Minimal Dispersion of Large Volume Boxes in the Cube

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Abstract

In this note we present a construction which improves the best known bound on the minimal dispersion of large volume boxes in the unit cube. The dispersion of a subset of the cube is the supremal volume over all axis parallel boxes in the cube which do not intersect the given subset. The minimal n-point dispersion is the infimal dispersion over all subsets of the cube containing n points. Define the large volume regime as the set of real volumes greater than $\frac{1}{4}$. In this note we work exclusively in the large volume setting. The construction presented in this paper yields a dimension independent upper bound which is an improvement on, and is proportional to the square root of the best known bound in this regime. We also show that some intermediate estimates are sharp, given that the dimension is taken to be larger than a specified volume-dependent constant.

Preface

This thesis is an original work by Kurt S. MacKay. The preprint "Minimal dispersion of large volume boxes in the cube" has been submitted for editorial review.

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Contents

1 Introduction

The dispersion of a subset $T \subset [0,1]^d$ is defined as the supremum of the volume over all axis parallel boxes in the cube which do not intersect T. Consider the class of all n point subsets of the cube. Define the minimal n-point dispersion as the infimum of the dispersion over all such subsets. The problem of estimating the minimal dispersion (defined originally in [14] as a modification of a concept in [6]) has been given attention in recent years in such contemporary works as [1],[3],[5],[7],[9],[10],[12],[16],[17]. We will refer to these works when we discuss the historical progress on the problem, and the best known bounds on the minimal dispersion. The dispersion of some particular sets has been studied in [8],[13],[15]. When the volume is "large" ($r > \frac{1}{4}$), the best known upper bound given in [12] on the minimal dispersion is of the order $(r - \frac{1}{4})^{-1}$. In this paper we present a construction which improves the best known bound on the minimal dispersion of large volume boxes in the cube. We construct a class of discrete configurations in the cube which can be employed to yield the result. Let $d > 1$, and let $r \in (\frac{1}{4}, 1]$. The inverse of the minimal dispersion is denoted as $N(r,d)$. The number $N(r,d)$ is the cardinality of the smallest set of points which intersects any axis parallel box with volume exceeding r. The topic of dispersion is of interest in studying topics in Discrete Geometry and Approximation Theory and in particular random point configurations as in [10]. For each $r \in (\frac{1}{4}, \frac{1}{2}]$ we construct a discrete configuration of points in the cube. The cardinality of such a configuration will be shown to be an upper bound for $N(r,d)$. First, consider the situation when $r \geq \frac{1}{2}$. The minimal dispersion is attained with one point at the center of the cube. Thus, it is clear that in the large volume regime we are interested in estimating the minimal dispersion when $r < \frac{1}{2}$. Theorem 1.1 improves the best known bound in the large volume setting and is given by.

Theorem 1.1. *Let $d > 1$, and let $r \in (\frac{1}{4}, \frac{1}{2}]$. Then*

$$N(r,d) \leq \left\lfloor \frac{\pi}{\sqrt{r - \frac{1}{4}}} \right\rfloor - 3. \tag{1}$$

Let $r \in (\frac{1}{4}, \frac{1}{2}]$, and let $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^d$. We construct a set of points on the diagonal in the following way. Consider the sequence

$$\mathfrak{Q}(r) = \left\{ r,\ \frac{r}{1-r},\ \frac{r}{1-\frac{r}{1-r}},\ \frac{r}{1-\frac{r}{1-\frac{r}{1-r}}},\ \frac{r}{1-\frac{r}{1-\frac{r}{1-\frac{r}{1-r}}}},\ \dots \right\}.$$

Denote the subsequent elements in the sequence as $q_1, q_2, \dots \in \mathfrak{Q}(r)$. We show that there exists a smallest number $n \geq 1$ such that

$$q_n \geq 1 - r.$$

Now construct the configuration

$$\mathfrak{q}(r) = \{q_i \mathbf{1} : 1 \leq i \leq n\}.$$

We define a monotone decreasing step function on $(\frac{1}{4}, \frac{1}{2}]$ as

$$r \longmapsto |\mathfrak{q}(r)|. \tag{2}$$

This function is right continuous, and induces a partition on the interval. We derive an explicit formula for the endpoints of the intervals, from which we obtain a formula for the function in (2). We show that this step function is an upper bound for the minimal dispersion. From this we obtain the estimate given in Theorem 1.1. After this we restrict the analysis to the diagonal

$$\{x\mathbf{1} :\ x \in [0, 1]\},$$

and diagonal analogues to obtain some properties of our configurations. Finally, we use the results on the diagonal analogues to show that some of our estimates are sharp, given that $d \geq C_r$, where C_r is a constant depending on r alone. We begin with the definitions and notation.

1.1 Definitions and Notation

Let $d \geq 1$. Denote the d-dimensional unit cube as $[0,1]^d$. By convention let $|\cdot|$ denote the cardinality, and let $\mathbf{1} := (1,1,\ldots,1) \in \mathbb{R}^d$. The set of all axis parallel boxes is denoted as

$$\mathfrak{B} := \left\{ \prod\nolimits_{i=1}^{d} I_i : \ I_i = [a_i, b_i) \subset [0,1] \right\}.$$

The dispersion of $T \subset [0,1]^d$ is denoted as

$$\operatorname{disp}(T) := \sup_{B \in \mathfrak{B}, \ B \cap T = \emptyset} \operatorname{Vol}(B).$$

The minimal dispersion is denoted as

$$\operatorname{disp}^*(n,d) := \inf_{T \subset [0,1]^d, \ |T| = n} \operatorname{disp}(T).$$

Let $r \in [0,1]$. The inverse of the minimal dispersion is denoted as

$$N(r,d) := \min\{n \in \mathbb{N} : \operatorname{disp}^*(n,d) \leq r\}.$$

Let $k \geq 0$. Inductively define a sequence of functions $\{f_k\}_{k \geq 0}$ in the following way. Let $\beta_0 = 1$. Define $f_0 : [0,1] \to [0,1]$ as the identity $f_0(x) = x$. Given functions $f_0, f_1, \ldots f_{k-1}$, and numbers $\beta_0, \beta_1, \ldots, \beta_{k-1}$, define

$$\beta_k = \inf\{x \geq 0 : x \in \operatorname{dom}(f_{k-1}), \ f_{k-1}(x) = 1\}, \tag{3}$$

and define $f_k : [0, \beta_k) \to \mathbb{R}$ by

$$f_k(x) = \frac{x}{1 - f_{k-1}(x)}. \tag{4}$$

Proposition 3.2 shows that the infimum in (3) is attained for all $k \geq 0$. We will see that the numbers in (3) are the endpoints of the interval partition described in the introduction. Let $k \geq 0$. It is clear that $\mathrm{dom}(f_k) \subset \mathrm{dom}(f_{k-1})$, hence $\beta_k < \beta_{k-1}$. Let $r \in (\frac{1}{4}, \frac{1}{2}]$. The following definition is a rigorous description of the step function given in (2). Define

$$\alpha(r) := \inf\{k \geq 0 : r \in \mathrm{dom}(f_k),\ f_k(r) \geq 1 - r\} + 1, \tag{5}$$

and define

$$n_r := \alpha(r) - 1. \tag{6}$$

In Remark 3.8 we show that the infimum in (5) is attained.

2 Previous Work

First some known results related to the dispersion problem will be discussed. In the paper [1] Aistleitner, Hinrichs, and Rudolf showed that for $r < \frac{1}{4}$,

$$(1 - 4r)\frac{\log_2 d}{4r} \leq N(r, d). \tag{7}$$

This lower bound gives a non-trivial estimate showing that the dispersion asymptotically increases with dimension in the $r < \frac{1}{4}$ regime. The upper bound

$$N(r, d) \leq \frac{2^{7d+1}}{r} \tag{8}$$

was given by Larcher, and is presented in [1]. This is an improvement on the bound given by Rote and Tichy in [10]. The inequality given by

$$N(r, d) \leq \frac{8d}{r} \log_2 \left(\frac{33}{r}\right) \tag{9}$$

is a consequence of a general result given in [2]. The authors present an argument which uses the $VC-$dimension of $\mathfrak{B}$ (which is $2d$) instead of the ambient dimension d. In the paper [11], Rudolf presented a probabilistic argument which yields the bound in (9). The bound in (9) is an improvement on (8) under the assumption that $r \geq \exp(-C^d)$ where $C > 1$ is an absolute constant. Sosnovec in [12] obtained another upper bound which is better when d grows to infinity

$$N(r, d) \leq C_r \log_2 d. \tag{10}$$

The constant C_r obtained in [12] grows extremely fast with r. This constant was improved by Ullrich and Vybìral [16] who showed that

$$C_r = \frac{2^7}{r^2} \log_2^2 \left(\frac{1}{r}\right).$$

Litvak in [9] gives an improvement on the known bounds in the regime $r \leq \exp(-d)$, and showed that

$$N(r, d) \leq \frac{C \ln d}{r} \ln \left(\frac{1}{r}\right).$$

This result is very close to the best possible in a probabilistic setting. Litvak also established that in the regime $r \geq (\ln^2 d)/(d \ln \ln(2d))$ that

$$N(r, d) \leq \frac{C \ln d}{r^2} \ln \left(\frac{1}{r}\right),$$

which is an improvement on the bound given by Ullrich and Vybìral. Now we turn our attention to the large volume regime $r > \frac{1}{4}$. Sosnovec in [12] gave dimension independent upper bound for $N(r, d)$. In particular, he proved that for $r \in (\frac{1}{4}, 1)$, and $d \geq 2$,

$$N(r, d) \leq \left\lfloor \frac{1}{r - \frac{1}{4}} \right\rfloor + 1.$$

The goal of this paper is to improve this bound. This is established in Theorem 1.1.

3 Auxiliary

Proposition 3.1. *Let $i \geq 0$. The function f_i is strictly increasing on its domain.*

Proof. Employ induction on i. Let $i = 0$. By definition $\mathrm{dom}(f_0) = [0, 1]$. For all $x \in [0, 1]$, $f_0(x) = x$. The base case is seen to be trivial. Assume that the Proposition holds for $i = 0, 1, 2, \ldots, k$. Let $x_1, x_2 \in \mathrm{dom}(f_{k+1})$ be such that $x_1 < x_2$. Then by domain inclusion $x_1, x_2 \in \mathrm{dom}(f_k)$. By the induction hypothesis $f_k(x_1) < f_k(x_2)$. It follows from the definition of f_{k+1} as in (4) that

$$f_{k+1}(x_1) = \frac{x_1}{1 - f_k(x_1)} < \frac{x_2}{1 - f_k(x_2)} = f_{k+1}(x_2).$$

This proves the Proposition. □

In Proposition 3.2 we show that the infimal definition of the endpoints in (3) is attained.

Proposition 3.2. *Let $r_0 = 1$. For each $i \geq 1$ there exists a unique number*

$$r_i \in \mathrm{dom}(f_i) = [0, \beta_i)$$

with the properties

$$\beta_i = r_{i-1} \tag{11}$$

$$f_{i-1}(r_i) = 1 - r_i \tag{12}$$

$$f_i(r_i) = 1. \tag{13}$$

Proof. Note that $\beta_1 = 1 = r_0$. Employ induction on n. For each $n > 0$ we produce a number r_n with the properties (11), (12), (13).

Let $n = 1$. Recall the definition of f_2 given in (4), by

$$f_2(x) = \frac{x}{1 - f_1(x)}.$$

We will show that there exists $r_1 \in \mathrm{dom}(f_1)$ such that

$$1 = f_1(r_1) = \frac{r_1}{1 - f_0(r_1)}.$$

Equivalently, we find the solution to the equation $f_0(x) - (1 - x) = 0$. It is clear that $r_1 = \frac{1}{2}$ is the solution, and that $r_1 < \beta_1$. Consequently, $\beta_2 = r_1$. This implies that $\mathrm{dom}(f_2) = [0, \beta_2)$. Hence, r_1 has properties (11), (12), (13).

Let $n = 2$. Recall the definition of f_3 given in (4), by

$$f_3(x) = \frac{x}{1 - f_2(x)}.$$

We show that there exists $r_2 \in \mathrm{dom}(f_2)$ such that

$$1 = f_2(r_2) = \frac{r_2}{1 - f_1(r_2)}.$$

Equivalently, we show that there exists a solution to the equation

$$f_1(x) - (1 - x) = 0.$$

The function f_1 is strictly increasing by Proposition 3.1. Note that $f_1(0) = 0$, and $f_1(r_1) = 1$. Apply the Intermediate Value Theorem to $f_1(x) - (1 - x)$. This yields a unique solution $r_2 < r_1$ such that

$$f_1(r_2) - (1 - r_2) = 0.$$

It follows that $\beta_3 = r_2$, and that $\mathrm{dom}(f_3) = [0, \beta_3)$. Hence r_2 has properties (11), (12), (13).

Let $k > 2$. Assume that there exist numbers $r_0, r_1, r_2, \ldots, r_{k-1}$ with the properties (11), (12), (13). Under the assumption that $\mathrm{dom}(f_k) = [0, \beta_k)$ where $\beta_k = r_{k-1}$, and $f_{k-1}(r_{k-1}) = 1$. Recall the definition of f_{k+1} given in (4), by

$$f_{k+1}(x) = \frac{x}{1 - f_k(x)}.$$

We show that there exists $r_k \in \mathrm{dom}(f_k)$, such that

$$1 = f_k(r_k) = \frac{r_k}{1 - f_{k-1}(r_k)}.$$

Equivalently, we show that there exists a solution to the equation

$$f_{k-1}(x) - (1 - x) = 0.$$

The function f_{k-1} is strictly increasing by Proposition 3.1. Note that $f_{k-1}(0) = 0$, and by the induction hypothesis, $f_{k-1}(r_{k-1}) = 1$. Apply the Intermediate Value Theorem to $f_{k-1}(x) - (1-x)$. This yields a unique solution $r_k < r_{k-1}$, such that

$$f_{k-1}(r_k) - (1 - r_k) = 0.$$

It follows that $\beta_{k+1} = r_k$, and that $\mathrm{dom}(f_{k+1}) = [0, \beta_{k+1})$. This yields a number r_{k+1} with the properties (11), (12), (13). This proves the Proposition. □

Remark 3.3. *For each $k \geq 1$, the infimum in the definition of β_k is attained at $\beta_k = r_{k-1}$. Proposition 3.2 justifies the assertion following the definition in (3). Fix the sequence*

$$\{r_m\}_{m \geq 0} = \{\beta_{m+1}\}_{m \geq 0}. \tag{14}$$

Proposition 3.4. *Let $n \geq 1$. Let $r \in (\frac{1}{4}, \frac{1}{2})$ be such that $f_i(r) < 1$ for all $i \leq n$. Then for all $i \leq n$,*

$$f_{i-1}(r) < f_i(r).$$

Proof. Fix $n \geq 1$. Employ induction on i. Let $r \in (\frac{1}{4}, \frac{1}{2})$ be such that $f_i(r) < 1$ for all $i \leq n$. Let $i = 1$. Recall the definition of f_0, f_1. Since $r < \frac{1}{2}$, it follows that

$$f_0(r) = r < \frac{r}{1-r} = f_1(r).$$

Let $1 \leq k < n$. Assume as the induction hypothesis that for all $1 \leq i \leq k$,

$$f_{i-1}(r) < f_i(r).$$

By assumption $f_{k-1}(r) < f_k(r)$, then by definition of f_k it follows that

$$f_k(r) = \frac{r}{1 - f_{k-1}(r)} < \frac{r}{1 - f_k(r)} = f_{k+1}(r).$$

This proves the Proposition. □

Corollary 3.5. *Let $r \in (\frac{1}{4}, \frac{1}{2}]$. Let $n \geq 1$ be such that $r < r_n$. Then for all $i \leq n$,*

$$f_{i-1}(r) < f_i(r).$$

Proof. Let $r \in (\frac{1}{4}, \frac{1}{2}]$. Let $n \geq 1$ be such that $r < r_n$. The sequence $\{r_m\}_{m>0}$ defined in (14) is decreasing. Hence, $r < r_n < r_{n-1} < \cdots < r_1$. Property (13) in Proposition 3.2 implies that $f_k(r_k) = 1$ for all $k \leq n$. Since $r < r_k$, it follows by Proposition 3.1 that $f_k(r) < f_k(r_k)$. Hence,

$$f_k(r) < f_k(r_k) = 1.$$

Now apply Proposition 3.4. □

Remark 3.6. *Corollary 3.5 gives the property that for all $n < k$, $f_n(r_k) < 1$. Property (13) in Proposition 3.2 gives that for all $k > 0$, $f_k(r_k) = 1$. Let $r \in (\frac{1}{4}, \frac{1}{2}]$, and recall the definition in (5) given by*

$$\alpha(r) := \inf\{k \geq 0 : r \in dom(f_k),\ \ f_k(r) \geq 1 - r\} + 1.$$

Then we have that for all $k > 0$,

$$\alpha(r_k) = k = n_{r_k} + 1.$$

This gives the integral values of the step function in (2) evaluated at the left endpoints of the interval partition given by

$$\cdots [r_3, r_2), [r_2, r_1), [r_1, 1). \tag{15}$$

The following Proposition shows that the function in (5) is constant over any interval in the partition (15).

Proposition 3.7. *Let $i \geq 1$. Let $r \in (\frac{1}{4}, \frac{1}{2}]$ be such that $r_i \leq r < r_{i-1}$. Then*

$$\alpha(r) = i.$$

Proof. First, let $i = 1$, and let $r \in (\frac{1}{4}, \frac{1}{2}]$ be such that $r_1 \leq r < 1$. Since $r_1 = \frac{1}{2}$, it follows that $r = \frac{1}{2}$. Hence, $\alpha(r) = 1$.

Let $i \geq 2$, and let $r \in (\frac{1}{4}, \frac{1}{2})$ be such that $r_i \leq r < r_{i-1}$. Since $r < r_{i-1}$, Corollary 3.5 implies that for all $k \leq i-1$, $f_{k-1}(r) < f_k(r)$. Apply Proposition 3.1 on f_{i-2} to get $f_{i-2}(r) < f_{i-2}(r_{i-1})$. By Proposition 3.2,

$$f_{i-2}(r_{i-1}) = 1 - r_{i-1}.$$

It follows that

$$f_{i-2}(r) < f_{i-2}(r_{i-1}) = 1 - r_i < 1 - r.$$

This means that for all $k \leq i - 2$,

$$f_{k-2}(r) < 1 - r.$$

It follows that $n_r > i - 2$. Since $r_i \leq r$, apply Proposition 3.1 on f_{i-1} to get $f_{i-1}(r_i) < f_{i-1}(r)$. Recall that by Proposition 3.2,

$$f_{i-1}(r_i) = 1 - r_i.$$

It follows that

$$1 - r \leq 1 - r_i = f_{i-1}(r_i) < f_{i-1}(r).$$

Thus, $f_{i-1}(r) > 1 - r$. It follows that $n_r \leq i - 1$. Therefore, $n_r = i - 1$. This proves the Proposition. □

Remark 3.8. *Proposition 3.7 shows that the function in (5) is constant over the intervals in the partition from (15). This shows that for each $r \in (\frac{1}{4}, \frac{1}{2}]$, the infimum in the definition of n_r as in (6) is attained.*

A brief discussion on Geometric Rational Sequences follows. We use the results herein to obtain explicit values for the numbers $\{r_k\}_{k\geq 1}$ as in (14).

The paper [4] provides results which can be applied to sequences of the form defined below.

Definition 3.9. *Let $r \in (\frac{1}{4}, \frac{1}{2}]$. A Geometric Rational Sequence $\{x_n(r)\}_{n\geq 0}$ is defined by setting an initial condition $x_0(r) = r$, and recursively defining*

$$x_{n+1} = \frac{r}{1 - x_n}.$$

If $x_n = 1$, then define $x_{n+1} = \infty$, $x_{n+2} = 0$, so that $x_{n+3} = r$.

Definition 3.10. *Let $r \in (\frac{1}{4}, \frac{1}{2}]$. The reduced form of a Geometric Rational Sequence $\{y_n(r)\}_{n\geq 0}$ is defined by setting an initial condition*

$$y_0(r) = -1 + r,$$

and recursively defining

$$y_{n+1}(r) = -1 - \frac{r}{y_n(r)}.$$

If $y_n(r) = 0$, then define $y_{n+1}(r) = \infty$, $y_{n+2}(r) = -1$, so that $y_{n+3}(r) = -1 + r$.

Remark 3.11. *Note that if $y_{n+3}(r) = -1 + r$, then $y_{n+2}(r) = -1$. Hence, $y_{n+1}(r) = \infty$ and $y_n(r) = 0$. This occurs if and only if the sequence $\{y_n(r)\}_{n\geq 0}$ is cyclical.*

Remark 3.12. *Let $i \geq 1$, and let $r \leq r_i$. Then,*

$$y_0(r) = -1 + r,$$

$$y_1(r) = -1 + \frac{r}{1 - r} = -1 + f_1(r),$$

$$y_2(r) = -1 + \frac{r}{1 - f_1(r)} = -1 + f_2(r).$$

Continuing in this way, we see that for all $k < i$ and $r \leq r_i$,

$$y_k(r) = -1 + f_k(r).$$

Proposition 3.13. *Let $m > 0$. Then the reduced sequence $\{y_n(r_m)\}_{n\geq 0}$ is cyclical with cycle length $m+3$.*

Proof. From Remark 3.12, we have that for all $k < m$,

$$y_k(r_m) = -1 + f_k(r_m).$$

Apply Proposition 3.4 to yield

$$f_{k-1}(r_m) < f_k(r_m)$$

for all $k < m$. This guarantees no repetition in the first $m-1$ terms of the reduced sequence $\{y_n(r_m)\}_{n\geq 0}$. By Proposition 3.2, $f_m(r_m) = 1$. It follows that

$$y_m(r_m) = f_m(r_m) - 1 = 1 - 1 = 0.$$

Recall the reduced sequence given in Definition 3.10. Then by definition

$$y_{m+1}(r_m) = \infty$$

$$y_{m+2}(r_m) = -1$$

$$y_{m+3}(r_m) = 1 - r_m.$$

This shows that $y_{m+3}(r_m) = 1 - r_m = y_0(r_m)$. It follows that the sequence is cyclical with cycle length $m+3$. $\square$

The following Theorem from [4] will be used. Note that there is a typographical error in the condition $\sigma^2 < 4\gamma$.

Theorem 3.14. *Let σ, $\gamma \in \mathbb{R}$ with $\sigma^2 < 4\gamma$, and $\theta = \arccos \frac{\sigma}{2\sqrt{\gamma}}$. A sequence satisfying $y_{n+1} = \sigma - \frac{\gamma}{y_n}$, $y_1 \in \mathbb{R}$, has a finite or infinite number of cluster points depending on whether or not $\frac{\theta}{\pi}$ is rational. Moreover when $\frac{\theta}{\pi} = \frac{k}{m} \in \mathbb{Q}$ is irreducible, the sequence takes on m distinct values $y_1, y_2, \ldots, y_m$ which are thereafter repeated in this order.*

Proposition 3.15. *Let $n \geq 1$. Let*

$$R_n = \frac{1}{4}\frac{1}{\cos^2(\frac{\pi}{n+3})}.$$

Then the reduced sequence $\{y_k(R_n)\}_{k\geq 0}$ has cycle length $n+3$.

Proof. Let $n \geq 1$. Let

$$R_n = \frac{1}{4}\frac{1}{\cos^2(\frac{\pi}{n+3})}.$$

In reference to Theorem 3.14, the sequence $\{y_k(R_n)\}_{k\geq 0}$ has the parameters $\sigma = -1$, and $\gamma = R_n$. Apply Theorem 3.14 with the given parameters, and set

$$\theta = \arccos\big(\frac{-1}{2\sqrt{R_n}}\big) = \arccos\big(-\cos(\frac{\pi}{n+3})\big) = \frac{\pi(n+2)}{n+3}.$$

Then

$$\frac{\theta}{\pi} \in \mathbb{Q}.$$

By Theorem 3.14, it follows that $\{y_k(R_n)\}_{k\geq 0}$ has cycle length $n+3$. □

Remark 3.16. *Proposition 3.15 gives a decreasing sequence of numbers $\{R_n\}_{n>0} \subset (\frac{1}{4}, \frac{1}{2}]$ such that as n goes to infinity $R_n \to \frac{1}{4}$. We show that these numbers correspond to the numbers $\{r_k\}_{k\geq 1}$ given in (14). These are exactly the values of the endpoints in the partition given in (15).*

Proposition 3.17. *Recall the sequence $\{r_m\}_{m>0}$ defined in (14). Then for all $n > 0$,*

$$r_n = R_n.$$

Proof. Apply induction on n. Let $n = 1$. It is easy to check that $r_1 = \frac{1}{2} = R_1$.

Let $n = 2$. Recall that $\{R_n\}_{n>0}$ is decreasing. Hence, $R_2 < R_1 = \beta_2$. By Proposition 3.2 it follows that $R_2 \in \mathrm{dom}(f_2)$. By Proposition 3.15, the sequence $\{y_k(R_2)\}_{k\geq 0}$ has cycle length $2+3$. From Remark 3.11, it follows that $y_2(R_2) = 0$. Then

$$y_2(R_2) = 0 = 1 - 1 = f_2(R_2) - 1.$$

This implies that $f_2(R_2) = 1$. Thus, $r_2 = R_2$.

Let $n = 3$. Note $R_3 < R_2 = \beta_3$. Then by Proposition 3.2, it follows that $R_3 \in \text{dom}(f_3)$. By Proposition 3.15, the sequence $\{y_k(R_3)\}_{k\geq 0}$ has cycle length $3+3$. From Remark 3.11, it follows that $y_3(R_3) = 0$. Then

$$y_3(R_3) = 0 = 1 - 1 = f_3(R_3) - 1.$$

This implies that $f_3(R_3) = 1$. Thus $r_3 = R_3$.

Fix $k > 3$. Assume that $r_n = R_n$ for $n = 1, 2, \ldots, k$. Note $R_{k+1} < R_k = \beta_{k+1}$. Then by Proposition 3.2, it follows that $R_{k+1} \in \text{dom}(f_{k+1})$. The sequence $\{y_l(R_{k+1})\}_{l\geq 0}$ has cycle length $(k+1)+3$. From Remark 3.11, it follows that $y_{k+1}(R_{k+1}) = 0$. Then

$$y_{k+1}(R_{k+1}) = 0 = 1 - 1 = f_{k+1}(R_{k+1}) - 1.$$

This implies that $f_{k+1}(R_{k+1}) = 1$. Thus, $r_{k+1} = R_{k+1}$. $\square$

4 Main Results

In this section we obtain an upper bound for the minimal dispersion in the large volume regime. We also use the results in the Auxiliary 3 to derive a closed form expression for the bound.

4.1 An Upper Bound for the Minimal Dispersion

Let $r \in (\frac{1}{4}, \frac{1}{2}]$, and let n_r be as in (6). Define the following configurations on the diagonal

$$\mathfrak{q}(r) = \{f_k(r)\mathbf{1} : 0 \leq k \leq n_r\}. \tag{16}$$

It is clear that $|\mathfrak{q}(r)| = n_r + 1$.

Definition 4.1. *Let $B = I_1 \times I_2 \times \cdots \times I_d \in \mathfrak{B}$. The box B is Type 1, if one of the following conditions holds. There exists $1 \leq j \leq d$, such that $I_j \subset [0, r]$, or such that $I_j \subset [f_{n_r}(r), 1]$. There exists $1 \leq j \leq d$, and $0 \leq k \leq n_r - 1$, such that $I_j \subset [f_k(r), f_{k+1}(r)]$.*

Definition 4.2. *Let $B = I_1 \times I_2 \times \cdots \times I_d \in \mathfrak{B}$. The box B is Type 2, if the following condition holds. There exist $1 \le j, l \le d$, and $0 \le k \le n_r - 1$, such that $I_j \subset [f_k(r), 1]$, and $I_l \subset [0, f_{k+1}(r)]$.*

Lemma 4.3. *Let $r \in (\frac{1}{4}, \frac{1}{2}]$. Let $B \in \mathfrak{B}$ be a box of Type 1 or Type 2. Then $\mathrm{Vol}(B) \le r$.*

Proof. Let $B = I_1 \times I_2 \times \cdots \times I_d \in \mathfrak{B}$. First assume that B is Type 1. Assume that $I_i \subset [0, r]$. Then the Lemma trivially holds. Assume that there exists $1 \le i \le d$, such that $I_i \subset [f_{n_r}(r), 1]$. Since n_r is the smallest integer such that $f_{n_r}(r) \ge 1 - r$, it follows that

$$\mathrm{Vol}(B) \le |I_i| \le 1 - f_{n_r}(r) \le r.$$

Assume that there exists $1 \le i \le d$, and $0 \le k < n_r$ such that $I_i \subset [f_k(r), f_{k+1}(r)]$. Then

$$\mathrm{Vol}(B) \le |I_i| = f_{k+1}(r) - f_k(r) = \frac{r}{1 - f_k(r)} - f_k(r) = \frac{r - (1 - f_k(r))f_k(r)}{1 - f_k(r)}.$$

Recall that n_r is the smallest integer such that $1 - r \le f_{n_r}(r)$. Since $k < n_r$, it follows that $f_k(r) < 1 - r$. Then

$$\mathrm{Vol}(B) \le \frac{r - (1 - f_k(r))f_k(r)}{1 - f_k(r)} \le \frac{r - r f_k(r)}{1 - f_k(r)} = r.$$

Hence, if $B \in \mathfrak{B}$ is Type 1, then $\mathrm{Vol}(B) \le r$.

Let $B \in \mathfrak{B}$ be a Type 2 box. By definition there exist $1 \le i, j \le d$ and $0 \le k \le n_r - 1$ such that $I_i \subset [f_k(r), 1]$ and $I_j \subset [0, f_{k+1}(r)]$. Recall the definition of f_{k+1} as in (4) given by

$$f_{k+1}(r) = \frac{r}{1 - f_k(r)}.$$

It follows that

$$\mathrm{Vol}(B) \le |I_i||I_j| = (1 - f_k(r))\frac{r}{1 - f_k(r)} = r.$$

This proves the claim. $\square$

Lemma 4.4. *Let $r \in (\frac{1}{4}, \frac{1}{2}]$. Let $B \in \mathfrak{B}$ be such that $\mathfrak{q}(r) \cap B = \emptyset$. Then B is either Type 1 or Type 2.*

Proof. Let

$$P(r) = \{p_i : p_i = f_i(r),\ 0 \leq i \leq n_r\}. \tag{17}$$

Then informally

$$\mathfrak{q}(r) = P(r)\mathbf{1}.$$

Let $B = I_1 \times I_2 \times \cdots \times I_d \in \mathfrak{B}$. Notice that $\mathfrak{q}(r) \neq \emptyset$ for all $r \in (\frac{1}{4}, \frac{1}{2}]$. Then it is clear that there exists $1 \leq i \leq d$ such that $I_i \neq [0, 1]$. Define

$$Q := I_i \cap P(r).$$

If $Q = \emptyset$, then B is Type 1. From here we list the remaining possible cases.

Case 1: In this case $Q = \{p_{n_r}\}$. Then $p_{n_r} \in I_i$. It is clear that $I_i \subset (p_{n_r-1}, 1]$. Since $B \cap \mathfrak{q}(r) = \emptyset$, there exists $1 \leq j \leq d$ such that $I_j \subset (p_{n_r}, 1]$ or $I_j \subset [0, p_{n_r})$. If $I_j \subset (p_{n_r}, 1]$, then by definition B is Type 1. If $I_j \subset [0, p_{n_r})$, then since $I_i \subset (p_{n_r-1}, 1]$, by definition B is Type 2.

Case 2: Denote $Q_0 = Q$. In the second case let

$$0 < m \leq n_r,$$

and define

$$Q_0 \subset \{p_m, p_{m+1}, \ldots, p_{n_r}\}.$$

The following algorithm shows that B is Type 1 or Type 2. Let $I_{m_0} = I_i$. Then it is clear that $I_{m_0} \subset (p_{m-1}, 1]$. Recall that $B \cap \mathfrak{q}(r) = \emptyset$. Then there exists I'_{m_0}, such that $I'_{m_0} \subset [0, p_m)$ or $I'_{m_0} \subset (p_m, 1]$. If $I'_{m_0} \cap Q_0 = \emptyset$, then B is Type 1. If $I'_{m_0} \subset [0, p_m)$, then since $I_{m_0} \subset (p_{m-1}, 1]$, B is Type 2. If $I'_{m_0} \subset (p_m, 1]$, then denote $I_{m_1} := I'_{m_0}$. Let

$$1 < m \leq n_r,$$

and define

$$Q_1 = Q_0 \cap I_{m_1} \subset \{p_m, p_{m+1}, \dots, p_{n_r}\}.$$

If $I_{m_1} \cap P(r) = \{p_{n_r}\}$, then appeal to Case 1. Since $B \cap \mathfrak{q}(r) = \emptyset$ there exists I'_{m_1}, such that $I'_{m_1} \subset [0, p_m)$ or $I'_{m_1} \subset (p_m, 1]$. If $I'_{m_1} \cap Q_1 = \emptyset$, then B is Type 1. If $I'_{m_1} \subset [0, p_m)$, then since $I_{m_1} \subset (p_{m-1}, 1]$ B is Type 2. If $I'_{m_1} \cap P(r) = \{p_{n_r}\}$, then appeal to Case 1. If $I'_{m_1} \subset (p_m, 1]$ denote $I_{m_2} := I'_{m_1} \subset (p_m, 1]$, and continue the algorithm. At step ℓ of the algorithm let

$$\ell < m \leq n_r,$$

and define

$$Q_\ell = Q_{m_{\ell-1}} \cap I_{m_\ell} \subset \{p_m, p_{m+1}, \dots, p_{n_r}\}.$$

If $Q_\ell = \emptyset$, then B is Type 1. Assume $Q_\ell \neq \emptyset$. If $I_{m_\ell} \cap P(r) = \{p_{n_r}\}$, then appeal to Case 1. If there exists an interval I'_{m_ℓ}, such that $I'_{m_\ell} \subset [0, p_m)$, then since $I_{m_\ell} \subset (p_{m-1}, 1]$ by definition B is Type 2. The algorithm terminates after at most n_r steps.

Case 3: In the last case $p_0 \in Q$. Since $B \cap \mathfrak{q}(r) = \emptyset$ there exists $I_j \subset [0, p_0)$ or $I_j \subset (p_0, 1]$. If $I_j \subset [0, p_0)$, then B is Type 1. Finally, if $I_j \subset (p_0, 1]$, then apply the results in Case 1 and Case 2. This proves the Lemma. □

Corollary 4.5. *Let $r \in (\frac{1}{4}, \frac{1}{2}]$. Then*

$$N(r, d) \leq \alpha(r).$$

Proof. Let $n \geq 1$. Let $r \in (\frac{1}{4}, \frac{1}{2}]$ be such that $r_n \leq r < r_{n-1}$. Recall the configuration $\mathfrak{q}(r)$ defined as in (16). By Lemma 4.3, and Lemma 4.4 we get that

$$\mathrm{disp}(\mathfrak{q}(r)) = r.$$

Since $|\mathfrak{q}(r)| = n_r + 1 = n$, it follows that $N(r, d) \leq n$. □

Corollary 4.5 gives an upper bound for the minimal dispersion.

4.2 A Formula for the Upper Bound

We derive a simple formula for the upper bound given in Corollary 4.5.

Corollary 4.6. *Let* $r \in (\frac{1}{4}, \frac{1}{2}]$. *Then*

$$\alpha(r) = \left\lfloor \frac{\pi}{\arccos\left(\frac{1}{2\sqrt{r}}\right)} \right\rfloor - 3.$$

Proof. Let $k > 0$. By Proposition 3.15 and the conclusion of Remark 3.17 it is clear that

$$\frac{\pi}{\arccos\left(\frac{1}{2\sqrt{r_k}}\right)} - 3 = \alpha(r_k) = k. \tag{18}$$

Notice that the function is strictly decreasing on $(\frac{1}{4}, \frac{1}{2}]$. Let $r \in (\frac{1}{4}, \frac{1}{2}]$, and let $k > 0$ be such that $r_k \leq r < r_{k-1}$. By Proposition 3.7 and (18) it follows that

$$\alpha(r) = k = \left\lfloor \frac{\pi}{\arccos\left(\frac{1}{2\sqrt{r}}\right)} \right\rfloor - 3.$$

□

Theorem 4.7. *Let* $r \in (\frac{1}{4}, \frac{1}{2}]$. *Then*

$$N(r, d) \leq \left\lfloor \frac{\pi}{\sqrt{r - \frac{1}{4}}} \right\rfloor - 3.$$

Proof. Let $r \in (\frac{1}{4}, \frac{1}{2}]$. Then

$$\arccos\left(\frac{1}{2\sqrt{r}}\right) \geq \sqrt{r - \frac{1}{4}}.$$

This combined with Corollary 4.5 and Corollary 4.6 gives

$$N(r, d) \leq \left\lfloor \frac{\pi}{\arccos\left(\frac{1}{2\sqrt{r}}\right)} \right\rfloor - 3 \leq \left\lfloor \frac{\pi}{\sqrt{r - \frac{1}{4}}} \right\rfloor - 3.$$

The Theorem is proved. □

5 Diagonal-Analogues and Sharpness

We prove some results about our configurations on the diagonal and diagonal analogues in the cube.

5.1 Dispersion of Subsets of the Diagonal

Proposition 5.1. *Let $r \in (\frac{1}{4}, \frac{1}{2}]$, and let $T \subset [0,1]^d$ be on the diagonal such that*

$$\mathrm{disp}(T) = r.$$

Then

$$|T| \geq n_r + 1.$$

Proof. Let $r \in (\frac{1}{4}, \frac{1}{2}]$. Let $T \subset [0,1]^d$ be on the diagonal such that

$$\mathrm{disp}(T) = r.$$

Let n_r be as in (6), and let $\mathfrak{q}(r)$ be the configuration as defined in (16). Let $P(r)$ be as defined in (17). Let

$$E = \{e_i \in [0,1] : e_1 < e_2 < \cdots < e_m,\ m = |T|\}$$

such that

$$T = \{e_i \mathbf{1} : e_i \in E\}.$$

Assume toward a contradiction that $|T| \leq n_r$. Partition $[0,1]$ into $n_r + 2$ intervals

$$[0, p_0], (p_0, p_1], (p_1, p_2], \ldots, (p_{n_r - 1}, p_{n_r}], (p_{n_r}, 1]. \tag{19}$$

Since $|T| \leq n_r$, there exist at least two intervals which do not intersect E. Assume $E \cap [0, p_0] = \emptyset$. Then $p_0 < e_1$. Construct the box

$$B = [0, e_1) \times [0,1]^{d-1}.$$

Then $B \cap T = \emptyset$. However, $\mathrm{Vol}(B) = e_1 > p_0 = r$ which contradicts the assumption that $\mathrm{disp}(T) = r$.

Now assume

$$E \cap [0, p_0] \neq \emptyset.$$

Then $e_1 \leq p_0$. Remove $[0, p_0]$ from the partition in (19). Then two intervals in the partition

$$(p_0, p_1], (p_1, p_2], \ldots, (p_{n_r-1}, p_{n_r}], (p_{n_r}, 1]$$

must not intersect E. If there exists $i < n_r$ such that $e_m < p_{i+1}$, then construct the box

$$B = [0, 1] \times [p_i, 1] \times [0, 1]^{d-2}.$$

It follows that $B \cap E = \emptyset$, however,

$$\mathrm{Vol}(B) = (1 - p_i) > p_{i+1}(1 - p_i) = r.$$

This contradicts the assumption that $\mathrm{disp}(T) = r$. Let $i < n_r$ be the smallest integer such that

$$E \cap (p_i, p_{i+1}] = \emptyset.$$

Assume $0 < k \leq m$ is the smallest integer such that $p_{i+1} < e_k$. Construct the box

$$B = [0, e_k) \times [p_i, 1] \times [0, 1]^{d-2}.$$

Since T is on the diagonal $B \cap E = \emptyset$, however,

$$\mathrm{Vol}(B) = e_k(1 - p_i) > p_{i+1}(1 - p_i) = r.$$

This contradicts the assumption that $\mathrm{disp}(T) = r$.
The Proposition follows. □

Proposition 5.2. *Let $i > 0$. Let r_i be as defined in (14). The configuration given by $\mathfrak{q}(r_i)$ in* (16) *are symmetric on the diagonal. That is, if $0 \leq j \leq i-1$, then*

$$1 - f_j(r_i) = f_{(i-1)-j}(r_i).$$

Proof. Let $i > 0$. Let r_i be as defined in (14). Employ an inductive argument on j. Let $j = 0$. Then by Proposition 3.2 it follows that

$$1 - f_0(r_i) = 1 - r_i = f_{i-1}(r_i).$$

Let $j = 1$. Then by definition of the functions, and by Proposition 3.2

$$f_{i-1}(r_i)(1 - f_{i-2}(r_i)) = \frac{r_i}{(1 - f_{i-2}(r_i))}(1 - f_{i-2}(r_i)) = r_i.$$

By Proposition 3.2, $f_{i-1}(r_i) = 1 - r_i$. It follows that

$$(1 - r_i)(1 - f_{i-2}(r_i)) = r_i.$$

Then

$$1 - f_{i-2}(r_i) = \frac{r_i}{1 - r_i} = f_1(r_i),$$

in particular

$$1 - f_1(r_i) = f_{i-2}(r_i) = f_{(i-1)-1}(r_i).$$

Fix $0 \leq j < i - 1$, and assume the induction hypothesis, for all $0 \leq k \leq j$. Namely that

$$1 - f_k(r_i) = f_{(i-1)-k}(r_i).$$

We show that

$$1 - f_{j+1}(r_i) = f_{(i-1)-(j+1)}(r_i).$$

By the induction hypothesis

$$1 - f_j(r_i) = f_{(i-1)-j}(r_i).$$

By construction of the functions, we have that

$$f_{(i-1)-j}(r_i)(1 - f_{(i-1)-j-1}(r_i)) = \frac{r_i}{(1 - f_{(i-1)-j-1}(r_i))}(1 - f_{(i-1)-j-1}(r_i)) = r_i.$$

Therefore,

$$1 - f_{(i-1)-j-1}(r_i) = \frac{r_i}{(1 - f_j(r_i))} = f_{j+1}(r_1).$$

It follows that

$$1 - f_{j+1}(r_1) = f_{(i-1)-(j+1)}(r_i).$$

The Proposition follows. $\square$

5.2 Dispersion of Subsets of the Extended Diagonal

Definition 5.3. *Let $r \in (\frac{1}{4}, \frac{1}{2}]$, and $d > 1$. Let $P(r)$ be given by (17), and denoted as $P(r) = \{p_i : p_i = f_i(r), 0 \leq i \leq n_r\}$. Define the Extended Diagonal as*

$$\mathfrak{D}(r, d) = [0, p_0]^d \cup (p_0, p_1]^d \cup \cdots \cup (p_{n_r}, 1]^d.$$

Definition 5.4. *Let $d > 1$. Let $x = (x_1, x_2, \ldots, x_d) \in [0, 1]^d$. Define*

$$s(x) = \min\{x_i : 1 \leq i \leq d\}.$$

Proposition 5.5. *Let $r \in (\frac{1}{4}, \frac{1}{2}]$, and $d > 1$. Let $A \subset \mathfrak{D}(r, d)$ be such that*

$$\operatorname{disp}(A) = r.$$

Then

$$|A| \geq n_r + 1.$$

Proof. Assume that the hypothesis holds. Let n_r be as in (6), and let $\mathfrak{q}(r)$ be the configuration as in (16). Define

$$C_0 = [0, p_0]^d, \; C_{n_r+1} = (p_{n_r}, 1]^d.$$

For $0 < i \leq n_r$ define

$$C_i = (p_{i-1}, p_i]^d.$$

Assume toward a contradiction that $|A| \leq n_r$. The n_r points contained in A must lie in the $n_r + 2$ disjoint sets in $\{C_i : 0 \leq i \leq n_r + 1\}$ composing $\mathfrak{D}(r, d)$. There exist at least two integers $i \leq n_r$, such that $A \cap C_i = \emptyset$.

First assume that $A \cap C_0 = \emptyset$. Since $A \subset \mathfrak{D}$, it follows that $s(p) > p_0$ for each $p \in A$. Let

$$t = \min\{s(p) : p \in A\}.$$

Construct the box $B = [0, t) \times [0, 1]^{d-1}$. The magnitude of the components of each $p \in A$ is bounded below by t. It follows that $A \cap B = \emptyset$, however, $\mathrm{Vol}(B) = t > p_0 = r$. This contradicts the assumption that $\mathrm{disp}(A) = r$.

Now assume that $A \cap C_0 \neq \emptyset$. Let $1 \leq i \leq n_r$ be the smallest number such that $A \cap C_i = \emptyset$. Assume that for all $j > i$, $A \cap C_j = \emptyset$. Then set $t = 1$. Construct a box

$$B = [0, t] \times (p_{i-1}, 1] \times [0, 1]^{d-2}.$$

Since $A \subset \mathfrak{D}$, it follows that $B \cap A = \emptyset$, however,

$$\mathrm{Vol}(B) = (1 - p_{i-1}) > p_i(1 - p_{i-1}) = r.$$

This contradicts the assumption that $\mathrm{disp}(A) = r$. Assume that $A \cap C_j \neq \emptyset$ for some smallest $j > i$. Then set

$$t = \min\{s(p) : p \in A \cap C_j\}.$$

Construct a box

$$B = [0, t) \times (p_{i-1}, 1] \times [0, 1]^{d-2}.$$

Since $A \subset \mathfrak{D}$, it follows that $A \cap B = \emptyset$, however,

$$\mathrm{Vol}(B) = t(1 - p_{i-1}) > p_i(1 - p_{i-1}) = r.$$

This contradicts the assumption that $\mathrm{disp}(A) = r$. It follows that

$$n_r + 1 \leq |A|.$$

$\square$

Remark 5.6. *The condition $d > 1$ in Proposition 5.5 is required. Let $r = \frac{1}{3}$ and $d = 1$. Note $\mathfrak{q}(r) = \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$. Define*

$$U_0 = [0, r], U_1 = (r, f(r)], U_2 = (f(r), 1 - r], U_3 = (1 - r, 1].$$

Then by definition
$$\mathfrak{D}(r,1) = U_0 \cup U_1 \cup U_2 \cup U_3.$$
Let $A \subset \mathfrak{D}(r,1)$ be the set $\{\frac{1}{3}, \frac{2}{3}\}$. Then
$$\mathrm{disp}(A) = \frac{1}{3}.$$
Since $|A| = 2$, and
$$n_r + 1 = |\mathfrak{q}(r)| = 3$$
the Proposition fails. This example shows that Proposition 5.5 only holds when $d > 1$.

5.3 Dispersion Dependent Configurations

Proposition 5.7. *Let $d \geq 2$. Let r_n be as in (14) and let $\mathfrak{q}(r_n)$ be as in (16). Let $A \subset \mathfrak{D}(r_n, d)$, be such that $|A| = n$, and*
$$\mathrm{disp}(A) = r_n. \tag{20}$$
Then
$$A = \mathfrak{q}(r_n).$$

Proof. Assume the hypothesis. Note $A \neq \emptyset$. For all $0 \leq i \leq n$ define C_i to be as in the proof of Proposition 5.5. Assume that for all $0 \leq i \leq n$, $A \cap C_i \neq \emptyset$. Then it follows that for all $0 \leq i \leq n-1$, $A \cap C_i = \{p_i\mathbf{1}\}$. Hence, $A = \mathfrak{q}(r_n)$. We state here that for all $0 \leq i \leq n-1$, $A \cap C_i \neq \emptyset$. This follows directly from the proof of Proposition 5.5: thus we shall omit the details to avoid repetition. Then for all $0 \leq i \leq n-1$, $|A \cap C_i| = 1$. Now apply induction on i to show that for all $1 \leq i \leq n$, $A \cap C_{n-i} = \{p_{n-i}\mathbf{1}\}$.

Let $i = 1$. Assume toward a contradiction that
$$A \cap C_{n-1} \neq \{p_{n-1}\mathbf{1}\}.$$

Let $p \in A \cap C_{n-1} \setminus \{p_{n-1}\mathbf{1}\}$. There exists a maximum component of p which is less than p_{n-1}. Without loss of generality, assume that this is the first component. Denote the magnitude of this component as $b(p) < p_{n-1}$. Construct the box

$$B = (b(p), 1] \times [0, 1]^{d-1}.$$

Since $A \subset \mathfrak{D}$, it follows that $A \cap B = \emptyset$. However,

$$\mathrm{Vol}(B) = 1 - b(p) > 1 - p_{n-1} = r_n.$$

This contradicts (20). Therefore, $A \cap C_{n-1} = \{p_{n-1}\mathbf{1}\}$.

Fix $0 < m \leq n - 1$. As an induction hypothesis assume that for all $m \leq k \leq n - 1$, $A \cap C_k = \{p_k\mathbf{1}\}$. Assume toward a contradiction that $A \cap C_{m-1} \neq \{p_{m-1}\mathbf{1}\}$. Let $p \in A \cap C_{m-1} \setminus \{p_{m-1}\mathbf{1}\}$. There exists a maximum component of p which is less than p_{m-1}. Without loss of generality, assume that this is the first component. Denote the magnitude of this component as $b(p) < p_{m-1}$. Construct the box

$$B = (b(p), 1] \times [0, p_m) \times [0, 1]^{d-1}.$$

Since $A \subset \mathfrak{D}$, it follows that $A \cap B = \emptyset$. Then

$$\mathrm{Vol}(B) = p_m(1 - b(p)) > p_m(1 - p_{m-1}) = r_n.$$

This contradicts (20). Therefore, it follows that $A \cap C_{m-1} = \{p_{m-1}\mathbf{1}\}$. Hence, for all $0 \leq k \leq n - 1$, $A \cap C_k = \{p_k\mathbf{1}\}$. It follows that $A = \mathfrak{q}(r_n)$. $\square$

5.4 Bound Sharpness

Now we show that the bound in Corollary 4.6 is sharp, given that d is large enough. Recall that for each $r \in (\frac{1}{4}, \frac{1}{2}]$, there exists $n > 0$ such that $r_n \leq r < r_{n-1}$, and

$$n_r + 1 = \alpha(r) = n.$$

Theorem 5.8. *Let $r \in (\frac{1}{4}, \frac{1}{2}]$. Then*

$$N(r, d) = n_r + 1,$$

given that $d \geq n^{n-1} + 1$, where $n = \alpha(r)$.

Proof. Let $r = \frac{1}{2} = r_1$, then for all $d \geq 2$,

$$N(r,d) = 1 = n_{r_1} + 1.$$

Let $r \in (\frac{1}{4}, \frac{1}{2})$ be such that $r_2 \leq r < r_1$. Let $d \geq 3$. Define $U_0 = [0, r_1]$, $U_1 = (r_1, 1]$, and denote $[0,1]^d = (U_0 \cup U_1)^d$. Assume toward a contradiction that there exists $q_1 \in [0,1]^d$ such that

$$\text{disp}(\{q_1\}) = r.$$

Then either $q_1 \in [0,1]^{d-1} \times U_0$ or $q_1 \notin [0,1]^{d-1} \times U_0$. Assume $q_1 \in [0,1]^{d-1} \times U_0$. Construct the box $B = [0,1]^{d-1} \times U_1$. Then $q_1 \notin B$, and

$$\text{Vol}(B) = r_1 = \frac{1}{2} > r.$$

This contradicts the assumption that $\text{disp}(\{q_1\}) = r$. Assume $q_1 \notin [0,1]^{d-1} \times U_0$. Construct the box $B = [0,1]^{d-1} \times U_0$. Then $q_1 \notin B$, however,

$$\text{Vol}(B) = r_1 > r.$$

This contradicts the assumption that $\text{disp}(\{q_1\}) = r$. Then $1 < N(r,d)$, and by Corollary 4.5, $N(r,d) \leq 2$. It follows that

$$N(r,d) = 2.$$

Let r be such that $r_3 \leq r < r_2$. Since $\alpha(r) = 3$, $d \geq 10$. Define $U_0 = [0, r_2]$, $U_1 = (r_2, 1-r_2]$, $U_2 = (1-r_2, 1]$. Select two points $q_1, q_2 \in [0,1]^d$. Assume toward a contradiction that $\text{disp}(\{q_1, q_2\}) = r$. The components of q_1 and q_2 are contained in the intervals U_0, U_1, U_2. Denote the components of q_1, q_2 as $\{q_{1,i}\}_{1 \leq i \leq d}$, $\{q_{2,i}\}_{1 \leq i \leq d}$. Let $M_1 \geq 4$ denote the largest number of components of $\{q_{1,i}\}_{1 \leq i \leq d}$ contained in a single interval, denoted as U_{m_1}. Denote the corresponding indices as $\{a_i\}_{1 \leq i \leq M_1} := \{a_i : a_1 < a_2 < \cdots < a_{M_1}\}$. Let $M_2 \geq 2$ denote the largest number of components of $\{q_{2,a_i}\}_{1 \leq i \leq M_1}$ contained in a single interval, denoted as U_{m_2}. Denote the corresponding indices as $\{b_i : 1 \leq i \leq M_2\} \subset \{a_i\}_{1\leq i\leq M_1}$. It follows that

$$q_{1,b_1}, q_{1,b_2} \in U_{m_1},$$

$$q_{2,b_1}, q_{2,b_2} \in U_{m_2}.$$

Project onto the components,

$$q_1 \to (0, 0, \ldots, 0, q_{1,b_1}, 0, 0, \ldots, 0, q_{1,b_2}, 0, 0, \ldots, 0) = q_1',$$

and

$$q_2 \to (0, 0, \ldots, 0, q_{2,b_1}, 0, 0, \ldots, 0, q_{2,b_2}, 0, 0, \ldots, 0) = q_2'.$$

The points are projected onto a 2-dimensional face of $[0, 1]^d$, given by

$$\{0\}^{b_1-1} \times [0, 1] \times \{0\}^{b_2-(1+b_1)} \times [0, 1] \times \{0\}^{d-b_2}.$$

Note that

$$q_1' \in \{0\}^{b_1-1} \times U_{m_1} \times \{0\}^{b_2-(1+b_1)} \times U_{m_1} \times \{0\}^{d-b_2},$$

and

$$q_2' \in \{0\}^{b_1-1} \times U_{m_2} \times \{0\}^{b_2-(1+b_1)} \times U_{m_2} \times \{0\}^{d-b_2}.$$

The components q_1', q_2' are contained in $\mathfrak{D}(r_2, 2)$. By Proposition 5.7 there exists $B' \in \mathfrak{B}$ such that

$$B' = \{0\}^{b_1-1} \times I_1 \times \{0\}^{b_2-(1+b_1)} \times I_2 \times \{0\}^{d-b_2},$$

where $q_1', q_2' \notin B'$. However, $\mathrm{Vol}(I_1 \times I_2) \geq r_2$. Let

$$B = [0, 1]^{b_1-1} \times I_1 \times [0, 1]^{b_2-(1+b_1)} \times I_2 \times [0, 1]^{d-b_2}.$$

It is clear that $q_1, q_2 \notin B$. However, $\mathrm{Vol}(B) \geq r_2 > r$. This contradicts the assumption that $\mathrm{disp}(\{q_1, q_2\}) = r$. Then $N(r, d) > 2$, and by Corollary 4.5,

$N(r,d) \leq 3$. It follows that

$$N(r,d) = 3.$$

Fix $n > 3$, and let $r_{n+1} \leq r < r_n$. Since $\alpha(r) = n+1$,

$$d \geq (n+1)^n + 1.$$

Let $q_1, q_2, \ldots, q_n$ be arbitrary points in the cube. Assume toward a contradiction that $\mathrm{disp}(\{\mathrm{q}_1, \mathrm{q}_2, \ldots, \mathrm{q}_\mathrm{n}\}) = r$. Define the partition

$$U_0 = [0, r_n], U_1 = (r_n, f_1(r_n)], \ldots, U_{n+1} = (1 - r_n, 1].$$

For all $1 \leq i \leq n$, denote the components of q_i as $(q_{i,1}, q_{i,2}, \ldots, q_{i,d})$. Denote $d_1 := (n+1)^n + 1$. Let

$$M_1 \geq d_2 := \frac{d_1 - 1}{(n+1)} + 1$$

denote the largest number of components of $\{q_{1,i}\}_{0<i\leq d}$ which are contained in a single interval, denoted as U_{m_1}. Denote the corresponding indices as

$$\{a_{1,i}\}_{1\leq i\leq M_1}.$$

Let

$$M_2 \geq d_3 := \frac{d_2 - 1}{(n+1)} + 1$$

denote the largest number of components of $\{q_{2,a_{1,i}}\}_{1\leq i\leq M_1}$ which are contained in a single interval, denoted as U_{m_2}. Denote the corresponding indices as $\{a_{2,i}\}_{1\leq i\leq M_2}$.

For all $1 \leq k \leq n$ let

$$M_k \geq d_{k+1} := \frac{d_k - 1}{(n+1)} + 1$$

denote the largest number of components of $\{q_{k,a_{k-1,i}}\}_{1\leq i\leq M_{k-1}}$ which are contained in U_{m_k}. Denote the corresponding indices as $\{a_{k,i}\}_{1\leq i\leq M_k}$. This guarantees that $M_n \geq 2$. Define a projection on $1 \leq j \leq n$, such that

$$q_j \to q_{j,a_{n,1}}, q_{j,a_{n,2}} \in U_{m_j}.$$

This embeds into $\mathfrak{D}(r_n, 2)$. Then by Proposition 5.7, there exists a box B' such that $q_j' \notin B'$ with I_1, I_2, such that

$$\mathrm{Vol}(I_1 \times I_2) \geq r_n.$$

This can be extended to a box

$$B = [0,1]^{a_{n,1}-1} \times I_1 \times [0,1]^{a_{n,2}-(1+a_{n,1})} \times I_2 \times [0,1]^{d_1 - a_{n,2}}.$$

It is clear that $q_1, q_2, \ldots, q_n \notin B$ and that $\mathrm{Vol}(B) \geq r_n > r$. This contradicts the assumption that $\mathrm{disp}(\{q_1, q_2, \ldots, q_n\}) = r$. Then $n < N(r, d)$, and by Corollary 4.5, $N(r, d) \leq n + 1$. It follows that

$$N(r, d) = n + 1.$$

□

6 Concluding Remarks

6.1 Construction when $r = \frac{1}{4}$

When $r = \frac{1}{4}$, and d is small the following configurations are better than the best known bound which asymptotically is $\log(d)$. We present a configuration which is easy to describe and visualize.

Proposition 6.1. *Let $r \geq \frac{1}{4}$. Then $N(r, d) \leq 2d$.*

Proof. Let $d = 2$ let

$$K = \{(\frac{1}{2}, \frac{1}{4}), (\frac{1}{2}, \frac{3}{4}), (\frac{1}{4}, \frac{1}{2}), (\frac{3}{4}, \frac{1}{2})\}.$$

Every box $B \cap K = \emptyset$ inside $[0,1]^2$ is contained in one of the following

$$
\begin{array}{llll}
[0,\frac{1}{2})\times[0,\frac{1}{2}), & [0,\frac{1}{2})\times(\frac{1}{2},1], & (\frac{1}{2},1]\times[0,\frac{1}{2}), & (\frac{1}{2},1]\times(\frac{1}{2},1], \\
(\frac{1}{4},\frac{3}{4})\times(\frac{1}{4},\frac{3}{4}), & [0,\frac{1}{4})\times[0,1], & (\frac{3}{4},1]\times[0,1], & [0,1]\times(\frac{3}{4},1], \\
[0,1]\times[0,\frac{1}{4}), & (\frac{1}{4},\frac{1}{2}),\times[0,1] & [0,1]\times(\frac{1}{4},\frac{1}{2}), & (\frac{1}{2},\frac{3}{4})\times[0,1], \\
[0,1]\times(\frac{1}{2},\frac{3}{4}), & & &
\end{array} \tag{21}
$$

This gives the result in the 2 dimensional case. Let $d > 2$. Let

$$K_1 = \left\{ \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, M_i, \frac{1}{2}, \dots, \frac{1}{2} \right) : M_i = \frac{1}{4},\ 1 \leq i \leq d \right\}$$

$$K_2 = \left\{ \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, M_i, \frac{1}{2}, \dots, \frac{1}{2} \right) : M_i = \frac{3}{4},\ 1 \leq i \leq d \right\}.$$

Let $K = K_1 \cup K_2$. Then each box in $B \in [0,1]^d$ such that $B \cap K = \emptyset$ is contained in a product of $d-2$ intervals $[0,1]$ with one of the boxes in (21). Each of the boxes have volume $\frac{1}{4}$. Therefore,

$$\mathrm{disp}(K) = \frac{1}{4},$$

and $|K| = 2d$. $\square$

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