UNIVERSITY OF ALBERTA

ON THE COMMUTATOR LENGTHS IN SOME FINITELY GENERATED GROUPS

MEHRI AKHAVAN MALAYERI

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

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Mehri Akhavan M.

Signature

Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **On the commutator lengths in some finitely generated groups** submitted by **Mehri Akhavan Malayeri** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematics.

Dr. A.H. Rhemtulla (Supervisor)

Dr. G.H. Cliff (Chair)

Dr. S.K. Sehgal

Dr. J.D. Lewis

Dr. H.J. Hoover

for Dr. C.C. Edmunds (External)

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DEDICATION

At 10:25, January 10, 1987 two long range ground to ground missiles launched by Iraqi regime struck their targets, two primary schools in the city of Boroujerd (west Iran) killing 67 students. This work is dedicated to the memory of those 67 innocent children.

And it is also dedicated to my husband, my supervisor, my parents, my son, and the other members of my family.

ABSTRACT

Let G be an arbitrary group, $w = w(x_1, \ldots, x_n)$ be a word in alphabets $x_1, \ldots, x_n$ and $G(w)$ the verbal subgroup of G, generated by $\{w(g_1, \ldots, g_n);\ g_i \in G\}$. The length $c(\gamma)$ of an element γ in $G(w)$ is the minimal number of $w^{\pm 1}$ words in a product equaling γ. When w is a commutator, we call $c(\gamma)$ as the "commutator length" of γ. This concept has been studied by many authors.

This thesis deals with the problems associated with the commutator length of certain classes of groups. The basic class of group which we consider is the class of free groups. We use simple and explicit algebraic methods to prove the following results:

1. Let w be a word in the free group $F(x, y)$,(freely generated by x and y) for which the total exponent of each alphabet is zero. We consider $F(w)$ and we find a suitable lower bound for the length of an arbitrary element γ of $F(w)$. And we discuss the interesting case $\gamma = w^n$ $(n \in \mathbb{N})$.

2. In $F(x, y)$ we introduce a formula to write $[x, y]^n$ as a product of $[n/2] + 1$ commutators which is the best lower bound for $[x, y]$. It is also the least upper bound, hence we find a simple algebraic proof for M. Culler's formula [C].

3. We also find another formula to write $([u_1, v_1] \ \ldots \ [u_k, v_k])^n$ as a product of $n(k-1) + [n/2] + 1$ commutators.

4. In $F(x, y)$ we show that $[x, y]^n$(where $n \in \mathbb{N}$ is an odd number) is never the product of two squares, although it is the product of three squares.

5. We discuss two questions rasied by Edmunds ([E-R],[L]) regarding commutators.

6. We show that if G is free nilpotent group of rank 2 and class 3 then the commutator length of G is equal to 2 and we calculate the commutator length of certain element of this group.

7. We show that if G is free abelian by nilpotent group of rank n then the commutator length of G is equal to n except in the trivial case where G is abelian.

8. We find lower and upper bounds for the commutator length of a free solvable group of rank n and class 3.

9. We know that if F is a free group then $c(F) = \infty$. We show the following interesting fact:

" *The commutator length of the wreath product of any group by the infinite cyclic group is less than or equal to 3.*"

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"Study from the cradle to the tomb.(Prophet Mohammad(PBUH))"

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CHAPTER 1
INTRODUCTION

In this thesis, we are mainly concerned with the commutator length of certain finitely generated groups. The basic class of the groups which we consider is the class of free groups. Each chapter contains the required definitions and notations. In this chapter we describe a brief survey of the main results of the other chapters.

In chapter 2, we consider the free group $F = F(x, y)$ of rank 2, freely generated by x, y. A word $w \in F$ is a commutator word, if there exist $k_i, l_i \in \mathbb{Z}$ such that

$$w = x^{k_1}y^{l_1}\dots x^{k_r}y^{l_r} \qquad \text{and} \qquad \sum_{i=1}^{n} k_i = \sum_{i=1}^{n} l_i = 0$$

Let $F(w) = \langle\ w(f_1, f_2)\ ; f_1, f_2 \in F\ \rangle$ be the verbal subgroup of F, defined by w. The fact that each element $\gamma \in F(w)$ can be written in different manners as a product of w-words or their inverses leads us to define "the minimal number of w-words or their inverses equaling γ " as the length of the element γ, and we denote it by $c(\gamma)$.

The main purpose of this chapter is to provide a suitable lower bound for $c(\gamma)$ in $F(w)$, and the following results will be obtained:

(i) $c(w^n) \to \infty$ as $n \to \infty$

(ii)$c(w^n) \neq 1$, for any $n \in \mathbb{N}, n \neq 1$.

Hence, it follows that no nontrivial commutator word in a free group is a proper power. This result for the case $w = [x, y]$ the commutator of x and y was obtained by Schützenberger [S], and it follows also from later more general result of Karrass, Magnus and Solitar [K-M-S] and G. Baumslag [B] and Steinberg [St].

In chapter 3, we continue considering free group $F = F(x, y)$ of rank 2 freely generated by x, y, and we assume $w = [x, y]$. In [C] M. Culler has proved that

$$c([x, y]^n) = [n/2] + 1 \tag{1}$$

In which $[n/2]$ denotes the greatest integer part of $n/2$, also in [Ba] C. Bavard, established formula (1) for $c([x, y]^n)$ and proved a generalization for $c([u_1, v_1] \ \dots \ [u_k, v_k])^n$ where $u_1, \dots, u_k, v_1, \dots, v_k$ belong to a basis of a free group F of rank $r, r \geq 2k$.

Lower bounds are estimated by a clever argument using a quasimorphism (which is defined in chapter 1) of the free group due to R. Brooks [Br]. If we apply the main theorem which is proved in chapter 1 for special case when $w = [x, y]$ we get the same lower bounds.

These estimates were shown to be upper bounds as well, by applying a topological calculus using surface topology to develop commutator identities. In chapter 3 we introduce a formula to write $[x, y]^n$ as a product of $[n/2] + 1$ commutators and also a formula for $([u_1, v_1] \ \dots \ [u_k, v_k])^n$, and finally in this chapter we discuss two questions raised byEdmundss ([E-R],[L]) regarding to the commutators, and we provide suitable answers to these questions.

In chapter 4 we establish the square length of $[x, y]^n$. Let G be a group,

$G' = \langle\ [h,g];\ h,g \in G\ \rangle$ be the derived subgroup of G and $G^2 = \langle\ g^2;\ g \in G\ \rangle$. It is known that $G' \subseteq G^2$, indeed, if h, g are elements of G, then:

$$[h,g] = hgh^{-1}g^{-1} = (hg)^2(g^{-1}h^{-1}g)^2(g^{-1})^2$$

Again assume $F = F(x,y)$, the free group of rank 2 freely generated by x, y. Let $\gamma \in F'$, "the minimal number of squares which is required to express γ as the product of squares" is called the square length of γ and denoted by $Sq\ (\gamma)$. It was shown by R.C. Lyndon and M. Newman [L-N] that $[x,y]$ is never the product of two squares in F, although it is always the product of three squares in F.

In chapter 4 we consider the more general case, i.e. $[x,y]^n$, $n \in \mathbb{N}$ an odd integer and we show that the equation $[x,y]^n = a^2b^2$ has no solution in F, although $Sq\ [x,y]^n = 3$. Two proofs of this theorem are given. The first one is by considering a suitable quotient of F' (which is much shorter than the second proof). The second proof is similar to the proof for the case $n = 1$ in [L-N] and it is based on a matrix argument.

In chapter 5 we consider $F_{(n,t)} = \langle x_1,\ \ldots,\ x_n \rangle$ the free nilpotent group of rank n and class t . P.W. Stroud in his Ph.D thesis [Str] in 1966, proved that for all t, every element of the commutator subgroup $F'_{(n,t)}$ can be expressed as a product of n commutators. In 1985 H. Allambergenov and V.A. Romankov [Al-R] proved that the minimal number $c\left(F_{(n,t)}\right)$ required to express an arbitrary element of $F'_{n,t}$ is precisely n provided $n \geq 2, t \geq 4$ or $n \geq 3, t \geq 3$. They proved this fact by providing an element d_n in $\gamma_t\left(F_{(n,t)}\right)$ that can not be written as a product of less than n commutators. For the case $n = 2, c = 3$ they proved that every element of $\gamma_3\left(F_{(n,t)}\right)$ is a commutator, and clamied that $c\left(F_{(2,3)}\right)$ is one. We show that the element $[x_1, x_2]^2$ can not be written as a commutator in the group $F_{(2,3)} = \langle x_1, x_2 \rangle$, this is done in the following theorem:

THEOREM 5.3. Let $F_{(2,3)} = \langle x_1, x_2 \rangle$ be the free nilpotent group of class 3 on 2 generators x_1, x_2. Then $c\left(F_{(2,3)}\right) = 2$. [A-R]

In [B-M] C. Bavard and G. Meigniez considered the same problem for the n-generator free metabelian group M_n. They showed that the minimal number $c(M_n)$ of commutators required to express an arbitrary element of the derived subgroup M_n' satisfies the inequality

$$[n/2] \leq c(M_n) \leq n$$

Where $[n/2]$ is the greatest integer part of $n/2$. Since $F_{(n,3)}$ groups are metabelian, the result of Allambergenov and Romankov [Al-R] shows that $c(M_n) \geq n$ for $n \geq 3$ and theorem 5.1 deals with the remaining case $n = 2$ and we have $c(M_n) = n$ for $n \geq 2$. We extend results in [B-M] and [Al-R] to the larger class of groups i.e. the class of abelian by nilpotent groups and we prove the following:

THEOREM 5.6. Let $G = \langle x_1, \ldots x_n \rangle$ be non-abelian free abelian by nilpotent group freely generated by $x_1, \ldots x_n$. Then $c(G) = n$. If A is an abelian normal subgroup of G and G/A is nilpotent, then every element of G' is a product of n commutators $[x_1, g_1]^{a_1}[x_2, g_2]^{a_2} \ldots [x_n, g_n]^{a_n}$ for suitable $g_1, \ldots, g_n$ in G and $a_1, \ldots, a_n$ in A. [A-R]

Then we prove the following theorem which establishes a lower as well as an upper bound for the commutator length of a finitely generated free solvable group of class 3.

THEOREM 5.9. Let $G = \langle x_1, \ldots x_n \rangle$ be a free solvable group of class 3, then

$$n \leq c(G) \leq n(n+3)/2$$

Then we prove a theorem about powers of certain commutators in $F_{(2,3)}$.

In chapter 6, we consider the wreath product of the free group by the infinite cyclic group, and we establish a suitable bound for the commutator length of this class of groups and we show it is a "c-group" (recall G is a c-group if there exists $n \in \mathbb{N}$ such that every element of G' can be expressed as a product of n commutators) and this is done in the following theorem

THEOREM.6.1. Let F be a free group and $W = F \mathrm{wr} C_\infty$ where C_∞ is the infinite cyclic group, then every element of W' is a product of three commutators.

CHAPTER 2
THE LENGTH OF THE VERBAL SUBGROUP OF THE FREE GROUP

Let F be a free group on a countably infinite set $\{x_1, x_2, \ldots\}$ and let $w = w(x_{i_1}, \cdots, x_{i_k})$ be a word in F. Given an arbitrary group G the subgroup, $G(w) = \langle\, w(g_1, \cdots, g_k)\ ;\ g_1, \ldots, g_k \in G\rangle$ is called the verbal subgroup of G defined by w. For example if $w = [x, y] = xyx^{-1}y^{-1}$ (called the commutator of x and y) then $G(w)$ is just the derived subgroup of G which is usually denoted by G'. As we mentioned in chapter 1, an element $\gamma \in G(w)$ can be written in a number of ways as the product of w-words, "the minimal number of w-words necessary to express γ" is called the w-length of γ and denoted by $c_w(\gamma)$ or simply the length of γ and denoted by $c(\gamma)$. We set

$$c(G) = \sup\{\ c(\gamma)\ ;\ \gamma \in G(w)\ \}$$

For example in case of the free group F of rank more than 1, length of F is infinity with respect to any proper, nontrivial word(the verbal subgroup is proper and nontrivial subgroup of G). In the case when w is a commutator word we show this fact in corollary 2.4 (see chapter 1 for definition).

Let $F = F(x, y)$ be the free group of rank 2, freely generated by x and y. The commutator length of elements of F has been studied by many authors, notably

M. Culler [C], who found the remarkable identity:

$$[x,y]^3 = [xyx^{-1}, y^{-1}xyx^{-2}][y^{-1}xy, y^2]$$

and proved that:

$$c([x,y]^n) = [n/2] + 1$$

where $[n/2]$ denotes the greatest integer part of $n/2$.

The idea of a quasi-morphism is very useful in finding a lower bound for c(F). C. Bavard in [Ba] estimated a lower bound for $c([x,y]^n)$ by using a quasi-morphism of the free group due to R. Brooks [Br]. We find the desired lower bounds for $c(\gamma)$ and $c(w^n)$ in $F(w)$, where w is any commutator word in x and y, by introducing a suitable quasi-morphism in F.

Definition: Let G be an arbitrary group, a quasi-morphism in G is a map $f : G \longrightarrow \mathbb{R}$ such that $|f(hg) - f(h) - f(g)|$, $(h, g \in G)$ is bounded and $\sup_{h,g\in G} |f(hg) - f(h) - f(g)| = D$ is called the defect of f. We say f is anti-symmetric if $f(h^{-1}) = -f(h)$.

Given a reduced word M in F, let $\Theta_M(X)$ be the number of occurrences of M in the reduced expression of an element $X \in G$ and set

$$f_M(X) = \Theta_M(X) - \Theta_{M^{-1}}(X)$$

First we check f_M is a quasi-morphism, let X, Y be two reduced words. Assume first XY is also reduced, then

$$f_M(XY) = f_M(X) + f_M(Y) + \delta_M(X,Y)$$

where $\delta_M(X,Y)$ is the number of occurrences of M "straddling" $X,\ Y$ minus the number of occurrences of M^{-1} "straddling" $X,\ Y$ (say $X = X'h,\ Y = gY'$ where $X',\ h,\ g$ and Y' are reduced words in F and length(h)=length(g)=length(M) -1 then $XY = X'hgY'$ and $\delta_M(X,Y) = f_M(hg)$) hence:

$$|\delta_M(X,Y)| \leq \text{length(M)} - 1$$

Example: Let $X = [x,y],\ Y = [x,y^2]$. If $M = y^{-1}x$ then $\delta_M(X,Y) = 1$; if $M = \left(y^{-1}x\right)^{-1}$ then $\delta_M(X,Y) = -1$; and finally, if length(M)=2 and $M \notin \{y^{-1}x, \left(y^{-1}x\right)^{-1}\}$ then $\delta_M(X,Y) = 0$.

More generally $X,\ Y$ can be written respectively as $X'A,\ A^{-1}Y'$ (where $X'Y'$ is reduced), then:

$$f_M(X'Y') = f_M(X) + f_M(Y) + \delta_M(X',Y') - \delta_M(X',A) - \delta_M(A^{-1},Y')$$

Hence:

$$|f_M(X'Y') - f_M(X) - f_M(Y)| = |\delta_M(X',Y') - \delta_M(X',A) - \delta_M(A^{-1},Y')|$$

Hence the defect of f_M is bounded above by 3(length(M)-1).

Now in the case of a commutator word $w = w(x,y) = x^{k_1}y^{l_1}\ldots x^{k_r}y^{l_r},\ \sum_{i=1}^{r} k_i = \sum_{i=1}^{r} l_i = 0$. The idea behind our proof is to find a good choice of the quasi-morphisms to give the desired lower bound for c(γ) in $F(w)$ and especially for $c(w^n)$ in $F(w)$. To do so, we first need some arrangements:

Let X'^{+} be a the descending sequence consisting of all positive powers of x in w;

$$X'^{+}:\quad k_{j_1},\ \ldots,\ k_{j_s}$$

where $k_{j_1} \geq \ldots \geq k_{j_s} \geq 0$ and k_{j_p} $(p = 1, \ldots, s)$ is a positive power of x in w. Similarly assume that X'^- is an ascending sequence consisting of all negative powers of x in w;

$$X'^- : \quad l_{m_1}, \ldots, l_{m_t}$$

where $l_{m_1} \leq \ldots \leq l_{m_t} \leq 0$ and l_{m_q} $(q = 1, \ldots, t)$ is a negative powers of x in w.

Delete from X'^+, X'^- those integers which are additive inverses of each other, and call the remaining sequence X^+, X^-. Hence in X^+, X^- we have $|k_{j_p}| \neq |l_{m_q}|$. Now let

$$\epsilon_1 = \max X^+, \; \epsilon_2 = \min X^- \text{ and } \epsilon_{1,2} = \epsilon_1 + \epsilon_2$$

Then either $|\epsilon_1| \gneq |\epsilon_2|$ or $|\epsilon_2| \gneq |\epsilon_1|$, in the first case replace ϵ_1 by $\epsilon_{1,2}$ in X^+ and call the new sequence X_1^+, and in the second case replace ϵ_2 by $\epsilon_{1,2}$ in X^- and call the new sequence X_1^- (in this way we break the powers such that they become additive inverses of each other, see the following example). We repeat in this way until all the powers of x are broken down such that they are additive inverses of each other, we do the same for the powers of y.

Example: If $w = x^5 y^2 x^{-2} y^{-1} x^{-1} y^3 x^{-2} y^{-4}$then:

$$\begin{cases} X^+ : \; 5 \\ X^- : -2 \, , \, -2 \, , \, -1 \end{cases} \qquad \begin{cases} Y^+ : \; 3 \, , \; 2 \\ Y^- : -4 \, , \, -1 \end{cases}$$

And;

$$\epsilon_1 = \max X^+ = 5, \qquad \epsilon_2 = \min X^- = -2$$

hence:

$$\epsilon_{1,2} = \max X^+ + minX^- = 3$$

Then;

$$\begin{cases} X_1^+ : & 3 \\ X_1^- : & -2\ ,\ -1 \end{cases} \qquad \text{and} \qquad w = x^2x^3y^2x^{-2}y^{-1}x^{-1}y^3x^{-2}y^{-4}$$

Now

$$\max X_1^+ + minX_1^- = 3 - 2 = 1$$

Then;

$$\begin{cases} X_2^+ : & 1 \\ X_2^- : & -1 \end{cases} \qquad \text{and} \qquad w = x^2x^2xy^2x^{-2}y^{-1}x^{-1}y^3x^{-2}y^{-4}$$

The process is finished for x. Now we repeat for y:

$$\max Y^+ + \min Y^- = 3 - 4 = -1$$

Then;

$$\begin{cases} Y_1^+ : & 2 \\ Y_1^- : & -1\ ,\ -1 \end{cases} \qquad \text{and} \qquad w = x^2x^2xy^2x^{-2}y^{-1}x^{-1}y^3x^{-2}y^{-3}y^{-1}$$

Again;

$$\max Y_1^+ + \min Y_1^- = 2 - 1 = 1$$

Then;

$$\begin{cases} Y_2^+ : & 1 \\ Y_2^- : & -1 \end{cases} \qquad \text{and} \qquad w = x^2x^2xyyx^{-2}y^{-1}x^{-1}y^3x^{-2}y^{-3}y^{-1}$$

Hence we break down the powers four times.

Now let

$\mathcal{N}_x(w) =$ the number of times we apply above process for the powers of x

$\mathcal{N}_y(w) =$ the number of times we apply above process for the powers of y

$\mathcal{N}(w) = \mathcal{N}_x(w) + \mathcal{N}_y(w) + 2r = r'$

In this way we write $w = z_1^{\epsilon_1} \ \ldots \ z_{r'}^{\epsilon_{r'}}$ such that $z_i^{\epsilon_i}$; $i = 1, \ \ldots, \ r'$ are in pair inverses of each other.

THEOREM 2.1. Let $F =< x, y >$ be a free group on free generators $x, \ y$, and let $w \in F'$ such that $w = z_1^{\epsilon_1} \ \ldots \ z_{r'}^{\epsilon_{r'}}$ represents an inverse pairing of subwords of w, let $\gamma \in F(w)$ with $c(\gamma) = k$, and let f be an anti-symmetric, quasi-morphism of F with defect D. Then

$$c(\gamma) \geq \frac{1}{\mathcal{N}(w)} \left(\frac{|f(\gamma)|}{D} + 1 \right)$$

Proof. Suppose $\gamma = w_1 w_2 \cdots w_k$, where each $w_i = u_{t+1} u_{t+2} \cdots u_{t+r'}$ $(t = (i-1)r')$ is the inverse pairing of w_i induced by the given pairing of w, above. Thus, letting $N = kr'$, we have $\gamma = u_1 u_2 \cdots u_N$. Note that since f is an anti-symmetric and the u_i's are in an inverse pairing, $\sum_{i=1}^{N} f(u_i) = 0$. Therefore,

$$|f(\gamma)| = |f(u_1 u_2 \cdots u_N)|$$

$$= |[f(u_1 u_2 \cdots u_N) - f(u_1) - f(u_2 u_3 \cdots u_N)] + [f(u_2 u_3 \cdots u_N) + f(u_1)]|$$

$$\leq D + |[f(u_2 u_3 \cdots u_N) + f(u_1)]|$$

$$= D + |[f(u_2 u_3 \cdots u_N) - f(u_2) - f(u_3 u_4 \cdots u_N)]$$

$$+ [f(u_3 u_4 \cdots u_N) + f(u_2) + f(u_1)]|$$

$$\leq 2D + |[f(u_3 u_4 \cdots u_N) + f(u_1) + f(u_2)]| =$$

$$\cdots$$

$$\leq (N-2)D + |[f(u_{N-1} u_N) + f(u_1) + f(u_2)] + \cdots + f(u_{N-2})|$$

$$= (N-2)D + |[f(u_{N-1} u_N) - f(u_{N-1}) - f(u_N)] + \sum_{i=1}^{N} f(u_i)|$$

$$\leq (N-1)D + |\sum_{i=1}^{N} f(u_i)|$$

$$= (N-1)D$$

$$= (kr' - 1)D$$

$$= (c(\gamma)\mathcal{N}(w) - 1)D.$$

The theorem follows immediately. □

Now let $w = x^{k_1}y^{l_1} \ \ldots \ x^{k_r}y^{l_r}$ be given and put:

$$wz = \begin{cases} wx, & \text{if } w \text{ starts with some positive power of } x \text{ i.e. } k_1 \gtrsim 0 \\ wx^{-1}, & \text{if } w \text{ starts with some negative power of } x \text{ i.e. } k_1 \lesssim 0 \end{cases}$$

we define:

$$n_1 = f_{x^2}(wz) = \Theta_{x^2}(wz) - \Theta_{x^{-2}}(wz)$$

And similarly

$$n_2 = f_{y^2}(wz),\ n_3 = f_{xy}(wz),\ n_4 = f_{yx}(wz),\ n_5 = f_{x^{-1}y}(wz),\ n_6 = f_{xy^{-1}}(wz)$$

In the previous example we have:

$$n_1 = 2,\ n_2 = n_3 = n_5 = 0,\ n_4 = n_6 = -2$$

Since $\mathbb{Z}$-linear combinations of (antisymmetric) quasi-morphisms are (antisymmetric) quasi-morphisms. We define :

$$f = n_1 f_{x^2} + n_2 f_{y^2} + n_3 f_{xy} + n_4 f_{yx} + n_5 f_{x^{-1}y} + n_6 f_{xy^{-1}}$$

and f is the desired quasi-morphism which we promised to introduce.

To prove the next theorem, we need to find the defect of f, which is done in the following lemma:

LEMMA 2.2. The defect of f is the maximum of the union of the following sets:

$$\{|n_1-(n_4+n_5)|,\ |n_4-(n_5+n_1)|,\ |n_5-(n_1+n_4)|\}$$

$$\{|n_1-(n_3+n_6)|,\ |n_3-(n_6+n_1)|,\ |n_6-(n_1+n_3)|\}$$

$$\{|n_2-(n_4+n_6)|,\ |n_4-(n_6+n_2)|,\ |n_6-(n_2+n_4)|\}$$

$$\{|n_2-(n_3+n_5)|,\ |n_3-(n_5+n_2)|,\ |n_5-(n_2+n_3)|\}$$

$$\{|n_1|, |n_2|, |n_3|, |n_4|, |n_5|, |n_6|\}$$

Proof: let $X = X'A,\ Y = A^{-1}Y'$ where $X,\ Y$ and $X'Y'$ are reduced words. First we calculate:

$$\Delta f_M = f_M(X'Y') - f_M(X) - f_M(Y) = \delta_M(X',\ Y') - \delta_M(X',\ A) - \delta_M(A^{-1},\ Y')$$

Where

$$M \in \{M_1 = x^2,\ M_2 = y^2,\ M_3 = xy,\ M_4 = x^{-1}y,\ M_5 = xy^{-1},\ M_6 = yx\} = \mathbb{B}$$

Then we find $\Delta f = f(X'Y') - f(X) - f(Y)$.

First case: If one of the words $A,\ X'$ or Y' is the empty word.

$$\begin{cases} \text{if} & X' = 1 & \text{then} & \Delta f_M = -\delta_M(A^{-1},\ Y') \\ \text{if} & Y' = 1 & \text{then} & \Delta f_M = -\delta_M(X',\ A) \\ \text{if} & A = 1 & \text{then} & \Delta f_M = \delta_M(X', Y') \end{cases}$$

Since $M \in \mathbb{B}$; $|\Delta f_M| \leq 1$

Now we have $f = \sum_{i=1}^{6} n_i f_{M_i}$, and in each case at most one of the $\Delta f_{M_i} \neq 0$, so in this case $D = \max\{|n_i|;\ i = 1,\ \ldots\ , 6\}$.

Second case: None of the words A, X' and Y' is the empty word. In the beginning it seems that there are 24 possibilities, but we observe in those cases which we have:

$$\begin{array}{ll} X' : \ldots\ g\ , & Y' : h\ \ \ldots \\ X'_1 : \ldots\ h^{-1}, & Y'_1 : g^{-1}\ \ldots \end{array}$$

where g, h are the generators of F or their inverses, and A is the same in both cases, we get:

$$|\delta_M(X',\, Y')-\delta_M(X',\, A)-\delta_M(A^{-1},\, Y')| = |\delta_M(X'_1,\, Y'_1)-\delta_M(A^{-1},\, Y'_1)-\delta_M(X'_1,\, A)|$$

hence $|\Delta f_M|$ is the same for both cases, and we left with only the following 12 cases:

(Case.2.1.) If we have:

$$\begin{array}{ll} X' : \ldots\ x & X'Y' : \ldots\ \ xy\ \ \ldots \\ A\ : x \ldots & X'A\ : \ldots\ \ x^2\ \ \ldots \\ Y'\ : y \ldots & A^{-1}Y' : \ldots\ x^{-1}y\ \ldots \end{array}$$

then:

$$\begin{array}{lll} \Delta f_{x^2} = -1 & \Delta f_{y^2} = 0 & \Delta f_{xy} = 1 \\ \Delta f_{yx} = 0 & \Delta f_{xy^{-1}} = 0 & \Delta f_{x^{-1}y} = -1 \end{array}$$

Therefore:

$$\Delta f = n_3 - (n_1 + n_6)$$

(Case.2.2.) If we have:

$$\begin{array}{l} X' : \ldots\ x \\ A\ : x \ldots \\ Y' : y^{-1} \ldots \end{array}$$

then, similar to the above case we have:

$$\Delta f = n_5 - (n_1 + n_4)$$

(Case.2.3.) If we have:

$$X' : \dots\ x$$
$$A\ : y \dots$$
$$Y' : x \dots$$

then:

$$\Delta f = n_1 - (n_3 + n_6)$$

(Case.2.4.) If we have:

$$X' : \dots\ x$$
$$A\ : y \dots$$
$$Y' : y^{-1} \dots$$

then:

$$\Delta f = n_5 - (n_2 + n_3)$$

(Case.2.5.) If we have:

$$X' : \dots\ x$$
$$A\ : y^{-1} \dots$$
$$Y' : y \dots$$

then:

$$\Delta f = n_3 - (n_2 + n_5)$$

(Case.2.6.) If we have:

$$X' : \dots\ x$$
$$A\ : y^{-1} \dots$$
$$Y' : x \dots$$

then:

$$\Delta f = n_1 - (n_4 + n_5)$$

(Case.2.7.) If we have:

$$X' : \dots\ x^{-1}$$
$$A\ : x^{-1} \dots$$
$$Y' : y \dots$$

then:

$$\Delta f = n_6 - (n_1 + n_3)$$

(Case.2.8.) If we have:

$$X' : \dots\ x^{-1}$$
$$A\ : x^{-1} \dots$$
$$Y' : y^{-1} \dots$$

then:

$$\Delta f = n_4 - (n_1 + n_5)$$

(Case.2.9.) If we have:

$$X' : \dots\ x^{-1}$$
$$A\ : y \dots$$
$$Y' : y^{-1} \dots$$

then:

$$\Delta f = n_4 - (n_2 + n_6)$$

(Case.2.10.) If we have:

$$X' : \dots\ x^{-1}$$
$$A\ : y^{-1} \dots$$
$$Y' : y \dots$$

then:

$$\Delta f = n_6 - (n_2 + n_4)$$

(Case.2.11.) If we have:

$$X' : \dots\ y$$
$$A\ : x \dots$$
$$Y' : y \dots$$

then:

$$\Delta f = n_2 - (n_4 + n_6)$$

And our final case is:

(Case.2.12.) If we have:

$$\begin{aligned} X' &: \ldots\ y \\ A &: x^{-1} \ldots \\ Y' &: y \ldots \end{aligned}$$

then:

$$\Delta f = n_2 - (n_3 + n_5)$$

hence from case 1 and case 2 we get the result. □

The case when γ is equal to w^n is interesting. This is done in the following theorem.

THEOREM.2.3. With the above notations:

$$c(w^n) \geq \frac{1}{\mathcal{N}(w)} \left(\frac{|f(w^n)|}{D} + 1 \right)$$

where

$$f(w^n) = \begin{cases} |n \sum_{i=1}^{6} n_i^2 + n_3|, & \text{for } w \ : \ x^{-1} \ \ldots \ y^{-1} \\ |n \sum_{i=1}^{6} n_i^2 - n_4|, & \text{for } w \ : \ x \ \ldots \ y \\ |n \sum_{i=1}^{6} n_i^2 + n_5|, & \text{for } w \ : \ x \ \ldots \ y^{-1} \\ |n \sum_{i=1}^{6} n_i^2 + n_6|, & \text{for } w \ : \ x^{-1} \ \ldots \ y. \end{cases}$$

Proof. By Theorem 2.1 we have:

$$c(w^n) \geq \frac{1}{\mathcal{N}(w)} \left(\frac{|f(w^n)|}{D} + 1 \right)$$

where D is obtained as in the last lemma. Let $f = \sum_{i=1}^{6} n_i f_{M_i}$ to complete the proof we need to find $f(w^n)$, and we have the following four cases:

(Case.1) If we have:

$$w \ : \ x \ \ldots \ y$$

then

$$f(w^n) = n_1(nn_1) + n_2(nn_2) + n_3(nn_3) + n_4(nn_4 - 1) + n_5(nn_5) + n_6(nn_6)$$

(Case.2) If we have:

$$w \; : \; x^{-1} \; \dots \; y$$

then

$$f(w^n) = n_1(nn_1) + n_2(nn_2) + n_3(nn_3) + n_4(nn_4) + n_5(nn_5) + n_6(nn_6 + 1)$$

(Case.3) If we have:

$$w \; : \; x \; \dots \; y^{-1}$$

then

$$f(w^n) = n_1(nn_1) + n_2(nn_2) + n_3(nn_3) + n_4(nn_4) + n_5(nn_5 + 1) + n_6(nn_6)$$

and finally

(Case.4) If we have:

$$w \; : \; x^{-1} \; \dots \; y^{-1}$$

then

$$f(w^n) = n_1(nn_1) + n_2(nn_2) + n_3(nn_3 + 1) + n_4(nn_4) + n_5(nn_5) + n_6(nn_6).$$

Hence the result is clear. □

If we consider the special case when $w = [x, y]$ is the commutator of x, y then $D = 2, \mathcal{N}(w) = 4$ and $f(w^n) = 4n - 1$. Hence we get the exact lower bound for $C([x,y])^n$ i.e. $[n/2]+1$ (which is found by M. Culler in [C]), and in the chapter 3 we introduce a formula to write $[x,y]^n$ as a product of $[n/2]+1$ commutators. Now from theorem 2.3 the following corollary is clear:

COROLLARY.2.4. With the above notations, We have:

(i) $c(w^n) \to \infty$ as $n \to \infty$

(ii) $c(w^n) \neq 1$, for any $n \in \mathbb{N}, n \neq 1$.

hence it follows that $C(F) = \infty$ with respect to any commutator word, and also it follows that no nontrivial commutator word in a free group is a proper power. This result for the case $w = [x,y]$ was obtained by Schützenberger [S], and it follows also from later more general result of Karrass, Magnus and Solitar [K-M-S] and G. Baumslag [B] and Steinberg [St].

CHAPTER 3
SOME COMMUTATOR EQUATIONS IN FREE GROUPS

In this chapter we use the following notations:

Let f, g be elements of a group $h^g = ghg^{-1}$, $[h, g] = hgh^{-1}g^{-1}$ and $[r]$ denotes the greatest integer part of $r \in \mathbb{R}$, and let $F = F(x, y)$ be a free group of rank 2 freely generated by x, y.

This chapter consists of two sections. In the first section, we consider the elements $[x, y]^n$ in F, as we have noted in chapter 2, M. Culler has proved that $c([x, y]^n) = [n/2] + 1$ by using the theory of surfaces [C], and also Theorem 2.3 in chapter 2 implies that $[n/2] + 1$ is a lower bound for $c([x, y]^n)$. In this section we introduce a formula to write $[x, y]^n$ as a product of $[n/2] + 1$ commutators. Then we consider the elements $([u_1, v_1] \ \ldots \ [u_k, v_k])^n$, where $u_1, \ \ldots, \ u_k, v_1, \ \ldots, \ v_k$ belong to a basis of a free group of rank r, $r \geq 2k$. C. Bavard in [Ba] has proved that $c([u_1, v_1] \ \ldots \ [u_k, v_k])^n = n(k-1) + [n/2] + 1$, and we also introduce a formula to write $([u_1, v_1] \ \ldots \ [u_k, v_k])^n$ as a product of $n(k-1) + [n/2] + 1$ commutators.

Finally in the second section of this chapter we discuss two questions raised byEdmundss ([E-S],[L])regarding commutators and we provide suitable answers to these questions.

§ **3.1.** To establish the required formula for $[x, y]^n$ we need only the following identities which hold in any group.

LEMMA.3.1.1 The following identities hold in any group G with $x,\ y \in G$.

(I) $[y^{-2}, yx]\,[x,y]^2 = [x, y^2\,[x^{-1}, y^{-1}]]^{y^{-1}}\,[x^{-2}, xy^{-1}]^{y}$

(II) $[x^{-2}, xy^{-1}]^{y}[x,y]^2 = [y^{-1}, x^2\,[y, x^{-1}]]^{yx^{-1}}\,[y^2, y^{-1}x^{-1}]^{yx}$

(III) $[y^2, y^{-1}x^{-1}]^{yx}[x,y]^2 = [x^{-1}, y^{-2}\,[x,y]]^{yxy}\,[x^2, x^{-1}y]^{yxy^{-1}}$

(IV) $[x^2, x^{-1}y]^{yxy^{-1}}[x,y]^2 = [y, x^{-2}\,[y^{-1}, x]]^{yxy^{-1}x}\,[y^{-2}, yx]^{[y,x]}$

Proof. To prove these identities we use the following easily verified rules for commutators. Let $a,\ b$ and c be arbitrary elements in any group we have:

$$abca^{-1}b^{-1}c^{-1} = [ac^{-1}, cb]\,, \qquad caba^{-1}b^{-1}c^{-1} = [a,b]^c$$

We start with proving I:

$$\begin{aligned}[y^{-2}, yx]\,[x,y]^2 &= (y^{-1})(x)(y^2x^{-1}y^{-1}xy)(x^{-1})y^{-1}\,[x,y] \\ &= (y^{-1})(x)(y^2\,[x^{-1}, y^{-1}])(x^{-1})\left([y^{-1}, x^{-1}]\,y^{-2}\right)(y) \\ &\quad (y \cdot x^{-1}y^{-1}x \cdot [x,y]) \\ &= [x, y^2\,[x^{-1}, y^{-1}]]^{y^{-1}}\left(y\left(x^{-1}y^{-1}x \cdot xyx^{-1}\right)y^{-1}\right) \\ &= [x, y^2\,[x^{-1}, y^{-1}]]^{y^{-1}}\,[x^{-2}, xy^{-1}]^{y}\end{aligned}$$

Now we prove II:

$$\begin{aligned}[x^{-2}, xy^{-1}]^{y}[x,y]^2 &= \left(\left(x^{-1} \cdot y^{-1} \cdot x^2yx^{-1}y^{-1}x \cdot y \cdot x^{-1}yxy^{-1}x^{-2} \cdot x\right)\left([x,y]\,[y^{-1}, x]\right)\right)^{y} \\ &= [y^{-1}, x^2\,[y, x^{-1}]]^{yx^{-1}}\,[y^2, y^{-1}x^{-1}]^{yx} \\ &\quad \text{(since: } [x,y]\,[y^{-1}, x] = [y^2, y^{-1}x^{-1}]^{x}\text{)}\end{aligned}$$

Now we prove III:

$$[y^2, y^{-1}x^{-1}]^{yx}[x,y]^2 = (yx)\left(y\cdot x^{-1}\cdot y^{-2}xyx^{-1}y^{-1}\cdot x\cdot yxy^{-1}x^{-1}y^2\cdot y^{-1}\right)\cdot$$

$$\left([y^{-1},x]\,[x^{-1},y^{-1}]\right)(yx)^{-1}$$

$$= [x^{-1}, y^{-2}[x,y]]^{yxy}\,[x^2, x^{-1}y]^{yxy^{-1}}$$

$$\text{(since: } [y^{-1},x]\,[x^{-1},y^{-1}] = [x^2, x^{-1}y]^{y^{-1}}$$

Finally we prove (IV):

$$[x^2, x^{-1}y]^{yxy^{-1}}[x,y]^2 = (yxy^{-1})\left(x\cdot y\cdot x^{-2}y^{-1}xyx^{-1}\cdot y^{-1}\cdot xy^{-1}x^{-1}yx^2\cdot x^{-1}\right)\cdot$$

$$\left([x^{-1},y^{-1}]\,[y,x^{-1}]\right)(yx^{-1}y^{-1})$$

$$= [y, x^{-2}[y^{-1},x]]^{yxy^{-1}x}[y^{-2}, yx]^{[y,x]}$$

the last equality is by: $[x^{-1},y^{-1}]\,[y,x^{-1}] = [y^{-2},yx]^{x^{-1}}$ and this complete the proof. □

Let us put:

$A = [y, x^{-1}[y^{-1},x]]^x$, $B = [x, y^2[x^{-1},y^{-1}]]^{y^{-1}}$

$C = [y^{-1}, x^2[y,x^{-1}]]^{yx^{-1}}$, $D = [x^{-1}, y^{-2}[x,y]]^{yxy}$

$E = [y, x^{-2}[y^{-1},x]]^{yxy^{-1}x}$, $F = [y^{-2}, yx]$

$G = [x^{-2}, xy^{-1}]^y$, $H = [y^2, y^{-1}x^{-1}]^{yx}$

$I = [x^2, x^{-1}y]^{yxy^{-1}}$, $J = B\cdot C\cdot D\cdot E$

$K = B\cdot G$, $L = B\cdot C\cdot H$

$M = B\cdot C\cdot D\cdot I$

Hence in the above lemma we proved that:

(*I*) $F\cdot[x,y]^2 = B\cdot G$

$(II)\ \ G \cdot [x,y]^2 = C \cdot H$

$(III)\ \ H \cdot [x,y]^2 = D \cdot I$

$(IV)\ \ I \cdot [x,y]^2 = E \cdot F^{[y,x]}$

Now we use the above identities to write $[x,y]^n$ as a product of $[n/2]+1$ commutators. It is enough, we consider the case when n is an odd number.

$$[x,y]^3 = (x \cdot y \cdot x^{-1}y^{-1}xyx^{-1} \cdot y^{-1} \cdot xy^{-1}x^{-1}yx \cdot x^{-1})y^{-1}xy \cdot yx^{-1}y^{-1}$$

$$= A \cdot \left[y^{-2}, yx\right] = A \cdot F$$

$$[x,y]^5 = [x,y]^3[x,y]^2 = A \cdot \left[y^{-2}, yx\right][x,y]^2$$

$$= A \cdot F \cdot [x,y]^2 = A \cdot B \cdot G = A \cdot K \qquad \text{by}(I)$$

$$[x,y]^7 = [x,y]^5[x,y]^2 = A \cdot B \cdot G \cdot [x,y]^2$$

$$= A \cdot B \cdot C \cdot H = A \cdot L \qquad \text{by}(II)$$

$$[x,y]^9 = [x,y]^7[x,y]^2 = A \cdot B \cdot C \cdot H \cdot [x,y]^2$$

$$= A \cdot B \cdot C \cdot D \cdot I = A \cdot M \qquad \text{by}(III)$$

$$[x,y]^{11} = [x,y]^9[x,y]^2 = A \cdot B \cdot C \cdot D \cdot I \cdot [x,y]^2$$

$$= A \cdot B \cdot C \cdot D \cdot E \cdot F^{[y,x]} = A \cdot J \cdot F^{[y,x]} \qquad \text{by}(IV)$$

$$[x,y]^{13} = [x,y]^{11}[x,y]^2 = A \cdot B \cdot C \cdot D \cdot E \cdot [y,x](F \cdot [x,y]^2)[x,y]$$

$$= A \cdot B \cdot C \cdot D \cdot E \cdot (B \cdot G)^{[y,x]} \qquad \text{by}(I)$$

$$= A \cdot J \cdot K^{[y,x]}$$

Similarly we obtain the following identities:

$$[x,y]^{15} = A \cdot J \cdot L^{[y,x]}$$

$[x,y]^{17} = A \cdot J \cdot M^{[y,x]}$

$[x,y]^{19} = A \cdot J \cdot J^{[y,x]} \cdot F^{[y,x]^2}$

$[x,y]^{21} = A \cdot J \cdot J^{[y,x]} \cdot K^{[y,x]^2}$

$[x,y]^{23} = A \cdot J \cdot J^{[y,x]} \cdot L^{[y,x]^2}$

$[x,y]^{25} = A \cdot J \cdot J^{[y,x]} \cdot M^{[y,x]^2}$

$[x,y]^{27} = A \cdot J \cdot J^{[y,x]} \cdot J^{[y,x]^2} \cdot F^{[y,x]^3}$

$[x,y]^{29} = A \cdot J \cdot J^{[y,x]} \cdot J^{[y,x]^2} \cdot K^{[y,x]^3}$

Now we are able to establish the general formula. Let $n \gtrsim 0$ be an odd integer, we have the following two cases.

(Case 1) If $n \leq 9$, then the commutator $A = \left[y, x^{-1}\left[y^{-1}, x\right]\right]^x$ is a common factor for $[x,y]^n$, and we have $c([x,y]^n) - 1 = [n/2] = 4N + R$ where $N \in \{0,\ 1\}$ and $R \in \{0,\ 1,\ 2,\ 3\}$, hence we have the following equations:

$$[x,y]^n = \begin{cases} A \cdot M^N & \text{for } R = 0 \\ A \cdot F^{N+1} & \text{for } R = 1 \\ A \cdot K^{N+1} & \text{for } R = 2 \\ A \cdot L^{N+1} & \text{for } R = 3 \end{cases}$$

(Case 2) If $n \gtrsim 9$, then $A \cdot B \cdot C \cdot D \cdot E = A \cdot J$ is a common factor for $[x,y]^n$ and $c([x,y]^n) - 5 = 4N + R$, where $R \in \{0,\ 1,\ 2,\ 3\}$ and $N \in \{0,\ 1,\ 2,\ 3,\ \ldots\}$. But $c([x,y]^n) = [n/2] + 1$, hence we get $[n/2] = 4(N+1) + R$ and we obtain the following result:

PROPOSITION.3.1.2. Let F be a free group and $x,\ y \in F$ and let:

$$A = \left[y, x^{-1}\left[y^{-1}, x\right]\right]^{x} \quad , \quad B = \left[x, y^{2}\left[x^{-1}, y^{-1}\right]\right]^{y^{-1}}$$

$$C = \left[y^{-1}, x^{2}\left[y, x^{-1}\right]\right]^{yx^{-1}} \quad , \quad D = \left[x^{-1}, y^{-2}\left[x, y\right]\right]^{yxy}$$

$$E = \left[y, x^{-2}\left[y^{-1}, x\right]\right]^{yxy^{-1}x} \quad , \quad F = \left[y^{-2}, yx\right]$$

$$G = \left[x^{-2}, xy^{-1}\right]^{y} \quad , \quad H = \left[y^{2}, y^{-1}x^{-1}\right]^{yx}$$

$$I = \left[x^{2}, x^{-1}y\right]^{yxy^{-1}} \quad , \quad J = B \cdot C \cdot D \cdot E$$

$$K = B \cdot G \quad , \quad L = B \cdot C \cdot H$$

$$M = B \cdot C \cdot D \cdot I.$$

Then:

$$[x,y]^{n} = \begin{cases} A \cdot J \cdot J^{[y,x]} \cdot J^{[y,x]^{2}} \ldots J^{[y,x]^{N-1}} \cdot M^{[y,x]^{N}} & \text{if } R = 0 \\ A \cdot J \cdot J^{[y,x]} \cdot J^{[y,x]^{2}} \ldots J^{[y,x]^{N}} \cdot F^{[y,x]^{N+1}} & \text{if } R = 1 \\ A \cdot J \cdot J^{[y,x]} \cdot J^{[y,x]^{2}} \ldots J^{[y,x]^{N}} \cdot K^{[y,x]^{N+1}} & \text{if } R = 2 \\ A \cdot J \cdot J^{[y,x]} \cdot J^{[y,x]^{2}} \ldots J^{[y,x]^{N}} \cdot L^{[y,x]^{N+1}} & \text{if } R = 3. \end{cases}$$

Now we consider more general case; let $u_1, \ldots, u_k, v_1, \ldots, v_k$ be elements of a free group. We show that how $([u_1, v_1] \ldots [u_k, v_k])^n$ may be written as a product of $n(k-1) + [n/2] + 1$ $(n,\ k \in \mathbb{N})$ commutators.

First we need to prove the following lemma:

LEMMA.3.1.3. With the above notations, and let $X = [u_1, v_1]$, $Y = [u_2, v_2]$, $Z = \left[v_1, (v_1 u_1)^{-1} Y v_1^{u_1}\right]^{u_1}$, $W = \left[v_1^{-1}, Y u_1\right]$ and $k \in \mathbb{N}$ then:

$I.\ (XY)^{3} = ZW\left(Y^{3}X\right)^{Y^{-1}} = ZWY^{2}XY$

$II.\ Y^{2k}(XY)^{3} = (ZW)^{Y^{2k}}\, Y^{2k+2}XY$

$III.\ Y^{2k}XY = X^{Y^{2k}}Y^{2k+1}$

Proof. First we prove I ;

$$(XY)^3 = u_1 \cdot v_1 \left(u_1^{-1}v_1^{-1}Yu_1v_1u_1^{-1}\right) v_1^{-1} \left(u_1v_1^{-1}u_1^{-1}Y^{-1}v_1u_1\right) u_1^{-1}$$
$$\left(v_1^{-1}Yu_1v_1u_1^{-1}\right)(YXY)$$
$$= \left[v_1, (v_1u_1)^{-1}Yv_1^{u_1}\right]^{u_1} \left[v_1^{-1}, Yu_1\right] Y^2XY$$
$$= ZWY^2XY$$

but $Y^2XY = Y^{-1}Y^3XY = \left(Y^3X\right)^{Y^{-1}}$ and we have already written Y^3 as a product of two commutators, hence $(XY)^3$ can be written as a product of 5 commutators.

Now we prove II

$$Y^{2k}(XY)^3 = Y^{2k}ZWY^2XY \qquad \text{(by(I))}$$
$$= Y^{2k}ZWY^{-2k}Y^{2k+2}XY$$
$$= (ZW)^{Y^{2k}}Y^{2k+2}XY$$

and III is clear.□

Let $n \gtrsim 0$ be an odd integer, we use the above lemma to write

$$A_{n,k} = ([u_1, v_1] \ \ldots \ [u_k, v_k])^n$$

as a product of $n(k-1) + [n/2] + 1$ commutators. If $k = 1$ then $A_{n,k} = [u_1, v_1]^n$ and we have already found the equations for this case. Hence fix $k = 2$, and let $n \gtrsim 0$ be an arbitrary odd integer. We find the required equation for $A_{n,2}$ as follows:

The case $n = 3$ is done in part I of above lemma and;

$$(XY)^5 = (XY)^3 (XY)^2 = ZWY^2XY(XY)^2 \qquad \text{(by (lemma.3.1.3.I))}$$

$$= (ZW)Y^2(XY)^3 = (ZW)(ZW)^{Y^2}Y^4(XY) \qquad \text{(by (lemma.3.1.3.II))}$$

and identity III of lemma 3.1.2 implies that:

$$(XY)^5 = (ZW)(ZW)^{Y^2}X^{Y^4}Y^5$$

But we know Y^5 can be written as a product of 3 commutators, hence $(XY)^5$ can be written as a product of 8 commutators.

Let $n = 7$ then we have:

$$(XY)^7 = (XY)^5(XY)^2 = (ZW)(ZW)^{Y^2}Y^4(XY)^3$$

$$= (ZW)(ZW)^{Y^2}(ZW)^{Y^4}Y^6(XY) \qquad \text{(by (lemma.3.1.3.II))}$$

$$= (ZW)(ZW)^{Y^2}(ZW)^{Y^4}X^{Y^6}Y^7 \qquad \text{(by (lemma.3.1.3.III))}$$

But Y^7 can be written as a product of 4 commutators. Similarly we have:

$$(XY)^9 = (ZW)(ZW)^{Y^2}(ZW)^{Y^4}(ZW)^{Y^6}X^{Y^8}Y^9$$

Now let $n = 2l - 1$. If we have;

$$(XY)^{2l-1} = (ZW)(ZW)^{Y^2}\cdots(ZW)^{Y^{2l-4}}X^{Y^{2l-2}}Y^{2l-1}$$

then we get:

$$(XY)^{2l+1} = (XY)^{2l-1}(XY)^2$$

$$= (ZW)(ZW)^{Y^2}\cdots(ZW)^{Y^{2l-4}}X^{Y^{2l-2}}Y^{2l-1}(XY)^2$$

Now since:

$$X^{Y^{2l-2}}Y^{2l-1}(XY)^2 = Y^{2l-2}XY^{-2l+2}Y^{2l-1}(XY)^2 = Y^{2l-2}(XY)^3$$

we imply that:

$$(XY)^{2l+1} = (ZW)(ZW)^{Y^2}\cdots(ZW)^{Y^{2l-4}}Y^{2l-2}(XY)^3$$

and by part II and III of lemma 3.1.2 we get:

$$(\star)\qquad \begin{aligned}(XY)^{2l+1} &= (ZW)(ZW)^{Y^2}\cdots(ZW)^{Y^{2l-4}}(ZW)^{Y^{2l-2}}Y^{2l}(XY)\\ &= (ZW)(ZW)^{Y^2}\ \ldots\ (ZW)^{Y^{2l-4}}(ZW)^{Y^{2l-2}}X^{Y^{2l}}Y^{2l+1}\end{aligned}$$

but Y^{2l+1} can be written as a product of $[(2l+1)/2]+1$ commutators, hence by the last equation $(XY)^{2l+1}$ can be written as a product of $2l+1+[(2l+1)/2]+1$ commutators.

So we have proved that if $k=2$, for any $n \in \mathbb{N}$, $(XY)^n$ can be written as a product of $n(k-1)+[n/2]+1 = n+[n/2]+1$ commutators. Now suppose we have the result for k. We prove the result for $k+1$, let

$$X_1 = [u_1, v_1]\,,\ X_2 = [u_2, v_2]\,,\ \ldots,\ X_{k+1} = [u_{k+1}, v_{k+1}]\,,$$

where u_i, v_i $(1 \le i \le k+1)$ are elements of a free group. Let $n = 2m+1$, and put $X_2X_3\ \ldots\ X_{k+1} = M$, $X_1 = N = [u_1, v_1]$, $Z_1 = [v_1, (v_1u_1)^{-1}Mv_1^{u_1}]^{u_1}$, and $W_1 = [v_1^{-1}, Mu_1]$. By apling $(\star)$ to N and M, we get:

$$(NM)^{2m+1} = (Z_1W_1)(Z_1W_1)^{M^2}(Z_1W_1)^{M^4}\cdots(Z_1W_1)^{M^{2m-2}}N^{M^{2m}}M^{2m+1},$$

and by induction M^{2m+1} can be written as a product of $(2m+1)(k-1)+[(2m+1)/2]+1$ commutators, hence $(NM)^{2m+1}$ can be written as a product of

$2+2m-2+1+(2m+1)(k-1)+[(2m+1)/2]+1=(2m+1)k+[(2m+1)/2]+1=$ $n(k+1-1)+[(2m+1)/2]+1$ commutators, hence we have:

PROPOSITION.3.1.3. Let u_i, v_i $(i = 1, \cdots, k)$ be elements of a free group . Then for any $n \in \mathbf{N}$ we can introduce a formula to write $([u_1, v_1] \cdots [u_k, v_k])^n$ as a product of $n(k-1)+[n/2]+1$ commutators.

§ 3.2. In this section we discuss two equations about commutators, raised byEdmundss. In 1974, in an attempt to test the power of the methods then available for solving equation in free groups,Edmundss ([E-R],[L]) asked the following questions:

(1) If $[v,w][x,y] = z^2$ $(\neq 1)$ in a free group, does it follow that z is a commutator, i.e. if $c(z^2) = 2$, does it follow that $c(z) = 1$?

(2) If $[v,w][x,y] = z^2$ $(\neq 1)$ in a free group and z is a commutator, does it follow that $[v,w] = [x,y]$? (i.e. If $c(z) = 1$ and $c(z^2) = 2$ with $[v,w][x,y] = z^2$ in a free group, does it follow that $[v,w] = [x,y]$?)

In 1990, question (1) was answered negatively by J. Comerford and Y. Lee [C-L] using a clever computer search based on the methods of D. Piollet combined with an algebraic translation of a representation of the generators for the mapping class groups due to J. Birman and D. Chillingworth. We also answer this question negatively just by finding suitable examples as follows.

We need the following result of M. J. Wicks [W]

"w is a commutator in a free group if and only if some conjugate of w is identically equal to $xyzx^{-1}y^{-1}z^{-1}$*."*

Regarding question (1) we find the following three examples:

The following identities are easily checked;

$$([x^{-1},y]\,[x,y^{-1}])^2 = [x^{-1},y\,[x^{-1},y]]^{[x^{-1},y]xy^{-1}}\,[yx^{-2},xy]^{y^{-1}}$$

$$([x,y]\,[x^{-2},y])^2 = [x^2,y^{-1}[x,y]]^{[x,y]x^{-2}y}\,[x,yx^{-1}y^{-1}]$$

$$([x,y]\,[x,y^{-2}])^2 = [y^{-2},x^{-1}y^2[x,y]xy^{-2}x^{-1}]^{[x,y]x}\,[xy^{-1}x^{-1},xy^2x^{-1}y]\,,$$

hence we conclude that:

$$c\left([x^{-1},y]\,[x,y^{-1}]\right)^2 \leq 2$$

$$c\left([x,y]\,[x^{-2},y]\right)^2 \leq 2$$

$$c\left([x,y]\,[x,y^{-2}]\right)^2 \leq 2.$$

By Wicks' result, we are able to check that a word in a free group is a commutator or not. Recall a word W is a "cycle" of the word X if and only if there are words X_1, X_2 such that $X = X_1X_2$ and $W = X_2X_1$. The (finite) set of cycles of X will be denoted by $\{X^*\}$. And the following negative transformations are defined in a free group:

(*i*) Deletion: $Xhh^{-1}Y \longrightarrow XY.$

(*ii*) Full deletion: $hXh^{-1} \longrightarrow X.$

These transformations are called negative since they decrease the length of the words.

Now suppose we are given an arbitrary word U in a free group. A sequence of negative transformations applied to U eventually yields a cyclically reduced conjugate U^*. We consider the set $\{U^*\}$ obtained from U in an effective manner and U will be a commutator just in the case some member of $\{U^*\}$ is in the form $XYZX^{-1}Y^{-1}Z^{-1}$ for some elements X, Y and Z of the free group. It is clearly possible to determine whether or not a given word is in the form $XYZX^{-1}Y^{-1}Z^{-1}$ in a finite number of steps(in fact this can be done in a number of steps which is a polynomial in the length of the word being tested).

Hence by using Wicks' result, one would be able to check that none of the following products of commutators is a commutator;

$$\left(\left[x^{-1},y\right]\left[x,y^{-1}\right]\right)^2, \qquad \left[x^{-1},y\right]\left[x,y^{-1}\right]$$

$$\left([x,y]\left[x^{-2},y\right]\right)^2, \qquad [x,y]\left[x^{-2},y\right]$$

$$\left([x,y]\left[x,y^{-2}\right]\right)^2, \qquad [x,y]\left[x,y^{-2}\right].$$

(For example let $U = \left[x^{-1},y\right]\left[x,y^{-1}\right]$ then

$$\{U^*\} = \{x^{-1}yxy^{-1}xy^{-1}x^{-1}y,\ yx^{-1}yxy^{-1}xy^{-1}x^{-1},\ x^{-1}yx^{-1}yxy^{-1}xy^{-1},$$
$$y^{-1}x^{-1}yx^{-1}yxy^{-1}x,\ xy^{-1}x^{-1}yx^{-1}yxy^{-1},\ y^{-1}xy^{-1}x^{-1}yx^{-1}yx,$$
$$xy^{-1}xy^{-1}x^{-1}yx^{-1}y,\ yxy^{-1}xy^{-1}x^{-1}yx^{-1}\}.$$

One can check that none of the elements of $\{U^*\}$ is in the form $XYZX^{-1}Y^{-1}Z^{-1}$, hence $U = \left[x^{-1},y\right]\left[x,y^{-1}\right]$ is not a commutator and $c(U) = 2$.)

Hence we have:

$$c\left(\left[x^{-1},y\right]\left[x,y^{-1}\right]\right)^2 = c\left(\left[x^{-1},y\right]\left[x,y^{-1}\right]\right) = 2$$

$$c\left([x,y]\left[x^{-2},y\right]\right)^2 = c\left([x,y]\left[x^{-2},y\right]\right) = 2$$

$$c\left([x,y]\left[x,y^{-2}\right]\right)^2 = c\left([x,y]\left[x,y^{-2}\right]\right) = 2$$

In [E-R] C. C.Edmundss and G. Rosenberger have answered question (2) negatively, by finding a suitable example, we also find the following example regarding to the second question:

One can easily check the following identity:

$$[x,y]^2 = \left[y^{-1},x\right]\left[x\left[y^{-1},x^{-1}\right],y\right]$$

and clearly $\left[y^{-1},x\right] \neq \left[x\left[y^{-1},x^{-1}\right],y\right]$.

CHAPTER 4
POWERS OF COMMUTATORS AS PRODUCT OF SQUARES

In this chapter we use the same notations as chapter 3. It has been shown [L-N] by R. C. Lyndon and M. Newman that in the free group $F = F(x, y)$, freely generated by x, y the commutator $[x, y] = xyx^{-1}y^{-1}$ is never the product of two squares in F, although it is always the product of three squares in F. Let $\gamma \in F'$, the "minimal number of squares which is required to write γ as a product of squares in F" is called the square length of γ and denoted by $Sq(\gamma)$. In this chapter we consider more general case, i.e. $Sq[x, y]^n$, $n \in \mathbb{N}$ and our object here is to prove the following theorem:

THEOREM.4.1. Let $F = F(x, y)$ be the free group of rank 2 freely generated by x, y, then $Sq[x, y]^n = 3$, if $n \in \mathbb{N}$ is odd and, $Sq[x, y]^n = 1$ if n is even.

Proof. The case when n is even the result is clear, hence let n be an odd integer.

First we show that $[x, y]^n$ can be written as a product of 3 squares in F. Put $[x, y] = W$, then one can check the following identity:

$$W^{2k+1} = [x, y]^{2k+1} = \left((W^k xy)^{W^k}\right)^2 (W^k y^{-1})^2 \left((W^{-k} x^{-1})^y\right)^2.$$

In the case $k = 0$ we get:

$$[x, y] = (xy)^2 (y^{-1})^2 \left((x^{-1})^y\right)^2,$$

hence

$$Sq[x,y]^n \leq 3,$$

hence to complete the proof it is enough to show that:

$$Sq[x,y]^n \neq 2,$$

so we prove that $W^{2k+1} \neq a^2b^2$ for any $k \in \mathbb{Z}$ and $a, b \in F$, we prove this part of theorem in two different ways as follows:

First proof. Let $a^2b^2 = W^r$ for some $r \in \mathbb{Z}$, then:

$$a^2b^2 \equiv (ab)^2 \qquad mod\ F'.$$

Since $a^2b^2 \in F'$, $(ab)^2 \in F'$, hence $ab \in F'$ and $a = ub^{-1}$ for some $u \in F'$. Now $a^2 = (ub^{-1})^2 = uu^{b^{-1}}b^{-2}$, hence $uu^{b^{-1}} = W^r$ and $W^r \equiv u^2 \pmod{\gamma_3(F)}$.

But $\gamma_2(F)/\gamma_3(F) \cong C_\infty$ and it is generated by $W = [x,y]$. Since W is the generator of $\gamma_2(F) \mod \gamma_3(F)$, $u^2 \equiv W^r$ has solution iff r is even, hence we proved that $W^{2k+1} \neq a^2b^2$ for any $k \in \mathbb{Z}$. □

Note: In a similar way $a^nb^n = W^r$ for some $r \in \mathbb{Z}$ implies that:

$$a^n = (ub^{-1})^n = uu^{b^{-1}}u^{b^{-2}} \ \ldots\ u^{b^{-(n-1)}}b^{-n}$$

$$a^nb^n = uu^{b^{-1}}u^{b^{-2}} \ \ldots\ u^{b^{-(n-1)}},$$

for some $u \in F'$. And we have:

$$u^n \equiv W^r \mod \gamma_3(F),$$

so

$$n \mid r,$$

hence

$$a^n b^n \neq W^r \quad \text{if} \quad n \nmid r.$$

Second proof. It is similar to the proof in [L-N] by R. C. Lyndon and M. Newman for the case $[x, y] \neq a^2 b^2$. Let

$$G = PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm I\}$$

We write elements of G as matrices instead of cosets, the group G is generated by

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and it is the free product of the cyclic group $\langle T \rangle$ of order 2 and cyclic group $\langle ST \rangle$ of order 3.

$$G = \langle\, S \,,\, T \,\rangle = \langle\, T \,\rangle \star \langle\, ST \,\rangle$$

G' is a free group of rank 2 and index 6 and is freely generated by

$$X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

and

$$[X, Y] = XYX^{-1}Y^{-1} = -S^6 = \begin{pmatrix} -1 & -6 \\ 0 & -1 \end{pmatrix}$$

The exponent of S modulo G' is six and by Kurosh Theorem two commuting elements of G are necessarily powers of the same elements of G (for more details see chapter 6 [Ro]).

We need the following two lemmas:

LEMMA.4.2. All solutions of the equation

$$A^2B^2 = -S^{6(2n+1)} \tag{1}$$

in $SL(2,\mathbb{Z})$ are given by $A = S^{3(2n+1)}M^{-1}$, $B = MS^{-3(2n+1)}N$, where M, N are any elements in $SL(2,\mathbb{Z})$ such that MN=NM, $\left(NS^{-3(2n+1)}\right)^2 = -I$.

Proof. Since $A^2B^2 = -S^{6(2n+1)}$ we get:

$$A^2B^2 - I = -S^{6(2n+1)} - I = -2S^{3(2n+1)}.$$

By multiplying on the left by A^{-1} and on the right by B^{-1} we have:

$$AB - (BA)^{-1} = -2A^{-1}S^{3(2n+1)}B^{-1},$$

hence:

$$0 = tr(AB) - tr(BA)^{-1} = tr\left(-2A^{-1}S^{3(2n+1)}B^{-1}\right).$$

Therefore:

$$tr\left(A^{-1}S^{3(2n+1)}B^{-1}\right) = 0,$$

hence:

$$\left(A^{-1}S^{3(2n+1)}B^{-1}\right)^2 = -I,$$

which we can rewrite as:

$$\left(ABS^{-3(2n+1)}\right)^2 = -I, \tag{2}$$

as well as:

$$-S^{3(2n+1)}B^{-1} = ABS^{-3(2n+1)}A, \tag{3}$$

from (1) it follows that

$$(4) \qquad AB = \left(A^{-1}S^{3(2n+1)}\right) \cdot \left(-S^{3(2n+1)}B^{-1}\right),$$

from (3) and (4) we get:

$$AB = \left(A^{-1}S^{3(2n+1)}\right)(AB)\left(S^{-3(2n+1)}A\right).$$

If we put

$$AB = N, \qquad A^{-1}S^{3(2n+1)} = M,$$

then MN=NM and;

$$A = S^{3(2n+1)}M^{-1},\ B = MS^{-3(2n+1)}N,\ ABS^{-3(2n+1)} = NS^{-3(2n+1)}.$$

So by (2), we have:

$$\left(NS^{-3(2n+1)}\right)^2 = -I,$$

hence:

$$(5) \qquad NS^{-3(2n+1)}N = -S^{3(2n+1)}.$$

Now we have:

$$\begin{aligned} A^2B^2 &= S^{3(2n+1)}M^{-1}S^{3(2n+1)}M^{-1}\left(MS^{-3(2n+1)}N\right)\left(MS^{-3(2n+1)}N\right) \\ &= S^{3(2n+1)}NS^{-3(2n+1)}N \\ &= S^{3(2n+1)}\left(-S^{3(2n+1)}\right) \qquad \text{by (5)} \\ &= -S^{6(2n+1)}. \end{aligned}$$

Therefore all the solutions of (1) are in the form indicated by the lemma. □

It is not difficult to prove the following lemma. (see [L-N])

LEMMA.4.3. Let V belong to $SL(2,\mathbb{Z})$, $V^2 = -I$. Then neither of congruences

$$W^2 \equiv \pm V \mod 3$$

has a solution W in $SL(2,\mathbb{Z})$.

Now we turn to the proof of the theorem.

It is enough to show that equation (1) considered as an equation over G' has no solution in G'. Suppose the contrary, then there exist $A, B \in G'$ such that:

$$N = AB \in G', \ MS^{-3(2n+1)} = A^{-1} \in G',$$

since MN=NM, we have:

$$M = U^a, \ \ N = U^b \quad (U \in G)\,.$$

Since by (5), $\left(NS^{-3(2n+1)}\right)^2 = -I$ in $SL(2,\mathbb{Z})$, in G we have:

$$\left(NS^{-3(2n+1)}\right)^2 = I,$$

and;

(6) $$N = VS^{3(2n+1)}, \ \ V^2 = I, \ \ V \in G.$$

Since

$$MS^{-3(2n+1)} = U^a S^{-3(2n+1)} \in G',$$

we also have:

$$M^b S^{-3(2n+1)b} = U^{ab} S^{-3(2n+1)b} \in G'.$$

(Generally, let K be an arbitrary group and $H \lhd K$. If for $x, y \in K$ we have $xy \in H$ then $x^n y^n$ also belong to H for any integer n.) So $S^{3(2n+1)b} \in G'$, since $U^b \in G'$.

Now since the exponent of S modulo G is 6, b must be even, thus for some $W \in G$ we have:

$$N = U^b = W^2,$$

and by (6) $W^2 = VS^{3(2n+1)}$, hence:

$$W^2 \equiv V \mod 3, \tag{7}$$

and by (5), as an element of $SL(2,\mathbb{Z})$, $V^2 = \left(NS^{-3(2n+1)}\right)^2 = -I$. But then Lemma 4.3 implies that there is no solution for (7), so we get a contradiction and this completes the proof. □

Note: In $F(x,y)$, $Sq[x,y]^n = 3$ for any odd number $n \in \mathbb{N}$. But there exist commutators with square length equal to two. Obviously $[h^2, g]$ and $[h, g^2]$ are product of two squares, and a nontrivial commutator is never a square([S] or see chapter 2). Thus $Sq[h^2, g] = Sq[h, g^2] = 2$.

But it is not the only case in which the square length of a commutator is two. As was shown by L.P. Comerford and C.C.Edmundss in[C-E]

CHAPTER 5
COMMUTATOR LENGTH OF CERTAIN FINITELY GENERATED GROUPS[1]

In this chapter we use the following notations:

$F_{n,t} = \langle x_1, \ldots, x_n \rangle$ denotes the free group of rank n and nilpotency class t freely generated by $x_1, \ldots, x_n$, $[x,y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$ for any x, y elements of a group.

In this chapter we show that if G is a free nilpotent group of rank 2 and class 3 then c(G)=2. We also show that if G is a free abelian by nilpotent group of rank n then $c(G) = n$ except in the trivial case where G is abelian. We prove some results about powers of $[x_1, x_2]$ in $F_{2,3} = \langle x_1, x_2 \rangle$ and we use these results to discuss question (2) in chapter 3 in a nilpotent group. Finally we find lower and upper bound for the commutator length of a finitely generated free solvable group of class 3.

We begin by establishing a technical result required in the proof of Theorem 5.3.

LEMMA.5.1. The following system of three equations in variables $s_1, s_2, r_1, r_2, \alpha$ and β has no integer solution.

(1) $$r_2 s_1 - r_1 s_2 = 2$$

[1]A version of pages 40-46 of this chapter will appear in [A-R]

(2) $$\frac{s_1r_2(r_2-1)}{2}-\frac{r_1s_2(s_2-1)}{2}+r_2s_2(s_1-r_1)-\alpha r_2+\beta s_2=0$$

(3) $$\frac{r_2s_1(s_1-1)}{2}-\frac{r_1s_2(r_1-1)}{2}-\alpha r_1+\beta s_1=0.$$

Proof. Put $c_1=\alpha r_2-\beta s_2,\quad c_2=\alpha r_1-\beta s_1$ then

$$\alpha=\frac{\begin{vmatrix}c_1 & -s_2\\ c_2 & -s_1\end{vmatrix}}{\begin{vmatrix}r_2 & -s_2\\ r_1 & -s_1\end{vmatrix}}=\frac{s_1c_1-s_2c_2}{2},\qquad \beta=\frac{\begin{vmatrix}r_2 & c_1\\ r_1 & c_2\end{vmatrix}}{-2}=\frac{r_1c_1-r_2c_2}{2}.$$

Hence we need $s_1c_1-s_2c_2$ and $r_1c_1-r_2c_2$ to be even.

$$c_1=\frac{s_1r_2(r_2-1)}{2}-\frac{r_1s_2(s_2-1)}{2}+r_2s_2(s_1-r_1)$$

$$c_2=\frac{r_2s_1(s_1-1)}{2}-\frac{r_1s_2(r_1-1)}{2}.$$

Hence we have:

$$2c_1=s_1r_2^2-s_1r_2-r_1s_2^2+r_1s_2+2s_1r_2s_2-2r_1r_2s_2$$

$$=r_2(s_1r_2-r_1s_2)-(s_1r_2-r_1s_2)-s_2(r_1s_2-s_1r_2)-r_1r_2s_2+s_1r_2s_2$$

$$=-2+2(r_2+s_2)-r_2s_2(r_1-s_1).$$

And we have:

$$2c_2=s_1^2r_2-s_1r_2-r_1^2s_2+r_1s_2=-2+s_1^2r_2-r_1^2s_2.$$

Hence we need to satisfy the following conditions:

(1) $$r_2s_1-r_1s_2=2$$

(4) $$2c_1=-2+2(r_2+s_2)-r_2s_2(r_1-s_1)$$

$$2c_2 = -2 + s_1^2 r_2 - r_1^2 s_2 \tag{5}$$

$$s_1 c_1 + s_2 c_2 \equiv r_1 c_1 + r_2 c_2 \equiv 0 \pmod 2 \tag{6}$$

Case 1. $r_1 s_2 = 2k$ for some integer k. Then

$$c_1 = -1 + (r_2 + s_2) - k r_2 + (1+k) s_2$$

$$c_2 = -1 + (1+k) s_1 - k r_1.$$

And modulo 2,

$$0 \equiv s_1 c_1 + s_2 c_2 \equiv s_1 + s_1 s_2 + s_2 \tag{7}$$

and

$$0 \equiv r_1 c_1 + r_2 c_2 \equiv r_1 + r_1 r_2 + r_2. \tag{8}$$

From (7) and (8) it follows that r_1, r_2, s_1 and s_2 are all even. But then $r_1 s_2 - r_2 s_1$ is divisible by 4 contradicting (1).

Case 2. $r_1 s_2$ is odd. It follows from (1) that r_1, r_2, s_1, s_2 are all odd.

$$\begin{aligned} s_1 c_1 + s_2 c_2 &= \tfrac{1}{2}\, s_1^2 r_2 (r_2 - 1) - \tfrac{1}{2}\, s_1 r_1 s_2 (s_2 - 1) + s_1 s_2 r_2 (s_1 - r_1) \\ &\quad + \tfrac{1}{2}\, s_2 r_2 s_1 (s_1 - 1) - \tfrac{1}{2}\, s_2^2 r_1 (r_1 - 1) \\ &= \tfrac{1}{2}\,(s_1 r_2 - s_2 r_1)(s_1 r_2 + s_2 r_1) - \tfrac{1}{2}\, s_1 (s_1 r_2 - r_1 s_2) - \tfrac{1}{2}\, s_1 s_2 (r_1 s_2 - r_2 s_1) \\ &\quad - \tfrac{1}{2}\, s_2 (r_2 s_1 - s_2 r_1) + s_1 s_2 r_2 (s_1 - r_1) \\ &= (s_1 r_2 + s_2 r_1) - s_1 + s_1 s_2 - s_2 + s_1 s_2 r_2 (s_1 - r_1) \end{aligned}$$

which is odd and contradicts (6).

We shall use the following well known identities regarding groups which are nilpotent of class 3.

LEMMA.5.2. Let $G = \langle x, y \rangle$ be nilpotent of class 3. Then, for all integers r, s the following hold.

$$[x^r, y] = [x, y]^r [x, y, x]^{r(r-1)/2}$$

$$[x^r, y^s] = [x, y]^{rs} [x, y, x]^{rs(r-1)/2} [x, y, y]^{rs(s-1)/2}.$$

THEOREM.5.3. Let $F_{2,3} = \langle x_1, x_2 \rangle$ be the free nilpotent group of class 3 on free generators x_1, x_2. Then $c(F_{2,3}) = 2$.

Proof. Let h, g be any two elements of $F_{2,3} \backslash \gamma_3(F_{2,3})$. We study the form of the element $[h, g]$. Since $\gamma_3(F_{2,3})$ lies in the center of $F_{2,3}$ we may express h as $x_1^{r_1} x_2^{r_2} [x_2, x_1]^\beta$ and g as $x_1^{s_1} x_2^{s_2} [x_2, x_1]^\alpha$. Put $z = [x_2, x_1]$, $\quad y_1 = z^\beta$ and $y_2 = z^\alpha$. Then

$$\begin{aligned}
[h, g] &= [x_1^{r_1} x_2^{r_2}, x_1^{s_1} x_2^{s_2}][x_1^{r_1} x_2^{r_2}, y_2][y_1, x_1^{s_1} x_2^{s_2}] \\
&= [x_1^{r_1} x_2^{r_2}, x_2^{s_2}][x_1^{r_1} x_2^{r_2}, x_1^{s_1}][x_1^{r_1} x_2^{r_2}, x_1^{s_1}, x_2^{s_2}][x_1^{r_1}, y_2][x_2^{r_2}, y_2][y_1, x_2^{s_2}][y_1, x_1^{s_1}] \\
&= [x_1^{r_1}, x_2^{s_2}][x_1^{r_1}, x_2^{s_2}, x_2^{r_2}][x_2^{r_2}, x_1^{s_1}][x_2^{r_2}, x_1^{s_1}, x_2^{s_2}][x_1, z]^{\alpha r_1} \\
&\quad \times [x_2, z]^{\alpha r_2} [z, x_2]^{\beta s_2} [z, x_1]^{\beta s_1} \\
&= [x_1, x_2]^{r_1 s_2} [x_1, x_2, x_1]^{\frac{s_2 r_1 (r_1 - 1)}{2}} [x_1, x_2, x_2]^{\frac{r_1 s_2 (s_2 - 1)}{2}} [x_1, x_2, x_2]^{r_1 r_2 s_2} \\
&\quad \times [x_2, x_1]^{r_2 s_1} [x_2, x_1, x_2]^{\frac{s_1 r_2 (r_2 - 1)}{2}} [x_2, x_1, x_1]^{\frac{r_2 s_1 (s_1 - 1)}{2}} [x_2, x_1, x_2]^{r_2 s_1 s_2} \\
&\quad \times [x_2, x_1, x_1]^{-\alpha r_1 + \beta s_1} [x_2, x_1, x_2]^{-\alpha r_2 + \beta s_2}
\end{aligned}$$

$$= [x_2, x_1]^{\lambda}[x_2, x_1, x_2]^{\mu}[x_2, x_1, x_1]^{\nu}$$

where

$$\lambda = r_2 s_1 - r_1 s_2$$

$$\mu = \frac{s_1 r_2(r_2 - 1)}{2} - \frac{r_1 s_2(s_2 - 1)}{2} - r_1 r_2 s_2 + r_2 s_1 s_2 + \beta s_2 - \alpha r_2$$

$$\nu = \frac{r_2 s_1(s_1 - 1)}{2} - \frac{s_2 r_1(r_1 - 1)}{2} + \beta s_1 - \alpha r_1.$$

Since $[x_2, x_1]$, $[x_2, x_1, x_2]$ and $[x_2, x_1, x_1]$ are basic commutators and the group under consideration is the free nilpotent class 3 group, it follows that if $[h, g] = [x_2, x_1]^2$ then $\lambda = 2, \quad \mu = \nu = 0$. But by Lemma 5.1 there are no integers $\alpha, \beta, r_1, s_1, r_2, s_2$ for this set of equations to hold and we conclude that $c(F_{2,3}) \geq 2$. Since $c(F_{2,3}) \leq 2$ by [Al-R] or [B-M], we obtain the equality.

The proof of next theorem makes use of the following two results. The first is elementary; the second is a result of Peter Stroud [Str]. We shall include the proofs since Stroud's result never got published, except in his Ph.D. thesis, due to his untimely death. In the case of a finite group G, Brian Hartley [H] has given bound for $c(G)$ in terms of the Fitting length of G. His proof incorporates Stroud's proof given below.

LEMMA.5.4. Let A be a normal subgroup of $G = \langle x_1, \ldots, x_n \rangle$. If A is abelian or A lies in the second center $\zeta_2(G)$ of G then every element of $[G, A]$ has the form $\prod_{i=1}^{n} [x_i, a_i], \quad a_i \in A$.

Proof. Consider $[g, d], \quad d \in A$ and $g = x_{i_1}^{\varepsilon_1} \cdots x_{i_r}^{\varepsilon_r}$ where $\varepsilon_i \in \{1, -1\}$. Write

$x_{i_1} = x, \quad \varepsilon_1 = \varepsilon$ and $g = x^\varepsilon y$. Then $[g,d] = [x^\varepsilon y, d] = [x, d^\varepsilon][y,d]$ if $A \leq \zeta_2(G)$ and $[g,d] = [x^\varepsilon, d][x^\varepsilon, d, y][y,d] = [x^\varepsilon, d]\,[y, d[d, x^\varepsilon]]$ if A is abelian. If $\varepsilon = -1$ then use $[x^{-1}, d] = [x, xd^{-1}x^{-1}]$.

Iterate the process r times to obtain $[g,d] = \prod_{j=1}^{r}[x_{ij}, d_j]$ with $d_j \in A$. Finally use the identity $[x,d_1][x,d_2] = [x, d_1 d_2]$ to see that every $\Pi[x_{ij}, d_j], \quad d_i \in A$ has the form $\prod_{i=1}^{n}[x_i, a_i], \quad a_i \in A$.

LEMMA.5.5. (P. Stroud) Let $G = \langle x_1, \ldots, x_n\rangle$ be a nilpotent group. Then every element of the commutator subgroup G' is a product of n commutators $[x_1, g_1]\cdots[x_n, g_n]$ for suitable g_i in G.

Proof. Use induction on the nilpotency class of G. If G is abelian then $G' = 1$ and the result is clear. So let $G \in \mathcal{N}_{r+1}$, nilpotent of class $r+1$, and assume the result for groups in the class $\mathcal{N}_r$. Let $\Gamma = \gamma_{r+1}(G) = [\gamma_r(G), G]$. Then an element g of G' has the form $g = [x_1, h_1]\cdots[x_n, h_n]d$ for some $d \in \Gamma$. By Lemma 5.3,

$$g = [x_1, h_1]\cdots[x_n, h_n][x_1, a_1]\cdots[x_n, a_n]$$
$$= \prod_{i=1}^{n}[x_i, h_1 a_i].$$

THEOREM.5.6. Let $G = \langle x_1, \ldots, x_n\rangle$ be non-abelian free abelian-by-nilpotent group freely generated by $x_1, \ldots, x_n$. Then $c(G) = n$. If A is an abelian normal subgroup of G and G/A is nilpotent, then every element of G' is a product of n commutators $[x_1, g_1]^{a_1}[x_2, g_2]^{a_2}\ldots[x_n, g_n]^{a_n}$ for suitable $g_1, \ldots, g_n$ in G and $a_1, \ldots, a_n$ in A.

Proof. By hypothesis, there exists a normal abelian subgroup A of G such that G/A is nilpotent. By Lemma 5.4, $[A,G] = \{[x_1,a_1]\cdots[x_n,a_n];\ a_i \in A\}$, and since $G/[A,G]$ is nilpotent, using Stroud's result, every element $g \in G'$ has the form

$$g = \Big(\prod_{i=1}^{n}[x_i,g_i]\Big)\Big(\prod_{i=1}^{n}[x_i,a_i]\Big) \qquad \text{with} \quad a_i \in A$$

$$= \prod_{i=1}^{n}\big([x_i,a_i][x_i,g_i]^{d_i}\big) \qquad \text{for suitable} \quad d_i \in A.$$

Now $[x_i,g_ia_i] = [x_i,a_i][x_i,g_i]^{a_i} = ([x_i,a_i][x_i,g_i])^{a_i}$. Thus $[x_i,a_i][x_i,g_i] = [x_i,g_ia_i]^{a_i^{-1}}$ and $g = \prod_{i=1}^{n}[x_i,g_ia_i]^{d_ia_i^{-1}}$, with $d_ia_i^{-1} \in A$. Thus $c(G) \le n$ and every element of G' has the required form. Since G is free abelian-by-nilpotent and non-abelian, the free metabelian group on n-generators is a quotient of G and hence so is the free nilpotent-class-three group on n generators. By Theorem 1 for the case when $n = 2$ and by [Al-R] for $n > 2$, $c(G) \ge n$. This shows that $c(G) = n$ and completes the proof.

Now we prove some result about powers of $[x_1,x_2]$ in $F_{2,3}$

THEOREM.5.7. Let $F_{2,3} = \langle x_1, x_2\rangle$ be a free nilpotent group of rank 2 and class 3 freely generated by $x_1,\ x_2$. Let $z = [x_2,x_1]$. Then

$$c(z^n) = \begin{cases} 1 & \text{if } n \equiv 0 \mod 4 \text{ and } z^n = \left[x_1^{n/2}x_2^{4}z^{n/4+2},\, x_1^{n/2}x_2^{2}z^{3n/4+1}\right] \\ 1 & \text{if } n \equiv \pm 1 \mod 4 \text{ and } z^n = \left[x_1x_2^{n+1}z^{n+1/2},\, x_1x_2z^{n+1/2}\right] \\ 2 & \text{if } n \equiv 2 \mod 4 \text{ and } z^n = [x_2^n, x_1]\left[x_1, x_2, x_2^{n(n-1)/2}\right] \end{cases}$$

Proof. We use the same notations as Lemma 5.1. In Theorem 5.3 we have proved that if $h = x_1^{r_1} x_2^{r_2} [x_2, x_1]^\beta$, $g = x_1^{s_1} x_2^{s_2} [x_2, x_1]^\alpha$ and;

$$\begin{cases} \lambda_1 & = r_2 s_1 - r_1 s_2 \\ \lambda_2 & = s_1 r_2 (r_2 - 1)/2 - r_1 s_2 (s_2 - 1)/2 + r_2 s_2 (s_1 - r_1) - \alpha r_2 + \beta s_2 \\ \lambda_3 & = r_2 s_1 (s_1 - 1)/2 - r_1 s_2 (r_1 - 1)/2 - \alpha r_1 + \beta s_1, \end{cases}$$

then:

$$(\star) \qquad [h, g] = [x_2, x_1]^{\lambda_1} [x_2, x_1, x_1]^{\lambda_2} [x_2, x_1, x_2]^{\lambda_3}.$$

We use this result in the following cases:

Case 1. Let n be an odd number, and choose

$$r_1 = s_1 = s_2 = 1, \quad r_2 = n + 1, \quad \alpha = \beta = n + 1/2,$$

then it follows that:

$$\begin{cases} r_2 s_1 - r_1 s_2 = n \\ s_1 r_2 (r_2 - 1)/2 - r_1 s_2 (s_2 - 1)/2 + r_2 s_2 (s_1 - r_1) - \alpha r_2 + \beta s_2 = 0 \\ r_2 s_1 (s_1 - 1)/2 - r_1 s_2 (r_1 - 1)/2 - \alpha r_1 + \beta s_1 = 0. \end{cases}$$

Hence by $(\star)$:

$$[x_2, x_1]^n = \left[x_1 x_2^{n+1} z^{n+1/2}, x_1 x_2 z^{n+1/2}\right]$$

it covers the case $n \equiv \pm 1 \mod 4$.

Case 2. Let $n \equiv 2 \mod 4$, we show that the following equations in variables s_1, s_2, r_1, r_2, α and β has no integer solution.

$$\begin{cases} r_2 s_1 - r_1 s_2 = n \\ s_1 r_2 (r_2 - 1)/2 - r_1 s_2 (s_2 - 1)/2 + r_2 s_2 (s_1 - r_1) - \alpha r_2 + \beta s_2 = 0 \\ r_2 s_1 (s_1 - 1)/2 - r_1 s_2 (r_1 - 1)/2 - \alpha r_1 + \beta s_1 = 0. \end{cases}$$

The proof is similar to the proof of Lemma 5.1, put;

$$c_1 = \alpha r_2 - \beta s_2, \quad c_2 = \alpha r_1 - \beta s_1,$$

then:

$$\alpha = \frac{s_1c_1 - s_2c_2}{n}, \quad \beta = \frac{r_1c_1 - r_2c_2}{n}.$$

So we need:

$$s_1c_1 - s_2c_2 \equiv 0 \mod n, \quad r_1c_1 - r_2c_2 \equiv 0 \mod n,$$

in which:

$$c_1 = \alpha r_2 - \beta s_2 = s_1r_2(r_2-1)/2 - r_1s_2(s_2-1)/2 + r_2s_2(s_1-r_1)$$

$$c_2 = \alpha r_1 - \beta s_1 = r_2s_1(s_1-1)/2 - r_1s_2(r_1-1)/2.$$

Hence we have:

$$2c_1 = s_1r_2^2 - s_1r_2 - r_1s_2^2 + r_1s_2 + 2s_1r_2s_2 - 2r_1r_2s_2$$

$$= r_2(s_1r_2 - r_1s_2) - (s_1r_2 - r_1s_2) - s_2(r_1s_2 - s_1r_2) - r_1r_2s_2 + s_1r_2s_2$$

$$= r_2n - n + ns_2 - r_2s_2(r_1 - s_1)$$

$$= -n + n(r_2 + s_2) - r_2s_2(r_1 - s_1).$$

And we have:

$$2c_2 = s_1^2r_2 - s_1r_2 - r_1^2s_2 + r_1s_2 = -n + s_1^2r_2 - r_1^2s_2.$$

Hence we get:

$$r_2s_1 - r_1s_2 = n \equiv 2(\mod 4)$$

$$2c_1 = -n + n(r_2 + s_2) - r_2s_2(r_1 - s_1)$$

$$2c_2 = -n + s_1^2r_2 - r_1^2s_2$$

$$s_1c_1 + s_2c_2 \equiv r_1c_1 + r_2c_2 \equiv 0 \mod n$$

Now we consider the following two cases:

Case 2.1. $r_1 s_2 = 2k$ for some integer k. Then $r_2 s_1 \equiv 0 \mod 2$, so

$$r_2 s_1 = n + 2k$$

$$2c_1 = -n + n(r_2 + s_2) - 2kr_2 + (n+2k)s_2$$

$$= -n + nr_2 + ns_2 - 2kr_2 + ns_2 + 2ks_2$$

$$= -n + r_2(n-2k) + 2s_2(n+k).$$

Let $n = 2k_1$, hence:

$$c_1 = -k_1 + r_2(k_1 - k) + s_2(2k_1 + k)$$

$$2c_2 = -n + (n+2k)s_1 - r_1(2k) = 2(-k_1 + (k+k_1)s_1 - kr_1)$$

$$c_2 = -k_1 - kr_1 + (k+k_1)s_1.$$

Hence we have:

$$0 \equiv s_1 c_1 + s_2 c_2 \mod n$$

$$= (-s_1 k_1 + r_2 s_1(k_1 - k) + s_1 s_2(k + 2k_1))$$

$$+ (-k_1 s_2 - kr_1 s_2 + ks_1 s_2 + k_1 s_1 s_2).$$

But in modulo 2,

$$n \equiv r_1 s_2 \equiv r_2 s_1 \equiv 0, \; k_1 \equiv 1,$$

hence it follows that:

$$s_1 s_2 - (s_1 + s_2) \equiv 0 \mod 2,$$

hence s_1, s_2 are even. We show r_1 and r_2 are also even.

We have:

$$0 \equiv r_1c_1 - r_2c_2 \mod n$$

$$= (-k_1r_1 + k_1r_1r_2 - kr_1r_2 + 2k_1r_1s_2 + kr_1s_2)$$

$$+ (k_1r_2 + kr_1r_2 - kr_2s_1 - k_1s_1r_2).$$

Since in modulo 2,

$$n \equiv r_1s_2 \equiv r_2s_1 \equiv 0, \; k_1 \equiv 1,$$

we get:

$$0 \equiv r_1c_1 - r_2c_2 \equiv (r_1 + r_2) + r_1r_2 \mod 2.$$

Hence $r_1, \; r_2$ are even, so in modulo 4;

$$n = r_2s_1 - r_1s_2 \equiv 0,$$

but we know $n \equiv 2$ in modulo 4, hence we get a contradiction.

Case 2.2. Let $r_2s_1 \equiv r_1s_2 \equiv 1$ in modulo 2, hence $r_1, \; r_2, \; s_1,$ and s_2 are odd and we have:

$$\begin{aligned}
s_1c_1 + s_2c_2 &= \frac{s_1^2r_2(r_2-1)}{2} - \frac{s_1r_1s_2(s_2-1)}{2} + s_1s_2r_2(s_1 - r_1) \\
&+ \frac{s_2r_2s_1(s_1-1)}{2} - \frac{r_2^2r_1(r_1-1)}{2} \\
&= \frac{s_1^2r_2^2 - s_2^2r_1^2}{2} - \frac{s_1^2r_2 - s_1r_1s_2}{2} - \frac{s_1r_1s_2^2 - s_1^2s_2r_2}{2} \\
&- \frac{s_1s_2r_2 - s_2^2r_1}{2} + s_1^2s_2r_2 - s_1r_1r_2s_2 \\
&= \frac{(r_2s_1 - r_1s_2)(r_2s_1 + r_1s_2)}{2} - \frac{s_1(r_2s_1 - r_1s_2)}{2} - \frac{s_1s_2(r_1s_2 - r_2s_1)}{2} \\
&- \frac{s_2(r_2s_1 - r_1s_2)}{2} + s_1r_2(s_1s_2) - (s_1r_2)(r_1s_2) \\
&= \frac{n(r_2s_1 + r_1s_2)}{2} - \frac{s_1n}{2} + \frac{s_1s_2n}{2} - \frac{s_2n}{2} + (s_1s_2)(s_1r_2) - (r_2s_1)(r_1s_2) \\
&\equiv 1 \mod 2. \quad (\text{since } n/2 \equiv s_1 \equiv s_2 \equiv r_1 \equiv r_2 \equiv 1 \mod 2)
\end{aligned}$$

But we know $s_1c_1 + s_2c_2 \equiv 0 \mod 2$, hence in the case $n \equiv 2$ in modulo 2 it is not possible to write $[x_2, x_1]^n$ as a commutator, so

$$c[x_2, x_1]^n = 2,$$

and we have:

$$[x_2, x_1]^n = [x_2^n, x_1]\left[x_1, x_2, x_2^{n(n-1)/2}\right].$$

Case 3. Let $n \equiv 0$ in modulo 4, we show that $c[x_2, x_1]^n = 1$. Let $n = 4k$ for some integer k, choose $s_1 = r_1 = 2k$, $r_2 = 4$, $s_2 = 2, \alpha = 3k + 1$, and $\beta = k + 2$ then:

$$\begin{cases} r_2s_1 - r_1s_2 = 4 \cdot 2k - 2k \cdot 2 = 4k = n \\ s_1r_2(r_2 - 1)/2 - r_1s_2(s_2 - 1)/2 + r_2s_2(s_1 - r_1) - \alpha r_2 + \beta s_2 = 0 \\ r_2s_1(s_1 - 1)/2 - r_1s_2(r_1 - 1)/2 - \alpha r_1 + \beta s_1 = 0, \end{cases}$$

hence by $(\star)$ we have:

$$\begin{aligned} [x_2, x_1]^n &= [x_2, x_1]^{4k} \\ &= \left[x_1^{2k}x_2^4z^{k+2}, x_1^{2k}x_2^2z^{3k+1}\right] \\ &= \left[x_1^{n/2}x_2^4z^{n/4+2}, x_1^{n/2}x_2^2z^{3n/4+1}\right], \end{aligned}$$

and this completes the proof. □

In chapter 3, we have discussed the following question, raised by Edmunds ([E-R],[L]) in a free group.

If $[v, w][x, y] = z^2$ ($\neq 1$) in a free group and z is a commutator, does it follow that $[v, w] = [x, y]$? (i.e. If $c(z) = 1$ and $c(z^2) = 2$ with $[v, w][x, y] = z^2$ in a free group, does it follow that $[v, w] = [x, y]$?)

As we know this question has been answered negatively in a free group. Now let $F_{2,3} = \langle x_1, x_2 \rangle$ be a free nilpotent group of rank 2 and class 3 freely generated by $x_1,\ x_2$, and let $z = [x_2, x_1]^3$ then $z^2 = [x_2, x_1]^5 [x_2, x_1]$, clearly $[x_2, x_1]^5 \neq [x_2, x_1]$ and by the last theorem

$$c(z^2) = 2, \quad c[x_2, x_1]^5 = 1$$

hence in this case also we have a negative answer to the above questions in $F_{2,3}$.

In 1990 in [E-R] C.C.Edmunds and G. Rosenberger made the following conjecture:

Conjecture: if $[v, w][x, y] = z^n$ *has a solution in a free group* F *with* $n \geq 2$ *and* $z \neq 1$, *then* $n \leq 3$.

Regarding the above conjecture in $F_{2,3}$, if $z = [x_2, x_1]$ then for any $n \equiv 2$ in modulo 4, $c(z^n) = 2$, hence in this case we have negative answer to this conjecture.

The proof of the next theorem requires the following lemma proved by Hartley in [H] and based on an argument of Rhemtulla[R2].

LEMMA.5.8. Let $G = \langle x_1, \ldots, x_n \rangle$ and H be a nilpotent normal subgroup of G. Suppose that $H = \langle y_1^G, \ldots, y_s^G \rangle$ is generated by conjugacy classes in G of elements $y_1, \ldots, y_s$. Then every element of $[H, G]$ can be expressed in the form:

$$\prod_{i=1}^{n} [h_i, x_i] \prod_{i=1}^{s} [h'_j, y_j] \quad (h_i, h'_j \in H)$$

where the product is taken in order of increasing suffices.

THEOREM.5.9. Let $G = \langle x_1, \ldots, x_n \rangle$ be a free solvable group of class 3,

then:

$$n \leq c(G) \leq \frac{n(n+3)}{2}$$

Proof. Let

$$M = \frac{G/\left[[G',G'],G\right]}{[G',G]/\left[[G',G'],G\right]}.$$

It is clear that M is a nilpotent group, by Lemma 5.5 every element of M' is congruent modulo $[G',G]/[[G',G'],G]$ to a product of n commutators of the form $[x_i,g_i]$ $(g_i \in G)$ modulo $[[G',G'],G]$. Since $G'/[[G',G'],G]$ is nilpotent, using Lemma 5.8 every element of $[G',G]/[[G',G'],G]$ has the following form modulo $[[G',G'],G]$,

$$\prod_{l=1}^{s}[X_l,k_l]\prod_{i=1}^{n}[x_i,h_i],$$

with $h_i, k_l \in G'$, $s = n(n-1)/2$ and $X_l \in \{[x_i,x_j];\ 1 \leq i \leq j \leq n\}$.

Finally use Lemma 5.3 to see that every element $g \in G$ has the following form:

$$g = \prod_{l=1}^{s}[X_l,k_l]\prod_{i=1}^{n}[x_i,h_i]\prod_{i=1}^{n}[x_i,m_i]\prod_{i=1}^{n}[x_i,g_i]\prod_{i=1}^{n}[x_i,m_i'],$$

with $m_i,\ m_i' \in G''$.

Now we have;

$$\begin{aligned}\prod_{i=1}^{n}[x_i,h_i]\prod_{i=1}^{n}[x_i,m_i] &= \prod_{i=1}^{n}([x_i,m_i][x_i,h_i])^{d_i} \quad \text{for suitable } d_i \in G'' \\ &= \prod_{i=1}^{n}[x_i,h_i m_i]^{d_i m_i^{-1}} \quad \text{with } d_i m_i^{-1} \in G''.\end{aligned}$$

And

$$\prod_{i=1}^{n}[x_i,g_i]\prod_{i=1}^{n}[x_i,m_i'] = \prod_{i=1}^{n}[x_i,g_i m_i']^{d_i' m_i'^{-1}} \quad \text{with } d_i' m_i'^{-1} \in G''.$$

Thus:

$$c(G) \leq n(n-1)/2 + 2n = n(n+3)/2.$$

Since the free metabelian group on n generators is a quotient of G and hence so is the free nilpotent-class-three group on n generators. By Theorem 1 for the case $n = 2$ and by [Al-R] for $n \gtrsim 2$, $c(G) \geq n$. This completes the proof. □

CHAPTER 6
COMMUTATOR LENGTH OF THE WREATH PRODUCT OF FREE GROUP BY INFINITE CYCLIC GROUP

In this chapter we use the same notations as chapter 5. Let F be a free group, as we know F is not a c-group (recall G is a c-group if there exists $n \in \mathbb{N}$ such that every element of G' can be expressed as a product of n commutators), but if we consider the group $W = F\mathrm{wr}C_\infty$, the wreath product of F by the infinite cyclic group we show that W is a c-group, this is done in the following theorem.

THEOREM.6.1. Let F be a free group and $W = F\mathrm{wr}C_\infty$ where C_∞ is the infinite cyclic group, then every element of W' is a product of three commutators. The proof of this theorem requires the following preparatory lemma.

LEMMA.6.2. Let A be a free abelian group and $W = A\mathrm{wr}C_\infty$ where C_∞ is the infinite cyclic group, then W is a c-group and furthermore the commutator length of W is equal to 1.

Proof. Let $B = \underset{i \in Z}{Dr} A_i$ where $A_i \simeq A$, be the base group of W and let $T = \langle t \rangle \simeq C_\infty$. Then W is the semidirect product of B by T;

$$W = BT.$$

Clearly,

$$W' = [W, W] = [BT, BT] = [B, T].$$

Now B is a normal abelian subgroup of W, Lemma 4.4 of Rhemtulla's thesis [R1] shows that

$$[B,T] = \{ \ [b,t] \ , \ b \in B \ \}.$$

(We sketch an analogous proof of this lemma:

It is clear that

$$\{ \ [b,t] \ , \ b \in B \ \} \subseteq [B,T].$$

Let $[b_1, t^{m_1}][b_2, t^{m_2}] \ \cdots \ [b_i, t^{m_i}]$ belong to [B,T]. Since $[b, t^{-1}] = [b^{-1}, t]^{t^{-1}}$ we may assume $m_k \geq 0$ $(1 \leq k \leq i)$. Then one can check that

$$[b_k, t^{m_k}] = [b'_k, t],$$

where $b'_k = b_k^{t^{m_k - 1}} \ \cdots \ b_k^t b_k$. And

$$[b'_1, t][b'_2, t] \ \cdots \ [b'_i, t] = [b'_1 b'_2 \ \cdots \ b'_i, t].$$

Hence $[B,T] = \{[b,t], \ b \in B \ \}$.)

And this completes the proof. □

Now we return to the proof of Theorem 6.1.

Proof. Let $B = \underset{i \in Z}{D} r F_i$ where $F_i \simeq F$, be the base group of W and let $T = \langle t \rangle \simeq C_\infty$. Then W is the semidirect product of B by T;

$$W = BT.$$

Now modulo B',

$$W \simeq A wr C_\infty,$$

where $A \simeq F/F'$.

Clearly by Lemma 6.2, modulo B',

$$c(W) = 1.$$

To get the result, it suffices to show that every element of $B' = \underset{i \in \mathbf{Z}}{D} r F_i'$ is a product of two commutator.

Let $\underline{g} = (g_i)_{i \in \mathbf{Z}} = (\cdots, 1, 1, 1, g_1, g_2, \cdots, g_k, 1, 1, 1, \cdots)$ be an arbitrary element of the base group of W, for convenient we establish the following notation:

$$\underline{g} = (g_i)_{i \in \mathbf{Z}} = (g_1, g_2, \cdots, g_k).$$

(It means $\underline{g}$ or its conjugate is equal to $(g_1, g_2, \cdots, g_k)$.)

Let $\underline{w} = (w_i)_{i \in \mathbf{Z}} = (w_1, w_2, \cdots, w_m)$ be an arbitrary element of B', where

$$w_i = [h_{i1}, g_{i1}][h_{i2}, g_{i2}] \cdots [h_{ir}, g_{ir}]; \quad i = 1, \cdots, m.$$

Our object is to find suitable elements $\underline{a}$, $\underline{b}$ of the base group of W such that:

$$\underline{w} = (w_1, w_2, \cdots, w_m) = [\underline{a}^{-1}, t^m] [\underline{b}^{-1}, t^{2m}].$$

Let $[\underline{a}^{-1}, t^m] = (x_k)_{k \in \mathbf{Z}}$, and let $[\underline{b}^{-1}, t^{2m}] = (y_k)_{k \in \mathbf{Z}}$.

Now if we find $\underline{a}$, $\underline{b}$ such that

$$\begin{aligned}[\underline{a}^{-1}, t^m] = (&h_{11}, h_{21}, \cdots, h_{m1}, g_{11}, g_{21}, \cdots, g_{m1},\\ &h_{11}^{-1}, h_{21}^{-1}, \cdots, h_{m1}^{-1}, g_{11}^{-1}, g_{21}^{-1}, \cdots, g_{m1}^{-1},\\ &h_{12}, h_{22}, \cdots, h_{m2}, g_{12}, g_{22}, \cdots, g_{m2},\end{aligned}$$

$h_{12}^{-1},\ h_{22}^{-1},\ \cdots,\ h_{m2}^{-1},\ g_{12}^{-1},\ g_{22}^{-1},\ \cdots,\ g_{m2}^{-1},$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$h_{1r-1},\ h_{2r-1},\ \cdots,\ h_{mr-1},\ g_{1r-1},\ g_{2r-1},\ \cdots,\ g_{mr-1},$

$h_{1r-1}^{-1},\ h_{2r-1}^{-1},\ \cdots,\ h_{mr-1}^{-1},\ g_{1r-1}^{-1},\ g_{2r-1}^{-1},\ \cdots,\ g_{mr-1}^{-1},$

$h_{1r},\ h_{2r},\ \cdots,\ h_{mr},\ g_{1r},\ g_{2r},\ \cdots,\ g_{mr},$

$h_{1r}^{-1},\ h_{2r}^{-1},\ \cdots,\ h_{mr}^{-1},\ g_{1r}^{-1},\ g_{2r}^{-1},\ \cdots,\ g_{mr}^{-1},$

$w_1,\ w_2,\ \cdots,\ w_m)$

$= (x_k)_{k\in \mathbf{Z}}$

$= \underline{a}\left(\underline{a}^{-1}\right)^{t^m} = (a_k a_{k-m}^{-1})_{k\in \mathbf{Z}},$

and,

$$\left[\underline{b}^{-1}, t^{2m}\right] = (h_{11}^{-1},\ h_{21}^{-1},\ \cdots,\ h_{m1}^{-1},\ g_{11}^{-1},\ g_{21}^{-1},\ \cdots,\ g_{m1}^{-1},$$

$h_{11},\ h_{21},\ \cdots,\ h_{m1},\ g_{11},\ g_{21},\ \cdots,\ g_{m1},$

$h_{12}^{-1},\ h_{22}^{-1},\ \cdots,\ h_{m2}^{-1},\ g_{12}^{-1},\ g_{22}^{-1},\ \cdots,\ g_{m2}^{-1},$

$h_{12},\ h_{22},\ \cdots,\ h_{m2},\ g_{12},\ g_{22},\ \cdots,\ g_{m2},$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$h_{1r}^{-1},\ h_{2r}^{-1},\ \cdots,\ h_{mr}^{-1},\ g_{1r}^{-1},\ g_{2r}^{-1},\ \cdots,\ g_{mr}^{-1},$

$h_{1r},\ h_{2r},\ \cdots,\ h_{mr},\ g_{1r},\ g_{2r},\ \cdots,\ g_{mr})$

$=(y_k)_{k\in \mathbf{Z}}$

$=\underline{b}\left(\underline{b}^{-1}\right)^{t^{2m}} = (b_k b_{k-2m}^{-1})_{k\in \mathbf{Z}}.$

Then we have $\underline{w} = (w_1,\ w_2,\ \cdots,\ w_m) = \left[\underline{a}^{-1}, t^m\right]\left[\underline{b}^{-1}, t^{2m}\right]$, and we are done.

First we find $\underline{a} = (a_k)_{k \in \mathbf{Z}}$. We have $x_k = a_k a_{k-m}^{-1}$, hence $a_k = x_k a_{k-m}$;

$$a_1 = x_1 = h_{11}$$

$$a_2 = x_2 = h_{21}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{m-1} = x_{m-1} = h_{m-11}$$

$$a_m = x_m = h_{m1}$$

$$a_{m+1} = x_{m+1} a_1 = g_{11} h_{11}$$

$$a_{m+2} = x_{m+2} a_1 = g_{21} h_{21}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2m} = x_{2m} a_m = g_{m1} h_{m1}$$

$$a_{2m+1} = x_{2m+1} a_{m+1} = h_{11}^{-1} g_{11} h_{11}$$

$$a_{2m+2} = x_{2m+2} a_{m+2} = h_{21}^{-1} g_{21} h_{21}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{3m} = x_{3m} a_{2m} = h_{m1}^{-1} g_{m1} h_{m1}$$

$$a_{3m+1} = x_{3m+1} a_{2m+1} = g_{11}^{-1} h_{11}^{-1} g_{11} h_{11} = [g_{11}, h_{11}]$$

$$a_{3m+2} = x_{3m+2} a_{2m+2} = [g_{21}, h_{21}]$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{4m} = x_{4m} a_{3m} = [g_{m1}, h_{m1}]$$

$$a_{4m+1} = h_{12}[g_{11}, h_{11}]$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$a_{5m} = h_{m2}[g_{m1}, h_{m1}]$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$a_{(4r-1)m+1} = x_{(4r-1)m+1}a_{(4r-2)m+1} = [g_{1r}, h_{1r}][g_{1r-1}, h_{1r-1}] \cdots [g_{11}, h_{11}]$

$a_{(4r-1)m+2} = x_{(4r-1)m+2}a_{(4r-2)m+2} = [g_{2r}, h_{2r}][g_{2r-1}, h_{2r-1}] \cdots [g_{21}, h_{21}]$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$a_{4rm} = x_{4rm}a_{(4r-1)m} = [g_{mr}, h_{mr}][g_{mr-1}, h_{mr-1}] \cdots [g_{m1}, h_{m1}]$.

Hence if we choose;

$$
\begin{aligned}
\underline{a} = (a_k)_{k\in\mathbf{Z}} = (&h_{11},\ h_{21},\ \cdots,\ h_{m1},\\
&g_{11}h_{11},\ g_{21}h_{21},\ \cdots,\ g_{m1}h_{m1},\\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots\\
&[g_{1r}, h_{1r}][g_{1r-1}, h_{1r-1}] \cdots [g_{11}, h_{11}],\\
&[g_{2r}, h_{2r}][g_{2r-1}, h_{2r-1}] \cdots [g_{21}, h_{21}],\\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots\\
&[g_{mr}, h_{mr}][g_{mr-1}, h_{mr-1}] \cdots [g_{m1}, h_{m1}]).
\end{aligned}
$$

Then $[\underline{a}^{-1}, t^m] = (x_k)_{k\in\mathbf{Z}}$ especially,

$x_{4rm+1} = a^{-1}_{(4r-1)m+1} = [h_{11}, g_{11}][h_{12}, g_{12}] \cdots [h_{1r}, g_{1r}] = w_1$

$x_{4rm+2} = a^{-1}_{(4r-1)m+2} = [h_{21}, g_{21}][h_{22}, g_{22}] \cdots [h_{2r}, g_{2r}] = w_2$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$x_{5rm} = a^{-1}_{4rm} = [h_{m1}, g_{m1}][h_{m2}, g_{m2}] \cdots [h_{mr}, g_{mr}] = w_m$.

Now we introduce $\underline{b} = (b_k)_{k\in\mathbf{Z}}$ such that:

$$[\underline{b}^{-1}, t^{2m}] = (y_k)_{k\in\mathbf{Z}} = (b_k b^{-1}_{k-2m})_{k\in\mathbf{Z}}.$$

Hence;

$$b_1 = y_1 = h_{11}^{-1}$$

$$b_2 = x_2 = h_{21}^{-1}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$b_m = y_m = h_{m1}^{-1}$$

$$b_{m+1} = y_{m+1} = g_{11}^{-1}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$b_{2m} = y_{2m} = g_{m1}^{-1}$$

$$b_{2m+1} = y_{2m+1} b_1 = 1$$

$$b_{2m+2} = y_{2m+2} b_2 = 1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$b_{4m} = y_{4m} b_{2m} = 1$$

$$b_{4m+1} = y_{4m+1} b_{2m+1} = h_{12}^{-1}$$

$$b_{4m+2} = y_{4m+2} b_{2m+2} = h_{22}^{-1}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$b_{5m} = y_{5m} b_{3m} = h_{m2}^{-1}$$

$$b_{5m+1} = y_{5m+1} b_{3m+1} = g_{12}^{-1}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$b_{6m} = y_{6m} b_{4m} = g_{m2}^{-1}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$b_{4rm-4m+1} = y_{4rm-4m+1} b_{4(r-1)m-2m+1} = h_{1r}^{-1}$$

$$b_{4rm-4m+2} = y_{4rm-4m+2} b_{4(r-1)m-2m+2} = h_{2r}^{-1}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$b_{4rm-2m} = y_{4rm-2m} b_{4(r-1)m} = g_{mr}^{-1}$$

Hence if we choose;

$$\begin{aligned} \underline{b} = (b_k)_{k\in Z} = (& h_{11}^{-1},\ h_{21}^{-1},\ \cdots,\ h_{m1}^{-1}, \\ & g_{11}^{-1},\ g_{21}^{-1},\ \cdots,\ g_{m1}^{-1}, \\ & 1,\ 1,\ 1,\ \cdots,\ 1,\ 1,\ 1,\ \cdots,1,\ 1,\ 1, \\ & 1,\ 1,\ 1,\ \cdots,\ 1,\ 1,\ 1,\ \cdots,1,\ 1,\ 1, \\ & h_{12}^{-1},\ h_{22}^{-1},\ \cdots,\ h_{m2}^{-1}, \\ & g_{12}^{-1},\ g_{22}^{-1},\ \cdots,\ g_{m2}^{-1}, \\ & 1,\ 1,\ 1,\ \cdots,\ 1,\ 1,\ 1,\ \cdots,1,\ 1,\ 1, \\ & 1,\ 1,\ 1,\ \cdots,\ 1,\ 1,\ 1,\ \cdots,1,\ 1,\ 1, \\ & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ & h_{1r}^{-1},\ h_{2r}^{-1},\ \cdots,\ h_{mr}^{-1}, \\ & g_{1r}^{-1},\ g_{2r}^{-1},\ \cdots,\ g_{mr}^{-1}, \\ & 1,\ 1,\ 1,\ \cdots,\ 1,\ 1,\ 1,\ \cdots,1,\ 1,\ 1, \\ & 1,\ 1,\ 1,\ \cdots,\ 1,\ 1,\ 1,\ \cdots,1,\ 1,\ 1), \end{aligned}$$

then we have, $[\underline{b}^{-1}, t^{2m}] = (y_k)_{k\in Z}$.

We may rewrite $\underline{a}$, $\underline{b}$ as follows:

Let $\underline{a} = (a_k)_{k\in\mathbf{Z}} = (a_1, \cdots, a_{4rm})$ (hence $a_k = 1$ for $k \neq 1, \cdots, 4rm$) and $\underline{b} = (b_k)_{k\in\mathbf{Z}} = (b_1, \cdots, b_{4rm-2m})$ (hence $b_k = 1$ for $k \neq 1, \cdots, 4rm-2m$). Then $k = 4lm + i$ for some integer numbers i and l $(1 \leq i \leq 4m,\ 0 \leq l \leq r-1)$. Hence we may rewrite a_k and b_k as following:

$$a_k = a_{4lm+i} = \begin{cases} h_{il+1}a_{k-m} & \text{for } 1 \leq i \leq m \\ g_{i-ml+1}a_{k-m} & \text{for } m+1 \leq i \leq 2m \\ h^{-1}_{i-2ml+1}a_{k-m} & \text{for } 2m+1 \leq i \leq 3m \\ g^{-1}_{i-3ml+1}a_{k-m} & \text{for } 3m+1 \leq i \leq 4m, \end{cases}$$

and,

$$b_k = b_{4lm+i} = \begin{cases} h^{-1}_{il+1} & \text{for } 1 \leq i \leq m \\ g^{-1}_{i-ml+1} & \text{for } m+1 \leq i \leq 2m \\ 1 & \text{for } 2m+1 \leq i \leq 4m. \end{cases}$$

Then we have:

$$\underline{w} = (w_1,\ w_2,\ \cdots,\ w_m) = \left[\underline{a}^{-1}, t^m\right]\left[\underline{b}^{-1}, t^{2m}\right],$$

and we are done. □

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