6).

2. *Mode transition operation between $m$ and $l$*

2.1 At a sampling time $k \in [m_r^{out} - N - N_{h_m} + 1,\ m_r^{out} - N + 1]$ when $l_q^{in}$ is within the prediction horizon $k + N + N_{h_m}$ but beyond the control horizon $k + N - 1$, EMPC takes the mode-switching operation of Eq. (4.7).

2.2 At a sampling time $k \in [m_r^{out} - N + 2,\ m_r^{out}]$ when $l_q^{in}$ is within the control horizon $k + N - 1$, EMPC takes the mode switching operation of Eq. (4.8).

3. *Infinite-time operation under mode $l$.* At a sampling time $k \in [l_q^{in},\ l_q^{out} - N - N_{h_l}]$, EMPC takes the infinite-time operation of Eq. (4.6) (with $m$ replaced with $l$).

A schematic of Algorithm 1 is illustrated in Fig. 4.1. It can be seen that when

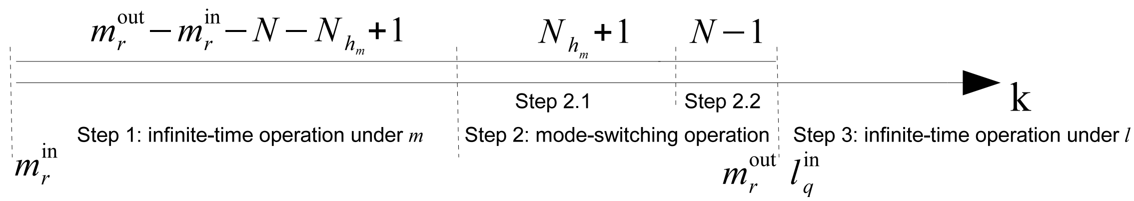


**Figure 4.1:** Time flow of Algorithm 1

an operating mode $m$ is switched in, the proposed EMPC first takes infinite-time operation for $m_r^{out} - m_r^{in} + 1 - N - N_{h_m}$ steps and then takes mode transition operations for $N + N_{h_m}$ steps before the system is switched to the subsequent operating mode $l$. Note that Algorithm 1 implies that the length of an operating mode, $m_r^{out} - m_r^{in} + 1$ is no shorter than the prediction horizon, $N + N_{h_m}$, of the infinite-time EMPC operation under mode $m$. This assumption is made explicit below.

**Assumption 12** *The length of each operating mode $m \in \mathbb{I}$ in the prescribed schedule $M(k)$ is no shorter than the prediction horizon of the infinite-time EMPC operation under the operating mode $m$. That is, $m_r^{out} - m_r^{in} + 1 \geq N + N_{h_m}$, $\forall m \in \mathbb{I}, \forall r \geq 1$.*

**Remark 7** *Assumption 12 assumes that the length of each operating mode under the prescribed schedule $M(k)$ is long enough such that each step outlined in Algorithm 1 is carried out. Note that it is a common practice in the control of chemical processes that the length of an scheduled plan is much longer than the transient settling time of the system. In this case, at the beginning of an operating mode, the scheduled mode switching is far away and has little effect on the current control decisions. It is then a natural strategy to assume infinite-time operation at the beginning and to deal with mode transition when the scheduled switching time approaches.*

### 4.2.2 Proposed EMPC design

At a sampling time $k \in [m_r^{in},\ m_r^{out} - N - N_{h_m}]$, EMPC takes infinite-time operation under mode $m$ and is formulated as the following optimization problem:

$$\min_{u(i)} \sum_{i=k}^{k+N-1} l_m(\tilde{x}(i), u(i)) + c_m(\tilde{x}(N), N_{h_m}) \tag{4.6a}$$

$$\text{s.t. } \tilde{x}(i+1) = f_m(\tilde{x}(i), u(i)), \quad i = k, ..., k+N-1 \tag{4.6b}$$

$$\tilde{x}(k) = x(k) \tag{4.6c}$$

$$\tilde{x}(i) \in \mathbb{X}_m, \quad i = k, ..., k+N-1 \tag{4.6d}$$

$$u(i) \in \mathbb{U}_m, \quad i = k, ..., k+N-1 \tag{4.6e}$$

$$\tilde{x}(k+N) \in \mathbb{D}_m \tag{4.6f}$$

where $\tilde{x}(i)$ denotes the predicted state trajectory, $x(k)$ is the state measurement at time instant $k$. In the above optimization problem, Eq. (4.6a) is the objective function which is the accumulated economic performance over the prediction horizon $N$ plus the performance when $h_m(x)$ takes over for another $N_{h_m}$ steps under the operating mode $m$. Eq. (4.6b) is the system model under the operating mode $m$. Eq. (4.6c) specifies the initial condition at time instant $k$. Eqs. (4.6d) and (4.6e) are the state and input constraints corresponding to the operating mode $m$. Eq. (4.6f) requires that the predicted system state is driven into the terminal region $\mathbb{D}_m$ at the end of the control horizon $N$.

The infinite-time operation under mode $m$ is carried out until the prediction horizon, $k + N + N_{h_m}$, reaches the mode switching time $m_r^{out}$. At the time instant $k = m_r^{out} - N - N_{h_m} + 1$, the EMPC operation is switched into a mode switching phase which is divided into two stages as outlined in step 2.1 and 2.2 of Algorithm 1. In the first stage when $k \in [m_r^{out} - N - N_{h_m} + 1,\ m_r^{out} - N + 1]$, the proposed EMPC is formulated as follows:

$$\min_{u(i)} \sum_{i=k}^{m_r^{out}} l_m(\tilde{x}(i), u(i)) + c_l(\tilde{x}(l_q^{in}), N_{h_l}) \tag{4.7a}$$

$$\text{s.t. } \tilde{x}(i+1) = f_m(\tilde{x}(i), u(i)), \quad i = k, ..., m_r^{out} \tag{4.7b}$$

$$\tilde{x}(k) = x(k) \tag{4.7c}$$

$$\tilde{x}(i) \in \mathbb{X}_m, \quad i = k, ..., k+N-1 \tag{4.7d}$$

$$u(i) \in \mathbb{U}_m, \quad i = k, ..., k+N-1 \tag{4.7e}$$

$$\tilde{x}(k+N) \in \mathbb{D}_m \tag{4.7f}$$

$$u(i) = h_m(\tilde{x}(i)), \quad i = k+N, ..., m_r^{out} \tag{4.7g}$$

$$\tilde{x}(l_q^{in}) \in \mathbb{D}_l \tag{4.7h}$$

In the above optimization problem, EMPC optimizes the economic performance under the operating mode $m$ for $N$ steps before the auxiliary controller $h_m(x)$ takes over till the end of the operating mode $m$ (Eq. (4.7g)). The constraint of Eq. (4.7h) is imposed to ensure recursive feasibility of the EMPC design for mode switching between the operating modes $m$ and $l$. The terminal cost $c_m(\tilde{x}(l_q^{in}), N_{h_l})$ is incorporated into the objective function to further extend the prediction horizon from the operating mode $m$ into its subsequent operating mode $l$ for $N_{h_l}$ steps. Note that in this stage, the number of free decision variables in the optimization problem of Eq. (4.7) is constantly $N \times n_u$, and the interval where the auxiliary controller $h_m(x)$ is implemented shrinks from $N_h$ to 0 as the sampling time $k$ increases.

In the second stage when $k \in [m_r^{out} - N + 2,\ m_r^{out}]$, the control horizon covers both the operating modes $m$ and $l$. The proposed EMPC design is formulated as follows:

$$\min_{u(i)} \sum_{i=k}^{m_r^{out}} l_m(\tilde{x}(i), u(i)) + \sum_{i=l_q^{in}}^{k+N-1} l_l(\tilde{x}(i), u(i)) + c_l(\tilde{x}(k+N), N_{h_l}) \tag{4.8a}$$

$$\text{s.t. } \tilde{x}(i+1) = f_m(\tilde{x}(i), u(i)), \quad i = k, ..., m_r^{out} \tag{4.8b}$$

$$\tilde{x}(i+1) = f_l(\tilde{x}(i), u(i)), \quad i = l_q^{in}, ..., k+N-1 \tag{4.8c}$$

$$\tilde{x}(k) = x(k) \tag{4.8d}$$

$$\tilde{x}(i) \in \mathbb{X}_m, \quad i = k, ..., k_r^{out} \tag{4.8e}$$

$$\tilde{x}(i) \in \mathbb{X}_l, \quad i = l_q^{in}, ..., k+N-1 \tag{4.8f}$$

$$u(i) \in \mathbb{U}_m, \quad i = k, ..., k_r^{out} \tag{4.8g}$$

$$u(i) \in \mathbb{U}_l, \quad i = l_q^{in}, ..., k+N-1 \tag{4.8h}$$

$$\tilde{x}(k+N) \in \mathbb{D}_l \tag{4.8i}$$

**Remark 8** *Note that the EMPC of Eq. (4.6) for infinite-time operation enjoys an enlarged operating region than the design in [45] in which the operating region is restricted to be the stability region of the auxiliary controller. Note also that in the*

*EMPC designs of Eqs. (4.6)-(4.8), the number of free decision variables are all $N \times n_u$ which means that the proposed EMPC scheme for switching operations can be very computationally efficient compared to that of [3].*

**Remark 9** *It is shown in [45] that the proposed EMPC design for infinite-time operation achieves practically improved asymptotic average performance than the optimal steady state operation. Similar analysis can be applied to the proposed EMPC of Eq. (4.6). That is, if the terminal cost horizon $N_{h_m}$ is long enough and the cost $l_m(x, h_m(x))$ is sufficiently smooth such that $l_m(x, h_m(x)) \approx l_m(x_m^{ss}, u_m^{ss})$ for all $x \in D_m^*$ where $D_m^* = \{x_{h_m}(N_{h_m}, x) : x \in \mathbb{D}_m\}$, then the EMPC of Eq. (4.6) achieves practically improved average performance than the optimal steady state operation in Step 2.1 of Algorithm 1 given that this phase is long enough.*

### 4.2.3 Recursive feasibility

In this subsection, we analyze the recursive feasibility of the proposed EMPC designs. Before presenting the results, we need the following assumptions on the length of each terminal cost horizon $N_{h_m}$ of each operating mode $m \in \mathbb{I}$.

**Assumption 13** *The terminal cost horizon for each operating mode $m \in \mathbb{I}$ is chosen such that $N_{h_m} \geq \max\{N_{ml}^*\}$ where $N_{ml}^*$ is defined in Eq. (4.4) and $l \in \mathbb{I}$ is any subsequent operating mode of $m$ according to the schedule $M(k)$.*

In the following, we show that under the proposed EMPC scheme for scheduled switching operations, initial feasibility implies recursive feasibility.

**Theorem 10** *Consider the system of Eq. (4.1) under the prescribed switching schedule $M(k)$ with $k \geq 0$. Assume that at a time instant $k_0 \geq 0$, the proposed EMPC implementation strategy outlined in Algorithm 1 is switched on and then implemented for all $k \geq k_0$. If Assumption 12 and Assumption 13 hold, and if the EMPC design at $k_0$ is feasible, then the proposed EMPC scheme for scheduled switching operations remains feasible for all $k \geq k_0$.*

**Proof.** We provide a sketch of the proof by showing that for any time instant $k \in [m_r^{in}, m_r^{out}]$, the feasibility of the EMPC design at $k$ implies the feasibility of the EMPC design at $k+1$.

(1) If the EMPC of Eq. (4.6) is feasible at $k = m_r^{in}$, it is recursively feasible for $k \in [m_r^{in},\ m_r^{out} - N - N_{h_m}]$. The recursive feasibility of the EMPC of Eq. (4.6) is a direct consequence of the forward-invariance property of the auxiliary controller $h_m(x)$ inside the terminal region $\mathbb{D}_m$.

(2) If the EMPC of Eq. (4.6) is feasible $k = m_r^{out} - N - N_{h_m}$, then the EMPC of Eq. (4.7) at $k = m_r^{out} - N - N_{h_m} + 1$ is also feasible. A feasible solution can be constructed based on the solution at $k = m_r^{out} - N - N_{h_m}$ in a standard way. Namely, by discarding the first element of the optimal input sequence at $k = m_r^{out} - N - N_{h_m}$ and augmenting the sequence by implementing $h_m(x)$ for another $N_{h_m} + 1$ steps at the end of the sequence. Specifically, it can be verified that the constraint of Eq. (4.7f) is satisfied because of the forward-invariant property of the auxiliary controller $h_m(x)$ inside $\mathbb{D}_m$. Also, the constraint of Eq. (4.7h) is satisfied because of Assumption 12 and that $h_m(x)$ is implemented for $N_{h_m} + 1$ steps in the constructed input sequence.

(3) If the EMPC of Eq. (4.7) is feasible at $k = m_r^{out} - N - N_{h_m} + 1$, it is recursively feasible for $k \in [m_r^{out} - N - N_{h_m} + 1,\ m_r^{out} - N + 1]$. In step 2.1, the EMPC design of Eq. (4.7) is implemented in a shrinking horizon fashion (the interval where $h_m(x)$ is implemented shrinks ). The solution to a current optimization problem will provide a feasible solution to the optimization at the next sampling instant by discarding its first element.

(4) If the EMPC of Eq. (4.7) is feasible at $k = m_r^{out} - N + 1$, then the EMPC of Eq. (4.8) at $k = [m_r^{out} - N + 2, m_r^{out}]$ is recursively feasible. Again, the standard approach to construct a feasible solution for the optimization problem at $k + 1$ can be applied which is to discard the first element of the optimal input sequence at $k$ and augment one more input at the end by implementing $h_l(x)$ one more step.

(5) If the EMPC of Eq. (4.8) at $k = m_r^{out}$ is feasible, then the EMPC of Eq. (4.6) at $k = l_q^{in}$ (with $m$ replace with $l$) is also feasible. Again, the same approach to construct a feasible solution based on $h_l(x)$ can be applied.

Thus, for any time instant $k \in [m_r^{in}, m_r^{out}]$, the feasibility of the EMPC design at $k$ implies the feasibility of the EMPC design at $k + 1$. Since the mode $m \in \mathbb{I}$ and $r \geq 1$ are arbitrarily chosen, the above analysis can be applied to every time instant $k \geq 0$ under the prescribed schedule $M(k)$. Therefore, under the proposed EMPC scheme, initial feasibility implies recursive feasibility. ■

## 4.3 Application to a chemical process

Consider a well-mixed, non-isothermal continuous stirred tank reactor (CSTR) in which two irreversible, second-order, endothermic reactions $A \rightarrow B$ and $B \rightarrow C$ take place. Due to the nonisothermal nature of the reactor, a jacket is used to provide heat to the reactor. The dynamic equations describing the behavior of the reactor, obtained through material and energy balances under standard modeling assumptions, are given below:

$$\begin{aligned}
\frac{\mathrm{d}C_A}{\mathrm{d}t} &= \frac{F}{V}(C_{A0} - C_A) - k_1 e^{\frac{-E_1}{RT}} C_A^2 \\
\frac{\mathrm{d}C_B}{\mathrm{d}t} &= -\frac{F}{V}C_B + k_1 e^{\frac{-E_1}{RT}} C_A^2 - k_2 e^{\frac{-E_2}{RT}} C_B^2 \\
\frac{\mathrm{d}C_C}{\mathrm{d}t} &= -\frac{F}{V}C_C + k_2 e^{\frac{-E_2}{RT}} C_B^2 \\
\frac{\mathrm{d}T}{\mathrm{d}t} &= -\frac{F}{V}(T_0 - T) + \frac{-\Delta H_1}{\rho C_p} k_1 e^{\frac{-E_1}{RT}} C_A^2 \\
& \quad + \frac{-\Delta H_2}{\rho C_p} k_2 e^{\frac{-E_2}{RT}} C_B^2 + \frac{Q}{\rho C_p V}
\end{aligned} \tag{4.9}$$

where $C_A$, $C_B$ and $C_C$ denote the concentrations of the reactant $A$, product $B$ and product $C$; $T$ denotes the temperature of the reactor; $Q$ denotes the heat supply to the reactor; $V$ represents the volume of the reactor, $\Delta H_1$, $k_1$, $E_1$ and $\Delta H_2$, $k_2$, $E_2$ denote the enthalpy, pre-exponential constant and activation energy of the reactions $A \rightarrow B$ and $B \rightarrow C$, respectively; $C_p$ and $\rho$ denote the heat capacity and the density of the fluid in the reactor, respectively. The values of the process parameters used in the simulations are shown in Table 6.2. The process model of Eq. (4.9) belongs to the following class of nonlinear systems: $\dot{x} = f(x, u)$ where the state vector is $x = [C_A, C_B, C_C, T]^T$, the control input is $u = Q$.

In this example, two operating modes are specified. In mode 1, the objective function is $l_1 = -C_B$, which is to maximize the product of $B$. In mode 2, the objective function is $l_2 = -C_C$, which is to maximize the product of $C$. Under the two operating modes, the system models and the input and state constraints remain the same. Specifically, $f_1 = f_2 = f$, $\mathbb{X}_1 = \mathbb{X}_2 = \mathbb{R}^4$, $\mathbb{U}_1 = \mathbb{U}_2 = \{u : [0, 5 \times 10^5] \ kJ/h\}$.

By solving the steady-state optimization problem of Eq. (4.2), we obtain the optimal steady states under the two operating modes as follows: $x_1^{ss} = [1.35, 1.98, 0.67, 188.92]$,

**Table 4.1:** Model parameters.

| | | | |
|---|---|---|---|
| $T_0 = 300$ | K | $k_1 = 176.94$ | $\frac{1}{\text{h}}$ |
| $C_{A0} = 4$ | $\frac{\text{kmol}}{\text{m}^3}$ | $k_2 = 10.88$ | $\frac{1}{\text{h}}$ |
| $V = 1$ | $\text{m}^3$ | $E_1 = 5000$ | $\frac{\text{kJ}}{\text{kmol}}$ |
| $F = 5$ | $\frac{\text{m}^3}{\text{h}}$ | $E_2 = 4000$ | $\frac{\text{kJ}}{\text{kmol}}$ |
| $C_p = 0.231$ | $\frac{\text{kJ}}{\text{kgK}}$ | $\Delta H_1 = 1.15 \times 10^4$ | $\frac{\text{kJ}}{\text{kmol}}$ |
| $\rho = 1000$ | $\frac{\text{kg}}{\text{m}^3}$ | $\Delta H_2 = 1.05 \times 10^4$ | $\frac{\text{kJ}}{\text{kmol}}$ |
| $R = 8.314$ | $\frac{\text{kJ}}{\text{kmolK}}$ | | |

$u_1^{ss} = 5.95 \times 10^4$ and $x_2^{ss} = [0.61, 1.59, 1.80, 448.63]$, $u_2^{ss} = 5 \times 10^5$. Since the system of Eq. (4.9) is open-loop stable, we use the open-loop optimal steady-state inputs as the auxiliary controllers with $h_1(x) = u_1^{ss}$ and $h_2(x) = u_2^{ss}$. The two auxiliary controllers are globally stabilizing with $\mathbb{D}_1 = \mathbb{D}_2 = \mathbb{R}^4$.

The system of Eq. (4.9) is discretized with a sampling time $\Delta = 0.1h$. The operating policy for the time interval of $t \in [0,\ 6h]$, or $k \in [0, 59]$, is scheduled as follows:

$$M(k) = \begin{cases} 1, & 0 \leq k \leq 29 \\ 2, & 30 \leq k \leq 59 \end{cases}$$

In the simulations, we compare the proposed EMPC scheme for scheduled switching operations with the approach in [3] as well as the auxiliary controllers. The initial state is $x(0) = [0.61, 1.59, 1.80, 448.63]$. The closed-loop state and input trajectories under the three approaches are shown in Fig. 4.2 and Fig. 4.3 respectively.

Let us define the following performance indexes to better compare the performance of each control configurations. The performance under operating mode 1 for $t \in [0h, 3h)$ is $J_1 = \sum_{k=0}^{29} C_B(k)$. The performance under operating mode 2 for $t \in [3h, 6h]$ is $J_2 = \sum_{k=30}^{59} C_C(k)$. The overall performance for $t \in [0h, 6h]$ is $J_{all} = J_1 + J_2$. The performance during the mode switching for $t \in [2.5h, 3.5h]$ is $J_{switch} = \sum_{k=25}^{29} C_B(k) + \sum_{k=30}^{34} C_C(k)$. These performance indexes under the three control configurations are shown in Table 4.2. In addition, the average controller evaluation times of the proposed approach and that of the approach in [3] are shown in Table 4.3

From these results, it can be seen that the three control configurations provide

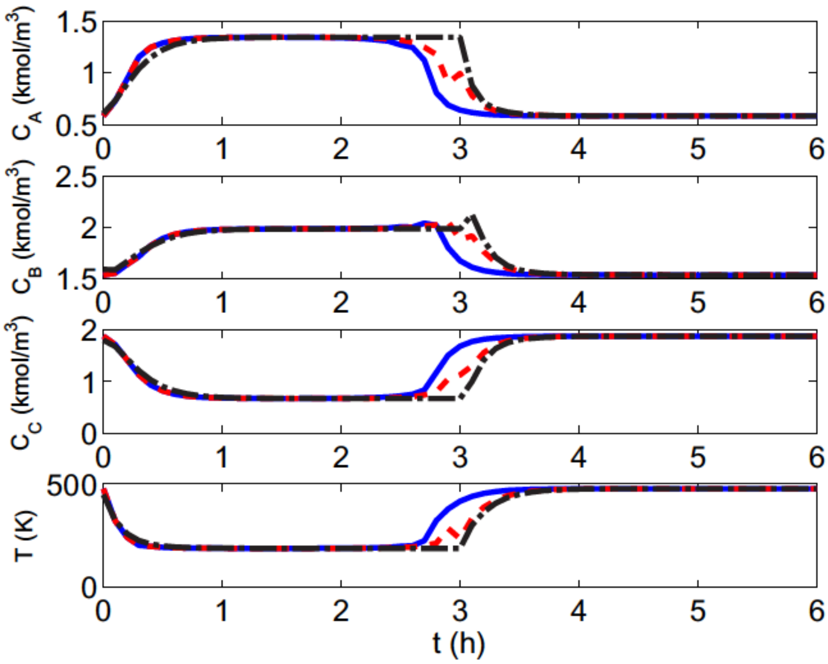


**Figure 4.2:** Closed-loop state trajectories of the proposed approach (solid lines), the approach in [3] (dashed lines) and the auxiliary controllers (dash-dotted line).

approximately the same performance for single-mode operation (i.e., when a current operating time is far away from the scheduled mode switching time) while the main differences occur during the mode transition operations. The EMPC proposed in [3] achieves the optimal performance under operating mode 1 for $t \in [0h, 3h)$ because it is initialized with a control horizon of the length of the operating mode 1 and is implemented in a shrinking horizon fashion. However, the performance between mode switching is compromised since it is not directly accounted for in the EMPC in [3]. The proposed EMPC design for scheduled mode switching operations provide the best performance during the mode switching operations and the best overall performance. This is mainly due to the extended prediction horizon based on the auxiliary controllers to account for the performance between mode transitions. It can be also seen from Table 4.3 that the proposed approach is more computationally efficient compared to the EMPC design in [3].

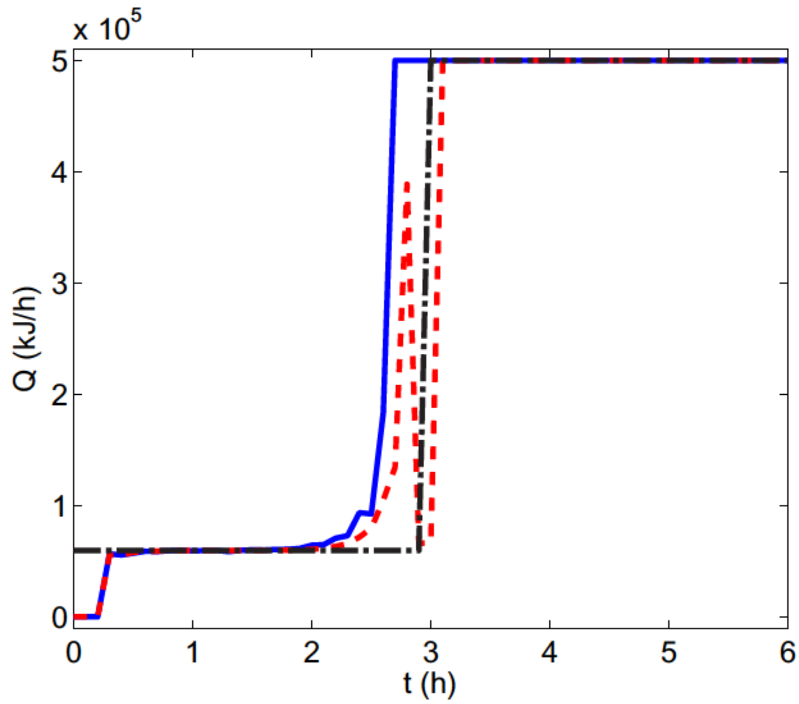


**Figure 4.3:** Closed-loop input trajectories of the proposed approach (solid line), the approach in [3] (dashed line) and the auxiliary controllers (dash-dotted line).

**Table 4.2:** Performance under different configurations

| | EMPC in [3] | Auxiliary controllers | Proposed |
|---|---|---|---|
| $J_1$ | 57.6 | 57.8 | 57.7 |
| $J_2$ | 55.9 | 54.3 | 53.2 |
| $J_{all}$ | 113.5 | 112.1 | 110.9 |
| $J_{switch}$ | 18.8 | 17.6 | 16.4 |

**Table 4.3:** Average controller evaluation time

| | Proposed | EMPC in [3] |
|---|---|---|
| $time(s)$ | 0.8 | 18.1 |

# Chapter 5

# Nonlinear MPC for zone tracking

## 5.1 Introduction

Model predictive control (MPC) is the most widely applied advanced control technique in the process industry due to its many advantages such as optimally handling process constraints and interaction. In the traditional paradigm, MPC fulfills the objective of set-point tracking by penalizing the deviation of the predicted state and input trajectories to the desired set-point. In practice, it is often beneficial to design MPC that tracks a zone region instead of a set-point. The superiority of zone control over set-point control are two-folds: first, it is capable of flexibly handling multiple objectives so that the degrees of freedom in the controller can be assigned to address more important objectives; second, it is inherently robust in the presence of model mismatch or process disturbances. As a matter of fact, all industrial MPCs are equipped with zone control option to some extent [4]. Successful applications of zone MPC have been reported in various areas such as diabetes treatment [46], control of building heating system [47], and control of the fluid catalytic cracking unit in oil refinery [48].

Despite the widely successful industrial applications, there has not been a systematic approach for the design and analysis of MPC for zone tracking. On the one hand, reported industrial zone MPC designs are heuristic in nature and lack stability guarantee. On the other hand, existing zone MPC designs with guaranteed stability [49], [50] essentially convert the zone tracking objective to tracking of the steady-state subspace in the target zone. The trick is to cast the steady-state set-point also as decision variables in the set-point tracking MPC scheme. However, tracking of a target zone is not equivalent to tracking of its steady-state subspace. As far as the satisfaction of a zone target is concerned, the system does not necessarily take steady-state operation.

Several difficulties arise when it comes to the design and analysis of MPC which tracks a zone target in a straightforward manner. First of all, the admissibility of a zone target needs to be carefully examined. Since the target zone is often user-specified based on the economic metrics of the process, it is not necessarily positive invariant. One might be tempted to find the largest positive invariant set in the target zone and convert the zone tracking objective to tracking of its largest positive invariant subset. Unfortunately, finding and characterizing the maximal positive invariant set in the target zone can be very difficult or expensive even for linear systems [51]. Even if the largest positive can be found, the aforementioned approach may not be desirable and may give deteriorated transient performance as shown by our simulation study.

In this work, we propose a general framework for nonlinear model predictive zone control. The proposed zone MPC tracks a target zone characterized by coupled system state and input. A control invariant subset of the target zone is incorporated in the proposed zone MPC as the terminal constraint to ensure closed-loop stability. We resort to the invariance principle and develop invariance-like theorem which is suitable for stability analysis of zone control. It is proved that under the proposed zone MPC design, the system converges to the largest control invariant subset of the target zone. In the stability analysis, we focus on the evolvement of the state-input pair $(x(n), u(n))$ instead of merely the state trajectory $x(n)$. This provides a better picture and more accurate description of the system behavior, especially when the input trajectory is also of interest. Further discussions are made on enlargement of the region of attraction by employing an auxiliary controller as well as handling a secondary economic objective via a second-step economic optimization. Two numerical examples are used to demonstrate the superiority of zone control over set-point control and the efficacy of the zone MPC design.

## 5.2 Problem setup

### 5.2.1 Notation

Throughout this work, the operator $|\cdot|$ denotes the Euclidean norm of a scalar or a vector. The symbol $|x|_{\mathbb{A}} := \inf_{z\in\mathbb{A}} |x-z|$ denotes the distance of a point $x$ to the set $\mathbb{A}$. The symbol $\mathcal{B}_\epsilon(\mathbb{X}) := \{x : |x|_{\mathbb{X}} < \epsilon\}$ denotes the set of points whose distance to the set $\mathbb{X}$ is less than $\epsilon$. The symbol '$\setminus$' denotes set substraction such that $\mathbb{A}\setminus\mathbb{B} := \{x \in \mathbb{A}, x \notin \mathbb{B}\}$. $V_\epsilon(x)$ denotes the $\epsilon$ sublevel set of the function $V(x)$ such that $V_\epsilon(x) = \{x : V(x) \leq \epsilon\}$. The symbol $\mathbb{I}_M^N$ denotes the set of integers $\{M, M+1, ..., N\}$. The symbol $\mathbb{I}_{\geq 0}$ denotes the set of non-negative integers $\{0, 1, 2, ...\}$. The symbol $proj_{\mathbb{X}}(\mathbb{O})$ denotes the projection of the set $\mathbb{O}$ onto its subspace $\mathbb{X}$. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and satisfies $\alpha(0) = 0$. A class $\mathcal{K}$ function $\alpha$ is called a class $\mathcal{K}_\infty$ function if $\alpha$ is unbounded.

### 5.2.2 System description and control objective

We consider the following nonlinear discrete time system:

$$x(n+1) = f(x(n), u(n)) \tag{5.1}$$

where $x(n) \in \mathbb{X} \subset \mathbb{R}^{n_x}$, $u(n) \in \mathbb{U} \subset \mathbb{R}^{n_u}$, $n \in \mathbb{I}_{\geq 0}$, denote the state and input at time $n$, respectively. The vector function $f(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ is continuous. The system is subject to coupled state and input constraint: $(x, u) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$, where $\mathbb{X}$, $\mathbb{U}$, $\mathbb{Z}$ are all compact sets. The control objective is to steer and maintain the system in a compact set $\mathbb{Z}_t \subset \mathbb{Z}$. The target set $\mathbb{Z}_t$ may correspond to a desired range of process state, input or output specified by user based on economic metrics of the process. We say that the target set $\mathbb{Z}_t$ is admissible if there exist complete state and input trajectories bounded in $\mathbb{Z}_t$ such that $(x(n), u(n)) \in \mathbb{Z}_t$, $n \in \mathbb{I}_{\geq 0}$. To address the admissibility of the zone control objective, we need the following definition:

**Definition 3** *(control invariant set) A set $\mathbb{O} \subseteq \mathbb{X} \times \mathbb{U}$ is called control invariant or a control invariant set if any $(x, u) \in \mathbb{O}$ implies $f(x, u) \in proj_{\mathbb{X}}(\mathbb{O})$.*

Since the target set $\mathbb{Z}_t$ is user-defined based on economic metrics of the process, it is not necessarily control invariant. A reasonable assumption we make in this work is that the target set $\mathbb{Z}_t$ is admissible. This is equivalent to assuming the existence of a nonempty control invariant subset of $\mathbb{Z}_t$. Apparently, any admissible steady-state pair $(x_s, u_s) \in \mathbb{Z}_t$ with $x_s = f(x_s, u_s)$ constitutes a control invariant subset of $\mathbb{Z}_t$. The conventional set-point control can be viewed as a special case of zone control where the target set degenerates to a singleton $\mathbb{Z}_t = \{(x_s,\ u_s)\}$.

**Remark 10** *In the classical definition, the control invariant set refers to a set of system state which is positive invariant under certain control policy (e.g., [32] Definition 2.10). In Definition 3, the control invariant set $\mathbb{O} \subseteq \mathbb{X} \times \mathbb{U}$ is defined in the augmented state-input space. We can think of the control invariant set $\mathbb{O}$ as induced by the control invariant state set $proj_{\mathbb{X}}(\mathbb{O})$ in the classical definition which is positive invariant under the set-valued control policy $(x, u) \in \mathbb{O}$. The reason to consider the augmented state-input space will become clear along the discussion as it provides a better picture and more accurate description of system behavior for zone control.*

## 5.3 Zone MPC formulation

At a sampling time $n$, the proposed zone MPC is formulated as the following optimization problem:

$$V_N^0(x(n)) = \min_{u_i, x_i^z, u_i^z} \sum_{i=0}^{N-1} |x_i - x_i^z|_Q^2 + |u_i - u_i^z|_R^2 \tag{5.2a}$$

$$\text{s.t. } x_{i+1} = f(x_i, u_i), \quad i \in \mathbb{I}_0^{N-1} \tag{5.2b}$$

$$x_0 = x(n) \tag{5.2c}$$

$$(x_i, u_i) \in \mathbb{Z}, \quad i \in \mathbb{I}_0^{N-1} \tag{5.2d}$$

$$(x_i^z, u_i^z) \in \mathbb{Z}_t, \quad i \in \mathbb{I}_0^{N-1} \tag{5.2e}$$

$$x_N \in \mathbb{X}_f \tag{5.2f}$$

The objective function Eq. (5.2a) penalizes the distance from the predicted state and input trajectories to the target zone $\mathbb{Z}_t$ by introducing artificial variables $x_i^z$ and $u_i^z$ bounded by Eq. (5.2e). $Q$, $R$ are positive-definite weighting matrices. The terminal set $\mathbb{X}_f \subseteq proj_{\mathbb{X}}(\mathbb{Z}_t)$ is designed to ensure recursive feasibility and stability of the zone MPC design. Specifically, $\mathbb{X}_f$ satisfies the following control invariant condition:

**Assumption 14** *For any $x \in \mathbb{X}_f$, there exists a $u$ such that $(x, u) \in \mathbb{Z}_t$ and $f(x, u) \in \mathbb{X}_f$*

Let $u^*(i+n|n)$, $i \in \mathbb{I}_0^{N-1}$, denote the optimal solution to Eq. (5.2). The corresponding state trajectory is $x^*(i+n|n)$, $i \in \mathbb{I}_0^{N-1}$. The input injected to the system at time $n$ is: $u(n) = u^*(n|n)$. At time $n+1$, the optimization of Eq. (5.2) is re-evaluated. Note that the solution of $u^*(n|n)$ may not be unique because of the nature of the zone tracking objective. This means that both the input and state trajectories of the closed-loop system may be set-valued.

## 5.4 Stability analysis

In this section, we discuss the stability of the proposed zone MPC design. Several difficulties arise when it comes to the stability analysis of zone MPC. First, the target set $\mathbb{Z}_t \subset \mathbb{X} \times \mathbb{U}$ involves zone regions for both system state and input. Unlike set-point control where asymptotic tracking of the steady state (i.e., $x(n) \to x_s$) usually implies asymptotic tracking of the corresponding steady-state input (i.e., $u(n) \to u_s$), in zone control it is either insufficient or inappropriate to only consider the evolvement of the state trajectory $x(n)$. The trajectory $(x(n), u(n))$ in the augmented state-input space should be investigated. Second, the target set $\mathbb{Z}_t$ may not be control invariant. This means that we cannot establish Lyapunov stability for the target set $\mathbb{Z}_t$ by finding a suitable Lyapunov function. Instead, we will resort to the more general tools by LaSalle's invariance principle.

In this section, we will first develop invariance-like theorem which generalizes LaSalle's invariance principle to cope with the analysis of control systems of Eq. (5.1) based on the control invariant set as defined in Definition 3. Then we establish stability of the proposed zone MPC design. It is proved that the system converges to the largest control invariant set in the target set $\mathbb{Z}_t$.

**Definition 4** *(Accumulation point and accumulation set) A point $x$ is called an accumulation point of the sequence $x(n)$ if there is an increasing sequence of integers $n_i$, $i \in \mathbb{I}_{\geq 0}$, such that $\lim_{i \to \infty} x(n_i) = x$. The accumulation set $\mathbb{S}_{x(n)}$ of a sequence $x(n)$ is the set of all accumulation points of $x(n)$.* *

*In the context of autonomous dynamical systems (i.e., $x_{n+1} = f(x_n)$), accumulation point and

**Definition 5** *(Attractive set) A set* $\mathbb{S}$ *is called attractive or an attractive set of a sequence* $x(n)$ *if:* $\lim_{n\to\infty} |x(n)|_{\mathbb{S}} = 0$.

In short, the accumulation point of a sequence $x(n)$ is the limit of a subsequence of $x(n)$. The attractive set of $x(n)$ is the set that $x(n)$ approaches as $n \to \infty$. For notational simplicity, we will use $x(n) \to \mathbb{S}$ to denote that $\mathbb{S}$ is attractive. The following result is a simplification of Theorem 5.2 in [52]. We provide the proof here for the completeness of this work.

**Lemma 17** *If the sequence* $x(n)$ *is bounded in a compact set* $\mathbb{X}$ *for all* $n \in \mathbb{I}_{\geq 0}$*, then the accumulation set* $\mathbb{S}_{x(n)}$ *is nonempty, compact, and is the smallest closed attractive set of* $x(n)$.

**Proof.** Since $x(n)$ is bounded in $\mathbb{X}$, by the Bolzano – Weierstrass Theorem, $x(n)$ contains a convergent subsequence. Hence $\mathbb{S}_{x(n)}$ is nonempty. It is also easy to show that $\mathbb{S}_{x(n)}$ is closed by showing that its complement $\mathbb{X} \setminus \mathbb{S}_{x(n)}$ is open. Since $\mathbb{S}_{x(n)}$ is bounded in $\mathbb{X}$ and closed, it is compact. Next Assume that $\mathbb{S}_{x(n)}$ is not attractive, then there exists $x \notin \mathbb{S}_{x(n)}$ such that $x$ is an accumulation point of $x(n)$. This is a contradiction because of the definition of $\mathbb{S}_{x(n)}$. Thus, $\mathbb{S}_{x(n)}$ is attractive. Assume that there is an attractive set $\hat{\mathbb{S}}$ of $x(n)$ which is a strict subset of $\mathbb{S}_{x(n)}$, then there exists an accumulation point $x \in \mathbb{S}_{x(n)}$ such that $|x|_{\hat{\mathbb{S}}} > 0$. This implies that $\hat{\mathbb{S}}$ is not attractive, also a contradiction. Therefore $\mathbb{S}_{x(n)}$ is the smallest closed attractive set of $x(n)$. ■

Lemma 17 holds for any bounded sequence which is not necessarily a state or input trajectory. For the state and input trajectories of the control system of Eq. (5.1), we have the following result:

**Theorem 11** *Consider the control system of Eq. (5.1), if the corresponding state and input trajectories are bounded such that* $(x(n), u(n)) \in \mathbb{Z}$ *for all* $n \in \mathbb{I}_{\geq 0}$*, then the accumulation set* $\mathbb{S}_{(x(n),u(n))}$ *is control invariant.*

---

accumulation set are often referred to as limit point and limit set. However, since we are considering control systems (i.e. $x_{n+1} = f(x_n, u_n)$) where the control law $u_n = \kappa(x_n)$ might be set-valued or discontinuous, we choose the terms accumulation point and accumulation set to avoid confusion with existing results such as LaSalle's invariance principle whose proof relies on the continuity of $f$ and the invariant limit set.

**Proof.** From Lemma 17 we know that the accumulation set $\mathbb{S}_{(x(n),u(n))}$ is non-empty and compact. We prove Theorem 11 by contradiction. Assume that $\mathbb{S}_{(x(n),u(n))}$ is not control invariant, then there is an accumulation point $(x^*, u^*) \in \mathbb{S}_{(x(n),u(n))}$ such that $|f(x^*, u^*)|_{\mathbb{S}_{x(n)}} > 0$ where $\mathbb{S}_{x(n)}$ is the limit set of $x(n)$ (i.e., the projection of $S_{(x(n),u(n))}$ onto $\mathbb{X}$). Let $\epsilon = |f(x^*, u^*)|_{\mathbb{S}_{x(n)}} > 0$. Because of the continuity of $f(\cdot)$, there exists $\delta > 0$ such that:

$$|f(x, u)|_{\mathbb{S}_{x(n)}} > \epsilon/2, \quad \forall (x, u) \in \mathcal{B}_\delta((x^*, u^*)) \tag{5.3}$$

Because $(x^*, u^*)$ is an accumulation point, there is an increasing sub-sequence $(x(n_i),\ u(n_i))$ such that $(x(n_i),\ u(n_i)) \in \mathcal{B}_\delta((x^*, u^*))$ for all $i \in \mathbb{I}_{\geq 0}$. However, from Eq. (5.3), this implies that $|x(n_i + 1)|_{\mathbb{S}_{x(n)}} > \epsilon/2$ for all $i \in \mathbb{I}_{\geq 0}$, which furthers implies the existence of an accumulation point outside $\mathbb{S}_{x(n)}$. This contradicts the definition of the accumulation set $\mathbb{S}_{x(n)}$. Therefore, the accumulation set $\mathbb{S}_{(x(n),u(n))}$ is control invariant. ■

Let $\mathbb{Z}_t^M$ denote the maximal control invariant subset of the target set $\mathbb{Z}_t \subset \mathbb{Z}$. That is,

$$\mathbb{Z}_t^M := \{(x, u) \mid (x, u) \in \mathbb{Z}_t, f(x, u) \in proj_{\mathbb{X}}(\mathbb{Z}_t)\}$$

By definition, $\mathbb{Z}_t^M$ is the union of all control invariant subsets of $\mathbb{Z}_t$. The following theorem generalizes LaSalle's invariance principle from autonomous systems (i.e., $x_{k+1} = f(x_k)$) to control systems (i.e., $x_{k+1} = f(x_k, u_k)$) based on the definition of control invariant set in the augmented state-input space in Definition 3.

**Theorem 12** *Consider the control system of Eq. (5.1) with $(x(n), u(n)) \in \mathbb{Z}$ for all $n \in \mathbb{I}_{\geq 0}$.*

*(i) If there is a bounded function $V(x) : \mathbb{X} \to \mathbb{R}$ and a continuous function $\rho(x, u) : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ which is positive-definite with respect to $\mathbb{Z}_t$ such that the following holds:*

$$V(x(n + 1)) - V(x(n)) \leq -\rho(x(n), u(n)) \tag{5.4}$$

*for all $n \in \mathbb{I}_{\geq 0}$, then $(x(n), u(n)) \to \mathbb{Z}_t^M$.*

*(ii) If, in addition, $V(x)$ is locally continuous on $proj_{\mathbb{X}}(\mathbb{Z}_t)$ and constant on $proj_{\mathbb{X}}(\mathbb{Z}_t^M)$, then $\mathbb{Z}_t^M$ is asymptotically stable.*

**Proof.** (i) First we prove by contradiction that $\mathbb{S}_{(x(n),u(n))} \subseteq \mathbb{Z}_t$. Assume that there is an accumulation point $(x^*, u^*) \notin \mathbb{Z}_t$ such that $\rho(x^*, u^*) = \epsilon > 0$. Using similar arguments as in the proof of Theorem 11, there is an increasing subsequence $(x(n_i),\ u(n_i)) \to (x^*, u^*)$, $i \in \mathbb{I}_{\geq 0}$, such that $\rho(x(n_i), u(n_i)) > \epsilon/2$. However, from Eq. (5.4), this implies that $V(\cdot)$ is decreased by $\epsilon/2$ infinitely many times which contradicts the boundedness of $V(\cdot)$. Therefore, the sequence $(x(n), u(n))$ cannot have an accumulation point outside $\mathbb{Z}_t$, which equivalently means that $\mathbb{S}_{(x(n),u(n))} \subseteq \mathbb{Z}_t$. Applying Theorem 11 and based on the definition of $\mathbb{Z}_t^M$, we have $\mathbb{S}_{(x(n),u(n))} \subseteq \mathbb{Z}_t^M$. From Lemma 17, $(x(n),\ u(n)) \to \mathbb{S}_{(x(n),u(n))}$, thus $(x(n),\ u(n)) \to \mathbb{Z}_t^M$.

(ii) Summing Eq. (5.4) for $n \in \mathbb{I}_0^{N-1}$, we have:

$$V(x(0)) \geq V(x(N)) + \sum_{n=0}^{N-1} \rho(x(n), u(n)) \tag{5.5}$$

Let $V(x) = c$ for $x \in \mathit{proj}_{\mathbb{X}}(\mathbb{Z}_t^M)$. Since $(x(n), u(n)) \to \mathbb{Z}_t^M$ and $V(x)$ is locally continuous on $\mathit{proj}_{\mathbb{X}}(\mathbb{Z}_t)$, Eq. (5.5) when taking limit as $N \to \infty$ becomes:

$$V(x(0)) \geq c + \sum_{n=0}^{\infty} \rho(x(n), u(n)) \tag{5.6}$$

Let us define a function $V'(x, u) : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ such that $V'(x, u) = V(x) - c$. $V'(x, u)$ is locally continuous on $\mathbb{Z}_t$ with $V'(x, u) = 0$ for $(x, u) \in \mathbb{Z}_t^M$. Substituting $V(x)$ by $V'(x, u)$, Eq. (5.6) becomes:

$$V'(x(0), u(0)) \geq \sum_{n=0}^{\infty} \rho(x(n), u(n))$$

It is easy to see from the above that $V'(x, u) > 0$ for $(x, u) \notin \mathbb{Z}_t^M$ because any $(x(0), u(0)) \notin \mathbb{Z}_t^M$ implies the existence of $(x(M), u(M)) \notin \mathbb{Z}_t$ for some $M$ so that $V'(x(0), u(0)) \geq \rho(x(M), u(M)) > 0$. Thus $V'(x, u)$ is positive-definite with respect to $\mathbb{Z}_t^M$. Moreover, substituting $V'(x, u)$ into Eq. (5.4), we have that $V'(x(n+1), u(n+1)) - V'(x(n), u(n)) \leq -\rho(x(n), u(n))$ where $\rho(x(n), u(n))$ is positive semidefinite with respect to $\mathbb{Z}_t^M$. These properties make $V'(x, u)$ a Lyapunov function for $\mathbb{Z}_t^M$. Therefore $\mathbb{Z}_t^M$ is stable in the sense of Lyapunov. Since it is also attractive from (i), $\mathbb{Z}_t^M$ is asymptotically stable. ■

Before stating and proving the stability results of the proposed zone MPC design, we need to define the $N$-step controllable set as follows:

**Definition 6** *(N-step controllable set [53]) We use $\mathbb{X}_N(\mathbb{Z}, \mathbb{X}_f)$ to denote the set of states in $\mathbb{X}$ that can be steered to $\mathbb{X}_f$ in $N$ steps while satisfying the state and input constraints $(x, u) \in \mathbb{Z}$. That is,*

$$\mathbb{X}_N(\mathbb{Z}, \mathbb{X}_f) = \left\{x(0) \in proj_{\mathbb{X}}(\mathbb{Z}) \mid \exists\, (x(n), u(n)) \in \mathbb{Z}, n \in \mathbb{I}_0^{N-1}, x(N) \in \mathbb{X}_f\right\}$$

**Theorem 13** *Consider the system of Eq. (5.1) under the zone MPC of Eq. (5.2). If Assumption 14 holds and $x(0) \in \mathbb{X}_N(\mathbb{Z}, \mathbb{X}_f)$, then:*

*(i) The zone MPC is recursively feasible with $(x(n), u(n)) \to \mathbb{Z}_t^M$.*

*(ii) If, in addition, the value function $V_N^0(x(n))$ is locally continuous on $proj_{\mathbb{X}}(\mathbb{Z}_t)$ and $\mathbb{X}_N(\mathbb{Z}_t, \mathbb{X}_f) = proj_{\mathbb{X}}(\mathbb{Z}_t^M)$, then $\mathbb{Z}_t^M$ is asymptotically stable.*

**Proof.** (i) From Assumption 14, the terminal set $\mathbb{X}_f$ is control invariant under the constraint $(x, u) \in \mathbb{Z}_t$ and therefore control invariant under the constraint $(x, u) \in \mathbb{Z}$. Given the optimal solution at time $n-1$, a feasible solution at time $n$ can be constructed by letting $u(i + n|n) = u^*(n + i|n - 1), i \in \mathbb{I}_0^{N-2}$ and by finding $u(N - 1 + n|n)$ based on Assumption 14. The result is standard in MPC literature (see e.g., [5], [32]) and is omitted here for brevity. Also following the standard analysis for optimal value function decrease, the following can be obtained:

$$V_N^0(x(n+1)) - V_N^0(x(n)) \leq -l^*(x(n), u(n)), \quad n \in \mathbb{I}_{\geq 0}$$

where $l^*(x(n), u(n))$ is the optimal objective function to the following optimization problem:

$$\begin{aligned} l^*(x(n), u(n)) = \min_{x^z, u^z} \quad & |x(n) - x^z|_Q^2 + |u(n) - u^z|_R^2 \\ \text{s.t.} \quad & (x^z, u^z) \in \mathbb{Z}_t \end{aligned}$$

It is easy to check that $l^*(x(n), u(n))$ is continuous and positive-definite with respect to the target set $\mathbb{Z}_t$. Thus Eq. (5.4) of Theorem 12 is satisfied with $V(x) = V_N^0(x)$ and $\rho(x, u) = l^*(x, u)$. Apply Theorem 12 (i), we have $(x(n), u(n)) \to \mathbb{Z}_t^M$.

(ii) It is easy to check from the definition of the $N$-step controllable set that $V_N^0(x) = 0$ for all $x \in \mathbb{X}_N(\mathbb{Z}_t, \mathbb{X}_f)$. Since $\mathbb{X}_N(\mathbb{Z}_t, \mathbb{X}_f) = proj_{\mathbb{X}}(\mathbb{Z}_t^M)$, $V_N^0(x) = 0$ for all $x \in proj_{\mathbb{X}}(\mathbb{Z}^M)$. Apply Theorem 12 (ii), $\mathbb{Z}_t^M$ is asymptotically stable. ■

**Remark 11** *In the stability analysis of MPC, (local) continuity of the value function $V_N^0(x(n))$ is usually required to establish asymptotic stability. Interested readers may* *refer to [54] or [32] Appendix C.3 for more detailed discussions on sufficient conditions for continuous value function. The key idea is to examine the continuity of the set-valued map* $\mathbb{U}_N(x) : \mathbb{X}_N(\mathbb{Z}_t, \mathbb{X}_f) \rightarrow \mathbb{R}^{N\times n_u}$ *where*

$$\mathbb{U}_N(x(n)) := \left\{u_0, u_1, ..., u_{N-1} \mid x(n) \in \mathbb{X}_N(\mathbb{Z}_t, \mathbb{X}_f), \textit{Eq. (5.2) is feasible} \right\}$$

*If* $\mathbb{U}_N(x(n))$ *is continuous on int* $(\mathbb{X}_N(\mathbb{Z}_t, \mathbb{X}_f))$*, then* $V_N^0(x(n))$ *is continuous on int* $(\mathbb{X}_N(\mathbb{Z}_t, \mathbb{X}_f))$*. A special case where* $\mathbb{U}_N(x(n))$ *is guaranteed to be continuous is linear system (f) with convex constraints* $(\mathbb{Z}, \mathbb{X}_f)$*. In this case, if* $proj_{\mathbb{X}}(\mathbb{Z}_t) \subset int(\mathbb{X}_N(\mathbb{Z}_t, \mathbb{X}_f))$*, then* $V_N^0(x(n))$ *is locally continuous on* $proj_{\mathbb{X}}(\mathbb{Z}_t)$*.*

## 5.5 Further discussions

### 5.5.1 Enlargement of the region of attraction

From Theorem 13, the largest control invariant set $\mathbb{Z}_t^M$ in the target zone $\mathbb{Z}_t$ is attractive with a region of attraction $\mathbb{X}_N(\mathbb{Z}, \mathbb{X}_f)$. From the definition of the $N$-step controllable set $\mathbb{X}_N(\mathbb{Z}, \mathbb{X}_f)$, two straightforward ways to enlarge the region of attraction are (a) to increase the control horizon $N$, and (b) to use a larger control invariant terminal set $\mathbb{X}_f$. The former is always a possible choice with its applicability limited by the online computation power. The latter amounts to finding the largest control invariant state set in the target zone (i.e., $proj_{\mathbb{X}}(\mathbb{Z}_t^M)$) which is difficult for generic nonlinear control systems. Interested readers may refer to [55], [56], [57], [58] for some existing results on the computation and approximation of the maximal invariant set.

Another approach which has proven very effective is to employ an auxiliary control law in the MPC design [59], [23]. By extending the prediction horizon based on the auxiliary controller, enlargement of the region of attraction as well as improvement of transient performance can be achieved without significantly increasing the computational complexity. In the following we discuss how the methodology can be applied in the MPC design for zone tracking. We make the following assumption about the auxiliary control law.

**Assumption 15** *There is a continuous control law* $u = h(x)$ *which uniformly asymptotically stabilizes a positive invariant set* $\mathbb{X}_{t,h} \subset \mathbb{X}$ *with* $(x, h(x)) \in int(\mathbb{Z}_t), \forall x \in \mathbb{X}_{t,h}$*. The region of attraction* $\mathbb{X}_h \subset \mathbb{X}$ *is such that* $proj_{\mathbb{X}}(\mathbb{Z}_t) \subset \mathbb{X}_h \subset proj_{\mathbb{X}}(\mathbb{Z})$*.*

Let $x_h(k,x)$, $u_h(k,x)$ denote the closed-loop state and input trajectories under the controller $h(x)$ at time instant $k$ with $x_h(0,x) = x$ and $u_h(k,x) = h(x_h(k,x))$. Moreover, let $\mathbb{X}_{t,h}^M \subseteq \mathbb{X}$ denote the largest positive invariant set under the controller $h(x)$ in the target zone $\mathbb{Z}_t$. That is,

$$\mathbb{X}_{t,h}^M := \{x \mid (x_h(k,x), u_h(k,x)) \in \mathbb{Z}_t, \quad \forall k \in \mathbb{I}_{\geq 0}\}$$

From the definition of $\mathbb{X}_{t,h}^M$ and $\mathbb{Z}_t^M$, we have $\mathbb{X}_{t,h}^M \subseteq proj_{\mathbb{X}}(\mathbb{Z}_t^M)$. The following result will be employed in the zone MPC design:

**Lemma 18** *If Assumption 15 holds, then there exists $N_h \in \mathbb{I}_{\geq 0}$ such that $x_h(k,x) \in \mathbb{X}_{t,h}^M$ for all $k \geq N_h$ and $x \in \mathbb{X}_h$.*

**Proof.** From the converse Lyapunov thoerem [60], there exists a smooth Lyapunov function $V(x_h(k,x))$ and $\mathcal{K}_\infty$ functions $\alpha_1$, $\alpha_2$ and $\alpha_3$ such that the following holds for all $k \geq 0$ and $x \in \mathbb{X}_h$:

$$\begin{aligned}
&\alpha_1(|x_h(k,x)|_{\mathbb{X}_{t,h}}) \leq V(x_h(k,x)) \leq \alpha_2(|x_h(k,x)|_{\mathbb{X}_{t,h}}) \\
&V(x_h(k+1,x)) - V(x_h(k,x)) \leq -\alpha_3(|x_h(k,x)|_{\mathbb{X}_{t,h}})
\end{aligned}$$

Since $V(\cdot)$ is continuous and $(x, h(x)) \in int(\mathbb{Z}_t)$ for $x \in \mathbb{X}_{t,h}$, there exists a $\epsilon$ sublevel set of the Lyapunov function, $V_\epsilon(x)$, with $\epsilon > 0$ such that $(x, h(t)) \in \mathbb{Z}_t$ for all $x \in V_\epsilon(x)$. Moreover, $V_\epsilon(x)$ is positive invariant under $h(x)$ (because of the non-increasing property of $V(\cdot)$) and is therefore a subset of $\mathbb{X}_{t,h}^M$. Since $\mathbb{X}_{t,h}$ is a strict subset of $V_\epsilon(x)$ and is uniformly asymptotically stable under $h(x)$, there is $N_h \in \mathbb{I}_{\geq 0}$ such that $x_h(k,x) \in V_\epsilon(x)$ for all $k \geq N_h$ and $x \in \mathbb{X}_h$. Thus, $x_h(k,x) \in V_\epsilon(x) \subseteq \mathbb{X}_{t,h}^M$ for all $k \geq N_h$ and $x \in \mathbb{X}_h$. ■

The results of Lemma 18 is illustrated in Figure 5.1. In the availability of the auxiliary controller $h(x)$, the zone MPC of Eq. (5.2) can be reinforced as follows:

$$V_N^0(x(n)) = \min_{u_i, x_i^z, u_i^z} \sum_{i=0}^{N+N_h-1} |x_i - x_i^z|_Q^2 + |u_i - u_i^z|_R^2 \tag{5.7a}$$

$$\text{s.t. } x_{i+1} = f(x_i, u_i), \quad i \in \mathbb{I}_0^{N-1} \tag{5.7b}$$

$$x_0 = x(n) \tag{5.7c}$$

$$(x_i, u_i) \in \mathbb{Z}, \quad i \in \mathbb{I}_0^{N-1} \tag{5.7d}$$

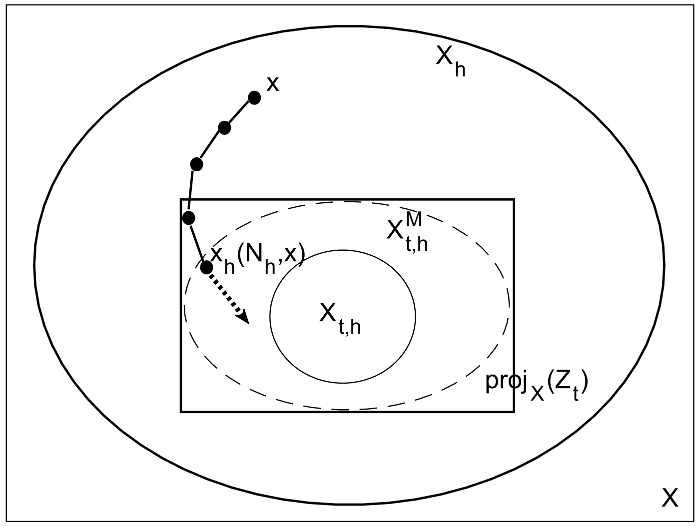


**Figure 5.1:** Closed-loop state trajectory under controller $u = h(x)$: for any initial state $x \in \mathbb{X}_h$, the trajectory $x_h(k,x)$ enters the positive invariant set $\mathbb{X}_{t,h}^M$ in $N_h$ steps and is maintained in it thereafter.

$$(x_i^z, u_i^z) \in \mathbb{Z}_t, \quad i \in \mathbb{I}_0^{N+N_h-1} \tag{5.7e}$$

$$x_N \in \mathbb{X}_h \tag{5.7f}$$

$$u_i = h(x_i), \quad i \in \mathbb{I}_N^{N+N_h-1} \tag{5.7g}$$

The above zone MPC design utilizes the auxiliary controller $h(x)$ to extend the prediction horizon of the zone MPC of Eq. (5.2) for another $N_h$ steps. If the terminal horizon $N_h$ is chosen such that it satisfies the condition of Lemma 18, then the objective function of Eq. (5.7a) is equivalent to an infinite horizon cost (with $(x_i, u_i) = (x_i^z, u_i^z) \in \mathbb{Z}_t$ for $i \geq N + N_h$). Stability of the zone MPC of Eq. (5.7) is summarized as follows:

**Theorem 14** *Consider the system of Eq. (5.1) under the zone MPC of Eq. (5.7) with $x(0) \in \mathbb{X}_N(\mathbb{Z}, \mathbb{X}_h)$. If Assumption 15 holds and the terminal horizon $N_h$ is chosen such that it satisfies the condition of Lemma 18, then:*

*(i) The zone MPC is recursively feasible with $(x(n), u(n)) \to \mathbb{Z}_t^M$.*

*(ii) If, in addition, the value function $V_N^0(x(n))$ is locally continuous on $proj_{\mathbb{X}}(\mathbb{Z}_t)$ and*
*$\mathbb{X}_N(\mathbb{Z}_t, \mathbb{X}_{t,h}^M) = proj_{\mathbb{X}}(\mathbb{Z}_t^M)$, then $\mathbb{Z}_t^M$ is asymptotically stable.*

**Proof.** The proof of Theorem 14 (i) is similar to the proof of Theorem 13 (i) with the feasible solution for $u(N-1+n|n)$ constructed by $u(N-1+n|n) = h(x^*(N-$

$1 + n|n-1)$) based on Assumption 15. The proof of Theorem 14 (ii) is similar to Theorem 13 (ii) with $\mathbb{X}_N(\mathbb{Z}_t, \mathbb{X}_f)$ replaced by $\mathbb{X}_N(\mathbb{Z}_t, \mathbb{X}_{t,h}^M)$. The details are omitted for brevity. ■

**Remark 12** *Comparing Theorem 13 (i) and Theorem 14 (i), we can see that the region of attraction is enlarged from $\mathbb{X}_N(\mathbb{Z}, \mathbb{X}_f)$ to $\mathbb{X}_N(\mathbb{Z}, \mathbb{X}_h)$ (because $\mathbb{X}_f \subset \mathbb{X}_h$). Moreover, from Theorem 13 (ii) and Theorem 14 (ii), we can see that larger terminal set $\mathbb{X}_f$ or $\mathbb{X}_{t,h}^M$ is desirable as far as the size of the stability region in the target set is concerned. We could theoretically use $\mathbb{X}_{t,h}^M$ as the terminal constraint set $\mathbb{X}_f$ in the zone MPC design of Eq. (5.2). Unfortunately, given the auxiliary controller $h(x)$ and the target set $\mathbb{Z}_t$, the largest positive invariant set $\mathbb{X}_{t,h}^M$ can be very difficult or computationally expensive to characterize. Employing the auxiliary controller explicitly in the zone MPC design as in Eq. (5.7) spares us the burden of explicitly characterizing $\mathbb{X}_{t,h}^M$.*

### 5.5.2 Handling a secondary economic objective

If the optimal solution to the optimization problem of Eq. (5.2) is non-unique, an economic optimization can be carried out subsequently to make use of the rest degrees of freedom. Specifically, the optimization problem is formulated as follows:

$$\min_{u_i, x_i^z, u_i^z} \sum_{i=0}^{N-1} l(x_i, u_i) \tag{5.8a}$$

$$\text{s.t. } (5.2b) - (5.2f) \tag{5.8b}$$

$$\sum_{i=0}^{N-1} |x_i - x_i^z|_Q^2 + |u_i - u_i^z|_R^2 = V_N^0(x(n)) \tag{5.8c}$$

where the cost function $l(x_i, u_i) : \mathbb{Z} \to \mathbb{R}$ characterizes process economic performance. Constraints (5.8b) and (5.8c) confine the solution space of the economic optimization to be in the solution space of the zone MPC of Eq. (5.2). Therefore the stability result of Theorem 13 is inherited.

It is also interesting to view the optimization problem of Eq. (5.8) as an economic MPC (EMPC). In particular, if $V_N^0(x(M)) = 0$ at some time instant $M \in \mathbb{I}_{\geq 0}$, then for all $n \geq M$, Eq. (5.8) is equivalent to the following EMPC design:

$$\begin{array}{ll} \min\limits_{u_i} & \sum\limits_{i=0}^{N-1} l(x_i, u_i) \\ \text{s.t.} & x_{i+1} = f(x_i, u_i), \quad i \in \mathbb{I}_0^{N-1} \\ & x_0 = x(n) \\ & (x_i, u_i) \in \mathbb{Z}_t, \quad i \in \mathbb{I}_0^{N-1} \\ & x_N \in \mathbb{X}_f \end{array} \tag{5.9}$$

In the above EMPC optimization problem, the predicted state and input trajectories are constrained by the target set $\mathbb{Z}_t$. Existing EMPC design and analysis methods (see e.g., [61], [10]) can be applied to the EMPC of Eq. (5.9). For example, one can find the optimal steady state with respect to the economic cost function $l(x, u)$ in the target set $\mathbb{Z}_t$ and verify the dissipativity condition of the optimal steady state in $\mathbb{Z}_t$. Likewise, if the auxiliary controller is utilized in the zone MPC design as in Eq. (5.7), the corresponding secondary economic optimization problem can be designed similarly. Existing results of EMPC based on an auxiliary controller [59], [45] can be applied.

## 5.6 Examples

**Example 1.** Consider the following linear scalar system:

$$x(n+1) = 1.25x(n) + u(n)$$

with state and input constraints $\mathbb{X} = [-5, 15]$, $\mathbb{U} = [-5, 5]$ respectively. The target set is $\mathbb{Z}_t = \{(x, u) \mid x \in [-5, 15], u \in [-1, 1]\}$. Using the algorithms in [53] [62], the largest control invariant set in the target zone can be obtained as follows: $\mathbb{Z}_t^M = \{(x, u) \mid Gx + Hu \leq \mathbf{1}\}$ where $G = [0, 0, 0.3125, -0.3125]^T$, $H = [1, -1, 0.25, -0.25]^T$ (Figure 5.2).

In the simulation, we compare four different zone MPC configurations: (a) a zone MPC that penalizes the distance to the target set $\mathbb{Z}_t$ (without terminal constraint), (b) a zone MPC that penalizes the distance to the largest control invariant set in the target zone $\mathbb{Z}_t^M$ (also without terminal constraint), (c) the proposed zone MPC of Eq. (5.2) with $\mathbb{X}_f = proj_{\mathbb{X}}(\mathbb{Z}_t^M) = [-4,\ 4]$, and (d) the proposed zone MPC of Eq. (5.7) with the auxiliary control law $h(x) = -0.3x$ (which asymptotically stabilizes the origin), $\mathbb{X}_h = [-5, 12]$, $N_h = 20$. It is easy to check that the condition of Lemma 18 is satisfied.

The control horizon of the zone MPCs (a), (b), (c) is $N = 10$ whereas for zone MPC (d), a short control horizon $N = 3$ is used. The simulation results are shown in Figure 5.2. The overall transient performance of the four different zone MPC designs over 30 sampling times $J_{0:29} = \sum_{k=0}^{29} |(x_k, u_k)|_{\mathbb{Z}_t}$ are shown in Table 5.1.

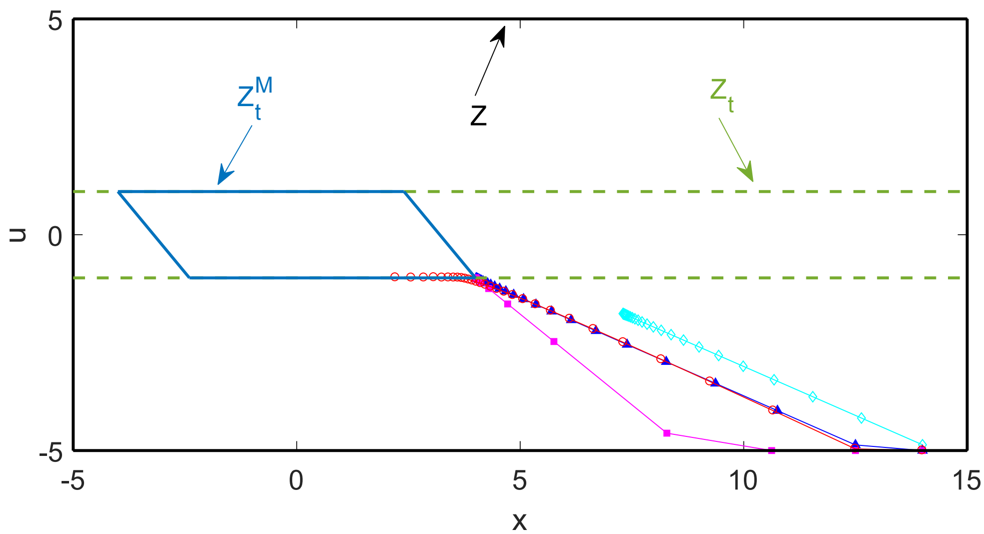


**Figure 5.2:** State and input trajectories of a): a zone MPC that tracks the target zone $\mathbb{Z}_t$ (diamonds), b): a zone MPC that tracks the largest control invariant set in the target zone $\mathbb{Z}_t^M$ (squares), c): the proposed zone MPC of Eq. (5.2) (triangles), and d): the proposed zone MPC of Eq. (5.7) (circles).

**Table 5.1:** The transient performance of the four zone MPC configurations over 30 sampling times

| | Zone MPC (a) | Zone MPC (b) | zone MPC (c) | zone MPC (d) |
|---|---|---|---|---|
| $J_{0:29}$ | 70.19 | 63.55 | 56.65 | 56.72 |

From these results, we can see that the zone MPC (a) which penalizes the distance to the target set $\mathbb{Z}_t$ without terminal conditions fails to asymptotically track the target set $\mathbb{Z}_t$. The zone MPC (b) that penalizes the distance to the largest control invariant set in the target zone $\mathbb{Z}_t^M$ achieves its tracking objective, but with deteriorated transient performance as compared to the proposed zone MPC design (c) and (d). Moreover, a closer look at the region of attractions of the proposed zone MPC designs (c) and (d) reveals that when $N = 3$, $x(0) \notin \mathbb{X}_N(\mathbb{Z}, \mathbb{X}_f)$ and $x(0) \in \mathbb{X}_N(\mathbb{Z}, \mathbb{X}_h)$. This indicates that by employing the auxiliary control law in the zone MPC design, we are allowed to use smaller control horizon $N$.

It is also worth pointing out that the above results resemble the comparison between set-point MPC and several popular economic MPC (EMPC) designs [28], [10], [59]. Specifically, zone MPC (a), (b) corresponds to EMPC without terminal constraint and set-point MPC, respectively [10]. Zone MPC (c) corresponds to EMPC with a point-wise terminal constraint [28]. Zone MPC (d) is analogous to EMPC with extended horizon based on an auxiliary controller [59].

**Example 2.** Consider the following nonlinear scalar system:

$$x(n+1) = x(n)^2 + u(n) \tag{5.10}$$

with initial condition $x(0) = -1$ and input constraint $\mathbb{U} = [-1.05, -0.95]$. The primary control objective is to maintain the system operation in the target zone: $\mathbb{Z}_t = \{(x, u) : x \in [-1.5, 0.5], u \in [-1.05, -0.95]\}$. There is also a secondary economic objective to minimize the control input $l(x, u) = u$. The corresponding optimal steady state is $(x_s, u_s) = (-0.6402, -1.05)$. We compare two control configurations: (a) a set-point MPC which tracks $(x_s, u_s)$ (without terminal constraint), and (b) the proposed zone MPC of Eq. (5.2) which handles the economic objective by taking the second step economic optimization of Eq. (5.8). The terminal constraint for the zone MPC is $\mathbb{X}_f = [-1.5, 0.5]$. Both MPCs take the same control horizon $N = 5$. The simulation results of cases (a) and (b) are shown in Figure 5.3 and Figure 5.4, respectively.

From Figure 5.3 and Figure 5.4, we can see that set-point MPC achieves its tracking objective at the expense of making excessive control moves. Note that this is not because of poor parameter tuning. The zone MPC resulted in a periodic operation in the target zone. While both MPC designs succeeded in maintaining the system operation in the target zone, the zone MPC apparently achieves better transient economic performance.

Further, we compared the performance of set-point MPC and zone MPC in the presence of small disturbances. An unknown normally distributed process disturbance $w(n) \sim \mathcal{N}(0, 0.1)$ is introduced to the system: $x(n+1) = x(n)^2 + u(n) + w(n)$.

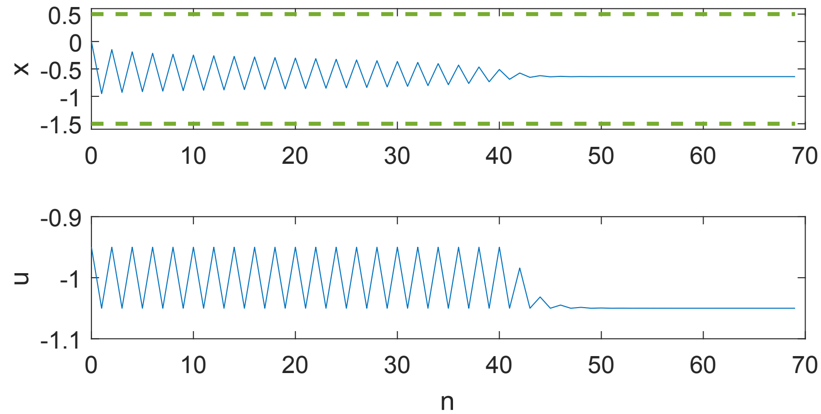


**Figure 5.3:** State and input trajectories of set-point MPC.

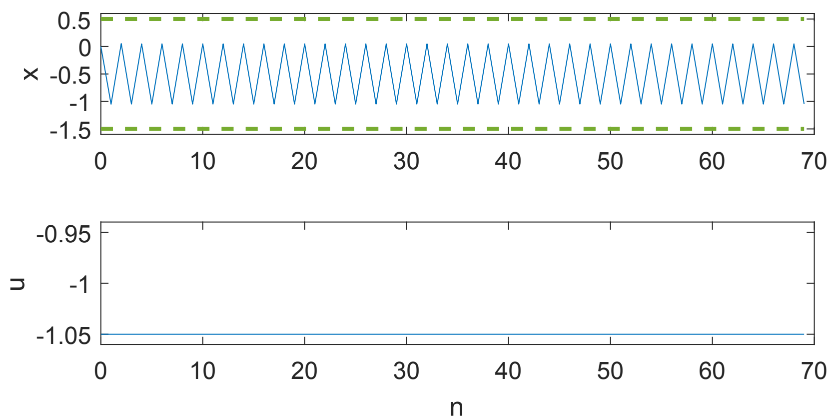


**Figure 5.4:** State and input trajectories of the zone MPC.

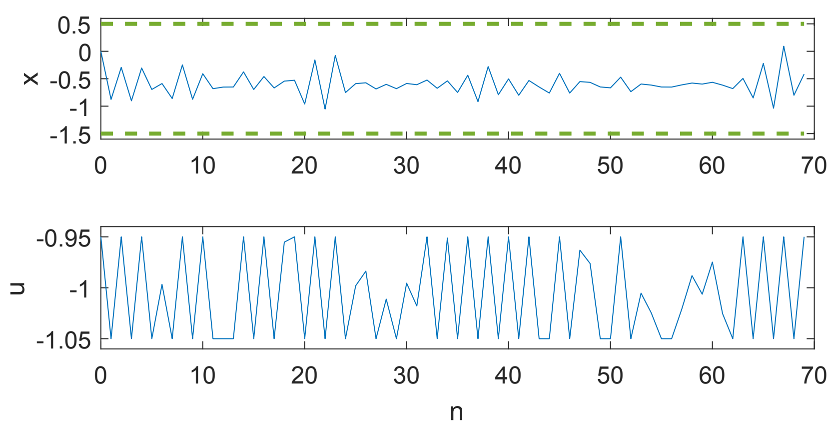


**Figure 5.5:** State and input trajectories of set-point MPC.

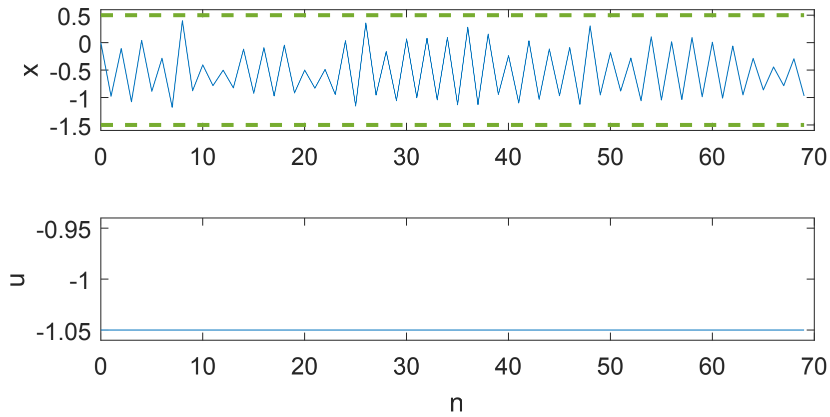


**Figure 5.6:** State and input trajectories of the zone MPC.

The state and input trajectories of the set-point MPC and zone MPC are shown in Figure 5.5 and Figure 5.6, respectively. It is seen that in the presence of small disturbances, zone MPC is still capable of achieving optimal economic performance while satisfying the zone objective, whereas set-point MPC inevitably suffers from fluctuations in the control input.

More insights into the superiority of zone control over set-point control can be acquired by observing the bifurcation diagram in Figure 5.7, which shows the stable steady states, periodic orbits and chaotic attractors of system (5.10). We can see that the target zone $\mathbb{Z}_t$ contains a set of unstable steady states as well as a set of stable periodic orbits with period 2. Thus, it is clear how relaxing a set-point target to a zone target can allow for more admissible operations, thereby releasing more degrees of freedom to achieve economic objectives.

## 5.7 Summary

In this work, we proposed a general framework of nonlinear model predictive control for zone tracking. The target zone is characterized by coupled system state and input, and is not necessarily control invariant. An invariance-like theorem is developed which naturally generalizes LaSalle's invariance principle from autonomous system to control systems with a zone target. Our results differ from the standard stability analysis for conventional set-point MPC in that we consider the evolvement of the state-input pair $(x(n), u(n))$ instead of merely the the state trajectory $x(n)$. This provides stronger

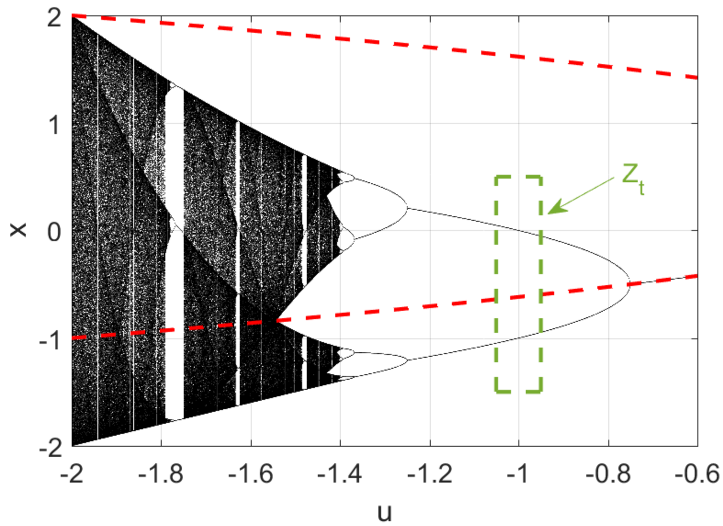


**Figure 5.7:** Bifurcation diagram of system (5.10). The bifurcation diagram shows the values of system state $x(n)$ that are visited or approached asymptotically under a constant $u(n)$. For $u \in (-0.75, -0.6]$, the system has stable steady-state operation. At $u = -0.75$, steady-state operation becomes unstable and the diagram bifurcates into two points which corresponds to a stable periodic operation of period two. As the value of $u$ decreases, more bifurcation occurs and at some point around $u = -1.4$ the system enters a chaotic operating region. The dashed line corresponds to steady-state operating points

results and more accurate description of system behavior. Two simulations examples demonstrate the superiority of zone control over set-point control and the efficacy of the proposed zone MPC framework.

# Chapter 6

# Application to an Oilsand Primary Separation Vessel

In this chapter, we apply the proposed EMPC algorithm to an oilsand primary separation vessel (PSV). We show how previously developed EMPC design and analysis results in the context of discrete-time system can be extended to continuous-time systems where the issue of sampling needs to be addressed. Both infinite-time and finite-time operations are considered which correspond to the result in Chapter 2 and Chapter 4.

## 6.1 Preliminaries

### 6.1.1 Notation

Throughout this chapter, the operator $|\cdot|$ denotes the Euclidean norm of a scalar or a vector. A continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and satisfies $\alpha(0) = 0$. The symbol $\Omega_r$ is used to denote a level set of the Lyapunov function $V(x)$ such that $\Omega_r := \{x \in \mathbb{R}^{n_x} : V(x) \leq r\}$. The symbol $\text{diag}(x_1, ..., x_i, ..., x_n)$ denotes a diagonal matrix with its $i$-th diagonal element being $x_i$. The symbol '$\setminus$' denotes set substraction such that $\mathbb{A} \setminus \mathbb{B} := \{x \in \mathbb{A}, x \notin \mathbb{B}\}$. The operator $\lceil\cdot\rceil$ is used to obtain the smallest integer greater than or equal to a real number $x \in \mathbb{R}$ such that $\lceil x \rceil := \min\{a \in \mathbb{Z} : a \geq x\}$ where $\mathbb{Z}$ denotes the set of integers.

### 6.1.2 System description

We consider a class of nonlinear systems which can be described by the following state-space model:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{6.1}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state vector and $u \in \mathbb{R}^{n_u}$ denotes the control input vector. The control input is restricted to be in a nonempty convex set $\mathbb{U} \subset \mathbb{R}^{n_u}$, such that:

$$\mathbb{U} := \{u \in \mathbb{R}^{n_u} : |u| \leq u^{\max}\} \tag{6.2}$$

where $u^{\max}$ is the magnitude of the input constraint. It is assumed that $f$ and $g$ are locally Lipschitz functions with respect to $x$, and that $f(0) = 0$ which implies that the origin is a steady state of the unforced nominal system. We assume that the system state are sampled periodically at time instants $\{t_{k\geq 0}\}$ such that $t_k = t_0 + k\Delta$ with $t_0 = 0$ being the initial time, $\Delta$ being a fixed sampling interval and $k$ being positive integers.

### 6.1.3 Stability assumption

We assume that there exists a continuous controller $u = h(x)$ which renders the origin of system (6.1) asymptotically stable while satisfying the input constraint. Based on converse Lyapunov theorem [63], the above assumption implies that there exists a continuous differentiable control Lyapunov function $V(x)$ and $\mathcal{K}$ functions $\alpha_j$, $j = 1, 2, 3, 4$, that satisfy the following conditions:

$$\begin{aligned}
&\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \\
&\frac{\partial V(x)}{\partial x}\Big(f(x) + g(x)h(x)\Big) \leq -\alpha_3(|x|) \\
&\left|\frac{\partial V(x)}{\partial x}\right| \leq \alpha_4(|x|) \\
&h(x) \in \mathbb{U}
\end{aligned} \tag{6.3}$$

for all $x \in \mathbb{D}$ where $\mathbb{D} \subset \mathbb{R}^{n_x}$ is a compact set containing the origin. We denote a level set of $V(x)$ in $\mathbb{D}$, $\Omega_\rho \subset \mathbb{D}$, as the stability region of the closed-loop system under the controller $h(x)$.

Using the Lipschitz properties assumed for $f$, $g$, and taking into account the boundedness of the input vector $u$ characterized in Eq. (6.2), there exits a positive constant $M$ such that

$$|f(x) + g(x)u| \leq M \tag{6.4}$$

for all $x \in \Omega_\rho$. In addition, by the continuous differentiable property of $V(x)$ and the Lipschitz properties of $f$ and $g$, there exist positive constants $L_f$, $L_g$, such that:

$$\begin{aligned}
\left| \frac{\partial V(x)}{\partial x} f(x) - \frac{\partial V(x')}{\partial x} f(x') \right| &\leq L_f \, |x - x'| \\
\left| \frac{\partial V(x)}{\partial x} g(x) - \frac{\partial V(x')}{\partial x} g(x') \right| &\leq L_g \, |x - x'|
\end{aligned} \tag{6.5}$$

for all $x, x' \in \Omega_\rho$.

## 6.2 EMPC with terminal cost

In this section, we present the proposed EMPC with terminal cost and focus on infinite-time operation. First, we highlight the stability properties of the closed-loop system under the controller $h(x)$ when it is implemented in a sample-and-hold fashion. Second, we discuss the design of the terminal cost and then introduce the proposed EMPC design. Nominal closed-loop stability and asymptotic average performance of the proposed EMPC design are subsequently analyzed.

### 6.2.1 Properties of the nonlinear controller $h(x)$

Since the nonlinear controller $h(x)$ ensures the asymptotic stability of the origin, it is natural to take advantage of it as the benchmark control system. We are aimed to develop an EMPC that gives closed-loop economic performance better than (or at least as good as) the one given by $h(x)$.

In this chapter, we consider that the controller $h(x)$ is implemented in a sample-and-hold fashion. The sample-and-hold implementation facilitates the construction of a terminal cost that will be used in the proposed EMPC. When $h(x)$ is implemented in sample-and-hold, the sampling time $\Delta$ plays an important role in ensuring the closed-loop stability. If the sampling time is not appropriately picked, the closed-loop stability under $h(x)$ may be lost. The following proposition summarizes the stability properties of the controller $h(x)$ implemented in sample-and-hold.

**Proposition 3** *Consider the state trajectory $\hat{x}(t)$ of the system of Eq. (6.1) in closed-loop under the controller $h(\hat{x})$ implemented in a sample-and-hold fashion with the initial state $\hat{x}(t_0) \in \Omega_\rho$. If the sampling time $\Delta$ is chosen such that $\Delta < \Delta_{\max}$ where $\Delta_{\max} \in (0, \sqrt{\frac{\rho}{(L_f+L_g u^{max})M}}]$ and is the solution to $d(\Delta_{\max}) = 0$ with the function $d(s)$ defined as follows:*

$$d(s) = -\alpha_3(\alpha_2^{-1}(\rho - (L_f + L_g u^{max})Ms^2)) + (L_f + L_g u^{max})Ms \tag{6.6}$$

*with class $\mathcal{K}$ functions $\alpha_2$, $\alpha_3$ defined in Eq. (6.3), $L_f$, $L_g$ defined in Eq. (6.5), and $M$ defined in Eq. (6.4), then $\hat{x}(t)$ will be driven into $\Omega_{\rho^*} \subset \Omega_\rho$ in $N^*$ steps without leaving the stability region, where*

$$\rho^* = \alpha_2(\alpha_3^{-1}((L_f + L_g u^{\max})M\Delta)) + (L_f + L_g u^{max})M\Delta^2 \tag{6.7}$$

*and*

$$N^* = \left\lceil \frac{\rho - \rho^*}{-\alpha_3(\alpha_2^{-1}(\rho^*))\Delta + (L_f + L_g u^{\max})M\Delta^2} \right\rceil. \tag{6.8}$$

*Moreover, once $\hat{x}(t)$ enters $\Omega_{\rho^*}$, it is maintained in it thereafter.*

**Proof.** First, we show that $\Delta_{\max}$ that satisfies Eq. (6.6) uniquely exists. It can be verified that $d(s)$ is a strictly increasing function for $s \in [0, \sqrt{\frac{\rho}{(L_f+L_g u^{max})M}}]$. It can also be easily checked that $d(0) < 0$ and $d(\sqrt{\frac{\rho}{(L_f+L_g u^{max})M}}) > 0$. Therefore the $\Delta_{\max}$ that satisfies $d(\Delta_{\max}) = 0$ for $\Delta_{\max} \in (0, \sqrt{\frac{\rho}{(L_f+L_g u^{max})M}}]$ uniquely exists.

Second, we show that when $\Delta \leq \Delta_{\max}$, the stability conditions established in [64] are satisfied and the system state will be driven into $\Omega_{\rho^*}$ in $N^*$ steps and is maintained in it thereafter. Note first that $\rho^* < \rho$ holds automatically which can be verified from the definition of $\Delta_{\max}$ and the fact that $\Delta < \Delta_{\max}$. From Eq. (6.7), the following inequality holds:

$$-\alpha_3(\alpha_2^{-1}(\rho^*)) + (L_f + L_g u^{\max})M\Delta < 0 \tag{6.9}$$

From the analysis in [64], the above equation implies that

$$\dot{V}(\hat{x}(t)) < -\alpha_3(\alpha_2^{-1}(\rho^*)) + (L_f + L_g u^{\max})M\Delta, \quad \forall\ \hat{x}(t) \in \Omega_\rho \setminus \Omega_{\rho^*} \tag{6.10}$$

which further implies that any $\hat{x}(t_0) \in \Omega_\rho \setminus \Omega_{\rho^*}$ will be driven into $\Omega_{\rho^*}$ in less than $N^*$ steps while $V(\hat{x})$ keeps decreasing. To show that once $\hat{x}$ enters $\Omega_{\rho^*}$ it is maintained in it thereafter, we note that for any $\rho_s \in (\rho_o, \rho]$, where $\rho_o = \alpha_2(\alpha_3^{-1}((L_f+L_g u^{\max})M\Delta))$, the following inequality holds

$$-\alpha_3(\alpha_2^{-1}(\rho_s)) + (L_f + L_g u^{\max})M\Delta < 0 \tag{6.11}$$

Using the result in [64] again, the above inequality implies that $\hat{x}$ is ultimately maintained in $\Omega_{\rho_o^*}$, where $\rho_o^* := \max\{V(x(t+\Delta)) : V(x(t)) \leq \rho_o\}$. Further, for any $\hat{x}(t_k) \in \Omega_{\rho_o}$, the change rate of the Lyapunov function $V(x)$ is bounded as follows for $t \in [t_k, t_{k+1}]$:

$$\dot{V}(t) < -\alpha_3(\alpha_1^{-1}(\rho_o)) + (L_f + L_g u^{\max})M\Delta < (L_f + L_g u^{\max})M\Delta \tag{6.12}$$

Taking into account the definition of $\rho_o^*$ and the above inequality, the following upper bound on $\rho_o^*$ can be obtained:

$$\rho_o^* < \rho_o + (L_f + L_g u^{\max})M\Delta^2 = \rho^* \tag{6.13}$$

which means that $\hat{x}$ is ultimately maintained in $\Omega_{\rho^*}$ ■

It can be seen from Proposition 3 that the nonlinear controller $h(x)$ implemented in sample-and-hold is capable of maintaining system state inside the stability region if the sampling time $\Delta$ is sufficiently small. Moreover, it will drive the system state into a level set $\Omega_{\rho^*} \subset \Omega_\rho$ in $N^*$ steps. Note that $\rho^*$ and $N^*$ are both monotonic functions of $\Delta$ for $\Delta \in (0, \Delta_{\max})$. Smaller sampling time $\Delta$ results in smaller $\rho^*$ and larger $N^*$. Moreover, arbitrarily small $\rho^*$ can be obtained if the sampling time $\Delta$ can be made arbitrarily small.

**Remark 13** *Note that the stability property of $h(x)$ implemented in sample-and-hold in Proposition 3 is made possible due to the continuous differentiability of the Lyapunov function $V(x)$ as well as the Lipschitz properties of $f$ and $g$. Note also that the result in Proposition 3 is conservative in the sense that the actual region in which $\hat{x}$ is maintained after $N^*$ steps might be much smaller than $\Omega_{\rho^*}$ characterized by Eq. (6.7) because worst case is assumed throughout the analysis.*

### 6.2.2 Proposed EMPC design

To proceed, we define $\hat{x}(t|t_k)$ and $\hat{u}(t|t_k)$ as the predicted state and input trajectories of the system of Eq. (6.1) under the sample-and-hold implementation of $h(\hat{x})$

evaluated at $t_k$ with the initial condition $\hat{x}(t_k|t_k) = x(t_k)$. Specifically, for a time interval $t \in [t_k, t_{k+N_h}]$, the predicted state and input trajectories can be obtained by integrating the following differential equation recursively for $l = 0, ..., N_h - 1$:

$$\dot{\hat{x}}(t|t_k) = f(\hat{x}(t|t_k)) + g(\hat{x}(t|t_k))\hat{u}(t|t_k), \quad t \in [t_{k+l}, t_{k+l+1}) \tag{6.14a}$$

$$\hat{u}(t|t_k) = h(\hat{x}(t_{k+l}|t_k)), \quad t \in [t_{k+l}, t_{k+l+1}) \tag{6.14b}$$

$$\hat{x}(t_k|t_k) = x(t_k) \tag{6.14c}$$

A function, $c(x(t_k), N_h)$, that characterizes the accumulated economic performance of the nominal system under $h(x)$ implemented in sample-and-hold over $N_h$ sampling periods with the initial state $\hat{x}(t_k|t_k) = x(t_k)$ can thus be defined as follows:

$$c(x(t_k), N_h) = \int_{t_k}^{t_{k+N_h}} l(\hat{x}(t|t_k), \hat{u}(t|t_k))\mathrm{d}t \tag{6.15}$$

where $l(x, u)$ is a bounded function defined on $\{(x, u) : x \in \Omega_\rho, u \in \mathbb{U}\}$ that characterizes the dynamic economic cost of the system at a specific time instant.

In the proposed design, EMPC minimizes the economic cost function $l(x, u)$ for a prediction horizon of $N$ sampling periods with the cost function $c(x, N_h)$ incorporated as the terminal cost. The system state is allowed to be operated only in the stability region $\Omega_\rho$. Specifically, the proposed EMPC at a sampling time $t_k$ is formulated as the following optimization problem:

$$\min_{u(\tau)\in S(\Delta)} \int_{t_k}^{t_{k+N}} l(\tilde{x}(\tau), u(\tau))\mathrm{d}\tau + c(\tilde{x}(t_{k+N}), N_h) \tag{6.16a}$$

$$\text{s.t. } \dot{\tilde{x}}(t) = f(\tilde{x}(t)) + g(\tilde{x}(t))u(t) \tag{6.16b}$$

$$\tilde{x}(t_k) = x(t_k) \tag{6.16c}$$

$$u(t) \in \mathbb{U} \tag{6.16d}$$

$$\tilde{x}(t) \in \Omega_\rho \tag{6.16e}$$

where $S(\Delta)$ denotes the family of continuous piece-wise functions with sampling time $\Delta$ and $\tilde{x}$ denotes the predicted state trajectory in this optimization problem.

In the above optimization problem, Eq. (6.16a) is the objective function involving the accumulated economic performance over the prediction horizon (i.e., the first term in the objective function) and a terminal cost $c(\tilde{x}(t_{k+N}), N_h)$. Eq. (6.16b) is the system model. Eq. (6.16c) specifies the initial condition at time $t_k$. Eq. (6.16d) is the constraint on control input $u$. Eq. (6.16e) specifies the region in which EMPC is allowed to operate which is used to ensure the closed-loop stability of the proposed EMPC.

The optimal solution to the optimization problem of Eq. (6.16) is denoted as $u^*(t|t_k)$, $t \in [t_k, t_{k+N})$, with the corresponding optimal state trajectory denoted as $x^*(t|t_k)$, $t \in [t_k, t_{k+N})$. The manipulated input of the closed-loop system under the proposed EMPC at a sampling instant $t_k$ is defined as follows:

$$u(t) = u^*(t|t_k),\ t \in [t_k, t_{k+1}) \tag{6.17}$$

which is the first value of the optimal input sequence. At the next sampling instant $t_{k+1}$, the optimization problem of Eq. (6.16) is re-evaluated.

### 6.2.3 Stability and performance analysis

In this subsection, we analyze the stability and economic performance of the system under the proposed EMPC design of Eq. (6.16). The stability property of the proposed EMPC design is presented in the following theorem.

**Theorem 15** *Consider the system of Eq. (6.1) in closed-loop under the proposed EMPC design of Eq. (6.16) with the initial state $x(t_0) \in \Omega_\rho$. If the sampling time $\Delta$ satisfies $\Delta < \Delta_{\max}$, then the optimization problem is feasible for all $t_k \geq t_0$, and the system state $x(t) \in \Omega_\rho$ for all $t \geq t_0$.*

**Proof.** We use induction to prove this theorem. At a sampling time $t_k$, let us assume that $x(t_k) \in \Omega_\rho$. If the sampling time $\Delta$ satisfies $\Delta < \Delta_{\max}$, it can be obtained from Proposition 1 that the predicted trajectory $\hat{x}(t|t_k)$ following Eq. (6.15) with $N_h = N$ is maintained in $\Omega_\rho$ for $t \in [t_k, t_{k+N})$ (i.e., constraint (6.16e) is satisfied). Also, the corresponding $\hat{u}(t|t_k)$ for $t \in [t_k, t_{k+N})$ satisfies the input constraint (6.16d). This implies that the optimization problem of Eq. (6.16) is feasible at $t_k$. Moreover, the feasibility of the optimization problem of Eq. (6.16) at $t_k$ implies that $x(t) \in \Omega_\rho$ for $t \in [t_k, t_{k+1}]$ since the first step input value is the actual input applied to the system. $x(t_{k+1}) \in \Omega_\rho$ further implies that the optimization problem of Eq. (6.16) at $t_{k+1}$ is also feasible. If $x(t_0) \in \Omega_\rho$, following the reasoning of induction, the optimization problem of Eq. (6.16) is feasible for all $t_k \geq t_0$, and $x(t) \in \Omega_\rho$ for all $t \geq t_0$. This proves Theorem 15. ■

Next, we analyze the closed-loop economic performance of the proposed EMPC. In particular, we focus on the asymptotic average economic performance since infinite-time operation is considered. The asymptotic average economic performance is defined as follows:

$$\bar{J}_{asy} := \lim_{F\to\infty} \frac{1}{F\Delta} \int_{t_0}^{t_F} l(x(t), u(t))\mathrm{d}t \tag{6.18}$$

Theorem 2 below provides an upper bound on the asymptotical average performance of the proposed EMPC design.

**Theorem 16** *Consider the system of Eq. (6.1) in closed-loop under the proposed EMPC design of Eq. (6.16) with the initial state $x(t_0) \in \Omega_\rho$ and sampling time $\Delta < \Delta_{\max}$. If $N_h$ is chosen such that $N_h \geq N^*$, then the asymptotical average economic performance of the closed-loop system under the EMPC, $\bar{J}_{asy}^{EMPC}$, is bounded as follows:*

$$\bar{J}_{asy}^{EMPC} \leq \bar{J}_\Delta^h \tag{6.19}$$

*where*

$$\bar{J}_\Delta^h := \max\left\{\frac{1}{\Delta}\int_0^\Delta l(x(t), h(x(0)))\mathrm{d}t : \; x(0) \in \Omega_{\rho^*}\right\} \tag{6.20}$$

*which denotes the maximum possible average economic performance over one sampling period $\Delta$ under the nonlinear controller $h(x)$ implemented in sample-and-hold when the closed-loop system state is within $\Omega_{\rho^*}$.*

**Proof.** Let us use $L(t_k)$ to denote the objective function value of the optimization problem of Eq. (6.16) evaluated at sampling time $t_k$. $L(t_k)$ corresponds to the accumulated economic performance of the open-loop system under the following augmented input sequence implemented in sample-and-hold for $t \in [t_k, t_{k+N+N_h}]$:

$$U^*(t_k) = [u^*(t_k|t_k), ..., u^*(t_{k+N-1}|t_k), h(\hat{x}(t_{k+N}|t_{k+N})), ..., h(\hat{x}(t_{k+N+N_h-1}|t_{k+N}))] \tag{6.21}$$

where $\hat{x}(t|t_{k+N})$ for $t \in [t_{k+N}, t_{k+N+N_h}]$ denotes the predicted state trajectory under the nonlinear controller $h(x)$ implemented in sample-and-hold starting from $x^*(t_{k+N}|t_k)$. We will refer to $U^*(t_k)$ as the optimal augmented solution to the optimization problem of Eq. (6.16) at $t_k$.

Since $x^*(t_{k+N}|t_k) \in \Omega_\rho$ (due to Eq. (6.16e)) and $N_h \geq N^*$, from Proposition 3, it can be obtained that the augmented optimal input sequence $U^*(t_k)$ will drive the predicted state trajectory into the invariant set $\Omega_{\rho^*}$, that is:

$$\hat{x}(t_{k+N+N_h}|t_{k+N}) \in \Omega_{\rho^*} \tag{6.22}$$

Let us further construct the following augmented input sequence:

$$\tilde{U}(t_{k+1}) = [u^*(t_{k+1}|t_k), ..., u^*(t_{k+N-1}|t_k), h(\hat{x}(t_{k+N}|t_{k+N})), ..., h(\hat{x}(t_{k+N+N_h}|t_{k+N}))] \tag{6.23}$$

which is obtained by discarding the first value in $U^*(t_k)$ and applying $h(\hat{x})$ for one more step at the end. It can be shown that $\tilde{U}(t_{k+1})$ is a feasible augmented solution to the optimization problem of Eq. (6.16) at $t_{k+1}$. To verify this, we only need to check whether $h(\hat{x}(t_{k+N}|t_{k+N}))$ satisfies the input constraint of Eq. (6.16d) and whether the state trajectory $\hat{x}(t|t_{k+N})$ for $t \in [t_{k+N}, t_{k+N+1}]$ satisfies the constraint of Eq. (6.16e). Both of these can be verified based on the properties of $h(\hat{x})$ and the fact that $x^*(t_{k+N}|t_k) \in \Omega_\rho$.

Let $\tilde{L}(t_{k+1})$ denote the objective function corresponding to $\tilde{U}(t_{k+1})$. From the definition of $\tilde{U}(t_{k+1})$, $\tilde{L}(t_{k+1})$ can be expressed as follows:

$$\begin{aligned} \tilde{L}(t_{k+1}) = \;& L(t_k) - \int_{t_k}^{t_{k+1}} l(x^*(t|t_k), u^*(t_k|t_k))\mathrm{d}t \\ + \;& \int_{t_{k+N+N_h}}^{t_{k+N+N_h+1}} l(\hat{x}(t|t_{k+N}),\ h(\hat{x}(t_{k+N+N_h}|t_{k+N}))\mathrm{d}t \end{aligned} \tag{6.24}$$

Since $u^*(t_k|t_k)$ is the actually implemented control input, the second term of the right-hand-side of the above equation can be replaced with the economic performance of the actual state and input trajectories

$$\int_{t_k}^{t_{k+1}} l(x^*(t|t_k), u^*(t_k|t_k))\mathrm{d}t = \int_{t_k}^{t_{k+1}} l(x(t), u(t))\mathrm{d}t. \tag{6.25}$$

Moreover, taking into account the definition of $\bar{J}^h_\Delta$ and the fact that $\hat{x}(t_{k+N+N_h}|t_{k+N}) \in \Omega_{\rho^*}$, the following inequality holds:

$$\int_{t_{k+N+N_h}}^{t_{k+N+N_h+1}} l(\hat{x}(t|t_{k+N}),\ h(\hat{x}(t_{k+N+N_h}|t_{k+N}))\mathrm{d}t \leq \bar{J}^h_\Delta \Delta \tag{6.26}$$

Substituting Eq. (6.25) and Eq. (6.26) into Eq. (6.24), the following inequality can be obtained:

$$\tilde{L}(t_{k+1}) \leq L(t_k) - \int_{t_k}^{t_{k+1}} l(x(t), u(t))\mathrm{d}t + \bar{J}^h_\Delta \Delta \tag{6.27}$$

The principle of optimality indicates that $L(t_{k+1}) \leq \tilde{L}(t_{k+1})$, which gives:

$$L(t_{k+1}) \leq L(t_k) - \int_{t_k}^{t_{k+1}} l(x(t), u(t))\mathrm{d}t + \bar{J}^h_\Delta \Delta \tag{6.28}$$

Summing the above inequality from $k = 0$ to $F - 1$ on both sides and combining the integration terms, the following inequality can be obtained:

$$\int_{t_0}^{t_F} l(x(\tau), u(\tau))\mathrm{d}\tau \leq L(t_0) - L(t_F) + \bar{J}^h_\Delta F\Delta \tag{6.29}$$

Dividing $F\Delta$ on both sides with $F$ approaching infinity, and taking into account that $L(t_k)$ at any specific sampling instant is a finite value because it is a finite time integration of the bounded cost function $l(x, u)$, the following inequality is obtained:

$$\bar{J}^{EMPC}_{asy} = \lim_{F\to\infty} \frac{1}{F\Delta} \int_{t_0}^{t_F} l(x(\tau), u(\tau))\mathrm{d}\tau \leq \lim_{F\to\infty} \frac{L(t_0) - L(t_F)}{F\Delta} + \bar{J}^h_\Delta = \bar{J}^h_\Delta \tag{6.30}$$

This proves Theorem 16. ■

**Remark 14** *In Theorem 16, $\bar{J}^h_\Delta$ characterizes the worst possible average economic performance over one sampling period $\Delta$ that may be obtained when $h(x)$ is implemented in sample-and-hold and the system state is inside the invariant set $\Omega_{\rho^*}$. When the region $\Omega_{\rho^*}$ is sufficiently small, if we further assume that the performance metric $l(x, u)$ is smooth, it is expected that different trajectories in $\Omega_{\rho^*}$ will give approximately the same economic performance. This further implies that:*

$$\bar{J}^h_\Delta \approx \lim_{F\to\infty} \frac{1}{F\Delta} \int_0^{F\Delta} l(x, h)\mathrm{d}t \tag{6.31}$$

*where $x(0) \in \Omega_{\rho^*}$. Since $h$ is able to drive the state to $\Omega_{\rho^*}$, the above approximation (6.31) also holds when $x(0) \in \Omega_\rho$. This implies that:*

$$\bar{J}^h_\Delta \approx \bar{J}^h_{asy} \tag{6.32}$$

*for $x(0) \in \Omega_\rho$. Therefore, the statement in Theorem 1 may also implies that the asymptotic average performance of the proposed LEMPC is no worse than the asymptotic average performance of the controller $h$.*

## 6.3 Fixed-time implementation

In the previous section, a terminal cost EMPC based on the auxiliary controller $h(x)$ is proposed in which the terminal cost $c(x, N_h)$ is predetermined. It is shown that the proposed terminal cost EMPC achieves guaranteed asymptotical average performance over infinite time operation. In this section, we focus on finite-time operation. We propose an revised implementation strategy of the proposed EMPC with terminal cost that achieves improved economic performance over the auxiliary controller $h(x)$ implemented in sample-and-hold for a fixed time duration longer than the prediction horizon.

### 6.3.1 Implementation strategy

We consider the performance of the EMPC in a fixed time interval $[t_0,\ t_F]$ where $F > N$. The proposed fixed-time EMPC implementation consists of two stages. In the first stage when $t_k \in [t_0, t_{F-N}]$, the proposed EMPC design of Eq. (6.16) is implemented with $N_h$ shrinking every sampling time from $F - N$ to 0; in the second stage when $t_k \in [t_{F-N+1}, t_F]$ the proposed EMPC design of Eq. (6.16) is implemented with $N_h = 0$ and the prediction horizon shrinking every sampling time from $N$ to 0. The specific EMPC formulations at the two stages are as follows:

*Stage 1* : At a sampling instant $t_k$, $t_k \in [t_0, t_{F-N}]$, the proposed EMPC design is formulated as follows:

$$\min_{u(\tau)\in S(\Delta)} \int_{t_k}^{t_{k+N}} l(\tilde{x}(\tau), u(\tau))\mathrm{d}\tau + c(\tilde{x}(t_{k+N}), F - k - N) \tag{6.33a}$$

$$\text{s.t. } \dot{\tilde{x}}(\tau) = f(\tilde{x}(\tau)) + g(\tilde{x}(\tau))u(\tau) \tag{6.33b}$$

$$\tilde{x}(t_k) = x(t_k) \tag{6.33c}$$

$$u(\tau) \in \mathbb{U} \tag{6.33d}$$

$$x(\tau) \in \Omega_\rho \tag{6.33e}$$

The first step value of the optimal solution to the optimization problem of Eq. (6.33) is implemented between $t_k$ and $t_{k+1}$. At the next sampling instant $t_{k+1}$, the optimization problem of Eq. (6.33) is re-evaluated until $t_k = t_{F-N}$.

*Stage 2* : At a sampling instant $t_k$, $t_k \in [t_{F-N+1}, t_F]$, the proposed EMPC design is formulated as follows:

$$\min_{u(\tau)\in S(\Delta)} \int_{t_k}^{t_F} l(\tilde{x}(\tau), u(\tau))\mathrm{d}\tau \tag{6.34a}$$

$$\text{s.t. } \dot{\tilde{x}}(\tau) = f(\tilde{x}(\tau)) + g(\tilde{x}(\tau))u(\tau) \tag{6.34b}$$

$$\tilde{x}(t_k) = x(t_k) \tag{6.34c}$$

$$u(\tau) \in \mathbb{U} \tag{6.34d}$$

$$x(\tau) \in \Omega_\rho \tag{6.34e}$$

Again, the first step value of the optimal solution to the optimization problem of Eq. (6.34) is implemented. At the next sampling instant $t_{k+1}$, the optimization problem of Eq. (6.34) is re-evaluated until $t_k = t_F$.

### 6.3.2 Stability and performance analysis

In this subsection, we analyze the stability and economic performance of the proposed fixed-time EMPC implementation. The stability property of the fixed-time EMPC implementation is presented in the following theorem.

**Theorem 17** *Consider the system of Eq. (6.1) in closed-loop under the fixed-time EMPC implementation of Eq. (6.33) and Eq. (6.34) with the initial state* $x(t_0) \in \Omega_\rho$. *If the sampling time* $\Delta$ *satisfies* $\Delta < \Delta_{\max}$, *then the optimization problems are feasible for all* $t_k \in [t_0, t_F]$, *and the system state* $x(t) \in \Omega_\rho$ *for all* $t \in [t_0, t_F]$.

**Proof.** The fixed-time EMPC designs of Eq. (6.33) and Eq. (6.34) are in the same form of the EMPC of Eq. (6.16). Following similar arguments as in the proof of Theorem 15, the closed-loop stability of the proposed fixed-time EMPC implementation can be proved. ■

The economic performance of the fixed-time EMPC implementation is summarized in Theorem 18 below.

**Theorem 18** *Consider the accumulated closed-loop economic performance of the system of Eq. (6.1) for a fixed time interval* $[t_0, t_F]$, *with the initial state* $x(t_0) \in \Omega_\rho$ *and sampling time* $\Delta < \Delta_{\max}$. *The accumulated economic performance of the closed-loop* *system under the fixed-time EMPC implementation of Eq. (6.33) and Eq. (6.34) is no worse than the accumulated economic performance of the closed-loop system under $h(x)$ implemented in sample-and-hold; that is:*

$$\int_{t_0}^{t_F} l(x(t), u(t))\mathrm{d}t \leq c(x(t_0), F) \tag{6.35}$$

*where $x$ and $u$ are the actual optimal state and input trajectories under the proposed fixed-time EMPC implementation strategy.*

**Proof.** Let us use $L(t_k)$ to denote the objective function value of the optimization problem of Eq. (6.33) or Eq. (6.34) evaluated at sampling time $t_k$. $L(t_k)$ corresponds to the accumulated economic performance of the open-loop predicted system state trajectory under the following input or augmented input sequence implemented in sample-and-hold for $t \in [t_k, t_F]$:

$$U^*(t_k) = [u^*(t_k|t_k), ..., u^*(t_{F-1}|t_k)]. \tag{6.36}$$

Note that when $t_k \leq t_{F-N}$, $U^*(t_k)$ for the optimization problem of Eq. (6.33) in stage 1 is the optimal augmented input sequence as follows:

$$U^*(t_k) = [u^*(t_k|t_k), ..., u^*(t_{k+N-1}|t_k), h(\hat{x}(t_{k+N}|t_{k+N})), ..., h(\hat{x}(t_{k+F-1}|t_{k+N}))] \tag{6.37}$$

It can be easily verified that the input sequence $\tilde{U}(t_{k+1}) = [u^*(t_{k+1}|t_k), ..., u^*(t_{F-1}|t_k)]$ is a feasible solution to the EMPC optimization problems at $t_{k+1}$. The objective function corresponding to this input sequence can be expressed as follows:

$$\tilde{L}(t_{k+1}) = L(t_k) - \int_{t_k}^{t_{k+1}} l(x(t), u(t))\mathrm{d}t \tag{6.38}$$

The principle of optimality indicates that $L(t_{k+1}) \leq \tilde{L}(t_{k+1})$, which gives:

$$L(t_{k+1}) \leq L(t_k) - \int_{t_k}^{t_{k+1}} l(x(t), u(t))\mathrm{d}t \tag{6.39}$$

Summing the above inequality from $k = 0$ to $F - 1$ on both sides and combining the integration terms, the following inequality can be obtained:

$$\int_{t_0}^{t_F} l(x(\tau), u(\tau))\mathrm{d}\tau \leq L(t_0) - L(t_F) \tag{6.40}$$

At the sampling time $t_F$,

$$L(t_F) = \int_{t_F}^{t_F} l(x(\tau), u(\tau))\mathrm{d}\tau = 0 \tag{6.41}$$

Because the input sequence generated by $h(\hat{x})$ in sample-and-hold from $t_0$ to $t_{F-1}$ with initial state $x(t_0) \in \rho$ is a feasible augmented solution to the optimization problem of Eq. (6.33) at $t_0$ , the following inequality holds:

$$L(t_0) \leq c(x(t_0), F) \tag{6.42}$$

Substituting Eq. (6.41) and Eq. (6.42) into Eq. (6.40):

$$\int_{t_0}^{t_F} l(x(t), u(t))\mathrm{d}t \leq c(x(t_0), F) \tag{6.43}$$

This proves Theorem 18. ■

**Remark 15** *We note that the shrinking horizon approach as used in Stage 2 may be used directly starting from $t_0$ with the initial horizon $N = F$ and the results in Theorem 18 still hold. Similar ideas have been used in [65, 30] to ensure finite-time performance of EMPC. However, the computational demand of such an implementation could be very high or even prohibitive for online implementation especially when $F$ is very large. Taking advantage of the terminal cost, the proposed fixed-time EMPC implementation is capable of achieving improved fixed-time performance over the auxiliary controller $h(x)$ with any predetermined prediction horizon $N$ which is an appealing feature for online implementation.*

**Remark 16** *At the second stage of the proposed fixed-time implementation strategy, the system state trajectory may be governed by the Turnpike property [66]. In this case, the system state will deviate from the steady state at the end of the fixed time window. An alternative implementation strategy to circumvent the dramatic deviation is to implement the auxiliary controller $h(x)$ instead of the shrinking horizon EMPC at stage 2. If $h(x)$ is implemented at stage 2, the stability and performance analysis in Theorem 17 and Theorem 18 still hold. This implementation strategy will be compared with the strategy outlined in Section 6.3.1 in Section 6.4.*

## 6.4 Application to a primary separation vessel

### 6.4.1 Process description and modeling

The primary separation vessel (PSV) is a key unit in the Clark hot water extraction process which is used for extraction of all commercial minable oil sands. As the first step in the extraction process, PSV facilitates the separation of aerated bitumen and coarse solid particles (clay, sand, etc.). Figure 6.1 shows a schematic of the PSV process. Oilsand ore is first fed into a mixing tank in which it is mixed with flood

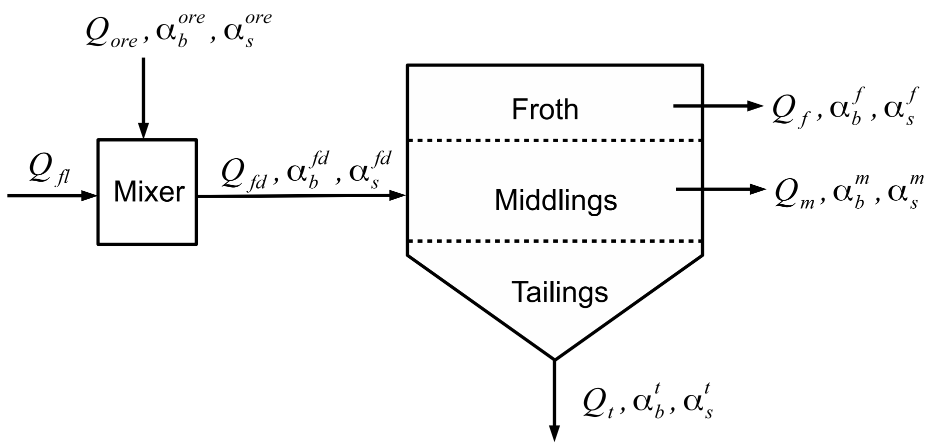


**Figure 6.1:** Schematic of the primary separation vessel

water, air and other process additives. The mixed oilsand slurry is then fed into the middle of the primary separation vessel in which gravity separation takes place due to the density difference of bitumen droplets, water and solids. A bitumen rich stream (froth) is withdrawn on top of the vessel and sent to downstream units for further processing. There are also streams withdrawn from the middle and the bottom layers. The stream withdrawn from the middle layer primarily contains water while the stream from the bottom layer mostly contains solids.

In the PSV process model, we assume that there are three typical bitumen particles and three solid particles in the feed stream. The mixer is modeled as a continuous stirred tank. The separation vessel is modeled into three distinguishable layers: top froth, middlings and tailings [67]. Each layer is assumed to be well mixed and all composition densities are considered constant. The froth/middlings interface level, which is a crucial control index [68], is dynamically modeled. The middlings/tailings interface is assumed to be stationary (i.e., volume of the tailings layer is constant). Based on mass balance and standard modeling assumptions, a set of ordinary differential equations is derived to describe the dynamics of the system [67]:

$$\frac{d\alpha_{bj}^{fd}}{dt} = \frac{1}{V_{mix}}(Q_{ore}(\alpha_{bj}^{ore} - \alpha_{bj}^{fd}) - Q_{fl}\alpha_{bj}^{fd}) \tag{6.44a}$$

$$\frac{d\alpha_{sj}^{fd}}{dt} = \frac{1}{V_{mix}}(Q_{ore}(\alpha_{sj}^{ore} - \alpha_{sj}^{fd}) - Q_{fl}\alpha_{sj}^{fd}) \tag{6.44b}$$

$$\frac{dV_f}{dt} = v_I A \tag{6.44c}$$

$$\frac{d\alpha_{bj}^{f}}{dt} = \begin{cases} \frac{1}{V_f}(-\alpha_{bj}^m v_{bj}^m A + v_I A(\alpha_{bj}^m - \alpha_{bj}^f) - Q_f \alpha_{bj}^f), & \text{if } v_{bj}^m - v_I < 0 \\ \frac{1}{V_f}(-\alpha_{bj}^f v_{bj}^m A - Q_f \alpha_{bj}^f), & \text{if } v_{bj}^m - v_I \geq 0 \end{cases} \tag{6.44d}$$

$$\frac{d\alpha_{sj}^{f}}{dt} = \begin{cases} \frac{1}{V_f}(-\alpha_{sj}^m v_{sj}^m A + v_I A(\alpha_{sj}^m - \alpha_{sj}^f) - Q_f \alpha_{sj}^f), & \text{if } v_{sj}^m - v_I < 0 \\ \frac{1}{V_f}(-\alpha_{sj}^f v_{sj}^m A - Q_f \alpha_{sj}^f), & \text{if } v_{sj}^m - v_I \geq 0 \end{cases} \tag{6.44e}$$

$$\frac{d\alpha_{bj}^{m}}{dt} = \begin{cases} \frac{1}{V_m}(Q_{fd}\alpha_{bj}^{fd} - Q_m \alpha_{bj}^m - \alpha_{bj}^m v_{bj}^t A + \alpha_{bj}^m v_{bj}^m A), & v_{bj}^m - v_I < 0 \\ \frac{1}{V_m}(Q_{fd}\alpha_{bj}^{fd} - Q_m \alpha_{bj}^m - \alpha_{bj}^m v_{bj}^t A + \alpha_{bj}^f v_{bj}^m A + v_I A(\alpha_{bj}^m - \alpha_{bj}^f)), & v_{bj}^m - v_I \geq 0 \end{cases} \tag{6.44f}$$

$$\frac{d\alpha_{sj}^{m}}{dt} = \begin{cases} \frac{1}{V_m}(Q_{fd}\alpha_{sj}^{fd} - Q_m \alpha_{sj}^m - \alpha_{sj}^m v_{sj}^t A + \alpha_{sj}^m v_{sj}^m A), & v_{sj}^m - v_I < 0 \\ \frac{1}{V_m}((Q_{fd}\alpha_{sj}^{fd} - Q_m \alpha_{sj}^m - \alpha_{sj}^m v_{sj}^t A + \alpha_{sj}^f v_{sj}^m A + v_I A(\alpha_{sj}^m - \alpha_{sj}^f)), & v_{sj}^m - v_I \geq 0 \end{cases} \tag{6.44g}$$

$$\frac{d\alpha_{bj}^{t}}{dt} = \frac{1}{V_t}(\alpha_{bj}^m v_{bj}^t A - Q_t \alpha_{bj}^t) \tag{6.44h}$$

$$\frac{d\alpha_{sj}^{t}}{dt} = \frac{1}{V_t}(\alpha_{sj}^m v_{sj}^t A - Q_t \alpha_{sj}^t) \tag{6.44i}$$

where $j = 1, 2, 3$ stands for the $j$-th species of bitumen or solids. The definition of model variables are given in Table 6.1. The values of constant model parameters are given in Table 6.2. In this model, the movement of all particles is assumed to be one-dimensional with the downward direction taken as the positive direction. The volumes of each layers satisfy:

$$V_m + V_f + V_t - V = 0 \tag{6.45}$$

The volumetric flows satisfy the following volume balance equations:

$$Q_{ore} + Q_{fl} = Q_{fd} = Q_f + Q_m + Q_t \tag{6.46}$$

The relative slip velocities of the particles are calculated using the following hindered settling model [69] :

$$v_{bj}^i = \frac{g d_{bj}^2 F(\alpha_w^i)(\rho_{bj} - \rho_i)}{18\mu_w} \tag{6.47a}$$

$$v_{sj}^i = \frac{g d_{sj}^2 F(\alpha_w^i)(\rho_{sj} - \rho_i)}{18\mu_w} \tag{6.47b}$$

where $i = f,\ m,\ t$, stands for the froth layer, middlings layer and tailings layer, respectively. The function $F(\alpha_w^i)$ is the Barnea and Mizrachi correlation [70]:

$$F(\alpha_w^i) = [1 + (1 - \alpha_w^i)^{\frac{1}{3}} e^{\frac{5(1-\alpha_w^i)}{3\alpha_w^i}}]^{-1} \tag{6.48}$$

The volumetric fraction of water in layer $i$, $i = f,\ m,\ t$, satisfies the overall volume balance in layer $i$:

$$\alpha_w^i = 1 - \sum_{j=1}^{3} \alpha_{bj}^i - \sum_{j=1}^{3} \alpha_{sj}^i \tag{6.49}$$

The suspension in layer $i$, $i = f,\ m,\ t$, is assumed to be uniform and its density is calculated as the weighted summation of component densities:

$$\rho_i = \rho_w \alpha_w^i + \sum_{j=1}^{3} \rho_{bj} \alpha_{bj}^i + \sum_{j=1}^{3} \rho_{sj} \alpha_{sj}^i \tag{6.50}$$

The froth/middlings interface velocity, $v_I$, is modeled by first order approximation of the shockwave equation [71]:

$$v_I = \frac{\sum_{j=1}^{3} \alpha_{bj}^m v_{bj}^m - \sum_{j=1}^{3} \alpha_{bj}^f v_{bj}^f}{\sum_{j=1}^{3} \alpha_{bj}^m - \sum_{j=1}^{3} \alpha_{bj}^f} \tag{6.51}$$

### 6.4.2 EMPC design

The system state vector which includes the volume of froth layer and the volumetric concentrations of all bitumen and solid species of the feed stream, froth layer, middlings layer and tailings layer, is defined as follows:

$$\begin{aligned} x = [\ & V_f \quad \alpha_{b1}^{fd} \quad \alpha_{b2}^{fd} \quad \alpha_{b3}^{fd} \quad \alpha_{s1}^{fd} \quad \alpha_{s2}^{fd} \quad \alpha_{s3}^{fd} \quad \alpha_{b1}^{f} \quad \alpha_{b2}^{f} \quad \alpha_{b3}^{f} \quad \alpha_{s1}^{f} \quad \alpha_{s2}^{f} \quad \alpha_{s3}^{f} \\ & \alpha_{b1}^{m} \quad \alpha_{b2}^{m} \quad \alpha_{b3}^{m} \quad \alpha_{s1}^{m} \quad \alpha_{s2}^{m} \quad \alpha_{s3}^{m} \quad \alpha_{b1}^{t} \quad \alpha_{b2}^{t} \quad \alpha_{b3}^{t} \quad \alpha_{s1}^{t} \quad \alpha_{s2}^{t} \quad \alpha_{s3}^{t} \ ]^T \end{aligned} \tag{6.52}$$

**Table 6.1:** Process variables.

| | |
|---|---|
| $Q_{ore},\ Q_{fl},\ Q_{fd},\ Q_f,\ Q_m,\ Q_t$ | volumetric flow rates of the oilsand ore, flood water, feed stream, froth stream, middlings stream and tailings stream |
| $V,\ V_f,\ V_m,\ V_t,\ V_{mix}$ | volumes of the vessel, froth layer, middlings layer, tailings layer and the feed flow mixer |
| $A$ | cross section area of the vessel |
| $v_I$ | froth/middlings interface velocity |
| $\alpha_{bj}^{fd},\ \alpha_{bj}^{f},\ \alpha_{bj}^{m},\ \alpha_{bj}^{t}$ | volumetric fractions of the $j$-th bitumen in feed stream, froth layer, middlings layer and tailings layer |
| $\alpha_{sj}^{fd},\ \alpha_{sj}^{f},\ \alpha_{sj}^{m},\ \alpha_{sj}^{t}$ | volumetric fractions of the $j$-th solid in feed stream, froth layer, middlings layer and tailings layer |
| $\alpha_{w}^{fd},\ \alpha_{w}^{f},\ \alpha_{w}^{m},\ \alpha_{w}^{t}$ | volumetric fractions of water in feed stream, froth layer, middlings layer and tailings layer |
| $\alpha_{bj}^{ore},\ \alpha_{sj}^{ore}$ | volumetric fractions of the $j$-th bitumen and sands in the ore |
| $v_{bj}^{f},\ v_{bj}^{m},\ v_{bj}^{t}$ | slip velocities of the $j$-th bitumen in froth layer, middlings layer and tailings layer |
| $v_{sj}^{f},\ v_{sj}^{m},\ v_{sj}^{t}$ | slip velocities of the $j$-th solid in froth layer, middlings layer and tailings layer |
| $\rho_{bj},\ \rho_{sj},\ \rho_w$ | the $j$-th bitumen particle, $j$-th solid particle and water densities |
| $d_{bj},\ d_{sj}$ | the $j$-th bitumen particle and solid particle diameters |
| $\rho_f,\ \rho_m,\ \rho_t$ | suspension densities of froth layer, middlings layer and tailings layer |
| $\mu_w$ | viscosity of water |
| $g$ | gravitation costant |

**Table 6.2:** Model parameters.

| | |
|---|---|
| $\rho_b = [800,\ 750,\ 700]\ kg/m^3$ | $A = 8.5\ m^2$ |
| $\rho_s = 2650\ kg/m^3$ | $\mu_w = 3.25 \times 10^{-4}\ kg/(m \cdot s)$ |
| $\rho_w = 971.8\ kg/m^3$ | $g = 9.81 m/s^2$ |
| $\alpha_b^{ore} = [0.021,\ 0.038,\ 0.051]$ | $d_b = [2.8,\ 5.8,\ 7.0] \times 10^{-5}\ m$ |
| $\alpha_s^{ore} = [0.0168,\ 0.042,\ 0.630]$ | $d_s = [0.78,\ 1.311,\ 12.5] \times 10^{-5}\ m$ |
| $V = 1000\ m^3$ | $V_t = 200\ m^3$ |
| $V_{mix} = 4.15\ m^3$ | $Q_{ore} = 0.0056 m^3/s$ |

**Table 6.3:** Steady-state operating point.

| | |
|---|---|
| $\alpha_b^{fd} = [0.0075,\ 0.0136,\ 0.0183]$ | $\alpha_s^{fd} = [0.0603,\ 0.0151,\ 0.2262]$ |
| $\alpha_b^{f} = [0.0648,\ 0.2019,\ 0.2918]$ | $\alpha_s^{f} = [0,\ 0,\ 0]$ |
| $\alpha_b^{m} = [0.0232,\ 0.0159,\ 0.0149]$ | $\alpha_s^{m} = [0.2982,\ 0.0598,\ 0.0315]$ |
| $\alpha_b^{t} = [0,\ 0,\ 0]$ | $\alpha_s^{t} = [0.0106,\ 0.0060,\ 0.2861]$ |
| $Q_{fl} = 0.0100\ m^3/s$ | $Q_f = 0.0008 m^3/s$ |
| $Q_m = 0.0027\ m^3/s$ | $Q_t = 0.0120 m^3/s$ |
| $V_f = 200.0\ m^3$ | |

The input vector which includes flow rates of flood water, middlings stream and tailings stream, is defined as follows:

$$u = [u_1 \quad u_2 \quad u_3]^T = [Q_{fl} \quad Q_m \quad Q_t]^T \tag{6.53}$$

The economic objective of the PSV process is to maximize the overall recovery rate of bitumen from feed ore, which is equivalent to minimizing the time integration of the following function:

$$\begin{aligned} r(x(t), u(t)) &= -\frac{\sum_{j=1}^{3} \alpha_{bj}^{f}(t) Q_f(t)}{\sum_{j=1}^{3} \alpha_{bj}^{ore} Q_{ore}} \\ &= -\frac{\big(x_8(t) + x_9(t) + x_{10}(t)\big)\big(Q_{ore} + u_1(t) - u_2(t) - u_3(t)\big)}{\sum_{j=1}^{3} \alpha_{bj}^{ore} Q_{ore}} \end{aligned} \tag{6.54}$$

The optimal steady state operating point is obtained by solving the following steady-state optimization problem:

$$
\begin{aligned}
(x_s, u_s) = \quad & \arg\min \;\; r(x,u) \\
\text{s.t.} \quad & f(x) = 0 \\
& u \in \mathbb{U}
\end{aligned}
\tag{6.55}
$$

where $f(x)$ is the right-hand-side of the process model of Eq. (6.44) and the control input is bounded inside the convex set $\mathbb{U}$ defined as follows:

$$
\mathbb{U} := \left\{ (u_1, u_2, u_3) : \begin{array}{l} 0 \leq u_1 \leq 0.01 \\ 0 \leq u_2 \leq 0.03 \\ 0 \leq u_3 \leq 0.03 \\ Q_f = Q_{ore} + u_1 - u_2 - u_3 \geq 0 \end{array} \right\}
\tag{6.56}
$$

The obtained optimal steady state is shown in Table 6.3. The economic cost function of EMPC is defined as follows:

$$
l(x,u) = r(x,u) + d(u)
\tag{6.57}
$$

where the term $d(u(t))$ penalizes the incremental of the inputs and is defined as follows:

$$
d(u(t)) = |u(t) - u(t_{k-1})|_R^2, \quad t \in (t_{k-1}, t_k]
\tag{6.58}
$$

where $R = \text{diag}(10^4, 10^4, 10^4)$. The incremental penalty term $d(u(t))$ is incorporated into the economic cost function to make the optimal solution unique while not changing the optimal steady state. The nonlinear auxiliary controller $h(x)$ is chosen as a single proportional controller paired between the froth volume $V_f$ and the middling outlet flow rate $Q_m$, specifically:

$$
h(x) = [u_{1s},\; u_{2s} + p(x_1 - x_{1s}),\; u_{3s}]^T
\tag{6.59}
$$

where the proportional gain is $p = -10^{-7}$. A quadratic Lyapunov function:

$$
V(x) = (x - x_s)^T P (x - x_s)
\tag{6.60}
$$

with $P = \text{diag}(10^{-4}, 1, 1, ..., 1)$ is used. A $\rho = 0.2$ level set of the quadratic Lyapunov function is chosen as the stability region in which the system state is allowed to evolve. The corresponding operating region for the froth layer volume is $155.3m^3 <$

**Table 6.4:** Transient and steady-state average bitumen recovery rates of the closed-loop system under the proposed EMPC with terminal cost, the EMPC in [1], the tracking MPC and the controller $h(x)$

| | Proposed EMPC | EMPC in [1] | Tracking MPC | $h(x)$ |
|---|---|---|---|---|
| Transient average | 0.8816 | 0.8123 | 0.7754 | 0.7690 |
| Steady state | 0.7514 | 0.7389 | 0.7611 | 0.7611 |

$V_f < 244.7m^3$. The sampling time is $\Delta = 1\ h$, the prediction horizon is $N = 5$, and the terminal cost evaluation step is $N_h = 30$.

In addition, the proposed EMPC design will be compared with the conventional tracking MPC. The stage cost of the tracking MPC is $l_t(x, u) = |x - x_s|_Q^2 + |u - u_s|_R^2$ where the weighting matrices are $Q = \text{diag}(10^{-4}, 1, 1, ..., 1)$ and $R = \text{diag}(10^4, 10^4, 10^4)$.

### 6.4.3 Simulation result

In this subsection, the performance of the proposed EMPC design will be compared with 1) an EMPC without terminal cost [1], 2) the conventional tracking MPC and 3) the controller $h(x)$. The initial state of all simulation runs is:

$$
\begin{aligned}
x(0) = [&190.0 \quad 0.0516 \quad 0.2106 \quad 0.3225 \quad 0 \quad 0 \quad 0 \quad 0.0168 \quad 0.0151 \\
&0.0151 \quad 0.1865 \quad 0.0459 \quad 0.2179 \quad 0 \quad 0 \quad 0 \quad 0.0020 \quad 0.0014 \\
&0.6097 \quad 0.0122 \quad 0.0220 \quad 0.0296 \quad 0.0975 \quad 0.0244 \quad 0.3656 \quad ]^T
\end{aligned} \tag{6.61}
$$

We first compare the performance of the proposed EMPC with terminal cost for infinite-time operation with the other control configurations. The closed-loop bitumen recovery rate trajectories, $r(x(t), u(t))$, under different controllers are shown in Fig. 6.2. The closed-loop froth layer volume trajectories are shown in Fig. 6.3. The closed-loop trajectories of the quadratic Lyapunov function of Eq. (6.60), which indicate the overall state evolutions, are shown in Fig. 6.4. The closed-loop input trajectories are shown in Fig. 6.5. The transient average recover rate for $t \in [0,\ 72h]$ and the steady state recovery rates achieved by different controllers are shown in Table 6.4.

From these results, it can be seen that the controller $h(x)$ and the tracking MPC eventually drive the system state to the optimal steady state. The two EMPC config-

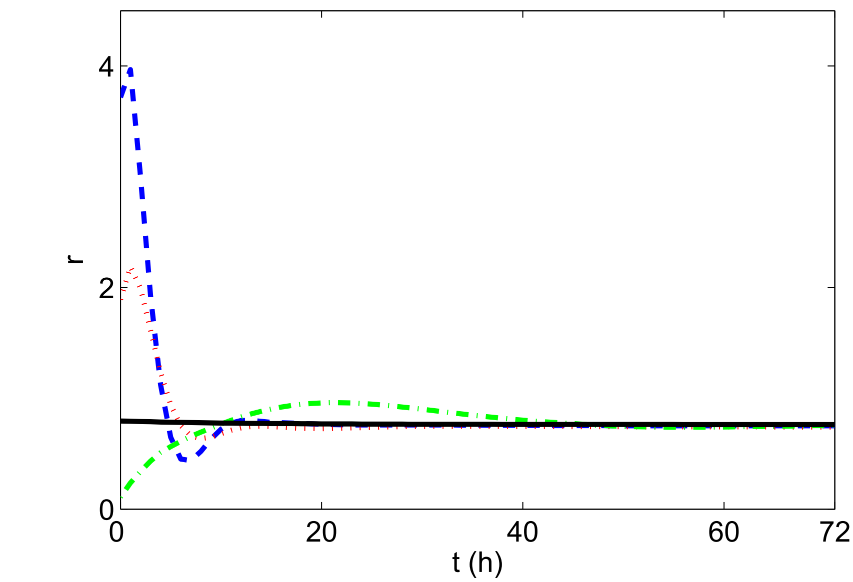


**Figure 6.2:** Bitumen recovery rate trajectores of the closed-loop system under the proposed EMPC with terminal cost (dashed line), the EMPC in [1] (dotted line), the tracking MPC (dash dotted line) and the controller $h(x)$ (solid line).

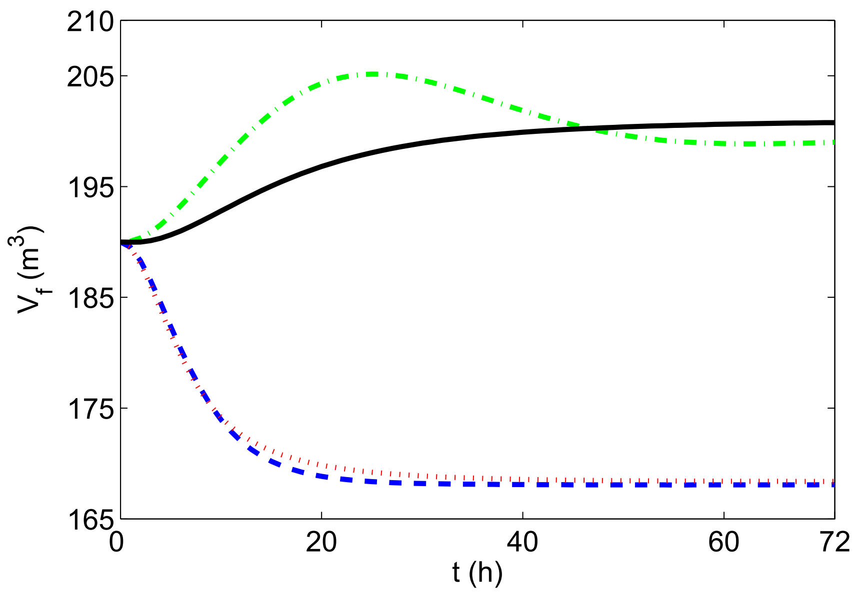


**Figure 6.3:** Froth layer volume trajectores of the closed-loop system under the proposed EMPC with terminal cost (dashed line), the EMPC in [1] (dotted line), the tracking MPC (dash dotted line) and the controller $h(x)$ (solid line).

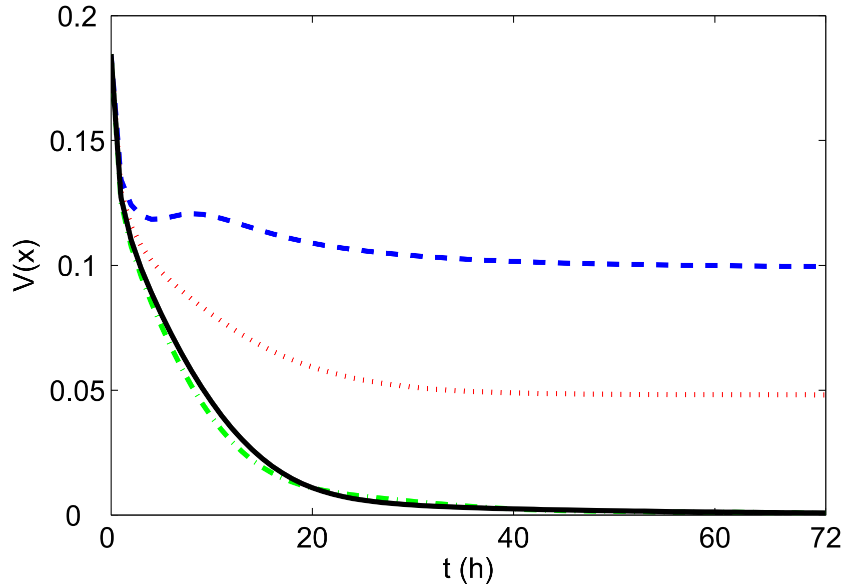


**Figure 6.4:** Lyapunov function trajectores of the closed-loop system under the proposed EMPC with terminal cost (dashed line), the EMPC in [1] (dotted line), the tracking MPC (dash dotted line) and the controller $h(x)$ (solid line).

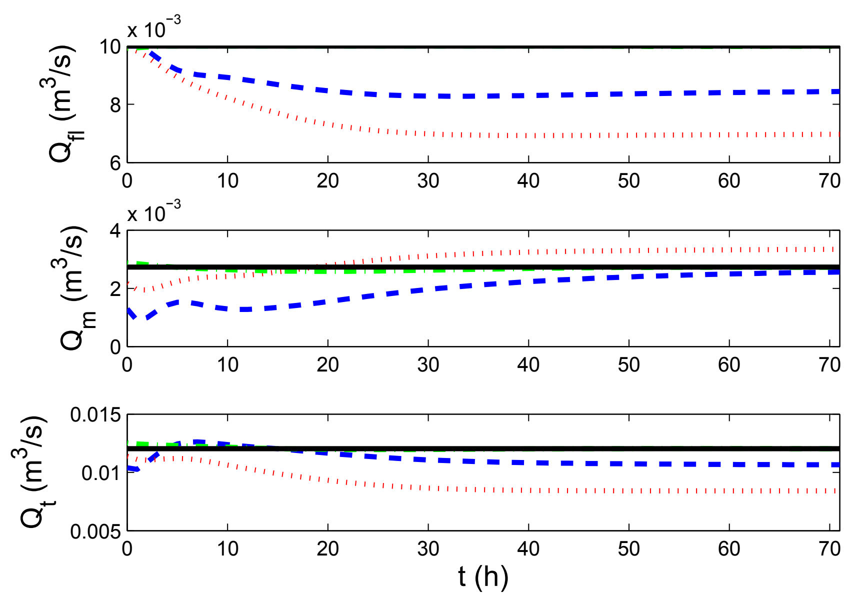


**Figure 6.5:** Input trajectories of the closed-loop system under the proposed EMPC with terminal cost (dashed line), the EMPC in [1] (dotted line), the tracking MPC (dash dotted line) and the controller $h(x)$ (solid line).

**Table 6.5:** Average bitumen recovery rates of the closed-loop system under (a): the infinite-time implementation, (b): fixed-time implementation, (c): alternative fixed-time implementation, and (d): the controller $h(x)$.

| | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| Average recovery rate | 0.8816 | 1.0191 | 0.8798 | 0.7690 |

urations (with and without terminal cost) drive the system state to new steady states slightly different from the optimal steady state while maintaining the system state inside the stability region. Note that this result agrees with the performance analysis in Theorem 2. It can be seen from Fig. 6.4 that in $N_h = 30$ steps, the controller $h(x)$ drives the system state to a small region around the optimal steady state with $\rho^* = 0.004$.

Note that the EMPCs achieve higher transient average recovery rates than the conventional tracking MPC or the controller $h(x)$ because recovery rate is directly accounted for in the EMPC cost functions. Note also that the proposed EMPC design achieves both higher transient recovery rate and higher steady-state recovery rate over the EMPC in [1] because of the enforcement of the proposed terminal cost.

Next, the performance of the fixed-time implementation strategy proposed in Section 4 is investigated. The fixed time interval is $t \in [0, 72h]$. Comparisons are made between the proposed EMPC with terminal cost for infinite-time operation, the proposed fixed-time implementation strategy, the alternative fixed-time implementation described in Remark 16, and the controller $h(x)$. The closed-loop bitumen recovery rate trajectories under different controllers and implementation strategies are shown in Fig. 6.6. The closed-loop froth layer volume trajectories are shown in Fig. 6.7. The closed-loop trajectories of the quadratic Lyapunov function are shown in Fig. 6.8. The closed-loop input trajectories are shown in Fig. 6.9. The average recovery rates and steady-state recovery rates achieved by different implementations for the fixed 72h time interval are shown in Table 6.5.

From these results, it can be seen that fixed-time implementation strategies achieve improved average bitumen recovery rates over the controller $h(x)$ for the fixed time interval, which agrees with the analysis of Theorem 4. It can be also seen that the

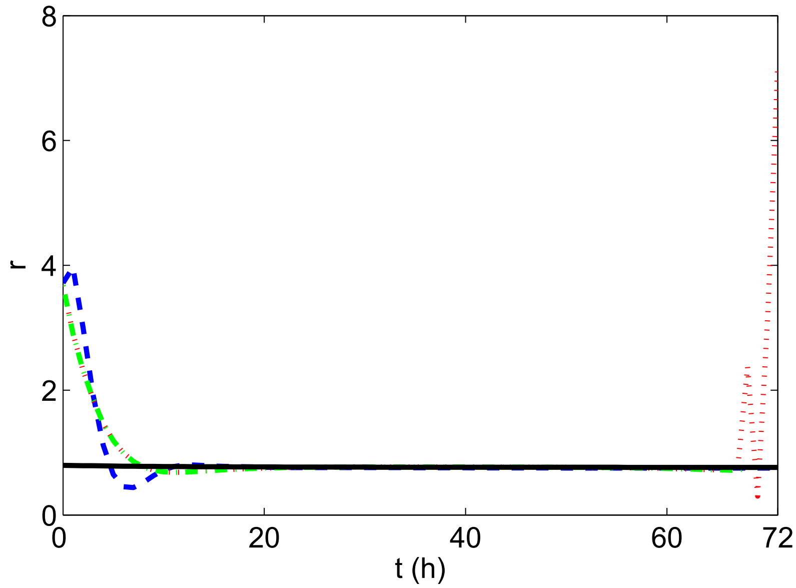


**Figure 6.6:** Bitumen recovery rates of the closed-loop system under the infinite-time implementation (dashed line), fixed-time implementation(dotted line), alternative fixed-time implementation (dash dotted line) and the controller $h(x)$ (solid line).

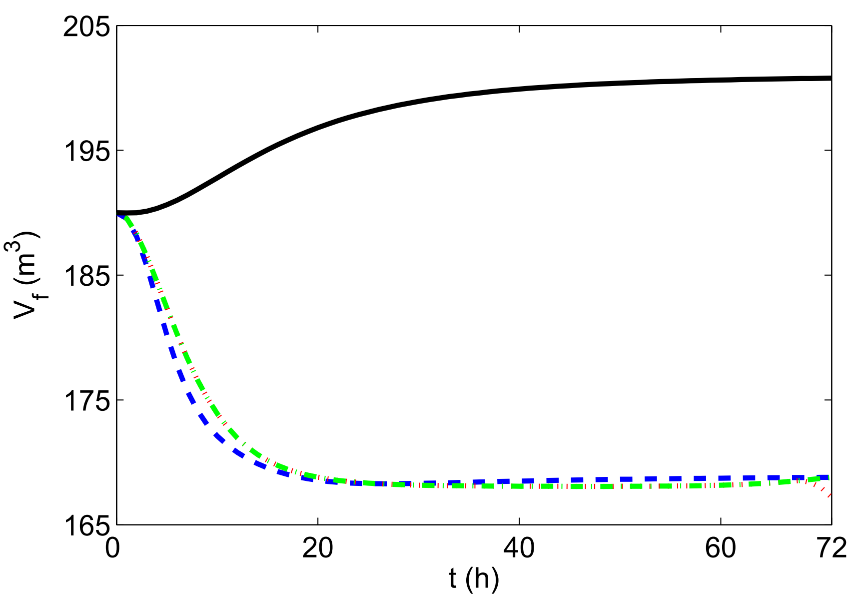


**Figure 6.7:** Froth layer volume trajectories of the closed-loop system under the infinite-time implementation (dashed line), fixed-time implementation(dotted line), alternative fixed-time implementation (dash dotted line) and the controller $h(x)$ (solid line).

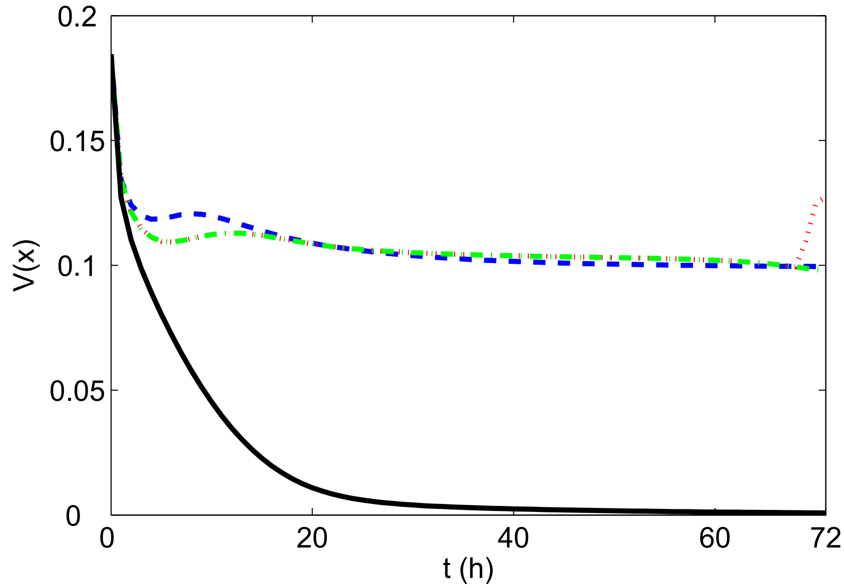


**Figure 6.8:** Lyapunov function trajectories of the closed-loop system under the infinite-time implementation (dashed line), fixed-time implementation(dotted line), alternative fixed-time implementation (dash dotted line) and the controller $h(x)$ (solid line).

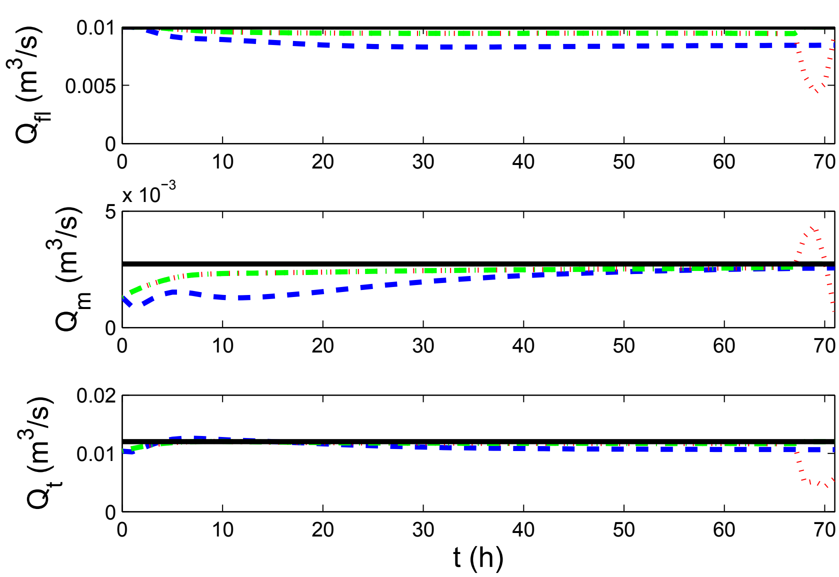


**Figure 6.9:** Input trajectories of the closed-loop system under the infinite-time implementation (dashed line), fixed-time implementation(dotted line), alternative fixed-time implementation (dash dotted line) and the controller $h(x)$ (solid line).

fixed-time implementation strategy proposed in Section 4.1 leads to highest average recovery rate for the fixed time interval with its system trajectory exhibiting the turnpike property. The dramatic deviation at the end is undesirable and is fixed by the alternative implementation described in Remark 16. In the first stage interval $t \in [0, 67h]$, the closed-loop system evolvements under the proposed EMPC design for infinite-time and finite-time operations remain basically the same until a slight divergence occurs at the end when $N_h$ for the finite-time operation gets smaller. This result indicates that the proposed EMPC design is insensitive to the value of $N_h$ when it is sufficiently large, which agrees with the properties of the controller $h(x)$. We would also like to note that it is not established that the fixed-time EMPC gives better performance than the infinite-time EMPC or vice versa. The performance depends on many different factors such as the duration of fix-time operation, the EMPC prediction horizon and the the auxiliary controller.

## 6.5 Summary

In this chapter, a terminal cost construction approach was developed for EMPC and EMPC algorithms were designed for both infinite-time and finite-time operations. In the proposed approach, an auxiliary nonlinear controller that renders the desired optimal steady state asymptotically stable was taken advantage of. The two EMPC algorithms give provable improved economic performance than the auxiliary controller with very flexible requirement on the length of the prediction horizon. This means that the proposed EMPC algorithms could be very computationally efficient. The proposed EMPC algorithms were applied to an oilsand primary separation vessel. The simulation results demonstrated the effectiveness of the proposed approaches.

# Chapter 7

# Conclusions and Future Work

## 7.1 Conclusions

In this thesis, we systematically discussed the stability and performance of the general EMPC scheme with extended horizon, and explored its extension or application to several specific scenarios.

We proposed the basic EMPC formulation with extended prediction horizon based on an auxiliary controller In Chapter 2. In the analysis, special attention was paid to the impact of the terminal horizon on the stability and performance of the proposed EMPC design. While a finite terminal horizon is in general not sufficient to ensure stability of the optimal steady state, it is sufficient to achieve practical stability for strictly dissipative systems under mild assumptions. Further conditions to ensure the exponential shrinkage of the practical stability region are provided. For a special case including conventional MPC with positive-definite stage costs, exponential stability can be achieved. Performance of the EMPC is also shown to be approximately upper-bounded by that of the auxiliary controller if a large terminal horizon is used. These results provide insights into the intrinsic properties of EMPC and also explain the computational efficiency of the EMPC design.

Chapters 3-6 are all extensions of the proposed EMPC design from Chapter 2. In Chapter 3, we design a terminal cost for economic model predictive control (EMPC) which preserves local optimality. We first showed, based on the strong duality and second order sufficient condition (SOSC) of the steady-state optimization problem, that the optimal operation of the system is locally equivalent to an infinite-horizon LQR controller. The proposed terminal cost is constructed as the value function of the LQR controller plus a linear term characterized by the Lagrange multiplier associated with the steady state constraint. EMPC with the proposed terminal cost is stabilizing with an appropriately chosen control horizon, and preserves the local optimality of the LQR controller.

In Chapter 4, we extended the proposed EMPC design to control systems with scheduled switching operations. The proposed EMPC scheme takes advantage of a set of auxiliary controllers that locally stabilizes the optimal steady state of each operating mode. In the proposed approach, EMPC operations are divided into two phases — an infinite-time operation phase and a mode transition phase, depending on the current sampling time and the scheduled mode switching time. Sufficient conditions to ensure recursive feasibility of the proposed EMPC design are established. The proposed EMPC design is computationally efficient and enjoys enlarged feasibility regions than the auxiliary controllers. Simulation results of a chemical process example demonstrate the superiority of our design over existing MPC designs for switched scheduling operations.

In Chapter 5, we proposed a general framework of nonlinear model predictive control for zone tracking. The target zone is characterized by coupled system state and input, and is not necessarily control invariant. An invariance-like theorem is developed which naturally generalizes LaSalle's invariance principle from autonomous system to control systems with a zone target. Our results differ from the standard stability analysis for conventional set-point MPC in that we consider the evolvement of the state-input pair $(x(n), u(n))$ instead of merely the the state trajectory $x(n)$. This provides stronger results and more accurate description of system behavior. Two simulations examples demonstrate the superiority of zone control over set-point control and the efficacy of the proposed zone MPC framework.

In Chapter 6, we applied the proposed EMPC algorithm to an oilsand primary separation vessel (PSV). We showed how previously developed EMPC design and analysis results in the context of discrete-time system can be extended to continuous-time systems where the issue of sampling were addressed.

## 7.2 Future research directions

*Distributed EMPC*. Distributed MPC has been a heated research area in recent years and is widely regarded as the most effective way to address organizational and computational issues of large-scale systems. Since EMPC involves general economic cost functions that are nonlinear and possibly non-convex, the computational complexity of EMPC is in general higher than conventional set-point tracking MPC. This makes the motivation for a distributed control framework even stronger for EMPC. Some existing results on distributed EMPC include [72, 73, 74, 75, 76]. It would be interesting to investigate EMPC with extended horizon in a distributed control framework. For example, based on the topology of the system and the communication network, a centralized auxiliary controller or a set of decentralized auxiliary controllers can be designed offline. In the online optimization, each EMPC unit will utilize the auxiliary controllers to predict the long term effects of their individual behavior on the overall networked control system. This could ensure the stability and asymptotic performance of the networked control system. Fundamental research on networked dissipative systems will be needed which requires knowledge from game theory, topology and graph theory. Practical issues encountered in the communication network such as asynchronous measurements and measurement delays and their influences on economic performance also deserve future research attention.

*EMPC with optimal non-steady-state operation*. In this thesis we have mainly focused on strictly dissipative systems whose infinite-time optimal operations are steady-state operation. However, for the general case, non-steady-state optimal operations are possible. On the one hand, generic nonlinear systems may feature non-steady optimal equilibrium solution such as optimal periodic solution. On the other hand, time-varying economic cost or disturbances may render the optimal operation non-steady. Recently, there are pioneering works on extending dissipativity analysis to periodic MPC design [77]. However, existing results rely on prior knowledge of the optimal steady state or the optimal periodic orbit. In the case of handling time-varying objectives and operating conditions, set-point independent EMPC design and analysis are needed.

*Robust EMPC*. Robustness is yet another untouched area in this thesis. It is conceivable that EMPC lack robustness compared to the conventional MPC, because conventional MPC is designed to stabilize the process while EMPC is designed to pursue economic performance. Earlier studies on the inherent robustness of MPC relied on the continuity of the value function [54] [32]. Recent development of MPC robustness analysis has been aided with the theory of input-to-state stability (ISS) [60], which offers an elegant way of proving robustness by finding an ISS-Lyapunov function. It will be rather straightforward to establish ISS stability of the proposed EMPC with extended horizon if the auxiliary controller is continuous and the terminal constraint set is inactive. Since the extended horizon is expected to add a certain degree of robustness to EMPC, it would also be interesting to investigate the extent to which extended horizon adds to the EMPC robustness.

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