Super Yangians and Quantum Loop Superalgebras

by

Peggy Jankovic

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Department of Mathematical and Statistical Sciences
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Abstract

The two families of quantum groups of Yangians and quantum loop algebras have a library of similar results on their structures and representation theories. Not until Gautam and Toledano-Laredo's works in the past decade [GTL13] [GTL16] has there been an explanation for these correspondences. We present a selection of results on super Yangians and quantum loop superalgebras for the Lie superalgebras $\mathfrak{gl}_{M|N}$ and $\mathfrak{q}_N$. These suggest that a similar connection may exist in the superalgebra case.

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CONTENTS

1

INTRODUCTION

The story behind quantum groups is one that began in physics literature in the early 1980s. They were described as "associative algebras whose defining relations are expressed in terms of a matrix of constants, called a quantum R-matrix" [CP94]. By 1985, V. G. Drinfeld and M. Jimbo independently noted that these quantum groups were Hopf algebras. In particular, many were deformations of the universal enveloping algebras of Lie algebras. This connection led to the study of quantum groups from the perspective of algebraic groups and Lie algebras without any theoretical physics involved. Meanwhile, the quantum Yang-Baxter Equation first surfaced in the mid 1960s as a condition for a quantum mechanical many-body problem. From there, it has been established as the "master equation in integrable models in statistical mechanics and quantum field theory" [Jim89], though its significance reaches other fields of study. In particular, interest in quantum groups has been fueled by the relationship between quantum groups, the quantum Yang-Baxter equation, and the theories of representations of Lie algebras in prime characteristic and of invariants of links and 3-manifolds. Notably, two families of infinite-dimensional quantum groups are *Yangians* and *quantum loop algebras*. Their historical importance lies in the ability to construct solutions to the quantum Yang-Baxter equation from their finite-dimensional representations.

In this thesis, the goal is to examine the similarities and connections in representation theory between two families of quantum groups in the superalgebra case: the quantum loop superalgebras and super Yangians. Recent work by Sachin Gautam and Valerio Toledano-Laredo [GTL13] [GTL16] has provided background for why these similarities exist in the non-super case, for semi-simple Lie algebras. These similarities have been noted for decades now. One can see an example of this in comparing the classification of finite dimensional representations in the two cases, as seen in Theorems 2.6 and 2.9 and the recurring use of Drinfeld polynomials. However, up until recently, the explanation behind these similarities had not been explained. Here, we investigate some results that work in two superalgebra cases. Theorem 4.14 and Theorem 4.18 specify the necessary and sufficient conditions for the irreducibility of Verma modules over $Y(\mathfrak{gl}_{M|N})$ and $U_q(\mathfrak{gl}_{M|N}[s])$ respectively. We show the degeneration process to obtain the super Yangian from its associated quantum loop superalgebra in Theorem 5.4 for $\mathfrak{gl}_{M|N}$ and Theorem 6.15 for $\mathfrak{q}_N$. These suggest that we may be able to assert that a link between quantum loop superalgebras and super Yangians exists in an analogous way to the link established by Gautam and Toledano-Laredo. This will not be explored in this thesis, but could be a future research goal.

In the second chapter, we establish the definitions of the structures used in this thesis, as well as preliminary results on them. This includes general definitions of Yangians and quantum loop (super)algebras. Further, we look at more specific examples for $\mathfrak{gl}_N, \mathfrak{gl}_{M|N}$, and $\mathfrak{q}_N$. We will also define Verma modules on some of these quantum groups.

Chapter 3 reviews three major works fundamental to the main ideas of this thesis [GM12] [GTL13] [GTL16]. This will provide background on the close ties between Yangians and quantum loop algebras. In particular, we will see how the Yangian can be derived as a limit form of a quantum loop algebra in Theorem 3.3, how an isomorphism between the completions of the two quantum groups is constructed in Theorem 3.6, and how certain categories of representations are equivalent in Theorem 3.12.

Next, Chapter 4 focusses on some aspects of the representation theory of the super Yangian $Y(\mathfrak{gl}_{M|N})$. In this chapter, the goal is to prove that a few of the results that apply to Yangians, specifically the results given by Billig, Futorny, and Molev [BFM06], also apply analogously to super Yangians. While first looking at the $\mathfrak{gl}_{1|1}$ case, we arrive at Theorem 4.9, which asserts that a Verma module over $Y(\mathfrak{gl}_{1|1})$ is irreducible if and only if its irreducible quotient is finite dimensional. This is followed by the significant Theorem 4.14, which allows us to parametrize the irreducibility of a Verma module by its highest weight. Further, we will find an analogous set of results for a subalgebra of the quantum loop superalgebra that we call the quantum current superalgebra. This centres around determining the dependency between irreducible Verma modules and their finite dimensional irreducible quotients (Theorem 4.17) for $U_q(\mathfrak{gl}_{1|1}[s])$ and showing when a Verma module over $U_q(\mathfrak{gl}_{M|N}[s])$ is irreducible (Theorem 4.18).

What follows in the fifth chapter is the degeneration process of obtaining the Yangian $Y(\mathfrak{gl}_{M|N})$ from the quantum loop superalgebra $Y_q(\mathcal{L}\mathfrak{gl}_{M|N})$. The bulk of this requires the creation of more convenient sets of generators and the manipulation of defining relations.

Finally, in Chapter 6, we follow the degeneration process again, but in the type $\mathfrak{q}$ setting. Using similar techniques as in the previous chapter, we define an isomorphism from $Y(\mathfrak{q}_N)$ to a sum of quotient subspaces of $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$.

The final two chapters heavily suggest that there exists an isomorphism between completions of super Yangians and quantum loop superalgebras. This would be an analogous result to the isomorphism constructed in the non-super case by Gautam and Toledano-Laredo, as seen as Theorem 3.6 in Chapter 3.

2

DEFINITIONS AND BASIC PROPERTIES

What follows are the necessary definitions and preliminary results for exploring the results presented later in this thesis. In some cases, we offer multiple presentations of a single structure; this is because we may need to choose one or the other depending on which set of relations best suits the situation.

2.1 GENERAL SETTING

2.1.1 *Yangians*

Originally, V.G. Drinfeld introduced the term *Yangian* to refer to quantum groups related to rational solutions of the Yang-Baxter Equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

for an R-matrix R, which is a tensor product of matrices. The latter form uses the notation $R_{ij} = \phi_{ij}(R)$ for the algebra morphisms $\phi_{ij} : A \otimes A \to A \otimes A \otimes A$, A a unital associative algebra. In this thesis, $A = \mathrm{End}(V)$ for a vector space V. The index i, j indicates where an element $a \otimes b$ is sent; $\phi_{12}(a \otimes b) = a \otimes b \otimes 1$, $\phi_{13}(a \otimes b) = a \otimes 1 \otimes b$, and $\phi_{23}(a \otimes b) = 1 \otimes a \otimes b$. These equations, named by Faddeev in the late 1970s, capture the properties of transformations in various fields of study within mathematics and physics, with applications in electric networks, braid groups, and spin models, among others [PA06].

Before examining more specific cases, let us first define the Yangian for a complex, semisimple Lie algebra $\mathfrak{g}$. First, we recall some basic associated objects and quantities for the Lie algebra. If we let $(\cdot, \cdot)$ be its non-degenerate invariant bilinear form, $\mathfrak{h}$ denote its Cartan algebra, then we can produce a Cartan matrix A. Its entries are defined

$$a_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

where the α_i are elements of $\mathfrak{h}^*$ that form a basis of simple roots of $\mathfrak{g}$. Let $d_i = \frac{(\alpha_i, \alpha_i)}{2}$. Recall that if we let the Cartan matrix be $A = (a_{ij})$, then $D = \sum_{i \in \mathbf{I}} d_i E_{ii}$ is a matrix such that DA is symmetric and the d_i are coprime integers. Then, we can define the Yangian of $\mathfrak{g}$, with the addition of a parameter $\hbar$.

Definition 2.1. Let the Yangian $Y_\hbar(\mathfrak{g})$ be the associative $\mathbb{C}[\hbar]$-algebra with identity generated by the elements $x_{i,r}^{\pm}$ and $h_{i,r}$ where i is an element of the set $\mathbf{I}$ of vertices of the Dynkin diagram of $\mathfrak{g}$ and r is a natural number. These are subject to the following relations for $i, j \in \mathbf{I}$ and $r, s \in \mathbb{N}$:

$$[h_{i,r}, h_{j,s}] = 0 \tag{2.1.1}$$

$$\left[h_{i,0}, x_{j,s}^{\pm}\right] = \pm d_i a_{ij} x_{j,s}^{\pm} \tag{2.1.2}$$

$$\left[h_{i,r+1}, x_{j,s}^{\pm}\right] - \left[h_{i,r}, x_{j,s+1}^{\pm}\right] = \pm \frac{d_i a_{ij} \hbar}{2} (h_{i,r} x_{j,s}^{\pm} + x_{j,s}^{\pm} h_{i,r}) \tag{2.1.3}$$

$$\left[x_{i,r+1}^{\pm}, x_{j,s}^{\pm}\right] - \left[x_{i,r+1}^{\pm}, x_{j,s}^{\pm}\right] = \pm \frac{d_i a_{ij} \hbar}{2} (x_{i,r}^{\pm} x_{j,s}^{\pm} + x_{j,s}^{\pm} x_{i,r}^{\pm}) \tag{2.1.4}$$

$$\left[x_{i,r}^{+}, x_{j,s}^{-}\right] = \delta_{ij} h_{i,r+s} \tag{2.1.5}$$

$$\sum_{\pi \in S_m} \left[x_{i,r_{\pi(1)}}^{\pm}, \left[x_{i,\pi(2)}^{\pm}, \cdots, \left[x_{i,r_{\pi(m)}}^{\pm}, x_{j,s}^{\pm}\right] \cdots\right]\right] = 0 \tag{2.1.6}$$

For the last relation (2.1.6), set $m = 1 - a_{ij}$ where $i \neq j$.

We equip the Yangian with a grading with the assignment of $\deg x^{\pm}_{i,r,} = r = \deg h_{i,r}$ and $\deg \hbar = 1$. For example, a monomial

$$\hbar^a x^+_{i_1,r_1} x^-_{i_2,r_2} \dots h_{i_k,r_k}$$

will have degree $a + r_1 + r_2 + \cdots + r_k$. We can also see that both sides of the relations above have the same degree or are zero; for instance, the left and right hand side of (2.1.4) are each of degree $r + s + 1$.

The generators of $Y_\hbar(\mathfrak{g})$ form the coefficients of the following elements of $Y_\hbar(\mathfrak{g})[[u^{-1}]]$

$$h_i(u) = 1 + \hbar \sum_{r \geq 0} h_{i,r} u^{-r-1} \tag{2.1.7}$$

$$x^{\pm}_i(u) = \hbar \sum_{r \geq 0} x^{\pm}_{i,r} u^{-r-1} \tag{2.1.8}$$

Note. A slightly different presentation of the Yangian exists, which is essentially equivalent to the above but with a change in notation. This one is more consistent with the notation used by Billig, Futorny, and Molev [BFM06]. We write $e_i^{(r)}$ for $x^+_{i,r}$, $f_i^{(r)}$ for $x^-_{i,r}$, and $h_i^{(r)}$ for $h_{i,r}$.

2.1.2 *Quantum Loop Algebras*

Definition 2.2. For a Lie algebra $\mathfrak{g}$, we let its *loop algebra* be $\mathcal{L}(\mathfrak{g}) = \mathfrak{g}[s, s^{-1}] = \mathfrak{g} \otimes \mathbb{C}[s, s^{-1}]$ for some indeterminate s.

$\mathcal{L}(\mathfrak{g})$ is isomorphic to the space of Laurent polynomial maps $\mathbb{C}^\times \to \mathfrak{g}$ [CP94]. Before presenting the definition of the quantum loop algebra of $\mathfrak{g}$, we introduce the following notation in order to increase readability of the relations. To denote quantum loop algebra, we can either write $U_\hbar(\mathcal{L}\mathfrak{g})$ or $U_q(\mathcal{L}\mathfrak{g})$; we relate the two variables $\hbar$ and q by the equation $q^2 = e^\hbar$. Further, we let $q_i := q^{d_i} = e^{\hbar d_i/2}$. Finally, we recall the *q-binomial coefficients* (sometimes called *Gaussian integers*) in $\mathbb{Q}(q)$:

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

which satisfy

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q \tag{2.1.9}$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}. \tag{2.1.10}$$

Definition 2.3. For a complex, semisimple Lie algebra $\mathfrak{g}$, let $U_\hbar(\mathcal{L}\mathfrak{g})$ be its quantum loop algebra. It is an associative algebra with identity over $\mathbb{C}[[\hbar]]$ with generators $E_{i,k}$, $F_{i,k}$, and $H_{i,k}$ for $i \in \mathbf{I}$ and k ranging over the integers. These generators are subject to the relations below, where we let $i, j \in \mathbf{I}$ and $r, s, k \in \mathbb{Z}$.

$$[H_{i,r}, H_{j,s}] = 0 \tag{2.1.11}$$

$$[H_{i,0}, E_{j,k}] = a_{ij} E_{j,k} \tag{2.1.12}$$

$$[H_{i,0}, F_{j,k}] = -a_{ij} F_{j,k} \tag{2.1.13}$$

$$[H_{i,r}, E_{j,k}] = \frac{[ra_{ij}]_{q_i}}{r} E_{j,r+k} \text{ for nonzero } r \tag{2.1.14}$$

$$[H_{i,r}, F_{j,k}] = -\frac{[ra_{ij}]_{q_i}}{r} F_{j,r+k} \text{ for nonzero } r \tag{2.1.15}$$

$$E_{i,k+1}E_{j,l} - q_i^{a_{ij}} E_{j,l}E_{i,k+1} = q_i^{a_{ij}} E_{i,k}E_{j,l+1} - E_{j,l+1}E_{i,k} \tag{2.1.16}$$

$$F_{i,k+1}F_{j,l} - q_i^{-a_{ij}} F_{j,l}F_{i,k+1} = q_i^{-a_{ij}} F_{i,k}F_{j,l+1} - F_{j,l+1}F_{i,k} \tag{2.1.17}$$

$$[E_{i,k}, F_{j,l}] = \delta_{ij} \frac{\psi_{i,k+l} - \phi_{i,k+1}}{q_i - q_i^{-1}} \tag{2.1.18}$$

For $i \neq j$, $m = 1 - a_{ij}$, and $k_1, \ldots, k_m, l \in \mathbb{Z}$,

$$\sum_{\pi \in S_m} \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} E_{i,k_{\pi(1)}} \cdots E_{i,k_{\pi(s)}} E_{j,l} E_{i,k_{\pi(s+1)}} \cdots E_{i,k_{\pi(m)}} = 0 \quad (2.1.19)$$

$$\sum_{\pi \in S_m} \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} F_{i,k_{\pi(1)}} \cdots F_{i,k_{\pi(s)}} F_{j,l} F_{i,k_{\pi(s+1)}} \cdots F_{i,k_{\pi(m)}} = 0 \quad (2.1.20)$$

where the elements $\psi_{i,r}$ and $\phi_{i,r}$ are defined by the series

$$\psi_i(z) = \sum_{r \geq 0} \psi_{i,r} z^{-r} = \exp\left(\frac{\hbar d_i}{2}\right) \exp\left((q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,s} z^{-s}\right) \quad (2.1.21)$$

$$\phi_i(z) = \sum_{r \geq 0} \phi_{i,-r} z^{r} = \exp\left(-\frac{\hbar d_i}{2}\right) \exp\left(-(q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,-s} z^{s}\right) \quad (2.1.22)$$

satisfying $\psi_{i,-k} = \phi_{i,k} = 0$ for all $k \geq 1$.

If given the quantum affine algebra instead, we can produce the corresponding quantum loop algebra by taking the quotient of the quantum affine algebra by the ideal generated by $\mathcal{C}^{1/2} - 1$, where $\mathcal{C}^{\pm 1/2}$ are central elements.

Note. We can also use the generators $\mathcal{X}_{i,r}^{\pm}$ and $\mathcal{H}_{i,r}$ instead, with $\mathcal{X}_{i,r}^{+}$ corresponding to $E_{i,r}$, $\mathcal{X}_{i,r}^{-}$ corresponding to $F_{i,r}$, and $\mathcal{H}_{i,r}$ corresponding to $H_{i,r}$. This is consistent with the notation used in [GM12] and will be used primarily in Section 3.1.

2.1.3 *Classification of finite-dimensional representations and Drinfeld polynomials*

This subsection introduces some of the most fundamental concepts to this thesis: highest weight representations and Drinfeld polynomials. In Chapter 4, we will revisit Verma modules in the $\mathfrak{gl}_{M|N}$ case.

Definition 2.4. Let $\mathbf{c} = \{c_{i,r}\}_{r\in\mathbb{N}, i\in\mathbf{I}}$ denote a collection of complex numbers. Take an element $\lambda \in \mathfrak{h}$ in the dual of the Cartan subalgebra such that $c_{i,0} = d_i\lambda(\alpha_i^\vee)$. A representation $M(\lambda, \mathbf{c})$ of $Y_\hbar(\mathfrak{g})$ is called a *highest weight representation* of highest weight $(\lambda, \mathbf{c})$ if there exists a highest weight vector $1_\lambda \in M(\lambda, \mathbf{c})$ such that $M(\lambda, \mathbf{c}) = Y_\hbar(\mathfrak{g})1_\lambda$, $x_{i,r}^+ 1_\lambda = 0$, and $h_{i,r}1_\lambda = c_{i,r}1_\lambda$ for all $i \in \mathbf{I}, r \in \mathbb{N}$. Further, for any $h \in \mathfrak{h}$, $h1_\lambda = \lambda(h)1_\lambda$.

Denote the irreducible quotient of $M(\lambda, \mathbf{c})$ by $L(\lambda, \mathbf{c})$. This is the quotient of the Verma module $M(\lambda, \mathbf{c})$ by its unique maximal proper submodule.

Theorem 2.5. *Every irreducible finite dimensional representation of $Y_\hbar(\mathfrak{g})$ is a highest weight representation for a unique highest weight $(\lambda, \mathbf{c})$.*

Finally, we reach the classification theorem for irreducible modules.

Theorem 2.6. *The irreducible representation $L(\lambda, \mathbf{c})$ is finite dimensional if and only if there exist unique monic polynomials $\{P_i(u) \in \mathbb{C}[u]\}_{i\in\mathbf{I}}$ such that*

$$1 + \hbar \sum_{r\geq 0} c_{i,r} u^{-r-1} = \frac{P_i(u + d_i\hbar)}{P_i(u)}$$

We call the polynomials $P_i(u)$ *Drinfeld polynomials*. Chari and Pressley [CP94] offer a proof of this theorem.

We also have a parallel set of definitions and results for the quantum loop algebra $U_\hbar(\mathcal{L}(\mathfrak{g}))$.

Definition 2.7. Let $\gamma = \{\gamma_{i,\pm m}^\pm\}_{m\in\mathbb{N}, i\in\mathbf{I}}$ denote a collection of complex numbers. Take an element $\lambda \in \mathfrak{h}$ in the dual of the Cartan subalgebra such that $\gamma_{i,0}^\pm = q^{d_i\lambda(\alpha_i^\vee)}$. A representation $M(\lambda, \gamma)$ of $U_\hbar(\mathcal{L}(\mathfrak{g}))$ is called an *l-highest weight representation* of l-highest weight (λ, γ) if there exists a highest weight vector $1_\lambda \in M(\lambda, \gamma)$ such that $M(\lambda, \gamma) = Y_\hbar(\mathfrak{g})1_\lambda$, $E_{i,r}1_\lambda = 0$, and $\psi_{i,\pm m}^\pm 1_\lambda = \gamma_{i,\pm m}^\pm 1_\lambda$ for all $i \in \mathbf{I}, r \in \mathbb{N}$.

Again, we denote the irreducible quotient of $M(\lambda, \mathbf{c})$ by $L(\lambda, \mathbf{c})$.

Theorem 2.8. *Every irreducible finite dimensional representation of $U_\hbar(\mathcal{L}(\mathfrak{g}))$ is a highest weight representation for a unique highest weight (λ, γ).*

Theorem 2.9. *The irreducible representation $L(\lambda, \gamma)$ is finite dimensional if and only if there exist unique monic polynomials $\{P_i(z) \in \mathbb{C}[z]\}_{i \in \mathbf{I}}$ with $P_i(0) \neq 0$ such that*

$$\sum_{m \geq 0} \gamma^{+}_{i,m} z^{-m} = q^{-d_i \deg(P_i)} \frac{P_i(q^2 z)}{P_i(z)} = \sum_{m \leq 0} \gamma^{-}_{i,m} z^{m}$$

As for Yangians, we call the polynomials $P_i(z)$ *Drinfeld polynomials*. Again, Chari and Pressley provide an outline of the proof of this theorem [CP94].

2.1.4 *PBW Bases*

Definition 2.10. From any ordered basis of a Lie algebra, we can build a basis for its universal enveloping algebra. This *Poincaré-Birkhoff-Witt (PBW) basis* consists of monomials in the Lie algebra's basis elements.

This rough idea of what the PBW Theorem is will suffice as a general definition, as we will see concrete, relevant examples to Yangians and quantum loop algebras. Throughout this thesis, we take advantage of the existence, or conjectured existence, of bases of this nature.

Theorem 2.11. ***(Proposition 12.1.8 [CP94])*** *Fix an ordering on the generators $x^{\pm}_{i,r}$ and $h_{i,r}$ of $Y_\hbar(\mathfrak{g})$. Then, the set of all ordered monomials in these generators is a vector space basis of $Y_\hbar(\mathfrak{g})$.*

Theorem 2.12. *Fix an ordering on the generators $E_{i,k}$, $F_{i,k}$ and $H_{i,k}$ of $U_\hbar(\mathcal{L}\mathfrak{g})$. Then, the set of all ordered monomials in these generators is a vector space basis of $U_\hbar(\mathcal{L}\mathfrak{g})$.*

The Yangian of $\mathfrak{gl}_N$ can certainly be presented as in the previous general presentation, but we can use an RTT presentation as well. This will be a more convenient presentation in some contexts. Let the generators of $U(\mathfrak{gl}_N)$ be the elementary matrices $E_{ij} \in \mathrm{End}(\mathbb{C}^N)$ following conventional notation.

Definition 2.13. The *Yangian* $Y(\mathfrak{gl}_N)$ is generated by the elements $t_{ij}^{(r)}$, $r \geq 1$ subject to the following relation:

$$R_{12}T_1T_2 = T_2T_1R_{12} \tag{2.2.1}$$

where we define

$$\begin{aligned}
T = T(u) &= \sum_{i,j} t_{ij}(u) \otimes E_{ij} \\
t_{ij}(u) &= \delta_{ij} + \sum_{r} t_{ij}^{(r)} u^{-r} \\
T_1 = T_1(u) &= \sum_{i,j} t_{ij}(u) \otimes E_{ij} \otimes 1 \\
T_2 = T_2(v) &= \sum_{i,j} t_{ij}(v) \otimes 1 \otimes E_{ij} \\
R = R(u,v) &= 1 - (u-v)^{-1} \sum_{i,j} E_{ij} \otimes E_{ji}.
\end{aligned}$$

The relation (2.2.1) is often referred to as the *ternary relation* or *RTT relation*. With some calculations and substitutions, the ternary relation can be shown to be equivalent to the relation:

$$\left[t_{ij}^{(M+1)}, t_{kl}^{(L)}\right] - \left[t_{ij}^{(M)}, t_{kl}^{(L+1)}\right] = t_{kj}^{(M)} t_{il}^{(L)} - t_{kj}^{(L)} t_{il}^{(M)} \tag{2.2.2}$$

Note that we have an embedding of $U(\mathfrak{gl}_N)$ into $Y(\mathfrak{gl}_N)$ given by $E_{ij} \mapsto t_{ij}^{(1)}$. As well, we have a homomorphism $Y(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N)$ given by $T(u) \mapsto E(u) = 1 + u^{-1} \cdot \sum_{i,j} E_{ij} \otimes E_{ij}$. Then, letting $E_1 = 1 + u^{-1} \cdot \sum_{i,j} E_{ij} \otimes E_{ij} \otimes 1$ and $E_2 = 1 + v^{-1} \cdot \sum_{i,j} E_{ij} \otimes 1 \otimes E_{ij}$, we can rewrite the defining relations in $U(\mathfrak{gl}_N)$ as $R_{12}E_1E_2 = E_2E_1R_{12}$.

2.2.1 *Quantum Loop Algebra of* $\mathfrak{gl}_N$

Incidentally, none of the results in this thesis directly concern the quantum loop algebra $U_q(\mathcal{L}\mathfrak{gl}_N)$. This presentation of it is included for the sake of completion and comparison to $Y(\mathfrak{gl}_N)$. We use that $\mathcal{L}\mathfrak{gl}_N = \mathfrak{gl}_N(\mathbb{C}[s, s^{-1}])$.

Definition 2.14. The quantum loop algebra $U_q(\mathcal{L}\mathfrak{gl}_N)$ is a $\mathbb{C}(q)$-algebra with generators $T_{ij}^{(r)}, \overline{T}_{ij}^{(r)}$ over $1 \leq i, j \leq N, r \in \mathbb{Z}_{\geq 0}$. These are subject to the relations:

$$T_{ij}^{(0)} = \overline{T}_{ij}^{(0)} = 0 \text{ if } 1 \leq j < i \leq N \tag{2.2.3}$$

$$T_{ii}^{(0)}\overline{T}_{ii}^{(0)} = \overline{T}_{ii}^{(0)}T_{ii}^{(0)} = 1 \text{ for } 1 \leq i \leq N \tag{2.2.4}$$

$$R(u,v)T_2(v)T_1(u) = T_1(u)T_2(v)R(u,v) \tag{2.2.5}$$

$$R(u,v)\overline{T}_2(v)\overline{T}_1(u) = \overline{T}_1(u)\overline{T}_2(v)R(u,v) \tag{2.2.6}$$

$$R(u,v)T_2(v)\overline{T}_1(u) = \overline{T}_1(u)T_2(v)R(u,v) \tag{2.2.7}$$

where we define

$$T(u) = \sum_{i,j=1}^{n} t_{ij}(u) \otimes E_{ij} \tag{2.2.8}$$

$$\overline{T}(u) = \sum_{i,j=1}^{n} \bar{t}_{ij}(u) \otimes E_{ij} \tag{2.2.9}$$

$$t_{ij}(u) = \sum_{r=0}^{\infty} t_{ij}^{(r)} u^{-r} \tag{2.2.10}$$

$$\bar{t}_{ij}(u) = \sum_{r=0}^{\infty} \bar{t}_{ij}^{(r)} u^r. \tag{2.2.11}$$

2.2.2 *Representations and Drinfeld Polynomials*

In this section, we present the fundamentals of representation theory of the Yangian $Y(\mathfrak{gl}_N)$. Let I be the left ideal generated by $e_i^{(r)}$ and $h_i^{(r)} - \mu_i^{(r)}$ for $i = 1, \ldots, N$, $r \geq 0$, and $\mu_i^{(r)} \in \mathbb{C}$.

Definition 2.15. The *Verma module* of highest weight $\mu(u)$ is $M(\mu(u)) = Y(\mathfrak{gl}_N)/I$. It has a basis consisting of monomials of the form $f_{\alpha^{(1)}}^{(r_1)} \cdots f_{\alpha^{(l)}}^{(r_l)} 1_\lambda$ for $\alpha^{(i)} \in \Delta^+, r_i \geq 0, l \geq 0$. These elements $f_{\alpha^{(i)}}^{(r_i)}$ are similar to those defined in Subsection 2.3.5.

Definition 2.16. Let $\eta = (\eta_1, \eta_2, \ldots, \eta_N) \in \mathbb{C}^N$. Define the *weight space* $M(\mu(u))_\eta$ corresponding to the *weight* μ to be

$$M(\mu(u))_\eta = \{X \in M(\lambda(u)) | E_{ii}X = \eta_i X, \ \ i = 1, \ldots, N\} \tag{2.2.12}$$

Now that we have defined the notion of a weight on $M(\mu(u))$, we can establish some connections between the representation theories of $Y(\mathfrak{gl}_N)$ and $\mathfrak{gl}_N$. In particular, if $\mathfrak{h}^*$ is the dual space to the Cartan subalgebra of $\mathfrak{gl}_N$, with basis $\varepsilon_1, \ldots, \varepsilon_N$, the weight η of $M(\mu(u))$ can be identified with the element $\eta_1\varepsilon_1 + \ldots \eta_N\varepsilon_N$ in the dual, where we let $\eta_i := \eta(E_{ii})$. Further, the set of weights of $M(\mu(u))$ coincides with the set of weights of the Verma module on $\mathfrak{gl}_N$ with highest weight $\mu^{(1)} = (\mu_1^{(1)}, \ldots, \mu_N^{(1)})$. This is largely a consequence of our definition of Verma modules over $Y(\mathfrak{gl}_N)$. The weight space decomposition of $M(\mu(u))$ with respect to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is given by

$$M(\mu(u)) = \bigoplus_{\mu} M(\mu(u))_\eta, \tag{2.2.13}$$

where η is taken over all weights of the Verma module. Now, consider some submodule K of $M(\mu(u))$. Acting on vectors in K by the E_{ii} yields the weight space decomposition of K given by $K = \bigoplus_{\eta} K_{\eta}$ where $K_{\eta} = K \cap M(\mu(u))_{\eta}$. Recall that the Verma module is generated by the vector $1_{\mu(u)}$. This means that any proper submodule of the Verma module has zero intersection with the weight space $M(\mu(u))_{\mu^{(1)}}$ As a consequence of this, the unique maximal proper submodule of $M(\lambda(u))$ is the sum of all of its proper submodules.

Definition 2.17. The *irreducible highest weight representation* $L(\mu(u))$ of $Y(\mathfrak{gl}_N)$ with the highest weight $\mu(u)$ is the quotient of the Verma module by its unique maximal proper submodule.

Observe that this irreducible highest weight representation is isomorphic to the irreducible quotient of an arbitrary highest weight representation L with highest weight μ. Further, it follows that two irreducible highest weight representations are isomorphic if and only if they have the same highest weight.

Theorem 2.18. *Every finite-dimensional irreducible representation L of the Yangian $Y(\mathfrak{gl}_N)$ is a highest weight representation. L contains a unique highest vector, up to a constant factor.*

From this theorem, we conclude that every finite dimensional irreducible representation of $Y(\mathfrak{gl}_N)$ is isomorphic to a unique irreducible highest weight representation. The power of this result lies in the fact that classifying all finite dimensional irreducible representations is reduced to determining when $L(\lambda(u))$ is finite dimensional, depending on $\lambda(u)$.

Theorem 2.19. *The irreducible highest weight representation $L(\lambda(u))$ of the Yangian $Y(\mathfrak{gl}_N)$ is finite dimensional if and only if the following relation holds:*

$$\lambda_1(u) \to \lambda_2(u) \to \cdots \to \lambda_N(u). \tag{2.2.14}$$

In the context of the theorem above, we say that a sequence $\lambda_1(u) \to \lambda_2(u) \to \cdots \to \lambda_N(u)$ *holds* if, for any $i = 1, 2, \ldots, N-1$ and certain monic polynomials $P_i(u)$, we have

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{P_i(u+1)}{P_i(u)}. \tag{2.2.15}$$

These polynomials are the *Drinfeld polynomials* of the corresponding representation of the Yangian.

Corollary 2.20. *Finite dimensional irreducible representations of $Y(\mathfrak{gl}_N)$ are parametrized by the tuples $(f(u), P_1(u), \ldots, P_{N-1}(u))$ for $f(u)$ a formal power series in u^{-1} with constant term 1 and the Drinfeld polynomials associated to the representation.*

2.3 SUPERALGEBRAS: $\mathfrak{gl}_{M|N}$

Now, we shift our focus to *Lie superalgebras*, structures that can be used to describe supersymmetry in theoretical physics. Since these superalgebras are equipped with a $\mathbb{Z}/2\mathbb{Z}$-grading, in most theories there is a correspondence between even elements and bosons, and odd elements and fermions. The superalgebras explored in this thesis are $\mathfrak{gl}_{M|N}$ and $\mathfrak{q}_N$; this section looks at the former.

2.3.1 *General Linear Lie Superalgebra*

Consider the $\mathbb{Z}/2\mathbb{Z}$-graded space $\mathbb{C}(M|N) = \mathbb{C}^M \oplus \mathbb{C}^N$ with standard basis $e_1, \ldots, e_M, e_{M+1}, \ldots e_{M+N}$. We implement the notion of parity by setting the parity of e_i to be

$$|i| := \begin{cases} 1 & i \leq M \\ 0 & i \geq M+1 \end{cases}.$$

The superalgebra of endomorphisms of this space, $\mathrm{End}_{\mathbb{C}}(\mathbb{C}(M|N))$, is also a $\mathbb{Z}/2\mathbb{Z}$-graded space. The parity of a basis matrix E_{ab} is $|E_{ab}| = |a| + |b|$ for all a, b between 1 and $M + N$. Then, if we equip this superalgebra with the superbracket $[X, Y] = XY - (-1)^{|X||Y|}YX$, we produce the *general linear Lie superalgebra* $\mathfrak{gl}_{M|N}$.

2.3.2 *Super Yangian of* $\mathfrak{gl}_{1|1}$

We will first give a presentation of the super Yangian $Y(\mathfrak{gl}_{1|1})$ following that of [Zha95]. Let V be a 2-dimensional $\mathbb{Z}_2$-graded vector space with homogeneous basis $\{v_1, v_2\}$ with v_2 even and v_1 odd. This is the vector module of $\mathfrak{gl}_{1|1}$, whose action on V is defined by $E_{ab}v_c = \delta_{cb}v_a$. We can denote the associated vector representation of $\mathfrak{gl}(1|1)$ by π, with $\pi(E_{ab}) = e_{ab}$, but for simplicity's sake we can neglect to distinguish between E_{ab} and e_{ab} and write E_{ab} instead. Next, define the permutation operator $P : V \otimes V \to V \otimes V$ by

$$P(v_a \otimes v_b) = (-1)^{ab} v_b \otimes v_a,$$

more explicitly given by

$$P = \sum_{a,b=1,2} e_{ab} \otimes e_{ba} (-1)^{|b|}.$$

Finally, we have a solution to the graded Yang-Baxter equation given by the R-matrix

$$R(u) = 1 + \frac{P}{u}$$

for a formal variable u.

Definition 2.21. $Y(\mathfrak{gl}_{1|1})$ is a $\mathbb{Z}_2$-graded associative algebra generated by the elements $t_{ab}^{(r)}$ for $r \in \mathbb{Z}_{\geq 0}$. The grading is given by setting $t_{ab}^{(r)}$ even if $a = b$ and odd otherwise. If we define the following series,

$$T(u) = \sum_{a,b} (-1)^{b+1} t_{ab}(u) \otimes e_{ba} \tag{2.3.1}$$

$$t_{ab}(u) = (-1)^{b+1}\delta_{ab} + \sum_{k=1}^{\infty} t_{ab}^{(k)} u^{-k} \tag{2.3.2}$$

we can then state the defining relations of $Y(\mathfrak{gl}_{1|1})$ in $Y(\mathfrak{gl}_{1|1}) \otimes End(V) \otimes End(V)$ as

$$T_{12}(u)T_{13}(v)R_{23}(v-u) = R_{23}(v-u)T_{13}(v)T_{12}(u) \tag{2.3.3}$$

The relations can be presented more explicitly as

$$[t_{ab}(u), t_{cd}(v)] = \frac{(-1)^{\eta(a,b;c,d)}}{u-v}\left(t_{cb}(u)t_{ad}(v) - t_{cb}(v)t_{ad}(u)\right) \tag{2.3.4}$$

$$\eta(a,b;c,d) \equiv 1 + d + (b+d)(a+d) + (a+b)(c+d) \text{mod} 2 \tag{2.3.5}$$

$$\left[t_{ab}^{(r)}, t_{cd}^{(s)}\right] = (-1)^{\eta(a,b;c,d)} \sum_{l=0}^{\min(r,s)-1} \left(t_{cb}^{(l)} t_{ad}^{(r+s-1-l)} - t_{cb}^{(r+s-1-l)} t_{ad}^{(l)}\right). \tag{2.3.6}$$

Theorem 2.22. ***PBW Theorem:*** *Consider the monomials of the form*

$$\left(t_{21}^{(p_1)}\right)^{a_1} \cdots \left(t_{21}^{(p_k)}\right)^{a_k} \left(t_{11}^{(q_1)}\right)^{b_1} \cdots \left(t_{11}^{(q_l)}\right)^{b_l} \left(t_{22}^{(r_1)}\right)^{c_1} \cdots \left(t_{22}^{(r_m)}\right)^{c_m} \left(t_{12}^{(s_1)}\right)^{d_1} \cdots \left(t_{12}^{(s_n)}\right)^{d_n}$$

where the p_i, q_i, r_i, s_i are respectively strictly increasing, $b_i, c_i \in \mathbb{Z}_{>0}$, and $a_i, d_i \in \{0, 1\}$. These elements form a basis of $Y(\mathfrak{gl}_{1|1})$.

We can use a *Gauss decomposition* of the matrix $T(u)$ to produce another presentation of $Y(\mathfrak{gl}_{1|1})$. To do so, one needs to introduce the notion of *quasideterminants*. This can be done for $Y(\mathfrak{gl}_{M|N})$ in general, but this subsection will include both the general theory and the new generators specific to $Y(\mathfrak{gl}_{1|1})$. It is worth including here as we will use this new set of generators for $Y(\mathfrak{gl}_{1|1})$ to describe $Y(\mathfrak{sl}_{1|1})$.

Definition 2.23. Consider an invertible $n \times n$ square matrix A over a ring with identity. If A satisfies the property that the (j,i)-th entry is invertible in the ring, then the (i,j)-th *quasideterminant* of A is defined to be

$$|A|_{ij} = ((A^{-1})_{ji})^{-1} =: \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & \boxed{a_{ij}} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} \tag{2.3.7}$$

$$= a_{ij} - \text{row}_i^j \cdot (A^{ij})^{-1} \cdot \text{col}_j^i \tag{2.3.8}$$

where row_i^j is the i^{th} row of A without the j^{th} entry, col_j^i is the j^{th} column of A without the i^{th} entry, and A^{ij} is A without the i^{th} row and j^{th} entry.

First, we need that the quasideterminants $|T|_{ii}$ of $T(u)$ over all $i = 1, \ldots, M+N$ to be defined and invertible. The ring we are working over here is the ring of power series in u^{-1}. Since the (i,i)-th entries of $T(u)$ are of the form $t_{ii}(u) = 1 + t_{ii}^{(1)}u^{-1} + t_{ii}^{(2)}u^{-2} + \cdots$, we know that it is invertible due to the nonzero constant term. Thus, the desired quasideterminants exist.

Now that this condition is met, we can apply Theorem 4.96 by Gelfand and Retakh [Gel+05] to obtain, as per Gow [Gow07]:

$$T(u) = F(u)D(u)E(u) \tag{2.3.9}$$

for

$$D(u) = \begin{pmatrix} d_1(u) & & \cdots & 0 \\ & d_2(u) & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & d_{M+N}(u) \end{pmatrix} \tag{2.3.10}$$

$$E(u) = \begin{pmatrix} 1 & e_{12}(u) & \cdots & e_{1,M+N}(u) \\ & \ddots & & e_{2,M+N}(u) \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix} \tag{2.3.11}$$

$$F(u) = \begin{pmatrix} 1 & & \cdots & 0 \\ f_{21}(u) & & \ddots & \vdots \\ \vdots & & & \ddots \\ f_{M+N,1}(u) & f_{M+N,2}(u) & \cdots & 1 \end{pmatrix} \tag{2.3.12}$$

where

$$d_i(u) = \begin{vmatrix} t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{1i}(u) \\ \vdots & \ddots & & \vdots \\ t_{il}(u) & \cdots & t_{i,i-1}(u) & \boxed{t_{ii}(u)} \end{vmatrix} \tag{2.3.13}$$

$$e_{ij}(u) = \begin{vmatrix} t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{1j}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1,i}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,j}(u) \\ t_{il}(u) & \cdots & t_{i,i-1}(u) & \boxed{t_{ij}(u)} \end{vmatrix} \tag{2.3.14}$$

$$f_{ij}(u) = \begin{vmatrix} t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{1i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1,i}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,j}(u) \\ t_{jl}(u) & \cdots & t_{j,i-1}(u) & \boxed{t_{ji}(u)} \end{vmatrix} \tag{2.3.15}$$

Explicitly calculating the $t_{ij}(u)$ for $1 \leq i, j \leq 2$ yields

$$t_{11}(u) = d_1(u) \tag{2.3.16}$$

$$t_{22}(u) = d_2(u) + f_1(u)d_1(u)e_1(u) \tag{2.3.17}$$

$$t_{12}(u) = d_1(u)e_{12}(u) = d_1(u)e_1(u) \tag{2.3.18}$$

$$t_{21}(u) = f_{21}(u)d_1(u) = f_1(u)d_1(u) \tag{2.3.19}$$

The formulas (2.3.13), (2.3.14), and (2.3.15) produce new generating formal series

$$d_1(u) = t_{11}(u) \tag{2.3.20}$$

$$d_2(u) = t_{22}(u) - t_{21}(u)(t_{11}(u))^{-1}t_{12}(u) \tag{2.3.21}$$

$$e(u) = (t_{11}(u))^{-1}t_{12}(u) \tag{2.3.22}$$

$$f(u) = t_{21}(u)(t_{11}(u))^{-1}. \tag{2.3.23}$$

Definition 2.24. Let $M(\lambda_1(u), \lambda_2(u))$ denote the *Verma module* of weight $\lambda(u) = (\lambda_1(u), \lambda_2(u))$ for a pair of power series in u^{-1} with complex coefficients. It is generated by a nonzero *highest weight vector* 1_λ. The generators of $Y(\mathfrak{gl}_{1|1})$ act on this highest weight vector in the following ways:

$$t_{12}^{(r)} 1_\lambda = 0, \tag{2.3.24}$$

$$t_{11}^{(r)} 1_\lambda = \lambda_1^{(r)} 1_\lambda, \tag{2.3.25}$$

$$t_{22}^{(r)} 1_\lambda = \lambda_2^{(r)} 1_\lambda, \tag{2.3.26}$$

where $r > 0$. The constant term of $\lambda_i(u)$ is $(-1)^{|i|+1}$.

Lemma 2.25. *For every power series $f(u) = 1 + f^{(1)}u^{-1} + \ldots$ in u^{-1}, there exists an automorphism of $Y(\mathfrak{gl}_{1|1})$.*

Proof. We define $\phi_f : Y(\mathfrak{gl}_{1|1}) \to Y(\mathfrak{gl}_{1|1})$ by $t_{ab}(u) \mapsto f(u)t_{ab}(u) =: \widetilde{t_{ab}(u)}$. These $\widetilde{t_{ab}(u)}$ satisfy the same relations as the $t_{ab}(u)$. In particular, if $[t_{ab}(u), t_{cd}(v)] = \frac{(-1)^{\eta(a,b,c,d)}}{u-v}(t_{cb}(u)t_{ad}(v) - t_{cb}(v)t_{ad}(u))$, then:

$$\begin{aligned}
\left[\widetilde{t_{ab}(u)}, \widetilde{t_{cd}(v)}\right] &= [f(u)t_{ab}(u), f(v)t_{cd}(u)] \\
&= f(u)t_{ab}(u)f(v)t_{cd}(v) - f(v)t_{cd}(v)f(u)t_{ab}(u) \\
&= f(u)f(v)\,[t_{ab}(u), t_{cd}(v)] \\
&= f(u)f(v)\left(\frac{(-1)^{\eta(a,b,c,d)}}{u-v}\right)(t_{cb}(u)t_{ad}(v) - t_{cb}(v)t_{ad}(u)) \\
&= \frac{(-1)^{\eta(a,b,c,d)}}{u-v}(f(u)t_{cb}(u)f(v)t_{ad}(v) - f(v)t_{cb}(v)f(u)t_{ad}(u)) \\
&= \frac{(-1)^{\eta(a,b,c,d)}}{u-v}(\widetilde{t_{cb}(u)}\widetilde{t_{ad}(v)} - \widetilde{t_{cb}(v)}\widetilde{t_{ad}(u)}).
\end{aligned}$$

Thus, ϕ_f is indeed an automorphism. $\square$

Lemma 2.25 will be applied later in order to use a power series to twist the action of $Y(\mathfrak{gl}_{1|1})$ on $M(\lambda_1(u), \lambda_2(u))$.

2.3.3 *Super Yangian of* $\mathfrak{sl}_{1|1}$

We define the special super Yangian subalgebra of the general linear super Yangian analogously to the nonsuper case. That is, $Y(\mathfrak{sl}_{1|1})$ consists of all elements of $Y(\mathfrak{gl}_{1|1})$ fixed by the automorphism ϕ_f for *any* power series f in u^{-1}.

Lemma 2.26. ***(Special case of [Gow07] Lemma 8.2)*** *$Y(\mathfrak{sl}_{1|1})$ is generated by the coefficients of the series $d_1(u)^{-1}d_2(u)$, $e_1(u)$, and $f_1(u)$.*

Theorem 2.27. ***(Special case of [Gow07] Proposition 8.1)*** *Let $Z(Y(\mathfrak{gl}_{1|1}))$ be the centre of $Y(\mathfrak{gl}_{1|1})$. Then,*

$$Y(\mathfrak{gl}_{1|1}) \simeq Z(Y(\mathfrak{gl}_{1|1})) \otimes Y(\mathfrak{sl}_{1|1}).$$

Lemma 2.28. ***(Special case of [Gow07] Theorem 4)*** *$Z(Y(\mathfrak{gl}_{1|1}))$ is generated by the coefficients of the* Quantum Berezinian

$$\begin{aligned} b_{1|1}(u) &= d_1(u)d_2(u-1) && (2.3.27)\\ &= 1 + \sum_{r\geq 1} b_r u^{-r} \\ b_r &= t_{11}^{(r)} - t_{22}^{(r)} + \textit{ terms of lower degree} && (2.3.28) \end{aligned}$$

2.3.4 *Quantum Loop Superalgebra for* $\mathfrak{gl}_{1|1}$

Here, we define $U_q(\mathcal{L}\mathfrak{gl}_{1|1})$ by specializing the more general definition below for $U_q(\mathcal{L}\mathfrak{gl}_{M|N})$.

Definition 2.29. The quantum affine superalgebra $U_q(\mathcal{L}\mathfrak{gl}_{1|1})$ is generated by the elements $T_{ij}^{(n)}, \overline{T_{ij}}^{(n)}$ for $i,j \in \{1,2\}$ and $n \in \mathbb{Z}_{\geq 0}$. These generators satisfy the relations

$$\begin{aligned}
R_{23}(z,w)T_{12}(z)T_{13}(w) &= T_{13}(w)T_{12}(z)R_{23}(z,w), && (2.3.29)\\
R_{23}(z,w)\overline{T}_{12}(z)\overline{T}_{13}(w) &= \overline{T}_{13}(w)\overline{T}_{12}(z)R_{23}(z,w), && (2.3.30)\\
R_{23}(z,w)T_{12}(z)\overline{T}_{13}(w) &= \overline{T}_{13}(w)T_{12}(z)R_{23}(z,w), && (2.3.31)\\
T_{12}^{(0)} = \overline{T}_{21}^{(0)} &= 0 && (2.3.32)\\
T_{11}^{(0)}\overline{T}_{11}^{(0)} &= 1 = \overline{T}_{11}^{(0)}T_{11}^{(0)} && (2.3.33)\\
T_{22}^{(0)}\overline{T}_{22}^{(0)} &= 1 = \overline{T}_{22}^{(0)}T_{22}^{(0)} && (2.3.34)
\end{aligned}$$

where $R(z,w)$ is the *Perk-Schultz R-matrix*

$$\begin{aligned}
R(z,w) &= \sum_{i=1}^{2}(zq_i - q_i^{-1})E_{ii}\otimes E_{ii} + (z-w)\sum_{i\neq j} E_{ii}\otimes E_{jj}\\
&\quad + z\sum_{i<j}(q_i - q_i^{-1})E_{ji}\otimes E_{ij} + w\sum_{i<j}(q_j - q_j^{-1})E_{ij}\otimes E_{ji}\\
&= (zq_1 - q_1^{-1})E_{11}\otimes E_{11} + (zq_2 - q_2^{-1})E_{22}\otimes E_{22} + (z-w)E_{11}\otimes E_{22} +\\
&\quad (z-w)E_{22}\otimes E_{11} + z(q_1 - q_1^{-1})E_{21}\otimes E_{12} + w(q_2 - q_2^{-1})E_{12}\otimes E_{21}\\
&= (zq - q^{-1})E_{11}\otimes E_{11} + (zq^{-1} - q)E_{22}\otimes E_{22} + (z-w)E_{11}\otimes E_{22} +\\
&\quad (z-w)E_{22}\otimes E_{11} + z(q - q^{-1})E_{21}\otimes E_{12} + w(q^{-1} - q)E_{12}\otimes E_{21}
\end{aligned}$$

with $q_i := q^{d_i}$ where $d_i = \begin{cases} 1 & i \leq 1 \\ -1 & i > 1 \end{cases} = \begin{cases} 1 & i = 1 \\ -1 & i = 2 \end{cases}$. So, $q_i = \begin{cases} q & i = 1 \\ q^{-1} & i = 2 \end{cases}$.

As well, we define $T(u)$ and $\overline{T}(u)$ as usual; $T(u) = \sum_{i,j=1}^{2} E_{ij}\otimes T_{ij}(u)$ and $\overline{T}(u) = \sum_{i,j=1}^{2} E_{ij}\otimes \overline{T}_{ij}(u)$. We let $T_{ij}(u) = \sum_{r\geq 0} T_{ij}^{(r)}u^{-r}$ and $\overline{T}_{ij}(u) = \sum_{r\geq 0} \overline{T}_{ij}^{(r)}u^{r}$ be formal power series in u^{-1} with coefficients being the generators of the superalgebra.

2.3.5 *Super Yangian of* $\mathfrak{gl}_{M|N}$

Now that the $\mathfrak{gl}_{1|1}$ case is familiar, we list the more general definitions of quantum groups for $\mathfrak{gl}_{M|N}$. Much like the way we presented $Y(\mathfrak{gl}_{1|1})$, we begin defining $Y(\mathfrak{gl}_{M|N})$ with an R-matrix and elements that satisfy a particular defining relation with it. Most objects presented here will be similar, only generalized to arbitrary M and N. It is most convenient to define parity slightly differently than in the $Y(\mathfrak{gl}_{1|1})$ and $Y(\mathfrak{gl}_{M|N})$ case by instead indexing the rows and columns of matrices using strictly positive indices. Then, $|a| := \begin{cases} 1 & a \leq M \\ 0 & a > M \end{cases}$.

For the vector module V of $\mathfrak{gl}_{M|N}$, spanned by homogenous elements $\{v_a | a = 1, 2, \ldots, M+N\}$, let P be the permutation operator acting on $V \otimes V$ given by $P(v_a \otimes v_b) = (-1)^{|a||b|} v_b \otimes v_a$. More explicitly, one can write

$$P = \sum_{a,b=1}^{M+N} e_{ab} \otimes e_{ba} (-1)^{|b|}.$$

We can use it to produce the matrix

$$R(u) = 1 + \frac{P}{u},$$

which is an R-matrix satisfying the graded Yang-Baxter equation, where u is a formal variable.

Definition 2.30. The $\mathbb{Z}_2$-graded associative algebra $Y(\mathfrak{gl}_{M|N})$ is generated by the elements $t_{ij}^{(r)}$ for $r \in \mathbb{Z}_{>0}$. Define

$$T(u) = \sum_{i,j=1}^{M+N} (-1)^{|j|} t_{ij}(u) \otimes e_{ji} \tag{2.3.35}$$

$$t_{ij}(u) = \sum_{n\geq 0} t_{ij}^{(n)} u^{-n} \tag{2.3.36}$$

In particular, $t_{ij}^{(0)} = (-1)^{|j|+1}$. Then, we can express the defining relations for the super Yangian as the equality

$$T_1(u)T_2(v)R_{12}(v-u) = R_{12}(v-u)T_2(v)T_1(u) \tag{2.3.37}$$

in $Y(\mathfrak{gl}_{M|N}) \otimes \mathrm{End}(V) \otimes \mathrm{End}(V)$. Expansion of this equality yields a more concrete family of relations on the generating power series

$$\left[t_{ij}(u), t_{kl}(v)\right] = \frac{(-1)^{\eta(i,j;k,l)}}{(u-v)}(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)) \tag{2.3.38}$$

$$\eta(i,j;k,l) \equiv 1 + l + (j+l)(i+l) + (i+j)(k+l) \text{ (mod 2)}. \tag{2.3.39}$$

The family of relations (2.3.38) on the power series yields two families of relations on the generators. First, we have the usual explicit definition of the superbracket.

$$\left[t_{ij}^{(r)}, t_{kl}^{(s)}\right] = (-1)^{\eta(i,j;k,l)} \sum_{a=0}^{\min(r,s)-1} \left(t_{kj}^{(a)} t_{il}^{(r+s-1-a)} - t_{kj}^{(r+s-1-a)} t_{il}^{(a)}\right) \tag{2.3.40}$$

Equivalently, we can write

$$\left[t_{ij}^{(r+1)}, t_{kl}^{(s)}\right] - \left[t_{ij}^{(r)}, t_{kl}^{(s+1)}\right] = (-1)^{\eta(i,j;k,l)} \left(t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}\right). \tag{2.3.41}$$

The relation (2.3.41) is produced by multiplying both sides of (2.3.38) by $u-v$, then expanding and equating the coefficient of $u^{-r}v^{-s}$ on each side of the equality.

Further, by the skew-super symmetry of the superbracket, we know that

$$\left[t_{ij}^{(r+1)}, t_{kl}^{(s)}\right] - \left[t_{ij}^{(r)}, t_{kl}^{(s+1)}\right] = (-1)^{(|i|+|j|)(|k|+|l|)} \left([t_{kl}^{(s+1)}, t_{ij}^{(r)}] - [t_{kl}^{(s)}, t_{ij}^{(r+1)}]\right).$$

As a result, after applying this, we see that (2.3.41) is equivalent to

$$\left[t_{ij}^{(r+1)}, t_{kl}^{(s)}\right] - \left[t_{ij}^{(r)}, t_{kl}^{(s+1)}\right] = (-1)^{|j||k|+|j||l|} \left(t_{il}^{(r)} t_{kj}^{(s)} - t_{il}^{(s)} t_{kj}^{(r)}\right). \qquad (2.3.42)$$

Alternatively, we can look at the super Yangian $\mathfrak{gl}_{M|N}$ using its presentation obtained using Gow's Gauss decomposition in [Gow07]. This uses the same matrices (2.3.9 - 2.3.15) as we did in the $\mathfrak{gl}_{1|1}$ case, only keeping the values of M and N general.

Definition 2.31. (Theorem 3 of [Gow07]) The Yangian $Y(\mathfrak{gl}_{M|N})$ is isomorphic as an associative superalgebra to the algebra with even generators $d_i^{(r)}, d_i'^{(r)}, f_j^{(r)}, e_j^{(r)}$ for $i \in \{1, \ldots, M+N\}, j \in \{1, \ldots, M+N-1\}, j \neq M, r \geq 1$ and odd generators $e_M^{(r)}, f_M^{(r)}$ for $r \geq 1$. These are subject to the following relations, where $r, s, t \geq 1$ and i, j, k range over the appropriate admissible values.

$$d_i^{(0)} = 1 \qquad (2.3.43)$$

$$\sum_{t=0}^{r} d_i^{(t)} d_i'^{(r-t)} = \delta_{r,0} \qquad (2.3.44)$$

$$\left[d_i^{(r)}, d_l^{(s)}\right] = 0 \qquad (2.3.45)$$

$$\left[d_i^{(r)}, e_j^{(s)}\right] = \begin{cases} (\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)} e_j^{(r+s-1-t)}, & 1 \leq j \leq M-1 \\ (\delta_{i,j} + \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)} e_j^{(r+s-1-t)}, & j = M \\ -(\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)} e_j^{(r+s-1-t)}, & M+1 \leq j \leq M+n-1 \end{cases} \qquad (2.3.46)$$

$$\left[d_i^{(r)}, f_j^{(s)}\right] = \begin{cases} -(\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} f_j^{(r+s-1-t)} d_i^{(t)}, & 1 \leq j \leq M-1 \\ -(\delta_{i,j} + \delta_{i,j+1}) \sum_{t=0}^{r-1} f_j^{(r+s-1-t)} d_i^{(t)}, & j = M \\ (\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} f_j^{(r+s-1-t)} d_i^{(t)}, & M+1 \leq j \leq M+n-1 \end{cases} \qquad (2.3.47)$$

$$\left[e_j^{(r)}, f_k^{(s)}\right] = \begin{cases} -\delta_{j,k} \sum_{t=0}^{r+s-1} d_j'^{(t)} d_{j+1}^{(r+s-1-t)}, & 1 \leq j \leq M-1 \\ +\delta_{j,k} \sum_{t=0}^{r+s-1} d_j'^{(t)} d_{j+1}^{(r+s-1-t)}, & M \leq j \leq M+n-1 \end{cases} \tag{2.3.48}$$

$$[e_M^{(r)}, e_M^{(s)}] = 0 \tag{2.3.49}$$

$$[f_M^{(r)}, f_M^{(s)}] = 0 \tag{2.3.50}$$

$$[e_j^{(r)}, e_j^{(s)}] = (-1)^{|j|} \left(\sum_{t=1}^{s-1} e_j^{(t)} e_j^{(r+s-1-t)} - \sum_{t=1}^{r-1} e_j^{(r)} e_j^{(r+s-1-t)} \right), \quad j \neq M \tag{2.3.51}$$

$$[f_j^{(r)}, f_j^{(s)}] = (-1)^{|j|} \left(\sum_{t=1}^{r-1} f_j^{(t)} f_j^{(r+s-1-t)} - \sum_{t=1}^{s-1} f_j^{(r)} f_j^{(r+s-1-t)} \right), \quad j \neq M \tag{2.3.52}$$

$$[e_j^{(r)}, e_{j+1}^{(s+1)}] - [e_j^{(r+1)}, e_{j+1}^{(s)}] = -(-1)^{|j|} e_j^{(r)} e_{j+1}^{(s)} \tag{2.3.53}$$

$$[f_j^{(r+1)}, f_{j+1}^{(s)}] - [f_j^{(r)}, f_{j+1}^{(s+1)}] = -(-1)^{|j|} f_{j+1}^{(r)} f_j^{(s)} \tag{2.3.54}$$

$$\text{if } |j-k| > 1, \text{ then } [e_j^{(r)}, e_k^{(s)}] = 0 \text{ and } [f_j^{(r)}, f_k^{(s)}] = 0 \tag{2.3.55}$$

$$\text{if } j \neq k, \text{ then } [[e_j^{(r)}, e_k^{(s)}], e_k^{(t)}] + [[e_j^{(r)}, e_k^{(t)}], e_k^{(s)}] = 0 \tag{2.3.56}$$

$$\text{if } j \neq k, \text{ then } [[f_j^{(r)}, f_k^{(s)}], f_k^{(t)}] + [[f_j^{(r)}, f_k^{(t)}], f_k^{(s)}] = 0 \tag{2.3.57}$$

$$[[e_{M-1}^{(r)}, e_M^{(1)}], [e_M^{(1)}, e_{M+1}^{(s)}]] = 0 \tag{2.3.58}$$

$$[[f_{M-1}^{(r)}, f_M^{(1)}], [f_M^{(1)}, f_{M+1}^{(s)}]] = 0 \tag{2.3.59}$$

These generators $d_i^{(r)}$, $d_i'^{(r)}$, $e_i^{(r)}$, and $f_i^{(r)}$ are coefficients of the following power series, respectively.

$$d_i(u) = \sum_{r \geq 0} d_i^{(r)} u^{-r} \tag{2.3.60}$$

$$(d_i(u))^{-1} = \sum_{r \geq 0} d_i'^{(r)} u^{-r} \tag{2.3.61}$$

$$e_i(u) = \sum_{r \geq 1} e_i^{(r)} u^{-r} \tag{2.3.62}$$

$$f_i(u) = \sum_{r \geq 1} f_i^{(r)} u^{-r} \tag{2.3.63}$$

Definition 2.32. The subalgebra $Y(\mathfrak{sl}_{M|N})$ is generated by the coefficients of the series $d_1(u)d_{i+1}(u)$, $e_i(u)$, and $f_i(u)$ for $i \in \{M+N-1\}$. If we define $h_i(u) := d_1(u)d_{i+1}(u)$, then our generators are $h_i^{(r)}, e_i^{(r)}$, and $f_i^{(r)}$.

Both sets of generators of the Yangian can be used to produce a PBW-type basis.

Theorem 2.33. ***PBW Theorem*** *Fix some ordering on the generators $t_{ij}^{(r)}$ of $Y(\mathfrak{gl}_{M|N})$. $Y(\mathfrak{gl}_{M|N})$ is equipped with a PBW basis formed by ordered products of these generators, where the odd generators (where $|i| + |j| \equiv 1 \pmod 2$) do not appear with powers of order greater than 1.*

Theorem 2.34. ***PBW Theorem*** *Fix some ordering on the generators $f_{ji}^{(r)}, d_i^{(r)}, e_{ij}^{(r)}$ of $Y(\mathfrak{gl}_{M|N})$ for $1 \leq i < j \leq m+n$ and $r \geq 1$. This ordering must be such that the f generators come first, then the d generators, and then the e generators. We define the $f_{ji}^{(r)}$ and $e_{ij}^{(r)}$ inductively using:*

$$
\begin{aligned}
f_{i+1,i}^{(r)} &= f_i^{(r)} \\
e_{i,i+1}^{(r)} &= e_i^{(r)} \\
f_{j,i}^{(r)} &= [f_{j,j-1}^{(1)}, f_{j-1,i}^{(r)}](-1)^{|j-1|} \\
e_{i,j}^{(r)} &= [e_{i,j-1}^{(r)}, e_{j-1,j}^{(1)}](-1)^{|j-1|}
\end{aligned}
$$

for $j > i+1$. Then, $Y(\mathfrak{gl}_{M|N})$ is equipped with a PBW basis formed by ordered products of these generators.

The proof of Proposition 4.13 will require using the generators $x_{i,r}^{\pm}$, $h_{i,r}$ for $Y(\mathfrak{sl}_{M|N})$, introduced in Proposition 9.1 of Gow's paper on Gauss Decomposition [Gow07]. These relations are essentially slight variants of the relations governing the similar generators for a general $Y(\mathfrak{g})$ in Definition 2.1. We can relate the generators $x^{\pm}_{i,r}$,$\mathrm{h}_{i,r}$ to the more familiar set of $h_i^{(r)}, e_i^{(r)}$, and $f_i^{(r)}$ through their associated power series in u^{-1}:

$$\begin{aligned}
h_i(u) &= d_i(u + \frac{1}{2}(-1)^{|i|}(M-i))^{-1} d_{i+1}(u + \frac{1}{2}(-1)^{|i|}(M-i)) \\
x_i^+(u) &= f_i(u + \frac{1}{2}(-1)^{|i|}(M-i)) \\
x_i^- &= (-1)^{|i|} e_i(u + \frac{1}{2}(-1)^{|i|}(M-i)).
\end{aligned}$$

The following is a simplified version of the PBW basis constructed by Gow.

Theorem 2.35. ***PBW Theorem*** *Fix any total ordering on the set of generators $h_{i,r}$ and $x^{\pm}_{i,r}$ for $Y(\mathfrak{sl}_{M|N})$. Then, ordered monomials in these elements (where odd elements $x^{\pm}_{M,s}$ may only occur in powers of 0 or 1) are a PBW basis for $Y(\mathfrak{sl}_{M|N})$.*

The Verma module $M(\mu(u))$ is defined to be the quotient of $Y(\mathfrak{sl}_{M|N})$ by the left ideal generated by $x^+_{i,r}$ and $h_{i,r} - \mu_i^{(r)}$ for $i = 1, 2, \ldots, M+N$ and $r \geq 0$. By Theorem 2.35 on $Y(\mathfrak{sl}_{M|N})$, we can say further that $M(\mu(u))$ has a basis comprising ordered monomials of the form

$$x^-_{\alpha^{(1)}, r_1} \cdots x^-_{\alpha^{(l)}, r_l}$$

for any ordering on the generators $x^-_{\alpha,r}$, α a positive root and $l \geq 0, r_i \geq 0, \alpha^{(i)} \in \Delta^+$. Further, $M(\mu(u))$ has a weight space decomposition with respect to $\mathfrak{h} \subset \mathfrak{sl}_{M|N}$ and these weights η are of the form $\eta = \mu^{(0)} - k_1\alpha_1 - \cdots - k_n\alpha_n$ for nonnegative integers k_i. Then, equip the set of weights with a partial ordering with η preceding η' if $\eta - \eta'$ is a linear combination of positive roots with nonnegative coefficients.

If we have a composite root $\beta = \alpha_{i_1} + \cdots \alpha_{i_p}$ for the simple roots α_i, then let $x_i^{\pm} = x^{\pm}_{\alpha_i}$ and we can express

$$x^{\pm}_{\beta} = [x^{\pm}_{i_1}, [x^{\pm}_{i_2}, \ldots, [x^{\pm}_{i_{p-1}}, x^{\pm}_{i_p}]...]].$$

Then, for some decomposition of s into a sum of p nonnegative integers $s_1 + \cdots + s_p$,

$$x^{\pm}_{\beta,s} = [x^{\pm}_{i_1,s_1}, [x^{\pm}_{i_2,s_2}, \ldots, [x^{\pm}_{i_{p-1},s_{p-1}}, x^{\pm}_{i_p,s_p}]...]].$$

Let $r_1 + \cdots + r_p$ be another decomposition of s and

$$\widetilde{x^{\pm}_{\beta,s}} = [x^{\pm}_{i_1,r_1}, [x^{\pm}_{i_2,r_2}, \ldots, [x^{\pm}_{i_{p-1},r_{p-1}}, x^{\pm}_{i_p,r_p}]...]].$$

Then, the difference $x^{\pm}_{\beta,s} - \widetilde{x^{\pm}_{\beta,s}}$ lies in $Y(\mathfrak{sl}_{M|N})_{s-1}$, the $s-1$-filtered part of the super Yangian. Observe that in the current algebra $\mathfrak{sl}_{M|N} \otimes_{\mathbb{C}} \mathbb{C}[u]$, $x^{\pm}_{\beta,s} = x^{\pm}_{\beta} \otimes u^s$. So, the difference of these two elements is $(x^{\pm}_{\beta} - \widetilde{x^{\pm}_{\beta}}) \otimes_{\mathbb{C}} u^s$. There exists an isomorphism from $U(\mathfrak{sl}_{M|N} \otimes_{\mathbb{C}} \mathbb{C}[u])$ to $\mathrm{gr}Y(\mathfrak{sl}_{M|N}) = \bigoplus_{r\geq 0} F_r/F_{r-1}$ sending $x^{\pm}_{\alpha} \otimes u^s$ to $x^{\pm}_{\alpha,s} \in F_s/F_{s-1}$. Then, what maps to $x^{\pm}_{\beta,s}$? If

$$x^{\pm}_{\beta} = [x^{\pm}_{\alpha_{i_1}}, [x^{\pm}_{\alpha_{i_2}}, \ldots, [x^{\pm}_{\alpha_{i_{p-1}}}, x^{\pm}_{\alpha_{i_p}}] \ldots]]$$

then its preimage is

$$\begin{aligned}
&[x^{\pm}{}_{i_1} \otimes u^{s_1}, [x^{\pm}_{i_2} \otimes u^{s_2}, \ldots [x^{\pm}_{i_{p-1}} \otimes u^{s_{p-1}}, x^{\pm}_{i_p} \otimes u^{s_p}] \cdots]] \\
&\quad = [x^{\pm}_{i_1}, [x^{\pm}_{i_2} ... [x^{\pm}_{i_{p-1}}, x^{\pm}_{i_p}]]] \otimes u^{s_1+\cdots+s_p}.
\end{aligned}$$

Similarly, the preimage of

$$\widetilde{x^{\pm}_{\beta,s}} = [x^{\pm}_{i_1,r_1}, [x^{\pm}_{i_2,r_2}, \ldots, [x^{\pm}_{i_{p-1},r_{p-1}}, x^{\pm}_{i_p,r_p}]...]]$$

is the element

$$\begin{aligned}
&[x^{\pm}{}_{i_1} \otimes u^{r_1}, [x^{\pm}_{i_2} \otimes u^{r_2}, \ldots [x^{\pm}_{i_{p-1}} \otimes u^{r_{p-1}}, x^{\pm}_{i_p} \otimes u^{r_p}] \cdots]] \\
&\quad = [x^{\pm}_{i_1}, [x^{\pm}_{i_2} ... [x^{\pm}_{i_{p-1}}, x^{\pm}_{i_p}]]] \otimes u^{r_1+\cdots+r_p}.
\end{aligned}$$

As a result, $x^{\pm}_{\beta,s} - \widetilde{x^{\pm}_{\beta,s}} + x = 0$ for some $x \in F_{s-1}$. This x vanishes in the quotient F_r/F_{r-1}. Thus, choice of decomposition of s is irrelevant.

2.3.6 *Quantum Loop Superalgebra*

Consider the loop algebra

$$\mathcal{L}\mathfrak{gl}_{M|N} = \mathfrak{gl}_{M|N} \otimes \mathbb{C}\left[u, u^{-1}\right] = \bigoplus_{1\le i,j\le M+N} E_{ij} \otimes \mathbb{C}[u, u^{-1}]. \tag{2.3.64}$$

Here, we use the definition of the quantum loop superalgebra $U_q(\mathcal{L}\mathfrak{gl}_{M|N})$ given by Zengo Tsuboi [Tsu14].

Definition 2.36. The quantum affine superalgebra $U_q(\mathcal{L}\mathfrak{gl}_{M|N})$ is generated by the elements $T_{ij}^{(n)}, \overline{T_{ij}}^{(n)}$ for $i, j \in \{1, 2, \ldots, M+N\}$ and $n \in \mathbb{Z}_{\ge 0}$. These generators satisfy the relations

$$R_{23}(z,w)T_{12}(z)T_{13}(w) = T_{13}(w)T_{12}(z)R_{23}(z,w), \tag{2.3.65}$$

$$R_{23}(z,w)\overline{T}_{12}(z)\overline{T}_{13}(w) = \overline{T}_{13}(w)\overline{T}_{12}(z)R_{23}(z,w), \tag{2.3.66}$$

$$R_{23}(z,w)T_{12}(z)\overline{T}_{13}(w) = \overline{T}_{13}(w)T_{12}(z)R_{23}(z,w), \tag{2.3.67}$$

$$T_{ij}^{(0)} = \overline{T}_{ji}^{(0)} = 0 \text{ for } 1 \le i < j \le M+N \tag{2.3.68}$$

$$T_{ii}^{(0)}\overline{T}_{ii}^{(0)} = 1 = \overline{T}_{ii}^{(0)}T_{ii}^{(0)} \text{ for } i = 1, \ldots, M+N \tag{2.3.69}$$

where $R(z,w)$ is the *Perk-Schultz R-matrix*

$$\begin{aligned} R(z,w) = & \sum_{i=1}^{M+N} (zq_i - wq_i^{-1})E_{ii} \otimes E_{ii} + (z-w)\sum_{i\ne j} E_{ii} \otimes E_{jj} \\ & + z\sum_{i<j}(q_i - q_i^{-1})E_{ji} \otimes E_{ij} + w\sum_{i<j}(q_j - q_j^{-1})E_{ij} \otimes E_{ji} \end{aligned} \tag{2.3.70}$$

with $q_i := q^{d_i}$ where $d_i = \begin{cases} 1 & i \leq M \\ -1 & i > M \end{cases}$. As well, we define $T(u)$ and $\overline{T}(u)$ as usual; $T(u) = \sum_{i,j=1}^{M+N} E_{ij} \otimes T_{ij}(u)$ and $\overline{T}(u) = \sum_{i,j=1}^{M+N} E_{ij} \otimes \overline{T}_{ij}(u)$. We let $T_{ij}(u) = \sum_{r\geq 0} T_{ij}^{(r)} u^{-r}$ (resp. $\overline{T}_{ij}(u) = \sum_{r\geq 0} \overline{T}_{ij}^{(r)} u^{r}$) be formal power series in u^{-1} (resp. u) with coefficients being the generators of the superalgebra. Then, the RTT relation (2.3.65) is equivalent to:

$$
\begin{aligned}
&(-1)^{|i||k|+|j||k|}(q^{-d_i\delta_{ik}}z - q^{d_i\delta_{ik}}w)T_{kl}(w)T_{ij}(z) \\
&\quad - (-1)^{|i||l|+|j||l|}(q^{-d_j\delta_{jl}}z - q^{d_j\delta_{jl}}w)T_{ij}(z)T_{kl}(w) \\
&=(-1)^{|i||j|}(q-q^{-1})\left[(\delta_{i>k}z + \delta_{i<k}w)T_{il}(w)T_{kj}(z) \ - (\delta_{j<l}z + \delta_{j>l}w)T_{il}(z)T_{kj}(w)\right].
\end{aligned}
$$

Equating the coefficients of $z^{-r}w^{-s}$ yields

$$
\begin{aligned}
&(-1)^{|i||k|+|j||k|}(q^{-d_i\delta_{ik}}T_{kl}^{(s)}T_{ij}^{(r+1)} - q^{d_i\delta_{ik}}T_{kl}^{(s+1)}T_{ij}^{(r)}) \\
&\quad - (-1)^{|i||l|+|j||l|}(q^{-d_j\delta_{jl}}T_{ij}^{(r+1)}T_{kl}^{(s)} - q^{d_j\delta_{jl}}T_{ij}^{(r)}T_{kl}^{(s+1)}) \\
&=(-1)^{|i||j|}(q-q^{-1})\Big[\delta_{i>k}T_{il}^{(s)}T_{kj}^{(r+1)} + \delta_{i<k}T_{il}^{(s+1)}T_{kj}^{(r)} \\
&\quad -\delta_{j<l}T_{il}^{(r+1)}T_{kj}^{(s)} - \delta_{j>l}T_{il}^{(r)}T_{kj}^{(s+1)}\Big].
\end{aligned}
$$

We multiply by -1 to align a later proof more closely with one in [CG15].

$$
\begin{aligned}
&(-1)^{|i||l|+|j||l|}(q^{-d_j\delta_{jl}}T_{ij}^{(r+1)}T_{kl}^{(s)} - q^{d_j\delta_{jl}}T_{ij}^{(r)}T_{kl}^{(s+1)}) \\
&\quad - (-1)^{|i||k|+|j||k|}(q^{-d_i\delta_{ik}}T_{kl}^{(s)}T_{ij}^{(r+1)} - q^{d_i\delta_{ik}}T_{kl}^{(s+1)}T_{ij}^{(r)}) \\
&=(-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}T_{il}^{(r+1)}T_{kj}^{(s)} + \delta_{j>l}T_{il}^{(r)}T_{kj}^{(s+1)}) \\
&\quad -(\delta_{i>k}T_{il}^{(s)}T_{kj}^{(r+1)} + \delta_{i<k}T_{il}^{(s+1)}T_{kj}^{(r)})\Big] \qquad (2.3.71)
\end{aligned}
$$

Further, $\mathcal{U}_q(\mathcal{L}\mathfrak{gl}_{M|N})$ is a $\mathbb{Z}_2$-graded superalgebra with parity of generators given by $|T_{ij}^{(n)}| = |\overline{T_{ij}}^{(n)}| = |i| + |j|$.

Note that the signs $(-1)^{|i||k|+|j||k|}$, $(-1)^{|i||l|+|j||l|}$, and $(-1)^{|i||j|}$ arise, while not being present in the original relation and its power series, due to the fact that we use the *graded tensor product*: $(A \otimes B)(C \otimes D) = (-1)^{|B||C|}(AC \otimes BD)$. In Zhang's introduction to the quantum loop algebra [Zha14b], the definition of the Perk-Schultz matrix $R(z,w)$ reflects this. It includes a $c_{\mathbf{V},\mathbf{V}}$, which is the map from $\mathbf{V} \otimes \mathbf{V}$ to itself sending $v_i \otimes v_j \mapsto (-1)^{|i||j|} v_j \otimes v_i$.

2.3.7 *Quantum Current Superalgebra*

Call the subalgebra of $U_q(\mathcal{L}\mathfrak{gl}_{M|N})$ generated by the $T_{ij}^{(r)}$ for all i, j and $r \geq 0$ as well as by all $\overline{T}_{ij}^{(0)}$ the *quantum current superalgebra*. This is a quantization of $U_q(\mathfrak{gl}_{M|N} \otimes \mathbb{C}[t])$. Denote it $U_q(\mathfrak{gl}_{M|N}[s])$.

2.4 SUPERALGEBRAS OF TYPE $\mathfrak{q}$

A new set of notation is used for $\mathfrak{gl}_{M|N}$ when $M = N$. Consider the $\mathbb{Z}/2\mathbb{Z}$-graded space $\mathbb{C}(N|N) = \mathbb{C}^N \oplus \mathbb{C}^N$ with standard basis $e_{-N}, \ldots e_{-1}, e_1, \ldots e_N$. We implement the notion of parity by setting the parity of e_i to be

$$|i| := \begin{cases} 1 & i < 0 \\ 0 & i > 0 \end{cases}.$$

The superalgebra of endomorphisms of this space, $\mathrm{End}_{\mathbb{C}}(\mathbb{C}(N|N))$, is also a $\mathbb{Z}/2\mathbb{Z}$-graded space. The parity of a basis matrix E_{ab} is $|E_{ab}| = |a| + |b|$ for all a, b between $-N$ and N, excluding 0. Then, if we equip this superalgebra with the superbracket $[X, Y] = XY - (-1)^{|X||Y|}YX$, we produce the *general linear Lie superalgebra* $\mathfrak{gl}_{N|N}$.

Then, the *Lie superalgebra of type* $\mathfrak{q}$, denoted $\mathfrak{q}_N$, is a subalgebra of $\mathfrak{gl}_{N|N} = \text{span}\{E_{ij} | -N \leq i, j \leq N \text{ for } i, j \neq 0\}$. In particular, let $\mathfrak{q}_N = \text{span}\{\underline{E}_{ij} := E_{ij} + E_{-i,-j} | -N < i, j \leq N\}$. Note that this means that $\underline{E}_{ij} = E_{ij} + E_{-i,-j} = E_{-i,-j} + E_{ij} = \underline{E}_{-i,-j}$. As a result, we only need a subset of the spanning set to produce a basis for $\mathfrak{q}_N$. Let this basis be

$$\{\underline{E}_{ij} := E_{ij} + E_{-i,-j} | -N \leq i < j \leq N \text{ or } 1 \leq i = j \leq N\}.$$

Let $\mathcal{L}\mathfrak{gl}_{N|N} = \mathfrak{gl}_{N|N} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$. The *twisted loop superalgebra of type* $\mathfrak{q}$ is the structure

$$\mathcal{L}_{tw}\mathfrak{q}_n = \{X(t) \in \mathcal{L}\mathfrak{gl}_{N|N} | \hat{\iota}(X(t)) = X(t)\}$$

for $\iota : \mathfrak{gl}_{N|N} \to \mathfrak{gl}_{N|N}$ the involution $\iota(E_{ij}) = E_{-i,-j}$ and $\hat{\iota} : \mathcal{L}\mathfrak{gl}_{N|N} \to \mathcal{L}\mathfrak{gl}_{N|N}$ mapping $X(t) \mapsto \iota(X)(t^{-1})$. Since if $\iota(X) = X, X \in \mathfrak{q}_N$, we can use ι to generate more elements fixed by the involution. For any $Y \in \mathfrak{gl}_{N|N}$, $\iota(Y + \iota(Y)) = \iota(Y) + \iota^2(Y) = Y + \iota(Y) \in \mathfrak{q}_N$. An alternative definition of the *twisted loop superalgebra of type* $\mathfrak{q}$ is

$$\mathcal{L}_{tw}\mathfrak{q}_N = \text{span}\{\underline{E}_{ij}^{(r)} := E_{ij} \otimes t^r + E_{-i,-j} \otimes t^{-r}, -N \leq i, j \leq N, r \geq 0\}.$$

When presenting the basis of $\mathfrak{q}_N$, we effectively needed to take half of the spanning set. Similarly, the basis of $\mathcal{L}_{tw}\mathfrak{q}_N$ is a subset of the spanning set, though not quite the same proportion. This is because if $r = 0$, then $\underline{E}_{ij}^{(0)} = E_{ij} + E_{-i,-j} = \underline{E}_{-i,-j}^{(r)}$, so we take $-N \leq i < j \leq n$ or $1 \leq i = j \leq N$. However, if $r \geq 1$, then

$\underline{E}_{ij}^{(r)} = E_{ij}t^r + E_{-i,-j}t^{-r} = \underline{E}_{-i,-j}^{(-r)}$, but we only take positive values of r. Thus, there is no additional restriction on i, j. Take the basis to be

$$\{\underline{E}_{ij}{}^{(r)} := E_{ij} \otimes t^r + E_{-i,-j} \otimes t^{-r} | -N \leq i, j \leq N \text{ for } r \geq 1\}$$
$$\cup \{\underline{E}_{ij}^{(0)} | -N \leq i < j \leq N \text{ or } 1 \leq i = j \leq N\}.$$

Consider also the twisted current superalgebra

$$\overline{\mathcal{L}}_{tw}\mathfrak{q}_N = \text{span}\{E_{ij} \otimes t^r + E_{-i,-j} \otimes (-t)^r\}$$

where $-N \leq i, j \leq N$, $r \geq 0$. It is contained in the current algebra $\mathfrak{gl}_{N|N} \otimes \mathbb{C}[t]$ and more closely related to the Super Yangian than $\mathcal{L}_{tw}\mathfrak{q}_N$. Here, the associated involution is $\hat{\iota}(X(t)) = \iota(X)(-t)$.

2.4.1 *Queer Yangian*

Finally, we present the queer Yangian, i.e. the Yangian of $\mathfrak{q}_N$. Like the Lie superalgebra, this Yangian is $\mathbb{Z}/2\mathbb{Z}$-graded.

Definition 2.37. The *Yangian of the queer Lie superalgebra* $\mathfrak{q}_N$, denoted $Y_\hbar(\mathfrak{q}_N)$, is a complex associative unital graded algebra. It is generated by the elements $t_{ij}^{(s)}$ for $s = 1, 2, \ldots$ and $i, j = -N, \ldots, N$; $ij \neq 0$. Parity of these elements is given by $|t_{ij}^{(s)}| = |i| + |j|$. The generators satisfy the relations

$$\begin{aligned}(u^2 - v^2) \cdot [t_{ij}, t_{kl}] \cdot (-1)^{|i||k|+|i||l|+|k||l|} = \\ \hbar(u+v) \cdot (t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)) \\ - \hbar(u-v) \cdot (t_{-k,j}(u)t_{-i,l}(v) - t_{k,-j}(v)t_{i,-l}(u)) \cdot (-1)^{|k|+|l|}\end{aligned} \tag{2.4.1}$$

where we define $t_{ij}(u)$ to be the formal power series in u^{-1}

$$t_{ij}(u) \quad = \quad \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \tag{2.4.2}$$

If we let $\hbar = 1$, we denote the super Yangian $Y(\mathfrak{q}_N)$. Note that (2.3) of [Naz99] tells us that $t_{ij}^{(r)}(-1)^r = t_{-i,-j}^{(r)}$, implying that one can simply take generators for values $i > 0$. Alternatively, let $-N \leq i < j \leq N$ or $1 \leq i = j \leq N$.

Observe the following. By dividing both sides of (2.4.1) by $(u^2 - v^2)$, we obtain

$$\begin{aligned}
[t_{ij}(u), t_{kl}(v)](-1)^{|i||k|+|i||l|+|k||l|} = & \frac{\hbar}{u-v}(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)) \\
& - \frac{\hbar}{u+v}(t_{-k,j}(u)t_{-i,l}(v) - t_{k,-j}(v)t_{i,-l}(u))(-1)^{|k|+|l|}
\end{aligned}$$

$$\begin{aligned}
&[t_{ij}(u), t_{kl}(v)](-1)^{|i||k|+|i||l|+|k||l|} \\
&= \frac{\hbar u^{-1}}{1 - vu^{-1}} \Big((\delta_{kj} + t_{kj}^{(1)} u^{-1} + \cdots)(\delta_{il} + t_{il}^{(1)} v^{-1} + \cdots) \\
&\quad -(\delta_{kj} + t_{kj}^{(1)} v^{-1} + \cdots)(\delta_{il} + t_{il}^{(1)} u^{-1} + \cdots) \Big) \\
&\quad - (-1)^{|k|+|l|} \frac{\hbar u^{-1}}{1 + vu^{-1}} \\
&\quad \Big((\delta_{-k,j} + t_{-k,j}^{(1)} u^{-1} + \cdots)(\delta_{-i,l} + t_{-i,l}^{(1)} v^{-1} + \cdots) \\
&\quad -(\delta_{k,-j} + t_{k,-j}^{(1)} v^{-1} + \cdots)(\delta_{i,-l} + t_{i,-l}^{(1)} u^{-1} + \cdots) \Big) \\
&= \hbar(u^{-1} + vu^{-2} + \cdots) \\
&\quad \Big((\delta_{kj} + t_{kj}^{(1)} u^{-1} + \cdots)(\delta_{il} + t_{il}^{(1)} v^{-1} + \cdots) \\
&\quad -(\delta_{kj} + t_{kj}^{(1)} v^{-1} + \cdots)(\delta_{il} + t_{il}^{(1)} u^{-1} + \cdots) \Big) \\
&\quad - (-1)^{|k|+|l|} \hbar(u^{-1} - vu^{-2} + v^2 u^{-3} + \cdots) \\
&\quad \Big((\delta_{-k,j} + t_{-k,j}^{(1)} u^{-1} + \cdots)(\delta_{-i,l} + t_{-i,l}^{(1)} v^{-1} + \cdots) \\
&\quad -(\delta_{k,-j} + t_{k,-j}^{(1)} v^{-1} + \cdots)(\delta_{i,-l} + t_{i,-l}^{(1)} u^{-1} + \cdots) \Big).
\end{aligned} \tag{2.4.3}$$

The coefficient of $u^{-r}v^{-s}$ on each side of the equation is

$$\begin{aligned}
&\left[t_{ij}^{(r)},\ t_{kl}^{(s)}\right] = \\
&\qquad \hbar \sum_{a=1}^{r} \Big(t_{kj}^{(r-a)} t_{il}^{(s+a-1)} - t_{kj}^{(s+a-1)} t_{il}^{(r-a)} - \\
&\qquad (-1)^{|k|+|l|+a+1} (t_{-k,j}^{(r-a)} t_{-i,l}^{(s+a-1)} - t_{-k,j}^{(s+a-1)} t_{-i,l}^{(r-a)}) \Big) .
\end{aligned}$$

This gives us another defining relation of the queer Yangian. Alternatively, we can start again with (2.4.1) and simply find the coefficient of $u^{-m}v^{-n}$ for $m, n > 0$ on each side of the equation. We produce an identity similar to the other defining relation of the Yangian $Y(\mathfrak{gl}_N)$ of the form (2.2.2). Using the shorthand $\theta(i,k,l) = (-1)^{|i||k|+|i||l|+|k||l|}$, the left hand side yields

$$\left(\left[t_{ij}^{(m+2)}, t_{kl}^{(n)} \right] - \left[t_{ij}^{(m)}, t_{kl}^{(n+2)} \right] \right) \theta(i,k,l)$$

and the right hand side yields

$$\begin{aligned}
&\hbar (t_{kj}^{(m+1)} t_{il}^{(n)} - t_{kj}^{(n)} t_{il}^{(m+1)} + t_{kj}^{(m)} t_{il}^{(n+1)} - t_{kj}^{(n+1)} t_{il}^{(m)}) \\
&\qquad - \hbar \left(t_{-k,j}^{(m+1)} t_{-i,l}^{(n)} - t_{k,-j}^{(n)} t_{i,-j}^{(m+1)} - t_{-k,j}^{(m)} t_{-i,l}^{(n+1)} + t_{k,-j}^{(n+1)} t_{i,-l}^{(m)} \right) (-1)^{|k|+|l|}.
\end{aligned}$$

As a result,

$$\begin{aligned}
&\left(\left[t_{ij}^{(m+2)}, t_{kl}^{(n)} \right] - \left[t_{ij}^{(m)}, t_{kl}^{(n+2)} \right] \right) \theta(i,k,l) = \\
&\qquad \hbar \left(t_{kj}^{(m+1)} t_{il}^{(n)} - t_{kj}^{(n)} t_{il}^{(m+1)} + t_{kj}^{(m)} t_{il}^{(n+1)} - t_{kj}^{(n+1)} t_{il}^{(m)} \right) \\
&\qquad - \hbar \left(t_{-k,j}^{(m+1)} t_{-i,l}^{(n)} - t_{k,-j}^{(n)} t_{i,-j}^{(m+1)} - t_{-k,j}^{(m)} t_{-i,l}^{(n+1)} + t_{k,-j}^{(n+1)} t_{i,-l}^{(m)} \right) (-1)^{|k|+|l|}.
\end{aligned}$$

Theorem 2.38. ***Theorem 2.3, [Naz99]*** *Let* $\mathrm{gr}\_Y(\mathfrak{q}_N)$ *be the* $\mathbb{Z}$*-graded algebra associated to the filtration on* $Y(\mathfrak{q}_N)$ *setting* $\deg(t_{ij}^{(r)}) = r-1$*. Then,* $\mathrm{gr}\_Y(\mathfrak{q}_N)$ *is isomorphic to* $U(\mathfrak{g})$ *for the twisted polynomial current superalgebra* $\overline{\mathcal{L}_{tw}}\mathfrak{q}_N$*.*

Effectively, Theorem 2.38 tells us that the PBW basis for the enveloping algebra provides us with a PBW basis for the Yangian.

2.4.2 *Twisted Quantum Loop Superalgebra*

The next algebraic player in this section is the twisted quantum loop superalgebra of type $\mathfrak{q}$, denoted $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$. This quantum group is defined as follows.

Definition 2.39. The twisted quantum loop superalgebra of type $\mathfrak{q}$ $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$ over complex rational functions in q is generated by the elements $T_{ij}^{(r)}$ for r nonnegative, i, j between $-N$ and N and nonzero. These generators behave according to the following relations:

$$\begin{aligned} T_{ij}^{(0)} &= 0 \text{ for } i > j && (2.4.4)\\ T_{ii}^{(0)}T_{-i,-i}^{(0)} &= 1 = T_{-i,-i}^{(0)}T_{ii}^{(0)} && (2.4.5)\\ T_{12}(w)T_{13}(z)S_{23}(w,z) &= S_{23}(w,z)T_{13}(z)T_{12}(w), && (2.4.6) \end{aligned}$$

using that

$$\begin{aligned} T(z) &= \sum_{i,j=-n,\ ij\neq 0}^{n} T_{ij}(z) \otimes E_{ij} && (2.4.7)\\ T_{ij}(z) &= \sum_{r\geq 0} T_{ij}^{(r)} z^{-r} && (2.4.8)\\ S(w,z) &= S + \frac{\varepsilon P}{w^{-1}z-1} + \frac{\varepsilon P J_1 J_2}{wz-1} && (2.4.9)\\ S &= \sum_{a=1}^{n}(1+(q-1)\mathsf{E}_{aa}^{0} \otimes E_{aa}) + \sum_{a=1}^{n}(1+(q^{-1}-1)\mathsf{E}_{aa}^{0}) \otimes E_{-a,-a} && (2.4.10) \end{aligned}$$

$$
\begin{aligned}
& +\varepsilon\left(\sum_{a,b=1,\ a>b}^{n} \mathsf{E}^0_{ab}\otimes E_{ba} - \sum_{a,b=1,\ a<b}^{n} \mathsf{E}^0_{ab}\otimes E_{-b,-a} - \sum_{a,b=1}^{n} \mathsf{E}^1_{ab}\otimes E_{-b,a}\right) \\
\mathsf{E}^0_{ab} &= E_{ab}+E_{-a,-b},\ \ \mathsf{E}^1_{ab}=E_{-a,b}+E_{a,-b},\ \ 1\le a,b\le n \qquad (2.4.11)\\
P &= \sum_{i,j=-n,\ ij\neq 0}^{n} (-1)^{|j|}E_{ij}\otimes E_{ji} \qquad (2.4.12)\\
J_1 &= J\otimes 1,\ \ J_2 = 1\otimes J. \qquad (2.4.13)
\end{aligned}
$$

The defining relation of $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$ is

$$
T_{12}(w)T_{13}(z)S_{23}(\mathcal{C}^{-1}w,\mathcal{C}^{-1}z) = S_{23}(\mathcal{C}w,\mathcal{C}z)T_{13}(z)T_{12}(w) \qquad (2.4.14)
$$

where $\mathcal{C}$ is an invertible, central, even element. Here, we let $\mathcal{C}=1$. Take this defining relation, substitute in (2.4.9) and expand to find the coefficient of $w^{-r}z^{-s}$ on both sides. This produces:

$$
\begin{aligned}
&(-1)^{p(i,j)p(k,l)}q^{\varphi(j,l)}(T^{(r)}_{ij}T^{(s+2)}_{kl} - T^{(r-1)}_{ij}T^{(s+1)}_{kl} - T^{(r+1)}_{ij}T^{(s+1)}_{kl} + T^{(r)}_{ij}T^{(s)}_{kl})\\
&\quad + \{j<l\}\theta(i,j,k)\varepsilon(T^{(r)}_{il}T^{(s+2)}_{kj} - T^{(r-1)}_{il}T^{(s+1)}_{kj} - T^{(r+1)}_{il}T^{(s+1)}_{kj} + T^{(r)}_{il}T^{(s)}_{kj})\\
&\quad + \{-l<j\}\theta(-i,-j,k)\varepsilon(T^{(r)}_{i,-l}T^{(s+2)}_{k,-j} - T^{(r-1)}_{i,-l}T^{(s+1)}_{k,-j} - T^{(r+1)}_{i,-l}T^{(s+1)}_{k,-j} + T^{(r)}_{i,-l}T^{(s)}_{k,-j})\\
&\quad + \varepsilon\theta(i,j,k)(T^{(r+1)}_{il}T^{(s+1)}_{kj} - T^{(r)}_{il}T^{(s)}_{kj}) - \varepsilon\theta(i,j,-k)(T^{(r-1)}_{i,-l}T^{(s+1)}_{k,-j} - T^{(r)}_{i,-l}T^{(s)}_{k,-j})\\
&=q^{\varphi(i,k)}(T^{(s+2)}_{kl}T^{(r)}_{ij} - T^{(s+1)}_{kl}T^{(r-1)}_{ij} - T^{(s+1)}_{kl}T^{(r+1)}_{ij} + T^{(s)}_{kl}T^{(r)}_{ij})\\
&\quad + \{k<i\}\theta(i,j,k)\varepsilon(T^{(s+2)}_{il}T^{(r)}_{kj} - T^{(s+1)}_{il}T^{(r-1)}_{kj} - T^{(s+1)}_{il}T^{(r+1)}_{kj} + T^{(s)}_{il}T^{(r)}_{kj})\\
&\quad + \{i<-k\}\theta(-i,-j,k)\varepsilon(T^{(s+2)}_{-i,l}T^{(r)}_{-k,j} - T^{(s+1)}_{-i,l}T^{(r-1)}_{-k,j} - T^{(s+1)}_{-i,l}T^{(r+1)}_{-k,j} + T^{(s)}_{-i,l}T^{(r)}_{-k,j})\\
&\quad + \varepsilon\theta(i,j,k)(T^{(s+1)}_{il}T^{(r+1)}_{kj} - T^{(s)}_{il}T^{(r)}_{kj}) - \varepsilon\theta(i,j,-k)(T^{(s+1)}_{-i,l}T^{(r-1)}_{-k,j} - T^{(s)}_{-i,l}T^{(r)}_{-k,j})
\end{aligned}
\qquad (2.4.15)
$$

We use the shorthand $p(i,j)=|i|+|j|$, $\varphi(i,j)=(\delta_{i,j}+\delta_{i,-j})\text{sign}(j)$, and $\theta(i,j,k)=(-1)^{|i||j|+|j||k|+|k||i|}$. The different superscripts appear depending on the powers of z and w in the expansion of $S(w,z)$. For instance, if finding the power of $w^{-r}z^{-s}$

in the term $-w^{-1}zS$, the corresponding terms from $T_{12}(w)T_{13}(z)$ need to provide $w^{-r+1}z^{-s-1}$, hence $T_{ij}^{(r-1)}T_{kl}^{(s+1)}$.

In this thesis, some alternative sets of generators for $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$ will be used. The first set is composed of the elements $\tau_{ij}^{(r)}$, where $\tau_{ij}^{(r)} = T_{ij}^{(r)}/(q-q^{-1})$, unless $i=j$ and $r=0$, in which case $\tau_{ii}^{(0)} = (T_{ii}^{(0)}-1)/(q-1)$. Then, we define another set of generators $T_{ij}^{(r,m)}$ given by $T_{ij}^{(r,0)} = \tau_{ij}^{(r)}$ and $T_{ij}^{(r,m+1)} = T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)}$.

This gives us equalities such as:

$$
\begin{aligned}
T_{ij}^{(r,1)} &= T_{ij}^{(r+1,0)} - T_{ij}^{(r,0)} && (2.4.16)\\
T_{ij}^{(r,2)} &= T_{ij}^{(r+1,1)} - T_{ij}^{(r,1)} \\
&= (T_{ij}^{(r+2,0)} - T_{ij}^{(r+1,0)}) - (T_{ij}^{(r+1,0)} - T_{ij}^{(r,0)}) \\
&= T_{ij}^{(r+2,0)} - 2T_{ij}^{(r+1,0)} + T_{ij}^{(r,0)} && (2.4.17)\\
T_{ij}^{(r,3)} &= T_{ij}^{(r+1,2)} - T_{ij}^{(r,2)} \\
&= (T_{ij}^{(r+2,1)} - T_{ij}^{(r+1,1)}) - (T_{ij}^{(r+1,1)} - T_{ij}^{(r,1)}) \\
&= T_{ij}^{(r+3,0)} - T_{ij}^{(r+2,0)} - T_{ij}^{(r+2,0)} + T_{ij}^{(r+1,0)} - T_{ij}^{(r+2,0)} \\
&\quad +T_{ij}^{(r+1,0)} + T_{ij}^{(r+1,0)} - T_{ij}^{(r,0)} \\
&= T_{ij}^{(r+3,0)} - 3T_{ij}^{(r+2,0)} + 3T_{ij}^{(r+1,0)} - T_{ij}^{(r,0)} && (2.4.18)
\end{aligned}
$$

After observing this pattern, we can assert the following lemma. This will be useful in Chapter 6.

Lemma 2.40. *For any r and m,*

$$T_{ij}^{(r,m)} = \sum_{a=0}^{m}(-1)^a \binom{m}{a} T_{ij}^{(r+m-a,0)} = \sum_{a=0}^{m}(-1)^{m-a}\binom{m}{a} T_{ij}^{(r+a,0)}$$

Proof. We prove this by induction. The base case is already proven in (2.4.16).

Then, we want to show that

$$T_{ij}^{(r,m+1)} = \sum_{a=0}^{m+1} (-1)^a \binom{m+1}{a} T_{ij}^{(r+m+1-a,0)} = \sum_{a=0}^{m+1} (-1)^{m-a+1} \binom{m+1}{a} T_{ij}^{(r+a,0)}$$

For this, we'll use that $T_{ij}^{(r,m+1)} = T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)}$ and the identity $\binom{m+1}{a} = \binom{m}{a} + \binom{m}{a-1}$. Then,

$$\begin{aligned}
T_{ij}^{(r,m+1)} &= T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)} \\
&= \sum_{a=0}^{m} (-1)^a \binom{m}{a} T_{ij}^{(r+m+1-a,0)} - \sum_{a=0}^{m} (-1)^a \binom{m}{a} T_{ij}^{(r+m-a,0)} \\
&= \sum_{a=1}^{m} (-1)^a \binom{m}{a} T_{ij}^{(r+m+1-a,0)} + (-1)^0 \binom{m}{0} T_{ij}^{(r+m+1-0,0)} \\
&\quad - \sum_{a=1}^{m} (-1)^{a-1} \binom{m}{a-1} T_{ij}^{(r+m+1-a,0)} - (-1)^m \binom{m}{m} T_{ij}^{(r,0)} \\
&= \sum_{a=1}^{m} (-1)^a \left(\binom{m}{a} + \binom{m}{a-1} \right) T_{ij}^{(r+m+1-a,0)} \\
&\quad + (-1)^0 \binom{m+1}{0} T_{ij}^{(r+m+1-0,0)} + (-1)^{m+1} \binom{m+1}{m+1} T_{ij}^{(r,0)} \\
&= \sum_{a=0}^{m+1} (-1)^a \binom{m+1}{a} T_{ij}^{(r+m+1-a,0)}
\end{aligned}$$

□

3

SIMILARITIES BETWEEN YANGIANS AND QUANTUM LOOP ALGEBRAS

Throughout this thesis, similar properties will be verified for Yangians and quantum loop algebras in various settings. The fact that these correspondences exist is no coincidence. Before exposition of those results, this chapter will provide an overview of three major results on Yangians and quantum loop algebras with the goal of providing a backdrop for what will follow. First, Theorem 3.3 outlines the process through which we can see the Yangian $Y(\mathfrak{g})$ as a limit form of the quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$. Then, we look at two important theorems that provide isomorphisms between the completions (Theorem 3.6) and categories of finite dimensional representations (Theorem 3.12) of the two families of quantum groups. In particular, many of my results take place in a superalgebra setting instead of an algebra setting; this hints that the theorems of this chapter could have their own analogues in the super case.

3.1 DEGENERATION

In this section, we use the presentation of the Yangian with generators $x_{i,r}^{\pm}$ and $h_{i,r}$ (see Definition 2.1) with $\hbar = 1$. Our goal is to show that the quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ degenerates to $Y(\mathfrak{g})$. This idea of Yangians as limit forms of quantum loop algebras underlies the connections between the two structures. Throughout the original paper, Guay and Ma use both $Y(\mathfrak{g}, \sigma)$ and $Y(\mathfrak{g})$ for some Dynkin diagram isomorphism σ, but they are isomorphic as Yangians so we will take σ to be trivial and only consider $Y(\mathfrak{g})$. Consider the following well-known proposition.

Proposition 3.1. *$U_\hbar(\mathcal{L}(\mathfrak{g}))/hU_\hbar(\mathcal{L}(\mathfrak{g})) \simeq U(\mathcal{L}(\mathfrak{g}))$ and $U_\hbar(\mathcal{L}(\mathfrak{g})) \simeq U(\mathcal{L}(\mathfrak{g}))[[h]]$ as $\mathbb{C}[[\hbar]]$-modules.*

Recall $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[s, s^{-1}]$. Take the sequence of algebra homomorphisms

$$U_\hbar(\mathcal{L}(\mathfrak{g})) \twoheadrightarrow U_\hbar(\mathcal{L}(\mathfrak{g}))/hU_\hbar(\mathcal{L}(\mathfrak{g})) \to U(\mathcal{L}(\mathfrak{g})) \twoheadrightarrow U(\mathfrak{g}) \tag{3.1.1}$$

where the last map $U(\mathcal{L}(\mathfrak{g})) \twoheadrightarrow U(\mathfrak{g})$ is defined by sending $s \mapsto 1$. Call the kernel of the composition of these maps $\mathbf{K}$. Let $\widetilde{Y}(\mathfrak{g})$ be the $\mathbb{C}[[\hbar]]$-algebra generated by $U_\hbar(\mathcal{L}(\mathfrak{g}))$ and $\frac{\mathbf{K}}{\hbar}$. It is a subalgebra of $\mathbb{C}((\hbar)) \otimes_{\mathbb{C}[[\hbar]]} U_\hbar(\mathcal{L}(\mathfrak{g}))$.

Finally, we reach the two statements of the main theorem of [GM12].

Theorem 3.2. *$\widetilde{Y}(\mathfrak{g})/\hbar\widetilde{Y}(\mathfrak{g}) \simeq Y(\mathfrak{g})$.*

Theorem 3.3. *$Y_\hbar(\mathfrak{g})$ is isomorphic to $gr_{\mathbf{K}}(U_\hbar(\mathcal{L}(\mathfrak{g})))$ where $gr_K(U_\hbar(\mathcal{L}(\mathfrak{g})))$ is the graded ring $\bigoplus_{n=0}^{\infty} \mathbf{K}^n/\mathbf{K}^{n+1}$.*

Proof. See [GM12] for a full proof. The main steps involve first showing that certain relations hold in the quantum loop algebra, then moving to the quotients $\mathbf{K}^r/\mathbf{K}^{r+1}$ given by its filtration. This then reproduces equivalences similar to the defining relations on the Yangian, leading to the defining relation-preserving algebra homomorphism between the Yangian and $gr_{\mathbf{K}}(U_\hbar(\mathcal{L}(\mathfrak{g})))$.

Note that it is enough to let $m = 1$, as we can write any difference $x^{\pm}_{i,r+m} - x^{\pm}_{i,r}$ as a telescopic sum:

$$x^{\pm}_{i,r+m} - x^{\pm}_{i,r} = x^{\pm}_{i,r+m} - x^{\pm}_{i,r+m-1} + x^{\pm}_{i,r+m-1} - x^{\pm}_{i,r+m-2} + \cdots + x^{\pm}_{i,r+1} - x^{\pm}_{i,r}$$

Thus, if we denote by φ the homomorphism between $Y_{\hbar}(\mathfrak{g})$ and $gr_{\mathbf{K}}(U_{\hbar}(\mathcal{L}(\mathfrak{g})))$, we can say that the image of φ is generated by $\overline{\mathcal{X}^{\pm}_{i,0}}$ and $\overline{\mathcal{X}^{\pm}_{i,1} - \mathcal{X}^{\pm}_{i,0}}$. The Yangian is generated by elements $x^{\pm}_{i,0}$ and $x^{\pm}_{i,1}$. $\mathbf{K}$ is generated by differences $\mathcal{X}^{\pm}_{i,r+1} - \mathcal{X}^{\pm}_{i,r}$, so their image $\overline{\mathcal{X}^{\pm}_{i,r+1} - \mathcal{X}^{\pm}_{i,r}}$ in the quotient space lies in $\mathbf{K}/\mathbf{K}^2 = \mathbb{C}$. Thus, these elements generate $\bigoplus \mathbf{K}^n/\mathbf{K}^{n+1}$. To prove surjectivity of φ, it suffices to justify that $\overline{\mathcal{X}^{\pm}_{i,r+1} - \mathcal{X}^{\pm}_{i,r}}$ lies in the image of φ.

When $r = 0$, we have $\overline{\mathcal{X}^{\pm}_{i,1} - \mathcal{X}^{\pm}_{i,0}} = \varphi(x^{\pm}_{i,1})$. Observe that $(\mathcal{X}^{\pm}_{i,r+1} - \mathcal{X}^{\pm}_{i,r}) - (\mathcal{X}^{\pm}_{i,1} - \mathcal{X}^{\pm}_{i,0}) \in \mathbf{K}^2$, for instance . Thus, $(\mathcal{X}^{\pm}_{i,2} - \mathcal{X}^{\pm}_{i,1}) - (\mathcal{X}^{\pm}_{i,1} - \mathcal{X}^{\pm}_{i,0}) \in \mathbf{K}^2$ and $\overline{\mathcal{X}^{\pm}_{i,2} - 2\mathcal{X}^{\pm}_{i,1} + \mathcal{X}^{\pm}_{i,0}} = \varphi(X_{i,2}) \in \varphi(Y(\mathfrak{g}))$. So, $\overline{\mathcal{X}^{\pm}_{i,2} - \mathcal{X}^{\pm}_{i,1}} = \varphi(X_{i,2}) + \varphi(X_{i,1})$. One can then follow a similar pattern to see that $\overline{\mathcal{X}^{\pm}_{i,r+1} - \mathcal{X}^{\pm}_{i,r}}$ lies in the image of φ for all r. Similar arguments apply for the generators $h_{i,r}, \mathcal{H}_{i,r}$. Thus, φ is surjective.

Injectivity follows from showing that we have a basis for $gr_{\mathbf{K}}(U_{\hbar}(\mathcal{L}(\mathfrak{g})))$ made up of of images of the Yangian. Specifically, each piece $\mathbf{K}^s/\mathbf{K}^{s+1}$ of $gr_{\mathbf{K}}(U_{\hbar}(\mathcal{L}(\mathfrak{g})))$ has a basis consisting of monomials in the elements $\varphi(x^{+}_{i,r}), \varphi(x^{-}_{i,r}), \varphi(h_{i,r})$ with coefficient $\hbar^t$, $0 \leq t \leq s$ where the sum of r-indices is $s - t$.

An alternate proof would prove surjectivity and injectivity at the same time. To do so, we show that the homomorphism is an isomorphism by using PBW basis of the quantum loop algebra to obtain a basis of $\mathbf{K}^n$ and thus also a basis of $\mathbf{K}^n/\mathbf{K}^{n+1}$. Then, one shows that the homomorphism sends a PBW basis of the Yangians to this basis of $\oplus \mathbf{K}^n/\mathbf{K}^{n+1}$. Thus, the map is an isomorphism. $\square$

In later chapters of this thesis, this proof showing degeneration of the Yangian to the quantum loop algebra will be imitated for the superalgebras $\mathfrak{gl}_{M|N}$ (Chapter 4) and $\mathfrak{q}_n$ (Chapter 6).

3.2 ISOMORPHISM OF COMPLETIONS

Next, we present a result of Gautam and Toledano Laredo [GTL13] that gives a map

$$U_q(\mathcal{L}\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})}$$

which leads to an isomorphism

$$\widehat{U_q(\mathcal{L}\mathfrak{g})} \xrightarrow{\sim} \widehat{Y_\hbar(\mathfrak{g})}$$

of completions, defined below. In essence, it is a stronger version of the main theorem in the previous section. The existence of this isomorphism has been conjectured for several decades, but was not proven or constructed until this pivotal article in 2010.

3.2.1 *Completion of the Yangian*

Consider the presentation of $Y_\hbar(\mathfrak{g})$ given in Definition 2.1. We also define a grading by $\deg x_{i,r}^{\pm} = r = \deg h_{i,r}$ and $\deg \hbar = 1$. Let $Y_\hbar(\mathfrak{g})[m]$ denote the span of all monomials of degree m. If $m = 0$, then $Y_\hbar(\mathfrak{g})[0]$ is equal to the enveloping algebra $U(\mathfrak{g})$. As a result, we can say that any element of the Yangian is the sum of its homogenous parts, i.e.

$$Y_\hbar(\mathfrak{g}) = \bigoplus_{m=0}^{\infty} Y_\hbar(\mathfrak{g})[m].$$

Critically, the sums of homogenous elements given by the right hand side of the equation are finite sums. More precisely, we can express an element of the Yangian as the finite sum $Y^{(0)} + \cdots + Y^{(k)}$ with $Y^{(m)} \in Y_\hbar(\mathfrak{g})[m]$ some finite sum of monomials of degree m. Thus, when we take the completion of the Yangian, we can say that

$$\widehat{Y_\hbar(\mathfrak{g})} = \prod_{m=0}^{\infty} Y_\hbar(\mathfrak{g})[m]$$

with the possibility of infinite sums of $Y^{(m)}$. Note that each $Y^{(m)}$ is still a finite sum of monomials of degree m; we can just take infinitely many $Y^{(m)}$ for different values of m. For this reason, the product of two infinite sums produces an infinite sum of finite graded pieces. More precisely,

$$(X^{(0)} + X^{(1)} + \cdots)(Y^{(0)} + Y^{(1)} + \cdots) = Z^{(0)} + Z^{(1)} + \cdots$$

where

$$Z^{(m)} = \sum_{i+j=m} X^{(i)} Y^{(j)}.$$

$Z^{(m)}$ is a finite sum due to the fact that $i, j \geq 0$ and only finitely many pairs of i, j can add to m.

3.2.2 *Statement of the theorem*

Let $q := e^{\pi i \hbar}$, where $i = \sqrt{-1}$. The relationship between q and $\hbar$ is relevant in that we use one or the other as a parameter depending on the context or what notation is the most concise.

Theorem 3.4. *[GTL13] There exists an algebra homomorphism $\phi : U_q(\mathcal{L}\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})}$ satisfying the following properties.*

1. *Let $\widehat{U_q(\mathcal{L}\mathfrak{g})}$ be the completion of $U_q(\mathcal{L}\mathfrak{g})$ with respect to the ideal of $z = 1$. Then, ϕ induces an isomorphism $\widehat{U_q(\mathcal{L}\mathfrak{g})} \xrightarrow{\sim} \widehat{Y_\hbar(\mathfrak{g})}$.*

2. *ϕ induces Drinfeld's degeneration of $U_q(\mathcal{L}\mathfrak{g})$ to $Y_\hbar(\mathfrak{g})$. This is Theorem 3.3 we saw in the previous section.*

ϕ is explicitly given by where it maps the loop generators of $U_q(\mathcal{L}\mathfrak{g})$.

$$\begin{aligned} \phi(H_{i,0}) &= d_i^{-1} t_{i,0} && (3.2.1)\\ \phi(H_{i,r}) &= \frac{\hbar}{q_i - q_i^{-1}} \sum_{m\geq 0} \tilde{h}_{i,m} \frac{r^m}{m!} && (3.2.2)\\ \phi(E_{i,k}) &= e^{k\sigma_i +} \sum_{m\geq 0} g_{i,m}^{+} x_{i,m}^{+} && (3.2.3)\\ \phi(F_{i,k}) &= e^{k\sigma_i -} \sum_{m\geq 0} g_{i,m}^{-} x_{i,m}^{-} && (3.2.4) \end{aligned}$$

Note. If our Lie algebra is of type A, D, or E, then $d_i = 1$.

The formulas (3.2.1- 3.2.4) introduce some new elements of either the Yangian or its completion. First, the elements $\tilde{h}_{i,m}$ are an alternative set of generators to the subalgebra of $Y_\hbar(\mathfrak{g})$ generated by the $h_{i,m}$. The relationship between the two sets of generators is given by

$$\hbar \sum_{m\geq 0} \tilde{h}_{i,m} u^{-m-1} = \log\left(1 + \hbar \sum_{m\geq 0} h_{i,m} u^{-m-1}\right) \qquad (3.2.5)$$

where both sides of the equation lie in $Y_\hbar(\mathfrak{g})[[u^{-1}]]$. We can be sure that the right hand side does indeed have coefficients that lie in $Y_\hbar(\mathfrak{g})$, i.e. are finite sums, by using the expansion $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$. Then, letting $x = \hbar \sum_{m\geq 0} h_{i,m} u^{-m-1}$, the k-th term of this expansion is

$$(-1)^{k+1}/k\left(\hbar\sum_{m\geq 0}h_{i,m}u^{-m-1}\right)^k=(-1)^{k+1}/k\left(\hbar\sum_{m\geq 0}h_{i,m}u^{-m}\right)^k u^{-k}$$
$$=(-1)^{k+1}/k\cdot\hbar^k(h_{i,0}+h_{i,1}u^{-1}+h_{i,2}u^{-2}+\cdots)^k u^{-k}.$$

The coefficient of u^{-l} for some $l\geq 0$ in the expansion of the k-th term is

$$(-1)^{k+1}/k\cdot\hbar^k\left(\sum_{j_1+j_2+\cdots+j_k=l-k}h_{i,j_1}h_{i,j_2}\cdots h_{i,j_k}\right).$$

As a result, the coefficient of u^{-l} for some $l\geq 0$ in the entire expansion of the left hand side of equation (3.2.5) is

$$\sum_{k=1}^{l}\left((-1)^{k+1}/k\cdot\hbar^k\left(\sum_{j_1+j_2+\cdots+j_k=l-k}h_{i,j_1}h_{i,j_2}\cdots h_{i,j_k}\right)\right),$$

which is a finite sum.

Second, we introduce the elements $g_{i,m}^{\pm}\in\widehat{Y_\hbar(\mathfrak{g})}$. This means adding several definitions to our arsenal. Let $G(u)$ be the formal power series

$$G(u)=\log\left(\frac{u}{e^{u/2}-e^{-u/2}}\right)$$

in $u\mathbb{Q}[[u]]$ and consider the elements γ_i in $\widehat{Y^0}[[u]]$ given by

$$\gamma_i(u)=\hbar\sum_{r\geq 0}\frac{\tilde{h}_{i,r}}{r!}\left(-\frac{d}{du}\right)^{r+1}G(u).$$

Thus, we arrive the elements $g_{i,m}^{\pm}\in\widehat{Y_\hbar(\mathfrak{g})}$ through the equality

$$\sum_{m\geq 0}g_{i,m}^{\pm}u^m=\left(\frac{\hbar}{q_i-q_i^{-1}}\right)^{1/2}\exp\left(\frac{\gamma_i(u)}{2}\right).$$

To justify that these elements do lie in $\widehat{Y_\hbar(\mathfrak{g})}$, we first show that $G(u) \in u\mathbb{Q}[[u]]$.

$$\begin{aligned} G(u) &= \log\left(\frac{u}{e^{u/2} - e^{-u/2}}\right) \\ &= \log\left(\frac{u}{2(\frac{u}{2} + (\frac{u}{2})^3\frac{1}{3!} + (\frac{u}{2})^5\frac{1}{5!} + \cdots)}\right) \\ &= \log\left(\frac{1}{1 + (\frac{u}{2})^2\frac{1}{3!} + (\frac{u}{2})^4\frac{1}{5!} + \cdots}\right) \end{aligned}$$

If we let $x = (\frac{u}{2})^2\frac{1}{3!} + (\frac{u}{2})^4\frac{1}{5!} + \cdots$, then

$$\begin{aligned} \log\left(\frac{1}{1 + (\frac{u}{2})^2\frac{1}{3!} + (\frac{u}{2})^4\frac{1}{5!} + \cdots}\right) &= -\log(1+x) \\ &= -\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right] \end{aligned}$$

which is clearly in $u\mathbb{Q}[[u]]$ after substituting out x in terms of u. Further, observe that

$$\begin{aligned} G(-u) &= \log\left(\frac{-u}{e^{-u/2} - e^{u/2}}\right) \\ &= \log\left(\frac{u}{e^{u/2} - e^{-u/2}}\right) \\ &= G(u) \end{aligned}$$

so $G(u)$ is in $u^2\mathbb{Q}[[u^2]] \subset u\mathbb{Q}[[u^2]]$. Next, $\gamma_i(u) \in \widehat{Y^0}[[u]]$ due to the fact that the derivatives preserve the nature of the series. Finally, we look at the elements $g_{i,m}$ in context of the expansion of $\phi(E_{i,k})$. Although $\phi(E_{i,k})$ will lie in the completion of the Yangian and thus contains infinite sums, we still require that each homogeneous graded piece is itself a finite sum of elements. We will now verify this for $\phi(E_{i,k}) = e^{k\sigma_i^+} \sum_{m\geq 0} g_{i,m}^+ x_{i,m}^+$.

Let $\sigma_i^{\pm}$ be the pair of homomorphisms on the subalgebra generated by the $x_{j,r}^{+}$ (respectively, the $x_{j,r}^{-}$) given in each case by

$$\sigma_i^{\pm} : x_{j,r}^{\pm} \mapsto x_{j,r+\delta_{ij}}^{\pm}$$

and look at $e^{k\sigma_i^{\pm}}$ using its Taylor expansion. Then,

$$\begin{aligned}
\phi(E_{i,k}) &= e^{k\sigma_i+} \sum_{m\geq 0} g_{i,m}^{+} x_{i,m}^{+} \\
&= \left(\mathrm{id} + k\sigma_i^{+} + \frac{k^2(\sigma_i^{+})^2}{2!} + \frac{k^3(\sigma_i^{+})^3}{3!} + \cdots\right) \sum_{m\geq 0} g_{i,m}^{+} x_{i,m}^{+} \\
&= \sum_{m\geq 0} g_{i,m}^{+} x_{i,m}^{+} + k \sum_{m\geq 0} g_{i,m}^{+} x_{i,m+1}^{+} + \frac{k^2}{2!} \sum_{m\geq 0} g_{i,m}^{+} x_{i,m+2}^{+} + \cdots
\end{aligned}$$

If the elements $g_{i,m}^{\pm}$ are homogeneous of degree 0, then the sum of elements of degree r in $\phi(E_{i,k})$ is

$$g_{i,r}^{+} x_{i,r}^{+} + k g_{i,r-1}^{+} x_{i,r}^{+} + \frac{k^2}{2!} g_{i,r-2}^{+} x_{i,r}^{+} + \cdots + \frac{k^r}{r!} g_{i,0}^{+} x_{i,r}^{+}.$$

If the elements $g_{i,m}^{\pm}$ are of degree 0 but not homogeneous, then

$$g_{i,m}^{+} = \sum g_{i,m}^{+}[s]$$

for homogeneous elements $g_{i,m}^{+}[s]$. Then, the sum of elements of degree r in $\phi(E_{i,k})$ is

$$g_{i,r}^{+}[r-m] + k g_{i,r-1}^{+}[r-1-m] + \frac{k^2}{2!} g_{i,r-2}^{+}[r-2-m] + \cdots + \frac{k^r}{r!} g_{i,0}^{+}[-m].$$

If the elements $g_{i,m}^{\pm}$ have some nonzero degree, then the terms we need are drawn from the sum

$$\sum g_{i,m}^{+} + \cdots + \sum g_{i,m}^{+} x_{i,m+r}^{+}.$$

In either case, we have a finite sum in each part of degree r in the expansion of $\phi(E_{i,k})$. Similar arguments show that the same is true for $\phi(F_{i,k})$.

Clearly, these formulas are not the most natural choices of maps. Gautam and Toledano Laredo arrived at them by determining certain conditions necessary on ϕ and ensuring they are compatible with the relations on both sets of generators. Throughout Chapter 3 of [GTL13], these necessary and sufficient conditions are broken down further and further until the precise formulas (3.2.1-3.2.4) surface.

Once ϕ is constructed, we can use it to produce more useful maps between pairs of simpler structures. The following proposition provides an example.

Proposition 3.5. *[GTL13] Take $\phi : U_q(\mathcal{L}\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})}$ as defined above. Then, the specialization of ϕ at $\hbar = 0$ is the homomorphism*

$$\exp^* : U(\mathfrak{g}[z, z^{-1}]) \to \widehat{U(\mathfrak{g}[s])}$$

defined on elements of $\mathfrak{g}[z, z^{-1}]$ by

$$\exp^*(X \otimes z^k) = X \otimes e^{ks}.$$

3.2.3 *Notes on completions*

Before presenting $\widehat{U_q(\mathcal{L}\mathfrak{g})}$, the completion of $U_q(\mathcal{L}\mathfrak{g})$, we will first review some of the fundamental concepts involved in the inverse limit definition. Suppose, for some algebra A, there exists an inverse system of homomorphisms

$$\cdots \leftarrow A_n \leftarrow A_{n+1} \leftarrow \cdots$$

with maps $\theta_{n+1} : A_{n+1} \to A_n$, where the A_n correspond to a filtration of A. A sequence (a_n) of elements is called *coherent* if for each a_n in the sequence, $a_n \in A_n$ and $\theta_{n+1} a_{n+1} = a_n$. Suppose we take the algebra of all such coherent sequences; call this the completion $\widehat{A}$ and denote this by

$$\widehat{A} = \varprojlim A/A_n.$$

We use this process to construct $\widehat{U_q(\mathcal{L}\mathfrak{g})}$.

3.2.4 *Completion of the quantum loop algebra*

Next, we look at $\widehat{U_q(\mathcal{L}\mathfrak{g})}$, the completion of $U_q(\mathcal{L}\mathfrak{g})$, and see how our map ϕ yields an isomorphism between the completions of the quantum loop algebra and the Yangian.

Consider the sequence

$$U_\hbar \mathcal{L}\mathfrak{g} \to U(\mathcal{L}\mathfrak{g}) \to U\mathfrak{g}$$

where the first map sends $\hbar \mapsto 0$ and the second maps sends $z \mapsto 1$. Let $\mathcal{J} \subset U_\hbar \mathcal{L}\mathfrak{g}$ be the kernel of the composition of these maps. The completion of $U_\hbar \mathcal{L}\mathfrak{g}$ with respect to $\mathcal{J}$ is

$$\widehat{U_q(\mathcal{L}\mathfrak{g})} = \varprojlim U_q(\mathcal{L}\mathfrak{g})/(\mathcal{J}^n U_q(\mathcal{L}\mathfrak{g})).$$

Theorem 3.6. *[GTL13] Take $\phi : U_q(\mathcal{L}\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})}$ as defined above. Then,*

1. *ϕ maps $\mathcal{J}$ to the ideal $\widehat{Y_\hbar(\mathfrak{g})}_+ = \prod_{n \geq 1} Y_\hbar(\mathfrak{g})_n$.*

2. *The corresponding homomorphism*

$$\widehat{\phi} : \widehat{U_q(\mathcal{L}\mathfrak{g})} \to \widehat{Y_\hbar(\mathfrak{g})}$$

is an isomorphism.

Proposition 3.7. *[GTL13]* $\mathrm{gr}(\phi)$ *is the inverse of the degeneration isomorphism* ι : $Y_\hbar(\mathfrak{g}) \xrightarrow{\sim} U_\hbar(\mathcal{L}\mathfrak{g})$ *given in [GM12].*

3.3 EQUIVALENCE OF CATEGORIES

In Gautam and Toldano Laredo's October 2013 article [GTL16], they establish an equivalence of categories Γ between the representations of Yangians and quantum loop algebras. This equivalence confirms that there is an underlying reason that the correspondences between the two structures and their representations exist.

For a representation V of $Y(\mathfrak{g})$, we need that $\Gamma(V)$ is a representation of $U_q(\mathcal{L}(\mathfrak{g}))$. As vector spaces, $V = \Gamma(V)$. We can make sense of what $\Gamma(V)$ by first taking it to correspond to some algebra homomorphism $\varphi : U_q(\mathcal{L}(\mathfrak{g})) \to \mathrm{End}_{\mathbb{C}}(V)$. Consider the generators $\mathcal{X}_{i,k}^{\pm}$ of $U_q(\mathcal{L}(\mathfrak{g}))$ and identify them with their image $\varphi(\mathcal{X}_{i,k}^{\pm})$. If V has weight space decomposition $V = \bigoplus_\mu V_\mu$, then the way $\mathcal{X}_{i,k}^{\pm}$ acts on each weight space of V should be that it maps $\mathcal{X}_{i,k}^{\pm} : V_\mu \to V_{\mu+\alpha_i}$ for the simple root α_i. Then, understanding the specific action of $\mathcal{X}_{i,k}^{\pm}$ on V_μ, denoted $(\mathcal{X}_{i,k}^{\pm})_\mu$, leads to the desired functor Γ. Each $\mathcal{X}_{i,k}^{\pm}$ belongs to $\mathrm{Hom}_{\mathbb{C}}(V_\mu, V_{\mu+\alpha_i})$. Thus, the series $\mathcal{X}_i^{\pm}(u)$ belongs to $\mathrm{Hom}_{\mathbb{C}}(V_\mu, V_{\mu+\alpha_i})[[u^{-1}]]$.

Gautam and Toledano Laredo work in the general case with certain categories $\mathcal{O}_{int}$ of integrable representations. These are representations of $U_q(\mathcal{L}\mathfrak{g})$ (resp. $Y_\hbar(\mathfrak{g})$) such that their restriction to $U_q(\mathfrak{g})$ (resp. $\mathfrak{g}$) are integrable representations in the corresponding category $\mathcal{O}$. This requires finite dimensionality, a weight space decomposition, and certain restrictions on these weights. However, when $\mathfrak{g}$ is a semisimple Lie algebra as is the case here, the categories $\mathcal{O}_{int}(U_q(\mathcal{L}\mathfrak{g}))$ and

$\mathcal{O}_{int}(Y_\hbar(\mathfrak{g}))$ are simply the categories of finite dimensional representations on the quantum loop algebra and Yangian. When studying finite dimensional representations of the Yangian, we can take advantage of the fact that the irreducible representations are parametrized by polynomials; this implies that a representation is essentially given by rank$\mathfrak{g}$-many sets of complex numbers. Unfortunately, as the Yangian is not semisimple, we cannot simply say that any finite dimensional representation of $Y_\hbar(\mathfrak{g})$ is the direct sum of some irreducible representations and study these irreducible building blocks. Nevertheless, the following remark produces a composition series that will be useful when determining what category of modules we need to focus on.

Remarks. Take a finite dimensional module M over $Y(\mathfrak{g})$. If it is not irreducible, then it contains an irreducible submodule $M_1 \subset M$. The quotient M/M_1 is also a finite dimensional $Y(\mathfrak{g})$-module. If it is not irreducible, then it contains an irreducible submodule $\overline{M_2} \subset M/M_1$. By the correspondence theorem, $\overline{M_2} = M_2/M_1$ for some module M_2 with $M_1 \subset M_2 \subset M$. After enough iterations of this process, we obtain a chain

$$M_1 \subset M_2 \subset \cdots \subset M_k \subset \cdots \subset M_r = M$$

such that M_{k+1}/M_k is an irreducible module for $Y_\hbar(\mathfrak{g})$. Such a finite chain must exist due to the finite dimensionality of M and the fact that dim$M_{k+1}/M_k \geq 1$.

Introduce the following notation for these relevant categories of modules:

- Mod$-Y_\hbar(\mathfrak{g})$ for all modules of $Y_\hbar(\mathfrak{g})$
- mod$-Y_\hbar(\mathfrak{g})$ for finite dimensional modules of $Y_\hbar(\mathfrak{g})$

We can use analogous notation for $U_q(\mathcal{L}\mathfrak{g})$. Next, we define a third category of modules over the Yangian of $\mathfrak{g}$. Consider a set $\Pi \in \mathbb{C}$ satisfying $\Pi \pm \frac{\hbar}{2} \subset \Pi$. It follows that Π must be infinite and is the union of sets $\{r+\frac{\hbar}{2}n|n\in\mathbb{Z}\}$ for roots $r\in\mathbb{C}$. Then, let $\mathrm{mod}^{\Pi}-Y_{\hbar}(\mathfrak{g})$ denote the category of finite dimensional modules such that the irreducible quotients M_{k+1}/M_k are parametrized by polynomials $P_1(u),\ldots,P_{\mathrm{rank}\mathfrak{g}}(u)$ with the property that their roots belong to Π.

Similarly, consider a set $\Omega\in\mathbb{C}$ of nonzero complex numbers such that $q^{\pm1}\Omega\subset\Omega$. Define the category $\mathrm{mod}^{\Omega}-U_q(\mathcal{L}\mathfrak{g})$ to be the set of finite dimensional modules whose irreducible quotients are parametrized by polynomials with roots in Ω.

Proposition 3.8. ***(Proposition 3.5 [GTL16])***

1. $\mathrm{mod}^{\Pi}-Y_{\hbar}(\mathfrak{g})$ *is a subcategory of* $\mathrm{mod}-Y_{\hbar}(\mathfrak{g})$*, closed under taking direct sums, subobjects, quotient, and extensions.*

2. $\mathrm{mod}^{\Omega}-U_q(\mathcal{L}\mathfrak{g})$ *is a subcategory of* $\mathrm{mod}-U_q(\mathcal{L}\mathfrak{g})$*, closed under taking direct sums, subobjects, quotient, and extensions.*

3. *If* $\mathfrak{g}$ *is a simple Lie algebra,* $\mathrm{mod}^{\Pi}-Y_{\hbar}(\mathfrak{g})$ *and* $\mathrm{mod}^{\Omega}-U_q(\mathcal{L}\mathfrak{g})$ *are closed under tensor products.*

Proof. This follows from the properties of finite dimensional modules. □

Next, our goal is to examine the action of the generators of the Yangians and quantum loop algebras on their respective categories of finite dimensional modules. We will show that these actions are given by rational functions.

Proposition 3.9. ***(Proposition 3.6 [GTL16])***

1. *Let V be a module over $Y_{\hbar}(\mathfrak{g})$ on which $\mathfrak{h}$ acts semisimply with finite dimensional weight spaces. Then, for every weight μ of V, the generating series $h_i(u)\in$* $\mathrm{End}(V_\mu)[[u^{-1}]]$ *and $x_i^{\pm}(u)\in$* $\mathrm{Hom}(V_\mu,V_{\mu\pm\alpha_i})[[u^{-1}]]$ *defined in (2.1.7) and (2.1.8) are the expansions at ∞ of rational functions of u. Further, let*

$$t_{i,1}=h_{i,1}-\frac{\hbar}{2}h_{i,0}^2\in Y_{\hbar}(\mathfrak{g})^{\mathfrak{h}}.$$

Then,

$$x_i^{\pm}(u) \quad = \quad \hbar u^{-1}\left(1 \mp \frac{\mathrm{ad}(t_{i,1})}{2d_i u}\right)^{-1} x_{i,0}^{\pm} \tag{3.3.1}$$

and $h_i(u) = 1 + [x_i^+(u), x_{i,0}^-]$.

2. ***Though we will not be using this quantum loop algebra version of this result, it is kept here for completeness and stress the argument that there should be a strong connection between the representation theories of the two structures.*** *Similarly, let* V *be a module over* $U_q(\mathcal{L}\mathfrak{g})$ *on which the operators* K_h *act semisimply with finite dimensional weight spaces. Then, for every weight* μ *of* V *and sign* $\varepsilon \in \{\pm\}$*, the generating series* $\psi_i(z)^{\pm} \in \mathrm{End}(V_\mu)[[z^{\mp 1}]]$ *and* $\mathcal{X}_i^{\varepsilon}(z)^{\pm} \in \mathrm{Hom}(V_\mu, V_{\mu\pm\alpha_i})[[z^{\mp 1}]]$ *are the expansions of rational functions at* $z = \infty$ *and* $z = 0$ *of rational functions in* u*. Further, let*

$$H_{i,\pm 1} = \pm\psi_{i,0}^{\mp}\psi_{i,\pm 1}^{\pm}/(q_i - q_i^{-1}).$$

Then,

$$\mathcal{X}_i^{\varepsilon}(z) = \left(1 - \varepsilon\frac{\mathrm{ad}(H_{i,1})}{[2]_i z}\right)^{-1}\mathcal{X}_{i,0}^{\varepsilon} = -z\left(1 - \varepsilon z\frac{\mathrm{ad}(H_{i,-1})}{[2]_i}\right)^{-1}\mathcal{X}_{i,-1}^{\varepsilon} \tag{3.3.2}$$

and $\psi_i(z) = \psi_{i,0}^- + (q_i - q_i^{-1})[\mathcal{X}_i^+(z), \mathcal{X}_{i,0}^-]$.

Proof. First, we justify how this expression for $x_i^{\pm}(u)$ is constructed. $\mathrm{ad}(t_{i,1})$ applied to $x_{i,0}^{\pm}$ yields $t_{i,1}x_{i,0}^{\pm} - x_{i,0}^{\pm}t_{i,1}$. So, the representation map $\phi : Y_\hbar(\mathfrak{g}) \to \mathrm{End}(V)$ corresponding to the module V satisfies

$$\phi(\mathrm{ad}(t_{i,1})x_{i,0}^{\pm}) \quad = \quad \phi(t_{i,1})\phi(x_{i,0}^{\pm}) - \phi(x_{i,0}^{\pm})\phi(t_{i,1}) \tag{3.3.3}$$

since ϕ is an algebra homomorphism. In particular, if we restrict the endomorphisms $\phi(t_{i,1})$ and $\phi(x_{i,0}^{\pm})$ to a weight space $V_\mu \subset V$ or $V_{\mu\pm\alpha_i} \subset V$, we get maps

$$\phi(t_{i,1}) \quad : \quad V_\mu \to V_\mu \tag{3.3.4}$$

$$\phi(t_{i,1}) \quad : \quad V_{\mu\pm\alpha_i} \to V_{\mu\pm\alpha_i} \tag{3.3.5}$$

$$\phi(x_{i,0}^{\pm}) \quad : \quad V_\mu \to V_{\mu\pm\alpha_i} \tag{3.3.6}$$

We can choose bases of V_μ and $V_{\mu\pm\alpha_i}$. Let $d_1 = \mathrm{dim} V_\mu$ and $d_2 = \mathrm{dim} V_{\mu\pm\alpha_i}$. Then, we can represent these restricted maps as matrices; (3.3.4) as a d_1 by d_1 matrix T_1, (3.3.5) as a d_2 by d_2 matrix T_2, and (3.3.6) as a d_2 by d_1 matrix X. As a result, the right hand side of (3.3.3) can be expressed as $T_2X - XT_1$.

Second, we explain why $x_i^{\pm}(u)$ is rational. The defining relations of the Yangian produce that $[t_{i,1}, x_{i,r}^{\pm}] = \pm 2d_i x_{i,r+1}^{\pm}$. Now, in general, for a vector v and matrix A, the product $(1 - A/u)^{-1}v$ will also be a vector and its entries will be rational functions of u. This applies to our current case. This is simple linear algebra; if we let $B = 1 - A/u$, then $B^{-1} = \mathrm{adjoint}(B)/\det(B)$. The adjoint of B will be composed of rational entries. $x_{i,0}^{\pm}$ is itself a matrix with rational entries as seen as an element of $\mathrm{Hom}(V_\mu, V_{mu+\alpha_i})$. Suppose this space of homomorphisms is made up of rectangular matrices of size $d_1 \times d_2$. Let $d = d_1 d_2$. We can then identify $\mathrm{Hom}(V_\mu, V_{mu+\alpha_i})$ with $\mathbb{C}^d$; $x_{i,0}^{\pm}$ is now seen as a column vector. Next, the adjoint action as seen in $\mathrm{ad}(t_{i,1})$ is a linear map from $\mathrm{Hom}(V_\mu, V_{\mu+\alpha_i})$ to itself, so after choosing a basis of this space and identifying it with $\mathbb{C}^d$, we can view $\mathrm{ad}(t_{i,1})$ as a $d \otimes d$ matrix. Thus, using that $A = ad(t_{i,1})v = x_{i,0}^{\pm}$ from the argument above and looking at the definition of $x_i^{\pm}(u)$, we see how it is a matrix with rational entries.

Third, we justify that $h_i(u)$ is rational. This is clear. Since $x_i^+(u)$ is rational and $h_i(u) = 1 + [x_i^+(u), x_{i,0}^-]$, $h_i(u)$ must be rational.

Similar arguments apply for the quantum loop algebra case. $\square$

As an aside, note that we can rewrite $x_i^-(u)$ in the following way:

$$\begin{aligned}
x_i^+(u) =& \hbar u^{-1}\left(1-\frac{\mathrm{ad}(t_{i,1})}{2d_iu}\right)^{-1} x_{i,0}^+ \\
=& \hbar u^{-1}\left(\frac{1}{1-\frac{\mathrm{ad}(t_{i,1})}{2d_iu}}\right) x_{i,0}^+ \\
=& \hbar u^{-1}\left(1+\frac{\mathrm{ad}(t_{i,1})}{2d_iu}+\frac{(\mathrm{ad}(t_{i,1}))^2}{(2d_iu)^2}+\frac{(\mathrm{ad}(t_{i,1}))^3}{(2d_iu)^3}+\cdots\right) x_{i,0}^+ \\
=& \hbar u^{-1}\left(1+\frac{\mathrm{ad}(t_{i,1})(x_{i,0}^+)}{2d_iu}+\frac{(\mathrm{ad}(t_{i,1}))(\mathrm{ad}(t_{i,1}))(x_{i,0}^+)}{(2d_iu)^2}\right. \\
&\left. +\frac{(\mathrm{ad}(t_{i,1}))^2(\mathrm{ad}(t_{i,1}))(x_{i,0}^+)}{(2d_iu)^3}+\cdots\right) \\
=& \hbar u^{-1}\left(1+\frac{[t_{i,1},x_{i,0}^+]}{2d_iu}+\frac{(\mathrm{ad}(t_{i,1}))[t_{i,1},x_{i,0}^+]}{(2d_iu)^2}+\frac{(\mathrm{ad}(t_{i,1}))^2[t_{i,1},x_{i,0}^+]}{(2d_iu)^3}+\cdots\right) \\
=& \hbar u^{-1}\left(1+\frac{2d_ix_{i,1}^+}{2d_iu}+\frac{(\mathrm{ad}(t_{i,1}))(2d_ix_{i,1}^+)}{(2d_iu)^2}+\frac{(\mathrm{ad}(t_{i,1}))^2(2d_ix_{i,1}^+)}{(2d_iu)^3}+\cdots\right) \\
=& \hbar u^{-1}\left(1+\frac{2d_ix_{i,1}^+}{2d_iu}+\frac{2d_i[t_{i,1},x_{i,1}^+]}{(2d_iu)^2}+\frac{(\mathrm{ad}(t_{i,1}))(2d_i)[t_{i,1},x_{i,1}^+]}{(2d_iu)^3}+\cdots\right) \\
=& \hbar u^{-1}\left(1+\frac{2d_ix_{i,1}^+}{2d_iu}+\frac{(2d_i)(2d_i)x_{i,2}^+}{(2d_i)^2u^2}+\frac{(2d_i)(2d_i)[t_{i,1},x_{i,2}^+]}{(2d_i)^3u^3}+\cdots\right) \\
=& \hbar u^{-1}\left(1+\frac{2d_ix_{i,1}^+}{2d_iu}+\frac{(2d_i)^2x_{i,2}^+}{(2d_i)^2u^2}+\frac{(2d_i)^3x_{i,3}^+}{(2d_i)^3u^3}+\cdots\right) \\
=& \hbar u^{-1}\left(1+\frac{x_{i,1}^+}{u}+\frac{x_{i,2}^+}{u^2}+\frac{x_{i,3}^+}{u^3}+\cdots\right) \\
=& \hbar u^{-1}\left(1+\sum_{a\geq 1}\frac{x_{i,a}^+}{u^a}\right) \\
=& \hbar u^{-1}+\hbar\sum_{a\geq 1}\frac{x_{i,a}^+}{u^{a+1}}
\end{aligned}$$

Similarly,

$$x_i^-(u) = \hbar u^{-1}\left(1+\frac{\mathrm{ad}(t_{i,1})}{2d_iu}\right)^{-1} x_{i,0}^-$$

$$
\begin{aligned}
&=\hbar u^{-1}\left(\frac{1}{1+\frac{\mathrm{ad}(t_{i,1})}{2d_i u}}\right)x_{i,0}^{-}\\
&=\hbar u^{-1}\left(\frac{1}{1-\frac{-\mathrm{ad}(t_{i,1})}{2d_i u}}\right)x_{i,0}^{-}\\
&=\hbar u^{-1}\left(1+\frac{-\mathrm{ad}(t_{i,1})}{2d_i u}+\left(\frac{-\mathrm{ad}(t_{i,1})}{2d_i u}\right)^2+\left(\frac{-\mathrm{ad}(t_{i,1})}{2d_i u}\right)^3+\cdots\right)x_{i,0}^{-}\\
&=\hbar u^{-1}\left(1+\frac{-\mathrm{ad}(t_{i,1})(x_{i,0}^{-})}{2d_i u}+\frac{(-1)^2\mathrm{ad}(t_{i,1})^2 x_{i,0}^{-}}{(2d_i u)^2}\right.\\
&\qquad\left.+\frac{(-1)^3(\mathrm{ad}(t_{i,1}))^3 x_{i,0}^{-}}{(2d_i u)^3}+\cdots\right)\\
&=\hbar u^{-1}\left(1+\frac{2d_i x_{i,1}^{-}}{2d_i u}+\frac{\mathrm{ad}(t_{i,1})(-2d_i x_{i,1}^{-})}{(2d_i u)^2}+\frac{-(\mathrm{ad}(t_{i,1}))^2(-2d_i x_{i,1}^{-})}{(2d_i u)^3}+\cdots\right)\\
&=\hbar u^{-1}\left(1+\frac{2d_i x_{i,1}^{-}}{2d_i u}+\frac{(-2d_i)^2 x_{i,2}^{-}}{(2d_i)^2 u^2}+\frac{-(\mathrm{ad}(t_{i,1}))(-2d_i)^2 x_{i,2}^{-}}{(2d_i)^3 u^3}+\cdots\right)\\
&=\hbar u^{-1}\left(1+\frac{x_{i,1}^{-}}{u}+\frac{x_{i,2}^{-}}{u^2}+\frac{-(-2d_i)^3 x_{i,3}^{-}}{(2d_i)^3 u^3}+\cdots\right)\\
&=\hbar u^{-1}\left(1+\frac{x_{i,1}^{-}}{u}+\frac{x_{i,2}^{-}}{u^2}+\frac{x_{i,3}^{-}}{u^3}+\cdots\right)\\
&=\hbar u^{-1}\left(1+\sum_{a\geq 1}\frac{x_{i,a}^{-}}{u^a}\right)\\
&=\hbar u^{-1}+\hbar\sum_{a\geq 1}\frac{x_{i,a}^{-}}{u^{a+1}}
\end{aligned}
$$

Thus, in general,

$$
x_i^{\pm}(u)=\hbar u^{-1}+\hbar\sum_{a\geq 1}\frac{x_{i,a}^{\pm}}{u^{a+1}}.
$$

Definition 3.10. ([GTL16])A finite dimensional representation of $Y_\hbar(\mathfrak{g})$ is said to be *non-congruent* if, for any $i\in\mathbf{I}$, the poles of $x_i^{+}(u)$ (resp. $x_i^{-}(u)$) are not congruent modulo $\mathbb{Z}$.

These non-congruent representations form a subcategory of mod$-Y_\hbar(\mathfrak{g})$. Our main goal in this section is to produce a functor from the category of non-congruent finite dimensional representations of the Yangian to the category of finite dimensional representations of the quantum loop algebra. This functor Γ has been shown to be exact, faithful, and compatible with shift automorphisms.

As in the previous section, arriving at this functor is not the most intuitive. Roughly speaking, Gautam and Toledano Laredo take the *commuting fields* $h_i(u)$ of $Y_\hbar(\mathfrak{g})$, note that each defines a difference equation on any finite dimensional representation of $Y_\hbar(\mathfrak{g})$, and determine how Γ must be compatible with these equations. Let A be a rational function $A : \mathbb{C} \to \mathrm{End}(V)$ that is regular at ∞ and satisfying $A(\infty) = 1$. Then, the system of difference equations we are concerned with is

$$\phi(u+1) \quad = \quad A(u)\phi(u), \tag{3.3.7}$$

where ϕ is some function $\mathbb{C} \to \mathrm{End}(V)$. If this system satisfies a particular condition (*non-resonance*, which won't be presented here), then there exist unique solutions $\phi^{\pm}$ to (3.3.7) that are meromorphic, among other properties. In fact, we can write them according to the formulas:

$$\phi^{+}(u) \quad = \quad e^{-\gamma A_0} A(u)^{-1} \overrightarrow{\prod_{n \geq 1}} A(u+n)^{-1} e^{A_0/n} \tag{3.3.8}$$

$$\phi^{-}(u) \quad = \quad e^{-\gamma A_0} \overrightarrow{\prod_{n \geq 1}} A(u-n) e^{A_0/n}, \tag{3.3.9}$$

where γ is the Euler-Mascheroni constant. Now, for any $i \in \mathbf{I}$, take the difference equation

$$\phi_i(u+1)_\mu \quad = \quad h_i(u)_\mu \phi_i(u)_\mu. \tag{3.3.10}$$

Then, we know that fundamental solutions $\phi_i^{\pm}(u)$ exist. The coefficient matrix $h_i(u)_\mu$ is simply the usual series $h_i(u)$, but with a subscript to remind us that we are working on one particular weight space V_μ of V. Introduce two new functions $\mathbb{C} \to \mathrm{GL}(V_\mu)$:

$$g_i^-(u)_\mu = \phi_i^-(u)_\mu$$

$$g_i^+(u)_\mu = \phi_i^+(u+1)_\mu^{-1}$$

Using (3.3.8) and (3.3.9), these become

$$\begin{aligned} g_i^-(u)_\mu &= e^{-\gamma\hbar h_{i,0}} \prod_{n\geq 1}^{\rightarrow} h_i(u-n)_\mu e^{\hbar h_{i,0}/n} && (3.3.11) \\ g_i^+(u)_\mu &= \left(\prod_{n\geq 1}^{\leftarrow} e^{\hbar h_{i,0}/n} h_i(u+n)_\mu \right) e^{\gamma\hbar h_{i,0}}. && (3.3.12) \end{aligned}$$

Since $\phi_i^\pm$ is meromorphic, evaluating the infinite products $\prod_{n\geq 1}^{\rightarrow} h_i(u-n)_\mu$ and $\prod_{n\geq 1}^{\leftarrow} h_i(u+n)_\mu$ at $u = z_0$ for some number z_0 should produce something that converges, except at finitely many choices at z_0. Thus, when z_0 is not a pole of $h_i(u-n)$ for any n, the infinite product converges to a number. Thus, we can make sense of $g_i^-(u)$, as well as of $g_i^+(u)$ using similar reasoning.

Finally, the last step before defining how $\mathcal{X}_{i,k}^\pm$ acts as an operator $V_\mu \to V_{\mu\pm\alpha_i}$ involves choosing a particular union of curves in $\mathbb{C}$ satisfying certain properties. As it happens, the definition of the action of $\mathcal{X}_{i,k}^\pm$ involves contour integration.

The series $x_i^\pm(u)_\mu$ and $x_i^\pm(u)_{\mu\pm\alpha_i}$ were shown in Proposition 3.9 to be rational. Thus, $x_i^\pm(u)_\mu$ and $x_i^\pm(u)_{\mu\pm\alpha_i}$ have finitely many poles. Then, take $\mathcal{C}_{i,\mu}^\pm$ to be some disjoint union of curves in $\mathbb{C}$ such that $\mathcal{C}_{i,\mu}^\pm$ contains all the poles of $x_i^\pm(u)_\mu$ but no $\mathbb{Z}_{\neq 0}$-translate of the poles of either $x_i^\pm(u)_\mu$ or $x_i^\pm(u)_{\mu\pm\alpha_i}$. As long as V is non-congruent, such a curve will always exist for any choice of i or μ.

Proposition 3.11. *The action of $\mathcal{X}_{i,k}^{\pm}$ as an operator $V_\mu \to V_{\mu\pm\alpha_i}$ is given by*

$$(\mathcal{X}_{i,k}^{\pm})_\mu = c_i^{\pm} \oint_{\mathcal{C}_{i,\mu}^{\pm}} e^{2\pi\imath k u} g_i^{\pm}(u)_{\mu\pm\alpha_i} x_i^{\pm}(u)_\mu du \qquad (3.3.13)$$

for scalars $c_i^{\pm} \in \mathbb{C}^\times$ satisfying

$$c_i^- c_i^+ = d_i \Gamma(\hbar d_i)^2.$$

In order to understand how to evaluate this contour integral, we can see how the operator $V_\mu \to V_{\mu\pm\alpha_i}$, $(\mathcal{X}_{i,k}^{\pm})_\mu$ is a matrix with complex entries. For one, $g_i^{\pm}(u)_{\mu\pm\alpha_i}$ is a matrix whose entries are meromorphic functions; not necessarily rational functions. As well, $x_i^{\pm}(u)_\mu$ can be viewed as a matrix, with rational functions on $\mathbb{C}$ as entries. Then, the product $g_i^{\pm}(u)_{\mu\pm\alpha_i} x_i^{\pm}(u)_\mu$ will be a matrix with entries that are meromorphic functions in u. Suppose one entry of $g_i^{\pm}(u)_{\mu\pm\alpha_i} x_i^{\pm}(u)_\mu$ is the function $f(u)$. Then, the corresponding entry of $(\mathcal{X}_{i,k}^{\pm})_\mu$ will be the given by the value of

$$c_i^{\pm} \oint_{\mathcal{C}_{i,\mu}^{\pm}} e^{2\pi\imath k u} f(u) du$$

which we can confirm is a complex number. Because of the definition of the $x_i^{\pm}(u)$ and the fact that $h_i(u) = 1 + [x_i^+(u).x_{i,0}^-]$, we know that $h_i(u)$ and $x_i^{\pm}(u)_\mu$ have the same poles. $g_i^{\pm}$ has poles contained in the set of all $\mathbb{Z}-$shifts of the poles of $x_i^{\pm}(u)$. As a result, the poles of $f(u)$ are the poles of $x_i^+(u)$ and all of its $\mathbb{Z}$-shifts. So, we can conclude that the contour integral

$$\frac{1}{2\pi i} \oint_{\mathcal{C}_{i,\mu}^{\pm}} e^{2\pi\imath k u} f(u) du$$

will be a sum of integrals

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} e^{2\pi\imath k u} f(u) du$$

where the interior of $\mathcal{C}$ contains exactly one pole of $f(u)$, which we know to be true due to the way $\mathcal{C}^{\pm}_{i,\mu}$ was chosen.

Naturally, $e^{2\pi\imath k u}$ is holomorphic; we can use its Taylor series expansion. For one integral $\frac{1}{2\pi i}\oint_{\mathcal{C}} e^{2\pi\imath k u} f(u)du$, if the pole contained in $\mathcal{C}$ is at $u = z_0$, then expand $f(u)$ as

$$f(u) = \frac{a_{-l}}{(u-z_0)^l} + \cdots + \frac{a_{-1}}{u-z_0} + a_0 + \cdots$$

and expand $e^{2\pi\imath k u}$ as

$$e^{2\pi\imath k u} = e^{2\pi\imath k z_0} + b_1(u-z_0) + b_2(u-z)^2 + \cdots .$$

Finally, the value of the integral is the coefficient of $\frac{1}{u-z_0}$ in the product of the two series.

Theorem 3.12. *The operator defined in Proposition 3.11 define an exact, faithful functor* Γ *from the category of non-congruent finite dimensional modules over the Yangian* $Y_\hbar(\mathfrak{g})$ *to the category of finite dimensional modules over the quantum loop algebra* $U_q(\mathcal{L}\mathfrak{g})$*.*

4

IRREDUCIBILITY OF VERMA MODULES FOR $Y(\mathfrak{gl}_{1|1})$ AND $Y(\mathfrak{gl}_{M|N})$

In their article on Verma modules for Yangians [BFM06], Billig, Futorny, and Molev arrive at several conclusions on Verma modules for the Yangians $Y(\mathfrak{gl}_2)$ and $Y(\mathfrak{gl}_N)$. Specifically, they designate the criteria for reducibility and what it says about the highest weight of the module. Here, we show that a subset of this same collection of propositions and theorems hold when applied to the super Yangians $Y(\mathfrak{gl}_{1|1})$ and $Y(\mathfrak{gl}_{M|N})$. In fact, the nature of super Yangians leads to an upshot not present in the non-super case: Theorem 4.9 asserts that a Verma module over $Y(\mathfrak{gl}_{1|1})$ is reducible if and only if the unique irreducible quotient is finite dimensional. In contrast, for $Y(\mathfrak{gl}_N)$, the most we can conclude is the sufficient and necessary condition for all weight subspaces of the irreducible quotient to be finite dimensional. Further, we see how we can parametrize the irreducibility of a Verma module over $Y(\mathfrak{gl}_{M|N})$ by the nature of the highest weight $\mu(u)$ (Theorem 4.14).

An earlier attempt to fit this set of results to the quantum loop superalgebras $U_q(\mathcal{L}\mathfrak{gl}_{1|1})$ and $U_q(\mathcal{L}\mathfrak{gl}_{M|N})$ was unsuccessful. This was due to the presence of barred $\overline{T}_{ij}^{(r)}$ generators. In the final section of this chapter, we choose a subalgebra, called the quantum current algebra, that satisfies the necessary properties for these results to hold, without any of the complications. This leads to Theorems 4.17 and 4.18 on irreducible Verma modules over $U_q(\mathfrak{gl}_{1|1}[s])$ and the general $U_q(\mathfrak{gl}_{M|N}[s])$.

4.1 REPRESENTATIONS OF $Y(\mathfrak{gl}_{1|1})$

First, we apply the notion of using an automorphism to twist the action of the Yangian by a power series from Lemma 2.25.

Lemma 4.1. *$M(\lambda_1(u), \lambda_2(u))$ is isomorphic to the Verma module $M(1, \nu(u))$ for $\nu(u) = \frac{\lambda_2(u)}{\lambda_1(u)}$.*

Proof. Use the automorphism ϕ_f corresponding to $f(u) = (\lambda_1(u))^{-1}$ to twist the action of $Y(\mathfrak{gl}_{1|1})$ on $M(\lambda_1(u), \lambda_2(u))$. We need to verify that two crucial aspects of our new Verma module are satisfied, namely that:

$$\begin{aligned}
\widetilde{t_{11}(u)}1_\lambda &= 1_\lambda \\
\widetilde{t_{22}(u)}1_\lambda &= \frac{\lambda_2(u)}{\lambda_1(u)}1_\lambda
\end{aligned}$$

Simple calculations show that both hold.

$$\begin{aligned}
\widetilde{t_{11}(u)}1_\lambda &= (\lambda_1(u))^{-1}t_{11}(u)1_\lambda = (\lambda_1(u))^{-1}\lambda_1(u)1_\lambda \\
&= 1_\lambda \\
\widetilde{t_{22}(u)}1_\lambda &= (\lambda_1(u))^{-1}t_{22}(u)1_\lambda = (\lambda_1(u))^{-1}\lambda_2(u)1_\lambda \\
&= \frac{\lambda_2(u)}{\lambda_1(u)}1_\lambda
\end{aligned}$$

□

Proposition 4.2. *Let $M(\lambda_1(u), \lambda_2(u))$ be a Verma module over $Y(\mathfrak{gl}_{1|1})$. Then, its restriction to $Y(\mathfrak{sl}_{1|1})$ is isomorphic to the Verma module $M(\nu(u))$ for $\nu(u) = \lambda_1(u)/\lambda_2(u)$. Further, $M(\lambda_1(u), \lambda_2(u))$ and $M(\nu(u))$ are irreducible if and only if the other is.*

Proof. We have that:

$$\begin{aligned} e(u)1_\nu = 0 =& e(u)1_\lambda, \\ (d_1(u))^{-1}d_2(u)1_\nu =& \nu(u)1_\nu, \\ (d_1(u))^{-1}d_2(u)1_\lambda =& \frac{\lambda_2(u)}{\lambda_1(u)}1_\lambda. \end{aligned}$$

As a result, we have a $Y(\mathfrak{sl}_{1|1})$-homomorphism $\varphi : M(\nu(u)) \to M(\lambda_1(u), \lambda_2(u))$ such that ϕ sends 1_ν to 1_λ. Next, we note that φ is surjective if $M(\lambda_1(u), \lambda_2(u))$ is generated by 1_λ as a $Y(\mathfrak{sl}_{1|1})$-module. This is satisfied by Theorem 2.27 since all elements of $Z(Y(\mathfrak{gl}_{1|1}))$ act as scalars on $M(\lambda_1(u), \lambda_2(u))$ (Lemma 2.28). Next, we extend $M(\nu(u))$ from a Verma module over $Y(\mathfrak{sl}_{1|1})$ to a module over $Y(\mathfrak{gl}_{1|1})$; define the action of $b_{1|1}(u)$ on $M(\nu(u))$ to be scalar multiplication by $\lambda_1(u)\lambda_2(u-1)$. Finally, Gauss decomposition gives:

$$\begin{aligned} t_{12}1_\nu &= 0 \\ t_{11}(u)1_\nu &= \lambda_1(u)1_\nu \\ t_{22}(u)1_\nu &= \lambda_2(u)1_\nu. \end{aligned}$$

As a result, we have a $Y(\mathfrak{gl}_{1|1})$-homomorphism $\psi : M(\lambda_1(u), \lambda_2(u)) \to M(\nu(u))$ such that $\psi(1_\lambda) = 1_\nu$. Injectivity of φ follows from $\psi \circ \varphi = \mathrm{id}_{M(\nu(u))}$. □

Before extending results from $Y(\mathfrak{gl}_n)$ to $Y(\mathfrak{gl}_{M|N})$, a helpful stepping stone is to take existing results for $Y(\mathfrak{gl}_2)$ and finding analogues for $Y(\mathfrak{gl}_{1|1})$.

Proposition 4.3. *Let λ_1, λ_2 be two formal power series in u^{-1} given by*

$$\lambda_i = \lambda_i(u) = (-1)^{|i|+1} + \lambda_i^{(1)} u^{-1} + \lambda_i^{(2)} u^{-2} + \cdots$$

If the ratio $\lambda_1(u)/\lambda_2(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u, say $\frac{P(u)}{Q(u)}$ for some polynomials P, Q, then the Verma module $M(\lambda_1(u), \lambda_2(u))$ over $Y(\mathfrak{gl}_{1|1})$ is reducible.

Before proving this proposition, we present several smaller, useful lemmas. This will be similar to the proof of Proposition 3.3 in the paper by Billig, Futorny, and Molev [BFM06] on Verma modules for Yangians.

Lemma 4.4. *$P(u), Q(u)$ are of the same degree. Their leading coefficients can be chosen to be $(-1)^{|1|+1} = -1$ and $(-1)^{|2|+1} = 1$ respectively.*

Proof. We let $P(u) = p_0 + p_1 u + p_2 u^2 + \cdots p_k u^k$, $Q(u) = q_0 + q_1 u + q_2 u^2 + \cdots q_l u^l$.

$$\frac{1 + \lambda_1^{(1)} u^{-1} + \lambda_1^{(2)} u^{-2} + \cdots}{1 + \lambda_2^{(1)} u^{-1} + \lambda_2^{(2)} u^{-2} + \cdots} = \frac{p_0 + p_1 u + p_2 u^2 + \cdots p_k u^k}{q_0 + q_1 u + q_2 u^2 + \cdots q_l u^l}$$

$$\iff$$

$$\begin{aligned}&(1 + \lambda_1^{(1)} u^{-1} + \lambda_1^{(2)} u^{-2} + \cdots)(q_0 + q_1 u + q_2 u^2 + \cdots q_l u^l) \\ &= (1 + \lambda_2^{(1)} u^{-1} + \lambda_2^{(2)} u^{-2} + \cdots)(p_0 + p_1 u + p_2 u^2 + \cdots p_k u^k)\end{aligned}$$

Multiply through both sides of the equation and compare terms for nonnegative powers of u. We can ignore the negative powers of u because there is no smallest power of u and these coefficients do not provide any more information. In fact, all we need to do is to look at the coefficients for the highest powers of u that appear. These coefficients are $q_l u^l$ and $p_k u^k$, thus we can conclude that $k = l$.

Set $d = k$. Further, without loss of generality we can let $p_d = -1$ and $q_d = 1$ as one can simply divide the polynomials accordingly to produce ones with these leading coefficients. $\square$

Remarks. By the defining relations of the super Yangian, $\left[t_{21}^{(r)}, t_{21}^{(s)}\right] = 0$. This is shown using (2.3.4); we have

$$\begin{aligned}
\eta(2,1;2,1) \equiv& 1 + 1 + (1+1)(2+1) + (2+1)(2+1) \text{mod} 2 \\
=& 2 + 6 + 9 \equiv 0 \\
[t_{21}(u), t_{21}(v)] =& \frac{(1)}{u-v} \left(t_{21}(u) t_{21}(v) - t_{21}(v) t_{21}(u) \right) \\
(u - v - 1) \left[t_{21}(u), t_{21}(v) \right] =& 0 \\
[t_{21}(u), t_{21}(v)] =& 0.
\end{aligned}$$

Now, define the vector subspace K of $M(\lambda_1(u), \lambda_2(u))$ to be the span of all vectors of the form $t_{21}^{(r_1)} t_{21}^{(r_2)} \cdots t_{21}^{(r_k)} 1_\lambda$ where at least one r_i exceeds $d = \text{deg} P(u) = \text{deg} Q(u)$, with $r_1 < \cdots < r_k$ as usual. Equivalently, we could define K to be the span of vectors $t_{21}^{(r_1)} t_{21}^{(r_2)} \cdots t_{21}^{(r_k)} 1_\lambda$ with $r_k > d$. In our proof of Proposition 4.3, we show that this is a nontrivial submodule of $M(\lambda_1(u), \lambda_2(u))$ implying the reducibility of $M(\lambda_1(u), \lambda_2(u))$. Next, we show that spanning elements of K remain in K under the action by $t_{11}^{(r)}$ for values of r greater than d.

Lemma 4.5. *If $r > d$, then $t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda \in K$.*

Proof. First, we show that $t_{11}^{(r)}t_{21}^{(r_1)} \in K$. By definition of K, this means that $r_1 \geq d$. By the defining relations of the Yangian,

$$\begin{aligned}
\left[t_{11}^{(r)}, t_{21}^{(r_1)}\right] 1_\lambda &= \sum_{a=0}^{\min(r,r_1)-1} \left(t_{21}^{(a)} t_{11}^{(r+r_1-1-a)} - t_{21}^{(r+r_1-1-a)} t_{11}^{(a)}\right) 1_\lambda \\
&= \sum_{a=0}^{\min(r,r_1)-1} \left(t_{21}^{(a)} t_{11}^{(r+r_1-1-a)} 1_\lambda - t_{21}^{(r+r_1-1-a)} t_{11}^{(a)} 1_\lambda\right) \\
t_{11}^{(r)} t_{21}^{(r_1)} 1_\lambda &= \sum_{a=0}^{\min(r,r_1)-1} \left(t_{21}^{(a)} t_{11}^{(r+r_1-1-a)} 1_\lambda - t_{21}^{(r+r_1-1-a)} t_{11}^{(a)} 1_\lambda\right) + t_{21}^{(r_1)} t_{11}^{(r)} 1_\lambda
\end{aligned}$$

However, since $r > d$, $t_{11}^{(r)} 1_\lambda = 0$ and $t_{11}^{(r+r_1-1-a)} 1_\lambda = 0$ for all a. Thus,

$$t_{11}^{(r)} t_{21}^{(r_1)} 1_\lambda = \sum_{a=0}^{\min(r,r_1)-1} -t_{21}^{(r+r_1-1-a)} t_{11}^{(a)} 1_\lambda = \sum_{a=0}^{\min(r,r_1)-1} -\lambda_1^{(a)} t_{21}^{(r+r_1-1-a)} 1_\lambda \in K.$$

Now, assume $t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda \in K$. We show that $t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} t_{21}^{(r_{k+1})} 1_\lambda \in K$.

$$\begin{aligned}
t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} t_{21}^{(r_{k+1})} 1_\lambda =& [t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)}, t_{21}^{(r_{k+1})}] 1_\lambda + t_{21}^{(r_{k+1})} t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda \\
=& t_{11}^{(r)} [t_{21}^{(r_1)} \cdots t_{21}^{(r_k)}, t_{21}^{(r_{k+1})}] 1_\lambda + [t_{11}^{(r)}, t_{21}^{(r_{k+1})}] t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda \\
& + t_{21}^{(r_{k+1})} t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda \\
=& t_{11}^{(r)} \cdot 0 \cdot 1_\lambda + [t_{11}^{(r)}, t_{21}^{(r_{k+1})}] t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda \\
& + t_{21}^{(r_{k+1})} t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda
\end{aligned}$$

By the induction assumption, $t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda \in K$. Thus, the term $t_{21}^{(r_{k+1})} t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda \in K$ as K is stable under the action of $t_{21}^{(s)}$ for all s. We have also used that the elements $t_{21}^{(s)}$ commute with each other. We are left to show that $[t_{11}^{(r)}, t_{21}^{(r_{k+1})}] t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda \in K$.

$$[t_{11}^{(r)}, t_{21}^{(r_{k+1})}]t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda =$$
$$\sum_{a=0}^{\min(r,r_{k+1})-1} \left(t_{21}^{(a)} t_{11}^{(r+r_{k+1}-1-a)} - t_{21}^{(r+r_{k+1}-1-a)} t_{11}^{(a)} \right) t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda$$
$$= \sum_{a=0}^{\min(r,r_{k+1})-1} t_{21}^{(a)} t_{11}^{(r+r_{k+1}-1-a)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda$$
$$- \sum_{a=0}^{\min(r,r_{k+1})-1} t_{21}^{(r+r_{k+1}-1-a)} t_{11}^{(a)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda$$

Both terms lie in K by the induction assumption and by the stability of K under the action of $t_{21}^{(s)}$. □

Analogous results hold for $t_{22}^{(r)}$ and $t_{12}^{(r)}$ in place of $t_{11}^{(r)}$.

Proof. Proof of Proposition 4.3

Recall that we can use an automorphism ϕ_f of $Y(\mathfrak{gl}_{1|1})$ to twist its action on $M(\lambda_1(u), \lambda_2(u))$. This produces an action isomorphic to that of $Y(\mathfrak{gl}_{1|1})$ on $M(f(u)\lambda_1(u), f(u)\lambda_2(u))$. This Verma module is reducible or irreducible exactly when $M(\lambda_1(u), \lambda_2(u))$ is. Choose $f(u) = \lambda_2(u)^{-1}u^{-d}Q(u)$. Then, our weight becomes

$$\begin{aligned} f(u)\lambda_1(u) &= \frac{\lambda_1(u)}{\lambda_2(u)} u^{-d} Q(u) \\ &= P(u)u^{-d} \\ &= p_0 u^{-d} + p_1 u^{-d+1} + \cdots + p_{d-1}u^{-1} - 1 \\ f(u)\lambda_2(u) &= u^{-d}Q(u) \\ &= q_0 u^{-d} + q_1 u^{-d+1} + \cdots q_{d-1}u^{-1} + 1 \end{aligned}$$

Thus, we can use these newly transformed values to assume without loss of generality that $\lambda_1(u)$ and $\lambda_2(u)$ are polynomials in u^{-1} of degree less than or equal to d (the $< d$ occurs if p_0 or q_0 is 0).

As briefly alluded to above, define the vector subspace K of $M(\lambda_1(u), \lambda_2(u))$ to be the span of all vectors of the form $t_{21}^{(r_1)} t_{21}^{(r_2)} \cdots t_{21}^{(r_k)} 1_\lambda$ where at least one r_i exceeds d, with $r_1 < \cdots < r_k$ as usual. We claim that K is a submodule of $M(\lambda_1(u), \lambda_2(u))$. To do this, we must show that K is stable under the actions of $t_{21}^{(r)}, t_{11}^{(r)}, t_{12}^{(r)}, t_{22}^{(r)}$. By the defining relations of the Yangian and the remark above, all generators of type $t_{21}^{(s)}$ pairwise commute, thus K remains stable.

Next, we need to show that $t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda \in K$. This can be done by induction. We know this to be true for $r > d$ by Lemma 4.5. Assume instead that $r \leq d$. The defining commutator relation yields

$$\left[t_{11}^{(r)}, t_{21}^{(r_1)}\right] = \sum_{a=0}^{\min(r,r_1)-1} \left(t_{21}^{(a)} t_{11}^{(r+r_1-1-a)} - t_{21}^{(r+r_1-1-a)} t_{11}^{(a)}\right).$$

If $k = 1$, $r_1 \geq d+1$. As a result, $r + r_1 - a - 1 \geq d + 1$ since $a + 1$ will never exceed the smaller of r and r_1, guaranteeing that the superscript remains greater than $d + 1$. Thus,

$$\begin{aligned}
\left[t_{11}^{(r)}, t_{21}^{(r_1)}\right] 1_\lambda &= \sum_{a=0}^{\min(r,r_1)-1} \left(t_{21}^{(a)} t_{11}^{(r+r_1-1-a)} - t_{21}^{(r+r_1-1-a)} t_{11}^{(a)}\right) 1_\lambda \\
&= \sum_{a=0}^{\min(r,r_1)-1} \left(t_{21}^{(a)} t_{11}^{(r+r_1-1-a)} 1_\lambda - t_{21}^{(r+r_1-1-a)} t_{11}^{(a)} 1_\lambda\right) \\
t_{11}^{(r)} t_{21}^{(r_1)} 1_\lambda &= \sum_{a=0}^{\min(r,r_1)-1} \left(t_{21}^{(a)} t_{11}^{(r+r_1-1-a)} 1_\lambda - t_{21}^{(r+r_1-1-a)} t_{11}^{(a)} 1_\lambda\right) + t_{21}^{(r_1)} t_{11}^{(r)} 1_\lambda
\end{aligned}$$

By examining each of these terms, we see why $t_{11}^{(r)} t_{21}^{(r_1)} 1_\lambda$ lies in K. First, for all a, $r + r_1 - 1 - a > d$, so $t_{11}^{(r+r_1-1-a)} 1_\lambda = 0$ and $t_{21}^{(a)} t_{11}^{(r+r_1-1-a)} 1_\lambda = 0 \in K$. Next, $t_{11}^{(a)} 1_\lambda =$

$\lambda_1^{(a)} 1_\lambda$ and $r + r_1 - 1 - a > d$, so $t_{21}^{(r+r_1-1-a)} t_{11}^{(a)} 1_\lambda = \lambda_1^{(a)} t_{21}^{(r+r_1-1-a)} 1_\lambda \in K$. Finally, $t_{21}^{(r_1)} t_{11}^{(r)} 1_\lambda \in K$ because $r_1 > d$ and $t_{11}^{(r)} 1_\lambda = \lambda_1^{(r)} 1_\lambda$, thus $t_{21}^{(r_1)} t_{11}^{(r)} 1_\lambda = \lambda_1^{(r)} t_{21}^{(r_1)} 1_\lambda \in K$.

Assume that $t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda \in K$ for all $k \leq p$. Then, we show that $t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda \in K$. First, note that

$$\begin{aligned}
&\left[t_{11}^{(r)}, t_{21}^{(r_1)} t_{21}^{(r_2)} \cdots \ t_{21}^{(r_p)} t_{21}^{(r_{p+1})}\right] 1_\lambda \\
&\qquad = t_{11}^{(r)} t_{21}^{(r_1)} t_{21}^{(r_2)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda - t_{21}^{(r_1)} t_{21}^{(r_2)} \cdot t_{21}^{(r_p)} t_{21}^{(r_{p+1})} t_{11}^{(r)} 1_\lambda
\end{aligned}$$

where $t_{21}^{(r_1)} t_{21}^{(r_2)} \cdot t_{21}^{(r_p)} t_{21}^{(r_{p+1})} t_{11}^{(r)} 1_\lambda = \lambda_1^{(r)} t_{21}^{(r_1)} t_{21}^{(r_2)} \cdot t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda \in K$. On the other hand,

$$\begin{aligned}
&t_{11}^{(r)} t_{21}^{(r_1)} t_{21}^{(r_2)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda \\
&\quad = \left[t_{11}^{(r)}, t_{21}^{(r_1)}\right] t_{21}^{(r_2)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda + t_{21}^{(r_1)} t_{11}^{(r)} t_{21}^{(r_2)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda \\
&\quad = \left(\sum_{a=0}^{\min(r,r_1)-1} t_{21}^{(a)} t_{11}^{(r+r_1-1-a)} - t_{21}^{(r+r_1-1-a)} t_{11}^{(a)} \right) t_{21}^{(r_2)} \cdots t_{21}^{(r_{p+1})} 1_\lambda \\
&\qquad + t_{21}^{(r_1)} t_{11}^{(r)} t_{21}^{(r_2)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda \\
&\quad = \sum_{a=0}^{\min(r,r_1)-1} \left(t_{21}^{(a)} t_{11}^{(r+r_1-1-a)} t_{21}^{(r_2)} \cdots t_{21}^{(r_{p+1})} 1_\lambda - t_{21}^{(r+r_1-1-a)} t_{11}^{(a)} t_{21}^{(r_2)} \cdots t_{21}^{(r_{p+1})} 1_\lambda \right) \\
&\qquad + t_{21}^{(r_1)} t_{11}^{(r)} t_{21}^{(r_2)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda
\end{aligned}$$

By the induction assumption, the terms above all lie in K. When considering the action of $t_{11}^{(s)}$ on K above, we used the fact that $t_{11}^{(s)} 1_\lambda = \lambda_1^{(s)} 1_\lambda = 0$ for $s > d$, since λ_1 is a polynomial of degree d so $\lambda_1^{(s)} = 0$ for all $s > p$.

An analogous result applies to the action of $t_{22}^{(s)}$; $t_{22}^{(s)} 1_\lambda = \lambda_2^{(s)} 1_\lambda = 0$ because $\lambda_2^{(s)} = 0$ for $s > d$. This is shown similarly. The defining commutator relation yields

$$\left[t_{21}^{(r_1)}, t_{22}^{(r)}\right] = - \sum_{a=0}^{\min(r,r_1)-1} \left(t_{21}^{(a)} t_{22}^{(r+r_1-1-a)} - t_{21}^{(r+r_1-1-a)} t_{22}^{(a)} \right) \tag{4.2.1}$$

since $\eta(2,2;2,1) \equiv 1+1+3\times 3+4\times 3 \equiv 2 \equiv 1 \bmod 2$. As a result,

$$t_{22}^{(r)}t_{21}^{(r_1)}1_\lambda = \lambda_2^{(r)}t_{21}^{(r_1)}1_\lambda + \sum_{a=0}^{\min(r,r_1)-1}\left(t_{21}^{(a)}t_{22}^{(r+r_1-1-a)} - t_{21}^{(r+r_1-1-a)}t_{22}^{(a)}\right)1_\lambda. \quad (4.2.2)$$

If $k=1$, then $r_1 \geq d+1$, so $r+r_1-1-a \geq d+1$; either $r < r_1$, in which case $r+r_1-1-a \geq r_1 \geq d+1$, or $r > r_1$, leaving $r+r_1-1-a \geq r \geq r_1 \geq d+1$. If $r = r_1, 2r-1-a \geq d+1$ trivially. Thus, $t_{21}^{(r+r_1-1-a)}1_\lambda \in K$ and $t_{22}^{(r+r_1-1-a)}1_\lambda = 0$. Repeated applications of (4.2.1) decrease the sum of the superscripts of each term in the sum by one: when looking at the maximum sum of superscripts, we move from $r+r_1$ (0 iterations) to $r+r_1-1$, to $r+r_1-2$, and so on, until we reach a sum of 1. This is produced by the step in the calculation

$$\begin{aligned} t_{22}^{(1)}t_{21}^{(1)}1_\lambda &= \lambda_2^{(1)}t_{21}^{(r_1)}1_\lambda - t_{21}^{(0)}t_{22}^{(1)}1_\lambda + t_{21}^{(1)}t_{22}^{(0)}1_\lambda \\ &= \lambda_2^{(1)}t_{21}^{(1)}1_\lambda + t_{21}^{(1)}1_\lambda. \end{aligned}$$

In any case, this shows that said iterations of (4.2.1) will eventually lead to an end where there are no more appearances of any $t_{22}^{(s)}$. Further, each term will lie in K because these decreasing sum lead to an accumulating increase in the $r+r_1-1-a$-type superscript, which either works in conjunction with $t_{22}^{(s)}1_\lambda = 0, s \geq d+1$ or the definition of elements in K to ensure that we get our desired elements. Then, let $k > 2$. Assume that $t_{22}^{(r)}t_{21}^{(r_1)}\cdots t_{21}^{(r_l)} \in K$ for all $l \leq p$. Then, our goal is to show that $t_{22}^{(r)}t_{21}^{(r_1)}\cdots t_{21}^{(r_p)}t_{21}^{(r_{p+1})} \in K$.

$$\begin{aligned} &\left[t_{22}^{(r)}, t_{21}^{(r_1)}t_{21}^{(r_2)}\ \cdots t_{21}^{(r_p)}t_{21}^{(r_{p+1})}\right]1_\lambda \\ &\qquad = \left[t_{22}^{(r)}, t_{21}^{(r_1)}\right]t_{21}^{(r_2)}\cdots t_{21}^{(r_p)}t_{21}^{(r_{p+1})}1_\lambda + t_{21}^{(r_1)}\left[t_{22}^{(r)}, t_{21}^{(r_2)}\cdots t_{21}^{(r_p)}t_{21}^{(r_{p+1})}\right]1_\lambda \\ &\qquad = -\left(\sum_{a=0}^{\min(r,r_1)-1} t_{21}^{(a)}t_{22}^{(r+r_1-1-a)} - t_{21}^{(r+r_1-1-a)}t_{22}^{(a)}\right)t_{21}^{(r_2)}\cdots t_{21}^{(r_p)}t_{21}^{(r_{p+1})}1_\lambda \end{aligned}$$

$$+\underbrace{t_{21}^{(r_1)}t_{22}^{(r)}t_{21}^{(r_2)}\cdots t_{21}^{(r_p)}t_{21}^{(r_{p+1})}1_\lambda}_{\in K}-\underbrace{\lambda_2^{(r)}t_{21}^{(r)}t_{21}^{(r_1)}\cdots t_{21}^{(r_{p+1})}1_\lambda}_{\in K}$$

The latter two terms are certainly in K by the induction hypothesis. As before, the terms in the sum have the property that the sum of superscripts decreases, meaning that iteration will move all $t_{22}^{(s)}$ generators to the right side of the sum. Then, we either produce a constant $\lambda_2^{(a)}$ in front of a term in K, or the fact that $r+r_1-1-a>d$ makes the terms not necessarily in K disappear. Thus, by expanding the bracket, we can conclude that $t_{22}^{(r)}t_{21}^{(r_1)}\cdots t_{21}^{(r_p)}t_{21}^{(r_{p+1})}\in K$.

Finally, we show that $t_{12}^{(r)}t_{21}^{(r_1)}\cdots t_{21}^{(r_p)}t_{21}^{(r_{p+1})}\in K$. First, the defining relation of the super Yangian yields

$$\left[t_{12}^{(r)},t_{21}^{(r_1)}\right]=\sum_{a=0}^{\min(r,r_1)-1}\left(t_{22}^{(a)}t_{11}^{(r+r_1-1-a)}-t_{22}^{(r+r_1-1-a)}t_{11}^{(a)}\right)$$

We use that $t_{11}^{(l)}1_\lambda=0=t_{22}^{(l)}1_\lambda$ for $l>p$ and $r+r_1-1-a\geq p+1$ to obtain

$$\begin{aligned}\left[t_{12}^{(r)},t_{21}^{(r_1)}\right]1_\lambda &= \sum_{a=0}^{\min(r,r_1)-1}\left(t_{22}^{(a)}t_{11}^{(r+r_1-1-a)}1_\lambda-t_{22}^{(r+r_1-1-a)}t_{11}^{(a)}1_\lambda\right)\\ &= \sum_{a=0}^{\min(r,r_1)-1}t_{22}^{(r+r_1-1-a)}t_{11}^{(a)}1_\lambda\\ &= \sum_{a=0}^{\min(r,r_1)-1}\lambda_1^{(a)}t_{22}^{(r+r_1-1-a)}1_\lambda\\ &= 0\end{aligned}$$

Thus, $t_{12}^{(r)}t_{21}^{(r_1)}1_\lambda=t_{21}^{(r_1)}t_{12}^{(r)}1_\lambda=0\in K$. Next, for the inductive step, we take the assumption that $t_{12}^{(r)}t_{21}^{(r_1)}\cdots t_{21}^{(r_l)}1_\lambda\in K$ for all $k\leq p$ and use it to show that $t_{12}^{(r)}t_{21}^{(r_1)}\cdots t_{21}^{(r_p)}t_{21}^{(r_{p+1})}1_\lambda\in K$.

$$\left[t_{12}^{(r)}, t_{21}^{(r_1)} t_{21}^{(r_2)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})}\right] 1_\lambda$$
$$= \left[t_{12}^{(r)}, t_{21}^{(r_1)}\right] t_{21}^{(r_2)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda + t_{21}^{(r_1)} \left[t_{12}^{(r)}, t_{21}^{(r_2)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})}\right] 1_\lambda$$
$$= \sum_{a=0}^{\min(r,r_1)-1} \underbrace{\left(t_{22}^{(a)} t_{11}^{(r+r_1-1-a)} t_{21}^{(r_2)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda - t_{22}^{(r+r_1-1-a)} t_{11}^{(a)} t_{21}^{(r_2)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda\right)}_{\in K \text{ by } K \text{ having stability under the action of } t_{11}^{(s)} \text{ and } t_{22}^{(s)}}$$
$$+ t_{21}^{(r_1)} t_{12}^{(r)} t_{21}^{(r_2)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda$$

We know the last term to belong to K because $t_{12}^{(r)} t_{21}^{(r_2)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda$ lies in K and K is stable under multiplication by $t_{21}^{(r_1)}$. Thus, $t_{12}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_p)} t_{21}^{(r_{p+1})} 1_\lambda \in K$. K is therefore a stable submodule of $M(\lambda_1(u), \lambda_2(u))$, which is now shown to be reducible. □

4.2.1 *Issues in Finding an Analogous Result for the Quantum Loop Algebra*

As we have just shown in the proof of Proposition 4.3, an analogous version of Proposition 3.3 of [BFM06] holds when adapting the result on $Y(\mathfrak{gl}_2)$ to $Y(\mathfrak{gl}_{1|1})$. However, an attempt at producing a parallel proposition for $U_q(\mathcal{L}\mathfrak{gl}_{1|1})$ was unsuccessful. In particular, defining a submodule K that is stable under the action of the generators of $U_q(\mathcal{L}\mathfrak{gl}_{1|1})$ proved to be difficult, mainly due to the incompatibilities between the $T_{ij}^{(r)}$ and the $\overline{T}_{ij}^{(r)}$ generators. For context, note that expanding the defining RTT relations of the quantum affine superalgebra produces the following triplet of equalities:

$$\begin{aligned}
&(-1)^{|i||l|+|j||l|}(q^{-d_j\delta_{jl}} T_{ij}^{(r+1)} T_{kl}^{(s)} - q^{d_j\delta_{jl}} T_{ij}^{(r)} T_{kl}^{(s+1)}) \\
&-(-1)^{|i||k|+|j||k|}(q^{-d_i\delta_{ik}} T_{kl}^{(s)} T_{ij}^{(r+1)} - q^{d_i\delta_{ik}} T_{kl}^{(s+1)} T_{ij}^{(r)})
\end{aligned} \tag{4.2.3}$$

$$
\begin{aligned}
= \quad & (-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}T_{il}^{(r+1)}T_{kj}^{(s)} - \delta_{j>l}T_{il}^{(r)}T_{kj}^{(s+1)}) \\
& -(\delta_{i>k}T_{il}^{(s)}T_{kj}^{(r+1)} + \delta_{i<k}T_{il}^{(s+1)}T_{kj}^{(r)})\Big]
\end{aligned}
$$

$$
\begin{aligned}
& (-1)^{|i||l|+|j||l|}(q^{-d_j\delta_{jl}}\overline{T}_{ij}^{(r-1)}\overline{T}_{kl}^{(s)} - q^{d_j\delta_{jl}}\overline{T}_{ij}^{(r)}\overline{T}_{kl}^{(s-1)}) \\
& -(-1)^{|i||k|+|j||k|}(q^{-d_i\delta_{ik}}\overline{T}_{kl}^{(s)}\overline{T}_{ij}^{(r-1)} - q^{d_i\delta_{ik}}\overline{T}_{kl}^{(s-1)}\overline{T}_{ij}^{(r)}) \qquad (4.2.4) \\
= \quad & (-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}\overline{T}_{il}^{(r-1)}\overline{T}_{kj}^{(s)} - \delta_{j>l}\overline{T}_{il}^{(r)}\overline{T}_{kj}^{(s-1)}) \\
& -(\delta_{i>k}\overline{T}_{il}^{(s)}\overline{T}_{kj}^{(r-1)} + \delta_{i<k}\overline{T}_{il}^{(s-1)}\overline{T}_{kj}^{(r)})\Big]
\end{aligned}
$$

$$
\begin{aligned}
& (-1)^{|i||l|+|j||l|}(q^{-d_j\delta_{jl}}T_{ij}^{(r+1)}\overline{T}_{kl}^{(s)} - q^{d_j\delta_{jl}}T_{ij}^{(r)}\overline{T}_{kl}^{(s-1)}) \\
& -(-1)^{|i||k|+|j||k|}(q^{-d_i\delta_{ik}}\overline{T}_{kl}^{(s)}T_{ij}^{(r+1)} - q^{d_i\delta_{ik}}\overline{T}_{kl}^{(s-1)}T_{ij}^{(r)}) \qquad (4.2.5) \\
= \quad & (-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}T_{il}^{(r+1)}\overline{T}_{kj}^{(s)} - \delta_{j>l}T_{il}^{(r)}\overline{T}_{kj}^{(s-1)}) \\
& -(\delta_{i>k}\overline{T}_{il}^{(s)}T_{kj}^{(r+1)} + \delta_{i<k}\overline{T}_{il}^{(s-1)}T_{kj}^{(r)})\Big]
\end{aligned}
$$

Theorem 5.1 will tell us that, with some rearrangement, quotient spaces, and setting $M = N = 1$, $Y(\mathfrak{gl}_{1|1})$ can be produced from $U_q(\mathcal{L}\mathfrak{gl}_{1|1})$ via a degeneration process. The identity (2.3.71) (repeated as (4.2.3) above) will be shown to correspond to the defining relations (2.3.40) and (2.3.41) of the Yangian. However, the $\overline{T}_{ij}^{(r)}$ generators do not have a natural analogue in the Yangian. As a result, this complicates defining some subspace K in order to show that it is a submodule, confirming that the Verma module given by a rational function is reducible.

Specifically, the Verma module $M(\nu(u), \overline{\nu}(u))$ over $U_q(\mathcal{L}\mathfrak{gl}_{1|1})$ is the quotient of $U_q(\mathcal{L}\mathfrak{gl}_{1|1})$ by the ideal $\mathcal{J}$, where

$$
\mathcal{J} =< T_{12}^{(r)}, \overline{T}_{12}^{(r)}, T_{11}(u) - \nu_1(u), \overline{T}_{11}(u) - \overline{\nu_1}(u), T_{22}(u) - \nu_2(u), \overline{T}_{22}(u) - \overline{\nu_2}(u) >
$$

and $\nu(u) = (\nu_1(u), \nu_2(u))$, $\overline{\nu}(u) = (\overline{\nu_1}(u), \overline{\nu_2}(u))$. Thus, $M(\nu(u), \overline{\nu}(u))$ is spanned by monomials in the $T_{21}^{(r)}$ and $\overline{T}_{21}^{(r)}$. If we assume that $\nu(u)$ and $\overline{\nu}(u)$ are expansions of rational functions, then problems arise when deciding how to define the subspace K we want to prove is a nontrivial submodule of $M(\nu(u), \overline{\nu}(u))$. Stability with respect to multiplication by $T_{11}^{(r)}, \overline{T}_{11}^{(r)}, T_{12}^{(r)}, \overline{T}_{12}^{(r)}, T_{22}^{(r)}, \overline{T}_{22}^{(r)}$, is a difficult condition to achieve with any natural choice of K. Thus, for the time being we set this result aside. Instead, we note that an analogue of Proposition 4.3 holds quite naturally on the quantum current subalgebra $U_q(\mathfrak{gl}_{M|N}[s])$ in Section 4.4.

4.2.2 *Reducible Verma Modules over $Y(\mathfrak{gl}_{1|1})$ and $Y(\mathfrak{sl}_{1|1})$*

What follows is a basic result on formal power series from Billig, Futorny, and Molev's paper on Verma modules for Yangian [BFM06]. Here, it is proven more explicitly and that will be useful in proving later propositions.

Lemma 4.6. *For a formal power series in u^{-1}, denoted $\nu(u) = 1 + \nu^{(1)}u^{-1} + \nu^{(2)}u^{-2} + \cdots$, if there exist a positive integer N and $(c_0, c_1, \ldots, c_m) \in \mathbb{C}^m - \{0\}$ such that $c_0\nu^{(r)} + c_1\nu^{(r+1)} + \cdots + c_m\nu^{r+m} = 0$ for all $r \geq N$, then $\nu(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u. Further, if this rational function is $P(u)/Q(u)$, then $P(u), Q(u)$ are monic polynomials of the same degree.*

Proof. Let $\widetilde{\nu(u)} = \nu^{(N)} + \nu^{(N+1)}u^{-1} + \cdots$. Multiply it by the polynomial $\sum_{i=0}^{m} c_i u^i$.

$$\begin{aligned}
&\widetilde{\nu(u)}(c_0 + c_1 u + c_2 u + \cdots c_m u^m) \\
&\qquad = (\nu^{(N)} + \nu^{(N+1)}u^{-1} + \cdots)(c_0 + c_1 u + c_2 u + \cdots c_m u^m)
\end{aligned}$$

Expand the right hand side and observe the following pattern in coefficients of $u^k, k \leq 0$. The constant term will have the coefficient $c_0\nu^{(N)} + c_1\nu^{(N+1)} +$

$c_2\nu^{(N+2)} + \cdots + c_m\nu^{(N+m)}$. Then, the u^{-1}-term will have the coefficient $c_0\nu^{(N+1)} + c_1\nu^{(N+2)} + c_2\nu^{(N+3)} + \cdots + c_m\nu^{(N+m+1)}$, the u^{-2}-term will have the coefficient $c_0\nu^{(N+2)} + c_1\nu^{(N+3)} + c_2\nu^{(N+4)} + \cdots + c_m\nu^{(N+m+2)}$, and so on. Ultimately, we conclude that the coefficient for $u^k, k \leq 0$ is $c_0\nu^{(N+k)} + c_1\nu^{(N+k+1)} + c_2\nu^{(N+k+2)} + \cdots + c_m\nu^{(N+m+k)}$. By applying that

$$c_0\nu^{(r)} + c_1\nu^{(r+1)} + \cdots + c_m\nu^{(r+m)} = 0$$

for all r greater than or equal to N, we see that all the above coefficients vanish. However, when we consider the coefficients of positive powers of u, we see that these do not necessarily vanish. For instance, the coefficient for u is $c_1\nu^{(N)} + c_2\nu^{(N+1)} + \cdots + c_m\nu^{(N+m-1)}$, the coefficient for u^2 is $c_2\nu^{(N)} + c_3\nu^{(N+1)} + \cdots + c_m\nu^{(N+m-2)}$, and so on. This ends with the coefficients $c_{m-1}\nu^{(N)} + c_m\nu^{(N+1)}$ for u^{m-1} and $c_m\nu^{(N)}$ for u^m. Thus, we obtain the general coefficient

$$b_k := \sum_{i=k}^{m} c_i\nu^{(N-k+i)}$$

for positive powers k of u. As a result,

$$\widetilde{\nu(u)}(c_0 + c_1u + c_2u + \cdots c_mu^m) = b_1u + b_2u^2 + \cdots + b_mu^m.$$

Note that we can reconstruct $\nu(u)$ from $\widetilde{\nu(u)}$ and use the calculations above to show that $\nu(u)$ is the desired rational function.

$$\begin{aligned}
\nu(u) &= 1 + \nu^{(1)}u^{-1} + \nu^{(2)}u^{-2} + \cdots + \nu^{(N-1)}u^{-N+1} + u^{-N}\widetilde{\nu(u)} \\
&= 1 + \nu^{(1)}u^{-1} + \nu^{(2)} + \cdots + \nu^{(N-1)}u^{-N+1} + \frac{b_1u + b_2u^2 + \cdots + b_mu^m}{c_0u^N + c_1u^{N+1} + \cdots + c_mu^{N+m}} \\
&= \frac{(1 + \nu^{(1)}u^{-1} + \cdots + \nu^{(N-1)}u^{-N+1})(c_0u^N + c_1u^{N+1} + \cdots + c_mu^{N+m})}{c_0u^N + c_1u^{N+1} + \cdots + c_mu^{N+m}}
\end{aligned}$$

$$+\frac{b_1u+\cdots+b_mu^m}{c_0u^N+c_1u^{N+1}+\cdots+c_mu^{N+m}}$$
$$=\frac{c_mu^{N+m}+\cdots+c_0\nu^{(N-1)}u+b_1u+\cdots b_mu^m}{c_mu^{N+m}+\cdots+c_0u^N}$$
$$=\frac{u^{N+m}+\frac{1}{c_m}\left((c_m\nu^{(1)}+c_{m-1})u^{N+m-1}+\cdots c_0\nu^{(N-1)}u+b_1u+\cdots b_mu^m\right)}{u^{N+m}+\frac{1}{c_m}\left(c_{m-1}u^{N+m-1}+\cdots+c_0u^N\right)}$$

□

Proposition 4.7. *Let $M(\lambda_1(u),\lambda_2(u))$ be a reducible Verma module over $Y(\mathfrak{gl}_{1|1})$. Then, $\frac{\lambda_1(u)}{\lambda_2(u)}$ is the Laurent expansion at $u=\infty$ of a rational function in u.*

Proof. First, we use that $M(\lambda_1(u),\lambda_2(u))$ is isomorphic to the Verma module $M(1,\nu(u))$ for $\nu(u)=\frac{\lambda_2(u)}{\lambda_1(u)}$ and work with $M(1,\nu(u))$ instead. We then need to show the result for $\nu(u)$. $M(1,\nu(u))$ has a weight space decomposition

$$M(1,\nu(u)) \quad = \quad \bigoplus_{\eta} M(1,\nu(u))_\eta \tag{4.2.6}$$

where $M(1,\nu(u))_\eta=\{X\in M(1,\nu(u))|(t_{11}^{(1)}-t_{22}^{(1)})X=\eta X\}$. This weight space is nonzero when η is of the form $\eta=-\nu^{(1)}-2k$ for a nonnegative integer k. This stems from the fact that $t_{11}^{(r)}1_\lambda=0$ in $M(1,\nu(u))$ for $r>0$ and $t_{22}^{(r)}1_\lambda=\nu^{(r)}1_\lambda$. If X is a monomial of the form $t_{21}^{(r_1)}\cdots t_{21}^{(r_k)}1_\lambda$, then $(t_{11}^{(1)}-t_{22}^{(1)})X=(-\nu^{(1)}-2k)X$. This can be shown by induction:

$$\begin{aligned}\left(t_{11}^{(1)}-t_{22}^{(1)}\right)t_{21}^{(r_1)}1_\lambda &= t_{11}^{(1)}t_{21}^{(r_1)}1_\lambda-t_{22}^{(1)}t_{21}^{(r_1)}1_\lambda\\ &=\left(-t_{21}^{(r_1)}+t_{21}^{(r_1)}t_{11}^{(1)}\right)1_\lambda-\left(t_{21}^{(r_1)}t_{22}^{(1)}+t_{21}^{(r_1)}\right)1_\lambda\\ &=\left(-t_{21}^{(r_1)}-t_{21}^{(r_1)}\nu^{(1)}-t_{21}^{(r_1)}\right)1_\lambda\\ &=\left(-\nu^{(1)}-2\right)t_{21}^{(r_1)}1_\lambda\end{aligned}$$

Then, if we assume that

$$\left(t_{11}^{(1)} - t_{22}^{(1)}\right) t_{21}^{(s_1)} \cdots t_{21}^{(s_{k-1})} 1_\lambda = \left(-\nu^{(1)} - 2(k-1)\right) t_{21}^{(s_1)} \cdots t_{21}^{(s_{k-1})} 1_\lambda,$$

we can show that $\left(t_{11}^{(1)} - t_{22}^{(1)}\right) t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda = \left(-\nu^{(1)} - 2k\right) t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda$:

$$\begin{aligned}
\left(t_{11}^{(1)} - t_{22}^{(1)}\right) t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda &= t_{11}^{(1)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda - t_{22}^{(1)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda \\
&= \left(-t_{21}^{(r_1)} + t_{21}^{(r_1)} t_{11}^{(1)}\right) t_{21}^{(r_2)} \cdots t_{21}^{(r_k)} 1_\lambda - \left(t_{21}^{(r_1)} t_{22}^{(1)} + t_{21}^{(r_1)}\right) t_{21}^{(r_2)} \cdots t_{21}^{(r_k)} 1_\lambda \\
&= -2 t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda + t_{21}^{(r_1)} \left(t_{11}^{(1)} - t_{22}^{(1)}\right) t_{21}^{(r_2)} \cdot t_{21}^{(r_k)} 1_\lambda \\
&= -2 t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda + \left(-\nu^{(1)} - 2(k-1)\right) t_{21}^{(r_1)} t_{21}^{(r_2)} \cdot t_{21}^{(r_k)} 1_\lambda \\
&= \left(-\nu^{(1)} - 2k\right) t_{21}^{(r_1)} t_{21}^{(r_2)} \cdot t_{21}^{(r_k)} 1_\lambda
\end{aligned}$$

If we fix some k, the monomials of the form

$$t_{21}^{(r_1)} t_{21}^{(r_2)} \cdots t_{21}^{(r_k)} 1_\lambda, \;\; 1 \le r_1 < r_2 < \cdots < r_k$$

form a basis for $M(1, \nu(u))_{-\nu^{(1)}-2k}$. Suppose K is a nontrivial submodule of $M(1, \nu(u))$, thus inheriting the weight space decomposition

$$K = \bigoplus_\eta K_\eta, \qquad K_\eta = K \cap M(1, \nu(u))_\eta. \tag{4.2.7}$$

Choose k to be the minimum positive integer such that $K_\eta \neq 0$ for $\eta = -\nu^{(1)} - 2k$; in other words, choose k in order to obtain the nonzero weight space of greatest weight. This condition implies that for any nonzero vector $\zeta \in K_\eta$, $t_{12}(u)\zeta = 0$.

Else, for at least one $r \geq 1$, $t_{12}^{(r)}\zeta$ would be nonzero and in the weight space $K_{\eta+2}$, contradicting the maximality of η. Let

$$\zeta = \sum_{\mathbf{r}} c_{\mathbf{r}} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda, \tag{4.2.8}$$

where we sum over finitely many k-tuples $\mathbf{r} = (r_1, \dots, r_k)$ with $1 \leq r_1 < r_2 < \cdots < r_k$ and $c_{\mathbf{r}} \in \mathbb{C}$.

Next, we determine the action of $t_{12}^{(r)}, t_{11}^{(r)}$, and $t_{22}^{(r)}$ on a monomial $t_{21}^{(r_1)} \cdots t_{21}^{(r_k)}$. Recall that, in general, we have

$$\left[t_{ab}^{(r)}, t_{cd}^{(s)}\right] \;=\; (-1)^{\eta(a,b;c,d)} \sum_{l=0}^{\min(r,s)-1} \left(t_{cb}^{(l)} t_{ad}^{(r+s-1-l)} - t_{cb}^{(r+s-1-l)} t_{ad}^{(l)}\right). \tag{4.2.9}$$

where $\eta(a,b;c,d) \equiv 1 + d + (b+d)(a+d) + (a+b)(c+d) \mathrm{mod} 2$.

In particular,

$$\begin{aligned}
t_{12}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda &= \left[t_{12}^{(r)}, t_{21}^{(r_1)}\right] t_{21}^{(r_2)} \cdots t_{21}^{(r_k)} 1_\lambda + t_{21}^{(r_1)} t_{12}^{(r)} t_{12}^{(r_2)} \cdots t_{21}^{(r_k)} 1_\lambda \\
&= \left(- \sum_{l=0}^{\min(r,r_1)} (t_{22}^{(l)} t_{11}^{(r+r_1-1-l)} - t_{22}^{(r+r_1-1-l)} t_{11}^{(l)})\right) t_{21}^{(r_2)} \cdots t_{21}^{(r_k)} 1_\lambda \\
&\quad + t_{21}^{(r_1)} t_{12}^{(r)} t_{12}^{(r_2)} \cdots t_{21}^{(r_k)} 1_\lambda \\
t_{12}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda &= \left(- \sum_{l=0}^{\min(r,r_1)} (t_{22}^{(l)} t_{11}^{(r+r_1-1-l)} - t_{22}^{(r+r_1-1-l)} t_{11}^{(l)})\right) t_{21}^{(r_2)} \cdots t_{21}^{(r_k)} 1_\lambda \\
&\quad + t_{21}^{(r_1)} \left(- \sum_{l=0}^{\min(r,r_2)} (t_{22}^{(l)} t_{11}^{(r+r_2-1-l)} - t_{22}^{(r+r_2-1-l)} t_{11}^{(l)})\right) t_{21}^{(r_3)} \cdots t_{21}^{(r_k)} 1_\lambda \\
&\quad + t_{21}^{(r_1)} t_{21}^{(r_2)} t_{12}^{(r)} t_{21}^{(r_3)} \cdots t_{21}^{(r_k)} 1_\lambda \\
t_{12}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda &= - \sum_{i=1}^{k} t_{21}^{(r_1)} \cdots t_{21}^{(r_{i-1})} \\
&\quad \left(- \sum_{a_i=0}^{\min(r,r_i)} (t_{22}^{(a_i)} t_{11}^{(r+r_i-1-a_i)} - t_{22}^{(r+r_i-1-a_i)} t_{11}^{(a_i)})\right) t_{21}^{(r_{i+1})} \cdots t_{21}^{(r_k)} 1_\lambda
\end{aligned} \tag{4.2.10}$$

In order to continue to analyze $t_{12}^{(r)}\zeta$, we need to break the above result down through the t_{11} and t_{22} terms. Similar calculations yield the necessary formulas:

$$\begin{aligned}
t_{11}^{(r)}t_{21}^{(r_1)}\cdots t_{21}^{(r_k)}1_\lambda &= \left[t_{11}^{(r)}, t_{21}^{(r_1)}\right] t_{21}^{(r_2)}\cdots t_{21}^{(r_k)}1_\lambda + t_{21}^{(r_1)}t_{11}^{(r)}t_{21}^{(r_2)}\cdots t_{21}^{(r_k)}1_\lambda \\
&= \left(\sum_{a_1=0}^{\min(r,r_1)-1}(t_{21}^{(a_1)}t_{11}^{(r+r_1-1-a_1)} - t_{21}^{(r+r_1-1-a_1)}t_{11}^{(a_1)})\right) t_{21}^{(r_2)}\cdots t_{21}^{(r_k)}1_\lambda \\
&\quad + t_{21}^{(r_1)}\left(\sum_{a_2=0}^{\min(r,r_2)-1}(t_{21}^{(a_2)}t_{11}^{(r+r_2-1-a_2)} - t_{21}^{(r+r_2-1-a_2)}t_{11}^{(a_2)})\right) t_{21}^{(r_3)}\cdots t_{21}^{(r_k)}1_\lambda \\
&\quad + t_{21}^{(r_1)}t_{21}^{(r_2)}t_{11}^{(r)}t_{21}^{(r_3)}\cdots t_{21}^{(r_k)}1_\lambda
\end{aligned}$$

$$t_{11}^{(r)}t_{21}^{(r_1)}\cdots t_{21}^{(r_k)}1_\lambda = \sum_{i=1}^{k} t_{21}^{(r_1)}\cdots t_{21}^{(r_{i-1})} \left(\sum_{a_i=0}^{\min(r,r_i)-1}(t_{21}^{(a_i)}t_{11}^{(r+r_i-1-a_i)} - t_{21}^{(r+r_i-1-a_i)}t_{11}^{(a_i)})\right) t_{21}^{(r_{i+1})}\cdots t_{21}^{(r_k)}1_\lambda \tag{4.2.11}$$

$$\begin{aligned}
\left[t_{22}^{(r)}, t_{21}^{(r_1)}\cdots t_{21}^{(r_k)}\right]1_\lambda &= \left[t_{22}^{(r)}, t_{21}^{(r_1)}\right] t_{21}^{(r_2)}\cdots t_{21}^{(r_k)}1_\lambda + t_{21}^{(r_1)}\left[t_{22}^{(r)}, t_{21}^{(r_2)}\cdots t_{21}^{(r_k)}\right]1_\lambda \\
&= -\left[t_{21}^{(r_1)}, t_{22}^{(r)}\right] t_{21}^{(r_2)}\cdots t_{21}^{(r_k)}1_\lambda + t_{21}^{(r_1)}\left[t_{22}^{(r)}, t_{21}^{(r_2)}\cdots t_{21}^{(r_k)}\right]1_\lambda
\end{aligned}$$

$$\begin{aligned}
t_{22}^{(r)}t_{21}^{(r_1)}\cdots t_{21}^{(r_k)}1_\lambda &= t_{21}^{(r_1)}\cdots t_{21}^{(r_k)}t_{22}^{(r)}1_\lambda \\
&\quad - \left(\sum_{a_1=0}^{\min(r,r_1)-1}(t_{21}^{(a_1)}t_{22}^{(r+r_1-1-a_1)} - t_{21}^{(r+r_1-1-a_1)}t_{22}^{(a_1)})\right) t_{21}^{(r_2)}\cdots t_{21}^{(r_k)}1_\lambda \\
&\quad + t_{21}^{(r_1)}\left[t_{22}^{(r)}, t_{21}^{(r_2)}\right] t_{21}^{(r_3)}\cdots t_{21}^{(r_k)}1_\lambda + t_{21}^{(r_1)}t_{21}^{(r_2)}\left[t_{22}^{(r)}, t_{21}^{(r_3)}\cdots t_{21}^{(r_k)}\right]1_\lambda \\
&= \nu^{(r)}t_{21}^{(r_1)}\cdots t_{21}^{(r_k)}1_\lambda - \left(\sum_{a_1=0}^{\min(r,r_1)-1}(t_{21}^{(a_1)}t_{22}^{(r+r_1-1-a_1)} - t_{21}^{(r+r_1-1-a_1)}t_{22}^{(a_1)})\right) \\
&\quad t_{21}^{(r_2)}\cdots t_{21}^{r_k}1_\lambda - t_{21}^{(r_1)}\left[t_{21}^{(r_2)}, t_{22}^{(r)}\right] t_{21}^{(r_3)}\cdots t_{21}^{(r_k)}1_\lambda + t_{21}^{(r_1)}t_{21}^{(r_2)}\left[t_{22}^{(r)}, t_{21}^{(r_3)}\cdots t_{21}^{r_k}\right]1_\lambda \\
&= \nu^{(r)}t_{21}^{(r_1)}\cdots t_{21}^{(r_k)}1_\lambda \\
&\quad - \left(\sum_{a_1=0}^{\min(r,r_1)-1}(t_{21}^{(a_1)}t_{22}^{(r+r_1-1-a_1)} - t_{21}^{(r+r_1-1-a_1)}t_{22}^{(a_1)})\right) t_{21}^{(r_2)}\cdots t_{21}^{(r_k)}1_\lambda
\end{aligned}$$

$$- t_{21}^{(r_1)} \left(\sum_{a_2=0}^{\min(r,r_2)-1} t_{21}^{(a_2)} (t_{22}^{(r+r_2-1-a_2)} - t_{21}^{(r+r_2-1-a_2)} t_{22}^{(a_2)}) \right) t_{21}^{(r_3)} \cdots t_{21}^{(r_k)} 1_\lambda$$

$$+ t_{21}^{(r_1)} t_{21}^{(r_2)} \left[t_{22}^{(r)}, t_{21}^{(r_3)} \right] t_{21}^{(r_4)} \cdots t_{21}^{(r_k)} 1_\lambda + t_{21}^{(r_1)} t_{21}^{(r_2)} t_{21}^{(r_3)} \left[t_{22}^{(r)}, t_{21}^{(r_4)} \cdots t_{21}^{r_k} \right] 1_\lambda$$

$$= \dots$$

$$= \nu^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda - \sum_{i=1}^{k} t_{21}^{(r_1)} \cdots t_{21}^{(r_{i-1})}$$

$$\left(\sum_{a_i=0}^{\min(r,r_i)-1} (t_{21}^{(a_i)} t_{22}^{(r+r_i-1-a_i)} - t_{21}^{(r+r_i-1-a_i)} t_{21}^{(a_i)}) \right) t_{21}^{(r_{i+1})} \cdots t_{21}^{(r_k)} 1_\lambda \tag{4.2.12}$$

Above, we've used that $t_{12}^{(r)} 1_\lambda = 0$, $t_{22}^{(r)} 1_\lambda = \nu^{(r)}$, and $t_{11}^{(r)} 1_\lambda = 0$ for $r \geq 1$. Thus, iterating the application of (4.2.11) and (4.2.12) produces a representation of (4.2.10) as linear combination of basis monomials $t_{21}^{(s_1)} \cdots t_{21}^{(s_{k-1})} 1_\lambda$, after finitely many steps. Let

$$N = \max\{r_1 + r_2 + \cdots + r_k | c_{\mathbf{r}} \neq 0 \text{ in the sum (4.2.8)}\}. \tag{4.2.13}$$

Henceforth, we will only consider values of r in $t_{12}^{(r)} \zeta$ that are greater than this N. Now, choose some monomial $t_{21}^{(s_1)} t_{21}^{(s_2)} \cdots t_{21}^{(s_{k-1})} 1_\lambda$ in the expansion of $t_{12}^{(r)} \zeta$ such that $1 < s_1 < s_2 < \cdots < s_{k-1}$ and $s_1 + s_2 + \cdots + s_{k-1} \leq N - 1$. What is its coefficient? By examining the formulas (4.2.10), (4.2.11), and (4.2.12), we pick out all such terms. Note that the purpose of (4.2.11) is effectively to push all $t_{11}^{(s)}$-type factors to the right of the term; the term is only nonzero if $s = 0$, producing $t_{11}^{(0)} = 1$.

First, we see that (4.2.10) produces two types of terms with $k - 1$ t_{21} factors with an inserted pair of t_{22} and t_{11}: ones where t_{22} has a lower index, then the ones with the inverse situation. The $t_{21}^{(r_1)} \cdots t_{21}^{(r_{i-1})} t_{22}^{(a_i)} t_{11}^{(r+r_i-1-a_i)} t_{21}^{(r_{i+1})} \cdots t_{21}^{(r_k)} 1_\lambda$ terms can be ignored; the high superscript on the t_{11} causes the production of terms with insufficiently low superscripts of t_{21} or, after repeated applications of (4.2.11), vanishing terms with a $t_{11}^{(s)}, s > 0$ immediately before 1_λ. Instead, we look at the terms of the form $t_{21}^{(r_1)} \cdots t_{21}^{(r_{i-1})} t_{22}^{(r+r_i-1-a_i)} t_{11}^{(a_i)} t_{21}^{(r_{i+1})} \cdots t_{21}^{(r_k)} 1_\lambda$. In the simpler case, $a_i = 0$, and we get

$$t_{21}^{(r_1)} \cdots t_{21}^{(r_{i-1})} t_{22}^{(r+r_i-1)} t_{21}^{(r_{i+1})} \cdots t_{21}^{(r_k)} 1_\lambda = \nu^{(r+r_i-1)} t_{21}^{(r_1)} \cdots t_{21}^{(r_{i-1})} t_{21}^{(r_{i+1})} \cdots t_{21}^{(r_k)} 1_\lambda$$
$$+ t_{21}^{(r_1)} \cdots t_{21}^{(r_{i-1})} \left(\sum_{j=i+1}^{k} t_{21}^{r_{i+1}} \cdots t_{21}^{(r_{j-1})} \left(\sum_{a_j=0}^{r_j-1} t_{21}^{(a_j)} t_{22}^{(r+r_j-1-a_j)} - t_{21}^{(r+r_j-1-a_j)} t_{22}^{(a_j)} \right) \right.$$
$$\left. t_{21}^{(r_{j+1})} \cdots t_{21}^{(r_k)} \right) 1_\lambda$$

This produces terms of the form we want with coefficient $\nu^{(r+r_i-1)}$, while further iterations will continually add more terms with coefficients $\nu^{(r+r_j-a_j)}$ from $a_j = 1, \ldots r_j - 1$ over all j. Further applications of (4.2.12) yield coefficients ranging $\nu^{(r)}, \nu^{(r+1)}, \ldots, \nu^{(r+r_1+r_2+\cdots+r_k-k)}$. In particular, the $\nu^{(r)}$ appears when $a_i = r_i - 1$ in (4.2.12), while we obtain the upper limit by observing that each time we apply the relation, the sum of all super scripts decreases by 1. The ν only appear when t_{22} has iteratively been pushed completely to the right of the term, immediately next to 1_λ. Taking a step back, suppose we start with the original result from (4.2.10) and apply (4.2.11) and (4.2.12) until we are left with a large sum of monomials of the form $t_{21}^{(s_1)} \cdots t_{21}^{(s_k)} t_{22}^{(a)} t_{11}^{(b)} 1_\lambda$. Of course, this monomial vanishes unless $b = 0$ since $t_{11}^{(b)} 1_\lambda = 0$ for $b > 0$. Thus, we assume our monomials are of the form $t_{21}^{(s_1)} \cdots t_{21}^{(s_k)} t_{22}^{(a)} 1_\lambda$. Obtaining these terms requires at most k applications of our three relations; each application pushes the non-t_{21} elements further to the right. With each iteration of all three relations, the superscript a of t_{22} increases by p with $0 \le p \le r_{i-1}$. Thus, a ranges from r to $r + \sum_{i=1}^{k} (r_i - 1) = r + N - k$.

Recall that

$$\zeta = \sum_{\mathbf{r}} c_{\mathbf{r}} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)}.$$

Then,

$$t_{12}^{(r)}\zeta = \sum_{\mathbf{r}}\sum_{i=1}^{k} c_{\mathbf{r}} t_{12}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)}.$$

So, once we pick out our desired terms, we obtain a coefficient that is the aforementioned linear combination of $\nu^{(r)}, \ldots, \nu^{(r+N-k)}$. The coefficients of these are a linear combination of the $c_{\mathbf{r}}$, where the coefficients there are effectively "counting" whether each $t_{21}^{(r_1)} \cdots t_{21}^{(r_k)}$ induces our desired $t_{21}^{(s_1)} \cdots t_{21}^{(s_{k-1})}$ monomial. More concretely, the coefficient of $t_{21}^{(s_1)} \cdots t_{21}^{(s_{k-1})}$ is $\sum_{i=0}^{N-k} \sum_{\mathbf{r}} c_{\mathbf{r}} a_{\mathbf{r},i} \nu^{(r+i)}$, where the $a_{\mathbf{r},i}$ are the "counting" coefficients. Note that they are entirely independent of r, since as long as $r > N$, any change in r does not cause more or fewer terms in (4.2.10) to contribute to the relevant coefficient.

Next, we need to show that the coefficient $\sum_{\mathbf{r}} c_{\mathbf{r}} a_{\mathbf{r},i}$ at $\nu^{(r+i)}$ is nonzero for some i; then, we can apply Lemma 4.6, thus completing the proof of Proposition 4.7. In particular, we will show that the coefficient of $\nu^{(r+s-1)}$ is nonzero, where $s = N - s_1 - s_2 - \cdots - s_{k-1}$. Upon inspecting the formulas (4.2.10), (4.2.11), and (4.2.12), we see that the superscript $r + s - 1$ can only occur when we have a $t_{22}^{(r+s-1)}$ moved to the right end of the monomial using (4.2.12). Critically, we recall that each application of (4.2.10), (4.2.11), and (4.2.12) decreases the sum of superscripts of the terms by 1, with the exception of the first term of (4.2.12) in which a weight arises. We can use this observation to determine how many iterations of (4.2.10), (4.2.11), and (4.2.12) should be applied in order to produce scalar multiples of $\nu^{(r+s-1)} t_{21}^{(s_1)} \cdots t_{21}^{(s_{k-1})}$. It follows that these formulas can only be applied once in total among them, again with the exception of the first term of (4.2.12) being permitted. As a result, to obtain the desired scalar multiple, one must apply (4.2.10) once then consider the first term from (4.2.12) and proceed no further. In turn, we get $t_{22}^{(r+s-1)}$ in (4.2.10) when $a_i = 0$ and we are summing over $\mathbf{r}$ with $\mathbf{s}_{(i)} = (s_1, s_2, \ldots, s_{i-1}, s, s_i, \ldots, s_{k-1})$ over $i = 1, \ldots k$, since those k-tuples are the ones that have $r_i = s$ in (4.2.10), thus producing the desired $t_{22}^{(r-s-1)}$. As a result, the coefficient of $\nu^{(r+s-1)}$ is $-c_{\mathbf{s}_{(1)}} - c_{\mathbf{s}_{(2)}} - \cdots - c_{\mathbf{s}_{(k-1)}}$. Note that $c_{\mathbf{s}_{(i)}} = 0$ if $\mathbf{s}_{(i)}$ does not satisfy $s_1 < s_2 < \cdots < s_{i-1} < s < s_{i+1} < \cdots < s_{k-1}$.

In particular, $c_{\mathbf{s}_{(i)}} = 0$ for all i if $s = s_j$ for some j. This is because the superscripts must be strictly increasing. As well $c_{\mathbf{s}_{(i)}} \neq 0$ for exactly one i; else, if both $c_{\mathbf{s}_{(i)}}$ and $c_{\mathbf{s}_{(j)}}$ are nonzero, that would mean that $s_1 < s_2 < \cdots < s_{i-1} < s < s_{i+1} < \cdots < s_{k-1}$ and $s_1 < s_2 < \cdots < s_{j-1} < s < s_{j+1} < \cdots s_{k-1}$, which can only hold if $i = j$. Otherwise, if $j > i$, say, then we would have the chain of inequalities $s_1 < s_2 < \cdots < s_{i-1} < s < s_{i+1} < \cdots s_{j-1} < s < s_j < \cdots < s_{k-1}$, which is impossible.

We now construct a nonzero coefficient $-c_{\mathbf{r}}$. First, choose $\mathbf{r} = (r_1, \ldots, r_k)$ such that $c_{\mathbf{r}}$ is nonzero. Take some index j such that $r_j > 0$ and $\sum_{i \neq j} r_i \leq N - 1$. Then, if we let $\mathbf{s} = (r_1, r_2, \ldots, r_{j-1}, r_{j+1}, \ldots, r_k)$, $\mathbf{s}_{(j)} = \mathbf{r}$, thus $c_{\mathbf{s}_{(j)}} \neq 0$ and the nonzero coefficient we need is $-c_{\mathbf{s}_{(j)}} = -c_{\mathbf{r}}$. □

The following corollary follows directly from Propositions 4.2, 4.3, and 4.7.

Corollary 4.8. *Let $M(\nu(u))$ be a Verma module over $Y(\mathfrak{sl}_{1|1})$. Then, it is reducible if and only if $\nu(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u.*

Theorem 4.9. *The Verma module $M(\lambda_1(u), \lambda_2(u))$ over $Y(\mathfrak{gl}_{1|1})$ is reducible if and only if the unique irreducible quotient $L(\lambda_1(u), \lambda_2(u))$ is finite dimensional.*

Proof. This is a consequence of Proposition 4.3. In the proof, we saw that the basis of $M(\lambda_1(u), \lambda_2(u))$ is made of monomials $t_{21}^{(r_1)} \cdots t_{21}^{(r_p)}$ with strictly increasing superscripts r_i. If $\lambda_1(u)/\lambda_2(u)$ is rational, then $M(\lambda_1(u), \lambda_2(u))$ has a nontrivial submodule K, spanned by monomials $t_{21}^{(r_1)} \cdots t_{21}^{(r_p)}$ where at least one $r_i > d$. Because these superscripts are strictly increasing, a monomial in the Verma module of length $p > d$ must have $r_p > d$, thus belongs to K. As a result, the basis for $M(\lambda_1(u), \lambda_2(u))/K$ is given by monomials in the $t_{21}^{(r_i)}$ with strictly increasing r_i

where the greatest superscript r_p is at most d. This means that one can have at most d generators in each monomial. Thus, the quotient is finite dimensional. If it isn't irreducible, then it will contain the irreducible $L(\lambda_1(u), \lambda_2(u))$, which must also be finite dimensional. $\square$

Note that the conclusion given by Theorem 4.9 holds for $Y(\mathfrak{gl}_{1|1})$ but does not hold in the original $Y(\mathfrak{gl}_2)$ case. For those non-super Verma modules, the requirement of strictly increasing superscripts is not present, meaning that a monomial can be of any arbitrary length and not have any $r_i > d$. Thus, we cannot use this restriction to draw the same conclusion.

Proposition 4.10. *Let $\nu(u) = P(u)/Q(u)$ with $P(u), Q(u)$ polynomials as before and of degree d. Take the Verma module $M(\nu(u))$ over $Y(\mathfrak{sl}_{1|1})$. Then, for any $s \geq d$, there exist constants $c_0, \ldots, c_s$ such that the vector*

$$\zeta = c_0 f^{(0)} 1_\nu + \cdots + c_s f^{(s)} 1_\nu$$

is annihilated by $e(u)$.

Proof. Recall that in (2.3.22) we defined $e(u) = (t_{11}(u))^{-1} t_{12}(u)$.

By Proposition 4.2, $M(\nu(u))$ is isomorphic to the restriction of the $Y(\mathfrak{gl}_{1|1})$-module $M(\lambda_1(u), \lambda_2(u))$ to $Y(\mathfrak{sl}_{1|1})$, where $\nu(u) = \lambda_1(u)/\lambda_2(u)$. Further, there exists a power series $g(u)$ such that $g(u)\lambda_1(u)$ and $g(u)\lambda_2(u)$ are both polynomials in u^{-1} of degree $\leq d$. This approach was taken previously in the proof of Proposition 4.3. Then, take some $s \geq d$. For this choice of s, $X = t_{21}^{(s+1)} 1_\lambda \in K$ where K is the submodule of $M(\lambda_1(u), \lambda_2(u))$ from the proof of the same proposition. As a result, $t_{12}^{(r)} X = 0$ for all r; else, calculations show that

$$t_{12}^{(r)} X = t_{12}^{(r)} t_{21}^{(s+1)} 1_\lambda$$

$$= \left(t_{21}^{(s+1)} t_{12}^{(r)} + \sum_{a=0}^{\min(r,s+1)-1} (t_{22}^{(a)} t_{11}^{(r+s-a)} - t_{22}^{(r+s-a)} t_{11}^{(a)}) \right) 1_\lambda$$
$$= t_{21}^{(s+1)} t_{12}^{(r)} 1_\lambda + \sum_{a=0}^{\min(r,s+1)-1} (t_{22}^{(a)} t_{11}^{(r+s-a)} 1_\lambda - t_{22}^{(r+s-a)} t_{11}^{(a)}) 1_\lambda$$
$$= \sum_{a=0}^{\min(r,s+1)-1} (\lambda_1^{(r+s-a)} t_{22}^{(a)} 1_\lambda - \lambda_1^{(a)} t_{22}^{(r+s-a)} 1_\lambda)$$
$$= \left(\sum_{a=0}^{\min(r,s+1)-1} \lambda_1^{(r+s-a)} \lambda_2^{(a)} - \lambda_1^{(a)} \lambda_2^{(r+s-a)} \right) 1_\lambda$$

From here, $t_{12}^{(r)} X \in K$ but $t_{12}^{(r)} X$ being a scalar multiple of the highest weight vector would imply that $1_\lambda \in K$, a contradiction. Now, let g be some power series in u^{-1} with constant term 1 that defines an automorphism ϕ_g. The image of $t_{21}^{(s+1)}$ under that automorphism is $t_{21}^{(s+1)} + f^{(1)} t_{21}^{(s)} + \cdots + f^{(s)} t_{21}^{(1)}$; this is found by picking out the coefficient of u^{-s-1} in the expansion of $\phi_g(t_{21}(u)) = g(u) t_{21}(u)$. Denote this coefficient as an element of the module by

$$\zeta := \phi_g(t_{21}^{(s+1)}) 1_\lambda = t_{21}^{(s+1)} 1_\lambda + g^{(1)} t_{21}^{(s)} 1_\lambda + \cdots + g^{(s)} t_{21}^{(1)} 1_\lambda \qquad (4.2.14)$$

and note that we still have $t_{12}^{(r)} \zeta = 0$. Finally, Gauss decomposition (see lines (2.3.9) through (2.3.15)) yields

$$t_{11}(u) = d_1(u) \qquad (4.2.15)$$
$$t_{22}(u) = d_2(u) + f_1(u) d_1(u) e_1(u) \qquad (4.2.16)$$
$$t_{12}(u) = d_1(u) e_{12}(u) = d_1(u) e_1(u) \qquad (4.2.17)$$
$$t_{21}(u) = f_{21}(u) d_1(u) = f_1(u) d_1(u) \qquad (4.2.18)$$

which allows us to write $f(u) := f_1(u) = t_{21}(u)\,(t_{11}(u))^{-1}$ and $e(u) := e_1(u) = (t_{11}(u))^{-1}\,t_{12}(u)$. Finally, we rewrite ζ in terms of $f(u)$. By (4.2.18),

$$\begin{aligned}\sum_k t_{21}^{(k)}u^{-k} &= \left(\sum_i f^{(i)}u^{-i}\right)\left(\sum_j t_{11}^{(j)}u^{-j}\right)\\ t_{21}^{(k)} &= \sum_{i+j=k} f^{(i)}t_{11}^{(j)}\end{aligned}$$

Thus,

$$\begin{aligned}\zeta = \left(\sum_{a+n=s+1} g^{(a)}t_{21}^{(b)}\right)1_\lambda &= \left(\sum_{a+b=s+1} g^{(a)}\left(\sum_{i+j=b} f^{(i)}t_{11}^{(j)}\right)\right)1_\lambda\\ &= \sum_{a+b=s+1} g^{(a)}\sum_{i+j=b}\lambda_1^{(j)}f^{(i)}1_\lambda\\ &= \sum_{0\le i\le a\le s+1}\lambda_1^{(a-i)}g^{(s+1-a)}f^{(i)}1_\lambda\end{aligned}$$

Apply that $e(u)\zeta = (t_{11}(u))^{-1}\,t_{12}(u)\zeta = 0$ to yield the final result. □

The following proposition is similar to the previous one, but its proof avoids Gauss decomposition.

Proposition 4.11. *Let $L(\nu)$ be the irreducible finite dimensional Verma module of weight $\nu(u)$ over $Y(\mathfrak{sl}_{1|1})$, where $\nu(u) = \frac{P(u)}{Q(u)}$. $P(u)$ and $Q(u)$ are polynomials of degree d as before. As in the case for $Y(\mathfrak{sl}_2)$, for sufficiently large s, there exist constants $c_0, \dots, c_s$ such that the vector*

$$\zeta = c_0 f^{(0)}1_\nu + \cdots + c_s f^{(s)}1_\nu$$

in the Verma module is annihilated by $e(u)$.

Proof. Let $N = \max\{\deg P(u), \deg Q(u)\}$. The dimension of $L(\nu)$ is 2^N ([Zha95], Theorem 4). Take the elements $f^{(0)}1_\nu, f^{(1)}1_\nu, \dots, f^{(s)}1_\nu$ in the Verma module $M(\nu)$. When passing to $L(\nu)$, these elements cannot be all linearly independent since $s > \dim L(\nu) - 1$. Thus, they are linearly dependent and there exist constants

$c_0, c_1, \ldots, c_s$ such that $c_0 f^{(0)} 1_\nu + \cdots + c_s f^{(s)} 1_\nu$ is a nonzero element of $M(\nu)$ that becomes zero in $L(\nu)$. Set $\zeta := c_0 f^{(0)} 1_\nu + \cdots + c_s f^{(s)} 1_\nu$ and let $\overline{\zeta}$ denote its class in $L(\nu)$. There is an exact sequence of the form

$$0 \longrightarrow \mathrm{rad}(M(\nu)) \longrightarrow M(\nu) \longrightarrow L(\nu) \longrightarrow 0$$

where $\mathrm{rad}(M(\nu))$ is the unique maximal submodule of $M(\nu)$. Thus, by exactness, if $\overline{\zeta} = \overline{0}$, then $\zeta \in \mathrm{rad}(M(\nu))$.

We can show that $e(u)\zeta = 0$ if we show that $e^{(i)}\zeta = 0$ for all i, i.e. $e^{(i)}\zeta \in \mathrm{rad}(M(\nu))$. If $e^{(i)}\zeta \neq 0$, then its weight should be the same as the weight of 1_ν with respect to the action of the Cartan subalgebra. Let $w(v)$ denote the weight of an element v. Thus, we have that

$$\begin{aligned} w(\zeta) = w(f^{(j)} 1_\nu) &= w(f^{(j)}) + w(1_\nu) \\ &= -(\varepsilon_1 - \varepsilon_2) + \mu^{(0)} \\ w(e^{(i)}\zeta) &= w(e^{(i)}) + w(\zeta) \\ &= (\varepsilon_1 - \varepsilon_2) - (\varepsilon_1 - \varepsilon_2)\mu^{(0)} \end{aligned}$$

This implies that $e^{(i)}\zeta$ and 1_ν have the same weight, thus $e^{(i)}\zeta = a1_\nu$ for some nonzero constant a. However, this is impossible because 1_ν is not an element of $\mathrm{rad}(M(\nu))$. Thus, $e^{(i)}\zeta = 0$ for all superscripts i and $e(u)\zeta = 0$. $\square$

4.3 GENERALIZING TO $\mathfrak{gl}_{M|N}$

In Billig, Futorny, and Molev's article on Yangians [BFM06], they first present results on $Y(\mathfrak{gl}_2)$ and $Y(\mathfrak{sl}_2)$ and their Verma modules. Then, they generalize to the case of $Y(\mathfrak{a})$, where $\mathfrak{a}$ is an arbitrary simple complex Lie algebra. Similarly, I will move from the results on $\mathfrak{gl}_{1|1}$ from above, many prompted by their analogous propositions on $Y(\mathfrak{gl}_2)$, and extend to results on $Y(\mathfrak{gl}_{M|N})$.

4.3.1 *Verma modules on $Y(\mathfrak{gl}_{M|N})$*

In the original version of the following proposition, we would let $\mathfrak{a}$ be an arbitrary simple complex Lie algebra. Here, we specifically use $\mathfrak{sl}_{M|N}$. This is because the proof relies on the e, f, and d (or h) generators given in Definition 2.31, along with the existing results on $\mathfrak{gl}_{1|1}$ and $\mathfrak{sl}_{1|1}$ from above using the same type of generators.

To define a Verma module over these generators, we use formulas that were derived from Gauss decomposition and apply what we know about Verma modules with the RTT presentation. For instance, if we return to $\mathfrak{gl}_{1|1}$, we use the formulas (2.3.20) to (2.3.23) to get that

$$\begin{aligned} e^{(r)}1_\lambda &= 0 \\ d_1^{(r)}1_\lambda &= \lambda_1^{(r)}1_\lambda \\ d_2^{(r)}1_\lambda &= \lambda_2^{(r)}1_\lambda. \end{aligned}$$

Proposition 4.12. ***(Analogue of Proposition 3.9 of [BFM06])***
Let $\mu(u) = (\mu_1(u), \ldots, \mu_{M+N-1}(u))$ be the highest weight of the Verma module $M(\mu(u))$ over the Yangian $Y(\mathfrak{sl}_{M|N})$ such that, for some $i \in \{1, \ldots, M+N\}$, the series $\mu_i(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u. Then, the module $M(\mu(u))$ is reducible.

Proof. Take the subalgebra of $Y(\mathfrak{sl}_{M|N})$ generated by the elements $\{e_i^{(r)}, f_i^{(r)}, h_i^{(r)}\}_{r=0}^{\infty}$; it is isomorphic to $Y(\mathfrak{sl}_{1|1})$. Further, if we let 1_μ be the highest weight vector of

$M(\mu(u))$, then the $Y(\mathfrak{sl}_{1|1})$-span of this highest weight vector is isomorphic to the Verma module $M(\mu_i(u))$ over $Y(\mathfrak{sl}_{1|1})$. Then, by Corollary 4.8 and Proposition 4.10, there exist constants $c_0, \ldots, c_p$ such that the vector

$$\zeta = c_0 f_i^{(0)} 1_\mu + \cdots + c_p f_i^{(p)} 1_\mu$$

is annihilated by $e_i(u)$. The relation (2.3.48) on generators $e_j^{(r)}$ and $f_j^{(s)}$ of $Y(\mathfrak{sl}_{M|N})$ imply that $e_j(u)\zeta = 0$ for all $j = 1, \ldots, M+N-1$. Thus, the submodule $Y(\mathfrak{sl}_{M|N})\zeta$ of $M(\mu(u))$ does not contain the highest weight vector 1_μ and we can conclude that $M(\mu(u))$ is reducible. □

The following proposition is the converse of Proposition 4.12.

Proposition 4.13. *Let $M(\mu(u))$ be a reducible Verma module over $Y(\mathfrak{sl}_{M|N})$. Then, for some $i \in \{1, \ldots, M+N\}$, $\mu_i(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u.*

Proof. $M(\mu(u))$ is assumed to be reducible; then, any nontrivial submodule K will inherit the weight space decomposition of $M(\mu(u))$:

$$K = \bigoplus_\eta K_\eta$$

where $K_\eta = K \cap M(\mu(u))_\eta$.

The standard ordering on the weights dictates that η precedes η' if $\eta' - \eta$ is a linear combination of the positive roots with nonnegative integer coefficients. Choose a maximal weight η; $M(\mu(u))$ must contain a nonzero weight vector $\zeta \in X_\eta$ that is annihilated by $e_i(u)$ for all i. We can write η in the form

$$\eta = \mu^{(0)} - k_1\alpha_1 - \cdots - k_n\alpha_n$$

for simple roots α_i. In general, the coefficients k_i are nonnegative integers, not all zero, but for this particular case, suppose that $k_1 = \cdots = k_{i-1} = 0$ and $k_i > 0$ for some index i. Our goal is to prove that $\mu_i(u)$, for this i, is the Laurent expansion at $u = \infty$ of a rational function in u. Without loss of generality, let $i = 1$.

Next, fix a total ordering on the set of positive roots Δ^+. Call it $\prec$ and let $\alpha_i \prec a_j$ if $i > j$. Further, let any composite precede any simple root. Use this ordering to define an ordering on the generators $\{x^-_{\alpha,r} | \alpha \in \Delta^+, r \geq 0\}$: $x^-_{\alpha,r}$ precedes $x^-_{\beta,s}$ if $\alpha \prec \beta$ or $\alpha = \beta$ and $r < s$.

We claim that ζ is a linear combination of monomials of the form

$$x^-_{\alpha_n,p_1} \cdots x^-_{\alpha_n,p_{k_n}} \cdots x^-_{\alpha_1,q_1} \cdots x^-_{\alpha_1,q_{k_1}} 1_\mu;$$

i.e., no generators $x^-_{\alpha,r}$ where α is a composite root can appear.

Write ζ as a sum of pieces with each ζ_k a linear combination of monomials of degree k.

$$\zeta = \zeta_0 + \zeta_1 + \zeta_2 + \cdots + \zeta_d$$

We prove by induction that each of these homogeneous parts is a linear combination of the desired monomials, without any generators associated to composite roots. Using reverse induction, assume that $\zeta_{k+1}, \ldots, \zeta_d$ satisfy this property, meaning that they are linear combinations of elements of the form $x^-_{\alpha_n,p_1} \cdots x^-_{\alpha_n,p_{k_n}} \cdots x^-_{\alpha_1,q_1} \cdots x^-_{\alpha_1,q_{k_1}} 1_\mu$. For a simple root α_1 and composite root β, denote $\gamma := \beta - \alpha_1$ for brevity. Let s be maximal such that $f^{(s)}_\beta$ appears in ζ_k.

The defining relations of $Y(\mathfrak{sl}_{M|N})$ tell us that

$$[x^+_{i,r}, x^-_{j,s}] = \delta_{ij} h_{i,r+s}$$

where we recall that the i and j in $x^+_{i,r}, x^-_{j,s}$ correspond to the simple roots α_i, α_j. Choose some decomposition of s and write

$$[x^+_{\alpha_1,r}, x^-_{\beta,s}] = [x^+_{1,r}, [x^-_{i_1,s_1}, [x^-_{i_2,s_2}, \dots, [x^-_{i_{p-1},s_{p-1}}, x^-_{i_p,s_p}]...]]].$$

One of $\alpha_{i_1}, \dots, \alpha_{i_p}$ will be equal to α_1; without loss of generality, let $i_1 = 1$.

$$\begin{aligned}
&[x^+_{1,r},[x^-_{1,s_1}, [x^-_{i_2,s_2}, \dots, [x^-_{i_{p-1},s_{p-1}}, x^-_{i_p,s_p}]...]]] \\
&\quad =[x^-_{1,s_1}, [x^+_{1,r}, [x^-_{i_2,s_2}, \dots, [x^-_{i_{p-1},s_{p-1}}, x^-_{i_p,s_p}]...]]] \\
&\qquad + [[x^+_{1,r}, x^-_{1,s_1}], [x^-_{i_2,s_2}, \dots, [x^-_{i_{p-1},s_{p-1}}, x^-_{i_p,s_p}]...]]] \\
&\quad =[x^-_{1,s_1}, [x^+_{1,r}, [x^-_{i_2,s_2}, \dots, [x^-_{i_{p-1},s_{p-1}}, x^-_{i_p,s_p}]...]]] \\
&\qquad + [[h_{1,r+s_1}, [x^-_{i_2,s_2}, \dots, [x^-_{i_{p-1},s_{p-1}}, x^-_{i_p,s_p}]...]]] \\
&\quad =[h_{1,r+s_1}, [x^-_{i_2,s_2}, \dots, [x^-_{i_{p-1},s_{p-1}}, x^-_{i_p,s_p}]...]]
\end{aligned}$$

We use that each i_j is unique, then the first term of nested superbrackets vanishes as only $i_1 = 1$. Next, take the defining relation

$$[h_{i,r+1}, x^\pm_{j,s}] - [h_{i,r}, x^\pm_{j,s+1}] = \pm\frac{a_{ij}}{2}(h_{i,r}x^\pm_{j,s} + x^\pm_{j,s}h_{i,r}),$$

and view each side of the equation as the coefficient of $u^{-r}v^{-s}$ in the expansion of some expression involving the series $h_i(u), x_j(v)$. In particular, take the equivalent identity

$$(u-v)[h_i(u), x^\pm_j(v)] = \pm\frac{a_{ij}}{2}(h_i(u)x^\pm_j(v) + x^\pm_j(v)h_i(u)) + [h_{i,0}, x^\pm_{j,0}] - [h_i(u), x_{j,0}]$$

and rewrite as

$$[h_i(u), x_j^{\pm}(v)] = \left(\frac{1}{u-v}\right)\frac{\pm a_{ij}}{2}(h_i(u)x_j^{\pm}(v) + x_j^{\pm}(v)h_i(u)).$$

Again, use that $\frac{1}{u-v} = (u^{-1} + vu^{-2} + v^2u^{-3} + \cdots$ to expand and pick out the coefficent of $u^{-r}v^s$. This yields an alternate defining relation for $Y(\mathfrak{sl}_{M|N})$:

$$[h_{i,r}, x_{j,s}^{\pm}] = \pm\frac{a_{ij}}{2}\sum_{l=1}^{r}\left(h_{i,r-l}x_{j,s-1+l}^{\pm} + x_{j,s-1+l}^{\pm}h_{i,r-l}\right)$$

for when $r \geq 1$. Thus,

$$\begin{aligned}
&[h_{1,r+s_1}, [x_{i_2,s_2}^-, \ldots, [x_{i_{p-1},s_{p-1}}^-, x_{i_p,s_p}^-]...]] \\
=&[x_{i_2,s_2}^-, [h_{1,r+s_1}, [x_{i_3,s_3}^-, \ldots, [x_{i_{p-1},s_{p-1}}^-, x_{i_p,s_p}^-]...]]] \\
&+ [[h_{1,r+s_1}, x_{i_2,s_2}^-,], [x_{i_3,s_3}^-, \ldots, [x_{i_{p-1},s_{p-1}}^-, x_{i_p,s_p}^-]...]] \\
=&[x_{i_2,s_2}^-, [h_{1,r+s_1}, [x_{i_3,s_3}^-, \ldots, [x_{i_{p-1},s_{p-1}}^-, x_{i_p,s_p}^-]...]]] \\
&- [-\frac{a_{1,i_2}}{2}\sum_{l=1}^{r+s_1}\left(h_{1,r+s_1-l}x_{i_2,s_1-1+l}^- + x_{i_2,s_2-1+l}^-h_{1,r+s_1-l}\right), \\
&[x_{i_3,s_3}^-, \ldots, [x_{i_{p-1},s_{p-1}}^-, x_{i_p,s_p}^-]...]]
\end{aligned}$$

Ultimately, by continuing to iterate this process and using that $[h_{i,0}, x_{j,s}^{\pm}] = \pm a_{ij}x_{j,s}^{\pm}$ we will obtain some constant multiple of $x_{\beta-\alpha_1,r+s}^- = x_{\gamma,r+s}$ and a sum of elements of degree smaller than $r+s$.

As a result, the expansion of $x_{\alpha_1,r}^+\zeta = x_{\alpha_1,r}^+\zeta_1 + x_{\alpha_1,r}^+\zeta_2 + ... + x_{\alpha_1,r}^+\zeta_k + x_{\alpha_1,r}^+\zeta_{k+1} + ...x_{\alpha_1,r}^+\zeta_d$ contains a monomial of degree $r+k$ where $x_{\gamma,r+s}^-$ appears as a factor. However, by the induction hypothesis, none of the components $\zeta_{k+1}, \ldots, \zeta_d$ contain generators associated to composite roots. This means that the expansion of $x_{\alpha_1,r}^+\zeta_j$ for $j = k+1, \ldots, r$ cannot contain $x_{\gamma,r+s}^-$. On the other hand, $x_{\alpha,r}^+\zeta = 0$. If

$x^-_{\gamma,r+s}$ appears in this expansion, then there must be more appearances of $x^-_{\gamma,r+s}$ with coefficients that all together add to 0. If the degree of one of these other monomials is $r+s+a$ for some a, then such a monomial can only occur in $\zeta_{r+s+a}, \zeta_{r+s+a+1}, \zeta_{r+s+a+2}, \ldots, \zeta_d$, but this cannot be. Thus, we can conclude that if β is a positive root such that $\beta - \alpha_1$ is a root as well, then a generator of the form $x^-_{\beta,s}$ cannot appear in the expansion of ζ_k.

Further, if β is such that $\beta - \alpha_1$ is not a root but $\beta - \alpha_2$ is, then we can similarly show that $x^-_{\beta,s}$ still cannot appear in the expansion of β_k by taking the expansion of $x_{\alpha_2,r}\zeta$. We can repeat this process until we conclude that β must be a simple root. Thus, we have shown that only generators associated to simple roots occur in the expansion of ζ.

Finally, we confirm that $\mu_1(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u. To do this, we first write ζ as a sum of vectors of the form

$$x^-_{\alpha_n,p_1} \cdots x^-_{\alpha_n,p_{k_n}} \cdots x^-_{\alpha_2,s_1} \ldots x^-_{\alpha_2,s_{k_2}} \zeta'$$

where ζ' is itself a linear combination of monomials $x^-_{\alpha_1,q_1} \cdots x^-_{\alpha_1,q_{k_1}} 1_\mu$. At least one of the vectors ζ' in ζ is nonzero. Since $x^+_{\alpha_1,r}\zeta = 0$, for each ζ' it must hold that $x^+_{\alpha_1,r}\zeta' = 0$.

The elements $x^+_{\alpha_1,r}, x^-_{\alpha_1,r}$, and $h_{\alpha_1,r}$ generate a subalgebra of $Y(\mathfrak{sl}_{M|N})$ that is isomorphic to $Y(\mathfrak{sl}_{1|1})$. We see that $M(\mu_1(u))$ is thus reducible over $Y(\mathfrak{sl}_{1|1})$. By Corollary 4.8, $\mu_1(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u. We chose $i = 1$ without loss of generality; we can conclude that this holds for all $\mu_i(u)$. □

This result can be extended to apply to Verma modules over $Y(\mathfrak{gl}_{M|N})$ as well; this presentation of $Y(\mathfrak{sl}_{M|N})$ uses generating series $h_i(u)$ and $x_i^\pm(u)$ which are produced by products of the generating series of $Y(\mathfrak{gl}_{M|N})$ obtained by Gauss decomposition. Specifically,

$$\begin{aligned}
h_i(u) &= d_i(u + \frac{1}{2}(-1)^{|i|}(M-i))^{-1} d_{i+1}(u + \frac{1}{2}(-1)^{|i|}(M-i)) \\
x_i^+(u) &= f_i(u + \frac{1}{2}(-1)^{|i|}(M-i)) \\
x_i^-(u) &= (-1)^{|i|} e_i(u + \frac{1}{2}(-1)^{|i|}(M-i)).
\end{aligned}$$

Together, Propositions 4.12 and 4.13 yield the following theorem.

Theorem 4.14. ***Analogue of Theorem 1.1 of [BFM06]*** *Let $M(\mu(u))$ be a Verma module over the Yangian $Y(\mathfrak{gl}_{M|N})$, where $\mu(u) = (\mu_1(u), \dots, \mu_{M+N}(u))$ and each $\mu_i(u)$ is a formal power series in u^{-1}*

$$\mu_i(u) = (-1)^{|i|+1} + \mu_i^{(0)} u^{-1} + \mu_i^{(1)} u^{-2} + \cdots$$

for $\mu_i^{(r)} \in \mathbb{C}$. Then, $M(\mu(u))$ is reducible if and only if for some index $i \in \{1, \dots, M+N\}$ the ratio of series $\frac{\mu_i(u)}{\mu_{i+1}(u)}$ is the Laurent expansion at $u = \infty$ of a rational function in u,

$$\frac{\mu_i(u)}{\mu_{i+1}(u)} = \frac{P(u)}{Q(u)},$$

where $P(u)$ and $Q(u)$ are polynomials in u of the same degree with leading coefficients $(-1)^{|i|+1}$ and $(-1)^{|i+1|+1}$ respectively.

4.4 QUANTUM CURRENT ALGEBRA

Because of the similarities between Yangians and quantum loop algebras, we would initially expect that many of the results that held on $Y(\mathfrak{gl}_{M|N})$ would also hold for $U_q(\mathcal{L}\mathfrak{gl}_{M|N})$. However, this was not exactly the case. For instance, we cannot show that an analogue of Proposition 4.3 holds; it most likely does hold for the quantum loop superalgebra, but it cannot be shown using exactly the same arguments as for the super Yangian case. The difficulties in defining a module K were examined in Subsection 4.2.1. In Subsection 2.3.7, we defined the quantum current superalgebra of $\mathfrak{gl}_{M|N}$; a subalgebra of the quantum loop algebra generated by $T_{ij}^{(r)}, \overline{T_{ij}}^{(0)}$ over all i, j, r. Thus, its generators are subject to the following pair of defining relations:

$$\begin{aligned}
&(-1)^{|i||l|+|j||l|}(q^{-d_j\delta_{jl}}T_{ij}^{(r+1)}T_{kl}^{(s)} - q^{d_j\delta_{jl}}T_{ij}^{(r)}T_{kl}^{(s+1)}) \\
&\qquad - (-1)^{|i||k|+|j||k|}(q^{-d_i\delta_{ik}}T_{kl}^{(s)}T_{ij}^{(r+1)} - q^{d_i\delta_{ik}}T_{kl}^{(s+1)}T_{ij}^{(r)}) \\
&\qquad =(-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}T_{il}^{(r+1)}T_{kj}^{(s)} + \delta_{j>l}T_{il}^{(r)}T_{kj}^{(s+1)}) \\
&\qquad -(\delta_{i>k}T_{il}^{(s)}T_{kj}^{(r+1)} + \delta_{i<k}T_{il}^{(s+1)}T_{kj}^{(r)})\Big]
\end{aligned} \tag{4.4.1}$$

$$\begin{aligned}
&(-1)^{|i||l|+|j||l|}(q^{-d_j\delta_{jl}}T_{ij}^{(r+1)}\overline{T}_{kl}^{(0)}) - (-1)^{|i||k|+|j||k|}(q^{-d_i\delta_{ik}}\overline{T}_{kl}^{(0)}T_{ij}^{(r+1)}) \\
&\qquad =(-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}T_{il}^{(r+1)}\overline{T}_{kj}^{(0)}) - (\delta_{i>k}\overline{T}_{il}^{(0)}T_{kj}^{(r+1)})\Big].
\end{aligned} \tag{4.4.2}$$

First, we keep to the $\mathfrak{gl}_{1|1}$ case, but we'll define Verma modules over the quantum current algebra in the general case. One can define an irreducible highest weight representation on $U_q(\mathcal{L}\mathfrak{gl}_{M|N})$ as follows: the highest weight is the pair of series $(\nu(u), \overline{\nu}(u))$ with $\nu(u) = (\nu_1(u), \ldots, \nu_{M+N}(u))$ an $M+N$-tuple of formal power series in u^{-1} and $\overline{\nu}(u) = (\overline{\nu}_1(u), \ldots, \overline{\nu}_{M+N}(u))$ an $M+N$-tuple of formal power series in u. The highest weight vector is denoted $1_{\nu,\overline{\nu}}$ with $L_{ii}(u)1_{\nu,\overline{\nu}} = \nu_i(u)1_{\nu,\overline{\nu}}$ and $\overline{L}_{ii}(u)1_{\nu,\overline{\nu}} = \overline{\nu}_i(u)1_{\nu,\overline{\nu}}$ for $1 \le i \le M+N$. As well, $T_{ij}(u)1_{\nu,\overline{\nu}} = 0 = \overline{T}_{ij}(u)1_{\nu,\overline{\nu}}$ for $i < j$.

When concerned only with the quantum current algebra $U_q(\mathfrak{gl}_{M|N}[s])$, a Verma module $M(\nu(u))$ will yield $L_{ii}(u)1_\nu = \nu_i(u)1_\nu$ and $\overline{T}_{ii}^{(0)}1_\nu = (\nu_i^{(0)})^{-1}1_\nu$ for $1 \le i \le$

$M+N$. As before, $L_{ij}(u)1_\nu = 0 = \overline{L}_{ij}(u)1_\nu$ for $i < j$. If we want to define the Verma module $M(\nu(u))$ as the quotient of $U_q(\mathfrak{gl}_{1|1}[s])$ by some ideal, we let $\mathcal{J} = \left\langle L_{ii} - \nu_i(u)), T_{ii}^{(0)} - (\nu_i^{(0)})^{-1}, L_{ij}(u) \text{ and } \overline{T}_{ij}^{(0)} \text{ for } i < j \right\rangle$. Since $U_q(\mathfrak{gl}_{1|1}[s])$ is a left module over itself, any left ideal will also be a submodule. Let $M(\nu(u)) = U_q(\mathfrak{gl}_{1|1}[s])/\mathcal{J}$.

Recall that Proposition 4.3 has the criteria that $\lambda_1(u)/\lambda_2(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u, which we call $P(u)/Q(u)$ for some polynomials P, Q. We have shown that these polynomials must be monic and of same degree. The rest of the proof for $U_q(\mathfrak{gl}_{1|1}[s])$ should follow that for $Y(\mathfrak{gl}_{1|1})$ closely; though the relations governing the generators of the structures are different, they are similar enough that one can define a subspace K and show that it is a submodule.

First, we show that we can produce commutator relations for $U_q(\mathfrak{gl}_{1|1}[s])$ similar to those of $Y(\mathfrak{gl}_{1|1})$. To do this, one needs to take (4.4.1) and (4.4.2), plug in certain values for i, j, k, l, and rearrange.

We start by calculating the commutator of $T_{11}^{(s)}$ and $T_{21}^{(r)}$. One distinction between the super Yangian and the quantum loop superalgebra is the definition of the commutator bracket. For the super Yangian, we let $[A, B] = AB - (-1)^{|A||B|}BA$. However, in the quantum loop superalgebra, we use the q-commutator $[A, B]_q = AB - (-1)^{|A||B|}qBA$. The quantum current algebra inherits this.

$$\begin{aligned}
&(-1)^{|2||1|+|1||1|}(q^{-1}T_{21}^{(r+1)}T_{11}^{(s)} - qT_{21}^{(r)}T_{11}^{(s+1)}) \\
&\quad - (-1)^{|2||1|+|1||1|}(T_{11}^{(s)}T_{21}^{(r+1)} - T_{11}^{(s+1)}T_{21}^{(r)}) \\
&= (-1)^{|2||1|}(q - q^{-1})(-T_{21}^{(s)}T_{11}^{(r+1)})
\end{aligned}$$

$\iff$

$$q^{-1}T_{21}^{(r+1)}T_{11}^{(s)} - qT_{21}^{(r)}T_{11}^{(s+1)} - T_{11}^{(s)}T_{21}^{(r+1)} + T_{11}^{(s+1)}T_{21}^{(r)}$$

$$= -qT_{21}^{(s)}T_{11}^{(r+1)} + q^{-1}T_{21}^{(s)}T_{11}^{(r+1)}$$

$\iff$

$$T_{11}^{(s+1)}T_{21}^{(r)} - qT_{21}^{(r)}T_{11}^{(s+1)} - T_{11}^{(s)}T_{21}^{(r+1)} + qT_{21}^{(r+1)}T_{11}^{(s)} - qT_{21}^{(r+1)}T_{11}^{(s)} + q^{-1}T_{21}^{(r+1)}T_{11}^{(s)}$$
$$= -qT_{21}^{(s)}T_{11}^{(r+1)} + q^{-1}T_{21}^{(s)}T_{11}^{(r+1)}$$

$\iff$

$$[T_{11}^{(s+1)}, T_{21}^{(r)}]_q - [T_{11}^{(s)}, T_{21}^{(r+1)}]_q = (q-q^{-1})\left(T_{21}^{(r+1)}T_{11}^{(s)} - T_{21}^{(s)}T_{11}^{(r+1)}\right)$$

We can form a telescoping sum:

$$\begin{aligned}
[T_{11}^{(s+1)}, T_{21}^{(r)}]_q - [T_{11}^{(s)}, T_{21}^{(r+1)}]_q &= (q-q^{-1})\left(T_{21}^{(r+1)}T_{11}^{(s)} - T_{21}^{(s)}T_{11}^{(r+1)}\right)\\
[T_{11}^{(s)}, T_{21}^{(r+1)}]_q - [T_{11}^{(s-1)}, T_{21}^{(r+2)}]_q &= (q-q^{-1})\left(T_{21}^{(r+2)}T_{11}^{(s-1)} - T_{21}^{(s-1)}T_{11}^{(r+2)}\right)\\
[T_{11}^{(s-1)}, T_{21}^{(r+2)}]_q - [T_{11}^{(s-2)}, T_{21}^{(r+3)}]_q &= (q-q^{-1})\left(T_{21}^{(r+3)}T_{11}^{(s-2)} - T_{21}^{(s-2)}T_{11}^{(r+3)}\right)\\
&\cdots\cdots\\
[T_{11}^{(1)}, T_{21}^{(r+s)}]_q - [T_{11}^{(0)}, T_{21}^{(r+s+1)}]_q &= (q-q^{-1})\left(T_{21}^{(r+s+1)}T_{11}^{(0)} - T_{21}^{(0)}T_{11}^{(r+s+1)}\right)\\
[T_{11}^{(0)}, T_{21}^{(r+s+1)}]_q - 0 &= 0\\
[T_{11}^{(s+1)}, T_{21}^{(r)}]_q &= (q-q^{-1})\sum_{a=0}^{\min(r,s)}\left(T_{21}^{(r+1+a)}T_{11}^{(s-a)} - T_{21}^{(s-a)}T_{11}^{(r+1+a)}\right)\\
[T_{11}^{(s)}, T_{21}^{(r)}]_q &= (q-q^{-1})\sum_{a=0}^{\min(r,s)-1}\left(T_{21}^{(r+s-a)}T_{11}^{(a)} - T_{21}^{(a)}T_{11}^{(r+s-a)}\right)
\end{aligned}$$

This is very similar to the version of this relation for the Yangian, other than no decrease to the sum of superscripts. Thus, we will be able to show that the action of $T_{11}^{(r)}$ on a monomial $T_{21}^{(r_1)}\cdots T_{21}^{(r_k)}1_\lambda$ produces another element in K. The relation we get for $[T_{22}^{(s)}, T_{21}^{(r)}]_q$ will be similar. Finally, we produce the commutator relation for $T_{12}^{(s)}$ and $T_{21}^{(r)}$:

$$(-1)^{|2||2|+|1||2|}(q^{-d_1\delta_{12}}T_{21}^{(r+1)}T_{12}^{(s)} - q^{d_1\delta_{12}}T_{21}^{(r)}T_{12}^{(s+1)})$$

$$
\begin{aligned}
&-(-1)^{|2||1|+|1||1|}(q^{-d_2\delta_{21}}T_{12}^{(s)}T_{21}^{(r+1)} - q^{d_2\delta_{21}}T_{12}^{(s+1)}T_{21}^{(r)}) \\
=&(-1)^{|2||1|}(q-q^{-1})\left[(\delta_{1<2}T_{22}^{(r+1)}T_{11}^{(s)} + \delta_{1>2}T_{22}^{(r)}T_{11}^{(s+1)})\right. \\
&\left.-(\delta_{2>1}T_{22}^{(s)}T_{11}^{(r+1)} + \delta_{2<1}T_{22}^{(s+1)}T_{11}^{(r)})\right]
\end{aligned}
$$

$$\iff$$

$$
\begin{aligned}
&-(T_{21}^{(r+1)}T_{12}^{(s)} - T_{21}^{(r)}T_{12}^{(s+1)}) - (T_{12}^{(s)}T_{21}^{(r+1)} - T_{12}^{(s+1)}T_{21}^{(r)}) \\
&= (q-q^{-1})\left(T_{22}^{(r+1)}T_{11}^{(s)} - T_{22}^{(s)}T_{11}^{(r+1)}\right)
\end{aligned}
$$

$$\iff$$

$$
\begin{aligned}
&-T_{12}^{(s)}T_{21}^{(r+1)} - T_{21}^{(r+1)}T_{12}^{(s)} + T_{12}^{(s+1)}T_{21}^{(r)} + T_{21}^{(r)}T_{12}^{(s+1)} \\
&= (q-q^{-1})\left(T_{22}^{(r+1)}T_{11}^{(s)} - T_{22}^{(s)}T_{11}^{(r+1)}\right)
\end{aligned}
$$

$$\iff$$

$$[T_{12}^{(s+1)}, T_{21}^{(r)}] - [T_{12}^{(s)}, T_{21}^{(r+1)}] = (q-q^{-1})\left(T_{22}^{(r+1)}T_{11}^{(s)} - T_{22}^{(s)}T_{11}^{(r+1)}\right)$$

Using the usual commutator instead of the q-commutator simplifies notation. Again, we can produce a telescoping sum to determine a formula for a single commutator bracket. This yields

$$[T_{12}^{(s)}, T_{21}^{(r)}] = (q-q^{-1})\sum_{a=0}^{\min(r,s)-1}\left(T_{22}^{(r+s-a)}T_{11}^{(a)} - T_{22}^{(a)}T_{11}^{(r+s-a)}\right)$$

When looking at the action of this commutator on the highest weight vector, we get

$$[T_{12}^{(s)}, T_{21}^{(r)}]1_\nu = (q-q^{-1})\sum_{a=0}^{\min(r,s)-1}\left(\nu_2^{(r+s-a)}\nu_1^{(a)} - \nu_2^{(a)}\nu_1^{(r+s-a)}\right)1_\nu.$$

Our goal is to show that the Verma module contains a nontrivial submodule K, which we define as the subspace spanned by elements of the form $T_{21}^{(r_1)} \cdots T_{21}^{(r_k)}$, where there is an index i with $r_i > d = \deg(\nu_1(u)/\nu_2(u))$.

If we prove that $T_{12}^{(s)}K \subset K$ by induction, then we would use the calculations above for the base case. Assuming $r > d = \deg P(u) = \deg Q(u)$, $r + s - a > d$ for any $a = 0, \ldots, \min(r,s) - 1$, so $\nu_2^{(r+s-a)} = 0$ and $\nu_1^{(r+s-a)} = 0$. Thus, $[T_{12}^{(s)}, T_{21}^{(r)}]1_\nu = 0$. Now, suppose we have the basis element $T_{21}^{(r_1)}T_{21}^{(r_2)}\cdots T_{21}^{(r_k)} \in K$ with $r_1 < r_2 < \cdots r_k$ and $r_i > d$ for some i. Then, we expand $T_{12}^{(s)}T_{21}^{(r_1)}T_{21}^{(r_2)}\cdots T_{21}^{(r_k)}$. Use that $T_{12}^{(s)}T_{21}^{(r_i)} = [T_{12}^{(s)}, T_{21}^{(r_i)}] + (-1)^{(|1|+|2|)(|2|+|1|)}T_{21}^{(r_i)}T_{12}^{(s)} = [T_{12}^{(s)}, T_{21}^{(r_i)}] - T_{21}^{(r_i)}T_{12}^{(s)}$.

$$
\begin{aligned}
&T_{12}^{(s)}T_{21}^{(r_1)}\cdots T_{21}^{(r_k)}1_\nu = [T_{12}^{(s)}, T_{21}^{(r_1)}]T_{21}^{(r_2)}\cdots T_{21}^{(r_k)}1_\nu - T_{21}^{(r_1)}T_{12}^{(s)}T_{21}^{(r_2)}\cdots T_{21}^{(r_k)}1_\nu \\
=&(q-q^{-1})\left(\sum_{a=0}^{\min(r_1,s)-1}\left(T_{22}^{(r_1+s-a)}T_{11}^{(a)} - T_{22}^{(a)}T_{11}^{(r_1+s-a)}\right)\right)T_{21}^{(r_2)}\cdots T_{21}^{(r_k)}1_\nu \\
&- T_{21}^{(r_1)}T_{12}^{(s)}T_{21}^{(r_2)}\cdots T_{21}^{(r_k)}1_\nu \\
=&(q-q^{-1})\left(\sum_{a=0}^{\min(r_1,s)-1}\left(T_{22}^{(r_1+s-a)}T_{11}^{(a)} - T_{22}^{(a)}T_{11}^{(r_1+s-a)}\right)\right)T_{21}^{(r_2)}\cdots T_{21}^{(r_k)}1_\nu \\
&- T_{21}^{(r_1)}[T_{12}^{(s)}, T_{21}^{(r_2)}]T_{21}^{(r_3)}\cdots T_{21}^{(r_k)}1_\nu + T_{21}^{(r_1)}T_{21}^{(r_2)}T_{12}^{(s)}T_{21}^{(r_3)}\cdots T_{21}^{(r_k)}1_\nu \\
=&(q-q^{-1})\left(\sum_{a=0}^{\min(r_1,s)-1}\left(T_{22}^{(r_1+s-a)}T_{11}^{(a)} - T_{22}^{(a)}T_{11}^{(r_1+s-a)}\right)\right)T_{21}^{(r_2)}\cdots T_{21}^{(r_k)}1_\nu \\
&- (q-q^{-1})T_{21}^{(r_1)}\left(\sum_{a=0}^{\min(r_2,s)-1}\left(T_{22}^{(r_2+s-a)}T_{11}^{(a)} - T_{22}^{(a)}T_{11}^{(r_2+s-a)}\right)\right)T_{21}^{(r_3)}\cdots T_{21}^{(r_k)}1_\nu \\
&+ T_{21}^{(r_1)}T_{21}^{(r_2)}[T_{12}^{(s)}, T_{21}^{(r_3)}]T_{21}^{(r_4)}\cdots T_{21}^{(r_k)}1_\nu - T_{21}^{(r_1)}T_{21}^{(r_2)}T_{21}^{(r_3)}T_{12}^{(s)}T_{21}^{(r_4)}\cdots T_{21}^{(r_k)}1_\nu \\
=&\sum_{i=1}^{k}(-1)^{i-1}(q-q^{-1})T_{21}^{(r_1)}\cdots T_{21}^{(r_{i-1})}\left(\sum_{a_i=0}^{\min(r_i,s)-1}\left(T_{22}^{(r_i+s-a_i)}T_{11}^{(a_i)} - T_{22}^{(a_i)}T_{11}^{(r_i+s-a_i)}\right)\right) \\
&T_{21}^{(r_{i+1})}\cdots T_{21}^{(r_k)}1_\nu
\end{aligned}
$$

After applying commutator relations repeatedly to move the $T_{22}^{(r_i+s-a_i)}T_{11}^{(a_i)}$ and $T_{22}^{(a_i)}T_{11}^{(r_i+s-a_i)}$ to the right of the product, we'll obtain basis elements of K with coefficients in $(q - q^{-1})$ and the $\nu^{(i)}$. How do we know these monomials in the $T_{21}^{(r)}$ to be in K? First, if i is the first index such that $r_i > d$, then all of $r_{i+1}, \ldots r_k$ are greater than d, so the missing $T_{21}^{(r_i)}$ does not mean the element no longer lies in K. The only potential problem would be when $r_{k-1} \leq d$ and $r_k > d$. Then, the term of the expansion we are concerned with is

$$(-1)^{k-1}(q-q^{-1})T_{21}^{(r_1)}\cdots T_{21}^{(r_{k-1})}\left(\sum_{a_k=0}^{\min(r_k,s)-1}\left(T_{22}^{(r_k+s-a_k)}T_{11}^{(a_k)}-T_{22}^{(a_k)}T_{11}^{(r_k+s-a_k)}\right)\right)1_\nu.$$

$r_k > d$, so $r_k + s - a_k > d$. Since we have assumed both $\nu_1(u)$ and $\nu_2(u)$ to be polynomials of degree d, $T_{22}^{(r_k+s-a_k)}1_\nu = 0 = T_{11}^{(r_k+s-a_k)}1_\nu$. Thus, this term disappears.

As a result, the proof of Proposition 4.3 and related lemmas will proceed almost exactly as before, other than insertions of $(q-q^{-1})$ in coefficients. Further, when necessary we use the q-commutator with $AB = [A,B] + (-1)^{|A||B|}qBA$, adding in q, whereas we used the superbracket in $Y(\mathfrak{gl}_{1|1})$ with $AB = [A,B] + (-1)^{|A||B|}BA$. Thus, we can conclude that if the quotient of weights is rational, then the Verma module is reducible.

The next result on $Y(\mathfrak{gl}_{1|1})$ that can be shown to hold on $U_q(\mathfrak{gl}_{1|1}[s])$ is Proposition 4.7, the converse statement to Proposition 4.3. Namely, we can show that if $M(\nu_1(u), \nu_2(u))$ is a reducible Verma module over $U_q(\mathfrak{gl}_{1|1}[s])$, then $\nu_1(u)/\nu_2(u)$ is a rational function. This uses Lemma 4.6, which is a general result on when a power series is the expansion of a rational function and needs no adjustment.

Proposition 4.15. *Let $M(\nu_1(u), \nu_2(u))$ be a Verma module over $U_q(\mathfrak{gl}_{1|1}[s])$. If it is reducible, then $\nu_1(u)/\nu_2(u)$ is a rational function.*

Proof. A full proof will not be presented here. Instead, what follows is an outline of the major steps of the proof and what changes in the quantum current case.

First, we use that $M(\nu_1(u), \nu_2(u)) \simeq M(1, \nu(u))$ for $\nu(u) = \frac{\nu_2(u)}{\nu_1(u)}$. This module has the weight space decomposition

$$M(1,\nu(u)) = \bigoplus_{\eta} M(1,\nu(u))_\eta$$

where the weight spaces are $M(1,\nu(u))_\eta = \{X \in M(1,\nu(u)) | T_{11}^{(0)} T_{22}^{(0)} X = \eta X\}$. A weight space is nonzero when $\eta = \nu^{(0)} \cdot q^{2k}$ for some integer k. We can prove that these are the only nonzero weight spaces by induction.

First, note that $T_{11}^{(0)} T_{21}^{(r)} = q T_{21}^{(r)} T_{11}^{(0)}$ and $T_{22}^{(0)} T_{21}^{(r)} = q T_{21}^{(r_1)} T_{22}^{(0)}$. This means that $T_{11}^{(0)} T_{22}^{(0)} T_{21}^{(r)} = q^2 T_{21}^{(r)} T_{11}^{(0)} T_{22}^{(0)}$, so $T_{11}^{(0)} T_{22}^{(0)} T_{21}^{(r)} 1_\nu = q^2 \nu^{(0)} T_{21}^{(r)} 1_\nu$. In general,

$$T_{11}^{(0)} T_{22}^{(0)} T_{21}^{(r_1)} \cdots T_{21}^{(r_k)} 1_\nu = q^{2k} \nu^{(0)} T_{21}^{(r_1)} \cdots T_{21}^{(r_k)} 1_\nu$$

Then, for some fixed k, the basis of $M(1,\nu(u))_{\nu^{(0)} q^{2k}}$ is made up of monomials of the form $T_{21}^{(r_1)} T_{21}^{(r_2)} \cdots T_{21}^{(r_k)} 1_\nu$ where $1 \leq r_1 < \cdots < r_k$. We assume that $M(1,\nu(u))$ is reducible. Call a nontrivial submodule K. K inherits the weight space decomposition of the Verma module:

$$K = \bigoplus_\eta K_\eta, \quad K_\eta = K \cap M(1,\nu(u))_\eta.$$

Choose k such that $\nu^{(0)} q^{2k} := \eta$ is the minimal weight with nonzero weight space. We can compare weights in terms of the power of q; this requires that q is not a root of unity. We say that $\nu^{(0)} q^{2k}$ is a greater weight than $\nu^{(0)} q^{2l}$ if $k > l$. We can then deduce that for any nonzero vector $\zeta \in K_\eta$, we must have $T_{12}(u)\zeta = 0$. Else, for at least one $r \geq 1$, $T_{12}^{(r)} \zeta \neq 0$. However, this would make $T_{12}^{(r)} \zeta \in K_{\eta \times q^{-2}}$, which contradicts the minimality of η. Since $\zeta \in K_\eta = K \cap M(1,\nu(u))_\eta$, we can write

$$\zeta = \sum_{\mathbf{r}} c_{\mathbf{r}} T_{21}^{(r_1)} \cdots T_{21}^{(r_k)}, \tag{4.4.3}$$

where the sum runs over finitely many k-tuples $\mathbf{r} = (r_1, \ldots, r_k)$ satisfying $1 \leq r_1 < r_2 < \cdots < r_k$. The coefficients $c_{\mathbf{r}}$ lie in $\mathbb{C}$.

Next, we look more closely at the equation $T_{12}^{(r)}\zeta = 0$ to see what this says about the coefficients $c_{\mathbf{r}}$. Ultimately, we want to be able to apply Lemma 4.6 to $\nu(u)$ to prove that it is the Laurent expansion at $u = \infty$ of a rational function in u.

First, we need to determine the action of $T_{11}^{(r)}, T_{12}^{(r)}$, and $T_{22}^{(r)}$ on a monomial $T_{21}^{(r_1)} \cdots T_{21}^{(r_k)}$. This has been done earlier in this section.

$$\begin{aligned}
&T_{12}^{(s)} T_{21}^{(r_1)} \cdots T_{21}^{(r_k)} 1_\nu \\
&= \sum_{i=1}^{k} (-1)^{i-1}(q - q^{-1}) T_{21}^{(r_1)} \cdots T_{21}^{(r_{i-1})} \\
&\left(\sum_{a_i=0}^{\min(r_i,s)-1} \left(T_{22}^{(r_i+s-a_i)} T_{11}^{(a_i)} - T_{22}^{(a_i)} T_{11}^{(r_i+s-a_i)} \right) \right) T_{21}^{(r_{i+1})} \cdots T_{21}^{(r_k)} 1_\nu
\end{aligned}$$

We then determine $T_{11}^{(r)} T_{21}^{(r_1)} \cdots T_{21}^{(r_k)} 1_\nu$ and $T_{22}^{(r)} T_{21}^{(r_1)} \cdots T_{21}^{(r_k)}$, which follows directly from earlier computations. Most importantly, as in the $Y(\mathfrak{gl}_{M|N})$ case, after iterating the above processes to eliminate $T_{11}^{(r)}$ and $T_{22}^{(r)}$, generators, we conclude that $T_{12}^{(r)} T_{21}^{(r_1)} \cdots T_{21}^{(r_k)} 1_\nu$ is a linear combination of basis monomials of $k-1$ generators, $T_{21}^{(s_1)} \cdots T_{21}^{(s_{k-1})} 1_\nu$. We will be picking out the coefficient of this polynomial.

The rest of the proof is identical to the $Y(\mathfrak{gl}_{M|N})$ case, with one exception; because the commutator on $U_q(\mathfrak{gl}_{M|N}[s])$ does not reduce the sum of superscripts unlike the Yangian which sees a decrease of 1 per iteration of the commutator relation, the coefficient for $T_{21}^{(s_1)} \cdots T_{21}^{(s_{k-1})} 1_\nu$ will be a linear combination of $\nu^{(r)}, \ldots, \nu^{(r+N-k)}, \ldots, \nu^{(r+N)}$ with the addition of powers of q. In contrast, the $Y(\mathfrak{gl}_{M|N})$ case only saw a linear combination of $\nu^{(r)}, \ldots, \nu^{(r+N-k)}$. □

The following corollary follows from the previous propositions.

Corollary 4.16. *The Verma module $M(\nu(u))$ over $U_q(\mathfrak{sl}_{1|1}[s])$ is reducible if and only if $\nu(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u.*

Likewise, our results on $U_q(\mathfrak{gl}_{1|1}[s])$ lead to this powerful theorem on reducibility of Verma module, a direct analogue of the result on $Y(\mathfrak{gl}_{M|N})$.

Theorem 4.17. *The Verma module $M(\nu_1(u), \nu_2(u))$ over $U_q(\mathfrak{gl}_{1|1}[s])$ is reducible if and only if the unique irreducible quotient $L(\nu_1(u), \nu_2(u))$ is finite dimensional.*

Proof. This applies for the same reason that it does for the Yangian. Because $(T_{21}^{(r)})^2 = 0$ for all r, as it is an odd generator, the superscripts in the basis monomials $T_{21}^{(r_1)} \cdot T_{21}^{(r_k)}$ must be distinct for the monomial to be nonzero. This means that any monomial of d or more generators, where d is the degree of each polynomial in the ratio between weights, must lie in K. As a result, the quotient of the Verma module by K is spanned by monomials with at most $d-1$ generators; thus, this quotient is finite dimensional. If it is irreducible, then we are done; else, the irreducible quotient is contained within and must be finite dimensional, so we are done. $\square$

As was true in the Yangian case, once we have proven results for the $\mathfrak{gl}_{1|1}$ setting we can extend to the general $\mathfrak{gl}_{M|N}$ setting. This is because the more general structure contains copies of the $\mathfrak{gl}_{1|1}$ structure through Gauss decomposition. These e_i, f_i, d_i, and d_i' generators (as we called them for $Y(\mathfrak{gl}_{M|N})$ have the property that, if $i \neq j$, $e_i^{(r)}$ and $f_j^{(s)}$ commute. This property is all we need to prove that if one series $\mu_i(u)$ of the highest weight $\mu(u) = (\mu_1(u), \mu_2(u), \ldots, \mu_{M+N-1}(u))$, then the Verma module $M(\mu(u))$ is reducible. Of course, this requires that we can produce these alternative generators through Gauss decomposition, which has been done by Cai, Wang,and Wu [Cai98] for $U_q(\mathcal{L}\mathfrak{gl}_{1|1})$.

Finally, supposing $M(\mu(u))$ is a reducible Verma module over $U_q(\mathfrak{sl}_{M|N}[s])$, we can show that there is an index $i \in \{1, 2, \ldots, M+N\}$ such that $\mu_i(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u. This is the converse of the previous result and the equivalent of Proposition 4.13 on $Y(\mathfrak{sl}_{M|N})$. To do this, we need a PBW basis for $U_q(\mathfrak{gl}_{M|N}[s])$, which we have by Zhang [Zha14a]. The other key component we need is a set of generators for $U_q(\mathfrak{gl}_{M|N}[s])$ satisfying certain properties. Suppose these generators are $X^+_{\alpha,r}, X^-_{\alpha,r}, H_{\alpha,r}$ over positive roots α and $r \in \mathbb{Z}$. Then, each triple $< X^+_{\alpha,r}, X^-_{\alpha,r}, H_{\alpha,r} >$ must generate a subalgebra of $U_q(\mathfrak{gl}_{M|N}[s])$ that is isomorphic to $U_q(\mathfrak{sl}_{1|1}[s])$. Again, Zhang provides such a representation, derived by the work of Yamane [Yam99] on the relations of affine Lie superalgebras and of affine quantized universal enveloping superalgebras. All this leads to the following theorem:

Theorem 4.18. *Let $M(\mu(u))$ be a Verma module over $U_q(\mathfrak{gl}_{M|N}[s])$. Then, $M(\mu(u))$ is reducible if and only if, for some index $i \in \{1, \dots M+N\}$ the ratio of series $\frac{\mu_i(u)}{\mu_{i+1}(u)}$ is the Laurent expansion at $u = \infty$ of a rational function in u, i.e.*

$$\frac{\mu_i(u)}{\mu_{i+1}(u)} = \frac{P(u)}{Q(u)},$$

where $P(u)$ and $Q(u)$ are monic polynomials in u of the same degree.

5

DEGENERATION IN THE $\mathfrak{gl}_{M|N}$ CASE

In this chapter, we will show that the Yangian of $\mathfrak{gl}_{M|N}$ can be produced by a degeneration process from the corresponding quantum loop algebra. This mimics the process for the analogous $\mathfrak{gl}_N$ case in [CG15] and the $\mathfrak{q}_N$ case that will be done in Chapter 6. The goal is to build up a set of results, using new bases and manipulated defining relations, such that the map in Theorem 5.4 is an isomorphism. The details of these proofs are similar enough that we will omit them in this chapter, while preserving them in the $\mathfrak{q}_N$ case.

Recall that we define $\mathcal{L}\mathfrak{gl}_{M|N} = \mathfrak{gl}_{M|N} \otimes \mathbb{C}[s, s^{-1}] = \bigoplus_{1 \le i,j \le M+N} E_{ij} \otimes \mathbb{C}[s, s^{-1}]$ for some indeterminate s.

5.1 DEFINING A HOMOMORPHISM

First, take the localization of $\mathbb{C}[q, q^{-1}]$ at the ideal $(q-1)$. Call it $\mathcal{A}$. Further, define the elements $\tau_{ij}^{(r)}, \overline{\tau_{ij}}^{(r)} \in U_q(\mathcal{L}(\mathfrak{gl}_{M|N})$ by

$$\tau_{ij}^{(r)} = \frac{T_{ij}^{(r)}}{q - q^{-1}}, \quad \overline{\tau}_{ij}^{(r)} = \frac{\overline{T}_{ij}^{(r)}}{q - q^{-1}}$$

for $r \geq 1, 1 \leq i, j \leq M+N$ unless $r = 0$ and $i = j$. In that case, define

$$\tau_{ii}^{(0)} = \frac{T_{ii}^{(0)} - 1}{q-1}, \quad \overline{\tau}_{ii}^{(0)} = \frac{\overline{T}_{ii}^{(0)} - 1}{q-1}.$$

Let $U_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_{M|N}))$ denote the $\mathcal{A}$-subalgebra of $U_q(\mathcal{L}(\mathfrak{gl}_{M|N})$ generated by the generators $\tau_{ij}^{(r)}, \overline{\tau}_{ij}^{(r)}$ for $r \geq 0$, $1 \leq i, j \leq M+N$. Consider this version of the sequence (3.1.1) of algebra homomorphisms, tailored to our setting:

$$\begin{aligned} U_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_{M|N})) \twoheadrightarrow U_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_{M|N}))/(q-1)U_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_{M|N})) \\ \to U(\mathcal{L}(\mathfrak{gl}_{M|N})) \twoheadrightarrow U(\mathfrak{gl}_{M|N}) \end{aligned}$$

where the last map $U(\mathcal{L}(\mathfrak{gl}_{M|N})) \twoheadrightarrow U(\mathfrak{gl}_{M|N})$ is defined by sending $s \mapsto 1$. Denote by ψ the composition of maps

$$U_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_{M|N})) \twoheadrightarrow U_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_{M|N}))/(q-1)U_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_{M|N})) \to U(\mathcal{L}(\mathfrak{gl}_{M|N})).$$

This map sends $\tau_{ij}^{(r)} \mapsto (-1)^{|i|} E_{ij} s^r$ and $\overline{\tau}_{ij}^{(r)} \mapsto -(-1)^{|i|} E_{ij} s^{-r}$.

Next, consider the subspace U of $U_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_{M|N})$ defined to be the span over $\mathbb{C}$ of $\tau_{ij}^{(r)}, \overline{\tau_{ij}^{(r)}}$ for all r. We also define the ideal K_m for $m \geq 0$ in $\mathcal{L}\mathfrak{gl}_{M|N}$ to be the span of elements of the type $X \otimes s^r(s-1)^m$, where $X \in \mathfrak{gl}_{M|N}$ and $r \in \mathbb{Z}$. We denote by $\mathbb{K}_m$ the two-sided ideal of $U_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_{M|N})$ generated by $\psi^{-1}(\mathsf{K}_m) \cap U$.

Finally, we define a third set of generators for $U(\mathcal{L}\mathfrak{gl}_{M|N})$. Set $T_{ij}^{(r,0)} = \tau_{ij}^{(r)}$ and define, recursively, $T_{ij}^{(r,m)} = T_{ij}^{(r+1,m-1)} - T_{ij}^{(r,m-1)}$. These will be the generators we will work with for the rest of this chapter. By induction, one can show that $\psi(T_{ij}^{(r,m)}) = (-1)^{|i|} E_{ij} s^r (s-1)^m$; one can start with $T_{ij}^{(r,1)} = T_{ij}^{(r+1)} - T_{ij}^{(r,0)}$ to see that $\psi(T_{ij}^{(r,1)}) = (-1)^{|i|} E_{ij} s^{r+1} - (-1)^{|i|} E_{ij} s^r = (-1)^{|i|} E_{ij} s^r (s-1)^1$.

Now, define $\mathbf{K}_m$ as the sum of the ideals $(q-q^{-1})^{m_0}\mathbb{K}_{m_1}\mathbb{K}_{m_2}\cdots\mathbb{K}_{m_k}$ with $\sum_{a=0}^{k} m_a \geq m$; i.e. it contains all elements of the form $(q-q^{-1})^{m_0}T_{i_1,j_1}^{(r_1,m_1)}\cdots T_{i_k,j_k}^{(r_k,m_k)}$ with $\sum_{a=0}^{k} m_a \geq m$. This will work naturally from the assignment $T_{ij}^{(r,m)} \in \mathbf{K}_m$. For example, $\mathbf{K}_0 = U_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_{M|N})$. Note that $\mathbf{K}_1$ contains $\varepsilon = q-q^{-1}$, sits inside of $\mathbf{K}_0$, and contains $(q-q^{-1})U_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_{M|N}) =: \mathcal{I}$. As a result, $\mathbf{K}_0/\mathbf{K}_1 \simeq \frac{\mathbf{K_0}/\mathcal{I}}{\mathbf{K_1}/\mathcal{I}}$ where $\mathbf{K_0}/\mathcal{I} \simeq U_{\mathbb{C}}(\mathcal{L}\mathfrak{gl}_{M|N})$ and $\mathbf{K_1}/\mathcal{I}$ is the ideal of $U_{\mathbb{C}}(\mathcal{L}\mathfrak{gl}_{M|N})$ that is the kernel of the map from $U(\mathcal{L}\mathfrak{gl}_{M|N})$ to $U(\mathfrak{gl}_{M|N})$ sending s to 1. Hence, $\mathbf{K}_0/\mathbf{K}_1 \simeq U(\mathfrak{gl}_{M|N})$.

Our ultimate goal is to produce an isomorphism

$$\varphi : Y(\mathfrak{gl}_{M|N}) \to \bigoplus_{m=0}^{\infty} \mathbf{K}_m/\mathbf{K}_{m+1},$$

so first we must ensure that this map respects the properties of the quotient spaces. Next, we use an existing relation on $U_q(\mathcal{L}\mathfrak{gl}_{M|N})$ to determine a natural choice for the image of a generator of the Yangian under φ. We will choose $\varphi(t_{ij}^{(m)}) = T_{ij}^{(1,m)}$, though in actuality one could choose $\varphi(t_{ij}^{(m)}) = T_{ij}^{(r,m)}$ and fix any $r \geq 0$.

Theorem 5.1. *The map $\varphi : Y(\mathfrak{gl}_{M|N}) \to \bigoplus_{m\geq 0} \mathbf{K}_m/\mathbf{K}_{m+1}$ is a homomorphism.*

Proof. We can rewrite (2.3.71) as

$$\begin{aligned}
&(-1)^{|i||l|+|j||l|}\left[q^{-d_j\delta_{jl}}(T_{ij}^{(r+1)} - T_{ij}^{(r)})T_{kl}^{(s)} - T_{ij}^{(r)}(q^{d_j\delta_{jl}}T_{kl}^{(s+1)} - q^{-d_j\delta_{jl}}T_{kl}^{(s)})\right] \\
&- (-1)^{|i||k|+|j||k|}\left[q^{-d_i\delta_{ik}}T_{kl}^{(s)}(T_{ij}^{(r+1)} - T_{ij}^{(r)}) - (q^{d_i\delta_{ik}}T_{kl}^{(s+1)} - q^{-d_i\delta_{ik}}T_{kl}^{(s)})T_{ij}^{(r)}\right] \\
=&(-1)^{|i||j|}(q-q^{-1})\left[(\delta_{j<l}T_{il}^{(r+1)}T_{kj}^{(s)} + \delta_{j>l}T_{il}^{(r)}T_{kj}^{(s+1)})\right. \\
&\left.-(\delta_{i>k}T_{il}^{(s)}T_{kj}^{(r+1)} + \delta_{i<k}T_{il}^{(s+1)}T_{kj}^{(r)})\right].
\end{aligned}$$

If we assume that $r, s \geq 1$, then for all generators $T_{ab}^{(p)}$ above, we can write $\tau_{ab}^{(p)} = \frac{T_{ab}^{(p)}}{q-q^{-1}}$. Then, using the recursively defined elements $T_{ab}^{(p,q)}$ we can rewrite our relation in terms of this second set of elements. For instance,

$$\begin{aligned}
\frac{T_{ij}^{(r+1)} - T_{ij}^{(r)}}{q-q^{-1}} &= \tau_{ij}^{(r+1)} - \tau_{ij}^{(r)} \\
&= T_{ij}^{(r+1,0)} - T_{ij}^{(r,0)} \\
&= T_{ij}^{(r,1)}.
\end{aligned}$$

Thus, dividing both sides of the equation by $\varepsilon^2 = (q-q^{-1})^2$ produces:

$$\begin{aligned}
& (-1)^{|i||l|+|j||l|} \left[q^{-d_j\delta_{jl}} (\tau_{ij}^{(r+1)} - \tau_{ij}^{(r)}) \tau_{kl}^{(s)} - \tau_{ij}^{(r)} (q^{d_j\delta_{jl}} \tau_{kl}^{(s+1)} - q^{-d_j\delta_{jl}} \tau_{kl}^{(s)}) \right] \\
& -(-1)^{|i||k|+|j||k|} \left[q^{-d_i\delta_{ik}} \tau_{kl}^{(s)} (\tau_{ij}^{(r+1)} - \tau_{ij}^{(r)}) - (q^{d_i\delta_{ik}} \tau_{kl}^{(s+1)} - q^{-d_i\delta_{ik}} \tau_{kl}^{(s)}) \tau_{ij}^{(r)} \right] \\
= \quad & (-1)^{|i||j|} \varepsilon \left[(\delta_{j<l} \tau_{il}^{(r+1)} \tau_{kj}^{(s)} + \delta_{j>l} \tau_{il}^{(r)} \tau_{kj}^{(s+1)}) \right. \\
& \left. -(\delta_{i>k} \tau_{il}^{(s)} \tau_{kj}^{(r+1)} + \delta_{i<k} \tau_{il}^{(s+1)} \tau_{kj}^{(r)}) \right] \\
& \iff \\
& (-1)^{|i||l|+|j||l|} \left[q^{-d_j\delta_{jl}} T_{ij}^{(r,1)} T_{kl}^{(s,0)} - T_{ij}^{(r,0)} (q^{d_j\delta_{jl}} T_{kl}^{(s+1,0)} - q^{-d_j\delta_{jl}} T_{kl}^{(s,0)}) \right] \\
& -(-1)^{|i||k|+|j||k|} \left[q^{-d_i\delta_{ik}} T_{kl}^{(s,0)} T_{ij}^{(r,1)} - (q^{d_i\delta_{ik}} T_{kl}^{(s+1,0)} - q^{-d_i\delta_{ik}} T_{kl}^{(s,0)}) T_{ij}^{(r,0)} \right] \\
= \quad & (-1)^{|i||j|} \varepsilon \left[(\delta_{j<l} T_{il}^{(r+1,0)} T_{kj}^{(s,0)} + \delta_{j>l} T_{il}^{(r,0)} T_{kj}^{(s+1,0)}) \right. \\
& \left. -(\delta_{i>k} T_{il}^{(s,0)} T_{kj}^{(r+1,0)} + \delta_{i<k} T_{il}^{(s+1,0)} T_{kj}^{(r,0)}) \right]. \qquad (5.1.1)
\end{aligned}$$

Now, note the following:

$$\begin{aligned}
& q^{-d_i\delta_{ik}} T_{kl}^{(s,0)} T_{ij}^{(r,1)} - q^{d_i\delta_{ik}} T_{kl}^{(s+1,0)} T_{ij}^{(r,0)} + q^{-d_i\delta_{ik}} T_{kl}^{(s,0)} T_{ij}^{(r,0)} \qquad (5.1.2) \\
= \quad & q^{-d_i\delta_{ik}} T_{kl}^{(s,0)} T_{ij}^{(r,1)} - q^{d_i\delta_{ik}} T_{kl}^{(s+1,0)} T_{ij}^{(r,0)} + q^{-d_i\delta_{ik}} T_{kl}^{(s,0)} T_{ij}^{(r,0)} \\
& -q^{-d_i\delta_{ik}} T_{kl}^{(s,1)} T_{ij}^{(r,0)} + q^{-d_i\delta_{ik}} T_{kl}^{(s,1)} T_{ij}^{(r,0)} \\
= \quad & q^{-d_i\delta_{ik}} \left(T_{kl}^{(s,0)} T_{ij}^{(r,1)} - T_{kl}^{(s,1)} T_{ij}^{(r,0)} \right) - q^{d_i\delta_{ik}} T_{kl}^{(s+1,0)} T_{ij}^{(r,0)} + q^{-d_i\delta_{ik}} T_{kl}^{(s,0)} T_{ij}^{(r,0)}
\end{aligned}$$

$$
\begin{aligned}
&+q^{-d_i\delta_{ik}}T_{kl}^{(s+1,0)}T_{ij}^{(r,0)} - q^{-d_i\delta_{ik}}T_{kl}^{(s,0)}T_{ij}^{(r,0)} \\
= \;& q^{-d_i\delta_{ik}}\left(T_{kl}^{(s,0)}T_{ij}^{(r,1)} - T_{kl}^{(s,1)}T_{ij}^{(r,0)}\right) - (q^{d_i\delta_{ik}} - q^{-d_i\delta_{ik}})T_{kl}^{(s+1,0)}T_{ij}^{(r,0)}
\end{aligned}
$$

Similar calculations yield that

$$
\begin{aligned}
&q^{-d_j\delta_{jl}}T_{ij}^{(r,1)}T_{kl}^{(s,0)} - T_{ij}^{(r,0)}(q^{d_j\delta_{jl}}T_{kl}^{(s+1,0)} - q^{-d_j\delta_{jl}}T_{kl}^{(s,0)}) \qquad (5.1.3)\\
= \;& q^{-d_j\delta_{jl}}\left(T_{ij}^{(r,1)}T_{kl}^{(s,0)} - T_{ij}^{(r,0)}T_{kl}^{(s,1)}\right) - (q^{d_j\delta_{jl}} - q^{-d_j\delta_{jl}})T_{ij}^{(r,0)}T_{kl}^{(s+1,0)}.
\end{aligned}
$$

Applying (5.1.2) and (5.1.3) to (5.1.1) gives:

$$
\begin{aligned}
&(-1)^{|i||l|+|j||l|}\Big[q^{-d_j\delta_{jl}}\left(T_{ij}^{(r,1)}T_{kl}^{(s,0)} - T_{ij}^{(r,0)}T_{kl}^{(s,1)}\right)\\
&-(q^{d_j\delta_{jl}} - q^{-d_j\delta_{jl}})T_{ij}^{(r,0)}T_{kl}^{(s+1,0)}\Big]\\
&-(-1)^{|i||k|+|j||k|}\Big[q^{-d_i\delta_{ik}}\left(T_{kl}^{(s,0)}T_{ij}^{(r,1)} - T_{kl}^{(s,1)}T_{ij}^{(r,0)}\right)\\
&-(q^{d_i\delta_{ik}} - q^{-d_i\delta_{ik}})T_{kl}^{(s+1,0)}T_{ij}^{(r,0)}\Big]\\
= \;&(-1)^{|i||j|}\varepsilon\Big[(\delta_{j<l}T_{il}^{(r+1,0)}T_{kj}^{(s,0)} + \delta_{j>l}T_{il}^{(r,0)}T_{kj}^{(s+1,0)})\\
&-(\delta_{i>k}T_{il}^{(s,0)}T_{kj}^{(r+1,0)} + \delta_{i<k}T_{il}^{(s+1,0)}T_{kj}^{(r,0)})\Big]. \qquad (5.1.4)
\end{aligned}
$$

Next, recall that $T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)} = T_{ij}^{(r,m+1)}$ and $T_{kl}^{(s+1,n)} - T_{kl}^{(s,n)} = T_{kl}^{(s,n+1)}$. We can use this for all $r, s \geq 1$ and $m, n \geq 0$. Below, in Lemma 5.2, we prove by induction that

$$
\begin{aligned}
&(-1)^{|i||l|+|j||l|}\Big[q^{-d_j\delta_{jl}}(T_{ij}^{(r,m+1)}T_{kl}^{(s,n)} - T_{ij}^{(r,m)}T_{kl}^{(s,n+1)})\\
&\quad-(q^{d_j\delta_{jl}} - q^{-d_j\delta_{jl}})T_{ij}^{(r,m)}T_{kl}^{(s+1,n)}\Big]\\
&\quad-(-1)^{|i||k|+|j||k|}\Big[q^{-d_i\delta_{ik}}(T_{kl}^{(s,n)}T_{ij}^{(r,m+1)} - T_{kl}^{(s,n+1)}T_{ij}^{(r,m)})\\
&\quad-(q^{d_i\delta_{ik}} - q^{-d_i\delta_{ik}})T_{kl}^{(s+1,n)}T_{ij}^{(r,m)}\Big]\\
=&(-1)^{|i||j|}(q - q^{-1})\Big[(\delta_{j<l}T_{il}^{(r+1,m)}T_{kj}^{(s,n)} + \delta_{j>l}T_{il}^{(r,m)}T_{kj}^{(s+1,n)}) \qquad (5.1.5)
\end{aligned}
$$

$$-(\delta_{i>k}T_{il}^{(s,n)}T_{kj}^{(r+1,m)}+\delta_{i<k}T_{il}^{(s+1,n)}T_{kj}^{(r,m)})\Big].$$

Specifically, we let $r=s=1$.

$$\begin{aligned}
&(-1)^{|i||l|+|j||l|}\Big[q^{-d_j\delta_{jl}}(T_{ij}^{(1,m+1)}T_{kl}^{(1,n)}-T_{ij}^{(1,m)}T_{kl}^{(1,n+1)})\\
&\quad-(q^{d_j\delta_{jl}}-q^{-d_j\delta_{jl}})T_{ij}^{(1,m)}T_{kl}^{(2,n)}\Big]\\
&\quad-(-1)^{|i||k|+|j||k|}\Big[q^{-d_i\delta_{ik}}(T_{kl}^{(1,n)}T_{ij}^{(1,m+1)}-T_{kl}^{(1,n+1)}T_{ij}^{(1,m)})\\
&\quad-(q^{d_i\delta_{ik}}-q^{-d_i\delta_{ik}})T_{kl}^{(2,n)}T_{ij}^{(1,m)}\Big]\\
=&(-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}T_{il}^{(2,m)}T_{kj}^{(1,n)}+\delta_{j>l}T_{il}^{(1,m)}T_{kj}^{(2,n)})\\
&\quad-(\delta_{i>k}T_{il}^{(1,n)}T_{kj}^{(2,m)}+\delta_{i<k}T_{il}^{(2,n)}T_{kj}^{(1,m)})\Big].
\end{aligned}\tag{5.1.6}$$

Multiply the equation by $(-1)^{|i||l|+|j||l|}$ and rearrange.

$$\begin{aligned}
&q^{-d_j\delta_{jl}}(T_{ij}^{(1,m+1)}T_{kl}^{(1,n)}-T_{ij}^{(1,m)}T_{kl}^{(1,n+1)})]\\
&-(-1)^{|i||k|+|j||k|+|i||l|+|j||l|}q^{-d_i\delta_{ik}}(T_{kl}^{(1,n)}T_{ij}^{(1,m+1)}-T_{kl}^{(1,n+1)}T_{ij}^{(1,m)})\\
=\quad&(-1)^{|i||j|+|i||l|+|j||l|}(q-q^{-1})(\delta_{j<l}T_{il}^{(2,m)}T_{kj}^{(1,n)}+\delta_{j>l}T_{il}^{(1,m)}T_{kj}^{(2,n)})\\
&-(-1)^{|i||j|+|i||l|+|j||l|}(q-q^{-1})(\delta_{i>k}T_{il}^{(1,n)}T_{kj}^{(2,m)}+\delta_{i<k}T_{il}^{(2,n)}T_{kj}^{(1,m)}).\\
&+(q^{d_j\delta_{jl}}-q^{-d_j\delta_{jl}})T_{ij}^{(1,m)}T_{kl}^{(2,n)}\\
&-(-1)^{|i||k|+|j||k|+|i||l|+|j||l|}(q^{d_i\delta_{ik}}-q^{-d_i\delta_{ik}})T_{kl}^{(2,n)}T_{ij}^{(1,m)}
\end{aligned}\tag{5.1.7}$$

Next, we look at (5.1.7) in the quotient $\mathbf{K}_{m+n+1}/\mathbf{K}_{m+n+2}$. Here, $\overline{q}=1$ and $\overline{q-q^{-1}}=\hbar\in\mathbf{K}_1$. For expressions like $(q-q^{-1})T_{il}^{(2,m)}T_{kj}^{(1,n)}$, we can see that $\overline{(q-q^{-1})T_{il}^{(2,m)}T_{kj}^{(1,n)}}=\overline{\hbar(T_{il}^{(1,m+1)}+T_{il}^{(1,m)})T_{kj}^{(1,n)}}=\overline{\hbar(T_{il}^{(1,m+1)}T_{kj}^{(1,n)}+T_{il}^{(1,m)}T_{kj}^{(1,n)})}$ $=\overline{\hbar T_{il}^{(1,m)}T_{kj}^{(1,n)}}$. As a result, each "2" in the primary superscript of generators in (5.1.7) can be replaced with a "1". The resulting equation below actually lies in $\mathbf{K}_{m+n+1}/\mathbf{K}_{m+n+2}$, but we omit the bars for readability.

$$\begin{aligned}
& (T_{ij}^{(1,m+1)} T_{kl}^{(1,n)} - T_{ij}^{(1,m)} T_{kl}^{(1,n+1)}) \\
& -(-1)^{|i||k|+|j||k|+|i||l|+|j||l|} (T_{kl}^{(1,n)} T_{ij}^{(1,m+1)} - T_{kl}^{(1,n+1)} T_{ij}^{(1,m)}) \\
= \quad & (-1)^{|i||j|+|i||l|+|j||l|} \hbar (\delta_{j<l} T_{il}^{(1,m)} T_{kj}^{(1,n)} + \delta_{j>l} T_{il}^{(1,m)} T_{kj}^{(1,n)}) \\
& -(-1)^{|i||j|+|i||l|+|j||l|} \hbar (\delta_{i>k} T_{il}^{(1,n)} T_{kj}^{(1,m)} + \delta_{i<k} T_{il}^{(1,n)} T_{kj}^{(1,m)}). \\
& +(q^{d_j \delta_{jl}} - q^{-d_j \delta_{jl}}) T_{ij}^{(1,m)} T_{kl}^{(1,n)} \\
& -(-1)^{|i||k|+|j||k|+|i||l|+|j||l|} (q^{d_i \delta_{ik}} - q^{-d_i \delta_{ik}}) T_{kl}^{(1,n)} T_{ij}^{(1,m)} \\
& \Longleftrightarrow \\
& T_{ij}^{(1,m+1)} T_{kl}^{(1,n)} - (-1)^{(|i|+|j|)(|k|+|l|)} T_{kl}^{(1,n)} T_{ij}^{(1,m+1)} \qquad (5.1.8) \\
& -(T_{ij}^{(1,m)} T_{kl}^{(1,n+1)} - (-1)^{(|i|+|j|)(|k|+|l|)} T_{kl}^{(1,n+1)} T_{ij}^{(1,m)}) \\
= \quad & (-1)^{|i||j|+|i||l|+|j||l|} \hbar (\delta_{j<l} + \delta_{j>l}) T_{il}^{(1,m)} T_{kj}^{(1,n)} \\
& -(-1)^{|i||j|+|i||l|+|j||l|} \hbar (\delta_{i>k} + \delta_{i<k}) T_{il}^{(1,n)} T_{kj}^{(1,m)}. \\
& +(q^{d_j \delta_{jl}} - q^{-d_j \delta_{jl}}) T_{ij}^{(1,m)} T_{kl}^{(1,n)} \\
& -(-1)^{|i||k|+|j||k|+|i||l|+|j||l|} (q^{d_i \delta_{ik}} - q^{-d_i \delta_{ik}}) T_{kl}^{(1,n)} T_{ij}^{(1,m)}
\end{aligned}$$

The left hand side of (5.1.8) is equal to $[T_{ij}^{(1,m+1)}, T_{kl}^{(1,n)}] - [T_{ij}^{(1,m)}, T_{kl}^{(1,n+1)}]$, which is identical to the left hand side of the defining relation of the Yangian if we interchange generators $T_{ij}^{(1,m)}$ with $t_{ij}^{(m)}$. Our goal now is to show that the right hand side of (5.1.8) looks like the right hand side of the defining relation on $Y(\mathfrak{gl}_{M|N})$:

$$(-1)^{\eta(i,j;k,l)} \left(t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)} \right) = (-1)^{|i||j|+|i||k|+|j||k|} \left(t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)} \right).$$

This is actually a more specific version of the expression, where we let $\hbar = 1$. We will instead pursue the general version:

$$(-1)^{|i||j|+|i||k|+|j||k|}\hbar \left(t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)} \right).$$

Observe that

$$q^{d_j \delta_{jl}} - q^{-d_j \delta_{jl}} = \begin{cases} 0 & i \neq j \\ q - q^{-1} & j = l \leq M \\ q^{-1} - q & j = l > M \end{cases}.$$

As a result, we can rewrite $q^{d_j \delta_{jl}} - q^{-d_j \delta_{jl}} = \delta_{jl} d_j (q - q^{-1})$, which becomes $\delta_{jl} d_j \hbar$ in $\mathbf{K}_{m+n+1}/\mathbf{K}_{m+n+2}$. Further, $\delta_{jl} T_{ij}^{(1,m)} T_{kl}^{(1,n)} = \delta_{jl} T_{il}^{(1,m)} T_{kj}^{(1,n)}$; whenever necessary, we may substitute one index for another if the term is only nonzero if the two indices are equal. Note also that $|i|^2 = |i|$ for any index i. Our expression now looks like

$$\begin{aligned}
& \left[(-1)^{|i||j|+|i||l|+|j||l|}\hbar(\delta_{j<l} + \delta_{j>l}) + \delta_{jl} d_j \hbar \right] T_{il}^{(1,m)} T_{kj}^{(1,n)} \\
& - \left[(-1)^{|i||j|+|i||l|+|j||l|}\hbar(\delta_{i>k} + \delta_{i<k}) \right. \\
& \left. +(-1)^{|i||i|+|j||i|+|i||l|+|j||l|} \delta_{ik} d_i \hbar \right] T_{il}^{(1,n)} T_{kj}^{(1,m)} \\
= \quad & (-1)^{|i||j|+|i||l|+|j||l|}\hbar \left[\left(\delta_{j<l} + \delta_{j>l} + (-1)^{|i||j|+|i|j|+|j||j|} \delta_{jl} d_j \right) T_{il}^{(1,m)} T_{kj}^{(1,n)} \right. \\
& \left. - \left(\delta_{i>k} + \delta_{i<k} + (-1)^{|i||k|} \delta_{ik} d_i \right) T_{il}^{(1,n)} T_{kj}^{(1,m)} \right] \\
= \quad & (-1)^{|i||j|+|i||l|+|j||l|}\hbar \left[\left(\delta_{j<l} + \delta_{j>l} + (-1)^{|j|} \delta_{jl} d_j \right) T_{il}^{(1,m)} T_{kj}^{(1,n)} \right. \\
& \left. - \left(\delta_{i>k} + \delta_{i<k} + (-1)^{|i|} \delta_{ik} d_i \right) T_{il}^{(1,n)} T_{kj}^{(1,m)} \right].
\end{aligned} \tag{5.1.9}$$

Finally, note that $(-1)^{|j|} = \begin{cases} 1 & j \leq M \\ -1 & j > M \end{cases} = d_j$ and $d_j^2 = 1$. $\delta_{j<l} + \delta_{j>l} + \delta_{jl}$ can simply be replaced by 1 as one, and only one, of those conditions must be true. Thus, the right hand side of (5.1.9) is equivalent to

$$(-1)^{|i||j|+|i||l|+|j||l|}\hbar\left(T_{il}^{(1,m)}T_{kj}^{(1,n)} - T_{il}^{(1,n)}T_{kj}^{(1,m)}\right). \tag{5.1.10}$$

Thus, we have shown that, in $\mathbf{K}_{m+n+1}/\mathbf{K}_{m+n+2}$,

$$\begin{aligned}
&[T_{ij}^{(1,m+1)}, T_{kl}^{(1,n)}] - [T_{ij}^{(1,m)}, T_{kl}^{(1,n+1)}] \\
&\qquad =(-1)^{|i||j|+|i||l|+|j||l|}\hbar\left(T_{il}^{(1,m)}T_{kj}^{(1,n)} - T_{il}^{(1,n)}T_{kj}^{(1,m)}\right).
\end{aligned}$$

If we set $\varphi(t_{ij}^{(m)}) = (-1)^{|i||j|}T_{ij}^{(1,m)}$ and apply φ to the defining relation (2.3.42), we obtain exactly (5.1.10). This confirms that φ is a homomorphism. $\square$

What follows is the proof by induction of the general version of a relation on $U_q(\mathcal{L}\mathfrak{gl}_{M|N})$.

Lemma 5.2. *The equality (5.1.5) holds for all $r, s \geq 1$ and $m, n \geq 0$.*

Proof. Let the result

$$\begin{aligned}
&(-1)^{|i||l|+|j||l|}\Big[q^{-d_j\delta_{jl}}(T_{ij}^{(r,m+1)}T_{kl}^{(s,n)} - T_{ij}^{(r,m)}T_{kl}^{(s,n+1)}) \\
&-(q^{d_j\delta_{jl}} - q^{-d_j\delta_{jl}})T_{ij}^{(r,m)}T_{kl}^{(s+1,n)}\Big] \\
&-(-1)^{|i||k|+|j||k|}\Big[q^{-d_i\delta_{ik}}(T_{kl}^{(s,n)}T_{ij}^{(r,m+1)} - T_{kl}^{(s,n+1)}T_{ij}^{(r,m)}) \\
&-(q^{d_i\delta_{ik}} - q^{-d_i\delta_{ik}})T_{kl}^{(s+1,n)}T_{ij}^{(r,m)}\Big] \\
=&(-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}T_{il}^{(r+1,m)}T_{kj}^{(s,n)} + \delta_{j>l}T_{il}^{(r,m)}T_{kj}^{(s+1,n)}) \\
&-(\delta_{i>k}T_{il}^{(s,n)}T_{kj}^{(r+1,m)} + \delta_{i<k}T_{il}^{(s+1,n)}T_{kj}^{(r,m)})\Big].
\end{aligned} \tag{5.1.11}$$

be labelled $R(r,m;s,n)$, where we say that the $R(r,m;s,n)$ holds if the equation holds. We prove this by induction. We know the base case $R(r,0;s,0)$ holds due to (5.1.4). The induction step involves first showing that $R(r,m+1;s,0)$ holds, then that $R(r,m;s,n+1)$ holds.

PROOF THAT $R(r,m;s,0)$ HOLDS FOR ALL m,r,s: If we assume that the relation $R(r,m;s,0)$ holds, we need to show that $R(r,m+1;s,0)$ does as well, i.e. that $R(r,m+1;s,0) \iff R(r+1,m;s,0) - R(r,m;s,0)$. This notation means that the left (respectively, right) hand side of $R(r,m+1;s,0)$ is the difference of the left (respectively, right) hand sides of $R(r+1,m;s,0)$ and $R(r,m;s,0)$.

$$
\begin{aligned}
& R(r,m+1;s,0) \\
& \iff \\
& (-1)^{|i||l|+|j||l|}\Big[q^{-d_j\delta_{jl}}(T_{ij}^{(r,m+2)}T_{kl}^{(s,0)} - T_{ij}^{(r,m+1)}T_{kl}^{(s,1)}) \\
& -(q^{d_j\delta_{jl}} - q^{-d_j\delta_{jl}})T_{ij}^{(r,m+1)}T_{kl}^{(s+1,0)}\Big] \\
& -(-1)^{|i||k|+|j||k|}\Big[q^{-d_i\delta_{ik}}(T_{kl}^{(s,0)}T_{ij}^{(r,m+2)} - T_{kl}^{(s,1)}T_{ij}^{(r,m+1)}) \\
& -(q^{d_i\delta_{ik}} - q^{-d_i\delta_{ik}})T_{kl}^{(s+1,0)}T_{ij}^{(r,m+1)}\Big] \\
= & (-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}T_{il}^{(r+1,m+1)}T_{kj}^{(s,0)} + \delta_{j>l}T_{il}^{(r,m+1)}T_{kj}^{(s+1,0)}) \\
& -(\delta_{i>k}T_{il}^{(s,0)}T_{kj}^{(r+1,m+1)} + \delta_{i<k}T_{il}^{(s+1,0)}T_{kj}^{(r,m+1)})\Big] \\
& \iff \\
& (-1)^{|i||l|+|j||l|}\Big[q^{-d_j\delta_{jl}}((T_{ij}^{(r+1,m+1)} - T_{ij}^{(r,m+1)})T_{kl}^{(s,0)} \\
& -(T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)})T_{kl}^{(s,1)}) \\
& -(q^{d_j\delta_{jl}} - q^{-d_j\delta_{jl}})(T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)})T_{kl}^{(s+1,0)}\Big] \\
& -(-1)^{|i||k|+|j||k|}\Big[q^{-d_i\delta_{ik}}(T_{kl}^{(s,0)}(T_{ij}^{(r+1,m+1)} - T_{ij}^{(r,m+1)}) \\
& -T_{kl}^{(s,1)}(T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)}))
\end{aligned}
$$

$$
\begin{aligned}
&-(q^{d_i\delta_{ik}}-q^{-d_i\delta_{ik}})T_{kl}^{(s+1,0)}(T_{ij}^{(r+1,m)}-T_{ij}^{(r,m)})\Big]\\
=&(-1)^{|i||j|}(q-q^{-1})\ \Big[(\delta_{j<l}(T_{il}^{(r+2,m)}-T_{il}^{(r+1,m)})T_{kj}^{(s,0)}\\
&+\delta_{j>l}(T_{il}^{(r+1,m)}-T_{il}^{(r,m)})T_{kj}^{(s+1,0)})\\
&-(\delta_{i>k}T_{il}^{(s,0)}(T_{kj}^{(r+2,m)}-T_{kj}^{(r+1,m)})+\delta_{i<k}T_{il}^{(s+1,0)}(T_{kj}^{(r+1,m)}-T_{kj}^{(r,m)}))\Big]\\
&\Longleftrightarrow\\
&(-1)^{|i||l|+|j||l|}\ \Big[q^{-d_j\delta_{jl}}(T_{ij}^{(r+1,m+1)}T_{kl}^{(s,0)}-T_{ij}^{(r+1,m)}T_{kl}^{(s,1)})\\
&-(q^{d_j\delta_{jl}}-q^{-d_j\delta_{jl}})T_{ij}^{(r+1,m)}T_{kl}^{(s+1,0)}\Big]\\
&-(-1)^{|i||k|+|j||k|}\ \Big[q^{-d_i\delta_{ik}}(T_{kl}^{(s,0)}T_{ij}^{(r+1,m+1)}-T_{kl}^{(s,1)}T_{ij}^{(r+1,m)})\\
&-(q^{d_i\delta_{ik}}-q^{-d_i\delta_{ik}})T_{kl}^{(s+1,0)}T_{ij}^{(r+1,m)}\Big]\\
&-(-1)^{|i||l|+|j||l|}\ \Big[q^{-d_j\delta_{jl}}(T_{ij}^{(r,m+1)}T_{kl}^{(s,0)}-T_{ij}^{(r,m)}T_{kl}^{(s,1)})\\
&-(q^{d_j\delta_{jl}}-q^{-d_j\delta_{jl}})T_{ij}^{(r,m)}T_{kl}^{(s+1,0)}\Big]\\
&+(-1)^{|i||k|+|j||k|}\ \Big[q^{-d_i\delta_{ik}}(T_{kl}^{(s,0)}T_{ij}^{(r,m+1)}-T_{kl}^{(s,1)}T_{ij}^{(r,m)})\\
&-(q^{d_i\delta_{ik}}-q^{-d_i\delta_{ik}})T_{kl}^{(s+1,0)}T_{ij}^{(r,m)}\Big]\\
=&(-1)^{|i||j|}(q-q^{-1})\ \Big[(\delta_{j<l}T_{il}^{(r+2,m)}T_{kj}^{(s,0)}+\delta_{j>l}T_{il}^{(r+1,m)}T_{kj}^{(s+1,0)})\\
&-(\delta_{i>k}T_{il}^{(s,0)}T_{kj}^{(r+2,m)}+\delta_{i<k}T_{il}^{(s+1,0)}T_{kj}^{(r+1,m)})\Big]\\
&+\Big[(-1)^{|i||j|}(q-q^{-1})\ \Big[(\delta_{j<l}(-T_{il}^{(r+1,m)})T_{kj}^{(s,0)}+\delta_{j>l}(-T_{il}^{(r,m)})T_{kj}^{(s+1,0)})\\
&-(\delta_{i>k}T_{il}^{(s,0)}(-T_{kj}^{(r+1,m)})+\delta_{i<k}T_{il}^{(s+1,0)}(-T_{kj}^{(r,m)})\Big]\Big]\\
&\Longleftrightarrow\\
&(-1)^{|i||l|+|j||l|}\ \Big[q^{-d_j\delta_{jl}}(T_{ij}^{(r+1,m+1)}T_{kl}^{(s,0)}-T_{ij}^{(r+1,m)}T_{kl}^{(s,1)})\\
&-(q^{d_j\delta_{jl}}-q^{-d_j\delta_{jl}})T_{ij}^{(r+1,m)}T_{kl}^{(s+1,0)}\Big]\\
&-(-1)^{|i||k|+|j||k|}\ \Big[q^{-d_i\delta_{ik}}(T_{kl}^{(s,0)}T_{ij}^{(r+1,m+1)}-T_{kl}^{(s,1)}T_{ij}^{(r+1,m)})\\
&-(q^{d_i\delta_{ik}}-q^{-d_i\delta_{ik}})T_{kl}^{(s+1,0)}T_{ij}^{(r+1,m)}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- \Big[(-1)^{|i||l|+|j||l|} \Big[q^{-d_j\delta_{jl}} (T_{ij}^{(r,m+1)} T_{kl}^{(s,0)} - T_{ij}^{(r,m)} T_{kl}^{(s,1)}) \\
&- (q^{d_j\delta_{jl}} - q^{-d_j\delta_{jl}}) T_{ij}^{(r,m)} T_{kl}^{(s+1,0)} \Big] \\
&- (-1)^{|i||k|+|j||k|} \Big[q^{-d_i\delta_{ik}} (T_{kl}^{(s,0)} T_{ij}^{(r,m+1)} - T_{kl}^{(s,1)} T_{ij}^{(r,m)}) \\
&- (q^{d_i\delta_{ik}} - q^{-d_i\delta_{ik}})\ T_{kl}^{(s+1,0)} T_{ij}^{(r,m)} \Big] \Big] \\
= &(-1)^{|i||j|} (q - q^{-1}) \Big[(\delta_{j<l} T_{il}^{(r+2,m)} T_{kj}^{(s,0)} + \delta_{j>l} T_{il}^{(r+1,m)} T_{kj}^{(s+1,0)}) \\
&- (\delta_{i>k} T_{il}^{(s,0)} T_{kj}^{(r+2,m)} + \delta_{i<k} T_{il}^{(s+1,0)} T_{kj}^{(r+1,m)}) \Big] \\
&- \Big[(-1)^{|i||j|} (q - q^{-1}) \Big[(\delta_{j<l} T_{il}^{(r+1,m)} T_{kj}^{(s,0)} + \delta_{j>l} T_{il}^{(r,m)}) T_{kj}^{(s+1,0)} \\
&- (\delta_{i>k} T_{il}^{(s,0)} T_{kj}^{(r+1,m)} + \delta_{i<k} T_{il}^{(s+1,0)} T_{kj}^{(r,m)} \Big] \Big] \\
&\Longleftrightarrow \\
&R(r+1, m; s, 0) - R(r, m; s, 0)
\end{aligned}
$$

Next, we show that $R(r,m;s,n+1) = R(r,m;s+1,n) - R(r,m;s,n)$ to complete the proof.

$$
\begin{aligned}
&R(r,m;s,n+1) \\
&\Longleftrightarrow \\
&(-1)^{|i||l|+|j||l|} \Big[q^{-d_j\delta_{jl}} (T_{ij}^{(r,m+1)} T_{kl}^{(s,n+1)} - T_{ij}^{(r,m)} T_{kl}^{(s,n+2)}) \\
&-(q^{d_j\delta_{jl}} - q^{-d_j\delta_{jl}}) T_{ij}^{(r,m)} T_{kl}^{(s+1,n+1)} \Big] \\
&- (-1)^{|i||k|+|j||k|} \Big[q^{-d_i\delta_{ik}} (T_{kl}^{(s,n+1)} T_{ij}^{(r,m+1)} - T_{kl}^{(s,n+2)} T_{ij}^{(r,m)}) \\
&-(q^{d_i\delta_{ik}} - q^{-d_i\delta_{ik}}) T_{kl}^{(s+1,n+1)} T_{ij}^{(r,m)} \Big] \\
&= (-1)^{|i||j|} (q - q^{-1}) \Big[(\delta_{j<l} T_{il}^{(r+1,m)} T_{kj}^{(s,n+1)} + \delta_{j>l} T_{il}^{(r,m)} T_{kj}^{(s+1,n+1)}) \\
&-(\delta_{i>k} T_{il}^{(s,n+1)} T_{kj}^{(r+1,m)} + \delta_{i<k} T_{il}^{(s+1,n+1)} T_{kj}^{(r,m)}) \Big] \\
&\Longleftrightarrow \\
&(-1)^{|i||l|+|j||l|} \Big[q^{-d_j\delta_{jl}} (T_{ij}^{(r,m+1)} (T_{kl}^{(s+1,n)} - T_{kl}^{(s,n)}) - T_{ij}^{(r,m)} (T_{kl}^{(s+1,n+1)} - T_{kl}^{(s,n+1)}))
\end{aligned}
$$

$$-(q^{d_j\delta_{jl}}-q^{-d_j\delta_{jl}})T_{ij}^{(r,m)}(T_{kl}^{(s+2,n)}-T_{kl}^{(s+1,n)})\Big]$$

$$-(-1)^{|i||k|+|j||k|}\Big[q^{-d_i\delta_{ik}}((T_{kl}^{(s+1,n)}-T_{kl}^{(s,n)})T_{ij}^{(r,m+1)}-(T_{kl}^{(s+1,n+1)}-T_{kl}^{(s,n+1)})T_{ij}^{(r,m)})$$

$$-(q^{d_i\delta_{ik}}-q^{-d_i\delta_{ik}})(T_{kl}^{(s+2,n)}-T_{kl}^{(s+1,n)})T_{ij}^{(r,m)}\Big]$$

$$=(-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}T_{il}^{(r+1,m)}(T_{kj}^{(s+1,n)}-T_{kj}^{(s,n)})$$

$$+\delta_{j>l}T_{il}^{(r,m)}(T_{kj}^{(s+2,n)}-T_{kj}^{(s+1,n)}))$$

$$-(\delta_{i>k}(T_{il}^{(s+1,n)}-T_{il}^{(s,n)})T_{kj}^{(r+1,m)}+\delta_{i<k}(T_{il}^{(s+2,n)}-T_{il}^{(s+1,n+1)})T_{kj}^{(r,m)})\Big]$$

$$\Longleftrightarrow$$

$$(-1)^{|i||l|+|j||l|}\Big[q^{-d_j\delta_{jl}}(T_{ij}^{(r,m+1)}T_{kl}^{(s+1,n)}-T_{ij}^{(r,m)}T_{kl}^{(s+1,n+1)})$$

$$-(q^{d_j\delta_{jl}}-q^{-d_j\delta_{jl}})T_{ij}^{(r,m)}T_{kl}^{(s+2,n)}\Big]$$

$$-(-1)^{|i||k|+|j||k|}\Big[q^{-d_i\delta_{ik}}(T_{kl}^{(s+1,n)}T_{ij}^{(r,m+1)}-T_{kl}^{(s+1,n+1)}T_{ij}^{(r,m)})$$

$$-(q^{d_i\delta_{ik}}-q^{-d_i\delta_{ik}})T_{kl}^{(s+2,n)}T_{ij}^{(r,m)}\Big]$$

$$+\Big[(-1)^{|i||l|+|j||l|}\Big[q^{-d_j\delta_{jl}}(T_{ij}^{(r,m+1)}(-T_{kl}^{(s,n)})-T_{ij}^{(r,m)}(-T_{kl}^{(s,n+1)}))$$

$$-(q^{d_j\delta_{jl}}-q^{-d_j\delta_{jl}})T_{ij}^{(r,m)}(-T_{kl}^{(s+1,n)})\Big]$$

$$-(-1)^{|i||k|+|j||k|}\Big[q^{-d_i\delta_{ik}}(-T_{kl}^{(s,n)})T_{ij}^{(r,m+1)}-(-T_{kl}^{(s,n+1)})T_{ij}^{(r,m)})$$

$$-(q^{d_i\delta_{ik}}-q^{-d_i\delta_{ik}})(-T_{kl}^{(s+1,n)})T_{ij}^{(r,m)}\Big]\Big]$$

$$=(-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}T_{il}^{(r+1,m)}T_{kj}^{(s+1,n)}+\delta_{j>l}T_{il}^{(r,m)}T_{kj}^{(s+2,n)})$$

$$-(\delta_{i>k}T_{il}^{(s+1,n)}T_{kj}^{(r+1,m)}+\delta_{i<k}T_{il}^{(s+2,n)}T_{kj}^{(r,m)})\Big]$$

$$+\Big[(-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}T_{il}^{(r+1,m)}(-T_{kj}^{(s,n)})+\delta_{j>l}T_{il}^{(r,m)}(-T_{kj}^{(s+1,n)}))$$

$$-(\delta_{i>k}(-T_{il}^{(s,n)})T_{kj}^{(r+1,m)}+\delta_{i<k}(-T_{il}^{(s+1,n+1)})T_{kj}^{(r,m)})\Big]\Big]$$

$$\Longleftrightarrow$$

$$(-1)^{|i||l|+|j||l|}\Big[q^{-d_j\delta_{jl}}(T_{ij}^{(r,m+1)}T_{kl}^{(s+1,n)}-T_{ij}^{(r,m)}T_{kl}^{(s+1,n+1)})$$

$$-(q^{d_j\delta_{jl}}-q^{-d_j\delta_{jl}})T_{ij}^{(r,m)}T_{kl}^{(s+2,n)}\Big]$$

$$
\begin{aligned}
&-(-1)^{|i||k|+|j||k|}\Big[q^{-d_i\delta_{ik}}(T_{kl}^{(s+1,n)}T_{ij}^{(r,m+1)}-T_{kl}^{(s+1,n+1)}T_{ij}^{(r,m)})\\
&-(q^{d_i\delta_{ik}}-q^{-d_i\delta_{ik}})T_{kl}^{(s+2,n)}T_{ij}^{(r,m)}\Big]\\
&-\Big[(-1)^{|i||l|+|j||l|}\Big[q^{-d_j\delta_{jl}}(T_{ij}^{(r,m+1)}T_{kl}^{(s,n)}-T_{ij}^{(r,m)}T_{kl}^{(s,n+1)})\\
&-(q^{d_j\delta_{jl}}-q^{-d_j\delta_{jl}})T_{ij}^{(r,m)}T_{kl}^{(s+1,n)}\Big]\\
&-(-1)^{|i||k|+|j||k|}\Big[q^{-d_i\delta_{ik}}T_{kl}^{(s,n)}T_{ij}^{(r,m+1)}-T_{kl}^{(s,n+1)}T_{ij}^{(r,m)})\\
&-(q^{d_i\delta_{ik}}-q^{-d_i\delta_{ik}})T_{kl}^{(s+1,n)}T_{ij}^{(r,m)}\Big]\Big]\\
&=(-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}T_{il}^{(r+1,m)}T_{kj}^{(s+1,n)}+\delta_{j>l}T_{il}^{(r,m)}T_{kj}^{(s+2,n)})\\
&-(\delta_{i>k}T_{il}^{(s+1,n)}T_{kj}^{(r+1,m)}+\delta_{i<k}T_{il}^{(s+2,n)}T_{kj}^{(r,m)})\Big]\\
&-\Big[(-1)^{|i||j|}(q-q^{-1})\Big[(\delta_{j<l}T_{il}^{(r+1,m)}T_{kj}^{(s,n)}+\delta_{j>l}T_{il}^{(r,m)}T_{kj}^{(s+1,n)})\\
&-(\delta_{i>k}T_{il}^{(s,n)}T_{kj}^{(r+1,m)}+\delta_{i<k}T_{il}^{(s+1,n+1)}T_{kj}^{(r,m)})\Big]\Big]\\
&\iff\\
&R(r,m;s+1,n)-R(r,m;s,n)
\end{aligned}
$$

□

5.2 BIJECTIVITY

Now that $\varphi : Y(\mathfrak{gl}_{M|N}) \to \bigoplus_{m=0}^{\infty} \mathbf{K}_m/\mathbf{K}_{m+1}$ has been shown to be a homomorphism, we will find also that it is an isomorphism under the assumption of Conjecture 5.3 below. To do this, we will show that it sends a basis of $Y(\mathfrak{gl}_{M|N})$ to a basis of $\bigoplus_{m=0}^{\infty} \mathbf{K}_m/\mathbf{K}_{m+1}$. We have a PBW basis for the Yangian, as was stated above in Theorem 2.33. As of yet, it is not known if there exists a PBW basis for our presentation of the quantum loop superalgebra $U_q(\mathcal{L}(\mathfrak{gl}_{M|N})$. However, one can expect that such a basis will exist, as it is a reasonable expectation given what else we know about $U_q(\mathcal{L}(\mathfrak{gl}_{M|N}))$. Thus, we present a PBW theorem as a conjecture.

Conjecture 5.3. *There exists a PBW basis for $U_q(\mathcal{L}(\mathfrak{gl}_{M|N}))$. Specifically, the basis is composed of ordered monomials in the generators $T_{ij}^{(r)}$ and $\overline{T}_{ij}^{(r)}$, where the odd generators (if $|i|+|j| = 1 \pmod 2$) may only occur in powers no greater than 1.*

One would expect the proof for Conjecture 5.3 to be similar to Gow and Molev's proof of a PBW theorem for $U_q(\mathcal{L}\mathfrak{gl}_N)$ [GM09]. From there, we need to produce a basis for the subspaces $\mathbf{K}_m$ and for the quotients $\mathbf{K}_m/\mathbf{K}_{m+1}$. These details will be omitted here, but the process would be very similar to what is done for the $\mathfrak{q}_n$ case in Section 6.2. One would also have to show that a basis for $U(\mathcal{L}_q(\mathfrak{gl}_{M|N}))$ in the $T_{ij}^{(r)}$ generators will yield a basis for $U_q(\mathcal{L}(\mathfrak{gl}_{M|N}))$ in the $T_{ij}^{(r,m)}$ generators.

Theorem 5.4. *The map $\varphi : Y(\mathfrak{gl}_{M|N}) \to \bigoplus_{m\geq 0} \mathbf{K}_m/\mathbf{K}_{m+1}$ with $\varphi(t_{ij}^{(m)}) = T_{ij}^{(1,m)}$ is an isomorphism.*

Proof. This is a consequence of Theorem 2.33, Conjecture 5.3, and the necessary steps outlined above. The details that have been omitted here will be included in the next chapter for the analogous $\mathfrak{q}_N$ case. □

6

SUPERALGEBRAS OF TYPE $\mathfrak{q}$

We will now shift focus to the queer Yangian $Y(\mathfrak{q}_N)$ and the twisted quantized loop algebra of type $\mathfrak{q}$ denoted $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$. Most importantly, we show that Drinfeld's result on Yangians as a degeneration of quantum loop algebras [Dri87], as shown by [GM12] holds in this case. Specifically, this isomorphism portraying queer Yangians as limit forms of the twisted quantized loop algebra is stated in Theorem 6.15. Showing this requires altering the appearance of the generators and defining relations of $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$ until they resemble those of the Yangian, as is done over the course of Section 6.1, then showing bijectivity between PBW bases in Section 6.2.

6.1 RELATIONS AND PRELIMINARY CALCULATIONS

Consider the complex associative unital algebra $U_q(\mathfrak{q}_N)$ as defined by Olshanski [Ols92].

Definition 6.1. Let $q \in \mathbb{C} - \{0\}$. The complex associative unital algebra $U_q(\mathfrak{q}_N)$ is generated by the elements L_{ij} for $i \leq j$. These generators satisfy the defining relations

$$L_{ii} L_{-i,-i} = L_{-i,-i} L_{ii} = 1 \tag{6.1.1}$$

$$L_{(1,2)} L_{(1,3)} S_{(2,3)} = S_{(2,3)} L_{(1,3)} L_{(1,2)} \tag{6.1.2}$$

where S is the R-matrix same as the S defined in (2.4.10) and we let

$$L =: \sum_{i \leq j} L_{ij} \otimes E_{ij} \in U_q(\mathfrak{q}_N) \otimes \mathrm{End}(\mathbb{C}^{N|N}).$$

Both sides of (6.1.2) lie in $U_q(\mathfrak{q}_N) \otimes \mathrm{End}(\mathbb{C}^{N|N}) \otimes \mathrm{End}(\mathbb{C}^{N|N})$.

By Nazarov [Naz99], there is an embedding of graded associative unital algebras $U(\mathfrak{q}_N) \to Y(\mathfrak{q}_N)$ given by

$$L_{ij} \mapsto -t_{ji}^{(1)} \cdot (-1)^{|i|}.$$

Likewise, we have an embedding of $U_q(\mathfrak{q}_N)$ into the twisted quantum loop algebra $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$.

We now produce another set of generators for $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$. Using the $\tau_{ij}^{(r)}$ generators introduced in Subsection 2.4.2, define $T_{ij}^{(r,m)}$ inductively on m by $T_{ij}^{(r,0)} = \tau_{ij}^{(r)}$ and $T_{ij}^{(r,m+1)} = T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)}$. The defining relation (2.4.15) of the quantum loop superalgebra $U_q(\mathcal{L}^{tw}\mathfrak{q}_n)$ can be rewritten in terms of these $T_{ij}^{(r,m)}$-type generators using that

$$\begin{aligned}
&{T_{ij}}^{(r)} T_{kl}^{(s+2)} - T_{ij}^{(r-1)} T_{kl}^{(s+1)} - T_{ij}^{(r+1)} T_{kl}^{(s+1)} + T_{ij}^{(r)} T_{kl}^{(s)} \\
=&T_{ij}^{(r)} (T_{kl}^{(s+2)} - 2T_{kl}^{(s+1)} + T_{kl}^{(s)}) + 2T_{ij}^{(r)} T_{kl}^{(s+1)}
\end{aligned}$$

$$
\begin{aligned}
&- T_{ij}^{(r)}T_{kl}^{(s)} - T_{ij}^{(r-1)}T_{kl}^{(s+1)} - T_{ij}^{(r+1)}T_{kl}^{(s+1)} + T_{ij}^{(r)}T_{kl}^{(s)} \\
=&T_{ij}^{(r)}(T_{kl}^{(s+2)} - 2T_{kl}^{(s+1)} + T_{kl}^{(s)}) \\
&+ (T_{ij}^{(r)} - T_{ij}^{(r-1)})T_{kl}^{(s+1)} - (T_{ij}^{(r+1)} - T_{ij}^{(r)})T_{kl}^{(s+1)} \\
=&T_{ij}^{(r)}(T_{kl}^{(s+2)} - 2T_{kl}^{(s+1)} + T_{kl}^{(s)}) - (T_{ij}^{(r+1)} - 2T_{ij}^{(r)} + T_{ij}^{(r-1)})T_{kl}^{(s+1)} \\
=&T_{ij}^{(r,0)}T_{kl}^{(s,2)} - T_{ij}^{(r-1,2)}T_{kl}^{(s+1,0)},
\end{aligned}
$$

$$
\begin{aligned}
&T_{il}{}^{(r+1)}T_{kj}^{(s+1)} - T_{il}^{(r)}T_{kj}^{(s)} = T_{il}^{(r+1)}(T_{kj}^{(s+1)} - T_{kj}^{(s)}) + (T_{il}^{(r+1)} - T_{il}^{(r)})T_{kj}^{(s)} \\
&= T_{il}^{(r+1,0)}T_{kj}^{(s,1)} + T_{il}^{(r,1)}T_{kj}^{(s,0)},
\end{aligned}
$$

$$
\begin{aligned}
&T_{i,-l}{}^{(r-1)}T_{k,-j}^{(s+1)} - T_{i,-l}^{(r)}T_{k,-j}^{(s)} = T_{i,-l}^{(r-1)}(T_{k,-j}^{(s+1)} - T_{k,-j}^{(s)}) + (T_{i,-l}^{(r-1)} - T_{i,-l}^{(r)})T_{k,-j}^{(s)} \\
=&T_{i,-l}^{(r-1,0)}T_{k,-j}^{(s,1)} - T_{i,-l}^{(r-1,1)}T_{k,-j}^{(s,0)}.
\end{aligned}
$$

Finally, the defining relation (2.4.15) becomes

$$
\begin{aligned}
(-1)^{p(i,j)p(k,l)}&q^{\varphi(j,l)}(T_{ij}^{(r,0)}T_{kl}^{(s,2)} - T_{ij}^{(r-1,2)}T_{kl}^{(s+1,0)}) \\
&+ \{j < l\}\theta(i,j,k)\varepsilon(T_{il}^{(r,0)}T_{kj}^{(s,2)} - T_{il}^{(r-1,2)}T_{kj}^{(s+1,0)}) \\
&+ \{-l < j\}\theta(-i,-j,k)\varepsilon(T_{i,-l}^{(r,0)}T_{k,-j}^{(s,2)} - T_{i,-l}^{(r-1,2)}T_{k,-j}^{(s+1,0)}) \\
&+ \varepsilon\theta(i,j,k)(T_{il}^{(r+1,0)}T_{kj}^{(s,1)} + T_{il}^{(r,1)}T_{kj}^{(s,0)}) \\
&- \varepsilon\theta(i,j,-k)(T_{i,-l}^{(r-1,0)}T_{k,-j}^{(s,1)} - T_{i,-l}^{(r-1,1)}T_{k,-j}^{(s,0)}) \\
=&q^{\varphi(i,k)}(T_{kl}^{(s,2)}T_{ij}^{(r,0)} - T_{kl}^{(s+1,0)}T_{ij}^{(r-1,2)}) \\
&+ \{k < i\}\theta(i,j,k)\varepsilon(T_{il}^{(s,2)}T_{kj}^{(r,0)} - T_{il}^{(s+1,0)}T_{kj}^{(r-1,2)}) \\
&+ \{i < -k\}\theta(-i,-j,k)\varepsilon(T_{-i,l}^{(s,2)}T_{-k,j}^{(r,0)} - T_{-i,l}^{(s+1,0)}T_{-k,j}^{(r-1,2)}) \\
&+ \varepsilon\theta(i,j,k)(T_{il}^{(s,1)}T_{kj}^{(r+1,0)} + T_{il}^{(s,0)}T_{kj}^{(r,1)}) \\
&- \varepsilon\theta(i,j,-k)(T_{-i,l}^{(s,1)}T_{-k,j}^{(r-1,0)} - T_{-i,l}^{(s,0)}T_{-k,j}^{(r-1,1)}).
\end{aligned}
$$

Name the relation above $R(r,0;s,0)$. By induction, we can produce a more general $R(r,m;s,n)$ using secondary superscripts other than zero.

Call the following generalization of $R(r,0;s,0)$ the relation $R(r,m;s,n)$. The base case in the proof by induction is $R(r,0;s,0)$ itself, which we know to be true.

$$\begin{aligned}
(-1)^{p(i,j)p(k,l)}q^{\varphi(j,l)}&(T_{ij}^{(r,m)}T_{kl}^{(s,n+2)} - T_{ij}^{(r-1,m+2)}T_{kl}^{(s+1,n)}) \\
&+ \{j<l\}\theta(i,j,k)\varepsilon(T_{il}^{(r,m)}T_{kj}^{(s,n+2)} - T_{il}^{(r-1,m+2)}T_{kj}^{(s+1,n)}) \\
&+ \{-l<j\}\theta(-i,-j,k)\varepsilon(T_{i,-l}^{(r,m)}T_{k,-j}^{(s,n+2)} - T_{i,-l}^{(r-1,m+2)}T_{k,-j}^{(s+1,n)}) \\
&+ \varepsilon\theta(i,j,k)(T_{il}^{(r+1,m)}T_{kj}^{(s,n+1)} + T_{il}^{(r,m+1)}T_{kj}^{(s,n)}) \\
&- \varepsilon\theta(i,j,-k)(T_{i,-l}^{(r-1,m)}T_{k,-j}^{(s,n+1)} - T_{i,-l}^{(r-1,m+1)}T_{k,-j}^{(s,n)}) \\
=q^{\varphi(i,k)}&(T_{kl}^{(s,n+2)}T_{ij}^{(r,m)} - T_{kl}^{(s+1,n)}T_{ij}^{(r-1,m+2)}) \\
&+ \{k<i\}\theta(i,j,k)\varepsilon(T_{il}^{(s,n+2)}T_{kj}^{(r,m)} - T_{il}^{(s+1,n)}T_{kj}^{(r-1,m+2)}) \\
&+ \{i<-k\}\theta(-i,-j,k)\varepsilon(T_{-i,l}^{(s,n+2)}T_{-k,j}^{(r,m)} - T_{-i,l}^{(s+1,n)}T_{-k,j}^{(r-1,m+2)}) \\
&+ \varepsilon\theta(i,j,k)(T_{il}^{(s,n+1)}T_{kj}^{(r+1,m)} + T_{il}^{(s,n)}T_{kj}^{(r,m+1)}) \\
&- \varepsilon\theta(i,j,-k)(T_{-i,l}^{(s,n+1)}T_{-k,j}^{(r-1,m)} - T_{-i,l}^{(s,n)}T_{-k,j}^{(r-1,m+1)})
\end{aligned}$$

Next, we use the basic identity $T_{ij}^{(r,m+1)} = T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)}$ to show $R(r,m+1;s,0)$holds, followed by $R(r,m;s,n+1)$. This is similar to the strategy in the proof of Lemma 5.2 above.

PROOF THAT $R(r,m;s,0)$ HOLDS FOR FIXED m AND ALL r,s: If we assume that $R(r,m;s,0)$ holds, we need to show that $R(r,m+1;s,0)$ does as well, i.e. that $R(r,m+1;s,0) \iff R(r+1,m;s,0) - R(r,m;s,0)$. Similar to its use above, this notation means that the left (respectively, right) hand side of $R(r,m+1;s,0)$ is the difference of the left (respectively, right) hand sides of $R(r+1,m;s,0)$ and $R(r,m;s,0)$.

$$
\begin{aligned}
&R(r,m+1;s,0)\\
&\Longleftrightarrow\\
&(-1)^{p(i,j)p(k,l)}q^{\varphi(j,l)}(T_{ij}^{(r,m+1)}T_{kl}^{(s,2)}-T_{ij}^{(r-1,m+3)}T_{kl}^{(s+1,0)})\\
&\quad+\{j<l\}\theta(i,j,k)\varepsilon(T_{il}^{(r,m+1)}T_{kj}^{(s,2)}-T_{il}^{(r-1,m+3)}T_{kj}^{(s+1,0)})\\
&\quad+\{-l<j\}\theta(-i,-j,k)\varepsilon(T_{i,-l}^{(r,m+1)}T_{k,-j}^{(s,2)}-T_{i,-l}^{(r-1,m+3)}T_{k,-j}^{(s+1,0)})\\
&\quad+\varepsilon\theta(i,j,k)(T_{il}^{(r+1,m+1)}T_{kj}^{(s,1)}+T_{il}^{(r,m+2)}T_{kj}^{(s,0)})\\
&\quad-\varepsilon\theta(i,j,-k)(T_{i,-l}^{(r-1,m+1)}T_{k,-j}^{(s,1)}-T_{i,-l}^{(r-1,m+2)}T_{k,-j}^{(s,0)})\\
&=q^{\varphi(i,k)}(T_{kl}^{(s,2)}T_{ij}^{(r,m+1)}-T_{kl}^{(s+1,0)}T_{ij}^{(r-1,m+3)})\\
&\quad+\{k<i\}\theta(i,j,k)\varepsilon(T_{il}^{(s,2)}T_{kj}^{(r,m+1)}-T_{il}^{(s+1,0)}T_{kj}^{(r-1,m+3)})\\
&\quad+\{i<-k\}\theta(-i,-j,k)\varepsilon(T_{-i,l}^{(s,2)}T_{-k,j}^{(r,m+1)}-T_{-i,l}^{(s+1,0)}T_{-k,j}^{(r-1,m+3)})\\
&\quad+\varepsilon\theta(i,j,k)(T_{il}^{(s,1)}T_{kj}^{(r+1,m+1)}+T_{il}^{(s,0)}T_{kj}^{(r,m+2)})\\
&\quad-\varepsilon\theta(i,j,-k)(T_{-i,l}^{(s,1)}T_{-k,j}^{(r-1,m+1)}-T_{-i,l}^{(s,0)}T_{-k,j}^{(r-1,m+2)})\\
&\Longleftrightarrow\\
&(-1)^{p(i,j)p(k,l)}q^{\varphi(j,l)}((T_{ij}^{(r+1,m)}-T_{ij}^{(r,m)})T_{kl}^{(s,2)}\\
&\quad-(T_{ij}^{(r,m+2)}-T_{ij}^{(r-1,m+2)})T_{kl}^{(s+1,0)})\\
&\quad+\{j<l\}\theta(i,j,k)\varepsilon((T_{il}^{(r+1,m)}-T_{il}^{(r,m)})T_{kj}^{(s,2)}\\
&\quad-(T_{il}^{(r,m+2)}-T_{il}^{(r-1,m+2)})T_{kj}^{(s+1,0)})\\
&\quad+\{-l<j\}\theta(-i,-j,k)\varepsilon((T_{i,-l}^{(r+1,m)}-T_{i,-l}^{(r,m)})T_{k,-j}^{(s,2)}\\
&\quad-(T_{i,-l}^{(r,m+2)}-T_{i,-l}^{(r-1,m+2)})T_{k,-j}^{(s+1,0)})\\
&\quad+\varepsilon\theta(i,j,k)((T_{il}^{(r+2,m)}-T_{il}^{(r+1,m)})T_{kj}^{(s,1)}\\
&\quad+(T_{il}^{(r+1,m+1)}-T_{il}^{(r,m+1)})T_{kj}^{(s,0)})
\end{aligned}
$$

$$
\begin{aligned}
&- \varepsilon\theta(i,j,-k)((T_{i,-l}^{(r,m)} - T_{i,-l}^{(r-1,m)})T_{k,-j}^{(s,1)} \\
&- (T_{i,-l}^{(r,m+1)} - T_{i,-l}^{(r-1,m+1)})T_{k,-j}^{(s,0)}) \\
=&q^{\varphi(i,k)}(T_{kl}^{(s,2)}(T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)}) \\
&- T_{kl}^{(s+1,0)}(T_{ij}^{(r,m+2)} - T_{ij}^{(r-1,m+2)})) \\
&+ \{k < i\}\theta(i,j,k)\varepsilon(T_{il}^{(s,2)}(T_{kj}^{(r+1,m)} - T_{kj}^{(r,m)}) \\
&- T_{il}^{(s+1,0)}(T_{kj}^{(r,m+2)} - T_{kj}^{(r-1,m+2)})) \\
&+ \{i < -k\}\theta(-i,-j,k)\varepsilon(T_{-i,l}^{(s,2)}(T_{-k,j}^{(r+1,m)} - T_{-k,j}^{(r,m)}) \\
&- T_{-i,l}^{(s+1,0)}(T_{-k,j}^{(r,m+2)} - T_{-k,j}^{(r-1,m+2)})) \\
&+ \varepsilon\theta(i,j,k)(T_{il}^{(s,1)}(T_{kj}^{(r+2,m)} - T_{kj}^{(r+1,m)}) \\
&+ T_{il}^{(s,0)}(T_{kj}^{(r+1,m+1)} - T_{kj}^{(r,m+q)})) \\
&- \varepsilon\theta(i,j,-k)(T_{-i,l}^{(s,1)}(T_{-k,j}^{(r,m)} - T_{-k,j}^{(r-1,m)}) \\
&- T_{-i,l}^{(s,0)}(T_{-k,j}^{(r,m+1)} - T_{-k,j}^{(r-1,m+1)})) \\
&\Longleftrightarrow
\end{aligned}
$$

$$
\begin{aligned}
[\ &(-1)^{p(i,j)p(k,l)}q^{\varphi(j,l)}(T_{ij}^{(r+1,m)}T_{kl}^{(s,2)} - T_{ij}^{(r,m+2)} - T_{kl}^{(s+1,0)}) \\
&+ \{j < l\}\theta(i,j,k)\varepsilon(T_{il}^{(r+1,m)}T_{kj}^{(s,2)} - T_{il}^{(r,m+2)}T_{kj}^{(s+1,0)}) \\
&+ \{-l < j\}\theta(-i,-j,k)\varepsilon(T_{i,-l}^{(r+1,m)}T_{k,-j}^{(s,2)} - T_{i,-l}^{(r,m+2)}T_{k,-j}^{(s+1,0)}) \\
&+ \varepsilon\theta(i,j,k)(T_{il}^{(r+2,m)}T_{kj}^{(s,1)} + T_{il}^{(r+1,m+1)}T_{kj}^{(s,0)}) \\
&-\varepsilon\theta(i,j,-k)(T_{i,-l}^{(r,m)}T_{k,-j}^{(s,1)} - T_{i,-l}^{(r,m+1)}T_{k,-j}^{(s,0)})\Big] \\
&- \Big[(-1)^{p(i,j)p(k,l)}q^{\varphi(j,l)}(T_{ij}^{(r,m)}T_{kl}^{(s,2)} - T_{ij}^{(r-1,m+2)}T_{kl}^{(s+1,0)}) \\
&+ \{j < l\}\theta(i,j,k)\varepsilon(T_{il}^{(r,m)}T_{kj}^{(s,2)} - T_{il}^{(r-1,m+2)}T_{kj}^{(s+1,0)}) \\
&+ \{-l < j\}\theta(-i,-j,k)\varepsilon(T_{i,-l}^{(r,m)}T_{k,-j}^{(s,2)} - T_{i,-l}^{(r-1,m+2)}T_{k,-j}^{(s+1,0)}) \\
&+ \varepsilon\theta(i,j,k)(T_{il}^{(r+1,m)}T_{kj}^{(s,1)} + T_{il}^{(r,m+1)}T_{kj}^{(s,0)}) \\
&- \varepsilon\theta(i,j,-k)(T_{i,-l}^{(r-1,m)}T_{k,-j}^{(s,1)} - T_{i,-l}^{(r-1,m+1)}T_{k,-j}^{(s,0)})\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[q^{\varphi(i,k)}(T_{kl}^{(s,2)}T_{ij}^{(r+1,m)} - T_{kl}^{(s+1,0)}T_{ij}^{(r,m+2)}) \\
&\quad + \{k < i\}\theta(i,j,k)\varepsilon(T_{il}^{(s,2)}T_{kj}^{(r+1,m)} - T_{il}^{(s+1,0)}T_{kj}^{(r,m+2)}) \\
&\quad + \{i < -k\}\theta(-i,-j,k)\varepsilon(T_{-i,l}^{(s,2)}T_{-k,j}^{(r+1,m)} - T_{-i,l}^{(s+1,0)}T_{-k,j}^{(r,m+2)}) \\
&\quad + \varepsilon\theta(i,j,k)(T_{il}^{(s,1)}T_{kj}^{(r+2,m)} + T_{il}^{(s,0)}T_{kj}^{(r+1,m+1)}) \\
&\quad - \varepsilon\theta(i,j,-k)(T_{-i,l}^{(s,1)}T_{-k,j}^{(r,m)} - T_{-i,l}^{(s,0)}T_{-k,j}^{(r,m+1)}) \\
&\quad - \Big[q^{\varphi(i,k)}(T_{kl}^{(s,2)}T_{ij}^{(r,m)} - T_{kl}^{(s+1,0)}T_{ij}^{(r-1,m+2)}) \\
&\quad + \{k < i\}\theta(i,j,k)\varepsilon(T_{il}^{(s,2)}T_{kj}^{(r,m)} - T_{il}^{(s+1,0)}T_{kj}^{(r-1,m+2)}) \\
&\quad + \{i < -k\}\theta(-i,-j,k)\varepsilon(T_{-i,l}^{(s,2)}T_{-k,j}^{(r,m)} - T_{-i,l}^{(s+1,0)}T_{-k,j}^{(r-1,m+2)}) \\
&\quad + \varepsilon\theta(i,j,k)(T_{il}^{(s,1)}T_{kj}^{(r+1,m)} + T_{il}^{(s,0)}T_{kj}^{(r,m+1)}) \\
&\quad -\varepsilon\theta(i,j,-k)(T_{-i,l}^{(s,1)}T_{-k,j}^{(r-1,m)} - T_{-i,l}^{(s,0)}T_{-k,j}^{(r-1,m+1)})\Big] \\
&\quad \Longleftrightarrow \\
&\quad R(r+1,m;s,0) - R(r,m;s,0)
\end{aligned}
$$

PROOF THAT $R(r,m;s,n)$ HOLDS FOR FIXED m,n AND ALL r,s A second induction process is now necessary to generalize for any n as well. Similar to the process in the last step, we justify that $R(r,m;s,n)$ holds by showing that $R(r,m;s,n+1) \Longleftrightarrow R(r,m;s+1,n) - R(r,m;s,n)$. We use that $T_{kl}^{(s,n+1)} = T_{kl}^{(s+1,n)} - T_{kl}^{(s,n)}$.

$$
\begin{aligned}
&R(r,m;s,n+1) \\
&\Longleftrightarrow \\
(-1)^{p(i,j)p(k,l)}&q^{\varphi(j,l)}(T_{ij}^{(r,m)}T_{kl}^{(s,n+3)} - T_{ij}^{(r-1,m+2)}T_{kl}^{(s+1,n+1)}) \\
&+ \{j < l\}\theta(i,j,k)\varepsilon(T_{il}^{(r,m)}T_{kj}^{(s,n+3)} - T_{il}^{(r-1,m+2)}T_{kj}^{(s+1,n+1)}) \\
&+ \{-l < j\}\theta(-i,-j,k)\varepsilon(T_{i,-l}^{(r,m)}T_{k,-j}^{(s,n+3)} - T_{i,-l}^{(r-1,m+2)}T_{k,-j}^{(s+1,n+1)})
\end{aligned}
$$

$$
\begin{aligned}
&+ \varepsilon\theta(i,j,k)(T_{il}^{(r+1,m)}T_{kj}^{(s,n+2)} + T_{il}^{(r,m+1)}T_{kj}^{(s,n+1)}) \\
&- \varepsilon\theta(i,j,-k)(T_{i,-l}^{(r-1,m)}T_{k,-j}^{(s,n+2)} - T_{i,-l}^{(r-1,m+1)}T_{k,-j}^{(s,n+1)}) \\
=&q^{\varphi(i,k)}(T_{kl}^{(s,n+3)}T_{ij}^{(r,m)} - T_{kl}^{(s+1,n+1)}T_{ij}^{(r-1,m+2)}) \\
&+ \{k<i\}\theta(i,j,k)\varepsilon(T_{il}^{(s,n+3)}T_{kj}^{(r,m)} - T_{il}^{(s+1,n+1)}T_{kj}^{(r-1,m+2)}) \\
&+ \{i<-k\}\theta(-i,-j,k)\varepsilon(T_{-i,l}^{(s,n+3)}T_{-k,j}^{(r,m)} - T_{-i,l}^{(s+1,n+1)}T_{-k,j}^{(r-1,m+2)}) \\
&+ \varepsilon\theta(i,j,k)(T_{il}^{(s,n+2)}T_{kj}^{(r+1,m)} + T_{il}^{(s,n+1)}T_{kj}^{(r,m+1)}) \\
&- \varepsilon\theta(i,j,-k)(T_{-i,l}^{(s,n+2)}T_{-k,j}^{(r-1,m)} - T_{-i,l}^{(s,n+1)}T_{-k,j}^{(r-1,m+1)}) \\
&\Longleftrightarrow \\
(-1)&^{p(i,j)p(k,l)}q^{\varphi(j,l)}(T_{ij}^{(r,m)}(T_{kl}^{(s+1,n+2)} - T_{kl}^{(s,n+2)}) \\
&- T_{ij}^{(r-1,m+2)}(T_{kl}^{(s+1,n)} - T_{kl}^{(s,n)})) \\
&+ \{j<l\}\theta(i,j,k)\varepsilon(T_{il}^{(r,m)}(T_{kj}^{(s+1,n+2)} - T_{kj}^{(s,n+2)}) \\
&- T_{il}^{(r-1,m+2)}(T_{kj}^{(s+2,n)} - T_{kj}^{(s+1,n)})) \\
&+ \{-l<j\}\theta(-i,-j,k)\varepsilon(T_{i,-l}^{(r,m)}(T_{k,-j}^{(s+1,n+2)} - T_{k,-j}^{(s,n+2)}) \\
&- T_{i,-l}^{(r-1,m+2)}(T_{k,-j}^{(s+2,n)} - T_{k,-j}^{(s+1,n)})) \\
&+ \varepsilon\theta(i,j,k)(T_{il}^{(r+1,m)}(T_{kj}^{(s+1,n+1)} - T_{kj}^{(s,n+1)}) \\
&+ T_{il}^{(r,m+1)}(T_{kj}^{(s+1,n)} - T_{kj}^{(s,n)})) \\
&- \varepsilon\theta(i,j,-k)(T_{i,-l}^{(r-1,m)}(T_{k,-j}^{(s+1,n+1)} - T_{k,-j}^{(s,n+1)}) \\
&- T_{i,-l}^{(r-1,m+1)}(T_{k,-j}^{(s+1,n)} - T_{k,-j}^{(s,n)})) \\
=&q^{\varphi(i,k)}((T_{kl}^{(s+1,n+2)} - T_{kl}^{(s,n+2)})T_{ij}^{(r,m)} \\
&- (T_{kl}^{(s+2,n)} - T_{kl}^{(s+1,n)})T_{ij}^{(r-1,m+2)}) \\
&+ \{k<i\}\theta(i,j,k)\varepsilon((T_{il}^{(s+1,n+2)} - T_{il}^{(s,n+2)})T_{kj}^{(r,m)} \\
&- (T_{il}^{(s+2,n)} - T_{il}^{(s+1,n)})T_{kj}^{(r-1,m+2)}) \\
&+ \{i<-k\}\theta(-i,-j,k)\varepsilon((T_{-i,l}^{(s+1,n+2)} - (T_{-i,l}^{(s,n+2)})T_{-k,j}^{(r,m)} \\
&- (T_{-i,l}^{(s+2,n)} - T_{-i,l}^{(s+1,n)})T_{-k,j}^{(r-1,m+2)})
\end{aligned}
$$

$$
\begin{aligned}
&+ \varepsilon\theta(i,j,k)((T_{il}^{(s+1,n+1)} - T_{il}^{(s,n+1)})T_{kj}^{(r+1,m)} \\
&+ (T_{il}^{(s+1,n)} - T_{il}^{(s,n)})T_{kj}^{(r,m+1)}) \\
&- \varepsilon\theta(i,j,-k)((T_{-i,l}^{(s+1,n+1)} - T_{-i,l}^{(s,n+1)})T_{-k,j}^{(r-1,m)} \\
&- (T_{-i,l}^{(s+1,n)} - T_{-i,l}^{(s,n)})T_{-k,j}^{(r-1,m+1)}) \\
&\iff
\end{aligned}
$$

$$
\begin{aligned}
[(-1)&^{p(i,j)p(k,l)}q^{\varphi(j,l)}(T_{ij}^{(r,m)}T_{kl}^{(s+1,n+2)} - T_{ij}^{(r-1,m+2)}T_{kl}^{(s+1,n)}) \\
&+ \{j<l\}\theta(i,j,k)\varepsilon(T_{il}^{(r,m)}T_{kj}^{(s+1,n+2)} - T_{il}^{(r-1,m+2)}T_{kj}^{(s+2,n)})) \\
&+ \{-l<j\}\theta(-i,-j,k)\varepsilon(T_{i,-l}^{(r,m)}T_{k,-j}^{(s+1,n+2)} - T_{i,-l}^{(r-1,m+2)}T_{k,-j}^{(s+2,n)}) \\
&+ \varepsilon\theta(i,j,k)(T_{il}^{(r+1,m)}T_{kj}^{(s+1,n+1)} + T_{il}^{(r,m+1)}T_{kj}^{(s+1,n)}) \\
&-\varepsilon\theta(i,j,-k)(T_{i,-l}^{(r-1,m)}T_{k,-j}^{(s+1,n+1)} - T_{i,-l}^{(r-1,m+1)}T_{k,-j}^{(s+1,n)})\Big] \\
&- \Big[(-1)^{p(i,j)p(k,l)}q^{\varphi(j,l)}(T_{ij}^{(r,m)}T_{kl}^{(s,n+2)} - T_{ij}^{(r-1,m+2)}T_{kl}^{(s,n)}) \\
&+ \{j<l\}\theta(i,j,k)\varepsilon(T_{il}^{(r,m)}T_{kj}^{(s,n+2)} - T_{il}^{(r-1,m+2)} - T_{kj}^{(s+1,n)}) \\
&+ \{-l<j\}\theta(-i,-j,k)\varepsilon(T_{i,-l}^{(r,m)}T_{k,-j}^{(s,n+2)} - T_{i,-l}^{(r-1,m+2)}T_{k,-j}^{(s+1,n)}) \\
&+ \varepsilon\theta(i,j,k)(T_{il}^{(r+1,m)}T_{kj}^{(s,n+1)} + T_{il}^{(r,m+1)}T_{kj}^{(s,n)}) \\
&-\varepsilon\theta(i,j,-k)(T_{i,-l}^{(r-1,m)}T_{k,-j}^{(s,n+1)} - T_{i,-l}^{(r-1,m+1)}T_{k,-j}^{(s,n)})\Big] \\
=& \Big[q^{\varphi(i,k)}(T_{kl}^{(s+1,n+2)}T_{ij}^{(r,m)} - T_{kl}^{(s+2,n)}T_{ij}^{(r-1,m+2)}) \\
&+ \{k<i\}\theta(i,j,k)\varepsilon(T_{il}^{(s+1,n+2)}T_{kj}^{(r,m)} - T_{il}^{(s+2,n)}T_{kj}^{(r-1,m+2)}) \\
&+ \{i<-k\}\theta(-i,-j,k)\varepsilon(T_{-i,l}^{(s+1,n+2)}T_{-k,j}^{(r,m)} - T_{-i,l}^{(s+2,n)}T_{-k,j}^{(r-1,m+2)}) \\
&+ \varepsilon\theta(i,j,k)(T_{il}^{(s+1,n+1)}T_{kj}^{(r+1,m)} + T_{il}^{(s+1,n)}T_{kj}^{(r,m+1)}) \\
&-\varepsilon\theta(i,j,-k)(T_{-i,l}^{(s,n+1)}T_{-k,j}^{(r-1,m)} - T_{-i,l}^{(s,n)}T_{-k,j}^{(r-1,m+1)})\Big] \\
&- \Big[q^{\varphi(i,k)}(T_{kl}^{(s,n+2)}T_{ij}^{(r,m)} - T_{kl}^{(s+1,n)}T_{ij}^{(r-1,m+2)}) \\
&+ \{k<i\}\theta(i,j,k)\varepsilon(T_{il}^{(s,n+2)}T_{kj}^{(r,m)} - T_{il}^{(s+1,n)}T_{kj}^{(r-1,m+2)}) \\
&+ \{i<-k\}\theta(-i,-j,k)\varepsilon((T_{-i,l}^{(s,n+2)}T_{-k,j}^{(r,m)} - T_{-i,l}^{(s+1,n)}T_{-k,j}^{(r-1,m+2)})
\end{aligned}
$$

$$+\,\varepsilon\theta(i,j,k)(T_{il}^{(s,n+1)}T_{kj}^{(r+1,m)} + T_{il}^{(s,n)}T_{kj}^{(r,m+1)})$$
$$-\,\varepsilon\theta(i,j,-k)(T_{-i,l}^{(s,n+1)}T_{-k,j}^{(r-1,m)} - T_{-i,l}^{(s,n)}T_{-k,j}^{(r-1,m+1)})\Big]$$
$$\Longleftrightarrow$$
$$R(r,m;s+1,n) - R(r,m;s,n)$$

Thus, we obtain the relation $R(r,m;s,n)$:

$$\begin{aligned}
&(-1)^{p(i,j)p(k,l)}q^{\varphi(j,l)}(T_{ij}^{(r,m)}T_{kl}^{(s,n+2)} - T_{ij}^{(r-1,m+2)}T_{kl}^{(s+1,n)})\\
&\quad + \{j<l\}\theta(i,j,k)\varepsilon(T_{il}^{(r,m)}T_{kj}^{(s,n+2)} - T_{il}^{(r-1,m+2)}T_{kj}^{(s+1,n)})\\
&\quad + \{-l<j\}\theta(-i,-j,k)\varepsilon(T_{i,-l}^{(r,m)}T_{k,-j}^{(s,n+2)} - T_{i,-l}^{(r-1,m+2)}T_{k,-j}^{(s+1,n)})\\
&\quad + \varepsilon\theta(i,j,k)(T_{il}^{(r+1,m)}T_{kj}^{(s,n+1)} + T_{il}^{(r,m+1)}T_{kj}^{(s,n)})\\
&\quad - \varepsilon\theta(i,j,-k)(T_{i,-l}^{(r-1,m)}T_{k,-j}^{(s,n+1)} - T_{i,-l}^{(r-1,m+1)}T_{k,-j}^{(s,n)})\\
&\;=q^{\varphi(i,k)}(T_{kl}^{(s,n+2)}T_{ij}^{(r,m)} - T_{kl}^{(s+1,n)}T_{ij}^{(r-1,m+2)})\\
&\quad + \{k<i\}\theta(i,j,k)\varepsilon(T_{il}^{(s,n+2)}T_{kj}^{(r,m)} - T_{il}^{(s+1,n)}T_{kj}^{(r-1,m+2)})\\
&\quad + \{i<-k\}\theta(-i,-j,k)\varepsilon(T_{-i,l}^{(s,n+2)}T_{-k,j}^{(r,m)} - T_{-i,l}^{(s+1,n)}T_{-k,j}^{(r-1,m+2)})\\
&\quad + \varepsilon\theta(i,j,k)(T_{il}^{(s,n+1)}T_{kj}^{(r+1,m)} + T_{il}^{(s,n)}T_{kj}^{(r,m+1)})\\
&\quad - \varepsilon\theta(i,j,-k)(T_{-i,l}^{(s,n+1)}T_{-k,j}^{(r-1,m)} - T_{-i,l}^{(s,n)}T_{-k,j}^{(r-1,m+1)})
\end{aligned}$$

This generalization to m,n is essential since those components of the generators are what determine how elements map from $Y(\mathfrak{q}_n)$ to $\mathbf{K}_m/\mathbf{K}_{m+1}$ for some ideals $\mathbf{K}$ in order to have an isomorphism, as will be seen in more detail below.

Given the above defining relations how do we define the $\mathbf{K}_m$? First, they must contain the $T_{ij}^{(r,\tilde{m})}$ and they must be nested as $\mathbf{K}_m \supset \mathbf{K}_{m+1}$. Further, as ideals we need $\mathbf{K}_{m_1}\cdot\mathbf{K}_{m_2} = \mathbf{K}_{m_1+m_2}$.

We reproduce the sequence (3.1.1) of algebra homomorphisms in a $\mathfrak{q}_n$-setting:

$$U_\hbar(\mathcal{L}^{tw}\mathfrak{q}_n) \twoheadrightarrow U_\hbar(\mathcal{L}^{tw}\mathfrak{q}_n)/\hbar U_\hbar(\mathcal{L}^{tw}\mathfrak{q}_n) \to U(\mathcal{L}^{tw}\mathfrak{q}_n) \twoheadrightarrow U(\mathfrak{q}_n) \tag{6.1.3}$$

where the last map $U(\mathcal{L}^{tw}\mathfrak{q}_n) \twoheadrightarrow U(\mathfrak{q}_n)$ is defined by sending $s \mapsto 1$. Denote by ψ the composition of maps

$$U_\hbar(\mathcal{L}^{tw}\mathfrak{q}_n) \twoheadrightarrow U_\hbar(\mathcal{L}^{tw}\mathfrak{q})_n/hU_\hbar(\mathcal{L}^{tw}\mathfrak{q}_n) \to U(\mathcal{L}\mathfrak{q}_n).$$

This map sends $\psi : \tau_{ij}^{(r)} \mapsto (-1)^{|i|}(E_{ij}s^r + E_{-i,-j}s^{-r})$. Next, consider the subspace U of $U_{\mathcal{A}}(\mathcal{L}^{tw}\mathfrak{q}_n)$ defined to be the span over $\mathbb{C}$ of $\tau_{ij}^{(r)}, \overline{\tau_{ij}^{(r)}}$ for all r. We also define the ideal K_m for $m \geq 0$ in $\mathcal{L}^{tw}\mathfrak{q}_n$ to be the span of elements of the type $X_+ \otimes s^r(s-1)^m + X_- \otimes s^{-r}(s^{-1}-1)^m$ over $r \in \mathbb{Z}$ and $X = X_+ + X_- \in \mathfrak{q}_n$, where $X_\pm = \sum_{i,j} c_{i,j}E_{\pm i,\pm j}$. Finally, we denote by $\mathbb{K}_m$ the two-sided ideal of $U_{\mathcal{A}}(\mathcal{L}\mathfrak{q}_n)$ generated by $\psi^{-1}(\mathsf{K}_m) \cap U$. If $\psi(\tau_{ij}^{(r)}) = (-1)^{|i|}(E_{ij}s^r + E_{-i,-j}s^{-r})$, then also $\psi(T_{ij}^{(r,0)}) = (-1)^{|i|}(E_{ij}s^r + E_{-i,-j}s^{-r})$. Then,

$$\begin{aligned}
\psi(T_{ij}^{(r,1)}) &= \psi(T_{ij}^{(r+1,0)} - T_{ij}^{(r,0)}) = \psi(T_{ij}^{(r+1,0)}) - \psi(T_{ij}^{(r,0)}) \\
&= (-1)^{|i|}(E_{ij}s^{r+1} + E_{-i,-j}s^{-r-1}) - (-1)^{|i|}(E_{ij}s^r + E_{-i,-j}s^{-r}) \\
&= (-1)^{|i|}\left(E_{ij}(s^{r+1} - s^r) + E_{-i,-j}(s^{-r-1} - s^{-r})\right) \\
&= (-1)^{|i|}\left(E_{ij}s^r(s-1) + E_{-i,-j}s^{-r}(s^{-1}-1)\right),
\end{aligned}$$

Similarly,

$$\psi(T_{ij}^{(r,2)}) = (-1)^{|i|}\left(E_{ij}s^r(s-1)^2 + E_{-i,-j}s^{-r}(s^{-1}-1)^2\right)$$

and from further iterations we can infer that

$$\psi(T_{ij}^{(r,m)}) = (-1)^{|i|}\left(E_{ij}s^r(s-1)^m + E_{-i,-j}s^{-r}(s^{-1}-1)^m\right) \in \mathsf{K}_m.$$

Thus, because it can be expressed in terms of τ generators, $T_{ij}^{(r,m)} \in \mathbb{K}_m$.

We can then define $\mathbf{K}_m$ as the sum of the ideals $(q-q^{-1})^{m_0}\mathbb{K}_{m_1}\mathbb{K}_{m_2}\cdots\mathbb{K}_{m_k}$ with $\sum_{a=0}^{k} m_a \geq m$; i.e. it contains all elements of the form $(q-q^{-1})^{m_0}T_{i_1,j_1}^{(r_1,m_1)}\cdots T_{i_k,j_k}^{(r_k,m_k)}$ with $\sum_{a=0}^{k} m_a \geq m$. This extends naturally from the assignment $T_{ij}^{(r,m)} \in \mathbf{K}_m$.

Thus, $\mathbf{K}_0 = U_{\mathcal{A}}(\mathcal{L}^{tw}\mathfrak{q}_n)$. Note that $\mathbf{K}_1$ contains ε, sits inside of $\mathbf{K}_0$, and contains $(q-q^{-1})U_{\mathcal{A}}(\mathcal{L}^{tw}\mathfrak{q}_n) =: \mathcal{I}$. As a result, $\mathbf{K}_0/\mathbf{K}_1 \simeq \frac{\mathbf{K_0}/\mathcal{I}}{\mathbf{K_1}/\mathcal{I}}$ where $\mathbf{K_0}/\mathcal{I} \simeq U_{\mathbb{C}}(\mathcal{L}^{tw}\mathfrak{q}_n)$ and $\mathbf{K_1}/\mathcal{I}$ is the ideal of $U_{\mathbb{C}}(\mathcal{L}^{tw}\mathfrak{q}_n)$ that is the kernel of the map from $U(\mathcal{L}^{tw}\mathfrak{q}_n)$ to $U(\mathfrak{q}_n)$ sending s to 1. Hence, $\mathbf{K}_0/\mathbf{K}_1 \simeq U(\mathfrak{q}_n)$.

Our ultimate goal is to produce an isomorphism $\varphi : Y(\mathfrak{q}_n) \to \bigoplus_{m=0}^{\infty}\mathbf{K}_m/\mathbf{K}_{m+1}$, so first we must ensure that this map respects the properties of the quotient spaces. Next, we use an existing relation on $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$ to determine a natural choice for the image of a generator of the Yangian under φ.

Upon examination, both sides of $R(r,m;s,n)$ are in $\mathbf{K}_{m+n+2}$. For instance, disregarding the constants, we see that the first line has $T_{ij}^{(r,m)}T_{kl}^{(s,n+2)} \in \mathbf{K}_m \cdot \mathbf{K}_{n+2} = \mathbf{K}_{m+n+2}$ and $T_{ij}^{(r-1,m+2)}T_{kl}^{(s+1,n)} \in \mathbf{K}_{m+2}\cdot\mathbf{K}_n \subset \mathbf{K}_{m+n+2}$. Similarly, since $\varepsilon \in \mathbf{K}_1$, we obtain $\varepsilon\theta(i,j,-k)T_{-i,l}^{(s,n+1)}T_{-k,j}^{(r-1,m)} \in \mathbf{K}_1\cdot\mathbf{K}_{n+1}\cdot\mathbf{K}_m \subset \mathbf{K}_{m+n+2}$.

For convenience, label each line of $R(r,m;s,n)$ as follows:

$$1\,(-1)^{p(i,j)p(k,l)}q^{\varphi(j,l)}(T_{ij}^{(r,m)}T_{kl}^{(s,n+2)} - T_{ij}^{(r-1,m+2)}T_{kl}^{(s+1,n)})$$
$$2 + \{j<l\}\theta(i,j,k)\varepsilon(T_{il}^{(r,m)}T_{kj}^{(s,n+2)} - T_{il}^{(r-1,m+2)}T_{kj}^{(s+1,n)})$$
$$3 + \{-l<j\}\theta(-i,-j,k)\varepsilon(T_{i,-l}^{(r,m)}T_{k,-j}^{(s,n+2)} - T_{i,-l}^{(r-1,m+2)}T_{k,-j}^{(s+1,n)})$$

$$④ + \varepsilon\theta(i,j,k)(T_{il}^{(r+1,m)}T_{kj}^{(s,n+1)} + T_{il}^{(r,m+1)}T_{kj}^{(s,n)})$$
$$⑤ - \varepsilon\theta(i,j,-k)(T_{i,-l}^{(r-1,m)}T_{k,-j}^{(s,n+1)} - T_{i,-l}^{(r-1,m+1)}T_{k,-j}^{(s,n)})$$
$$=$$
$$⑥ q^{\varphi(i,k)}(T_{kl}^{(s,n+2)}T_{ij}^{(r,m)} - T_{kl}^{(s+1,n)}T_{ij}^{(r-1,m+2)})$$
$$⑦ + \{k < i\}\theta(i,j,k)\varepsilon(T_{il}^{(s,n+2)}T_{kj}^{(r,m)} - T_{il}^{(s+1,n)}T_{kj}^{(r-1,m+2)})$$
$$⑧ + \{i < -k\}\theta(-i,-j,k)\varepsilon(T_{-i,l}^{(s,n+2)}T_{-k,j}^{(r,m)} - T_{-i,l}^{(s+1,n)}T_{-k,j}^{(r-1,m+2)})$$
$$⑨ + \varepsilon\theta(i,j,k)(T_{il}^{(s,n+1)}T_{kj}^{(r+1,m)} + T_{il}^{(s,n)}T_{kj}^{(r,m+1)})$$
$$⑩ - \varepsilon\theta(i,j,-k)(T_{-i,l}^{(s,n+1)}T_{-k,j}^{(r-1,m)} - T_{-i,l}^{(s,n)}T_{-k,j}^{(r-1,m+1)})$$

Lines ①,④,⑤,⑥,⑨, and ⑩ are in $\mathbf{K}_{m+n+2}$, while ②,③,⑦, and ⑧ lie in $\mathbf{K}_{m+n+3}$.

Thus, viewing both sides of $R(r,m;s,n)$ as elements in $\mathbf{K}_{m+n+2}/\mathbf{K}_{m+n+3}$ reduces the relation to the equality ①′ + ④′ + ⑤′ = ⑥′ + ⑨′ + ⑩′.

Note that the possible values for $\varphi(j,l)$ are $\varphi(j,l) = 0, \pm 1$. We have that $q^{\varphi(j,l)} = q^{\varphi(j,l)} - 1 + 1 = \frac{q^{2\varphi(j,l)}-1}{q^{\varphi(j,l)}+1} + 1 = \frac{q^{\varphi(j,l)}(q^{\varphi(j,l)}-q^{-\varphi(j,l)})}{q^{\varphi(j,l)}+1} + 1$. As a result, if $\varphi(j,l) = 0$, then $q^{\varphi(j,l)}$ is equal to 1. If $\varphi(j,l) = \pm 1$, then we obtain $\pm\frac{q^{\pm 1}(q-q^{-1})}{q^{\pm 1}+1} + 1 \equiv 1 \mathrm{mod} \mathbf{K}_1$.

Using this, the following equality holds in $\mathbf{K}_{m+n+2}/\mathbf{K}_{m+n+3}$. Set $\hbar = \bar{\varepsilon} \in \mathbf{K}_1/\mathbf{K}_2$.

$$①' \quad (-1)^{p(i,j)p(k,l)}(T_{ij}^{(r,m)}T_{kl}^{(s,n+2)} - T_{ij}^{(r-1,m+2)}T_{kl}^{(s+1,n)})$$
$$④' \quad + \hbar\theta(i,j,k)(T_{il}^{(r+1,m)}T_{kj}^{(s,n+1)} + T_{il}^{(r,m+1)}T_{kj}^{(s,n)})$$
$$⑤' \quad - \hbar\theta(i,j,-k)(T_{i,-l}^{(r-1,m)}T_{k,-j}^{(s,n+1)} - T_{i,-l}^{(r-1,m+1)}T_{k,-j}^{(s,n)})$$
$$=$$
$$⑥' \quad (T_{kl}^{(s,n+2)}T_{ij}^{(r,m)} - T_{kl}^{(s+1,n)}T_{ij}^{(r-1,m+2)})$$

$$\begin{aligned}
&(9)' \quad +\hbar\theta(i,j,k)(T_{il}^{(s,n+1)}T_{kj}^{(r+1,m)} + T_{il}^{(s,n)}T_{kj}^{(r,m+1)}) \\
&(10)' \quad -\hbar\theta(i,j,-k)(T_{-i,l}^{(s,n+1)}T_{-k,j}^{(r-1,m)} - T_{-i,l}^{(s,n)}T_{-k,j}^{(r-1,m+1)})
\end{aligned}$$

Next, the goal is to eliminate appearances of $r \pm 1$ in generators in $\mathbf{K}_{m+2}$ and analogous situations; we want to reduce to only r, s in the first superscript. For example, note that

$$T_{ij}^{(r-1,m+2)} = T_{ij}^{(r,m+2)} + T_{ij}^{(r-1,m+3)},$$

where $T_{ij}^{(r-1,m+2)} \in \mathbf{K}_{m+2}$, $T_{ij}^{(r,m+2)} \in \mathbf{K}_{m+2}$, and $T_{ij}^{(r-1,m+3)} \in \mathbf{K}_{m+3}$. Similarly,

$$T_{kl}^{(s+1,n)} = T_{kl}^{(s,n+1)} + T_{kl}^{(s,n)}.$$

As a result, line (1)′ becomes

$$\begin{aligned}
&(-1)^{p(i,j)p(k,l)}(T_{ij}^{(r,m)}T_{kl}^{(s,n+2)} - (T_{ij}^{(r,m+2)} + T_{ij}^{(r-1,m+3)})(T_{kl}^{(s,n+1)} + T_{kl}^{(s,n)})) \\
&\equiv (-1)^{p(i,j)p(k,l)}(T_{ij}^{(r,m)}T_{kl}^{(s,n+1)} - T_{ij}^{(r,m+2)}T_{kl}^{(s,n)}) \in \mathbf{K}_{m+n+2}/\mathbf{K}_{m+n+3}.
\end{aligned}$$

Applying similar substitutions to all lines yields

$$\begin{aligned}
&(-1)^{p(i,j)p(k,l)}(T_{ij}^{(r,m)}T_{kl}^{(s,n+2)} - T_{ij}^{(r,m+2)}T_{kl}^{(s,n)}) \\
&+\hbar\theta(i,j,k)(T_{il}^{(r,m)}T_{kj}^{(s,n+1)} + T_{il}^{(r,m+1)}T_{kj}^{(s,n)}) \\
&-\hbar\theta(i,j,-k)(T_{i,-l}^{(r,m)}T_{k,-j}^{(s,n+1)} - T_{i,-l}^{(r,m+1)}T_{k,-j}^{(s,n)}) \\
&\equiv \\
&(T_{kl}^{(s,n+2)}T_{ij}^{(r,m)} - T_{kl}^{(s,n)}T_{ij}^{(r,m+2)}) \\
&+\hbar\theta(i,j,k)(T_{il}^{(s,n+1)}T_{kj}^{(r,m)} + T_{il}^{(s,n)}T_{kj}^{(r,m+1)}) \\
&-\hbar\theta(i,j,-k)(T_{-i,l}^{(s,n+1)}T_{-k,j}^{(r,m)} - T_{-i,l}^{(s,n)}T_{-k,j}^{(r,m+1)})
\end{aligned} \tag{6.1.4}$$

We want to compare this to the defining relation of $Y_\hbar(\mathfrak{q}_n)$, repeated here.

$$
\begin{aligned}
&\left(\left[t_{ij}^{(m+2)}, t_{kl}^{(n)}\right] - \left[t_{ij}^{(m)}, t_{kl}^{(n+2)}\right]\right)\theta(i,k,l) \\
=&\hbar\left(t_{kj}^{(m+1)}t_{il}^{(n)} - t_{kj}^{(n)}t_{il}^{(m+1)} + t_{kj}^{(m)}t_{il}^{(n+1)} - t_{kj}^{(n+1)}t_{il}^{(m)}\right) \\
&-\hbar\left(t_{-k,j}^{(m+1)}t_{-i,l}^{(n)} - t_{k,-j}^{(n)}t_{i,-j}^{(m+1)} - t_{-k,j}^{(m)}t_{-i,l}^{(n+1)} + t_{k,-j}^{(n+1)}t_{i,-l}^{(m)}\right)(-1)^{|k|+|l|} \\
&\Longleftrightarrow \\
&\left(t_{ij}^{(m+2)}t_{kl}^{(n)} - (-1)^{p(i,j)p(k,l)}t_{kl}^{(n)}t_{ij}^{(m+2)}\right. \\
&\left.-t_{ij}^{(m)}t_{kl}^{(n+2)} + (-1)^{p(i,j)p(k,l)}t_{kl}^{(n+2)}t_{ij}^{(m)}\right)\theta(i,k,l) \\
=&\hbar\left(t_{kj}^{(m+1)}t_{il}^{(n)} - t_{kj}^{(n)}t_{il}^{(m+1)} + t_{kj}^{(m)}t_{il}^{(n+1)} - t_{kj}^{(n+1)}t_{il}^{(m)}\right) \\
&-\hbar\left(t_{-k,j}^{(m+1)}t_{-i,l}^{(n)} - t_{k,-j}^{(n)}t_{i,-j}^{(m+1)} - t_{-k,j}^{(m)}t_{-i,l}^{(n+1)} + t_{k,-j}^{(n+1)}t_{i,-l}^{(m)}\right)(-1)^{|k|+|l|}
\end{aligned}
$$

After switching (i, j, m) with (k, l, n), we produce

$$
\begin{aligned}
&\left(t_{kl}^{(n+2)}t_{ij}^{(m)} - (-1)^{p(k,l)p(i,j)}t_{ij}^{(m)}t_{kl}^{(n+2)}\right. \\
&\left.-t_{kl}^{(n)}t_{ij}^{(m+2)} + (-1)^{p(k,l)p(i,j)}t_{ij}^{(m+m)}t_{kl}^{(n)}\right)\theta(i,j,k) \\
=&\hbar\left(t_{il}^{(n+1)}t_{kj}^{(m)} - t_{il}^{(m)}t_{kj}^{(n+1)} + t_{il}^{(n)}t_{kj}^{(m+1)} - t_{il}^{(m+1)}t_{kj}^{(n)}\right) \\
&-\hbar\left(t_{-i,l}^{(n+1)}t_{-k,j}^{(m)} - t_{i,-l}^{(m)}t_{k,-l}^{(n+1)} - t_{-i,l}^{(n)}t_{-k,j}^{(m+1)} + t_{i,-l}^{(m+1)}t_{k,-j}^{(n)}\right)(-1)^{|i|+|j|}.
\end{aligned}
$$

Multiplying both sides by $-\theta(i, j, k)$ and using that

$$
\begin{aligned}
-\theta(i,j,k)(-1)^{|i|+|j|} &= -(-1)^{|i||j|+|i||k|+|k||j|+|i|+|j|} \\
&= -(-1)^{|i|(|k|+1)+|j|(|k|+1)+|i||j|} \\
&= -\theta(i,j,-k)
\end{aligned}
$$

yields (6.1.4) after identifying $T_{ij}^{(r,m)}$ with $t_{ij}^{(m)}$. This confirms that the map φ described in (6.1.5) below is a homomorphism.

Recall that $U_{\mathcal{A}}(\mathcal{L}^{tw}\mathfrak{q}_n)$ is generated by the $\tau_{ij}^{(r)}$, where $\tau_{ij}^{(r)} = T_{ij}^{(r)}/(q-q^{-1})$, unless $i=j$ and $r=0$, in which case $\tau_{ii}^{(0)} = (T_{ii}^{(0)}-1)/(q-1)$. Set $T_{ij}^{(r,0)} = \tau_{ij}^{(r)}$ and, inductively, $T_{ij}^{(r,m+1)} = T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)}$. We will refer to these elements as having double superscripts to distinguish them from the elements $T_{ij}^{(r)}$. Then, (6.1.4) holds for the $T_{ij}^{(r,m)}$ with $m \geq 0$ if $r \geq 2$. Below, we show that (6.1.4) in fact holds for $r \geq 1$, though further changes must be made to the computations to accommodate the $r = 0$ case.

Define the map

$$\varphi : Y(\mathfrak{q}_n) \to \bigoplus_{m=0}^{\infty} \mathbf{K}_m/\mathbf{K}_{m+1} \tag{6.1.5}$$
$$\varphi(t_{ij}^{(m)}) = T_{ij}^{(r,m)}$$

for some fixed choice of $r \geq 2$. By the computations above it preserves the defining relations of the Yangian of type $\mathfrak{q}$, so we can conclude that φ is an algebra homomorphism. Ultimately, we will show that it is in fact an isomorphism in Section 6.2.

Note that since one could chose any other integer greater than 2 and the relations still follow, we may as well choose the simpler route and generally only use $r = 2$. We can see that $r = 1$ will not cause problems by taking $R(2, m; s, n)$ and manipulating the equation in $\mathbf{K}_{m+n+2}/\mathbf{K}_{m+n+3}$; we will eventually arrive at (6.1.4) with $r = 1$. One can use $s = 1$ in (6.1.4) with the same justification.

6.2 ISOMORPHISM

In this section, we prove that the map φ above in (6.1.5) is an isomorphism, implying that the queer Yangian is a degenerate form of its associated quantum loop superalgebra. To show that φ is biective, we can prove that it sends a basis of ordered monomials to a basis of ordered monomials. This will first require defining an ordering on generators and monomials of $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$.

Recall that our presentation of $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$ consists of generators $T_{ij}^{(r)}$ for $-n \leq i \leq j \leq n$, $r \geq 0$ and $T_{ij}^{(r)}$ for $-n \leq j < i \leq n$, $r \geq 1$. Note that we can use $\tau_{ij}^{(r)}$ and $\tau_{ii}^{(0)}$ elements in place of these when necessary. Alternatively, we can use the generators $T_{ij}^{(0,m)}$ for $-n \leq i \leq j \leq n$, $m \geq 0$ and $T_{ij}^{(1,m)}$ for $-n \leq j < i \leq n$, $m \geq 0$. We produce this generating set using that $T_{ij}^{(0,1)} = \tau_{ij}^{(1)} - \tau_{ij}^{(0)}$ and $T_{ij}^{(1,0)} = \tau_{ij}^{(1)}$, then using that $T_{ij}^{(r,m+1)} = T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)}$ for larger values of m. Consider the pattern that arises when attempting to keep all r-values in $T_{ij}^{(r,m)}$ to either 0 or 1. These computations assume that $i < j$. If $j < i$, then we stop once we have an expression with all first superscripts, i.e. r-values, equal to 1.

$$\begin{aligned}
\tau_{ij}^{(1)} =& T_{ij}^{(1,0)} = T_{ij}^{(0,1)} + T_{ij}^{(0,0)} \\
\tau_{ij}^{(2)} =& T_{ij}^{(2,0)} = T_{ij}^{(1,0)} + T_{ij}^{(1,1)} = T_{ij}^{(0,1)} + T_{ij}^{(0,0)} + T_{ij}^{(0,2)} + T_{ij}^{(0,1)} \\
=& T_{ij}^{(0,2)} + 2T_{ij}^{(0,1)} + T_{ij}^{(0,0)} \\
\tau_{ij}^{(3)} =& T_{ij}^{(3,0)} = T_{ij}^{(2,1)} + T_{ij}^{(2,0)} \\
=& T_{ij}^{(1,2)} + T_{ij}^{(1,1)} + T_{ij}^{(1,0)} + T_{ij}^{(1,1)} \\
=& T_{ij}^{(1,2)} + 2T_{ij}^{(1,1)} + T_{ij}^{(1,0)} \\
=& T_{ij}^{(0,3)} + T_{ij}^{(0,2)} + 2T_{ij}^{(0,2)} + 2T_{ij}^{(0,1)} + T_{ij}^{(0,1)} + T_{ij}^{(0,0)} \\
=& T_{ij}^{(0,3)} + 3T_{ij}^{(0,2)} + 3T_{ij}^{(0,1)} + T_{ij}^{(0,0)} \\
\tau_{ij}^{(4)} =& T_{ij}^{(4,0)} = T_{ij}^{(3,1)} + T_{ij}^{(3,0)} = T_{ij}^{(2,2)} + T_{ij}^{(2,1)} + T_{ij}^{(1,2)} + 2T_{ij}^{(1,1)} + T_{ij}^{(1,0)} \\
=& T_{ij}^{(1,3)} + T_{ij}^{(1,2)} + T_{ij}^{(1,2)} + T_{ij}^{(1,1)} + T_{ij}^{(1,2)} + 2T_{ij}^{(1,1)} + T_{ij}^{(1,0)} \\
=& T_{ij}^{(1,3)} + 3T_{ij}^{(1,2)} + 3T_{ij}^{(1,1)} + T_{ij}^{(1,0)} = T_{ij}^{(0,4)} + T_{ij}^{(0,3)} + 3T_{ij}^{(0,3)} \\
&+ 3T_{ij}^{(0,2)} + 3T_{ij}^{(0,2)} + 3T_{ij}^{(0,1)} + T_{ij}^{(0,1)} + T_{ij}^{(0,0)} \\
=& T_{ij}^{(0,4)} + 4T_{ij}^{(0,3)} + 6T_{ij}^{(0,2)} + 4T_{ij}^{(0,1)} + T_{ij}^{(0,0)}
\end{aligned}$$

From this, we infer the following lemmas that allows us to replace sets of generators. The above equalities effectively act as our base cases.

Lemma 6.2. *For fixed m and $r \geq 1$,*

$$T_{ij}^{(r,m)} = \sum_{a=0}^{r-1} \binom{r-1}{a} T_{ij}^{(1,r-1+m-a)} = \sum_{a=0}^{r-1} \binom{r-1}{a} T_{ij}^{(1,m+a)}.$$

Proof. We prove the result by induction on r. The base case is clear, when $r = 1$, we have $T_{ij}^{(1,m)} = \binom{0}{0} T_{ij}^{(1,1-1+m-0)} = T_{ij}^{(1,m)}$. Thus, assume the result holds for all $n \leq r$. Then, we aim to show that

$$T_{ij}^{(r+1,m)} = \sum_{a=0}^{r} \binom{r}{a} T_{ij}^{(1,m+a)}.$$

$$\begin{aligned}
T_{ij}^{(r+1,m)} &= T_{ij}^{(r,m+1)} + T_{ij}^{(r,m)} \\
&= \sum_{a=0}^{r-1} \binom{r-1}{a} T_{ij}^{(1,m+a+1)} + \sum_{a=0}^{r-1} \binom{r-1}{a} T_{ij}^{(1,m+a)} \\
&= \sum_{a=1}^{r} \binom{r-1}{a-1} T_{ij}^{(1,m+a)} + \sum_{a=0}^{r-1} \binom{r-1}{a} T_{ij}^{(1,m+a)} \\
&= \sum_{a=1}^{r-1} \left[\left(\binom{r-1}{a-1} + \binom{r-1}{a} \right) T_{ij}^{(1,m+a)} \right] + T_{ij}^{(1,m+r)} + T_{ij}^{(1,m)} \\
&= \sum_{a=1}^{r-1} \left[\binom{r}{a} T_{ij}^{(1,m+a)} \right] + T_{ij}^{(1,m+r)} + T_{ij}^{(1,m)} \\
&= \sum_{a=0}^{r} \binom{r}{a} T_{ij}^{(1,m+a)}
\end{aligned}$$

□

Lemma 6.3. *For $-n \leq i < j \leq n$, fixed m, and $r \geq 0$,*

$$T_{ij}^{(r,m)} = \sum_{a=0}^{r} \binom{r}{a} T_{ij}^{(0,m+a)} = \sum_{a=0}^{r} \binom{r}{a} T_{ij}^{(0,r+m-a)}.$$

Proof. We prove this by induction on r. Base cases appear above.

$$\begin{aligned}
T_{ij}^{(r+1,m)} &= T_{ij}^{(r,m+1)} + T_{ij}^{(r,m)} \\
&= \sum_{a=0}^{r} \binom{r}{a} T_{ij}^{(0,a+m+1)} + \sum_{a=0}^{r} \binom{r}{a} T_{ij}^{(0,a+m)} \\
&= \sum_{a=1}^{r+1} \binom{r}{a-1} T_{ij}^{(0,a+m)} + \sum_{a=0}^{r} \binom{r}{a} T_{ij}^{(0,a+m)} \\
&= \sum_{a=1}^{r} \left[\left(\binom{r}{a-1} + \binom{r}{a} \right) T_{ij}^{(0,a+m)} \right] + T_{ij}^{(0,r+m+1)} + T_{ij}^{(0,m)} \\
&= \sum_{a=0}^{r+1} \left(\binom{r}{a-1} + \binom{r}{a} \right) T_{ij}^{(0,a+m)} \\
&= \sum_{a=0}^{r+1} \binom{r+1}{a} T_{ij}^{(0,a+m)}
\end{aligned}$$

□

Lemma 6.4. *For $-n \leq i \leq j \leq n$, and $r \geq 1$,*

$$\tau_{ij}^{(r)} = \sum_{a=0}^{r} \binom{r}{a} T_{ij}^{(0,r-a)} = \sum_{a=0}^{r} \binom{r}{a} T_{ij}^{(0,a)}.$$

Proof. This is a special case of the previous lemma; $\tau_{ij}^{(r)} = T_{ij}^{(r,0)}$. □

Lemma 6.5. *For all r, m,*

$$T_{ij}^{(r,m)} = \sum_{a=r}^{m+r} \binom{m}{a-r} (-1)^{m+r-a} \tau_{ij}^{(a)}.$$

Proof. Prove by induction. First, we prove the assertion for fixed r with induction on m. We see that the base case holds; $T_{ij}^{(r,0)} = \tau_{ij}^{(r)}, T_{ij}^{(r,1)} = T_{ij}^{(r+1,0)} - T_{ij}^{(r,0)} = \binom{1}{0}(-1)^{0+r+1-r-1}\tau_{ij}^{(r+1)} + \binom{1}{1}(-1)^{0+r-r}\tau_{ij}^{(r)}$. Then,

$$T_{ij}^{(r,m+1)} = T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)}$$

$$
\begin{aligned}
&= \sum_{a=r+1}^{m+r+1} \binom{m}{a-r-1} (-1)^{m+r+1-a} \tau_{ij}^{(a)} - \sum_{a=r}^{m+r} \binom{m}{a-r} (-1)^{m+r-a} \tau_{ij}^{(a)} \\
&= - \sum_{a=r+1}^{m+r} \left(\binom{m}{a-r-1} (-1)^{m+r-a} \tau_{ij}^{(a)} \right) + \binom{m}{m} (-1)^0 \tau_{ij}^{(m+r+1)} \\
&\quad - \sum_{a=r+1}^{m+r} \left(\binom{m}{a-r} (-1)^{m+r-a} \tau_{ij}^{(a)} \right) - \binom{m}{0} (-1)^m \tau_{ij}^{(r)} \\
&= - \sum_{a=r+1}^{m+r} \binom{m+1}{a-r} (-1)^{m+r-a} \tau_{ij}^{(a)} + \binom{m}{0} (-1)^{m+1} \tau_{ij}^{(r)} \\
&\quad + \binom{m}{m} (-1)^0 \tau_{ij}^{(m+r+1)} \\
&= \sum_{a=r}^{m+r+1} \binom{m+1}{a-r} (-1)^{m+r+1-a} \tau_{ij}^{(a)}
\end{aligned}
$$

□

When considering an ordering on monomials built up from generators, we will use the notion of the *height* of a generator. For a generator $T_{ij}^{(r,m)}$, we define $\text{height}(T_{ij}^{(r,m)}) = j - i$. Extend this to monomials; the height of a monomial is the sum of the weights of each generator. For instance,

$$
\text{height}(T_{i_1,j_1}^{(r_1,m_1)} T_{i_2,j_2}^{(r_2,m_2)} \cdots T_{i_p,j_p}^{(r_p,m_p)}) = \sum_{k=1}^{p} (j_k - i_k)
$$

Proposition 6.6. *Consider monomials formed by products of $T_{ij}^{(r)}$. Define the following ordering on these single-superscript generators:*

$$
\begin{aligned}
T_{ij}^{(r)} < T_{kl}^{(s)} \quad & \text{if:} \quad i > k \\
& \text{or:} \quad i = k, j > l \\
& \text{or:} \quad i = k, j = l, r > s
\end{aligned}
$$

Then, ordered monomials in these generators span $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$.

In this proof and throughout this section, if the generators in a monomial are in the correct order, we can refer to the monomial as *reduced*.

Proof. Take the monomial $M = c_M \prod_{i,j,r} T_{ij}^{(r)}$ where $c_M \in \mathbb{C}$. First, we show that if a monomial with two terms is not reduced, it can be rewritten as a reduced monomial along with the sum of other monomials either with generators in correct order or, if in incorrect order, with smaller sums of superscripts. Thus, by induction, these incorrect monomials can themselves be rewritten correctly (as *reduced* monomials) since those sums of superscripts can only decrease finitely, to 0.

Suppose we have the incorrectly ordered monomial $T_{kl}^{(s+2)} T_{ij}^{(r)}$, where the issue lies specifically in that $i > k$. Then, by (2.4.15),

$$
\begin{aligned}
& q^{\varphi(i,k)} T_{kl}^{(s+2)} T_{ij}^{(r)} \\
&= (-1)^{p(i,j)p(k,l)} q^{\varphi(j,l)} (T_{ij}^{(r)} T_{kl}^{(s+2)}) \\
&+ (-1)^{p(i,j)p(k,l)} q^{\varphi(j,l)} (T_{ij}^{(r)} T_{kl}^{(s)} - T_{ij}^{(r-1)} T_{kl}^{(s+1)} - T_{ij}^{(r+1)} T_{kl}^{(s+1)}) \\
&+ \{j < l\} \theta(i,j,k) \varepsilon (T_{il}^{(r)} T_{kj}^{(s+2)} - T_{il}^{(r-1)} T_{kj}^{(s+1)} - T_{il}^{(r+1)} T_{kj}^{(s+1)} + T_{il}^{(r)} T_{kj}^{(s)}) \\
&+ \{-l < j\} \theta(-i,-j,k) \varepsilon (T_{i,-l}^{(r)} T_{k,-j}^{(s+2)} - T_{i,-l}^{(r-1)} T_{k,-j}^{(s+1)} - T_{i,-l}^{(r+1)} T_{k,-j}^{(s+1)} + T_{i,-l}^{(r)} T_{k,-j}^{(s)}) \\
&+ \varepsilon \theta(i,j,k) (T_{il}^{(r+1)} T_{kj}^{(s+1)} - T_{il}^{(r)} T_{kj}^{(s)}) \\
&- \varepsilon \theta(i,j,-k) (T_{i,-l}^{(r-1)} T_{k,-j}^{(s+1)} - T_{i,-l}^{(r)} T_{k,-j}^{(s)}) \\
&- q^{\varphi(i,k)} (T_{kl}^{(s)} T_{ij}^{(r)} - T_{kl}^{(s+1)} T_{ij}^{(r-1)} - T_{kl}^{(s+1)} T_{ij}^{(r+1)}) \qquad (6.2.1) \\
&- \{k < i\} \theta(i,j,k) \varepsilon (T_{il}^{(s+2)} T_{kj}^{(r)} - T_{il}^{(s+1)} T_{kj}^{(r-1)} - T_{il}^{(s+1)} T_{kj}^{(r+1)} + T_{il}^{(s)} T_{kj}^{(r)}) \\
&- \{i < -k\} \theta(-i,-j,k) \varepsilon (T_{-i,l}^{(s+2)} T_{-k,j}^{(r)} - T_{-i,l}^{(s+1)} T_{-k,j}^{(r-1)} - T_{-i,l}^{(s+1)} T_{-k,j}^{(r+1)} + T_{-i,l}^{(s)} T_{-k,j}^{(r)}) \\
&- \varepsilon \theta(i,j,k) (T_{il}^{(s+1)} T_{kj}^{(r+1)} - T_{il}^{(s)} T_{kj}^{(r)}) \\
&+ \varepsilon \theta(i,j,-k) (T_{-i,l}^{(s+1)} T_{-k,j}^{(r-1)} - T_{-i,l}^{(s)} T_{-k,j}^{(r)}).
\end{aligned}
$$

Next, we show that all terms from line (6.2.1) onwards satisfy the condition of either being properly ordered or of smaller sum of superscripts. First, since $i > k$, any monomial in the form $T_{ij}^{(r)}T_{kl}^{(s)}$ where j, r, l, s can be anything is in the proper ordering. This leaves the following terms to consider:

$$\begin{aligned}
&- q^{\varphi(i,k)}(T_{kl}^{(s)}T_{ij}^{(r)} - T_{kl}^{(s+1)}T_{ij}^{(r-1)} - T_{kl}^{(s+1)}T_{ij}^{(r+1)}) \\
&- \{i < -k\}\theta(-i,-j,k)\varepsilon(T_{-i,l}^{(s+2)}T_{-k,j}^{(r)} - T_{-i,l}^{(s+1)}T_{-k,j}^{(r-1)} - T_{-i,l}^{(s+1)}T_{-k,j}^{(r+1)} + T_{-i,l}^{(s)}T_{-k,j}^{(r)}) \\
&+ \varepsilon\theta(i,j,-k)(T_{-i,l}^{(s+1)}T_{-k,j}^{(r-1)} - T_{-i,l}^{(s)}T_{-k,j}^{(r)}).
\end{aligned}$$

We then use induction on either the heights, sums of superscripts, or first superscripts to show that any of these incorrectly ordered terms can, eventually, be written as a sum of reduced monomials. In the first line, the terms

$$q^{\varphi(i,k)}(T_{kl}^{(s)}T_{ij}^{(r)} - T_{kl}^{(s+1)}T_{ij}^{(r-1)})$$

have a sum of superscripts lesser than our original monomial and the term with $T_{kl}^{(s+1)}T_{ij}^{(r+1)}$, while having the same sum of superscripts, sees a decrease in the superscript in the first generator. Thus, this will continue to decrease with each attempt at obtaining reduced monomials. Eventually, if $s = -2$, say, we get $T_{kl}^{(-1)}T_{ij}^{(r+1)} = 0$. Next, the terms

$$\{i < -k\}\theta(-i,-j,k)\varepsilon(T_{-i,l}^{(s+2)}T_{-k,j}^{(r)} - T_{-i,l}^{(s+1)}T_{-k,j}^{(r-1)} - T_{-i,l}^{(s+1)}T_{-k,j}^{(r+1)} + T_{-i,l}^{(s)}T_{-k,j}^{(r)})$$

will also eventually vanish because of a decrease in height. When $i < -k$, $i + k < 0$, so

$$\text{height}(T_{-i,l}^{(s+2)}T_{-k,j}^{(r)}) = j + l + i + k < j + l - i - k = \text{height}(T_{kl}^{(s+2)}T_{ij}^{(r)}).$$

This means that those terms will also vanish after enough attempts at rewriting all elements in terms of reduced monomials, as height cannot decrease indefinitely.

Finally, we can use induction on the sum of superscripts on

$$\varepsilon\theta(i,j,-k)(T^{(s+1)}_{-i,l}T^{(r-1)}_{-k,j} - T^{(s)}_{-i,l}T^{(r)}_{-k,j}).$$

These terms have a sum of superscripts $s+r$, a decrease in comparison to our original sum of superscripts $s+r+2$. Thus, repeated iterations of attempts at obtaining properly ordered monomials will decrease the sum of superscripts until one or both generators vanish. We use that $T^{(m)}_{ij} = 0$ for all $m < 0$ and for $m = 0$ if $i \neq j$.

No details will be given to prove the second case as it is very similar to the first. This is when the two generators are in the wrong order but $i = k, l < j$, i.e. the monomial is $T^{(s+2)}_{il}T^{(r)}_{ij}$.

Finally, we have the case when the two generators are in the wrong order but $i = k, j = l$ and we differentiate using superscripts. Let $r+1 < s+1$, i.e. the incorrectly ordered monomial is $T^{(r+1)}_{ij}T^{(s+1)}_{ij}$. Here, the approach is slightly different. Start by isolating the term $T^{(r+1)}_{ij}T^{(s+1)}_{ij}$ in (2.4.15). Use that $\varphi(i,i) = (\delta_{i,i} + \delta_{i,-i})\mathrm{sign}(i) = \mathrm{sign}(i)$, $p(i,j)^2 = (|i|+|j|)^2 = |i|^2 + 2|i||j| + |j|^2 = |i| + 2|i||j| + |j|$ so $(-1)^{p(i,j)p(i,j)} = (-1)^{|i|+|j|}$, and $\theta(i,j,i) = (-1)^{|i||j|+|j||i|+|i||i|} = (-1)^{2|i||j|+|i|} = (-1)^{|i|}, \theta(i,j,-i) = (-1)^{|i||j|+|j||-i|+|-i||i|} = (-1)^{|j|(|i|+|-i|)} = (-1)^{|j|}$.

$$\begin{aligned}
&\varepsilon\theta(i,j,i)T^{(r+1)}_{ij}T^{(s+1)}_{ij}\\
&\quad= -(-1)^{p(i,j)p(i,j)}(q^{\varphi(j,j)}T^{(r)}_{ij}T^{(s+2)}_{ij} - q^{\varphi(i,i)}T^{(s+2)}_{ij}T^{(r)}_{ij})\\
&\qquad -(-1)^{p(i,j)p(i,j)}q^{\varphi(j,j)}(T^{(r)}_{ij}T^{(s)}_{ij} - T^{(r-1)}_{ij}T^{(s+1)}_{ij} - T^{(r+1)}_{ij}T^{(s+1)}_{ij})\\
&\qquad -\{j<j\}\theta(i,j,i)\varepsilon(T^{(r)}_{ij}T^{(s+2)}_{ij} - T^{(r-1)}_{ij}T^{(s+1)}_{ij} - T^{(r+1)}_{ij}T^{(s+1)}_{ij} + T^{(r)}_{ij}T^{(s)}_{ij})\\
&\qquad -\{-j<j\}\theta(-i,-j,i)\varepsilon(T^{(r)}_{i,-j}T^{(s+2)}_{i,-j} - T^{(r-1)}_{i,-j}T^{(s+1)}_{i,-j}\\
&\qquad - T^{(r+1)}_{i,-j}T^{(s+1)}_{i,-j} + T^{(r)}_{i,-j}T^{(s)}_{i,-j})\\
&\qquad -\varepsilon\theta(i,j,i)(-T^{(r)}_{ij}T^{(s)}_{ij}) + \varepsilon\theta(i,j,-i)(T^{(r-1)}_{i,-j}T^{(s+1)}_{i,-j} - T^{(r)}_{i,-j}T^{(s)}_{i,-j})
\end{aligned}$$

$$
\begin{aligned}
&+ q^{\varphi(i,i)}(T_{ij}^{(s)}T_{ij}^{(r)} - T_{ij}^{(s+1)}T_{ij}^{(r-1)} - T_{ij}^{(s+1)}T_{ij}^{(r+1)}) \\
&+ \{i < i\}\theta(i,j,i)\varepsilon(T_{ij}^{(s+2)}T_{ij}^{(r)} - T_{ij}^{(s+1)}T_{ij}^{(r-1)} \\
&- T_{ij}^{(s+1)}T_{ij}^{(r+1)} + T_{ij}^{(s)}T_{ij}^{(r)}) \\
&+ \{i < -i\}\theta(-i,-j,i)\varepsilon(T_{-i,j}^{(s+2)}T_{-i,j}^{(r)} - T_{-i,j}^{(s+1)}T_{-i,j}^{(r-1)} \\
&- T_{-i,j}^{(s+1)}T_{-i,j}^{(r+1)} + T_{-i,j}^{(s)}T_{-i,j}^{(r)}) \\
&+ \varepsilon\theta(i,j,i)(T_{ij}^{(s+1)}T_{ij}^{(r+1)} - T_{ij}^{(s)}T_{ij}^{(r)}) \\
&- \varepsilon\theta(i,j,-i)(T_{-i,j}^{(s+1)}T_{-i,j}^{(r-1)} - T_{-i,j}^{(s)}T_{-i,j}^{(r)}). \\
= &- (-1)^{|i|+|j|}(q^{\mathrm{sign}(j)}T_{ij}^{(r)}T_{ij}^{(s+2)} - q^{\mathrm{sign}(i)}T_{ij}^{(s+2)}T_{ij}^{(r)}) \\
&- (-1)^{|i|+|j|}q^{\mathrm{sign}(j)}(T_{ij}^{(r)}T_{ij}^{(s)} - T_{ij}^{(r-1)}T_{ij}^{(s+1)} - T_{ij}^{(r+1)}T_{ij}^{(s+1)}) \\
&- \varepsilon(-1)^{|i|}(-T_{ij}^{(r)}T_{ij}^{(s)}) + \varepsilon(-1)^{|j|}(T_{i,-j}^{(r-1)}T_{i,-j}^{(s+1)} - T_{i,-j}^{(r)}T_{i,-j}^{(s)}) \\
&- \{-j < j\}\theta(-i,-j,i)\varepsilon(T_{i,-j}^{(r)}T_{i,-j}^{(s+2)} - T_{i,-j}^{(r-1)}T_{i,-j}^{(s+1)} \\
&- T_{i,-j}^{(r+1)}T_{i,-j}^{(s+1)} + T_{i,-j}^{(r)}T_{i,-j}^{(s)}) \\
&+ \{i < -i\}\theta(-i,-j,i)\varepsilon(T_{-i,j}^{(s+2)}T_{-i,j}^{(r)} - T_{-i,j}^{(s+1)}T_{-i,j}^{(r-1)} \\
&- T_{-i,j}^{(s+1)}T_{-i,j}^{(r+1)} + T_{-i,j}^{(s)}T_{-i,j}^{(r)}) \\
&+ q^{\mathrm{sign}(i)}(T_{ij}^{(s)}T_{ij}^{(r)} - T_{ij}^{(s+1)}T_{ij}^{(r-1)} - T_{ij}^{(s+1)}T_{ij}^{(r+1)}) \\
&+ \varepsilon(-1)^{|i|}(T_{ij}^{(s+1)}T_{ij}^{(r+1)} - T_{ij}^{(s)}T_{ij}^{(r)}) - \varepsilon(-1)^{|j|}(T_{-i,j}^{(s+1)}T_{-i,j}^{(r-1)} - T_{-i,j}^{(s)}T_{-i,j}^{(r)}).
\end{aligned}
$$

All terms in the last three rows are correct as is $(-1)^{|i|+|j|}q^{\mathrm{sign}(i)}T_{ij}^{(s+2)}T_{ij}^{(r)}$ in the first row. The incorrectly ordered terms, up to a coefficient, are $T_{ij}^{(r)}T_{ij}^{(s+2)}$, $T_{ij}^{(r)}T_{ij}^{(s)}$, $T_{ij}^{(r-1)}T_{ij}^{(s+1)}$, $T_{ij}^{(r+1)}T_{ij}^{(s+1)}$, $T_{i,-j}^{(r-1)}T_{i,-j}^{(s+1)}$, $T_{i,-j}^{(r)}T_{i,-j}^{(s+2)}$, $T_{i,-j}^{(r+1)}T_{i,-j}^{(s+1)}$, and $T_{i,-j}^{(r)}T_{i,-j}^{(s)}$. In most of these cases, we can use the argument on induction on the sum of superscripts. If $j < -j$ or, equivalently, $j < 0$, then the only remaining term is $(-1)^{|i|+|j|}q^{\mathrm{sign}(j)}(T_{ij}^{(r+1)}T_{ij}^{(s+1)})$, which we can simply move to the left hand side and consolidate with our original monomial. If $j > -j$ or, equivalently, $j > 0$, then the two remaining terms are $(-1)^{|i|+|j|}q^{\mathrm{sign}(j)}(T_{ij}^{(r+1)}T_{ij}^{(s+1)})$, which can be consolidated as just stated, and $\{-j < j\}\theta(-i,-j,i)\varepsilon T_{i,-j}^{(r+1)} T_{i,-j}^{(s+1)}$. To deal with this second term, we simply replace j with $-j$ in (2.4.15) and isolate for $T_{i,-j}^{(r+1)} T_{i,-j}^{(s+1)}$. The right hand side of the equation will only contain reduced monomials or ones that can be reduced by induction. We do not have any problematic $\{j < -j\}\theta(-i,j,i)\varepsilon T_{ij}^{(r+1)} T_{ij}^{(s+1)}$ term because of the assumption that $j > -j$.

Thus, we know that any monomial in two generators can be corrected according to our chosen ordering. It follows that we can do this with a monomial in any arbitrary length, as the reordering process comprises swapping consecutive generators in the way described above. □

Proposition 6.7. *Consider monomials formed by products of $\tau_{ij}^{(r)}$. Define the following ordering on generators:*

$$\tau_{ij}^{(r)} < \tau_{kl}^{(s)} \quad \text{if:} \quad i > k$$
$$\text{or:} \quad i = k, j > l$$
$$\text{or:} \quad i = k, j = l, r > s$$

Then, ordered monomials in these generators span $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$.

Proof. This follows from the previous proposition. We use the relation

$$\begin{aligned}
&q^{\varphi(i,k)} T_{kl}^{(s+2)} T_{ij}^{(r)} \\
&\quad = (-1)^{p(i,j)p(k,l)} q^{\varphi(j,l)} (T_{ij}^{(r)} T_{kl}^{(s+2)}) \\
&\qquad + (-1)^{p(i,j)p(k,l)} q^{\varphi(j,l)} (T_{ij}^{(r)} T_{kl}^{(s)} - T_{ij}^{(r-1)} T_{kl}^{(s+1)} - T_{ij}^{(r+1)} T_{kl}^{(s+1)}) \\
&\qquad + \{j < l\}\theta(i,j,k)\varepsilon (T_{il}^{(r)} T_{kj}^{(s+2)} - T_{il}^{(r-1)} T_{kj}^{(s+1)} - T_{il}^{(r+1)} T_{kj}^{(s+1)} + T_{il}^{(r)} T_{kj}^{(s)}) \\
&\qquad + \{-l < j\}\theta(-i,-j,k)\varepsilon (T_{i,-l}^{(r)} T_{k,-j}^{(s+2)} - T_{i,-l}^{(r-1)} T_{k,-j}^{(s+1)} - T_{i,-l}^{(r+1)} T_{k,-j}^{(s+1)} + T_{i,-l}^{(r)} T_{k,-j}^{(s)}) \\
&\qquad + \varepsilon\theta(i,j,k) (T_{il}^{(r+1)} T_{kj}^{(s+1)} - T_{il}^{(r)} T_{kj}^{(s)}) \\
&\qquad - \varepsilon\theta(i,j,-k) (T_{i,-l}^{(r-1)} T_{k,-j}^{(s+1)} - T_{i,-l}^{(r)} T_{k,-j}^{(s)})
\end{aligned}$$

$$
\begin{aligned}
&- q^{\varphi(i,k)}(T_{kl}^{(s)}T_{ij}^{(r)} - T_{kl}^{(s+1)}T_{ij}^{(r-1)} - T_{kl}^{(s+1)}T_{ij}^{(r+1)})\\
&- \{k<i\}\theta(i,j,k)\varepsilon(T_{il}^{(s+2)}T_{kj}^{(r)} - T_{il}^{(s+1)}T_{kj}^{(r-1)} - T_{il}^{(s+1)}T_{kj}^{(r+1)} + T_{il}^{(s)}T_{kj}^{(r)})\\
&- \{i<-k\}\theta(-i,-j,k)\varepsilon(T_{-i,l}^{(s+2)}T_{-k,j}^{(r)} - T_{-i,l}^{(s+1)}T_{-k,j}^{(r-1)} - T_{-i,l}^{(s+1)}T_{-k,j}^{(r+1)} + T_{-i,l}^{(s)}T_{-k,j}^{(r)})\\
&- \varepsilon\theta(i,j,k)(T_{il}^{(s+1)}T_{kj}^{(r+1)} - T_{il}^{(s)}T_{kj}^{(r)})\\
&+ \varepsilon\theta(i,j,-k)(T_{-i,l}^{(s+1)}T_{-k,j}^{(r-1)} - T_{-i,l}^{(s)}T_{-k,j}^{(r)}). \qquad (6.2.2)
\end{aligned}
$$

Note that there is the possibility that $i = j$ and $r = 0$. Then, we must use that $\tau_{ii}^{(0)} = \frac{T_{ii}^{(0)}}{q-1}$. To work with this, take $T_{ii}^{(0)}$ and rewrite it as $T_{ii}^{(0)} = T_{ii}^{(0)} - 1 + 1$. After a few more algebraic manipulations, we can proceed. The same would apply if $k = l$ and $s + 2 = 0$. Here, we restrict to the simpler case and assume no rearranging is needed. Divide by $(q - q^{-1})^2$, use that $\tau_{ij}^{(r)} = \frac{T_{ij}^{(r)}}{q-q^{-1}}$, and it becomes

$$
\begin{aligned}
&q^{\varphi(i,k)}\tau_{kl}^{(s+2)}\tau_{ij}^{(r)}\\
&\quad =(-1)^{p(i,j)p(k,l)}q^{\varphi(j,l)}(\tau_{ij}^{(r)}\tau_{kl}^{(s)+2})\\
&\qquad + (-1)^{p(i,j)p(k,l)}q^{\varphi(j,l)}(\tau_{ij}^{(r)}\tau_{kl}^{(s)} - \tau_{ij}^{(r-1)}\tau_{kl}^{(s+1)} - \tau_{ij}^{(r+1)}\tau_{kl}^{(s+1)})\\
&\qquad + \{j<l\}\theta(i,j,k)\varepsilon(\tau_{il}^{(r)}\tau_{kj}^{(s+2)} - \tau_{il}^{(r-1)}\tau_{kj}^{(s+1)} - \tau_{il}^{(r+1)}\tau_{kj}^{(s+1)} + \tau_{il}^{(r)}\tau_{kj}^{(s)})\\
&\qquad + \{-l<j\}\theta(-i,-j,k)\varepsilon(\tau_{i,-l}^{(r)}\tau_{k,-j}^{(s+2)} - \tau_{i,-l}^{(r-1)}\tau_{k,-j}^{(s+1)} - \tau_{i,-l}^{(r+1)}\tau_{k,-j}^{(s+1)} + \tau_{i,-l}^{(r)}\tau_{k,-j}^{(s)})\\
&\qquad + \varepsilon\theta(i,j,k)(\tau_{il}^{(r+1)}\tau_{kj}^{(s+1)} - \tau_{il}^{(r)}\tau_{kj}^{(s)})\\
&\qquad - \varepsilon\theta(i,j,-k)(\tau_{i,-l}^{(r-1)}\tau_{k,-j}^{(s+1)} - \tau_{i,-l}^{(r)}\tau_{k,-j}^{(s)})\\
&\qquad - q^{\varphi(i,k)}(\tau_{kl}^{(s)}\tau_{ij}^{(r)} - \tau_{kl}^{(s+1)}\tau_{ij}^{(r-1)} - \tau_{kl}^{(s+1)}\tau_{ij}^{(r+1)})\\
&\qquad - \{k<i\}\theta(i,j,k)\varepsilon(\tau_{il}^{(s+2)}\tau_{kj}^{(r)} - \tau_{il}^{(s+1)}\tau_{kj}^{(r-1)} - \tau_{il}^{(s+1)}\tau_{kj}^{(r+1)} + \tau_{il}^{(s)}\tau_{kj}^{(r)})\\
&\qquad - \{i<-k\}\theta(-i,-j,k)\varepsilon(\tau_{-i,l}^{(s+2)}\tau_{-k,j}^{(r)} - \tau_{-i,l}^{(s+1)}\tau_{-k,j}^{(r-1)} - \tau_{-i,l}^{(s+1)}\tau_{-k,j}^{(r+1)} + \tau_{-i,l}^{(s)}\tau_{-k,j}^{(r)})\\
&\qquad - \varepsilon\theta(i,j,k)(\tau_{il}^{(s+1)}\tau_{kj}^{(r+1)} - \tau_{il}^{(s)}\tau_{kj}^{(r)})\\
&\qquad + \varepsilon\theta(i,j,-k)(\tau_{-i,l}^{(s+1)}\tau_{-k,j}^{(r-1)} - \tau_{-i,l}^{(s)}\tau_{-k,j}^{(r)}) \qquad (6.2.3)
\end{aligned}
$$

when rewritten using $\tau_{ij}^{(r)}-$ type generators. □

Finally, we make one last change of generators and use the third set, in the form $T_{ij}^{(r,m)}$.

Proposition 6.8. *Consider the monomials formed by products of $T_{ij}^{(0,m)}$ for $-n \leq i \leq j \leq n$, $m \geq 0$ and $T_{ij}^{(1,m)}$ for $-n \leq j < i \leq n$, $m \geq 0$. Define the following ordering on these double-superscript generators:*

$$\begin{aligned} T_{ij}^{(0,m)} < T_{kl}^{(0,n)} \quad & \text{if:} \quad i > k \\ & \text{or:} \quad i = k, j > l \\ & \text{or:} \quad i = k, j = l, m > n \\ T_{ij}^{(1,m)} < T_{kl}^{(1,n)} \quad & \text{if:} \quad i > k \\ & \text{or:} \quad i = k, j > l \\ & \text{or:} \quad i = k, j = l, m > n \end{aligned}$$

Rewriting $T_{ij}^{(1,m)}$ as $T_{ij}^{(0,m+1)} + T_{ij}^{(0,m)}$ allows us to compare the two types of generators. For instance, $T_{ij}^{(1,m)} < T_{ij}^{(0,m)}$ because $T_{ij}^{(1,m)} = T_{ij}^{(0,m+1)} + T_{ij}^{(0,m)}$ and $T_{ij}^{(0,m+1)} < T_{ij}^{(0,m)}$. Thus, $T_{ij}^{(1,n)} < T_{ij}^{(0,m)}$ if $n \geq m$.

Then, using this partial ordering and comparing monomials using lexicographical ordering, these ordered monomials span $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$.

Proof. Lemma 6.4 tells us that

$$\tau_{ij}^{(r)} = \sum_{a=0}^{r} \binom{r}{a} T_{ij}^{(0,a)}.$$

Suppose we have a reduced monomial $\tau_{ij}^{(r)}\tau_{kl}^{(s)}$, i.e. $\tau_{ij}^{(r)} < \tau_{kl}^{(s)}$ according to the ordering defined in Proposition 6.7. Then,

$$\tau_{ij}^{(r)}\tau_{kl}^{(s)} = \left(\sum_{a=0}^{r} \binom{r}{a} T_{ij}^{(0,a)}\right)\left(\sum_{b=0}^{s} \binom{s}{b} T_{kl}^{(0,b)}\right)$$

$$= \sum_{a=0}^{r} \sum_{b=0}^{s} \binom{r}{a} \binom{s}{b} T_{ij}^{(0,a)} T_{kl}^{(0,b)}.$$

Examine the different flavours of terms from this sum. Note that every term preserves the order of the subscripts in the generators, hence any possible nonreduced monomials that surface can only have been produced because of the superscripts. If $a \geq b$, then the term must be a reduced monomial. If $a < b$, the term may not be reduced. However, the sum of secondary superscripts of the generators in the monomial is less than or equal to $r + s$. If equal, we know that the monomial is reduced. Else, this lower upper bound on the sum of secondary superscripts means that, by induction, after enough attempts at obtaining reduced monomials this sum will eventually decrease to zero and we are done.

For a monomial of arbitrary length, the reducing process comprises considering pairs of adjacent generators, which we know can be done. Thus, we have an ordering on the generators $T_{ij}^{(0,r)}$ and $T_{ij}^{(1,r)}$. $\square$

Proposition 6.9. *The ordered monomials in the $T_{ij}^{(r)}$ for $i \leq j, r \geq 0$ or $i > j, r \geq 1$ are linearly independent over $\mathbb{C}(q)$.*

Proof. Suppose $\sum c_M M = 0$, where $c_M \in \mathbb{C}(q)$ and M is a monomial in the $\tau_{ij}^{(r)}$. Clear any denominators in the coefficients c_M so that we effectively have that $c_M \in \mathbb{C}[q, q^{-1}]$. Then, if each of these c_M is equal to $(q-1)^{e_M} p(q, q^{-1})$ for some exponent e_M. After dividing through by one of these $(q-1)^{e_M}$, we can assume that at least one c_M does not lie in the ideal $(q-1)\mathbb{C}[q, q^{-1}]$. Next, let $\mathcal{A} = \mathbb{C}[q, q^{-1}]$. Then, if $U_{\mathcal{A}}(\mathcal{L}^{tw}\mathfrak{q}_n)$ is the $\mathcal{A}$-subalgebra of $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$ generated by the $\tau_{ij}^{(r)}$, we have that $0 = \sum c_M M \in U_{\mathcal{A}}(\mathcal{L}^{tw}\mathfrak{q}_n)$. We have that

$$U_{\mathcal{A}}(\mathcal{L}^{tw}\mathfrak{q}_n)/(q-1)U_{\mathcal{A}}(\mathcal{L}^{tw}\mathfrak{q}_n) \simeq U(\mathcal{L}^{tw}\mathfrak{q}_n).$$

As a result, passing to this quotient produces

$$\sum \overline{c_M}\,\overline{M} = 0, \quad \overline{c_M} \in \mathbb{C}[q, q^{-1}]/(q-1)\mathbb{C}[q, q^{-1}] \simeq \mathbb{C}$$

Here, $\overline{M}$ belongs to the classical PBW basis for $U(\mathcal{L}^{tw}\mathfrak{q}_n)$, which is made up of monomials in the generators of $\mathcal{L}^{tw}\mathfrak{q}_n$. As a result, this forces that $\overline{c_M} = 0$ for all M, thus each c_M is an element of $(q-1)\mathbb{C}[q, q^{-1}]$. However, this is a direct contradiction to the assumption that at least one c_M has no factor of $(q-1)$. As a result, each $c_M = 0$ and thus the monomials M are linearly independent. □

Corollary 6.10. *The ordered monomials in the $T_{ij}^{(r)}$ form a basis of $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$.*

Proof. This follows directly from Propositions 6.8 and 6.9. □

In particular, these monomials may have positive and negative powers of $T_{ii}^{(0)}$; by [Ols92] (6.4), diagonal elements have exponents in $\mathbb{Z}$. However, since $T_{ii}^{(0)} T_{-i,-i}^{(0)} = 1 = T_{-i,-i}^{(0)} T_{ii}^{(0)}$ by defining relations, we can choose to only take generators with $i > 0$ and replace $\left(T_{ii}^{(0)}\right)^d$ with $\left(T_{-i,-i}^{(0)}\right)^{-d}$ where necessary. Further, $\left(T_{ij}^{(0)}\right)^2 = 0$ when $T_{ij}^{(0)}$ is odd, i.e. $|i| + |j| = 1$. This means that the ordered monomials will contain these generators only with exponents of 0 or 1. Again, the same held in [Ols92] (6.4).

Corollary 6.11. *The ordered monomials in the $\tau_{ij}^{(r)}$ form a basis of $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$.*

Proof. This follows directly from Propositions 6.10 due to the simple relationship between the two types of generators. □

Proposition 6.12. *The ordered monomials in the $T_{ij}^{(0,m)}$ for $-n \le i \le j \le n$, $m \ge 0$ and $T_{ij}^{(1,m)}$ for $-n \le j < i \le n$, $m \ge 0$ are linearly independent over $\mathbb{C}(q)$.*

Proof. Take a linear combination $\sum c_M M$ in these ordered monomials in $T_{ij}^{(0,m)}$ and $T_{ij}^{(1,m)}$ with coefficients $c_M \in \mathbb{C}(q)$. Suppose $\sum c_M M = 0$. Denote by M^{max}

the maximal monomial with $c_{M^{max}} \neq 0$. By Lemma 6.5, we know that $T_{ij}^{(0,m)} = \sum_{a=0}^{m} \binom{m}{a}(-1)^{m-a}\tau_{ij}^{(a)}$ and $T_{ij}^{(1,m)} = \sum_{a=1}^{m+1} \binom{m}{a-1}(-1)^{m+1-a}\tau_{ij}^{(a)}$. Thus, replace every $T_{ij}^{(0,m)}$ in each M with $\tau_{ij}^{(m)} + \sum_{a=0}^{m-1} \binom{m}{a}(-1)^{m-a}\tau_{ij}^{(a)}$ and every $T_{ij}^{(1,m)}$ with $\tau_{ij}^{(m+1)} + \sum_{a=1}^{m} \binom{m}{a-1}(-1)^{m+1-a}\tau_{ij}^{(a)}$. Then, write

$$\sum c_M M = \sum \widetilde{c_M}\widetilde{M},$$

where the $\widetilde{M}$ are monomials in the τ generators. If M^{max} is some product of generators $T_{i_1,j_1}^{(0,m_1)}, \ldots, T_{i_{s_0},j_{s_0}}^{(0,m_{s_0})}, T_{k_1,l_1}^{(1,n_1)}, \ldots, T_{k_{s_1},l_{s_1}}^{(1,n_{s_1})}$ with coefficient $c_{M^{max}}$, then $\widetilde{M}^{max}$ is the product of $\tau_{i_1,j_1}^{(m_1)}, \ldots, \tau_{i_{s_0},j_{s_0}}^{(m_{s_0})}, \tau_{k_1,l_1}^{(n_1+1)}, \ldots, \tau_{k_{s_1},l_{s_1}}^{(n_{s_1}+1)}$. Notably, since these τ-generators each have a coefficient of 1, the coefficient $\widetilde{c_{\widetilde{M}^{max}}}$ is equal to $c_{M^{max}}$. However, since $\sum c_M M = \sum \widetilde{c_M}\widetilde{M} = 0$, by the linear independence of the τ generators, all $\widetilde{c_{\widetilde{M}}} = 0$. In particular, $\widetilde{c_{\widetilde{M}^{max}}} = 0$, thus $c_{M^{max}} = 0$, which is a contradiction to the fact that $c_{M^{max}} \neq 0$. We can then conclude that the double-superscript generators form linearly independent monomials over $\mathbb{C}(q)$. $\square$

The following Corollary follows from Propositions 6.8 and 6.12.

Corollary 6.13. *The ordered monomials in the $T_{ij}^{(1,r)}$ for $i > j$ and $T_{ij}^{(0,r)}$ for $i < j$ form a basis of $U_q(\mathcal{L}^{tw}\mathfrak{q}_N)$.*

In particular, these monomials may have positive and negative powers of $T_{ii}^{(0,0)}$; by [Ols92] (6.4), diagonal elements have exponents in $\mathbb{Z}$. However, since $T_{ii}^{(0)}T_{-i,-i}^{(0)} = 1 = T_{-i,-i}^{(0)}T_{ii}^{(0)}$ by defining relations, we can choose to only take such generators with $i > 0$ and replace $\left(T_{ii}^{(0)}\right)^d$ with $\left(T_{-i,-i}^{(0)}\right)^{-d}$ where necessary, then use that $T_{-i,-i}^{(0)} = (q-1)\tau_{-i,-i}^{(0)} + 1 = (q-1)T_{-i,-i}^{(0,0)} + 1$. Further, $\left(T_{ij}^{(r)}\right)^2 = 0$ if $|T_{ij}^{(r)}| = 1$.

Proposition 6.14. *A basis for $\mathbf{K}_m/\mathbf{K}_{m+1}$ is given by the classes of ordered monomials of the form $(q-q^{-1})^{m_0}T_{ij}^{(r_1,m_1)}\cdots T_{ij}^{(r_k,m_k)}$ with $\sum_{a=0}^{k} m_a = m$ and $i \leq j$ if $r_i = 0$, $i > j$ if $r_i = 1$.*

Proof. As was observed above, the basis for each $\mathbf{K}_m$ is given by monomials of the form $(q-q^{-1})^{m_0}T_{ij}^{(r_1,m_1)}\cdots T_{ij}^{(r_k,m_k)}$ with $\sum_{a=0}^{k} m_a \geq m$. In general, we have that $T_{ij}^{(1,m)} + (-1)^m T_{-i,-j}^{(1,m)} \in \mathbb{K}_{m+1} \subset \mathbf{K}_{m+1}$. The same applies for $T_{ij}^{(0,m)}$. As a result, in $\mathbf{K}_m/\mathbf{K}_{m+1}$, we have $T_{ij}^{(1,m)} = (-1)^{m+1}T_{-i,-j}^{(1,m)}$. Thus, it is enough to choose only half of all $T_{ij}^{(0,m)}$ and $T_{ij}^{(1,m)}$. Further, $\psi(T_{ij}^{(0,0)}) = E_{ij} + E_{-i,-j} = \psi(T_{-i,-j}^{(0,0)})$; thus, for $T_{ij}^{(0,0)}$, we use $0 < i \leq j$. $\square$

Theorem 6.15. *The map $\varphi : Y(\mathfrak{q}_n) \to \bigoplus_{m=0}^{\infty} \mathbf{K}_m/\mathbf{K}_{m+1}$ with $\varphi(t_{ij}^{(m)}) = T_{ij}^{(1,m)}$ is an isomorphism.*

Proof. This follows from the fact that φ maps a basis of the Yangian to a basis of $\bigoplus_{m=0}^{\infty} \mathbf{K}_m/\mathbf{K}_{m+1}$. $\square$

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