Artificial Piezoelectricity in Silicon Phononic Crystals

by

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Abstract

Certain materials, like quartz, are electrically polarized when they are strained. This is called the direct piezoelectric effect. It manifests naturally by the appearance of bound electrical charges at the surfaces of a strained medium. This is always accompanied by the converse effect, whereby a solid becomes strained when placed in an electric field. Piezoelectricity has important applications in acoustic transducers, acousto-optic and RF devices, precision positioning instrumentation, to name a few. However, since piezoelectric behavior is expressed only in non-centro symmetric crystals, such as quartz and lithium niobate, they have limited utility for on-chip applications. In addition, the piezoelectric constants of these materials are typically weak, which limits the electromechanical conversion efficiency that can be achieved, with typical values of $\sim 5.5\%$ for single-crystal lithium niobate ($LiNbO_3$) and $\sim 20.8\%$ for lead zirconium titanate (PZT).

Here we propose to develop a new class of meta-materials based on heterogeneous metal-on-silicon phononic crystals to realize artificial piezoelectricity in silicon, which is a centro-symmetric material. We have developed analytical models in one-dimensional meta-atom structures to mimic piezoelectric behaviors, derive constitutive relations for the direct and converse piezoelectric effects, and determine the effective electromechanical coupling coefficient. We found that near unity electromechanical coupling coefficient can in principle be achieved by

driving the system near resonance, with the added advantage of low voltage operation. Moreover, our structure permits scalable frequency operation up to tens of giga hertz (GHz). We have also designed and simulated realistic 2D metal-on-silicon phononic crystal structures on the Silicon-on-Insulator (SOI) platform and demonstrated artificial piezoelectricity by numerical experiment. By tailoring both the electromagnetic and phononic band structures of these periodic structures, efficient excitation of coherent phononic modes can be achieved, which can potentially have novel applications in acousto-optics, acousto-electromagnetics, transducers, and quantum phononics.

To my parents.

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Contents

Symbols

$\mathbf{B}$	Magnetic induction field 23, 91
β	Bloch wavevector 72
C	Capacitance 34
$\mathbf{D}$	Electric displacement 7, 23, 91
$\mathbf{E}$	Electric field 7, 23, 91
F_0	Constant force (not time dependent) 32, 37
$\mathbf{H}$	Magnetic field 23, 91
H	Height 32
$\mathbf{J}$	Free current density 91
K	Spring constant 17
L	Natural length in the absence of an applied voltage or external mechanical force 32
Λ	Period of length 73
L_{eq}	Changed length of the bar $L_{\text{eq}} = L + \Delta L$ 35
$[M]$	(6×6) Bond transformation matrix, See [2], as well as book inside cover and Chapter 3, §D of vol. 1 [3]. 8
$[N]$	(6×6) Bond transformation matrix, See [2], as well as book inside cover and Chapter 3, §D of vol. 1 [3]. 8
$\mathbf{P}$	Polarization 26, 91
ν	Poisson's ratio 104
$\langle P_n \rangle$	Time-averaged acoustic Poynting vector 64
$[S]$	Strain matrix 9
$\mathbf{S}$	Poynting vector 92
T^{Cauchy}	Cauchy stress $T^{\text{Cauchy}} = c_{11} S_{xx}$ 79
$\mathbf{T}^{\text{Maxwell}}$	Maxwell stress tensor 93
T_a	Total stress, subscript a means applied 39, 41
$\mathcal{T}$	Traction vector is simply the force vector on a cross-section divided by that cross-section's area 32
U	Acoustic amplitude 47
U	Electrostatic energy in the capacitor 34
V_0	DC voltage 34, 37, 40

$V(x, y)$	Voltage map 90
W	Width 32
E	Young's modulus of elasticity 104
Z	Acoustic impedance 68
$[a]$	Coordinate transformation matrix 8
W	Acoustic Power 70
κ	Acoustic wavevector 87
α	Direction as in Fig. 2.3 27, 28
α	Dummy variable 41
β	Direction as in Fig. 2.3 27, 28
β	Dummy variable 41
K	Bulk modulus 104
B	Bulk modulus 104
c_{11}	Material stiffnes component 33
$\mathbf{c}^D$	Stiffness at constant or zero electric displacement 25
$\mathbf{c}^E$	Stiffness at constant or zero electric field 25, 97
$\mathbf{c}$	Stiffness 8, 22, 23, 95
ΔL	Change in the equilibrium/natural length due to an not time varying applied force 34, 37, 40
ρ	Free charge density 91
χ_m	Magnetic susceptibility 91
χ_e	Electric susceptibility 91
A	Cross-section's area 32
δF_{e}	Perturbative electrostatic force 37, 38
δF_{m}	External modulating mechanical force 37, 38
δL	Dynamical change in length 40
$\mathbf{d}$	Piezoelectric charge constant 8, 24, 25
η	Damping 107
$\mathbf{e}$	Piezoelectric stress constant 8, 25
$U(x)$	Eigenmode 75
k	electromagnetic wavevector 87
ϵ_0	Permittivity of free space 19, 91
ϵ^S	Permittivity at constant or zero strain 25, 97
ϵ^T	Electric permittivity at constant stress 24
ϵ	Electric permittivity 7, 8, 23, 94
$\mathbf{L}$	Equilibrium position 22
ϵ_r	Relative permittivity 91
$\mathbf{f}$	Body force density 97, 107
$\mathbf{f}$	Lorentz's force per unit volume (force density) 91

$\mathbf{g}$	Strain constant 25
$\mathbf{h}$	Stress constant 25
β^S	Impermeability at constant or zero strain 25
β^T	Impermeability at constant or zero stress 25
β	Impermeability 25
V_0	Initial volume 104
R	The intensity reflection coefficient 69
I	Intensity 69
T	The intensity transmission coefficient 70
κ^2	Electromechanical coupling coefficient 64
k	Composite wavevector 46
k	Continuous wavevector 39
k_n	Discrete wavevector 59
l_0	Unstrained length 17
l_{equil}	Equilibrium distance of charge 17
ρ	Material density 22, 95, 97
$\delta V(t)$	Modulation voltage 37
μ_0	Magnetic permeability in vacuum 91
μ_r	Relative permeability 91
ℓ	New displaced position 22
N	Normal force 32, 33
σ	Normal stress on a surface 32
q	Normalized wavevector $q = kL$ 47
p	Electric dipole moment 19
μ	Permeability 23
ϕ	Voltage 97, 109
p	Pressure 68
P	Pressure 104
q	Electric charge magnitude 17, 91
r	Pressure reflection coefficient 69
$\mathbf{s}^D$	Compliance at constant electric displacement 25
$\mathbf{s}^E$	Compliance at constant electric field 24
$\mathbf{s}$	Compliance 22, 24
$x^{'}$	Scaled Co-ordinate 47
G	Shear modulus 105
μ	Shear modulus 104
$\mathbf{S}$	Mechanical strain 22
$\mathbf{T}$	Mechanical stress, primarily means Cauchy stress in this Thesis 8, 22, 23, 32, 97, 107

∇_{s}	Symmetric part of the gradient 22, 97
t	Pressure transmission coefficient 69
$\mathbf{u}$	Particle displacement field 22, 95, 97
$\hat{n}$	Unit normal to the surface, which is inclined at an angle θ with respect to the positive x-axis, that $\mathcal{T}$ is acting on 32
$\mathbf{v}$	Velocity of charge 91
η	Viscosity 23

Acronyms

CMOS	complementary metal-oxide-semiconductor 1, 3, 14
CMUT	capacitive micromachined ultrasonic transducer 1, 13
IDT	inter-digital transducer 11, 89
MEMS	micro-electro-mechanical systems 1, 13, 14
PnC	phononic crystal 10, 11, 31
PZT	lead zirconium titanate 1, 2
SAW	surface acoustic wave 3
SOI	silicon-on-insulator 1

Author's Contribution

I declare that I have written this thesis completely by myself and that I have not used any sources other than those referenced. When material was taken from other works – whether modified or unmodified – this is clearly indicated and referenced. I have used the 30 days Trial License of Professional FlexPDE [1] software, provided by PDE Solutions Inc. with generosity, associated with the Computer ID: 4c8c 3bba 7cc5 66ef, twice for all the simulation results in this Thesis, and later a legal licensed copy by Prof. Dr. André Costopoulos, followed by a 3D upgrade by Prof. Dr. Ravina Sanghera, the Office of Dean of Students at the University of Alberta.

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Chapter 1

1 Prologue

This project started with the aim of generation and detection of acoustic waves of GHz *and beyond in the silicon platform using the complementary metal-oxide-semiconductor (CMOS)-compatible processes. The objective of our research is to develop a new platform of metamaterial to exhibit piezoelectricity. In particular, we aim to realize artificial piezoelectricity in metal-silicon phononic crystals. Gold is used merely as electrical contact in this thesis. Gold easily diffuses in silicon and it acts as deep-level trap and recombination center in silicon. For this reason, gold is not a CMOS compatible material. We can use aluminum, copper, instead of gold, if we would like to fabricate the devices using CMOS-compatible process flows. Schottky diode is formed at the interface of metal-silicon. We illustrated electric field induced stress using gold/silicon in this thesis. Without loss of generality, the theoretical analysis is valid in other material combinations, such as gold/glass material. We also explore the feasibility of extending the acoustic frequencies from hundreds of* MHz *to tens of* GHz *range. Applications of interest include acousto-optic devices, acoustic RF, microwave and millimeter-wave devices, transducers and sensors.*

Generation of acoustic waves in silicon – which is non-piezoelectric naturally – was first achieved using micro-electro-mechanical systems (MEMS) structures. In 1994, the group of B. T. Khuri-Yakub from Stanford University invented a silicon device called capacitive micro-machined ultrasonic transducer (CMUT) [4]. However, the acoustic transducers typically operate in the MHz *range. The higher the center frequency, the smaller is the lateral and axial resolutions in ultrasound imaging. The acousto-optic devices typically rely on piezoelectric materials, e. g., lead zirconium titanate (PZT), lithium niobate ($LiNbO_3$) for acoustic wave transduction. However, for on-chip acoustic wave transduction, they have important drawbacks of cost, incompatibility of integration of piezoelectric materials, limited flexibility in electromechanical coupling and frequency response in silicon-on-insulator (SOI) platform for silicon phononics [5]. Silicon is a centro-symmetric crystal and thus does not exhibit piezoelectric behavior naturally. In this thesis, we propose and develop general framework to induce artificial piezoelectricity in centro-*

symmetric crystals and apply it in silicon using heterogeneous metal-on-silicon phononic crystal, as silicon is an important platform. We performed numerical experiments to show GHz *transduction, near unity electromechanical coupling coefficient in silicon.*

We know that in piezoelectric materials, the piezoelectric constants, let us say the e-constants, are fixed by the microscopic material structure. By using the special electrical and mechanical arrangements presented in this thesis, one can use a suitable control voltage to adjust the effective piezoelectric constants. A resonator is a narrow band filter. If an acoustic wave is presented as an input to a periodic array of unit cell elements, a filtered signal will be present at the output of the structure. In this PhD thesis, the artificial piezoelectricity is exemplified by creating a voltage tunable acoustic wave resonator. In one formulation, it can be designed to have performance similar to the Fabry-Perot interferometer. The more elements in the periodic structure, the higher the finesse or Q-factor. In a unit cell, we can vary the electrode width of gold with respect to the silicon width and observe different kinds of filtering effects. The DC bias that we have in our analysis in Chap. 3 and Chap. 4 acts like a control knob to tune this resonator.

1.1. Background

Carl Linnaeus and Franz Aepinus in the mid-18th century studied the relationship between the temperature change and the generation of electric potential–the pyroelectric effect. René Just Haüy and Antoine César Becquerel wanted to find a relationship between the mechanical stress and the electric charge, but their experiment was not successful. In 1880, Pierre Curie and his elder brother Jacques Curie with deep understanding of the underlying crystal structures and their knowledge of the pyroelectric effect, experimentally demonstrated the direct piezoelectric effect, i. e., certain kind of materials generate voltage when squeezed or stretched. In 1881, Gabriel Lippmann mathematically deduced the converse piezoelectric effect, whereby a solid becomes longer or shorter when placed in an electric field. The Curie brothers immediately experimentally validated the existence of the converse piezoelectric effect. Conventional piezoelectric material must lack inversion symmetry in its crystal structure, see for example the widely used PZT crystal in Fig. 1.1. In other words, isotropic materials do not exhibit piezoelectricity naturally.

One of the first applications of the inter-digital circuit, reported in 1965, was to use interlocking comb shaped fingers to generate surface acoustic wave (SAW) on a piezoelectric substrate [6]. The alternating direction of the electric field between fingers creates regions of tension and compression alternating between fingers. This results in the generation of a mechanical wave–the surface acoustic wave. Devices based on SAW can be used for a variety of applications, one being on-chip RF signal processing. The key point is that in conventional SAW devices, the substrate must be made of piezoelectric materials, like quartz, lithium niobate etc. We would like to replace this substrate material with silicon phononic crystals, which can be fabricated using

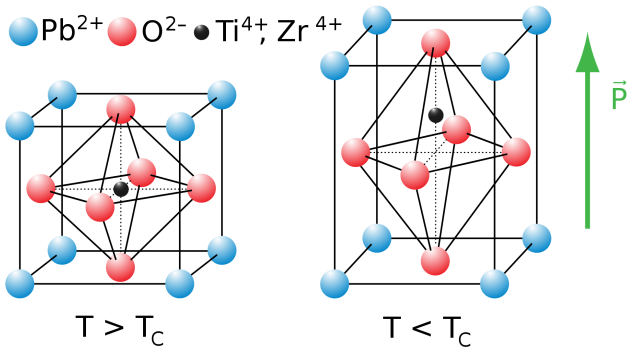


Figure 1.1.: *Tetragonal unit cell of lead titanate. Lead zirconium titanate ($Pb[Zr_x Ti_{(1-x)}]O_3$ with $0 \leq x \leq 1$), commonly known as PZT, is the most common piezoelectric ceramic in use today. The PZT crystal has dipole moment, $\mathbf{P}$, below the Curie temperature T_c. Above the Curie point, the crystal structure exhibits inversion symmetry and the piezoelectric behavior is lost, see the black ball in both the pictures. (Adapted from Wikimedia Commons with GNU Free Documentation License, Version 1.3 or later.)*

standard CMOS-compatible processes, of course gold can easily be replaced with aluminum or copper and the formulation presented in this thesis is still valid.

Silicon is an important material in the semiconductor industry. Substantial amount of resources has been invested to optimize the process flows to fabricate integrated circuits using this material. The integrated circuit industry does not prefer to introduce exotic materials, like lithium niobate, Hafnium oxide etc. [7] in the standard process flows. It will be very useful if we can engineer the piezoelectric behavior in silicon using existing well-established CMOS-compatible processes. This will create an entirely new material platform in centro-symmetric crystals, like silicon, to exhibit piezoelectricity, allowing phononic devices to be fabricated in this platform without requiring the substrate to be piezoelectric. Potential applications include acoustic RF, microwave and millimeter-wave devices, acousto-optic devices, bio-sensors, etc. There is also growing interest in employing these devices for quantum information applications at cryogenic temperatures, such as coupling surface acoustic wave resonators with superconducting qubits [8].

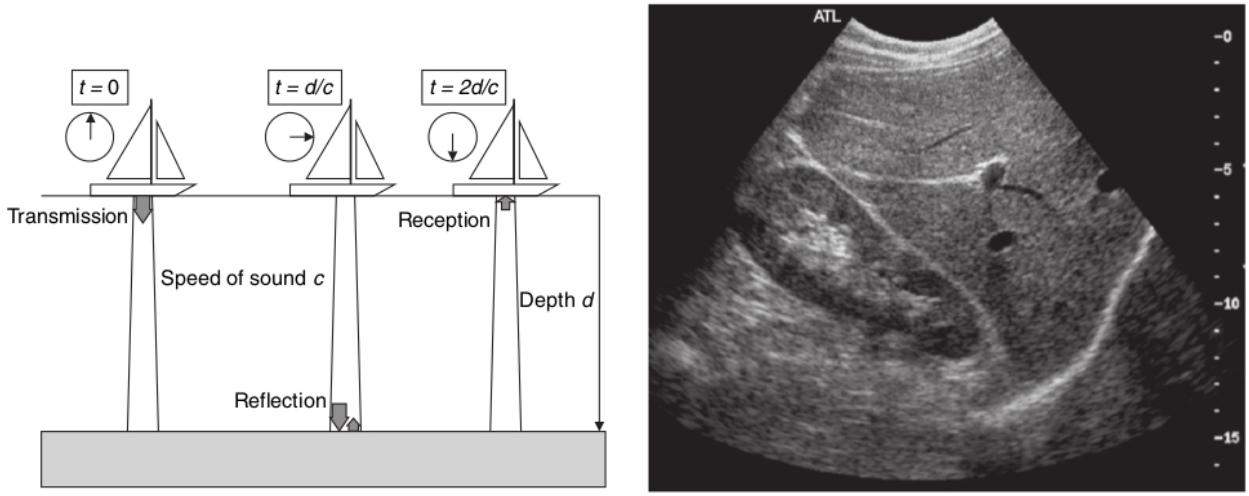
This PhD thesis details our progress in designing silicon phononic crystals and developing models for evaluating their performance in realizing artificial piezoelectricity. In particular, we have developed analytical model of acoustic wave transduction to extract key parameters in a 1D phononic lattice, including the electromechanical coupling coefficient, which quantifies the efficiency of transduction of piezoelectric material between mechanical and electrical energy and vice versa. We have also performed numerical experiment of 2D silicon phononic crystals using

FlexPDE [1] and verified the numerical simulation using analytical model. Our aim is to achieve acoustic transducers working at frequencies from hundreds of MHz to even tens of GHz.

1.1.1. Why High Frequency Transducer? A Concrete Example

In this Section, we shall explain the rationale of having a piezoelectric transducer of higher frequency. We illustrate by using a concrete example in ultrasound imaging. This will allow the reader to appreciate why we pursue research in the hundreds of MHz (to GHz regime). The center frequency of a piezoelectric transducer is typically in the MHz range, e. g.., commercially 1–5 MHz, in some laboratory it is up to 70 MHz.

It would be useful to define some key parameters in ultrasound for the purpose of illustration. We form images using "pulse–echo" mode in ultrasound, meaning sound has to travel and come back after reflection, i. e., traversing twice the distance to form an image, see Fig. 1.2.



(a) Pulse-Echo **(b)** B-Mode Imaging

Figure 1.2.: *Illustration of the pulse–echo principle, and the B-mode imaging in ultrasound.* (a) *Measurement of water depth using the pulse–echo principle. The depth is worked out by measuring the time from transmission of the pulse to reception of the echo. The speed of sound must be known.* (b) *An example of a B-mode image showing reflections from organ and blood vessel boundaries and scattering from tissues. (Adapted from [9]).*

Lateral Resolution

The lateral resolution is the ability to separate structures side by side. If the transducer center frequency can be increased, we can obtain higher lateral resolution. The lateral resolution, at the focal point, after p. 41 in [10] is,

$$\Delta x = 1.2 f_{\#} \lambda_{\text{acoustic}}, \tag{1.1}$$

where the f-number is

$$f_{\#} = \frac{\text{focal length (f)}}{\text{Diameter of pupil (D) or image sensor footprint size}} \propto \frac{1}{2\text{NA}}. \tag{1.2}$$

Here the numerical aperture NA is the acoustic wave gathering capability. The acoustic wavelength in tissue, as an example, for ultrasound transducer center frequency at 10 MHz is

$$v_{\text{sound}} \sim 1540\,\text{m}\,\text{s}^{-1}, \tag{1.3}$$

$$\lambda_{\text{acoustic}} = \frac{v_{\text{sound}}}{f_{\text{transducer}}} = \frac{1540\,\text{m}\,\text{s}^{-1}}{10\,\text{MHz}} \approx 154\,\mu\text{m}. \tag{1.4}$$

Let us say, the $f_{\#} = 1.5$ for 12 mm focal length and 4 mm transducer radius. The lateral resolution is

$$\Delta x = 1.2 \times 1.5 \times 154\,\mu\text{m} \approx 277\,\mu\text{m}. \tag{1.5}$$

Axial Resolution

The axial resolution is the ability to separate structures parallel to the ultrasound beam. In tissue imaging, the different components, like liver, fat, bone etc, have varying degrees of attenuation [11], [12]. Assuming constant damping factor, the higher frequency provides better axial resolution [11], but the imaging depth is reduced. The axial resolution is [13]

$$\Delta z = \frac{1}{2} \lambda_{\text{acoustic}} N_{\text{number of pulses}}. \tag{1.6}$$

Increasing the center frequency and reducing the number of scanning lines or pulses can increase the axial resolution of the transducer, see Tab. 1.1.

The important factor in determining the axial resolution is spatial pulse length. It is the minimum reflector separation required along the direction of sound travel (along the B-scan line) to produce separate echoes. We cannot distinguish structures smaller than the spatial pulse length. It has very high importance in diagnostics, if we can diagnose a disease in the early-stage, meaning detecting tiny malignant structures, say cancer, we can probably do something

Frequency	Axial Resolution (2λ)
1 MHz	$\sim$ 3 mm
5 MHz	$\sim$ 0.6 mm
100 MHz	$\sim$ 31 μm

Table 1.1.: *Scaling of axial resolution with frequency, assuming constant speed of sound of* $\sim 1540\ \mathrm{m\,s^{-1}}$ *in tissue [13]. This is* 2λ *resolution along beam, since* $2d = v_{\mathrm{sound}}t$*, we can get half of this.*

to prevent the spread, i. e., metastasis, which could be life saving.

Bandwidth

The bandwidth of an acoustic transducer is

$$\text{Bandwidth of the transducer} \sim \frac{1}{\text{Pulse length}}. \tag{1.7}$$

Frame Rate

The frame rate of image acquisition is increased if high frequency is used, this formula comes in handy [14]

$$\text{Frame rate} \times \text{depth of penetration} \times \text{Number of transmit pulses} = \frac{v_{\mathrm{sound}}}{2} \tag{1.8}$$

Temporal resolution of a two-dimensional image is improved when the frame rate is high [14].

Center Frequency, Bandwidth, Sensitivity

The signal bandwidth is computed using [15]

$$\mathrm{dB} = 20 \log_{10}\left(\frac{A_1}{A_2}\right), \tag{1.9}$$

where A_1 and A_2 are amplitudes of signal 1 and 2 respectively.

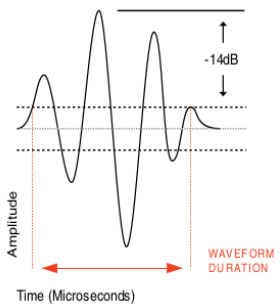
The approximate relations, shown in Fig. 1.3, can be used to assist in transducer selection. For example, if a -14 dB waveform duration of one microsecond is needed, what frequency transducer should be selected? From the graph, a bandwidth of approximately 1 to 1.2 MHz corresponds to approximately 1 microsecond -14 dB waveform duration. Assuming a nominal

50% fractional bandwidth transducer, this calculates to a nominal center frequency of 2 to 2.4 MHz. Typically, we do not have exact frequency transducer. Therefore, a transducer of 2.25 MHz or 3.5 MHz may be applicable.

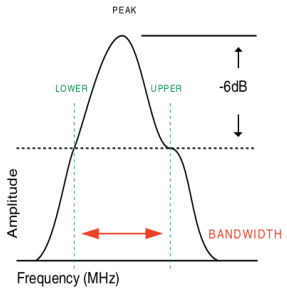
A_1 = Amplitude of signal 1
A_2 = Amplitude of signal 2

$\frac{A_1}{A_2}$	Ratio	dB
$\frac{100\%}{70.71\%}$	1.4142	3
$\frac{100\%}{50\%}$	2	6
$\frac{100\%}{25\%}$	4	12
$\frac{100\%}{10\%}$	10	20
$\frac{100\%}{1\%}$	100	40

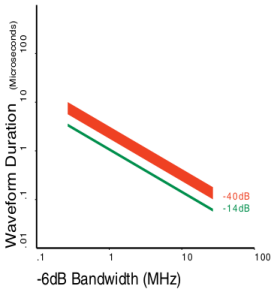
(a) dB definition



(b) Waveform duration



(c) Bandwidth



(d) Waveform duration & Bandwidth

Figure 1.3.: (a) dB *definition,* (b) *Illustrates waveform duration at the -14* dB *level or 20% amplitude of peak,* (c) *Illustrates peak frequency, upper and lower -6* dB *frequencies and* MHz *bandwidth measurements,* (d) *Relation between* MHz *bandwidth and waveform duration. The scatter is wider at -40* dB *because the 1% trailing end of the waveform contains very little energy and so has very little effect on the analysis of bandwidth. Because of the scatter it is most appropriate to specify waveforms in the time domain (microseconds) and spectra in the frequency domain. (Adapted from [15].)*

Imaging Depth

As the center frequency of the acoustic transducer is increased, the loss is increased, hence the penetration depth is lowered. It is mentioned earlier this heuristic relation between frequency and the penetration depth. The exact frequency will depend on applications, such as diagnostic imaging, or nondestructive imaging.

Sensitivity

Sensitivity is the ability of an ultrasonic system to detect reflectors (or defects) at a given depth in a test material. The greater the signal that is received from these reflectors, the more sensitive the transducer system.

1.1.2. Proof of Nonexistence of Piezoelectricity in Natural Silicon

In a linear material, the electric displacement $\mathbf{D}$, is related to the electric field $\mathbf{E}$ via

$$\mathbf{D} = \epsilon \cdot \mathbf{E}, \tag{1.10}$$

where ϵ is the electric permittivity. In a piezoelectric medium, the electric displacement $\mathbf{D}$ has an additional term

$$\mathbf{D} = \epsilon^T \cdot \mathbf{E} + \mathbf{d} : \mathbf{T}, \tag{1.11}$$

where the superscript, T, over ϵ is the permittivity at a fixed stress $\mathbf{T}$ in a mechanical medium, and $\mathbf{d}$ (unit $\mathrm{C\,N^{-1}}$), is a parameter called the piezoelectric charge constant matrix. Another commonly used piezoelectric constant is the piezoelectric stress constant $\mathbf{e}$, which is related to $\mathbf{d}$ via the relation

$$\mathbf{e} = \mathbf{d} : \mathbf{c}, \tag{1.12}$$

where $\mathbf{c}$ is the material stiffness.

We informally mentioned earlier that there cannot be piezoelectric effect in natural silicon, meaning both $\mathbf{d}$ and $\mathbf{e}$ are zero in silicon, i. e., the application of mechanical stress does not affect the electric displacement, or when a mechanical medium is strained, it does not affect the electric displacement. In this Section, we provide mathematical proof of this. This is useful in the sense that we can apply similar principles in this thesis for centro-symmetric materials other than silicon by appropriate electrode arrangements to make them exhibit piezoelectricity. The content of this Section is literally inspired from §B and §E, Chapter 8 of [3]; I preserve the notation [ ] used to denote matrices in [3] in this Section to preserve consistency with the book.

If there is no electric field, we can write in a piezoelectric material,

$$[D] = [d][T]. \tag{1.13}$$

Multiplication of the above by the coordinate transformation matrix $[a]$ provides

$$[D^{'}] = [a][D] = [a][d][T], \tag{1.14}$$

where the superscript $(^{'})$ denotes the parameter in the new coordinate system. From the transformation laws in Chapter 1 in [3],

$$[T] = [M]^{-1}[T^{'}]. \tag{1.15}$$

Here the $[M]$ and later $[N]$ are the (6×6) Bond transformation matrices [1]. This is equivalent to

$$[D^{'}] = [a][D] = [a][d][M]^{-1}[T^{'}]. \tag{1.16}$$

[1] See [2], as well as book inside cover and Chapter 3, §D of vol. 1 [3].

The piezoelectric strain matrix in the new coordinate system is

$$[d^{'}] = [a][d][M]^{-1} = [a][d]\widetilde{[N]}, \tag{1.17}$$

where tilde ($\sim$) is the transposed operator and we have used the relation $[M]^{-1} = \widetilde{[N]}$, see §D of Chapter 3 in [3].

Now if the transformation is a symmetry operation of the crystal, then

$$[d^{'}] = [d]. \tag{1.18}$$

and the symmetry condition is

$$[d][M] = [a][d]. \tag{1.19}$$

Similarly, using $[e^{'}] = [a][e][N]^{-1} = [a][e]\widetilde{[M]}$, it is shown in p. 280, Chapter 8 in [3]

$$[e][N] = [a][e]. \tag{1.20}$$

Inversion of the co-ordinate axis means

$$\begin{bmatrix} \hat{x}^{'} = -\hat{x} \\ \hat{y}^{'} = -\hat{y} \\ \hat{z}^{'} = -\hat{z} \end{bmatrix}. \tag{1.21}$$

An examination of the symmetry groups for piezoelectric crystal classes shows that none of them contains the inversion symmetry operator

$$[a] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{1.22}$$

As it is shown in §D of Chapter 1 in [3], that the strain matrix $[S]$, under the inversion operation

$$[S^{'}] = [a][S]\widetilde{[a]}, \tag{1.23}$$

is unchanged. In component form, the strain transformation law is

$$S^{'}_{ij} = a_{ik}a_{jl}S_{kl} \tag{1.24}$$

where $i, j, k, l = 1, 2, 3$. We can write

$$S^{'}_{Kj} = N_{KJ}S_J, \tag{1.25}$$

where $K, J = 1, 2, 3, 4, 5, 6$. This means that the (6×6) Bond matrix $[N]$ *corresponding to* the inversion $[a]$ is $[N] = I_6$, where I is an identity matrix .

$$[I] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.26}$$

This leads to the piezoelectric stress matrix in Eq. (1.20)

$$[e] = -[e]. \tag{1.27}$$

This is satisfied *if and only if* all e_{jI} are zero. This shows that piezoelectricity cannot exist in an isotropic solid. *Hence there cannot be piezoelectricity in naturally occurring silicon. In this thesis, by special electrode arrangements and in some geometric configuration(s), we create non-vanishing e_{jI} constant(s) in silicon.*

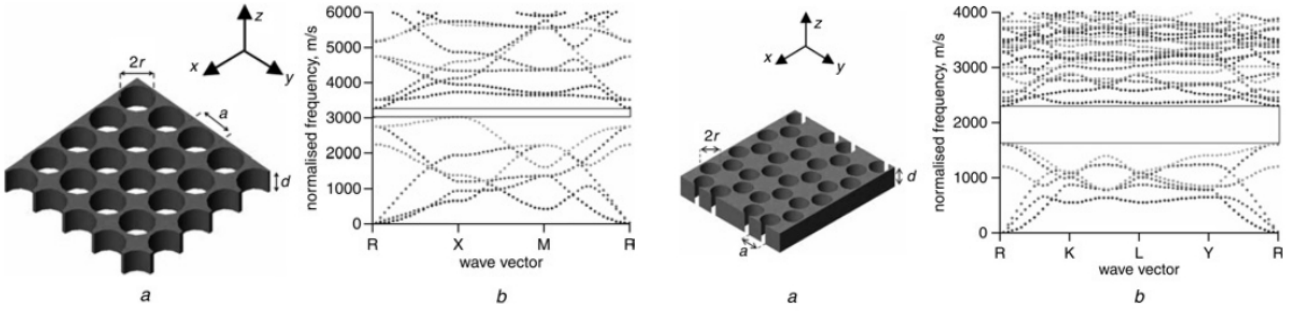
As in the cubic case, certain classes of hexagonal crystals are non-piezoelectric and exhibit no linear interaction between electromagnetic and acoustic fields [16]. Of the 32 crystal classes, 21 are non-centro symmetric and of these, 20 exhibit direct piezoelectricity (the 21st is the cubic class 432), see Appendix B.1 and B.2, pp. 372–374 of [3] and Table 3.1 in p. 47 of [16]. Ferroelectric materials have permanent electric polarization and they exhibit piezoelectricity.

1.1.3. Why the Phononic Crystal (PnC)?

The mechanical motion is induced in silicon structures by application of voltage in various devices such as in CMUT [4], [17], silicon resonators in [18]. The purpose to have a phononic crystal (PnC) is to tailor the stiffness of bulk material (in this case bulk silicon), as the mechanical amplitude is inversely proportional to the stiffness of the material, cf. Eq. (3.72) and Eq. (3.165). Another way to achieve this is to use porous silicon, [19], [20]. We illustrate it by creating a voltage tunable mechanical resonator structure that can be designed to have a variety of useful transfer functions, when acoustic waves are passed through it.

1.2. Literature Review

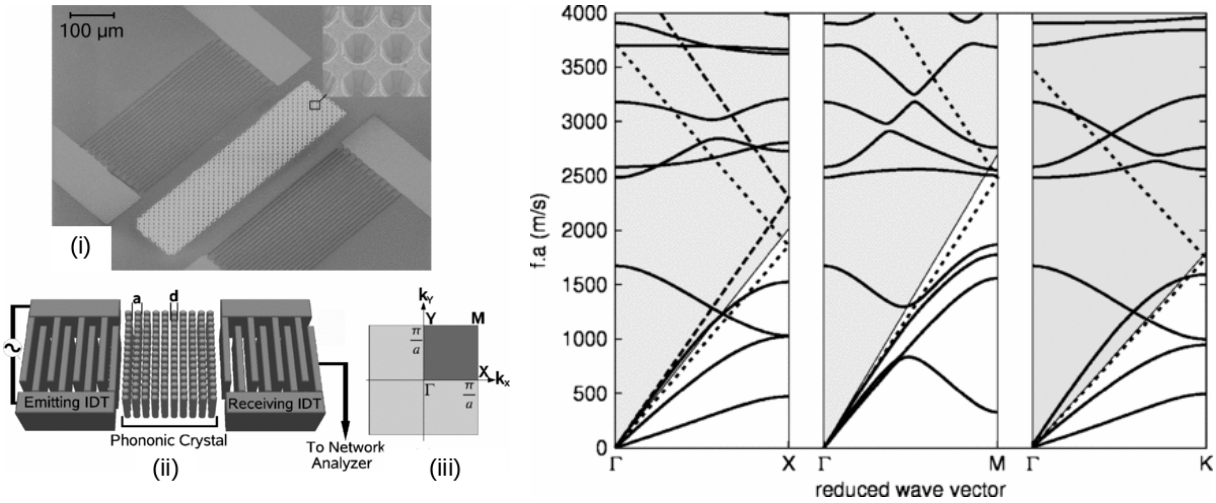
Kushwaha *et al.*[21] first theoretically calculated the PnC band structure in 1993. In 2006, Benchabane *et al.*[22] experimentally demonstrated complete surface acoustic wave bandgap from 203 to 226 MHz in 2D square lattice based on lithium niobate platform. In 2007, Mohammadi *et al.*[23] demonstrated complete phononic bandgap using square and honeycomb lattices by making holes in silicon membrane, see Fig. 1.4. They can tailor the gap to mid-gap ratio by varying size of hole, spacing between adjacent holes and the thickness of the plate. The authors in [23] tailored the stiffness constant c in silicon platform, but they did not utilize it to demonstrate any piezoelectric behavior, whereas the authors in [22] engineered the stiffness constant c in the already piezoelectric material lithium niobate, see Fig. 1.5.



(a) Square lattice **(b)** Hexagonal lattice

Figure 1.4.: (a) *Square lattice PnC plate schematic and band structure for* $r/a = 0.45$ *and* $d/a = 0.5$. (b) *Hexagonal lattice PnC plate schematic and band structure for* $r/a = 0.45$ *and* $d/a = 1$. *(Adapted from [23]).*

In 2017, Zhang *et al.*[24] provided topological design of 2D solid/solid (with epoxy and gold) hexagonal latticed phononic crystal as in Fig. 1.6. For in-plane acoustic wave excitation the various gap-midgap ratio with varying 2D designs are shown in Fig. 1.7.



(a) Surface acoustic wave **(b)** Bandstructure

Figure 1.5.: (a) *(i) Scanning electron microscope image of a phononic crystal for surface acoustic waves. The structure, consisting of 9* µm *diameter air holes with a 10* µm *pitch, is surrounded by a pair of inter-digital transducers (IDTs) to ensure surface wave transduction. The inset in the top-right corner shows a more detailed view of the holes etched in the lithium niobate single crystal substrate. (ii) Schematic of the experimental setup used for an electrical characterization of the crystal. (iii) Sketch of the first Brillouin zone.* (b) *Theoretical band diagrams for bulk waves propagating in the plane of a square lattice void/lithium niobate phononic crystal with a 64% filling fraction ($d/a = 0.9$) along the ΓX, ΓM, and ΓY directions of the first Brillouin zone. (Adapted from [22]).*

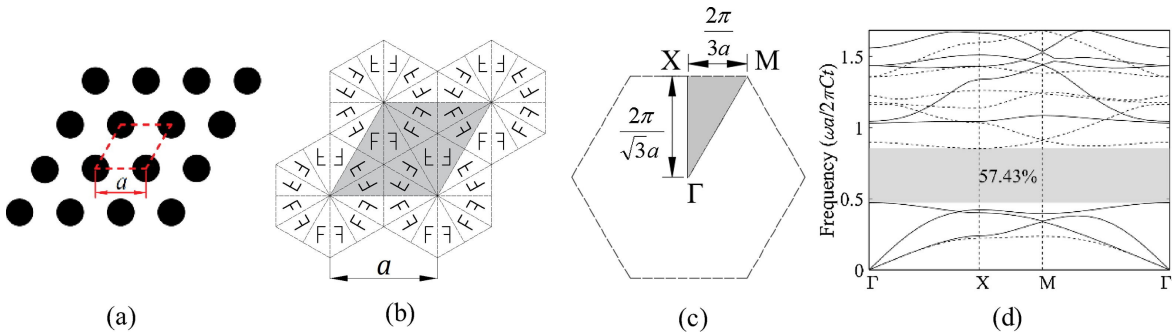


Figure 1.6.: *(a) A typical 2D phononic band gap crystal with the hexagonal lattice constant a, and the area within dashed red rhombus denotes its primitive unit cell. (b) Sixfold symmetry and selection of the primitive unit cell (shaded area). (c) Irreducible Brillouin zone of the hexagonal lattice with sixfold symmetry ($\Gamma - X - M - \Gamma$). (d) The band diagram of the phononic band gap crystal. The dashed and solid lines denote out-of-plane and in-plane phononic bands, respectively. (Adapted from [24]).*

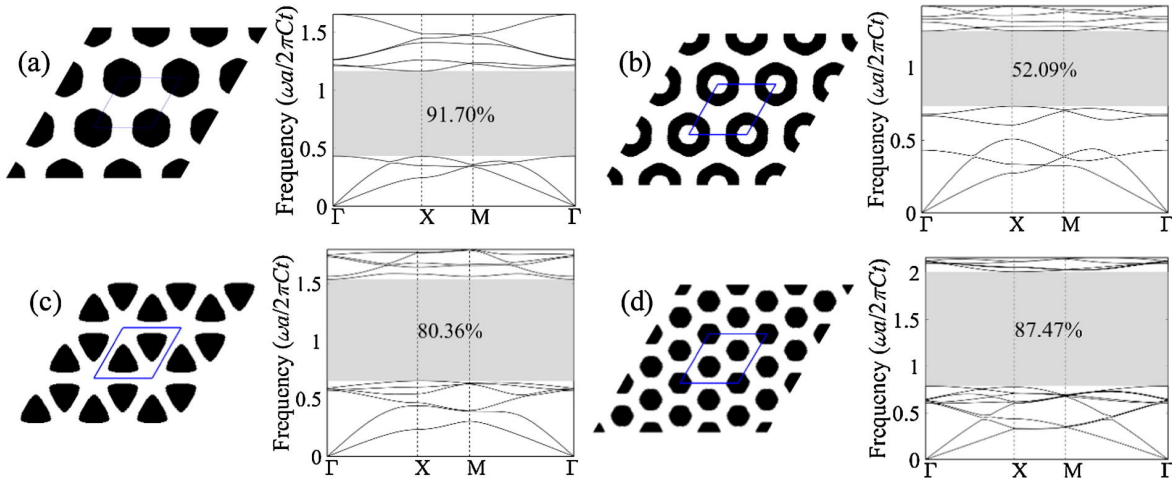


Figure 1.7.: *Optimized hexagonal-latticed phononic bandgap crystals and their band diagrams for in-plane waves; (a) the third band; (b) the fifth band; (c) the sixth band; (d) the ninth band. (Adapted from [24]).*

Wang *et al.*[25] worked on tuning phononic bandgap in 2D rectangular lattice, but they used piezoelectric materials of PZT-4 polymer. An application in microfluidics in provided by Ding *et al.*in [26]. But they used lithium niobate as the underlying piezo material, see Fig. 1.8.

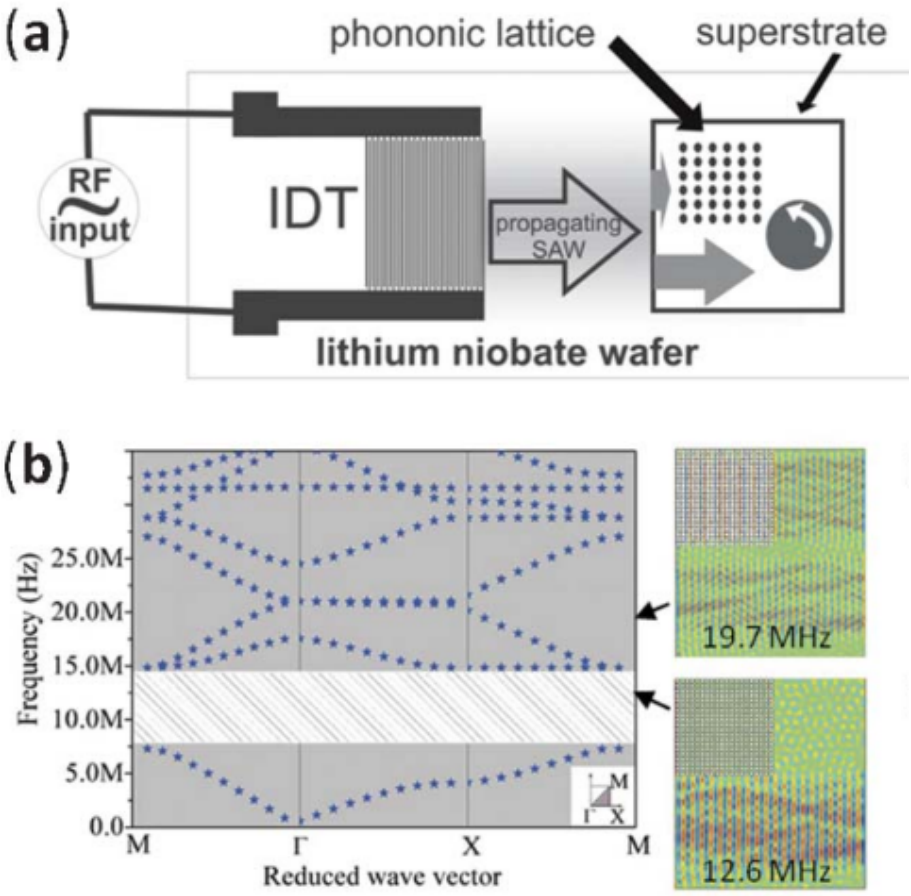


Figure 1.8.: *The RF signal is applied to 2D surface acoustic wave structure to create waves, see the calculated bandstructure. It is clearly seen that they used lithium niobate as piezo material in micro-fluidics. (a) Schematic of SAW device using a superstrate with embedded PC. Traveling SAWs generated by IDT on the $LiNbO_3$ substrate leak into a water-coupling layer, inducing Lamb waves in the silicon superstrate that are then spatially filtered by the phononic lattice to provide asymmetric exposure to the droplet. (b) Left panel shows the band structure of the designed phononic lattice; the shaded area depicts the absolute phononic band gap. Right panel shows simulations at two different frequencies. (Adapted from [26]).*

Generation of acoustic waves in silicon was first achieved using MEMS structures. In 1994, the group of B. T. Khuri-Yakub from Stanford University invented a silicon device called CMUT [4]. They were able to transmit and receive ultrasound waves, which are sound waves with frequencies above 20 kHz, using the same device. Stiction in MEMS architecture due to charging effects (and large CMUT array fabrication) affects the reliability of CMUT for use in medical applications [17]. The resonant frequency of CMUT is typically limited to tens of MHz, whereas the resonant frequency is scalable using silicon phononic crystals from hundreds of MHz (up to GHz range). For example, in medical imaging, higher transducer frequency entails smaller axial resolution and large frame rate. In other arena, higher frequency may be desirable for other purposes.

In bulk ferroelectric[2] piezoelectric materials, the thickness of the active element is determined by the desired frequency of the transducer. A thin plate vibrates with a wavelength λ that is twice its thickness d, i. e., $\lambda = 2d$. Therefore, piezoelectric crystals are cut to a thickness that is $1/2$ the desired acoustic wavelength. The higher the frequency of the transducer, the thinner the active element. The primary reason that high frequency contact transducers are not produced is because the element is very thin and too fragile [17], [28]. The typical thickness of $6''$ silicon wafer is 500 µm, and that of $8''$ silicon wafer is 600 µm. By lapping and polishing, the thickness can be reduced to about 100 µm.

Zinc oxide (ZnO) and aluminum nitride (AlN) are non-ferroelectric piezoelectrics, i. e., have no permanent electric polarization. They have wurtzite structure and they exhibit piezoelectric response along [0001]; crystal orientation is important in this case. For MEMS applications, ZnO and AlN are commonly deposited by sputtering [29], but this is not CMOS-compatible.

Recently, the authors in [30] has written various formulations of the forcing functions in an electrically induced medium and they applied in elastomer applications. In [31], the authors used the existing piezoelectric materials and were able to demonstrate piezoelectric behavior in meta-atoms in directions $d_{\alpha\beta}$ where there is no natural piezoelectric constant. In [32], the authors performed experiment of strain resulting from both electrostriction and piezoelectricity in GaN, their plots and relations provide very useful tool for our analysis. The article by Lasha Tkeshelashvili [33] provided a good theoretical analysis of artificial ferroelectricity. The effective electromechanical coefficient in thin film resonators is calculated in [34]. Very useful and clear concepts on piezoelectricity are provided by Uchino in [35], [36]. If we would like to do homogenization of our artificial piezostructure, a very relevant monograph is provided in [37] and more detailed in book [38]. A good resource that was instrumental for my research was the by Mainprice [39], preferably his online presentation at [40]. The IEEE standard [41] on surface acoustic wave devices contains important standard and it was helpful to verify my 1D design. The electro-mechanical responses of stacked dielectric actuators is described in [42]. Recently the authors in [43] used square shaped structure as phononic crystals for acoustic

[2]Ferroelectric materials have permanent electric polarization and they exhibit piezoelectricity, see recent publication in polymer [27].

wave propagation. This brief technical paper [44] by COMSOL provided a good clarification on the electrostrictive effect. Recently piezoelectric like behavior in silicon, exposed to air oxidation, is observed in [45]. Researchers in [46] have shown pseudo-piezoelectric behavior using electrostrictive effect. Researchers in [47] studied the doping concentration dependent piezoelectric behavior of Hafnium oxide in silicon.

1.3. Numerical Experiment Platform

FlexPDE [1] is a *scripted* finite element model builder, having community platform in the form of bulletin boards in [48]. It takes systems of partial differential equations provided by the user and creates a finite element model, builds a mesh, solves the problem and presents graphical output. As such, it has no built-in knowledge of any specific area of application. As long as the user can construct a well-posed PDE system, FlexPDE has a good chance of solving it. Couplings among disparate fields of analysis offer it no difficulty, because it has no concept of what fields belong together.

In order to perform post-processing, we need to create another script to use a saved transfer file (one is automatically created by default.) Difficult geometries can be imported using DXF boundaries (2D) or OBJ mesh (3D).

1.4. Summary of the Chapters

In Chap. 3 we develop detailed analytical derivation of 1D artificial phononic crystal, followed by numerical experiment using the FlexPDE [1] finite element method in strictly 1D. We compute the bandstructure using both the transfer matrix method and the plane wave expansion method. In Chap. 4 we advance our analytical derivation to 3D/2D artificial phononic crystal, followed by numerical experiments using the FlexPDE [1] finite element method. Finally we wrap the thesis in Chap. 5 with potential applications and further works.

Chapter 2

2 Introduction to Piezoelectricity

In this chapter, the field of piezoelectricity will be introduced briefly. A toy model will be developed to explain the origin of piezoelectricity. The various forms of the constitutive laws governing piezoelectricity will be written in a matrix form for reference. A brief introduction to tensor and contracted notations will be provided as the field of piezoelectricity is rife with such notations. Few key terminologies in this field will be explained to enable the reader to appreciate the parameters in a data sheet in industry.

2.1. Piezoelectric Toy Model

Using a static model, we shall try to illustrate the importance of lack of the inversion symmetry condition on piezoelectricity [1]. The constitutive law will be formed in such a toy model. This will assist our understanding of the origin of piezoelectricity and design artificial piezoelectric materials. In Fig. 2.1, two equal but opposite charges q are placed at equal distance from the pivot point at $x = 0$. This toy model is an electrically-neutral system of charged particles connected together by rigid and elastic bonds. The charges representing positive and negative ions in an actual solid are constrained to move along the x axis. The equilibrium distance l_{equil} of charge a and charge b are $l_a = l_b = l_{\text{equil}}$. Without loss of generality, let us assume that the spring constants K are identical for both the springs, and their unstrained lengths l_0 are equal to l_0. The restoring forces on the ions/electrodes a and b are

$$F_a^{\text{spring}} = -K(l_a - l_0), \tag{2.1}$$

$$F_b^{\text{spring}} = K(l_b - l_0). \tag{2.2}$$

[1] This Section is inspired from §A, pp. 265–270, Chap. 8 in [3].

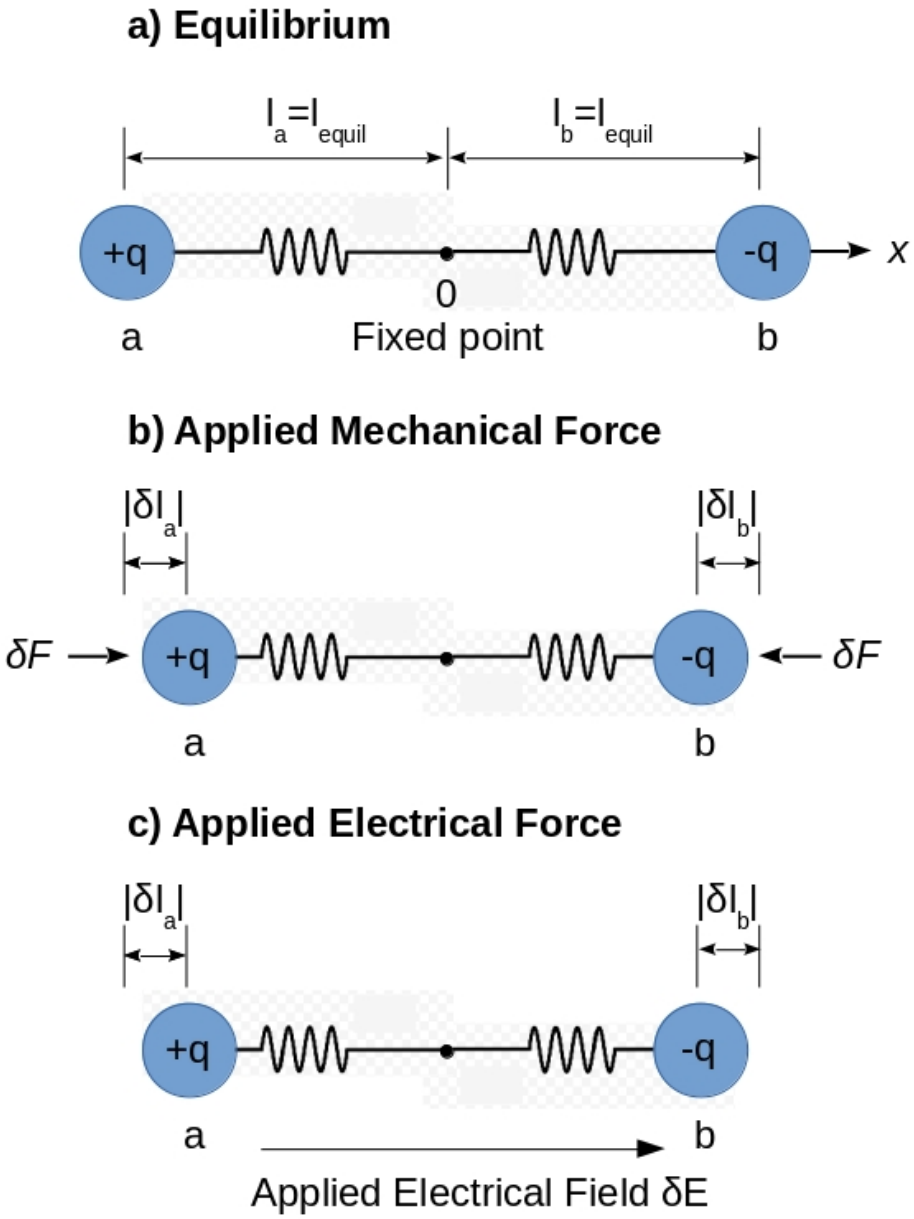


Figure 2.1.: *a) Two charges of equal but opposite polarity are symmetrically placed on either side of the pivot at $x = 0$. b) The charges are mechanically squeezed via an external force δF. c) An electric field δE displaces the two charges by an equal amount towards each other. (The spring diagram to draw the figure is adapted from Wikimedia Commons with GNU Free Documentation License, Version 1.2 or later.)*

The total forces acting on the particles a and b are

$$F_a = -K(l_a - l_0) + \frac{q^2}{4\pi\epsilon_0(l_a + l_b)^2}, \tag{2.3}$$

$$F_b = K(l_b - l_0) - \frac{q^2}{4\pi\epsilon_0(l_a + l_b)^2}, \tag{2.4}$$

where ϵ_0 is the permittivity of free space. The electric dipole moment p with respect to a pivotal reference point is

$$p = \sum_n q_n x_n. \tag{2.5}$$

The equilibrium dipole moment is

$$p_{\text{equil}} = -2ql_{\text{equil}}. \tag{2.6}$$

We have spontaneous electrical polarization in this configuration.

The mechanical response is

$$\delta L = \delta l_a + \delta l_b. \tag{2.7}$$

Mechanical Response due to the Applied Mechanical Force

What happens to the system when a mechanical stress (here modeled as mechanical force) is applied? We expect both a mechanical response and an electrical response. For the sake of simplicity, we assume the system is symmetric, the system is compressed by an equal amount $\left|\delta l_a\right| = \left|\delta l_b\right|$. The mechanical response to the applied mechanical force is

$$\delta L = -2\left|\delta l_a\right|, \tag{2.8}$$

where the minus sign is for compression.

Electrical Response due to the Applied Mechanical Force

The dipole moment is

$$p_{\text{new}} = -2ql_{\text{equil}} + 2q\left|\delta l_a\right|. \tag{2.9}$$

The electrical response, measured in terms of change in the total electric dipole moment, is

$$\delta p = p_{\text{old}} - p_{\text{new}} = -2q\left|\delta l_a\right|. \tag{2.10}$$

This is expected intuitively, as a piezoelectric material is mechanically deformed, it produces potential difference or re-distributes charges.

With the derivation so far, we can write a matrix equation

$$\begin{bmatrix} \delta p \\ \delta L \end{bmatrix} = \begin{bmatrix} \boxtimes & -2q\left|\delta l_a\right| \\ \boxtimes & -2\left|\delta l_a\right| \end{bmatrix} \begin{bmatrix} \delta E \\ \delta F \end{bmatrix}, \tag{2.11}$$

where the entries $\boxtimes$ will be determined due the responses of electrical field.

When a perturbation electrical field is applied, the charges a and b are displaced until the total forces on each charge are balanced again. This means

$$F_a + q\delta E = 0, \tag{2.12}$$
$$F_b + q\delta E = 0. \tag{2.13}$$

For small perturbation from equilibrium, we can use the following approximations [3]

$$F_a = F_a^{\text{equil}} + \frac{\partial F_a}{\partial l_a}\Big|_{\text{equil}} \delta l_a + \frac{\partial F_a}{\partial l_b}\Big|_{\text{equil}} \delta l_b, \tag{2.14}$$
$$F_b = F_b^{\text{equil}} + \frac{\partial F_b}{\partial l_a}\Big|_{\text{equil}} \delta l_a + \frac{\partial F_b}{\partial l_b}\Big|_{\text{equil}} \delta l_b, \tag{2.15}$$

where the equilibrium forces equal zero. Since $l_a = l_b$ at equilibrium, using Eqs. (2.3)–(2.4), we can write

$$\left(\frac{\delta F_a}{\delta l_a}\right)_{\text{equil}} = -\left(\frac{\delta F_b}{\delta l_b}\right)_{\text{equil}} = A, \tag{2.16}$$
$$\left(\frac{\delta F_a}{\delta l_b}\right)_{\text{equil}} = -\left(\frac{\delta F_b}{\delta l_a}\right)_{\text{equil}} = B. \tag{2.17}$$

Substitution of Eqs. (2.14)–(2.15) and Eqs. (2.16)–(2.17) into Eqs. (2.12)–(2.13) results

$$A\,\delta l_a + B\,\delta l_b = -q\delta E, \tag{2.18}$$
$$-B\,\delta l_a - A\,\delta l_b = -q\delta E. \tag{2.19}$$

The solutions are

$$\delta l_b = -\delta l_a = \frac{q\delta E}{A - B}, \tag{2.20}$$

where

$$A = -K - \frac{q^2}{8\pi\epsilon_0 l_a^3}, \tag{2.21}$$

$$B = -\frac{q^2}{8\pi\epsilon_0 l_a^3}. \tag{2.22}$$

Electrical Response due to the Electric Field

The electrical response is

$$\delta p = -q\delta l_a + q\delta l_b = \frac{2q^2\delta E}{A-B} = -\frac{2q^2\delta E}{K} \tag{2.23}$$

Mechanical Response due to the Electric Field

The mechanical response is

$$\delta L = -2\left|\delta l_a\right| = -\frac{2q}{K}\delta E, \tag{2.24}$$

where the minus sign is for compression.

The charged ions in Fig. 2.1 move in the direction with the application of an applied electric field just like the applied mechanical force. Using the above analysis, we can fill up the system matrix as

Constitutive Relations (Toy Model)

$$\begin{bmatrix}\delta p\\ \delta L\end{bmatrix} = \begin{bmatrix}-2q^2/K & -2q\left|\delta l_a\right|\\ -2q/K & -2\left|\delta l_a\right|\end{bmatrix}\begin{bmatrix}\delta E\\ \delta F\end{bmatrix}. \tag{2.25}$$

In this toy model, the electrical response is defined as the change in the total electric dipole moment of the system, and the mechanical response is defined as the change in the length of the system. The excitation mechanical field is provided by mechanical force and the excitation electrical stimulus is the electrical field. We can cast it similar to the constitutive relation matrix in Eq. (2.37)

$$\begin{bmatrix}\delta p\\ \delta L\end{bmatrix} = \begin{bmatrix}\chi & d\\ d & s\end{bmatrix}\begin{bmatrix}\delta E\\ \delta F\end{bmatrix}. \tag{2.26}$$

2.2. Constitutive Relations

Piezoelectricity is the field of physics that deals with the linear coupling[2] of the electric field with the acoustic field. Hooke's law states that the strain field, $\mathbf{S}$, dimensionless, is linearly proportional to the stress field, $\mathbf{T}$, in $\mathrm{N\,m^{-2}}$, or conversely that the stress is linearly proportional to the strain. In general, the stress field component is

$$T_{ij} = c_{ijkl} S_{kl}, \qquad i,j,k,l = x,y,z, \tag{2.27}$$

where summation over the repeated subscripts k and l is assumed [3]. The microscopic spring constants c_{ijkl} are called the elastic stiffness constants $\mathbf{c}$. Alternatively, the strains can be expressed as general linear function of the stresses

$$S_{ij} = s_{ijkl} T_{kl}, \qquad i,j,k,l = x,y,z. \tag{2.28}$$

The constants s_{ijkl} are called the compliance constants $\mathbf{s}$, which are measures of deformability of a medium and have large values for easily deformed materials, small values for rigid materials. Hooke's law can be written succinctly in tensor notation

$$\mathbf{T} = \mathbf{c} : \mathbf{S}, \tag{2.29}$$

$$\mathbf{S} = \mathbf{s} : \mathbf{T}. \tag{2.30}$$

Here the double scalar (or double dot) product of a fourth rank and a second rank tensor is defined by the summation over pairs of subscripts in Eq. (2.27) and Eq. (2.28).

If an elastic medium is envisioned as a collection of discrete particles, material deformation will displace an acoustic particle to the position vector ℓ from the equilibrium position $\mathbf{L}$ with respect to a reference point, see Fig. 2.2. The acoustic theory deals with macroscopic phenomena and is formulated as if matter is continuum. The particle displacement field, $\mathbf{u}$, in an elastic medium is defined as $\mathbf{u}(t,\mathbf{L}) = \ell(t,\mathbf{L}) - \mathbf{L}$. The strain $\mathbf{S}$ is a measure of material deformation and is defined in terms of particle displacement field as $\mathbf{S} = \frac{1}{2}(\nabla\mathbf{u} + \widetilde{\nabla}\mathbf{u}) = \nabla_{\mathrm{s}}\mathbf{u}$, where tilde ($\sim$) is the transposed operator and ∇_{s} is the symmetric part of the gradient of particle displacement field. Thus, the strain is the symmetric component of the the acoustic particle displacement field gradient.

Acoustic waves in a medium are described by time-varying patterns of the particle displacement field $\mathbf{u}$. The elastic wave equation is governed by the wave equation $\rho\partial_t^2\mathbf{u} = \nabla\cdot\mathbf{T}$, where ρ

[2] In electrostrictive coupling, the mechanical stress is a quadratic function of the electric field, it occurs in *all* materials.

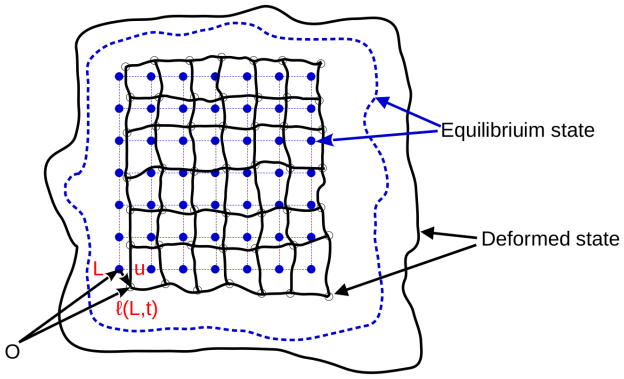


Figure 2.2.: *Equilibrium positions of acoustic particles are denoted by solid dots and object by dashed line (blue color), displaced positions are denoted by circles and deformed body by solid line (black color). Adapted from Ref. [3].*

is the material density. From Hooke's law we know $\mathbf{T} = \mathbf{c} : \mathbf{S}$, where $\mathbf{c}$ is the material stiffness tensor. We can write the acoustic wave equation as

$$\rho\partial_t^2\mathbf{u} = \nabla\cdot\mathbf{T} = \nabla\cdot(\mathbf{c} : \mathbf{S}) = \nabla\cdot(\mathbf{c} : \nabla_s\mathbf{u}). \tag{2.31}$$

Damping in a solid can be accounted for by modifying Hooke's law in the acoustic wave equation as

$$\mathbf{T} = \mathbf{c} : \mathbf{S} + \eta : \frac{\partial\mathbf{S}}{\partial t}, \tag{2.32}$$

where η is the viscosity in $\mathrm{N\,s\,m^{-2}}$.

It is well known in optics that the constitutive relations are

$$\mathbf{D}(\omega) = \epsilon(\omega)\mathbf{E}(\omega), \tag{2.33}$$
$$\mathbf{B}(\omega) = \mu(\omega)\mathbf{H}(\omega), \tag{2.34}$$

where $\mathbf{D}$ is the electric displacement, $\mathbf{E}$ the electric field, $\mathbf{B}$ the magnetic induction field, $\mathbf{H}$ the magnetic field, ϵ the permittivity, and μ is the permeability. We will not consider the magnetic constitutive relation[3] in this work but will instead assume non-magnetic materials.

In some special class of crystal symmetry, the stress field $\mathbf{T}$ of Eq. (2.32) affects the electric polarization, and hence the electric displacement $\mathbf{D}$. In piezoelectric material, it turns out that [3], [49]

$$\mathbf{D} = \epsilon^T\cdot\mathbf{E} + \mathbf{d} : \mathbf{T}, \tag{2.35}$$

[3] A similar phenomenon is dubbed as piezomagnetism.

where in ϵ^T the superscript, T, over ϵ is the permittivity at fixed stress in a mechanical medium, and $\mathbf{d}$ (unit $\mathrm{C\,N^{-1}}$), is a parameter called the piezoelectric charge constant matrix.

The strain of Eq. (2.30), in a piezoelectric medium is additionally affected by the electric field [3], [49]

$$\mathbf{S} = \mathbf{d}^t \cdot \mathbf{E} + \mathbf{s}^E : \mathbf{T}, \tag{2.36}$$

where in $\mathbf{s}^E$ the superscript E over compliance $\mathbf{s}$ is compliance at a constant electric field, and t over $\mathbf{d}$ is the transpose of the piezoelectric charge constant matrix in Voigt notation.

We found the signs, see [50], and proper notations, especially in terms of transpose, of the constitutive relations for piezoelectricity in many reference books and papers contain typos/errors, including [3], [51], except [49]. This is useful to rule out any potential errors, especially when implementing numerical model. Hence we would like write the constitutive relations handy for reference. There are four variations of the piezoelectric relation. We can cast the constitutive relations in a piezoelectric material in the self-explanatory heading in matrix form as [3], [49],

Constitutive Relations (Homogeneous Set 1)

$$\begin{bmatrix} \mathbf{D} \\ \mathbf{S} \end{bmatrix} = \begin{bmatrix} \epsilon^T & \mathbf{d} \\ \mathbf{d}^t & \mathbf{s}^E \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{T} \end{bmatrix}. \tag{2.37}$$

Constitutive Relations (Homogeneous Set 2)

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} \beta^S & -\mathbf{h} \\ -\mathbf{h}^t & \mathbf{c}^D \end{bmatrix} \begin{bmatrix} \mathbf{D} \\ \mathbf{S} \end{bmatrix}. \tag{2.38}$$

Constitutive Relations (Mixed Set 1)

$$\begin{bmatrix} \mathbf{D} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} \epsilon^S & \mathbf{e} \\ -\mathbf{e}^t & \mathbf{c}^E \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{S} \end{bmatrix}. \tag{2.39}$$

Constitutive Relations (Mixed Set 2)

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{S} \end{bmatrix} = \begin{bmatrix} \beta^T & -\mathbf{g} \\ \mathbf{g}^t & \mathbf{s}^D \end{bmatrix} \begin{bmatrix} \mathbf{D} \\ \mathbf{T} \end{bmatrix}. \tag{2.40}$$

Here $\mathbf{d}$ and $\mathbf{g}$ are strain constants; $\mathbf{e}$ and $\mathbf{h}$ are stress constants, β impermeability, β^S impermeability at constant strain, β^T impermeability at constant stress, $\mathbf{c}^D$ stiffness at constant or zero electric displacement, $\mathbf{c}^E$ stiffness at constant or zero electric field, ϵ^S permittivity at constant or zero strain, $\mathbf{s}^D$ compliance at constant electric displacement.

2.3. Tensor and Voigt Notations

We like to clearly state the notations used in this work. In other branch of physics, mathematics and engineering, these notations could cause great deal of confusion, if not clarified properly.

- Double dot product of two matrices produces a scalar result

$$A : B = A_{ij} B_{ij} \tag{2.41}$$

  One immediate usage of this is the calculation of strain energy density in small scale linear elasticity, as given by

$$\frac{1}{2}\sigma : \epsilon \tag{2.42}$$

  where σ is the stress and ϵ is the strain [52] [4].

- Gradient of a scalar

$$\nabla f(\mathbf{x}) = \frac{\partial f}{\partial x_i} = f, i \tag{2.43}$$

  It tells how much a scalar function increases along each direction, meaning the direction of greatest rate of increase.

- Gradient of a vector, results a second rank tensor

$$\nabla \mathbf{v} = v_{i,j} \tag{2.44}$$

- Gradient of a second rank tensor

$$\nabla \mathbf{A} = \frac{\partial A_{ij}}{\partial x_k} \tag{2.45}$$

- Divergence of a vector is a scalar

$$\nabla \cdot \mathbf{v} = v_{i,i} \tag{2.46}$$

[4] Typically in continuum mechanics stress is denoted by σ and strain by ϵ. See [52] for standard and well researched notation.

- Divergence of second rank tensor is a vector

$$\nabla \cdot \mathbf{A} = \frac{A_{ij}}{\partial x_j} e_i \tag{2.47}$$

- Dyadic product of two vectors is a matrix

$$\mathbf{a} \otimes \mathbf{b} = a_i b_j \tag{2.48}$$

The tensor notation is phenomenal in the sense that it is concise and it is impossible to find some relations, as a concrete example, when want to invert Hooke's Law from strain-as-a-function-of-stress to stress-as-a-function-of-strain. Such an inversion is all but impossible using matrix notation [52]. As we can see in Eqs. (2.37), Eqs. (2.38), Eqs. (2.39), and Eqs. (2.40), we have to take the transpose of d, h, e and g matrices. The concept of transpose of third rank and fourth rank tensors do not exist even. Voigt made simplifications of converting the description of stress and strain from three-dimensional, second rank tensors to six-dimensional vectors (first rank tensors), as well as considering the symmetry property the stiffness etc. matrices [53]. A concrete example of this notational rearrangement in our case is provided in §4.4. We used both the tensor and matrix notations to derive the piezoelectric wave equations in Chap. 4.

2.4. Key Terminology and Definitions in Piezoelectricity

In this Section, we collect the definitions of key parameters in piezo field. This will be helpful to gain quantitative understanding of key metrics of piezo. The strain is negative for compression and positive for tension by convention. Similarly the normal stress is negative for compression and positive for tensile normal stress.

Piezoelectric Charge Constant d

$$d = \frac{\text{Polarization generated}}{\text{Mechanical stress applied}} \tag{2.49}$$

We know the electric displacement

$$\mathbf{D} = \epsilon \mathbf{E} + \mathbf{P} \tag{2.50}$$

where $\mathbf{P}$ is permanent polarization. By applying stress, we create additional polarization in a piezo medium by Eq. (2.37).

Piezoelectric Strain Constant d

The d has another name, called the piezoelectric strain constant.

$$d = \frac{\text{Mechanical strain experienced}}{\text{Electric field applied}} \tag{2.51}$$

It will be very useful to appreciate this definition in practical term. By dimensional analysis, we find,

$$[d] = \frac{[S]}{[E]} = \frac{1}{\frac{V}{m}} = \frac{m}{V}. \tag{2.52}$$

This means how much displacement we can obtain per 1 V of applied bias.

As a piezoelectric element is anisotropic, physical constants relate to both the direction of the applied mechanical or electric force and the directions perpendicular to the applied force. Consequently, each constant generally has two subscripts that indicate the directions of the two related quantities [54]. In Fig. 2.3, $d_{\alpha\beta}$, where α is the direction of polarization generated when electric field is zero, β direction of applied stress. Alternatively, α can mean the direction of applied electric field strength and β is the direction of induced strain. As an example, d_{15} means induced polarization in direction 1 per unit shear stress applied about direction 2, or induced shear strain about direction 2 per unit electric field applied in direction 1.

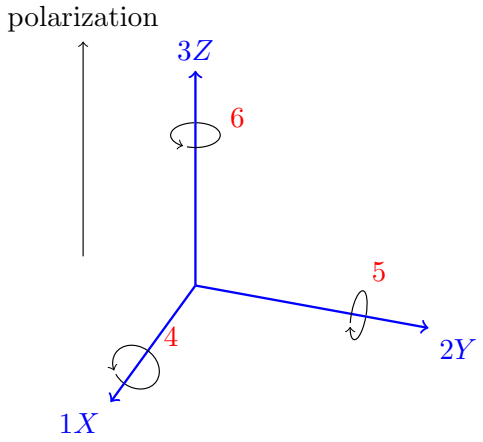


Figure 2.3.: *Direction of forces affecting a piezoelectric element. Adapted from [54], drawn with the Community Support of TEXStackExchange, [55].*

Piezoelectric Stress Constant e

$$e = \frac{\text{Stress developed}}{\text{Applied electric field}} \tag{2.53}$$

We also know that e is related to d via the relation

Relation between Piezoelectric Stress & Strain Constants

$$\mathbf{e} = \mathbf{d} : \mathbf{c} \tag{2.54}$$

We shall use this relation extensively in this thesis.

Piezoelectric Voltage Constant g

$$g = \frac{\text{Electric field generated}}{\text{Mechanical stress applied}} \tag{2.55}$$

Another definition is

$$g = \frac{\text{Mechanical strain experienced}}{\text{Electric displacement applied}} \tag{2.56}$$

The piezoelectric voltage constant has two subscripts $g_{\alpha\beta}$, where α direction of electric field generated, β is direction of stress applied. It can also mean α direction of applied electric displacement and β direction of induced strain. As an example, g_{15} means induced electric field in direction 1 per unit shear stress applied about direction 2, or induced shear strain about direction 2 per unit electric displacement indirection 1.

Piezoelectric Stress Constant h

$$\frac{1}{h} = \frac{\text{Strain developed}}{\text{Applied electric field}} \tag{2.57}$$

We find it useful to the definitions of the four piezo constants handy for reference, as shown in Eqs. (2.58)– (2.61).

Definitions of Piezoelectric Constants at a Glance

$$d_{ij} = \left(\frac{\partial D_i}{\partial T_j}\right)^E = \left(\frac{\partial S_j}{\partial E_i}\right)^T \quad \text{Strain - Charge,} \tag{2.58}$$

$$e_{ij} = \left(\frac{\partial D_i}{\partial S_j}\right)^E = -\left(\frac{\partial T_j}{\partial E_i}\right)^S \quad \text{Stress - Charge,} \tag{2.59}$$

$$g_{ij} = -\left(\frac{\partial E_i}{\partial T_j}\right)^D = \left(\frac{\partial S_j}{\partial D_i}\right)^T \quad \text{Strain - Voltage,} \tag{2.60}$$

$$h_{ij} = -\left(\frac{\partial E_i}{\partial S_j}\right)^D = -\left(\frac{\partial T_j}{\partial D_i}\right)^S \quad \text{Stress - Voltage.} \tag{2.61}$$

We list the ranks, and symbols of each physical quantity used in this thesis (with some exceptions) in Tab. 2.1 and Tab. 2.2 in the case of 3D and Tab. 2.3 in the case of 2D.

Tensor	Rank	Symbol	SI Units
Elastic strain	2nd	$\mathcal{S}_{ij}$	Dimensionless
Elastic stress	2nd	$\mathcal{T}_{ij}$	Pa
Electric field	1st	E_i	$\mathrm{V\,m^{-1}}$
Electric displacement	1st	D_i	$\mathrm{C\,m^{-2}}$
Dielectric permittivity	2nd	ϵ_{ij}	$\mathrm{F\,m^{-1}}$
Dielectric polarization	1st	P_i	$\mathrm{C\,m^{-1}}$
Elastic stiffness	4th	c_{ijkl}	Pa
Elastic compliance	4th	s_{ijkl}	$\mathrm{Pa^{-1}}$
Viscosity	4th	η_{ijkl}	$\mathrm{N\,s\,m^{-1}}$
Piezoelectric strain	3rd	d_{ijk}	$\mathrm{C\,N^{-1}}$ or $\mathrm{m\,V^{-1}}$
Piezoelectric stress	3rd	e_{ijk}	$\mathrm{C\,m^{-2}}$ or $\mathrm{N\,V^{-1}\,m^{-1}}$
Piezoelectric strain	3rd	g_{ijk}	$\mathrm{V\,m\,N^{-1}}$ or $\mathrm{m^2\,C^{-1}}$
Piezoelectric stress	3rd	h_{ijk}	$\mathrm{V\,m^{-1}}$ or $\mathrm{N\,C^{-1}}$

Table 2.1.: *Tensor ranks and symbols in 3D.*

Physical Quantity	Matrix/Vector	Symbol	SI Units
Elastic strain	(3×3)	$\mathbf{S}$	Dimensionless
Elastic stress	(3×3)	$\mathbf{T}$	Pa
Electric field	(3×1)	$\mathbf{E}$	$\mathrm{V\,m^{-1}}$
Electric displacement	(3×1)	$\mathbf{D}$	$\mathrm{C\,m^{-2}}$
Mechanical displacement	(3×1)	$\mathbf{u}$	m
Dielectric permittivity	(3×3)	ϵ	$\mathrm{F\,m^{-1}}$
Dielectric polarization	(3×1)	$\mathbf{P}$	$\mathrm{C\,m^{-1}}$
Elastic stiffness	(6×6)	$\mathbf{c}$	Pa
Elastic compliance	(6×6)	$\mathbf{s}$	$\mathrm{Pa^{-1}}$
Viscosity	(6×6)	η	$\mathrm{Pa^{-1}}$
Piezoelectric strain	(3×6)	$\mathbf{d}$	$\mathrm{C\,N^{-1}}$ or $\mathrm{m\,V^{-1}}$
Piezoelectric stress	(3×6)	$\mathbf{e}$	$\mathrm{C\,m^{-2}}$ or $\mathrm{N\,V^{-1}\,m^{-1}}$
Piezoelectric strain	(3×6)	$\mathbf{g}$	$\mathrm{V\,m\,N^{-1}}$ or $\mathrm{m^2\,C^{-1}}$
Piezoelectric stress	(3×6)	$\mathbf{h}$	$\mathrm{V\,m^{-1}}$ or $\mathrm{N\,C^{-1}}$

Table 2.2.: *Physical quantities in 3D.*

Physical Quantity	Matrix/Vector	Symbol	SI Units
Elastic strain	(2×2)	$\mathbf{S}$	Dimensionless
Elastic stress	(2×2)	$\mathbf{T}$	Pa
Electric field	(2×1)	$\mathbf{E}$	$\mathrm{V\,m^{-1}}$
Electric displacement	(2×1)	$\mathbf{D}$	$\mathrm{C\,m^{-2}}$
Mechanical displacement	(2×1)	$\mathbf{u}$	m
Dielectric permittivity	(2×2)	ϵ	$\mathrm{F\,m^{-1}}$
Dielectric polarization	(2×1)	$\mathbf{P}$	$\mathrm{C\,m^{-1}}$
Elastic stiffness	(3×3)	$\mathbf{c}$	Pa
Elastic compliance	(3×3)	$\mathbf{s}$	$\mathrm{Pa^{-1}}$
Viscosity	(3×3)	η	$\mathrm{Pa^{-1}}$
Piezoelectric strain	(2×3)	$\mathbf{d}$	$\mathrm{C\,N^{-1}}$ or $\mathrm{m\,V^{-1}}$
Piezoelectric stress	(2×3)	$\mathbf{e}$	$\mathrm{C\,m^{-2}}$ or $\mathrm{N\,V^{-1}\,m^{-1}}$
Piezoelectric strain	(2×3)	$\mathbf{g}$	$\mathrm{V\,m\,N^{-1}}$ or $\mathrm{m^2\,C^{-1}}$
Piezoelectric stress	(2×3)	$\mathbf{h}$	$\mathrm{V\,m^{-1}}$ or $\mathrm{N\,C^{-1}}$

Table 2.3.: *Physical quantities in 2D.*

Chapter 3

3 One-Dimensional Piezorope

In this chapter, the artificial piezoelectric behavior will be devised from the first principles in one-dimension. The equilibrium mechanical displacement of a finite length piezorope due to DC bias will be formulated, followed by dynamical equation of motion. The electromechanical responses of a finite length piezorope will be derived to form constitutive relations under harmonic excitation. We analyze acoustic wave transduction in 1D phononic crystal, made of centro-symmetric materials, e. g., silicon, by constructing two such artificial dipoles in opposing configuration. The electromechanical coupling coefficient will be deduced as a key metric to quantify the system efficiency. The bandstructure of infinite periodic structures will be calculated. In order to compare the artificial piezoelectricity with the natural piezoelectricity, the piezoelectric stress e-constant will be derived for the artificial case. The 1D structure will be simulated by the finite element software FlexPDE [1].

The piezoelectric behavior is expressed by the constitutive relations in Eq. (2.37). In centro-symmetric crystals, e. g., silicon, there are no in-built direct and converse piezoelectric effects. Here we attempt to emulate these effects in a phononic crystal (PnC) by overlaying it with a network of metal electrodes to generate a built-in field within each unit cell of the PnC. This built-in field along with the strain field in the PnC give rise to electromechanical responses which mimic the piezoelectric effects.

We shall start in §3.1 by analyzing the mechanical and electric responses of a 1D piezorope to applied DC bias (electric stimulus) and stress (mechanical stimulus). This simple model allows us to establish both the converse and direct piezo effects in a 1D piezorope in an electric potential. We then derive the responses of a 1D piezorope to harmonic excitation using the acoustic wave equation in §3.2. For such a harmonic excitation, we shall develop constitutive relations to express artificial piezoelectricity in a one dimensional piezorope made of generic centro-symmetric crystals in §3.3.

3.1. Change in Equilibrium Length of a 1D Bar due to External Force

We consider a 1D bar with natural length L in the absence of an applied voltage or external mechanical force. The bar has length L, width W, and height H. The bar can be considered 1D if $L \gg W$, and $L \gg H$; hence such a 1D bar is dubbed as (piezo)rope. Suppose the bar is subject to external forces F_0 applied on the left and right facets of the bar, as shown in Fig. 3.1a. These forces may represent electrostatic forces arising from an applied voltage across the bar, or mechanical forces. We wish to determine the change in length of the bar due to these applied forces.

The traction vector, $\mathcal{T}$, is simply the force vector on a cross-section divided by that cross-section's area, A, [52],

$$\mathcal{T} = \frac{\mathbf{F}}{A}. \tag{3.1}$$

The traction vector is related to the stress tensor $\mathbf{T}$ via

$$\mathcal{T} = \mathbf{T} \cdot \hat{n}, \tag{3.2}$$

where $\hat{n} = (\cos(\theta), \sin(\theta))$ is a unit normal to the surface, which is inclined at an angle θ with respect to the positive x-axis, that $\mathcal{T}$ is acting on. In tensor notation,

$$\mathcal{T}_i = T_{ij} n_j. \tag{3.3}$$

The normal stress σ on a surface is related to the traction vector by

$$\sigma = \mathcal{T} \cdot \hat{n}. \tag{3.4}$$

It has to be borne in mind that the normal stresses are just the scalar quantities on the surface, not a full tensor. Substituting Eq. (3.2) into Eq. (3.4), we obtain

$$\sigma = \hat{n} \cdot \mathbf{T} \cdot \hat{n}. \tag{3.5}$$

In tensor notation,

$$\sigma = T_{ij} n_i n_j. \tag{3.6}$$

The applied normal forces, N, cause an internal stress field to develop inside the bar. By convention, the stress field is negative for compression and positive for tension. Consider now an internal segment of length dx inside the bar, as shown in Fig. 3.1b. The direction of the traction vector is always the same as internal force vector [52]. The normal forces $N(x)$ and

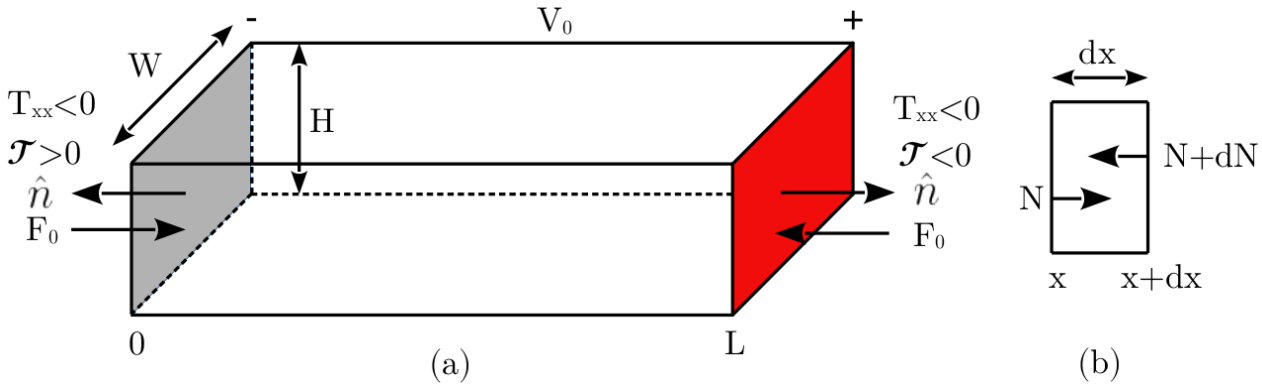


Figure 3.1.: *One-dimensional piezoelectric model, a bar of length L, height H, and width W. In the case of 1D, $L \gg H$, $L \gg W$. (a) 1D bar of length L, DC voltage V_0 applied across two ends of the bar. The correct signs of traction vector $\mathcal{T}$, normal stress component T_{xx}, unit normal $\hat{\mathbf{n}}$, and (DC Coulomb) force F_0 on each facet are labelled. (b) normal forces acting on an internal segment dx.*

$N(x+dx)$ acting on the left and right facets of the segment are given by

$$N(x) = \mathcal{T}_x(x)A = -T_{xx}(x)A, \qquad \text{as } \sigma = \mathcal{T}_x = -T_{xx} \text{ at the left surface.} \tag{3.7}$$

$$N(x+dx) = \mathcal{T}_x(x+dx)A = T_{xx}(x+dx)A, \qquad \text{as } \sigma = \mathcal{T}_x = T_{xx} \text{ at the right surface.} \tag{3.8}$$

At equilibrium, the net normal force N acting on the segment is zero:

$$\begin{aligned} & N(x) - N(x+dx) = 0 \\ \Rightarrow\ & N(x) - \left[N(x) + dx\frac{dN}{dx}\right] = 0 \\ \Rightarrow\ & \frac{dN}{dx} = 0 \\ \therefore\ & N(x) = \text{constant} = F_0. \end{aligned} \tag{3.9}$$

From Hooke's law relating the stress T_{xx} to the strain S_{xx}

$$T_{xx} = c_{11}S_{xx} = c_{11}\frac{\partial u}{\partial x}, \tag{3.10}$$

where c_{11} is the elastic stiffness constant, we obtain the constitutive relation

$$S_{xx} = \frac{T_{xx}}{c_{11}} = \frac{N}{c_{11}A}. \tag{3.11}$$

Now since the strain S_{xx} is related to the displacement field as

$$S_{xx} = \frac{\partial u}{\partial x}, \Rightarrow \int du = \int S_{xx}\, dx, \Rightarrow u(x) - u(0) = \int_0^x S_{xx}\, dx, \tag{3.12}$$

the net elongation/contraction of the 1D piezo bar is

$$\Delta L = u(L) - u(0) = \int_0^L S_{xx}\, dx. \tag{3.13}$$

Using Eq. (3.11) and Eq. (3.9), we obtain the change in the equilibrium length of the bar due to the applied force F_0 as

$$\Delta L = \int_0^L \frac{N}{c_{11}A}\, dx = \int_0^L \frac{F_0}{c_{11}A}\, dx = \frac{F_0 L}{c_{11}A}. \tag{3.14}$$

Therefore, the change in the equilibrium/natural length, ΔL, due to an applied force is

Change in Natural Length due to Applied Force F_0

$$\Delta L = \frac{F_0 L}{c_{11}A}. \tag{3.15}$$

This simply implies that the larger the force, the higher the shrinkage of the bar, the less stiff the material is, the higher the compliance, resulting in larger shrinkage of the bar.

3.1.1. Change in Length due to Electrostatic Force

We next consider the case where the applied force is due to a DC voltage difference V_0 applied between the left and right ends of the bar. Treating the bar as a parallel plate capacitor with the two facets as the electrodes, the capacitance C is given by

$$C = \frac{\epsilon A}{\ell}, \tag{3.16}$$

where ℓ is the length of the bar ($\ell = L + \Delta L$) and ϵ is the electric permittivity of the material. The electrostatic energy U in the capacitor is

$$U = \frac{1}{2}CV^2 = \frac{\epsilon A}{2\ell}V^2. \tag{3.17}$$

The electrostatic force acting on the facets of the bar can be calculated from

$$F_0 = \frac{\partial U}{\partial \ell} = -\frac{\epsilon A}{2\ell^2} V^2. \tag{3.18}$$

At equilibrium, we have

$$F_0 = -\frac{\epsilon A V_0^2}{2(L+\Delta L)^2}. \tag{3.19}$$

From Eq. (3.15), the change in length at equilibrium is

$$\Delta L = \frac{F_0 L}{c_{11} A} = -\frac{\epsilon A V_0^2}{2(L+\Delta L)^2} \cdot \frac{L}{c_{11} A} = -\frac{\epsilon L}{2c_{11}(L+\Delta L)^2} \cdot V_0^2, \tag{3.20}$$

which can be expressed as

$$\left(\frac{\Delta L}{L}\right)^3 + 2\left(\frac{\Delta L}{L}\right)^2 + \frac{\Delta L}{L} + \frac{\epsilon}{2c_{11}} \cdot \frac{V_0^2}{L^2} = 0. \tag{3.21}$$

Solution of the above cubic equation gives the equilibrium length L_{eq} of the bar, $L_{\text{eq}} = L + \Delta L$, in the presence of an applied voltage.

We can obtain an approximate expression for the change in length, ΔL, of the bar by linearizing the electrostatic force around the natural length L to get

$$F_0 = -\frac{\epsilon A V_0^2}{2(L+\Delta L)^2} = -\frac{\epsilon A V_0^2}{2L^2\left(1+\frac{\Delta L}{L}\right)^2} \approx -\frac{\epsilon A V_0^2}{2L^2}\left(1 - 2\frac{\Delta L}{L}\right), \tag{3.22}$$

The change in length given by Eq. (3.15) becomes

$$\begin{aligned} \Delta L = \frac{F_0 L}{c_{11} A} &\approx -\frac{\epsilon A V_0^2}{2L^2}\left(1 - 2\frac{\Delta L}{L}\right) \cdot \frac{L}{c_{11} A} \\ \Rightarrow \frac{\Delta L}{L} &= \frac{1}{2\left(1 - \frac{c_{11} L^2}{\epsilon V_0^2}\right)}. \\ \Rightarrow \Delta L &\approx -\frac{1}{2}\frac{\epsilon V_0^2}{c_{11} L}, \quad \text{when } \frac{c_{11} L^2}{\epsilon V_0^2} \gg 1. \end{aligned} \tag{3.23}$$

We summarize above results at equilibrium condition explicitly as function of DC voltage V_0

Equilibrium Parameters due to DC Voltage V_0

$$\Delta L \approx -\frac{1}{2}\frac{\epsilon V_0^2}{c_{11}L}, \tag{3.24a}$$

$$L_{\text{eq}} \approx L - \frac{1}{2}\frac{\epsilon V_0^2}{c_{11}L}, \tag{3.24b}$$

$$F_0 \approx -\frac{\epsilon A}{2L^2}V_0^2\left(1 + \frac{\epsilon V_0^2}{c_{11}L^2}\right). \tag{3.24c}$$

To a good approximation, the change in length of a bar, ΔL, varies with the square of DC voltage and a negative sign corresponds to compression of the bar. This initial DC bias is evidently a strict electrostriction effect, meaning the change in length follows parabolic voltage dependence and it occurs in *all* materials. We create artificial dipole using such electrode scheme in 1D, although there could be other electrode arrangements in 2D or 3D. We bias the bar at V_0, and operate around the neighborhood of V_0, thus linearize the response, see Fig. 2 in [32].

3.1.2. Change in Electric Field due to Change in Length (Backaction)

Due to the change in length ΔL of the bar, there is a corresponding change in the electric field between the two facets of the bar given by

$$E = -\frac{V_0}{L + \Delta L} = -\frac{V_0}{L\left(1 + \frac{\Delta L}{L}\right)} \approx -\frac{V_0}{L}\left(1 - \frac{\Delta L}{L}\right). \tag{3.25}$$

Writing the total electric field as $E = E_0 + \Delta E$, where the E_0 is the electric field before the length change,

$$E_0 = -\frac{V_0}{L}, \tag{3.26}$$

we obtain the change in the electric field due to the change in the bar length,

$$\Delta E = -\frac{\Delta L}{L}E_0. \tag{3.27}$$

Hence we can express

Change in Electric Field due to Change in Length (Backaction)

$$\frac{\Delta E}{E_0} = -\frac{\Delta L}{L}. \tag{3.28}$$

Thus the change in the electric field is proportional to the change in length of the 1D bar. The above equation expresses the direct piezoelectric effect of the bar: a mechanical change in the length of the bar generates a change in the electric field in the bar.

3.2. Dynamical Equation of Motion of Acoustic Field Displacement

Up to now we have only considered the electromechanical responses of the 1D bar to a static stimulus (DC voltage or a constant change in length). In this Section, we shall derive the dynamical equation of motion for the acoustic displacement field inside the bar due to harmonic voltage excitation. We shall solve this acoustic wave equation in later sections for the acoustic wave transduction problem.

Fig. 3.2a shows a 1D bar subject to a DC bias voltage V_0 and a modulation voltage $\delta V(t)$. The equilibrium electrostatic force F_0 due to the DC bias voltage V_0 is already accounted for by the equilibrium length $L_{\text{eq}} = L + \Delta L$, where ΔL is given by Eq. (3.24a). The modulation voltage $\delta V(t)$ gives rise to a perturbative electrostatic force δF_{e}. We also allow for an external modulating mechanical force acting at the end faces of the bar denoted by δF_{m}.

The equation of motion for an internal segment dx in Fig. 3.2b, using Newton's second law of motion, is

$$\begin{aligned} & N(x+dx,t) - N(x,t) = \rho\, dx A \frac{\partial^2 u}{\partial t^2} \\ \Rightarrow\ & \frac{N(x+dx,t) - N(x,t)}{dxA} = \rho \frac{\partial^2 u}{\partial t^2} \\ \Rightarrow\ & \frac{1}{A}\frac{\partial N}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}. \end{aligned} \tag{3.29}$$

Here N is the normal force acting on an internal segment dx, and the signs of $N(x+dx,t)$ and $N(x,t)$ are shown in Fig. 3.2b for tensile force. Using $N = T_{xx}A$, Eq. (3.29) becomes

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial T_{xx}}{\partial x}. \tag{3.30}$$

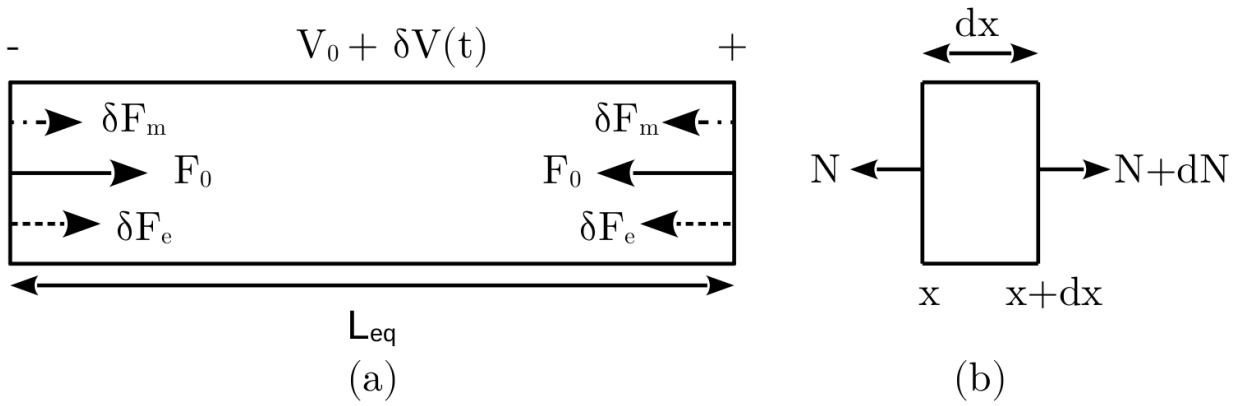


Figure 3.2.: *(a) A 1D bar with an applied DC bias voltage V_0 and modulation voltage $\delta V(t)$. F_0 is the electrostatic force resulting from the DC bias voltage V_0, δF_m is the applied mechanical force, and δF_e is the electrostatic force resulting from the modulation voltage $\delta V(t)$. (b) Internal segment dx of the bar.*

Finally with the use of Hooke's law in Eq. (3.10), we obtain

$$\rho \frac{\partial^2 u}{\partial t^2} = c_{11} \frac{\partial^2 u}{\partial x^2}. \tag{3.31}$$

The above equation describes the motion of the acoustic displacement field inside the bar. The external forces δF_e and δF_m can be accounted for by adding appropriate body force terms to the above equation or they can be treated as forcing boundary conditions on the solution.

3.3. Electromechanical Responses of the 1D Piezorope

For a 1D bar of finite length, we wish to establish the piezoelectric constitutive relations, analogous to Eq. (2.35) and Eq. (2.36), under sinusoidal excitation.

We solve Eq. (3.31) subject to the boundary condition that the stress at the both ends of the bar is equal to the applied force per unit area, $(\delta F_\mathrm{m} + \delta F_\mathrm{e})/A$. If we assume harmonic response at frequency ω [1], then we can write

$$u(x,t) = u(x) e^{i\omega t}. \tag{3.32}$$

[1] Valid as we operate around the neighborhood of the DC bias V_0 and thus linearized the system output, i. e., strain or displacement.

Eq. (3.31) becomes

$$\begin{aligned} -\rho\omega^2 u &= c_{11}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \\ \Rightarrow \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} &+ \frac{\rho\omega^2}{c_{11}}u(x) = 0. \end{aligned} \tag{3.33}$$

The general solution is

$$u(x) = A\cos(kx) + B\sin(kx), \tag{3.34}$$

where the continuous wavevector k is

$$k = \omega\sqrt{\frac{\rho}{c_{11}}}. \tag{3.35}$$

Using Eq. (3.10), the stress T_{xx} is

$$T_{xx} = c_{11}\frac{\partial u}{\partial x} = -c_{11}kA\sin(kx) + c_{11}kB\cos(kx). \tag{3.36}$$

The boundary conditions for a compressive stress field applied at the ends of the bar, see Fig. 3.3, are

$$\text{at } x = 0, \qquad T_{xx}(0) = -T_{\mathrm{a}}, \tag{3.37}$$

$$\text{at } x = L_{\mathrm{eq}}, \qquad T_{xx}(L_{\mathrm{eq}}) = -T_{\mathrm{a}}, \tag{3.38}$$

where T_a is the total stress at the end facets due to both electric force and mechanical force on cross-sectional area A of the bar,

$$T_{\mathrm{a}} = \frac{\delta F_{\mathrm{e}} + \delta F_{\mathrm{m}}}{A}, \tag{3.39}$$

Using Eq. (3.36) and the boundary conditions, the stress at the boundaries are

$$T_{xx}(0) = c_{11}kB = -T_{\mathrm{a}}, \tag{3.40}$$

$$T_{xx}(L_{\mathrm{eq}}) = -c_{11}kA\sin(kL_{\mathrm{eq}}) + c_{11}kB\cos(kL_{\mathrm{eq}}) = -T_{\mathrm{a}}. \tag{3.41}$$

Solving for the coefficients A and B, we get

$$B = -\frac{T_{\mathrm{a}}}{c_{11}k}, \tag{3.42}$$

$$A = \frac{T_{\mathrm{a}}}{c_{11}k}\left[\csc(kL_{\mathrm{eq}}) - \cot(kL_{\mathrm{eq}})\right]. \tag{3.43}$$

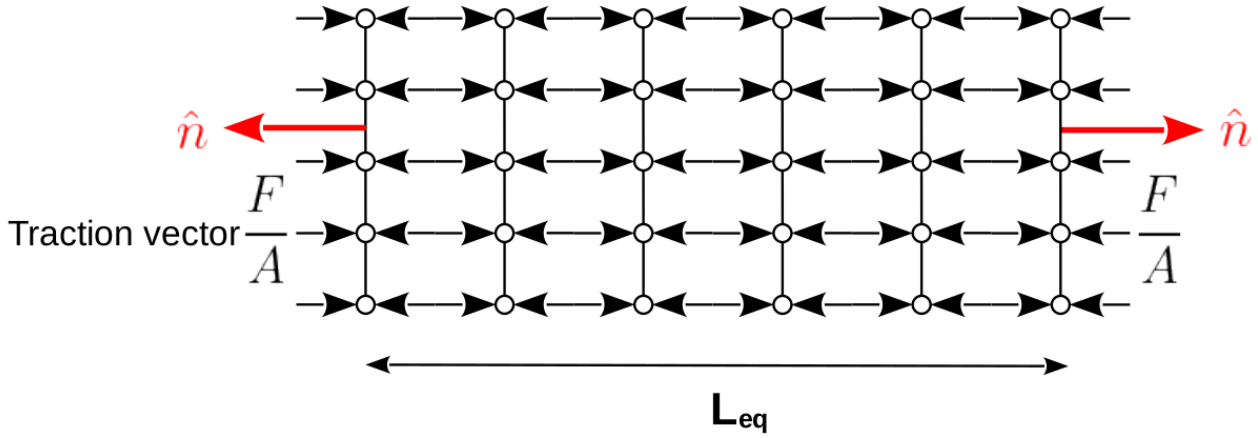


Figure 3.3.: *The circles denote the acoustic particles in an elastic medium, and for illustration purpose, the compressive traction vector is used here. The internal traction vectors (having unit of force per unit area) cancel out each other, only the traction vectors at the end facets survive which give rise to the boundary conditions.*

Thus the acoustic particle displacement field for a time-harmonic excitation is

Acoustic Displacement for Harmonic Excitation

$$u(x) = \frac{T_\mathrm{a}}{c_{11}k}\left[\alpha\cos(kx) - \sin(kx)\right], \tag{3.44}$$

where

$$\alpha = \tan\left(\frac{kL_\mathrm{eq}}{2}\right). \tag{3.45}$$

The dynamical change in length δL of the bar due to a time-harmonic excitation (as opposed to the length change ΔL due to a DC bias voltage V_0 in Eq. (3.24a)) is

$$\begin{aligned}\delta L &= u(L_\mathrm{eq}) - u(0)\\ &= \frac{T_\mathrm{a}}{c_{11}k}\left[\alpha\cos(kL_\mathrm{eq}) - \sin(kL_\mathrm{eq})\right] - \frac{T_\mathrm{a}}{c_{11}k}\alpha\\ &= \frac{T_\mathrm{a}\beta}{c_{11}k},\end{aligned} \tag{3.46}$$

where [2]

$$\beta = \alpha \left(\cos(kL_{\rm eq}) - 1\right) - \sin(kL_{\rm eq}). \tag{3.47}$$

Therefore, the change in length of the 1D piezorope due to a sinusoidal stress is

$$\delta L = \frac{T_a \beta}{c_{11} k}. \tag{3.48}$$

In the above expression, the stress T_a can be due to an applied mechanical force or an applied electric force, or both, at the two ends of the bar.

3.3.1. Responses due to Applied Mechanical Force

Mechanical response:

If only mechanical force is applied, i. e., $\delta F_{\rm e} = 0$, in which case the applied stress is

$$T_{\rm a} = \frac{\delta F_{\rm m}}{A}. \tag{3.49}$$

The mechanical response using Eq. (3.48) is

$$\delta L = \frac{\beta}{c_{11} k A} \delta F_{\rm m}. \tag{3.50}$$

This is the constitutive relation for the mechanical response due to an applied mechanical force on the bar.

Electrical response:

We define the electrical response as the change in the electric field, δE, when there is an applied mechanical force $\delta F_{\rm m}$ or applied modulation voltage δV. Here we calculate the electrical response due to mechanical force only, i. e., $\delta F_{\rm e} = 0$. The total electric field is (assuming $L \ll \lambda$)

$$E = -\frac{V_0}{\ell} = -\frac{V_0}{L_{\rm eq} + \delta L} = -\frac{V_0}{L_{\rm eq}\left(1 + \frac{\delta L}{L_{\rm eq}}\right)} \approx -\frac{V_0}{L_{\rm eq}} \left(1 - \frac{\delta L}{L_{\rm eq}}\right). \tag{3.51}$$

[2] Here α, β are dummy variables, in contrast to §2.4.

Writing the total electric field as $E = E_{\mathrm{dc}} + \delta E$, where the static electric field is

$$E_{\mathrm{dc}} = -\frac{V_0}{L_{\mathrm{eq}}}, \tag{3.52}$$

the electric field perturbation is then

$$\delta E = \frac{V_0}{L_{\mathrm{eq}}^2}\delta L. \tag{3.53}$$

The dynamical change in length δL in this case is provided by Eq. (3.50). The electrical response due to mechanical excitation in explicit form is

$$\delta E = \frac{V_0}{L_{\mathrm{eq}}^2}\frac{\beta}{c_{11}kA}\delta F_{\mathrm{m}}. \tag{3.54}$$

The above relation expresses the direct piezoelectric effect, i. e., the application of a mechanical stress or force induces a change in the internal electric field. We note that in order to observe piezoelectricity, we *must* require a DC bias voltage V_0, we can say that this is electrostriction-induced linear piezoelectric effect. If the DC bias voltage V_0 vanishes, there is no piezo effect, even if we apply any amount of mechanical force or stress. By using an appropriate phononic crystal design with an applied DC potential distribution over the lattice to mimic the built-in field in natural piezoelectric materials, we should be able to generate the piezoelectric effect in centro-symmetric materials such as silicon.

3.3.2. Responses due to Applied Electric Force

We consider the responses of the 1D bar to the electrical force perturbation only, i. e., by setting the mechanical force $\delta F_{\mathrm{m}} = 0$. The applied stress in this case is

$$T_{\mathrm{a}} = \frac{\delta F_{\mathrm{e}}}{A}. \tag{3.55}$$

The electric force depends on the length change as well as the potential difference. From Eq. (3.18), the total electric force is

$$\begin{aligned} F_{\mathrm{e}} &= -\frac{\epsilon A}{2\ell^2}V^2, \\ \Rightarrow F_0 + \delta F_{\mathrm{e}} &= -\frac{\epsilon A}{2\left(L_{\mathrm{eq}} + \delta L\right)^2}\left(V_0 + \delta V\right)^2. \end{aligned} \tag{3.56}$$

By linearizing the above equation in terms of the small perturbations δL, and only keeping terms linear in δL and δV, we obtain

$$F_0 + \delta F_\mathrm{e} \approx -\frac{\epsilon A}{2L_\mathrm{eq}^2}\left(V_0^2 - 2\frac{\delta L}{L_\mathrm{eq}}V_0^2 + 2V_0\delta V\right). \tag{3.57}$$

The first term on the right hand side is the equilibrium force F_0. The perturbation in the electrical force is thus

$$\delta F_\mathrm{e} = 2F_0\left(\frac{\delta V}{V_0} - \frac{\delta L}{L_\mathrm{eq}}\right). \tag{3.58}$$

Mechanical response:

Using Eq. (3.48), the mechanical response of the 1D piezobar due to the voltage perturbation is

$$\begin{aligned}
\delta L &= \frac{T_\mathrm{a}\beta}{c_{11}k} \\
&= \frac{\beta}{c_{11}kA}\delta F_\mathrm{e} \\
&= \frac{\beta}{c_{11}k}\cdot\frac{2F_0}{A}\cdot\left(\frac{\delta V}{V_0} - \frac{\delta L}{L_\mathrm{eq}}\right) \\
\Rightarrow \delta L\left(1 + \frac{2\beta}{c_{11}k}\cdot\frac{F_0}{AL_\mathrm{eq}}\right) &= \frac{2\beta}{c_{11}k}\cdot\frac{F_0}{A}\cdot\frac{\delta V}{V_0}.
\end{aligned}$$

Hence the change in length due to the applied voltage modulation is

$$\delta L = \frac{\frac{2\beta}{c_{11}k}\cdot\frac{F_0}{A}}{\left(1 + \frac{2\beta}{c_{11}k}\cdot\frac{F_0}{AL_\mathrm{eq}}\right)}\cdot\frac{\delta V}{V_0}. \tag{3.59}$$

The above equation directly relates δL to δV and provides the basis for the converse piezoelectric effect of the bar, i. e., by the application of an electric field or voltage there is an elongation or contraction of a piezo material. It has to be noted that if there is vanishing DC bias voltage V_0, there is no converse piezo effect. Again, we have to have a DC bias in our meta-atom to mimic piezoelectricity behavior.

Electrical response:

For the 1D bar, an applied voltage will affect the electric field via two mechanisms: the change in the voltage and the induced change in the length of bar due to the electric force:

$$E = -\frac{V}{\ell} = -\frac{V_0 + \delta V}{L_{\text{eq}} + \delta L} \tag{3.60}$$

Linearizing the above equation in δV and δL, we get

$$E \approx -\frac{1}{L_{\text{eq}}}\left(V_0 + \delta V - \frac{V_0 \delta L}{L_{\text{eq}}}\right). \tag{3.61}$$

Using Eq. (3.52), the electrical response due to a voltage perturbation is

$$\delta E = -\frac{\delta V}{L_{\text{eq}}} + \frac{V_0}{L_{\text{eq}}^2}\delta L. \tag{3.62}$$

Using Eq. (3.59) to relate δL to δV, we obtain

$$\delta E = \left(\frac{1}{L_{\text{eq}}^2} \cdot \frac{\frac{2\beta}{c_{11}k} \cdot \frac{F_0}{A}}{1 + \frac{2\beta}{c_{11}k} \cdot \frac{F_0}{AL_{\text{eq}}}} - \frac{1}{L_{\text{eq}}}\right)\delta V. \tag{3.63}$$

3.3.3. Artificial Piezo Matrix (under Harmonic Excitation)

We can summarize the above electromechanical responses of the 1D bar in the form of a piezoelectric matrix,

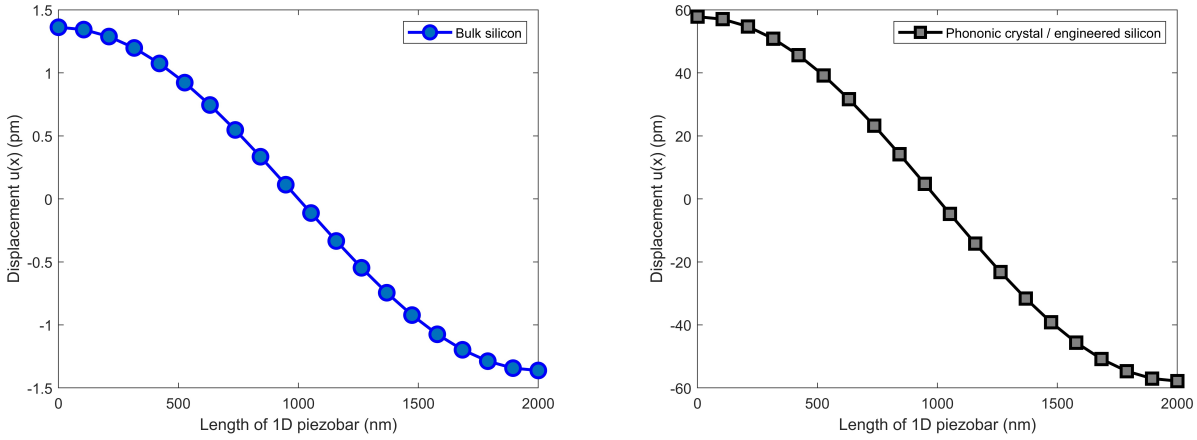
Constitutive Relations for Artificial Piezo in 1D

$$\begin{pmatrix} \delta E \\ \delta L \end{pmatrix} = \begin{pmatrix} \frac{1}{L_{\text{eq}}^2} \cdot \frac{\frac{2\beta}{c_{11}k} \cdot \frac{F_0}{A}}{1 + \frac{2\beta}{c_{11}k} \cdot \frac{F_0}{AL_{\text{eq}}}} - \frac{1}{L_{\text{eq}}} & \frac{V_0}{L_{\text{eq}}^2} \cdot \frac{\beta}{c_{11}kA} \\ \frac{\frac{2\beta}{c_{11}k} \cdot \frac{F_0}{A}}{1 + \frac{2\beta}{c_{11}k} \cdot \frac{F_0}{AL_{\text{eq}}}} \cdot \frac{1}{V_0} & \frac{\beta}{c_{11}kA} \end{pmatrix} \begin{pmatrix} \delta V \\ \delta F_m \end{pmatrix} \tag{3.64}$$

The above matrix is analogous to the natural piezoelectric constitutive tensor in Eq. (2.37).

3.3.4. Numerical Calculations

We consider a bar of silicon with physical length $L = 2$ µm. For a DC voltage of $V_0 = 10$ V, modulation voltage $\delta V = 0.1$ V, the electrostatic force is $F_0 \sim 12.94$ pN for the cross-sectional area of 100 nm by 100 nm. The displacement along the length of the silicon bar at $f = 2.1297$ GHz, driven slightly off the resonance frequency $f_1 = v_{\text{sound}}/2L$ is shown in Fig. 3.4a using bulk silicon $c_{11} = 169$ GPa, $\rho = 2329$ kg m^{-3}. In Fig. 3.4b, using the stiffness of silicon one order of magnitude lower by phononic engineering, i. e., $c_{11} = 16.9$ GPa, and $\rho = 2000$ kg m^{-3} (to account for the holey phononic structure as an illustration), the acoustic displacement is shown at $f_1 = v_{\text{sound}}/2L = 0.72673$ GHz, the frequency decreases as $v_{\text{sound}} = \sqrt{c_{11}/\rho}$. The rest of the (bulk) silicon material property parameters are provided in Chap. A. Since we have applied a compressive force, we expect the left side of the bar to be squeezed in the right direction and the right side to be displaced toward the left, while the middle of the bar experience zero displacement. This is exactly what the computed displacement profile shows in Fig. 3.4. We clearly observe that the magnitude is increased by ~ 40 fold in phononic crystal in compared to the bulk silicon. We can also use porous silicon to tailor the stiffness of silicon to obtain lower stiffness.



(a) $c_{11} = 169$ GPa, $\rho = 2329$ kg m^{-3} **(b)** $c_{11} = 16.9$ GPa, $\rho = 2000$ kg m^{-3}

Figure 3.4.: *The displacement along the artificial piezo bar, driven slightly off the resonance frequency. The DC voltage* $V_0 = 10$ V*, modulation voltage* $\delta V = 0.1$ V *driven at* (a) $f = 2.1297$ GHz, $c_{11} = 169$ GPa, $\rho = 2329$ kg m^{-3}, (b) $f = 0.72673$ GHz, $c_{11} = 16.9$ GPa , $\rho = 2000$ kg m^{-3}. *The rest of the (bulk) silicon properties are given in Chap. A.*

3.4. Solutions of 1D Piezobar

Here we provide solutions of the 1D piezobar using surface force method, natural modes of the 1D var, which one of these natural modes can be excited, the body force method, and how waves will propagate from such a 1D infinitely in both directions. The surface forces are exerted to the surfaces of an object, e. g., normal force, shear force. The body forces act throughout the volume of a body, e. g., gravity, electric field, magnetic field, fictitious forces, i. e., Coriolis effect are also considered body forces.

3.4.1. Composite Solution of 1D Piezobar using Boundary Force Method

In this Section, we wish to solve the acoustic wave equation, Eq. (3.30), in a 1D bar of finite length by a composite solution due to voltage perturbation, see Fig. 3.5. The boundary forces arise from small-signal voltage perturbation by electrodes on the end facets of the bar.

The equation of motion for the displacement field $u(x,t)$ is

$$\rho\frac{\partial^2 u}{\partial t^2} = c_{11}\frac{\partial^2 u}{\partial x^2}, \tag{3.65}$$

subject to the boundary conditions

$$\text{at } x = 0, \qquad T_{xx}(0) = \frac{\delta F}{A} = T_\text{a}, \tag{3.66}$$

$$\text{at } x = L_\text{eq}, \qquad T_{xx}(L_\text{eq}) = \frac{\delta F}{A} = T_\text{a}. \tag{3.67}$$

Here A is the area of the end facets of the 1D bar, and force δF arises from voltage perturbation δV at an angular frequency ω. Using the ansatz $u(x,t) = u(x)e^{i\omega t}$, Eq. (3.65) becomes

$$\begin{aligned} -\rho\omega^2 u &= c_{11}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \\ \Rightarrow \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} &= -\frac{\rho\omega^2}{c_{11}}u(x). \end{aligned} \tag{3.68}$$

This eigenvalue problem has a general solution

$$u(x) = A\cos(kx) + B\sin(kx), \tag{3.69}$$

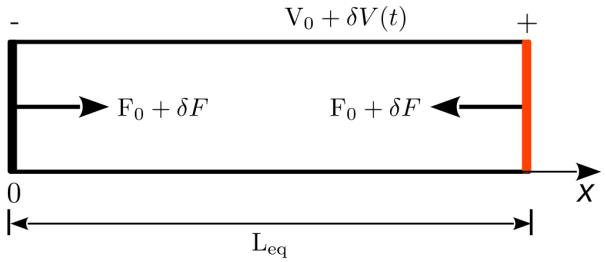


Figure 3.5.: *1D bar of finite length excited by a time-varying voltage perturbation.*

where the composite wavevector, k, is

$$k = \omega\sqrt{\frac{\rho}{c_{11}}}. \tag{3.70}$$

We know the relation between stress and acoustic particle displacement field from Eq. (3.10)

$$T_{xx} = c_{11}\frac{\mathrm{d}u}{\mathrm{d}x} = -c_{11}kA\sin(kx) + c_{11}kB\cos(kx). \tag{3.71}$$

Applying the boundary conditions in Eqs. (3.66)–(3.67), we find the coefficients $B = T_\mathrm{a}/c_{11}k$ and $A = -T_\mathrm{a}/c_{11}k \cdot \tan(kL_\mathrm{eq}/2)$. Therefore, the displacement for all the allowed modes with composite k is

Acoustic Displacement of 1D Bar

$$u(x) = \frac{1}{c_{11}k} \cdot \frac{\delta F}{A} \cdot \left[\sin(kx) - \tan\left(\frac{kL_\mathrm{eq}}{2}\right)\cos(kx)\right], \tag{3.72}$$

where $k = \omega\sqrt{\rho/c_{11}} = \omega/v_s$, and the mechanical wave velocity $v_s = \sqrt{c_{11}/\rho}$. The equilibrium displacement ΔL of a 1D bar is very small, on the order of pm for the 1D case we are considering here with typical parameters; hence we can appropriate $L_\mathrm{eq} \approx L$. If we scale the position, $x^{'}$, by the length of a bar, i. e., $x^{'} = x/L$, the composite displacement can be casted in the form

Acoustic Displacement of 1D Bar (Scaled Co-ordinate)

$$u(x^{'}) = \frac{U}{\cos\left(q/2\right)}\sin\left[q\left(x^{'} - 1/2\right)\right], \tag{3.73}$$

where the normalized wavevector $q = kL = \omega L\sqrt{\rho/c_{11}}$, and amplitude $U = 1/c_{11}k \cdot \delta F/A$. The maximum displacement occurs at $x^{'} = 0, 1$, where $u_{\max}(x^{'}) = U\tan(q/2)$.

The electrostatic force between the two plates is

$$F = -\frac{\epsilon A}{2L_{\text{eq}}^2}V^2 = -\frac{\epsilon A}{2L_{\text{eq}}^2}(V_0 + \delta V)^2 = -\frac{\epsilon A}{2L_{\text{eq}}^2}(V_0^2 + \delta V^2 + 2V_0\delta V). \tag{3.74}$$

The electrostatic force due to the DC bias is $F_0 = -\epsilon A V_0^2/2L_{\text{eq}}^2$. We ignore the term $F_2 = -\epsilon A\delta V^2/2L_{\text{eq}}^2$ driving at twice the frequency. Therefore the driving force rotating at fundamental frequency ω at the boundaries of the slab is $F_1 = -\epsilon A V_0\delta V/L_{\text{eq}}^2$. Hence

$$U = \frac{1}{c_{11}k}\frac{\delta F}{A} = U_1 = \frac{1}{c_{11}k}\frac{F_1}{A} = -\frac{\epsilon}{c_{11}kL^2}\cdot V_0\cdot\delta V = -\frac{\epsilon}{c_{11}L}\cdot\frac{1}{q}\cdot V_0\delta V. \tag{3.75}$$

The maximum displacement amplitude is

Maximum Displacement

$$u_{\max}(x^{'}) = \frac{\epsilon}{c_{11}L}\cdot\frac{1}{q}\cdot V_0\delta V\cdot\tan(q/2), \tag{3.76}$$

where $q = \omega L\sqrt{\rho/c_{11}} = \frac{\omega}{v_s}L$. For low frequencies, $q \ll 1$, i. e., $(\omega \ll v_s/L)$, we have $\tan(q/2) \approx q/2$. The maximum amplitude is $u_{\max}(x^{'}) \approx \frac{\epsilon}{c_{11}L}\cdot\frac{1}{q}\cdot V_0\delta V\cdot q/2 = \frac{\epsilon}{2c_{11}L}\cdot V_0\delta V$, which is the same as DC displacement with $V_0^2 = V_0\delta V$ in Eq. (3.24a). As $q \to \pi$, $\tan(q/2) \to \tan(\pi/2) \to \infty$, the maximum displacement amplitude becomes very large. The frequency corresponding to $q = \pi$ is $q_{\max} = \frac{2\pi f_{\max}}{v_s}L = \pi$. Hence

Lowest Frequency for Maximum Displacement

$$f_{\max} = \frac{v_s}{2L}. \tag{3.77}$$

This corresponds to the 1st mode of the bar. More generally, infinite mode amplitudes occur at $q_n/2 = n\pi/2$, where $n = 1, 3, 5, \cdots$, i. e., $q_n = \frac{2\pi f_n}{v_s}L = n\pi$. The excited allowed frequencies are

Modes

$$f_n = n\frac{v_s}{2L}, \qquad n = 1, 3, 5, \cdots. \tag{3.78}$$

The mechanical displacement curve for a 1D bar made of bulk silicon is plotted in Fig. 3.6 with the associated parameters mentioned in the figure caption.

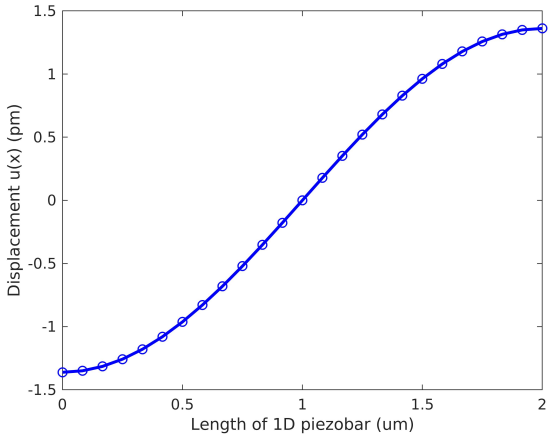


Figure 3.6.: *The displacement along the artificial piezo bar in tension.* $V_0 = 10$ V, $\delta V = 0.1$ V, *driving* $f = 2.1297$ GHz.

3.4.2. Natural Modes of 1D Bar

In this Section, we wish to compute the natural modes of a bar by solving the equation of motion for the displacement field $u(x,t)$

$$\rho \frac{\partial^2 u}{\partial t^2} = c_{11} \frac{\partial^2 u}{\partial x^2}. \tag{3.79}$$

However, now this is subjected to the free boundary conditions

$$\text{at } x = 0, \qquad T_{xx}(0) = 0, \tag{3.80}$$

$$\text{at } x = L_{\text{eq}}, \qquad T_{xx}(L_{\text{eq}}) = 0. \tag{3.81}$$

Since $T_{xx} = c_{11}\frac{\partial u}{\partial x}$, the Neumann boundary condition for the free ends renders $\frac{\partial u}{\partial x} = 0$ at $x = 0$ and $x = L$. We use the method of separation of variables and let $u(x,t) = X(x)T(t)$. By substituting in Eq. (3.79), we obtain

$$X \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = \frac{c_{11}}{\rho} T \frac{\mathrm{d}^2 X}{\mathrm{d}x^2} \tag{3.82}$$

$$\Rightarrow \frac{\rho}{c_{11}} \frac{1}{T} \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = \frac{1}{X} \frac{\mathrm{d}^2 X}{\mathrm{d}x^2}. \tag{3.83}$$

By letting $\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -k_n^2$, we obtain $\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + k_n^2 X = 0$. It has solution of the form $X(x) = A\cos(k_n x) + B\sin(k_n x)$. By applying the boundary conditions, we find $B = 0$, the eigen function has the form $X(x) = \cos(k_n x)$ and the eigenvalues are $k_n = n\pi/L$, where $n = 0, 1, 2, \cdots$. The

natural modes of the bar are thus

Natural Eigenfunctions and Eigenvalues for 1D Bar

$$X_n(x) = \cos(k_n x), \qquad k_n = \frac{n\pi}{L}, \qquad n = 0, 1, 2, \cdots . \tag{3.84}$$

The time dependence of the modes is

$$\frac{\rho}{c_{11}} \frac{1}{T_n} \frac{\mathrm{d}^2 T_n}{\mathrm{d}t^2} = -k_n^2 \tag{3.85}$$

$$\Rightarrow \frac{\mathrm{d}^2 T_n}{\mathrm{d}t^2} + k_n^2 \left(\frac{c_{11}}{\rho} \right) T_n = 0. \tag{3.86}$$

This has solution of the form $T_n = e^{\pm i\omega_n t}$ with $\omega_n = k_n \sqrt{c_{11}/\rho} = v_s k_n$. The natural frequencies are

Natural Modes for 1D Bar

$$f_n = v_s \frac{n}{2L}. \qquad n = 0, 1, 2, \cdots . \tag{3.87}$$

This is *similar* to the frequencies derived in §3.4.1 in Eq. (3.78), though additionally it contains the even modes. We will see in the next Section that the even modes die down by eigen mode expansion method.

3.4.3. Excitation of Natural Modes of 1D Bar by Electric Force

In this Section, we want to solve the acoustic wave equation for a single bar of finite length using the eigenseries expansion method. We repeat the equation of motion here

$$\rho \frac{\partial^2 u}{\partial t^2} = c_{11} \frac{\partial^2 u}{\partial x^2}. \tag{3.88}$$

We assume that the displacement field is of the form

$$u(x) = \sum_{n=0}^{\infty} A_n e^{ik_n x} + \sum_{n=0}^{\infty} B_n e^{-ik_n x}. \tag{3.89}$$

By applying the Neumann boundary condition, i. e., $\frac{\partial u}{\partial x} = 0$ at both ends, we find $A_n = B_n$, spatial frequencies are $k_n = n\pi/L_{\mathrm{eq}}$, where $n \in \mathbb{N}_0$ and the eigenfunctions are co-sinusoidal.

Therefore we can express the solution $u(x)$ (in the scaled unit) in terms of eigenmode expansion

$$u(x^{'}) = \frac{U_1}{\cos\Big(q/2\Big)} \sin\left[q\left(x^{'} - 1/2\right)\right] = \sum_{n=0}^{\infty} A_n \cos(q_n x^{'}), \tag{3.90}$$

where $q = \frac{\omega}{v_s} L$, $q_n = \frac{\omega_n}{v_s} L = k_n L = n\pi$. By multiplying Eq. (3.90) with $\cos(q_m x^{'})$, and integrating over $x^{'}$, we obtain

$$\int_0^1 \frac{U_1}{\cos\Big(q/2\Big)} \sin\Big[q(x^{'} - 1/2)\Big] \cos(q_m x^{'})\, dx^{'} = \sum_{n=0}^{\infty} A_n \int_0^1 \cos(q_n x^{'}) \cos(q_m x^{'})\, dx^{'} \tag{3.91}$$

The right hand side integral is

$$\int_0^1 \cos(q_n x^{'}) \cos(q_m x^{'})\, dx^{'} = \begin{cases} 1, & n = m = 0. \\ 1/2, & n = m \neq 0. \\ 0, & n \neq m. \end{cases} \tag{3.92}$$

We thus have, for $m = 0$, $A_0 = 0$, for $m > 0$,

$$A_m = \frac{4U_1}{\cos(q/2)} \cdot \frac{\sin(q_m/2)}{q^2 - q_m^2} \Big[q\cos(q/2)\sin(q_m/2) - q_m \sin(q/2)\cos(q_m/2)\Big], \tag{3.93}$$

where $q_m = k_m L = m\pi$ and $m = 1, 2, 3, \cdots$. Since $\sin(q_m/2) = \sin(m\pi/2) = 0$ for even m, only odd modes are excited. Hence for odd m, $\sin(q_m/2) = \sin(m\pi/2) = \pm 1$ and $\cos(q_m/2) = \cos(m\pi/2) = 0$. This is consistent with the frequencies derived in §3.4.1 in Eq. (3.78) and the remark that only the odd natural modes in Eq. (3.87) are excited in §3.4.2. Hence the coefficient $A_m = \frac{4qU_1}{q^2 - q_m^2}$, where $q = \frac{\omega}{v_s} L$, $q_m = m\pi$, $U_1 = -\frac{\epsilon}{c_{11} L} \cdot \frac{1}{q} \cdot V_0 \delta V$, as given in Eq. (3.75). Therefore,

$$A_m = -\frac{4\epsilon}{c_{11} L} \cdot \frac{V_0 \delta V}{q^2 - q_m^2} = -\frac{4\epsilon}{c_{11} L^3} \cdot \frac{V_0 \delta V}{k^2 - k_m^2}. \tag{3.94}$$

The displacement amplitudes A_m hitting exactly at the odd resonances are shown in Fig. 3.7. A measurable magnitude can be detected in bulk silicon and realistic damping parameters. The damping is incorporated by replacing the denominator $(k^2 - k_m^2)$ with $(k^2 - k_m^2 - i\eta_{11}\omega k_m^2 / c_{11})$.

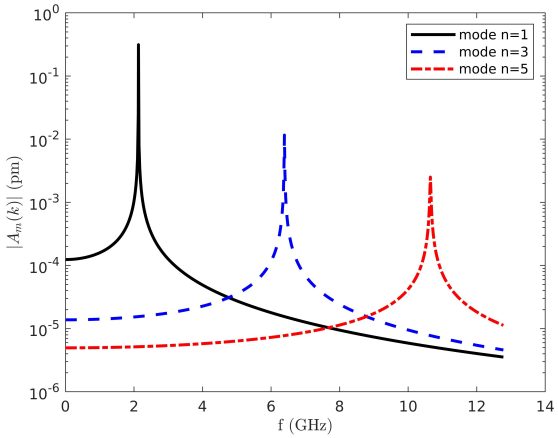


Figure 3.7.: *The amplitude spectrum exciting the odd modes. The bulk silicon damping* $\eta_{11} = 0.005$ N s m^{-2} *[3],* $V_0 = 10$ V, $\delta V = 0.1$ V.

Thus we have

Displacement Using Eigenseries Expansion Method

$$u(x') = \frac{U_1}{\cos\left(q/2\right)} \sin\left[q\left(x' - 1/2\right)\right] = \sum_{m=\text{odd}} \frac{4qU_1}{q^2 - q_m^2} \cos(m\pi x'). \tag{3.95}$$

The excitation amplitude A_m is infinite for $q = q_m = m\pi$ for $m = 1, 3, 5, \cdots$. The ratio of successive amplitudes is

$$\frac{A_{m+1}}{A_m} = \frac{q^2 - q_m^2}{q^2 - q_{m+1}^2} = \frac{q^2 - m^2\pi^2}{q^2 - (m+1)^2\pi^2}. \tag{3.96}$$

Alternatively,

$$\frac{A_m}{A_{m+1}} = \frac{q^2 - (m^2 + 2m + 1)\pi^2}{q^2 - m^2\pi^2} = 1 - \frac{(2m+1)\pi^2}{q^2 - m^2\pi^2}. \tag{3.97}$$

This means that the amplitude of the fundamental mode is largest among the allowed modes. We observe this kind of behavior when we will plot the acoustic displacement amplitude spectrum.

3.4.4. Eigenseries Expansion Solution of 1D Bar by Body Force Method

In this Section, we wish to solve the acoustic wave equation for a single bar of finite length using the body force method. The equation we wish to solve is

$$\rho\frac{\partial^2 u}{\partial t^2} = c_{11}\frac{\partial^2 u}{\partial x^2} + F(x,t), \tag{3.98}$$

where $F(x,t)$ is the body force distribution, unit $\mathrm{N\,m^{-3}}$. The boundary conditions are $T_{xx} = c_{11}\frac{\partial u}{\partial x} = 0$ at $x = 0$ and $x = L$. Assuming the force varies as $\sim e^{i\omega t}$, the solution for u also has the same time dependence. Therefore

$$-\rho\omega^2 u(x) = c_{11}\frac{d^2u(x)}{dx^2} + F(x) \tag{3.99}$$

$$\Rightarrow \frac{d^2u(x)}{dx^2} + k^2 u(x) = -\frac{1}{c_{11}}F(x), \tag{3.100}$$

where $k = \omega/v_s$ and the mechanical wave velocity $v_s = \sqrt{c_{11}/\rho}$. We solve the above equation by the method of eigenfunction expansion. The eigenmodes of the bar are

$$X_n(x) = \cos(k_n x), \qquad k_n = \frac{n\pi}{L}, \qquad \forall n = 0, 1, 2, \cdots . \tag{3.101}$$

Let

$$u(x) = \sum_{n=0}^{\infty} A_n \cos(k_n x) \tag{3.102}$$

$$F(x) = \sum_{n=0}^{\infty} F_n \cos(k_n x). \tag{3.103}$$

We have

$$-\sum_n k_n^2 A_n \cos(k_n x) + \sum_n k^2 A_n \cos(k_n x) = -\frac{1}{c_{11}}\sum_n F_n \cos(k_n x). \tag{3.104}$$

By matching the coefficients,

$$(k^2 - k_n^2)A_n = -F_n/c_{11}. \tag{3.105}$$

Hence the coefficients are

$$A_n = -\frac{1}{c_{11}} \cdot \frac{F_n}{k^2 - k_n^2}. \tag{3.106}$$

The force amplitude distribution in a slab is

$$F(x) = \frac{1}{A}\Big[F_1\delta(x) - F_1\delta(x-L)\Big] \tag{3.107}$$

where, as derived earlier in §3.4.1, the driving force rotating at fundamental frequency ω is $F_1 = \epsilon A V_0 \delta V/L^2$. By projecting $F_n = \langle F, \cos(k_n x)\rangle$, we find that $F_n = 0$ for $n = 0$, and even n. For odd n, the Fourier amplitudes are $F_n = \frac{4}{L}\cdot\frac{F_1}{A}$. Hence $F_n = \frac{4}{AL}\cdot\frac{\epsilon A}{L^2}V_0\delta V = \frac{4\epsilon}{L^3}\cdot V_0\delta V$ for odd n. Thus the excitation mode amplitudes, which agrees with the solution Eq. (3.94), based on the forced boundary conditions in §3.4.3, are

$$A_n = -\frac{1}{c_{11}}\cdot\frac{F_n}{k^2-k_n^2} = -\frac{4\epsilon}{c_{11}L^3}\cdot\frac{V_0\delta V}{k^2-k_n^2}. \tag{3.108}$$

3.4.5. 1D Infinite Bar

In this Section, we wish to find the acoustic modes in an infinite bar, the modes are excited by two electrodes L distance apart, see Fig. 3.8. We wish to solve the wave equation in Eq. (3.109) with the domain extended to infinity in both directions

$$\rho\frac{\partial^2 u}{\partial t^2} = c_{11}\frac{\partial^2 u}{\partial x^2} + F(x)e^{i\omega t}, \qquad -\infty < x < \infty. \tag{3.109}$$

Choosing origin of the co-ordinate system in middle of the bar, the force distribution is

$$F(x) = \frac{F_1}{A}\Big[\delta(x+L/2) - \delta(x-L/2)\Big] \tag{3.110}$$

Assuming solution of the form $u \sim e^{i\omega t}$, Eq. (3.109) is

$$-\rho\omega^2 u(x) = c_{11}\frac{d^2u(x)}{dx^2} + F(x) \tag{3.111}$$

$$\Rightarrow \frac{d^2u(x)}{dx^2} + k_0^2 u(x) = -\frac{1}{c_{11}}F(x), \tag{3.112}$$

where $k_0 = \omega\sqrt{\rho/c_{11}}$. By taking Fourier transform of Eq. (3.112), we obtain

$$-k^2U(k) + k_0^2U(k) = -\frac{1}{c_{11}}F(k) \tag{3.113}$$

$$\Rightarrow U(k) = -\frac{1}{c_{11}}\cdot\frac{F(k)}{k_0^2-k^2}, \tag{3.114}$$

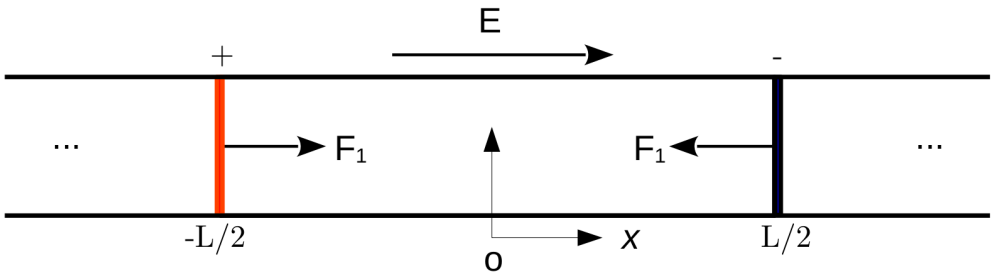


Figure 3.8.: *Two electrodes are placed L distance apart in a 1D infinite bar. A time-varying voltage is applied by the electrodes. The acoustic modes are allowed to propagate to infinity on both ends, hence infinite bar.*

The Fourier transform of force distribution $F(x)$ is $F(k) = 2i \cdot F_1/A \cdot \sin(kL/2)$. The Fourier amplitude for displacement $U(k) = -2i/c_{11} \cdot F_1/A \cdot \sin(kL/2)/(k_0^2 - k^2)$. We derived earlier in §3.4.1 that the driving force rotating at fundamental frequency ω is $F_1 = \epsilon A V_0 \delta V / L^2$. Hence

$$U(k) = -i\frac{2\epsilon}{c_{11}L^2} \cdot \frac{\sin(kL/2)}{k_0^2 - k^2} \cdot V_0 \delta V. \tag{3.115}$$

By taking inverse Fourier transform of Eq. (3.115), we obtain

$$u(x) = -\frac{\epsilon}{c_{11}L^2} \cdot V_0 \delta V \cdot \frac{i}{2k_0} \left(e^{-ik_0|x+L/2|} - e^{-ik_0|x-L/2|} \right), \tag{3.116}$$

where we used $\mathcal{F}^{-1}\{e^{\pm ikL/2}/(k_0^2 - k^2)\} = i/(2k_0)e^{-ik_0(x \pm L/2)}$. By letting $u_0 = 1/(2ik_0) \cdot \epsilon/(c_{11}L^2) \cdot V_0 \delta V$, we can express the solution in three different regions,

$$x < -L/2, \qquad u(x) = u_0 \left[e^{ik_0(x+L/2)} - e^{ik_0(x-L/2)} \right] \tag{3.117}$$

$$-L/2 < x < L/2, \qquad u(x) = u_0 \left[e^{-ik_0(x+L/2)} - e^{ik_0(x-L/2)} \right] \tag{3.118}$$

$$x > L/2, \qquad u(x) = u_0 \left[e^{-ik_0(x+L/2)} - e^{-ik_0(x-L/2)} \right] \tag{3.119}$$

Thus in each region, the solution consists of 2 propagating acoustic waves, see Fig. 3.9. The outgoing wave in region $x > L/2$ is

$$u(x) = u_0 \left[e^{-ik_0(x+L/2)} - e^{-ik_0(x-L/2)} \right] \tag{3.120}$$

$$= u_0 \left(e^{-ik_0 L/2} - e^{ik_0 L/2} \right) e^{-ik_0 x} \tag{3.121}$$

$$= -2iu_0 \sin(k_0 L/2) e^{-ik_0 x}. \tag{3.122}$$

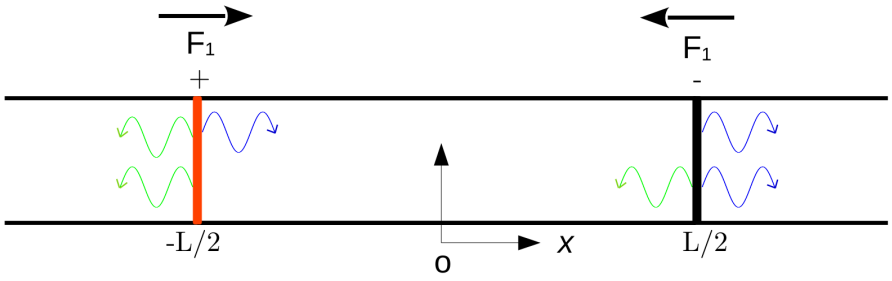


Figure 3.9.: *There are two counter propagating waves inside the bar, emanating from both the electrodes. However, outside the bar, waves propagate away from the electrodes to infinity.*

Hence the closed form solution of displacement for the right propagating wave outside the bar is

$$u(x) = -\frac{\epsilon}{c_{11}L^2} \cdot V_0\delta V \cdot \frac{\sin(k_0L/2)}{k_0} \cdot e^{-ik_0x}, \tag{3.123}$$

and is plotted in Fig. 3.10 for a 1D bar made of bulk silicon. The maximum displacement amplitude is obtained when $\sin(k_0L/2) = \pm 1$, which means $k_0L/2 = n\pi/2$, where $n = 1, 3, 5, \cdots$. Hence

$$k_0 = \frac{n\pi}{L}, \qquad \forall n = 1, 3, 5, \cdots . \tag{3.124}$$

which corresponds to the natural modes of a finite bar of length L in Eq. (3.84). The maximum displacement amplitude in this case is

$$u_{\max} = \frac{\epsilon}{c_{11}L} \cdot \frac{V_0\delta V}{k_0L} = \frac{\epsilon}{c_{11}L} \cdot \frac{V_0\delta V}{n\pi}. \tag{3.125}$$

This is the same order as the DC displacement $u_{\text{DC}} = \epsilon/(c_{11}L) \cdot V_0^2/2$, as given in Eq. (3.24a) in §3.1.

3.5. 1D Infinite Periodic Structure (1D Phononic Crystal)

In this Section, we wish to solve the acoustic displacement problem for an infinite periodic structure, i. e., a 1D phononic crystal.

Fig. 3.11 shows one period of length L of a 1D phononic crystal. The equation of the displacement field $u(x, t)$ along the 1D bar subject to periodic electrostatic driving force $F(x, t)$ is

$$\rho\frac{\partial^2 u}{\partial t^2} = c_{11}\frac{\partial^2 u}{\partial x^2} + F(x, t), \qquad -L/2 < x < L/2, \tag{3.126}$$

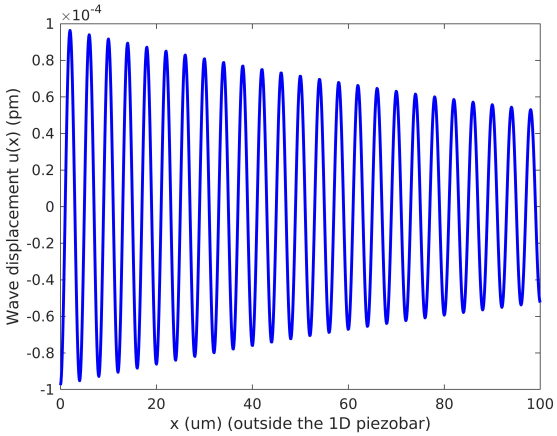


Figure 3.10.: *The acoustic wave displacement for the right propagating wave outside the 1D infinite bar using bulk silicon* $c_{11} = 169 \cdot 10^9$ N m^{-2}*, bulk density of silicon* $\rho = 2329$ kg m^{-3} *[56],* $V_0 = 10$ V*, perturbation voltage* $\delta V = 0.1$ V*,* $L = 2$ µm *in an intentionally highly damped silicon infinite bar with* $\eta_{11} = 0.1$ N s m^{-2}*, driven at the fundamental frequency* $f = 2.1297$ GHz*.*

where $F(t, x) = F(x)e^{i\omega t}$, ω being the driving frequency and $F(x)$ is the periodic force distribution along the 1D bar. Assuming the solution for u also varies harmonically at frequency ω, the above equation reduces to

$$\frac{d^2u(x)}{dx^2} + k^2u(x) = -\frac{1}{c_{11}}F(x), \tag{3.127}$$

where

$$k = \omega/v_s. \tag{3.128}$$

The periodic boundary conditions are

$$u(-L/2, t) = u(L/2, t), \tag{3.129}$$

$$\frac{\partial u}{\partial x}(-L/2, t) = \frac{\partial u}{\partial x}(L/2, t). \tag{3.130}$$

We solve this problem by the method of eigenseries expansion. We first determine the eigenfunctions by solving the homogeneous equation

$$\rho\frac{\partial^2 u}{\partial t^2} = c_{11}\frac{\partial^2 u}{\partial x^2}, \tag{3.131}$$

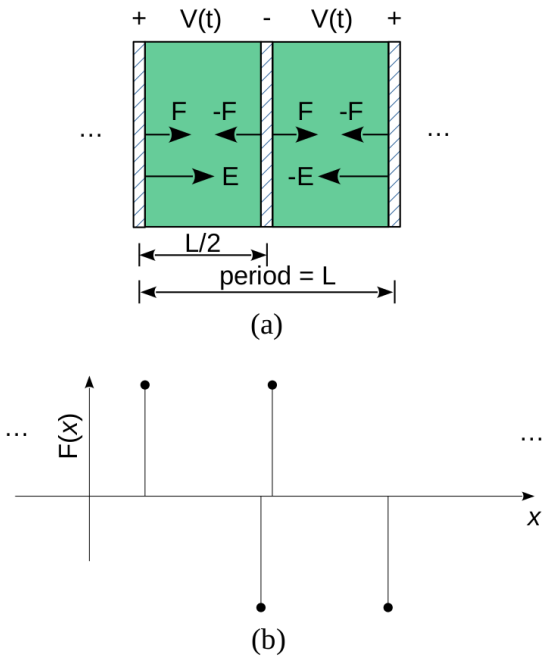


Figure 3.11.: *(a) One period of length L of a 1D phononic crystal. (b) The applied body force distribution $F(x)$ within each period.*

subject to the periodic boundary conditions Eq. (3.129)–(3.130). Using the separation of variables, $u(x,t) = X(x)T(t)$ into Eq. (3.131), we get

$$\rho \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} X(x) = c_{11} \frac{\mathrm{d}^2 X}{\mathrm{d}x^2} T(t). \tag{3.132}$$

Dividing both sides by ρXT, we have

$$\frac{1}{T}\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = \frac{c_{11}}{\rho}\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -\omega_n^2 \,(\text{constant}). \tag{3.133}$$

We separate the two functions

$$\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = -\omega_n^2 T. \tag{3.134}$$

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -\frac{\rho\omega_n^2}{c_{11}} X. \tag{3.135}$$

The solutions of Eq. (3.134) are

$$T_n(t) = e^{\pm i\omega_n t}. \tag{3.136}$$

The general solution of Eq. (3.135) is of the form

$$X_n = A_n \cos(k_n x) + B_n \sin(k_n x). \tag{3.137}$$

By applying the boundary conditions in Eq. (3.129)–(3.130), we obtain the spatial frequency eigenvalues, k_n,

$$k_n = \frac{2n\pi}{L}, \qquad \forall n = 0, 1, 2, \cdots. \tag{3.138}$$

The general solution for $u(x)$ can thus be expressed as

$$u(x) = \sum_{n=0}^{\infty} \Big[A_n \cos(k_n x) + B_n \sin(k_n x) \Big] \tag{3.139}$$

By substituting Eq. (3.139) into the equation of motion in Eq. (3.127) we obtain,

$$\sum_n (k^2 - k_n^2) \Big[A_n \cos(k_n x) + B_n \sin(k_n x) \Big] = -\frac{1}{c_{11}} F(x) \tag{3.140}$$

For $n = 0$,

$$B_0 = 0, \tag{3.141}$$

$$A_0 = -\frac{1}{c_{11}(k^2 - k_n^2)} \cdot \frac{1}{L} \int_{-L/2}^{L/2} F(x)\, dx. \tag{3.142}$$

For $n > 0$,

$$A_n = -\frac{2}{c_{11}(k^2 - k_n^2)} \cdot \frac{1}{L} \int_{-L/2}^{L/2} F(x) \cos(k_n x)\, dx, \tag{3.143}$$

$$B_n = -\frac{2}{c_{11}(k^2 - k_n^2)} \cdot \frac{1}{L} \int_{-L/2}^{L/2} F(x) \sin(k_n x)\, dx. \tag{3.144}$$

For a given force distribution, we can compute the coefficients A_n and B_n.

3.5.1. Optimum Electrode Design for 1D Phononic Crystal

The electric force distribution within each period is generated by suitable arrangement of electrodes and applied voltages. We consider the simplest electrode arrangement as shown in Fig. 3.12, which consists of a positive electrode of width δx placed at location x_0 and a negative

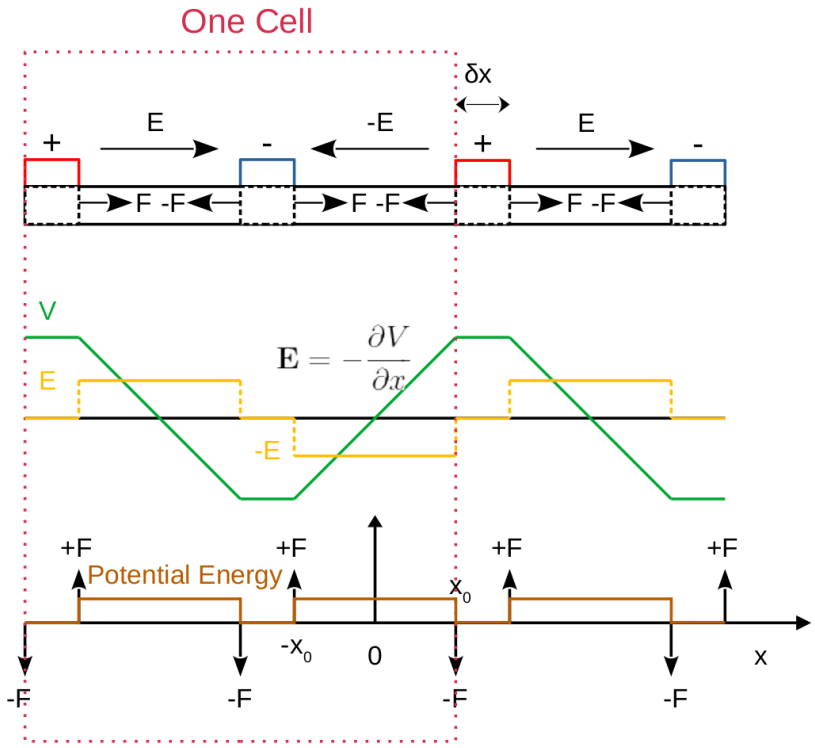


Figure 3.12.: *Schematic drawing of the electrode placement, electric field, voltage, electrostatic force distributions, and electrostatic potential energy in 1D phononic crystal. Electrodes of width δx placed x_0 from the middle of one lattice in an infinite periodic phononic crystal. The associated force distribution is shown.*

electrode of the same width symmetrically placed at location $-x_0$.

The electrostatic force between two electrodes can be computed from $F(x) = -\partial U/\partial x$, where U is the electrostatic energy in the capacitive system formed by the two electrodes (assuming no dispersion):

$$U(x) = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\epsilon A}{L/2}V^2 = \frac{1}{4}\epsilon(AL)E^2, \tag{3.145}$$

where, the electric field $|E| = V/(L/2)$ and A is the area of the 1D slab.

The electrostatic force between the two electrodes is thus

$$F = -\frac{\epsilon A}{2(L/2)^2}V^2 = -\frac{2\epsilon A}{L^2}(V_0 + \delta V)^2 = -\frac{2\epsilon A}{L^2}(V_0^2 + \delta V^2 + 2V_0\delta V). \tag{3.146}$$

The electrostatic force due to the DC bias is

$$F_0 = -2\epsilon A V_0^2/L^2. \tag{3.147}$$

We ignore the term

$$F_2 = -2\epsilon A \delta V^2/L^2, \tag{3.148}$$

driving at twice the frequency. Therefore the driving force rotating at fundamental frequency ω at the boundaries of the slab is

$$F_1 = -4\epsilon A V_0 \delta V/L^2. \tag{3.149}$$

The force distribution for the electrode arrangement in Fig. 3.12 is

$$F(x) = \frac{F_1}{A}\left[-\delta(x + x_0 + \delta x) + \delta(x + x_0) - \delta(x - x_0) + \delta(x - x_0 - \delta x)\right], \tag{3.150}$$

where A is the cross-sectional area of each electrode. The unknown coefficients for this electrode configuration are

$$A_0 = B_0 = 0, \tag{3.151}$$

$$A_n = 0, \tag{3.152}$$

$$B_n \propto \frac{4F_1}{A}\cos\left[k_n(x_0 + \delta x/2)\right]\sin(k_n \delta x/2). \tag{3.153}$$

Thus odd modes are excited with amplitudes

$$B_n = -\frac{2}{c_{11}L(k^2 - k_n^2)}\frac{4F_1}{A}\cos\left[k_n(x_0 + \delta x/2)\right]\sin(k_n \delta x/2). \tag{3.154}$$

We note that in the limit of electrodes with infinitesimally small width, i. e., $\delta x \to 0$, the coefficient $B_n \to 0$. To maximize B_n, we choose

$$\begin{aligned} &\sin(k_1 \delta x/2) = \pm 1 \\ \Rightarrow\ &\frac{2\pi}{L}\frac{\delta x}{2} = \frac{m'\pi}{2}, \qquad m' \text{ is odd} \\ \Rightarrow\ &\delta x = \frac{m'}{2}L. \end{aligned} \tag{3.155}$$

and

$$\begin{aligned} &\cos\Big[k_1 x_0 + \delta x/2)\Big] = \pm 1 \\ \Rightarrow\ &\frac{2\pi}{L}(x_0 + \delta x/2) = m\pi, \qquad \text{m is even or odd} \\ \Rightarrow\ &x_0 + \delta x/2 = mL/2 \\ \Rightarrow\ &x_0 = (m - m^{'}/2)L/2. \end{aligned} \tag{3.156}$$

For $m^{'} = 1$ and $m = 1$, $\delta x = L/2$ and $x_0 = L/4$, meaning the width of an electrode is one-quarter of the length of the lattice and the each electrode is placed at one-quarter distance from the middle of the lattice, as schematically shown in Fig. 3.12. Therefore the optimized amplitude is

$$B_n = -\frac{2}{c_{11}L(k^2 - k_n^2)}\frac{4F_1}{A}. \tag{3.157}$$

By substituting $\frac{F_1}{A} = -\frac{4\epsilon}{L^2}V_0\delta V$ in Eq. (3.157), for maximum displacement, the amplitude coefficient is

$$B_n = \frac{32\epsilon}{c_{11}L^3} \cdot \frac{V_0\delta V}{(k^2 - k_n^2)}. \tag{3.158}$$

Acoustic Wave With Damping

Since the amplitude in Eq. (3.158) has singularity exactly at resonance, and we prefer to drive the system as close as possible to the resonance, we need to incorporate damping; this also represents realistic material properties. This can be done via two ways, one quick way is to replace the stiffness constant c_{11} by $(c_{11} + i\omega\eta_{11})$ and use the formulae for no damping. We present here derivation with few additional lines, as it entails insightful analytical solution.

The loss in a material can be incorporated via Eq. (2.32) into the equation of motion provided by Eq. (3.30). For the one-dimensional case, we can write

$$T_{xx} = c_{11}S_{xx} + \eta_{11}\frac{\partial S_{xx}}{\partial t}, \tag{3.159}$$

where η_{11} is damping having unit $\mathrm{N\,s\,m^{-2}}$. By plugging into the equation of motion, and adding the forcing function $F(x,t)$, we arrive at

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial T_{xx}}{\partial x} + F(x,t). \tag{3.160}$$

Considering the time harmonic case, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + \frac{i\eta_{11}\omega}{c_{11}}\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + k^2 u(x) = -\frac{1}{c_{11}}F(x), \tag{3.161}$$

where k is provided by Eq. (3.35) or Eq. (3.128). Using the ansatz Eq. (3.139), we obtain

$$\sum_n (k^2 - k_n^2 - i\eta_{11}\omega k_n^2/c_{11})\left[A_n\cos(k_n x) + B_n\sin(k_n x)\right] = -\frac{1}{c_{11}}F(x), \tag{3.162}$$

where discrete k_n is provided by Eq. (3.138). In the case of electrode configuration in Fig. 3.12, $A_0 = B_0 = 0$, $A_n = 0$ as before, and

$$B_n = -\frac{1}{c_{11}} \cdot \frac{2}{(k^2 - k_n^2) - i\eta_{11}\omega k_n^2/c_{11}} \cdot \frac{1}{L}\int_{-L/2}^{L/2} F(x)\sin(k_n x)\,dx. \tag{3.163}$$

Following the same approach as in the lossless case, after some manipulation, the complex amplitude coefficient with damping is

$$B_n = \frac{32\epsilon}{c_{11}L^3} \cdot \frac{V_0\delta V}{(k^2 - k_n^2) - i\eta_{11}\omega k_n^2/c_{11}}. \tag{3.164}$$

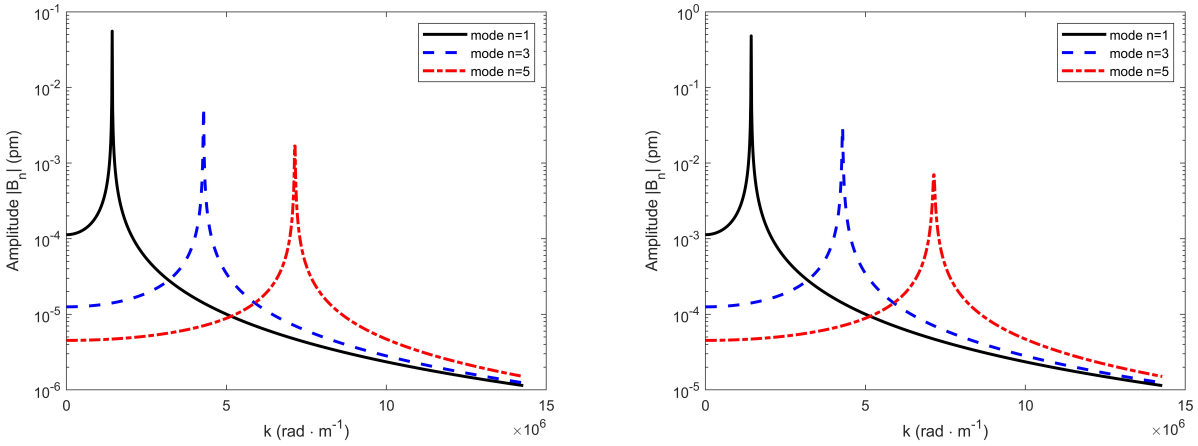
Hence the acoustic displacement field for maximum response is

Acoustic Displacement Field for Maximum Response

$$u(x) = \sum_{n=0}^{\infty} \frac{32\epsilon}{c_{11}L^3} \cdot \frac{V_0\delta V}{(k^2 - k_n^2) - i\eta_{11}\omega k_n^2/c_{11}}\sin(k_n x). \tag{3.165}$$

The amplitude of the acoustic wave transduction in such a 1D phononic crystal is shown in Fig. 3.13a using bulk silicon stiffness, and in Fig. 3.13b using phononic engineered silicon with stiffness c_{11} one order lower in magnitude. We can observe that the magnitude is correspondingly higher in engineered silicon with lower stiffness, as intuitively expected.

One way to explain this setup is that we can adjust the thermometer inside a room independent of whether the window is open or not, see the illustration in Fig. 3.14. We can say DC is the window and AC is the thermometer. However, in our case we must have the window closed in order to control the temperature. This is like saying we have to have DC turned on. In other words, the device being modeled is a voltage tunable mechanical resonator structure that can be designed to have a variety of useful transfer functions, when acoustic waves are passed through it.



(a) Bulk silicon **(b)** Phononic engineered silicon

Figure 3.13.: *The magnitude of acoustic wave transduction for 1D infinite periodic crystal. In* (a) *bulk silicon* $c_{11} = 169 \cdot 10^9$ N m^{-2}*, bulk density of silicon* $\rho = 2329$ kg m^{-3} *[56], in* (b) *phononic engineered silicon* $c_{11} = 16.9 \cdot 10^9$ N m^{-2}*, one order of magnitude lower than the bulk silicon* c_{11} *constant, and density of holey/patterned silicon* $\rho = 2000$ kg m^{-3} *are used. The 1D phononic crystal is operated at* $V_0 = 10$ V*,* $\delta V = 0.1$ V*, and its damping factor* $\eta = 0.005$ N s m^{-2} *is used.*

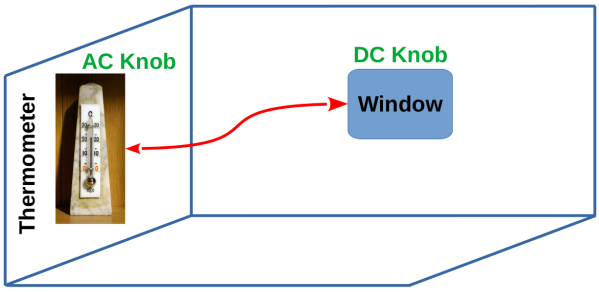


Figure 3.14.: *The window and the thermometer in a room can be controlled independently. However, we artificially create a scenario, where the thermostat can only be changed if the window is closed, meaning we must have the DC voltage on, i. e., stress the piezo material via DC voltage to create artificial dipole. (The thermostat figure is adapted from Wikimedia Commons with GNU Free Documentation License, Version 1.3 or later.)*

3.6. Electromechanical Coupling Coefficient

The electromechanical coupling κ^2 indicates how efficiently electrical energy is converted into acoustic energy in the phononic crystal and is defined as

$$\kappa^2 = \frac{\text{acoustic energy generated}}{\text{applied electrical energy}} \tag{3.166}$$

Generated Acoustic Energy

For an acoustic mode at frequency ω_n

$$u(x,t) = B_n e^{i(\omega_n t - k_n x)}. \tag{3.167}$$

The time-averaged acoustic Poynting vector, $\langle P_n \rangle$, unit is $\mathrm{W\,m^{-2}}$, is defined as [3]

$$\langle P_n \rangle = -\Re \frac{1}{2} v^* T_{xx}, \tag{3.168}$$

where the velocity is

$$v = \frac{\partial u}{\partial t} = i\omega_n B_n e^{i(\omega_n t - k_n x)}. \tag{3.169}$$

Using Eq. (3.10), the stress is

$$T_{xx} = -ik_n c_{11} B_n e^{i(\omega_n t - k_n x)}. \tag{3.170}$$

Therefore, the time-averaged acoustic Poynting vector for mode n is

$$\langle P_n \rangle = \frac{1}{2}\omega_n k_n c_{11} \left|B_n\right|^2. \tag{3.171}$$

The average acoustic energy density per unit area in mode n in one period is

$$\langle E_n \rangle = \langle P_n \rangle \cdot T_n = \pi k_n c_{11} \left|B_n\right|^2. \tag{3.172}$$

In Eq. (3.172), the (complex) amplitude B_n of the displacement field u, is provided by Eq. (3.164) incorporating damping. The average acoustic energy density per unit area is

$$\langle E_n \rangle = \pi k_n c_{11} \cdot \frac{1024\epsilon^2}{c_{11}^2 L^6} \cdot \frac{V_0^2 \delta V^2}{\left(k^2 - k_n^2\right)^2 + \left(\omega k_n^2 \eta_{11}/c_{11}\right)^2}. \tag{3.173}$$

The time-averaged acoustic energy in one lattice period is

$$\langle E_n \rangle_{\text{acoustic}} = \pi k_n c_{11} \cdot A \cdot \frac{1024\epsilon^2}{c_{11}^2 L^6} \cdot \frac{V_0^2 \delta V^2}{\left(k^2 - k_n^2\right)^2 + \left(\omega k_n^2 \eta_{11}/c_{11}\right)^2}. \tag{3.174}$$

Applied Electrical Energy

The electrostatic energy stored in a parallel plate capacitive system is $U = \frac{1}{2}CV^2(t)$, where the applied voltage is $V(t) = V_0 + \delta V \cos(\omega t)$. The electrostatic energy is

$$U = \frac{1}{2}CV_0^2 + \frac{1}{2}C\delta V^2 \cos^2(\omega t) + CV_0 \delta V \cos(\omega t). \tag{3.175}$$

The time-averaged input electrical energy is

$$\langle U \rangle = \frac{1}{2}CV_0^2 + \frac{1}{4}C\delta V^2. \tag{3.176}$$

We require only the perturbative input energy component in the calculation of the electromechanical coupling coefficient

$$\langle E \rangle_{\text{input}} = \frac{1}{4}C\delta V^2. \tag{3.177}$$

The electromechanical coupling coefficient is

$$\kappa^2 = \frac{\langle E_n \rangle_{\text{acoustic}}}{\langle E \rangle_{\text{input}}} = 4096\pi k_n c_{11} \cdot A \cdot \frac{\epsilon^2}{c_{11}^2 L^6} \cdot \frac{1}{C\delta V^2} \cdot \frac{V_0^2 \delta V^2}{\left(k^2 - k_n^2\right)^2 + \left(\omega k_n^2 \eta_{11}/c_{11}\right)^2}. \tag{3.178}$$

Using the spatial eigen frequency k_n relation in Eq. (3.138), and capacitance $C = \epsilon A/(L/2)$ we obtain the expression for the electromechanical coupling factor for each mode n

Electromechanical Coupling Coefficient

$$\kappa^2 = 4096\pi^2 n \cdot \frac{\epsilon}{c_{11}} \cdot \frac{1}{L^6} \cdot \frac{V_0^2}{\left(k^2 - k_n^2\right)^2 + \left(\omega k_n^2 \eta_{11}/c_{11}\right)^2}. \tag{3.179}$$

The stiffness measures rigidity of an elastic material, it has small values for easily deformed materials and large values for rigid materials. For a composite material having air holes, we expect to obtain lower stiffness constant as well as density. In addition, by phononic band gap engineering, we can get lower sound velocity, and hence lower stiffness constant, unattainable in conventional bulk materials. We derived earlier that the electromechanical transduction efficiency scales inversely with the material stiffness constant, and scales with the square of the

DC bias voltage, i. e.,

$$\kappa^2 \propto \frac{V_0^2}{c_{11}}. \tag{3.180}$$

Hence we can achieve higher κ^2 using phononic crystal by engineering the stiffness constant. This is an added advantage for low power operation of a device. As a reminder, piezoelectric behavior is not manifested in centro-symmetric crystals, like silicon naturally. We used phononic crystals made of centro-symmetric materials like silicon, and heterogeneous metal to emulate piezoelectricity – this is the salient feature of this project.

Table 3.1.: *The electromechanical coupling coefficient in surface acoustic wave (SAW) devices with various piezo materials, after [35].*

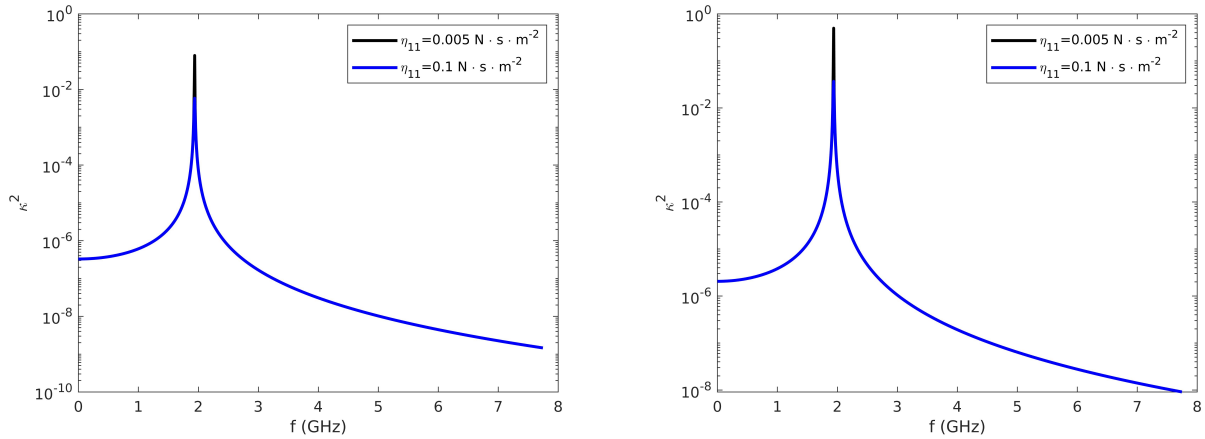
	Material	Cut – Propagation direction	κ^2 (%)
Single crystal	Quartz	ST-X	0.16
	$LiNbO_3$	128°Y-X	5.5
	$LiTaO_3$	X112°-Y	0.75
	$Li_2B_4O_7$	$(110) - < 001 >$	0.8
Ceramic	PZT-In$(Li_{3/5}W_{2/5})O_3$		1
	(Pb, Nd)(Ti, Mn, In)O_3		2.6
Thin Film	ZnO/glass		0.64
	ZnO/Sapphire		1

The (effective) [3] electromechanical coupling constant κ^2 using optimized gold electrode and PZT material at GHz frequency regime is 20.8% [34]. In the literature, typical values of κ^2 are provided in Table 3.1. We can potentially achieve near unity electromechanical transduction efficiency at resonance with lower driving voltage.

In Fig. 3.15, we used the lattice length $L = 4.4$ µm, $f_1 \approx 1.9360$ GHz. In Fig. 3.15a, using the DC voltage $V_0 = 20$ V, the maximum electromechanical coupling coefficient is $\kappa^2 \approx 7.93\%$ at damping factor $\eta = 0.005\ \mathrm{N\,s\,m^{-2}}$ and $\kappa^2 \approx 0.59\%$ at damping factor $\eta = 0.59\ \mathrm{N\,s\,m^{-2}}$. In Fig. 3.15b, keeping the rest of the parameters the same, while applying the DC voltage to be $V_0 = 50$ V, the maximum $\kappa^2 \approx 49.57\%$ at damping factor $\eta = 0.005\ \mathrm{N\,s\,m^{-2}}$ and $\kappa^2 \approx 3.67\%$ at damping factor $\eta = 0.1\ \mathrm{N\,s\,m^{-2}}$. The electromechanical coupling coefficient approaches unity, as we drive the system nearer to the resonance. It has to be borne mind that our model does not take into account nonlinear effects, which could become prominent as the electromechanical coupling approaches unity and the calculated results in Fig. 3.15 may no longer be accurate.

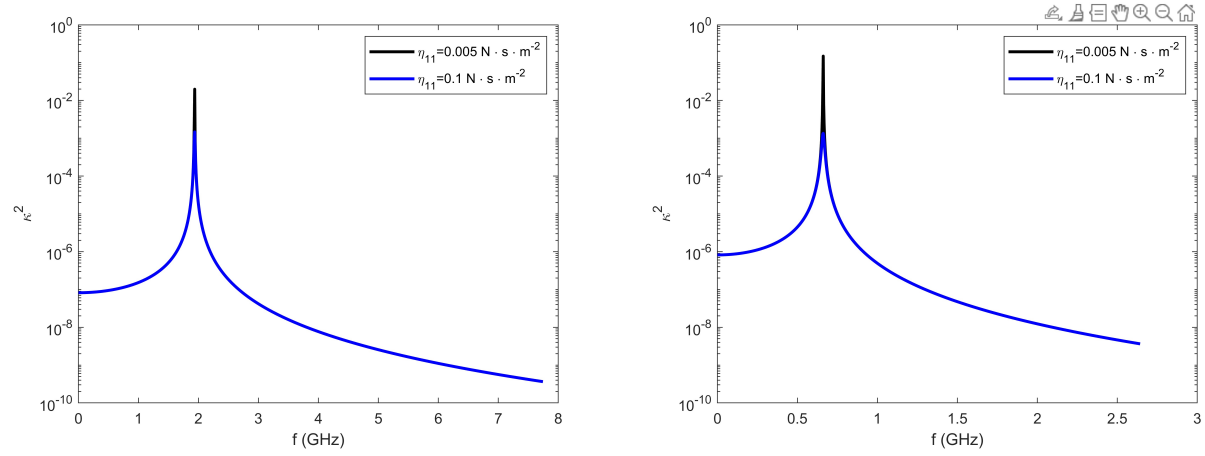
In Fig. 3.16a, the electromechanical coupling coefficient κ^2 is computed at $V_0 = 10$ V using bulk silicon properties, $c_{11} = 169 \cdot 10^9\ \mathrm{N\,m^{-2}}$, silicon mass density $\rho = 2329\ \mathrm{kg\,m^{-3}}$, at the DC

[3] meaning including loss in [34].



(a) $V_0 = 20$ V **(b)** $V_0 = 50$ V

Figure 3.15.: *The electromechanical coupling coefficient exciting the fundamental mode having lattice length* $L = 4.4$ µm. (a) $V_0 = 20$ V, (b) $V_0 = 50$ V.



(a) Bulk silicon **(b)** Phononic engineered silicon

Figure 3.16.: *The electromechanical coupling coefficient exciting the fundamental mode having lattice length* $L = 4.4$ µm, *DC bias* $V_0 = 10$ V. (a) *Bulk silicon properties* $c_{11} = 169 \cdot 10^9$ N m^{-2}, *bulk silicon mass density* $\rho = 2329$ kg m^{-3}. (b) *Phononic engineered silicon with* $c_{11} = 16.9 \cdot 10^9$ N m^{-2} *one order lower than the bulk silicon* c_{11}, *holey/porous silicon mass density* $\rho = 2000$ kg m^{-3}.

voltage $V_0 = 10$ V, and lattice length $L = 4.4$ µm. At fundamental frequency $f_1 = 1.9360$ GHz, the electromechanical coupling coefficient $\kappa^2 \approx 1.98\%$ at damping factor $\eta = 0.005\ \mathrm{N\,s\,m^{-2}}$ and $\kappa^2 \approx 0.15\%$ at damping factor $\eta = 0.1\ \mathrm{N\,s\,m^{-2}}$. Now, in Fig. 3.16b, using phononic engineered silicon with $c_{11} = 16.9 \cdot 10^9\ \mathrm{N\,m^{-2}}$ one order lower than the bulk silicon c_{11}, silicon mass density $\rho = 2000\ \mathrm{kg\,m^{-3}}$, having the same DC voltage $V_0 = 10$ V, and lattice length L as before, at fundamental frequency $f_1 = 0.6607$ GHz, the electromechanical coupling factor is increased to $\kappa^2 \approx 14.87\%$ at damping factor $\eta = 0.005\ \mathrm{N\,s\,m^{-2}}$ and $\kappa^2 \approx 0.13\%$ at damping factor $\eta = 0.1\ \mathrm{N\,s\,m^{-2}}$. The electromechanical coupling coefficient approaches unity, as we drive the system nearer to the resonance. It has to be borne mind that our model does not take into account nonlinear effects, which could become prominent as the electromechanical coupling approaches unity and the calculated results in Fig. 3.16 may no longer be accurate.

3.7. 1D Phononic Band Structure Calculation using Transfer Matrix Method

In this Section, we compute the band structure of 1D phononic crystal. The acoustic impedance is $Z = \rho v_{\mathrm{sound}}$, where ρ is the volumetric mass density and v_{sound} is the velocity of sound. In linear media, the frequency is unaltered upon reflection and transmission, however the wavenumber is changed. Let us say we have two media with ρ_A, c_A and ρ_B, c_B, where c is the stiffness of a medium. In medium A, the wavenumber is $k_A = \omega/v_A$, whereas in medium B, the wavenumber is $k_B = \omega/v_B$.

If a pressure wave is impinged from medium A to B, the acoustic pressure waves, p, for normal incidence, reflection and transmission are

$$p_A^i = A_A^i e^{j(\omega t - k_A x)}, \tag{3.181}$$

$$p_A^r = A_A^r e^{j(\omega t + k_A x)}, \tag{3.182}$$

$$p_B^t = A_B^t e^{j(\omega t - k_B x)}. \tag{3.183}$$

In medium A, the total pressure wave is

$$p_A = A_A^i e^{j(\omega t - k_A x)} + A_A^r e^{j(\omega t + k_A x)}. \tag{3.184}$$

In medium B, the total pressure wave is

$$p_B = A_B^t e^{j(\omega t - k_B x)}. \tag{3.185}$$

The particle velocities can be expressed in terms of acoustic pressure wave as

$$v_A^i = \frac{p_A^i}{\rho_A c_A}, \tag{3.186}$$

$$v_A^r = -\frac{p_A^r}{\rho_A c_A}, \tag{3.187}$$

$$v_B^t = \frac{p_B^t}{\rho_B c_B}. \tag{3.188}$$

At the interface between two media, the waves of propagation are in equilibrium for all time. Hence they must satisfy the boundary conditions that the pressure waves and the net particle velocity must be continuous across the boundary.

$$p_B = p_A, \tag{3.189}$$

$$\Rightarrow p_B^t = p_A^i + p_A^r, \tag{3.190}$$

$$\Rightarrow A_B^t e^{j(\omega t - k_B x)} = A_A^i e^{j(\omega t - k_A x)} + A_A^r e^{j(\omega t + k_A x)} \tag{3.191}$$

At $x = 0$ for all time, the pressure continuity equation implies

$$A_B^t = A_A^i + A_A^r. \tag{3.192}$$

The velocity continuity equation implies that

$$v_B = v_A, \tag{3.193}$$

$$\Rightarrow v_B^t = v_A^i + v_A^r, \tag{3.194}$$

$$\Rightarrow \frac{A_B^t e^{j(\omega t - k_B x)}}{\rho_B c_B} = \frac{A_A^i e^{j(\omega t - k_A x)}}{\rho_A c_A} - \frac{A_A^r e^{j(\omega t + k_A x)}}{\rho_A c_A}. \tag{3.195}$$

At $x = 0$ for all time, the velocity continuity equation implies

$$\frac{A_B^t}{\rho_B c_B} = \frac{A_A^i - A_A^r}{\rho_A c_A}. \tag{3.196}$$

By using Eq. (3.192) in Eq. (3.196), the pressure reflection coefficient r is

$$r_{AB} = \frac{A_A^r}{A_A^i} = \frac{\rho_B c_B - \rho_A c_A}{\rho_B c_B + \rho_A c_A} = \frac{Z_B - Z_A}{Z_B + Z_A}. \tag{3.197}$$

The pressure transmission coefficient t is

$$t_{AB} = \frac{A_B^t}{A_A^i} = \frac{2\rho_B c_B}{\rho_B c_B + \rho_A c_A} = \frac{2Z_B}{Z_B + Z_A}. \tag{3.198}$$

The intensity I has to be continuous across material boundary. Hence

$$I_B = I_A, \tag{3.199}$$
$$\Rightarrow I_B^t = I_A^i - I_A^r, \tag{3.200}$$
$$\Rightarrow \frac{|A_B^t|^2}{\rho_B c_B} = \frac{|A_A^i|^2}{\rho_A c_A} - \frac{|A_A^r|^2}{\rho_A c_A}. \tag{3.201}$$

The (intensity) reflection coefficient R is the reflected acoustic power over the incident acoustic power

$$R = |r_{AB}|^2 = \frac{W_A^r}{W_A^i} = \frac{I_A^r \,\text{area}}{I_A^i \,\text{area}} = \left|\frac{A_A^r}{A_A^i}\right|^2 = \left|\frac{\rho_B c_B - \rho_A c_A}{\rho_B c_B + \rho_A c_A}\right|^2 = \left|\frac{Z_B - Z_A}{Z_B + Z_A}\right|^2. \tag{3.202}$$

The (intensity) transmission coefficient T is the transmitted acoustic power, W, over the incident acoustic power

$$T = \frac{W_B^t}{W_A^i} = \frac{I_B^t \,\text{area}}{I_A^i \,\text{area}} = 4\frac{\rho_A c_A \rho_B c_B}{\Big(\rho_B c_B + \rho_A c_A\Big)^2} = 4\frac{Z_A Z_B}{\Big(Z_B + Z_A\Big)^2}. \tag{3.203}$$

We summarize the pressure amplitude and intensity coefficients for both reflection and transmission

$$t_{AB} = 1 + r_{AB}, \tag{3.204}$$
$$R + T = 1. \tag{3.205}$$

We note that $r_{AB} = -r_{BA}$.

We consider a general acoustic slab. At the left of an interface, the acoustic wave consists of a forward-propagating component denoted by a_1 and a reflected component b_1. At the right interface, the corresponding forward and backward propagating components are a_2 and b_2, respectively. The transfer matrix for waves traversing from medium A to medium B is [57]

$$\mathcal{M}_{\text{interface}} = \frac{1}{t_{BA}}\begin{bmatrix} t_{AB}t_{BA} - r_{AB}r_{BA} & r_{BA} \\ -r_{AB} & 1 \end{bmatrix}, \tag{3.206}$$

and for waves traveling a distance d is

$$\mathcal{M}_{\text{propagation}} = \begin{bmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{bmatrix}, \tag{3.207}$$

where $\phi = kd = (\omega/v)d$.

A schematic diagram of a 1D phononic crystal having a lattice constant Λ with two different materials of ρ_g, c_g and ρ_s, c_s, where subscripts g means gold, and s means silicon, is shown in Fig. 3.17. One period consists of half of an electrode, let us say half of positive electrode in Fig. 3.17, followed by interface reflection and transmission, followed by propagation in silicon etc as depicted in Fig. 3.17.

The transfer matrix for a sound wave traveling a unit cell, as shown in Fig. 3.17, is

$$\mathcal{M} = \mathcal{M}_9\mathcal{M}_8\mathcal{M}_7\mathcal{M}_6\mathcal{M}_5\mathcal{M}_4\mathcal{M}_3\mathcal{M}_2\mathcal{M}_1, \tag{3.208}$$

where the transfer matrices are explicitly written here for the sake of clarity

$$\mathcal{M}_1 = \mathcal{M}_9 = \begin{bmatrix} e^{-i\phi_{½}^{\text{gold}}} & 0 \\ 0 & e^{i\phi_{½}^{\text{gold}}} \end{bmatrix}, \tag{3.209}$$

where $\phi_{½}^{\text{gold}} = kL_{\text{electrode}}/2 = (\omega/v_{\text{gold}})L_{\text{electrode}}/2$.

$$\mathcal{M}_2 = \mathcal{M}_6 = \frac{1}{t_{sg}} \begin{bmatrix} t_{gs}t_{sg} - r_{gs}r_{sg} & r_{sg} \\ -r_{gs} & 1 \end{bmatrix}, \tag{3.210}$$

$$\mathcal{M}_3 = \mathcal{M}_7 = \begin{bmatrix} e^{-i\phi^{\text{silicon}}} & 0 \\ 0 & e^{i\phi^{\text{silicon}}} \end{bmatrix}, \tag{3.211}$$

where $\phi^{\text{silicon}} = kL_{\text{silicon}} = (\omega/v_{\text{silicon}})L_{\text{silicon}}$.

$$\mathcal{M}_4 = \mathcal{M}_8 = \frac{1}{t_{gs}} \begin{bmatrix} t_{sg}t_{gs} - r_{sg}r_{gs} & r_{gs} \\ -r_{sg} & 1 \end{bmatrix}, \tag{3.212}$$

$$\mathcal{M}_5 = \begin{bmatrix} e^{-i\phi^{\text{gold}}} & 0 \\ 0 & e^{i\phi^{\text{gold}}} \end{bmatrix}, \tag{3.213}$$

where $\phi^{\text{gold}} = kL_{\text{electrode}} = (\omega/v_{\text{gold}})L_{\text{electrode}}$.

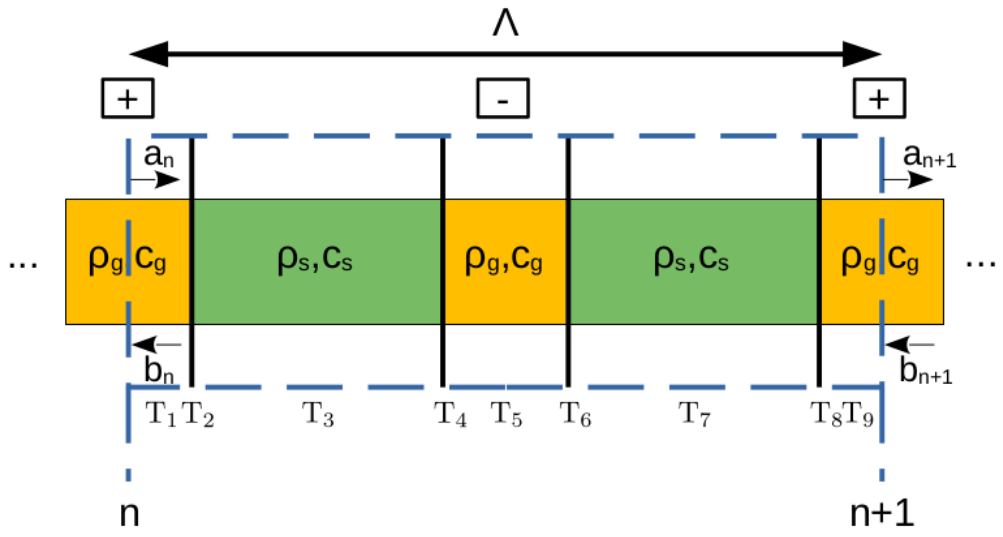


Figure 3.17.: *1D phononic crystal with the period of length Λ, which consists of two materials with parameters {ρ_i, c_i}, with $i = g, s$, where g means gold, s means silicon.*

Therefore, we can write the amplitudes using the transfer matrix notation

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \mathcal{M} \begin{bmatrix} a_n \\ b_n \end{bmatrix}. \tag{3.214}$$

Since this is an infinite periodic structure, we can invoke the Bloch's theorem and write

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = e^{-i\beta\Lambda} \begin{bmatrix} a_n \\ b_n \end{bmatrix}, \tag{3.215}$$

where β is the Bloch wave vector or effective propagation constant. Hence

$$\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} = e^{-i\beta\Lambda} \begin{bmatrix} a_n \\ b_n \end{bmatrix}. \tag{3.216}$$

The eigenvalues of the above equation are $\lambda = e^{-i\beta\Lambda}$, and these are the roots of the characteristic equation [58]

$$\Delta_T e^{i\beta\Lambda} + e^{-i\beta\Lambda} = \mathcal{M}_{11} + \mathcal{M}_{22}. \tag{3.217}$$

For lossless system, the matrix $\mathcal{M}$ is unitary, and the determinant of $\mathcal{M}$, i. e., $\Delta_{\mathcal{M}} = \mathcal{M}_{11}\mathcal{M}_{22} -$

$\mathcal{M}_{12}\mathcal{M}_{21} = 1$. Hence

$$\cos(\beta\Lambda) = \frac{1}{2}(\mathcal{M}_{11} + \mathcal{M}_{22}). \tag{3.218}$$

Moreover, for symmetrical network $\mathcal{M}_{22} = \mathcal{M}_{11}^*$ and this implies

$$\cos(\beta\Lambda) = \Re[\mathcal{M}_{11}]. \tag{3.219}$$

The dispersion relation in such an infinite 1D periodic phononic crystal is

$$\beta = \frac{1}{\Lambda}\cos^{-1}(\Re[\mathcal{M}_{11}(\omega)]). \tag{3.220}$$

Let us consider a 1D phononic crystal made up of 200 nm electrode width and 2 µm silicon width. The material properties of silicon and gold are provided in Chap. A. The computed band structure [4] is shown in Fig. 3.18.

We know that the phase velocity $v_{\text{phase}} = \omega/k$ is the slope [5] of the dispersion curve in Fig. 3.18. The sound wave velocity is related to material parameters via $v_{\text{sound}} = \sqrt{c_{11}/\rho}$. We take arithmetic average of the densities of the two different materials in one lattice as the effective density. From this we can extract the stiffness parameter of such a composite infinite phononic crystal.

The stiffness measures rigidity of an elastic material, it has small values for easily deformed materials and large values for rigid materials. For a composite material having air holes, we expect to obtain lower stiffness constant as well as density. In addition, by phononic band gap engineering, we can get tailor sound velocity, and hence tailor stiffness constant, unattainable in conventional bulk materials. In the bandgap region, no wave can propagate, hence the acoustic velocity is zero, it means that waves feel zero material stiffness as it travels through, although there are gold and silicon layers present. We derived earlier that the electromechanical transduction efficiency scales inversely with the material stiffness constant, and scales with the square of the DC bias voltage, i. e.,

$$\kappa^2 \propto \frac{V_0^2}{c_{11}}. \tag{3.221}$$

Hence we can achieve higher κ^2 using phononic crystal by engineering the stiffness constant. This is an added advantage for low power operation of a device.

The Matlab code for the 1D Phononic bandstructure calculation using the Transfer Matrix Method in this Section is attached in Chap. B.

[4] It has to be noted that the band structure in Fig. 3.18 is not electrically stressed.

[5] Multiply by 2π for correctness as per diagram in Fig. 3.18.

3.8. Derivation of Artificial Piezo-constants

In this Section, we shall first derive the effective (stiffened) stiffness in a 1D piezo medium. For such a medium with periodic piezoelectric profile, we use plane wave expansion method to find the eigenmodes and dispersion diagram. Later we shall find the connecting artificial piezoelectric stress e-constant to compare with the naturally occurring piezo e-constant.

3.8.1. Effective Stiffness

In a 1D medium, the acoustic displacement is related to the electric potential by the following equations [3]

$$\frac{\partial}{\partial x}\left(c_{11}\frac{\partial u}{\partial x}\right) - \rho\frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial x}\left[e(x)\frac{\partial \phi}{\partial x}\right], \tag{3.222}$$

$$\mu_0\frac{\partial}{\partial x}\left(\epsilon\frac{\partial \phi}{\partial x}\right) = \mu_0\frac{\partial}{\partial x}\left(e\frac{\partial u}{\partial x}\right). \tag{3.223}$$

where u is the acoustic displacement and ϕ is the electric potential. The e-constant is defined as stress developed over applied electric field. The electric field is

$$E = -\frac{\partial \phi}{\partial x} = -\frac{e(x)}{\epsilon(x)}\frac{\partial u}{\partial x}. \tag{3.224}$$

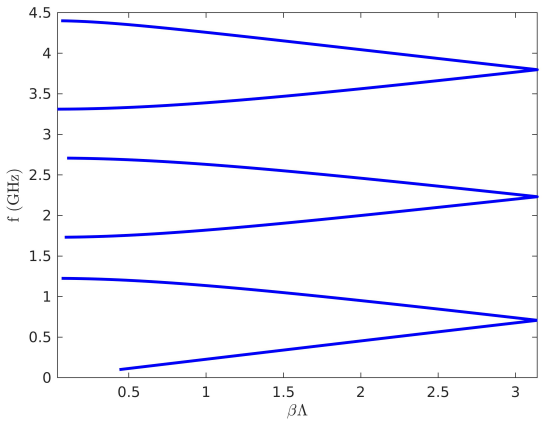


Figure 3.18.: *1D phononic crystal with the period of length Λ, which consists of two materials with parameters $\{\rho_i,\ c_i\}$, with $i = g, s$, where g means gold, s means silicon.*

After manipulation of Eq. (3.222) using Eq. (3.224), we arrive at

$$\frac{\partial}{\partial x}\left(c_{\text{eff}}(x)\frac{\partial u}{\partial x}\right) = \rho\frac{\partial^2 u}{\partial t^2}, \tag{3.225}$$

where the stiffened stiffness profile is

Stiffened Stiffness

$$c_{\text{eff}}(x) = c_{11} + \frac{e^2(x)}{\epsilon(x)}. \tag{3.226}$$

3.8.2. 1D Bandstructure using Plane Wave Expansion Method

If we assume harmonic time variation $u \sim e^{j\omega t}$ in Eq. (3.225), we arrive at

$$\frac{\partial}{\partial x}\left(c_{\text{eff}}(x)\frac{\partial u}{\partial x}\right) = -\rho\omega^2 u(x). \tag{3.227}$$

We make $c_{\text{eff}}(x)$ periodic by spatially periodic voltage distribution through the $e(x)$-constant. We solve the above eigenvalue equation for the case of periodic piezoelectric profile $c_{\text{eff}}(x)$ using the plane wave expansion method. If the stiffness is periodic, we can write

$$c_{\text{eff}}(x) = c_{\text{eff}}(x+L), \tag{3.228}$$

$$c_{\text{eff}}(x) = \sum_{n=-\infty}^{\infty} c_n e^{-jk_n x}, \tag{3.229}$$

$$c_n = \frac{1}{L}\int_0^L c_{\text{eff}}(x) e^{jk_n x}\, dx. \tag{3.230}$$

Using the Floquet's theorem assuming that the wave propagates in the x-direction, we can write the acoustic displacement as

$$u(x) = e^{-ikx}U(x), \tag{3.231}$$

where the eigenmode $U(x)$ and has the periodicity of the lattice,

$$U(x) = U(x+L), \tag{3.232}$$

$$U(x) = \sum_{m=-\infty}^{\infty} u_m e^{-ik_m x}, \qquad k_m = \frac{2m\pi}{L}. \tag{3.233}$$

Hence we can write

$$u(x) = e^{-ikx}U(x) = e^{-ikx}\sum_m u_m e^{-ik_m x} = \sum_m u_m e^{-i(k+k_m)x}. \tag{3.234}$$

Substituting Eq. (3.229) and Eq. (3.234) in Eq. (3.227), we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_n c_n e^{-jk_n x}\frac{\mathrm{d}}{\mathrm{d}x}\sum_m u_m e^{-j(k+k_m)x}\right\} = -\rho\omega^2\sum_m u_m e^{-j(k+k_m)x}, \tag{3.235}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_n\sum_m -j(k+k_m)c_n u_m e^{-j(k+k_m+k_n)x}\right\} = -\rho\omega^2\sum_m u_m e^{-j(k+k_m)x}, \tag{3.236}$$

$$\Rightarrow \sum_n\sum_m (k+k_m)(k+k_m+k_n)c_n u_m e^{-j(k+k_m+k_n)x} = \rho\omega^2\sum_m u_m e^{-j(k+k_m)x}. \tag{3.237}$$

Dropping e^{-kx} from both sides, we get

$$\sum_n\sum_m (k+k_m)(k+k_m+k_n)c_n u_m e^{-j(k_m+k_n)x} = \rho\omega^2\sum_m u_m e^{-jk_m x}. \tag{3.238}$$

Multiplying both sides by $e^{jk_l x}$ and integrating over $0 < x < L$, we get

$$\sum_n\sum_m (k+k_m)(k+k_m+k_n)c_n u_m \int_0^L e^{-j(k_m+k_n-k_l)x}\,dx = \rho\omega^2\sum_m u_m\int_0^L e^{-j(k_m-k_l)x}\,dx \tag{3.239}$$

We know

$$\int_0^L e^{-j(k_m-k_l)x}\,dx = \begin{cases} L, & k_m = k_l, \quad (m=l) \\ 0 & k_m \neq k_l. \end{cases} \tag{3.240}$$

The left hand side term of Eq. (3.239) is nonzero for $m+n-l=0$, meaning $n=l-m$, the right hand side term is non-zero for $m-l=0$, meaning $m=l$. The Eq. (3.239) becomes for nonzero terms

$$\sum_m (k+k_m)(k+k_m+k_{l-m})c_{l-m}u_m = \rho\omega^2 u_l, \tag{3.241}$$

where

$$k_m = \frac{2m\pi}{L}, \quad k_{l-m} = \frac{2(l-m)\pi}{L}, \quad k_m + k_{l-m} = \frac{2l\pi}{L}. \tag{3.242}$$

The m and n are truncated symmetrically about zero for this type of Fourier expansion. The

Eq. (3.241) becomes (relabeling ℓ as n)

$$\sum_{m=-\infty}^{\infty} \left(k + \frac{2m\pi}{L}\right)\left(k + \frac{2n\pi}{L}\right) c_{n-m} u_m = \rho\omega^2 u_n. \tag{3.243}$$

We can cast it as eigenvalue equation,

$$\mathcal{A}u = \lambda u, \tag{3.244}$$

where the eigenvalues are $\lambda = \rho\omega^2$, and the matrix entries are

$$A_{nm} = \left(k + \frac{2n\pi}{L}\right)\left(k + \frac{2m\pi}{L}\right) c_{n-m}. \tag{3.245}$$

The only remaining parameter is the Fourier coefficients c_{n-m}. For the square stiffness profile in Fig. 3.19,

$$c_{(n-m)} = \begin{cases} \frac{2}{L}\left[c_g d + c_s\left(\frac{L}{2} - d\right)\right], & \text{if } (n-m) = 0, \\ 0, & \text{if } (n-m) \text{ is odd}, \\ \frac{\cos((n-m)\pi/2)}{(n-m)\pi/2} \sin\left(\frac{k_{(n-m)}d}{2} - \frac{(n-m)\pi}{2}\right)\left(c_g - c_s\right), & \text{if } (n-m) \text{ is even } \backslash\{0\}. \end{cases} \tag{3.246}$$

Using plane wave expansion, the bandstructure of 1D phononic crystal made up of 200 nm electrode width and 2 µm silicon width is shown in Fig. 3.20. The material properties of silicon and gold are provided in Chap. A.

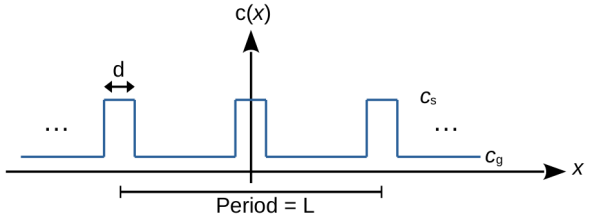


Figure 3.19.: *An even square wave stiffness profile. The width of an electrode is d, one period has two opposing electric fields, as shown in Fig. 3.17. The silicon and gold stiffnesses are denoted by c_s, c_g respectively.*

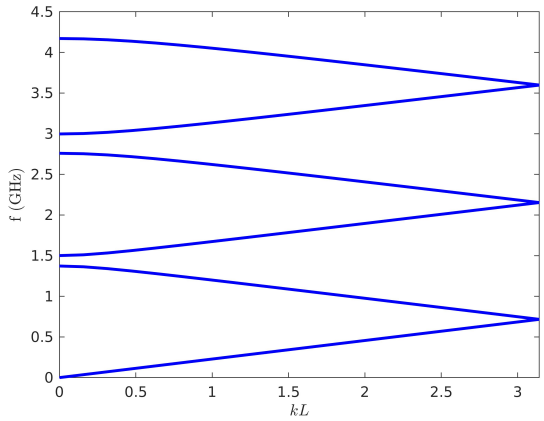


Figure 3.20.: *Bandstructure of 1D phononic crystal having stiffness profile in Fig. 3.19, electrode width* $d = 200$ nm*, period* $L = 4.4$ µm*.*

3.8.3. Artificial Piezoelectric Constant

From Eq. (2.35) and Eq. (2.36), the linear constitutive relations in a 1D piezo material are [3], [49]

$$S = s^{\mathrm{E}}\, T + d\, E, \tag{3.247}$$

$$D = d\, T + \epsilon^{\mathrm{T}}\, E. \tag{3.248}$$

The small static perturbation are denoted by subscript δS, δE, δT from the equilibrium values denoted by S_0, E_0, T_0 respectively. We can write

$$\delta S = s^{\mathrm{E}}\, \delta T + d\, \delta E, \tag{3.249}$$

$$\delta D = d\, \delta T + \epsilon^{\mathrm{T}}\, \delta E. \tag{3.250}$$

We can determine the piezoelectric charge constant d from

$$d = \frac{\delta S}{\delta E}\Big|_{\delta T = 0} \tag{3.251}$$

For 1D phononic crystal, let the applied electric field between the two electrodes is

$$E_x = E_0 + \delta E = \frac{V_0 + \delta V}{d_{\mathrm{spacing}}}. \tag{3.252}$$

From Eq. (4.36) in §4.2, the Maxwell stress is

$$T^E_{xx} = \frac{1}{2}\epsilon E_x^2 = \frac{1}{2}\epsilon\left(E_0 + \delta E\right)^2, \tag{3.253}$$

$$= \frac{1}{2}\epsilon\left(E_0^2 + 2E_0\delta E + \delta E^2\right), \tag{3.254}$$

$$\approx \frac{1}{2}\epsilon\left(E_0^2 + 2E_0\delta E\right), \tag{3.255}$$

$$= T_0 + \delta T. \tag{3.256}$$

where

$$T_0 = \frac{1}{2}\epsilon E_0^2, \tag{3.257}$$

$$\delta T = \epsilon E_0 \delta E. \tag{3.258}$$

The Maxwell stress gives rise to the electric force density $\mathrm{N\,m^{-3}}$,

$$f^E = \frac{\partial}{\partial x} T^E_{xx}. \tag{3.259}$$

We know from the continuum mechanics,

$$\frac{\partial}{\partial x} T^{\text{Cauchy}} + f = \rho \frac{\partial^2 u}{\partial t^2}, \tag{3.260}$$

where $T^{\text{Cauchy}} = c_{11} S_{xx} = c_{11}\partial u/\partial x$.

The equation of motion is

$$c_{11}\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} = -f^E. \tag{3.261}$$

At steady state

$$c_{11}\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} T^{\text{Cauchy}} = -f^E. \tag{3.262}$$

Hence we can write,

$$\frac{\partial}{\partial x} T^{\text{Cauchy}} = -\frac{\partial}{\partial x} T^E_{xx}. \tag{3.263}$$

We arrive at an important conclusion that the Cauchy stress is the negative of the electric stress,

$$T^{\text{Cauchy}} = -T^E_{xx}. \tag{3.264}$$

Now we can write

$$S_{xx} = -\frac{T^E_{xx}}{c_{11}} = -\frac{T_0 + \delta T}{c_{11}} = S_0 + \delta S, \tag{3.265}$$

where

$$S_0 = -\frac{T_0}{c_{11}}, \tag{3.266}$$

$$\delta S = -\frac{\delta T}{c_{11}} = -\frac{\epsilon E_0 \delta E}{c_{11}}, \qquad \text{using Eq. (3.258).} \tag{3.267}$$

We can write the piezoelectric strain constant as

$$d = -\frac{\delta S}{\delta E} = -\frac{\epsilon E_0}{c_{11}}. \tag{3.268}$$

It is well-known that piezoelectric strain constant, d, is related to the piezoelectric stress constant, e via the material stiffness as

$$e = c\, d \tag{3.269}$$

Finally we arrive at

Electric field Induced Artificial e-constant

$$e = -\epsilon E_0. \tag{3.270}$$

This will be the connection between the artificial piezo e-constant with the naturally occurring e-constant. A schematic of the artificial e-constant is provided in Fig. 3.21.

3.8.4. Voltage Induced Phononic Bandstructure Tuning

We derived earlier in Eq. (3.226) that the effective stiffness depends on voltage via the e-constant as

$$c_{\text{eff}}(x) = c_{11} + \frac{e^2(x)}{\epsilon(x)}. \tag{3.271}$$

This paves a way to control the bandstructure using the DC voltage. We note that this is far below the breakdown electric field of silicon which is 35 GV m^{-1}, i. e., 35,000 V/μm; we have a (pure) silicon layer of 2 μm and we apply 2000 V. The authors in [59] used 0-20,000 V, albeit in dielectric elastomer, see §2.3 in [59]. In Fig. 3.22, we compute the bandstructure using zero bias and a bias of $V_0 = 2000$ V for show the effect of voltage controlled bandstructure tuning (the b). We can clearly observe an (upward) change in the bandstructure. This provides the DC voltage as a knob to tune the bandstructure.

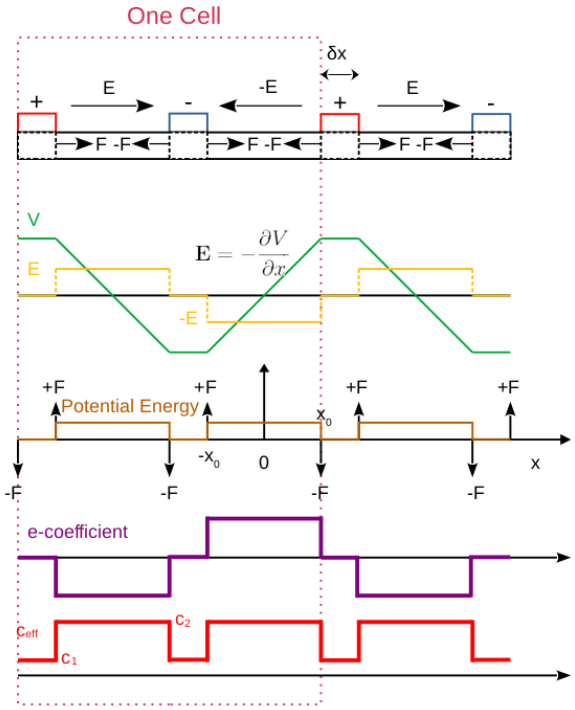


Figure 3.21.: *A schematic diagram of the artificial e-constant in a 1D lattice. The e-constant is zero on metal electrodes, whereas it has finite value on silicon.*

The Matlab code for the 1D Phononic bandstructure calculation using the Plane Wave Expansion Method in this Section is attached in Chap. C.

3.9. Numerical Experiment using FlexPDE

In this Section, we shall simulate the 1D artificial piezorope numerically using finite element method via the commercial software FlexPDE [1].

The Cauchy's equation of continuum mechanics is

$$\frac{\partial T_{xx}}{\partial x} + f(x,t) = \rho \frac{\partial^2 u}{\partial t^2}, \tag{3.272}$$

where T_{xx} is the normal stress, $f(x,t)$ is the forcing function, ρ is the mass density and u is the

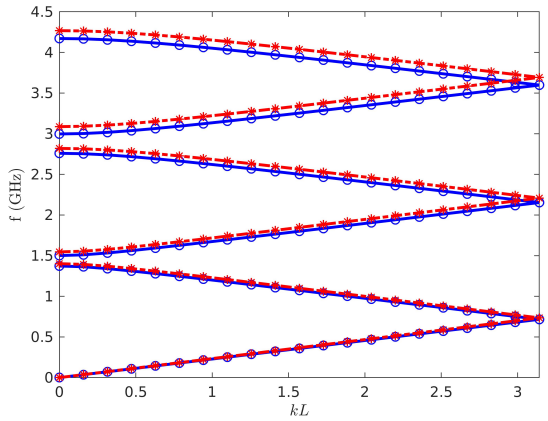


Figure 3.22.: *1D phononic crystal made up of 200* nm *electrode width and 2* µm *silicon width. The circle solid line (blue color) is the dispersion graph without applied voltage, whereas the * dashed line (red color) is for the bandstructure with* $V_0 = 2000$ V*. The appropriate permittivities of gold and silicon are taken into account in respective sections. It is clearly observed that the dispersion graph is shifted upwards with the application of voltage. This gives the possibility to tune the bandstructure.*

acoustic displacement. Using Hooke's law, incorporating loss in the material, we can write

$$\frac{\partial}{\partial x}\left(c_{11}S_{xx} + \eta_{11}\frac{\partial S_{xx}}{\partial t}\right) + f(x,t) = \rho\frac{\partial^2 u}{\partial t^2}, \tag{3.273}$$

where S_{xx} is the strain and η_{11} is the damping constant. Using the relation of strain and acoustic displacement, we arrive at

$$\frac{\partial}{\partial x}\left(c_{11}\frac{\partial u}{\partial x} + \eta_{11}\frac{\partial}{\partial t}\frac{\partial u}{\partial x}\right) + f(x,t) = \rho\frac{\partial^2 u}{\partial t^2}. \tag{3.274}$$

The transient Eq. (3.274) contains both the DC voltage and excitation signal via the forcing function $f(x,t)$. We can use the following forcing function from Eq. (3.222) in §3.8.1,

$$f(x,t) = \frac{\partial}{\partial x}\left[e(x)\frac{\partial\phi(x,t)}{\partial x}\right], \tag{3.275}$$

where $\phi(x,t)$ is the combined DC voltage and excitation signal, i. e., $\phi(x,t) = V_{\text{dc}} + V_{\text{signal}}e^{i\omega t}$.

The artificial piezoelectric e-constant is not constant like in the naturally occurring piezo material, rather it is design/geometry dependent. Alternatively, we can use the forcing function using the Maxwell stress tensor, we shall derive it for the generic 3D in §4.2. A detailed analysis of various analytical solutions is compared in recent publication [30]. Since it is simple geometry we can use the well-known analytical formula for force due to Coulomb attraction.

If we operate in the neighborhood of the DC voltage V_{dc}, we can linearize the response due to small perturbation of excitation signal. Since the time-domain simulation entails a long computation time, we can view the response through the frequency domain. Assuming $u(x,t) = u(x)e^{i\omega t}$, we arrive at

$$\frac{\mathrm{d}u}{\mathrm{d}x}\left(c_{11}\frac{\mathrm{d}u}{\mathrm{d}x} + i\omega\eta_{11}\frac{\mathrm{d}u}{\mathrm{d}x}\right) + f(x) = -\rho\omega^2 u(x). \tag{3.276}$$

Here the DC voltage in the forcing function and the mechanical response u due to the DC bias is completely washed out. We can incorporate the DC bias mimicking the physics, where the charge lies in thin space-charge regions in capacitor plates. We artificially create such thin charge regions and apply DC bias, as shown via blue strips in Fig. 3.23. We can apply the excitation signal to the place of interest. As an example, we apply the excitation signal on the leftmost boundary, as marked in black line in Fig. 3.23.

We derived in Eq. (3.24c), that the Coulomb force due to DC bias is

$$|f(x)| = |F_0| \approx \frac{\epsilon A}{2d^2_{\text{silicon-strip}}} V_0^2 \left(1 + \frac{\epsilon V_0^2}{c_{11} d^2_{\text{silicon-strip}}}\right). \tag{3.277}$$

The mechanical perturbation due to excitation signal δV after creation of the dipole, as we

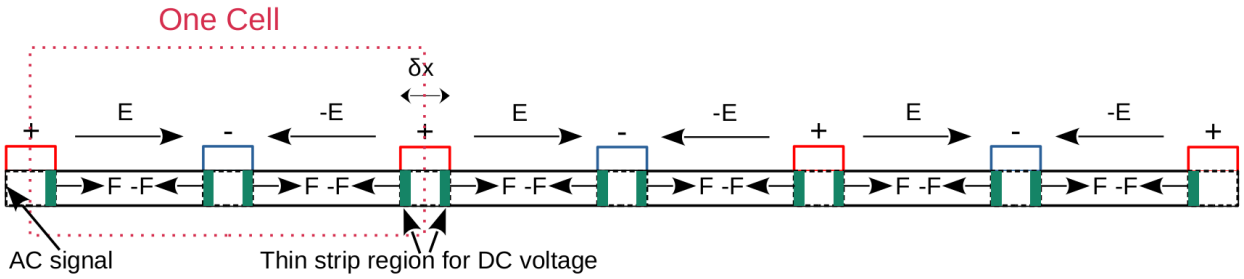


Figure 3.23.: *A schematic diagram of 1D artificial piezorope showing three units. The DC voltage is shown via blue strips, the excitation signal is applied at the leftmost boundary. The natural boundary condition is applied for signal to propagate.*

derived in Eq. (3.59) is

$$\delta L = \frac{\frac{2\beta}{c_{11}k} \cdot \frac{F_0}{A}}{\left(1 + \frac{2\beta}{c_{11}k} \cdot \frac{F_0}{AL_{\text{eq}}}\right)} \cdot \frac{\delta V}{V_0}. \tag{3.278}$$

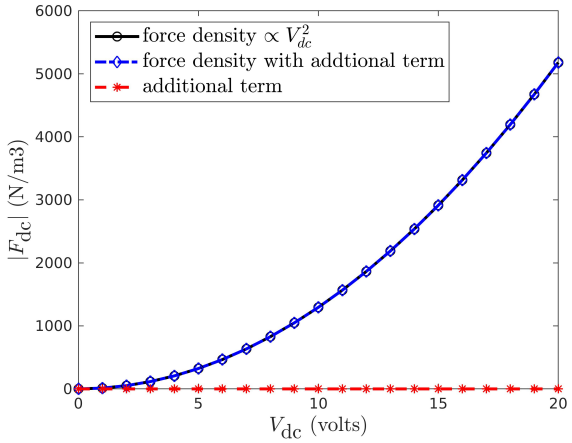
We plot Eq. (3.277), Eq. (3.278) in Fig. 3.24. As can be seen from Fig. 3.24a, the additional term $\frac{\epsilon^2 V_0^4}{2c_{11}d_{\text{silicon}}^4}$ in Eq. (3.277) is insignificant, hence we can approximate the force due to DC bias as

$$|f(x)| = |F_0| \approx \frac{\epsilon A}{2d_{\text{silicon-strip}}^2} V_0^2. \tag{3.279}$$

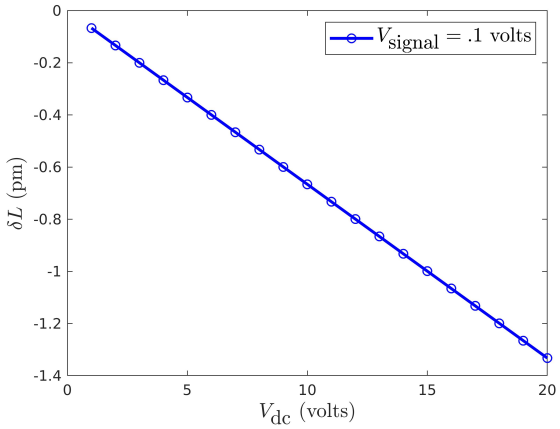
The δL for varying DC voltage with an application of excitation signal of 0.1 V is plotted in Fig. 3.24b. For a fixed DC bias at 10 volt, δL is plotted for varying signal voltage in Fig. 3.24c. In reality the signal voltage is few orders of magnitude lower than the DC bias. The mechanical displacement for a 1D rope with a DC bias $V_0 = 10$ V, modulation voltage $\delta V = 0.1$ V is plotted in Fig. 3.24d. We observe a resonance behavior at ≈ 2.13 GHz. It has to be borne in mind that this rope does not have any finite width metal electrodes. However, these plots provide both a qualitative and quantitative understanding of what we can expect when we numerically solve the piezorope.

We numerically solve the Eq. (3.276) using the DC voltage in Eq. (3.277) and the perturbation displacement δL due to the signal δV using Eq. (3.278). Since this is a resonance phenomenon and we are interested to operate very close to the resonance peak, the signal voltage will be very small in compare to the DC bias, but the acoustic displacement will be very large. We apply a DC voltage $V_0 = 10$ V, modulation voltage $\delta V = 0.1$ V. The purpose of the metal electrodes is purely for electrical connection. Since we expect a large acoustic displacement near/at resonance and we are interested to operate at this regime, we can make the electrical connection as thin we can practically fabricate, rather than $\lambda/4$ wide we derived in §3.5.1. We choose width of the metal electrode to be 200 nm, the width of the space charge region for DC bias to be 50 nm for numerical computation. We can pattern 200 nm gold electrode with ease in the fabrication facilities nanoFAB at the University of Alberta. In the numerical implementation, we tucked in the 50 nm space charge region inside the metal electrode. The width of silicon region is 2 µm. The permittivity, stiffness, mass density, damping for gold and silicon are implemented in the respective region. The boundary condition for u FlexPDE is set to *natural* for signal to propagate from the leftmost boundary, where I apply the δL due to the signal δV using Eq. (3.278). For a finite three-cell, the mechanical displacement is shown in Fig. 3.25 near resonance and the frequency response is shown in Fig. 3.26. The computation is performed strictly in 1D – not finite width/truncated 2D – using FlexPDE [1].

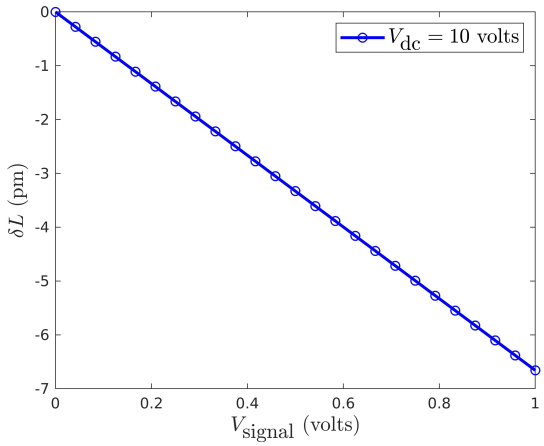
The FlexPDE code of all the plots in this Section is attached in Chap. D.



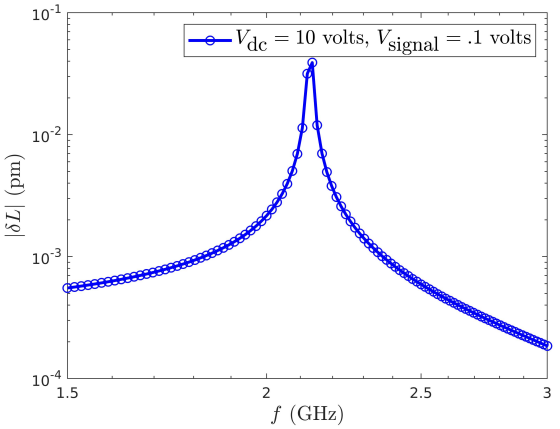
(a) The force density ($\mathrm{N\,m^{-3}}$).



(b) δL versus DC bias.



(c) δL versus excitation signal.



(d) Frequency response of u in a 1D rope.

Figure 3.24.: *These figures are for a finite-width 1D rope when stressed by DC voltage, excitation signal.* (a) *The Coulomb force due to varying DC bias. The additional force density term* $\epsilon^2 V_0^4 / \left(2c_{11} d_{silicon}^4\right)$ *in Eq.* (3.277) *is insignificant. We can clearly observe a parabolic force with voltage. We shall bias the bar in the neighborhood of a DC bias and linearize the response.* (b) *The perturbation* δL *due to excitation signal with a DC bias for an excitation signal of 0.1* V. (c) *The perturbation* δL *due to excitation signal as a function of excitation signal with 10* V *DC bias.* (d) *The frequency dependence of the small perturbation. We can observe a resonance behavior around 2.13* GHz.

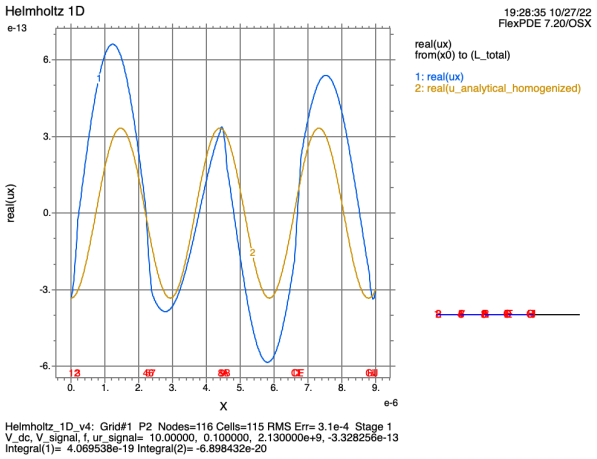


Figure 3.25.: *(Blue color) The mechanical displacement at* $f = 2.13$ GHz *for the DC voltage* $V_0 = 10$V*, modulation voltage* $\delta V = 0.1$ V *in a three-cell piezorope made of silicon and gold with the silicon width of 2* µm *and gold electrode width of 200* nm *with 50* nm *space charge region for emulating DC voltage. (Yellow color) The homogenized wave equation in such a 1D piezorope after Eq.* (3.289)*.*

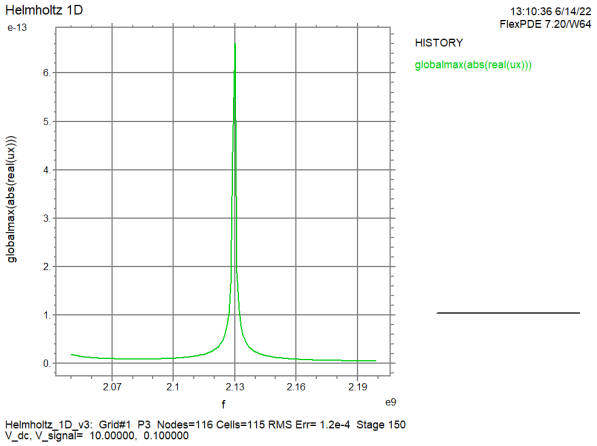


Figure 3.26.: *The frequency response of the acoustic displacement, a DC voltage* $V_0 = 10$V*, modulation voltage* $\delta V = 0.1$ V*.*

3.9.1. Analytical Solution of Piezoelectric Wave Equation

The coupled equations to be solved in 1D are [50]

$$\frac{\partial^2 u}{\partial x^2} - \frac{\rho}{c}\frac{\partial^2 u}{\partial t^2} = -\frac{e}{c}\frac{\partial E}{\partial x}, \tag{3.280}$$

$$\frac{\partial^2 E}{\partial x^2} - \mu\epsilon\frac{\partial^2 E}{\partial t^2} = \mu e\frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial x}\right). \tag{3.281}$$

The complimentary homogeneous equations are

$$\frac{\partial^2 u}{\partial x^2} - \frac{\rho}{c}\frac{\partial^2 u}{\partial t^2} = 0, \tag{3.282}$$

$$\frac{\partial^2 E}{\partial x^2} - \mu\epsilon\frac{\partial^2 E}{\partial t^2} = 0. \tag{3.283}$$

The homogeneous solutions are

$$u(x,t) = u_0 e^{i(\omega t - \kappa x)}, \quad \kappa = \omega\sqrt{\rho/c}, \tag{3.284}$$

$$E(x,t) = E_0 e^{i(\omega t - kx)}, \quad k = \omega\sqrt{\mu\epsilon}. \tag{3.285}$$

The particular solution for the acoustic field is

$$u_p(x,t) = Ae^{i(\omega t - kx)}. \tag{3.286}$$

The amplitude of the particular solution is derived to be

$$A = \frac{i\frac{e}{c}\sqrt{\mu\epsilon}E_0}{\omega\left[\frac{\rho}{c} - \mu\epsilon\right]}. \tag{3.287}$$

Hence the total solution for the acoustic field is

$$u_{\text{total}} = u_{\text{homogeneous}} + u_{\text{particular}} \tag{3.288}$$

The analytical expression for the plane wave displacement field, after Eq. (II.53) in [50], is

$$u(x,t) = \left[u_0 e^{-i\kappa x} + i\frac{e/c\sqrt{\mu\epsilon}}{\omega\left(\rho/c - \mu\epsilon\right)} \cdot \delta E \cdot e^{-ikx}\right] e^{i\omega t}, \tag{3.289}$$

where the acoustic wavevector $\kappa = \omega\sqrt{\rho/c}$, electromagnetic wavevector $k = \omega\sqrt{\mu\epsilon}$.

For the inverse problem, the total electric field, $E_{\text{total}} = E_{\text{homogeneous}} + E_{\text{particular}}$, is

$$E(x,t) = \left[E_0 e^{-ikx} + i \frac{\mu e \omega \sqrt{\frac{\rho}{c}}}{\mu\epsilon - \frac{\rho}{c}} u_0 e^{-i\kappa x} \right] e^{i\omega t}, \tag{3.290}$$

3.9.2. Homogenization of Acoustic & Electromagnetic Parameters

The proper way to homogenize the structure in Fig. 3.17 would be to compute both the acoustic and electromagnetic bandstructure and extract the parameters as *felt* by the acoustic wave or by the electric field for the inverse operation. We computed the acoustic bandstructure using the transfer matrix method in §3.7 and using the plane wave expansion method in §3.8.2. We computed the electromagnetic bandstructure as well, as shown in Fig. 3.27, using the well written monograph in [60].

If we homogenize the structure using effective mean theory (without further consideration), we find the homogenized density to be

$$\rho_{\text{homo}} = (2n+1)\frac{L_{\text{gold}}}{L_{\text{total}}}\rho_{\text{gold}} + 2n\frac{L_{\text{silicon}}}{L_{\text{total}}}\rho_{\text{silicon}}, \tag{3.291}$$

where there are n number of unit cells. In similar fashion, we homogenize (without further consideration) relative permittivity ϵ_r, stiffness c_{11}, viscosity η_{11}, piezo-electric e-constant in gold and silicon. In gold, the e-constant is zero, whereas in silicon, it has finite value dictated by the DC bias V_0, as shown schematically in Fig. 3.21.

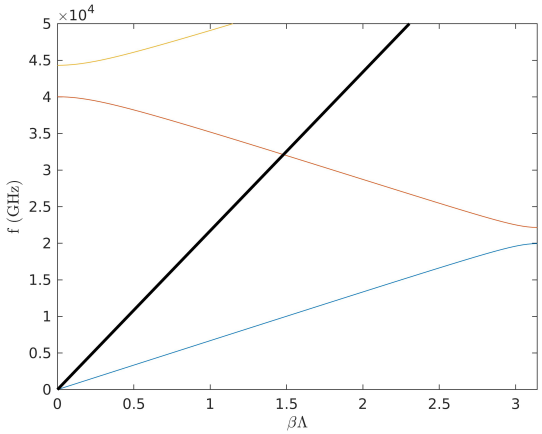


Figure 3.27.: *The 1D electromagnetic bandstructure of Fig. 3.17 using the programs in [60]. The black line is the lightline.*

For comparison, if we view the structure in Fig. 3.23 as a homogeneous slab of material, the traveling acoustic displacement is shown via yellow color plot using Eq. (3.289) in Fig. 3.25.

Please note that in this 1D case, for a given frequency f, we can extract the acoustic wavevector κ, and the electromagnetic wavevector k, if we take a horizontal slice in Fig. 3.20 and Fig. 3.27 respectively. The only parameters remaining to compare the numerical non-homogenized result to the homogenized analytical *equivalent* expression in Eq. (3.289) are the electromagnetic stress constant e and the stiffness c. Since the electric field vanishes inside metal electrodes, and non-zero on silicon layer, as shown schematically in Fig. 3.21, I am unaware of any definite way to take the average or homogenized e-constant and the stiffness c, even in such simple 1D case to consider as the effective e-constant and effective stiffness c. It is to be borne in mind that we are performing the simulation in the frequency domain, in essence we are taking a phase snapshot. We have taken the arithmetic mean value of the e-constant and the stiffness c in gold and silicon in Fig. 3.21, and compared the homogenized result in Fig. 3.25. The comparison of the numerical result using such a homogenized *equivalent* piezoelectric e-constant and the stiffness c is a curve fitting problem. If someone would like to perform detailed comparison, one might resort to running a number of e and c values or better adopt the simulated annealing genetic algorithm. The same argument holds true for the 2D structures in Chap. 4, and the phase snapshot in 2D might be difficult to compare too.

Chapter 4

4 Two-Dimensional Piezoslab

In this chapter, the forcing force density due to the electric field will be formulated from the Maxwell stress tensor. This provides a generic formulation for the force density term that can be applied to an arbitrary topology. The effective stress tensor and the artificial piezoelectric e-constant will be derived. We shall explore voltage induced strain, Poisson's effect, and acoustic responses of a 2D piezosheet with finite number of cells using finite element method via the software FlexPDE [1] in a simple 2D design.

4.1. Design of 2D Silicon Phononic Crystal with Artificial Piezoelectricity

In this Section, we shall design a simple 2D lattice that exhibits artificial piezoelectricity.

In the surface acoustic wave device, the inter-digital transducers (IDTs) electrodes are deposited on top of an inherently piezoelectric substrate, e. g., lithium niobate in Fig. 1.8 [26]. The electric field is coupled via the inherent piezoelectric constants – one form is Eq. (2.36). We create artificial dipoles by using two opposing electric fields due to DC voltage to a create compression and rarefaction in a mechanical medium, as shown schematically in Fig. 4.1.

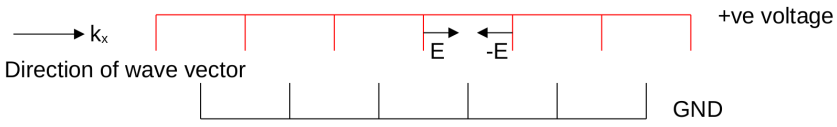


Figure 4.1.: *The DC voltage in two inter-penetrating comb structure creates electric dipoles in opposing directions. These dipoles stretch and compress the mechanical medium and the wave propagates.*

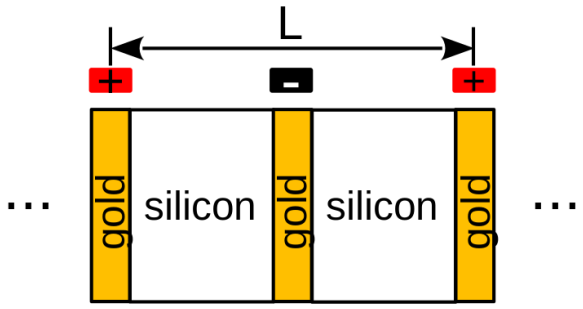


Figure 4.2.: *The (two interdigital) metal electrodes are deposited on top of non-piezoelectric substrate, e. g., silicon. A DC bias applied across a silicon slab. One cell of phononic lattice with period L. The two opposing electrical dipoles can be discerned.*

This allows us to design a 2D structure with DC bias as shown in Fig. 4.2. Since the metal electrodes are for electrical connection, and we wish to operate near/at resonance, when the signal voltage is much smaller than the DC bias, we can make the electrodes less wide, meaning not $\lambda/4$. The electrode with could be 200 nm, the width of the silicon is 2 µm. Using Eq. (3.138), a powerful rule of thumb for choosing the size of the unit is

Scalable Eigenfrequency

$$f_n = n\frac{v_{\text{sound}}}{L}, \qquad \forall n = 1, 2, \cdots, \tag{4.1}$$

where L is the length of the unit cell. We learned in §3.4.3 through Eq. (3.97), that the higher order amplitude decreases, so we would like to operate in the fundamental mode. We can appreciate that the frequency is scalable, meaning in order to have high frequency transducers we can just scale the length of the structure. We can easily fabricate high frequency transducers with the current planar silicon technology without the introduction of exotic materials in the process flow, such as silicon doped hafnium oxide (Si:HfO_2) thin films as done in [7], [47]. We can choose 200 nm electrode width and 2 µm silicon width.

4.2. Maxwell Stress Tensor

In this Section, we shall derive the force density due to electric field that we require in order to solve the Cauchy's equation of continuum mechanics in Eq. (4.52) in an arbitrary geometry. Once the DC voltage distribution is provided, we can solve the Poisson's equation $\nabla^2 V(x, y) = g$ to find out the voltage map $V(x, y)$, where g is the arbitrary voltage distribution. We can use

this to find the electric field and then the driving force density due this electric field.

The Maxwell's equations are

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}, \tag{4.2}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{4.3}$$

$$\nabla\cdot\mathbf{D} = \rho, \tag{4.4}$$

$$\nabla\cdot\mathbf{B} = 0. \tag{4.5}$$

Here $\mathbf{E}$ denotes the electric field, $\mathbf{D}$ the electric displacement, $\mathbf{B}$ the magnetic induction and $\mathbf{H}$ the magnetic field. The free charge density is denoted by ρ, while the free current density by $\mathbf{J}$. The constitutive relations are $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0\left(1+\chi_e\right)\mathbf{E} = \epsilon_0\epsilon_r\mathbf{E} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M} = \mu_0\left(1+\chi_m\right)\mathbf{H} = \mu_0\mu_r\mathbf{H} = \mu\mathbf{H}$ for linear media for a particular frequency. The vacuum electric permittivity is denoted by ϵ_0, the relative permittivity by ϵ_r, the electric susceptibility by χ_e and the (linear) polarization by $\mathbf{P}$. The magnetic permeability in vacuum is denoted by μ_0, relative permeability by μ_r, the magnetic susceptibility by χ_m. Using these constitutive relations and considering vacuum, we can write the Maxwell's equations as

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}, \tag{4.6}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{4.7}$$

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{4.8}$$

$$\nabla\cdot\mathbf{B} = 0. \tag{4.9}$$

The Lorentz's force law, which is not part of the Maxwell's equations, is

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}, \tag{4.10}$$

where $\mathbf{v}$ is the velocity of charge q. The Lorentz's force per unit volume (force density), $\mathbf{f}$, [1] is

$$\mathbf{f} = \rho\mathbf{E} + \rho\mathbf{v}\times\mathbf{B} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}, \tag{4.11}$$

[1] Denoted by small $\mathbf{f}$.

where the charge density is denoted by ρ [2]. We can write

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}, \tag{4.12}$$

$$= \epsilon_0 \left(\nabla\cdot\mathbf{E}\right)\mathbf{E} + \left[\frac{1}{\mu_0}\nabla\times\mathbf{B} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right]\times\mathbf{B}, \tag{4.13}$$

$$= \epsilon_0 \left(\nabla\cdot\mathbf{E}\right)\mathbf{E} + \frac{1}{\mu_0}\left(\nabla\times\mathbf{B}\right)\times\mathbf{B} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}. \tag{4.14}$$

The Poynting vector $\mathbf{S}$ is the energy flowing per unit time per unit area. One form of the Poynting vector is $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$. We can write

$$\frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right) = \frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} + \mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t} = \frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} - \mathbf{E}\times\left(\nabla\times\mathbf{E}\right). \tag{4.15}$$

The volume Lorentz force is

$$\mathbf{f} = \epsilon_0 \left(\nabla\cdot\mathbf{E}\right)\mathbf{E} + \frac{1}{\mu_0}\left(\nabla\times\mathbf{B}\right)\times\mathbf{B} - \epsilon_0\left[\frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right) + \mathbf{E}\times\left(\nabla\times\mathbf{E}\right)\right], \tag{4.16}$$

$$= \epsilon_0 \left(\nabla\cdot\mathbf{E}\right)\mathbf{E} + \frac{1}{\mu_0}\left(\nabla\times\mathbf{B}\right)\times\mathbf{B} - \epsilon_0\frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right) - \epsilon_0\mathbf{E}\times\left(\nabla\times\mathbf{E}\right), \tag{4.17}$$

$$= \epsilon_0\left[\left(\nabla\cdot\mathbf{E}\right)\mathbf{E} - \mathbf{E}\times\left(\nabla\times\mathbf{E}\right)\right] - \frac{1}{\mu_0}\mathbf{B}\times\left(\nabla\times\mathbf{B}\right) - \epsilon_0\frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right). \tag{4.18}$$

The magnetic induction $\mathbf{B}$ field is always solenoidal. In order to look the electric and magnetic field terms symmetric, we can cast the force density as

$$\mathbf{f} = \epsilon_0\left[\left(\nabla\cdot\mathbf{E}\right)\mathbf{E} - \mathbf{E}\times\left(\nabla\times\mathbf{E}\right)\right] + \frac{1}{\mu_0}\left[\left(\nabla\cdot\mathbf{B}\right)\mathbf{B} - \mathbf{B}\times\left(\nabla\times\mathbf{B}\right)\right] - \epsilon_0\frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right). \tag{4.19}$$

We wish to eliminate the curls of the electric and magnetic induction fields. We know from the vector identity

$$\nabla\left(\mathbf{A}\cdot\mathbf{B}\right) = \left(\mathbf{A}\cdot\nabla\right)\mathbf{B} + \left(\mathbf{B}\cdot\nabla\right)\mathbf{A} + \mathbf{A}\times\left(\nabla\times\mathbf{B}\right) + \mathbf{B}\times\left(\nabla\times\mathbf{A}\right). \tag{4.20}$$

If $\mathbf{A} = \mathbf{B}$, we have

$$\frac{1}{2}\nabla\left(\mathbf{A}\cdot\mathbf{A}\right) = \mathbf{A}\times\left(\nabla\times\mathbf{A}\right) + \left(\mathbf{A}\cdot\nabla\right)\mathbf{A}. \tag{4.21}$$

[2] Not to be confused with the mass density throughout the thesis. We prefer not to introduce unique symbol for each quantity, rather follow the notation in the standard literature and introduce new symbols when necessary.

Therefore, the force density is

$$\mathbf{f} = \epsilon_0 \left[\Big(\nabla\cdot\mathbf{E}\Big)\,\mathbf{E} + \Big(\mathbf{E}\cdot\nabla\Big)\,\mathbf{E} - \frac{1}{2}\nabla\mathbf{E}^2 \right] + \frac{1}{\mu_0}\left[\Big(\nabla\cdot\mathbf{B}\Big)\,\mathbf{B} + \Big(\mathbf{B}\cdot\nabla\Big)\,\mathbf{B} - \frac{1}{2}\nabla\mathbf{B}^2 \right] - \epsilon_0 \frac{\partial}{\partial t}\Big(\mathbf{E}\times\mathbf{B}\Big) \tag{4.22}$$

The momentum per unit volume of the electro-magnetic field is $\frac{1}{c^2}\mathbf{S}$. The divergence of the outer/dyadic product is

$$\nabla\cdot\Big(\mathbf{B}\otimes\mathbf{A}\Big) = \mathbf{A}\,\Big(\nabla\cdot\mathbf{B}\Big) + \Big(\mathbf{B}\cdot\nabla\Big)\,\mathbf{A} \tag{4.23}$$

We can make use of the above definition and write

$$\nabla\cdot\Big(\mathbf{E}\otimes\mathbf{E}\Big) = \mathbf{E}\,\Big(\nabla\cdot\mathbf{E}\Big) + \Big(\mathbf{E}\cdot\nabla\Big)\,\mathbf{E} \tag{4.24}$$

$$\nabla\cdot\Big(\mathbf{B}\otimes\mathbf{B}\Big) = \mathbf{B}\,\Big(\nabla\cdot\mathbf{B}\Big) + \Big(\mathbf{B}\cdot\nabla\Big)\,\mathbf{B} \tag{4.25}$$

The Lorentz force per unit volume is

$$\mathbf{f} = \epsilon_0\nabla\cdot\Big(\mathbf{E}\otimes\mathbf{E}\Big) + \frac{1}{\mu_0}\nabla\cdot\Big(\mathbf{B}\otimes\mathbf{B}\Big) - \left(\frac{1}{2}\epsilon_0\nabla\mathbf{E}^2 + \frac{1}{2\mu_0}\nabla\mathbf{B}^2\right) - \epsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t}, \tag{4.26}$$

$$= \nabla\cdot\left[\epsilon_0\mathbf{E}\otimes\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\otimes\mathbf{B} - \frac{1}{2}\left(\epsilon_0\mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2\right)\mathbb{I}\right] - \frac{1}{c^2}\frac{\partial\mathbf{S}}{\partial t}, \tag{4.27}$$

where $\mathbb{I}$ is a unit dyad. We can succinctly write the force density as

$$\mathbf{f} = \nabla\cdot\mathbf{T}^{\text{Maxwell}} - \frac{1}{c^2}\frac{\partial\mathbf{S}}{\partial t}, \tag{4.28}$$

where the Maxwell stress tensor $\mathbf{T}^{\text{Maxwell}}$ is

$$\mathbf{T}^{\text{Maxwell}} = \epsilon_0\mathbf{E}\otimes\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\otimes\mathbf{B} - \frac{1}{2}\left(\epsilon_0\mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2\right)\mathbb{I}. \tag{4.29}$$

As a reminder, the stress $\mathbf{T}^{\text{Maxwell}}$ has units of $\mathrm{N\,m^{-2}}$. Hence the divergence of the stress $\nabla\cdot\mathbf{T}^{\text{Maxwell}}$ has the unit of $\mathrm{N\,m^{-3}}$; it is the momentum density generated unit time. The temporal derivative of the electro-magnetic field's momentum density is denoted by $\epsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t}$.

For the electro-statics and magneto-statics cases,

$$\mathbf{f} = \nabla\cdot\mathbf{T}^{\text{Maxwell}}. \tag{4.30}$$

In the case of electro-statics only,

$$\mathbf{T}^{\text{Maxwell}} = \epsilon_0 \mathbf{E} \otimes \mathbf{E} - \frac{1}{2}\epsilon_0 \mathbf{E}^2 \mathbb{I}. \tag{4.31}$$

The outer product is

$$\mathbf{E} \otimes \mathbf{E} = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \begin{bmatrix} E_x & E_y & E_z \end{bmatrix} \tag{4.32}$$

Now, by changing from vacuum ϵ_0 to material permittivity ϵ, we can incorporate material medium and write the Maxwell stress as

$$\mathbf{T}^{\text{Maxwell}} = \epsilon \begin{bmatrix} E_x^2 & E_x E_y & E_x E_z \\ E_y E_x & E_y^2 & E_y E_z \\ E_z E_x & E_z E_y & E_z^2 \end{bmatrix} - \frac{1}{2}\epsilon \begin{bmatrix} E^2 & 0 & 0 \\ 0 & E^2 & 0 \\ 0 & 0 & E^2 \end{bmatrix}, \tag{4.33}$$

where

$$E^2 = E_x^2 + E_y^2 + E_z^2. \tag{4.34}$$

3D case

For the generic case of 3D, the Maxwell stress tensor is

$$\mathbf{T}^{\text{Maxwell}} = \epsilon \begin{bmatrix} E_x^2 - \frac{E^2}{2} & E_x E_y & E_x E_z \\ E_y E_x & E_y^2 - \frac{E^2}{2} & E_y E_z \\ E_z E_x & E_z E_y & E_z^2 - \frac{E^2}{2} \end{bmatrix}. \tag{4.35}$$

1D case

For 1D, we can write

$$T^{\text{Maxwell}} = \epsilon \frac{1}{2} E_x^2. \tag{4.36}$$

2D case

In the case of 2D, once we have the voltage distribution $V(x, y)$, the electric field is

$$\mathbf{E} = -\nabla V. \tag{4.37}$$

The outer product is

$$\mathbf{E} \otimes \mathbf{E} = \begin{bmatrix} E_x \\ E_y \end{bmatrix} \begin{bmatrix} E_x & E_y \end{bmatrix}. \tag{4.38}$$

The Maxwell stress is

$$\mathbf{T}^{\text{Maxwell}} = \epsilon \begin{bmatrix} E_x^2 & E_x E_y \\ E_y E_x & E_y^2 \end{bmatrix} - \frac{1}{2}\epsilon \begin{bmatrix} E^2 & 0 \\ 0 & E^2 \end{bmatrix} = \epsilon \begin{bmatrix} \frac{E_x^2 - E_y^2}{2} & E_x E_y \\ E_y E_x & -\frac{E_x^2 - E_y^2}{2} \end{bmatrix}. \tag{4.39}$$

The force density is

$$\mathbf{f} = \nabla \cdot \mathbf{T}^{\text{Maxwell}}. \tag{4.40}$$

We can write each component of the force density using

$$f_i = \frac{\partial T_{ij}^{\text{Maxwell}}}{\partial x_j} = T_{ij,j}^{\text{Maxwell}}. \tag{4.41}$$

Finally, we can explicitly write each force density component as

$$f_x = \frac{\partial}{\partial x}\left(\epsilon \frac{E_x^2 - E_y^2}{2}\right) + \frac{\partial}{\partial y}\left(\epsilon E_x E_y\right), \tag{4.42}$$

$$f_y = \frac{\partial}{\partial x}\left(\epsilon E_x E_y\right) - \frac{\partial}{\partial y}\left(\epsilon \frac{E_x^2 - E_y^2}{2}\right). \tag{4.43}$$

We use the Lorentz force density as the driving force in the Cauchy's equation

$$\mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathbf{c} : \nabla^2 \mathbf{u}, \tag{4.44}$$

where ρ is the mass density, $\mathbf{u}$ is the acoustic displacement in meter. In electrostatic case (assuming isotropic stiffness constant $\mathbf{c}$), we have

$$\mathbf{f} = -c\nabla^2\mathbf{u}, \tag{4.45}$$

$$\Rightarrow \begin{bmatrix} f_x \\ f_y \end{bmatrix} = -c \begin{bmatrix} \nabla^2 u_x \\ \nabla^2 u_y \end{bmatrix}, \tag{4.46}$$

$$\Rightarrow \begin{bmatrix} f_x \\ f_y \end{bmatrix} = -c \begin{bmatrix} \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \\ \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \end{bmatrix}. \tag{4.47}$$

We can separately extract each component as,

$$f_x = -c\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right), \tag{4.48}$$

$$f_y = -c\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right). \tag{4.49}$$

To be more explicit, given an arbitrary voltage distribution, $g(x, y)$, in a design we can find the voltage map $V(x, y)$ by solving the Poisson's equation $\nabla^2 V(x, y) = g$; then we can extract the E_x and E_y field components using $\mathbf{E} = -\nabla V(x, y)$. With these, we can make use of numerics to solve the following two decoupled Poisson's equations to find the acoustic displacement,

$$\frac{\partial}{\partial x}\left(\epsilon\frac{E_x^2 - E_y^2}{2}\right) + \frac{\partial}{\partial y}\left(\epsilon E_x E_y\right) = -c\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right), \tag{4.50}$$

$$\frac{\partial}{\partial x}\left(\epsilon E_x E_y\right) - \frac{\partial}{\partial y}\left(\epsilon\frac{E_x^2 - E_y^2}{2}\right) = -c\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right). \tag{4.51}$$

4.3. Mechanical Displacement - Potential Formulation

In this Section, we shall form the relationship between the mechanical displacement and the electrical potential from the Cauchy's equation of motion. With the help of the auxiliary equation, we shall derive the acoustic wave equation in a piezoelectric medium. We shall derive the stiffened stiffness and the artificial piezoelectric constant. For the simple 2D geometry, we shall further simplify the formulae and provide values for the piezoelectric e-constant in this artificial piezoelectric case using silicon.

The Cauchy's equation of continuum mechanics is

$$\nabla\cdot\mathbf{T}+\mathbf{f}=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{4.52}$$

where $\mathbf{T}$ is the Cauchy stress in $\mathrm{N\,m^{-2}}$, $\mathbf{f}$ is the body force density in $\mathrm{N\,m^{-3}}$, ρ the mass density, and $\mathbf{u}$ is the acoustic displacement vector in m. In a piezoelectric medium, the stress is,

$$\mathbf{T}=\mathbf{c}^E:\nabla_s\mathbf{u}-\mathbf{e}^t\cdot\mathbf{E}, \tag{4.53}$$

where $\mathbf{c}^E$ is the stiffness tensor at constant electric field, and ∇_s is the symmetric part of the displacement gradient, which relates to strain tensor. We will embed the Poisson's ratio, and the damping inside the stiffness for the moment. The $\mathbf{e}$ represents the piezoelectric tensor expressing the ratio of the stress developed to the applied electric field, see §2.4. Therefore, we can write

$$\nabla\cdot\left(\mathbf{c}^E:\nabla_s\mathbf{u}-\mathbf{e}^t\cdot\mathbf{E}\right)+\mathbf{f}=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}. \tag{4.54}$$

The electric field is $\mathbf{E}=-\nabla\phi$, where ϕ is the voltage. Hence

$$\nabla\cdot\left(\mathbf{c}^E:\nabla_s\mathbf{u}+\mathbf{e}^t\cdot\nabla\phi\right)+\mathbf{f}=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{4.55}$$

If there is no body force, we can write

$$\nabla\cdot\left(\mathbf{c}^E:\nabla_s\mathbf{u}+\mathbf{e}^t\cdot\nabla\phi\right)=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{4.56}$$

We have from Eq. (8.139) in [3], the auxiliary equation connecting the potential and the mechanical displacement in a piezomaterial,

$$0=-\mu_0\nabla\cdot\left(\epsilon^S\cdot\nabla\phi\right)+\mu_0\nabla\cdot\left(\mathbf{e}:\nabla_s\mathbf{u}\right), \tag{4.57}$$

where ϵ^S is the electric permittivity tensor at constant strain. We can rearrange to write the potential gradient as

$$\nabla\phi=\left(\epsilon^S\right)^{-1}\left(\mathbf{e}:\nabla_s\mathbf{u}\right). \tag{4.58}$$

Substituting the gradient of potential in Eq. (4.58) to Eq. (4.56), we get

$$\nabla\cdot\left\{\mathbf{c}^{E}:\nabla_s\mathbf{u}+\mathbf{e}^{t}\cdot\left[\left(\epsilon^{S}\right)^{-1}\left(\mathbf{e}:\nabla_s\mathbf{u}\right)\right]\right\}=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{4.59}$$

$$\Rightarrow\nabla\cdot\left\{\left[\mathbf{c}^{E}+\mathbf{e}^{t}\cdot\left(\left(\epsilon^{S}\right)^{-1}\mathbf{e}\right)\right]:\nabla_s\mathbf{u}\right\}=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}. \tag{4.60}$$

We derive the effective stiffness as

Stiffened Stiffness

$$\mathbf{c}^{\text{eff}}=\mathbf{c}^{E}+\mathbf{e}^{t}\cdot\left[\left(\epsilon^{S}\right)^{-1}\mathbf{e}\right] \tag{4.61}$$

This allows us to write the wave equation in a piezoelectric medium succinctly as

$$\nabla\cdot\left(\mathbf{c}^{\text{eff}}:\nabla_s\mathbf{u}\right)=\rho\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{4.62}$$

4.4. Derivation of Artificial Piezo-constants

In this Section, we shall derive the explicit formulae for the stiffened stiffness and the artificial piezoelectric stress constant.

In the literature and industry for quoting parameters, the more useful Voigt notation [53], [49] is used, which is a contraction for composite and symmetric indices, as shown in Tab. 4.1. Moreover, this allows for the higher rank tensor for manipulation, i. e., taking transpose of the piezoelectric stress e-constant in Eq. (4.61). Without this contraction, the concept of transpose of a tensor of rank more than two does not exist.

I or J	ij or kl
1	xx
2	yy
3	zz
4	yz, zy
5	xz, zx
6	xy, yx

Table 4.1.: *Voigt notation for contraction.*

4.4.1. Piezoelectric Stress Constant

In this Section, we shall explicitly derive the components of the artificial e-constants in 2D.

From Eq. (2.39), we can write

$$D_i = e_{iI} S_I. \tag{4.63}$$

It is well-known in piezoelectricity [3],

$$\mathbf{e} = \mathbf{d} : \mathbf{c}^E, \tag{4.64}$$

$$\epsilon^T = \epsilon^S + \mathbf{e} : (\mathbf{c}^E)^{-1} \cdot \mathbf{e}^t. \tag{4.65}$$

Using the Voigt notation, we can write

$$e_{iI} = d_{iJ} c^E_{JI}. \tag{4.66}$$

The d is a third rank tensor, utilizing the symmetric property of the d-tensor, we can write $d_{ijk} = d_{ikj}$ [61]. From Eq. (2.36), we have

$$S_I = d^t_{iI} E_i + s_{IJ} T_J, \tag{4.67}$$

$$= d_{Ii} E_i + s_{IJ} T_J. \tag{4.68}$$

As in §3.8.3, denoting the small static perturbation by subscript δS, δE, δT from the equilibrium values by S_0, E_0, T_0 respectively, we can write

$$\delta S_I = d_{Ii} \delta E_i + s_{IJ} \delta T_J \tag{4.69}$$

$$\Rightarrow d_{Ii} = \frac{\delta S_I}{\delta E_i}, \quad \text{if no stress perturbation, i. e., } \delta T_J = 0. \tag{4.70}$$

$$\tag{4.71}$$

Expounding upon the derivation in Eq. (3.264) in 1D, the relationship between the Cauchy stress and the Maxwell stress tensor is

$$T = -T^{\text{Maxwell}}. \tag{4.72}$$

In index notation, we can write

$$T_I = -T_I^{\text{Maxwell}}, \tag{4.73}$$
$$\Rightarrow c_{IJ} S_J = -T_I^{\text{Maxwell}}, \tag{4.74}$$
$$\Rightarrow S_J = -(c_{IJ})^{-1} T_I^{\text{Maxwell}}, \tag{4.75}$$
$$\Rightarrow \delta S_J = -(c_{IJ})^{-1} \delta T_I^{\text{Maxwell}}. \tag{4.76}$$
$$\tag{4.77}$$

We derived $d_{Ii} = \frac{\delta S_I}{\delta E_i}$ and $\delta S_J = -(c_{IJ})^{-1} \delta T_I^{\text{Maxwell}}$. By change of index,

$$\delta S_I = -(c_{JI})^{-1} \delta T_J^{\text{Maxwell}}. \tag{4.78}$$

We obtain the d-constant as

$$d_{Ii} = \frac{\delta S_I}{\delta E_i} = -\frac{1}{\delta E_i} (c_{JI})^{-1} \delta T_J^{\text{Maxwell}}. \tag{4.79}$$

Transposing and changing of index of Eq. (4.66) results

$$e_{Ji} = c_{JI}^E d_{Ii} = -c_{JI}^E (c_{JI})^{-1} \frac{\delta T_J^{\text{Maxwell}}}{\delta E_i} = -\frac{\delta T_J^{\text{Maxwell}}}{\delta E_i} \tag{4.80}$$

Finally we derive the artificial e-constant as,

Artificial e-constant

$$e_{Ji} = -\frac{\delta T_J^{\text{Maxwell}}}{\delta E_i} \tag{4.81}$$

As proved in §1.1.2, there is no linear coupling between acoustic and electromagnetic wave in silicon, meaning all $e_{Ji} = 0$ in natural silicon; however, by special electrode arrangement and biasing, we are able to induce non-vanishing e_{Ji} in silicon or any centro-symmetric materials per se. In this sense, we call these e_{Ji} constants artificial.

2D Case

We should have $2^3 = 8$ independent quantities in the e-tensor in 2D, but due to symmetry, we have only 6. The electric displacement is

$$\begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} e_{x1} & e_{x2} & e_{x6} \\ e_{y1} & e_{y2} & e_{y6} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ 2S_6 \end{bmatrix}. \tag{4.82}$$

For intrinsic silicon, the e-constant is a null tensor. However, by appropriate electrode arrangement, we can have non-zero e-constant in silicon. In practice, we measure the d-matrix, and is related via $\mathbf{d} = \mathbf{e} : \left(\mathbf{c}^E\right)^{-1}$.

We decompose the electric field in 2D into components,

$$\mathbf{E} = -\nabla V = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} E_{x0} + \delta E_x \\ E_{y0} + \delta E_y \end{bmatrix}. \tag{4.83}$$

The Maxwell stress is

$$T^{\text{Maxwell}} = \epsilon \mathbf{E} \otimes \mathbf{E} - \frac{1}{2}\epsilon E^2 \mathcal{I}, \tag{4.84}$$

where $E^2 = E_x^2 + E_y^2$. We can write the Maxwell stress as,

$$T^{\text{Maxwell}} = \epsilon \begin{bmatrix} \frac{1}{2}\left(E_x^2 - E_y^2\right) & E_x E_y \\ E_x E_y & -\frac{1}{2}\left(E_x^2 - E_y^2\right) \end{bmatrix}, \tag{4.85}$$

$$= T_0^{\text{Maxwell}} + \delta T^{\text{Maxwell}}, \tag{4.86}$$

where the perturbation in the Maxwell stress is

$$\delta T^{\text{Maxwell}} = \epsilon \begin{bmatrix} E_{x0}\delta E_x - E_{y0}\delta E_y & E_{x0}\delta E_y + E_{y0}\delta E_x \\ E_{x0}\delta E_y + E_{y0}\delta E_x & -\left(E_{x0}\delta E_x - E_{y0}\delta E_y\right) \end{bmatrix}, \tag{4.87}$$

$$= \begin{bmatrix} \delta T_1^{\text{Maxwell}} & \delta T_6^{\text{Maxwell}} \\ \delta T_6^{\text{Maxwell}} & \delta T_2^{\text{Maxwell}} \end{bmatrix}. \tag{4.88}$$

From Eq. (4.82), the $\mathbf{e}$-matrix in 2D is

$$\mathbf{e} = \begin{bmatrix} e_{x1} & e_{x2} & e_{x6} \\ e_{y1} & e_{y2} & e_{y6} \end{bmatrix}. \tag{4.89}$$

In component form $e_{iI} = e_{Ii}$, so we can write

$$e_{x1} = -\delta T_1^{\text{Maxwell}}/\delta E_x, \qquad e_{y1} = -\delta T_1^{\text{Maxwell}}/\delta E_y, \tag{4.90}$$
$$e_{x2} = -\delta T_2^{\text{Maxwell}}/\delta E_x, \qquad e_{y2} = -\delta T_2^{\text{Maxwell}}/\delta E_y, \tag{4.91}$$
$$e_{x6} = -\delta T_6^{\text{Maxwell}}/\delta E_x, \qquad e_{y6} = -\delta T_6^{\text{Maxwell}}/\delta E_y. \tag{4.92}$$
$$\tag{4.93}$$

Finally, we can write the e-constant explicitly in 2D as,

Explicit Expressions of the Artificial e-constants in 2D

$$e_{x1} = -\epsilon \left(E_{x0} - E_{y0} \frac{\delta E_y}{\delta E_x} \right), \tag{4.94}$$
$$e_{x2} = \epsilon \left(E_{x0} - E_{y0} \frac{\delta E_y}{\delta E_x} \right), \tag{4.95}$$
$$e_{x6} = -\epsilon \left(E_{x0} \frac{\delta E_y}{\delta E_x} + E_{y0} \right), \tag{4.96}$$
$$e_{y1} = -\epsilon \left(E_{x0} \frac{\delta E_x}{\delta E_y} - E_{y0} \right), \tag{4.97}$$
$$e_{y2} = \epsilon \left(E_{x0} \frac{\delta E_x}{\delta E_y} - E_{y0} \right), \tag{4.98}$$
$$e_{y6} = -\epsilon \left(E_{x0} + E_{y0} \frac{\delta E_x}{\delta E_y} \right). \tag{4.99}$$

4.4.2. Effective Stiffness

Here we shall find expressions for the stiffness using both tensor notation and the more useful Voigt notation.

Using Eq. (4.61), we can write the stiffness in full tensorial index notation,

$$c_{ijkl}^{\text{eff}} = c_{ijkl}^{E} + e_{ijm}(\epsilon_{mn}^{S})^{-1} e_{nkl}, \quad \forall i, j, k, l, m, n \in (x, y, z). \tag{4.100}$$

Using regular [3] and symmetric index [4] notations, the 6×6 stiffness matrix in 3D is

$$c_{IJ}^{\text{eff}} = c_{IJ}^{E} + e_{jI}^{t}(\epsilon^{S})_{ji}^{-1} e_{iJ}, \tag{4.101}$$

$$= c_{IJ}^{E} + e_{Ij}(\epsilon^{S})_{ji}^{-1} e_{iJ}. \tag{4.102}$$

In 1D, $i = j = k = l = x$. In 2D, i, j, k, l can be from set (x, y), provided that we are considering on the plane $x-y$. In general the number of quantities = dimension$^{\text{number of indices}}$. We considered the following assumptions in our contracted notation,

1. The Cauchy stress tensor is symmetric provided that we are considering the object in rotational equilibrium. This provides the minor symmetry of the stiffness tensor $c_{ijkl} = c_{jikl}$
2. Small-scale strain is defined as $S_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = S_{yx}$. Due to symmetry of the small-scale strain, $c_{ijkl} = c_{ijlk}$.
3. Major symmetry is defined as $c_{ijkl} = c_{klij}$.

Considering all these symmetries, we can succinctly write as

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij} = c_{jilk}. \tag{4.103}$$

1D Case:

In 1D, the only permitted indices are $i = j = k = l = x$. Therefore,

$$c_{xxxx}^{\text{eff}} = c_{xxxx}^{E} + e_{xxx}(\epsilon_{xx}^{S})^{-1} e_{xxx}. \tag{4.104}$$

In Voigt notation,

$$c_{11}^{\text{eff}} = c_{11}^{E} + e_{1x}(\epsilon_{1}^{S})^{-1} e_{x1}, \tag{4.105}$$

$$= c_{11}^{E} + \frac{e_{1x}^{2}}{\epsilon_{1}^{S}}. \tag{4.106}$$

This is exactly we derived in Eq. (3.226) in §3.8.1. In our case of interest, silicon has symmetry class cubic and m3m ($4/m\bar{3}2/m$). Silicon has cubic diamond crystal structure. In the case of silicon material, this $e_{1x} = 0$, meaning there is no intrinsic piezoelectricity can be expressed in silicon. However by appropriate electrode configuration in 1D, we showed in Chap. 3 that it is possible to have non-zero $e_{1x} = -\epsilon E_0$ constant in silicon, where E_0 is the DC electric field.

[3] Meaning i,j,k,l
[4] Meaning I,J

2D Case:

Using the relation $T_{ij} = c_{ijkl}S_{kl}$, and expanding out in 2D, we arrive at the Hooke's law in an anisotropic medium

$$\begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{16} \\ c_{12} & c_{22} & c_{26} \\ c_{16} & c_{26} & c_{66} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ 2S_6 \end{bmatrix}. \tag{4.107}$$

We keep the composite symmetric indices numbering consistent, in line with the 3D case, this is useful when quoting/comparing the parameters from industry.

The concept of the Poisson's ratio ν does not even exist in 1D. If the Poisson's ratio is not included in 2D, the numerical results do not make sense, even if we consider our simple structure in Fig. 4.2 to be quasi-2D. By making powerful use of the tensor notation, we arrive at [5] [6]

$$c_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \tag{4.108}$$

$$c_{12} = \frac{E\nu}{(1+\nu)(1-2\nu)}, \tag{4.109}$$

$$c_{44} = \frac{E}{2(1+\nu)}. \tag{4.110}$$

Here E is Young's modulus of elasticity, $c_{12} = \lambda$ and $c_{44} = \mu$, where $\lambda = 2\mu\nu/(1-2\nu)$ and μ are the Lamé parameters, μ is the shear modulus.

We summarize the various moduli of elasticity, as this will be useful when we extract the correct parameters from measurement data in the literature. The Young's modulus of elasticity, E, is defined as the normal stress over the axial strain,

$$E = \frac{T_{xx}}{S_{xx}}. \tag{4.111}$$

The bulk modulus, K or sometimes B, is defined as pressure P increase over normalized decrease in volume dV, where the initial volume is V_0,

$$K = \frac{dP}{-dV/V_0}. \tag{4.112}$$

[5] p. 48 of US Letter Edition in [52], also available online [62].
[6] Eq. (2.14), p. 38 in [63]

The shear modulus, G, is the ratio of the shear stress to the shear strain.

$$G = \frac{T_{xy}}{S_{xy} + S_{yx}}. \tag{4.113}$$

The Poisson's ratio is defined as the sensitivity of strain in two stresses, one in line with the strain and the other perpendicular to it,

$$S_{xx} = \frac{1}{E}\left[T_{xx} - \nu(T_{yy} + T_{zz})\right]. \tag{4.114}$$

The Poisson's ratio is $-1 < \nu < 0.5$. For silicon it is $\nu \sim 0.28$, hence it behaves like plastics. The Poisson's ratio $\nu = 0.5$ means the volumetric strain is zero, meaning, the material is incompressible, e. g., water. The Poisson's ratio $\nu = 0$ means no change in lateral strain when we apply tensile/compressive force, e. g., cork. Another way to look at the Poisson's ratio is that assuming we have uniaxial stress and the strain in the z-direction equals the strain in the y-direction, the strain the y-direction is proportional to the strain in the x-direction, and the strain the z-direction is proportional to the strain in the x-direction; this proportionality is called the Poisson's ratio.

Once we find any two of parameters in the literature, the rest can be found from these relationship,

$$E = 2G(1 + \nu) = 3B(1 - 2\nu). \tag{4.115}$$

In 2D, the permittivity tensor is

$$\epsilon^S = \begin{bmatrix} \epsilon^S_{xx} & \epsilon^S_{xy} \\ \epsilon^S_{yx} & \epsilon^S_{yy} \end{bmatrix} = \begin{bmatrix} \epsilon_1 & \epsilon_6 \\ \epsilon_6 & \epsilon_2 \end{bmatrix}. \tag{4.116}$$

From Eq. (4.107), the stiffness matrix is

$$\mathbf{c}^E = \begin{bmatrix} c_{11} & c_{12} & c_{16} \\ c_{12} & c_{22} & c_{26} \\ c_{16} & c_{26} & c_{66} \end{bmatrix}. \tag{4.117}$$

From Eq. (4.82), the e-matrix is

$$\mathbf{e} = \begin{bmatrix} e_{x1} & e_{x2} & e_{x6} \\ e_{y1} & e_{y2} & e_{y6} \end{bmatrix}. \tag{4.118}$$

We derived in Eq. (4.61) that the effective stiffness is

$$\mathbf{c}^{\text{eff}} = \mathbf{c}^{E} + \mathbf{e}^{t} \cdot \left[\left(\epsilon^{S} \right)^{-1} \mathbf{e} \right]. \tag{4.119}$$

Hence the stiffened stiffness for the artificial piezoelectric case in 2D is

Stiffened Stiffness for Artificial Piezoelectricity in 2D

$$\mathbf{c}^{\text{eff}} = \begin{bmatrix} c_{11} + \frac{1}{\epsilon_1}\left(e_{x1}^2 + e_{y1}^2\right) & c_{12} + \frac{1}{\epsilon_1}\left(e_{x1}e_{x2} + e_{y1}e_{y2}\right) & \frac{1}{\epsilon_1}\left(e_{x1}e_{x6} + e_{y1}e_{y6}\right) \\ & c_{11} + \frac{1}{\epsilon_1}\left(e_{x2}^2 + e_{y2}^2\right) & \frac{1}{\epsilon_1}\left(e_{x2}e_{x6} + e_{y2}e_{y6}\right) \\ \text{sym} & & c_{44} + \frac{1}{\epsilon_1}\left(e_{x6}^2 + e_{y6}^2\right) \end{bmatrix} \tag{4.120}$$

4.4.3. Design Specific Effective Stiffness & Piezoelectric Constant

In this Section, we shall compute the e-matrix for the design in Fig. 4.2.

Since we apply DC voltage along the horizontal direction in Fig. 4.2, the e-matrix components are reduced to

$$e_{x1} = -\epsilon E_{x0}, \tag{4.121}$$
$$e_{x2} = \epsilon E_{x0}, \tag{4.122}$$
$$e_{x6} = 0, \tag{4.123}$$
$$e_{y1} = 0, \tag{4.124}$$
$$e_{y2} = 0, \tag{4.125}$$
$$e_{y6} = -\epsilon E_{x0}, \tag{4.126}$$

where $|E_{x0}| = V_{\text{dc}}/L_{\text{silicon}}$. For a 2µm wide silicon strip, and 10V DC bias, the electric field is $E_{x0} = 5\ \text{MV}\,\text{m}^{-1}$. We note that this is far below the breakdown electric field of silicon which is $35\ \text{GV}\,\text{m}^{-1}$. Finally the component e_{x1} is $-5.179 \cdot 10^{-4}\ \text{C}\,\text{m}^{-2}$, which is about two orders of magnitude lower than the quartz's natural e-value. As we ramp up DC bias voltage, we linearly increase the e-constant value in the artificial piezoelectric configuration. As a comparison, the quartz has $e_{x4} = -4.36e^{-2}\ \text{C}\,\text{m}^{-2}$, $e_{x1} = 0.171\ \text{C}\,\text{m}^{-2}$, p. 377 in [3], the PZT-5H has $e_{z3} = 23.3\ \text{C}\,\text{m}^{-2}$, p. 377 in [3]. The artificial piezoelectric stress e-matrix in the 2D design in Fig. 4.2 with 10V DC voltage and 2µm wide silicon strip is

$$\mathbf{e} = \begin{bmatrix} -5.179 \cdot 10^{-4} & 5.179 \cdot 10^{-4} & 0 \\ 0 & 0 & -5.179 \cdot 10^{-4} \end{bmatrix} (\text{C}\,\text{m}^{-2}). \tag{4.127}$$

In this simple geometric configuration, the difference between the stiffened and natural stiffnesses, $\Delta\mathbf{c}^E = \mathbf{c}^{\text{eff}} - \mathbf{c}^E$ turns out to be,

$$\Delta\mathbf{c}^E = \begin{bmatrix} 2589.9 & -2589.9 & 0 \\ & 2589.9 & 0 \\ \text{sym} & & 2589.9 \end{bmatrix} (\text{N m}^{-2}). \tag{4.128}$$

This is to be expected, as the stiffness perturbation is minute – in compared to the GN m^{-2} scale of the values of the natural stiffness – see the exaggerated drawing in Fig. 3.21. Moreover, the stiffness alone can be tailored by appropriate patterning, such as holey structures in [23], and widely used patterning in photonic structures, e. g., see [64]. We require to tailor both the electromagnetic and phononic band structures of these periodic structures for the efficient excitation of coherent phononic modes.

4.4.4. Piezoelectric Wave Equation

We shall explicitly write the components of the piezoelectric wave equation that we developed in Eq. (4.62) in §4.3. Since we are interested to operate near/at resonance, we must include damping so that the mechanical amplitude response does not blow up. We shall express the wave equation in the index notation, as well as, the more convenient matrix notation. The index notation appears in some piezoelectric literature, p. 299 in [3]. We can use the force density either from the Maxwell stress tensor, or the analytical Coulomb law for our simple design in Fig. 4.2.

The Cauchy's equation of continuum mechanics is

$$\nabla\cdot\mathbf{T} + \mathbf{f} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{4.129}$$

where $\mathbf{f}$ is the body force density. The Cauchy stress, $\mathbf{T}$, for piezoelectric material, including damping η, is

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \nabla\cdot\left(\mathbf{c}^E : \nabla_s\mathbf{u} + \eta : \frac{\partial}{\partial t}\nabla_s\mathbf{u} + \mathbf{e}^t\cdot\nabla\phi\right) + \mathbf{f}, \tag{4.130}$$

$$\Rightarrow \rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \nabla\cdot\left(\mathbf{c}^E : \nabla_s\mathbf{u} + \eta : \frac{\partial}{\partial t}\nabla_s\mathbf{u}\right) + \mathbf{f}_{\text{voltage}} + \mathbf{f}, \tag{4.131}$$

where $\mathbf{f}_{\text{voltage}} = \nabla\cdot\Big(\mathbf{e}^t\cdot\nabla\phi\Big)$. Neglecting the body force, we have

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \nabla\cdot\left(\mathbf{c}^E:\nabla_s\mathbf{u} + \eta:\frac{\partial}{\partial t}\nabla_s\mathbf{u}\right) + \nabla\cdot\Big(\mathbf{e}^t\cdot\nabla\phi\Big) \tag{4.132}$$

Tensor Notation

Using the index notation, we write Eq. (4.132) for each component as,

$$\rho u_{i,tt} = \Big(c^E_{ijkl}S_{kl}\Big)_{,j} + \Big(\eta_{ijkl}S_{kl,t}\Big)_{,j} + \Big(e_{ijk}\phi_{,k}\Big)_{,j} \tag{4.133}$$

$$\Rightarrow \rho u_{i,tt} = \Big(c^E_{ijkl}S_{lk}\Big)_{,j} + \Big(\eta_{ijkl}S_{lk,t}\Big)_{,j} + e_{ijk}\phi_{,jk} \quad \text{Strain is symmetric} \tag{4.134}$$

$$\Rightarrow \rho u_{i,tt} = \Big(c^E_{ijkl}u_{l,k}\Big)_{,j} + \Big(\eta_{ijkl}u_{l,kt}\Big)_{,j} + e_{ijk}\phi_{,jk} \tag{4.135}$$

$$\Rightarrow \rho u_{i,tt} = c^E_{ijkl}u_{l,jk} + \eta_{ijkl}u_{l,jkt} + e_{ijk}\phi_{,jk} \tag{4.136}$$

Finally, we can write the individual component compactly as

Piezoelectric Wave Equation (Index Notation)

$$\rho u_{i,tt} = c^E_{ijkl}u_{l,jk} + \eta_{ijkl}u_{l,jkt} + e_{ijk}\phi_{,jk} \tag{4.137}$$

For 2D, we can limit the indices to $\forall i,j,l,k \in \{x,y\}$.

Matrix Notation

We write Eq. (4.132) using the regular and symmetric index notation,

$$\rho u_{i,tt} = \nabla_{iK}c^E_{KL}\nabla_{Lj}u_j + \nabla_{iK}\eta_{KL}\nabla_{Lj}u_{j,t} + \nabla_{iK}e^t_{jK}\phi_{,j} \tag{4.138}$$

Finally, we can write the equation using the matrix notation as,

Piezoelectric Wave Equation (Matrix Notation)

$$\rho u_{i,tt} = \nabla_{iK}c^E_{KL}\nabla_{Lj}u_j + \nabla_{iK}\eta_{KL}\nabla_{Lj}u_{j,t} + \nabla_{iK}e_{Kj}\phi_{,j} \tag{4.139}$$

The divergence and gradient operators in 3D are [3][7]

$$\nabla\cdot = \nabla_{iJ} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}, \nabla_s = \nabla_{Ij} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \tag{4.140}$$

In the case of 2D, we have the operators

$$\nabla\cdot = \nabla_{iJ} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}, \nabla_s = \nabla_{Ij} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \tag{4.141}$$

Frequency Domain Piezoelectric Wave Equation

The voltage, ϕ, below contains both the DC and time-dependent excitation signal term

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \nabla\cdot\left(\mathbf{c}^E : \nabla_s\mathbf{u} + \eta : \frac{\partial}{\partial t}\nabla_s\mathbf{u}\right) + \mathbf{f}_{\text{voltage}} \tag{4.142}$$

where $\phi = V_{\text{dc}} + V_{\text{signal}}e^{i\omega t}$. The electric force density term $\mathbf{f}_{\text{voltage}}$ can be the $\nabla\cdot\left(\mathbf{e}^t\cdot\nabla\phi\right)$ or we can obtain it from the Maxwell stress tensor via Eq. (4.40) or we can use some other analytical form, if available. Since time-dependent simulation is computationally expensive, assuming $e^{i\omega t}$ dependency, we can alternatively view the solution in the frequency domain, where we write Eq. (4.142) in the frequency domain as ,

$$-\rho\omega^2\mathbf{u} = \nabla\cdot\left(\mathbf{c}^E : \nabla_s\mathbf{u} + i\omega\eta : \nabla_s\mathbf{u}\right) + \mathbf{f}_{\text{voltage}} \tag{4.143}$$

By doing so, the DC terms are completely washed out and only the sinusoidal terms survive. We must cater for the DC voltage separately that is required to create the dipoles in the artificial

[7]The subscript notation used in [3] for the divergence and gradient operators are unusual, but serves a good purpose for record keeping.

piezoelectric case. We denote the complex stiffness as

$$\varsigma_{11} = c_{11} + i\omega\eta_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} + i\omega\eta_{11}, \tag{4.144}$$

$$\varsigma_{12} = c_{12} + i\omega\eta_{12} = \frac{E\nu}{(1+\nu)(1-2\nu)} + i\omega\eta_{12}, \tag{4.145}$$

$$\varsigma_{44} = c_{44} + i\omega\eta_{44} = \frac{E}{2(1+\nu)} + i\omega\eta_{44}. \tag{4.146}$$

We can write the two coupled equations using the e-components as

$$\begin{aligned}\frac{\partial}{\partial x}\left[\varsigma_{11}\frac{\partial u_x}{\partial x} + \varsigma_{12}\frac{\partial u_y}{\partial y}\right] + \frac{\partial}{\partial y}\left[\varsigma_{44}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\right] + \rho\omega^2 u_x \\ + \frac{\partial}{\partial x}\left(e_{x1}\frac{\partial \phi}{\partial x} + e_{y1}\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(e_{x6}\frac{\partial \phi}{\partial x} + e_{y6}\frac{\partial \phi}{\partial y}\right) = 0\end{aligned} \tag{4.147}$$

$$\begin{aligned}\frac{\partial}{\partial y}\left[\varsigma_{11}\frac{\partial u_y}{\partial y} + \varsigma_{12}\frac{\partial u_x}{\partial x}\right] + \frac{\partial}{\partial x}\left[\varsigma_{44}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\right] + \rho\omega^2 u_y \\ + \frac{\partial}{\partial y}\left(e_{x2}\frac{\partial \phi}{\partial x} + e_{y2}\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial x}\left(e_{x6}\frac{\partial \phi}{\partial x} + e_{y6}\frac{\partial \phi}{\partial y}\right) = 0\end{aligned} \tag{4.148}$$

We note that in our case of artificial piezoelectricity, the e-constants are not constant across the structure, like the bulk natural piezoelectric case. Instead, in the metal region, it is strictly zero, whereas in silicon it is nonzero and depends on the applied electric field, see Fig. 3.21. Hence we must use some form of homogenization, if we were to compare the numerical results, and by doing so, we may expect shifts/changes from the actual behavior.

The other powerful form of the forcing function is the usage of the Maxwell stress tensor that we developed in §4.2; this form is generic that can be applied in any topology. Using Eq. (4.42), we can write

$$\begin{aligned}\frac{\partial}{\partial x}\left[\varsigma_{11}\frac{\partial u_x}{\partial x} + \varsigma_{12}\frac{\partial u_y}{\partial y}\right] + \frac{\partial}{\partial y}\left[\varsigma_{44}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\right] + \rho\omega^2 u_x \\ + \frac{1}{2}\epsilon\frac{\partial}{\partial x}\left(E_x^2 - E_y^2\right) + \epsilon\frac{\partial}{\partial y}E_x E_y = 0\end{aligned} \tag{4.149}$$

$$\frac{\partial}{\partial y}\left[\underset{\sim}{c}_{11}\frac{\partial u_y}{\partial y}+\underset{\sim}{c}_{12}\frac{\partial u_x}{\partial x}\right]+\frac{\partial}{\partial x}\left[\underset{\sim}{c}_{44}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)\right]+\rho\omega^2 u_y - \frac{1}{2}\epsilon\frac{\partial}{\partial y}\left(E_x^2-E_y^2\right)+\epsilon\frac{\partial}{\partial x}E_xE_y=0 \tag{4.150}$$

For the simple design in Fig. 4.2, we can make good of the Coulomb force law and write

$$\frac{\partial}{\partial x}\left[\underset{\sim}{c}_{11}\frac{\partial u_x}{\partial x}+\underset{\sim}{c}_{12}\frac{\partial u_y}{\partial y}\right]+\frac{\partial}{\partial y}\left[\underset{\sim}{c}_{44}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)\right]+\rho\omega^2 u_x+f_{\text{Coulomb}}(x)=0. \tag{4.151}$$

$$\frac{\partial}{\partial y}\left[\underset{\sim}{c}_{11}\frac{\partial u_y}{\partial y}+\underset{\sim}{c}_{12}\frac{\partial u_x}{\partial x}\right]+\frac{\partial}{\partial x}\left[\underset{\sim}{c}_{44}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)\right]+\rho\omega^2 u_y=0, \tag{4.152}$$

where we apply the Coulomb force density at the narrow strip of length d_{Si} across the silicon is,

$$|f_{\text{Coulomb}}|=\frac{\epsilon V_0^2}{2d_{\text{Si}}^2}\quad(\text{N m}^{-3}), \tag{4.153}$$

and zero elsewhere. The excitation signal is placed the place of interest, e. g., we place at the left most boundary of a finite lattice.

4.5. Numerical Experiment using FlexPDE

In this Section, using the commercial software FlexPDE [1], we script the various formulations that we developed in §4.4.4 to solve for mechanical displacement in the 2D design in Fig. 4.2 in the frequency domain. In essence, we set up virtual test bench using the finite element method.

We can explicitly set the polynomial order in the FlexPDE via **order = 1, 2, 3** in the descriptor file or in the program settings. It is preferable to start with the lowest order and observe, if there is spurious modes. Currently with the rise of multicore and multithreaded processors, we can take advantage of (typically) faster computation using the **threads** in the SELECT section in the descriptor or in the program settings, where n is the number of threads of the microprocessor. In FlexPDE, the accuracy can be controlled via **errlim** in the descriptor [8]. The maximum error is displayed in each run, while the RMS error is displayed at the bottom

[8] "FlexPDE applies a consistency check to integrals of the PDE's over the mesh cells. From this it estimates the relative uncertainty in the solution variables and compares this to an accuracy tolerance. If any mesh cell exceeds the tolerance, that cell is split, and the solution is recomputed. The error tolerance is called **ERRLIM**,

of each plot.

Although FlexPDE can handle complex number manipulation, we can cast our governing equations into purely real variables and let the FlexPDE solve the equations. In case of damping, we can decompose the displacement into real and imaginary components as

$$u_x = u_x^R + iu_x^I, \tag{4.154}$$

$$u_y = u_y^R + iu_y^I. \tag{4.155}$$

The Eq. (4.151), Eq. (4.152) can be explicitly written as

$$\begin{aligned}&\frac{\partial}{\partial x}\left[c_{11}\frac{\partial u_x^R}{\partial x} - \omega\eta_{11}\frac{\partial u_x^I}{\partial x} + c_{12}\frac{\partial u_y^R}{\partial y} - \omega\eta_{12}\frac{\partial u_y^I}{\partial y}\right]\\ &+\frac{\partial}{\partial y}\left[c_{44}\frac{\partial u_y^R}{\partial x} - \omega\eta_{44}\frac{\partial u_y^I}{\partial x} + c_{44}\frac{\partial u_x^R}{\partial y} - \omega\eta_{44}\frac{\partial u_x^I}{\partial y}\right] + \rho\omega^2 u_x^R + f_{\text{Coulomb}}(x) = 0\end{aligned} \tag{4.156}$$

$$\begin{aligned}&\frac{\partial}{\partial x}\left[c_{11}\frac{\partial u_x^I}{\partial x} + \omega\eta_{11}\frac{\partial u_x^R}{\partial x} + c_{12}\frac{\partial u_y^I}{\partial y} + \omega\eta_{12}\frac{\partial u_y^R}{\partial y}\right]\\ &+\frac{\partial}{\partial y}\left[c_{44}\frac{\partial u_y^I}{\partial x} + \omega\eta_{44}\frac{\partial u_y^R}{\partial x} + c_{44}\frac{\partial u_x^I}{\partial y} + \omega\eta_{44}\frac{\partial u_x^R}{\partial y}\right] + \rho\omega^2 u_x^I = 0\end{aligned} \tag{4.157}$$

$$\begin{aligned}&\frac{\partial}{\partial y}\left[c_{11}\frac{\partial u_y^R}{\partial y} - \omega\eta_{11}\frac{\partial u_y^I}{\partial y} + c_{12}\frac{\partial u_x^R}{\partial x} - \omega\eta_{12}\frac{\partial u_x^I}{\partial x}\right]\\ &+\frac{\partial}{\partial x}\left[c_{44}\frac{\partial u_y^R}{\partial x} - \omega\eta_{44}\frac{\partial u_y^I}{\partial x} + c_{44}\frac{\partial u_x^R}{\partial y} - \omega\eta_{44}\frac{\partial u_x^I}{\partial y}\right] + \rho\omega^2 u_y^R = 0\end{aligned} \tag{4.158}$$

$$\begin{aligned}&\frac{\partial}{\partial y}\left[c_{11}\frac{\partial u_y^I}{\partial y} + \omega\eta_{11}\frac{\partial u_y^R}{\partial y} + c_{12}\frac{\partial u_x^I}{\partial x} + \omega\eta_{12}\frac{\partial u_x^R}{\partial x}\right]\\ &+\frac{\partial}{\partial x}\left[c_{44}\frac{\partial u_y^I}{\partial x} + \omega\eta_{44}\frac{\partial u_y^R}{\partial x} + c_{44}\frac{\partial u_x^I}{\partial y} + \omega\eta_{44}\frac{\partial u_x^R}{\partial y}\right] + \rho\omega^2 u_y^I = 0\end{aligned} \tag{4.159}$$

and can be set in the SELECT section of the script.". See, p. 80 of Version 7.20 User Manual in [65].

4.5.1. Voltage-induced Strain & Effect of Poisson's ratio

The purpose of this numerical experiment is to study the DC voltage induced stress and the resulting deformation, its magnitude in (the unit cell) of our 2D lattice in Fig. 4.2, as well as the important effect of Poisson's ratio. This also tests if we have setup the model correctly in the FlexPDE. We note that we have not introduced the AC perturbation yet in this Section.

The structure we like to study is shown in Fig. 4.3. This is nothing but two capacitor plates joined back-to-back. Each electrode is 2 µm wide and each silicon layer is 2 µm wide. The height of the structure is 10 µm. Each silicon layer is compressed by DC voltage. We have two opposing polarizations in this slab. We stress the structure in steps of 25 V. As a result, we observe squeezing of the slab, as shown in Fig. 4.4. As we squeeze in the x-direction, the slab is bulged in the y-direction, as expected from the Poisson's effect. This is important, if we do not include the Poisson's ratio in the definition of the stiffness in Eqs. (4.108)–(4.110), Eqs. (4.145)–(4.145), the subsequent numerical results do not make sense. The concept of Poisson's ratio even does not exist in 1D and hence we did not require to introduce Poisson's ratio in strictly 1D simulation in Chap. 3. This is markedly important when setting up the numerical simulation.

As the viscosity of solid gold is more than that of crystal silicon in the GHz range of interest, [9], we expect the gold electrode to be more sticky and deform less in compare to silicon. This is exactly what we observe in Fig. 4.4. We have magnified the displacement by a factor of $4 \cdot 10^5$ in Fig. 4.4 for visual purposes.

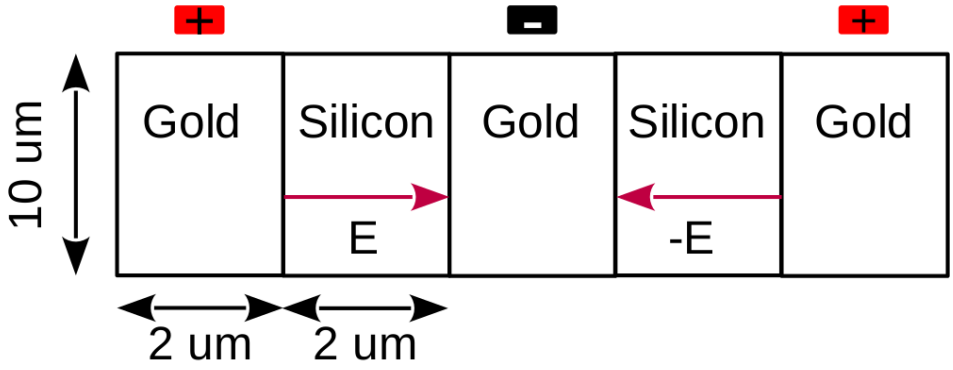
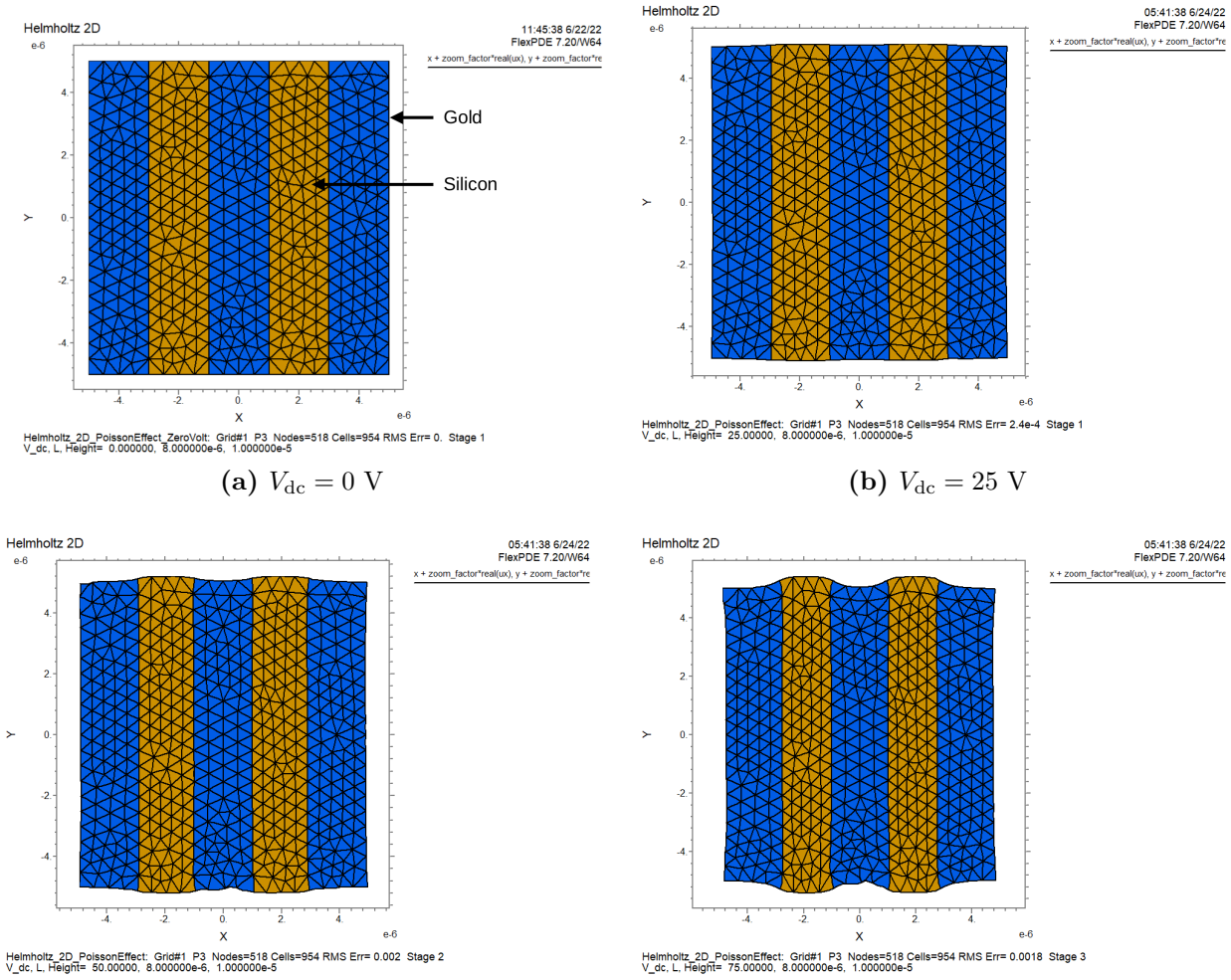


Figure 4.3.: *DC voltage is applied at the two end gold electrodes, the middle electrode has 0 voltage. There are two opposing electric fields/polarization in the two silicon layers. These silicon layers act as artificial piezoelectric medium. The height is chosen to be 10* µm*, the width of each layer is chosen to be 2* µm*.*

[9] Usually in the literature, the viscosity of gold is given in the liquid phase [66], and the viscosity of silicon in the molten silicon [67]. See Appendix Chap. A on how we calculated the material parameters in solid phase, after Ref. [68] in [3].

The x-displacement is plotted in Fig. 4.5a-Fig. 4.5b for a DC voltage of 25 V. The magnitude of this displacement is of the same order as we calculated with Eq. (3.24a) in 1D, which is reassuring. The leftmost section experiences displacement in the positive x-direction, while the rightmost section experiences displacement in the negative x-direction, while the middle region experiences no displacement. This is what we intuitively expect if we squeeze a slab from the both ends either via electrical means or mechanical means. The FlexPDE automatically marks the highest value in a plot, here the x-displacement by $\times$ and the lowest value by o. We also plot the y-displacement in Fig. 4.5c-Fig. 4.5d. Moreover, we observe that the field values are continuous across the material boundaries, meaning our numerical results are trustworthy.

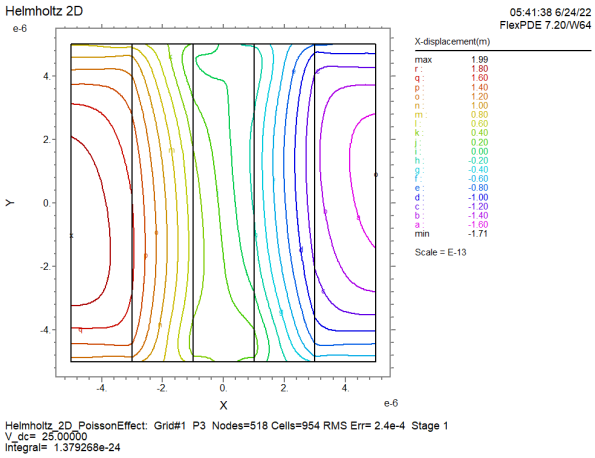
The voltage should be constant (at least in our scale) on metal electrodes, while it varies linearly across silicon. This is exactly observed in Fig. 4.6a-Fig. 4.6b by solving the Poisson's equation. The electric field should be constant across silicon and zero on metal surface. This is indeed we observe in Fig. 4.6c. The fringing effect is observed near the corners. We also see the two opposing electric fields in Fig. 4.6d. This creates the needed polarization in our artificial piezoelectric case. We note that the arrow means field existing at the base not the tip (See p.29, in [69]), hence there is no field in the zero voltage middle electrode.



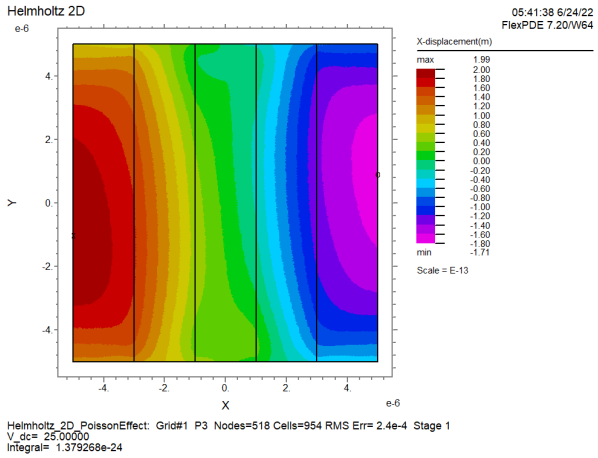
(a) $V_{\mathrm{dc}} = 0$ V **(b)** $V_{\mathrm{dc}} = 25$ V

(c) $V_{\mathrm{dc}} = 50$ V **(d)** $V_{\mathrm{dc}} = 75$ V

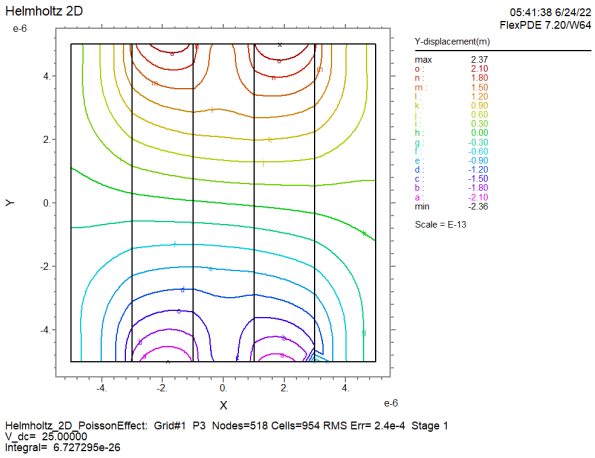
Figure 4.4.: *The voltage induced stress in the structure, having the width of the electrode as* 2 µm *and the width of the silicon layer as* 2 µm*. As we increase the voltage, we stress the silicon layer and hence the silicon slab is squeezed.* (a) *No stress. The zoom factor of* $4 \cdot 10^5$ *is used in* (b) $V_{dc} = 25$ V, (c) $V_{dc} = 50$ V, *and* (d) $V_{dc} = 75$ V*. The Poisson's effect is self-evident, in addition gold deforms less and more sticky, as gold is more viscous than silicon.*



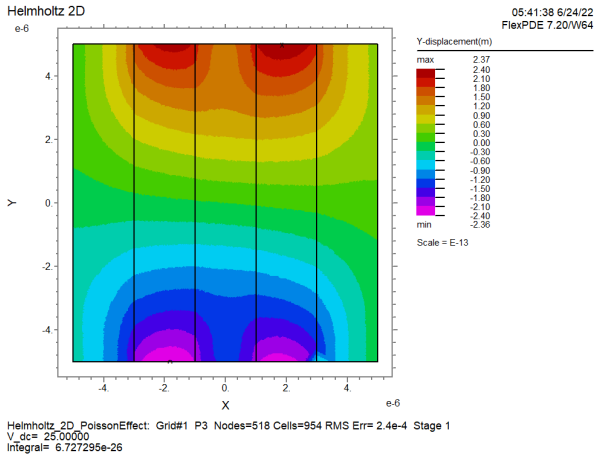
(a) Contour plot of x-displacement



(b) Painted plot of x-displacement

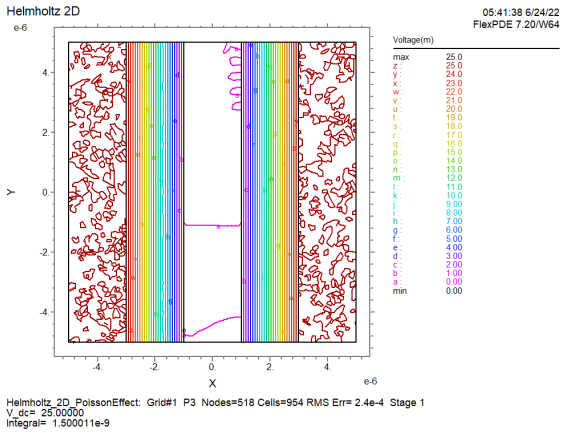


(c) Contour plot of y-displacement

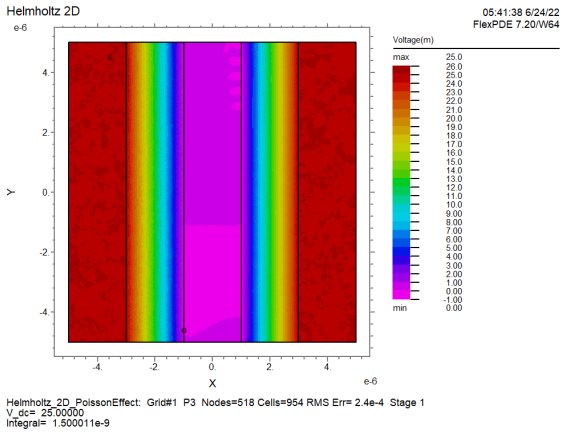


(d) Painted plot of y-displacement

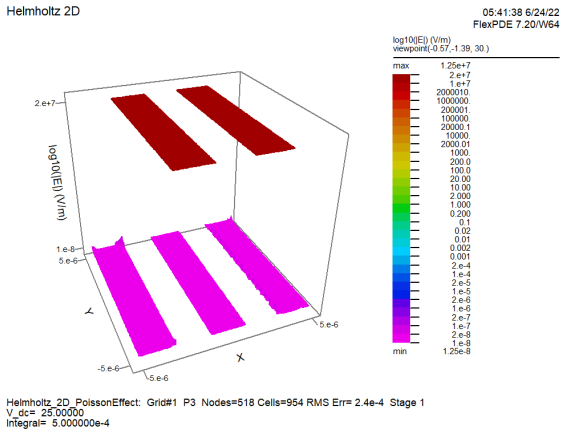
Figure 4.5.: *DC voltage of* 25 V *is applied across each silicon slab along the* x*-direction,* (a)–(b) x*-displacement. The magnitude of* x*-displacement is of the same order as we computed in the 1D case in Chap. 3. We observe the leftmost side has* $+x$ *directed displacement, the rightmost side has* $-x$ *directed displacement, while the center of the structure experiences (zero) displacement. This is what we intuitively expect, if we squeeze a bar from the both ends.* (c)–(d) y*-displacement.*



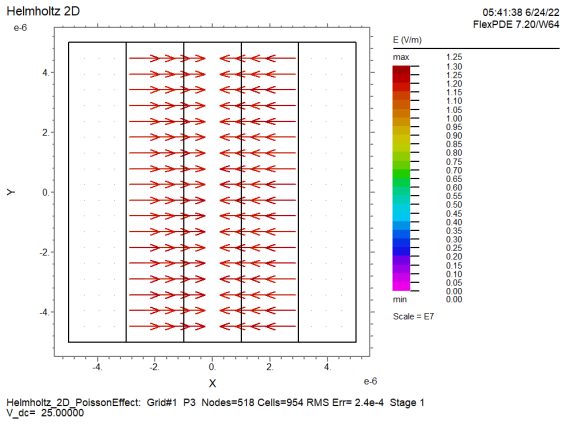
(a) Contour plot of voltage



(b) Painted plot of voltage



(c) Magnitude of E-field in log scale



(d) Electric-field

Figure 4.6.: (a)–(b) *The voltage distribution, which will be linear across silicon and constant in metal.* (c) *The magnitude of the electric field in* $\log_{10}$ *scale. We notice the fringing effect near the corners.* (d) *Vectorial electric field with arrows (the vector tail begins at the starting position in FlexPDE, hence the vector is not outside the domain, see p. 29 in [69]).*

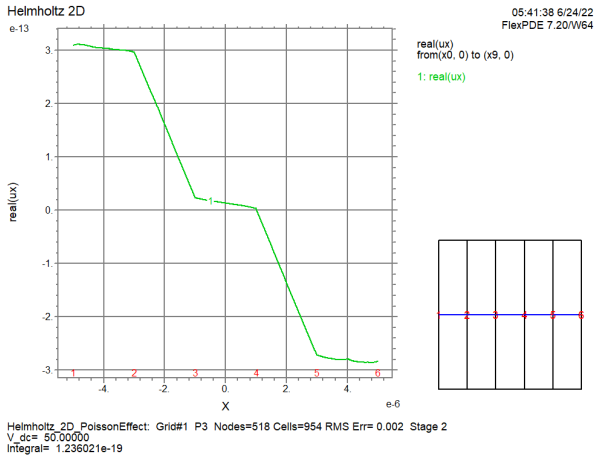


Figure 4.7.: *The displacement at the cross-section in the middle of the slab. The leftmost interface has highest positive displacement, the rightmost section has highest negative displacement, while the middle section has (nearly) zero displacement. We observe kinks at the material interfaces.*

We can plot the field variables between any two points in FlexPDE and in Fig. 4.7, we took a cross section across the middle of the structure. As the structure is compressed, we can expect and indeed observe positive displacement at the leftmost boundary and negative displacement at the rightmost boundary and (near)-zero displacement at the center. There are kinks at the material boundaries. These results prove that our numerical results provide correct outputs in the FlexPDE.

We have demonstrated the DC voltage induced stress and the resulting strain in the media. This knowledge buttress our understanding of how we can create two opposing polarizations in a cell using electrode arrangement. We will use this kind of cell to create artificial piezoelectricity in a phononic crystal.

We have incorporated gold and silicon material parameters of Chap. A. The FlexPDE code of all the plots in this Section is attached in Chap. E.

4.5.2. One Bar with Electrodes

The purpose of this numerical experiment is to generate a running wave in an already DC stressed 2D bar. To start with, we simulate the mechanical responses of a 2D bar due to harmonic excitation and DC bias in the frequency domain. This will essentially form one subunit of the unit cell in our artificial phononic crystal in Fig. 4.2. This serves a crucial foundation in building and testing the descriptor using FlexPDE.

The 2D bar that we would like to simulate is shown in Fig. 4.8. This is nothing but a parallel

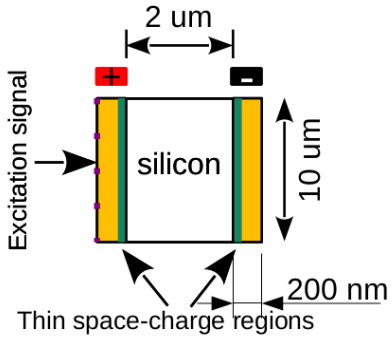


Figure 4.8.: *A 2D bar of silicon of 2* µm *acting as piezomaterial, sandwiched by gold electrodes of 200* nm*. A thin space-charge region of 50* nm *is artificially created to act as DC Coulomb force due to DC bias. Time-harmonic excitation signal is applied at the left most boundary.*

plate capacitor where we impress both the DC and AC voltages on the electrode. We like to operate in the linearized regime, ignoring higher harmonics. In this case, the time-domain simulation is not appealing per se. Moreover, the time-domain simulation is computationally expensive to reach steady state. Therefore, we view the mechanical response in the frequency domain.

However, the DC voltage term is completely washed out in the frequency domain. We introduce a concept of thin space-charge layer to mimic the DC voltage. In Fig. 4.8, a silicon bar of 2 µm is sandwiched between 200 nm gold electrodes. A thin space-charge region of 50 nm is created to mimic the boundary DC Coulomb force. In a (parallel plate) capacitor, the bound charges line up near the electrode boundaries of metal/dielectric interfaces. In numerics, it is difficult to implement such thin charge distribution. The computational cost in terms of computer RAM, number of interpolation nodes, cells using finite element method is high if this region is chosen very thin, e. g., 1 nm. Practically it is good enough to choose a thin region of tens of nm. This motivated us to choose 50 nm thin space charge region. We tuck-in this 50 nm artificial layer inside the 200 nm gold electrode. The total length of the structure is 2.4 µm, but we should only use 2 µm the length that will generate the eigenfrequency using Eq. (4.1). We apply DC bias of 10 V and $V_{\text{signal}} = 0.1$ V. We expect squeezing of silicon due to DC bias, and then the responses of mechanical displacement due to odd multiples of half-wavelength modes along the x-direction. Time-harmonic signal will propagate from the leftmost boundary; we record a snapshot of mechanical displacement at a given frequency.

The Coulomb attraction DC force density due to DC bias is shown in the thin strip regions in Fig. 4.9. We have only two opposing forces, but of the same magnitude here. The rest of the silicon and gold electrode regions have zero force densities.

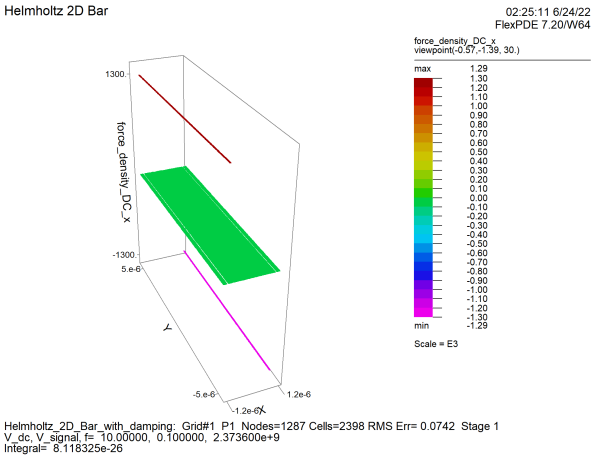


Figure 4.9.: *The electrostatic attraction DC force densities in* $\mathrm{N\,m^{-3}}$ *due to DC voltage. The left strip has force in the positive* x*-direction, the right strip has force in the negative* x*-direction.*

In Fig. 4.10, we have a snapshot of mechanically strained structure at resonant frequency $f_1 = 2.3806$ GHz, we have used a zoom factor of $3 \cdot 10^3$.

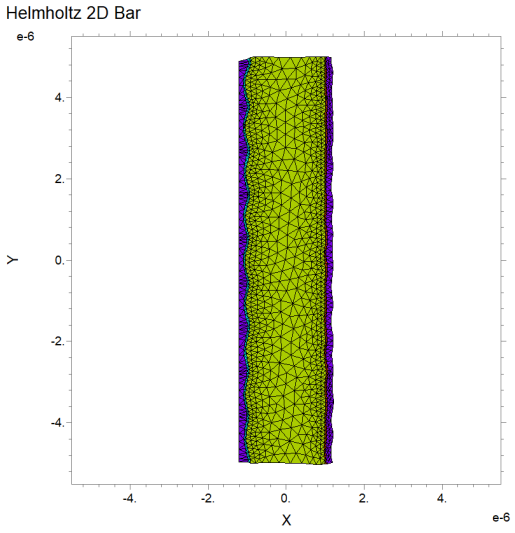
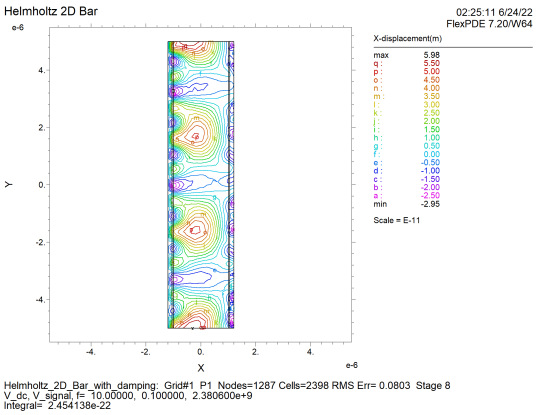
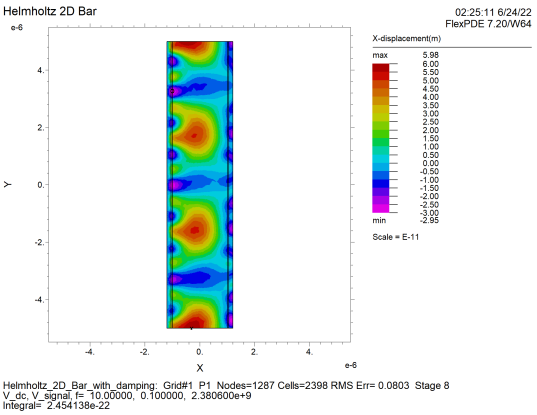


Figure 4.10.: *A snapshot of the 2D bar due to fundamental* $f_1 = 2.3806$ GHz*. Signal is applied at the leftmost boundary and travels through the bar, as can be seen by wiggles.*

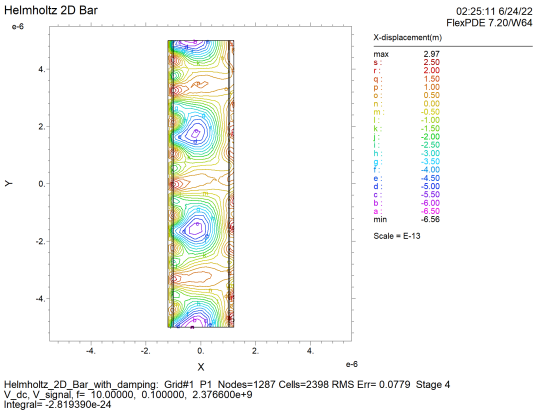
The x-displacement mode profile at resonant $f_1 = 2.3806$ GHz, as well as, at the off-resonant $f = 2.3766$ GHz are shown in Fig. 4.11. By comparing the magnitudes, the structure vibrates vigorously at the resonant frequency, as expected. We can observe three resonance modes in the vertical direction, this is due to choice of integer aspect ratio of 2D bar of 5:1. These modes will go away if we choose non-integer aspect ratio. We also notice lookalike resonances in the gold electrodes, we believe these are due to reflection. We will address this in §4.5.4.



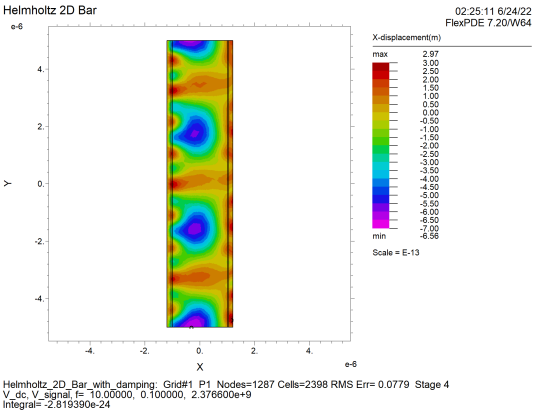
(a) Mode profile of x-displacement at resonant $f_1 = 2.3806$ GHz (Contour plot)



(b) Mode profile of x-displacement at resonant $f_1 = 2.3806$ GHz (Painted plot)

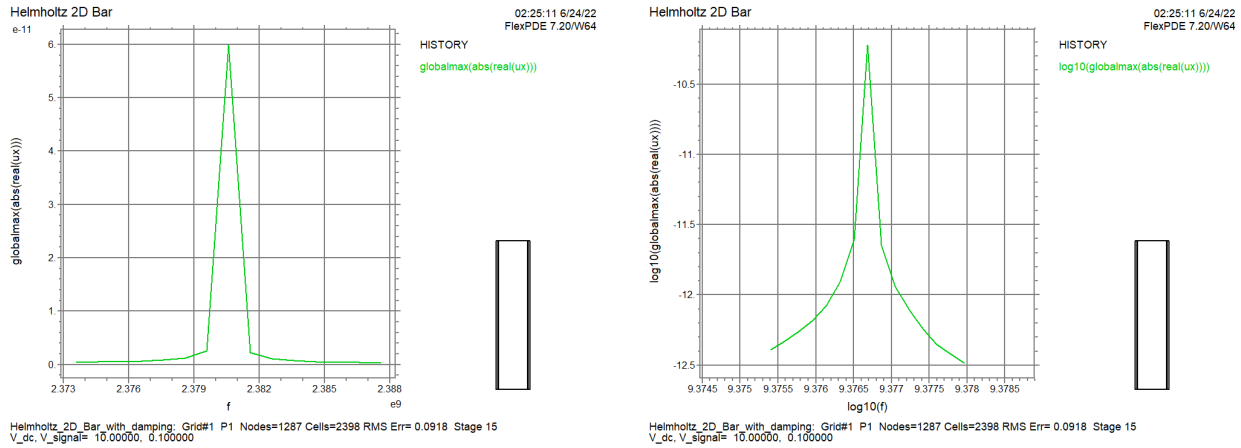


(c) Mode profile of x-displacement at off-resonant $f = 2.3766$ GHz (Contour plot)



(d) Mode profile of x-displacement at off-resonant $f = 2.3766$ GHz (Painted plot)

Figure 4.11.: (a)-(b) *The mode profile of x-displacement at $f_1 = 2.3806$ GHz. At resonance we observe very high mechanical displacement in magnitude in compare to off-resonance modes, one such is* (c)-(d) *at $f = 2.3766$ GHz. The resonances/patterns in the gold electrodes are due to reflections.*



(a) Frequency response

(b) Frequency response in $\log - \log$ scale.

Figure 4.12.: *We conduct 15 frequency sweeps at a step of* $df = 1$ MHz. *The x-displacement mechanical responses in linear scale in* (a)*, and in* $\log - \log$ *scale in* (b).

As we sweep the frequency, we can observe the acoustic displacement, this acts as virtual spectrum analyzer. We record in total 15 frequency sweeps at an interval of $df = 1$ MHz, centering at $f_1 = 2.3806$ GHz, as shown in Fig. 4.12. We observe a resonance peak at $f_1 = 2.3806$ GHz, in congruence with numerical value of Eq. (4.1).

We have numerically simulated the mechanical displacement in a 2D bar in the frequency domain due to both DC bias and harmonic excitation. The frequency response of our 2D bar matches with the deceptively simple analytical formula in Eq. (4.1). This gives us confidence that we have successfully implemented a numerical simulator incorporating both the DC and AC terms in the frequency domain.

We have incorporated gold and silicon material parameters, including damping of Chap. A. The FlexPDE code of all the plots in this Section is attached in Chap. F, in addition we have included explicit formulations of Eqs. (4.156)–Eq. (4.159) in Chap. G.

4.5.3. Response of One Unit of Piezoslab

The purpose of this study is to generate a running wave in a standalone one unit of lattice due to harmonic excitation and DC bias in the frequency domain. We explained the rationale of simulation in the frequency domain in §4.5.2, and will not repeat here.

The structure that we would like to simulate is shown in Fig. 4.13. The total width of this structure is wider by one electrode width, in compare to the unit cell of lattice in artificial

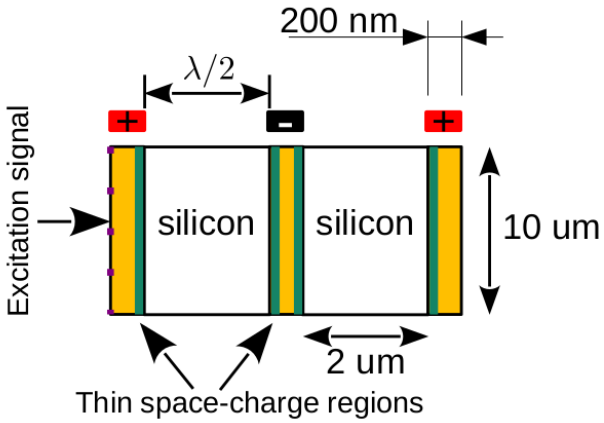


Figure 4.13.: *One unit in the artificial phononic lattice. Each silicon slab is 2* µm *wide, gold electrodes are 200* nm *wide, we create 50* nm *space-charge regions. The length of total structure is 4.6* µm *and its height is 10* µm.

phononic crystal in Fig. 4.2. We call the structure in Fig. 4.13 as one unit in this Section. This is acceptable, as the purpose of the electrodes is for electrical connection, and this should not alter the eigenfrequency, which is determined by the width of the silicon layer that acts as the artificial piezo material. This is our hypothesis, we want to find out this as well from our numerical test bench.

In Fig. 4.13, each silicon slab is 2 µm wide and 10 µm in height, each gold electrode is 200 nm wide and 10 µm in height. We create 50 nm space-charge region where we apply the DC voltage and we tuck-in this inside the electrode. We have two opposing polarization/electric fields. We apply excitation signal at the leftmost boundary. We apply 10 volt DC voltage and signal voltage of 0.1 V. The total length of the structure is 4.6 µm. However, we should only use 2 µm silicon width that will generate the eigenfrequency using analytical formula in Eq. (4.1).

We display the DC force density in Fig. 4.14. The snapshot of the mechanical displacement of the unit cell at resonant $f_1 = 2.3806$ GHz is shown in Fig. 4.15, where we used a zoom factor of $4 \cdot 10^3$ for visual purposes. We can clearly observe that the excitation AC signal at the leftmost boundary has travelled to the right, the wiggles confirm at least visually.

The x-displacement mode profile at the resonant $f_1 = 2.3806$ GHz, as well as, at the off-resonant $f = 2.3766$ GHz are shown in Fig. 4.16. By comparing the magnitudes, the structure vibrates vigorously at the resonant frequency, as expected. We can observe one mode in the horizontal direction in Fig. 4.16b in the resonant case. We also notice lookalike resonance modes in the gold electrodes, we believe these are due to reflection. We will address this in §4.5.4.

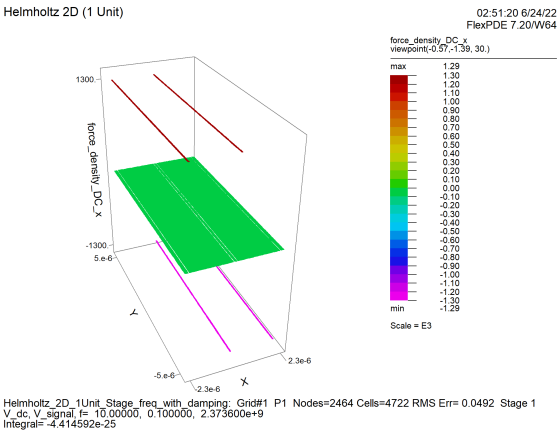


Figure 4.14.: *The DC Coulomb force densities* ($\mathrm{N\,m^{-3}}$) *in the space-charge regions. We inject Coulomb attraction force (densities) across the two silicon slabs with correct signs in the space-charge regions.*

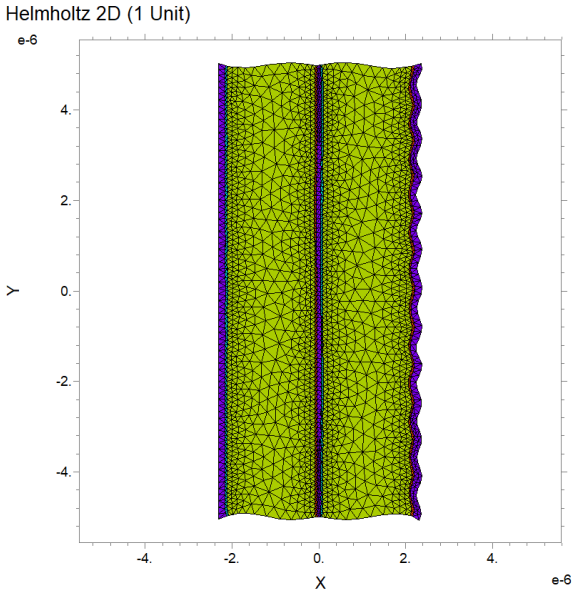
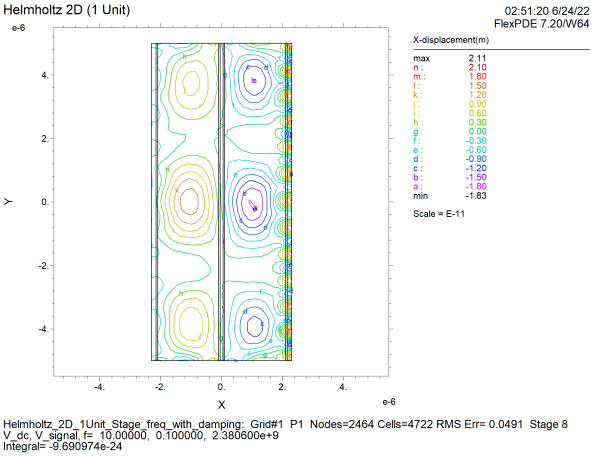
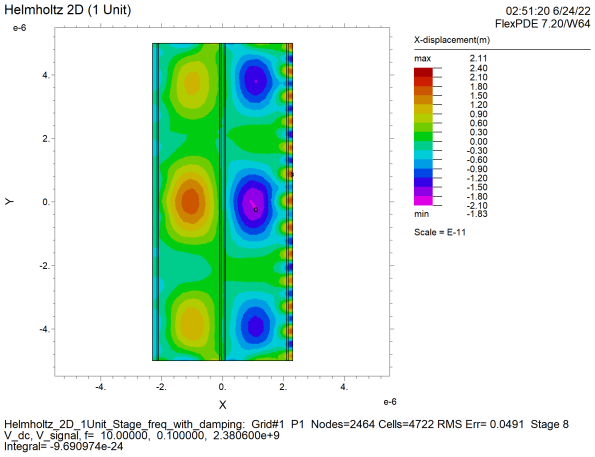


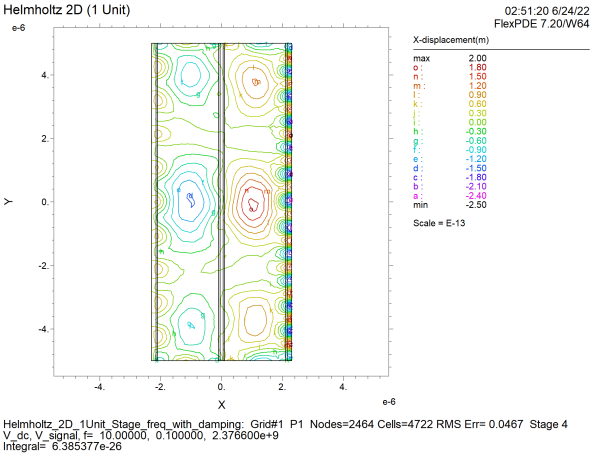
Figure 4.15.: *Mechanical vibration snapshot at resonant* $f_1 = 2.3806$ GHz. *Signal is applied at the leftmost boundary and travels through the unit cell, as can be seen by wiggles.*



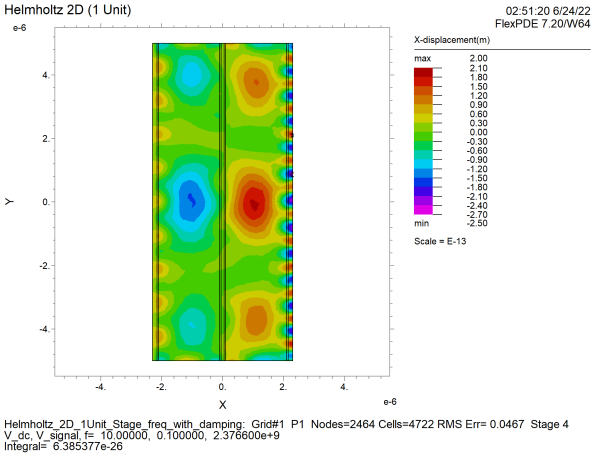
(a) Mode profile of x-displacement at resonant $f_1 = 2.3806$ GHz (Contour plot)



(b) Mode profile of x-displacement at resonant $f_1 = 2.3806$ GHz (Painted plot)

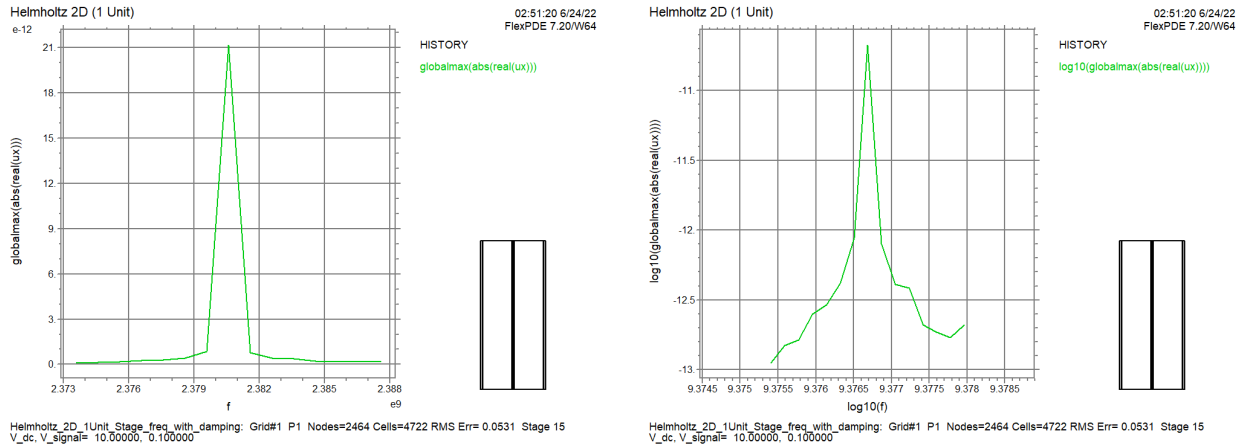


(c) Mode profile of x-displacement at off-resonant $f = 2.3766$ GHz (Contour plot)



(d) Mode profile of x-displacement at off-resonant $f = 2.3766$ GHz (Painted plot)

Figure 4.16.: (a)-(b) *The mode profile of x-displacement at $f_1 = 2.3806$ GHz. At resonance we observe very high mechanical displacement in magnitude in compare to off-resonance modes, one such is* (c)-(d) *at $f = 2.3766$ GHz. The resonances/patterns in the gold electrodes are due to reflections.*



(a) Linear scale **(b)** log – log scale

Figure 4.17.: *We perform 15 frequency sweeps at steps of 1* MHz *centered at the lowest order resonant frequency of* $f_1 = 2.3806$ GHz. *We clearly observe a resonance peak.* (a) *Linear scale.* (b) log – log *scale.*

We record in total 15 frequency sweeps at an interval of $df = 1$ MHz, centering at $f_1 = 2.3806$ GHz, as shown in Fig. 4.17. We observe a resonance peak at $f_1 = 2.3806$ GHz, in congruence with numerical value of Eq. (4.1).

We have numerically simulated the mechanical displacement in a 2D unit cell of the lattice in the frequency domain due to both DC bias and harmonic excitation. The frequency response of our 2D unit cell matches with the deceptively simple analytical formula in Eq. (4.1). This gives us further confidence that we have successfully implemented a numerical simulator incorporating both the DC and AC terms in the frequency domain.

We have incorporated gold and silicon material parameters, including damping of Chap. A. The FlexPDE code of all the plots in this Section is attached in Chap. H.

4.5.4. Terminating Absorber

The purpose of this numerical experiment is to show/prove our hypothesis that the resonance type modes, primarily at the rightmost gold electrodes, are due to reflection from the boundary, and how to mitigate them.

We noted resonance type modes in the gold electrodes, predominantly at the rightmost electrode in Fig. 4.11, Fig. 4.16. We hypothesized that these are due to acoustic impedance mismatch, as the wave reaches the rightmost boundary and then reflects from the rightmost boundary. If we have an infinite length medium, a running wave will not bounce back. Alternatively,

we can place an absorbing medium to the rightmost interface of finite length medium and be reasonably assured that minimal/no wave is reflected to the medium. In radiation boundary condition/Sommerfeld boundary condition, the distance from the source has to be infinite for the wave to vanish at infinity. In numerics, due to finite limitation, we can approximate it well using an absorber of finite length and (moderately) high absorption coefficient. We use a kind pseudo-silicon type material with high absorption and test our hypothesis.

In our case of running wave, if $\nabla U \cdot \mathbf{n} = \frac{\partial U}{\partial n}$, where $\mathbf{n}$ is the direction perpendicular to the boundary, and outwards from the solution domain, [10] the radiation boundary condition is [11]

$$\frac{\partial u_x^R}{\partial n} = \omega\sqrt{\rho}u_x^I, \quad \frac{\partial u_x^I}{\partial n} = -\omega\sqrt{\rho}u_x^R, \tag{4.160}$$

$$\frac{\partial u_y^R}{\partial n} = 0, \quad \frac{\partial u_y^I}{\partial n} = 0. \tag{4.161}$$

2D Bar with Absorber

We place an absorber with high damping, e. g., $\eta_{11} = \eta_{12} = \eta_{44} = 100$ poise ($1\ \mathrm{P} = 0.1\ \mathrm{N\,s\,m^{-2}}$) and length of 23 µm and same height, i. e., 10 µm at the end facet of the 2D bar of Fig. 4.8, as shown in Fig. 4.18. From numerical experiment, as shown in Fig. 4.19, waves do get transferred and energy flows from the bar to the load and the field value, here acoustic displacement diminishes, see the log contour and painted plots in Fig. 4.19. If we take a cross section at the middle of the structure, we clearly observe the attenuation of the acoustic displacement, as shown in Fig. 4.20.

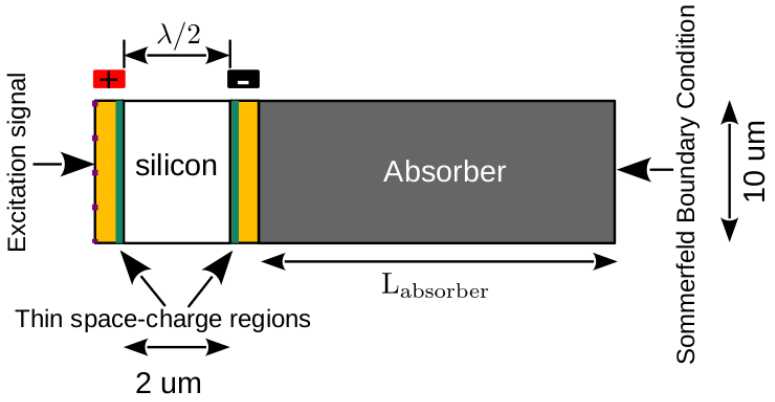


Figure 4.18.: *Schematic of adding absorber at the end as a load.*

[10] See, p. 50 in [69].
[11] See, p. 109 in [70], [71]

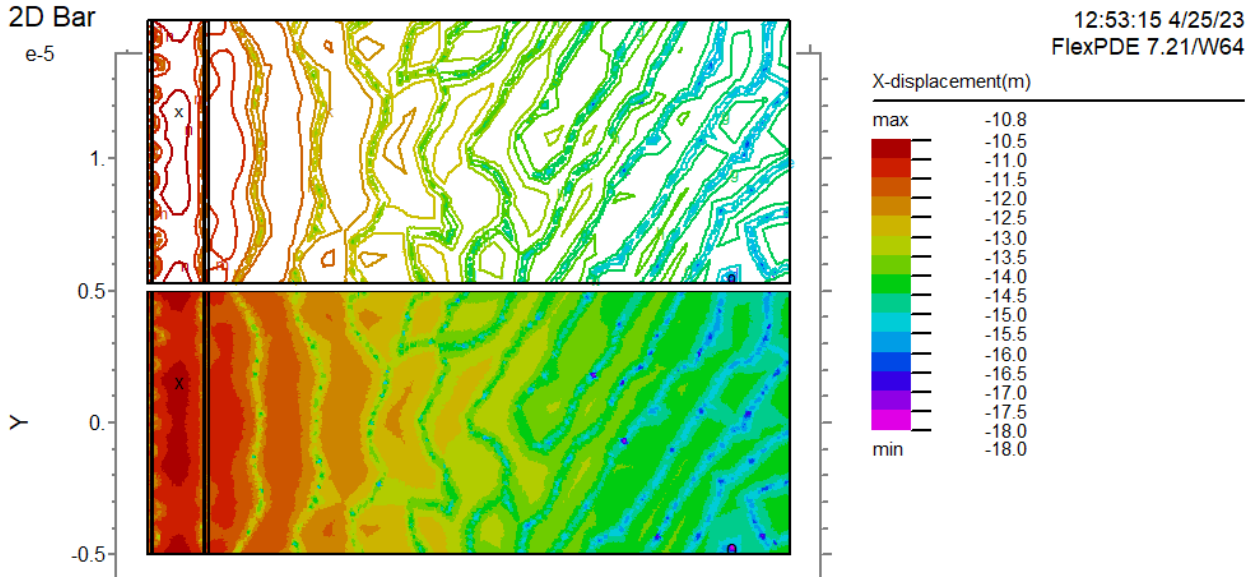


Figure 4.19.: *Contour and painted plot of wave propagation from the 2D in* log *scale. See the practically vanishing magnitude of* 10^{-18} m *at the far end of the load.*

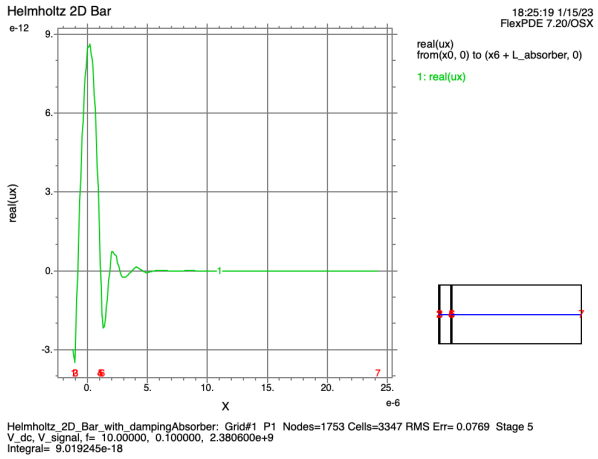


Figure 4.20.: *Cross section at the middle of the 2D bar. Waves reaching the end facet do not bounce back to the structure and attenuates in the load.*

2D Bar with Top & Bottom Absorbers

Up until now, we have used the reflective boundary conditions on the top and bottom surfaces of the structures. In the Lite version of the FlexPDE [1] software, due to the limitation of the number of nodes to be solved, this natural or Neumann boundary condition serves the purpose to a good degree. However, the proper way to enforce the boundary condition would be the absorbing boundary conditions on the top and bottom surfaces, impedance matched to the respective materials. The short-circuit equivalent mechanical boundary condition can be specified via **value(.)** in FlexPDE, while the open circuit is specified via **natural(.)**. Both the short-circuit, and open-circuit equivalent boundary conditions are totally reflective. The 50 Ω equivalent non-reflective boundary condition can be emulated by using an absorbing boundary condition, as shown in Fig. 4.21, Fig. 4.22 and Fig. 4.23, where the height of the top and bottom absorber layers are 10 μm.

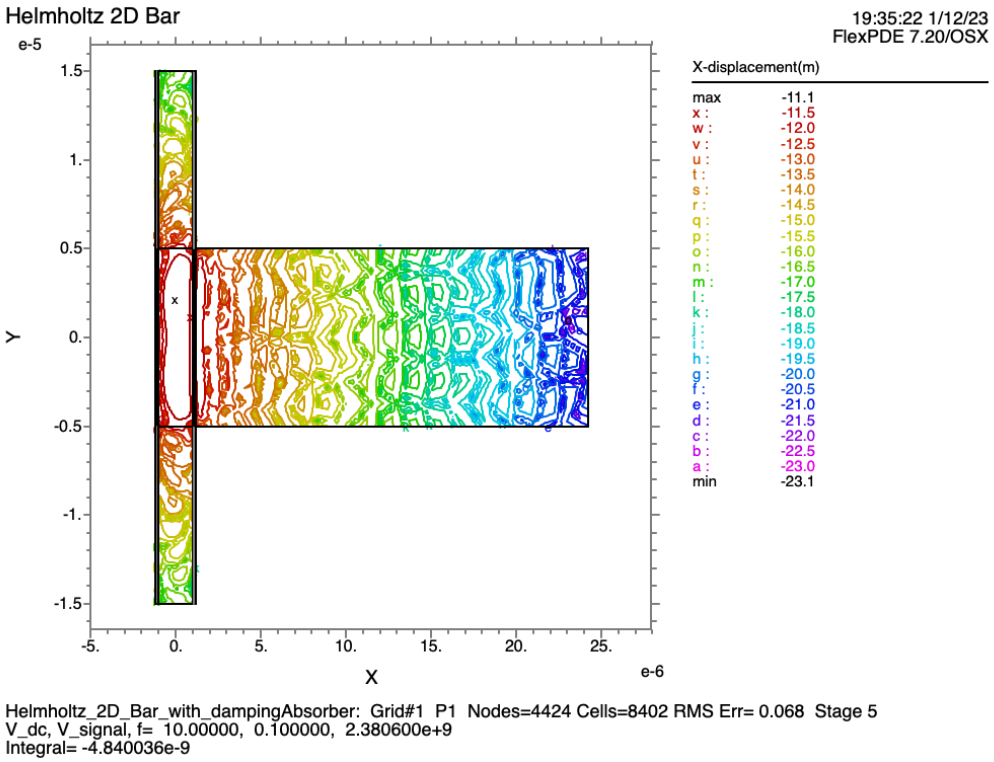


Figure 4.21.: *Contour plot of wave propagation from the 2D in* log *scale. See the practically vanishing magnitude of* 10^{-23} m *at the far end of the load.*

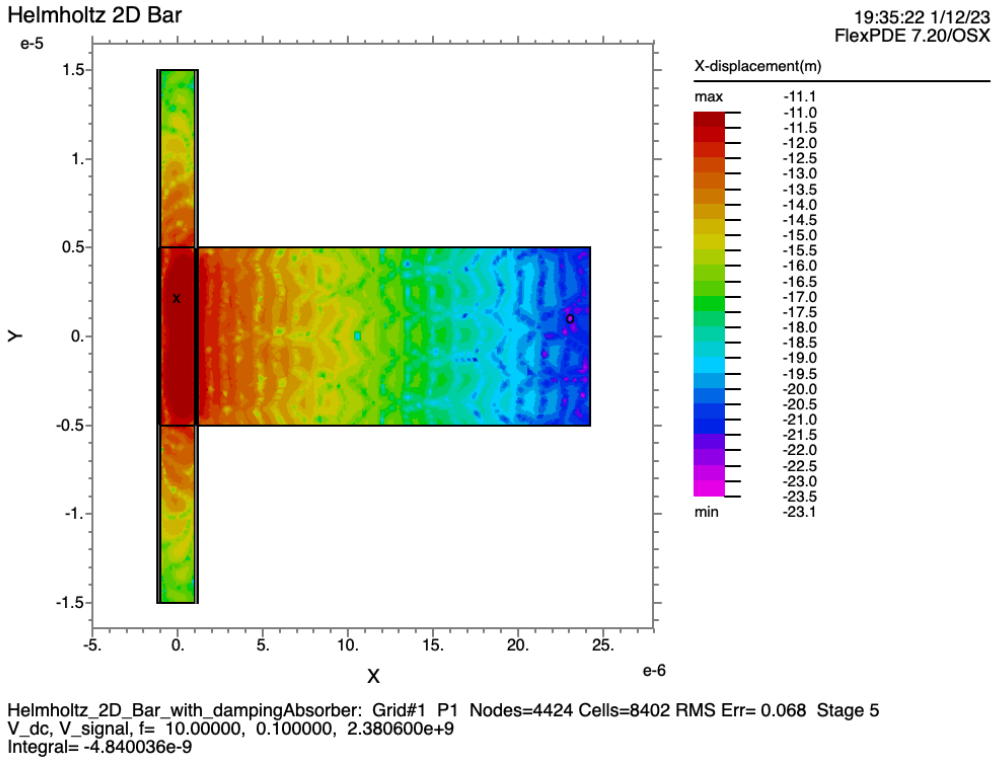


Figure 4.22.: *Painted plot of wave propagation from the 2D in* log *scale. See the practically vanishing magnitude of* 10^{-23} m *at the far end of the load.*

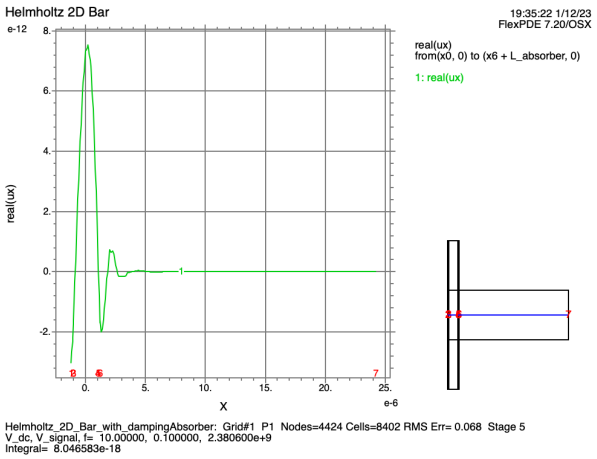


Figure 4.23.: *Cross section at the middle of the 2D bar. Waves reaching the end facet do not bounce back to the structure and attenuates in the load.*

4.5.5. Effect of the Bias Voltage on the Generated Acoustic Wave

The purpose of this numerical experiment in 2D is to validate our hypothesis that the generated acoustic wave is linearly proportional to the DC bias that is used the (pre)-stress the material, as well as it depends linearly on the amplitude of the signal voltage, as we derived analytically for the 1D case in Eq. (3.76) in §3.4.1.

The structure we like to simulate is schematically drawn in Fig. 4.13, with the addition of the right absorbing layer to mimic a running wave. The absorbing layers are placed at the top and the bottom to account for the non-reflecting boundary condition, this is like the equivalent 50-ohm resistor in coaxial cable. We record the global of the maximum mechanical displacement using.

$$u_{\max} = \sqrt{u_x^2 + u_y^2} = \sqrt{\Re[u_x]^2 + \Im[u_x]^2 + \Re[u_y]^2 + \Im[u_y]^2} \tag{4.162}$$

At 10 V DC bias and 0.1 V excitation signal at the left, the phase snapshot at the resonant frequency $f_1 = 2.3806$ GHz is shown in Fig. 4.24. We increase the DC bias from 0 V in steps of 1 V and record the global maximum in Fig. 4.25a. We clearly observe a linear dependence as expected from the 1D analysis by Eq. (3.76) in §3.4.1. At a fixed stress of 10 V, we increase the excitation signal from 0 V in step of 0.1 V and record the global maximum of the mechanical displacement in Fig. 4.25b. This is in agreement with what we expected in 1D in Chap. 3.

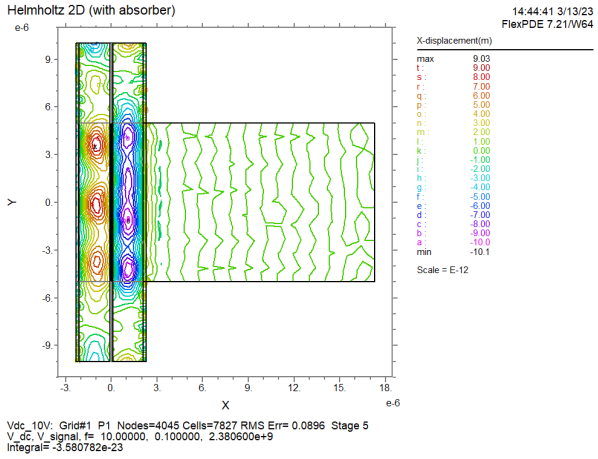
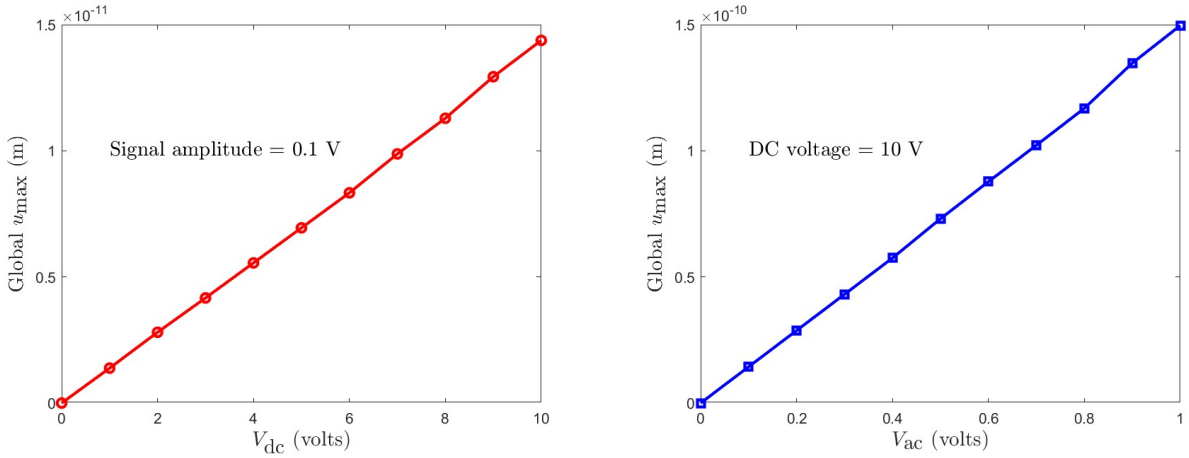


Figure 4.24.: *The x-component of the mechanical displacement in a one unit of piezoslab, right terminated by the absorbing boundary condition. The absorbing layers are placed at the top and bottom to account for the non-reflecting equivalent 50 ohm coaxial cable.*



(a) Varying DC bias **(b)** Varying signal voltage

Figure 4.25.: *The global maximum of the generated mechanical amplitude, as we sweep the DC bias and the excitation signal.* (a) *At a fixed excitation signal amplitude of 0.1* V*, the DC bias is increased from 0* volt *(when no acoustic wave is generated) in a step of 1* V. (b) *At a fixed DC voltage of 10* V*, we place excitation signal voltage from 0* V *in steps of 0.1* V.

4.5.6. Electromechanical Coupling in 2D

The electromechanical coupling κ^2 is extracted from the numerical simulation using the formula in this section. The time-average acoustic energy in one lattice period is

$$\langle E \rangle_{\text{acoustic}} = \pi k c_{11} A \left| u_{\max} \right|^2 , \tag{4.163}$$

where A is the area. We record the global maximum of the mechanical displacement using Eq. (4.162). The time-averaged input electrical energy, including the DC voltage to create the polarization dipoles, is

$$\langle E \rangle_{\text{input}} = \frac{1}{2} C V_0^2 + \frac{1}{4} C \delta V^2 . \tag{4.164}$$

The electromechanical coupling coefficient is

$$\kappa^2 = \frac{\langle E \rangle_{\text{acoustic}}}{\langle E \rangle_{\text{input}}} . \tag{4.165}$$

For the structure in Fig. 4.13, $\kappa^2 \approx 7.5$ % at DC voltage of 10 V and signal voltage of 0.1 V.

4.5.7. Homogenized Piezoelectric Structure using Effective Piezoelectric Constant

The purpose of this numerical experiment is so simulate acoustic wave propagation in 2D homogenized piezoelectric structure using the effective piezoelectric constant computed from the model in this Chapter, and compare how faithfully this homogenized artificial constants reproduce the non-homogenized structure in Fig. 4.26.

The two coupled equations using the e-components are written as

$$\frac{\partial}{\partial x}\left[\alpha_{11}\frac{\partial u_x}{\partial x}+\alpha_{12}\frac{\partial u_y}{\partial y}\right]+\frac{\partial}{\partial y}\left[\alpha_{44}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)\right]+\rho\omega^2 u_x=0, \tag{4.166}$$

$$\frac{\partial}{\partial y}\left[\alpha_{12}\frac{\partial u_x}{\partial x}+\alpha_{22}\frac{\partial u_y}{\partial y}\right]+\frac{\partial}{\partial x}\left[\alpha_{44}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)\right]+\rho\omega^2 u_y=0, \tag{4.167}$$

where

$$\alpha_{11}=c_{11}^{\text{homo}}+i\omega\eta_{11}^{\text{homo}}+\frac{1}{\epsilon_1^{\text{homo}}}(e_{\text{x1}}^{\text{homo}})^2 \tag{4.168}$$

$$=\frac{E^{\text{homo}}(1-\nu^{\text{homo}})}{(1+\nu^{\text{homo}})(1-2\nu^{\text{homo}})}+i\omega\eta_{11}^{\text{homo}}+\frac{1}{\epsilon_1^{\text{homo}}}(e_{\text{x1}}^{\text{homo}})^2, \tag{4.169}$$

$$\alpha_{12}=c_{12}^{\text{homo}}+i\omega\eta_{12}^{\text{homo}}+\frac{1}{\epsilon_1^{\text{homo}}}e_{\text{x1}}^{\text{homo}}e_{\text{x2}}^{\text{homo}} \tag{4.170}$$

$$=\frac{E^{\text{homo}}\nu^{\text{homo}}}{(1+\nu^{\text{homo}})(1-2\nu^{\text{homo}})}+i\omega\eta_{12}^{\text{homo}}+\frac{1}{\epsilon_1^{\text{homo}}}e_{\text{x1}}^{\text{homo}}e_{\text{x2}}^{\text{homo}}, \tag{4.171}$$

$$\alpha_{22}=c_{11}^{\text{homo}}+i\omega\eta_{22}^{\text{homo}}+\frac{1}{\epsilon_1^{\text{homo}}}(e_{\text{x2}}^{\text{homo}})^2 \tag{4.172}$$

$$=\frac{E^{\text{homo}}(1-\nu^{\text{homo}})}{(1+\nu^{\text{homo}})(1-2\nu^{\text{homo}})}+i\omega\eta_{22}^{\text{homo}}+\frac{1}{\epsilon_1^{\text{homo}}}(e_{\text{x2}}^{\text{homo}})^2, \tag{4.173}$$

$$\alpha_{44}=c_{44}^{\text{homo}}+i\omega\eta_{44}^{\text{homo}}+\frac{1}{\epsilon_1^{\text{homo}}}(e_{\text{y6}}^{\text{homo}})^2 \tag{4.174}$$

$$=\frac{E^{\text{homo}}}{2(1+\nu^{\text{homo}})}+i\omega\eta_{44}^{\text{homo}}+\frac{1}{\epsilon_1^{\text{homo}}}(e_{\text{y6}}^{\text{homo}})^2. \tag{4.175}$$

The artificial piezo-constants e_{x1} etc for the design specific case are provided in §4.4.3. There is a plethora of homogenization techniques. One must choose the appropriate one that provides as close as possible faithful representation of the non-homogenized structure. We homogenize using the mean theory in Eq. (3.291) without further consideration.

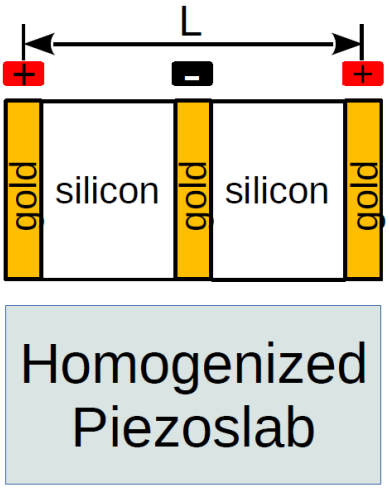


Figure 4.26.: *The (two interdigital) metal electrodes are deposited on top of non-piezoelectric substrate, e. g., silicon. A DC bias applied across a silicon slab. The two opposing electrical dipoles can be discerned. This can be considered as one unit of the artificial piezo structure in 2D (though one cell of phononic lattice with period L). The bottom figure represents the homogenized piezoslab.*

The structure we would like to simulate is shown in the top figure in Fig. 4.26. The parameters are exactly the same in §4.5.3. There could be myriad number of homogenization techniques. We have to find one that faithfully represents the non-homogenized structure. Assuming that we can find the proper homogenized artificial piezo constants, we can simulate an equivalent piezo slab – this is shown schematically in the bottom figure in Fig. 4.26. We impinge an excitation signal the leftmost boundary as shown in Fig. 4.26. We include top and bottom layers to mimic non-reflecting boundary condition, as well as the right absorber where we place the Sommerfeld boundary condition at the rightmost boundary. The damping of the top, bottom and the right layers is $\eta_{11} = \eta_{12} = \eta_{22} = \eta_{44} = 4\ \mathrm{N\,s\,m^{-2}}$. A DC voltage is 10 V is placed with an excitation signal amplitude of 0.1 V at the leftmost boundary in Fig. 4.26. The generated acoustic wave in 2D non-homogenized structure is shown in Fig. 4.27. Using the nominal effective mean values of homogenized parameters, the acoustic wave in such an equivalent piezoslab is shown in Fig. 4.28a. If we replace the homogenized mass density and the homogenized relative permittivity by that of silicon, the acoustic wave generated in such homogenized 2D slab turns out to be Fig. 4.28b. As clearly mentioned in §3.9.2, a proper homogenization technique will ultimately resemble the non-homogenized mode profile. The mode profile in the non-homogenized structure in Fig. 4.27 and the homogenized structure in Fig. 4.28 are not the same, as there are resonance effects in the non-homogenized structure.

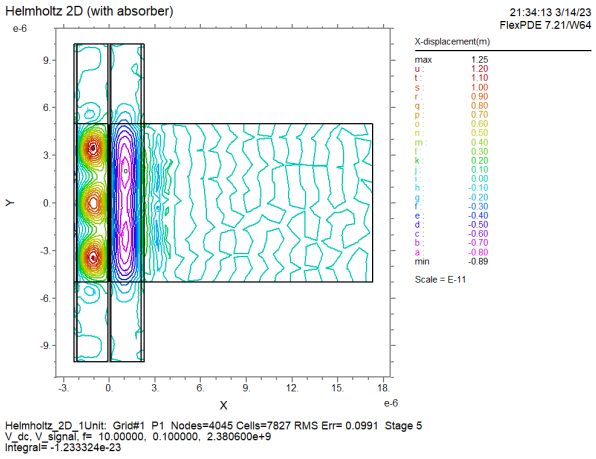
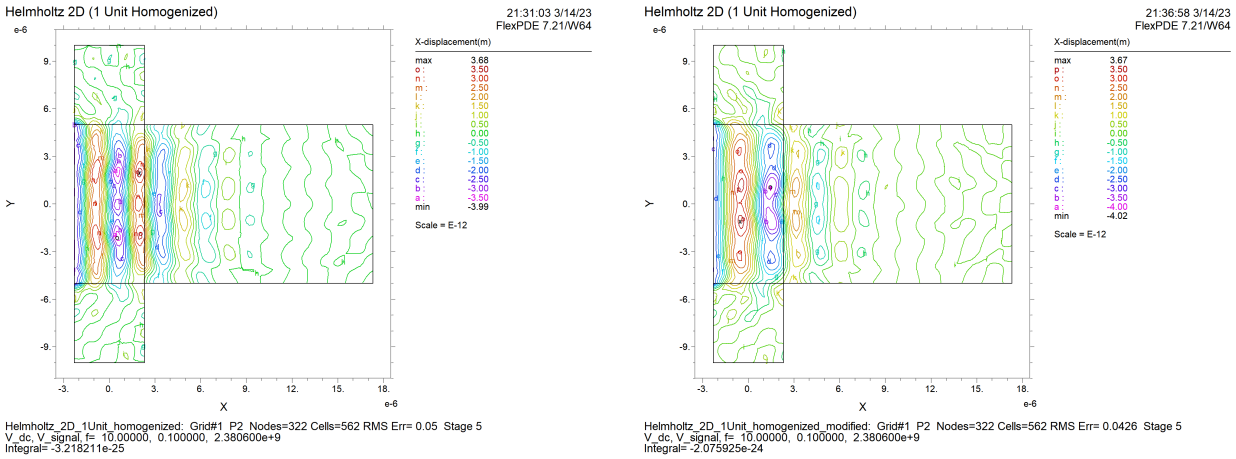


Figure 4.27.: *The x-component of the mechanical displacement in a one unit of non-homogenized piezoslab, right terminated by the absorbing boundary condition. The absorbing layers are placed at the top and bottom to account for the non-reflecting equivalent 50 ohm coaxial cable.*



(a) Mean value homogenization **(b)** Modified homogenization

Figure 4.28.: *The x-displacement in 2D homogenized piezoslab using mean value in* (a). *The mode profile is clearly not the same as the non-homogenized structure. If we replace the mass density and the relative permittivity by silicon, we obtain acoustic wave in such a homogenized structure in* (b), *which might be closer to the non-homogenized structure in Fig. 4.27.*

Chapter 5
5 Epilogue

Naturally occurring silicon does not exhibit piezoelectricity. In this thesis, we created dipole using DC bias in a sub-unit of a simple geometric structure made of silicon material. We showed non-vanishing piezoelectric constants in silicon. This same principle can be applied to other centro-symmetric materials which do not possess natural piezoelectricity. We join these two sub-units in back-to-back configuration and create a phononic crystal of such units, and provide one example of such artificial piezoelectricity as electrically tunable mechanical resonator in silicon, having a mechanical Q-factor in the order of 1000. In our artificial piezoelectric phononic crystal, the frequency is scalable and can be up to hundreds of MHz *range which is immensely beneficial to the ultrasound community. Depending on the application, the frequency can be easily pushed to* GHz *range. Moreover, the group velocity and hence the stiffness of bulk silicon can be tailored using phononic crystal or porous silicon to achieve higher mechanical amplitude and thus higher electromechanical coupling coefficient using low voltage.*

5.1. Application # 1: On-Chip Mechanical Signal Generator

The purpose of this small numerical experiment is to have an on-chip mechanical signal generator in silicon by application of both DC and AC sinusoidal voltage. A filter does not typically have its own built-in/on-board signal generator. Typically a filter takes a signal generated from some source, then applies a transfer function to the input signal, thus allowing an output signal with the desired characteristics. In §5.2, we can use such a mechanical signal generator for a mechanical filter in artificial piezo-silicon.

A schematic of an on-board mechanical signal generator is shown in Fig. 5.1. The DC Coulomb force is applied via the narrow thin space-charge region. An AC voltage that is small compared to the DC voltage is applied to the right and left sides. The potentials applied on the left and right sides are 180° out of phase with each other. This set setup is different from the boundary

conditions in Fig. 4.8. We expect the left and right capacitor plates to move in the opposite directions in unison, around the quiescent point of the DC biased compressed silicon layer. Thus we expect to create a symmetric mechanical response, as the electrical voltage is the same but opposite at the desired frequency of operation.

A 10 V DC bias is applied across the silicon layer, while the magnitude of the AC sinusoidal voltage is 0.1 V. The frequency of operation is $f_1 = 2.3806$ GHz, a little bit detuned from the resonance frequency. The dimension of the structure is shown in Fig. 5.1. The silicon layer is compressed due to DC bias. If the left side AC voltage enforces additional compression along the positive x-direction, the right side AC voltage will cause similar amount of compression along the negative x-direction. This is exactly we observe in Fig. 5.2. For this simple geometry, we used the change in mechanical displacement, δL due to AC voltage perturbation δV, as given by Eq. (3.278), this is a powerful trick we deployed in our FlexPDE code.

Thus we have created a mechanical signal generator using artificial piezo-electric effect in silicon. The input is the DC voltage, coupled with AC voltage, while the output is the mechanical displacement. This excitation structure can be used as electrically excited mechanical signal generator.

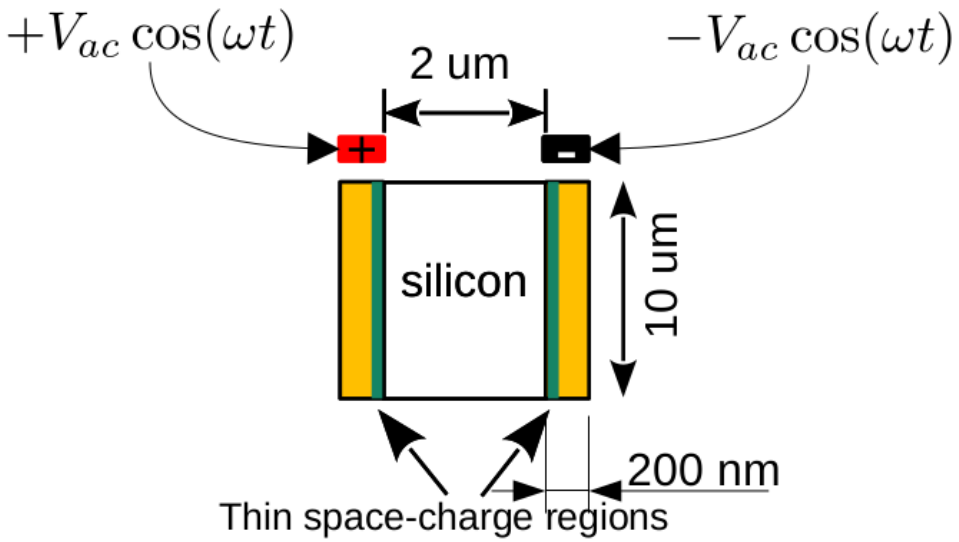
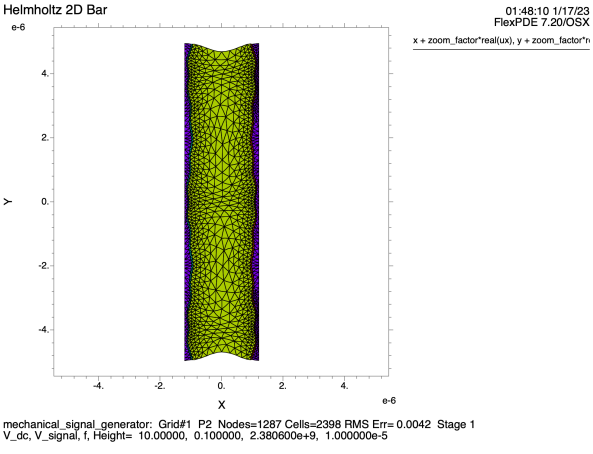
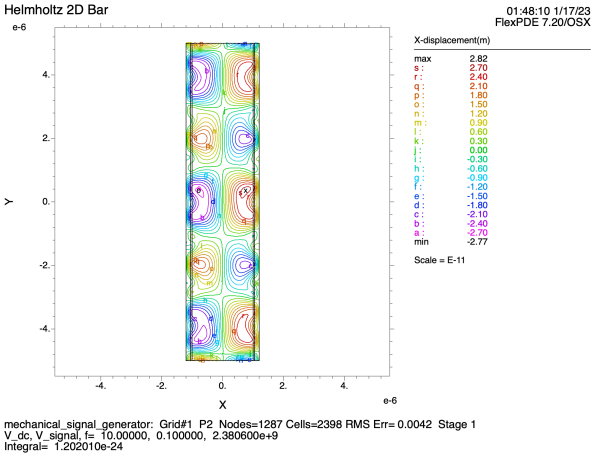


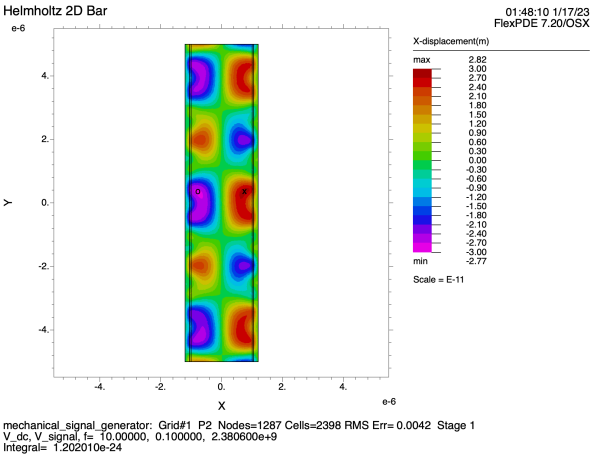
Figure 5.1.: *A 2D bar of silicon of 2* µm *acting as piezomaterial, sandwiched by gold electrodes of 200* nm*. A thin space-charge region of 50* nm *is artificially created to act as DC Coulomb force due to DC bias. The left electrode is applied a positive sinusoidal AC voltage, while the right side is negative AC voltage, cf.Fig. 4.8.*



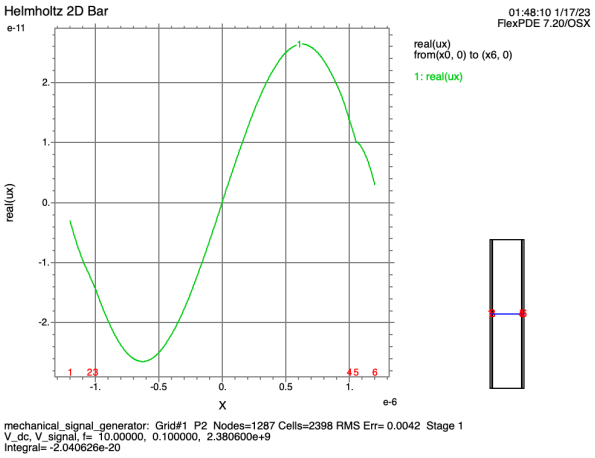
(a) Mechanical vibration snapshot.



(b) Mode profile of x-displacement (Contour plot)



(c) Mode profile of x-displacement (Painted plot)



(d) Cross section of x-displacement across the middle of the structure.

Figure 5.2.: *We apply DC bias of 10* V *and* $V_{signal} = 0.1$ V *at* $f_1 = 2.3806$ GHz. (a) *The mechanical vibration snapshot. The mode profile of* x*-displacement as contour plot in* (c) *and as painted plot in* (c). (d) *A cross-sectional displacement across the middle of the structure.*

5.2. Application # 2: Electrically Tunable Mechanical Resonator

The purpose of this numerical experiment is to generate a running wave in a finite 2D lattice and simulate the mechanical responses in that finite lattice due to both harmonic excitation and DC bias in the frequency domain. We explained the rationale of simulation in the frequency domain in §4.5.2, and will not repeat here. Due to computational cost, we limit the number of cells to three, but the FlexPDE code listing in Chap. I is scalable for any n number of units via the parameter nUnit in the **DEFINITION** section.

The structure we like to simulate is shown in Fig. 5.3. The silicon slab is 2 µm wide, gold electrodes are 200 nm wide. We create 50 nm space-charge regions and tuck them inside the electrodes. The length of total structure is 13.4 µm and its height is 10 µm. We create dipoles using DC voltage of 10 V and apply excitation signal of 0.1 V at the leftmost boundary. The applied DC force densities are shown in Fig. 5.4.

The acoustic vibration snapshot at resonant $f_1 = 2.3806$ GHz is shown in Fig. 5.5, we used zoom factor of $4 \cdot 10^3$ for visualization purpose. The acoustic particle displacement on the far right boundary is the signal that has traveled through the structure.

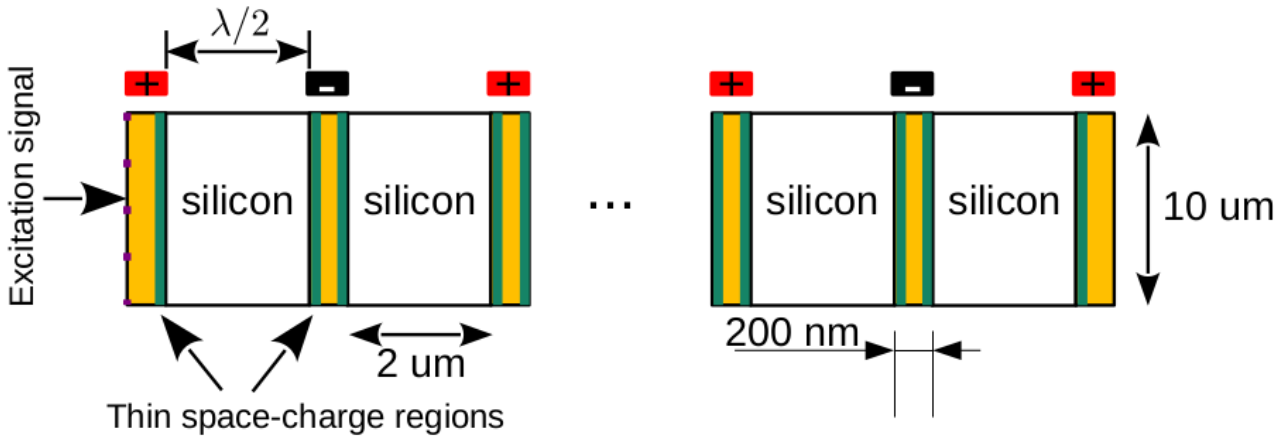


Figure 5.3.: *Finite lattice of artificial phononic crystal. We note the starting and ending electrodes and the associated configuration of space-charge regions. We apply signal at the leftmost boundary, and will travel along the lattice. The size of each silicon slab is 2* µm*, gold electrodes 200* nm*. We create 50* nm *space-charge region where we apply only DC voltage, the height of the entire structure is 10* µm*.*

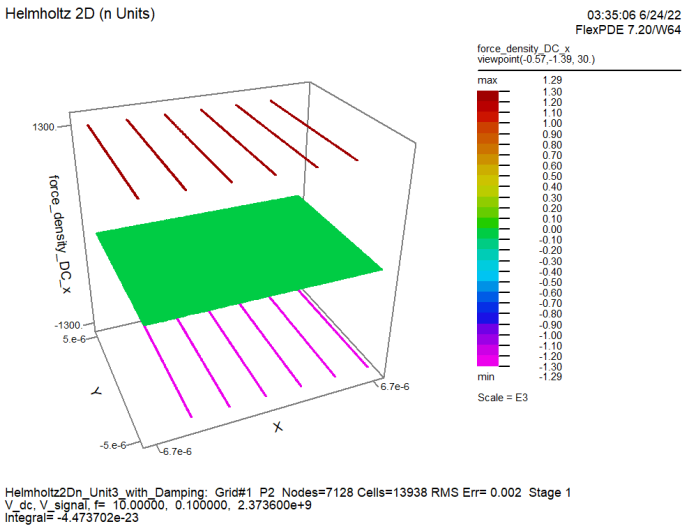


Figure 5.4.: *DC force densities in 3-cell finite lattice.*

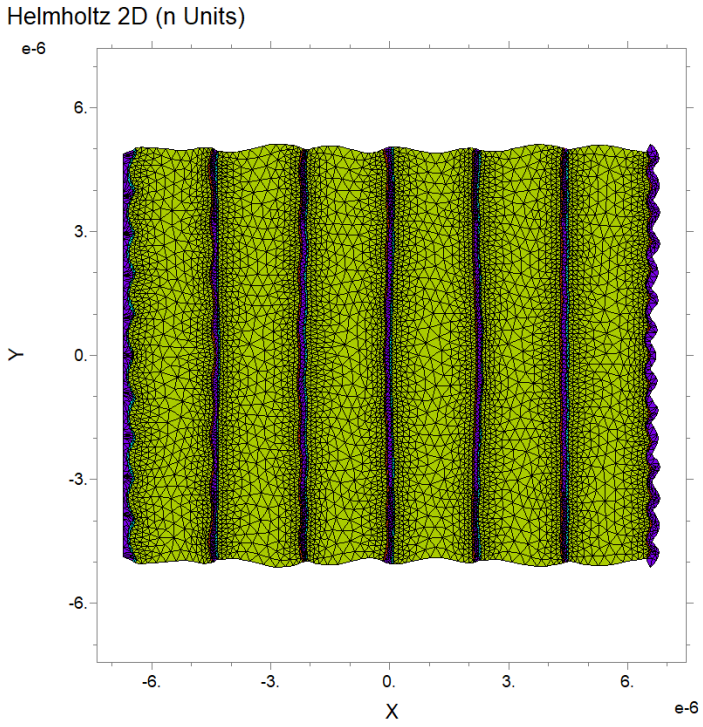
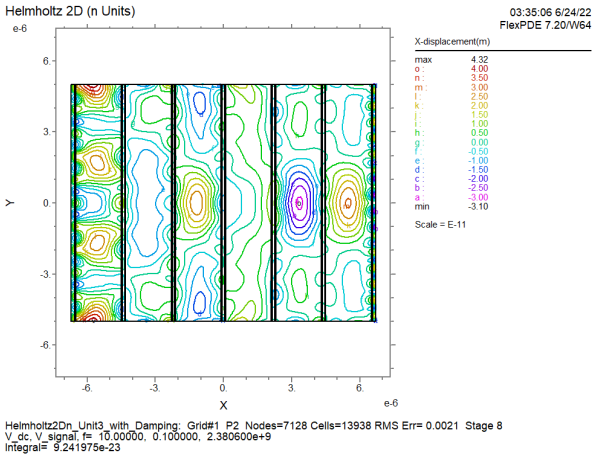


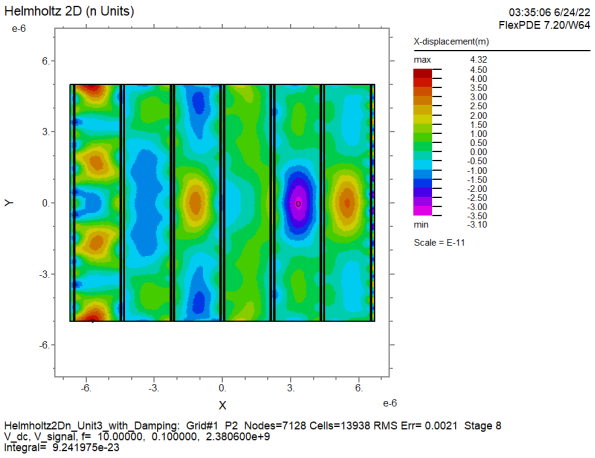
Figure 5.5.: *In-plane vibration snapshot at resonant* $f_1 = 2.3806$ GHz*, we used zoom factor of* $4 \cdot 10^3$*.*

We next compute and display the x-mode resonance at the fundamental $f_1 = 2.3806$ GHz in and off-resonance x-mode at $f = 2.3766$ GHz in Fig. 5.6. We can clearly notice the magnitude between the resonance and off-resonant modes.

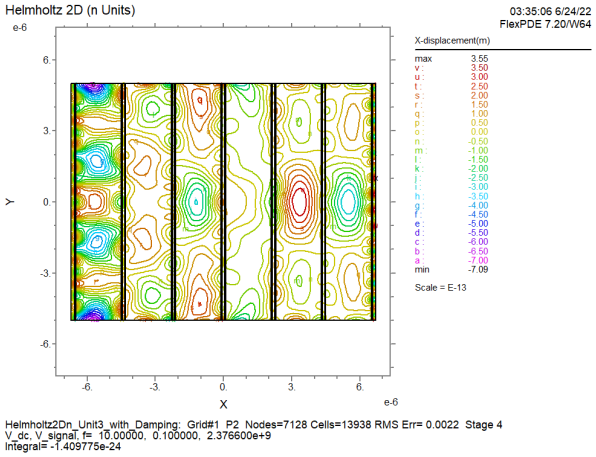
The x-displacement mode profile at the resonant $f_1 = 2.3806$ GHz, as well as, at the off-resonant $f = 2.3766$ GHz are shown in Fig. 5.6. By comparing magnitudes, the structure vibrates vigorously at resonant frequency, as expected. In Fig. 5.6b, we observe one mode along the horizontal direction from the rightmost boundary spanning two silicon bars and believe this is due to back reflection. We also observe three modes in the first silicon slab and one mode in the third slab along the vertical direction. We also notice lookalike resonance modes in the gold electrodes, and believe these are due to reflection. We will address this in §5.2.1.



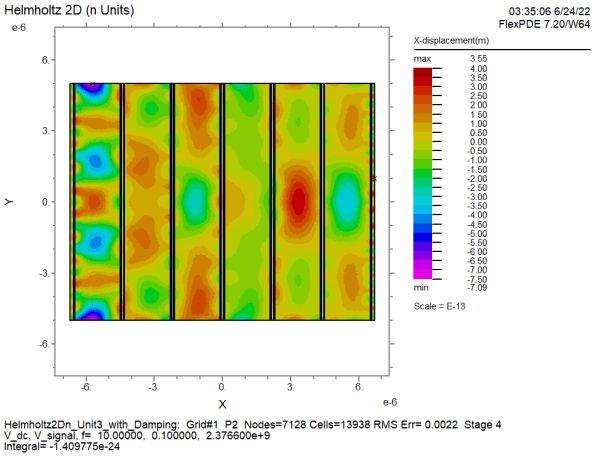
(a) Mode profile of x-displacement at resonant $f_1 = 2.3806$ GHz (Contour plot)



(b) Mode profile of x-displacement at resonant $f_1 = 2.3806$ GHz (Painted plot)



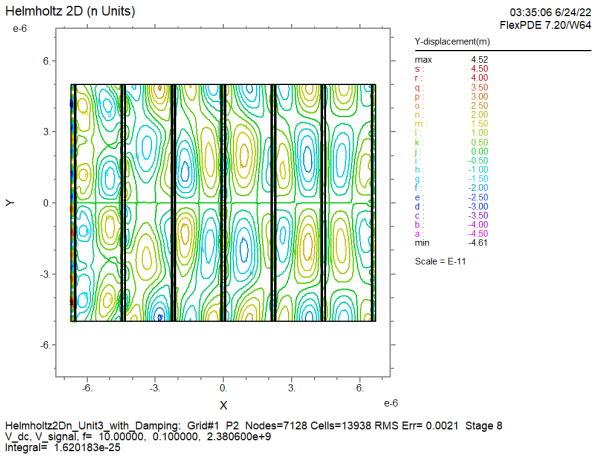
(c) Mode profile of x-displacement at off-resonant $f = 2.3766$ GHz (Contour plot)



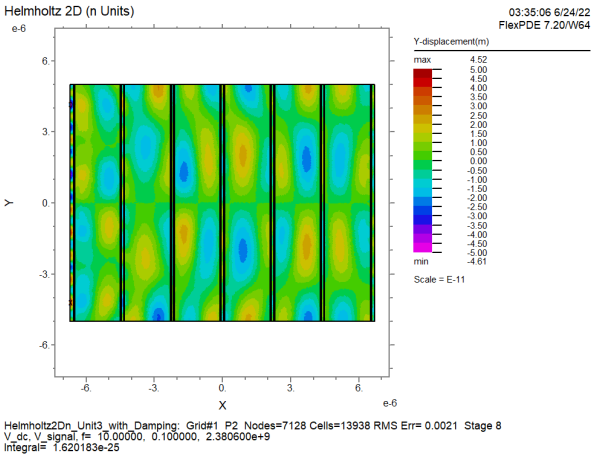
(d) Mode profile of x-displacement at off-resonant $f = 2.3766$ GHz (Painted plot)

Figure 5.6.: (a)-(b) *The mode profile of x-displacement at $f_1 = 2.3806$ GHz. At resonance we observe very high mechanical displacement in magnitude in compare to off-resonance modes, one such is* (c)-(d) *at $f = 2.3766$ GHz. The resonances/patterns in the gold electrodes are due to reflections.*

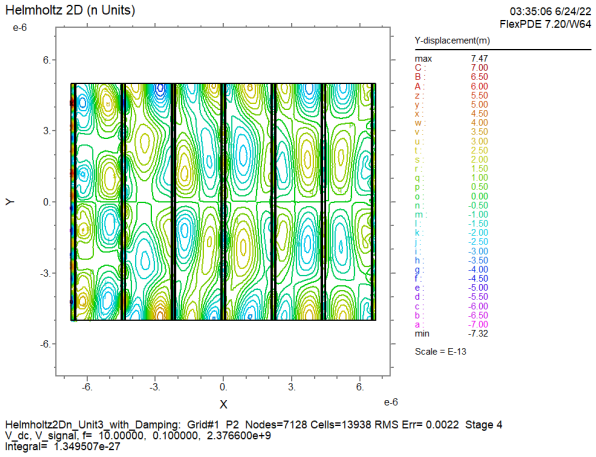
We notice rather interesting mode profile for the y-displacement, as shown in Fig. 5.7. In reality, we will observe mixed modes of x and y- displacements, i. e., u_x and u_y, of Fig. 5.6 and Fig. 5.7, and will be difficult to disentangle them. However, in numerics, we can easily separate these modes.



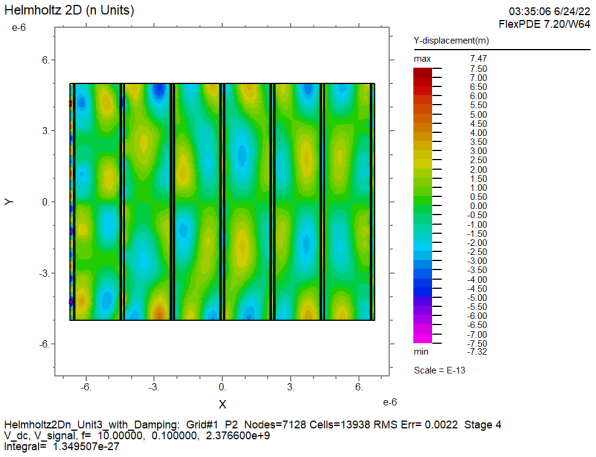
(a) Mode profile of y-displacement at resonant $f_1 = 2.3806$ GHz (Contour plot)



(b) Mode profile of y-displacement at resonant $f_1 = 2.3806$ GHz (Painted plot)

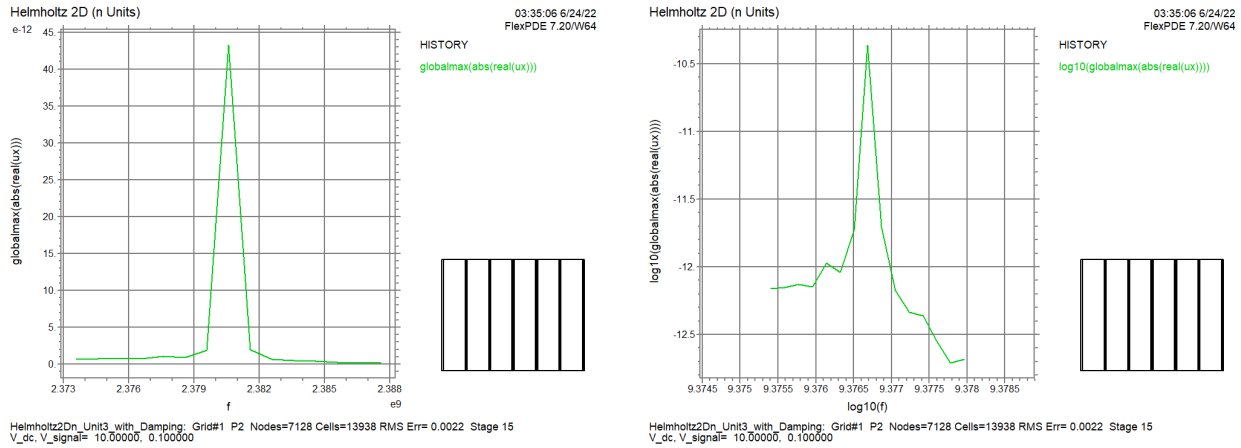


(c) Mode profile of y-displacement at off-resonant $f = 2.3766$ GHz (Contour plot)



(d) Mode profile of y-displacement at off-resonant $f = 2.3766$ GHz (Painted plot)

Figure 5.7.: (a)-(b) *The mode profile of y-displacement at* $f_1 = 2.3806$ GHz. (c)-(d) *The mode profile of y-displacement at* $f = 2.3766$ GHz.



(a) Linear scale **(b)** log – log scale

Figure 5.8.: *Frequency response of x-displacement in* (a) *linear scale,* (b) log – log *scale.*

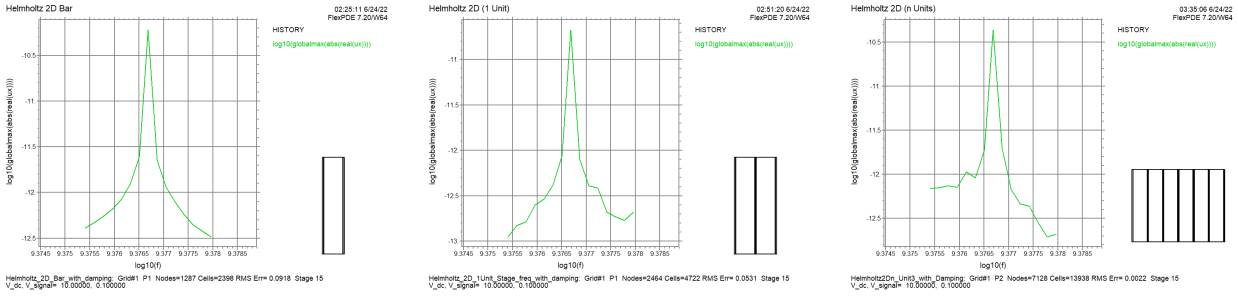
As we sweep the frequency, we can observe the acoustic displacement, this acts as virtual spectrum analyzer. We record in total 15 frequency sweeps at an interval of $df = 1$ MHz, centering at $f_1 = 2.3806$ GHz, as shown in Fig. 5.8. We observe a resonance peak at $f_1 = 2.3806$ GHz, in congruence with numerical value of Eq. (4.1).

We have numerically simulated the mechanical displacement in three cells of the finite lattice of 2D structure in Fig. 4.2 in the frequency domain due to both DC bias and harmonic excitation. The frequency response of our 2D unit cells matches with the deceptively simple analytical formula in Eq. (4.1). This gives us firm confidence that we have successfully implemented a numerical simulator incorporating both the DC and AC terms in the frequency domain.

We have incorporated gold and silicon material parameters, including damping of Chap. A. The FlexPDE code of all the plots in this Section is attached in Chap. I.

Filtering: Effect of Addition of n Cells

We cannot have bandgap in a bar or single unit cell. The concept of bandgap formation occurs as we stack up infinite number of these unit cells. We truncated to 3 units in this thesis for computational cost. As we add more cells, we can expect narrowing of the frequency response, or increasing the Q-factor, as shown in Fig. 5.9. The $\Delta f \sim 2$ MHz, with $f_0 = 2.3806$ GHz, providing a mechanical Q-factor of $Q = f_0/\Delta f = 1190$, i. e., in the order of 1000.



(a) Bar **(b)** 1 Cell **(c)** 3 Cells

Figure 5.9.: *The x-displacement frequency response of* (a) *Bar.* (b) *1 unit.* (c) *three units. We intuitively expect and observe narrowing of the line, in other words, the quality factor of the filter is increased as we we add more cells.*

5.2.1. 3-Unit Cells of Lattice with Absorber

Now, we like to add a terminating absorber load to the three cell finite lattice, as shown in Fig. 5.10. We employ radiation boundary condition at the far right interface of the load. The contour and painted plots of the x-displacement is shown in Fig. 5.11 in linear scale. We notice that the displacement field is continuous across the boundary of the structure and load, which is reassuring that energy is transferred to the absorber. We take a cross section of the structure and observe that waves do get attenuated, see Fig. 5.12.

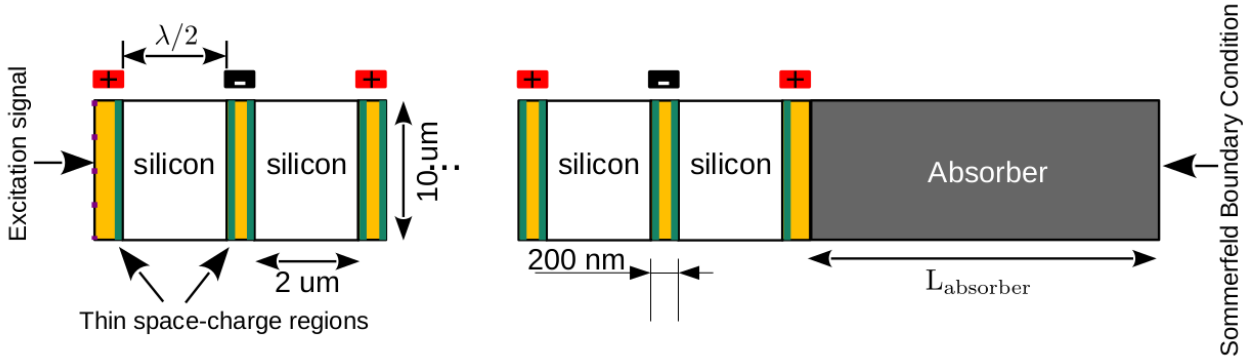


Figure 5.10.: *Schematics of adding a high absorption load to the three unit finite lattice.*

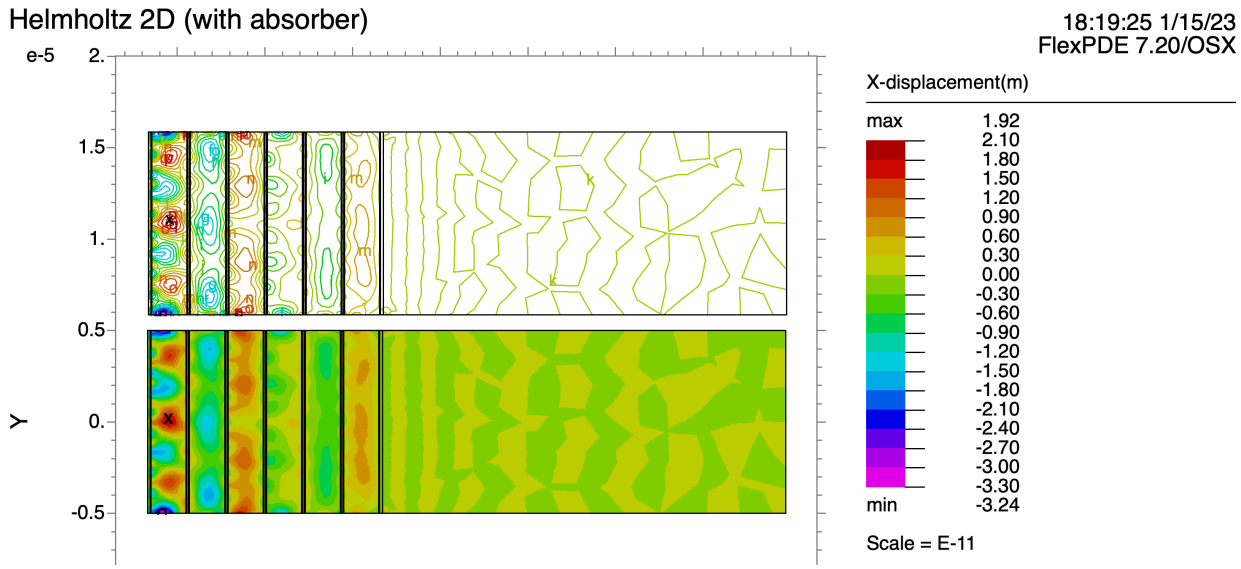


Figure 5.11.: *The acoustic displacement is continuous across the structure and the load, energy gets transferred to load and minimally reflected back to the structure.*

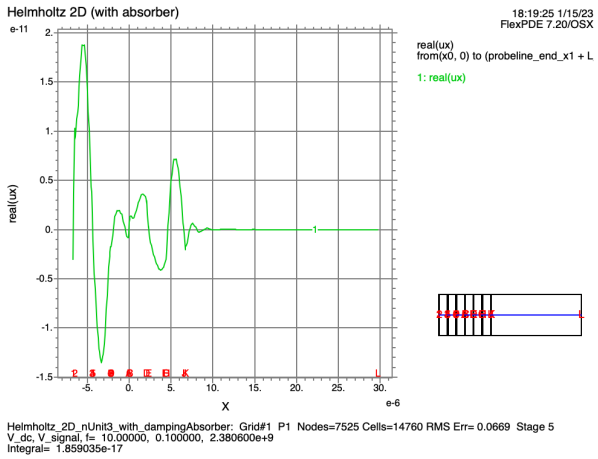


Figure 5.12.: *Cross section at the middle of the three cells and absorber. Waves reaching the end facet do not bounce back to the structure and attenuates in the load.*

In this Section, by setting up numerical test bench, we demonstrated that if we place an absorber at the end of our acoustic structure, a running wave minimally or do not bounce back to the structure. This effectively emulates an infinite medium. This is like 50 Ω resistor in

coaxial cable. The FlexPDE code of all the plots in this Section are attached in Chap. J and Chap. K.

5.2.2. Quarter Wave Layers

The purpose of this study is to observe the effect of quarter wavelength gold and silicon layers on the filter. We intuitively expect the resonance modes to be localized either in the silicon layers or gold layers in analogy of air modes and dielectric modes in photonic crystals.

The structure has silicon and gold layers of 2 µm each, the length of three units is 39 µm, the height is 10 µm. We pad an absorber with high damping, e. g., $\eta_{11} = \eta_{12} = \eta_{44} = 100$ poise (1 P = 0.1 $\mathrm{N\,s\,m^{-2}}$) and length of 23 µm and same height, i. e., 10 µm at the end facet of 3 cells. We keep the DC voltage the same in line with the rest of our experiments, i. e., $V_{\mathrm{dc}} = 10$ V and the signal voltage to be 0.1 V.

In fact, we do observe localized resonance behavior in silicon in phase snapshot diagram of Fig. 5.13. We also notice virtually no signal reflects back to the domain after properly applying the Sommerfeld boundary condition, in Fig. 5.14.

In practice, comparing the acoustic displacement amplitudes in non-quarter wavelength and quarter wavelength, one has to make good judgement on what would be the width of the gold layers, DC bias, whereas the excitation signal is always small but its effect is high due to resonance phenomenon. We have taken a simpler approach here. We could synthesize different kinds of filters by varying the silicon and gold layers. In reality, gold can be easily deposited by sputtering, has good electrical properties but it is lossy, hence the acoustic modes in Fig. 5.13 gets attenuated. We could replace it with steel, but bonding in silicon is a challenge. A better electrical bonding material could be used in-line with CMOS compatible process flows, such as aluminum, copper.

5.2.3. 3-Unit Cells of Lattice with Top & Bottom Absorbers

We explained the rationale of using the top and bottom absorbers in §4.5.4 and hence do not wish to repeat here. In order to properly cater for the non-reflective boundary condition, we used the absorber layers, impedance matched to each material, and 5 μm high and the length of the right side absorber layer is 15 μm. At 10 V DC bias, and 0.1 V excitation signal voltage at 2.3806 GHz, the x-component of the acoustic displacement is shown via contour plot in Fig. 5.15a and via painted plot in Fig. 5.15b. A cross-section at the middle of the structure is shown in Fig. 5.16.

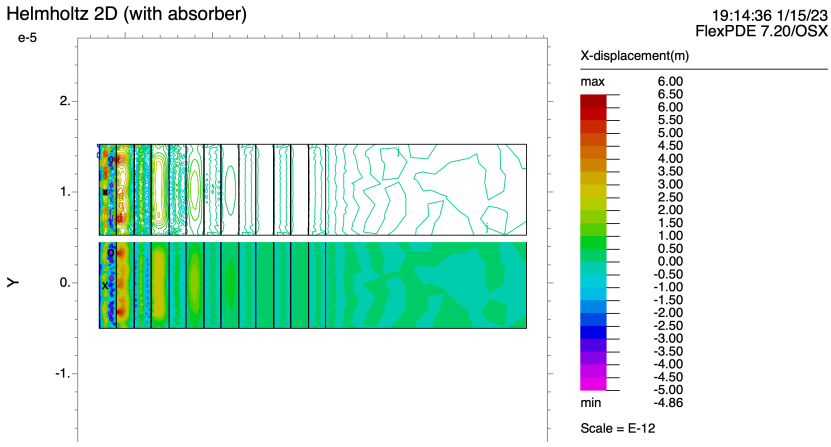


Figure 5.13.: *We used quarter wave gold and silicon layers. The acoustic displacement is continuous across the structure and the load, energy gets transferred to load and minimally reflected back to the structure.*

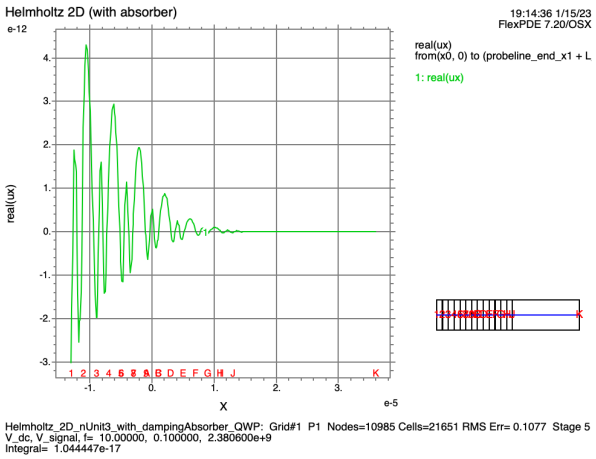
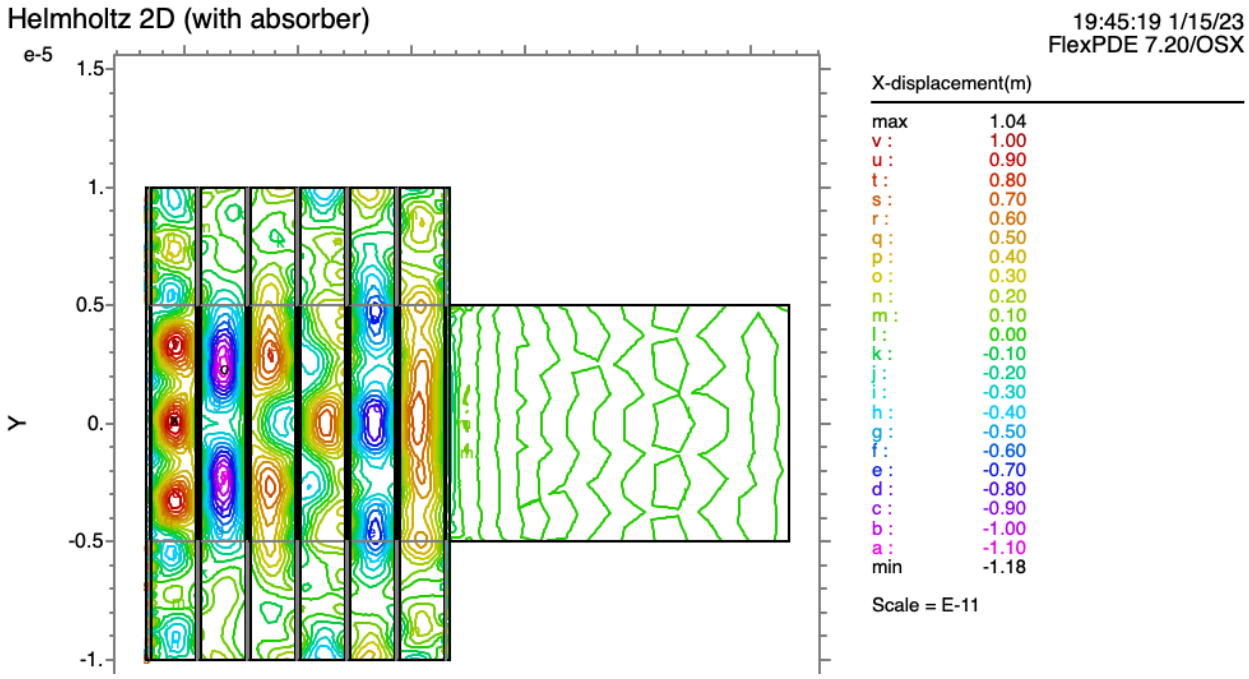
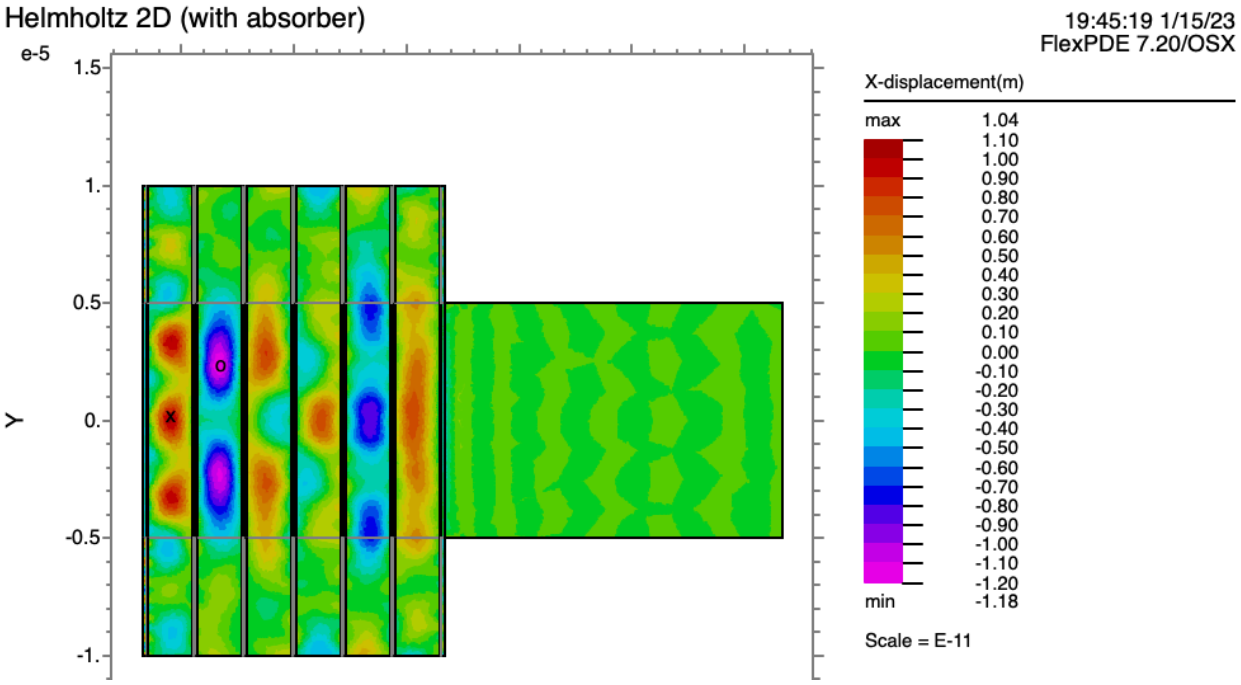


Figure 5.14.: *Cross section at the middle of the three cells and absorber. Waves reaching the end facet do not bounce back to the structure and attenuates in the load.*



(a) Contour plot



(b) Painted plot

Figure 5.15.: *The acoustic x-displacement is continuous across the structure and the load, energy gets transferred to load and minimally reflected back to the structure.*

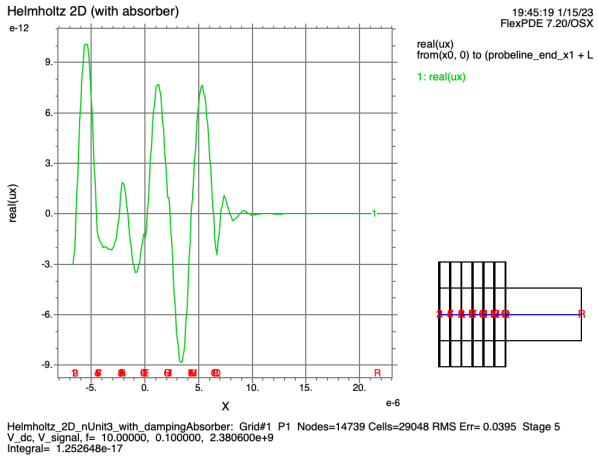


Figure 5.16.: *Cross section at the middle of the three cells and absorber. Waves reaching the end facet do not bounce back to the structure and attenuates in the load.*

5.3. Discussion

This thesis is concerned with the theoretical analysis of device physics. Hence it would not do proper justice to outline exact steps to fabricate and characterize the structure. We are concerned about electrical field induced stress and deformation of material, and how this mechanical deformation propagates. An oscillator will start generating a signal without stimulus. It requires a feedback mechanism of some sort. In my thesis, the structure is electrically excited. Once the (electrical) stimulus is stopped, the energy will die out and the response in the form of mechanical vibration will die. Hence it is viewed as a resonator, rather than oscillator. A resonator is a narrow band filter.

The electrostrictive effect is used to (pre)-stress the silicon and hence mimicked the behavior of piezoelectricity by linearizing the strain response at the neighborhood of the DC bias, cf. Eq. (3.24a), and drawn in Fig. 5.17 after [32]. An electrostriction effect is illustrated using gold/silicon material. Without loss of generality, the theoretical analysis is valid for other material combinations such as gold/glass.

Homogenization is discussed in §3.9.2, §4.5.7. If the two material system parameters, in this case, gold/silicon, are homogenized, the bandgap information is lost. Homogenization is always possible, when we are interested in the average properties of a region of interest. There are various homogenization techniques available. However, since we are interested in detailed bandgap information and frequency response, the homogenization methods are not appropriate for the purpose of analysis presented in the thesis.

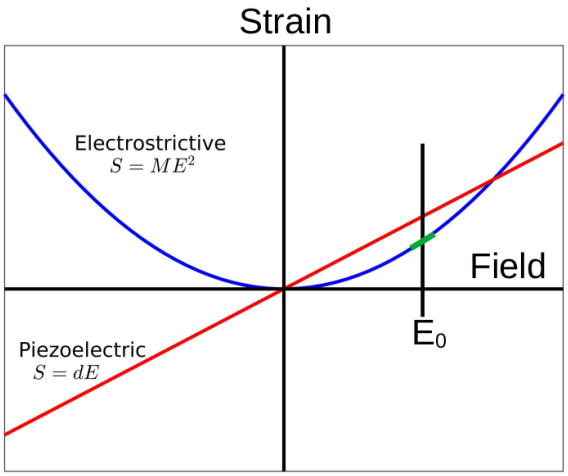


Figure 5.17.: *Shows equivalence in piezo, drawn after [32]. Here E denotes electric field, S strain. Piezoelectric strain is strictly linear in the electric field, whereas the electrostrictive effect is nonlinear in the electric field. However, no physical system is linear across a wide stimulus. By linearizing the system, away from zero, along the parabolic curve, we can show the equivalence of the piezo effect.*

The structures that are simulated in Chap. 4 are in fact 3D. The 2D version of the problem that we solved in Chap. 4 is commonly referred to as plane stress in mechanics, see, Chap. 7 of [71]. If we restrict the motion along the y-direction, but mechanical deformation is allowed along the x-direction, this problem collapses to 1D.

5.4. Summary

In this thesis, we have developed the analytical formulations of the artificial piezoelectric phononic crystal from the first principles. Our formulation is generic in the sense that it can be applied not only to silicon, but many other insulating materials such as plastics. We choose silicon as this is an important material for on-chip integrated applications and the process-steps in fabrication using CMOS planar technology is at a very mature stage, we can replace gold with aluminum or copper for CMOS compatibility. Although gold is not CMOS compatible, it is used in wire-bonding in integrated circuits and routinely in plasmonics in semiconductor fabrication facilities. We have validated our analytical prediction using in-house developed programming scripts for finite element method via numerical experiments using FlexPDE [1].

5.5. Novel Application Arenas

We could explore potential applications such as electrostrictive actuator/vibrator. There is difference between a (voltage controlled) filter and a (voltage controlled) actuator/vibrator. Lightwaves have large radiation loss [58], using air between two acoustic waveguides. This situation can be improved to a great extent. We could explore acousto-optic sensors, active control of diffractive elements, coherent control of light-sound interaction by generating exact acoustic mode etc. One could use flexible plastic substrate based surface acoustic wave transducer. This is a leap from the work done in [72], where the authors used ZnO (which is already piezoelectric) nano crystals. Other areas could be inexpensive disposable biosensors, see a very useful review in [73], mobile haptic sensor, on-chip acousto optics modulator, programmable 1D diffraction grating, inverse operation: detection of voltage due to mechanical wave, like large scale earthquake sensors. As the number of earthquakes across the globe have increased, see the data from the United States Geological Survey (USGS) [74], we potentially deploy thousands of sensors across the globe fabricated using silicon platform. Generally speaking, piezoelectric materials are much more expensive and problematic to manufacture than simple dielectric materials like silicon.

5.6. Outlook

The work could be further extended to

- 3D structures (for other geometric configuration).
- Perform proper simulations of hexagonal lattice and explore other possible topology. I have attached the FlexPDE scripts with this thesis to study and build up.
- Inverse operation.
- Implement various types of filtering effects.
- Fabricate and perform measurements to characterize performance of fabricated piezoelectric phononic crystal samples.

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Appendix A

A Material Properties

The following material properties for silicon and gold are used throughout the thesis, unless otherwise stated.

In the literature, the viscosity is usually quoted in centipose (cP), where 1 cP is equal to $1 \cdot 10^{-3}$ N s m^{-2}. We shall use the viscosity or damping of silicon and gold in solid phase. The acoustic attenuation in single crystal silicon and gold is [1]

$$\alpha = 4.343 \left(\frac{\rho}{c_{mn}} \right)^{1/2} \left(\frac{\eta_{mn}}{c_{mn}} \right) \omega^2, \tag{A.1}$$

where α is the attenuation factor in dB m^{-1}, c_{mn} stiffness, and η_{mn} damping. For silicon, $\alpha = 650$ dB m^{-1} for longitudinal polarization (along the body diagonal) at $f = 1$ GHz, p.95 in [3]. For solid gold $\alpha = 20,000$ dB m^{-1} for longitudinal polarization (along the face diagonal) at $f = 1$ GHz, p.95 in [3]. Solid gold is malleable, so it is expected to be more squishy than silicon crystal. However, it turns out to be the contrary. As a general trend, the acoustic attenuation factor increases in the order INSULATOR $\longrightarrow$ SEMICONDUCTOR $\longrightarrow$ METAL. The damping of silicon and gold are calculated based on this formula in this thesis and considered as constant rather than frequency dependent to simplify the computation, in addition to the fact that the quoted the attenuation factor in [3] is at $f = 1$ GHz.

Silicon

The Young's modulus, Poisson ratio etc. along the various silicon crystal orientations (such as <100>, <110>, <111>) vary, see [56]. Albeit very minute, this affects the computed artificial

[1] Although the formula provided in p. 95 of [3] is for shear wave and is for the 44 component of the stiffness and attenuation matrices.

piezoelectric constants via the formulae as derived in Chap. 3 and Chap. 4.

- Volumetric mass density $\rho = 2329\ \mathrm{kg\,m^{-3}}$ [56].
- Young's modulus = 169 GPa [56].
- Isotropic bulk elastic stiffness constant $c_{11} = 169$ GPa (for 1D calculation) [56].
- Relative permittivity $\epsilon_r = 11.7$ in the $1 \sim 15$ GHz range [75].
- Poisson's ratio = 0.27 (along [100] direction) [56].
- Viscosity $\eta_{11} = 0.005\ \mathrm{N\,s\,m^{-2}}$.
- Viscosity $\eta_{12} = 0.005\ \mathrm{N\,s\,m^{-2}}$.
- Viscosity $\eta_{44} = 0.005\ \mathrm{N\,s\,m^{-2}}$.

For silicon, $c_{22} = c_{11}$, $c_{66} = c_{44}$, $c_{16} = c_{26} = 0$, the stiffness matrix after p. 364 [3] is

$$\begin{aligned}
\mathbf{c}^E &= \begin{bmatrix} c_{11} & c_{12} & c_{16} \\ c_{12} & c_{22} & c_{26} \\ c_{16} & c_{26} & c_{66} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{11} & 0 \\ 0 & 0 & c_{44} \end{bmatrix} \\
&= \begin{bmatrix} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & 0 \\ \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \\
&= \begin{bmatrix} 2.1118 \cdot 10^{11} & 7.8107 \cdot 10^{10} & 0 \\ 7.8107 \cdot 10^{10} & 2.1118 \cdot 10^{11} & 0 \\ 0 & 0 & 6.6535 \cdot 10^{10} \end{bmatrix} (\mathrm{N\,m^{-2}}).
\end{aligned} \tag{A.2}$$

We note that the damping matrix also follows the same structrue as the stiffness matrix [3],

$$\eta = \begin{bmatrix} 0.005 & 0.005 & 0 \\ 0.005 & 0.005 & 0 \\ 0 & 0 & 0.005 \end{bmatrix} (\mathrm{N\,s\,m^{-2}}). \tag{A.3}$$

The compliance matrix entries respect the following relations with the stiffness matrix elements

[3]

$$c_{11} = \frac{s_{11} + s_{12}}{(s_{11} - s_{12})(s_{11} + 2s_{12})}, \tag{A.4}$$

$$c_{12} = \frac{-s_{12}}{(s_{11} - s_{12})(s_{11} + 2s_{12})}, \tag{A.5}$$

$$c_{44} = 1/s_{44}. \tag{A.6}$$

For silicon, the permittivity matrix at the $1 \sim 15$ GHz range is [75], p. 382 in [3],

$$\epsilon^S = \begin{bmatrix} 11.7 & 0 \\ 0 & 11.7 \end{bmatrix}. \tag{A.7}$$

Gold

- Volumetric mass density $\rho = 19300\ \mathrm{kg\,m^{-3}}$ [76]–[77].
- Young's modulus = 79 GPa [76]–[77].
- Isotropic bulk elastic stiffness constant $c_{11} = 79$ GPa (for 1D calculation) [76]–[77].
- Relative permittivity $\epsilon_r = 1$ at low and RF frequencies, [78].
- Poisson's ratio = 0.42 [76]–[77].
- Viscosity $\eta_{11} = 0.0186\ \mathrm{N\,s\,m^{-2}}$.
- Viscosity $\eta_{12} = 0.0186\ \mathrm{N\,s\,m^{-2}}$.
- Viscosity $\eta_{44} = 0.0186\ \mathrm{N\,s\,m^{-2}}$.

$$\mathbf{c}^E = \begin{bmatrix} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & 0 \\ \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} = \begin{bmatrix} 2.0167 \cdot 10^{11} & 1.4604e \cdot 10^{10} & 0 \\ 1.4604e \cdot 10^{10} & 2.0167 \cdot 10^{11} & 0 \\ 0 & 0 & 2.7817 \cdot 10^{10} \end{bmatrix} (\mathrm{N\,m^{-2}}).$$

$$\eta = \begin{bmatrix} 0.0186 & 0.0186 & 0 \\ 0.0186 & 0.0186 & 0 \\ 0 & 0 & 0.0186 \end{bmatrix} (\mathrm{N\,s\,m^{-2}}). \tag{A.8}$$

For gold, the permittivity matrix at the low and RF frequency range is,

$$\epsilon^S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{A.9}$$

Artificial Absorber Materials

The material parameters for the artificial absorber materials, gold and silicon, where applicable, are the same of the gold and silicon respectively, except the damping is increased to 10 $\mathrm{N\,s\,m^{-2}}$.

B Appendix B

Matlab Code for 1D Phononic Bandstructure using Transfer Matrix Method

```
% ---------------Bandgap Calculation (Transfer Matrix Method)---------------
% Late Modified: January 4, 2023

clc
clear
% close all

[permittivity_silicon, massDensity_silicon, C11_silicon] =
    ↪ MaterialParametersSilicon();
[permittivity_gold, massDensity_gold, C11_gold] = MaterialParametersGold();

f_array = linspace(1e8, 5.4e9, 5e3)';

Bloch_vector = zeros(length(f_array), 1);

electrode_width = 200e-9;
silicon_width = 2e-6;

velocity_gold = sqrt(C11_gold/massDensity_gold);
Z_gold = massDensity_gold*velocity_gold;
```

```
velocity_silicon = sqrt(C11_silicon/massDensity_silicon);
Z_silicon = massDensity_silicon*velocity_silicon;

lattice_period = 2*silicon_width + 2*electrode_width;
massDensity_average = (2*silicon_width/lattice_period) * massDensity_silicon +
    ↪  (2*electrode_width/lattice_period) * massDensity_gold;

branch_num = 1;

for ii=1:length(f_array)

    f = f_array(ii);
    w = 2*pi*f;

    %% 1.) Travel half of positive electrode
    L_travel = electrode_width/2;
    phi_gold = w/velocity_gold*L_travel;

    T1 = [exp(-1i*phi_gold), 0; 0, exp(1i*phi_gold)];

    %% 2.) Interface from positive electrode to silicon
    rAB = (Z_silicon - Z_gold)/(Z_silicon + Z_gold);
    rBA = -rAB;
    tAB = 2*Z_silicon/(Z_silicon + Z_gold);
    tBA = 2*Z_gold/(Z_silicon + Z_gold);

    T2 = 1/tBA*[tAB*tBA-rAB*rBA, rBA ; -rAB, 1];

    %% 3.) Travel in Silicon
    L_travel = silicon_width;
    phi_silicon = w/velocity_silicon*L_travel;

    T3 = [exp(-1i*phi_silicon), 0; 0, exp(1i*phi_silicon)];

    %% 4.) Interface from Si to negative electrode
    rAB = (Z_gold - Z_silicon)/(Z_gold + Z_silicon);
    rBA = -rAB;
    tAB = 2*Z_gold/(Z_silicon + Z_gold);
    tBA = 2*Z_silicon/(Z_silicon + Z_gold);
```

```
T4 = 1/tBA*[tAB*tBA-rAB*rBA, rBA ; -rAB, 1];

%% 5.) Travel in negative electrode
L_travel = electrode_width;
phi_gold = w/velocity_gold*L_travel;

T5 = [exp(-1i*phi_gold), 0; 0, exp(1i*phi_gold)];

%% 6.) Interface from negative electrode to silicon
rAB = (Z_silicon - Z_gold)/(Z_silicon + Z_gold);
rBA = -rAB;
tAB = 2*Z_silicon/(Z_silicon + Z_gold);
tBA = 2*Z_gold/(Z_silicon + Z_gold);

T6 = 1/tBA*[tAB*tBA-rAB*rBA, rBA ; -rAB, 1];

%% 7.) Travel in Silicon
L_travel = silicon_width;
phi_silicon = w/velocity_silicon*L_travel;

T7 = [exp(-1i*phi_silicon), 0; 0, exp(1i*phi_silicon)];

%% 8.) Interface from Si to positive electrode
rAB = (Z_gold - Z_silicon)/(Z_gold + Z_silicon);
rBA = -rAB;
tAB = 2*Z_gold/(Z_silicon + Z_gold);
tBA = 2*Z_silicon/(Z_silicon + Z_gold);

T8 = 1/tBA*[tAB*tBA-rAB*rBA, rBA ; -rAB, 1];

%% 9.) Travel half of positive electrode
L_travel = electrode_width/2;
phi_gold = w/velocity_gold*L_travel;

T9 = [exp(-1i*phi_gold), 0; 0, exp(1i*phi_gold)];

%% Total Matrix
T = T9*T8*T7*T6*T5*T4*T3*T2*T1;
```

```
    %% Compute Bloch vector

    Bloch_vector(ii,1) = 1/lattice_period * acos(real(T(1,1)));

    if (abs(imag(Bloch_vector(ii,1))) >= 1/1000)
        % Bloch_vector(ii,1) = 0;
        % f_array(ii) = 0;
        BlochMatrix(ii,1) = 0;
        BlochMatrix(ii,2) = 0;
        BlochMatrix(ii,3) = false;
    else
        % Bloch_vector(ii,1) = Bloch_vector(ii,1);
        % f_array(ii) = f;
        BlochMatrix(ii,1) = f;
        BlochMatrix(ii,2) = Bloch_vector(ii,1);
        BlochMatrix(ii,3) = true;
    end

end

for ii = 1:length(f_array) - 1
    if ( BlochMatrix(ii,3) == false ) && ( BlochMatrix(ii+1,3) == true )
        BlochMatrix(ii, 4) = 0;
    elseif ( BlochMatrix(ii,3) == true ) && ( BlochMatrix(ii+1,3) == true )
        BlochMatrix(ii, 4) = branch_num;
    elseif ( BlochMatrix(ii,3) == true ) && ( BlochMatrix(ii+1,3) == false )
        BlochMatrix(ii, 4) = branch_num;
        branch_num = branch_num + 1;
    end

end

%%

for jj = 1:(branch_num-1)
    for ii = 1:length(f_array)
        if BlochMatrix(ii,4) == jj
            Bandnumber(ii,1,jj) = BlochMatrix(ii,1);
```

```
            Bandnumber(ii,2,jj) = BlochMatrix(ii,2);
        end

    end

end

%%
figure(1)
clf

for ii = 1:branch_num-1
    x = nonzeros(Bandnumber(:,2,ii));
    y = nonzeros(Bandnumber(:,1,ii))*1e-9;
    plot(x*lattice_period, y,'-b', 'linewidth', 2)
    hold on
end

hold off
xlabel('$\beta␣\Lambda$','Interpreter','latex')
ylabel('f␣(GHz)' ,'Interpreter','latex')
xlim([min(x*lattice_period) max(x*lattice_period)])
print -djpeg ComputedImages/Bandgap_1D.jpg -r600

%% Define constants and material parameters of silicon
function [permittivity_silicon, massDensity_silicon, C11_silicon] =
    ↪ MaterialParametersSilicon()

permittivity_0 = 8.8542e-12; % free space permittivity. F/m
% Breakdown electric field of silicon is 35 GV/m
permittivityRelative_silicon = 11.7; % relative permittivity of silicon.

permittivity_silicon = permittivityRelative_silicon * permittivity_0;

massDensity_silicon = 2329; % kg/m^3
C11_silicon = 169e9; % N/m2

end
```

```
%% Define constants and material parameters of gold
function [permittivity_gold, massDensity_gold, C11_gold] =
    ↪ MaterialParametersGold()

permittivity_0 = 8.8542e-12; % free space permittivity. F/m
% Breakdown electric field of silicon is 35 GV/m
permittivityRelative_gold = 1; % relative permittivity of silicon.

permittivity_gold = permittivityRelative_gold * permittivity_0;

massDensity_gold = 19300; % kg/m^3
C11_gold = 79e9; % N/m2

end
```

Appendix C

C Matlab Code for 1D Bandstructure using Plane Wave Expansion Method

```
clc
clear
% close all
% clf

%%
[permittivity_silicon, massDensity_silicon, C11_silicon] =
    ↪ MaterialParametersSilicon();
[permittivity_gold, massDensity_gold, C11_gold] = MaterialParametersGold();

% V0 = 2e4;
V0 = 0;
nMax = 20;
%%

electrode_width = 200e-9;
silicon_width = 2e-6;

d = electrode_width;
L = 2*silicon_width + 2*electrode_width;

massDensity_average = (2*silicon_width/L) * massDensity_silicon + (2*
    ↪ electrode_width/L) * massDensity_gold;
```

```
e_coeff_silicon = -permittivity_silicon*(-V0/silicon_width);
C11_silicon = C11_silicon + (e_coeff_silicon^2)/(permittivity_silicon);

e_coeff_gold = 0;
%%

kVec = [0: pi/(20*L): pi/L]';
kNum = length(kVec);

totalBandNumbers = (2*nMax + 1);

nVec = [-nMax:1:nMax];
mVec = [-nMax:1:nMax];

f = [];

for kCounter = 1:kNum

    k = kVec(kCounter);
    Q = zeros(totalBandNumbers);

    for nCounter = 1:totalBandNumbers
        for mCounter = 1:totalBandNumbers
            n = nVec(nCounter);
            m = mVec(mCounter);

            if ((m-n) == 0)
                c_eff = 2/L*(C11_gold*d + C11_silicon*(L/2-d));
            elseif ( mod((m-n),2) == 1 ) % odd
                c_eff = 0;
            else
                c_eff = cos((m-n)*pi/2)/((m-n)*pi/2) * sin(2*pi*(m-n)/L/2*d - (m
                    ↪ -n)*pi/2) * (C11_gold - C11_silicon);
            end

            Q(nCounter, mCounter) = (2*pi*n/L + k)* (2*pi*m/L + k) * c_eff;

        end
```

```
    end

    [V, D] = eig(Q);
    numerical_eigenvalues = diag(D, 0);
    [numerical_eigenvalues, ~] = sort(numerical_eigenvalues, 'ascend');

    V = fliplr(V);
    VAggregate(:,:, kCounter) = V(:,:);

    numerical_eigenvalues = 1/(2*pi)*sqrt(numerical_eigenvalues/
        ↪ massDensity_average);

    for jj = 1:totalBandNumbers
        if imag(numerical_eigenvalues(jj)) >= 1e-2
            numerical_eigenvalues(jj) = 0;
        end
    end

    f = [f; numerical_eigenvalues.'];
end

bandNumbersToPlot = 6;

% figure(1)
% for idx = 1:bandNumbersToPlot
% plot(kVec*L, f(:, idx)*1e-9, '-b','LineWidth', 2)
% % plot(kVec*L, f(:, idx)*1e-9, '-r','LineWidth', 2)
% hold on
% end
% hold off
% xlabel('$k L$','Interpreter','latex')
% ylabel('f (GHz)','Interpreter','latex')
% xlim([kVec(1)*L kVec(end)*L])
%
% print -djpeg ComputedImages/Bandgap_PWE.jpeg -r600

%%
figure(2)
for idx = 1:bandNumbersToPlot
```

```
    plot(kVec*L, f(:, idx)*1e-9, '-ob','LineWidth', 2)
% plot(kVec*L, f(:, idx)*1e-9, '-.*r','LineWidth', 2)
    hold on
end
hold off
xlabel('$k␣L$','Interpreter','latex')
ylabel('f␣(GHz)','Interpreter','latex')
xlim([kVec(1)*L kVec(end)*L])

figure(2)
hold on

print -djpeg ComputedImages/Bandgap_PWE_tuning.jpeg -r600


%% Define constants and material parameters
function [permittivity_silicon, massDensity_silicon, C11_silicon] =
    ↪ MaterialParametersSilicon()

permittivity_0 = 8.8542e-12; % free space permittivity. F/m
% Breakdown electric field of silicon is 35 GV/m
permittivityRelative_silicon = 11.7; % relative permittivity of silicon.

permittivity_silicon = permittivityRelative_silicon * permittivity_0;

massDensity_silicon = 2329; % kg/m^3
C11_silicon = 169e9; % N/m2

end

%% Define constants and material parameters
function [permittivity_gold, massDensity_gold, C11_gold] =
    ↪ MaterialParametersGold()

permittivity_0 = 8.8542e-12; % free space permittivity. F/m
% Breakdown electric field of silicon is 35 GV/m
permittivityRelative_gold = 1; % relative permittivity of silicon.

permittivity_gold = permittivityRelative_gold * permittivity_0;
```

```
massDensity_gold = 19300; % kg/m^3
C11_gold = 79e9; % N/m2

end
```

D Appendix D

FlexPDE Code Listing for Response of n-Units of Piezorope

```
{ Many units 1D
1.) Stage frequency
2.) DC voltage ON
3.) AC signal ON
4.) Periodic condition OFF
5.) Specify the number of units via nUnit
6.) We expect the phase snapshot in mechanical displacement
    due to AC perturbation coupled with DC voltage at a given frequency.
7.) We compare with the homogenized analytical solution.
}

TITLE 'Helmholtz 1D' { Helmholtz_1D.pde }
SELECT errlim = 1e-4 order = 2 threads = 1 stages = 1 regrid = off featureplot
   ↪  = on !vandenberg=on !ngrid = 20
COORDINATES cartesian1
VARIABLES ux = complex(ux_real, ux_imag)
DEFINITIONS
       nUnit = 3

   zoom_factor = 7e5 !df = 1e6 f = 2.05e9 + df * (stage-1)
   f = 2.13e9
   omega = 2*pi*f { rad-Hz }
   !In matlab, stage = 9; fc = 3.86e8; df = 1e8; fl = fc - df *(stage-1)/2
```

```
mass_density

permittivity_relative permittivity_0 = 8.85e-12 { free space permittivity F
    ↪ /m } permittivity = permittivity_relative * permittivity_0
force_density_DC_x

{ ===== Stiffness ===== }
stiffness_11_real viscosity_11 stiffness_11_imag = omega*viscosity_11
stiffness_11 = complex(stiffness_11_real, stiffness_11_imag)

{ ===== Silicon ===== }
    mass_density_silicon = 2329 { kg/m3 } stiffness_11_real_silicon = 169e9
        ↪  { N/m2 }
permittivity_relative_silicon = 11.7
{ http://www.ioffe.ru/SVA/NSM/Semicond/Si/mechanic.html#Elastic }

viscosity_11_silicon = 0.005 { N.s/m^2 }
{ B.A. Auld, vol 1, chapter 3, a representative value }

{ ===== Gold ===== }
mass_density_gold = 19300 { kg/m3 } stiffness_11_real_gold = 79e9 { N/m2 }
permittivity_relative_gold = 1 { At low and RF frequencies }
{ https://www.azom.com/properties.aspx?ArticleID=598 }
{ http://chemistry.elmhurst.edu/vchembook/125Adensitygold.html }

viscosity_11_gold = 0.0186 { N.s/m^2 }

{ ===== Structure ===== }
!The charge width is tucked inside the electrode width.

silicon_width = 2e-6 {m} electrode_width = 200e-9 {m} charge_width = 50e-9
    ↪ {m}

L_repeat = 2*electrode_width + 2*silicon_width { m }
L = 3*electrode_width + 2*silicon_width { m }

L_shifter_last_unit = L + (nUnit-2)*L_repeat
L_total = L + (nUnit-2)*L_repeat
```

```
![electrode][Space-charge region][silicon][Space-charge region][electrode][
    ↪ Space-charge region][silicon][Space-charge region][electrode]
!x_centering = (L + (nUnit-2)*L_repeat + L_shifter_last_unit)/2

! Starting unit
x0 = 0
x1 = electrode_width - charge_width
x2 = electrode_width
x3 = electrode_width + silicon_width
x4 = electrode_width + silicon_width + charge_width
x5 = electrode_width + silicon_width + electrode_width - charge_width
x6 = 2*electrode_width + silicon_width
x7 = 2*electrode_width + 2*silicon_width
x8 = 2*electrode_width + 2*silicon_width + charge_width
x9_recessed = 3*electrode_width + 2*silicon_width - charge_width
x9 = 3*electrode_width + 2*silicon_width

! Repeated units
x2r = 0
x3r = silicon_width
x4r = silicon_width + charge_width
x5r = silicon_width + electrode_width - charge_width
x6r = silicon_width + electrode_width
x7r = 2*silicon_width + electrode_width
x8r = 2*silicon_width + electrode_width + charge_width
x9r_recessed = 2*electrode_width + 2*silicon_width - charge_width
x9r = 2*electrode_width + 2*silicon_width

! Last unit
x2l = 0
x3l = silicon_width
x4l = silicon_width + charge_width
x5l = silicon_width + electrode_width - charge_width
x6l = electrode_width + silicon_width
x7l = electrode_width + 2*silicon_width
x8l = electrode_width + 2*silicon_width + charge_width
x9l = 2*electrode_width + 2*silicon_width
```

```
probeline_end_x = x9 + (nUnit-2)*L_repeat

{ ===== Voltage ===== }
V_dc = 10 { volts }
V_signal = 0.1 { volts }

{ ===== Artificial piezo-material ===== }
width_artificial_piezo = silicon_width
permittivity_artificial_piezo = permittivity_relative_silicon *
   ↪ permittivity_0
mass_density_artificial_piezo = mass_density_silicon
c11_artificial_piezo = stiffness_11_real_silicon

{ ===== Force density due to DC bias ===== }
force_density_DC_x_calculated = permittivity_artificial_piezo * V_dc^2/(2*
   ↪ width_artificial_piezo^2) * ( 1 + permittivity_artificial_piezo *
   ↪ V_dc^2/( c11_artificial_piezo * width_artificial_piezo^2) ) !{ N/m^3
   ↪ }

!force_density_DC_x_calculated = 1e4 { N/m3 }
!force_density_DC_x_calculated = 1e4 + 3e3*(stage-1) { N/m3 }

{ ===== Change in length due to AC voltage perturbation ===== }
DeltaL = -0.5 * permittivity_artificial_piezo * V_dc^2/(
   ↪ c11_artificial_piezo*width_artificial_piezo)
Leq = width_artificial_piezo + DeltaL
k = omega * sqrt(mass_density_artificial_piezo/c11_artificial_piezo)
alpha_here = tan(k*Leq/2)
beta_here = alpha_here*(cos(k*Leq) - 1) - sin(k*Leq)
force_DC_per_unit_area = -0.5 * permittivity_artificial_piezo * V_dc^2 /
   ↪ (2*width_artificial_piezo^2) * (1 - 2*DeltaL/width_artificial_piezo)
ur_signal = 2*beta_here/(c11_artificial_piezo*k) * force_DC_per_unit_area /
   ↪  ( 1 + 2*beta_here/(c11_artificial_piezo*k) * (force_DC_per_unit_area
   ↪ /Leq) ) * V_signal/V_dc

    !ur_signal = 0
!ur_signal = 1e-14 { m }

{ ===== Energy density ===== }
```

```
![energy_density] = N/m^2
!energy_density = 0.5*( stiffness_11*dx(ux) * dx(ux)

{ ===== Homogenization ===== }
mass_density_homo = (2*nUnit + 1)*electrode_width/L_total *
    ↪ mass_density_gold + 2*nUnit*silicon_width/L_total *
    ↪ mass_density_silicon

stiffness_real_homo = (2*nUnit + 1)*electrode_width/L_total *
    ↪ stiffness_11_real_gold + 2*nUnit*silicon_width/L_total *
    ↪ stiffness_11_real_silicon
stiffness_imag_homo = (2*nUnit + 1)*electrode_width/L_total * (omega*
    ↪ viscosity_11_gold) + 2*nUnit*silicon_width/L_total * (omega*
    ↪ viscosity_11_silicon)
stiffness_homo = complex(stiffness_real_homo, stiffness_imag_homo)

e_gold = 0
!Caution: The sign of e changes depending on voltage polarity across the
    ↪ silicon slab.
!Average across one unit is obviously zero, but not across one sub-unit.
e_silicon = permittivity_relative_silicon * permittivity_0 * (V_dc/
    ↪ silicon_width)
e_homo = (2*nUnit + 1)*electrode_width/L_total * e_gold + 2*nUnit*
    ↪ silicon_width/L_total * e_silicon

!Permeability
permeability_0 = 4*pi*1e-7 ! henry/m

!Permittivity
permittivity_relative_homo = (2*nUnit + 1)*electrode_width/L_total *
    ↪ permittivity_relative_gold + 2*nUnit*silicon_width/L_total *
    ↪ permittivity_relative_silicon
permittivity_homo = permittivity_relative_homo*permittivity_0

k_EM = omega*sqrt(permeability_0*permittivity_homo)
k_acoustic = omega*sqrt(mass_density_homo/stiffness_homo)

u_analytical_homogenized = ur_signal*(cos(k_acoustic*x) - complex(0,1)*sin(
    ↪ k_acoustic*x)) + complex(0,1) * e_homo/stiffness_homo*sqrt(
```

```
      ↪ permeability_0*permittivity_homo) *(V_signal/silicon_width) / (omega
      ↪ *(mass_density_homo/stiffness_homo - permeability_0*permittivity_homo
      ↪ )) * (cos(k_EM*x) - complex(0,1)*sin(k_EM*x))

   !u_analytical_homogenized = ur_signal*exp(-complex(0,1)*k_acoustic*x) +
      ↪ complex(0,1) * e_homo/stiffness_homo*sqrt(permeability_0*
      ↪ permittivity_homo) *(V_signal/silicon_width) / (omega*(
      ↪ mass_density_homo/stiffness_homo - permeability_0*permittivity_homo))
      ↪  * exp(-complex(0,1)*k_EM*x)
   !u_analytical = ur_signal*( cos(k_acoustic*x) - complex(0,1)*sin(k_acoustic
      ↪ *x) )
   !u_analytical = ur_signal*cexp(-complex(0,1)*k_acoustic*x)


EQUATIONS
       ux: dx( stiffness_11*dx(ux) ) + mass_density*omega^2*ux +
          ↪ force_density_DC_x = 0

BOUNDARIES
! =====Initial Unit=====
       region '(1) positive electrode left'
       mass_density = mass_density_gold  permittivity_relative =
          ↪ permittivity_relative_gold
       stiffness_11_real = stiffness_11_real_gold
       viscosity_11 = viscosity_11_gold { N.s/m^2 }
           force_density_DC_x = 0

       start (x0)   point value(ux) = complex(ur_signal,0) ! left
       line to (x1)  point natural(ux) = complex(0,0) ! right

   region '(2) THIN SPACE CHARGE - LEFT ELECTRODE'
       mass_density = mass_density_gold  permittivity_relative =
          ↪ permittivity_relative_gold
       stiffness_11_real = stiffness_11_real_gold
       viscosity_11 = viscosity_11_gold { N.s/m^2 }
              force_density_DC_x = force_density_DC_x_calculated

       start (x1)   point natural(ux) = complex(0,0) ! left
       line to (x2)  point natural(ux) = complex(0,0) ! right
```

```
    region '(3) silicon left'
    mass_density = mass_density_silicon permittivity_relative =
       ↪ permittivity_relative_silicon
    stiffness_11_real = stiffness_11_real_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 }
           force_density_DC_x = 0

    start (x2)   point natural(ux) = complex(0,0) ! left
    line to (x3)  point natural(ux) = complex(0,0) ! right

region '(4) THIN SPACE CHARGE - CENTER ELECTRODE LEFT'
    mass_density = mass_density_gold  permittivity_relative =
       ↪ permittivity_relative_gold
    stiffness_11_real = stiffness_11_real_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 }
           force_density_DC_x = -force_density_DC_x_calculated

    start (x3)   point natural(ux) = complex(0,0) ! left
    line to (x4)  point natural(ux) = complex(0,0) ! right

    region '(5) negative electrode'
    mass_density = mass_density_gold  permittivity_relative =
       ↪ permittivity_relative_gold
    stiffness_11_real = stiffness_11_real_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 }
           force_density_DC_x = 0

    start (x4)   point natural(ux) = complex(0,0) ! left
    line to (x5)  point natural(ux) = complex(0,0) ! right

region '(6) THIN SPACE CHARGE - CENTER ELECTRODE RIGHT'
    mass_density = mass_density_gold  permittivity_relative =
       ↪ permittivity_relative_gold
    stiffness_11_real = stiffness_11_real_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 }
           force_density_DC_x = force_density_DC_x_calculated

    start (x5)   point natural(ux) = complex(0,0) ! left
```

```
    line to (x6)  point natural(ux) = complex(0,0) ! right

    region '(7) silicon right'
    mass_density = mass_density_silicon permittivity_relative =
       ↪ permittivity_relative_silicon
    stiffness_11_real = stiffness_11_real_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 }
           force_density_DC_x = 0

    start (x6)   point natural(ux) = complex(0,0) ! left
    line to (x7)  point natural(ux) = complex(0,0) ! right

region '(8) THIN SPACE CHARGE - LEFT ELECTRODE'
    mass_density = mass_density_gold  permittivity_relative =
       ↪ permittivity_relative_gold
    stiffness_11_real = stiffness_11_real_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 }
           force_density_DC_x = -force_density_DC_x_calculated

    start (x7)   point natural(ux) = complex(0,0) ! left
    line to (x8)  point natural(ux) = complex(0,0) ! right

region '(9) positive electrode right'
    mass_density = mass_density_gold  permittivity_relative =
       ↪ permittivity_relative_gold
    stiffness_11_real = stiffness_11_real_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 }
           force_density_DC_x = 0

    start (x8)       point natural(ux) = complex(0,0) ! left
    line to (x9_recessed)  point natural(ux) = complex(0,0) ! right

region '(10) THIN SPACE CHARGE - RIGHT ELECTRODE'
    mass_density = mass_density_gold  permittivity_relative =
       ↪ permittivity_relative_gold
    stiffness_11_real = stiffness_11_real_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 }
           force_density_DC_x = force_density_DC_x_calculated
```

```
      start (x9_recessed)  point natural(ux) = complex(0,0) ! left
      line to (x9)     point natural(ux) = complex(0,0) ! right

! =====Repeating Units=====
   repeat i = 1 to (nUnit-2)

      region '(3r) silicon left'
      mass_density = mass_density_silicon permittivity_relative =
         ↪ permittivity_relative_silicon
      stiffness_11_real = stiffness_11_real_silicon
      viscosity_11 = viscosity_11_silicon { N.s/m^2 }
             force_density_DC_x = 0

      start (x2r + L + (i-1)*L_repeat)   point natural(ux) = complex(0,0) !
         ↪ left
      line to (x3r + L + (i-1)*L_repeat)  point natural(ux) = complex(0,0) !
         ↪ right

   region '(4r) THIN SPACE CHARGE - CENTER ELECTRODE LEFT'
      mass_density = mass_density_gold  permittivity_relative =
         ↪ permittivity_relative_gold
      stiffness_11_real = stiffness_11_real_gold
      viscosity_11 = viscosity_11_gold { N.s/m^2 }
             force_density_DC_x = -force_density_DC_x_calculated

      start (x3r + L + (i-1)*L_repeat)   point natural(ux) = complex(0,0) !
         ↪ left
      line to (x4r + L + (i-1)*L_repeat)  point natural(ux) = complex(0,0) !
         ↪ right

      region '(5r) negative electrode'
      mass_density = mass_density_gold  permittivity_relative =
         ↪ permittivity_relative_gold
      stiffness_11_real = stiffness_11_real_gold
      viscosity_11 = viscosity_11_gold { N.s/m^2 }
             force_density_DC_x = 0

      start (x4r + L + (i-1)*L_repeat)   point natural(ux) = complex(0,0) !
         ↪ left
```

```
    line to (x5r + L + (i-1)*L_repeat)  point natural(ux) = complex(0,0) !
       ↪ right

region '(6r) THIN SPACE CHARGE - CENTER ELECTRODE RIGHT'
    mass_density = mass_density_gold  permittivity_relative =
       ↪ permittivity_relative_gold
    stiffness_11_real = stiffness_11_real_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 }
           force_density_DC_x = force_density_DC_x_calculated

    start (x5r + L + (i-1)*L_repeat)   point natural(ux) = complex(0,0) !
       ↪ left
    line to (x6r + L + (i-1)*L_repeat)  point natural(ux) = complex(0,0) !
       ↪ right

    region '(7r) silicon right'
    mass_density = mass_density_silicon permittivity_relative =
       ↪ permittivity_relative_silicon
    stiffness_11_real = stiffness_11_real_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 }
           force_density_DC_x = 0

    start (x6r + L + (i-1)*L_repeat)   point natural(ux) = complex(0,0) !
       ↪ left
    line to (x7r + L + (i-1)*L_repeat)  point natural(ux) = complex(0,0) !
       ↪ right

region '(8r) THIN SPACE CHARGE - RIGHT ELECTRODE'
    mass_density = mass_density_gold  permittivity_relative =
       ↪ permittivity_relative_gold
    stiffness_11_real = stiffness_11_real_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 }
           force_density_DC_x = -force_density_DC_x_calculated

    start (x7r + L + (i-1)*L_repeat)   point natural(ux) = complex(0,0) !
       ↪ left
    line to (x8r + L + (i-1)*L_repeat)  point natural(ux) = complex(0,0) !
       ↪ right
```

```
   region '(9r) positive electrode right'
       mass_density = mass_density_gold  permittivity_relative =
          ↪ permittivity_relative_gold
       stiffness_11_real = stiffness_11_real_gold
       viscosity_11 = viscosity_11_gold { N.s/m^2 }
              force_density_DC_x = 0

       start (x8r + L + (i-1)*L_repeat)       point natural(ux) = complex(0,0)
          ↪  ! left
       line to (x9r_recessed + L + (i-1)*L_repeat)  point natural(ux) =
          ↪ complex(0,0) ! right

   region '(10r) positive electrode right'
       mass_density = mass_density_gold  permittivity_relative =
          ↪ permittivity_relative_gold
       stiffness_11_real = stiffness_11_real_gold
       viscosity_11 = viscosity_11_gold { N.s/m^2 }
              force_density_DC_x = force_density_DC_x_calculated

       start (x9r_recessed + L + (i-1)*L_repeat) point natural(ux) = complex
          ↪ (0,0) ! left
       line to (x9r + L + (i-1)*L_repeat)     point natural(ux) = complex(0,0)
          ↪  ! right

   endrepeat
! =====Last Unit=====
       region '(3e) silicon left'
       mass_density = mass_density_silicon permittivity_relative =
          ↪ permittivity_relative_silicon
       stiffness_11_real = stiffness_11_real_silicon
       viscosity_11 = viscosity_11_silicon { N.s/m^2 }
              force_density_DC_x = 0

       start (x2l + L_shifter_last_unit)  point natural(ux) = complex(0,0) !
          ↪ left
       line to (x3l + L_shifter_last_unit)  point natural(ux) = complex(0,0) !
          ↪  right

   region '(4e) THIN SPACE CHARGE - CENTER ELECTRODE LEFT'
```

```
    mass_density = mass_density_gold  permittivity_relative =
       ↪ permittivity_relative_gold
    stiffness_11_real = stiffness_11_real_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 }
           force_density_DC_x = -force_density_DC_x_calculated

    start (x3l + L_shifter_last_unit)  point natural(ux) = complex(0,0) !
       ↪ left
    line to (x4l + L_shifter_last_unit)  point natural(ux) = complex(0,0) !
       ↪  right

    region '(5e) negative electrode'
    mass_density = mass_density_gold  permittivity_relative =
       ↪ permittivity_relative_gold
    stiffness_11_real = stiffness_11_real_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 }
           force_density_DC_x = 0

    start (x4l + L_shifter_last_unit)   point natural(ux) = complex(0,0) !
       ↪ left
    line to (x5l + L_shifter_last_unit)   point natural(ux) = complex(0,0)
       ↪ ! right

region '(6e) THIN SPACE CHARGE - CENTER ELECTRODE RIGHT'
    mass_density = mass_density_gold  permittivity_relative =
       ↪ permittivity_relative_gold
    stiffness_11_real = stiffness_11_real_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 }
           force_density_DC_x = force_density_DC_x_calculated

    start (x5l + L_shifter_last_unit)  point natural(ux) = complex(0,0) !
       ↪ left
    line to (x6l + L_shifter_last_unit)  point natural(ux) = complex(0,0) !
       ↪  right

    region '(7e) silicon right'
    mass_density = mass_density_silicon permittivity_relative =
       ↪ permittivity_relative_silicon
    stiffness_11_real = stiffness_11_real_silicon
```

```
        viscosity_11 = viscosity_11_silicon { N.s/m^2 }
               force_density_DC_x = 0

        start (x6l + L_shifter_last_unit)   point natural(ux) = complex(0,0) !
           ↪ left
        line to (x7l + L_shifter_last_unit)   point natural(ux) = complex(0,0)
           ↪ ! right

    region '(8e) THIN SPACE CHARGE - RIGHT ELECTRODE'
        mass_density = mass_density_gold  permittivity_relative =
           ↪ permittivity_relative_gold
        stiffness_11_real = stiffness_11_real_gold
        viscosity_11 = viscosity_11_gold { N.s/m^2 }
               force_density_DC_x = -force_density_DC_x_calculated

        start (x7l + L_shifter_last_unit)  point natural(ux) = complex(0,0) !
           ↪ left
        line to (x8l + L_shifter_last_unit)  point natural(ux) = complex(0,0) !
           ↪  right

    region '(9e) positive electrode right'
        mass_density = mass_density_gold  permittivity_relative =
           ↪ permittivity_relative_gold
        stiffness_11_real = stiffness_11_real_gold
        viscosity_11 = viscosity_11_gold { N.s/m^2 }
               force_density_DC_x = 0

        start (x8l + L_shifter_last_unit)  point natural(ux) = complex(0,0) !
           ↪ left
        line to (x9l + L_shifter_last_unit)  point natural(ux) = complex(0,0) !
           ↪  right

    !feature 'Signal line'
        !point value(ux) = complex(ur_signal, 0)

{
MONITORS
        grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy))
```

```
        contour(real(ux)) as "X-displacement(m)" contour(real(ux)) as "X-
            ↪ displacement(m)" painted
        elevation(real(ux), real(uy)) from(x0, height/2) to (x9 + (nUnit-1)*L,
            ↪ height/2)
}

PLOTS
    !elevation(real(ux)) from(x0) to (probeline_end_x) PNG(1024, 2)
    elevation(real(ux), real(u_analytical_homogenized)) from(x0) to (L_total)
        ↪ PNG(1024, 2)
        !export FILE = 'ux_real.txt' FORMAT"#X #1"
        report(V_dc, V_signal, f, ur_signal)

    elevation(real(ux), imag(u_analytical_homogenized)) from(x0) to (L_total)
        ↪ PNG(1024, 2)
        !export FILE = 'ux_real.txt' FORMAT"#X #1"
        report(V_dc, V_signal, f, ur_signal)

    !elevation(real(ux)) from(x0) to (L_total) PNG(1024, 2)
        !export FORMAT"#X #1" FILE = 'ExtractedData/ux_real.txt'
        !report(V_dc, V_signal, f, ur_signal)

    !elevation(imag(ux)) from(x0) to (L_total) PNG(1024, 2)
        !export FILE = 'ExtractedData/ux_imag.txt' FORMAT"#X #1"
        !report(V_dc, V_signal, f)

HISTORIES
        history(globalmax(abs(real(ux)))) PNG(1024, 2) report(V_dc, V_signal, f
            ↪ )
    history(globalmax(abs(real(ux)))) versus f PNG(1024, 2) report(V_dc,
        ↪ V_signal)
        history( log10(globalmax(abs(real(ux)))) ) PNG(1024, 2) versus log10(f)
            ↪  report(V_dc, V_signal)
END
```

Appendix E

E FlexPDE Code Listing for Voltage-induced Strain & Effect of Poisson's ratio

```
{ One Unit
1.) Stage DC voltage
2.) DC voltage ON
3.) AC signal OFF
4.) Periodic condition OFF
5.) We expect electric field induced stress/strain of the medium
     and Poisson effect.
}

TITLE 'Helmholtz 2D'
SELECT errlim = 1e-4 order = 3 threads = 1 stages = 3 regrid = off !
   ↪ featureplot = on !ngrid = 20
VARIABLES ux = complex(ux_real, ux_imag) uy = complex(uy_real, uy_imag)
   ↪ voltage (threshold = 1e-2)
DEFINITIONS
    zoom_factor = 4e5

    mass_density Young_modulus Poisson_ratio
    { Poisson ratio = 0.5 means volumetric strain is zero => material is
       ↪ incompressible, e.g., water
```

```
  Poisson ratio = 0 means no change in lateral strain when we apply tensile
     ↪ /compressive force, e.g., cork
  Strain_z = Strain_y in uniaxial stress
  Strain_y is proportional to Strain_x
  Strain_z is proportional to Strain_x
  This proportionality is called Poisson ratio.
}

permittivity_relative permittivity_0 = 8.8542e-12 { free space permittivity
   ↪  F/m } permittivity = permittivity_relative * permittivity_0

{ ===== Plane Strain ===== }
C = Young_modulus / ( (1 + Poisson_ratio ) * ( 1 - 2*Poisson_ratio ) )
G = Young_modulus / ( 2*(1 + Poisson_ratio ) )

ex1 = dx(ux) ! Strain
ey1 = dy(uy) ! Strain
exy1 = dx(uy) + dy(ux) ! Strain

! Excluding damping
sx1 = C*(ex1 - Poisson_ratio*ex1 + Poisson_ratio*ey1) ! Plane stress
sy1 = C*(ey1 - Poisson_ratio*ey1 + Poisson_ratio*ex1) ! Plane stress
sxy1 = G*exy1 ! Plane stress

{ ===== Silicon ===== }
mass_density_silicon = 2329 { kg/m3 } Young_modulus_silicon = 169e9 { N/m2
   ↪ } Poisson_ratio_silicon = 0.27 { along [100] direction }
permittivity_relative_silicon = 11.7
{ http://www.ioffe.ru/SVA/NSM/Semicond/Si/mechanic.html#Elastic }

{ ===== Gold ===== }
mass_density_gold = 19300 { kg/m3 } Young_modulus_gold = 79e9 { N/m2 }
   ↪ Poisson_ratio_gold = 0.42
permittivity_relative_gold = 1 { At low and RF frequencies }
{ https://www.azom.com/properties.aspx?ArticleID=598 }
{ http://chemistry.elmhurst.edu/vchembook/125Adensitygold.html }

{ ===== Structure ===== }
!The charge width is tucked inside the electrode width.
```

```
    silicon_width = 2e-6 {m} electrode_width = 2e-6 {m}
    L = 2*electrode_width + 2*silicon_width { m } Height = 10e-6 {m}

    ! [electrode][silicon][electrode][silicon][electrode]
    x0 = -(electrode_width + silicon_width + electrode_width/2)
    x2 = -(silicon_width + electrode_width/2)
    x3 = -(electrode_width/2)
    x6 = electrode_width/2
    x7 = electrode_width/2 + silicon_width
    x9 = electrode_width/2 + silicon_width + electrode_width

    { ===== Voltage ===== }
    V_dc = 25 + 25*(stage-1){ volts }
    !V_dc = 25
    electric_field_x = -dx(voltage) electric_field_y = -dy(voltage)
    electric_field = vector(electric_field_x, electric_field_y) { volts/meter }
    electric_field_magnitude = sqrt(electric_field_x^2 + electric_field_y^2)
    electric_field_vector_angle = sign(electric_field_y)*arccos(
       ↪ electric_field_x/electric_field_magnitude)/pi*180

    { ===== Artificial piezo-material ===== }
    width_artificial_piezo = silicon_width
    permittivity_artificial_piezo = permittivity_relative_silicon *
       ↪ permittivity_0
    mass_density_artificial_piezo = mass_density_silicon
       ↪ Young_modulus_artificial_piezo = Young_modulus_silicon
       ↪ Poisson_ratio_artificial_piezo = Poisson_ratio_silicon
    C_artificial_piezo = Young_modulus_artificial_piezo/( (1 +
       ↪ Poisson_ratio_artificial_piezo) * (1 - 2*
       ↪ Poisson_ratio_artificial_piezo) )
    c11_artificial_piezo = C_artificial_piezo * (1 -
       ↪ Poisson_ratio_artificial_piezo)

    { ===== Energy density ===== }
    ![energy_density] = N/m^2
    energy_density = 0.5*(sx1*ex1 + sy1*ey1 + sxy1*exy1)

EQUATIONS
    voltage: div(permittivity*grad(voltage)) = 0 { potential equation }
```

```
    ux: dx(sx1) + dy(sxy1) + 0.5*permittivity*dx(electric_field_x^2 -
        ↪ electric_field_y^2) + permittivity*dy(electric_field_x*
        ↪ electric_field_y) = 0
    uy: dx(sxy1) + dy(sy1) - 0.5*permittivity*dy(electric_field_x^2 -
        ↪ electric_field_y^2) + permittivity*dx(electric_field_x*
        ↪ electric_field_y) = 0

BOUNDARIES
    region '(1) positive electrode left'
        mass_density = mass_density_gold Young_modulus = Young_modulus_gold
            ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
            ↪ permittivity_relative_gold

        start (x0, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
            ↪ complex(0, 0) ! bottom
        line to (x2, -Height/2)  natural(ux) = complex(0, 0)   natural(uy) =
            ↪ complex(0, 0)  value(voltage) = V_dc ! right
        line to (x2, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
            ↪ complex(0, 0) ! top
        line to (x0, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
            ↪ complex(0, 0)  value(voltage) = V_dc ! left
        line to close

    region '(3) silicon left'
        mass_density = mass_density_silicon Young_modulus =
            ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
            ↪ permittivity_relative = permittivity_relative_silicon

        start (x2, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
            ↪ complex(0, 0) ! bottom
        line to (x3, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
            ↪ complex(0, 0) ! right
        line to (x3, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
            ↪ complex(0, 0) ! top
        line to (x2, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
            ↪ complex(0, 0) ! left
        line to close

    region '(5) negative electrode'
```

```
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold

    start (x3, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x6, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0)  value(voltage) = 0 ! right
    line to (x6, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x3, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0)  value(voltage) = 0 ! left
    line to close

region '(7) silicon right'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon

    start (x6, -Height/2)    natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x7, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x7, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x6, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(9) positive electrode right'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold

    start (x7, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x9, -Height/2)  natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) value(voltage) = V_dc ! right
        !periodic (x - L, y) map(ux) = -1 map(uy) = -1
```

```
        line to (x9, Height/2)  natural(ux) = complex(0, 0)   natural(uy) =
            ↪ complex(0, 0) ! top
        line to (x7, Height/2)  natural(ux) = complex(0, 0)   natural(uy) =
            ↪ complex(0, 0) value(voltage) = V_dc ! left
        line to close

{
MONITORS
        grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy))
        contour(real(ux)) as "X-displacement(m)" contour(real(ux)) as "X-
            ↪ displacement(m)" painted
        elevation(real(ux), real(uy)) from(x0, height/2) to (x9, height/2)
}

PLOTS
    grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy)) PNG(1024, 2)
        report(V_dc, L, Height)

    contour(real(ux)) PNG(1024, 2) as "X-displacement(m)" report(V_dc)
        !vtk(real(ux))
    contour(real(ux)) PNG(1024, 2) as "X-displacement(m)" painted report(V_dc)
        !vtk(real(ux))

    contour(real(uy)) PNG(1024, 2) as "Y-displacement(m)" report(V_dc)
        !vtk(real(uy))
    contour(real(uy)) PNG(1024, 2) as "Y-displacement(m)" painted report(V_dc)
        !vtk(real(uy))

    contour(voltage) PNG(1024, 2) as "Voltage(m)" report(V_dc)
    contour(voltage) PNG(1024, 2) as "Voltage(m)" Painted report(V_dc)

    contour(electric_field_magnitude) PNG(1024, 2) as "|E| (V/m)" report(V_dc)
    contour(electric_field_magnitude) log PNG(1024, 2) as "log10(|E|) (V/m)"
        ↪ report(V_dc)
    surface(electric_field_magnitude) log PNG(1024, 2) as "log10(|E|) (V/m)"
        ↪ report(V_dc)
    vector(electric_field) points=20 PNG(1024, 2) as "E (V/m)" report(V_dc)
    contour(electric_field_vector_angle) PNG(1024, 2) as "E angle (degrees)"
        ↪ report(V_dc)
```

```
    elevation(real(ux)) from(x0, 0) to (x9, 0) PNG(1024, 2)
        !export FILE = 'ux_real.txt' FORMAT"#X #1"
        report(V_dc)

    elevation(real(uy)) from(x0, 0) to (x9, 0) PNG(1024, 2) !fixed range (- 1e
        ↪ -19, 1e-19 )
        !export FILE = 'uy_real.txt' FORMAT"#X #1"
        report(V_dc)

    elevation(real(ux)) from((electrode_width/2 + silicon_width/2), -height/2)
        ↪ to ((electrode_width/2 + silicon_width/2), height/2) PNG(1024, 2)
        !export FILE = 'ux_real_probe_line.txt' FORMAT"#Y #1"
        report(V_dc)

    elevation(real(uy)) from((electrode_width/2 + silicon_width/2), -height/2)
        ↪ to ((electrode_width/2 + silicon_width/2), height/2) PNG(1024, 2)
        !export FILE = 'uy_real_probe_line.txt' FORMAT"#Y #1"
        report(V_dc)

END
```

Appendix F

F FlexPDE Code Listing for One Bar with Electrodes

```
{
=====Bar with electrodes=====
1.) Stage frequency
2.) DC voltage ON
3.) AC signal ON
4.) Periodic condition OFF
5.) We expect the phase snapshot of mechanical displacement
    due to AC signal perturbation coupled with DC voltage at a given frequency
      ↪ .
}

TITLE 'Helmholtz 2D Bar'
SELECT errlim = 1e-4 order = 1 threads = 4 stages = 15 regrid = off
   ↪ featureplot = on !ngrid = 20
VARIABLES ux = complex(ux_real, ux_imag) uy = complex(uy_real, uy_imag)
DEFINITIONS
    zoom_factor = 2e3 df = 1e6 f = 2.3736e9 + df * (stage-1)
    !zoom_factor = 2e3 f = 2.3806e9
    omega = 2*pi*f { rad-Hz }
    !In MATLAB, type, stage = 15; fc = 2.3806e9; df = 1e6; fl = fc - df *(stage
      ↪ -1)/2

    mass_density Young_modulus Poisson_ratio
```

```
{ Poisson ratio = 0.5 means volumetric strain is zero => material is
    ↪ incompressible, e.g., water
  Poisson ratio = 0 means no change in lateral strain when we apply tensile
     ↪ /compressive force, e.g., cork
  Strain_z = Strain_y in uniaxial stress
  Strain_y is proportional to Strain_x
  Strain_z is proportional to Strain_x
  This proportionality is called Poisson ratio.
}

permittivity_relative permittivity_0 = 8.8542e-12 { free space permittivity
    ↪  F/m } permittivity = permittivity_relative * permittivity_0
force_density_DC_x

{ ===== Plane Strain ===== }
C = Young_modulus / ( (1 + Poisson_ratio ) * ( 1 - 2*Poisson_ratio ) )
G = Young_modulus / ( 2*(1 + Poisson_ratio ) )

viscosity_11 viscosity_12 viscosity_44

ex1 = dx(ux) ! Strain
ey1 = dy(uy) ! Strain
exy1 = dx(uy) + dy(ux) ! Strain

! Excluding damping
{
sx1 = C*(ex1 - Poisson_ratio*ex1 + Poisson_ratio*ey1) ! Plane stress
sy1 = C*(ey1 - Poisson_ratio*ey1 + Poisson_ratio*ex1) ! Plane stress
sxy1 = G*exy1 ! Plane stress
}

! Including damping

sx1 = (C*(1 - Poisson_ratio) + complex(0,1)*omega*viscosity_11)*ex1 + (C*
    ↪ Poisson_ratio + complex(0,1)*omega*viscosity_12)*ey1 ! Plane stress
sy1 = (C*(1 - Poisson_ratio) + complex(0,1)*omega*viscosity_11)*ey1 + (C*
    ↪ Poisson_ratio + complex(0,1)*omega*viscosity_12)*ex1 ! Plane stress
sxy1 = (G + complex(0,1)*omega*viscosity_44)*exy1 ! Plane stress
```

```
{ ===== Silicon ===== }
mass_density_silicon = 2329 { kg/m3 } Young_modulus_silicon = 169e9 { N/m2
    ↪ } Poisson_ratio_silicon = 0.27 { along [100] direction }
permittivity_relative_silicon = 11.7
{ http://www.ioffe.ru/SVA/NSM/Semicond/Si/mechanic.html#Elastic }

viscosity_11_silicon = 0.005 { N.s/m^2 }
viscosity_12_silicon = 0.005 { N.s/m^2 }
viscosity_44_silicon = 0.005 { N.s/m^2 }
{ B.A. Auld, vol 1, chapter 3, a representative value }

{ ===== Gold ===== }
mass_density_gold = 19300 { kg/m3 } Young_modulus_gold = 79e9 { N/m2 }
    ↪ Poisson_ratio_gold = 0.42
permittivity_relative_gold = 1 { At low and RF frequencies }
{ https://www.azom.com/properties.aspx?ArticleID=598 }
{ http://chemistry.elmhurst.edu/vchembook/125Adensitygold.html }

viscosity_11_gold = 0.0186 { N.s/m^2 }
viscosity_12_gold = 0.0186 { N.s/m^2 }
viscosity_44_gold = 0.0186 { N.s/m^2 }

{ ===== Structure ===== }
!The charge width is tucked inside the electrode width.
silicon_width = 2e-6 {m} electrode_width = 200e-9 {m} charge_width = 50e-9
    ↪ {m}
Height = 10e-6 {m}

! [electrode][Space-charge region][silicon][Space-charge region][electrode]
x0 = -(electrode_width + silicon_width/2)
x1 = -(charge_width + silicon_width/2)
x2 = -(silicon_width/2)
x3 = silicon_width/2
x4 = charge_width + silicon_width/2
x6 = electrode_width + silicon_width/2

{ ===== Voltage ===== }
V_dc = 10 { volts }
```

```
V_signal = 0.1 { volts }

{ ===== Artificial piezo-material ===== }
width_artificial_piezo = silicon_width
permittivity_artificial_piezo = permittivity_relative_silicon *
    ↪ permittivity_0
mass_density_artificial_piezo = mass_density_silicon
    ↪ Young_modulus_artificial_piezo = Young_modulus_silicon
    ↪ Poisson_ratio_artificial_piezo = Poisson_ratio_silicon
C_artificial_piezo = Young_modulus_artificial_piezo/( (1 +
    ↪ Poisson_ratio_artificial_piezo) * (1 - 2*
    ↪ Poisson_ratio_artificial_piezo) )
c11_artificial_piezo = C_artificial_piezo * (1 -
    ↪ Poisson_ratio_artificial_piezo)

{ ===== Force density due to DC bias ===== }
force_density_DC_x_calculated = permittivity_artificial_piezo * V_dc^2/(2*
    ↪ width_artificial_piezo^2) * ( 1 + permittivity_artificial_piezo *
    ↪ V_dc^2/( c11_artificial_piezo * width_artificial_piezo^2) ) { N/m^3 }

!force_density_DC_x_calculated = 1e4 { N/m3 }
!force_density_DC_x_calculated = 1e4 + 3e3*(stage-1) { N/m3 }

{ ===== Change in length due to AC signal perturbation ===== }
DeltaL = -0.5 * permittivity_artificial_piezo * V_dc^2/(
    ↪ c11_artificial_piezo*width_artificial_piezo)
Leq = width_artificial_piezo + DeltaL
k = omega * sqrt(mass_density_artificial_piezo/c11_artificial_piezo)
alpha = tan(k*Leq/2)
beta = alpha*(cos(k*Leq) - 1) - sin(k*Leq)
force_DC_per_unit_area = -0.5 * permittivity_artificial_piezo * V_dc^2 /
    ↪ (2*width_artificial_piezo^2) * (1 - 2*DeltaL/width_artificial_piezo)
ux_signal = 2*beta/(c11_artificial_piezo*k) * force_DC_per_unit_area / ( 1
    ↪ + 2*beta/(c11_artificial_piezo*k) * (force_DC_per_unit_area/Leq) ) *
    ↪ V_signal/V_dc

!ux_signal = 0
!ux_signal = 1e-14 { m }
```

```
    { ===== Energy density ===== }
    ![energy_density] = N/m^2
    energy_density = 0.5*(sx1*ex1 + sy1*ey1 + sxy1*exy1)

EQUATIONS
    ux: dx(sx1) + dy(sxy1) + mass_density*omega^2*ux + force_density_DC_x = 0
    uy: dx(sxy1) + dy(sy1) + mass_density*omega^2*uy = 0

BOUNDARIES
    region '(1) positive electrode left'
        mass_density = mass_density_gold Young_modulus = Young_modulus_gold
            ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
            ↪ permittivity_relative_gold
        viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
            ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
            ↪ N.s/m^2 }
        force_density_DC_x = 0

        start (x0, -Height/2)   natural(ux) = complex(0, 0)     natural(uy) =
            ↪ complex(0, 0) ! bottom
        line to (x1, -Height/2)   natural(ux) = complex(0, 0)     natural(uy) =
            ↪  complex(0, 0) ! right
        line to (x1, Height/2)   natural(ux) = complex(0, 0)    natural(uy) =
            ↪ complex(0, 0) ! top
        line to (x0, Height/2)   value(ux) = complex(ux_signal, 0)  natural(uy)
            ↪  = complex(0, 0) ! left
        line to close

    region '(2) THIN SPACE CHARGE - LEFT ELECTRODE'
        mass_density = mass_density_gold Young_modulus = Young_modulus_gold
            ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
            ↪ permittivity_relative_gold
        viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
            ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
            ↪ N.s/m^2 }
        force_density_DC_x = force_density_DC_x_calculated

        start (x1, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
            ↪ complex(0, 0) ! bottom
```

```
    line to (x2, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x2, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
           line to (x1, Height/2)   natural(ux) = complex(0, 0)  natural(uy
              ↪ ) = complex(0, 0) ! left
    line to close

region '(3) silicon'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
       ↪ viscosity_44_silicon { N.s/m^2 }
    force_density_DC_x = 0

    start (x2, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x3, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x3, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x2, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(4) THIN SPACE CHARGE - RIGHT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x3, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
```

```
       line to (x4, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
          ↪ complex(0, 0) ! right
       line to (x4, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
          ↪ complex(0, 0) ! top
       line to (x3, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
          ↪ complex(0, 0) ! left
       line to close

   region '(5) negative electrode'
       mass_density = mass_density_gold Young_modulus = Young_modulus_gold
          ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
          ↪ permittivity_relative_gold
       viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
          ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
          ↪ N.s/m^2 }
       force_density_DC_x = 0

       start (x4, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
          ↪ complex(0, 0) ! bottom
       line to (x6, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
          ↪ complex(0, 0) ! right
       line to (x6, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
          ↪ complex(0, 0) ! top
       line to (x4, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
          ↪ complex(0, 0) ! left
       line to close

   !feature 'Signal line'
       !start(x0, -Height/2) value(ux) = complex(ux_signal, 0) line to (x0,
          ↪ Height/2)

{
MONITORS
       grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy))
       contour(real(ux)) as "X-displacement(m)" contour(real(ux)) as "X-
          ↪ displacement(m)" painted
       elevation(real(ux), real(uy)) from(x0, height/2) to (x6, height/2)
}
```

```
PLOTS
    grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy)) PNG
        report(V_dc, V_signal, f, Height)

    contour(real(ux)) PNG as "X-displacement(m)" report(V_dc, V_signal, f)
        !vtk(real(ux))
    contour(real(ux)) PNG as "X-displacement(m)" painted report(V_dc, V_signal,
       ↪  f)
        !vtk(real(ux))

    contour(real(uy)) PNG as "Y-displacement(m)" report(V_dc, V_signal, f)
        !vtk(real(uy))
    contour(real(uy)) PNG as "Y-displacement(m)" painted report(V_dc, V_signal,
       ↪  f)
        !vtk(real(uy))

    surface(force_density_DC_x) PNG painted report(V_dc, V_signal, f)
        !vtk(force_density_DC_x)

    elevation(real(ux)) from(x0, 0) to (x6, 0) PNG(1024, 2)
        !export FILE = 'ux_real.txt' FORMAT"#X #1"
        report(V_dc, V_signal, f)

    elevation(real(uy)) from(x0, 0) to (x6, 0) PNG(1024, 2) !fixed range (- 1e
       ↪ -19, 1e-19 )
        !export FILE = 'uy_real.txt' FORMAT"#X #1"
        report(V_dc, V_signal, f)

    elevation(real(ux)) from((electrode_width/2 + silicon_width/2), -Height/2)
       ↪ to ((electrode_width/2 + silicon_width/2), Height/2) PNG(1024, 2)
        !export FILE = 'ux_real_probe_line.txt' FORMAT"#Y #1"
        report(V_dc, V_signal, f)

    elevation(real(uy)) from((electrode_width/2 + silicon_width/2), -Height/2)
       ↪ to ((electrode_width/2 + silicon_width/2), Height/2) PNG(1024, 2)
        !export FILE = 'uy_real_probe_line.txt' FORMAT"#Y #1"
        report(V_dc, V_signal, f)

HISTORIES
```

```
        history(globalmax(abs(real(ux)))) PNG(1024, 2) report(V_dc, V_signal, f
            ↪ )
        history(globalmax(abs(real(ux)))) versus f PNG(1024, 2) report(V_dc,
            ↪ V_signal)
        history( log10(globalmax(abs(real(ux)))) ) PNG(1024, 2) versus log10(f)
            ↪  report(V_dc, V_signal)

        history(globalmax(abs(real(uy)))) PNG(1024, 2) report(V_dc, V_signal, f
            ↪ )
        history(globalmax(abs(real(uy)))) versus f PNG(1024, 2) report(V_dc,
            ↪ V_signal)
        history( log10(globalmax(abs(real(uy)))) ) PNG(1024, 2) versus log10(f)
            ↪  report(V_dc, V_signal)

END
```

Appendix G

G FlexPDE Code Listing for One Bar with Electrodes (Explicit Formulation)

```
{ Explicit Formulation
=====Bar with electrodes=====
1.) Stage frequency
2.) DC voltage ON
3.) AC signal ON
4.) Periodic condition OFF
5.) We expect the phase snapshot of mechanical displacement
     due to AC signal perturbation coupled with DC voltage at a given frequency
        ↪ .
}

TITLE 'Helmholtz 2D Bar'
SELECT errlim = 1e-4 order = 1 threads = 4 stages = 15 regrid = off
    ↪ featureplot = on !ngrid = 20
VARIABLES ux_real ux_imag uy_real uy_imag
DEFINITIONS
    zoom_factor = 2e3 df = 1e6 f = 2.3736e9 + df * (stage-1)
        !zoom_factor = 2e3 f = 2.3806e9
    omega = 2*pi*f { rad-Hz }
    !In MATLAB, type, stage = 15; fc = 2.3806e9; df = 1e7; fl = fc - df *(stage
       ↪ -1)/2
```

```
mass_density Young_modulus Poisson_ratio
{ Poisson ratio = 0.5 means volumetric strain is zero => material is
   ↪ incompressible, e.g., water
  Poisson ratio = 0 means no change in lateral strain when we apply tensile
     ↪ /compressive force, e.g., cork
  Strain_z = Strain_y in uniaxial stress
  Strain_y is proportional to Strain_x
  Strain_z is proportional to Strain_x
  This proportionality is called Poisson ratio.
}

permittivity_relative permittivity_0 = 8.8542e-12 { free space permittivity
   ↪  F/m } permittivity = permittivity_relative * permittivity_0
force_density_DC_x

{ ===== Plane Strain ===== }
C = Young_modulus / ( (1 + Poisson_ratio ) * ( 1 - 2*Poisson_ratio ) )
G = Young_modulus / ( 2*(1 + Poisson_ratio ) )

viscosity_11 viscosity_12 viscosity_44

! For explicit forms
c11 = C*(1-Poisson_ratio)
c12 = C*Poisson_ratio
c44 = G

{ ===== Silicon ===== }
mass_density_silicon = 2329 { kg/m3 } Young_modulus_silicon = 169e9 { N/m2
   ↪ } Poisson_ratio_silicon = 0.27 { along [100] direction }
permittivity_relative_silicon = 11.7
{ http://www.ioffe.ru/SVA/NSM/Semicond/Si/mechanic.html#Elastic }

viscosity_11_silicon = 0.005 { N.s/m^2 }
viscosity_12_silicon = 0.005 { N.s/m^2 }
viscosity_44_silicon = 0.005 { N.s/m^2 }
{ B.A. Auld, vol 1, chapter 3, a representative value }

{ ===== Gold ===== }
```

```
mass_density_gold = 19300 { kg/m3 } Young_modulus_gold = 79e9 { N/m2 }
    ↪ Poisson_ratio_gold = 0.42
permittivity_relative_gold = 1 { At low and RF frequencies }
{ https://www.azom.com/properties.aspx?ArticleID=598 }
{ http://chemistry.elmhurst.edu/vchembook/125Adensitygold.html }

viscosity_11_gold = 0.0186 { N.s/m^2 }
viscosity_12_gold = 0.0186 { N.s/m^2 }
viscosity_44_gold = 0.0186 { N.s/m^2 }

{ ===== Structure ===== }
!The charge width is tucked inside the electrode width.
silicon_width = 2e-6 {m} electrode_width = 200e-9 {m} charge_width = 50e-9
    ↪ {m}
Height = 10e-6 {m}

! [electrode][Space-charge region][silicon][Space-charge region][electrode]
x0 = -(electrode_width + silicon_width/2)
x1 = -(charge_width + silicon_width/2)
x2 = -(silicon_width/2)
x3 = silicon_width/2
x4 = charge_width + silicon_width/2
x6 = electrode_width + silicon_width/2

{ ===== Voltage ===== }
V_dc = 10 { volts }
V_signal = 0.1 { volts }

{ ===== Artificial piezo-material ===== }
width_artificial_piezo = silicon_width
permittivity_artificial_piezo = permittivity_relative_silicon *
    ↪ permittivity_0
mass_density_artificial_piezo = mass_density_silicon
    ↪ Young_modulus_artificial_piezo = Young_modulus_silicon
    ↪ Poisson_ratio_artificial_piezo = Poisson_ratio_silicon
C_artificial_piezo = Young_modulus_artificial_piezo/( (1 +
    ↪ Poisson_ratio_artificial_piezo) * (1 - 2*
    ↪ Poisson_ratio_artificial_piezo) )
```

```
    c11_artificial_piezo = C_artificial_piezo * (1 -
        ↪ Poisson_ratio_artificial_piezo)

    { ===== Force density due to DC bias ===== }
    force_density_DC_x_calculated = permittivity_artificial_piezo * V_dc^2/(2*
        ↪ width_artificial_piezo^2) * ( 1 + permittivity_artificial_piezo *
        ↪ V_dc^2/( c11_artificial_piezo * width_artificial_piezo^2) ) { N/m^3 }

    !force_density_DC_x_calculated = 1e4 { N/m3 }
    !force_density_DC_x_calculated = 1e4 + 3e3*(stage-1) { N/m3 }

    { ===== Change in length due to AC signal perturbation ===== }
    DeltaL = -0.5 * permittivity_artificial_piezo * V_dc^2/(
        ↪ c11_artificial_piezo*width_artificial_piezo)
    Leq = width_artificial_piezo + DeltaL
    k = omega * sqrt(mass_density_artificial_piezo/c11_artificial_piezo)
    alpha = tan(k*Leq/2)
    beta = alpha*(cos(k*Leq) - 1) - sin(k*Leq)
    force_DC_per_unit_area = -0.5 * permittivity_artificial_piezo * V_dc^2 /
        ↪ (2*width_artificial_piezo^2) * (1 - 2*DeltaL/width_artificial_piezo)
    ux_signal = 2*beta/(c11_artificial_piezo*k) * force_DC_per_unit_area / ( 1
        ↪ + 2*beta/(c11_artificial_piezo*k) * (force_DC_per_unit_area/Leq) ) *
        ↪ V_signal/V_dc

        !ux_signal = 0
    !ux_signal = 1e-14 { m }

    { ===== Energy density ===== }
    ![energy_density] = N/m^2

EQUATIONS
    ux_real: dx( c11*dx(ux_real) - omega*viscosity_11*dx(ux_imag) + c12*dy(
        ↪ uy_real) - omega*viscosity_12*dy(uy_imag) ) + dy( c44*dx(uy_real) -
        ↪ omega*viscosity_44*dx(uy_imag) + c44*dy(ux_real) - omega*viscosity_44
        ↪ *dy(ux_imag) ) + mass_density*omega^2*ux_real + force_density_DC_x =
        ↪ 0
    ux_imag: dx( c11*dx(ux_imag) + omega*viscosity_11*dx(ux_real) + c12*dy(
        ↪ uy_imag) + omega*viscosity_12*dy(uy_real) ) + dy( c44*dx(uy_imag) +
        ↪ omega*viscosity_44*dx(uy_real) + c44*dy(ux_imag) + omega*viscosity_44
```

```
      ↪ *dy(ux_real) ) + mass_density*omega^2*ux_imag = 0

   uy_real: dy( c11*dy(uy_real) - omega*viscosity_11*dy(uy_imag) + c12*dx(
      ↪ ux_real) - omega*viscosity_12*dx(ux_imag) ) + dx( c44*dx(uy_real) -
      ↪ omega*viscosity_44*dx(uy_imag) + c44*dy(ux_real) - omega*viscosity_44
      ↪ *dy(ux_imag) ) + mass_density*omega^2*uy_real = 0
   uy_imag: dy( c11*dy(uy_imag) + omega*viscosity_11*dy(uy_real) + c12*dx(
      ↪ ux_imag) + omega*viscosity_12*dx(ux_real) ) + dx( c44*dx(uy_imag) +
      ↪ omega*viscosity_44*dx(uy_real) + c44*dy(ux_imag) + omega*viscosity_44
      ↪ *dy(ux_real) ) + mass_density*omega^2*uy_imag = 0

BOUNDARIES
   region '(1) positive electrode left'
      mass_density = mass_density_gold Young_modulus = Young_modulus_gold
         ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
         ↪ permittivity_relative_gold
      viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
         ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
         ↪ N.s/m^2 }
      force_density_DC_x = 0

      start (x0, -Height/2)  natural(ux_real) = 0  natural(ux_imag) = 0
         ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! bottom
      line to (x1, -Height/2) natural(ux_real) = 0  natural(ux_imag) = 0
         ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! right
      line to (x1, Height/2)  natural(ux_real) = 0  natural(ux_imag) = 0
         ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! top
      line to (x0, Height/2)  value(ux_real) = ux_signal  natural(ux_imag) =
         ↪ 0 natural(uy_real) = 0  natural(uy_imag) = 0 ! left
      line to close

   region '(2) THIN SPACE CHARGE - LEFT ELECTRODE'
      mass_density = mass_density_gold Young_modulus = Young_modulus_gold
         ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
         ↪ permittivity_relative_gold
      viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
         ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
         ↪ N.s/m^2 }
      force_density_DC_x = force_density_DC_x_calculated
```

```
    start (x1, -Height/2)  natural(ux_real) = 0  natural(ux_imag) = 0
       ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! bottom
    line to (x2, -Height/2) natural(ux_real) = 0  natural(ux_imag) = 0
       ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! right
    line to (x2, Height/2)  natural(ux_real) = 0  natural(ux_imag) = 0
       ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! top
           line to (x1, Height/2)  natural(ux_real) = 0  natural(ux_imag) =
              ↪  0 natural(uy_real) = 0  natural(uy_imag) = 0 ! left
    line to close

region '(3) silicon left'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
       ↪ viscosity_44_silicon { N.s/m^2 }
    force_density_DC_x = 0

    start (x2, -Height/2)  natural(ux_real) = 0  natural(ux_imag) = 0
       ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! bottom
    line to (x3, -Height/2) natural(ux_real) = 0  natural(ux_imag) = 0
       ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! right
    line to (x3, Height/2)  natural(ux_real) = 0  natural(ux_imag) = 0
       ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! top
    line to (x2, Height/2)  natural(ux_real) = 0  natural(ux_imag) = 0
       ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! left
    line to close

region '(4) THIN SPACE CHARGE - CENTER ELECTRODE LEFT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated
```

```
        start (x3, -Height/2)  natural(ux_real) = 0  natural(ux_imag) = 0
           ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! bottom
        line to (x4, -Height/2) natural(ux_real) = 0  natural(ux_imag) = 0
           ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! right
        line to (x4, Height/2)  natural(ux_real) = 0  natural(ux_imag) = 0
           ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! top
        line to (x3, Height/2)  natural(ux_real) = 0  natural(ux_imag) = 0
           ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! left
        line to close

    region '(5) negative electrode'
        mass_density = mass_density_gold Young_modulus = Young_modulus_gold
           ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
           ↪ permittivity_relative_gold
        viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
           ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
           ↪ N.s/m^2 }
        force_density_DC_x = 0

        start (x4, -Height/2)  natural(ux_real) = 0  natural(ux_imag) = 0
           ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! bottom
        line to (x6, -Height/2) natural(ux_real) = 0  natural(ux_imag) = 0
           ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! right
        line to (x6, Height/2)  natural(ux_real) = 0  natural(ux_imag) = 0
           ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! top
        line to (x4, Height/2)  natural(ux_real) = 0  natural(ux_imag) = 0
           ↪ natural(uy_real) = 0  natural(uy_imag) = 0 ! left
        line to close

    !feature 'Signal line'
        !start(x0, -Height/2) value(ux) = complex(ux_signal, 0) line to (x0,
           ↪ Height/2)

{
MONITORS
        grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy))
        contour(real(ux)) as "X-displacement(m)" contour(real(ux)) as "X-
           ↪ displacement(m)" painted
        elevation(real(ux), real(uy)) from(x0, 0) to (x9 + (nUnit-1)*L, 0
```

```
}

PLOTS
    grid(x + zoom_factor*ux_real, y + zoom_factor*uy_real) PNG(1024, 2)
        !report(probeline_end_x1, probeline_end_x2)
        !report(V_dc, V_signal, f, L, Height)

    contour(ux_real) PNG(1024, 2) as "X-displacement(m)" report(V_dc, V_signal,
        ↪  f)
        !vtk(ux_real)
    contour(ux_real) PNG(1024, 2) as "X-displacement(m)" painted report(V_dc,
        ↪ V_signal, f)
        !vtk(ux_real)

    contour(uy_real) PNG(1024, 2) as "Y-displacement(m)" report(V_dc, V_signal,
        ↪  f)
        !vtk(uy_real)
    contour(uy_real) PNG(1024, 2) as "Y-displacement(m)" painted report(V_dc,
        ↪ V_signal, f)
        !vtk(uy_real)

    surface(force_density_DC_x) PNG(1024, 2) painted report(V_dc, V_signal, f)
        !vtk(force_density_DC_x)

    elevation(ux_real) from(x0, 0) to (x6, 0) PNG(1024, 2)
        !export FILE = 'ux_real.txt' FORMAT"#X #1"
        report(V_dc, V_signal, f)

    elevation(uy_real) from(x0, 0) to (x6, 0) PNG(1024, 2) !fixed range (- 1e
        ↪ -19, 1e-19 )
        !export FILE = 'uy_real.txt' FORMAT"#X #1"
        report(V_dc, V_signal, f)

    elevation(ux_real) from((electrode_width/2 + silicon_width/2), -Height/2)
        ↪ to ((electrode_width/2 + silicon_width/2), Height/2) PNG(1024, 2)
        !export FILE = 'ux_real_probe_line.txt' FORMAT"#Y #1"
        report(V_dc, V_signal, f)
```

```
    elevation(uy_real) from((electrode_width/2 + silicon_width/2), -Height/2)
       ↪ to ((electrode_width/2 + silicon_width/2), Height/2) PNG(1024, 2)
       !export FILE = 'uy_real_probe_line.txt' FORMAT"#Y #1"
       report(V_dc, V_signal, f)

HISTORIES
       history(globalmax(abs(ux_real))) PNG(1024, 2) report(V_dc, V_signal, f)
       history(globalmax(abs(ux_real))) versus f PNG(1024, 2) report(V_dc,
          ↪ V_signal)
       history( log10(globalmax(abs(ux_real))) ) PNG(1024, 2) versus log10(f)
          ↪ report(V_dc, V_signal)

       history(globalmax(abs(uy_real))) PNG(1024, 2) report(V_dc, V_signal, f)
       history(globalmax(abs(uy_real))) versus f PNG(1024, 2) report(V_dc,
          ↪ V_signal)
       history( log10(globalmax(abs(uy_real))) ) PNG(1024, 2) versus log10(f)
          ↪ report(V_dc, V_signal)

END
```

Appendix H

H FlexPDE Code Listing for Response of One Unit of Piezoslab

```
{ One Unit
1.) Stage frequency
2.) DC voltage ON
3.) AC signal ON
4.) Periodic condition OFF
5.) We expect the phase snapshot in mechanical displacement
    due to AC perturbation coupled with DC voltage at a given frequency.
}

TITLE 'Helmholtz 2D (1 Unit)'
SELECT errlim = 1e-4 order = 1 threads = 4 stages = 15 regrid = off
   ↪ featureplot = on !ngrid = 20
VARIABLES ux = complex(ux_real, ux_imag) uy = complex(uy_real, uy_imag)
DEFINITIONS
   zoom_factor = 4e3 df = 1e6 f = 2.3736e9 + df * (stage-1)
   !zoom_factor = 4e3 f = 2.3806e9
   omega = 2*pi*f { rad-Hz }
   !In MATLAB, type, stage = 15; fc = 2.3806e9; df = 1e7; fl = fc - df *(stage
      ↪ -1)/2

   mass_density Young_modulus Poisson_ratio
   { Poisson ratio = 0.5 means volumetric strain is zero => material is
      ↪ incompressible, e.g., water
```

```
  Poisson ratio = 0 means no change in lateral strain when we apply tensile
     ↪ /compressive force, e.g., cork
  Strain_z = Strain_y in uniaxial stress
  Strain_y is proportional to Strain_x
  Strain_z is proportional to Strain_x
  This proportionality is called Poisson ratio.
}

permittivity_relative permittivity_0 = 8.8542e-12 { free space permittivity
    ↪  F/m } permittivity = permittivity_relative * permittivity_0
force_density_DC_x

{ ===== Plane Strain ===== }
C = Young_modulus / ( (1 + Poisson_ratio ) * ( 1 - 2*Poisson_ratio ) )
G = Young_modulus / ( 2*(1 + Poisson_ratio ) )

viscosity_11 viscosity_12 viscosity_44

ex1 = dx(ux) ! Strain
ey1 = dy(uy) ! Strain
exy1 = dx(uy) + dy(ux) ! Strain

! Excluding damping
{
sx1 = C*(ex1 - Poisson_ratio*ex1 + Poisson_ratio*ey1) ! Plane stress
sy1 = C*(ey1 - Poisson_ratio*ey1 + Poisson_ratio*ex1) ! Plane stress
sxy1 = G*exy1 ! Plane stress
}

! Including damping

sx1 = (C*(1 - Poisson_ratio) + complex(0,1)*omega*viscosity_11)*ex1 + (C*
    ↪ Poisson_ratio + complex(0,1)*omega*viscosity_12)*ey1 ! Plane stress
sy1 = (C*(1 - Poisson_ratio) + complex(0,1)*omega*viscosity_11)*ey1 + (C*
    ↪ Poisson_ratio + complex(0,1)*omega*viscosity_12)*ex1 ! Plane stress
sxy1 = (G + complex(0,1)*omega*viscosity_44)*exy1 ! Plane stress


{ ===== Silicon ===== }
```

```
mass_density_silicon = 2329 { kg/m3 } Young_modulus_silicon = 169e9 { N/m2
    ↪ } Poisson_ratio_silicon = 0.27 { along [100] direction }
permittivity_relative_silicon = 11.7
{ http://www.ioffe.ru/SVA/NSM/Semicond/Si/mechanic.html#Elastic }

viscosity_11_silicon = 0.005 { N.s/m^2 }
viscosity_12_silicon = 0.005 { N.s/m^2 }
viscosity_44_silicon = 0.005 { N.s/m^2 }
{ B.A. Auld, vol 1, chapter 3, a representative value }

{ ===== Gold ===== }
mass_density_gold = 19300 { kg/m3 } Young_modulus_gold = 79e9 { N/m2 }
    ↪ Poisson_ratio_gold = 0.42
permittivity_relative_gold = 1 { At low and RF frequencies }
{ https://www.azom.com/properties.aspx?ArticleID=598 }
{ http://chemistry.elmhurst.edu/vchembook/125Adensitygold.html }

viscosity_11_gold = 0.0186 { N.s/m^2 }
viscosity_12_gold = 0.0186 { N.s/m^2 }
viscosity_44_gold = 0.0186 { N.s/m^2 }

{ ===== Structure ===== }
!The charge width is tucked inside the electrode width.
silicon_width = 2e-6 {m} electrode_width = 200e-9 {m} charge_width = 50e-9
    ↪ {m}
L = 3*electrode_width + 2*silicon_width { m } Height = 10e-6 {m}

! [electrode][Space-charge region][silicon][Space-charge region][electrode
    ↪ ][Space-charge region][silicon][Space-charge region][electrode]
x0 = -(electrode_width + silicon_width + electrode_width/2)
x1 = -(charge_width + silicon_width + electrode_width/2)
x2 = -(silicon_width + electrode_width/2)
x3 = -(electrode_width/2)
x4 = -(electrode_width/2 - charge_width)
x5 = electrode_width/2 - charge_width
x6 = electrode_width/2
x7 = electrode_width/2 + silicon_width
x8 = electrode_width/2 + silicon_width + charge_width
x9 = electrode_width/2 + silicon_width + electrode_width
```

```
{ ===== Voltage ===== }
V_dc = 10 { volts }
V_signal = 0.1 { volts }

{ ===== Artificial piezo-material ===== }
width_artificial_piezo = silicon_width
permittivity_artificial_piezo = permittivity_relative_silicon *
    ↪ permittivity_0
mass_density_artificial_piezo = mass_density_silicon
    ↪ Young_modulus_artificial_piezo = Young_modulus_silicon
    ↪ Poisson_ratio_artificial_piezo = Poisson_ratio_silicon
C_artificial_piezo = Young_modulus_artificial_piezo/( (1 +
    ↪ Poisson_ratio_artificial_piezo) * (1 - 2*
    ↪ Poisson_ratio_artificial_piezo) )
c11_artificial_piezo = C_artificial_piezo * (1 -
    ↪ Poisson_ratio_artificial_piezo)

{ ===== Force density due to DC bias ===== }
force_density_DC_x_calculated = permittivity_artificial_piezo * V_dc^2/(2*
    ↪ width_artificial_piezo^2) * ( 1 + permittivity_artificial_piezo *
    ↪ V_dc^2/( c11_artificial_piezo * width_artificial_piezo^2) ) { N/m^3 }

!force_density_DC_x_calculated = 1e4 { N/m3 }
!force_density_DC_x_calculated = 1e4 + 3e3*(stage-1) { N/m3 }

{ ===== Change in length due to AC signal perturbation ===== }
DeltaL = -0.5 * permittivity_artificial_piezo * V_dc^2/(
    ↪ c11_artificial_piezo*width_artificial_piezo)
Leq = width_artificial_piezo + DeltaL
k = omega * sqrt(mass_density_artificial_piezo/c11_artificial_piezo)
alpha = tan(k*Leq/2)
beta = alpha*(cos(k*Leq) - 1) - sin(k*Leq)
force_DC_per_unit_area = -0.5 * permittivity_artificial_piezo * V_dc^2 /
    ↪ (2*width_artificial_piezo^2) * (1 - 2*DeltaL/width_artificial_piezo)
ux_signal = 2*beta/(c11_artificial_piezo*k) * force_DC_per_unit_area / ( 1
    ↪ + 2*beta/(c11_artificial_piezo*k) * (force_DC_per_unit_area/Leq) ) *
    ↪ V_signal/V_dc
```

```
    !ux_signal = 0
    !ux_signal = 1e-14 { m }

    { ===== Energy density ===== }
    ![energy_density] = N/m^2
    energy_density = 0.5*(sx1*ex1 + sy1*ey1 + sxy1*exy1)

EQUATIONS
    ux: dx(sx1) + dy(sxy1) + mass_density*omega^2*ux + force_density_DC_x = 0
    uy: dx(sxy1) + dy(sy1) + mass_density*omega^2*uy = 0

BOUNDARIES
    region '(1) positive electrode left'
        mass_density = mass_density_gold Young_modulus = Young_modulus_gold
            ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
            ↪ permittivity_relative_gold
        viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
            ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
            ↪ N.s/m^2 }
        force_density_DC_x = 0

        start (x0, -Height/2)    natural(ux) = complex(0, 0)    natural(uy) =
            ↪ complex(0, 0) ! bottom
        line to (x1, -Height/2)   natural(ux) = complex(0, 0)    natural(uy) =
            ↪ complex(0, 0) ! right
        line to (x1, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
            ↪ complex(0, 0) ! top
        line to (x0, Height/2)   value(ux) = complex(ux_signal, 0) natural(uy)
            ↪ = complex(0, 0) ! left
        line to close

    region '(2) THIN SPACE CHARGE - LEFT ELECTRODE'
        mass_density = mass_density_gold Young_modulus = Young_modulus_gold
            ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
            ↪ permittivity_relative_gold
        viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
            ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
            ↪ N.s/m^2 }
        force_density_DC_x = force_density_DC_x_calculated
```

```
    start (x1, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x2, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x2, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
        line to (x1, Height/2)   natural(ux) = complex(0, 0)  natural(uy
           ↪ ) = complex(0, 0) ! left
    line to close

region '(3) silicon left'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
       ↪ viscosity_44_silicon { N.s/m^2 }
    force_density_DC_x = 0

    start (x2, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x3, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x3, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x2, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(4) THIN SPACE CHARGE - CENTER ELECTRODE LEFT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated
```

```
    start (x3, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x4, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x4, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x3, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(5) negative electrode'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0

    start (x4, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x5, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x5, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x4, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(6) THIN SPACE CHARGE - CENTER ELECTRODE RIGHT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = force_density_DC_x_calculated
```

```
    start (x5, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x6, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x6, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x5, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(7) silicon right'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
       ↪ viscosity_44_silicon { N.s/m^2 }
    force_density_DC_x = 0

    start (x6, -Height/2)    natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x7, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x7, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x6, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(8) THIN SPACE CHARGE - RIGHT ELECTRODE'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated
```

```
      start (x7, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
         ↪ complex(0, 0) ! bottom
      line to (x8, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
         ↪ complex(0, 0) ! right
      line to (x8, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
         ↪ complex(0, 0) ! top
      line to (x7, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
         ↪ complex(0, 0) ! left
      line to close

   region '(9) positive electrode right'
      mass_density = mass_density_gold Young_modulus = Young_modulus_gold
         ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
         ↪ permittivity_relative_gold
      viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
         ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
         ↪ N.s/m^2 }
      force_density_DC_x = 0

      start (x8, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
         ↪ complex(0, 0) ! bottom
      line to (x9, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
         ↪ complex(0, 0) ! right
          !periodic (x - L, y) map(ux) = -1 map(uy) = -1
      line to (x9, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
         ↪ complex(0, 0) ! top
      line to (x8, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
         ↪ complex(0, 0) ! left
      line to close

   !feature 'Signal line'
      !start(x0, -Height/2) value(ux) = complex(ux_signal, 0) line to (x0,
         ↪ Height/2)

{
MONITORS
      grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy))
      contour(real(ux)) as "X-displacement(m)" contour(real(ux)) as "X-
         ↪ displacement(m)" painted
```

```
        elevation(real(ux), real(uy)) from(x0, height/2) to (x6, height/2)
}

PLOTS
    grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy)) PNG
        report(V_dc, V_signal, f, Height)

    contour(real(ux)) PNG as "X-displacement(m)" report(V_dc, V_signal, f)
        !vtk(real(ux))
    contour(real(ux)) PNG as "X-displacement(m)" painted report(V_dc, V_signal,
        ↪ f)
        !vtk(real(ux))

    contour(real(uy)) PNG as "Y-displacement(m)" report(V_dc, V_signal, f)
        !vtk(real(uy))
    contour(real(uy)) PNG as "Y-displacement(m)" painted report(V_dc, V_signal,
        ↪ f)
        !vtk(real(uy))

    surface(force_density_DC_x) PNG painted report(V_dc, V_signal, f)
        !vtk(force_density_DC_x)

    elevation(real(ux)) from(x0, 0) to (x9, 0) PNG(1024, 2)
        !export FILE = 'ux_real.txt' FORMAT"#X #1"
        report(V_dc, V_signal, f)

    elevation(real(uy)) from(x0, 0) to (x9, 0) PNG(1024, 2) !fixed range (- 1e
        ↪ -19, 1e-19 )
        !export FILE = 'uy_real.txt' FORMAT"#X #1"
        report(V_dc, V_signal, f)

    elevation(real(ux)) from(silicon_width/2, -Height/2) to (silicon_width/2,
        ↪ Height/2) PNG(1024, 2)
        !export FILE = 'ux_real_probe_line.txt' FORMAT"#Y #1"
        report(V_dc, V_signal, f)

    elevation(real(uy)) from(silicon_width/2, -Height/2) to (silicon_width/2,
        ↪ Height/2) PNG(1024, 2)
        !export FILE = 'uy_real_probe_line.txt' FORMAT"#Y #1"
```

```
       report(V_dc, V_signal, f)

HISTORIES
       history(globalmax(abs(real(ux)))) PNG(1024, 2) report(V_dc, V_signal, f
          ↪ )
       history(globalmax(abs(real(ux)))) versus f PNG(1024, 2) report(V_dc,
          ↪ V_signal)
       history( log10(globalmax(abs(real(ux)))) ) PNG(1024, 2) versus log10(f)
          ↪  report(V_dc, V_signal)

       history(globalmax(abs(real(uy)))) PNG(1024, 2) report(V_dc, V_signal, f
          ↪ )
       history(globalmax(abs(real(uy)))) versus f PNG(1024, 2) report(V_dc,
          ↪ V_signal)
       history( log10(globalmax(abs(real(uy)))) ) PNG(1024, 2) versus log10(f)
          ↪  report(V_dc, V_signal)

END
```

Appendix I

FlexPDE Code Listing for Response of Many Units of Piezoslab

```
{ Many units
1.) Stage frequency
2.) DC voltage ON
3.) AC signal ON
4.) Periodic condition OFF
5.) Specify the number of units via nUnit
6.) We expect the phase snapshot in mechanical displacement
     due to AC perturbation coupled with DC voltage at a given frequency.
}

TITLE 'Helmholtz 2D (n Units)'
SELECT errlim = 1e-4 order = 2 threads = 4 stages = 15 regrid = off
   ↪ featureplot = on !ngrid = 20
VARIABLES ux = complex(ux_real, ux_imag) uy = complex(uy_real, uy_imag)
DEFINITIONS
    nUnit = 3

    zoom_factor = 5e3 df = 1e6 f = 2.3736e9 + df * (stage-1)
    !zoom_factor = 5e3 f = 2.3806e9
    omega = 2*pi*f { rad-Hz }
    !In MATLAB, type, stage = 15; fc = 2.3806e9; df = 1e7; fl = fc - df *(stage
       ↪ -1)/2
```

```
mass_density Young_modulus Poisson_ratio
{ Poisson ratio = 0.5 means volumetric strain is zero => material is
   ↪ incompressible, e.g., water
  Poisson ratio = 0 means no change in lateral strain when we apply tensile
     ↪ /compressive force, e.g., cork
  Strain_z = Strain_y in uniaxial stress
  Strain_y is proportional to Strain_x
  Strain_z is proportional to Strain_x
  This proportionality is called Poisson ratio.
}

permittivity_relative permittivity_0 = 8.8542e-12 { free space permittivity
   ↪  F/m } permittivity = permittivity_relative * permittivity_0
force_density_DC_x

{ ===== Plane Strain ===== }
C = Young_modulus / ( (1 + Poisson_ratio ) * ( 1 - 2*Poisson_ratio ) )
G = Young_modulus / ( 2*(1 + Poisson_ratio ) )

viscosity_11 viscosity_12 viscosity_44

ex1 = dx(ux) ! Strain
ey1 = dy(uy) ! Strain
exy1 = dx(uy) + dy(ux) ! Strain

! Excluding damping

sx1 = C*(ex1 - Poisson_ratio*ex1 + Poisson_ratio*ey1) ! Plane stress
sy1 = C*(ey1 - Poisson_ratio*ey1 + Poisson_ratio*ex1) ! Plane stress
sxy1 = G*exy1 ! Plane stress


! Including damping
{
sx1 = (C*(1 - Poisson_ratio) + complex(0,1)*omega*viscosity_11)*ex1 + (C*
   ↪ Poisson_ratio + complex(0,1)*omega*viscosity_12)*ey1 ! Plane stress
sy1 = (C*(1 - Poisson_ratio) + complex(0,1)*omega*viscosity_11)*ey1 + (C*
   ↪ Poisson_ratio + complex(0,1)*omega*viscosity_12)*ex1 ! Plane stress
sxy1 = (G + complex(0,1)*omega*viscosity_44)*exy1 ! Plane stress
```

```
}

{ ===== Silicon ===== }
mass_density_silicon = 2329 { kg/m3 } Young_modulus_silicon = 169e9 { N/m2
   ↪ } Poisson_ratio_silicon = 0.27 { along [100] direction }
permittivity_relative_silicon = 11.7
{ http://www.ioffe.ru/SVA/NSM/Semicond/Si/mechanic.html#Elastic }

viscosity_11_silicon = 0.005 { N.s/m^2 }
viscosity_12_silicon = 0.005 { N.s/m^2 }
viscosity_44_silicon = 0.005 { N.s/m^2 }
{ B.A. Auld, vol 1, chapter 3, a representative value }

{ ===== Gold ===== }
mass_density_gold = 19300 { kg/m3 } Young_modulus_gold = 79e9 { N/m2 }
   ↪ Poisson_ratio_gold = 0.42
permittivity_relative_gold = 1 { At low and RF frequencies }
{ https://www.azom.com/properties.aspx?ArticleID=598 }
{ http://chemistry.elmhurst.edu/vchembook/125Adensitygold.html }

viscosity_11_gold = 0.0186 { N.s/m^2 }
viscosity_12_gold = 0.0186 { N.s/m^2 }
viscosity_44_gold = 0.0186 { N.s/m^2 }

{ ===== Structure ===== }
!The charge width is tucked inside the electrode width.

silicon_width = 2e-6 {m} electrode_width = 200e-9 {m} charge_width = 50e-9
   ↪ {m}
L_repeat = 2*electrode_width + 2*silicon_width { m } Height = 10e-6 {m}
L = 3*electrode_width + 2*silicon_width { m }

L_shifter_last_unit = L + (nUnit-2)*L_repeat

![electrode][Space-charge region][silicon][Space-charge region][electrode][
   ↪ Space-charge region][silicon][Space-charge region][electrode]
x_centering = (L + (nUnit-1)*L_repeat)/2

! Starting unit
```

```
x0 = 0 - (x_centering)
x1 = electrode_width - charge_width - (x_centering)
x2 = electrode_width - (x_centering)
x3 = electrode_width + silicon_width - (x_centering)
x4 = electrode_width + silicon_width + charge_width - (x_centering)
x5 = electrode_width + silicon_width + electrode_width - charge_width - (
   ↪ x_centering)
x6 = 2*electrode_width + silicon_width - (x_centering)
x7 = 2*electrode_width + 2*silicon_width - (x_centering)
x8 = 2*electrode_width + 2*silicon_width + charge_width - (x_centering)
x9_recessed = 3*electrode_width + 2*silicon_width - charge_width - (
   ↪ x_centering)
x9 = 3*electrode_width + 2*silicon_width - (x_centering)

! Repeated units
x2r = 0 - (x_centering)
x3r = silicon_width - (x_centering)
x4r = silicon_width + charge_width - (x_centering)
x5r = silicon_width + electrode_width - charge_width - (x_centering)
x6r = silicon_width + electrode_width - (x_centering)
x7r = 2*silicon_width + electrode_width - (x_centering)
x8r = 2*silicon_width + electrode_width + charge_width - (x_centering)
x9r_recessed = 2*electrode_width + 2*silicon_width - charge_width - (
   ↪ x_centering)
x9r = 2*electrode_width + 2*silicon_width - (x_centering)

! Last unit
x2l = 0 - (x_centering)
x3l = silicon_width - (x_centering)
x4l = silicon_width + charge_width - (x_centering)
x5l = silicon_width + electrode_width - charge_width - (x_centering)
x6l = electrode_width + silicon_width - (x_centering)
x7l = electrode_width + 2*silicon_width - (x_centering)
x8l = electrode_width + 2*silicon_width + charge_width - (x_centering)
x9l = 2*electrode_width + 2*silicon_width - (x_centering)

!probeline_end_x1 = x9 + (nUnit-2)*L_repeat + x9l - (x_centering)
!probeline_end_x2 = x9 + (nUnit-2)*L_repeat + (silicon_width +
   ↪ electrode_width + silicon_width/2) - (x_centering)
```

```
probeline_end_x1 = x_centering
probeline_end_x2 = x_centering - (electrode_width + silicon_width/2)

{ ===== Voltage ===== }
V_dc = 10 { volts }
V_signal = 0.1 { volts }

{ ===== Artificial piezo-material ===== }
width_artificial_piezo = silicon_width
permittivity_artificial_piezo = permittivity_relative_silicon *
   ↪ permittivity_0
mass_density_artificial_piezo = mass_density_silicon
   ↪ Young_modulus_artificial_piezo = Young_modulus_silicon
   ↪ Poisson_ratio_artificial_piezo = Poisson_ratio_silicon
C_artificial_piezo = Young_modulus_artificial_piezo/( (1 +
   ↪ Poisson_ratio_artificial_piezo) * (1 - 2*
   ↪ Poisson_ratio_artificial_piezo) )
c11_artificial_piezo = C_artificial_piezo * (1 -
   ↪ Poisson_ratio_artificial_piezo)

{ ===== Force density due to DC bias ===== }
force_density_DC_x_calculated = permittivity_artificial_piezo * V_dc^2/(2*
   ↪ width_artificial_piezo^2) * ( 1 + permittivity_artificial_piezo *
   ↪ V_dc^2/( c11_artificial_piezo * width_artificial_piezo^2) ) { N/m^3 }

!force_density_DC_x_calculated = 1e4 { N/m3 }
!force_density_DC_x_calculated = 1e4 + 3e3*(stage-1) { N/m3 }

{ ===== Change in length due to AC signal perturbation ===== }
DeltaL = -0.5 * permittivity_artificial_piezo * V_dc^2/(
   ↪ c11_artificial_piezo*width_artificial_piezo)
Leq = width_artificial_piezo + DeltaL
k = omega * sqrt(mass_density_artificial_piezo/c11_artificial_piezo)
alpha = tan(k*Leq/2)
beta = alpha*(cos(k*Leq) - 1) - sin(k*Leq)
force_DC_per_unit_area = -0.5 * permittivity_artificial_piezo * V_dc^2 /
   ↪ (2*width_artificial_piezo^2) * (1 - 2*DeltaL/width_artificial_piezo)
ux_signal = 2*beta/(c11_artificial_piezo*k) * force_DC_per_unit_area / ( 1
   ↪ + 2*beta/(c11_artificial_piezo*k) * (force_DC_per_unit_area/Leq) ) *
```

```
      ↪ V_signal/V_dc

   !ux_signal = 0
   !ux_signal = 1e-14 { m }

   { ===== Energy density ===== }
   ![energy_density] = N/m^2
   energy_density = 0.5*(sx1*ex1 + sy1*ey1 + sxy1*exy1)

EQUATIONS
   ux: dx(sx1) + dy(sxy1) + mass_density*omega^2*ux + force_density_DC_x = 0
   uy: dx(sxy1) + dy(sy1) + mass_density*omega^2*uy = 0

BOUNDARIES
! =====Initial Unit=====
   region '(1) positive electrode left'
      mass_density = mass_density_gold Young_modulus = Young_modulus_gold
         ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
         ↪ permittivity_relative_gold
      viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
         ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
         ↪ N.s/m^2 }
      force_density_DC_x = 0

      start (x0, -Height/2)    natural(ux) = complex(0, 0)     natural(uy) =
         ↪ complex(0, 0) ! bottom
      line to (x1, -Height/2)  natural(ux) = complex(0, 0)     natural(uy) =
         ↪ complex(0, 0) ! right
      line to (x1, Height/2)   natural(ux) = complex(0, 0)    natural(uy) =
         ↪ complex(0, 0) ! top
      line to (x0, Height/2)   value(ux) = complex(ux_signal, 0) natural(uy)
         ↪ = complex(0, 0) ! left
      line to close

   region '(2) THIN SPACE CHARGE - LEFT ELECTRODE'
      mass_density = mass_density_gold Young_modulus = Young_modulus_gold
         ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
         ↪ permittivity_relative_gold
```

```
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = force_density_DC_x_calculated

    start (x1, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x2, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x2, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
        line to (x1, Height/2)   natural(ux) = complex(0, 0)  natural(uy
           ↪ ) = complex(0, 0) ! left
    line to close

region '(3) silicon left'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
       ↪ viscosity_44_silicon { N.s/m^2 }
    force_density_DC_x = 0

    start (x2, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x3, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x3, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x2, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(4) THIN SPACE CHARGE - CENTER ELECTRODE LEFT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
```

```
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x3, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x4, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x4, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x3, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(5) negative electrode'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0

    start (x4, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x5, -Height/2)  natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x5, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x4, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(6) THIN SPACE CHARGE - CENTER ELECTRODE RIGHT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
```

```
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = force_density_DC_x_calculated

    start (x5, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x6, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x6, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x5, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(7) silicon right'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
       ↪ viscosity_44_silicon { N.s/m^2 }
    force_density_DC_x = 0

    start (x6, -Height/2)    natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x7, -Height/2)  natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x7, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x6, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(8) THIN SPACE CHARGE - LEFT ELECTRODE'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
```

```
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x7, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x8, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x8, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x7, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(9) positive electrode right'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0

    start (x8, -Height/2)      natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x9_recessed, -Height/2) natural(ux) = complex(0, 0)   natural(
       ↪ uy) = complex(0, 0) ! right
    line to (x9_recessed, Height/2)   natural(ux) = complex(0, 0)   natural
       ↪ (uy) = complex(0, 0) ! top
    line to (x8, Height/2)      natural(ux) = complex(0, 0)   natural(uy) =
       ↪  complex(0, 0) ! left
    line to close

region '(10) THIN SPACE CHARGE - RIGHT ELECTRODE'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
```

```
        viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
            ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
            ↪ N.s/m^2 }
        force_density_DC_x = force_density_DC_x_calculated

        start (x9_recessed, -Height/2)   natural(ux) = complex(0, 0)   natural(
            ↪ uy) = complex(0, 0) ! bottom
        line to (x9, -Height/2)     natural(ux) = complex(0, 0)   natural(uy) =
            ↪  complex(0, 0) ! right
        line to (x9, Height/2)      natural(ux) = complex(0, 0)   natural(uy) =
            ↪  complex(0, 0) ! top
        line to (x9_recessed, Height/2)   natural(ux) = complex(0, 0)   natural
            ↪ (uy) = complex(0, 0) ! left
        line to close

! =====Repeating Units=====
    repeat i = 1 to (nUnit-2)

    region '(3r) silicon left'
        mass_density = mass_density_silicon Young_modulus =
            ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
            ↪ permittivity_relative = permittivity_relative_silicon
        viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
            ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
            ↪ viscosity_44_silicon { N.s/m^2 }
        force_density_DC_x = 0

        start (x2r + L + (i-1)*L_repeat, -Height/2)    natural(ux) = complex(0,
            ↪  0)   natural(uy) = complex(0, 0) ! bottom
        line to (x3r + L + (i-1)*L_repeat, -Height/2)   natural(ux) = complex
            ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
        line to (x3r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
            ↪  0)   natural(uy) = complex(0, 0) ! top
        line to (x2r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
            ↪  0)   natural(uy) = complex(0, 0) ! left
        line to close

    region '(4r) THIN SPACE CHARGE - CENTER ELECTRODE LEFT'
```

```
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x3r + L + (i-1)*L_repeat, -Height/2)   natural(ux) = complex(0,
       ↪ 0)   natural(uy) = complex(0, 0) ! bottom
    line to (x4r + L + (i-1)*L_repeat, -Height/2)  natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! right
    line to (x4r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! top
    line to (x3r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(5r) negative electrode'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0

    start (x4r + L + (i-1)*L_repeat, -Height/2)    natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! bottom
    line to (x5r + L + (i-1)*L_repeat, -Height/2)  natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! right
    line to (x5r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! top
    line to (x4r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! left
    line to close

region '(6r) THIN SPACE CHARGE - CENTER ELECTRODE RIGHT'
```

```
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = force_density_DC_x_calculated

    start (x5r + L + (i-1)*L_repeat, -Height/2)    natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! bottom
    line to (x6r + L + (i-1)*L_repeat, -Height/2)  natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! right
    line to (x6r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! top
    line to (x5r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! left
    line to close

region '(7r) silicon right'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
       ↪ viscosity_44_silicon { N.s/m^2 }
    force_density_DC_x = 0

    start (x6r + L + (i-1)*L_repeat, -Height/2)    natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! bottom
    line to (x7r + L + (i-1)*L_repeat, -Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x7r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! top
    line to (x6r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(8r) THIN SPACE CHARGE - RIGHT ELECTRODE'
```

```
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x7r + L + (i-1)*L_repeat, -Height/2)    natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! bottom
    line to (x8r + L + (i-1)*L_repeat, -Height/2)  natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! right
    line to (x8r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! top
    line to (x7r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(9r) positive electrode right'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0

    start (x8r + L + (i-1)*L_repeat, -Height/2)      natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! bottom
    line to (x9r_recessed + L + (i-1)*L_repeat, -Height/2) natural(ux) =
       ↪ complex(0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x9r_recessed + L + (i-1)*L_repeat, Height/2)  natural(ux) =
       ↪ complex(0, 0)   natural(uy) = complex(0, 0) ! top
    line to (x8r + L + (i-1)*L_repeat, Height/2)      natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(10r) positive electrode right'
```

```
      mass_density = mass_density_gold Young_modulus = Young_modulus_gold
         ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
         ↪ permittivity_relative_gold
      viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
         ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
         ↪ N.s/m^2 }
      force_density_DC_x = force_density_DC_x_calculated

      start (x9r_recessed + L + (i-1)*L_repeat, -Height/2)   natural(ux) =
         ↪ complex(0, 0)   natural(uy) = complex(0, 0) ! bottom
      line to (x9r + L + (i-1)*L_repeat, -Height/2)     natural(ux) = complex
         ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
      line to (x9r + L + (i-1)*L_repeat, Height/2)     natural(ux) = complex
         ↪ (0, 0)   natural(uy) = complex(0, 0) ! top
      line to (x9r_recessed + L + (i-1)*L_repeat, Height/2)   natural(ux) =
         ↪ complex(0, 0)   natural(uy) = complex(0, 0) ! left
      line to close

   endrepeat

! =====Last Unit=====
   region '(3e) silicon left'
      mass_density = mass_density_silicon Young_modulus =
         ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
         ↪ permittivity_relative = permittivity_relative_silicon
      viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
         ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
         ↪ viscosity_44_silicon { N.s/m^2 }
      force_density_DC_x = 0

      start (x2l + L_shifter_last_unit, -Height/2)    natural(ux) = complex
         ↪ (0, 0)   natural(uy) = complex(0, 0) ! bottom
      line to (x3l + L_shifter_last_unit, -Height/2)   natural(ux) = complex
         ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
      line to (x3l + L_shifter_last_unit, Height/2)   natural(ux) = complex
         ↪ (0, 0)   natural(uy) = complex(0, 0) ! top
      line to (x2l + L_shifter_last_unit, Height/2)   natural(ux) = complex
         ↪ (0, 0)   natural(uy) = complex(0, 0) ! left
      line to close
```

```
region '(4e) THIN SPACE CHARGE - CENTER ELECTRODE LEFT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x3l + L_shifter_last_unit, -Height/2)    natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! bottom
    line to (x4l + L_shifter_last_unit, -Height/2)  natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x4l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! top
    line to (x3l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(5e) negative electrode'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0

    start (x4l + L_shifter_last_unit, -Height/2)   natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! bottom
    line to (x5l + L_shifter_last_unit, -Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x5l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! top
    line to (x4l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! left
    line to close
```

```
region '(6e) THIN SPACE CHARGE - CENTER ELECTRODE RIGHT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = force_density_DC_x_calculated

    start (x5l + L_shifter_last_unit, -Height/2)   natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! bottom
    line to (x6l + L_shifter_last_unit, -Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x6l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! top
    line to (x5l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! left
    line to close

region '(7e) silicon right'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
       ↪ viscosity_44_silicon { N.s/m^2 }
    force_density_DC_x = 0

    start (x6l + L_shifter_last_unit, -Height/2)    natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! bottom
    line to (x7l + L_shifter_last_unit, -Height/2)  natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x7l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! top
    line to (x6l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(8e) THIN SPACE CHARGE - RIGHT ELECTRODE'
```

```
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x7l + L_shifter_last_unit, -Height/2)    natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! bottom
    line to (x8l + L_shifter_last_unit, -Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x8l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! top
    line to (x7l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(9e) positive electrode right'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0

    start (x8l + L_shifter_last_unit, -Height/2)   natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! bottom
    line to (x9l + L_shifter_last_unit, -Height/2)  natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x9l + L_shifter_last_unit, Height/2)  natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! top
    line to (x8l + L_shifter_last_unit, Height/2)  natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! left
    line to close

!feature 'Signal line'
```

```
       !start(x0, -Height/2) value(ux) = complex(ux_signal, 0) line to (x0,
          ↪ Height/2)

{
MONITORS
       grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy))
       contour(real(ux)) as "X-displacement(m)" contour(real(ux)) as "X-
          ↪ displacement(m)" painted
       elevation(real(ux), real(uy)) from(x0, 0) to (x9 + (nUnit-1)*L, 0
}

PLOTS
   grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy)) PNG(1024, 2)
       !report(probeline_end_x1, probeline_end_x2)
       !report(V_dc, V_signal, f, L, Height)

   contour(real(ux)) PNG(1024, 2) as "X-displacement(m)" report(V_dc, V_signal
      ↪ , f)
       !vtk(real(ux))
   contour(real(ux)) PNG(1024, 2) as "X-displacement(m)" painted report(V_dc,
      ↪ V_signal, f)
       !vtk(real(ux))

   contour(real(uy)) PNG(1024, 2) as "Y-displacement(m)" report(V_dc, V_signal
      ↪ , f)
       !vtk(real(uy))
   contour(real(uy)) PNG(1024, 2) as "Y-displacement(m)" painted report(V_dc,
      ↪ V_signal, f)
       !vtk(real(uy))

   surface(force_density_DC_x) PNG(1024, 2) painted report(V_dc, V_signal, f)
       !vtk(force_density_DC_x)

   elevation(real(ux)) from(x0, 0) to (probeline_end_x1, 0) PNG(1024, 2)
       !export FILE = 'ux_real.txt' FORMAT"#X #1"
       report(V_dc, V_signal, f)

   elevation(real(uy)) from(x0, 0) to (probeline_end_x1, 0) PNG(1024, 2) !
      ↪ fixed range (- 1e-19, 1e-19 )
```

```
        !export FILE = 'uy_real.txt' FORMAT"#X #1"
        report(V_dc, V_signal, f)

    elevation(real(ux)) from(probeline_end_x2, -Height/2) to (probeline_end_x2,
        ↪  Height/2) PNG(1024, 2)
        !export FILE = 'ux_real_probe_line.txt' FORMAT"#Y #1"
        report(V_dc, V_signal, f)

    elevation(real(uy)) from(probeline_end_x2, -Height/2) to (probeline_end_x2,
        ↪  Height/2) PNG(1024, 2)
        !export FILE = 'uy_real_probe_line.txt' FORMAT"#Y #1"
        report(V_dc, V_signal, f)

HISTORIES
        history(globalmax(abs(real(ux)))) PNG(1024, 2) report(V_dc, V_signal, f
            ↪ )
        history(globalmax(abs(real(ux)))) versus f PNG(1024, 2) report(V_dc,
            ↪ V_signal)
        history( log10(globalmax(abs(real(ux)))) ) PNG(1024, 2) versus log10(f)
            ↪  report(V_dc, V_signal)

        history(globalmax(abs(real(uy)))) PNG(1024, 2) report(V_dc, V_signal, f
            ↪ )
        history(globalmax(abs(real(uy)))) versus f PNG(1024, 2) report(V_dc,
            ↪ V_signal)
        history( log10(globalmax(abs(real(uy)))) ) PNG(1024, 2) versus log10(f)
            ↪  report(V_dc, V_signal)

END
```

Appendix J

J FlexPDE Code Listing for 2D Bar with Absorber

```
{
=====Bar with electrodes ending in absorber=====
1.) Stage frequency
2.) DC voltage ON
3.) AC signal ON
4.) Periodic condition OFF
5.) We expect the phase snapshot of mechanical displacement
    due to AC signal perturbation coupled with DC voltage at a given frequency
      ↪ .
    The wave propagates and we mimic the 50 ohm terminator as absorber.
}

TITLE 'Helmholtz 2D Bar'
SELECT errlim = 1e-4 order = 1 threads = 4 stages = 5 regrid = off featureplot
   ↪  = on vandenberg=on !ngrid = 20
VARIABLES ux = complex(ux_real, ux_imag) uy = complex(uy_real, uy_imag)
DEFINITIONS
    !zoom_factor = 2e3 df = 1e6 f = 2.3736e9 + df * (stage-1)
    zoom_factor = 1e2 f = 2.3806e9
    omega = 2*pi*f { rad-Hz }
    !In MATLAB, type, stage = 15; fc = 2.3806e9; df = 1e6; fl = fc - df *(stage
      ↪ -1)/2
```

```
mass_density Young_modulus Poisson_ratio
{ Poisson ratio = 0.5 means volumetric strain is zero => material is
   ↪ incompressible, e.g., water
  Poisson ratio = 0 means no change in lateral strain when we apply tensile
     ↪ /compressive force, e.g., cork
  Strain_z = Strain_y in uniaxial stress
  Strain_y is proportional to Strain_x
  Strain_z is proportional to Strain_x
  This proportionality is called Poisson ratio.
}

permittivity_relative permittivity_0 = 8.8542e-12 { free space permittivity
   ↪  F/m } permittivity = permittivity_relative * permittivity_0
force_density_DC_x

{ ===== Plane Strain ===== }
C = Young_modulus / ( (1 + Poisson_ratio ) * ( 1 - 2*Poisson_ratio ) )
G = Young_modulus / ( 2*(1 + Poisson_ratio ) )

viscosity_11 viscosity_12 viscosity_44

ex1 = dx(ux) ! Strain
ey1 = dy(uy) ! Strain
exy1 = dx(uy) + dy(ux) ! Strain

! Excluding damping
{
sx1 = C*(ex1 - Poisson_ratio*ex1 + Poisson_ratio*ey1) ! Plane stress
sy1 = C*(ey1 - Poisson_ratio*ey1 + Poisson_ratio*ex1) ! Plane stress
sxy1 = G*exy1 ! Plane stress
}

! Including damping

sx1 = (C*(1 - Poisson_ratio) + complex(0,1)*omega*viscosity_11)*ex1 + (C*
   ↪ Poisson_ratio + complex(0,1)*omega*viscosity_12)*ey1 ! Plane stress
sy1 = (C*(1 - Poisson_ratio) + complex(0,1)*omega*viscosity_11)*ey1 + (C*
   ↪ Poisson_ratio + complex(0,1)*omega*viscosity_12)*ex1 ! Plane stress
sxy1 = (G + complex(0,1)*omega*viscosity_44)*exy1 ! Plane stress
```

```
{ ===== Silicon ===== }
mass_density_silicon = 2329 { kg/m3 } Young_modulus_silicon = 169e9 { N/m2
    ↪ } Poisson_ratio_silicon = 0.27 { along [100] direction }
permittivity_relative_silicon = 11.7
{ http://www.ioffe.ru/SVA/NSM/Semicond/Si/mechanic.html#Elastic }

viscosity_11_silicon = 0.005 { N.s/m^2 }
viscosity_12_silicon = 0.005 { N.s/m^2 }
viscosity_44_silicon = 0.005 { N.s/m^2 }
{ B.A. Auld, vol 1, chapter 3, a representative value }

{ ===== Gold ===== }
mass_density_gold = 19300 { kg/m3 } Young_modulus_gold = 79e9 { N/m2 }
    ↪ Poisson_ratio_gold = 0.42
permittivity_relative_gold = 1 { At low and RF frequencies }
{ https://www.azom.com/properties.aspx?ArticleID=598 }
{ http://chemistry.elmhurst.edu/vchembook/125Adensitygold.html }

viscosity_11_gold = 0.0186 { N.s/m^2 }
viscosity_12_gold = 0.0186 { N.s/m^2 }
viscosity_44_gold = 0.0186 { N.s/m^2 }

{ ===== Absorber ===== }
mass_density_absorber = 19300 { kg/m3 } Young_modulus_absorber = 79e9 { N/
    ↪ m2 } Poisson_ratio_absorber = 0.42
permittivity_relative_absorber = 1

viscosity_11_absorber = 10 { N.s/m^2 }
viscosity_12_absorber = 10 { N.s/m^2 }
viscosity_44_absorber = 10 { N.s/m^2 }

{ ===== Structure ===== }
!The charge width is tucked inside the electrode width.
silicon_width = 2e-6 {m} electrode_width = 200e-9 {m} charge_width = 50e-9
    ↪ {m}
Height = 10e-6 {m}
```

```
L_absorber = 23e-6

! [electrode][Space-charge region][silicon][Space-charge region][electrode]
x0 = -(electrode_width + silicon_width/2)
x1 = -(charge_width + silicon_width/2)
x2 = -(silicon_width/2)
x3 = silicon_width/2
x4 = charge_width + silicon_width/2
x6 = electrode_width + silicon_width/2

{ ===== Voltage ===== }
V_dc = 10 { volts }
V_signal = 0.1 { volts }

{ ===== Artificial piezo-material ===== }
width_artificial_piezo = silicon_width
permittivity_artificial_piezo = permittivity_relative_silicon *
   ↪ permittivity_0
mass_density_artificial_piezo = mass_density_silicon
   ↪ Young_modulus_artificial_piezo = Young_modulus_silicon
   ↪ Poisson_ratio_artificial_piezo = Poisson_ratio_silicon
C_artificial_piezo = Young_modulus_artificial_piezo/( (1 +
   ↪ Poisson_ratio_artificial_piezo) * (1 - 2*
   ↪ Poisson_ratio_artificial_piezo) )
c11_artificial_piezo = C_artificial_piezo * (1 -
   ↪ Poisson_ratio_artificial_piezo)

{ ===== Force density due to DC bias ===== }
force_density_DC_x_calculated = permittivity_artificial_piezo * V_dc^2/(2*
   ↪ width_artificial_piezo^2) * ( 1 + permittivity_artificial_piezo *
   ↪ V_dc^2/( c11_artificial_piezo * width_artificial_piezo^2) ) { N/m^3 }

!force_density_DC_x_calculated = 1e4 { N/m3 }
!force_density_DC_x_calculated = 1e4 + 3e3*(stage-1) { N/m3 }

{ ===== Change in length due to AC signal perturbation ===== }
DeltaL = -0.5 * permittivity_artificial_piezo * V_dc^2/(
   ↪ c11_artificial_piezo*width_artificial_piezo)
Leq = width_artificial_piezo + DeltaL
```

```
    k = omega * sqrt(mass_density_artificial_piezo/c11_artificial_piezo)
    alpha = tan(k*Leq/2)
    beta = alpha*(cos(k*Leq) - 1) - sin(k*Leq)
    force_DC_per_unit_area = -0.5 * permittivity_artificial_piezo * V_dc^2 /
        ↪ (2*width_artificial_piezo^2) * (1 - 2*DeltaL/width_artificial_piezo)
    ux_signal = 2*beta/(c11_artificial_piezo*k) * force_DC_per_unit_area / ( 1
        ↪ + 2*beta/(c11_artificial_piezo*k) * (force_DC_per_unit_area/Leq) ) *
        ↪ V_signal/V_dc

    !ux_signal = 0
    !ux_signal = 1e-14 { m }

    { ===== Energy density ===== }
    ![energy_density] = N/m^2
    energy_density = 0.5*(sx1*ex1 + sy1*ey1 + sxy1*exy1)

EQUATIONS
    ux: dx(sx1) + dy(sxy1) + mass_density*omega^2*ux + force_density_DC_x = 0
    uy: dx(sxy1) + dy(sy1) + mass_density*omega^2*uy = 0

BOUNDARIES
    region '(1) positive electrode left'
        mass_density = mass_density_gold Young_modulus = Young_modulus_gold
            ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
            ↪ permittivity_relative_gold
        viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
            ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
            ↪ N.s/m^2 }
        force_density_DC_x = 0

        start (x0, -Height/2)   natural(ux) = complex(0, 0)     natural(uy) =
            ↪ complex(0, 0) ! bottom
        line to (x1, -Height/2)   natural(ux) = complex(0, 0)     natural(uy) =
            ↪  complex(0, 0) ! right
        line to (x1, Height/2)   natural(ux) = complex(0, 0)    natural(uy) =
            ↪ complex(0, 0) ! top
        line to (x0, Height/2)   value(ux) = complex(ux_signal, 0)  natural(uy)
            ↪  = complex(0, 0) ! left
        line to close
```

```
region '(2) THIN SPACE CHARGE - LEFT ELECTRODE'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = force_density_DC_x_calculated

    start (x1, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x2, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x2, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
           line to (x1, Height/2)   natural(ux) = complex(0, 0)  natural(uy
              ↪ ) = complex(0, 0) ! left
    line to close

region '(3) silicon'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
       ↪ viscosity_44_silicon { N.s/m^2 }
    force_density_DC_x = 0

    start (x2, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x3, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x3, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x2, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close
```

```
region '(4) THIN SPACE CHARGE - RIGHT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x3, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x4, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x4, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x3, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(5) negative electrode'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0

    start (x4, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x6, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x6, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x4, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(a) Absorber'
```

```
        mass_density = mass_density_absorber Young_modulus =
            ↪ Young_modulus_absorber Poisson_ratio = Poisson_ratio_absorber
            ↪ permittivity_relative = permittivity_relative_absorber
        viscosity_11 = viscosity_11_absorber { N.s/m^2 } viscosity_12 =
            ↪ viscosity_12_absorber { N.s/m^2 } viscosity_44 =
            ↪ viscosity_44_absorber { N.s/m^2 }
        force_density_DC_x = 0

        start (x6, -Height/2)          natural(ux) = complex(0, 0)   natural(uy)
            ↪  = complex(0, 0) ! bottom
        line to (x6 + L_absorber, -Height/2)  natural(ux_real) = omega*sqrt(
            ↪ mass_density)*ux_imag  natural(ux_imag) = -omega*sqrt(
            ↪ mass_density)*ux_real natural(uy) = complex(0, 0) ! right
        line to (x6 + L_absorber, Height/2)   natural(ux) = complex(0, 0)
            ↪ natural(uy) = complex(0, 0) ! top
        line to (x6, Height/2)         natural(ux) = complex(0, 0)   natural(uy)
            ↪  = complex(0, 0) ! left
        line to close

    !feature 'Signal line'
        !start(x0, -Height/2) value(ux) = complex(ux_signal, 0) line to (x0,
            ↪ Height/2)

{
MONITORS
        grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy))
        contour(real(ux)) as "X-displacement(m)" contour(real(ux)) as "X-
            ↪ displacement(m)" painted
        elevation(real(ux), real(uy)) from(x0, height/2) to (x6, height/2)
}

PLOTS
    grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy)) PNG(1024, 2)
        report(V_dc, V_signal, f, Height)

    contour(log10(abs(real(ux)))) PNG(1024, 2) as "X-displacement(m)" report(
        ↪ V_dc, V_signal, f)
        !vtk(real(ux))
```

```
    contour(log10(abs(real(ux)))) PNG(1024, 2) as "X-displacement(m)" painted
        ↪ report(V_dc, V_signal, f)
        !vtk(real(ux))

    contour(log10(abs(real(uy)))) PNG as "Y-displacement(m)" report(V_dc,
        ↪ V_signal, f)
        !vtk(real(uy))
    contour(log10(abs(real(uy)))) PNG as "Y-displacement(m)" painted report(
        ↪ V_dc, V_signal, f)
        !vtk(real(uy))

    surface(force_density_DC_x) PNG painted report(V_dc, V_signal, f)
        !vtk(force_density_DC_x)

    elevation(real(ux)) from(x0, 0) to (x6 + L_absorber, 0) PNG(1024, 2)
        !export FILE = 'ux_real.txt' FORMAT"#X #1"
        report(V_dc, V_signal, f)

    elevation(real(uy)) from(x0, 0) to (x6 + L_absorber, 0) PNG(1024, 2) !fixed
        ↪  range (- 1e-19, 1e-19 )
        !export FILE = 'uy_real.txt' FORMAT"#X #1"
        report(V_dc, V_signal, f)

    elevation(real(ux)) from((electrode_width/2 + silicon_width/2), -Height/2)
        ↪ to ((electrode_width/2 + silicon_width/2), Height/2) PNG(1024, 2)
        !export FILE = 'ux_real_probe_line.txt' FORMAT"#Y #1"
        report(V_dc, V_signal, f)

    elevation(real(uy)) from((electrode_width/2 + silicon_width/2), -Height/2)
        ↪ to ((electrode_width/2 + silicon_width/2), Height/2) PNG(1024, 2)
        !export FILE = 'uy_real_probe_line.txt' FORMAT"#Y #1"
        report(V_dc, V_signal, f)

END
```

Appendix K

FlexPDE Code Listing for n-Unit Cells of Lattice with Absorber

```
{ Many units with absorber to act as 50 ohm resistor
1.) Stage frequency
2.) DC voltage ON
3.) AC signal ON
4.) Periodic condition OFF
5.) Specify the number of units via nUnit
6.) We expect the phase snapshot in mechanical displacement
    due to AC perturbation coupled with DC voltage at a given frequency.
    We also expect no or little reflection as the wave travels from left to
       ↪ right.
    We implement infinity distance of Sommerfeld boundary condition.
}

TITLE 'Helmholtz 2D (with absorber)'
SELECT errlim = 1e-4 order = 1 threads = 4 stages = 10 regrid = off
   ↪ featureplot = on vandenberg=on !ngrid = 20
VARIABLES ux = complex(ux_real, ux_imag) uy = complex(uy_real, uy_imag)
DEFINITIONS
   nUnit = 3

   !zoom_factor = 5e3 df = 1e6 f = 2.3736e9 + df * (stage-1)
   zoom_factor = 5e1 f = 2.3806e9
   omega = 2*pi*f { rad-Hz }
```

```
!In MATLAB, type, stage = 15; fc = 2.3806e9; df = 1e7; fl = fc - df *(stage
   ↪ -1)/2

mass_density Young_modulus Poisson_ratio
{ Poisson ratio = 0.5 means volumetric strain is zero => material is
   ↪ incompressible, e.g., water
  Poisson ratio = 0 means no change in lateral strain when we apply tensile
     ↪ /compressive force, e.g., cork
  Strain_z = Strain_y in uniaxial stress
  Strain_y is proportional to Strain_x
  Strain_z is proportional to Strain_x
  This proportionality is called Poisson ratio.
}

permittivity_relative permittivity_0 = 8.8542e-12 { free space permittivity
   ↪  F/m } permittivity = permittivity_relative * permittivity_0
force_density_DC_x

{ ===== Plane Strain ===== }
C = Young_modulus / ( (1 + Poisson_ratio ) * ( 1 - 2*Poisson_ratio ) )
G = Young_modulus / ( 2*(1 + Poisson_ratio ) )

viscosity_11 viscosity_12 viscosity_44

ex1 = dx(ux) ! Strain
ey1 = dy(uy) ! Strain
exy1 = dx(uy) + dy(ux) ! Strain

! Excluding damping
{
sx1 = C*(ex1 - Poisson_ratio*ex1 + Poisson_ratio*ey1) ! Plane stress
sy1 = C*(ey1 - Poisson_ratio*ey1 + Poisson_ratio*ex1) ! Plane stress
sxy1 = G*exy1 ! Plane stress
}

! Including damping
sx1 = (C*(1 - Poisson_ratio) + complex(0,1)*omega*viscosity_11)*ex1 + (C*
   ↪ Poisson_ratio + complex(0,1)*omega*viscosity_12)*ey1 ! Plane stress
```

```
sy1 = (C*(1 - Poisson_ratio) + complex(0,1)*omega*viscosity_11)*ey1 + (C*
    ↪ Poisson_ratio + complex(0,1)*omega*viscosity_12)*ex1 ! Plane stress
sxy1 = (G + complex(0,1)*omega*viscosity_44)*exy1 ! Plane stress

{ ===== Silicon ===== }
mass_density_silicon = 2329 { kg/m3 } Young_modulus_silicon = 169e9 { N/m2
    ↪ } Poisson_ratio_silicon = 0.27 { along [100] direction }
permittivity_relative_silicon = 11.7
{ http://www.ioffe.ru/SVA/NSM/Semicond/Si/mechanic.html#Elastic }

viscosity_11_silicon = 0.005 { N.s/m^2 }
viscosity_12_silicon = 0.005 { N.s/m^2 }
viscosity_44_silicon = 0.005 { N.s/m^2 }
{ B.A. Auld, vol 1, chapter 3, a representative value }

{ ===== Gold ===== }
mass_density_gold = 19300 { kg/m3 } Young_modulus_gold = 79e9 { N/m2 }
    ↪ Poisson_ratio_gold = 0.42
permittivity_relative_gold = 1 { At low and RF frequencies }
{ https://www.azom.com/properties.aspx?ArticleID=598 }
{ http://chemistry.elmhurst.edu/vchembook/125Adensitygold.html }

viscosity_11_gold = 0.0186 { N.s/m^2 }
viscosity_12_gold = 0.0186 { N.s/m^2 }
viscosity_44_gold = 0.0186 { N.s/m^2 }

{ ===== Absorber ===== }
mass_density_absorber = 19300 { kg/m3 } Young_modulus_absorber = 79e9 { N/
    ↪ m2 } Poisson_ratio_absorber = 0.42
permittivity_relative_absorber = 1

viscosity_11_absorber = 10 { N.s/m^2 }
viscosity_12_absorber = 10 { N.s/m^2 }
viscosity_44_absorber = 10 { N.s/m^2 }

{ ===== Structure ===== }
!The charge width is tucked inside the electrode width.
```

```
silicon_width = 2e-6 {m} electrode_width = 200e-9 {m} charge_width = 50e-9
   ↪ {m}
L_repeat = 2*electrode_width + 2*silicon_width { m } Height = 10e-6 {m}
L = 3*electrode_width + 2*silicon_width { m }

L_shifter_last_unit = L + (nUnit-2)*L_repeat
L_absorber = 23e-6

![electrode][Space-charge region][silicon][Space-charge region][electrode][
   ↪ Space-charge region][silicon][Space-charge region][electrode]
x_centering = (L + (nUnit-1)*L_repeat)/2

! Starting unit
x0 = 0 - (x_centering)
x1 = electrode_width - charge_width - (x_centering)
x2 = electrode_width - (x_centering)
x3 = electrode_width + silicon_width - (x_centering)
x4 = electrode_width + silicon_width + charge_width - (x_centering)
x5 = electrode_width + silicon_width + electrode_width - charge_width - (
   ↪ x_centering)
x6 = 2*electrode_width + silicon_width - (x_centering)
x7 = 2*electrode_width + 2*silicon_width - (x_centering)
x8 = 2*electrode_width + 2*silicon_width + charge_width - (x_centering)
x9_recessed = 3*electrode_width + 2*silicon_width - charge_width - (
   ↪ x_centering)
x9 = 3*electrode_width + 2*silicon_width - (x_centering)

! Repeated units
x2r = 0 - (x_centering)
x3r = silicon_width - (x_centering)
x4r = silicon_width + charge_width - (x_centering)
x5r = silicon_width + electrode_width - charge_width - (x_centering)
x6r = silicon_width + electrode_width - (x_centering)
x7r = 2*silicon_width + electrode_width - (x_centering)
x8r = 2*silicon_width + electrode_width + charge_width - (x_centering)
x9r_recessed = 2*electrode_width + 2*silicon_width - charge_width - (
   ↪ x_centering)
x9r = 2*electrode_width + 2*silicon_width - (x_centering)
```

```
! Last unit
x2l = 0 - (x_centering)
x3l = silicon_width - (x_centering)
x4l = silicon_width + charge_width - (x_centering)
x5l = silicon_width + electrode_width - charge_width - (x_centering)
x6l = electrode_width + silicon_width - (x_centering)
x7l = electrode_width + 2*silicon_width - (x_centering)
x8l = electrode_width + 2*silicon_width + charge_width - (x_centering)
x9l = 2*electrode_width + 2*silicon_width - (x_centering)

probeline_end_x1 = x_centering
probeline_end_x2 = x_centering

{ ===== Voltage ===== }
V_dc = 10 { volts }
V_signal = 0.1 { volts }

{ ===== Artificial piezo-material ===== }
width_artificial_piezo = silicon_width
permittivity_artificial_piezo = permittivity_relative_silicon *
    ↪ permittivity_0
mass_density_artificial_piezo = mass_density_silicon
    ↪ Young_modulus_artificial_piezo = Young_modulus_silicon
    ↪ Poisson_ratio_artificial_piezo = Poisson_ratio_silicon
C_artificial_piezo = Young_modulus_artificial_piezo/( (1 +
    ↪ Poisson_ratio_artificial_piezo) * (1 - 2*
    ↪ Poisson_ratio_artificial_piezo) )
c11_artificial_piezo = C_artificial_piezo * (1 -
    ↪ Poisson_ratio_artificial_piezo)

{ ===== Force density due to DC bias ===== }
force_density_DC_x_calculated = permittivity_artificial_piezo * V_dc^2/(2*
    ↪ width_artificial_piezo^2) * ( 1 + permittivity_artificial_piezo *
    ↪ V_dc^2/( c11_artificial_piezo * width_artificial_piezo^2) ) { N/m^3 }

!force_density_DC_x_calculated = 1e4 { N/m3 }
!force_density_DC_x_calculated = 1e4 + 3e3*(stage-1) { N/m3 }

{ ===== Change in length due to AC signal perturbation ===== }
```

```
    DeltaL = -0.5 * permittivity_artificial_piezo * V_dc^2/(
        ↪ c11_artificial_piezo*width_artificial_piezo)
    Leq = width_artificial_piezo + DeltaL
    k = omega * sqrt(mass_density_artificial_piezo/c11_artificial_piezo)
    alpha = tan(k*Leq/2)
    beta = alpha*(cos(k*Leq) - 1) - sin(k*Leq)
    force_DC_per_unit_area = -0.5 * permittivity_artificial_piezo * V_dc^2 /
        ↪ (2*width_artificial_piezo^2) * (1 - 2*DeltaL/width_artificial_piezo)
    ux_signal = 2*beta/(c11_artificial_piezo*k) * force_DC_per_unit_area / ( 1
        ↪ + 2*beta/(c11_artificial_piezo*k) * (force_DC_per_unit_area/Leq) ) *
        ↪ V_signal/V_dc

    !ux_signal = 0
    !ux_signal = 1e-14 { m }

    { ===== Energy density ===== }
    ![energy_density] = N/m^2
    energy_density = 0.5*(sx1*ex1 + sy1*ey1 + sxy1*exy1)

EQUATIONS
    ux: dx(sx1) + dy(sxy1) + mass_density*omega^2*ux + force_density_DC_x = 0
    uy: dx(sxy1) + dy(sy1) + mass_density*omega^2*uy = 0

BOUNDARIES
! =====Initial Unit=====
    region '(1) positive electrode left'
        mass_density = mass_density_gold Young_modulus = Young_modulus_gold
            ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
            ↪ permittivity_relative_gold
        viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
            ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
            ↪ N.s/m^2 }
        force_density_DC_x = 0

        start (x0, -Height/2)    natural(ux) = complex(0, 0)     natural(uy) =
            ↪ complex(0, 0) ! bottom
        line to (x1, -Height/2)  natural(ux) = complex(0, 0)     natural(uy) =
            ↪ complex(0, 0) ! right
```

```
    line to (x1, Height/2)   natural(ux) = complex(0, 0)    natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x0, Height/2)   value(ux) = complex(ux_signal, 0) natural(uy)
       ↪ = complex(0, 0) ! left
    line to close

region '(2) THIN SPACE CHARGE - LEFT ELECTRODE'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = force_density_DC_x_calculated

    start (x1, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x2, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x2, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
           line to (x1, Height/2)   natural(ux) = complex(0, 0)  natural(uy
              ↪ ) = complex(0, 0) ! left
    line to close

region '(3) silicon left'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
       ↪ viscosity_44_silicon { N.s/m^2 }
    force_density_DC_x = 0

    start (x2, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x3, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
```

```
    line to (x3, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x2, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(4) THIN SPACE CHARGE - CENTER ELECTRODE LEFT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x3, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x4, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x4, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x3, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(5) negative electrode'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0

    start (x4, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x5, -Height/2)  natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
```

```
    line to (x5, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x4, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(6) THIN SPACE CHARGE - CENTER ELECTRODE RIGHT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = force_density_DC_x_calculated

    start (x5, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x6, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x6, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x5, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(7) silicon right'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
       ↪ viscosity_44_silicon { N.s/m^2 }
    force_density_DC_x = 0

    start (x6, -Height/2)    natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x7, -Height/2)  natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
```

```
    line to (x7, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x6, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(8) THIN SPACE CHARGE - LEFT ELECTRODE'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x7, -Height/2)    natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x8, -Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! right
    line to (x8, Height/2)   natural(ux) = complex(0, 0)  natural(uy) =
       ↪ complex(0, 0) ! top
    line to (x7, Height/2)   natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! left
    line to close

region '(9) positive electrode right'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0

    start (x8, -Height/2)      natural(ux) = complex(0, 0)   natural(uy) =
       ↪ complex(0, 0) ! bottom
    line to (x9_recessed, -Height/2) natural(ux) = complex(0, 0)   natural(
       ↪ uy) = complex(0, 0) ! right
```

```
      line to (x9_recessed, Height/2)   natural(ux) = complex(0, 0)   natural
         ↪ (uy) = complex(0, 0) ! top
      line to (x8, Height/2)      natural(ux) = complex(0, 0)   natural(uy) =
         ↪  complex(0, 0) ! left
      line to close

   region '(10) THIN SPACE CHARGE - RIGHT ELECTRODE'
      mass_density = mass_density_gold Young_modulus = Young_modulus_gold
         ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
         ↪ permittivity_relative_gold
      viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
         ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
         ↪ N.s/m^2 }
      force_density_DC_x = force_density_DC_x_calculated

      start (x9_recessed, -Height/2)   natural(ux) = complex(0, 0)   natural(
         ↪ uy) = complex(0, 0) ! bottom
      line to (x9, -Height/2)     natural(ux) = complex(0, 0)   natural(uy) =
         ↪  complex(0, 0) ! right
      line to (x9, Height/2)      natural(ux) = complex(0, 0)   natural(uy) =
         ↪  complex(0, 0) ! top
      line to (x9_recessed, Height/2)   natural(ux) = complex(0, 0)   natural
         ↪ (uy) = complex(0, 0) ! left
      line to close

! =====Repeating Units=====
   repeat i = 1 to (nUnit-2)

   region '(3r) silicon left'
      mass_density = mass_density_silicon Young_modulus =
         ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
         ↪ permittivity_relative = permittivity_relative_silicon
      viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
         ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
         ↪ viscosity_44_silicon { N.s/m^2 }
      force_density_DC_x = 0

      start (x2r + L + (i-1)*L_repeat, -Height/2)    natural(ux) = complex(0,
         ↪  0)   natural(uy) = complex(0, 0) ! bottom
```

```
    line to (x3r + L + (i-1)*L_repeat, -Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x3r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! top
    line to (x2r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(4r) THIN SPACE CHARGE - CENTER ELECTRODE LEFT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x3r + L + (i-1)*L_repeat, -Height/2)   natural(ux) = complex(0,
       ↪ 0)   natural(uy) = complex(0, 0) ! bottom
    line to (x4r + L + (i-1)*L_repeat, -Height/2)  natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! right
    line to (x4r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! top
    line to (x3r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(5r) negative electrode'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0

    start (x4r + L + (i-1)*L_repeat, -Height/2)    natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! bottom
```

```
    line to (x5r + L + (i-1)*L_repeat, -Height/2)  natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! right
    line to (x5r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! top
    line to (x4r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! left
    line to close

region '(6r) THIN SPACE CHARGE - CENTER ELECTRODE RIGHT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = force_density_DC_x_calculated

    start (x5r + L + (i-1)*L_repeat, -Height/2)    natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! bottom
    line to (x6r + L + (i-1)*L_repeat, -Height/2)  natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! right
    line to (x6r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! top
    line to (x5r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! left
    line to close

region '(7r) silicon right'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
       ↪ viscosity_44_silicon { N.s/m^2 }
    force_density_DC_x = 0

    start (x6r + L + (i-1)*L_repeat, -Height/2)    natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! bottom
```

```
    line to (x7r + L + (i-1)*L_repeat, -Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x7r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! top
    line to (x6r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(8r) THIN SPACE CHARGE - RIGHT ELECTRODE'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x7r + L + (i-1)*L_repeat, -Height/2)    natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! bottom
    line to (x8r + L + (i-1)*L_repeat, -Height/2)  natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! right
    line to (x8r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)  natural(uy) = complex(0, 0) ! top
    line to (x7r + L + (i-1)*L_repeat, Height/2)   natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(9r) positive electrode right'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0

    start (x8r + L + (i-1)*L_repeat, -Height/2)      natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! bottom
```

```
      line to (x9r_recessed + L + (i-1)*L_repeat, -Height/2) natural(ux) =
         ↪ complex(0, 0)   natural(uy) = complex(0, 0) ! right
      line to (x9r_recessed + L + (i-1)*L_repeat, Height/2)  natural(ux) =
         ↪ complex(0, 0)   natural(uy) = complex(0, 0) ! top
      line to (x8r + L + (i-1)*L_repeat, Height/2)      natural(ux) = complex
         ↪ (0, 0)   natural(uy) = complex(0, 0) ! left
      line to close

   region '(10r) positive electrode right'
      mass_density = mass_density_gold Young_modulus = Young_modulus_gold
         ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
         ↪ permittivity_relative_gold
      viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
         ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
         ↪ N.s/m^2 }
      force_density_DC_x = force_density_DC_x_calculated

      start (x9r_recessed + L + (i-1)*L_repeat, -Height/2)   natural(ux) =
         ↪ complex(0, 0)   natural(uy) = complex(0, 0) ! bottom
      line to (x9r + L + (i-1)*L_repeat, -Height/2)     natural(ux) = complex
         ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
      line to (x9r + L + (i-1)*L_repeat, Height/2)     natural(ux) = complex
         ↪ (0, 0)   natural(uy) = complex(0, 0) ! top
      line to (x9r_recessed + L + (i-1)*L_repeat, Height/2)   natural(ux) =
         ↪ complex(0, 0)   natural(uy) = complex(0, 0) ! left
      line to close

   endrepeat

! =====Last Unit=====
   region '(3e) silicon left'
      mass_density = mass_density_silicon Young_modulus =
         ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
         ↪ permittivity_relative = permittivity_relative_silicon
      viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
         ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
         ↪ viscosity_44_silicon { N.s/m^2 }
      force_density_DC_x = 0
```

```
    start (x2l + L_shifter_last_unit, -Height/2)    natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! bottom
    line to (x3l + L_shifter_last_unit, -Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x3l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! top
    line to (x2l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(4e) THIN SPACE CHARGE - CENTER ELECTRODE LEFT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x3l + L_shifter_last_unit, -Height/2)    natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! bottom
    line to (x4l + L_shifter_last_unit, -Height/2)  natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x4l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! top
    line to (x3l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(5e) negative electrode'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0
```

```
    start (x4l + L_shifter_last_unit, -Height/2)   natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! bottom
    line to (x5l + L_shifter_last_unit, -Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x5l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! top
    line to (x4l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! left
    line to close

region '(6e) THIN SPACE CHARGE - CENTER ELECTRODE RIGHT'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = force_density_DC_x_calculated

    start (x5l + L_shifter_last_unit, -Height/2)   natural(ux) = complex(0,
       ↪  0)   natural(uy) = complex(0, 0) ! bottom
    line to (x6l + L_shifter_last_unit, -Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x6l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! top
    line to (x5l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! left
    line to close

region '(7e) silicon right'
    mass_density = mass_density_silicon Young_modulus =
       ↪ Young_modulus_silicon Poisson_ratio = Poisson_ratio_silicon
       ↪ permittivity_relative = permittivity_relative_silicon
    viscosity_11 = viscosity_11_silicon { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_silicon { N.s/m^2 } viscosity_44 =
       ↪ viscosity_44_silicon { N.s/m^2 }
    force_density_DC_x = 0
```

```
    start (x6l + L_shifter_last_unit, -Height/2)    natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! bottom
    line to (x7l + L_shifter_last_unit, -Height/2)  natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x7l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! top
    line to (x6l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(8e) THIN SPACE CHARGE - RIGHT ELECTRODE'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = -force_density_DC_x_calculated

    start (x7l + L_shifter_last_unit, -Height/2)    natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! bottom
    line to (x8l + L_shifter_last_unit, -Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
    line to (x8l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)  natural(uy) = complex(0, 0) ! top
    line to (x7l + L_shifter_last_unit, Height/2)   natural(ux) = complex
       ↪ (0, 0)   natural(uy) = complex(0, 0) ! left
    line to close

region '(9e) positive electrode right'
    mass_density = mass_density_gold Young_modulus = Young_modulus_gold
       ↪ Poisson_ratio = Poisson_ratio_gold permittivity_relative =
       ↪ permittivity_relative_gold
    viscosity_11 = viscosity_11_gold { N.s/m^2 } viscosity_12 =
       ↪ viscosity_12_gold { N.s/m^2 } viscosity_44 = viscosity_44_gold {
       ↪ N.s/m^2 }
    force_density_DC_x = 0
```

```
      start (x8l + L_shifter_last_unit, -Height/2)   natural(ux) = complex(0,
         ↪  0)   natural(uy) = complex(0, 0) ! bottom
      line to (x9l + L_shifter_last_unit, -Height/2)  natural(ux) = complex
         ↪ (0, 0)   natural(uy) = complex(0, 0) ! right
      line to (x9l + L_shifter_last_unit, Height/2)  natural(ux) = complex(0,
         ↪  0)   natural(uy) = complex(0, 0) ! top
      line to (x8l + L_shifter_last_unit, Height/2)  natural(ux) = complex(0,
         ↪  0)   natural(uy) = complex(0, 0) ! left
      line to close

   region '(a) Absorber'
      mass_density = mass_density_absorber Young_modulus =
         ↪ Young_modulus_absorber Poisson_ratio = Poisson_ratio_absorber
         ↪ permittivity_relative = permittivity_relative_absorber
      viscosity_11 = viscosity_11_absorber { N.s/m^2 } viscosity_12 =
         ↪ viscosity_12_absorber { N.s/m^2 } viscosity_44 =
         ↪ viscosity_44_absorber { N.s/m^2 }
      force_density_DC_x = 0

      start (x9l + L_shifter_last_unit, -Height/2)         natural(ux) =
         ↪ complex(0, 0)   natural(uy) = complex(0, 0) ! bottom
      line to (x9l + L_shifter_last_unit + L_absorber, -Height/2)  natural(
         ↪ ux_real) = omega*sqrt(mass_density)*ux_imag  natural(ux_imag) = -
         ↪ omega*sqrt(mass_density)*ux_real natural(uy) = complex(0,0) !
         ↪ right
      line to (x9l + L_shifter_last_unit + L_absorber, Height/2)   natural(ux
         ↪ ) = complex(0, 0)   natural(uy) = complex(0, 0) ! top
      line to (x9l + L_shifter_last_unit, Height/2)        natural(ux) =
         ↪ complex(0, 0)   natural(uy) = complex(0, 0) ! left
      line to close

   !feature 'Signal line'
      !start(x0, -Height/2) value(ux) = complex(ux_signal, 0) line to (x0,
         ↪ Height/2)

{
MONITORS
      grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy))
```

```
        contour(real(ux)) as "X-displacement(m)" contour(real(ux)) as "X-
            ↪ displacement(m)" painted
        elevation(real(ux), real(uy)) from(x0, 0) to (x9 + (nUnit-1)*L, 0
}

PLOTS
    grid(x + zoom_factor*real(ux), y + zoom_factor*real(uy)) PNG(1024, 2)
        !report(probeline_end_x1, probeline_end_x2)
        !report(V_dc, V_signal, f, L, Height)

    contour(real(ux)) PNG(1024, 2) as "X-displacement(m)" report(V_dc, V_signal
        ↪ , f)
        !vtk(real(ux))
    contour(real(ux)) PNG(1024, 2) as "X-displacement(m)" painted report(V_dc,
        ↪ V_signal, f)
        !vtk(real(ux))

    contour(real(uy)) PNG(1024, 2) as "Y-displacement(m)" report(V_dc, V_signal
        ↪ , f)
        !vtk(real(uy))
    contour(real(uy)) PNG(1024, 2) as "Y-displacement(m)" painted report(V_dc,
        ↪ V_signal, f)
        !vtk(real(uy))

    surface(force_density_DC_x) PNG(1024, 2) painted report(V_dc, V_signal, f)
        !vtk(force_density_DC_x)

    elevation(real(ux)) from(x0, 0) to (probeline_end_x1 + L_absorber, 0) PNG
        ↪ (1024, 2)
        !export FILE = 'ux_real.txt' FORMAT"#X #1"
        report(V_dc, V_signal, f)

    elevation(real(uy)) from(x0, 0) to (probeline_end_x1 + L_absorber, 0) PNG
        ↪ (1024, 2) !fixed range (- 1e-19, 1e-19 )
        !export FILE = 'uy_real.txt' FORMAT"#X #1"
        report(V_dc, V_signal, f)

    ! Input
    elevation(real(ux)) from(x0, -Height/2) to (x0, Height/2) PNG(1024, 2)
```

```
        !export FILE = 'ux_real_signal_line.txt' FORMAT"#Y #1"
        report(V_dc, V_signal, f)

    ! Output
    elevation(real(ux)) from(probeline_end_x2, -Height/2) to (probeline_end_x2,
        ↪ Height/2) PNG(1024, 2)
        !export FILE = 'ux_real_probe_line.txt' FORMAT"#Y #1"
        report(V_dc, V_signal, f)

    elevation(real(uy)) from(probeline_end_x2, -Height/2) to (probeline_end_x2,
        ↪ Height/2) PNG(1024, 2)
        !export FILE = 'uy_real_probe_line.txt' FORMAT"#Y #1"
        report(V_dc, V_signal, f)

END
```