Higher Categorical Structures as Universal Fixed Points

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

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Abstract

Let $\mathbf{Cat}_{(\infty,\infty)}$ denote the $(\infty,1)$-category of (∞,∞)-categories with weakly inductive equivalences. The main objective of this thesis is to demonstrate that $\mathbf{Cat}_{(\infty,\infty)}$ satisfies universal properties with respect to homotopy-coherent internalisation and enrichment. To achieve these universal properties, we extend the theory of endofunctor algebras to the $(\infty,1)$-categorical setting, and establish an analogue of Adámek's free algebra construction. For any ∞-topos $\mathcal{X}$, we define an $(\infty,1)$-category $\mathbf{Sh}_{(n,r)}(\mathcal{X})$ of sheaves of (n,r)-categories over $\mathcal{X}$, where $0 \leq n \leq \infty$, and $0 \leq r \leq n+2$, and relate these categories through a general construction of complete Segal space objects over $\mathcal{X}$, and observe that presheaves of (n,r)-categories admit a well-defined notion of sheafification. By realising the construction of complete Segal space objects as an endofunctor over an appropriately-defined $(\infty,1)$-category of distributors, we use our generalised theory of endofunctor algebras to prove that $\mathbf{Sh}_{(\infty,\infty)}(\mathcal{X})$ is the universal distributor that is invariant under the construction of complete Segal space objects. We then study the theory of $(\infty,1)$-categorical enrichment and analyse the continuity of this construction to prove similarly that $\mathbf{Cat}_{(\infty,\infty)}$ is the initial object among presentably symmetric monoidal $(\infty,1)$-categories that are invariant under enrichment.

Preface

This thesis is an original work by Z. Goldthorpe.

Chapter 2 is an overview of existing literature. The mathematical content in this chapter is unoriginal, with the only exception being Section 2.1.1: the notion and limit characterisation of marked strict ω-categories is original, but has not been published.

Chapter 3 is based on the original work [13]. The work is available on the arXiv, and has been submitted for publication. Chapters 4 and 5 are mostly adapted—with corrections—from the published work [11]. The only exception is Section 5.2.2, which proves [11, Conjecture 3.4.3], and is based on the original work [12]. This work is available on the arXiv, but has not been submitted for publication.

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Chapter 1

Introduction

Classically, the structures of many mathematical objects admit similar general forms: such an object typically consists of a set of elements, and is equipped with various structure supported on this set. In general, elements of a set are distinct from one another, but they do not have any identifying characteristics; it is the structures on the set that give individual elements significance.

For example, in the two-dimensional vector space $\mathbb{R}^2 = \mathrm{Span}_{\mathbb{R}}\{x, y\}$, the x- and y-axes are distinct, but interchangeable. However, if we endow $\mathbb{R}^2$ with the structure of a multiplication $(a, b) \times (c, d) := (ac - bd, ad + bc)$, the result is the field $\mathbb{C}$ of complex numbers, wherein the "real" x-axis and "imaginary" y-axis carry distinct meaning.

Since the characteristics of elements of a mathematical object are completely determined by the object's structure, we are typically less interested in an object's underlying set of elements. In particular, objects with "the same structure" should be considered to be the same, even if their underlying sets are different. In other words, we are often only interested in mathematical objects up to *isomorphism*: a 1-to-1 correspondence between the underlying sets of objects that preserves the structure of the objects.

Category theory provides a general framework for the holistic study of mathematical objects with the same type of structure. Specifically, a *category* consists of:

- a class of *objects*, and
- between two objects A and B, a collection of *morphisms* $f : A \to B$.

such that morphisms can be composed like functions; see Definition 2.0.0.1. The objects of a category represent the mathematical objects of a certain type of structure, and the morphisms denote structure-preserving functions between these objects. For example, there is a category $\mathbf{Vect}_{\mathbb{R}}$ whose objects are real vector spaces, and whose morphisms are the linear transformations between them.

What is curious about a category is that its objects—like elements of a set—do not carry any intrinsic identifying characteristics: they are nothing but abstract points, connected between each other via the morphisms of the category. Reducing elaborate mathematical structure to mere points seems like a gross oversimplification, but all of the structural information of these objects is encoded in the morphisms. Indeed, one of the most fundamental insights of category theory—a consequence of the Yoneda Lemma—is that an object A of a category is completely determined *up to isomorphism* by its morphisms $S \to A$.

As a result, categories provide a natural context within which to study objects up to isomorphism, rather than building objects from their underlying sets. In particular, categorical constructions of objects are defined in terms of *universal properties*, which uniquely characterise an object's structure up to isomorphism. For example, the cartesian product $A \times B$ of two objects A and B is defined by the universal property that a morphism $S \to A \times B$ naturally corresponds to a pair of morphisms $S \to A$ and $S \to B$. It is then automatic that if A' is isomorphic to A, and B' is isomorphic to B, then $A' \times B'$ is isomorphic to $A \times B$ also.

Despite the broad success of category theory, particularly in algebra and geometry, isomorphisms are sometimes too rigid a notion of equivalence between objects: an isomorphism always asserts a 1-to-1 correspondence between the underlying sets. The prototypical example of a weaker identification than isomorphism is *homotopy equivalence*, which identifies topological spaces that can be continuously deformed into one another. For instance, a solid disk can be continuously contracted into a point, and this exhibits a homotopy equivalence between the disk and a singleton. This cannot be an isomorphism: the disk has (uncountably) infinitely many underlying points, whereas the singleton has just one. Therefore, homotopy equivalence cannot be meaningfully expressed as an isomorphism of structure.

Higher category theory is a generalisation of category theory developed with the intention to address these weaker notions of equivalence. In particular, a *higher category* consists of

- a class of objects,
- between two objects A and B, a collection of 1-*morphisms* $f : A \to B$,
- between 1-morphisms $f, g : A \to B$, a collection of 2-*morphisms* $\alpha : f \Rightarrow g$,
- between 2-morphisms $\alpha, \beta : f \Rightarrow g$, a collection of 3-*morphisms* $\Gamma : \alpha \Rrightarrow \beta$,
- and so on, *ad infinitum.*

Just as morphisms determine the structure of objects, $(n+1)$-morphisms determine structure on n-morphisms for every $n \geq 1$. For example, given topological spaces X and Y, we

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\alpha\Downarrow\overset{\Gamma}{\Rrightarrow}\Downarrow\beta"] & B \end{tikzcd}$$

Figure 1.1: The shape of a 3-morphism $\Gamma : \alpha \Rrightarrow \beta$.

can use 2-morphisms to describe continuous deformations between two continuous functions $f, g : X \to Y$.

The catch is that higher categories are significantly more difficult to define—so much so that it is impractical to define them algebraically. Instead, higher categories are presented via *geometric models*.

For example, say that a higher category is an ∞-*groupoid* if its k-morphisms are invertible for all $k > 0$. Then, the Homotopy Hypothesis (see Hypothesis 2.3.2.1) stipulates that topological spaces serve as a geometric model for ∞-groupoids: objects correspond to points of the space, 1-morphisms correspond to continuous paths between points, 2-morphisms correspond to continuous deformations of paths (which are called *homotopies*), 3-morphisms correspond to continuous deformations of homotopies, and so on.

There are several geometric models for various classes of higher categories. Among these models are the n-fold Segal spaces of [5], the Segal n-categories of [33], $(n+k, n)$-$\mathbf{\Theta}$-spaces of [29], n-quasicategories of [2], and complicial sets of [34]. Fortunately, these models have been proven to be suitably equivalent; see also [27] and [19]. In fact, if we call a higher category an (∞, r)-*category* if its k-morphisms are invertible for all $k > r$, then the Unicity Theorem of [6] gives a unique characterisation of the theory of (∞, r)-categories for each finite $r \geq 0$.

One iterative approach to higher categories is based on the following observation: given two objects x and y of a higher category $\mathcal{C}$, the collection $\mathrm{Hom}_{\mathcal{C}}(x, y)$ of 1-morphisms $x \to y$, and the k-morphisms between them, forms another higher category! In particular, if every $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is an (∞, r)-category, then $\mathcal{C}$ is an $(\infty, r+1)$-category. This observation motivates studying higher categories through *enrichment*.

Roughly speaking, a higher category $\mathcal{C}$ is said to be *enriched* in a higher category $\mathcal{V}$ if every $\mathrm{Hom}_{\mathcal{C}}(x, y)$ can be viewed as an object of $\mathcal{V}$. A general theory of enrichment is developed in [10], associating to any monoidal $(\infty, 1)$-category $\mathcal{V}$ an $(\infty, 1)$-category $\mathcal{V}\mathbf{Cat}$ of higher categories enriched in $\mathcal{V}$.

This allows us to construct the $(\infty, 1)$-category $\mathbf{Cat}_{(\infty,r)}$ of (∞, r)-categories by induction on r. Indeed, by the Homotopy Hypothesis, take $\mathbf{Cat}_{(\infty,0)}$ to be the $(\infty, 1)$-category of spaces. Then, given $\mathbf{Cat}_{(\infty,r)}$, define $\mathbf{Cat}_{(\infty,r+1)} := (\mathbf{Cat}_{(\infty,r)})\mathbf{Cat}$.

In order to obtain fully general (∞, ∞)-categories, we need to take an appropriate "limit" as $r \to \infty$. Although the Unicity Theorem ensures satisfactory uniqueness of $\mathbf{Cat}_{(\infty,r)}$ when r is finite, there is inevitable ambiguity when extending the theory to $r = \infty$.

We have an infinite tower of inclusions

$$\mathbf{Cat}_{(\infty,0)} \subseteq \mathbf{Cat}_{(\infty,1)} \subseteq \mathbf{Cat}_{(\infty,2)} \subseteq \cdots$$

where each inclusion $\mathbf{Cat}_{(\infty,r)} \subseteq \mathbf{Cat}_{(\infty,r+1)}$ admits a left adjoint $\pi_{\leq r}$ and a right adjoint $\kappa_{\leq r}$. Given an $(\infty, r+1)$-category $\mathcal{C}$, the right adjoint yields the sub-(∞, r)-category $\kappa_{\leq r}\mathcal{C}$ of $\mathcal{C}$ obtained by discarding all non-invertible $(r+1)$-morphisms. On the other hand, the left adjoint yields the (∞, r)-category $\pi_{\leq r}\mathcal{C}$ obtained by formally inverting all of the $(r+1)$-morphisms in $\mathcal{C}$.

Taking limits along either family of adjoints yields two distinct models of (∞, ∞)-categories:

$$\mathbf{Cat}_{\omega} := \varprojlim \left(\cdots \to \mathbf{Cat}_{(\infty,2)} \xrightarrow{\pi_{\leq 1}} \mathbf{Cat}_{(\infty,1)} \xrightarrow{\pi_{\leq 0}} \mathbf{Cat}_{(\infty,0)} \right)$$
$$\mathbf{Cat}_{(\infty,\infty)} := \varprojlim \left(\cdots \to \mathbf{Cat}_{(\infty,2)} \xrightarrow{\kappa_{\leq 1}} \mathbf{Cat}_{(\infty,1)} \xrightarrow{\kappa_{\leq 0}} \mathbf{Cat}_{(\infty,0)} \right)$$

The discrepancy arises from an ambiguity in the notion of equivalence in the fully general setting of (∞, ∞)-categories. Roughly speaking the notion of equivalence is the strongest possible in $\mathbf{Cat}_{\omega}$, whereas it is the weakest possible in $\mathbf{Cat}_{(\infty,\infty)}$; see Remarks 3.2.3.6 and 3.2.3.10. Note that there are also intermediate notions of equivalence; see Definitions 2.1.1.4 and 2.1.1.5 and Remark 2.1.1.14. We are particularly interested in $\mathbf{Cat}_{(\infty,\infty)}$, wherein the equivalences are the most flexible.

Any theory of (∞, ∞)-categories should be self-contained with respect to enrichment: a higher category $\mathcal{C}$ where $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is an (∞, ∞)-category for all objects x, y of $\mathcal{C}$ should be an (∞, ∞)-category itself. The main objective of this thesis is to prove not only that $\mathbf{Cat}_{(\infty,\infty)}$ is self-contained with respect to enrichment, but it is universally characterised by this fact. More precisely, for any well-behaved $(\infty, 1)$-category $\mathcal{V}$ with an equivalence $\mathcal{V} \simeq \mathcal{V}\mathbf{Cat}$, there is an essentially unique map $\mathbf{Cat}_{(\infty,\infty)} \to \mathcal{V}$; see Theorem 5.2.0.1.

1.1 Outline

Chapter 2 sets the stage for the thesis. In Section 2.1, we define strict n-categories for $0 \leq n \leq \omega$, which are higher categories where the composition of higher morphisms are strictly associative in all dimensions. This serves as a convenient toy model for gaining some intuition for the general case. We explore in Section 2.1.1 the various notions of equivalence in a strict ω-category. Lemma 2.1.0.2 implies that the 1-category $\omega\mathbf{Cat}$ of strict ω-categories is analogous to the $(\infty, 1)$-category $\mathbf{Cat}_{\omega}$, and we note in Remark 2.1.1.6 that the natural notion of equivalence in $\omega\mathbf{Cat}$ induced from its realisation as a limit of

$\pi_{\leq r}$ maps corresponds to weakly coinductive equivalences. On the other hand, $\mathbf{Cat}_{(\infty,\infty)}$ is analogous to a variant of $\omega\mathbf{Cat}$ where strict ω-categories are endowed with a marking, which indicates which k-morphisms of a strict ω-category are equivalences; this is demonstrated in Proposition 2.1.1.13.

The remaining sections of Chapter 2 are expository, briefly touching on the history leading up to $(\infty, 1)$-category theory. In Section 2.2, we review the connection between equivalences in higher categories and homotopy theory. We justify weakening the strictly associative composition in higher categories in Section 2.3 by discussing the Homotopy Hypothesis, relating n-groupoids and homotopy n-types. We then discuss the combinatorial model of homotopy types via simplicial sets, and discuss Kan complexes as our ambient model of ∞-groupoids. In Section 2.4, we discuss quasicategories as our ambient model of $(\infty, 1)$-categories.

In Chapter 3, we study an explicit model of sheaves of (n, r)-categories relative to an $(\infty, 1)$-topos, for arbitrary n and r, including $n = r = \infty$. In particular, by specialising to the $(\infty, 1)$-topos of spaces, we obtain our model of (n, r)-categories. The construction is heavily inspired by iterated complete Segal spaces as in [5]. We motivate the axioms of a complete Segal space in Section 3.1. The sheaves of (n, r)-categories, as well as the sheaves of ω-categories, are defined in Section 3.2.

In Section 3.3, we relate our model with the theory of complete Segal spaces in distributors as described in [27]. In Section 3.3.2, we construct $(\infty, 1)$-categories of distributors over a fixed $(\infty, 1)$-topos, and prove that these categories are complete and cocomplete. Then, we show that taking complete Segal spaces is functorial in Section 3.3.3. In Theorem 3.3.3.7, we prove that the functor preserves weakly contractible limits of right adjoints.

We employ the above theory in Section 3.3.4 to prove that sheaves of (∞, r)-categories and of ω-categories form distributors. This is already known when r is finite, so the interesting case is when $r = \infty$. We moreover prove in Theorem 3.3.4.5 that sheaves of (∞, ∞)-categories are invariant under the formation of complete Segal spaces. In Section 3.3.5, we briefly explore an analogue of geometric morphisms for distributors. We demonstrate the functoriality of the construction of sheaves of (n, r)-categories and of ω-categories with respect to these geometric morphisms in Corollary 3.3.5.5. We then prove in Proposition 3.3.5.8 that all $\pi_{\leq r}$ are geometric morphisms of distributors. Finally, in Theorem 3.3.5.10, we show that if $\mathcal{C}$ is an $(\infty, 1)$-site, then there is a well-defined sheafification on $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_{(n,r)})$ and $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_{\omega})$ yielding sheaves of higher categories in $\mathcal{C}$.

In Chapter 4, we extend the theory of endofunctor algebras and coalgebras to the $(\infty, 1)$-categorical setting, in order to use this theory to study universal fixed points. We review the 1-categorical theory in Section 4.1, and motivate the theory with the toy example of induction and coinduction. In Section 4.2, we generalise endofunctor $(\infty, 1)$-algebras to lax algebras, to simplify (and generalise) the proof that Adámek's construction in the $(\infty, 1)$-categorical setting yields free endofunctor algebras. We study Adámek's construction in Section 4.3, and also briefly explore the ramifications of this construction in the context of understanding free endofunctor fixed points.

Finally, in Chapter 5, we study higher categories from the perspective of enrichment. We review the prerequisite theory for enrichment in Section 5.1.1, and prove that enrichment preserves suitably nice weakly contractible limits of monoidal $(\infty, 1)$-categories in Section 5.1.2. We then apply the fixed point theory of Chapter 4 to enrichment in Section 5.2 to prove our main results. We show in Section 5.2.1 that $\mathbf{Cat}_{(\infty,\infty)}$ is initial among the presentably symmetric monoidal fixed points of enrichment. In Section 5.2.2, for completeness, we calculate the initial non-presentably symmetric monoidal fixed point of enrichment, and characterise it as a full subcategory of $\mathbf{Cat}_{(\infty,\infty)}$.

Chapter 2

Background and Motivation

Definition 2.0.0.1. A (locally small) *category* $\mathcal{C}$ consists of:

- a class of *objects* $A \in \mathcal{C}$,
- for objects $A, B \in \mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* $f : A \to B$,
- for $A, B, C \in \mathcal{C}$, a *composition map*

$$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

- for $A \in \mathcal{C}$, an *identity* morphism $\mathrm{id}_A : A \to A$,

subject to the following axioms:

- Composition is *associative*: for $f : A \to B$, $g : B \to C$, and $h : C \to D$, we have

$$(h \circ g) \circ f = h \circ (g \circ f)$$

- Composition is *unital*: for $f : A \to B$, we have

$$f \circ \mathrm{id}_A = f; \qquad \mathrm{id}_B \circ f = f$$

Through the Yoneda embedding, we can view any object A of a category $\mathcal{C}$ as a "set" with structure—more precisely, every object A is determined up to isomorphism by the set-valued presheaf h_A it represents.

However, in some cases, viewing objects of a category as mere structured sets is too reductive to encapsulate the particular behaviour of these objects. For example, the notions of product $A \times B$ and coproduct $A \sqcup B$ of objects A and B make sense in a general category, but are typically distinct since the product corresponds to the cartesian product of sets, whereas the coprodct corresponds to the disjoint union of sets:

$$A \times B = \{(a, b) \mid a \in A; b \in B\}; \qquad A \sqcup B = \{(a, 0) \mid a \in A\} \cup \{(0, b) \mid b \in B\}$$

On the other hand, for categories such as the category of abelian groups, the category of modules of a ring, and the category of chain complexes, (finite) products and coproducts coincide and yield the direct sum $A \oplus B$. The key is that, in each of the aforementioned categories with direct sums, we can view the objects therein as *abelian groups* with additional structure, so that we can express the key identity

$$(a, b) = (a, 0) + (0, b)$$

that allows us to identify $A \times B$ and $A \sqcup B$.

We can modify the notion of a category to capture additional "built-in" structure of its objects through a generalisation called *enriched category theory*, developed extensively in [24].

Definition 2.0.0.2. Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a monoidal category (see [24, §1.1]). Then, a $\mathcal{V}$-*enriched category* $\mathcal{C}$ consists of:

- a class of *objects* $A \in \mathcal{C}$,
- for objects $A, B \in \mathcal{C}$, an object $\mathrm{Hom}_{\mathcal{C}}(A, B) \in \mathcal{V}$,
- for $A, B, C \in \mathcal{C}$, a *composition morphism*

$$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \otimes \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

- for $A \in \mathcal{C}$, an *identity* $j_A : \mathbb{1} \to \mathrm{Hom}_{\mathcal{C}}(A, A)$,

subject to the following axioms:

- Composition is *associative*: for all $A, B, C, D \in \mathcal{C}$, the diagram

$$\begin{tikzcd}
(\mathrm{Hom}_{\mathcal{C}}(C,D) \otimes \mathrm{Hom}_{\mathcal{C}}(B,C)) \otimes \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[rr, "\sim"] \arrow[d, "(\circ)\otimes\mathrm{id}"'] & & \mathrm{Hom}_{\mathcal{C}}(C,D) \otimes (\mathrm{Hom}_{\mathcal{C}}(B,C) \otimes \mathrm{Hom}_{\mathcal{C}}(A,B)) \arrow[d, "\mathrm{id}\otimes(\circ)"] \\
\mathrm{Hom}_{\mathcal{C}}(B,D) \otimes \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ"'] & \mathrm{Hom}_{\mathcal{C}}(A,D) & \mathrm{Hom}_{\mathcal{C}}(C,D) \otimes \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[l, "\circ"]
\end{tikzcd}$$

  commutes, and

- Composition is *unital*: for all $A, B \in \mathcal{C}$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A,B) \otimes \mathbb{1} \arrow[r, "\sim"] \arrow[d, "\mathrm{id}\otimes j_A"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) & \mathbb{1} \otimes \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[l, "\sim"'] \arrow[d, "j_B\otimes\mathrm{id}"] \\
\mathrm{Hom}_{\mathcal{C}}(A,B) \otimes \mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[ur, "\circ"'] & & \mathrm{Hom}_{\mathcal{C}}(B,B) \otimes \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[ul, "\circ"]
\end{tikzcd}$$

  commutes.

From [24, §1.2], every monoidal category $(\mathcal{V}, \otimes, \mathbb{1})$ defines a category $\mathcal{V}\mathbf{Cat}$ of small $\mathcal{V}$-enriched categories and the $\mathcal{V}$-enriched functors between them. If $\mathcal{V}$ is moreover symmetric monoidal, then $\mathcal{V}\mathbf{Cat}$ inherits a symmetric monoidal tensor product as well; see [24, §1.4]. Given $\mathcal{V}$-enriched categories $\mathcal{C}$ and $\mathcal{D}$, the objects of $\mathcal{C} \otimes \mathcal{D}$ are given by pairs (C, D) where $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Between two objects $(C, D), (C', D') \in \mathcal{C} \otimes \mathcal{D}$, the hom-object is given by

$$\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}((C, D), (C', D')) := \mathrm{Hom}_{\mathcal{C}}(C, C') \otimes \mathrm{Hom}_{\mathcal{D}}(D, D')$$

The symmetric monoidal structure on $\mathcal{V}$ ensures that $\mathcal{C} \otimes \mathcal{D}$ has well-defined composition.

For example, if $(\mathcal{V}, \times, *)$ is a cartesian monoidal category—that is, if the tensor product is given by the cartesian product—then the induced tensor product on $\mathcal{V}\mathbf{Cat}$ is cartesian monoidal as well.

Moreover, a lax monoidal functor $F : \mathcal{V} \to \mathcal{W}$ between monoidal categories $\mathcal{V}$ and $\mathcal{W}$ induces a functor $F_* : \mathcal{V}\mathbf{Cat} \to \mathcal{W}\mathbf{Cat}$, which associates to a $\mathcal{V}$-enriched category $\mathcal{C}$ the $\mathcal{W}$-enriched category $F_*\mathcal{C}$ that has the same objects of $\mathcal{C}$, but for $C, D \in F_*\mathcal{C}$ has the hom-object $\mathrm{Hom}_{F_*\mathcal{C}}(C, D) := F(\mathrm{Hom}_{\mathcal{C}}(C, D))$. The lax monoidal structure on F ensures that $F_*\mathcal{C}$ has well-defined composition.

If the functor F is a symmetric monoidal functor between symmetric monoidal categories, then the induced functor F_* is likewise symmetric monoidal with respect to the induced tensor products on $\mathcal{V}\mathbf{Cat}$ and $\mathcal{W}\mathbf{Cat}$.

2.1 Strict higher categories

Let **Set** denote the category of sets and the functions between them. By design, categories enriched in **Set** with its cartesian monoidal tensor product are precisely the (locally small) categories of Definition 2.0.0.1. Therefore, $(\mathbf{Set})\mathbf{Cat} \cong 1\mathbf{Cat}$ recovers the usual category of small categories and the functors between them.

Enriching in $1\mathbf{Cat}$ yields categories $\mathcal{C}$ where each $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a category. The objects of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ serve as the morphisms for $\mathcal{C}$, so the morphisms of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ map between the morphisms of $\mathcal{C}$ and can therefore be thought of as 2-*morphisms* of $\mathcal{C}$. This suggests that enrichment provides a means for constructing higher categories.

Definition 2.1.0.1. For $n \geq 0$, define the category $n\mathbf{Cat}$ of small *strict n-categories* inductively as follows.

Define $0\mathbf{Cat} := \mathbf{Set}$ to be the category of sets. Then, given the category $n\mathbf{Cat}$, define $(n+1)\mathbf{Cat} := (n\mathbf{Cat})\mathbf{Cat}$ to be the category of small categories enriched in $n\mathbf{Cat}$ with its cartesian monoidal tensor product.

In a strict $(n+1)$-category $\mathcal{C}$ and $k \geq 0$, define the 1-morphisms of $\mathcal{C}$ to be the objects of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for $A, B \in \mathcal{C}$. Recursively, define the $(k+1)$-morphisms of $\mathcal{C}$, for $k > 0$, to be the k-morphisms of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for $A, B \in \mathcal{C}$.

We have an adjoint triple

$$\pi \dashv i \dashv \mathrm{Ob} : \ \mathbf{Set} \begin{array}{c} \xleftarrow{\pi} \\ \xhookrightarrow{i} \\ \xleftarrow[\mathrm{Ob}]{} \end{array} 1\mathbf{Cat}$$

where $i : \mathbf{Set} \hookrightarrow \mathbf{Cat}$ identifies a set X with a category whose objects are given by the elements of X, and whose only morphisms are identities. For a small category $\mathcal{C}$, the right adjoint yields the underlying set $\mathrm{Ob}(\mathcal{C})$ of objects of $\mathcal{C}$. On the other hand, $\pi\mathcal{C}$ is the set of equivalence classes in $\mathrm{Ob}(\mathcal{C})$ under the equivalence generated by asserting $A \sim B$ whenever there exists a morphism $A \to B$; that is, $\pi\mathcal{C}$ is the set of path-connected components on the underlying graph of $\mathcal{C}$.

All of these functors preserve the cartesian product. By iteratively applying enrichment, we obtain an adjoint triple

$$\pi_{\leq n} \dashv i_{\leq n} \dashv u_{\leq n} : \ n\mathbf{Cat} \begin{array}{c} \xleftarrow{\pi_{\leq n}} \\ \xhookrightarrow{i_{\leq n}} \\ \xleftarrow[u_{\leq n}]{} \end{array} (n+1)\mathbf{Cat}$$

for every $n \geq 0$.

Lemma 2.1.0.2. *In the huge* 1*-category* $\widehat{1\mathbf{Cat}}$ *of large categories, the limits*

$$\varprojlim \left(\cdots \to 3\mathbf{Cat} \xrightarrow{u_{\leq 2}} 2\mathbf{Cat} \xrightarrow{u_{\leq 1}} 1\mathbf{Cat} \xrightarrow{u_{\leq 0}} 0\mathbf{Cat} \right),$$
$$\varprojlim \left(\cdots \to 3\mathbf{Cat} \xrightarrow{\pi_{\leq 2}} 2\mathbf{Cat} \xrightarrow{\pi_{\leq 1}} 1\mathbf{Cat} \xrightarrow{\pi_{\leq 0}} 0\mathbf{Cat} \right)$$

are isomorphic.

Proof. Note that the following diagrams commute:

$$\begin{tikzcd} 2\mathbf{Cat} \arrow[r, "\pi_{\leq 1}"] \arrow[dr, "u_{\leq 1}"'] & 1\mathbf{Cat} \arrow[dr, "u_{\leq 0}"] & \\ & 1\mathbf{Cat} \arrow[r, "u_{\leq 0}"'] & 0\mathbf{Cat} \end{tikzcd} \qquad \begin{tikzcd} & 1\mathbf{Cat} \arrow[r, "\pi_{\leq 0}"] & 0\mathbf{Cat} \\ 2\mathbf{Cat} \arrow[ur, "\pi_{\leq 1}"] \arrow[r, "u_{\leq 1}"'] & 1\mathbf{Cat} \arrow[ur, "\pi_{\leq 0}"'] & \end{tikzcd}$$

By iterating enrichment and overlaying these diagrams, we obtain the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & 4\mathbf{Cat} \arrow[r, "\pi_{\leq 3}"] \arrow[dr, "u_{\leq 3}"'] & 3\mathbf{Cat} \arrow[r, "\pi_{\leq 2}"] \arrow[dr, "u_{\leq 2}"'] & 2\mathbf{Cat} \arrow[r, "\pi_{\leq 1}"] \arrow[dr, "u_{\leq 1}"'] & 1\mathbf{Cat} \arrow[r, "\pi_{\leq 0}"] \arrow[dr, "u_{\leq 0}"'] & 0\mathbf{Cat} \\ \cdots \arrow[r] & 4\mathbf{Cat} \arrow[r, "u_{\leq 3}"'] \arrow[ur, "\pi_{\leq 3}"'] & 3\mathbf{Cat} \arrow[r, "u_{\leq 2}"'] \arrow[ur, "\pi_{\leq 2}"'] & 2\mathbf{Cat} \arrow[r, "u_{\leq 1}"'] \arrow[ur, "\pi_{\leq 1}"'] & 1\mathbf{Cat} \arrow[r, "u_{\leq 0}"'] \arrow[ur, "\pi_{\leq 0}"'] & 0\mathbf{Cat} \end{tikzcd}$$

The lemma now follows from observing that the inclusion of either of the rows in the above diagram is coinitial (dual to cofinality); see [23, §2.5]. $\square$

Definition 2.1.0.3. Define the category of small *strict ω-categories* to be the category $\omega\mathbf{Cat}$ isomorphic to either (and thus both) of the limits in Lemma 2.1.0.2. It follows for any $n \geq 0$ that we have an adjoint triple

$$\pi_{\leq n} \dashv i_{\leq n} \dashv u_{\leq n} : \; n\mathbf{Cat} \underset{\xleftarrow{u_{\leq n}}}{\overset{\xleftarrow{\pi_{\leq n}}}{\xhookrightarrow{i_{\leq n}}}} \omega\mathbf{Cat}$$

For a strict ω-category $\mathcal{C}$, define its k-morphisms to be the k-morphisms of $u_{\leq k}\mathcal{C}$. In particular, a strict ω-category has k-morphisms for all finite $k \geq 1$.

2.1.1 Equivalences in a higher category

One of the main purposes of the higher-dimensional morphisms in a higher category is to express more nuanced relationships between mathematical objects. An important example is when we wish to express weaker notions of equivalence than isomorphisms. For a concrete example, consider the large strict 2-category $\mathbf{Cat}$ where

- the objects of $\mathbf{Cat}$ are the small categories,
- the 1-morphisms are the functors $F : \mathcal{C} \to \mathcal{D}$,
- the 2-morphisms are the natural transformations $\alpha : F \Rightarrow G$.

Recall that the purpose of ordinary categories is to study mathematical objects up to isomorphism, rather than equality. In particular, when studying a category $\mathcal{C}$, we are not interested in the underlying set $\mathrm{Ob}(\mathcal{C})$ of its objects, but rather the set $\tau\mathcal{C}$ of *isomorphism classes* of its objects.

On the other hand, a functor $F : \mathcal{C} \to \mathcal{D}$ between categories is an isomorphism (in the 1-category $\widehat{1\mathbf{Cat}}$) if and only if it is fully faithful, and induces a bijection on objects. In particular, isomorphisms of categories do not account for isomorphism classes in the domain and codomain.

A functor $F : \mathcal{C} \to \mathcal{D}$ more appropriately exhibits an equivalence of categories if it is fully faithful, and is *essentially surjective* in the sense that it induces a surjection $\tau\mathcal{C} \to \tau\mathcal{D}$ (and thus a bijection, if the functor is also fully faithful). With the axiom of choice, a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories in the above sense if and only if it admits a *pseudo-inverse*: a functor $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms

$$G \circ F \overset{\sim}{\Longrightarrow} \mathrm{id}_{\mathcal{C}}; \qquad\qquad \mathrm{id}_{\mathcal{D}} \overset{\sim}{\Longrightarrow} F \circ G$$

which is readily expressible in the strict 2-category $\mathbf{Cat}$ of categories.

Remark 2.1.1.1. In fact, reasoning recursively, since the 1-morphisms of a strict 2-category are precisely the objects of its hom-categories, we should only be interested in the structure of 1-morphisms up to 2-isomorphism. In particular, when defining a 1-morphism $f : A \to B$ to be an equivalence of objects, it is inconsistent to ask for the existence of an inverse $g : B \to A$ that admits *equalities* $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Instead, the composites $g \circ f$ and $f \circ g$ should only be 2-*isomorphic* to the respective identity 1-morphisms. In the strict 2-category **Cat**, this recovers precisely the above notion of an equivalence of categories.

Definition 2.1.1.2. We define the equivalence relation $(\simeq)$ in a strict n-category $\mathcal{C}$ by induction.

Equivalence in a strict 0-category (that is, a set) is given by equality of elements. Given a notion of equivalence in strict n-categories, say that two objects A and B in a strict $(n+1)$-category $\mathcal{C}$ are equivalent if there exists a pair of 1-morphisms $f : A \to B$ and $g : B \to A$ such that $g \circ f \simeq \mathrm{id}_A$ in $\mathrm{Hom}_{\mathcal{C}}(A, A)$, and $f \circ g \simeq \mathrm{id}_B$ in $\mathrm{Hom}_{\mathcal{C}}(B, B)$. In this situation, call $f : A \to B$ an *equivalence*.

Remark 2.1.1.3. The functor $\pi_{\leq n} : (n+1)\mathbf{Cat} \to n\mathbf{Cat}$ preserves equivalence for all $n \geq 0$ precisely because when $n = 0$, the functor sends isomorphisms to identities. On the other hand, the functor $u_{\leq n}$ does not preserve equivalence, since the underlying set of a category forgets the isomorphisms altogether.

We therefore have a well-defined notion of equivalence in any strict n-category so long as n is finite. However, it is not obvious how an equivalence should be defined in the context of a strict ω-category. Two desiderata are clear:

(EQ1) Identity k-morphisms should be equivalences.

(EQ2) A k-morphism $\alpha : f \to g$ should be an equivalence if and only if there exists another k-morphism $\beta : g \to f$, and $(k+1)$-morphisms

$$\eta : \mathrm{id}_g \xrightarrow{\sim} \alpha \circ \beta; \qquad \epsilon : \beta \circ \alpha \xrightarrow{\sim} \mathrm{id}_f$$

such that η and ϵ are equivalences.

In a strict n-category with n finite, this uniquely characterises the class of equivalences in the sense of Definition 2.1.1.2: (EQ2) eventually terminates at identities of n-morphisms.

For strict ω-categories, there are already two distinct means of generating equivalences:

Definition 2.1.1.4. Fix a strict ω-category $\mathcal{C}$. For a k-morphism α in $\mathcal{C}$, a collection I of higher morphisms in $\mathcal{C}$ is said to be *invertibility data* for α if:

- $\alpha \in I$, and

- for all ℓ-morphisms $(\lambda : p \to q) \in I$, there exist an ℓ-morphism $(\rho : q \to p) \in I$ and $(\ell+1)$-morphisms $(\mathrm{id}_p \to \rho \circ \lambda) \in I$ and $(\lambda \circ \rho \to \mathrm{id}_q) \in I$.

Say that a k-morphism α of $\mathcal{C}$ is a *coinductive equivalence* if α admits invertibility data.

On the other hand, say that a k-morphism α of $\mathcal{C}$ is an *inductive equivalence* if α admits invertibility data I wherein all ℓ-morphisms in I are identities for $\ell \gg 0$. These are precisely the higher morphisms generated by induction from (EQ1) and (EQ2).

Note that k-isomorphisms are inductive equivalences, and all inductive equivalences are coinductive equivalences. Moreover, the projections $\pi_{\le n} : \omega\mathbf{Cat} \to n\mathbf{Cat}$ send all coinductive equivalences to equivalences.

However, the inductive and coinductive notions of equivalence in $\omega\mathbf{Cat}$ are not the only natural notions of equivalence that extend Definition 2.1.1.2.

Definition 2.1.1.5. Using the realisation of $\omega\mathbf{Cat}$ as the limit

$$\varprojlim\left(\cdots \to 3\mathbf{Cat} \xrightarrow{\pi_{\le 2}} 2\mathbf{Cat} \xrightarrow{\pi_{\le 1}} 1\mathbf{Cat} \xrightarrow{\pi_{\le 0}} 0\mathbf{Cat}\right)$$

say that a k-morphism in a strict ω-category $\mathcal{C}$ is a *weakly coinductive equivalence* if it maps to an equivalence in $\pi_{\le n}\mathcal{C}$ for every finite $n \ge 0$.

Remark 2.1.1.6. Every coinductive equivalence is weakly coinductive. However, the converse is not true. For a k-morphism $\alpha : f \to g$ to be a weakly coinductive equivalence, we only need to ensure for every finite $n \ge 1$ that there exists a set I_n of higher morphisms such that

- $\alpha \in I_n$, and
- for all ℓ-morphisms $(\lambda : p \to q) \in I_n$ with $\ell < n$, there exist $(\rho : q \to p) \in I$ and $(\ell+1)$-morphisms $(\mathrm{id}_p \to \rho \circ \lambda) \in I$ and $(\lambda \circ \rho \to \mathrm{id}_q) \in I$.

In particular, the n-morphisms in I_n need not be invertible in any sense. Therefore, $\bigcup_n I_n$ does not necessarily exhibit α as a coinductive equivalence.

An explicit counterexample demonstrating that these two notions of equivalence are distinct can be found in [20, Construction 4.29].

One might argue that weakly coinductive equivalences are too weak a notion of equivalence. After all, the functor $\pi_{\le n} : \omega\mathbf{Cat} \to n\mathbf{Cat}$ collapses a strict ω-category $\mathcal{C}$ into a strict n-category by taking the n-morphisms of $\pi_{\le n}\mathcal{C}$ to be classes of n-morphisms of $\mathcal{C}$ connected by zig-zags of $(n+1)$-morphisms. In other words, $\pi_{\le n}$ sends all $(n+1)$-morphisms to identities, even if they are non-invertible in any reasonable sense.

This motivates considering a less destructive truncation functor:

Definition 2.1.1.7. Let $\tau : 1\mathbf{Cat} \to \mathbf{Set}$ denote the functor that associates to a category $\mathcal{C}$ the set $\tau\mathcal{C}$ of isomorphism classes of $\mathcal{C}$. This functor preserves products, and therefore induces truncations $\tau_{\leq n} : (n+1)\mathbf{Cat} \to n\mathbf{Cat}$ for every $n \geq 0$.

Remark 2.1.1.8. For a strict n-category $\mathcal{C}$, a k-morphism in $\mathcal{C}$ is an equivalence if and only if it corresponds to a k-isomorphism in $\tau_{\leq k}\mathcal{C}$.

Remark 2.1.1.9. From the above remark, we might be tempted to define a k-morphism of a strict ω-category $\mathcal{C}$ to be an equivalence if and only if it descends to a k-isomorphism in $\tau_{\leq k}\mathcal{C}$. However, this approach is circular: without establishing a notion of equivalence, we cannot define functors $\tau_{\leq n} : \omega\mathbf{Cat} \to n\mathbf{Cat}$. In other words, a choice of definition of equivalence on strict ω-categories corresponds to a cone from $\omega\mathbf{Cat}$ to the diagram

$$\cdots \to 3\mathbf{Cat} \xrightarrow{\tau_{\leq 2}} 2\mathbf{Cat} \xrightarrow{\tau_{\leq 1}} 1\mathbf{Cat} \xrightarrow{\tau_{\leq 0}} 0\mathbf{Cat}$$

In particular, a limit of the above diagram would, in some sense, classify the notions of equivalence on strict ω-categories.

Definition 2.1.1.10. A *marked strict ω-category* is a pair $(\mathcal{C}, W)$, where $\mathcal{C}$ is a strict ω-category, and W is a class of k-morphisms such that

- W contains all identity k-morphisms for $k \geq 1$, and
- W is *saturated* in the sense that a k-morphism $\alpha : f \to g$ is in W if and only if there exists a k-morphism $\beta : g \to f$ and $(k+1)$-morphisms

  $$\epsilon : \beta \circ \alpha \to \mathrm{id}_f; \qquad \eta : \mathrm{id}_g \to \alpha \circ \beta$$

  such that $\epsilon, \eta \in W$.

Call a k-morphism *marked* if it lies in W.

Say that a functor $F : \mathcal{C} \to \mathcal{C}'$ between marked strict ω-categories preserves the marking if $F(\alpha)$ is marked in $\mathcal{C}'$ whenever α is marked in $\mathcal{C}$. Let $\omega\mathbf{Cat}^+$ denote the category of small marked strict ω-categories and marked functors between them.

Remark 2.1.1.11. Fix a marked strict ω-category $(\mathcal{C}, W)$. If $W_{\geq n}$ denotes the set of marked k-morphisms with $k \geq n$, then the saturation condition on W implies that W is completely determined by $W_{\geq n}$ for any finite n.

In particular, if $\mathcal{C}$ is a strict n-category, in that its only k-morphisms for $k > n$ are identities, then $\mathcal{C}$ admits a unique marking consisting of precisely the equivalences of Definition 2.1.1.2.

Remark 2.1.1.12. The marking on a strict ω-category is analogous to the stratification on weak complicial sets as in [34]. In particular, the saturation condition on the marking on a strict ω-category corresponds to the saturation condition on weak complicial sets given in [31, Definition 3.2.7].

Proposition 2.1.1.13. *We have an isomorphism of* 1*-categories*

$$\omega\mathbf{Cat}^+ \cong \varprojlim\left(\cdots \to 3\mathbf{Cat} \xrightarrow{\tau_{\leq 2}} 2\mathbf{Cat} \xrightarrow{\tau_{\leq 1}} 1\mathbf{Cat} \xrightarrow{\tau_{\leq 0}} 0\mathbf{Cat}\right)$$

Proof. Define the projection $\tau_{\leq n} : \omega\mathbf{Cat}^+ \to n\mathbf{Cat}$ as follows. For a marked strict ω-category $(\mathcal{C}, W)$, take $\tau_{\leq n}\mathcal{C}$ to be strict n-category obtained by discarding the $(n+1)$-morphisms from the quotient of the strict $(n+1)$-category $u_{\leq(n+1)}\mathcal{C}$ that identifies n-morphisms that are connected by a marked $(n+1)$-morphism. The saturation condition ensures that we have a commutative triangle

$$\begin{tikzcd}
\omega\mathbf{Cat}^+ \arrow[r, "\tau_{\leq(n+1)}"] \arrow[dr, "\tau_{\leq n}"'] & (n+1)\mathbf{Cat} \arrow[d, "\tau_{\leq n}"] \\
& n\mathbf{Cat}
\end{tikzcd}$$

for every $n \geq 0$.

Suppose we have a cone of functors $F_n : \mathcal{X} \to n\mathbf{Cat}$ for every $n \geq 0$. From the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 4\mathbf{Cat} \arrow[r, "\tau_{\leq 3}"] \arrow[dr, "u_{\leq 3}"'] & 3\mathbf{Cat} \arrow[r, "\tau_{\leq 2}"] \arrow[dr, "u_{\leq 2}"'] & 2\mathbf{Cat} \arrow[r, "\tau_{\leq 1}"] \arrow[dr, "u_{\leq 1}"'] & 1\mathbf{Cat} \arrow[r, "\tau_{\leq 0}"] \arrow[dr, "u_{\leq 0}"'] & 0\mathbf{Cat} \\
\cdots \arrow[r] & 4\mathbf{Cat} \arrow[r, "u_{\leq 3}"'] & 3\mathbf{Cat} \arrow[r, "u_{\leq 2}"'] & 2\mathbf{Cat} \arrow[r, "u_{\leq 1}"'] & 1\mathbf{Cat} \arrow[r, "u_{\leq 0}"'] & 0\mathbf{Cat}
\end{tikzcd}$$

the above cone induces a unique functor $F : \mathcal{X} \to \omega\mathbf{Cat}$. Explicitly, the k-morphisms of FX are given by the k-morphisms of $F_{k+1}X$.

Since the F_n define a cone, we have for every $n \geq 0$ that $F_nX = \tau_{\leq n}(F_{n+1}X)$. In order to endow FX with a suitable marking such that $\tau_{\leq n}(FX) = F_nX$ for every $n \geq 0$, we therefore need an n-morphism of FX to be marked if and only if its corresponding n-morphism in $F_{n+1}X$ is an equivalence in the sense of Definition 2.1.1.2.

That this choice of marking contains the identities and is saturated is by design, and the marking is moreover completely determined by the cone. It follows that $F : \mathcal{X} \to \omega\mathbf{Cat}$ lifts uniquely to a functor $F : \mathcal{X} \to \omega\mathbf{Cat}^+$ such that $\tau_{\leq n}F = F_n$ for every $n \geq 0$, proving that $\omega\mathbf{Cat}^+$ with the projections defined above is a universal cone, proving the proposition. $\square$

Remark 2.1.1.14. The above proposition makes precise the claim in Remark 2.1.1.9. We have an obvious forgetful functor $\omega\mathbf{Cat}^+ \to \omega\mathbf{Cat}$, and the sections of this functor are precisely functorial choices of markings (*i.e.*, equivalences) on strict ω-categories.

Note that the forgetful functor $\omega\mathbf{Cat}^+ \to \omega\mathbf{Cat}$ admits a left and a right adjoint. The left adjoint associates to any strict ω-category the marking given by the inductive equivalences. On the other hand, the right adjoint associates to any strict ω-category the marking given by the coinductive equivalences: indeed, for a marked strict ω-category $(\mathcal{C}, W)$, the class W serves as invertibility data for every $f \in W$.

2.2 Abstract homotopy theory

By the Yoneda Lemma, objects A of a category $\mathcal{C}$ are completely determined, up to isomorphism, by the class of morphisms $S \to A$, where S varies over all objects of $\mathcal{C}$. This enables us to readily define isomorphism-invariant constructions in a category by characterising how morphisms map into (or out of) these constructions; that is, define constructions using *universal properties.*

With weaker notions of equivalence available in (strict) n-categories, we expect constructions in n-categories to be invariant under these notions of equivalence. Note that 1-categorical universal properties are insufficient to this end.

Example 2.2.0.1. Let $\mathbb{I}$ denote the "walking isomorphism"; that is, the unique category with two objects $\bot, \top$, where the only non-identity morphisms are $f : \bot \to \top$ and $g : \top \to \bot$. Then, the commutative square

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{\top\} \arrow[d] \\ \{\bot\} \arrow[r] & \mathbb{I} \end{tikzcd}$$

exhibits $\varnothing$ as the fibre product $\{\bot\} \times_{\mathbb{I}} \{\top\}$ in the 1-category $1\mathbf{Cat}$.

However, $\mathbb{I}$ is contractible; that is, the map $\mathbb{I} \to *$ into the singleton is an equivalence of categories, and the corresponding commutative square

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \{\top\} \arrow[d] \\ \{\bot\} \arrow[r] & * \end{tikzcd}$$

is not a pullback square.

Following the philosophy of Remark 2.1.1.1, the issue with 1-categorical universal properties in preserving the weaker notion of equivalence in a higher category is the fact that 1-categorical universal properties completely characterise incoming 1-morphisms up to equality, rather than up to 2-isomorphism.

An example of a 2-categorical universal property is given below:

Definition 2.2.0.2. Given a cospan $A \xrightarrow{f} C \xleftarrow{g} B$ of 1-morphisms in a (strict) 2-category $\mathcal{C}$, the corresponding 2-*fibre product* is any object $A \times_C B$ with projection 1-morphisms $\pi_A : A \times_C B \to A$ and $\pi_B : A \times_C B \to B$ as well as a 2-isomorphism

$$\begin{tikzcd}
A \times_C B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\overset{\phi}{\underset{\sim}{\Nearrow}}"] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

with the following 2-universal properties:

- Any diagram

$$\begin{tikzcd}
L \arrow[drr, bend left, "\ell_B"] \arrow[ddr, bend right, "\ell_A"'] & & \\
& \arrow[r, phantom, "\underset{\sim}{\Nearrow}"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

  decomposes (not necessarily uniquely) into a pasting diagram of the form

$$\begin{tikzcd}
L \arrow[drr, bend left, "\ell_B"] \arrow[ddr, bend right, "\ell_A"'] \arrow[dr, dashed] & & \\
& A \times_C B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\overset{\phi}{\underset{\sim}{\Nearrow}}"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

- For all 1-morphisms $p, q : L \to A \times_C B$ and 2-morphisms $\alpha : \pi_A \circ p \Rightarrow \pi_B \circ q$ and $\beta : \pi_B \circ p \Rightarrow \pi_B \circ q$ such that

$$\begin{tikzcd}[column sep=small]
& L \arrow[dl, "p"'] \arrow[dr, "q"] & \\
A \times_C B \arrow[d, "\pi_A"'] \arrow[rr, phantom, "\overset{\alpha}{\Rightarrow}"] & & A \times_C B \arrow[d, "\pi_B"] \arrow[dll, "\pi_A"'] \\
A \arrow[dr, "f"'] \arrow[rr, phantom, "\overset{\phi}{\underset{\sim}{\Rightarrow}}"] & & B \arrow[dl, "g"] \\
& C &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& L \arrow[dl, "p"'] \arrow[dr, "q"] & \\
A \times_C B \arrow[d, "\pi_A"'] \arrow[drr, "\pi_B"] \arrow[rr, phantom, "\overset{\beta}{\Rightarrow}"] & & A \times_C B \arrow[d, "\pi_B"] \\
A \arrow[dr, "f"'] \arrow[rr, phantom, "\overset{\phi}{\underset{\sim}{\Rightarrow}}"] & & B \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

  there exists a unique 2-morphism $\psi : p \Rightarrow q$ such that $\alpha = \pi_A * \psi$ and $\beta = \pi_B * \psi$.

Evidently, even 2-categorical universal properties are significantly more involved than their 1-categorical counterparts, all to ensure that the result is invariant under weaker equivalences.

2.2.1 Model categories

The fibre product of a cospan $A \xrightarrow{f} C \xleftarrow{g} B$ of sets is the set

$$A \times_C B = \{(a,b) \mid f(a) = g(b)\}$$

of pairs of elements living over the same basepoint.

A closer inspection of 2-fibre products in the strict 2-category **Cat** offers a more intuitive understanding of the failure of 1-categorical limits in being invariant under equivalence.

Example 2.2.1.1. The 2-fibre product of a cospan of categories $\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$ can be described explicitly as follows. The objects of $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ are given by triples (A, B, ϕ) where $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $\phi : FA \xrightarrow{\sim} GB$ is an isomorphism in $\mathcal{C}$. The morphisms $(A, B, \phi) \to (A', B', \phi')$ are given by pairs (f, g) where $f : A \to A'$ in $\mathcal{A}$ and $g : B \to B'$ in $\mathcal{B}$ such that

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\phi"', "\wr"] & FA' \arrow[d, "\phi'", "\wr"'] \\ FB \arrow[r, "Fg"'] & FB' \end{tikzcd}$$

commutes in $\mathcal{C}$.

Therefore, the 2-fibre product is the category of pairs of elements living over *isomorphic* basepoints (with an explicit witness for the isomorphism of the basepoints). In particular, the 1-categorical fibre product in Example 2.2.0.1 fails to exhibit a 2-fibre product precisely because the fibres of the functor $\{\top\} \to \mathbb{I}$ are not equivalent, despite the objects of $\mathbb{I}$ being isomorphic.

This suggests that we might be able to use 1-categorical fibre products to compute 2-fibre products, so long as the fibres of the morphism we pull back along behave well under isomorphisms in the base. Indeed, this can be verified directly in the context of **Cat**:

Definition 2.2.1.2. Say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an *isofibration* if for all isomorphisms $f : F(C) \xrightarrow{\sim} D$ in $\mathcal{D}$, there exists a lift $\widetilde{f} : C \xrightarrow{\sim} C'$ in $\mathcal{C}$ such that $F\widetilde{f} = f$.

Proposition 2.2.1.3. *Suppose $\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$ is a cospan of categories such that G is an isofibration. If $\mathcal{A} \times^1_{\mathcal{C}} \mathcal{B}$ denotes the fibre product computed in the 1-category $\widehat{1\mathbf{Cat}}$, then the canonical functor $\mathcal{A} \times^1_{\mathcal{C}} \mathcal{B} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ into the 2-fibre product is an equivalence of categories.*

In order to make effective use of the above proposition, we also need a canonical way of "resolving" any given functor into an isofibration:

Proposition 2.2.1.4. *Every functor $F : \mathcal{C} \to \mathcal{D}$ admits a factorisation $\mathcal{C} \xrightarrow{\widetilde{G}} \widetilde{\mathcal{C}} \xrightarrow{\widetilde{F}} \mathcal{D}$ where $\widetilde{F}$ is an isofibration, and $\widetilde{G}$ is an equivalence of categories.*

Proof. Let $(\mathcal{C} \downarrow^{\cong} \mathcal{D})$ denote the following category:

- The objects are triples (C, D, ϕ), where $C \in \mathcal{C}$, $D \in \mathcal{D}$, and $\phi : FC \xrightarrow{\sim} D$ is an isomorphism in $\mathcal{D}$.

- The morphisms $(C, D, \phi) \to (C', D', \phi')$ are given by pairs (f, g), where $f : C \to C'$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$ such that

$$\begin{tikzcd} FC \arrow[r, "Ff"] \arrow[d, "\phi", "\wr"'] & FC' \arrow[d, "\phi'"', "\wr"] \\ D \arrow[r, "g"'] & D' \end{tikzcd}$$

  commutes in $\mathcal{D}$.

Then, the canonical functor $\mathcal{C} \to (\mathcal{C} \downarrow^{\cong} \mathcal{D})$ sending $C \mapsto (C, F(C), \mathrm{id}_{F(C)})$ is an equivalence of categories, the functor $(\mathcal{C} \downarrow^{\cong} \mathcal{D}) \to \mathcal{D}$ sending $(C, D, \phi) \mapsto D$ is an isofibration, and the composite of these two is precisely $F : \mathcal{C} \to \mathcal{D}$. $\square$

This phenomenon generalises to many higher categories of interest, typically arising from homotopy theory or derived categories, leading to the notion of a model category.

Definition 2.2.1.5. A *lifting problem* in a category $\mathcal{C}$ is the problem of finding a lift k that fits in the commutative square below:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\ell"'] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "\ell"'] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r] \arrow[ur, dashed, "k"] & \bullet \end{tikzcd} \tag{2.1}$$

such that both triangles in the diagram commute.

Given a class $\mathfrak{R}$ of morphisms of $\mathcal{C}$, denote by $\mathrm{llp}(\mathfrak{R})$ the class of morphisms of $\mathcal{C}$ that satisfy the *left lifting property* with respect to $\mathfrak{R}$: a morphism ℓ lies in $\mathrm{llp}(\mathfrak{R})$ if and only if a lift k in Eq. (2.1) exists for all $r \in \mathfrak{R}$.

Dually, given a class $\mathfrak{L}$ of morphisms of $\mathcal{C}$, denote by $\mathrm{rlp}(\mathfrak{L})$ the class of morphisms of $\mathcal{C}$ that satisfy the *right lifting property* with respect to $\mathfrak{L}$.

A pair $(\mathfrak{L}, \mathfrak{R})$ of classes of morphisms of $\mathcal{C}$ is called a *weak factorisation system* if:

- Every morphism f in $\mathcal{C}$ admits a factorisation $f = r \circ \ell$ with $\ell \in \mathfrak{L}$ and $r \in \mathfrak{R}$, and

- $\mathfrak{R} = \mathrm{rlp}(\mathfrak{L})$ and $\mathfrak{L} = \mathrm{llp}(\mathfrak{R})$.

Definition 2.2.1.6. [30, Definition 2.1] A *model category* is a complete and cocomplete category $\mathcal{M}$ equipped with three classes of morphisms:

- A class W of *weak equivalences*, denoted $f : A \xrightarrow{\sim} B$,
- A class F of *fibrations*, denoted $f : A \twoheadrightarrow B$,
- A class C of *cofibrations*, denoted $f : A \rightarrowtail B$,

subject to the following axioms:

- Both $(C \cap W, F)$ and $(C, W \cap F)$ form weak factorisation systems on $\mathcal{K}$, and
- W satisfies *2-out-of-3*: if two of the morphisms in a commutative triangle

$$\begin{tikzcd} & B \arrow[dr] & \\ A \arrow[ur] \arrow[rr] & & C \end{tikzcd}$$

  lie in W, then so does the third.

Remark 2.2.1.7. A morphism is a *trivial* or *acyclic fibration* if it is both a fibration and a weak equivalence; dually, a morphism is a *trivial* or *acyclic cofibration* if it is both a cofibration and a weak equivalence.

Remark 2.2.1.8. Call an object A in a model category $\mathcal{M}$ *fibrant* if the unique map $A \to *$ is a fibration. By the weak factorisation property, every object A is connected via acyclic cofibration to a fibrant object $A \overset{\sim}{\rightarrowtail} \mathsf{R}A$. Call any such object a *fibrant resolution* of A.

Dually, call an object A *cofibrant* if the unique map $\emptyset \to A$ is a cofibration. By the weak factorisation property, every object A is connected via acyclic fibration to a cofibrant object $\mathsf{Q}A \overset{\sim}{\twoheadrightarrow} A$. Call any such object a *cofibrant resolution* of A.

Example 2.2.1.9. The 1-category 1**Cat** admits a model structure called the *canonical model structure* where

(W) The weak equivalences are the equivalences of categories,

(F) The fibrations are the isofibrations,

(C) The cofibrations are the functors that are (strictly) injective on objects.

In particular, all categories are cofibrant and fibrant in this model structure.

Example 2.2.1.10. The category **Top** of topological spaces and continuous functions between them admits a model called the *Quillen model structure* where

(W) The weak equivalences are the weak homotopy equivalences,

(F) The fibrations are the *Serre fibrations*; that is, maps with the right lifting property against the inclusions $\mathbb{D}^n \times \{0\} \hookrightarrow \mathbb{D}^n \times [0,1]$, where $\mathbb{D}^n$ is the n-disk, and $[0,1]$ is the unit interval,

(C) The cofibrations are the retracts of relative cell complexes.

In particular, all topological spaces are fibrant, and the cofibrant spaces are precisely the retracts of cell complexes.

Example 2.2.1.11. The category $\mathbf{Ch}_{\geq 0}(R)$ of bounded chain complexes of R-modules for a ring R admits a model structure called the *projective model structure* where

(W) The weak equivalences are the quasi-isomorphisms,

(F) The fibrations are the chain maps that are epimorphisms in all positive degrees,

(C) The cofibrations are the chain maps that are levelwise monomorphisms with projective cokernel.

In particular, all bounded chain complexes are fibrant, and the cofibrant complexes are precisely the complexes of projective modules.

Despite the abstract nature of the weak equivalences in a general model category, every model category has an intrinsic homotopy theory; see [21, Chapter 7].

Definition 2.2.1.12. Fix a model category $\mathcal{M}$. A *cylinder object* of an object A is any factorisation

$$\nabla : \; A \sqcup A \xrightarrow{[\partial^0, \partial^1]} \mathbb{I} \otimes A \xrightarrow{\sim} A$$

of the fold map.

Then, a *left homotopy* between two morphisms $f, g : A \to B$ is a morphism $H : \mathbb{I} \otimes A \to B$ such that $H \circ \partial^0 = f$ and $H \circ \partial^1 = g$.

If A is cofibrant, then left homotopy defines an equivalence relation on $\mathrm{Hom}_{\mathcal{M}}(A, B)$ for any B. Let $\mathrm{Ho}(\mathcal{M})$ denote the category where the objects are the cofibrant-fibrant objects of $\mathcal{M}$, and the morphisms are left homotopy classes of morphisms in $\mathcal{M}$. This is called the *homotopy category* of $\mathcal{M}$.

Theorem 2.2.1.13 (Whitehead Theorem for model categories)**.** *For a model category $\mathcal{M}$, the homotopy category* $\mathrm{Ho}(\mathcal{M})$ *is equivalent to the category obtained by formally inverting the weak equivalences of $\mathcal{M}$. In particular, a morphism in $\mathcal{M}$ between cofibrant-fibrant objects is a weak equivalence if and only if its corresponding morphism in* $\mathrm{Ho}(\mathcal{M})$ *is an isomorphism.*

Remark 2.2.1.14. As every object is connected to a cofibrant-fibrant object via a zig-zag of weak equivalences, and cofibrant-fibrant objects are precisely the objects for which the homotopy theory encoded by a model category is most well-defined, we can think of the non-cofibrant or non-fibrant objects as scaffolding necessary for the homotopy theory of $\mathrm{Ho}(\mathcal{M})$ to be studied using solely 1-categorical language. In particular, the objects of particular interest in a model category are just the cofibrant-fibrant ones.

2.2.2 Homotopy limits and colimits

With model categories as a context for abstract homotopy theory, we briefly discuss how to compute homotopy-coherent limits and colimits using appropriately-prepared 1-categorical limits and colimits; see also [21, Chapters 8 and 18].

Definition 2.2.2.1. Let $\mathcal{M}, \mathcal{M}'$ be model categories. An adjunction $F : \mathcal{M} \rightleftarrows \mathcal{M}' : U$ is a *Quillen adjunction* if F preserves cofibrations and U preserves fibrations. A Quillen adjunction induces an adjunction $\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M}') : \mathbb{R}U$ of *total derived functors* between the corresponding homotopy categories.

A Quillen adjunction is called a *Quillen equivalence* if the induced adjunction of total derived functors is an equivalence of homotopy categories.

Remark 2.2.2.2. For an arbitrary object $A \in \mathcal{M}$, the total left derived functor $\mathbb{L}F$ acts by computing F on a (functorially chosen) cofibrant resolution $\mathsf{Q}A$ of A; that is we have a weak equivalence $\mathbb{L}F(A) \simeq F(\mathsf{Q}A)$. By Ken Brown's Lemma, left Quillen functors preserve weak equivalences between cofibrant objects, so the cofibrant resolutions ensure that F is "corrected" to be invariant under weak equivalence. This is analogous to derived functors in homological algebra.

We want to ensure that if two diagrams $F, G : \mathcal{J} \to \mathcal{M}$ into a model category are levelwise weakly equivalent—for instance, there exists a natural transformation $F \Rightarrow G$ where $F(A) \to G(A)$ is a weak equivalence for every $A \in \mathcal{M}$—then the corresponding homotopy limits of F and G are weakly equivalent in $\mathcal{M}$.

If $\mathcal{M}$ is any complete and cocomplete category, then for all small categories $\mathcal{J}$, the diagonal functor $\Delta : \mathcal{M} \to \mathbf{Fun}(\mathcal{J}, \mathcal{M})$ admits left and right adjoints $\varinjlim \dashv \Delta \dashv \varprojlim$ given by taking colimits and limits, respectively. In particular, if we can show that the diagonal is both a left and a right Quillen functor, then we can compute homotopy limits and colimits by taking the derived functors associated to the adjoints of Δ.

Proposition 2.2.2.3. [26, Proposition A.2.8.2] *Let $\mathcal{M}$ be a combinatorial model category, and $\mathcal{J}$ a small category. Then,*

- *There exists a* projective model structure *on* $\mathbf{Fun}(\mathcal{J}, \mathcal{M})$ *where the weak equivalences and the fibrations are determined pointwise on* $\mathcal{M}$.

- *There exists an* injective model structure *on* $\mathbf{Fun}(\mathcal{J}, \mathcal{M})$ *where the weak equivalences and the cofibrations are determined pointwise on* $\mathcal{M}$.

Remark 2.2.2.4. If $\mathcal{M}$ is a combinatorial model category, and $\mathcal{J}$ is a small category, then the identity functor on $\mathbf{Fun}(\mathcal{J}, \mathcal{M})$ induces a Quillen equivalence

$$\mathrm{id} : \mathbf{Fun}(\mathcal{J}, \mathcal{M})_{\mathrm{proj}} \rightleftarrows \mathbf{Fun}(\mathcal{J}, \mathcal{M})_{\mathrm{inj}} : \mathrm{id}$$

Let $\mathcal{M}$ be a combinatorial model category, and $\mathcal{J}$ a small category. Since the diagonal functor $\Delta : \mathcal{M} \to \mathbf{Fun}(\mathcal{J}, \mathcal{M})$ preserves projective fibrations and projective acyclic fibrations, it defines a right Quillen functor $\mathcal{M} \to \mathbf{Fun}(\mathcal{J}, \mathcal{M})_{\mathrm{proj}}$.

Likewise, the diagonal functor also preserves injective cofibrations and injective acyclic cofibrations, and so defines a left Quillen functor $\mathcal{M} \to \mathbf{Fun}(\mathcal{J}, \mathcal{M})_{\mathrm{inj}}$. Following Remark 2.2.2.2, this justifies the following definition.

Definition 2.2.2.5. Let $\mathcal{M}$ be a combinatorial model category. Then, a *homotopy limit* of a functor $F : \mathcal{J} \to \mathcal{M}$ for $\mathcal{J}$ a small category is any object in $\mathcal{M}$ that is weakly equivalent to the limit $\varprojlim \mathsf{R}_{\mathrm{inj}} F$ of an injectively fibrant resolution of F.

Dually, a *homotopy colimit* of F is any object in $\mathcal{M}$ that is weakly equivalent to the colimit $\varinjlim \mathsf{Q}_{\mathrm{proj}} F$ of a projectively cofibrant resolution of F.

Example 2.2.2.6. By [26, Proposition A.2.4.4(i)], if a cospan $A \xrightarrow{f} C \xleftarrow{g} B$ in a model category consists of fibrant objects, and g is a fibration, then the (1-categorical) fibre product $A \times_C B$ is weakly equivalent to the homotopy fibre product. This generalises the phenomenon observed in Proposition 2.2.1.3.

2.3 The fundamental ∞-groupoid

To any topological space X, we can associate a category $\Pi_1 X$ where the objects are the points of X, and the morphisms $x \to y$ are continuous paths from x to y in X modulo endpoint-preserving homotopy. As all of the morphisms in $\Pi_1 X$ are invertible, it defines a groupoid, and is called the *fundamental groupoid* of X. This construction is moreover functorial: any continuous function $f : X \to Y$ induces a functor $f_* : \Pi_1 X \to \Pi_1 Y$ via postcomposition of paths with f.

The fundamental groupoid $\Pi_1 X$ encodes all of the 1-homotopical data of the space X. The set $\pi_0 X$ of path-connected components is precisely the set $\pi_{\leq 0} \Pi_1 X$ of isomorphism classes of $\Pi_1 X$, and for any $x \in X$, the fundamental group $\pi_1(X, x)$ is precisely the automorphism group $\mathrm{Aut}_{\Pi_1 X}(x)$ of the corresponding point in $\Pi_1 X$.

One might hope that this construction generalises: can we encode the 2-homotopical data of a space X with a 2-category $\Pi_2 X$? A natural guess is to take the objects of $\Pi_2 X$ to be the points of X, the 1-morphisms $x \to y$ to be continuous paths from x to y, and then take the 2-morphisms $f \Rightarrow g$ of paths to be endpoint-preserving homotopies $f \simeq g$ modulo boundary-preserving higher homotopies.

However, this construction fails to define a strict 2-category: part of the reason why $\Pi_1 X$ takes homotopy classes of paths as morphisms is to ensure that concatenation of paths induces a well-defined and associative composition operation. Defining concatenation $g \cdot f$ of paths $f, g : [0,1] \to X$ usually requires choosing a homeomorphism $[0,1] \sqcup_* [0,1] \cong [0,1]$. If this homeomorphism is fixed, iterated composition is no longer associative: for compatible paths f, g, h, we instead only have a canonical (endpoint-preserving) homotopy equivalence

$$(h \cdot g) \cdot f \simeq h \cdot (g \cdot f)$$

In particular, our putative 2-category $\Pi_2 X$ does not immediately admit an associative composition induced by path composition.

If X is Hausdorff, then [16] shows how the naïve idea above can be modified to define a fundamental strict 2-groupoid for X by identifying paths if they are connected by *thin* homotopies. In fact, a fundamental strict 2-groupoid can be achieved in general; see [15, §2.4.3]. However, it is impossible to generalise further—we cannot even encode the 3-homotopical data of cell complexes with strict 3-categories; see [14, Remark 8.8].

In order to define the fundamental 2-groupoid in a more natural and generalisable way, we need a notion of 2-category that allows the composition to be associative in a weaker sense—that is, up to canonical 2-isomorphism. This is consistent with the philosophy of Remark 2.1.1.1: we should not be asserting associativity through equality of 1-morphisms in a 2-category in the first place! This adjustment leads to a notion that is historically known as a *bicategory* in [7].

Definition 2.3.0.1. A 2-*category* $\mathcal{B}$ consists of the following data:

- A class of *objects* $A \in \mathcal{B}$,
- For objects $A, B \in \mathcal{B}$, a category $\mathrm{Hom}_{\mathcal{B}}(A, B)$ of 1-*morphisms* and 2-*morphisms* between them,
- For $A, B, C \in \mathcal{B}$, a *composition functor*

$$\circ : \mathrm{Hom}_{\mathcal{B}}(B, C) \times \mathrm{Hom}_{\mathcal{B}}(A, B) \to \mathrm{Hom}_{\mathcal{B}}(A, B)$$

- For $A \in \mathcal{B}$, an *identity* morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{B}}(A, A)$,
- For 1-morphisms $f : A \to B$, $g : B \to C$, $h : C \to D$, an *associator*

$$\alpha_{h,g,f} : (h \circ g) \circ f \overset{\sim}{\Rightarrow} h \circ (g \circ f)$$

  natural in f, g, h, and

- For 1-morphisms $f : A \to B$, left and right *unitors*

$$\lambda_f : \mathrm{id}_B \circ f \overset{\sim}{\Rightarrow} f, \qquad \rho_f : f \circ \mathrm{id}_A \overset{\sim}{\Rightarrow} f$$

  natural in f,

subject to the following coherence axioms:

- The associators satisfy the *pentagon axiom*: for all $f : A \to B$, $g : B \to C$, $h : C \to D$, $i : D \to E$, the diagram

$$\begin{tikzcd}
 & & (i \circ h) \circ (g \circ f) \arrow[drr, Rightarrow, "\alpha_{i,h,g\circ f}", "\sim"'] & & \\
((i \circ h) \circ g) \circ f \arrow[urr, Rightarrow, "\alpha_{i\circ h,g,f}", "\sim"'] \arrow[dr, Rightarrow, "\sim", "\alpha_{i,h,g} * f"'] & & & & i \circ (h \circ (g \circ f)) \\
 & (i \circ (h \circ g)) \circ f \arrow[rr, Rightarrow, "\sim", "\alpha_{i,h\circ g,f}"'] & & i \circ ((h \circ g) \circ f) \arrow[ur, Rightarrow, "\sim", "i * \alpha_{h,g,f}"'] &
\end{tikzcd}$$

  commutes in $\mathrm{Hom}_{\mathcal{B}}(A, E)$, and

- The associators and unitors satisfy the *triangle axiom*: for all 1-morphisms $f : A \to B$ and $g : B \to C$, the diagram

$$\begin{tikzcd}
(g \circ \mathrm{id}_B) \circ f \arrow[rr, Rightarrow, "\alpha_{g,\mathrm{id}_B,f}", "\sim"'] \arrow[dr, Rightarrow, "\sim", "\rho_g * f"'] & & g \circ (\mathrm{id}_B \circ f) \arrow[dl, Rightarrow, "\sim"', "g * \lambda_f"] \\
 & g \circ f &
\end{tikzcd}$$

  commutes in $\mathrm{Hom}_{\mathcal{B}}(A, C)$.

Now, the naïve definition of $\Pi_2 X$ yields a well-defined 2-category in general, which we may call the *fundamental 2-groupoid* of X; see [17]. In particular, the fundamental 2-groupoid $\Pi_2 X$ recovers the second homotopy groups $\pi_2(X, x)$ for $x \in X$ as the automorphism groups of the 1-morphisms id_x in $\Pi_2 X$.

Unfortunately, higher categories defined with fully weak associativity are remarkably more complicated than their strict analogues. For instance, in a 3-category—historically called a *tricategory*—the pentagon and triangle axioms are replaced with 3-isomorphisms (called the *pentagonator* and *triangulator*), which are then subject to their own coherence axioms. A fully algebraic definition of a 3-category can be found in [15, Definition 3.1.2].

2.3.1 Simplicial sets

Revisiting the fundamental 1-groupoid $\Pi_1 X$ of a space, it is natural to wonder if we can classify the essential image of this construction: which groupoids are equivalent to the funamental 1-groupoid of a space? It turns out that this functor is essentially surjective:

Proposition 2.3.1.1. *For a small groupoid $\mathcal{G}$, there exists a topological space $K(\mathcal{G})$ such that $\Pi_1 K(\mathcal{G}) \simeq \mathcal{G}$ and $\pi_n(K(\mathcal{G}), x)$ is trivial for all $x \in K(\mathcal{G})$ and $n \geq 2$.*

The construction of $K(\mathcal{G})$ is quite intuitive, and is discussed in [4, §1.4]:

- Every object of $\mathcal{G}$ should correspond to a point in $K(\mathcal{G})$, so introduce a point in $K(\mathcal{G})$ for every object of $\mathcal{G}$.

- For every morphism $x \to y$ in $\mathcal{G}$, glue a path from x to y in $K(\mathcal{G})$; that is, introduce a copy of the unit interval $[0, 1]$, and glue its endpoints to the points x and y already present in $K(\mathcal{G})$.

- To ensure that the composition of paths in $\Pi_1 K(\mathcal{G})$ is consistent with that of $\mathcal{G}$, introduce for every commutative triangle

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

  in $\mathcal{G}$ a topological triangle in $K(\mathcal{G})$, and glue its boundaries to the paths corresponding to f, g, and h according to the above diagram.

- The above three steps suffice in ensuring that $\Pi_1 K(\mathcal{G}) \simeq \mathcal{G}$. To kill the second homotopy groups, however, we need to introduce for every commutative tetrahedron

$$\begin{tikzcd} & y \arrow[d, "j"] \arrow[ddr, "g"] & \\ & w & \\ x \arrow[uur, "f"] \arrow[ur, "i"] \arrow[rr, "h"'] & & z \arrow[ul, "k"'] \end{tikzcd}$$

  a topological 3-simplex, glued to the topological triangles associated to each of the four faces of the above diagram.

- Likewise, to kill the higher homotopy groups, we need to introduce topological n-simplices for every commutative n-simplex in $\mathcal{G}$.

In particular, the construction of $K(\mathcal{G})$ can be divided into two distinct constructions: first, we extract from $\mathcal{G}$ a complex of n-simplices for every $n \geq 0$; then, we use the complex of n-simplices as a blueprint for realising a topological space.

We can combinatorially encode a complex of simplices using *simplicial sets*.

Definition 2.3.1.2. Define the *simplex category* $\mathbf{\Delta}$ to be the category whose objects are the finite chains

$$[n] := \{0 < 1 < \cdots < n\}$$

and whose morphisms are the order-preserving functions between them. Then, define the category of *simplicial sets* to be the functor category $\mathbf{sSet} := \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set})$. For a simplicial set $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$, write $X_n := X([n])$ for the set of *n-cells* of X.

Remark 2.3.1.3. The morphisms of the simplex category are generated by

- the *coface map* $\delta^i : [n] \to [n+1]$ for $0 \leq i \leq n+1$, which is the unique monotone injection whose image does not contain $i \in [n+1]$, and
- the *codegeneracy map* $\sigma^i : [n] \to [n-1]$ for $0 \leq i \leq n-1$, where σ^i is the unique monotone surjection such that $\sigma^i(i) = \sigma^i(i+1)$.

In particular, the structure of a simplicial set X is completely determined by its *face maps* $d_i := X(\delta^i) : X_{n+1} \to X_n$ and its *degeneracy maps* $s_i : X(\sigma^i) : X_{n-1} \to X_n$.

Call an n-cell of X *degenerate* if it lies in the image of one of the degeneracy maps of X.

Example 2.3.1.4. Define the *standard n-simplex* $\Delta[n]$ to be the simplicial set given by the presheaf represented by $[n]$; that is, the k-cells of $\Delta[n]$ correspond to order-preserving maps $[k] \to [n]$.

In particular, $\Delta[n]$ has exactly n vertices (*i.e.*, 0-cells), which we may identify with the elements $\{0, 1, \ldots, n\}$. We then view any k-cell of $\Delta[n]$ as being supported on the vertices in the image of its corresponding map $[k] \to [n]$. By convention, we also view a 1-cell $\alpha : [1] \to [n]$ as a directed edge from $\alpha(0)$ to $\alpha(1)$.

The co-Yoneda Lemma implies that every simplicial set can be realised as a canonical colimit of the representable simplicial sets; that is, every simplicial set is obtained by formally gluing together standard simplices. In particular, if we specify how to realise each standard simplex as a topological space, we can then follow the same gluing procedure to realise an arbitrary simplicial set as a topological space as well.

$$\left\{\ 0\ \right\}$$

(a) $\Delta[0]$

$$\left\{\ 0 \longrightarrow 1\ \right\}$$

(b) $\Delta[1]$

$$\left\{\begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}\right\}$$

(c) $\Delta[2]$

Figure 2.1: The non-degenerate cells of the first few standard simplices.

Definition 2.3.1.5. For $n \geq 0$, define the *topological n-simplex* to be the subspace

$$\Delta_{\mathbf{Top}}[n] := \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_{\geq 0} :\middle|: x_0 + \cdots + x_n = 1\right\} \subset \mathbb{R}^{n+1}$$

This construction defines a functor $\Delta_{\mathbf{Top}}[-] : \mathbf{\Delta} \to \mathbf{Top}$, where

$$\begin{aligned} &\delta^i : \Delta_{\mathbf{Top}}[n] \to \Delta_{\mathbf{Top}}[n+1], && \sigma^i : \Delta_{\mathbf{Top}}[n] \to \Delta_{\mathbf{Top}}[n-1], \\ &(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_n) && (x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_i + x_{i+1}, \ldots, x_n) \end{aligned}$$

In particular, this functor induces an adjunction

$$|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$$

The right adjoint is the *singular nerve*, which associates to a topological space X the simplicial set $\mathrm{Sing}(X)$ whose n-cells correspond to continuous functions $\Delta_{\mathbf{Top}}[n] \to X$. On the other hand, the left adjoint is the *geometric realisation*, which is given by the left Kan extension of $\Delta_{\mathbf{Top}}[-]$ along the Yoneda embedding $\mathbf{\Delta} \hookrightarrow \mathbf{sSet}$.

On the other hand, we can also realise the standard simplices as commutative diagrams in $1\mathbf{Cat}$, through which we can probe groupoids to extract their simplicial structure.

Definition 2.3.1.6. We have a fully faithful embedding $\mathbf{\Delta} \subset 1\mathbf{Cat}$ that identifies $[n]$ with the category $\{0 \to 1 \to \cdots \to n\}$. Through this functor, we can define the *nerve* of any small category $\mathcal{C}$ as the simplicial set $N\mathcal{C}$ whose n-cells correspond to functors $[n] \to \mathcal{C}$.

As every category can be written as a canonical colimit of these chains (that is, the inclusion $\mathbf{\Delta} \subset 1\mathbf{Cat}$ is *dense* in the sense of [24, Chapter 5]), it follows that the nerve defines a fully faithful functor $1\mathbf{Cat} \hookrightarrow \mathbf{sSet}$.

Remark 2.3.1.7. In particular, we can make the construction of $K(\mathcal{G})$ in Proposition 2.3.1.1 precise: the topological space is precisely the geometric realisation of the nerve of the groupoid $\mathcal{G}$; that is, $K(\mathcal{G}) = |N\mathcal{G}|$.

2.3.2 The Homotopy Hypothesis

Although the definition of a general n-category is impractically technical, the naïve construction of a fundamental n-groupoid $\Pi_n X$ is very straightforward. Specifically, the fundamental ∞-groupoid $\Pi_\infty X$ should be an ∞-category where:

- the objects of $\Pi_\infty X$ are the points of X,
- the 1-morphisms of $\Pi_\infty X$ are the continuous paths in X,
- the 2-morphisms of $\Pi_\infty X$ are the endpoint-preserving homotopies in X,
- the 3-morphisms of $\Pi_\infty X$ are the boundary-preserving higher homotopies in X,
- and so on.

Composition of k-morphisms should be given by concatenation, which is well-defined up to canonical homotopy. In some sense, all of the required coherence should already be built into the topological structure of the space X. The fundamental n-groupoid can then be obtained by truncating $\Pi_\infty X$.

This construction should also be functorial: a continuous function $f : X \to Y$ should induce a functor of ∞-groupoids $f_* : \Pi_\infty X \to \Pi_\infty Y$. Moreover, homotopies between continuous functions should induce natural transformations between fundamental ∞-groupoids, and likewise for higher homotopies. In particular, a continuous function should induce an equivalence of fundamental ∞-groupoids whenever the function is a homotopy equivalence of spaces. Conversely, since the fundamental ∞-groupoid should encode all homotopical data of the space, if a continuous function induces an equivalence of fundamental ∞-groupoids, it should induce an isomorphism of homotopy groups in all directions; that is, such a continuous function should be a *weak homotopy equivalence*.

The construction of Proposition 2.3.1.1 should generalise to any appropriate notion of ∞-groupoid: given an ∞-groupoid $\mathcal{G}$, there should exist a space $K(\mathcal{G})$ such that $\Pi_\infty K(\mathcal{G}) \simeq \mathcal{G}$. From the previous discussion, $K(\mathcal{G})$ should be completely determined up to weak homotopy equivalence. Altogether, these musings lead us to the following:

Hypothesis 2.3.2.1 (The Homotopy Hypothesis)**.** [4, §2.3] *There is an equivalence between ∞-groupoids and* homotopy types *(topological spaces modulo weak homotopy equivalence).*

Assuming the Homotopy Hypothesis, we can use topological spaces as a model for ∞-groupoids, circumventing the need to formalise higher categories. However, the category of topological spaces is poorly behaved, and topological spaces do not reflect the expected combinatorial nature of (higher) categories. Recall that the construction of $K(\mathcal{G})$ in Proposition 2.3.1.1 factors through simplicial sets, which are combinatorial structures that form a presheaf category. This motivates exploring a simplicial model for ∞-groupoids instead.

Definition 2.3.2.2. Define the *boundary* $\partial\Delta[n]$ of the standard n-simplex to be the simplicial subset of $\Delta[n]$ obtained by discarding the unique non-degenerate n-cell corresponding to the identity $[n] \to [n]$ (and therefore also discarding all k-cells corresponding to surjections $[k] \to [n]$).

For $0 \leq i \leq n$, define the ith *horn* $\Lambda^i[n]$ of $\Delta[n]$ to be the simplicial subset of $\partial\Delta[n]$ obtained by discarding the $(n-1)$-cell corresponding to $\delta^i : [n-1] \to [n]$ (and therefore also discarding all k-cells corresponding to maps $[k] \to [n]$ whose image contains $[n] \setminus \{i\}$).

$$\left\{\begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}\right\}$$

(a) $\Lambda^0[2]$

$$\left\{\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}\right\}$$

(b) $\Lambda^1[2]$

$$\left\{\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd}\right\}$$

(c) $\Lambda^2[2]$

Figure 2.2: The horns of $\Delta[2]$.

Proposition 2.3.2.3. [26, Proposition 1.1.2.2] *The essential image of the fully faithful functor $N : 1\mathbf{Cat} \hookrightarrow \mathbf{sSet}$ consists of those simplicial sets $K : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ such that every map $\Lambda^i[n] \to K$ for $0 < i < n$ admits a unique extension*

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d, hook] & K \\ \Delta[n] \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

If extensions also exist when $i = 0$ and $i = n$, even if the extensions are not unique a priori, then K is moreover isomorphic to the nerve of a groupoid.

Proof sketch. Given a simplicial set K with the unique extension property with respect to $\Lambda^i[n]$ for $0 < i < n$, we construct a category with set of objects given by K_0 and set of morphisms given by K_1. Referring to Fig. 2.2, a map $\Lambda^1[2] \to K$ corresponds to a choice of a composable pair of 1-cells $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$. We therefore define the composite $g \circ f$ to be the third face of the unique extension of $\Lambda^1[2] \to K$ along $\Lambda^1[2] \subset \Delta[2]$; that is, the unique choice of dashed 1-cell in the diagram

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "\exists!"'] & & z \end{tikzcd}$$

On the other hand, identity morphisms are selected via $s_0 : K_0 \to K_1$.

The unique extensions along the horn inclusions $\Lambda^i[3] \subset \Delta[3]$ for $i = 1, 2$ assert that iterated composites $(h \circ g) \circ f$ and $h \circ (g \circ f)$ coincide for all composable 1-cells f, g, h. This proves that the 0-cells and 1-cells with the above composition does indeed define a category. The unique extensions in higher dimensions are necessary to ensure that the nerve of this category recovers all of K.

If the nerve $N\mathcal{C}$ of a category admits extensions along $\Lambda^i[n] \subset \Delta[n]$ for $i = 0, n$, then taking $n = 2$ implies in particular that we can always find dashed arrows in the diagrams

$$\begin{tikzcd} & y \arrow[dr, dashed] & \\ x \arrow[ur, "f"] \arrow[rr, "\mathrm{id}"'] & & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, dashed] \arrow[rr, "\mathrm{id}"'] & & y \end{tikzcd}$$

making the triangles commute. In other words, the lifts imply that every morphism in $\mathcal{C}$ admits a section and a retraction, meaning that every morphism is an isomorphism. □

In the singular nerve $\mathrm{Sing}(X)$ of a topological space X, the 0-cells are the points of the space X, and the 1-cells are the continuous paths in X. If $\alpha : \Delta_{\mathbf{Top}}[2] \to X$ is a 2-cell of $\mathrm{Sing}(X)$, then α can be viewed as tracing an endpoint-preserving homotopy

$$\begin{tikzcd} & \alpha(0,1,0) \arrow[dr, squiggly, "d_0\alpha"] \arrow[d, phantom, "\wr|"] & \\ \alpha(1,0,0) \arrow[ur, squiggly, "d_2\alpha"] \arrow[rr, squiggly, "d_1\alpha"'] & {} & \alpha(0,0,1) \end{tikzcd}$$

from the concatenation of paths $d_0\alpha \cdot d_2\alpha$ to the path $d_1\alpha$, exhibiting $d_1\alpha$ as the composite of $d_2\alpha$ with $d_0\alpha$. The higher-dimensional cells of $\mathrm{Sing}(X)$ can be viewed similarly as higher homotopies as well.

In particular, the singular nerve construction seems to be completely analogous to the desired construction of the fundamental ∞-groupoid of a space. Moreover, the singular nerve behaves very similarly to the nerve of a groupoid: since every $|\Lambda^i[n]|$ is a retract of $|\Delta[n]| \cong \Delta_{\mathbf{Top}}[n]$, we can extend any map $\Lambda^i[n] \to \mathrm{Sing}(X)$ along $\Lambda^i[n] \subset \Delta[n]$, albeit not uniquely.

Definition 2.3.2.4. A simplicial set K is a *Kan complex* if every map $\Lambda^i[n] \to K$ for $0 \leq i \leq n$ admits an extension

$$\begin{tikzcd}
\Lambda^i[n] \arrow[r] \arrow[d, hook] & K \\
\Delta[n] \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

Analogous to the proof sketch of Proposition 2.3.2.3, given a composable pair of 1-cells $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$ exhibited by a map $\Lambda^1[2] \to K$, an extension along $\Lambda^1[2] \subset \Delta[2]$ produces a 2-simplex

$$\begin{tikzcd}
& y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow"] & \\
x \arrow[ur, "f"] \arrow[rr, dashed, "\exists h"'] & {} & z
\end{tikzcd}$$

which we can view as exhibiting h as a composite of f and g. Although h is not unique, any two choices h and h' of composites are connected by a canonical 2-cell of K using extensions of appropriate maps $\Lambda^1[3] \to K$ along $\Lambda^1[3] \subset \Delta[3]$ suggested by the diagram below

$$\begin{tikzcd}
& y \arrow[d, "g"] \arrow[ddr, "g"] & \\
& z \arrow[dr, equal] \arrow[d, phantom, "\Downarrow"] & \\
x \arrow[uur, "f"] \arrow[ur, "h"] \arrow[rr, "h'"'] & {} & z
\end{tikzcd}$$

Likewise, the extensions along horn inclusions $\Lambda^i[3] \subset \Delta[3]$ for $i = 1, 2$ connect iterated composites $(h \circ g) \circ f$ and $h \circ (g \circ f)$ via zig-zags of 2-cells, demonstrating that composition in K (which is already ambiguous up to 2-cells) is associative up to 2-cells. Extensions along higher-dimensional (inner) horn inclusions imply that the 2-cells exhibiting associativity are coherent up to higher-dimensional cells in K. In particular, the structure of a Kan complex encodes homotopy-coherent compositional structure analogous to that of Definition 2.3.0.1.

On the other hand, the extensions along $\Lambda^0[2] \subset \Delta[2]$ and $\Lambda^2[2] \subset \Delta[2]$ imply that 1-cells in K admit left and right inverses up to canonical 2-cell, and extensions along higher-dimensional (outer) horn inclusions imply that these inverses are homotopically coherent. This suggests that Kan complexes may serve as a geometric model of ∞-groupoids. Assuming the Homotopy Hypothesis, we can formalise this suggestion with the following theorem:

Theorem 2.3.2.5 (Quillen)**.** [22, Theorems 3.6.4 and 3.6.7] *There is a model structure on* **sSet** *(called the* Kan-Quillen model structure*) where*

(W) The weak equivalences are those $f : X \to Y$ such that the induced map $|X| \to |Y|$ of geometric realisations is a weak homotopy equivalence of spaces,

(F) The fibrations are those $f : X \to Y$ that satisfy the right lifting property with respect to all horn inclusions $\Lambda^i[n] \subset \Delta[n]$ for $0 \leq i \leq n$, which are called Kan fibrations. *In particular, the fibrant objects are precisely the Kan complexes,*

(C) The cofibrations are the monomorphisms. In particular, all simplicial sets are cofibrant.

Moreover, the geometric realisation and singular nerve define a Quillen equivalence between $\mathbf{sSet}_{\mathrm{Quillen}}$ *and* $\mathbf{Top}_{\mathrm{Quillen}}$ *(see Example 2.2.1.10).*

2.4 Quasicategories

Comparing the nerve of 1-groupoids with Kan complexes, the characterisation of the nerve of arbitrary 1-categories motivates the following relaxation of Definition 2.3.2.4:

Definition 2.4.0.1. Say that a simplicial set $\mathcal{C}$ is a *quasicategory* if every map $\Lambda^i[n] \to \mathcal{C}$ for $0 < i < n$ admits an extension

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta[n] \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

The extension property with respect to inner horns implies, just as with Kan complexes, that a quasicategory carries a homotopy-coherent compositional structure. Excluding extensions along outer horn inclusions removes the additional constraint on Kan complexes that this compositional structure is invertible. In particular, quasicategories should correspond to a more general class of higher categories where 1-morphisms are not necessarily invertible.

Note that the homotopy coherence of this compositional structure relies on the interpretation of k-cells for $k \geq 2$ as encoding invertible k-morphisms of $\mathcal{C}$; in particular, quasicategories cannot hope to encode fully general higher categories, but rather just those whose k-morphisms are invertible for $k \geq 2$.

Definition 2.4.0.2. Fix a notion of higher category, with k-morphisms for all $k \geq 1$, and a corresponding notion of when a k-morphism is an equivalence. For integers $0 \leq n, r \leq \infty$, say that such a higher category is an (n, r)*-category* if:

- for $k > n$, any parallel k-morphisms are equivalent, and
- for $k > r$, any k-morphism is an equivalence.

Here, two k-morphisms are called *parallel* if they map from the same source to the same target.

Example 2.4.0.3. For any reasonable notion of higher category, the following identifications should hold:

- Vacuously, every higher category is an (∞, ∞)-category.
- A small $(0, 0)$-category is a set, a $(1, 1)$-category is an ordinary category, and a $(2, 2)$-category is a 2-category in the sense of Definition 2.3.0.1. More generally, an (n, n)-category is precisely an n-category.
- An $(n, 0)$-category is, by definition, an n-groupoid.
- A $(0, 1)$-category is a set with a *preorder* (a reflexive and transitive relation). Likewise, a $(1, 2)$-category is a 2-category whose hom-categories are preordered sets.

  Note that every preordered set is equivalent as a category to a partially ordered set.

Remark 2.4.0.4. In an (n, r)-category, every k-endomorphism for $k > n$ is equivalent to an identity, and is therefore an equivalence. This implies that any k-morphism for $k > n$ with equivalent source and target must already be an equivalence. As k-morphisms for $k > 1$ necessarily map between parallel $(k - 1)$-morphisms, it follows in an (n, r)-category that every k-morphism is an equivalence for $k > n + 1$. In particular, every (n, ∞)-category is an $(n, n + 1)$-category.

Remark 2.4.0.5. The notion of an (n, r)-category can be extended to allow for n or r to be negative; see [4, §2].

Remark 2.4.0.6. The term "∞-category" commonly refers specifically to $(\infty, 1)$-categories, rather than the fully general (∞, ∞)-categories. This is the convention in [26], for instance.

We want to view quasicategories as a model for $(\infty, 1)$-categories.

2.4.1 Simplicially-enriched categories

A more direct approach to defining $(\infty, 1)$-categories is to rely on the observation that an $(\infty, 1)$-category is a higher category $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is an ∞-groupoid for all $x, y \in \mathcal{C}$. Through the Homotopy Hypothesis, we are led to studying categories enriched in $(\mathbf{sSet}, \times, \Delta[0])$, where we also endow $\mathbf{sSet}$ with the Kan-Quillen model structure.

As we are only interested in simplicial sets up to weak equivalence, we must similarly weaken our notion of equivalence between $\mathbf{sSet}$-enriched categories. Note that the tensor product on $(\mathbf{sSet}, \times, \Delta[0])$ is compatible with the model structure on $\mathbf{sSet}$ in that it defines a *monoidal model category* in the sense of [26, Definition A.3.1.2]. This implies that $\mathrm{Ho}(\mathbf{sSet})$ inherits a total derived tensor product, and any $\mathbf{sSet}$-enriched category $\mathcal{C}$ induces a $\mathrm{Ho}(\mathbf{sSet})$-enriched category $h(\mathcal{C})$. Since the isomorphisms in $\mathrm{Ho}(\mathbf{sSet})$ correspond to the weak equivalences in **sSet**, we want to consider **sSet**-enriched categories as weakly equivalent whenever their induced Ho(**sSet**)-enriched categories are equivalent.

Theorem 2.4.1.1. [26, Proposition A.3.2.4 and Theorem A.3.2.24] *The category* (**sSet**)**Cat** *admits a left proper combinatorial model structure uniquely determined by the following:*

(W) The weak equivalences are those functors $F : \mathcal{C} \to \mathcal{D}$ of **sSet***-enriched categories such that the induced functor $h(\mathcal{C}) \to h(\mathcal{D})$ of* Ho(**sSet**)*-enriched categories is an equivalence in the strict* 2*-category* (Ho(**sSet**))**Cat***,*

(F) The fibrant objects are those **sSet***-enriched categories $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a Kan complex for all $X, Y \in \mathcal{C}$.*

(C) The cofibrations are generated by $\varnothing \hookrightarrow *$ and the inclusions $\Sigma(\partial\Delta[n]) \hookrightarrow \Sigma(\Delta[n])$, where ΣK is the **sSet***-enriched category with two objects $\perp, \top$ such that*

$$\mathrm{Hom}_{\Sigma K}(x, y) = \begin{cases} *, & \text{if } x = y, \\ K, & \text{if } x = \perp \text{ and } y = \top, \\ \varnothing, & \text{if } x = \top \text{ and } y = \perp \end{cases}$$

Remark 2.4.1.2. Bergner characterises the fibrations in the above model structure in [8] as those functors $F : \mathcal{C} \to \mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ is a Kan fibration for all $X, Y \in \mathcal{C}$, and the ordinary functor underlying $h(\mathcal{C}) \to h(\mathcal{D})$ is an isofibration in the sense of Definition 2.2.1.2.

The Homotopy Hypothesis implies that the model category (**sSet**)**Cat** already provides a model of higher categories. We can therefore use this model to transfer homotopy theory to the subcategory of **sSet** spanned by quasicategories.

Definition 2.4.1.3. [26, Definitions 1.1.5.1, 1.1.5.3] For $[n] \in \mathbf{\Delta}$, define the **sSet**-enriched category $\mathfrak{C}[n]$ as follows:

- Take the objects of $\mathfrak{C}[n]$ to be the elements of $\{0, 1, \ldots, n\}$,
- For $i, j \in [n]$, define $\mathrm{Hom}_{\mathfrak{C}[n]}(i, j)$ to be the nerve of the category of subsets of the interval $[i, j] := \{i, i+1, \ldots, j-1, j\}$ that contain i and j.
- Composition $\mathrm{Hom}_{\mathfrak{C}[n]}(j, k) \times \mathrm{Hom}_{\mathfrak{C}[n]}(i, j) \to \mathrm{Hom}_{\mathfrak{C}[n]}(i, k)$ is induced by the monotone map that sends a pair of subsets $S \subseteq [i, j]$ and $T \subseteq [j, k]$ to their union $S \cup T \subseteq [i, k]$.

An order-preserving map $\phi : [n] \to [m]$ induces a functor $\mathfrak{C}[n] \to \mathfrak{C}[m]$ by sending $i \mapsto \phi(i)$, and the simplicial map $\mathrm{Hom}_{\mathfrak{C}[n]}(i, j) \to \mathrm{Hom}_{\mathfrak{C}[m]}(\phi(i), \phi(j))$ is induced by the monotone map that sends a subset $S \subseteq [i, j]$ to its image $\phi(S) \subseteq [\phi(i), \phi(j)]$.

We therefore have a functor $\mathfrak{C} : \mathbf{\Delta} \to (\mathbf{sSet})\mathbf{Cat}$, which induces an adjunction

$$\mathfrak{C} : \mathbf{sSet} \rightleftarrows (\mathbf{sSet})\mathbf{Cat} : N$$

The right adjoint is the *homotopy-coherent nerve*, which associates to a **sSet**-enriched category $\mathcal{C}$ the simplicial set $N\mathcal{C}$ whose n-cells correspond to functors $\mathfrak{C}[n] \to \mathcal{C}$. On the other hand, the left adjoint is the left Kan extension of $\mathfrak{C}$ along the Yoneda embedding $\mathbf{\Delta} \hookrightarrow \mathbf{sSet}$.

Remark 2.4.1.4. If $\mathcal{C}$ is an ordinary category, where its hom-sets are viewed as discrete (constant) simplicial sets, then the homotopy-coherent nerve of $\mathcal{C}$ coincides with the nerve of $\mathcal{C}$ in the sense of Definition 2.3.1.6.

Theorem 2.4.1.5. [26, Theorem 2.2.5.1] *There is a model structure on* **sSet** *(called the* Joyal model structure*) uniquely determined by the following:*

(W) The weak equivalences are the categorical equivalences*: those maps $f : X \to Y$ such that the induced map $\mathfrak{C}[X] \to \mathfrak{C}[Y]$ is a weak equivalence in the model structure of Theorem 2.4.1.1,*

(F) The fibrant objects are the quasicategories,

(C) The cofibrations are the monomorphisms. In particular, all simplicial sets are cofibrant.

Moreover, the adjunction $\mathfrak{C} \dashv N$ defines a Quillen equivalence between the model categories $\mathbf{sSet}_{\mathrm{Joyal}}$ and $(\mathbf{sSet}_{\mathrm{Quillen}})\mathbf{Cat}$.

In particular, we can view quasicategories as a model for $(\infty, 1)$-categories.

2.4.2 The theory of $(\infty, 1)$-categories

An $(\infty, 1)$-category is a homotopy-coherent generalisation of an ordinary 1-category. In particular, most basic results of ordinary category theory have a homotopy-coherent analogue in the theory of $(\infty, 1)$-categories—such as the theory of limits or colimits, adjunctions, Kan extensions, local presentability, and toposes—most of which is explained in-depth in [26].

Since $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is an ∞-groupoid—and thus a homotopy type—for all objects x, y in an $(\infty, 1)$-category $\mathcal{C}$, the Yoneda Lemma implies that we can view objects of $\mathcal{C}$ as homotopy types with additional structure. In other words, $\mathcal{C}$ is a context within which one can study abstract homotopy theory. On the other hand, model categories are designed specifically to study abstract homotopy theories with 1-categorical language.

We can reconcile the two languages quite readily, at least assuming that the model categories are enriched in $\mathbf{sSet}_{\mathrm{Quillen}}$ in the sense of [26, Definition A.3.1.5].

Definition 2.4.2.1. Suppose $\mathcal{M}$ is a $\mathbf{sSet}_{\text{Quillen}}$-enriched model category. Then, for cofibrant X and fibrant Y, the simplicial set $\mathrm{Hom}_{\mathcal{M}}(X, Y)$ is a Kan complex. In particular, the full subcategory $\mathcal{M}^\circ$ of $\mathcal{M}$ spanned by the cofibrant-fibrant objects is fibrant in the huge model category of $\mathbf{sSet}_{\text{Quillen}}$-enriched categories.

Since the homotopy-coherent nerve is a right Quillen functor, it follows that $N(\mathcal{M}^\circ)$ is a large quasicategory, which is called the *underlying $(\infty, 1)$-category* of $\mathcal{M}$.

Definition 2.4.2.2. $\mathbf{sSet}_{\text{Quillen}}$ is self-enriched from being a monoidal model category with respect to its cartesian tensor product. Denote its underlying $(\infty, 1)$-category by $\mathcal{S}$. By the Homotopy Hypothesis, this is also the $(\infty, 1)$-category of small ∞-groupoids.

Definition 2.4.2.3. $\mathbf{sSet}_{\text{Joyal}}$ is a monoidal model category, and therefore self-enriched. However, it is *not* enriched over $\mathbf{sSet}_{\text{Quillen}}$.

Nonetheless, we can define an $(\infty, 1)$-category $\mathbf{Cat}_\infty$ of small $(\infty, 1)$-categories as the homotopy-coherent nerve of the $\mathbf{sSet}$-enriched category $\mathbf{qCat}$ whose objects are the quasicategories, and whose simplicial hom-set $\mathrm{Hom}_{\mathbf{qCat}}(\mathcal{C}, \mathcal{D})$ is given by the maximal Kan complex contained in the quasicategory $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$.

Remark 2.4.2.4. There is a *Cartesian model structure* $\mathbf{sSet}^+_{\text{Cart}}$ on marked simplicial sets that is Quillen equivalent to $\mathbf{sSet}_{\text{Joyal}}$ defined in [26, Proposition 3.1.3.7]. By [26, Corollary 3.1.4.4], $\mathbf{sSet}^+_{\text{Cart}}$ is a simplicial model category, so we can equivalently define $\mathbf{Cat}_\infty$ as the $(\infty, 1)$-category underlying $\mathbf{sSet}^+_{\text{Cart}}$.

Remark 2.4.2.5. By [26, Proposition 5.2.4.6, Corollary A.3.1.12], every simplicial Quillen adjunction $F : \mathcal{M} \rightleftarrows \mathcal{M}' : U$ between simplicial model categories induces an adjunction $\mathbb{L}F : N(\mathcal{M}^\circ) \rightleftarrows N(\mathcal{M}'^\circ) : \mathbb{R}U$ of underlying $(\infty, 1)$-categories. If $F \dashv U$ is moreover a Quillen equivalence, then $\mathbb{L}F \dashv \mathbb{R}U$ is an adjoint equivalence of $(\infty, 1)$-categories.

Similarly, [26, Theorem 4.2.4.1] implies that homotopy limits and colimits in a simplicial model category agree with the limits and colimits in the underlying $(\infty, 1)$-category.

Chapter 3

Towards Higher Categories

Although $(\infty, 1)$-categories are not fully general higher categories, they form the smallest class of higher categories that simultaneously encapsulate ordinary category theory and provide access to higher-dimensional morphisms to encode weaker notions of equivalence. In particular, every higher category has an underlying $(\infty, 1)$-category obtained by discarding the non-invertible k-morphisms for $k > 1$ (analogous to Definition 2.1.1.7).

This makes $(\infty, 1)$-category theory a convenient framework in which to formalise models of (n, r)-categories for arbitrary n and r. As discussed in the introduction, there are several models of higher categories in the literature; see [2, 5, 29, 33, 34] for examples. The Unicity Theorem of [6] characterises the $(\infty, 1)$-category of small (∞, r)-categories for $0 \leq r < \infty$, up to an action of $(\mathbb{Z}/2)^r$ given by reversing k-morphisms in each dimension, giving a means to prove that any two "reasonable" models of (∞, r)-categories are equivalent in a precise sense. Note, however, that the "versality" condition asserted in the Unicity Theorem is quite strong, and requires the majority of the effort when invoking that a putative theory of (∞, r)-categories is correct.

We can characterise (n, r)-categories recursively by the observation that a higher category $\mathcal{C}$ is an $(n+1, r+1)$-category precisely if $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is an (n, r)-category for all $x, y \in \mathcal{C}$. In particular, this suggests that we can approach general higher categories using iterated enrichment, as in Section 2.1. However, it is highly nontrivial to describe enrichment in the $(\infty, 1)$-categorical setting: the compositional structure needs to be coherently associative up to the notion of higher homotopy encoded in the enriching monoidal $(\infty, 1)$-category.

Such a homotopy-coherent theory of enrichment is described in [10], and we will study higher categories through this lens in Chapter 5. However, the level of generality of this theory obfuscates the resulting presentation of higher categories; the model proposed in this chapter is much more elementary.

The purpose of this chapter is to describe a uniform presentation of the $(\infty, 1)$-category of small (n, r)-categories for any fixed $-2 \leq n \leq \infty$ and $0 \leq r \leq n+2$ in terms of localisations of presheaf $(\infty, 1)$-categories. In fact, we construct an $(\infty, 1)$-category of *sheaves* of (n, r)-categories over any $(\infty, 1)$-topos. The definition is based on the iterated construction of *complete Segal space* objects as in [27], which we make precise in Section 3.3. This allows us to prove that, when $n = \infty$ and r is finite, our construction fits in the context of the Unicity Theorem (and is therefore "correct"); see Corollary 3.3.4.4.

Convention 3.0.0.1. Throughout this chapter, and the remaining chapters, we omit the "$(\infty, 1)$" prefix. That is, a *category* refers to an $(\infty, 1)$-category, and likewise for other categorical notions such as groupoids, limits and colimits, et cetera. We will refer to the classical variants as *ordinary categories* or 1-*categories*. If misinterpretation is possible, we may also refer to $(\infty, 1)$-categories as ∞-*categories*; see Remark 2.4.0.6.

3.1 The Segal condition

The purpose of this section is to motivate the definition provided in Section 3.2. In particular, the material presented in this section is well-known.

Recall the construction of the nerve $N\mathcal{C}$ of an ordinary category $\mathcal{C}$ from Definition 2.3.1.6: the n-cells of $N\mathcal{C}$ correspond to the functors $[n] \to \mathcal{C}$. Since $[n]$ is the free category generated by the graph

$$0 \to 1 \to 2 \to \cdots \to n$$

it follows that functors $[n] \to \mathcal{C}$ correspond to chains of composable morphisms

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \to \ldots \xrightarrow{f_n} x_n$$

in $\mathcal{C}$. This leads to the following alternative characterisation of the nerve of an ordinary category:

Proposition 3.1.0.1. *A simplicial set $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ is isomorphic to the nerve of a small 1-category if and only if X satisfies the* Segal condition*: for every $n \geq 0$, the map*

$$X_n \to \underbrace{X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n}$$

induced by the inclusions $[1] \cong \{i, i+1\} \subset [n]$ is an isomorphism.

Proof. That the nerve of a 1-category satisfies the Segal condition follows from observing that the inclusions $\{i, i+1\} \subset [n]$ induce an isomorphism of 1-categories

$$[n] \cong \underbrace{[1] \sqcup_{[0]} [1] \sqcup_{[0]} \cdots \sqcup_{[0]} [1]}_{n}$$

Conversely, suppose X satisfies the above condition, then we construct a 1-category $\mathcal{C}$ by taking its set of objects to be X_0, and its set of morphisms to be X_1, where $f \in X_1$ is viewed as a morphism from $d_1 f$ to $d_0 f$.

For $x \in X$, the corresponding identity morphism is given by $s_0 x$, and composition in $\mathcal{C}$ is induced by

$$c : X_1 \times_{X_0} X_1 \xleftarrow{\sim} X_2 \xrightarrow{d_1} X_1$$

That this composition is associative follows from the commutativity of the diagram

$$\begin{tikzcd}
X_3 \arrow[rr, "d_1"] \arrow[dr, "\sim"] \arrow[dd, "d_2"'] & & X_2 \arrow[d, "\wr"'] \arrow[dd, bend left=60, "d_1"] \\
& X_1 \times_{X_0} X_1 \times_{X_0} X_1 \arrow[r, "c \times_{X_0} X_1"] \arrow[d, "X_1 \times_{X_0} c"'] & X_1 \times_{X_0} X_1 \arrow[d, "c"'] \\
X_2 \arrow[r, "\sim"] \arrow[rr, bend right, "d_1"'] & X_1 \times_{X_0} X_1 \arrow[r, "c"] & X_1
\end{tikzcd}$$

A similar diagram demonstrates that the composition is also unital. Therefore, $\mathcal{C}$ is a well-defined 1-category.

One can then readily check that the maps $X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ induce an isomorphism $X \cong N\mathcal{C}$. $\square$

From the above argument, any simplicial object $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{X}$ into a finitely complete 1-category $\mathcal{X}$ satisfying the Segal condition is completely determined by the maps

$$d_0, d_1 : X_1 \rightrightarrows X_0; \qquad s_0 : X_0 \to X_1; \qquad c : X_1 \times_{X_0} X_1 \to X_1$$

and the simplicial structure implies that $(X_0, X_1, d_0, d_1, s_0, c)$ defines a *category object* in $\mathcal{X}$, in the sense of [3, Definition 2.1]. This suggests that a possible approach to higher categories is through *internalisation*: the idea is to characterise $(n+1, r+1)$-categories as category objects in the category of (n, r)-categories. However, arbitrary category objects are too general.

For example, a category object $\mathcal{D}_1 \rightrightarrows \mathcal{D}_0$ in the 1-category $1\mathbf{Cat}$ of small 1-categories yields a *double category* in the sense of [9, Définition 10]:

- The objects of $\mathcal{D}_0$ are taken to be the objects of the double category $\mathcal{D}_\bullet$,
- The morphisms of $\mathcal{D}_0$ are the *vertical morphisms* of $\mathcal{D}_\bullet$, denoted $v : A \to B$,
- The objects of $\mathcal{D}_1$ are the *horizontal morphisms* of $\mathcal{D}_\bullet$, denoted $f : A \nrightarrow B$,

- The morphisms of $\mathcal{D}_1$ are the 2-*cells* of $\mathcal{D}$, and can be visualised as squares

$$\begin{tikzcd} A \arrow[r, "f", "\shortmid"{marking}] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "v"] \\ C \arrow[r, "g"', "\shortmid"{marking}] & D \end{tikzcd}$$

Nonetheless, we can view strict 2-categories as those double categories $\mathcal{D}_1 \rightrightarrows \mathcal{D}_0$ such that $\mathcal{D}_0$ is a set, ensuring that there are no nontrivial vertical morphisms.

This suggestion leads to a characterisation of $(n+1, r+1)$-categories as functors $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}_{(n,r)}$ such that

- X_0 is discrete; that is, X_0 is equivalent to a set, and
- $X_\bullet$ satisfies the *Segal condition* in that for $k \geq 0$, the canonical map

$$X_k \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{k}$$

  is an equivalence of (n, r)-categories.

This leads to the notion of a *Segal category*, as studied in [33]. While this approach offers a relatively elementary characterisation of (n, r)-categories, the tradeoff of this approach is that the appropriate notion of equivalence is more complicated. This is because the "underlying set" of a higher category is not invariant under equivalence, and therefore an equivalence between Segal categories $X_\bullet$ and $Y_\bullet$ needs not induce a bijection between X_0 and Y_0.

We therefore take an adjacent approach where the equivalences remain simple (nothing but equivalences of the underlying functors) in exchange for a more technical condition on the (n, r)-categories themselves. Rather than taking X_0 to be a set, we instead ask that X_0 is an ∞-groupoid, and that X_1 is an (n, r)-category. This introduces two distinct notions of equivalence between objects of the higher category: the horizontal equivalences from the categorical structure of $X_\bullet$, and the vertical equivalences from the ∞-groupoid X_0.

We reconcile this by asserting a univalence condition that the horizontal and vertical equivalences coincide, which we refer to as *Rezk-completeness*. This implies that X_0 is the underlying $(n+1)$-groupoid of $X_\bullet$ obtained by discarding all non-invertible k-morphisms in $X_\bullet$ for all $k \geq 1$. Such objects are called *complete Segal spaces*, which were first introduced in [29] as a model for $(\infty, 1)$-categories.

3.2 Higher category objects

In this section, we give an explicit definition of the category of (n, r)-category-valued sheaves for any $-2 \leq n \leq \infty$ and $0 \leq r \leq n+2$. In particular, taking sheaves over a point, we obtain an explicit definition of the category of (n, r)-categories. These categories are obtained as accessible localisations of functor categories valued in an ∞-topos, so we recall some preliminary notions below.

Definition 3.2.0.1. [26, Definition 5.5.4.1] Let $\mathcal{C}$ be a category, and S a collection of morphisms in $\mathcal{C}$. An object $x \in \mathcal{C}$ is *S-local* if for all $f : y \to z$ in S, the induced map $\mathrm{Hom}_{\mathcal{C}}(z, x) \to \mathrm{Hom}_{\mathcal{C}}(y, x)$ is an equivalence.

Remark 3.2.0.2. [26, Proposition 5.5.4.15] If $\mathcal{C}$ is locally presentable, and S is a small set of morphisms in $\mathcal{C}$, then the full subcategory $S^{-1}\mathcal{C}$ of $\mathcal{C}$ spanned by the S-local objects is locally presentable also, and the inclusion $S^{-1}\mathcal{C} \subseteq \mathcal{C}$ is accessible.

Let $\overline{S}$ denote the *strongly saturated* class of morphisms generated by S; that is, the smallest class of morphisms containing S and closed under:

- pushouts along arbitrary morphisms of $\mathcal{C}$,
- small colimits in $\mathbf{Fun}([1], \mathcal{C})$,
- 2-out-of-3.

The fully faithful inclusion $S^{-1}\mathcal{C} \subseteq \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to S^{-1}\mathcal{C}$, and for f a morphism in $\mathcal{C}$, its image Lf in $S^{-1}\mathcal{C}$ is an equivalence if and only if $f \in \overline{S}$. In particular, the left adjoint L exhibits $S^{-1}\mathcal{C}$ as the localisation of $\mathcal{C}$ at the morphisms in S; that is, $S^{-1}\mathcal{C}$ is the category obtained by formally inverting the morphisms in S. By [26, Proposition 5.5.4.2], every accessible localisation of a locally presentable category arises in this way.

Definition 3.2.0.3. [26, Definition 6.1.0.4] Define a *site of sheafification* to be a small category $\mathcal{C}$ equipped with an accessible left exact localisation functor $(-)^{\#} : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$. For such a site of sheafification $\mathcal{C}$, define its corresponding *category of sheaves* $\mathbf{Sh}(\mathcal{C})$ to be the essential image of $(-)^{\#}$. Then, a category $\mathcal{X}$ is an *∞-topos* if it is of the form $\mathcal{X} \simeq \mathbf{Sh}(\mathcal{C})$ for some site of sheafification $\mathcal{C}$.

Remark 3.2.0.4. Every small category $\mathcal{C}$ equipped with a Grothendieck topology defines a site of sheafification. However, not every ∞-topos can be written as a category of sheaves over a Grothendieck topology; see [26, §6.2.2].

Moreover, although every 1-category is an $(\infty, 1)$-category, note that 1-categorical topoi do not form a subclass of ∞-topoi: ∞-categorical sheaves on a 1-categorical site do not form a 1-topos, but rather a 1-*localic* ∞-*topos*.

3.2.1 Polysimplicial sheaves

Our definition of (n, r)-categories is obtained by unfolding iterated complete Segal spaces. We can therefore build the theory in terms of presheaves on $\mathbf{\Delta}^{\times r}$. Since we assert that the 0-cells form a groupoid, we can preemptively quotient $\mathbf{\Delta}^{\times r}$ by this assertion. In this subsection, we establish all of the preliminary notions necessary for defining (n, r)-categories with this approach.

Definition 3.2.1.1. Define the *polysimplex category* $\mathbf{\Sigma}_\infty$ as follows:

- The objects of $\mathbf{\Sigma}_\infty$ are infinite sequences $[\vec{k}] = (k_0, k_1, k_2, \dots)$ such that there exists some $r \geq 0$ for which $k_n = 0$ if and only if $n \geq r$. Call r the *dimension* of $[\vec{k}]$, denoted $\dim[\vec{k}] := r$.

- The morphisms $\vec{\phi} : [\vec{k}] \to [\vec{\ell}]$ are sequences of morphisms $\phi_n : [k_n] \to [\ell_n]$ in the simplex category $\mathbf{\Delta}$ such that there exists some $r \geq 0$ for which ϕ_n is constant if and only if $n \geq r$.

  The composite of two morphisms $[\vec{k}] \xrightarrow{\vec{\phi}} [\vec{\ell}] \xrightarrow{\vec{\psi}} [\vec{m}]$ is given as follows. Let $r \geq 0$ be minimal such that $\psi_r \phi_r$ is constant. Then,

$$(\vec{\psi} \circ \vec{\phi})_n = \begin{cases} \psi_n \phi_n, & \text{if } n \leq r \\ \psi_r \phi_r, & \text{otherwise} \end{cases}$$

For $0 \leq r \leq \infty$, denote by $\mathbf{\Sigma}_r$ the full subcategory of $\mathbf{\Sigma}_\infty$ spanned by the polysimplices $[\vec{k}]$ with $\dim[\vec{k}] \leq r$.

Remark 3.2.1.2. The category $\mathbf{\Sigma}_r$ is equivalent to the quotient category $\mathbf{\Theta}^r$ of $\mathbf{\Delta}^{\times r}$ defined in [32, §2]. We choose different notation so as to not confuse the polysimplex category with Joyal's disk category.

Definition 3.2.1.3. For a polysimplex $[\vec{k}]$ and $r \geq 0$, define the polysimplices

$$[\vec{k}]_{<r} := (k_0, k_1, k_2, \dots, k_{r-1}, 0, \dots)$$
$$[\vec{k}]_{\geq r} := (k_r, k_{r+1}, k_{r+2}, \dots)$$

These constructions extend to define endofunctors on $\mathbf{\Sigma}_\infty$. Note that $[-]_{<r}$ restricts to a right adjoint to the inclusion $\mathbf{\Sigma}_r \subseteq \mathbf{\Sigma}_\infty$.

Given two polysimplices $[\vec{k}]$ and $[\vec{\ell}]$, denote their concatenation by $[\vec{k}, \vec{\ell}]$; that is, $[\vec{k}, \vec{\ell}]$ is the unique polysimplex such that $[\vec{k}, \vec{\ell}]_{<r} = [\vec{k}]$ and $[\vec{k}, \vec{\ell}]_{\geq r} = [\vec{\ell}]$, where $r = \dim[\vec{k}]$.

Definition 3.2.1.4. For $0 \leq r \leq \infty$, we have a canonical projection map

$$\begin{aligned} \mathbf{\Delta} \times \mathbf{\Sigma}_r &\to \mathbf{\Sigma}_{r+1} \\ ([n], [\vec{k}]) &\mapsto \begin{cases} [0], & n = 0 \\ [n, \vec{k}], & n > 0 \end{cases} \end{aligned}$$

For any category $\mathcal{X}$, this induces a fully faithful inclusion

$$\mathbf{Fun}(\mathbf{\Sigma}_{r+1}^{\mathrm{op}}, \mathcal{X}) \subseteq \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Sigma}_r^{\mathrm{op}}, \mathcal{X}) \simeq \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Fun}(\mathbf{\Sigma}_r^{\mathrm{op}}, \mathcal{X}))$$

whose essential image consists of those $F : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Fun}(\mathbf{\Sigma}_r^{\mathrm{op}}, \mathcal{X})$ such that $F_0 : \mathbf{\Sigma}_r^{\mathrm{op}} \to \mathcal{X}$ is essentially constant.

Definition 3.2.1.5. Fix a site of sheafification $\mathcal{C}$, and let $\mathcal{X} := \mathbf{Sh}(\mathcal{C})$. Then, a *polysimplicial sheaf* over $\mathcal{C}$ is defined to be a functor $\mathbf{\Sigma}_\infty^{\mathrm{op}} \to \mathcal{X}$.

For a polysimplicial sheaf $X : \mathbf{\Sigma}_\infty^{\mathrm{op}} \to \mathcal{X}$ and a polysimplex $[\vec{k}]$, let $\Delta[\vec{k}, X] : \mathbf{\Sigma}_\infty^{\mathrm{op}} \to \mathcal{X}$ denote the polysimplicial sheaf given by the left Kan extension of X along $[\vec{k}, -] : \mathbf{\Sigma}_\infty \to \mathbf{\Sigma}_\infty$. By definition, it follows for all polysimplicial sheaves X' that we have a natural equivalence

$$\mathrm{Map}(\Delta[\vec{k}, X], X') \simeq \mathrm{Map}(X, X'_{\vec{k}, \bullet})$$

We are particularly interested in a couple of special cases:

- For a polysimplex $[\vec{k}]$ and an object $U \in \mathcal{C}$, define the *U-local $\vec{k}$-polysimplex* to be the object $\Delta_U[\vec{k}] := \Delta[\vec{k}, h_U^\#]$ that corepresents the functor $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X}) \to \mathcal{S}$ mapping $X_\bullet \mapsto X_{\vec{k}}(U)$.

- Define the *suspension* of a polysimplicial sheaf X to be $\Sigma X := \Delta[1, X]$, which is left adjoint to the *desuspension* $\Omega X' := X'_{1,\bullet}$.

Remark 3.2.1.6. The functor $[0, -] : \mathbf{\Sigma}_\infty \to \mathbf{\Sigma}_\infty$ is constant on the terminal object $[\vec{0}]$. In particular, the polysimplicial sheaf $\Delta[0, X]$ is independent of X, so we denote this object by $\Delta[0]$.

On the other hand, the U-local 0-polysimplex $\Delta_U[0]$ is *not* the terminal object; rather, it is the corepresenting object for the functor mapping $X_\bullet \mapsto X_0(U)$. This is precisely the representable sheaf $\Delta_U[0] \simeq h_U^\#$.

Remark 3.2.1.7. As $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X})$ is a reflective localisation of $\mathcal{P}(\mathbf{\Sigma}_\infty \times \mathcal{C})$, every polysimplicial sheaf is a canonical colimit of the local polysimplices $\Delta_U[\vec{k}]$.

Remark 3.2.1.8. Left Kan extension along the functor $\mathbf{\Sigma}_r \subseteq \mathbf{\Sigma}_\infty$ defines a fully faithful inclusion $\mathbf{Fun}(\mathbf{\Sigma}_r^{\mathrm{op}}, \mathcal{X}) \subseteq \mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X})$. We refer to objects in the essential image of this inclusion as r-polysimplicial sheaves. Explicitly, a functor $X : \mathbf{\Sigma}_r^{\mathrm{op}} \to \mathcal{X}$ is viewed as a polysimplicial sheaf by taking $X_{\vec{k}} := X_{\vec{k}_{<r}}$ for every $[\vec{k}] \in \mathbf{\Sigma}_\infty$.

Definition 3.2.1.9. Let $\mathcal{X} = \mathbf{Sh}(\mathcal{C})$ be an ∞-topos of sheaves. For $U \in \mathcal{C}$, left Kan extension of $\Delta_U[-] : \mathbf{\Delta} \to \mathbf{Fun}(\mathbf{\Sigma}_r^{\mathrm{op}}, \mathcal{X})$ along the Yoneda embedding $\mathbf{\Delta} \hookrightarrow \mathcal{P}(\mathbf{\Delta})$ induces a *U-local realisation*

$$|-|_U^{\#} : \mathcal{P}(\mathbf{\Delta}) \to \mathbf{Fun}(\mathbf{\Sigma}_r^{\mathrm{op}}, \mathcal{X})$$

which is left adjoint to the functor $\mathbf{Fun}(\mathbf{\Sigma}_r^{\mathrm{op}}, \mathcal{X}) \to \mathcal{P}(\mathbf{\Delta})$ sending $X_\bullet \mapsto X_\bullet(U)$.

Remark 3.2.1.10. The above construction generalises readily to define, for any $X : \mathbf{\Sigma}_r^{\mathrm{op}} \to \mathcal{X}$, an action $(-) \odot X : \mathcal{P}(\mathbf{\Delta}) \to \mathbf{Fun}(\mathbf{\Sigma}_{r+1}^{\mathrm{op}}, \mathcal{X})$ via left Kan extension along $\Delta[-, X]$.

Remark 3.2.1.11. For a simplicial set S, the maps $|S|_U^{\#} \to X_\bullet$ of polysimplicial sheaves correspond to maps $S_\bullet \to X_\bullet(U)$ in $\mathcal{P}(\mathbf{\Delta})$ by viewing S as a simplicial space via

$$\mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}) \subseteq \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}) = \mathcal{P}(\mathbf{\Delta})$$

On the other hand, the functor $\mathbf{Fun}(\mathbf{\Sigma}_r^{\mathrm{op}}, \mathcal{X}) \to \mathcal{S}$ sending $X \mapsto \mathrm{Hom}_{\mathcal{S}}(S, X(U))$ is corepresented by $S \times \Delta_U[0]$, where $S \times (-)$ acts pointwise as a product of spaces on $\Delta_U[0]$.

3.2.2 Sheaves of higher categories

We say that a simplicial space $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}$ satisfies the *Segal condition* if, for all $k \geq 0$, the induced map

$$X_k \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{k}$$

is an equivalence. In particular, X satisfies the Segal condition if and only if it is local in the sense of Definition 3.2.0.1 with respect to the canonical inclusions

$$\underbrace{\Delta[1] \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \Delta[1]}_{k} \subseteq \Delta[k]$$

for all $k \geq 0$. The simplicial space on the left is called the *spine* of $\Delta[k]$, which we generalise below.

Definition 3.2.2.1. Let $\mathcal{X} = \mathbf{Sh}(\mathcal{C})$ be an ∞-topos of sheaves. For a polysimplex $[\vec{k}]$ and an object $U \in \mathcal{C}$, define the *U-local $\vec{k}$-spine* to be the pushout

$$\mathrm{Sp}_U[\vec{k}] := \underbrace{\Delta_U[1, \vec{k}_{\geq 1}] \sqcup_{\Delta_U[0]} \cdots \sqcup_{\Delta_U[0]} \Delta_U[1, \vec{k}_{\geq 1}]}_{k_0}$$

in $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X})$. Note that the U-local $\vec{k}$-spine admits a canonical inclusion $\mathrm{Sp}_U[\vec{k}] \subseteq \Delta_U[\vec{k}]$.

For Rezk-completeness, we need to probe the categorical equivalences in a polysimplicial set satisfying the Segal condition.

Notation 3.2.2.2. Let $\mathbb{I}$ denote the *walking isomorphism*, which is the (1-)groupoidification of the 1-category [1]. We tacitly identify $\mathbb{I}$ with its nerve, which means explicitly that $\mathbb{I}_k = \{\perp, \top\}^{\times(k+1)}$, where the face maps are given by projections, and the degeneracy maps are given by diagonal inclusions.

Finally, to truncate theories of higher categories to n-categories, we appeal to the Homotopy Hypothesis. Specifically, recall that a space X is a *homotopy n-type*—that is, $\pi_m(X, x)$ is trivial for all $x \in X$ and $m > n$—if and only if every map $\mathbb{S}^m \to X$ is homotopic to a constant map for $m > n$.

Notation 3.2.2.3. For $m > -2$, define the *simplicial m-sphere* to be $\mathbb{S}^m := \partial\Delta[m+1]$.

We can now provide our definition of (n, r)-categories in terms of polysimplicial sheaves.

Definition 3.2.2.4. Let $\mathcal{X} = \mathbf{Sh}(\mathcal{C})$ be an ∞-topos of sheaves.

For $-2 \leq n \leq \infty$ and $0 \leq r \leq n+2$, let $W_{(n,r)}$ denote the strongly saturated class of morphisms in $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X})$ generated by:

(W1) $\Sigma^m \operatorname{Sp}_U[\vec{k}] \hookrightarrow \Sigma^m \Delta_U[\vec{k}]$ for $U \in \mathcal{C}$, $m \geq 0$, and $[\vec{k}] \in \mathbf{\Sigma}_\infty$,

(W2) $\Sigma^m |\mathbb{I}|_U^\# \to \Sigma^m \Delta_U[0]$ for $U \in \mathcal{C}$, and $m \geq 0$,

(W3) $\Sigma^m(\mathbb{S}^{k-m} \times \Delta_U[0]) \to \Sigma^m \Delta_U[0]$ for $U \in \mathcal{C}$, and finite $m \geq r$ and $k > n$, and

(W4) $\Sigma^m \Delta_U[1] \to \Sigma^m \Delta_U[0]$ for $U \in \mathcal{C}$, and finite $m \geq r$.

Call a polysimplicial sheaf a *sheaf of (n, r)-categories* if it is $W_{(n,r)}$-local. Denote by $\mathbf{Sh}_{(n,r)}(\mathcal{C})$ the full subcategory of $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X})$ spanned by the sheaves of (n, r)-categories.

Notation 3.2.2.5. Over the ∞-topos $\mathcal{S} \simeq \mathbf{Sh}(*)$ of spaces, let $-2 \leq n \leq \infty$ and $0 \leq r \leq n+2$. Then, we take the category of small (n, r)-categories to be $\mathbf{Cat}_{(n,r)} := \mathbf{Sh}_{(n,r)}(*)$.

Remark 3.2.2.6. Suppose $\mathcal{M}$ is a left proper combinatorial simplicial model category whose underlying ∞-category is an ∞-topos $\mathcal{X}$. For $-2 \leq n \leq \infty$ and $0 \leq r \leq n+2$, we can endow the 1-category $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{M})$ with a simplicial model structure such that

(C) A morphism $f : X_\bullet \to X'_\bullet$ is a cofibration if and only if the map $X_{\vec{k}} \to X'_{\vec{k}}$ is a cofibration in $\mathcal{M}$ for every $[\vec{k}] \in \mathbf{\Sigma}_\infty$,

(W) The levelwise weak equivalences are among the weak equivalences in $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{M})$, and

(F) An object $X_\bullet$ is fibrant if and only if it is injectively fibrant (see Proposition 2.2.2.3), and the induced map $N(\mathbf{\Sigma}_\infty)^{\mathrm{op}} \to N(X)$ in the underlying ∞-category $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X})$ is a sheaf of (n, r)-categories.

Indeed, this follows the same argument as [27, Proposition 1.5.4].

By Lemma 3.2.3.3, we can moreover restrict $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{M})$ to a Quillen equivalent model structure on the subcategory $\mathbf{Fun}(\mathbf{\Sigma}_r^{\mathrm{op}}, \mathcal{M})$.

Lemma 3.2.2.7. *For an ∞-topos $\mathcal{X} = \mathbf{Sh}(\mathcal{C})$, fix $0 \leq r \leq \infty$ and suppose $[\ell] \cong [\ell^1] \sqcup_{[0]} [\ell^2]$ in $\mathbf{\Delta}$. Then, for all $X : \mathbf{\Sigma}_r^{\mathrm{op}} \to \mathcal{X}$, the inclusion*

$$j : \Delta[\ell^1, X] \sqcup_{\Delta[0]} \Delta[\ell^2, X] \hookrightarrow \Delta[\ell, X]$$

lies in the strongly saturated class of morphisms generated by the local spine inclusions in $\mathbf{Fun}(\mathbf{\Sigma}_{r+1}^{\mathrm{op}}, \mathcal{X})$.

Proof. As the strongly saturated class of morphisms is closed under colimits in the arrow category $\mathbf{Fun}([1], \mathbf{Fun}(\mathbf{\Sigma}_{r+1}^{\mathrm{op}}, \mathcal{X}))$, it follows from Remark 3.2.1.7 that it suffices to prove the case where $X \simeq \Delta_U[\vec{k}]$. In this situation, consider the factorisation of the local spine inclusion $\mathrm{Sp}_U[\ell, \vec{k}] \subseteq \Delta_U[\ell, \vec{k}]$ given by

$$\mathrm{Sp}_U[\ell, \vec{k}] \xrightarrow{\sim} \mathrm{Sp}_U[\ell^1, \vec{k}] \sqcup_{\Delta[0]} \mathrm{Sp}_U[\ell^2, \vec{k}] \xhookrightarrow{i} \Delta_U[\ell^1, \vec{k}] \sqcup_{\Delta[0]} \Delta_U[\ell^2, \vec{k}] \xhookrightarrow{j} \Delta_U[\ell, \vec{k}]$$

Since i is a pushout of local spine inclusions, it follows from 2-out-of-3 that j lies in the strongly saturated class of morphisms, as desired. $\square$

Corollary 3.2.2.8. *Fix an ∞-topos $\mathcal{X} = \mathbf{Sh}(\mathcal{C})$ and $0 \leq r \leq \infty$. For any class W of morphisms in $\mathbf{Fun}(\mathbf{\Sigma}_r^{\mathrm{op}}, \mathcal{X})$. Then, the following classes of morphisms generate the same strongly saturated class of morphisms in $\mathbf{Fun}(\mathbf{\Sigma}_{r+1}^{\mathrm{op}}, \mathcal{X})$:*

(1) $\{\Delta[\ell^1, X] \sqcup_{\Delta[0]} \Delta[\ell^2, X] \hookrightarrow \Delta[\ell, X] :|: [\ell] = [\ell^1] \sqcup_{[0]} [\ell^2]; X : \mathbf{\Sigma}_r^{\mathrm{op}} \to \mathcal{X}\} \cup \{\Delta[\ell, f] :|: f \in W; \ell > 0\}$

(2) $\left\{\mathrm{Sp}_U[\vec{k}] \hookrightarrow \Delta_U[\vec{k}] :\middle|: [\vec{k}] \in \mathbf{\Sigma}_{r+1}; U \in \mathcal{C}\right\} \cup \{\Sigma f :|: f \in W\}$

Proof. Let W_1 and W_2 denote the strongly saturated classes of morphisms generated by (1) and (2), respectively, then clearly $W_2 \subseteq W_1$.

Conversely, any $\Delta[\ell^1, X] \sqcup_{\Delta[0]} \Delta[\ell^2, X] \hookrightarrow \Delta[\ell, X]$ lies in W_2 by Lemma 3.2.2.7. For a polysimplicial sheaf $X_\bullet$, let

$$\mathrm{Sp}[\ell, X] := \underbrace{\Sigma X \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \Sigma X}_{\ell} \hookrightarrow \Delta[\ell, X]$$

then this inclusion also lies in W_2.

For $f : X_\bullet \to X'_\bullet$ in W, the induced map $\mathrm{Sp}[\ell, X] \to \mathrm{Sp}[\ell, X']$ is a pushout of elements of W_2 and is thus an element of W_2 as well. From the commutative square

$$\begin{tikzcd}
\mathrm{Sp}[\ell, X] \arrow[r, hook, "\in W_2"] \arrow[d, "\in W_2"'] & \Delta[\ell, X] \arrow[d, "{\Delta[\ell,f]}"] \\
\mathrm{Sp}[\ell, X'] \arrow[r, hook, "\in W_2"'] & \Delta[\ell, X']
\end{tikzcd}$$

it follows from 2-out-of-3 that $\Delta[\ell, f] \in W_2$, as desired. $\square$

Corollary 3.2.2.9. *If $\mathcal{C}$ and $\mathcal{C}'$ are two sites of sheafification such that $\mathbf{Sh}(\mathcal{C}) \simeq \mathbf{Sh}(\mathcal{C}')$, then $\mathbf{Sh}_{(n,r)}(\mathcal{C}) \simeq \mathbf{Sh}_{(n,r)}(\mathcal{C}')$ for all $-2 \le n \le \infty$ and $0 \le r \le n+2$.*

Proof. Let $\mathcal{X} \simeq \mathbf{Sh}(\mathcal{C})$. Lemma 3.2.2.7 implies that the strongly saturated class of morphisms generated by (W1) is independent of the choice of site $\mathcal{C}$. The independence of the class $W_{(n,r)}$ then follows from observing that for any simplicial space S, locality in $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X})$ with respect to $\Sigma^m |S|_U^\# \to \Sigma^m \Delta_U[0]$ for all $U \in \mathcal{C}$ is equivalent to locality with respect to $\Sigma^m(S \odot X) \to \Sigma^m X$ for all $X : \mathbf{\Sigma}_\infty^{\mathrm{op}} \to \mathcal{X}$. $\square$

We conclude this subsection with a recursive description of sheaves of (n, r)-categories.

Proposition 3.2.2.10. *Fix an ∞-topos $\mathcal{X} = \mathbf{Sh}(\mathcal{C})$. For $-2 \le n \le \infty$ and $0 \le r \le n+2$, the essential image of the fully faithful inclusion*

$$\mathbf{Sh}_{(n+1,r+1)}(\mathcal{C}) \subseteq \mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X}) \subseteq \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X}))$$

induced by Definition 3.2.1.4 consists of those $F : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X})$ such that:

(S1) $F_0 : \mathbf{\Sigma}_\infty^{\mathrm{op}} \to \mathcal{X}$ is essentially constant,

(S2) F_1 is a sheaf of (n, r)-categories over $\mathcal{C}$,

(S3) F satisfies the Segal condition*: for every $k \ge 2$, the canonical map*

$$F_k \to \underbrace{F_1 \times_{F_0} \cdots \times_{F_0} F_1}_{k}$$

is an equivalence in $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X})$, and

(S4) F is Rezk-complete*: it is local with respect to $|\mathbb{I}|_U^\# \to \Delta_U[0]$ for every $U \in \mathcal{C}$.*

Remark 3.2.2.11. If F satisfies (S1), (S2), and (S3) and $n = \infty$, then F_k is a sheaf of (∞, r)-categories over $\mathcal{C}$ for every $k \ge 0$. In particular, we can characterise $\mathbf{Sh}_{(\infty,r+1)}(\mathcal{C})$ as the full subcategory of $\mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Sh}_{(\infty,r)}(\mathcal{C}))$ spanned by those $F : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Sh}_{(\infty,r)}(\mathcal{C})$ such that

- F_0 is essentially constant,
- F satisfies the Segal condition, and
- F is Rezk-complete.

This recovers the model of (∞, r)-categories as r-fold complete Segal spaces described in [5].

Note that if n is finite, then F_0 is a sheaf of $(n+1)$-groupoids, and therefore F cannot be identified with a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Sh}_{(n,r)}(\mathcal{C})$.

Proof of Proposition 3.2.2.10. A functor $F : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Sigma}_\infty^{\mathrm{op}} \to \mathcal{X}$ factors through the functor $\mathbf{\Delta} \times \mathbf{\Sigma}_\infty \to \mathbf{\Sigma}_\infty$ of Definition 3.2.1.4 if and only if F satisfies (S1). Assuming F satisfies (S1), we therefore tacitly identify F with its underlying functor $F : \mathbf{\Sigma}_\infty^{\mathrm{op}} \to \mathcal{X}$.

By definition, (S2) is equivalent to asserting that F is local with respect to $\Sigma(W_{(n,r)})$; that is, F is local with respect to:

(W1') $\Sigma^{m+1} \mathrm{Sp}_U[\vec{k}] \hookrightarrow \Sigma^{m+1}\Delta_U[\vec{k}]$ for $U \in \mathcal{C}$, $m \geq 0$, and $[\vec{k}] \in \mathbf{\Sigma}_\infty$,

(W2') $\Sigma^{m+1}|\mathbb{I}|_U^\# \to \Sigma^{m+1}\Delta_U[0]$ for $U \in \mathcal{C}$, and $m \geq 0$,

(W3') $\Sigma^{m+1}(\mathbb{S}^{k-m} \times \Delta_U[0]) \to \Sigma^{m+1}\Delta_U[0]$ for $U \in \mathcal{C}$, and finite $m \geq r$ and $k > n$, and

(W4') $\Sigma^{m+1}\Delta_U[1] \to \Sigma^{m+1}\Delta_U[0]$ for $U \in \mathcal{C}$, and finite $m \geq r$.

all of which are a subset of the generators for $W_{(n+1,r+1)}$. In particular, F is the image of a sheaf of $(n+1, r+1)$-categories if and only if F satisfies (S1) and (S2), and is moreover local with respect to

- $\mathrm{Sp}_U[\vec{k}] \hookrightarrow \Delta_U[\vec{k}]$ for $[\vec{k}] \in \mathbf{\Sigma}_r$ and $U \in \mathcal{C}$, and
- $|\mathbb{I}|_U^\# \to \Delta_U[0]$ for $U \in \mathcal{C}$.

Locality with respect to $|\mathbb{I}|_U^\# \to \Delta_U[0]$ is precisely condition (S4). It remains to show that (S3) is equivalent to locality with respect to the remaining spine inclusions.

The Segal condition (S3) is equivalent to locality with respect to $\mathrm{Sp}[k, X] \hookrightarrow \Delta[k, X]$ for every $k \geq 2$ and all polysimplicial sheaves X. By Remark 3.2.1.7, it suffices to assert locality with respect to $\mathrm{Sp}_U[\vec{k}] \hookrightarrow \Delta_U[\vec{k}]$ for $U \in \mathcal{C}$ and all $[\vec{k}] \in \mathbf{\Sigma}_\infty$. Assuming (S2), locality with respect to $\Sigma^{m+1}\Delta_U[1] \to \Sigma^{m+1}\Delta_U[0]$ for all finite $m \geq r$ implies that the Segal condition is equivalent to locality with respect to $\mathrm{Sp}_U[\vec{k}] \hookrightarrow \Delta_U[\vec{k}]$ for $U \in \mathcal{C}$ and $[\vec{k}] \in \mathbf{\Sigma}_r$, as desired. $\square$

3.2.3 Sheaves of (∞, ∞)-categories

Recall from the context of strict higher categories that we have three sections of the inclusion $i_{\leq n} : n\mathbf{Cat} \hookrightarrow (n+1)\mathbf{Cat}$:

- The left adjoint $\pi_{\leq n}$, which acts by trivialising all $(n+1)$-morphisms,
- The right adjoint $u_{\leq n}$, which discards all $(n+1)$-morphisms,
- The truncation functor $\tau_{\leq n}$ from Definition 2.1.1.7, which trivialises invertible $(n+1)$-morphisms, and discards the non-invertible $(n+1)$-morphisms.

The goal of this section is to construct analogous functors for (n,r)-categories, and establish analogues of Lemma 2.1.0.2 and Proposition 2.1.1.13. Throughout the subsection, fix an ∞-topos $\mathcal{X} = \mathbf{Sh}(\mathcal{C})$.

Definition 3.2.3.1. For $n \leq n'$ and $r \leq n'$, we have an inclusion $W_{(n',r')} \subseteq W_{(n,r)}$ of classes of morphisms in $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X})$. We therefore have a fully faithful inclusion

$$\mathbf{Sh}_{(n,r)}(\mathcal{C}) \subseteq \mathbf{Sh}_{(n',r')}(\mathcal{C})$$

of reflective subcategories of $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X})$. In particular, this inclusion admits a left adjoint, which we may denote $\pi_{\leq(n,r)}$.

Remark 3.2.3.2. The inclusion $\mathbf{Sh}_{(n,r)}(\mathcal{C}) \subseteq \mathbf{Sh}_{(n',r')}(\mathcal{C})$ does not admit a right adjoint in general. For instance, a right adjoint fails to exist for the inclusion $\mathbf{Cat}_{(0,0)} \subseteq \mathbf{Cat}_{(1,0)}$ of the 1-category of sets into the $(2,1)$-category of small groupoids. If such a right adjoint u were to exist, then we would have for any small groupoid $\mathcal{G}$ an equivalence

$$u(\mathcal{G}) \cong \mathrm{Hom}_{\mathbf{Set}}(*, u(\mathcal{G})) \simeq \mathrm{Hom}_{\mathbf{Grpd}}(*, \mathcal{G}) \simeq \mathcal{G}$$

of groupoids, implying that every groupoid is essentially discrete.

Although right adjoints to general inclusions $\mathbf{Sh}_{(n,r)}(\mathcal{C}) \subseteq \mathbf{Sh}_{(n',r')}(\mathcal{C})$ fail to exist, they do exist whenever $n = n'$.

Lemma 3.2.3.3. *For $0 \leq r < \infty$, the category $\mathbf{Sh}_{(\infty,r)}(\mathcal{C})$ is precisely the full subcategory of $\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})$ spanned by those $X : \mathbf{\Sigma}_\infty^{\mathrm{op}} \to \mathcal{X}$ that factor through $[-]_{<r} : \mathbf{\Sigma}_\infty^{\mathrm{op}} \to \mathbf{\Sigma}_r^{\mathrm{op}}$.*

Proof. We prove the case where $r = 0$; the rest follow by induction with Proposition 3.2.2.10. When $r = 0$, we are showing that $X : \mathbf{\Sigma}_\infty^{\mathrm{op}} \to \mathcal{X}$ is a sheaf of $(\infty,0)$-categories precisely if it is essentially constant.

Note that $W_{(\infty,0)}$ is generated by $\Sigma^m \operatorname{Sp}_U[\vec{k}] \hookrightarrow \Sigma^m \Delta_U[\vec{k}]$ and $\Sigma^m \Delta_U[1] \to \Sigma^m \Delta_U[0]$ for $U \in \mathcal{C}$, $m \geq 0$, and $[\vec{k}] \in \mathbf{\Sigma}_\infty$. By induction with Corollary 3.2.2.8, $W_{(\infty,0)}$ therefore contains the morphisms $\Delta_U[\vec{k},1] \to \Delta_U[\vec{k}]$ for all $[\vec{k}] \in \mathbf{\Sigma}_\infty$, as well as the morphisms

$$\underbrace{\Delta_U[\vec{k},1] \sqcup_{\Delta_U[\vec{k}]} \cdots \sqcup_{\Delta_U[\vec{k}]} \Delta_U[\vec{k},1]}_{\ell} \simeq \Delta[\vec{k}, \operatorname{Sp}_U[\ell]] \to \Delta_U[\vec{k},\ell]$$

for all $[\vec{k}] \in \mathbf{\Sigma}_\infty$ and $\ell \geq 0$. This implies that $W_{(\infty,0)}$ contains $\Delta_U[\vec{k},\ell] \to \Delta_U[\vec{k}]$ for every $U \in \mathcal{C}$ and $[\vec{k},\ell] \in \mathbf{\Sigma}_\infty$, implying that X is $W_{(\infty,0)}$-local if and only if X is essentially constant, as desired. $\square$

Proposition 3.2.3.4. *For $-2 \leq n \leq \infty$ and $0 \leq r < r' \leq n+2$, the fully faithful inclusion $\mathbf{Sh}_{(n,r)}(\mathcal{C}) \subseteq \mathbf{Sh}_{(n,r')}(\mathcal{C})$ admits a right adjoint $\kappa_{\leq r}$.*

Proof. First suppose $n = \infty$. The right adjoint to the inclusion $\mathbf{Fun}(\mathbf{\Sigma}_r^{\mathrm{op}}, \mathcal{X}) \to \mathbf{Fun}(\mathbf{\Sigma}_{r'}^{\mathrm{op}}, \mathcal{X})$ is given by restriction to $\mathbf{\Sigma}_r \subseteq \mathbf{\Sigma}_{r'}$ (see Remark 3.2.1.8). By Lemma 3.2.3.3, restriction descends to a right adjoint $\kappa_{\leq r} : \mathbf{Sh}_{(\infty,r')}(\mathcal{C}) \to \mathbf{Sh}_{(\infty,r)}(\mathcal{C})$.

The general case follows from observing that the above right adjoint descends further to a right adjoint $\mathbf{Sh}_{(n,r')}(\mathcal{C}) \to \mathbf{Sh}_{(n,r)}(\mathcal{C})$ for $n < \infty$. $\square$

As $\mathbf{Cat}_{(n,r)}$ is the full subcategory of $\mathbf{Cat}_{(n,r+1)}$ consisting of those higher categories whose $(r+1)$-morphisms are invertible, the left adjoint $\pi_{\leq r}$ acts on an $(n, r+1)$-category by formally inverting all $(r+1)$-morphisms, whereas $\kappa_{\leq r}$ acts by discarding all non-invertible $(r+1)$-morphisms. In particular, the right adjoint $\kappa_{\leq r}$ behaves like a homotopy-coherent analogue of $\tau_{\leq r}$ from Definition 2.1.1.7.

Proposition 3.2.3.5. *The functors* $\kappa_{\leq r} : \mathbf{Sh}_{(\infty,\infty)}(\mathcal{C}) \to \mathbf{Sh}_{(\infty,r)}(\mathcal{C})$ *exhibit* $\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})$ *as the limit*

$$\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C}) \simeq \varprojlim \left(\cdots \to \mathbf{Sh}_{(\infty,3)}(\mathcal{C}) \xrightarrow{\kappa_{\leq 2}} \mathbf{Sh}_{(\infty,2)}(\mathcal{C}) \xrightarrow{\kappa_{\leq 1}} \mathbf{Sh}_{(\infty,1)}(\mathcal{C}) \xrightarrow{\kappa_{\leq 0}} \mathbf{Sh}_{(\infty,0)}(\mathcal{C}) \right)$$

in $\widehat{\mathbf{Cat}_\infty}$.

Proof. Since $\mathbf{\Sigma}_\infty \simeq \varinjlim_r \mathbf{\Sigma}_r$, we have an equivalence $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X}) \simeq \varprojlim_r \mathbf{Fun}(\mathbf{\Sigma}_r^{\mathrm{op}}, \mathcal{X})$. As fully faithful functors are stable under limits, this allows us to identify the limit $\varprojlim_r \mathbf{Sh}_{(\infty,r)}(\mathcal{C})$ of right adjoints with the full subcategory of $\mathbf{Fun}(\mathbf{\Sigma}_\infty^{\mathrm{op}}, \mathcal{X})$ on those $X : \mathbf{\Sigma}_\infty^{\mathrm{op}} \to \mathcal{X}$ that restrict on $\mathbf{\Sigma}_r$ to a sheaf of (∞, r)-categories for every $0 \leq r < \infty$, but these are precisely the sheaves of (∞, ∞)-categories. $\square$

Remark 3.2.3.6. As $\kappa_{\leq r}$ is a homotopy-coherent analogue of $\tau_{\leq r}$ (see also Proposition 5.2.1.3), Propositions 2.1.1.13 and 3.2.3.5 suggest that $\mathbf{Cat}_{(\infty,\infty)}$ is a homotopy-coherent analogue of the 1-category $\omega\mathbf{Cat}^+$ of *marked* strict ω-categories.

The "marked 1-morphisms" in an (∞, ∞)-category $\mathcal{C} : \mathbf{\Sigma}_\infty^{\mathrm{op}} \to \mathcal{S}$ are implicit in the choice of space $\mathcal{C}_0$ which, by Rezk-completeness, encodes the underlying ∞-groupoid of $\mathcal{C}$.

This result motivates studying the complementary construction below, which should be a homotopy-coherent analogue of the 1-category $\omega\mathbf{Cat}$ of (unmarked) strict ω-categories of Definition 2.1.0.3:

Definition 3.2.3.7. Define the category of *sheaves of ω-categories* over $\mathcal{C}$ to be the limit

$$\mathbf{Sh}_\omega(\mathcal{C}) := \varprojlim \left(\cdots \to \mathbf{Sh}_{(\infty,3)}(\mathcal{C}) \xrightarrow{\pi_{\leq 2}} \mathbf{Sh}_{(\infty,2)}(\mathcal{C}) \xrightarrow{\pi_{\leq 1}} \mathbf{Sh}_{(\infty,1)}(\mathcal{C}) \xrightarrow{\pi_{\leq 0}} \mathbf{Sh}_{(\infty,0)}(\mathcal{C}) \right)$$

in $\widehat{\mathbf{Cat}_\infty}$.

Remark 3.2.3.8. As $\mathbf{Sh}_\omega(\mathcal{C})$ is given as a limit of left adjoints between locally presentable categories, it is locally presentable. Moreover, we have by [26, Proposition 5.5.3.13 and Corollary 5.5.3.4] that it can be computed equivalently as the colimit

$$\mathbf{Sh}_\omega(\mathcal{C}) \simeq \varinjlim^R \left(\mathbf{Sh}_{(\infty,0)}(\mathcal{C}) \subseteq \mathbf{Sh}_{(\infty,1)}(\mathcal{C}) \subseteq \mathbf{Sh}_{(\infty,2)}(\mathcal{C}) \subseteq \cdots\right)$$

in $\mathbf{Pres}^R_\infty$.

Moreover, the right adjoint inclusions $\mathbf{Sh}_{(\infty,r)}(\mathcal{C}) \subseteq \mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})$ induce a canonical right adjoint functor $\mathbf{Sh}_\omega(\mathcal{C}) \to \mathbf{Sh}_{(\infty,\infty)}$, whose left adjoint may be denoted by $\pi_{\leq\omega}$.

Notation 3.2.3.9. We similarly define the category of ω-categories to be $\mathbf{Cat}_\omega := \mathbf{Sh}_\omega(*)$, as in Notation 3.2.2.5.

Remark 3.2.3.10. Intuitively, $\mathbf{Cat}_\omega$ consists of (∞,∞)-categories wherein the equivalences are precisely the *weakly coinductive equivalences*; see Remark 2.1.1.6.

3.3 Distributors and complete Segal spaces

Lurie axiomatises the construction of complete Segal spaces in [27]: given a suitable category $\mathcal{Y}$ and a subcategory $\mathcal{X} \subseteq \mathcal{Y}$ of "spaces", there exists a category $\mathbf{CSS}_{\mathcal{X}}(\mathcal{Y})$ of simplicial objects in $\mathcal{Y}$ satisfying the axioms of a complete Segal space.

The goal of this section is to show that our model of (sheaves of) (∞,r)-categories fits into this abstract framework of complete Segal space objects, extending Remark 3.2.2.11. This will imply, for instance, that our model defines a theory of (∞,r)-categories in the sense of [6]; see Corollary 3.3.4.4. We moreover prove in Theorem 3.3.4.5 that $\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})$ is closed under the construction of complete Segal spaces.

3.3.1 Preliminaries

In this subsection, we recall the theory of distributors in the sense of [27], and the construction of complete Segal space objects. In particular, none of the content here is original.

Definition 3.3.1.1. [27, Definition 1.2.1] Fix a category $\mathcal{Y}$ and a full subcategory $\mathcal{X}$. Say that $\mathcal{Y}$ is an $\mathcal{X}$-*distributor* if the following hold:

(D1) $\mathcal{X}$ and $\mathcal{Y}$ are locally presentable,

(D2) The inclusion $\mathcal{X} \subseteq \mathcal{Y}$ admits both a left adjoint $\pi_{\mathcal{X}}$ called $\mathcal{X}$-*truncation*, and a right adjoint $\kappa_{\mathcal{X}}$ called $\mathcal{X}$-*core*,

(D3) For all $y \to x$ in $\mathcal{Y}$ with $x \in \mathcal{X}$, the pullback functor $\mathcal{X}_{/x} \to \mathcal{Y}_{/y}$ preserves small colimits, and

(D4) The functor $\chi : \mathcal{X} \to (\widehat{\mathbf{Cat}_\infty})^{\mathrm{op}}, x \mapsto \mathcal{Y}_{/x}$, which sends morphisms in $\mathcal{X}$ to pullback functors, preserves small limits.

Remark 3.3.1.2. [27, Example 1.2.3 and Remark 1.2.6] If $\mathcal{Y}$ is an $\mathcal{X}$-distributor, then $\mathcal{X}$ is necessarily an ∞-topos. Conversely, every ∞-topos $\mathcal{X}$ is a distributor relative to itself.

Definition 3.3.1.3. For $\mathcal{X}$ a finitely complete category, let $\mathbf{Cat}(\mathcal{X})$ denote the full subcategory of $\mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{X})$ spanned by those $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{X}$ satisfying the *Segal condition*; that is, the canonical maps

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are equivalences for every $n \geq 0$.

Further, define $\mathbf{Grpd}(\mathcal{X})$ to be the full subcategory of $\mathbf{Cat}(\mathcal{X})$ spanned by those $X_\bullet$ such that whenever $[n] = S \cup S'$ with $S \cap S' = \{i\}$, the induced map

$$X_n \to X_S \times_{X_{\{i\}}} X_{S'}$$

is an equivalence.

Remark 3.3.1.4. If $\mathcal{X}$ is locally presentable, then $\mathbf{Cat}(\mathcal{X})$ and $\mathbf{Grpd}(\mathcal{X})$ are strongly reflective subcategories of $\mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{X})$; in particular, they are locally presentable as well.

Definition 3.3.1.5. [27, Definition 1.2.7] Fix an ∞-topos $\mathcal{X}$ and an $\mathcal{X}$-distributor $\mathcal{Y}$. Then, define $\mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$ to be the full subcategory of $\mathbf{Cat}(\mathcal{Y})$ spanned by those $Y \in \mathbf{Cat}(\mathcal{Y})$ such that $Y_0 \in \mathcal{X}$. Such objects are called *Segal space objects* in $\mathcal{Y}$.

Remark 3.3.1.6. As $\mathbf{SS}_{\mathcal{X}}(\mathcal{Y}) \simeq \mathbf{Cat}(\mathcal{Y}) \times_{\mathcal{X}} \mathcal{Y}$ is a fibre product of right adjoints between locally presentable categories, $\mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$ is also locally presentable, and $\mathbf{SS}_{\mathcal{X}}(\mathcal{Y}) \subseteq \mathbf{Cat}(\mathcal{Y})$ is a right adjoint.

Lemma 3.3.1.7. [27, Proposition 1.1.14] *If $\mathcal{X}$ is a category with all finite limits, then the inclusion $\mathbf{Grpd}(\mathcal{X}) \subseteq \mathbf{Cat}(\mathcal{X})$ admits a right adjoint $X_\bullet \mapsto X_\bullet^\sim$.*

Proof sketch. If $\mathcal{X} \simeq \mathcal{S}$, then we construct the right adjoint explicitly. For $X_\bullet \in \mathbf{Cat}(\mathcal{S})$, define the $\mathrm{Ho}(\mathcal{S})$-enriched category hX where its objects are the underlying points of X_0, and the mapping space $\mathrm{Hom}_{hX}(x, y)$ is the homotopy class of the fibre product $\{x\} \times_{X_0} X_1 \times_{X_0} \{y\}$ in $\mathrm{Ho}(\mathcal{S})$. Then, we define $X_n^\sim$ to be the full subspace of X_n spanned by those cells $\alpha \in X_n$ for which each $d_i\alpha$ descends to an isomorphism in hX.

If $\mathcal{X} \simeq \mathcal{P}(\mathcal{C})$, then $\mathbf{Cat}(\mathcal{X}) \simeq \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}(\mathcal{S}))$ and $\mathbf{Grpd}(\mathcal{X}) \simeq \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Grpd}(\mathcal{X}))$, so the right adjoint in this case is given pointwise by the right adjoint for $\mathcal{S}$.

For a general $\mathcal{X}$, let $X_\bullet \in \mathbf{Cat}(\mathcal{X})$, and consider the object $jX_\bullet \in \mathbf{Cat}(\mathcal{P}(\mathcal{X}))$ obtained by pointwise post-composition with the Yoneda embedding $j : \mathcal{X} \hookrightarrow \mathcal{P}(\mathcal{X})$. Then, the reflection $(jX_\bullet)^\sim \in \mathbf{Grpd}(\mathcal{P}(\mathcal{X}))$ turns out to be pointwise representable, allowing it to descend to an object $X_\bullet^\sim \in \mathbf{Grpd}(\mathcal{X})$. $\square$

Corollary 3.3.1.8. [27, Notation 1.2.9] *For an ∞-topos $\mathcal{X}$ and an $\mathcal{X}$-distributor $\mathcal{Y}$, the inclusion* $\mathbf{Grpd}(\mathcal{X}) \subseteq \mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$ *admits a right adjoint* $\mathrm{Gp} : \mathbf{SS}_{\mathcal{X}}(\mathcal{Y}) \to \mathbf{Grpd}(\mathcal{X})$.

Remark 3.3.1.9. Explicitly, $\mathrm{Gp}_\bullet Y \simeq (\kappa_{\mathcal{X},*} Y_\bullet)^\sim$, where $\kappa_{\mathcal{X},*}$ applies the core pointwise. By the proof sketch of Lemma 3.3.1.7, the canonical map $\mathrm{Gp}_0 Y \to Y_0$ is an equivalence for all $Y_\bullet \in \mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$.

Definition 3.3.1.10. Fix an $\mathcal{X}$-distributor $\mathcal{Y}$. Say that a morphism $f : Y_\bullet \to Y'_\bullet$ in $\mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$ is a *Segal equivalence* if the following conditions hold:

(E1) f is *fully faithful*, in that

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & Y'_1 \arrow[d] \\ Y_0 \times Y_0 \arrow[r] & Y'_0 \times Y'_0 \end{tikzcd}$$

is a pullback square, and

(E2) f is *essentially surjective* in that the induced map $|\mathrm{Gp}_\bullet Y| \to |\mathrm{Gp}_\bullet Y'|$ is an equivalence in $\mathcal{X}$.

Note for $Y_\bullet$ a Segal space object, $|Y_\bullet| := \varinjlim Y$ denotes its *geometric realisation.*

Say that $Y_\bullet \in \mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$ is a *complete Segal space* if it is local with respect to the Segal equivalences. Denote by $\mathbf{CSS}_{\mathcal{X}}(\mathcal{Y})$ the full subcategory of $\mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$ spanned by the complete Segal spaces.

Remark 3.3.1.11. By [27, Theorem 1.2.13], $Y_\bullet \in \mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$ is a complete Segal space if and only if $\mathrm{Gp}_\bullet Y$ is essentially constant; that is, $\mathrm{Gp}_\bullet Y$ lies in the essential image of the diagonal $\mathcal{X} \subseteq \mathbf{Grpd}(\mathcal{X})$.

Proposition 3.3.1.12. [27, Proposition 1.3.2] *If $\mathcal{Y}$ is an $\mathcal{X}$-distributor, then the fully faithful diagonal functor $\mathcal{X} \subseteq \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y})$ exhibits $\mathbf{CSS}_{\mathcal{X}}(\mathcal{Y})$ as an $\mathcal{X}$-distributor as well.*

Remark 3.3.1.13. The truncation functor $\pi_{\mathcal{X}} : \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}) \to \mathcal{X}$, being left adjoint to the diagonal, is given by geometric realisation $\pi_{\mathcal{X}} Y \simeq |Y_\bullet|$. On the other hand, the core functor $\kappa_{\mathcal{X}} : \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}) \to \mathcal{X}$ is given by $\kappa_{\mathcal{X}} Y \simeq Y_0$.

3.3.2 Limits and colimits of distributors

In preparation for proving that $\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})$ is invariant under the construction of complete Segal space objects, we lay the necessary groundwork to demonstrate the functoriality of $\mathbf{CSS}_{\mathcal{X}}(\mathcal{Y})$ in $\mathcal{Y}$.

Definition 3.3.2.1. Fix an ∞-topos $\mathcal{X}$. Then, define $\mathbf{Dist}_{\mathcal{X}}^{L}$ to be the subcategory of $(\widehat{\mathbf{Cat}_{\infty}})_{\mathcal{X}/}$ where the objects are $\mathcal{X}$-distributors $\mathcal{X} \subseteq \mathcal{Y}$, and a functor $\psi : \mathcal{Z} \to \mathcal{Y}$ under $\mathcal{X}$ is a morphism of $\mathbf{Dist}_{\mathcal{X}}^{L}$ if and only if

(L1) ψ preserves colimits, and

(L2) ψ preserves truncation, in that $\pi_{\mathcal{X}}\psi \simeq \pi_{\mathcal{X}}$.

On the other hand, define $\mathbf{Dist}_{\mathcal{X}}^{R}$ to be the subcategory of $(\widehat{\mathbf{Cat}_{\infty}})_{\mathcal{X}/}$ where the objects are again the $\mathcal{X}$-distributors $\mathcal{X} \subseteq \mathcal{Y}$, and a functor $\phi : \mathcal{Y} \to \mathcal{Z}$ under $\mathcal{X}$ is a morphism of $\mathbf{Dist}_{\mathcal{X}}^{R}$ if and only if

(R1) ϕ preserves limits and λ-filtered colimits for some $\lambda \gg 0$, and

(R2) ϕ preserves cores, in that $\kappa_{\mathcal{X}}\phi \simeq \kappa_{\mathcal{X}}$.

Remark 3.3.2.2. Since $\mathcal{X}$-distributors are necessarily locally presentable, (L1) is equivalent by [26, Corollary 5.5.2.9] to asserting that ψ admits a right adjoint ϕ. As truncation is left adjoint to the inclusion of $\mathcal{X}$, (L2) is then equivalent to asserting that the right adjoint ϕ lies under $\mathcal{X}$; that is, $\phi|_{\mathcal{X}} \simeq \mathrm{id}_{\mathcal{X}}$.

Dually, (R1) is likewise equivalent to asserting that ϕ admits a left adjoint ψ. As core is right adjoint to the inclusion of $\mathcal{X}$, (R2) is equivalent to asserting that the left adjoint ψ lies under $\mathcal{X}$; that is $\psi|_{\mathcal{X}} \simeq \mathrm{id}_{\mathcal{X}}$. This demonstrates that the two categories of Definition 3.3.2.1 are formally dual: $\mathbf{Dist}_{\mathcal{X}}^{L} \simeq (\mathbf{Dist}_{\mathcal{X}}^{R})^{\mathrm{op}}$.

Proposition 3.3.2.3. *For any ∞-topos $\mathcal{X}$, the functor $\mathbf{Dist}_{\mathcal{X}}^{R} \to (\widehat{\mathbf{Cat}_{\infty}})_{/\mathcal{X}}$ sending an $\mathcal{X}$-distributor $\mathcal{Y}$ to its core $\kappa_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ creates all small limits.*

Proof. Let $\mathcal{Y}_{\bullet} : K \to \mathbf{Dist}_{\mathcal{X}}^{R}$ be a small diagram indexed by a simplicial set K. The limit of the diagram $K \to (\widehat{\mathbf{Cat}_{\infty}})_{/\mathcal{X}}$ can be computed as the limit of the corresponding diagram $K^{\triangleright} \to \widehat{\mathbf{Cat}_{\infty}}$ that sends the cocone point of $K^{\triangleright}$ to $\mathcal{X}$ via core maps $\kappa_{\mathcal{X}} : \mathcal{Y}_k \to \mathcal{X}$. By [26, Theorem 5.5.3.18], the limit is equivalent to the limit of the diagram $\widetilde{\mathcal{Y}}_{\bullet} : K^{\triangleright} \to \mathbf{Pres}_{\infty}^{R}$.

Let $\mathcal{Y} := \varprojlim_k \widetilde{\mathcal{Y}}_k$, then $\mathcal{Y}$ is locally presentable, and the projection map $\kappa_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is a right adjoint. The left adjoint coincides with the functor $\mathcal{X} \to \mathcal{Y} \simeq \varprojlim_k \widetilde{\mathcal{Y}}_k$ in $\widehat{\mathbf{Cat}_{\infty}}$ induced by the inclusions $\mathcal{X} \subseteq \mathcal{Y}_k$ for $k \in K^{\triangleright}$. In particular, since these inclusions are fully faithful right adjoints, it follows that the left adjoint $\mathcal{X} \to \mathcal{Y}$ is also a fully faithful right adjoint. This proves that $\mathcal{Y}$ satisfies (D1) and (D2).

Let $y \to x$ in $\mathcal{Y}$ with $x \in \mathcal{X}$. Note that $\mathcal{Y}_{/y} \simeq \varprojlim_k (\widetilde{\mathcal{Y}}_k)_{/y_k}$, where y_k is the image of y under the projection $\mathcal{Y} \to \mathcal{Y}_k$. Indeed, this follows from the right adjoint property of slicing (see [26, Proposition 4.2.1.5], and just above it) which induces a right Quillen functor (as the alternative join of [26, Definition 4.2.1.1] is a left Quillen functor by design), and the limit creation property of functor categories in the second argument. Now, the map $\mathcal{X}_{/x} \to \mathcal{Y}_{/y}$ is the canonical map induced by the functors $\mathcal{X}_{/x} \to (\widetilde{\mathcal{Y}}_k)_{/y_k}$ for each $k \in K^{\triangleright}$.

Since each $\mathcal{Y}_k$ is an $\mathcal{X}$-distributor, the functor $\mathcal{X}_{/x} \to (\widetilde{\mathcal{Y}}_k)_{/y_k}$ is a left adjoint for every $k \in K^{\triangleright}$. By [26, Proposition 5.5.3.13], $\mathcal{X}_{/x} \to \mathcal{Y}_{/y}$ must be a left adjoint as well, establishing (D3).

For $k \in K^{\triangleright}$, let $\chi^k : \mathcal{X} \to (\widehat{\mathbf{Cat}_\infty})^{\mathrm{op}}$ denote the functor mapping $x \in \mathcal{X}$ to the category $(\widetilde{\mathcal{Y}}_k)_{/x}$. Since each χ^k preserves limits from $\widetilde{\mathcal{Y}}_k$ being an $\mathcal{X}$-distributor, the same is true for the induced functor $\vec{\chi} : \mathcal{X} \to \mathbf{Fun}(K^{\triangleright}, (\widehat{\mathbf{Cat}_\infty})^{\mathrm{op}})$ that sends each $x \in \mathcal{X}$ to the functor mapping $k \mapsto (\widetilde{\mathcal{Y}}_k)_{/x}$. Now, the functor $\chi : \mathcal{X} \to (\widehat{\mathbf{Cat}_\infty})^{\mathrm{op}}$ sending $x \mapsto \mathcal{Y}_{/x} \simeq \varprojlim_k (\widetilde{\mathcal{Y}}_k)_{/x}$ is precisely the composite of limit-preserving functors

$$\mathcal{X} \xrightarrow{\vec{\chi}} \mathbf{Fun}(K^{\triangleright}, (\widehat{\mathbf{Cat}_\infty})^{\mathrm{op}}) \xrightarrow{\varprojlim} (\widehat{\mathbf{Cat}_\infty})^{\mathrm{op}}$$

and thus preserves limits, establishing (D4).

Therefore, the limit $\mathcal{Y}$ is an $\mathcal{X}$-distributor, and the canonical projections $\mathcal{Y} \to \mathcal{Y}_k$ are core-preserving right adjoints. Suppose we have a cone from an $\mathcal{X}$-distributor $\mathcal{Z}$ to $\mathcal{Y}_\bullet$ in $\mathbf{Dist}^R_{\mathcal{X}}$. This induces a cone from $\mathcal{Z}$ to $\widetilde{\mathcal{Y}}_\bullet$ in $\mathbf{Pres}^R_\infty$, inducing an essentially unique right adjoint $\mathcal{Z} \to \mathcal{Y}$. Moreover, since the functors $\mathcal{Z} \to \mathcal{Y}_k$ lie under $\mathcal{X}$, the same is true for $\mathcal{Z} \to \mathcal{Y}$, ensuring that the canonical functor $\mathcal{Z} \to \mathcal{Y}$ is a morphism of $\mathbf{Dist}^R_{\mathcal{X}}$, proving that $\mathcal{Y}$ is indeed a limit of $\mathcal{Y}_\bullet$ in $\mathbf{Dist}^R_{\mathcal{X}}$. □

In particular, most limits in $\mathbf{Dist}^R_{\mathcal{X}}$ can be computed on the underlying categories.

Lemma 3.3.2.4. *For any category $\mathcal{D}$ and functor $p : S \to \mathcal{D}$, the forgetful functor $\mathcal{D}_{p/} \to \mathcal{D}$ creates colimits indexed by weakly contractible simplicial sets.*

Proof. By [26, Corollary 2.1.2.2], the forgetful functor $\mathcal{D}_{p/} \to \mathcal{D}$ is a left fibration. Therefore, by the dual of [26, Proposition 2.4.2.4], the forgetful functor is a cocartesian fibration where every morphism of $\mathcal{D}_{p/}$ is cocartesian. In particular, if K is a weakly contractible simplicial set, then any functor $K^{\triangleright} \to \mathcal{D}_{p/}$ is a colimit diagram relative to the forgetful functor $\mathcal{D}_{p/} \to \mathcal{D}$ by [26, Proposition 4.3.1.12]. By [26, Proposition 4.3.1.5(2)], this means that such a diagram $K^{\triangleright} \to \mathcal{D}_{p/}$ is a colimit diagram if and only if its composite $K^{\triangleright} \to \mathcal{D}$ is a colimit diagram, as desired. □

Corollary 3.3.2.5. *For any ∞-topos $\mathcal{X}$, the forgetful functor $\mathbf{Dist}^R_{\mathcal{X}} \to \widehat{\mathbf{Cat}_\infty}$ creates all small weakly contractible limits.*

Proposition 3.3.2.6. *For any ∞-topos $\mathcal{X}$, the functor $\mathbf{Dist}^L_{\mathcal{X}} \to (\widehat{\mathbf{Cat}_\infty})_{/\mathcal{X}}$ sending an $\mathcal{X}$-distributor $\mathcal{Y}$ to its truncation $\pi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ creates all small limits.*

Proof. Let $\mathcal{Y}_\bullet : K \to \mathbf{Dist}_{\mathcal{X}}^L$ be a small diagram indexed by a simplicial set K. The limit of the diagram $K \to (\widehat{\mathbf{Cat}_\infty})_{/\mathcal{X}}$ can be computed as a limit of the corresponding diagram $K^\triangleright \to \widehat{\mathbf{Cat}_\infty}$ that sends the cocone point of $K^\triangleright$ to $\mathcal{X}$ via truncation maps $\pi_{\mathcal{X}} : \mathcal{Y}_k \to \mathcal{X}$. By [26, Proposition 5.5.3.13], we can compute this limit instead as a limit of the diagram $\widetilde{\mathcal{Y}}_\bullet : K^\triangleright \to \mathbf{Pres}_\infty^L$.

Let $\mathcal{Y} := \varprojlim_k \widetilde{\mathcal{Y}}_k$, then $\mathcal{Y}$ is locally presentable, and the projection map $\pi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is a left adjoint. The right adjoint coincides with the functor $\mathcal{X} \to \mathcal{Y} \simeq \varprojlim_k \widetilde{\mathcal{Y}}_k$ in $\widehat{\mathbf{Cat}_\infty}$ induced by the inclusions $\mathcal{X} \subseteq \mathcal{Y}_k$ for $k \in K^\triangleright$. In particular, since these inclusions are fully faithful left adjoints, the same is true for the right adjoint $\mathcal{X} \to \mathcal{Y}$. This proves that $\mathcal{Y}$ satisfies (D1) and (D2). Moreover, that $\mathcal{Y}$ satisfies (D3) and (D4) follows the argument in Proposition 3.3.2.3 verbatim.

Therefore, the limit $\mathcal{Y}$ is an $\mathcal{X}$-distributor, and the canonical projections $\mathcal{Y} \to \mathcal{Y}_k$ are truncation-preserving left adjoints. Suppose we have a cone from an $\mathcal{X}$-distributor $\mathcal{Z}$ to $\mathcal{Y}_\bullet$ in $\mathbf{Dist}_{\mathcal{X}}^L$. This induces a cone from $\mathcal{Z}$ to $\widetilde{\mathcal{Y}}_\bullet$ in $\mathbf{Pres}_\infty^L$, inducing an essentially unique left adjoint $\mathcal{Z} \to \mathcal{Y}$. Moreover, since the functors $\mathcal{Z} \to \mathcal{Y}_k$ lie under $\mathcal{X}$, the same is true for $\mathcal{Z} \to \mathcal{Y}$, ensuring that the canonical functor $\mathcal{Z} \to \mathcal{Y}$ is a morphism of $\mathbf{Dist}_{\mathcal{X}}^L$, proving that $\mathcal{Y}$ is indeed a limit of $\mathcal{Y}_\bullet$ in $\mathbf{Dist}_{\mathcal{X}}^L$. □

Corollary 3.3.2.7. *For any ∞-topos $\mathcal{X}$, the forgetful functor $\mathbf{Dist}_{\mathcal{X}}^L \to \widehat{\mathbf{Cat}_\infty}$ creates all small weakly contractible limits.*

Remark 3.3.2.8. From the duality in Remark 3.3.2.2, $\mathbf{Dist}_{\mathcal{X}}^R$ has all small limits and colimits (and likewise for $\mathbf{Dist}_{\mathcal{X}}^L$).

3.3.3 The complete Segal space functor

In this subsection, we prove that $\mathbf{CSS}_{\mathcal{X}}$ is functorial over $\mathbf{Dist}_{\mathcal{X}}^R$, and moreover prove that the construction preserves certain limits and colimits.

Definition 3.3.3.1. Let $\mathcal{Y}$ be a category with full subcategory $\mathcal{X} \subseteq \mathcal{Y}$. Say that a functor $\phi : \mathcal{Y} \to \mathcal{Z}$ *preserves fibre products over* $\mathcal{X}$ if for every fibre product $y_1 \times_x y_2$ in $\mathcal{Y}$ with $x \in \mathcal{X}$, the induced map $\phi(y_1 \times_x y_2) \to \phi(y_1) \times_{\phi(x)} \phi(y_2)$ is an equivalence in $\mathcal{Z}$.

If $\mathcal{X}$ has finite limits, and $\mathcal{X} \subseteq \mathcal{Y}$ preserves these limits, then say that a functor $\phi : \mathcal{Y} \to \mathcal{Z}$ is *relatively left exact* (relative to $\mathcal{X}$) if $\phi| : \mathcal{X} \to \mathcal{Z}$ is left exact, and ϕ preserves fibre products over $\mathcal{X}$.

The functorial nature of $\mathbf{CSS}_{\mathcal{X}}$ can be summarised as follows:

Lemma 3.3.3.2. *Let $g_* : \mathcal{X} \to \mathcal{X}'$ be a geometric morphism of ∞-topoi. Fix an $\mathcal{X}$-distributor $\mathcal{Y}$ and an $\mathcal{X}'$-distributor $\mathcal{Y}'$, and let $\phi : \mathcal{Y} \to \mathcal{Y}'$ be a functor such that*

- *ϕ extends g_*, in that $\phi|_{\mathcal{X}} \simeq g_*$,*
- *ϕ commutes with cores, in that $\phi\kappa_{\mathcal{X}} \simeq \kappa_{\mathcal{X}'}\phi$, and*
- *ϕ preserves fibre products over $\mathcal{X}$.*

Then, $\phi_* : \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{Y}) \to \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{Y}')$ restricts to a functor $\phi_* : \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}) \to \mathbf{CSS}_{\mathcal{X}'}(\mathcal{Y}')$.

Remark 3.3.3.3. In the situation of Lemma 3.3.3.2, $\phi_* : \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}) \to \mathbf{CSS}_{\mathcal{X}'}(\mathcal{Y}')$ acts pointwise on the underlying functors. Since limits of complete Segal spaces are computed pointwise, this ensures that ϕ_* preserves fibre products over $\mathcal{X}$. Likewise, Remark 3.3.1.13 ensures that ϕ_* commutes with cores as well. Therefore, Lemma 3.3.3.2 combined with Proposition 3.3.1.12 implies an *endofunctorial* nature of the construction of complete Segal spaces.

Proof of Lemma 3.3.3.2. Since ϕ is relatively left exact, it follows that ϕ_* restricts to a functor $\mathbf{SS}_{\mathcal{X}}(\mathcal{Y}) \to \mathbf{SS}_{\mathcal{X}'}(\mathcal{Y}')$. Similarly, the left exactness of the left adjoint $g^* \dashv g_*$ induces a commutative square of left adjoints

$$\begin{tikzcd}
\mathbf{Grpd}(\mathcal{X}') \arrow[r, hook] \arrow[d, "g^*"'] & \mathbf{Cat}(\mathcal{X}') \arrow[d, "g^*"] \\
\mathbf{Grpd}(\mathcal{X}) \arrow[r, hook] & \mathbf{Cat}(\mathcal{X})
\end{tikzcd}$$

Taking right adjoints implies for any $X_\bullet \in \mathbf{Cat}(\mathcal{X})$ that $(g_*X_\bullet)^\sim \simeq g_*(X_\bullet^\sim)$. It therefore follows for any $Y_\bullet \in \mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$ that

$$\begin{aligned}
\mathrm{Gp}_\bullet(\phi_*Y) &\simeq ((\kappa_{\mathcal{X}'} \circ \phi)_*Y)^\sim && \text{(Remark 3.3.1.9)} \\
&\simeq ((\phi \circ \kappa_{\mathcal{X}})_*Y)^\sim && \text{(ϕ commutes with cores)} \\
&\simeq ((g_* \circ \kappa_{\mathcal{X}})_*Y)^\sim && \text{(ϕ extends g_*)} \\
&\simeq g_*(\kappa_{\mathcal{X},*}Y)^\sim \simeq g_*(\mathrm{Gp}_\bullet Y) && \text{(Remark 3.3.1.9)}
\end{aligned}$$

In particular, if $Y_\bullet$ is a complete Segal space, then $\mathrm{Gp}_\bullet(\phi_*Y) \simeq g_* \mathrm{Gp}_\bullet Y$ is essentially constant, ensuring that $\phi_*Y_\bullet$ is a complete Segal space by Remark 3.3.1.11. □

Lemma 3.3.3.4. *Fix an ∞-topos $\mathcal{X}$, and let $\mathcal{Y}$ be an $\mathcal{X}$-distributor. Choose $\lambda \gg 0$ such that $\mathcal{Y}$ is locally λ-presentable and the core functor $\kappa_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is λ-accessible. Then,*

(i) $\mathbf{CSS}_{\mathcal{X}}(\mathcal{Y})$ is stable under λ-filtered colimits in $\mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{Y})$.

(ii) λ-filtered colimits in $\mathbf{CSS}_{\mathcal{X}}(\mathcal{Y})$ commute with λ-small limits.

(iii) The core functor $\kappa_{\mathcal{X}} : \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}) \to \mathcal{X}$ preserves λ-filtered colimits.

Proof. This follows the proof of [27, Proposition 1.2.29] *mutatis mutandis*, noting that λ-filtered colimits commute with λ-small limits in any locally λ-presentable category. $\square$

Proposition 3.3.3.5. *For an ∞-topos $\mathcal{X}$, the construction of complete Segal spaces defines a functor $\mathbf{CSS}_{\mathcal{X}} : \mathbf{Dist}^R_{\mathcal{X}} \to \mathbf{Dist}^R_{\mathcal{X}}$, where the functor $\phi_* : \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}) \to \mathbf{CSS}_{\mathcal{X}}(\mathcal{Z})$ induced by any $\phi : \mathcal{Y} \to \mathcal{Z}$ in $\mathbf{Dist}^R_{\mathcal{X}}$ acts pointwise.*

Proof. Let $\phi : \mathcal{Y} \to \mathcal{Z}$ be a morphism of $\mathbf{Dist}^R_{\mathcal{X}}$. Then, Lemma 3.3.3.2 implies that pointwise application of ϕ defines a functor $\phi_* : \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}) \to \mathbf{CSS}_{\mathcal{X}}(\mathcal{Z})$. Moreover, this functor preserves cores and small limits.

Choose $\lambda \gg 0$ so that $\phi : \mathcal{Y} \to \mathcal{Z}$ is a λ-accessible functor between locally λ-presentable categories. Then, Lemma 3.3.3.4 implies that λ-filtered colimits in $\mathbf{CSS}_{\mathcal{X}}(\mathcal{Y})$ and $\mathbf{CSS}_{\mathcal{X}}(\mathcal{Z})$ are computed pointwise, and are thus preserved by ϕ_*. Therefore, ϕ_* is indeed a morphism of $\mathbf{Dist}^R_{\mathcal{X}}$. $\square$

Remark 3.3.3.6. By Remark 3.3.2.2, the construction of complete Segal spaces also defines an endofunctor on $\mathbf{Dist}^L_{\mathcal{X}}$. However, the functor $\psi_! : \mathbf{CSS}_{\mathcal{X}}(\mathcal{Z}) \to \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y})$ induced by $\psi : \mathcal{Z} \to \mathcal{Y}$ does not act pointwise; rather, if $\mathrm{Seg}_{\mathcal{X}} : \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{Y}) \to \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y})$ denotes a left adjoint to the inclusion, then $\psi_!$ is given on $Z_\bullet \in \mathbf{CSS}_{\mathcal{X}}(\mathcal{Z})$ by $(\psi_! Z)_\bullet \simeq \mathrm{Seg}_{\mathcal{X}}(\psi_* Z_\bullet)$.

We can now establish the continuity of the construction of complete Segal spaces:

Theorem 3.3.3.7. *For an ∞-topos $\mathcal{X}$, the functor $\mathbf{CSS}_{\mathcal{X}} : \mathbf{Dist}^R_{\mathcal{X}} \to \mathbf{Dist}^R_{\mathcal{X}}$ preserves small weakly contractible limits.*

Proof. Let $\mathcal{Y}_\bullet : K \to \mathbf{Dist}^R_{\mathcal{X}}$ be a diagram indexed by a weakly contractible simplicial set K, and let $\mathcal{Y} := \varprojlim_k \mathcal{Y}_k$, with canonical projection maps $\varpi_k : \mathcal{Y} \to \mathcal{Y}_k$. Consider the commutative square

$$\begin{tikzcd}
\mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}) \arrow[r] \arrow[d, hook] & \varprojlim_k \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}_k) \arrow[d, hook] \\
\mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{Y}) \arrow[r, "\sim"'] & \varprojlim_k \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{Y}_k)
\end{tikzcd}$$

where the limits are computed in $\widehat{\mathbf{Cat}}_\infty$. By Corollary 3.3.2.5, the limit $\mathcal{Y} = \varprojlim_k \mathcal{Y}_k$ can be computed on underlying categories in $\widehat{\mathbf{Cat}}_\infty$, implying that the bottom arrow in the above diagram is an equivalence. Corollary 3.3.2.5 also implies that it suffices to show that the top arrow in the above diagram is an equivalence as well. As all other functors are fully faithful, the top arrow is fully faithful. Therefore, it remains to show that the top arrow is essentially surjective.

From the commutative diagram, we can identify $\varprojlim_k \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}_k)$ with the full subcategory of $\mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{Y})$ spanned by those $Y : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{Y}$ for which $\varpi_k Y : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{Y}_k$ lies in $\mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}_k)$ for every $k \in K$. It suffices to show that any such functor $Y : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{Y}$ is already a complete Segal space in $\mathcal{Y}$.

Certainly $Y_0 \in \mathcal{X}$, since $\varpi_k Y_0 \in \mathcal{X}$ for all $k \in K$. Since ϖ_k is a morphism of $\mathbf{Dist}^R_{\mathcal{X}}$, it preserves limits. In particular, for $n \geq 0$, the canonical map

$$Y_n \to \underbrace{Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1}_{n}$$

in $\mathcal{Y}$ descends via ϖ_k to an equivalence in $\mathcal{Y}_k$ for every $k \in K$, and is therefore also an equivalence. In particular, $Y_\bullet \in \mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$.

As K is weakly contractible, it is nonempty, so fix some $k_0 \in K$. Since ϖ_{k_0} preserves cores and limits, the proof of Lemma 3.3.3.2 implies that $\mathrm{Gp}_\bullet(\varpi_{k_0,*} Y_\bullet) \simeq \mathrm{Gp}_\bullet Y$. Since $\mathrm{Gp}_\bullet(\varpi_{k_0,*} Y_\bullet)$ is essentially constant, this proves that $Y_\bullet$ is a complete Segal space, as desired. $\square$

3.3.4 The distributor of sheaves of (∞, r)-categories

Throughout this subsection, fix an ∞-topos of sheaves $\mathcal{X} = \mathbf{Sh}(\mathcal{C})$. In this section, we prove that the category $\mathbf{Sh}_{(\infty,r)}(\mathcal{C})$ defines an $\mathcal{X}$-distributor for every $0 \leq r \leq \infty$. Note that Corollary 3.3.2.7 then ensures that the category $\mathbf{Sh}_\omega(\mathcal{C})$ defines an $\mathcal{X}$-distributor as well.

Lemma 3.3.4.1. *Let $\mathcal{Y}$ be an $\mathcal{X}$-distributor. Then, $Y_\bullet \in \mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$ is a complete Segal space if and only if it is local with respect to $|\mathbb{I}|^\#_U \to \Delta_U[0]$ for every $U \in \mathcal{C}$.*

Proof. Since $\mathrm{Gp}_\bullet Y \in \mathbf{Grpd}(\mathcal{X})$ satisfies the Segal condition, it is essentially constant if and only if the map $\mathrm{Gp}_1 Y \to \mathrm{Gp}_0 Y$ is an equivalence, which is equivalent to asserting that the underlying functor $\mathrm{Gp}_\bullet Y : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{X}$ is local with respect to $\Delta_U[1] \to \Delta_U[0]$ for every $U \in \mathcal{C}$. Now, note that

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathcal{X})}(\Delta_U[1], \mathrm{Gp}_\bullet Y) \\
&\simeq \mathrm{Hom}_{\mathcal{P}(\mathbf{\Delta})}(\Delta[1], \mathrm{Gp}_\bullet Y(U)) && \text{(Definition 3.2.1.5)} \\
&\simeq \mathrm{Hom}_{\mathbf{Cat}(\mathcal{S})}(\Delta[1], \mathrm{Gp}_\bullet Y(U)) && \text{(both } \Delta[1] \text{ and } \mathrm{Gp}_\bullet Y(U) \text{ satisfy the Segal condition)} \\
&\simeq \mathrm{Hom}_{\mathbf{Grpd}(\mathcal{S})}(\mathbb{I}, \mathrm{Gp}_\bullet Y(U)) && (\mathbb{I} \text{ is the groupoidification of } \Delta[1]) \\
&\simeq \mathrm{Hom}_{\mathcal{P}(\mathbf{\Delta})}(\mathbb{I}, \mathrm{Gp}_\bullet Y(U)) && (\mathbf{Grpd}(\mathcal{S}) \subseteq \mathcal{P}(\mathbf{\Delta}) \text{ is fully faithful}) \\
&\simeq \mathrm{Hom}_{\mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathcal{X})}(|\mathbb{I}|^\#_U, \mathrm{Gp}_\bullet Y) && \text{(Definition 3.2.1.9)} \\
&\simeq \mathrm{Hom}_{\mathbf{Grpd}(\mathcal{X})}(|\mathbb{I}|^\#_U, \mathrm{Gp}_\bullet Y) && \text{(both } |\mathbb{I}|^\#_U \text{ and } \mathrm{Gp}_\bullet Y \text{ are groupoid objects in } \mathcal{X}) \\
&\simeq \mathrm{Hom}_{\mathbf{SS}_{\mathcal{X}}(\mathcal{Y})}(|\mathbb{I}|^\#_U, Y_\bullet) && \text{(Corollary 3.3.1.8)}
\end{aligned}$$

and similarly for $\Delta_U[0]$. Therefore, $\mathrm{Gp}_\bullet Y$ is essentially constant if and only if $Y_\bullet$ is local with respect to $|\mathbb{I}|^\#_U \to \Delta_U[0]$ for every $U \in \mathcal{C}$, as desired. □

Corollary 3.3.4.2. *The category $\mathbf{Sh}_{(\infty,r)}(\mathcal{C})$ is an $\mathcal{X}$-distributor for every $0 \le r < \infty$. Moreover, we have a canonical equivalence $\mathbf{Sh}_{(\infty,r+1)}(\mathcal{C}) \simeq \mathbf{CSS}_\mathcal{X}(\mathbf{Sh}_{(\infty,r)}(\mathcal{C}))$.*

Proof. This is clear if $r = 0$, since $\mathbf{Sh}_{(\infty,0)}(\mathcal{C}) \simeq \mathcal{X}$. The result for $r > 0$ follows by induction from combining Lemma 3.3.4.1 with Proposition 3.2.2.10. □

Remark 3.3.4.3. That iterated application of $\mathbf{CSS}_\mathcal{X}$ to $\mathcal{X}$ yields sheaves of (∞, r)-categories is already suggested in [27, Variant 1.3.8].

Corollary 3.3.4.4. $\mathbf{Cat}_{(\infty,r)}$ *defines a theory of (∞, r)-categories in the sense of* [6] *for all $r \ge 0$.*

Proof. This follows from Corollary 3.3.4.2 and [6, Theorem 14.6]. □

We are particularly interested in the limit cases.

Theorem 3.3.4.5. *The category $\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})$ is an $\mathcal{X}$-distributor. Moreover, there is a canonical equivalence $\mathbf{CSS}_\mathcal{X}(\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})) \xrightarrow{\sim} \mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})$.*

Proof. By Proposition 3.2.3.5 and Corollary 3.3.4.2, we can compute $\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})$ as the limit of categories

$$\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C}) \simeq \varprojlim \left(\cdots \to \mathbf{CSS}^3_\mathcal{X}(\mathcal{X}) \xrightarrow{\kappa_{\mathcal{X},*}} \mathbf{CSS}^2_\mathcal{X}(\mathcal{X}) \xrightarrow{\kappa_{\mathcal{X},*}} \mathbf{CSS}_\mathcal{X}(\mathcal{X}) \xrightarrow{\kappa_\mathcal{X}} \mathcal{X} \right)$$

By Corollary 3.3.2.5, this limit can be lifted to a limit in $\mathbf{Dist}^R_\mathcal{X}$, proving that $\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})$ is indeed an $\mathcal{X}$-distributor. Moreover, Theorem 3.3.3.7 proves that the induced functor $\mathbf{CSS}_\mathcal{X}(\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})) \to \mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})$ is an equivalence, as desired. □

Remark 3.3.4.6. By Corollary 4.3.1.8, which we will prove in Section 4.3, and the observation that $\mathcal{X}$ is a terminal object in $\mathbf{Dist}^R_\mathcal{X}$, the above proof moreover implies that $\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})$ is a *terminal coalgebra* for $\mathbf{CSS}_\mathcal{X}$. In particular, $\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C})$ is the terminal object of the full subcategory $\mathbf{Fix}^R_{\mathbf{CSS}}$ of $\mathbf{Dist}^R_\mathcal{X}$ spanned by those $\mathcal{X}$-distributors $\mathcal{Y}$ equipped with an equivalence $\mathcal{Y} \xrightarrow{\sim} \mathbf{CSS}_\mathcal{X}(\mathcal{Y})$.

3.3.5 Geometric morphisms and sheafification

Throughout this subsection, fix a sheaf ∞-topos $\mathcal{X} = \mathbf{Sh}(\mathcal{C})$.

The goal of this subsection is to prove various comparison results between categories of sheaves. We prove in Corollary 3.3.5.5 that the construction $\mathbf{Sh}_{(n,r)}(-)$ or $\mathbf{Sh}_\omega(-)$ is functorial with respect to geometric morphisms, sending geometric morphisms to right adjoints with relatively left exact left adjoints. In Proposition 3.3.5.8 and Corollary 3.3.5.9, we prove that $\pi_{\leq r} : \mathbf{Sh}_{(\infty,r')}(\mathcal{C}) \to \mathbf{Sh}_{(\infty,r)}(\mathcal{C})$ and $\pi_{\leq \omega} : \mathbf{Sh}_{(\infty,\infty)}(\mathcal{C}) \to \mathbf{Sh}_{\omega}(\mathcal{C})$ are relatively left exact. Finally, we show in Theorem 3.3.5.10 that $\mathbf{Sh}_{(n,r)}(\mathcal{C})$ is a relatively left exact accessible localisation of $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_{(n,r)})$, and likewise $\mathbf{Sh}_{\omega}(\mathcal{C})$ is a relatively left exact accessible localisation of $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{SS}_{\omega})$.

Lemma 3.3.5.1. *Suppose K is sifted in the sense of* [26, Definition 5.5.8.1]. *Then, the functor $\varinjlim : \mathbf{Fun}(K, \mathcal{X}) \to \mathcal{X}$ preserves fibre products over $\mathcal{X}$, where $\mathcal{X}$ is viewed as a full subcategory of $\mathbf{Fun}(K, \mathcal{X})$ spanned by the essentially constant functors.*

Proof. Suppose $F \to x \leftarrow G$ is a cospan of functors $K \to \mathcal{X}$, where $x \in \mathcal{X}$ is viewed as a constant functor. Since K is sifted, the diagonal $K \to K \times K$ is cofinal. Therefore,

$$\begin{aligned}\left(\varinjlim_{k\in K} F(k)\right) \times_x \left(\varinjlim_{\ell\in K} G(\ell)\right) &\simeq \varinjlim_{(k,\ell)\in K\times K} \Big(F(k) \times_x G(\ell)\Big) && \text{(colimits are universal in } \mathcal{X}\text{)} \\ &\simeq \varinjlim_{j\in K} \Big(F(j) \times_x G(j)\Big) && (K \to K \times K \text{ is cofinal})\end{aligned}$$

showing that $\varinjlim : \mathbf{Fun}(K, \mathcal{X}) \to \mathcal{X}$ preserves the fibre product $F \times_x G$. □

Lemma 3.3.5.2. *Fix an $\mathcal{X}$-distributor $\mathcal{Y}$. Suppose a morphism $Y_\bullet \to Z_\bullet$ in $\mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$ is a Segal equivalence. Then, for any cospan $Z_\bullet \to x \leftarrow Z'_\bullet$ in $\mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$ with $x \in \mathcal{X}$, the induced map $Y \times_x Z' \to Z \times_x Z'$ is a Segal equivalence.*

Proof. The functor $Y \times_x Z' \to Z \times_x Z'$ is certainly fully faithful in the sense of (E1), since limits commute with limits. We need to show that the map is also essentially surjective in the sense of (E2); that is, $|\operatorname{Gp}_\bullet(Y \times_x Z')| \to |\operatorname{Gp}_\bullet(Z \times_x Z')|$ is an equivalence in $\mathcal{X}$.

Since $\operatorname{Gp}_\bullet$ is a right adjoint that restricts to the identity on $\mathcal{X}$, we have an equivalence $\operatorname{Gp}_\bullet(Y \times_x Z') \simeq (\operatorname{Gp}_\bullet Y) \times_x (\operatorname{Gp}_\bullet Z')$, with a similar equivalence for $Z \times_x Z'$. On the other hand, since $\mathbf{\Delta}^{\mathrm{op}}$ is sifted by [26, Lemma 5.5.8.4], geometric realisation also preserves fibre products over $\mathcal{X}$ by Lemma 3.3.5.1.

Therefore, $Y \times_x Z' \to Z \times_x Z'$ satisfies (E2) if and only if the corresponding functor $|\operatorname{Gp}_\bullet Y| \times_x |\operatorname{Gp}_\bullet Z'| \to |\operatorname{Gp}_\bullet Z| \times_x |\operatorname{Gp}_\bullet Z'|$ is an equivalence, which follows from the fact that the is the base change of the map $|\operatorname{Gp}_\bullet Y| \to |\operatorname{Gp}_\bullet Z|$ along $|\operatorname{Gp}_\bullet Z'| \to x$, which is an equivalence since $Y \to Z$ is a Segal equivalence. □

Proposition 3.3.5.3. *For an $\mathcal{X}$-distributor $\mathcal{Y}$, the left adjoint $\operatorname{Seg}_{\mathcal{X}} : \mathbf{SS}_{\mathcal{X}}(\mathcal{Y}) \to \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y})$ to inclusion preserves fibre products over $\mathcal{X}$.*

Proof. Suppose $Y \to x \leftarrow Y'$ is a cospan in $\mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$, where $x \in \mathcal{X}$. Since complete Segal spaces are closed under limits in $\mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$, $(\operatorname{Seg}_{\mathcal{X}} Y) \times_x (\operatorname{Seg}_{\mathcal{X}} Y')$ is a complete Segal space.

Therefore, it suffices to show that the map $Y \times_x Y' \to (\mathrm{Seg}_{\mathcal{X}} Y) \times_x (\mathrm{Seg}_{\mathcal{X}} Y')$ is a Segal equivalence, as it would then induce an equivalence $\mathrm{Seg}_{\mathcal{X}}(Y \times_x Y') \to (\mathrm{Seg}_{\mathcal{X}} Y) \times_x (\mathrm{Seg}_{\mathcal{X}} Y')$.

Since $Y \times_x Y' \to (\mathrm{Seg}_{\mathcal{X}} Y) \times_x (\mathrm{Seg}_{\mathcal{X}} Y')$ factors as

$$Y \times_x Y' \to (\mathrm{Seg}_{\mathcal{X}} Y) \times_x Y' \to (\mathrm{Seg}_{\mathcal{X}} Y) \times_x (\mathrm{Seg}_{\mathcal{X}} Y')$$

the fact that it is a Segal equivalence follows from Lemma 3.3.5.2, since $Y \to \mathrm{Seg}_{\mathcal{X}} Y$ and $Y' \to \mathrm{Seg}_{\mathcal{X}} Y'$ are Segal equivalences. $\square$

Lemma 3.3.5.4. *In the situation of Lemma 3.3.3.2, suppose $\phi : \mathcal{Y} \to \mathcal{Y}'$ admits a left adjoint ψ. Then, the induced functor $\phi_* : \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}) \to \mathbf{CSS}_{\mathcal{X}'}(\mathcal{Y}')$ admits a left adjoint $\psi_!$. If moreover $\psi : \mathcal{Y}' \to \mathcal{Y}$ preserves fibre products over $\mathcal{X}'$, then the same is true for $\psi_!$.*

Proof. Choose $\lambda \gg 0$ so that $\phi : \mathcal{Y} \to \mathcal{Z}$ is a λ-accessible functor between locally λ-presentable categories. Then, Lemma 3.3.3.4 implies that λ-filtered colimits in $\mathbf{CSS}_{\mathcal{X}}(\mathcal{Y})$ and $\mathbf{CSS}_{\mathcal{X}}(\mathcal{Z})$ are computed pointwise, and are thus preserved by ϕ_*. In particular, [26, Corollary 5.5.2.9] implies that ϕ_* admits a left adjoint $\psi_!$.

If ψ preserves fibre products over $\mathcal{X}'$, then the induced functor $\psi_* : \mathbf{SS}_{\mathcal{X}'}(\mathcal{Y}') \to \mathbf{SS}_{\mathcal{X}}(\mathcal{Y})$ is left adjoint to $\phi_* : \mathbf{SS}_{\mathcal{X}}(\mathcal{Y}) \to \mathbf{SS}_{\mathcal{X}'}(\mathcal{Y}')$. Taking left adjoints of the functors in the commutative square

$$\begin{tikzcd}
\mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}) \arrow[r, "\phi_*"] \arrow[d, hook] & \mathbf{CSS}_{\mathcal{X}'}(\mathcal{Y}') \arrow[d, hook] \\
\mathbf{SS}_{\mathcal{X}}(\mathcal{Y}) \arrow[r, "\phi_*"'] & \mathbf{SS}_{\mathcal{X}'}(\mathcal{Y}')
\end{tikzcd}$$

yields the commutative square in the diagram

$$\begin{tikzcd}
\mathbf{CSS}_{\mathcal{X}'}(\mathcal{Y}) \arrow[r, hook] \arrow[dr, "\mathrm{id}"'] & \mathbf{SS}_{\mathcal{X}'}(\mathcal{Y}') \arrow[r, "\psi_*"] \arrow[d, "\mathrm{Seg}_{\mathcal{X}'}"'] & \mathbf{SS}_{\mathcal{X}}(\mathcal{Y}) \arrow[d, "\mathrm{Seg}_{\mathcal{X}}"] \\
& \mathbf{CSS}_{\mathcal{X}'}(\mathcal{Y}') \arrow[r, "\psi_!"'] & \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y})
\end{tikzcd}$$

where the vertical functors preserve fibre products over $\mathcal{X}'$ by Proposition 3.3.5.3, and the inclusion preserves all limits from being a right adjoint. Therefore, $\psi_!$ preserves fibre products over $\mathcal{X}'$ as well. $\square$

Corollary 3.3.5.5. *Given a geometric morphism $g_* : \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}')$ of ∞-topoi of sheaves, we have for all $-2 \le n \le \infty$ and $0 \le r \le n$ an adjunction*

$$g^* : \mathbf{Sh}_{(n,r)}(\mathcal{C}') \rightleftarrows : \mathbf{Sh}_{(n,r)}(\mathcal{C}) : g_*$$

and similarly an adjunction

$$g^* : \mathbf{Sh}_{\omega}(\mathcal{C}') \rightleftarrows : \mathbf{Sh}_{\omega}(\mathcal{C}) : g_*$$

where each g_* acts pointwise, and the corresponding left adjoints g^* are relatively left exact.

Lemma 3.3.5.6. *Let K be a filtered simplicial set, and $(\mathcal{A}^k \to \mathcal{B}^k)_{k\in K}$ a diagram in* $\mathbf{Fun}(\Delta[1], \mathbf{Cat}_\infty)$ *consisting of cofinal functors. Then, the colimit of this diagram is again a cofinal functor.*

Proof. Let $\mathcal{A} \to \mathcal{B}$ denote the colimit of the diagram of cofinal functors. By Quillen's Theorem A [26, Theorem 4.1.3.1], we show for any $B \in \mathcal{B}$ that $\mathcal{A}_{B/} := \mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{B/}$ is weakly contractible.

Choose $k \in K$ so that $B \in \mathcal{B}^k$; that is, so that B is in the essential image of the coprojection $\mathcal{B}^k \to \mathcal{B}$. Since K is filtered, the canonical functor $K_{k/} \to K$ is cofinal. Therefore,

$$\begin{aligned}
\mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{B/} &\simeq \left(\varinjlim_{\ell} \mathcal{A}^\ell\right) \times_{\left(\varinjlim_{\ell} \mathcal{B}^\ell\right)} \left(\varinjlim_{\ell} \mathcal{B}^\ell\right)_{B/} \\
&\simeq \left(\varinjlim_{k\to\ell} \mathcal{A}^\ell\right) \times_{\left(\varinjlim_{k\to\ell} \mathcal{B}^\ell\right)} \left(\varinjlim_{k\to\ell} \mathcal{B}^\ell\right)_{B/} && \text{(cofinality of } K_{k/} \to K) \\
&\simeq \left(\varinjlim_{k\to\ell} \mathcal{A}^\ell\right) \times_{\left(\varinjlim_{k\to\ell} \mathcal{B}^\ell\right)} \left(\varinjlim_{k\to\ell} \mathcal{B}^\ell_{B/}\right) && (B \in \mathcal{B}^k) \\
&\simeq \varinjlim_{k\to\ell} \left(\mathcal{A}^\ell \times_{\mathcal{B}^\ell} \mathcal{B}^\ell_{B/}\right) && \text{(filtered colimits are left exact in } \mathbf{Cat}_\infty)
\end{aligned}$$

By assumption, every $\mathcal{A}^\ell \times_{\mathcal{B}^\ell} \mathcal{B}^\ell_{B/}$ is weakly contractible for every $k \to \ell$, implying the same for $\mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{B/}$, as desired. □

Corollary 3.3.5.7. *The full subcategory of* $\mathbf{Cat}_\infty$ *spanned by sifted categories is closed under filtered colimits.*

Proposition 3.3.5.8. *For $0 \leq r < r' \leq \infty$, the functor $\pi_{\leq r} : \mathbf{Sh}_{(\infty,r')}(\mathcal{C}) \to \mathbf{Sh}_{(\infty,r)}(\mathcal{C})$ is relatively left exact.*

Proof. We prove the proposition by induction on r, so suppose first that $r = 0$.

Let $\mathbf{\Delta}_\infty$ denote the full subcategory of $\mathbf{\Delta}^{\times\infty}$ spanned by those $[\vec{k}] = (k_0, k_1, k_2, \dots)$ such that $k_n = 0$ for all $n \gg 0$. For $0 \leq r' \leq \infty$, denote by $\mathbf{\Delta}_{r'}$ the full subcategory of $\mathbf{\Delta}_\infty$ spanned by those $[\vec{k}]$ such that $k_n = 0$ for all $n \geq r'$. Then, we have a fully faithful inclusion $\mathbf{\Sigma}_{r'} \subseteq \mathbf{\Delta}_{r'}$ for every $0 \leq r' \leq \infty$.

The functor $\pi_{\leq 0} : \mathbf{Sh}_{(\infty,r')}(\mathcal{C}) \to \mathbf{Sh}_{(\infty,0)}(\mathcal{C}) \simeq \mathcal{X}$ is left adjoint to the diagonal inclusion $\mathcal{X} \subseteq \mathbf{Sh}_{(\infty,r')}(\mathcal{C})$. The composite of right adjoints

$$\mathcal{X} \subseteq \mathbf{Sh}_{(\infty,r')}(\mathcal{C}) \subseteq \mathbf{Fun}(\mathbf{\Sigma}_{r'}^{\mathrm{op}}, \mathcal{X}) \subseteq \mathbf{Fun}(\mathbf{\Delta}_{r'}^{\mathrm{op}}, \mathcal{X})$$

where the rightmost inclusion is right Kan extension along $\mathbf{\Sigma}_{r'} \subseteq \mathbf{\Delta}_{r'}$ is precisely the diagonal inclusion, which implies that $\pi_{\leq 0}$ is the restriction of the left adjoint $\varinjlim : \mathbf{Fun}(\mathbf{\Delta}_{r'}^{\mathrm{op}}, \mathcal{X}) \to \mathcal{X}$ to the diagonal. By Lemma 3.3.5.1, it suffices to prove that $\mathbf{\Delta}_{r'}^{\mathrm{op}}$ is sifted for every $0 \leq r' \leq \infty$.

For $r' < \infty$, note that $\mathbf{\Delta}_{r'} \simeq \mathbf{\Delta}^{\times r'}$. Since $\mathbf{\Delta}^{\mathrm{op}}$ is sifted by [26, Lemma 5.5.8.4], and [26, Corollary 4.1.1.13 and Proposition 4.1.1.3(2)] imply that sifted categories are closed under finite products, it follows that $\mathbf{\Delta}_{r'}^{\mathrm{op}}$ is sifted for r' finite. Since

$$\mathbf{\Delta}_\infty \simeq \varinjlim(\mathbf{\Delta}_0 \subseteq \mathbf{\Delta}_1 \subseteq \mathbf{\Delta}_2 \subseteq \dots)$$

it follows from Corollary 3.3.5.7 that $\mathbf{\Delta}_\infty^{\mathrm{op}}$ is sifted as well.

Now, let $r > 0$, and suppose all $\pi_{\leq(r-1)}$ preserve fibre products over $\mathcal{X}$. Then,

$$\pi_{\leq(r-1),*} : \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Sh}_{(\infty,r'-1)}(\mathcal{C})) \to \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Sh}_{(\infty,r-1)}(\mathcal{C}))$$

restricts to a functor $\mathbf{SS}_{\mathcal{X}}(\mathbf{Sh}_{(\infty,r'-1)}(\mathcal{C})) \to \mathbf{SS}_{\mathcal{X}}(\mathbf{Sh}_{(\infty,r-1)}(\mathcal{C}))$ that also preserves fibre products over $\mathcal{X}$.

As $\pi_{\leq r} \simeq \pi_{\leq(r-1),!}$ is the image of $\pi_{\leq(r-1)}$ under $\mathbf{CSS}_{\mathcal{X}} : \mathbf{Dist}_{\mathcal{X}}^L \to \mathbf{Dist}_{\mathcal{X}}^L$, and $\pi_{\leq(r-1)}$ preserves fibre products over $\mathcal{X}$ by assumption, it follows by Lemma 3.3.5.4 that $\pi_{\leq r}$ preserves fibre products over $\mathcal{X}$. □

Corollary 3.3.5.9. *The localisation functor $\pi_{\leq\omega} : \mathbf{Sh}_{(\infty,\infty)}(\mathcal{C}) \to \mathbf{Sh}_\omega(\mathcal{C})$ is relatively left exact.*

The remainder of this section is dedicated to proving the following result:

Theorem 3.3.5.10 (Sheafification)**.** *For $-2 \leq n \leq \infty$ and $0 \leq r \leq n+2$, the category $\mathbf{Sh}_{(n,r)}(\mathcal{C})$ includes fully faithfully into $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_{(n,r)})$, and this inclusion admits a relatively left exact left adjoint.*

Likewise, $\mathbf{Sh}_\omega(\mathcal{C})$ includes fully faithfully into $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_\omega)$, and the inclusion admits a relatively left exact left adjoint.

Proof. Follows from combining Proposition 3.3.5.12 below with Lemma 3.3.5.4. □

Lemma 3.3.5.11. *Let $\mathcal{Y}$ be an $\mathcal{X}$-distributor. For every small category $\mathcal{K}$, the category $\mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{Y})$ is a distributor over $\mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{X})$. Moreover, we have an equivalence*

$$\mathbf{CSS}_{\mathbf{Fun}(\mathcal{K}^{\mathrm{op}},\mathcal{X})}(\mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{Y})) \simeq \mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}))$$

Proof. $\mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{X})$ and $\mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{Y})$ are certainly locally presentable, verifying (D1). The inclusion $\mathcal{X} \subseteq \mathcal{Y}$ preserves small limits and colimits, and limits and colimits in functor categories are computed pointwise, which proves (D2). By [27, Corollary 1.2.5], $\mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{Y})$ is then a distributor over $\mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{X})$ if and only if the following holds:

(∗) For any simplicial set J and natural transformation $\overline{\alpha} : \overline{p} \Rightarrow \overline{q} : J^{\triangleright} \to \mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{Y})$ such that $\overline{q}$ is a colimit diagram in $\mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{X})$ and the naturality squares of $\alpha = \overline{\alpha}|_K$ are pullback squares, then $\overline{p}$ is a colimit diagram in $\mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{Y})$ if and only if the naturality squares of $\overline{\alpha}$ are pullback squares.

This condition can be checked pointwise, and [27, Corollary 1.2.5] implies that the analogous result holds for $\mathcal{X} \subseteq \mathcal{Y}$. Therefore, $\mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{Y})$ is then a distributor over $\mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{X})$, as desired.

Since $\mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{Y})) \simeq \mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{Y}))$ and fibre products are computed pointwise in functor categories, we have $\mathbf{SS}_{\mathbf{Fun}(\mathcal{K}^{\mathrm{op}},\mathcal{X})}(\mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{Y})) \simeq \mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathbf{SS}_{\mathcal{X}}(\mathcal{Y}))$. Finally, since the conditions for a transformation to be a Segal equivalence can be checked pointwise, this equivalence descends to an equivalence

$$\mathbf{CSS}_{\mathbf{Fun}(\mathcal{K}^{\mathrm{op}},\mathcal{X})}(\mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{Y})) \simeq \mathbf{Fun}(\mathcal{K}^{\mathrm{op}}, \mathbf{CSS}_{\mathcal{X}}(\mathcal{Y}))$$

as desired. □

Proposition 3.3.5.12. *Let $-2 \leq n \leq \infty$ and $0 \leq r \leq n+2$. If $\mathcal{X} \simeq \mathcal{P}(\mathcal{C})$, then we have an equivalence $\mathbf{Sh}_{(n,r)}(\mathcal{C}) \simeq \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_{(n,r)})$. Similarly, $\mathbf{Sh}_{\omega}(\mathcal{C}) \simeq \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{SS}_{\omega})$.*

Proof. Suppose first that $n = \infty$. If $r = 0$, then $\mathbf{Sh}_{(\infty,0)}(\mathcal{C}) \simeq \mathcal{P}(\mathcal{C})$ and $\mathbf{Cat}_{(\infty,0)} \simeq \mathcal{S}$. Iteratively applying Lemma 3.3.5.11 then proves the proposition if r is finite. For $r = \infty$, we have

$$\begin{aligned}
\mathbf{Sh}_{(\infty,\infty)}(\mathcal{C}) &\simeq \varprojlim \left(\dots \xrightarrow{\kappa_{\leq 1}} \mathbf{Sh}_{(\infty,1)}(\mathcal{C}) \xrightarrow{\kappa_{\leq 0}} \mathbf{Sh}_{(\infty,0)}(\mathcal{C}) \right) \\
&\simeq \varprojlim \left(\dots \xrightarrow{\kappa_{\leq 1,*}} \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_{(\infty,1)}) \xrightarrow{\kappa_{\leq 0,*}} \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_{(\infty,0)}) \right) \\
&\simeq \mathbf{Fun}\left(\mathcal{C}^{\mathrm{op}}, \varprojlim \left(\dots \xrightarrow{\kappa_{\leq 2}} \mathbf{Cat}_{(\infty,1)} \xrightarrow{\kappa_{\leq 1}} \mathbf{Cat}_{(\infty,0)} \right) \right) \\
&\simeq \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_{(\infty,\infty)})
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{Sh}_{\omega}(\mathcal{C}) &\simeq \varprojlim \left(\dots \xrightarrow{\pi_{\leq 1}} \mathbf{Sh}_{(\infty,1)}(\mathcal{C}) \xrightarrow{\pi_{\leq 0}} \mathbf{Sh}_{(\infty,0)}(\mathcal{C}) \right) \\
&\simeq \varprojlim \left(\dots \xrightarrow{\pi_{\leq 1,*}} \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_{(\infty,1)}) \xrightarrow{\pi_{\leq 0,*}} \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_{(\infty,0)}) \right) \\
&\simeq \mathbf{Fun}\left(\mathcal{C}^{\mathrm{op}}, \varprojlim \left(\dots \xrightarrow{\pi_{\leq 2}} \mathbf{Cat}_{(\infty,1)} \xrightarrow{\pi_{\leq 1}} \mathbf{Cat}_{(\infty,0)} \right) \right) \\
&\simeq \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{SS}_{\omega})
\end{aligned}$$

Now suppose $n < \infty$. Then, $\mathbf{Sh}_{(n,r)}(\mathcal{C})$ is the localisation of $\mathbf{Sh}_{(\infty,r)}(\mathcal{C}) \simeq \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_{(\infty,r)})$ at the maps $\Sigma^m(\mathbb{S}^{k-m} \times \Delta_U[0]) \to \Sigma^m \Delta_U[0]$ for $m \geq r$, $k > n$, and $U \in \mathcal{C}$. Now, the result follows from the observation that a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}_{(\infty,r)}$ is local with respect to the maps $\Sigma^m(\mathbb{S}^{k-m} \times \Delta_U[0]) \to \Sigma^m \Delta_U[0]$ if and only if $F(U) \in \mathbf{Cat}_{(\infty,r)}$ is local with respect to $\Sigma^m \mathbb{S}^{k-m} \to \Sigma^m *$. □

Chapter 4

Endofunctor Fixed Points and Algebras

In Theorem 3.3.4.5, we demonstrated that $\mathbf{Cat}_{(\infty,\infty)}$ is closed under the formation of complete Segal space objects; that is, $\mathbf{Cat}_{(\infty,\infty)}$ is a fixed point (up to equivalence) of a suitable functor $\mathbf{CSS} : \mathbf{Dist}^R \to \mathbf{Dist}^R$. The crux of the proof is the observation that the category arises as the result of an indefinite iterative application of the functor $\mathbf{CSS}$ to an object of $\mathbf{Dist}^R$.

The construction of fixed points through iterative function application is certainly not novel. The theory of fixed points in ordinary category theory was already developed by the 1970s; see [1, 25]. We review this theory in Section 4.1.

The goal of this chapter is to provide an ∞-categorical generalisation of this theory. We lay the necessary technical groundwork in Section 4.2, introducing the notion of (lax) algebras of an endofunctor. In Section 4.3, we employ the language of lax endofunctor algebras to establish an ∞-categorical analogue of Adámek's construction, and then apply this theory to study fixed points of a general class of endofunctors.

4.1 Classical theory

Suppose that we are interested in studying the fixed points of the functor $S : \mathbf{Set} \to \mathbf{Set}$ that maps $X \mapsto X \sqcup \{\perp\}$. Starting with the empty set $\varnothing$, we can construct a fixed point through infinite application of S as a colimit

$$\varinjlim \left(\varnothing \xrightarrow{!} S(\varnothing) \xrightarrow{S(!)} S^2(\varnothing) \to \cdots \right)$$

By identifying $S^n(\varnothing)$ with the set $\{0, 1, \ldots, n-1\}$, the canonical map $S^n(\varnothing) \to S^{n+1}(\varnothing)$ is given by inclusion, and the colimit is isomorphic to the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers.

The canonical map $\varinjlim_n S^{n+1}(\varnothing) \to S(\varinjlim_n S^n(\varnothing))$ is inverse to the map

$$\mathrm{succ} : \mathbb{N} \sqcup \{\perp\} \to \mathbb{N}$$
$$\mathrm{succ}(n) = \begin{cases} n+1, & n \in \mathbb{N} \\ 0, & n = \perp \end{cases}$$

which is bijective. In particular, $\mathbb{N}$ defines a fixed point of S.

This bijection is, in some sense, reflective of the inductive structure of $\mathbb{N}$: the formally-adjoined point maps to the base 0, and the rest of the function encodes the successor operation on $\mathbb{N}$. This structure allows us to exhibit the induction axiom on $\mathbb{N}$ as a universal property: given a set X with a chosen point $x_\perp \in X$ and a function $f : X \to X$, we have by induction a unique sequence $x_\bullet : \mathbb{N} \to X$ such that $x_0 = x_\perp$, and $x_{n+1} = f(x_n)$. Note that a choice of point $x_\perp$ and a function $f : X \to X$ is equivalent to defining a function $f : X \sqcup \{\perp\} \to X$. Therefore, we have

Proposition 4.1.0.1. *For any function $f : S(X) \to X$, there exists a unique function $x_\bullet : \mathbb{N} \to X$ such that*

$$\begin{tikzcd} S(\mathbb{N}) \arrow[r, "S(x_\bullet)"] \arrow[d, "\mathrm{succ}"', "\wr"] & S(X) \arrow[d, "f"] \\ \mathbb{N} \arrow[r, "x_\bullet"'] & X \end{tikzcd}$$

commutes.

Alternatively, if we start with the singleton set $*$, we can construct a fixed point through infinite application of S as a limit

$$\varprojlim \left(\cdots \to S^2(*) \xrightarrow{S(!)} S(*) \xrightarrow{!} * \right)$$

By identifying $S^n(*)$ with the set $\{0, 1, \ldots, n-1, \infty\}$, the map $S^{n+1}(*) \to S^n(*)$ is the projection that sends $(n-1) \mapsto \infty$. The limit is then isomorphic to the set $\overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}$ of extended natural numbers, where the projection $\overline{\mathbb{N}} \to \{0, 1, \ldots, n-1, \infty\}$ sends all $k \geq n$ to ∞.

The canonical map $S(\varprojlim_n S^n(*)) \to \varprojlim_n S^{n+1}(*)$ is then inverse to the map

$$\mathrm{pred} : \overline{\mathbb{N}} \to \overline{\mathbb{N}} \sqcup \{\perp\}$$
$$\mathrm{pred}(m) = \begin{cases} m-1, & 0 < m < \infty \\ \perp, & m = 0 \\ \infty, & m = \infty \end{cases}$$

In general, a function $p : X \to X \sqcup \{\perp\}$ can be thought of as a partial function $X \to X$ that sends a point to its "predecessor", where root elements in X with no predecessor are mapped to $\perp$. We can then define a function $d : X \to \overline{\mathbb{N}}$ that associates to each point $x \in X$ its "depth": the number of predecessors of x. Indeed, such a function is defined by *coinduction*: if x has no predecessor, then $d(x) = 0$; otherwise, $d(x) = d(p(x)) + 1$.

Proposition 4.1.0.2. *For any function $p : X \to S(X)$, there exists a unique function $d : X \to \overline{\mathbb{N}}$ such that*

$$\begin{tikzcd} X \arrow[r, "d"] \arrow[d, "p"'] & \overline{\mathbb{N}} \arrow[d, "\mathrm{pred}", "\wr"'] \\ S(X) \arrow[r, "S(d)"'] & S(\overline{\mathbb{N}}) \end{tikzcd}$$

commutes.

In particular, Propositions 4.1.0.1 and 4.1.0.2 imply that $\mathbb{N}$ and $\overline{\mathbb{N}}$ are universal fixed points of S: specifically, $\mathbb{N}$ is the initial fixed point, and $\overline{\mathbb{N}}$ is the terminal fixed point.

4.1.1 Adámek's construction

This subsection reviews the theory studied in [1]; in particular, none of the content here is original. Throughout this subsection, fix a 1-category $\mathcal{K}$ and a functor $F : \mathcal{K} \to \mathcal{K}$.

Definition 4.1.1.1. An F-*algebra* is a pair (A, μ), where $A \in \mathcal{K}$, and $\mu : FA \to A$ is the *action* on A.

Given two F-algebras (A, μ), (A', μ'), a morphism $\varphi : A \to A'$ is an F-*algebra homomorphism* if

$$\begin{tikzcd} FA \arrow[r, "F\varphi"] \arrow[d, "\mu"'] & FA' \arrow[d, "\mu'"] \\ A \arrow[r, "\varphi"'] & A' \end{tikzcd}$$

commutes. Let $\mathcal{K}(F)$ denote the 1-category of F-algebras and F-algebra homomorphisms.

Moreover, let $\mathbf{Fix}(F)$ denote the full subcategory of $\mathcal{K}(F)$ spanned by those F-algebras (A, μ) where μ is an isomorphism.

Remark 4.1.1.2. We can similarly define an F-*coalgebra* as a pair (C, ν) where $C \in \mathcal{K}$ and $\nu : C \to FC$ is the *coaction*. Denote the 1-category of F-coalgebras by $\mathcal{K}_{\mathrm{co}}(F)$. The category $\mathbf{Fix}(F)$ is then equivalent to the full subcategory of $\mathcal{K}_{\mathrm{co}}(F)$ spanned by those F-coalgebras whose coactions are isomorphisms.

The theory of F-coalgebras is entirely dual to the theory of F-algebras. In particular, we have an equivalence $\mathcal{K}_{\mathrm{co}}(F) \simeq (\mathcal{K}^{\mathrm{op}}(F))^{\mathrm{op}}$. Therefore, we focus on the theory of F-algebras in this subsection.

Remark 4.1.1.3. Let $S : \mathbf{Set} \to \mathbf{Set}, X \mapsto X \sqcup \{\perp\}$ as in the beginning of Section 4.1. Proposition 4.1.0.1 shows that $(\mathbb{N}, \mathrm{succ})$ defines an initial S-algebra; that is, an initial object in $\mathbf{Set}(S)$. On the other hand, Proposition 4.1.0.2 shows that $(\overline{\mathbb{N}}, \mathrm{pred})$ defines a terminal S-coalgebra.

Lemma 4.1.1.4 (Lambek)**.** [25, Lemma 2.2] *If (I, i) is an initial object in $\mathcal{K}(F)$, then the action $i : FI \to I$ is an isomorphism. In particular, (I, i) is also an initial object in $\mathbf{Fix}(F)$.*

Remark 4.1.1.5. Conversely, we will see in Theorem 4.3.2.5 that the inclusion $\mathbf{Fix}(F) \subseteq \mathcal{K}(F)$ preserves small colimits if F is moderately well-behaved.

In [1], Adámek studies the construction of free F-algebras generated by objects of $\mathcal{K}$, and provides a general-purpose algorithm for constructing them.

Definition 4.1.1.6. Let $U : \mathcal{K}(F) \to \mathcal{K}$ denote the forgetful functor. A *free F-algebra* generated by an object $K \in \mathcal{K}$ is, if it exists, a corepresenting object for the functor $\mathrm{Hom}_{\mathcal{K}}(K, U(-)) : \mathcal{K}(F) \to \mathbf{Set}$.

If $\mathcal{K}$ has an initial object $\emptyset$, then note that the free F-algebra on $\emptyset$ is precisely the initial F-algebra. In this case, Adámek's construction reduces to the following:

Proposition 4.1.1.7. *Suppose $\mathcal{K}$ has an initial object $\emptyset$, and consider the diagram*

$$\emptyset \xrightarrow{!} F(\emptyset) \xrightarrow{F(!)} F^2(\emptyset) \xrightarrow{F^2(!)} F^3(\emptyset) \to \cdots$$

If this diagram has a colimit I, and the canonical map $j : FI \to I$ is an isomorphism, then (I, j^{-1}) is an initial F-algebra.

Remark 4.1.1.8. Adámek's construction allows for the number of applications of F on $\emptyset$ to be transfinite if necessary, assuming the relevant intermediate colimits exist. We only present the countable case above since it is the easiest to state, and is the most relevant case for our purposes (once generalised to ∞-categories).

Remark 4.1.1.9. The general construction of a free F-algebra generated by an object $K \in \mathcal{K}$ is based on a similarly iterative construction of the form

$$K \to K \sqcup FK \to K \sqcup F(K \sqcup FK) \to K \sqcup F(K \sqcup F(K \sqcup FK)) \to \cdots$$

If the above construction "stabilises" (after possibly transfinitely many steps), then it stabilises on the free F-algebra generated by K; see [1, p. 592] for the precise construction.

4.2 Algebras of $(\infty, 1)$-endofunctors

Throughout this section, fix an ∞-category $\mathcal{K}$ and an endofunctor $F : \mathcal{K} \to \mathcal{K}$. We continue to follow Convention 3.0.0.1 and omit the "$(\infty, 1)$" prefix in categorical notions.

We can readily generalise Definition 4.1.1.1 as follows.

Definition 4.2.0.1. Define the category $\mathcal{K}(F)$ of F-algebras as the pullback

$$\begin{tikzcd} \mathcal{K}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{K}^{\Delta[1]} \arrow[d, two heads] \\ \mathcal{K} \arrow[r, "{(F,\mathrm{id})}"'] & \mathcal{K} \times \mathcal{K} \end{tikzcd}$$

Note that the vertical map on the right is induced by the inclusion $\{0, 1\} \hookrightarrow \Delta[1]$, and is thus a categorical fibration. Therefore, the pullback is a homotopy pullback of ∞-categories.

Remark 4.2.0.2. We obtain the category of F-coalgebras as $\mathcal{K}_{\mathrm{co}}(F) \simeq (\mathcal{K}^{\mathrm{op}}(F))^{\mathrm{op}}$, as in Remark 4.1.1.2.

We can also prove Lambek's lemma in this more general setting:

Lemma 4.2.0.3 (Lambek)**.** *Suppose (I, i) is an initial object in $\mathcal{K}(F)$. Then, the action $i : FI \to I$ is an equivalence.*

Proof. Consider the F-algebra (FI, Fi). Since (I, i) is initial, there is an essentially unique F-algebra homomorphism $u : (I, i) \to (FI, Fi)$. The composite $i \circ u$ thus defines an F-algebra endomorphism of (I, i), and must therefore be homotopic to the identity; that is, $i \circ u \simeq \mathrm{id}_I$.

On the other hand, consider the diagram

$$\begin{tikzcd} FI \arrow[r, "Fu"] \arrow[d, "i"'] \arrow[dr, "F(i\circ u)"] & FFI \arrow[d, "Fi"] \\ I \arrow[r, "u"'] & FI \end{tikzcd}$$

The perimeter commutes up to homotopy because u is an F-algebra homomorphism, and the upper triangle commutes by the functoriality of F. Since $i \circ u \simeq \mathrm{id}_I$, it follows that also $u \circ i \simeq F(i \circ u) \simeq \mathrm{id}_{FI}$. Therefore, $u \simeq i^{-1}$, proving that i is an equivalence, as desired. $\square$

However, Adámek's construction is a bit more difficult to reproduce.

4.2.1 Lax algebras

In Proposition 4.1.1.7, the action of the initial F-algebra I is given by an inverse to the canonical map $j : I \to FI$. In an ∞-categorical setting, inverses are only unique up to higher homotopy, and the universal property of an initial F-algebra is much more involved. Therefore, we avoid choosing an inverse of the canonical map by studying lax F-algebras.

Definition 4.2.1.1. Define the category $\mathcal{K}^{\mathrm{lax}}(F)$ of *lax F-algebras* as the pullback

$$\begin{tikzcd} \mathcal{K}^{\mathrm{lax}}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{K}^{\{1\leftarrow 0\rightarrow 2\}} \arrow[d, two heads] \\ \mathcal{K} \arrow[r, "(F,\mathrm{id})"'] & \mathcal{K}^{\{1\}} \times \mathcal{K}^{\{2\}} \end{tikzcd}$$

The vertical map on the right is induced by the cofibration $\{1, 2\} \hookrightarrow \{1 \leftarrow 0 \rightarrow 2\}$, and is thus a categorical fibration, showing that the pullback square is a homotopy pullback of ∞-categories.

Concretely, a lax F-algebra is given by a span $FB \xleftarrow{r} E \xrightarrow{a} B$, where a is a *lax action*, and r is a *resolution*.

Remark 4.2.1.2. Every square in the tower below is a pullback square:

$$\begin{tikzcd} \mathcal{K}^{\mathrm{lax}}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{K}^{\{1\leftarrow 0\rightarrow 2\}} \arrow[d, two heads] \\ \mathcal{K}^{\{0\rightarrow 2\}} \arrow[r, "(F(2),\mathrm{id})"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{K}^{\{1\}} \times \mathcal{K}^{\{0\rightarrow 2\}} \arrow[d, two heads] \\ \mathcal{K}^{\{0\}} \times \mathcal{K}^{\{2\}} \arrow[r, "(F(2),\mathrm{id})"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{K}^{\{1\}} \times \mathcal{K}^{\{0\}} \times \mathcal{K}^{\{2\}} \arrow[d] \\ \mathcal{K}^{\{2\}} \arrow[r, "(F,\mathrm{id})"'] & \mathcal{K}^{\{1\}} \times \mathcal{K}^{\{2\}} \end{tikzcd}$$

As the vertical maps on the right are induced by inclusions of simplicial sets, it follows that they are categorical fibrations, showing that all of these pullback squares are also homotopy pullback squares of ∞-categories.

Proposition 4.2.1.3. *Let $\widehat{\mathcal{K}}(F)$ denote the full subcategory of $\mathcal{K}^{\mathrm{lax}}(F)$ spanned by those lax F-algebras $FB \leftarrow E \rightarrow B$ where the resolution $E \rightarrow FB$ is invertible. Then, there is a canonical equivalence $\mathcal{K}(F) \xrightarrow{\sim} \widehat{\mathcal{K}}(F)$.*

Proof. Note that $\widehat{\mathcal{K}}(F)$ can be defined equivalently as follows, using the cartesian model structure on marked simplicial sets of [26, §3.1]. Let $\Lambda^0_+[2] := \{1 \xleftarrow{+} 0 \rightarrow 2\}$ denote the marked simplicial set obtained by taking the walking span $\Lambda^0[2]$ and marking the left-pointing edge. Then, $\widehat{\mathcal{K}}(F)$ is the pullback

$$\begin{tikzcd}
\widehat{\mathcal{K}}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}^\flat(\Lambda^0_+[2], \mathcal{K}^\natural) \arrow[d, two heads] \\
\mathcal{K} \arrow[r, "{(F,\mathrm{id})}"'] & \mathcal{K} \times \mathcal{K}
\end{tikzcd}$$

where $\mathrm{Map}^\flat(X, Y)$ is the underlying simplicial set of the internal hom of marked simplicial sets, and $\mathcal{K}^\natural$ is the category $\mathcal{K}$ marked at the equivalences. By [26, Remark 3.1.4.5], $\mathrm{Map}^\flat$ provides the cartesian model structure with an enrichment in Joyal's model structure on simplicial sets. Note that $\mathcal{K}^\natural$ is fibrant in the cartesian model structure, and so the vertical map induced by an inclusion of marked simplicial sets is therefore a categorical fibration.

Let $\Delta[1]^\flat$ denote the simplicial set $\Delta[1]$ marked only at the degenerate edges. Then, the inclusion $\Delta[1]^\flat \to \Lambda^0_+[2]$ that picks out the unmarked edge of $\Lambda^0_+[2]$ is marked anodyne by [26, Proposition 3.1.1.5], and admits a retraction $\Lambda^0_+[2] \to \Delta[1]$. By 2-out-of-3, it follows that this retraction is a cartesian equivalence. Therefore, since $\mathcal{K}^\natural$ is fibrant, the retraction induces a categorical equivalence $\mathcal{K}^{\Delta[1]} = \mathrm{Map}^\flat(\Delta[1]^\flat, \mathcal{K}^\natural) \to \mathrm{Map}^\flat(\Lambda^0_+[2], \mathcal{K}^\natural)$. In particular, we have the following diagram:

$$\begin{tikzcd}
\mathcal{K}(F) \arrow[rr] \arrow[dr, dashed] \arrow[ddr, bend right] & & \mathcal{K}^{\Delta[1]} \arrow[d, "\wr"] \\
& \widehat{\mathcal{K}}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}^\flat(\Lambda^0_+[2], \mathcal{K}^\natural) \arrow[d, two heads] \\
& \mathcal{K} \arrow[r, "{(F,\mathrm{id})}"'] & \mathcal{K} \times \mathcal{K}
\end{tikzcd}$$

The perimeter is the definitional pullback square for $\mathcal{K}(F)$, so both the perimeter and the inner pullback square are homotopy pullbacks. Since the objects of the two pullback diagrams are connected by categorical equivalences, the induced map $\mathcal{K}(F) \to \widehat{\mathcal{K}}(F)$ is a categorical equivalence as well. $\square$

The embedding identifies an F-algebra (A, α) with the lax F-algebra $FA = FA \xrightarrow{\alpha} A$, and conversely any lax F-algebra $FB \xleftarrow{r} E \xrightarrow{a} B$ with an invertible resolution induces an F-algerba (B, ar^{-1}). We therefore tacitly identify $\mathcal{K}(F)$ with its essential image in $\mathcal{K}^{\mathrm{lax}}(F)$.

In thet context of Adámek's construction, we can now avoid explicitly inverting the canonical map $I \to FI$ by instead proving that the lax F-algebra $FI \xleftarrow{\sim} I \xrightarrow{\mathrm{id}} I$ is initial in $\widehat{\mathcal{K}}(F)$.

4.2.2 Propagation pushout

As mentioned in Remark 4.1.1.9, Adámek's general free F-algebra construction on an object K is given as the colimit of a possibly transfinite sequence of the form

$$K \to K \sqcup FK \to K \sqcup F(K \sqcup FK) \to K \sqcup F(K \sqcup F(K \sqcup FK)) \to \cdots$$

The morphisms in this sequence become more evident when presented as a transfinite sequence of pushout squares:

$$\begin{tikzcd}
\emptyset \arrow[r, "!"] \arrow[d, "!"'] & FK \arrow[r, "Fi_1"] \arrow[d] & F(K \sqcup FK) \arrow[r, "Fi_2"] \arrow[d] & F(K \sqcup F(K \sqcup FK)) \arrow[r, "Fi_3"] \arrow[d] & \dots \\
K \arrow[r, "i_1"'] \arrow[ur, phantom, "\ulcorner" very near end] & K \sqcup FK \arrow[r, "i_2"'] \arrow[ur, phantom, "\ulcorner" very near end] & K \sqcup F(K \sqcup FK) \arrow[r, "i_3"'] \arrow[ur, phantom, "\ulcorner" very near end] & K \sqcup F(K \sqcup F(K \sqcup FK)) \arrow[r, "i_4"'] & \dots
\end{tikzcd}$$

Note that the first stage is precisely the pushout of the free lax F-algebra generated by K. This suggests a natural generalisation of this "propagation" construction for lax F-algebras.

Definition 4.2.2.1. Suppose $\mathcal{K}$ has finite colimits. For any lax F-algebra $FB \xleftarrow{r} E \xrightarrow{a} B$, consider the following diagram:

$$\begin{tikzcd}
FB \arrow[r, "Fi"] & F(B \sqcup_E FB) \\
E \arrow[u, "r"] \arrow[r, "r"] \arrow[d, "a"'] & FB \arrow[u, "Fi"'] \arrow[d] \\
B \arrow[r, "i"'] \arrow[ur, phantom, "\ulcorner" very near end] & B \sqcup_E FB
\end{tikzcd}$$

The vertical arrows on the right define a lax F-algebra $\Pi(FB \leftarrow E \to B)$. This construction extends to an endofunctor $\Pi : \mathcal{K}^{\mathrm{lax}}(F) \to \mathcal{K}^{\mathrm{lax}}(F)$, and the horizontal arrows define a canonical natural transformation $\eta : \mathrm{Id} \Rightarrow \Pi$.

We refer to Π as the *propagation* of lax F-algebras, and η as the *unit* of the endofunctor.

The remainder of this subsection makes the above definition more precise; the reader may safely skip to the next subsection.

Definition 4.2.2.2. Define the category $\mathcal{K}^{\square}(F)$ to be the (homotopy) pullback

$$\begin{tikzcd}
\mathcal{K}^{\square}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathcal{K}^{\Lambda^0[2]^{\triangleright}} \arrow[d, two heads] \\
\mathcal{K} \arrow[r, "(F, \mathrm{id})"'] & \mathcal{K} \times \mathcal{K}
\end{tikzcd}$$

Note that $\Lambda^0[2]^\triangleright \cong \Delta[1] \times \Delta[1]$ is the walking commutative square, so $\mathcal{K}^\square(F)$ consists of commutative squares of the form

$$\begin{tikzcd} E \arrow[r] \arrow[d] & FB \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

There is an evident forgetful functor $U : \mathcal{K}^\square(F) \to \mathcal{K}^{\mathrm{lax}}(F)$ given on objects by the mapping

$$\left\{ \begin{tikzcd} E \arrow[r] \arrow[d] & FB \arrow[d] \\ B \arrow[r] & C \end{tikzcd} \right\} \quad \mapsto \quad \left\{ \begin{tikzcd} FB \\ E \arrow[u] \arrow[d] \\ B \end{tikzcd} \right\}$$

We can construct a precursor to the propagation endofunctor on $\mathcal{K}^{\mathrm{lax}}(F)$ through $\mathcal{K}^\square(F)$, giving a functor $\Pi^\square : \mathcal{K}^\square(F) \to \mathcal{K}^{\mathrm{lax}}(F)$ and a natural transformation $\eta^\square : U \Rightarrow \Pi^\square$. Intuitively, this functor and transformation come from the mapping

$$\left\{ \begin{tikzcd} E \arrow[r, "r"] \arrow[d] \arrow[dr, gray] & FB \arrow[d] \\ B \arrow[r, "c"'] & C \end{tikzcd} \right\} \quad \mapsto \quad \left\{ \begin{tikzcd} FB \arrow[r, "Fc"] & FC \\ E \arrow[u, "r"] \arrow[ur, gray] \arrow[r, "r"] \arrow[d] \arrow[dr, gray] & FB \arrow[u, "Fc"'] \arrow[d] \\ B \arrow[r, "c"'] & C \end{tikzcd} \right\} \tag{4.1}$$

To formalise this construction, consider the maps $\overline{r}, \overline{c} : \Delta[1] \to \Lambda^0[2]^\triangleright$ which classify the upper edge (corresponding to $E \to FB$ in the diagram) and lower edge (corresponding to $c : B \to C$ in the diagram), respectively, then we can define a functor $\mathcal{K}^\square(F) \to \mathcal{K}^{\Lambda^1[2]}$ via

$$\begin{tikzcd} \mathcal{K}^\square(F) \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "F\overline{c}^*"] \arrow[ddr, bend right, "\overline{r}^*"'] & & \\ & \mathcal{K}^{\Lambda^1[2]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{K}^{\Delta[1]} \arrow[d] \\ & \mathcal{K}^{\Delta[1]} \arrow[r] & \mathcal{K} \end{tikzcd}$$

Explicitly, the dashed arrow describes the mapping

$$\left\{ \begin{tikzcd} E \arrow[r, "r"] \arrow[d] \arrow[dr, gray] & FB \arrow[d] \\ B \arrow[r, "c"'] & C \end{tikzcd} \right\} \quad \mapsto \quad \left\{ \begin{tikzcd} & FC \\ E \arrow[r, "r"] & FB \arrow[u, "Fc"'] \end{tikzcd} \right\}$$

However, this mapping does *not* define a composite for this sequence of arrows. Since $\Lambda^1[2] \hookrightarrow \Delta[2]$ is inner anodyne, and $\mathcal{K}$ is a quasicategory, the map $\mathcal{K}^{\Delta[2]} \to \mathcal{K}^{\Lambda^1[2]}$ is a trivial inner fibration. Therefore, we can find a section $\mathcal{K}^{\Lambda^1[2]} \to \mathcal{K}^{\Delta[2]}$ by solving the lifting problem

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, tail] & \mathcal{K}^{\Delta[2]} \arrow[d, two heads, "\wr"] \\
\mathcal{K}^{\Lambda^1[2]} \arrow[r, equal] \arrow[ur, dashed, "\exists"] & \mathcal{K}^{\Lambda^1[2]}
\end{tikzcd}$$

This provides a functorial choice of composites to the diagram above, and particular provides a functor $\mathcal{K}^{\square}(F) \to \mathcal{K}^{\Delta[2]}$.

In particular, since $\Delta[1] \times \Delta[1] \cong \Lambda^0[2]^{\triangleright} = (\Delta[1]^{\triangleright}) \sqcup_{\Delta[0]^{\triangleright}} (\Delta[1]^{\triangleright}) = \Delta[2] \sqcup_{\Delta[1]} \Delta[2]$ is obtained by gluing two triangles along their hypotenuse, we can extend the above section to define a map $\mathcal{K}^{\square}(F) \to \mathcal{K}^{\Delta[1]\times\Delta[1]}$ corresponding to the mapping

$$\left\{\begin{tikzcd}
E \arrow[r, "r"] \arrow[d] \arrow[dr] & FB \arrow[d] \\
B \arrow[r, "c"'] & C
\end{tikzcd}\right\} \mapsto \left\{\begin{tikzcd}
FB \arrow[r, "Fc"] & FC \\
E \arrow[u, "r"] \arrow[r, "r"] \arrow[ur] & FB \arrow[u, "Fc"']
\end{tikzcd}\right\}$$

which is precisely the upper square in the mapping sketched in (4.1). By gluing this square with the forgetful functor $\mathcal{K}^{\square}(F) \to \mathcal{K}^{\Lambda^0[2]^{\triangleright}} = \mathcal{K}^{\Delta[1]\times\Delta[1]}$, which describes the lower square in (4.1), we obtain a functor

$$\mathcal{K}^{\square}(F) \to \mathcal{K}^{\Delta[1]\times\Delta[1]} \times_{\mathcal{K}^{\Delta[1]}} \mathcal{K}^{\Delta[1]\times\Delta[1]} \cong \mathcal{K}^{\Lambda^0[2]\times\Delta[1]} = \mathbf{Fun}(\Delta[1], \mathcal{K}^{\Lambda^0[2]})$$

describing precisely the mapping sketched in (4.1). Moreover, it follows that the adjunct $\mathcal{K}^{\square}(F) \times \Delta[1] \to \mathcal{K}^{\Lambda^0[2]}$ factors through the forgetful functor $\mathcal{K}^{\mathrm{lax}}(F) \to \mathcal{K}^{\Lambda^0[2]}$.

The resulting functor $\mathcal{K}^{\square}(F) \times \Delta[1] \to \mathcal{K}^{\mathrm{lax}}(F)$ corresponds to a map

$$\eta^{\square} : \Delta[1] \to \mathbf{Fun}(\mathcal{K}^{\square}(F), \mathcal{K}^{\mathrm{lax}}(F))$$

In particular, $\eta^{\square}$ classifies a natural transformation between functors $\mathcal{K}^{\square}(F) \to \mathcal{K}^{\mathrm{lax}}(F)$ whose domain, by construction, is precisely the forgetful functor U.

Definition 4.2.2.3. Let $\Pi^{\square}$ be the codomain of the natural transformation constructed above, and denote the natural transformation itself by $\eta^{\square} : U \Rightarrow \Pi^{\square}$.

We can now use the above construction to create the propagation endofunctor on $\mathcal{K}^{\mathrm{lax}}(F)$, as well as its unit.

Definition 4.2.2.4. Let $\mathcal{K}$ be finitely cocomplete. By [26, Proposition 4.2.2.7], taking colimits defines a functor $\varinjlim : \mathcal{K}^{\Lambda^0[2]} \to \mathcal{K}^{\Lambda^0[2]^{\triangleright}}$, which restricts to $\varinjlim : \mathcal{K}^{\mathrm{lax}}(F) \to \mathcal{K}^{\square}(F)$. This is a section of the forgetful functor $U : \mathcal{K}^{\square}(F) \to \mathcal{K}^{\mathrm{lax}}(F)$.

Define the *propagation endofunctor* Π on $\mathcal{K}^{\mathrm{lax}}(F)$ to be the composite

$$\mathcal{K}^{\mathrm{lax}}(F) \xrightarrow{\varinjlim} \mathcal{K}^{\square}(F) \xrightarrow{\Pi^{\square}} \mathcal{K}^{\mathrm{lax}}(F)$$

This functor then admits a *unit* given by the composite

$$\eta : \Delta[1] \xrightarrow{\eta^{\square}} \mathbf{Fun}(\mathcal{K}^{\square}(F), \mathcal{K}^{\mathrm{lax}}(F)) \xrightarrow{\varinjlim^{*}} \mathbf{Fun}(\mathcal{K}^{\mathrm{lax}}(F), \mathcal{K}^{\mathrm{lax}}(F))$$

Indeed, this classifies a natural transformation $\mathrm{Id} \Rightarrow \Pi$ because $\varinjlim$ is a section of U.

4.2.3 Colimits of lax algebras

Recall from Remark 4.2.1.2 that $\mathcal{K}^{\mathrm{lax}}(F)$ fits into the pullback square

$$\begin{tikzcd}
\mathcal{K}^{\mathrm{lax}}(F) \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{K}^{\Lambda^0[2]} \arrow[d] \\
\mathcal{K}^{\Delta[1]} \arrow[r, "{(F(2),\mathrm{id})}"'] & \mathcal{K} \times \mathcal{K}^{\Delta[1]}
\end{tikzcd}$$

The goal of this subsection is to prove that $\mathcal{K}^{\mathrm{lax}}(F)$ has all colimits that $\mathcal{K}$ does.

Proposition 4.2.3.1. *The forgetful functor $\mathcal{K}^{\mathrm{lax}}(F) \xrightarrow{u} \mathcal{K}^{\Delta[1]}$ reflects colimits.*

Specifically, let $p : J \to \mathcal{K}^{\mathrm{lax}}(F)$ be a map of simplicial sets, and say that the lax F-algebra at p_j is given by $FB_j \leftarrow E_j \to B_j$. Suppose $up : J \to \mathcal{K}^{\Delta[1]}$ admits a colimit $E_\infty \to B_\infty$. Then, we have a cocone of maps $E_j \to FB_j \to FB_\infty$, and so by the universal property of $E_\infty = \varinjlim_j E_j$, there is an essentially unique map $E_\infty \to FB_\infty$. The resulting lax F-algebra $FB_\infty \leftarrow E_\infty \to B_\infty$ is then a colimit of $p : J \to \mathcal{K}^{\mathrm{lax}}(F)$.

In order to prove this, we will rely on the following technical results:

Lemma 4.2.3.2. *Let $p : J \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ be a map of simplicial sets into a strict fibre product of quasicategories, and suppose that the composite $\pi_{\mathcal{A}} p : J \to \mathcal{A}$ admits a colimit $\overline{\pi} : J^{\triangleright} \to \mathcal{A}$. Then, p admits a colimit in $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ if and only if we can always solve the lifting problem*

$$\begin{tikzcd}
J \star T \arrow[rr] \arrow[d] & & \mathcal{B} \arrow[d] \\
J^{\triangleright} \star T \arrow[r] \arrow[urr, dashed] & \mathcal{A} \arrow[r] & \mathcal{C}
\end{tikzcd} \tag{4.2}$$

where T is any simplicial set, $J \star T \to \mathcal{B}$ extends $\pi_{\mathcal{B}} p : J \to \mathcal{B}$, and $J^{\triangleright} \star T \to \mathcal{A}$ extends $\overline{\pi}$.

Proof. If p admits a colimit, then certainly every such lifting problem (4.2) can be solved. Conversely, suppose every lifting problem (4.2) can be solved. By taking $T = \emptyset$, we obtain a cocone $\overline{p} : J^{\triangleright} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ extending p. The goal is to show that $\overline{p}$ is indeed a colimit cocone for p. Therefore, we need to find a dotted arrow fitting in any diagram

$$\begin{tikzcd}
J \star S \arrow[r] \arrow[d, hook] & J \star T \arrow[d] \arrow[dr] & & \\
J^{\triangleright} \star S \arrow[r] \arrow[rr, bend right] & J^{\triangleright} \star T \arrow[r, dotted] \arrow[dr, dashed] & \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d] \\
& & \mathcal{A} \arrow[r] & \mathcal{C}
\end{tikzcd}$$

where the map $J \star T \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ extends p, and $J^{\triangleright} \star S \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ extends $\overline{p}$. Since we have a colimit cocone $\overline{\pi}$ in $\mathcal{A}$, we can find an arrow $J^{\triangleright} \star T \to \mathcal{A}$ fitting as the dashed arrow in the above diagram. This reduces the problem for finding a dotted arrow into solving a lifting problem (4.2), which can be done by assumption. $\square$

Lemma 4.2.3.3. *For all simplicial sets S, T, the diagram*

$$\begin{tikzcd}
(S \times \{0\}) \star (T \times \{1\}) \arrow[r] \arrow[d] & (S \times \Delta[1]) \star (T \times \{1\}) \arrow[d] \\
(S \times \{0\}) \star (T \times \Delta[1]) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & (S \star T) \times \Delta[1]
\end{tikzcd}$$

is a pushout square, which is moreover a homotopy pushout square as the maps are cofibrations.

Proof. On n-cells, the square is given by

$$\begin{tikzcd}
(S_n \times \{0\}) \sqcup (T_n \times \{1\}) \sqcup \coprod_{i+j=n-1} (S_i \times \{0\}) \times (T_j \times \{1\}) \arrow[r] \arrow[d] & (S_n \times \Delta[1]_n) \sqcup (T_n \times \{1\}) \sqcup \coprod_{i+j=n-1} (S_i \times \Delta[1]_i) \times (T_j \times \{1\}) \arrow[d] \\
(S_n \times \{0\}) \sqcup (T_n \times \Delta[1]_n) \sqcup \coprod_{i+j=n-1} (S_i \times \{0\}) \times (T_j \times \Delta[1]_j) \arrow[r] & (S_n \times \Delta[1]_n) \sqcup (T_n \times \Delta[1]_n) \sqcup \coprod_{i+j=n-1} (S_i \times T_j \times \Delta[1]_n)
\end{tikzcd}$$

Since colimits commute with colimits, it suffices to show that the diagram restricted to each set of coproduct summands forms a pushout square. In other words, it suffices to show for all $i + j = n - 1$ (where $i, j \geq -1$ and we take $K_{-1} := *$ for any simplicial set K) that the square

$$\begin{tikzcd}
(S_i \times \{0\}) \times (T_j \times \{1\}) \arrow[r] \arrow[d] & (S_i \times \Delta[1]_i) \times (T_j \times \{1\}) \arrow[d] \\
(S_i \times \{0\}) \times (T_j \times \Delta[1]_j) \arrow[r] & S_i \times T_j \times \Delta[1]_{i+j+1}
\end{tikzcd}$$

is a pushout square. This is trivial if $i = -1$ or $j = -1$, so suppose $i, j \geq 0$. Since **sSet** is cartesian closed, products commute with colimits, which allows us to reduce further to showing that

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \mathrm{Hom}_{\boldsymbol{\Delta}}([i],[1]) \arrow[d] \\
\mathrm{Hom}_{\boldsymbol{\Delta}}([j],[1]) \arrow[r] & \mathrm{Hom}_{\boldsymbol{\Delta}}([i+j+1],[1])
\end{tikzcd}$$

is a pushout square. Note that maps $[i] \to [1]$ correspond to integers $0 \leq c \leq i+1$, where c indicates the first index of the map that is sent to 1 (and $c = i+1$ means that the map is constant at zero). With this interpretation, the top map picks out the morphism $[i] \to [1]$ corresponding to the integer $c = 0$, while the vertical map on the left picks out the zero map $[j] \to [1]$. Observing that we have a pushout square amounts to observing that a morphism $[i+j+1] \to [1]$ falls into one of the following three cases:

- it corresponds to a cut $0 \leq c < j+1$, in which case it comes from a nonzero morphism $[j] \to [1]$
- it corresponds to a cut $j+1 < c \leq i+j+2$, in which case it comes from a morphism $[i] \to [1]$ that starts at zero (the vertical map on the right shifts the index of the cut up by $j+1$)
- it corresponds to the cut $c = j+1$, in which case it simultaneously comes from the constant zero morphism $[j] \to [1]$ and the constant one morphism $[i] \to [1]$

□

We moreover recall the following result:

Lemma 4.2.3.4. [26, Lemma 2.1.2.3] *Let $A_0 \subseteq A$ and $B_0 \subseteq B$ be inclusions such that either $A_0 \subseteq A$ is right anodyne, or $B_0 \subseteq B$ is left anodyne. Then, the inclusion*

$$(A_0 \star B) \sqcup_{A_0 \star B_0} (A \star B_0) \hookrightarrow A \star B$$

is inner anodyne.

Proof of Proposition 4.2.3.1. Suppose we have a diagram $p : J \to \mathcal{K}^{\mathrm{lax}}(F)$ such that the composite $up : J \to \mathcal{K}^{\Delta[1]}$ admits a colimit $\overline{p}_u : J^{\triangleright} \to \mathcal{K}^{\Delta[1]}$. By Lemma 4.2.3.2, it suffices to show for any simplicial set T that we can find a lift for any problem

$$\begin{tikzcd}
J \star T \arrow[r, "q"] \arrow[d, hook] & \mathcal{K}^{\mathrm{lax}}(F) \arrow[r] & \mathcal{K}^{\{1 \leftarrow 0 \to 2\}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{K}^{\{1 \leftarrow 0\}} \arrow[d] \\
J^{\triangleright} \star T \arrow[r] \arrow[urr, dashed] & \mathcal{K}^{\{0 \to 2\}} \arrow[r, "{(F(2),\mathrm{id})}"'] & \mathcal{K}^{\{1\}} \times \mathcal{K}^{\{0 \to 2\}} \arrow[r] & \mathcal{K}^{\{1\}} \times \mathcal{K}^{\{0\}}
\end{tikzcd} \tag{4.3}$$

where q extends the composite $J \xrightarrow{p} \mathcal{K}^{\mathrm{lax}}(F)$, and the leftmost arrow on the bottom extends $\overline{p}_u : J^{\triangleright} \to \mathcal{K}^{\Delta[1]}$. Since we have the pullback square on the right, it suffices to find a lift $J^{\triangleright} \star T \to \mathcal{K}^{\{0\to 1\}}$ to the upper right corner. By currying, we are finding a suitable map $(J^{\triangleright} \star T) \times \Delta[1] \to \mathcal{K}$.

For the sake of clarity, we will refer to maps of simplicial sets based on an intuitive diagram that they reflect. For this purpose, we will denote the lax F-algebra q_j at $j \in J \subseteq J \star T$ by $FB_j \leftarrow E_j \to B_j$, and we will denote the lax F-algebra q_t at $t \in T \subseteq J \star T$ by $FC_t \leftarrow D_t \to C_t$. Similarly, denote the colimit $p_u(\infty) \in \mathcal{K}^{\{0\to 2\}}$ by $E_\infty \to B_\infty$. Then, the desired lift $(J^{\triangleright} \star T) \times \Delta[1] \to \mathcal{K}$ reflects the diagram

$$\left\{ \begin{tikzcd} FB_j \arrow[rr] \arrow[dr] & & FC_t \\ & FB_\infty \arrow[ur] & \\ E_j \arrow[uu] \arrow[rr] \arrow[dr] & & D_t \arrow[uu] \\ & E_\infty \arrow[uu] \arrow[ur] & \end{tikzcd} \right\} \tag{4.4}$$

The first goal is to produce a map reflecting the diagram

$$\left\{ \begin{tikzcd} & FB_\infty \arrow[dr] & \\ E_j \arrow[ur] \arrow[r, dashed] \arrow[dr] & E_\infty \arrow[u, dashed] \arrow[r, dashed] \arrow[d, dashed] & FC_t \\ & D_t \arrow[ur] & \end{tikzcd} \right\} \tag{4.5}$$

from which the dashed arrows can be recovered by the universal property of $E_\infty = \varinjlim_j E_j$ (note that the dashed arrows $E_\infty \to D_t$ are already provided from the bottom row of (4.3) via the map $\{\infty\} \star T \subseteq J^{\triangleright} \star T \to \mathcal{K}^{\{0\to 2\}} \to \mathcal{K}^{\{0\}}$).

The commutative diagram

$$\left\{ \begin{tikzcd} FB_j \arrow[r] & FC_t \\ E_j \arrow[u] \arrow[r] \arrow[ur] & D_t \arrow[u] \end{tikzcd} \right\} \tag{4.6}$$

is obtained by the map $(J \star T) \times \Delta[1] \to \mathcal{K}$ given as the adjunct of the top row of (4.3). By Lemma 4.2.3.3, we can write

$$(J \star T) \times \Delta[1] = \big((J \times \{0\}) \star (T \times \Delta[1])\big) \sqcup_{(J\times\{0\})\star(T\times\{1\})} \big((J \times \Delta[1]) \star (T \times \{1\})\big)$$

In particular, we can isolate the upper-left triangle of (4.6) by restricting to the simplicial subset $(J \times \Delta[1]) \star (T \times \{1\}) \hookrightarrow (J \star T) \times \Delta[1] \to \mathcal{K}$. We also have a map $J \star \{\infty\} \star T \to \mathcal{K}$

obtained from the bottom row of (4.3) as the composite $J^{\triangleright} \star T \to \mathcal{K}^{\{1\}} \times \mathcal{K}^{\{0\to 2\}} \to \mathcal{K}^{\{1\}}$, which reflects the diagram

$$\left\{ \begin{tikzcd} & FB_\infty \arrow[dr] & \\ FB_j \arrow[ur] \arrow[rr] & & FC_t \end{tikzcd} \right\}$$

Gluing with the upper left triangle of (4.6), we can produce a map

$$(J \times \Delta[1]) \star \{\infty\} \star (T \times \{1\}) \to \mathcal{K}$$

corresponding to the diagram

$$\left\{ \begin{tikzcd} FB_j \arrow[r] \arrow[drr, bend right=15] & FB_\infty \arrow[dr] & \\ E_j \arrow[u] \arrow[ur] \arrow[rr] & & FC_t \end{tikzcd} \right\} \tag{4.7}$$

as a solution to the lifting problem

$$\begin{tikzcd} \big((J \times \Delta[1]) \star (T \times \{1\})\big) \sqcup_{(J\times\{1\})\star(T\times\{1\})} \big((J \times \{1\}) \star \{\infty\} \star (T \times \{1\})\big) \arrow[r] \arrow[d, hook] & \mathcal{K} \\ (J \times \Delta[1]) \star \{\infty\} \star (T \times \{1\}) \arrow[ur, dashed] & \end{tikzcd}$$

By applying Lemma 4.2.3.4 to the right anodyne map $J \times \{1\} = J \times \Lambda^1[1] \hookrightarrow J \times \Delta[1]$ and the inclusion $T \times \{1\} \hookrightarrow \{\infty\} \star (T \times \{1\})$, we see that the vertical map is inner anodyne. Therefore, since $\mathcal{K}$ is a quasicategory, it follows that such a lift indeed exists.

Now, glue the map for (4.7) to the bottom-right triangle of (4.6) to produce a map

$$\big((J \times \{0\}) \star (T \times \Delta[1])\big) \sqcup_{(J\times\{0\})\star(T\times\{1\})} \big((J \times \Delta[1]) \star \{\infty\} \star (T \times \{1\})\big) \to \mathcal{K}$$

In particular, we can restrict this map to the simplicial subset

$$\begin{aligned} &\big((J \times \{0\}) \star (T \times \Delta[1])\big) \sqcup_{(J\times\{0\})\star(T\times\{1\})} \big((J \times \{0\}) \star \{\infty\} \star (T \times \{1\})\big) \\ &\cong (J \times \{0\}) \star \Big((T \times \Delta[1]) \sqcup_{T\times\{1\}} \big(\{\infty\} \star (T \times \{1\})\big)\Big) \end{aligned}$$

reflecting the subdiagram

$$\left\{ \begin{tikzcd} & FB_\infty \arrow[dr] & \\ E_j \arrow[ur] \arrow[rr] \arrow[dr] & & FC_t \\ & D_t \arrow[ur] & \end{tikzcd} \right\}$$

which is precisely the perimeter of (4.5) required to invoke the universal property of E_∞. Indeed, since E_∞ is a colimit of the diagram $J \xrightarrow{p} \mathcal{K}^{\mathrm{lax}}(F) \to \mathcal{K}^{\{1 \leftarrow 0 \to 2\}} \to \mathcal{K}^{\{0\}}$, it follows that we can find a dashed morphism fitting in the diagram

$$\begin{tikzcd}
(J \times \{0\}) \star (T \times \{0\}) \arrow[r] \arrow[d] & (J \times \{0\}) \star \Big((T \times \Delta[1]) \sqcup_{T \times \{1\}} \big(\{\infty\} \star (T \times \{1\}) \big) \Big) \arrow[d] \arrow[ddr, bend left] & \\
(J^{\triangleright} \times \{0\}) \star (T \times \{0\}) \arrow[r] \arrow[drr, bend right] & (J^{\triangleright} \times \{0\}) \star \Big((T \times \Delta[1]) \sqcup_{T \times \{1\}} \big(\{\infty\} \star (T \times \{1\}) \big) \Big) \arrow[dr, dashed] & \\
& & \mathcal{K}
\end{tikzcd}$$

where the bottom map $(J^{\triangleright} \times \{0\}) \star (T \times \{0\}) \to \mathcal{K}$ is the projection of the bottom row of (4.3) onto $\mathcal{K}^{\{0\}}$ that describes the complex of morphisms

$$\left\{ \begin{tikzcd}
& E_\infty \arrow[dr] & \\
E_j \arrow[ur] \arrow[rr] & & D_t
\end{tikzcd} \right\}$$

This dashed morphism precisely recovers the diagram (4.5). To obtain the desired diagram (4.4), we glue this morphism with the map $(J \times \Delta[1]) \star \{\infty\} \star (T \times \{1\}) \to \mathcal{K}$ describing (4.7). Indeed, we get a pushout square

$$\begin{tikzcd}
(J \times \{0\}) \star \{\infty\} \star (T \times \{1\}) \arrow[r] \arrow[d] & (J \times \Delta[1]) \star \{\infty\} \star (T \times \{1\}) \arrow[d] \arrow[ddr, bend left] & \\
(J^{\triangleright} \times \{0\}) \star \Big((T \times \Delta[1]) \sqcup_{T \times \{1\}} \big(\{\infty\} \star (T \times \{1\}) \big) \Big) \arrow[r] \arrow[drr, bend right] \arrow[ur, phantom, very near end, "\ulcorner"] & (J^{\triangleright} \star T) \times \Delta[1] \arrow[dr, dashed] & \\
& & \mathcal{K}
\end{tikzcd} \tag{4.8}$$

and the dashed arrow precisely reflects the diagram (4.4), meaning that its adjunct map $J^{\triangleright} \star T \to \mathcal{K}^{\{0 \to 1\}}$ pulls back to give precisely a lift in (4.3), as desired.

To see that (4.8) is indeed a pushout square, note that by expanding the pushout on the bottom left corner and using that joins preserve pushouts, this is equivalent to showing that the diagram

$$\begin{tikzcd}
& (J \times \{0\}) \star \{\infty_1\} \star (T \times \{1\}) \arrow[r] \arrow[d] & (J \times \Delta[1]) \star \{\infty_1\} \star (T \times \{1\}) \arrow[dd] \\
(J^{\triangleright} \times \{0\}) \star (T \times \{1\}) \arrow[r] \arrow[d] & (J^{\triangleright} \times \{0\}) \star \{\infty_1\} \star (T \times \{1\}) \arrow[dr] & \\
(J^{\triangleright} \times \{0\}) \star (T \times \Delta[1]) \arrow[rr] & & (J^{\triangleright} \star T) \times \Delta[1]
\end{tikzcd}$$

is a universal cocone diagram, where I have tacitly replaced $\{\infty\}$ with $\{\infty_1\}$, to indicate that its image in $(J^{\triangleright} \star T) \times \Delta[1]$ lies in the top cell $(J \star \{\infty_1\} \star T) \times \{1\}$.

Note that $(J\times\{0\})^{\triangleright}\star\{\infty_1\}\star(T\times\{1\})$ is isomorphic to $(J\times\{0\})\star(\{\infty\}\times\Delta[1])\star(T\times\{1\})$ using the associativity of the join operation and how $\{\infty_0\}\star\{\infty_1\} \cong \{\infty\}\times\Delta[1]$. Therefore, the above diagram is precisely the result of pasting the following two pushout squares:

$$\begin{tikzcd}
 & (J \times \{0\}) \star \{\infty_1\} \star (T \times \{1\}) \arrow[r] \arrow[d] & (J \times \Delta[1]) \star \{\infty_1\} \star (T \times \{1\}) \arrow[d] \\
(J^{\triangleright} \star \times \{0\}) \star (T \times \{1\}) \arrow[r] \arrow[d] & (J \times \{0\}) \star (\{\infty\} \times \Delta[1]) \star (T \times \{1\}) \arrow[r] \arrow[ur, phantom, "\ulcorner" near end] & (J^{\triangleright} \times \Delta[1]) \star (T \times \{1\}) \arrow[d] \\
(J^{\triangleright} \times \{0\}) \star (T \times \Delta[1]) \arrow[rr] \arrow[urr, phantom, "\ulcorner" near end] & & (J^{\triangleright} \star T) \times \Delta[1]
\end{tikzcd}$$

Indeed, the bottom square is precisely an instance of Lemma 4.2.3.3, and the upper square is the result of applying $(-) \star (T \times \{1\})$ to another instance of Lemma 4.2.3.3. This proves that (4.8) is indeed a pushout square diagram, and thus we have our desired lift of (4.3). $\square$

A very similar result holds for colimits of coalgebras:

Proposition 4.2.3.5. *The forgetful functor $\mathcal{K}_{\mathrm{co}}(F) \to \mathcal{K}$ sending an F-coalgebra to its underlying object reflects colimits.*

Proof. Note that $\mathcal{K}_{\mathrm{co}}(F)$ is equivalent to the full subcategory of $\mathcal{K}^{\mathrm{lax}}(F)$ spanned by those lax F-algebras where the lax action is invertible. Since the colimit of equivalences in $\mathcal{K}^{\Delta[1]}$ is an equivalence, the result follows from Proposition 4.2.3.1. $\square$

Corollary 4.2.3.6. *The forgetful functor $\mathcal{K}(F) \to \mathcal{K}$ reflects limits.*

4.3 Universal fixed points

As in Section 4.2, we continue to fix a category $\mathcal{K}$ and an endofunctor $F : \mathcal{K} \to \mathcal{K}$. The goal of this section is to prove that Adámek's construction also yields free F-algebras in the ∞-categorical setting. In particular, this establishes when transfinite application of F to an initial object $\emptyset$ of $\mathcal{K}$ yields an initial fixed point of F; see Corollary 4.3.1.8.

In Section 4.3.2, we focus on the construction of fixed points. Note that a free F-algebra is generally not a fixed point of F.

Example 4.3.0.1. If $S : \mathbf{Set} \to \mathbf{Set}, X \mapsto X \sqcup \{\perp\}$, then it turns out that free S-algebras are fixed points of S. Indeed, the free S-algebra generated by a set X is given by $X \sqcup \mathbb{N}$, where the S-algebra action is inherited from that of $\mathbb{N}$, and is thus a bijection.

Example 4.3.0.2. Let $* : \mathbf{Set} \to \mathbf{Set}$ be the terminal endofunctor on **Set**; that is, the functor that sends $X \mapsto *$ for all sets X. Then, the category of $*$-algebras is precisely the 1-category $\mathbf{Set}_*$ of pointed sets.

The only fixed point of $*$ is the singleton. On the other hand, the free pointed set generated by a set X is given by formally adjoining a basepoint $X \sqcup \{*\}$. In particular, X generates a fixed point of $*$ if and only if $X = \varnothing$.

Instead, we briefly study an orthogonal generalisation of Adámek's initial algebra construction focused on yielding fixed points. Specifically, we study free fixed points of F generated by F-coalgebras. This is based on the fact that an initial object $\emptyset$ of K has a unique F-coalgebra structure given by the coaction $! : \emptyset \to F(\emptyset)$, which can be iterated to produce a fixed point of F.

4.3.1 Free lax algebras

By the discussion in Section 4.2.2, Adámek's free F-algebra construction is given by transfinite application of a propagation pushout endofunctor Π via its unit $\eta : \mathrm{Id} \Rightarrow \Pi$. From this perspective, Adámek's construction can be thought of as a special case of a more abstract result:

Theorem 4.3.1.1 (Free fixed point construction)**.** *Let $\mathcal{L}$ be a category, and $\Pi : \mathcal{L} \to \mathcal{L}$ an endofunctor with a unit; that is, with a natural transformation $\eta : \mathrm{Id} \Rightarrow \Pi$. Denote by $\mathcal{L}^\Pi$ the full subcategory of $\mathcal{L}$ spanned by objects K such that $\eta_K : K \to \Pi K$ is an equivalence.*

For an ordinal θ, let $[\theta]$ denote the nerve of the poset of all ordinals $0 \leq \xi \leq \theta$. For any object $L \in \mathcal{L}$, construct the diagrams $D_L^\theta : [\theta] \to \mathcal{L}$ by transfinite induction as follows:

- *Define $D_L^0 : [0] \to \mathcal{L}$ to be the diagram picking out the object L.*
- *Given D_L^θ, define $D_L^{\theta+1}$ to be the extension of D_L^θ that sends the morphism $\theta \leq \theta + 1$ in $[\theta + 1]$ to $\eta_{D_L^\theta(\theta)} : D_L^\theta(\theta) \to \Pi D_L^\theta(\theta)$. Note that this construction is well-defined by, for example, identifying ∞-categories with strictly* **sSet***-enriched categories as in Theorem 2.4.1.5.*
- *Given D_L^θ for all $\theta < \lambda$ with λ a limit ordinal, let $\underline{\lambda} = \varinjlim_{\theta<\lambda}[\theta]$ be the nerve of the poset of all ordinals $0 \leq \xi < \lambda$, which induces a functor $D_L^{<\lambda} : \underline{\lambda} \to \mathcal{L}$. Since $[\lambda] \cong \underline{\lambda}^\triangleright$, define D_L^λ to be a colimit cocone for $D_L^{<\lambda}$, if it exists.*

Suppose for some limit ordinal λ that the diagram $D_L^\lambda : [\lambda] \to \mathcal{L}$ is well-defined, and let $\widehat{L} := D_L^\lambda(\lambda)$. Then, the following are equivalent:

(i) $\widehat{L} \in \mathcal{L}^\Pi$.

(ii) $\widehat{L}$ *corepresents the functor* $\mathrm{Hom}_{\mathcal{L}}(L,-)| : \mathcal{L}^{\Pi} \to \mathcal{S}$*; that is,* $\widehat{L}$ *is the free object in* $\mathcal{L}^{\Pi}$ *generated by* L*.*

Proof. Since (ii) certainly implies (i), we need to show that (i) implies (ii), for which it is enough to prove that the coprojection $L \to \widehat{L}$ induces a homotopy equivalence of mapping spaces $\mathrm{Hom}_{\mathcal{L}}(\widehat{L}, K) \to \mathrm{Hom}_{\mathcal{L}}(L, K)$ whenever $K \in \mathcal{L}^{\Pi}$. Indeed, if $\widehat{L} \in \mathcal{L}^{\Pi}$, this would prove that $\mathrm{Hom}_{\mathcal{L}^{\Pi}}(\widehat{L}, K) = \mathrm{Hom}_{\mathcal{L}}(L, K)$ for all $K \in \mathcal{L}^{\Pi}$.

Fix $K \in \mathcal{L}^{\Pi}$ and let $\delta^{\theta}_K : [\theta] \to \mathcal{L}$ denote the constant diagram on K. By transfinite induction, we can define a natural transformation $\delta^{\theta}_K \Rightarrow D^{\theta}_K$ for every ordinal θ, where the component $K = \delta^{\theta}_K(\xi) \to D^{\theta}_K(\xi)$ is given by the transfinite composite

$$K \xrightarrow{\eta} \Pi K \xrightarrow{\Pi\eta} \Pi^2 K \to \cdots \to D^{\theta}_K(\xi)$$

Since $K \in \mathcal{L}^{\Pi}$, the map $\eta_K : K \to \Pi K$ is an equivalence, which ensures that the diagrams D^{θ}_K are well-defined for all ordinals θ, and moreover that the natural transformation $\delta^{\theta}_K \Rightarrow D^{\theta}_K$ is a natural equivalence.

Now, consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{L}}(\widehat{L}, K) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Hom}_{\mathcal{L}}(L, K) \arrow[d, hook] \\
\mathrm{Nat}(D^{\lambda}_L, \delta^{\lambda}_K) \arrow[r, "\sim"'] & \mathrm{Nat}(D^{\lambda}_L, D^{\lambda}_K) \arrow[u, dashed, bend right]
\end{tikzcd}$$

The vertical map on the left is an equivalence by [26, Lemma 4.2.4.3(ii)], and the horizontal map on the bottom is an equivalence since $\delta^{\lambda}_K \Rightarrow D^{\lambda}_K$ is a natural equivalence. The vertical map on the right is given by the functoriality of the construction of the diagram $D^{\lambda}_{(-)}$, and admits a retraction (denoted by the dashed arrow) that acts by projecting a natural transformation $D^{\lambda}_L \Rightarrow D^{\lambda}_K$ to the zeroth component $L \to K$. Since the square commutes, it follows from the 2-out-of-6 property that all of the arrows in the diagram are equivalences. In particular, the map $\mathrm{Hom}_{\mathcal{L}}(\widehat{L}, K) \to \mathrm{Hom}_{\mathcal{L}}(L, K)$ is an equivalence, as desired. $\square$

The following lemma explicitly bridges Adámek's construction to the fixed point construction above.

Lemma 4.3.1.2. *Let* $\mathcal{K}$ *be finitely cocomplete so that we have the propagation endofunctor and unit on* $\mathcal{K}^{\mathrm{lax}}(F)$*. Then, the inclusion* $\mathcal{K}(F) \hookrightarrow \mathcal{K}^{\mathrm{lax}}(F)$ *factors through the full subcategory* $\mathcal{K}^{\mathrm{lax}}(F)^{\Pi} \subset \mathcal{K}^{\mathrm{lax}}(F)$ *of* Π*-fixed points, and the corestriction* $\mathcal{K}(F) \to \mathcal{K}^{\mathrm{lax}}(F)^{\Pi}$ *is an equivalence.*

Proof. Recall that the propagation unit at a lax F-algebra $FB \xleftarrow{r} E \xrightarrow{a} B$ is the morphism consisting of the horizontal arrows in the diagram

$$\begin{tikzcd}
FB \arrow[r, "Fi"] & F(B \sqcup_E FB) \\
E \arrow[u, "r"] \arrow[r, "r"] \arrow[d, "a"'] \arrow[dr, phantom, "\ulcorner" very near end] & FB \arrow[u, "Fi"'] \arrow[d] \\
B \arrow[r, "i"'] & B \sqcup_E FB
\end{tikzcd} \tag{4.9}$$

By Proposition 4.2.1.3, $\mathcal{K}(F)$ can be identified with the full subcategory of $\mathcal{K}^{\text{lax}}(F)$ on the lax F-algebras $FB \xleftarrow{r} E \xrightarrow{a} B$ where the resolution r is an equivalence. In particular, this implies that the pushout morphism $i : B \to B \sqcup_E FB$ is an equivalence (since pushouts preserve equivalences), and thus that the top morphism $Fi : FB \to F(B \sqcup_E FB)$ is an equivalence also. This shows that the inclusion $\mathcal{K}(F) \hookrightarrow \mathcal{K}^{\text{lax}}(F)$ indeed factors through $\mathcal{K}^{\text{lax}}(F)^{\Pi}$.

Conversely, if a lax F-algebra $FB \xleftarrow{r} E \xrightarrow{a} B$ lies in $\mathcal{K}^{\text{lax}}(F)^{\Pi}$, then the middle component $r : E \to FB$ in (4.9) in particular is an equivalence. This implies that the lax F-algebra lies in the essential image of $\mathcal{K}(F)$, showing that the fully faithful inclusion $\mathcal{K}(F) \to \mathcal{K}^{\text{lax}}(F)^{\Pi}$ is essentially surjective, thus completing the proof. $\square$

Remark 4.3.1.3. By [26, Corollary 4.4.2.4], finite cocompleteness follows from assuming $\mathcal{K}$ has pushouts and an initial object, which is necessary to ensure that the propagation endofunctor is well-defined for the entire category $\mathcal{K}^{\text{lax}}(F)$. This assumption is not strictly necessary: we can instead choose any full subcategory $\mathcal{L} \subseteq \mathcal{K}^{\text{lax}}(F)$ that contains $\mathcal{K}(F)$ and has enough pushouts to construct a propagation functor $\Pi : \mathcal{L} \to \mathcal{K}^{\text{lax}}(F)$. If Π corestricts to an endofunctor on $\mathcal{L}$, then the above lemma can be adapted to show that $\mathcal{K}(F) \hookrightarrow \mathcal{L}^{\Pi}$ is an equivalence also.

Theorem 4.3.1.4 (Adámek's construction on lax algebras)**.** *For a category $\mathcal{K}$ and endofunctor $F : \mathcal{K} \to \mathcal{K}$, fix a lax F-algebra $FB \xleftarrow{r} E \xrightarrow{a} B$. Construct the diagrams $D^{\theta} : [\theta] \to \mathcal{K}^{\Delta[1]}$ by transfinite induction, where θ is an ordinal:*

- *Take $D^0 : [0] \to \mathcal{K}^{\Delta[1]}$ to be the diagram that picks out the arrow $a : E \to B$. Note that the resolution map provides an arrow $r_0 := r : E \to FB$.*

- *Given $D^{\theta} : [\theta] \to \mathcal{K}^{\Delta[1]}$, denote by $E^{\theta} \to B^{\theta}$ the arrow of $\mathcal{K}$ picked out by $D^{\theta}(\theta)$. Suppose we have chosen an arrow $r_{\theta} : E^{\theta} \to FB^{\theta}$. Then, define $D^{\theta+1}$ to be the* *extension of D^θ that sends the morphism $\theta \le \theta+1$ in $[\theta+1]$ to the pushout square*

$$\begin{tikzcd} E^\theta \arrow[r, "r_\theta"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & FB^\theta \arrow[d] \\ B^\theta \arrow[r, "i_{\theta+1}"'] & B^{\theta+1} \end{tikzcd}$$

In particular, $D^{\theta+1}(\theta+1)$ picks out the arrow $E^{\theta+1} \to B^{\theta+1}$ where $E^{\theta+1} := FB^\theta$. Moreover, choose $r_{\theta+1} := Fi_{\theta+1} : E^{\theta+1} \to FB^{\theta+1}$.

- *For a limit ordinal λ, and given D^θ for all $\theta < \lambda$, let $\underline{\lambda} = \varinjlim_{\theta<\lambda}[\theta]$ be the nerve of the poset of all ordinals $0 \le \xi < \lambda$, so that the provided diagrams induce $D^{<\lambda} : \underline{\lambda} \to \mathcal{K}^{\Delta[1]}$.*

  Then, define D^λ to be a colimit cocone for $D^{<\lambda}$. If the colimit point $D^\lambda(\lambda)$ is the arrow $E^\lambda \to B^\lambda$, then the choice of r_θ for every $\theta < \lambda$ induces a canonical map $r_\lambda : E^\lambda \to FB^\lambda$ by the universal property of E^λ.

Suppose for some limit ordinal λ that the diagram $D^\lambda : [\lambda] \to \mathcal{K}^{\Delta[1]}$ is well-defined, and let $E^* \to B^*$ be the arrow picked out by $D^\lambda(\lambda)$ in $\mathcal{K}^{\Delta[1]}$. If the canonical map $r_* : E^* \to FB^*$ induced by the r_θ chosen in the construction is invertible, then $FB^* \xrightarrow{r_*^{-1}} E^* \to B^*$ defines an action that realises B^* as the free F-algebra generated by $FB \leftarrow E \to B$.

Remark 4.3.1.5. Adámek's construction for a lax F-algebra $FB \xleftarrow{r} E \xrightarrow{a} B$ can be described more explicitly if the construction terminates after countably many steps (that is, $\lambda = \omega$). In this case, we are assuming that the pushout squares in the diagram

$$\begin{tikzcd} E \arrow[r, "r"] \arrow[d, "a"'] \arrow[dr, phantom, "\ulcorner", very near end] & FB \arrow[r, "Fi_1"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & F(B \sqcup_E FB) \arrow[r, "Fi_2"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & F(B \sqcup_E F(B \sqcup_E FB)) \arrow[r, "Fi_3"] \arrow[d] & \dots \\ B \arrow[r, "i_1"'] & B \sqcup_E FB \arrow[r, "i_2"'] & B \sqcup_E F(B \sqcup_E FB) \arrow[r, "i_3"'] & B \sqcup_E F(B \sqcup_E F(B \sqcup_E FB)) \arrow[r] & \dots \end{tikzcd}$$

exist, and moreover that this diagram has a colimit $E^* \to B^*$ (in $\mathcal{K}^{\Delta[1]}$). If the canonical map $E^* \to FB^*$ induced by the top row is invertible, then composing an inverse with the colimit arrow defines an action $FB^* \xrightarrow{\sim} E^* \to B^*$ that realises B^* as the free F-algebra generated by $FB \leftarrow E \to B$.

Proof of Theorem 4.3.1.4. For every $\theta \le \lambda$, let $E^\theta \to B^\theta$ denote the arrow picked out by $D^\lambda(\theta)$. With the $r_\theta : E^\theta \to FB^\theta$, we obtain lax F-algebras $A^\theta := \{FB^\theta \leftarrow E^\theta \to B^\theta\}$, where A^0 is the original lax F-algebra $A^0 = A := \{FB \leftarrow E \to B\}$.

Let $\mathcal{L}$ denote the full subcategory of $\mathcal{K}^{\mathrm{lax}}(F)$ spanned by $\mathcal{K}(F)$ and the lax F-algebras A^θ for $\theta \le \lambda$. Assuming that D^λ is well-defined ensures that we have the pushouts in $\mathcal{K}$ necessary to define the unital propagation functor $\Pi : \mathcal{L} \to \mathcal{K}^{\mathrm{lax}}(F)$ as in Definition 4.2.2.1.

Moreover, we have by design that $\Pi(A^\theta) = A^{\theta+1}$ for every $\theta < \lambda$, and $\Pi(A^\lambda) \simeq A^\lambda$ since we assume that the map $r_\lambda = r_*$ is invertible. Therefore, Π corestricts to an endofunctor $\mathcal{L} \to \mathcal{L}$.

For $\theta \le \lambda$, let $D_A^\theta : [\theta] \to \mathcal{L}$ denote the diagram constructed in Theorem 4.3.1.1 with the endofunctor Π and the lax F-algebra A. We can see by transfinite induction that the diagrams D_A^θ are indeed well-defined, and moreover that the lax F-algebra $D_A^\theta(\theta)$ is precisely A^θ.

- This is immediate if $\theta = 0$.
- Given that D_A^θ is well-defined and $D_A^\theta(\theta) = A^\theta$, it follows that $D_A^{\theta+1}$ exists and maps $(\theta+1)$ to $A^{\theta+1}$ because $A^{\theta+1} = \Pi(A^\theta)$.
- Suppose for a limit ordinal ξ that D_A^θ is well-defined for all $\theta < \xi$, and $D_A^\theta(\theta) = A^\theta$. It would follow that D_A^ξ is well-defined and $D_A^\xi(\xi) = A^\xi$ if we can show that the colimit of $D_A^{<\xi} : \underline{\xi} \to \mathcal{L}$ is A^ξ.

  To see this, recall that $E^\xi \to B^\xi$ is defined to be the colimit of $D^{<\xi} : \underline{\xi} \to \mathcal{K}^{\Delta[1]}$, and the universal property of E^ξ then canonically induces the map r_ξ from the maps r_θ for $\theta < \xi$. This is precisely how the colimit $\varinjlim D_A^{<\xi}$ of lax F-algebras is constructed by Proposition 4.2.3.1.

By assumption, the lax F-algebra $A^\lambda = \{FB^* \leftarrow E^* \to B^*\}$ has an invertible resolution r_*, so $A^\lambda = D_A^\lambda(\lambda)$ lies in $\mathcal{L}^\Pi$. Therefore, the conclusion follows from Theorem 4.3.1.1 and Lemma 4.3.1.2. □

Corollary 4.3.1.6 (Adámek's free algebra construction)**.** *Fix an object $K \in \mathcal{K}$. Construct the objects K^θ and morphisms $i_\theta : K^\theta \to K \sqcup FK^\theta$, for θ an ordinal, by transfinite induction:*

- *Take $K^0 := K$ and $i_0 : K \to K \sqcup FK$ to be the first coprojection.*
- *Given $K^\theta \to K \sqcup FK^\theta$, we have a pushout square*

$$\begin{tikzcd} FK^\theta \arrow[r, "Fi_\theta"] \arrow[d] & F(K \sqcup FK^\theta) \arrow[d] \arrow[dl, phantom, "\ulcorner", very near start] \\ K \sqcup FK^\theta \arrow[r] & K \sqcup F(K \sqcup FK^\theta) \end{tikzcd}$$

  where the vertical arrows are given by second coprojections for the respective coproducts. Define $K^{\theta+1} := K \sqcup FK^\theta$ and take $i_{\theta+1} : K^{\theta+1} \to K \sqcup FK^{\theta+1}$ to be the bottom row of the above pushout square.

- *Suppose $i_\theta : K^\theta \to K \sqcup FK^\theta = K^{\theta+1}$ is well-defined for every $\theta < \lambda$ with λ a limit ordinal. Define $K^\lambda := \varinjlim_{\theta<\lambda} K^\theta$. Since $K^{\theta+1} = K \sqcup FK^\theta$, we obtain a canonical map $i_\lambda : K^\lambda \to K \sqcup FK^\lambda$.*

Suppose for some limit ordinal λ that K^λ is well-defined, and $i_\lambda : K^\lambda \to K \sqcup FK^\lambda$ is invertible. Then, the map $FK^\lambda \to K^\lambda$ induced by the coprojections $FK^\theta \to K \sqcup FK^\theta = K^{\theta+1}$ for $\theta < \lambda$ defines an action that realises K^λ as the free F-algebra generated by the object K.

Proof. Assume first that $\mathcal{K}$ has an initial object $\emptyset$. Then, the result follows from applying Theorem 4.3.1.4 to the free lax F-algebra $FK \leftarrow \emptyset \to K$ generated by the object K.

Now, suppose $\mathcal{K}$ does not have an initial object. Extend F to an endofunctor on $\mathcal{K}^\triangleleft$ by fixing the cone point, then $\mathcal{K}(F)$ is a full subcategory of $\mathcal{K}^\triangleleft(F)$. Then, K^λ is a free F-algebra in $\mathcal{K}^\triangleleft(F)$ generated by K by the previous paragraph. Since K^λ lives in $\mathcal{K}(F)$ as well, it restricts to a free F-algebra in $\mathcal{K}(F)$ generated by K also. $\square$

Remark 4.3.1.7. As in Remark 4.3.1.5, Adámek's construction of free F-algebras can be described more succinctly if the construction terminates after countably many steps. In this case, we suppose the coproducts in the diagram

$$\begin{tikzcd}
 & FK \arrow[r, "Fi_1"] \arrow[d] & F(K \sqcup FK) \arrow[r, "Fi_2"] \arrow[d] & F(K \sqcup F(K \sqcup FK)) \arrow[r, "Fi_3"] \arrow[d] & \dots \\
K \arrow[r, "i_1"'] & K \sqcup FK \arrow[r, "i_2"'] \arrow[ur, phantom, "\ulcorner" very near end] & K \sqcup F(K \sqcup FK) \arrow[r, "i_3"'] \arrow[ur, phantom, "\ulcorner" very near end] & K \sqcup F(K \sqcup F(K \sqcup FK)) \arrow[r, "i_4"'] & \dots
\end{tikzcd}$$

exist (note that the top row is obtained from the bottom row by applying F). If the bottom row has a colimit K^* in $\mathcal{K}$ that is preserved by the functor $K \sqcup F(-) : \mathcal{K} \to \mathcal{K}$, then the canonical map $FK^* \to K^*$ induced by the vertical arrows in the above diagram realises K^* as the free F-algebra generated by K.

Corollary 4.3.1.8 (Adámek's initial algebra construction)**.** *Let $\emptyset$ be an initial object of $\mathcal{K}$. Construct objects I^θ and maps $j_\xi : I^\xi \to FI^\theta$ for $\xi \leq \theta$ by transfinite induction:*

- *Define $I^0 := \emptyset$, with i_0 the unique map $I^0 \to FI^0$.*
- *Given j_θ, define $I^{\theta+1} := FI^\theta$ and $j_{\theta+1} := Fj_\theta$.*
- *For a limit ordinal λ, given the maps $j_\theta : I^\theta \to FI^\theta = I^{\theta+1}$ for every $\theta < \lambda$, define $I^\lambda := \varinjlim_{\theta<\lambda} I^\theta$. Since $FI^\theta = I^{\theta+1}$, we then obtain a canonical map $j_\lambda : I^\lambda \to FI^\lambda$.*

Suppose for some limit ordinal λ that I^λ is well-defined, and $j_\lambda : I^\lambda \to FI^\lambda$ is invertible. Then, the pair $(I^\lambda, j_\lambda^{-1})$ defines an initial F-algebra.

Proof. Follows from Corollary 4.3.1.6 by taking $K = \emptyset$. $\square$

Remark 4.3.1.9. The corollary implies in particular that if

$$\emptyset \xrightarrow{!} F\emptyset \xrightarrow{F(!)} F^2\emptyset \xrightarrow{F^2(!)} F^3\emptyset \to \dots$$

admits a colimit I, and the canonical map $j : I \to FI$ is an equivalence, then (I, j^{-1}) is an initial F-algebra.

4.3.2 Free fixed points

In this subsection, we study the fixed points K of F, exhibited by an explicit choice of equivalence $K \xrightarrow{\sim} FK$.

Definition 4.3.2.1. Let $\mathbb{I}$ denote the nerve of the walking isomorphism defined in Notation 3.2.2.2. Then, define the category $\mathbf{Fix}(F)$ as the pullback

$$\begin{tikzcd} \mathbf{Fix}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{K}^{\mathbb{I}} \arrow[d, two heads] \\ \mathcal{K} \arrow[r, "(F,\mathrm{id})"'] & \mathcal{K} \times \mathcal{K} \end{tikzcd}$$

Note that the vertical map on the right is a categorical fibration, as it is induced by an inclusion of simplicial sets, meaning that the pullback is a homotopy pullback of ∞-categories.

Recall that we denote by $\mathcal{K}(F)$ the category of F-algebras, and $\mathcal{K}_{\mathrm{co}}(F) \simeq (\mathcal{K}^{\mathrm{op}}(F))^{\mathrm{op}}$ the category of F-coalgebras.

Proposition 4.3.2.2. *The category* $\mathbf{Fix}(F)$ *is equivalent to the full subcategory of* $\mathcal{K}(F)$ *spanned by the* F*-algebras with trivial (*i.e. *invertible) action. Dually,* $\mathbf{Fix}(F)$ *is also equivalent to the full subcategory of* $\mathcal{K}_{\mathrm{co}}(F)$ *spanned by the* F*-coalgebras with trivial coaction.*

Proof. Let $\mathbb{I}^{\natural} = \mathbb{I}^{\sharp}$ denote the walking isomorphism as a marked simplicial set, marked at every edge. This is fibrant in the cartesian model structure on marked simplicial sets. Let $\mathcal{K}_{\mathrm{co}}(F)^F$ denote the full subcategory of $\mathcal{K}_{\mathrm{co}}(F)$ spanned by the F-coalgebras with trivial coaction. We will establish that the forgetful functor $\mathbf{Fix}(F) \to \mathcal{K}_{\mathrm{co}}(F)$ restricts to an equivalence $\mathbf{Fix}(F) \to \mathcal{K}_{\mathrm{co}}(F)^F$. The statement regarding F-algebras is similar.

Let $\Delta[1]^{\sharp}$ denote the simplicial set $\Delta[1]$ marked at all edges. Then, either inclusion $\Delta[1]^{\sharp} \to \mathbb{I}^{\natural}$ is marked anodyne. In particular, one of these inclusions induces a trivial categorical fibration $\mathcal{K}^{\mathbb{I}} = \mathrm{Map}^{\flat}(\mathbb{I}^{\natural}, \mathcal{K}^{\natural}) \to \mathrm{Map}^{\flat}(\Delta[1]^{\sharp}, \mathcal{K}^{\natural})$ fitting in the diagram

$$\begin{tikzcd}
\mathbf{Fix}(F) \arrow[rr] \arrow[dr, dashed] \arrow[ddr, bend right] & & \mathcal{K}^{\mathbb{I}} \arrow[d, two heads, "\wr"] \\
& \mathcal{K}_{\mathrm{co}}(F)^F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Map}^{\flat}(\Delta[1]^{\sharp}, \mathcal{K}^{\natural}) \arrow[d, two heads] \\
& \mathcal{K} \arrow[r, "(\mathrm{id}, F)"'] & \mathcal{K} \times \mathcal{K}
\end{tikzcd}$$

The perimeter commutes for a suitably chosen inclusion $\Delta[1]^{\sharp} \to \mathbb{I}^{\natural}$, and is the definitional pullback square for $\mathbf{Fix}(F)$. As $\mathcal{K}^{\natural}$ is marked at the equivalences, the inner square is also a pullback square.

All of the vertical maps on the right are categorical fibrations, so the pullback diagrams are both homotopy pullback diagrams in the Joyal model structure on simplicial sets. In particular, since the corresponding objects of the two pullback diagrams are connected by categorical equivalences, it follows that the induced map $\mathbf{Fix}(F) \to \mathcal{K}_{\mathrm{co}}(F)^F$ is a categorical equivalence as well. $\square$

Remark 4.3.2.3. By Lemma 4.2.0.3, any initial F-algebra is then an initial object in $\mathbf{Fix}(F)$. Dually, any terminal F-coalgebra is a terminal object in $\mathbf{Fix}(F)$.

Definition 4.3.2.4. For a limit ordinal λ, denote by $\underline{\lambda}$ the nerve of the poset of ordinals $0 \leq \xi < \lambda$. Then, define a *λ-sequence* in $\mathcal{K}$ to be a functor $\underline{\lambda} \to \mathcal{K}$.

Say that the pair $(\mathcal{K}, F)$ is *compatible with λ-sequences* if:

- $\mathcal{K}$ is closed under colimits of θ-sequences for all limit ordinals $0 < \theta \leq \lambda$, and
- F preserves colimits of λ-sequences.

Note that $(\mathcal{K}, F)$ is automatically a compatible with λ-sequences for some regular cardinal $\lambda \gg 0$ if $\mathcal{K}$ is accessible, and F is an accessible functor. The remainder of this subsection provides a cursory study of $\mathbf{Fix}(F)$ in the case where $(\mathcal{K}, F)$ is compatible with λ-sequences for some limit ordinal λ.

The key observation is the following:

Theorem 4.3.2.5. *Suppose $(\mathcal{K}, F)$ is compatible with λ-sequences for some limit ordinal λ. Then, the fully faithful inclusion $\mathbf{Fix}(F) \hookrightarrow \mathcal{K}_{\mathrm{co}}(F)$ admits a left adjoint, realising $\mathbf{Fix}(F)$ as a reflective localisation of the category of F-coalgebras.*

Proof. Note that F induces a unital endofunctor on $\mathcal{K}_{\mathrm{co}}(F)$. Explicitly, the image under F of an F-coalgebra is another F-coalgebra, and we have a unit $\eta : \mathrm{Id} \Rightarrow F$ whose component on any coalgebra (C, ν) is its coaction $\eta_C = \nu$. Moreover, the full subcategory $\mathcal{K}_{\mathrm{co}}(F)^F$ of the F-fixed points in $\mathcal{K}_{\mathrm{co}}(F)$ is precisely the full subcategory of F-coalgebras with trivial coaction by definition. By Proposition 4.3.2.2, it therefore suffices to show that the inclusion $\mathcal{K}_{\mathrm{co}}(F)^F \hookrightarrow \mathcal{K}_{\mathrm{co}}(F)$ admits a left adjoint.

By Proposition 4.2.3.5, colimits of θ-sequences in $\mathcal{K}_{\mathrm{co}}(F)$ for $\theta \leq \lambda$ exist and are computed on the underlying objects in $\mathcal{K}$. In particular, for any F-coalgebra (C, ν), the colimit I_C of the λ-sequence

$$C \xrightarrow{\nu} FC \xrightarrow{F\nu} F^2C \xrightarrow{F^2\nu} F^3C \to \dots$$

in $\mathcal{K}_{\mathrm{co}}(F)$ exists, with coaction given by the canonical map $\varinjlim_n F(F^nC) \to F(\varinjlim_n F^nC)$. Since F preserves colimits of λ-sequences, this coaction is an equivalence. By Theorem 4.3.1.1, it follows that the functor $\mathrm{Hom}_{\mathcal{K}_{\mathrm{co}}(F)}(C, -)| : \mathcal{K}_{\mathrm{co}}(F)^F \to \mathcal{S}$ is corepresentable. Since this is true for any F-coalgebra (C, ν), it follows that the inclusion $\mathcal{K}_{\mathrm{co}}(F)^F \hookrightarrow \mathcal{K}_{\mathrm{co}}(F)$ admits a left adjoint, as desired. $\square$

Definition 4.3.2.6. Let $(\mathcal{K}, F)$ be compatible with λ-sequences for some limit ordinal λ. Denote the left adjoint of the inclusion $\mathbf{Fix}(F) \hookrightarrow \mathcal{K}_{\mathrm{co}}(F)$ by $I_{(-)} : \mathcal{K}_{\mathrm{co}}(F) \to \mathbf{Fix}(F)$. Call an F-coalgebra homomorphism F-*local* if its image under $I_{(-)}$ is an equivalence in $\mathbf{Fix}(F)$.

Example 4.3.2.7. If (C, ν) is an F-coalgebra, then the coaction ν trivially defines an F-coalgebra homomorphism $(C, \nu) \to (FC, F\nu)$. As F-coalgebra homomorphisms, all coactions are F-local.

We now provide a complete characterisation of the F-local morphisms:

Proposition 4.3.2.8. *Suppose $(\mathcal{K}, F)$ is compatible with λ-sequences for some limit ordinal λ. Let $I_{(-)} : \mathcal{K}_{\mathrm{co}}(F) \to \mathbf{Fix}(F)$ denote the left adjoint to the inclusion. Then, an F-coalgebra homomorphism $F : (C, \nu) \to (D, \mu)$ is F-local if and only if there exists a morphism $s : D \to I_C$ such that*

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d] & D \arrow[d] \arrow[dl, dashed, "s"] \\ I_C \arrow[r, "I_\varphi"'] & I_D \end{tikzcd}$$

commutes up to homotopy.

Proof. If φ is F-local, then $I_C \to I_D$ is an equivalence, which allows us to construct the morphism s. Conversely, any $s : D \to I_C$ induces a morphism $s_* : I_D \to I_{I_C} \simeq I_C$ fitting in the diagram

$$\begin{tikzcd} I_C \arrow[r, "\varphi_*"] \arrow[d, "\wr"'] & I_D \arrow[d, "\wr"] \arrow[dl, "s_*"'] \\ I_C \arrow[r, "\varphi_*"'] & I_D \end{tikzcd}$$

showing that s_* is a weak inverse of φ_*. $\square$

Corollary 4.3.2.9. *Suppose $(\mathcal{K}, F)$ is compatible with λ-sequences for some limit ordinal λ. Let $\varphi : (C, \nu) \to (D, \mu)$ be an F-coalgebra homomorphism. If there exists a morphism $s : D \to FC$ in $\mathcal{K}$ such that the diagram*

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\nu"'] & D \arrow[d, "\mu"] \arrow[dl, dashed, "s"] \\ FC \arrow[r, "F\varphi"'] & FD \end{tikzcd}$$

commutes up to homotopy, then φ is F-local.

Remark 4.3.2.10. By Proposition 4.3.2.2, we have a completely dual theory as well. In particular, if $\mathcal{K}$ is closed under limits of inverse λ-sequences (that is, functors $\underline{\lambda}^{\mathrm{op}} \to \mathcal{K}$), and F preserves these limits, then $\mathbf{Fix}(F)$ is a coreflective subcategory of $\mathcal{K}(F)$; that is, the fully faithful inclusion admits a right adjoint $T_{(-)} : \mathcal{K}(F) \to \mathbf{Fix}(F)$.

If we refer to an algebra homomorphism as F-*colocal* if its image under $T_{(-)}$ is an equivalence, then we have in particular that an algebra homomorphism $\varphi : (A, \alpha) \to (B, \beta)$ is F-colocal whenever we can find a map $s : FB \to A$ such that the diagram

$$\begin{tikzcd} FA \arrow[r, "F\varphi"] \arrow[d, "\alpha"'] & FB \arrow[d, "\beta"] \arrow[dl, dashed, "s"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

commutes up to homotopy.

We conclude this subsection with another universal property satisfied by the free fixed point I_C associated to an F-coalgebra C realised by studying F-algebras *relative* to C. More specifically, let (C, ν) be an F-coalgebra. Then, we can define an endofunctor F_C on the undercategory $\mathcal{K}_{C/}$ by the composite

$$F_C : \mathcal{K}_{C/} \xrightarrow{F} \mathcal{K}_{FC/} \xrightarrow{\nu^*} \mathcal{K}_{C/}$$

where ν^* acts by precomposition with the coaction $\nu : C \to FC$.

Lemma 4.3.2.11. *Given a category $\mathcal{C}$ and a morphism $f : x \to y$ in $\mathcal{C}$, the precomposition functor $\mathcal{C}_{y/} \xrightarrow{f^*} \mathcal{C}_{x/}$ creates colimits indexed by weakly contractible simplicial sets.*

Proof. We have a commutative triangle

$$\begin{tikzcd} \mathcal{C}_{y/} \arrow[rr, "f^*"] \arrow[dr] & & \mathcal{C}_{x/} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

where the downward maps create colimits indexed by weakly contractible simplicial sets by Lemma 3.3.2.4. □

Proposition 4.3.2.12. *Suppose $(\mathcal{K}, F)$ is compatible with λ-sequences for some limit ordinal λ. By Theorem 4.3.2.5, denote by $I_{(-)} : \mathcal{K}_{\mathrm{co}}(F) \to \mathbf{Fix}(F)$ the left adjoint to the inclusion. For any F-coalgebra (C, ν), the free fixed point I_C and the inverse of the induced equivalence $I_C \xrightarrow{\sim} FI_C$ define an initial F_C-algebra in $\mathcal{K}_{C/}$; that is, an initial object in $\mathcal{K}_{C/}(F_C)$.*

Proof. By assumption, $F : \mathcal{K} \to \mathcal{K}$ preserves colimits of λ-sequences. Therefore, the induced functor $F_C : \mathcal{K}_{C/} \to \mathcal{K}_{C/}$ preserves colimits of λ-sequences by Lemma 4.3.2.11. In particular, using Corollary 4.3.1.8, we can construct the initial F_C-algebra by Adámek's construction. The initial object in $\mathcal{K}_{C/}$ is given by the identity on C, so Adámek's construction builds the λ-sequence

$$C \xrightarrow{\nu} FC \xrightarrow{F\nu} F^2C \xrightarrow{F^2\nu} F^3C \to \dots$$

in $\mathcal{K}_{C/}$. By Lemma 3.3.2.4, this colimit can be calculated in $\mathcal{K}$, which is precisely the colimit defining the free fixed point I_C in Theorem 4.3.2.5. □

Chapter 5

Fixed Points of Enrichment

In Chapter 3, we defined the category of (n, r)-categories for $-2 \leq n \leq \infty$ and $0 \leq r \leq n+2$ in terms of polysimplicial sheaves satisfying the axioms of a complete Segal space. This offers a relatively concise model of these categories through internalisation. In this chapter, we study the relationship between higher categories and enriched categories, analogous to the development of strict higher categories in Section 2.1.

The theory of enrichment in a monoidal ∞-category $\mathcal{V}$ is extensively developed in [10]. In particular, they define a functorial construction of the ∞-category $\mathcal{V}\mathbf{Cat}$ of $\mathcal{V}$-enriched categories. If $\mathcal{V}$ is symmetric monoidal, then $\mathcal{V}\mathbf{Cat}$ inherits a symmetric monoidal structure analogous to that of [24, §1.4], which allows for the construction $\mathcal{V} \mapsto \mathcal{V}\mathbf{Cat}$ to be iterated. Note that if $\mathcal{V}$ is cartesian monoidal, then the tensor product on $\mathcal{V}\mathbf{Cat}$ is cartesian monoidal as well.

Proposition 5.0.0.1. *For $-2 \leq n \leq \infty$ and $0 \leq r \leq n+2$, we have an equivalence*

$$\mathbf{Cat}_{(n+1,r+1)} \simeq (\mathbf{Cat}_{(n,r)})\mathbf{Cat}$$

between $(n+1, r+1)$-categories and categories enriched in $\mathbf{Cat}_{(n,r)}$.

In particular, $\mathbf{Cat}_{(\infty,\infty)} \simeq (\mathbf{Cat}_{(\infty,\infty)})\mathbf{Cat}$.

Proof. The crux of this argument is [18, Theorem 7.18], which proves for every $\mathcal{S}$-distributor $\mathcal{Y}$ that the category $\mathbf{CSS}(\mathcal{Y})$ of complete Segal spaces over $\mathcal{Y}$ is equivalent to the category $\mathcal{Y}\mathbf{Cat}$ of categories enriched in $\mathcal{Y}$ with its cartesian monoidal tensor product.

By Theorem 3.3.4.5, it then follows that $\mathbf{Cat}_{(\infty,\infty)} \simeq (\mathbf{Cat}_{(\infty,\infty)})\mathbf{Cat}$. For finite $r \geq 0$, Corollary 3.3.4.2 likewise implies by induction that $\mathbf{Cat}_{(\infty,r+1)} \simeq (\mathbf{Cat}_{(\infty,r)})\mathbf{Cat}$.

The general case follows from [10, Theorem 6.1.8] by observing for any (∞, r)-category $\mathcal{C}$ that the suspension $\Sigma\mathcal{C}$ in the sense of Definition 3.2.1.5 satisfies the same universal property as the $\mathbf{Cat}_{(\infty,r)}$-enriched category $\Sigma\mathcal{C}$ in the sense of [10, Definition 4.3.21]. $\square$

Remark 5.0.0.2. This proposition demonstrates that our construction of $\mathbf{Cat}_{(n,r)}$ is equivalent to the category of (n,r)-categories defined in [10, §6.1] for all $-2 \leq n \leq \infty$ and finite $0 \leq r \leq n+2$.

This shows, in particular, that $\mathbf{Cat}_{(\infty,\infty)}$ is invariant under enrichment. Remark 3.3.4.6 moreover implies that $\mathbf{Cat}_{(\infty,\infty)}$ is a universal $\mathcal{S}$-distributor that is invariant under enrichment in their cartesian monoidal structure.

The goal of this chapter is to prove that $\mathbf{Cat}_{(\infty,\infty)}$ satisfies a much stronger universal property with respect to being invariant under enrichment; namely, that it is the initial locally presentable fixed point of enrichment. This is made precise in Theorem 5.2.0.1.

The approach is as follows. Enrichment defines an endofunctor on the large category $\mathbf{SymMon}_\infty$ of symmetric monoidal ∞-categories and symmetric monoidal functors between them, and moreover restricts to an endofunctor on the large category $\mathbf{SymMon}_\infty^{\mathbf{Pres}}$ of presentably symmetric monoidal ∞-categories and symmetric monoidal left adjoints between them. We prove in Proposition 5.2.1.4 that $\mathbf{Cat}_{(\infty,\infty)}$ is the initial $(-)\mathbf{Cat}$-algebra in $\mathbf{SymMon}_\infty^{\mathbf{Pres}}$.

Note that the universal property of $\mathbf{Cat}_{(\infty,\infty)}$ is restricted to the presentably symmetric monoidal categories. In Theorem 5.2.2.14, we show that the initial $(-)\mathbf{Cat}$-algebra in $\mathbf{SymMon}_\infty$ is given by the full subcategory of $\mathbf{Cat}_{(\infty,\infty)}$ spanned by *Noetherian* (∞,∞)-categories; that is, (∞,∞)-categories with a weak local finiteness condition.

5.1 The operadic approach to enrichment

In this section, we study the enrichment endofunctor on (presentably) symmetric monoidal categories. The main goal of the section is to prove that enrichment preserves a broad class of limits: namely, all limits of diagrams indexed by a weakly contractible simplicial set.

5.1.1 Nonsymmetric operads

In this subsection, we provide a brief overview of the necessary details regarding (generalised) nonsymmetric coloured operads of [10] and symmetric coloured operads of [28]. In particular, none of the content in this subsection is original.

Unless otherwise specified, operads are taken to refer to nonsymmetric coloured operads. For an explanation behind the definitions of operads and their connection to enrichment, see [10, §2].

Definition 5.1.1.1. A *categorical pattern* $\mathfrak{P}$ in the sense of [10, Definition 3.2.1] consists of a quasicategory $\mathcal{C}$, a family of diagrams $p_\alpha : K_\alpha^\triangleleft \to \mathcal{C}$, and a marking on $\mathcal{C}$ such that every edge in $K_\alpha^\triangleleft$ is sent to a marked edge of $\mathcal{C}$ via p_α.

A map of categorical patterns from $\mathfrak{P} = (\mathcal{C}, \{p_\alpha\})$to $\mathfrak{P}' = (\mathcal{C}', \{p'_\beta\})$ is a map $f : \mathcal{C} \to \mathcal{C}'$ of marked simplicial sets such that for every index α, there is an index β such that $f \circ p_\alpha = p'_\beta$.

Theorem 5.1.1.2. [28, Theorem B.0.20, Proposition B.2.9] *For a categorical pattern $\mathfrak{P} = (\mathcal{C}, \{p_\alpha : K_\alpha^\triangleleft \to \mathcal{C}\})$, there is a unique left proper combinatorial simplicial model structure on $\mathbf{sSet}^+_{/\mathcal{C}}$ such that the cofibrations are the morphisms of $\mathbf{sSet}^+$ whose underlying maps of simplicial sets are monomorphisms, and whose fibrant objects are those $\pi : X \to \mathcal{C}$ such that*

- *π is an inner fibration,*
- *every marked edge of $\mathcal{C}$ admits a π-cocartesian lift in X, and these π-cocartesian lifts of marked edges of $\mathcal{C}$ are precisely the marked edges of X,*
- *for every index α, the pullback $\pi_\alpha : X \times_{\mathcal{C}} K_\alpha^\triangleleft \to K_\alpha^\triangleleft$ along p_α is the cocartesian fibration associated to a limit cone $K_\alpha^\triangleleft \to \mathbf{Cat}_\infty$,*
- *for every index α and any section $s : K_\alpha^\triangleleft \to X \times_{\mathcal{C}} K_\alpha^\triangleleft$ of π_α, the composite diagram $K_\alpha^\triangleleft \xrightarrow{s} X \times_{\mathcal{C}} K_\alpha^\triangleleft \xrightarrow{\pi_\alpha} X$ is a π-limit cone.*

Denote this model structure by $\mathbf{sSet}^+_{\mathfrak{P}}$.

Moreover, given a map $f : \mathfrak{P} \to \mathfrak{P}'$ of categorical patterns, composition with f induces a left Quillen functor $f_! : \mathbf{sSet}^+_{\mathfrak{P}} \to \mathbf{sSet}^+_{\mathfrak{P}'}$.

Remark 5.1.1.3. The model structure induced by the trivial categorical pattern on $\Delta[0]$ (with the unique marking, and no diagrams are chosen) on $\mathbf{sSet}^+$ has as fibrant objects the ∞-categories marked at their equivalences. Moreover, the ∞-category underlying $\mathbf{sSet}^+$ is equivalent to the category $\mathbf{Cat}_\infty$ of small ∞-categories.

By [28, Remark B.2.5], every model category $\mathbf{sSet}^+_{\mathfrak{P}}$ induced by a categorical pattern $\mathfrak{P}$ is canonically enriched over $\mathbf{sSet}^+$ endowed with the above model structure. In particular, if X is $\mathfrak{P}$-fibrant, then $\mathrm{Hom}_{\mathfrak{P}}(-, X) : (\mathbf{sSet}^+_{\mathfrak{P}})^{\mathrm{op}} \to \mathbf{sSet}^+$ is a right Quillen functor.

Definition 5.1.1.4. Say that a morphism $\phi : [n] \to [m]$ in $\mathbf{\Delta}$ is *inert* if it is a subinterval inclusion, meaning $\phi(i) = \phi(0) + i$ for every $0 \leq i \leq n$. We then construct the following categories:

- Let $\mathfrak{O}^{\mathrm{gen}}$ denote the categorical pattern on $\mathbf{\Delta}^{\mathrm{op}}$ obtained by marking $\mathbf{\Delta}^{\mathrm{op}}$ at the inert morphisms, and choosing as diagrams the subcategory inclusions $G^\triangleleft_{[n]} \to \mathbf{\Delta}^{\mathrm{op}}$ for $n \geq 0$, where $G^\triangleleft_{[n]}$ is spanned by the objects $[0]$, $[1]$, $[n]$, and the inert morphisms between them. Then, define the category $\mathbf{Opd}^{\mathrm{gen}}_\infty$ of *generalised operads* to be the ∞-category underlying the model category $\mathbf{sSet}^+_{\mathfrak{O}^{\mathrm{gen}}}$.

- Let $\mathfrak{O}$ denote the categorical pattern on $\mathbf{\Delta}^{\mathrm{op}}$ obtained by marking the inert morphisms, and choosing as diagrams the subcategory inclusions $K_{[n]}^{\triangleleft} \to \mathbf{\Delta}^{\mathrm{op}}$ for $n \geq 0$, where $K_{[n]}^{\triangleleft}$ is spanned by the inert morphisms $[1] \to [n]$. Then, define the category $\mathbf{Opd}_\infty$ of *operads* to be the ∞-category underlying the model category $\mathbf{sSet}_{\mathfrak{O}}^+$.

- Let $\mathfrak{M}$ denote the categorical pattern on $\mathbf{\Delta}^{\mathrm{op}}$ obtained by marking all morphisms, and taking the diagrams $K_{[n]}^{\triangleleft} \to \mathbf{\Delta}^{\mathrm{op}}$ as above. Then, define the category $\mathbf{Mon}_\infty$ of *monoidal categories* as the ∞-category underlying the model category $\mathbf{sSet}_{\mathfrak{M}}^+$.

The identity functor on $\mathbf{\Delta}^{\mathrm{op}}$ induces maps of categorical patterns $\mathfrak{O}^{\mathrm{gen}} \to \mathfrak{O} \to \mathfrak{M}$, and thus left Quillen functors $\mathbf{sSet}_{\mathfrak{O}^{\mathrm{gen}}}^+ \to \mathbf{sSet}_{\mathfrak{O}}^+ \to \mathbf{sSet}_{\mathfrak{M}}^+$. Therefore, we have adjunctions

$$\begin{tikzcd} \mathbf{Opd}_\infty^{\mathrm{gen}} \arrow[r, bend left, "L_{\mathrm{gen}}"] \arrow[r, phantom, "\perp"] & \mathbf{Opd}_\infty \arrow[l, bend left] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathbf{Mon}_\infty \arrow[l, bend left, "(-)^{\otimes}"] \end{tikzcd}$$

Remark 5.1.1.5. The morphisms in $\mathbf{Mon}_\infty$ are the strong monoidal functors. Let $\mathbf{Mon}_\infty^{\mathrm{lax}}$ denote the full subcategory of $\mathbf{Opd}_\infty$ spanned by the image of $(-)^{\otimes} : \mathbf{Mon}_\infty \to \mathbf{Opd}_\infty$, then $\mathbf{Mon}_\infty^{\mathrm{lax}}$ is the category of monoidal small categories and *lax* monoidal functors.

Definition 5.1.1.6. Let $\mathbf{\Gamma}^{\mathrm{op}}$ denote the category of finite pointed sets, generated by the representatives $\langle n \rangle := \{*, 1, \ldots, n\}$ for $n \geq 0$. Call a map $\phi : \langle n \rangle \to \langle m \rangle$ *inert* if every $1 \leq j \leq m$ in $\langle m \rangle$ is the image of a unique element of $\langle n \rangle$. We now construct the following categories:

- Let $\mathfrak{O}^{\Sigma}$ denote the categorical pattern on $\mathbf{\Gamma}^{\mathrm{op}}$ obtained by marking $\mathbf{\Gamma}^{\mathrm{op}}$ at the inert morphisms, and choosing as diagrams the subcategory inclusions $K_{\langle n \rangle}^{\triangleleft} \to \mathbf{\Gamma}^{\mathrm{op}}$ for $n \geq 0$, where $K_{\langle n \rangle}^{\triangleleft}$ is spanned by the inert maps $\langle 1 \rangle \to \langle n \rangle$. Then, define the category $\mathbf{Opd}_\infty^{\Sigma}$ of *symmetric operads* to be the ∞-category underlying the model category $\mathbf{sSet}_{\mathfrak{O}^{\Sigma}}^+$.

- Fix a symmetric operad $\pi : \mathcal{O} \to \mathbf{\Gamma}^{\mathrm{op}}$ (that is, an object of $\mathbf{Opd}_\infty^{\Sigma}$). Let $\mathfrak{M}_{\mathcal{O}}$ denote the categorical pattern on $\mathcal{O}$ obtained by marking all morphisms, and choosing as diagrams all functors $K_{\langle n \rangle}^{\triangleleft} \to \mathcal{O}$ for $n \geq 0$ that associate to each inert $\langle 1 \rangle \to \langle n \rangle$ in $K_{\langle n \rangle}^{\triangleleft}$ a π-cocartesian lift in $\mathcal{O}$. Then, define the category $\mathbf{Mon}_{\mathcal{O}}$ of $\mathcal{O}$-*monoidal categories* to be the ∞-category underlying the model category $\mathbf{sSet}_{\mathfrak{M}_{\mathcal{O}}}^+$.

Remark 5.1.1.7. We have from [28, Construction 4.1.2.9] a functor $c : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Gamma}^{\mathrm{op}}$ defined by sending $[n]$ to the set of partitions ("cuts") of $[n]$ into at most two contiguous pieces. This functor defines a map of categorical patterns $c : \mathfrak{O} \to \mathfrak{O}^{\Sigma}$, and thus an adjunction $c_! : \mathbf{Opd}_\infty \rightleftarrows \mathbf{Opd}_\infty^{\Sigma} : c^*$, where the right adjoint is the forgetful functor.

Example 5.1.1.8. Consider the *commutative operad* $\mathbb{E}_\infty$ given by the identity $\mathbf{\Gamma}^{\mathrm{op}} \to \mathbf{\Gamma}^{\mathrm{op}}$, and let $\mathbf{SymMon}_\infty := \mathbf{Mon}_{\mathbb{E}_\infty}$ denote the category of *symmetric monoidal categories*. By definition, $\mathbf{SymMon}_\infty$ is the category associated to the model category given by the categorical pattern obtained by marking $\mathbf{\Gamma}^{\mathrm{op}}$ at all edges, and taking as diagrams the subcategory inclusions $K^{\triangleleft}_{\langle n \rangle} \to \mathbf{\Gamma}^{\mathrm{op}}$ for $n \geq 0$. In particular, the identity functor on $\mathbf{\Gamma}^{\mathrm{op}}$ induces a map of categorical patterns $\mathfrak{O}^{\Sigma} \to \mathfrak{M}_{\mathbb{E}_\infty}$ and thus an adjunction $\mathbf{Opd}^{\Sigma}_\infty \rightleftarrows \mathbf{SymMon}_\infty : (-)^{\otimes}$, where the right adjoint is a (non-full) inclusion.

Definition 5.1.1.9. For a monoidal ∞-category $\mathcal{V}^{\otimes}$ (viewed as an operad), we have from Remark 5.1.1.3 a right Quillen functor $\mathrm{Hom}_{\mathfrak{O}}(-, \mathcal{V}^{\otimes}) : (\mathbf{sSet}^{+}_{\mathfrak{O}})^{\mathrm{op}} \to \mathbf{sSet}^{+}$. Denote the associated functor by $\mathbf{Alg}_{(-)}(\mathcal{V}) : \mathbf{Opd}^{\mathrm{op}}_\infty \to \mathbf{Cat}_\infty$. In particular, for any operad $\mathcal{O}$, we have an category $\mathbf{Alg}_{\mathcal{O}}(\mathcal{V})$ of $\mathcal{O}$-algebras in $\mathcal{V}$. Denote the cartesian fibration associated to $\mathbf{Alg}_{(-)}(\mathcal{V})$ by $\mathbf{Alg}(\mathcal{V}) \to \mathbf{Opd}_\infty$. This is the *algebra fibration* associated to $\mathcal{V}$.

Remark 5.1.1.10. For a symmetric monoidal category $\mathcal{V}^{\otimes}$, we also have a right Quillen functor $\mathrm{Hom}_{\mathfrak{O}^{\Sigma}}(-, \mathcal{V}^{\otimes}) : (\mathbf{sSet}^{+}_{\mathfrak{O}^{\Sigma}})^{\mathrm{op}} \to \mathbf{sSet}^{+}$ that induces $\mathbf{Alg}^{\Sigma}_{(-)}(\mathcal{V}) : (\mathbf{Opd}^{\Sigma}_\infty)^{\mathrm{op}} \to \mathbf{Cat}_\infty$.

Example 5.1.1.11. Any category $\mathcal{C}$ with finite products induces a symmetric monoidal structure $\mathcal{C}^{\times}$ under the cartesian product by [28, Corollary 2.4.1.9]. In particular, we have a cartesian symmetric monoidal category $\mathbf{Cat}^{\times}_\infty$. Now, [28, Remark 2.4.2.6] establishes an equivalence $\mathbf{Alg}^{\Sigma}_{\mathcal{O}}(\mathbf{Cat}^{\times}_\infty) \simeq \mathbf{Mon}_{\mathcal{O}}$ for any symmetric operad $\mathcal{O}$. In particular, $\mathbf{SymMon}_\infty \simeq \mathbf{Alg}^{\Sigma}_{\mathbb{E}_\infty}(\mathbf{Cat}^{\times}_\infty)$ establishes an equivalence between symmetric monoidal categories and commutative algebra objects in $\mathbf{Cat}^{\times}_\infty$.

Remark 5.1.1.12. Let $\mathbb{E}_1$ denote the *associative operad* given in [28, Definition 4.1.1.3]. We have by [10, Corollary 4.3.12] that $\mathbf{Mon}_\infty \simeq \mathbf{Mon}_{\mathbb{E}_1} \simeq \mathbf{Alg}^{\Sigma}_{\mathbb{E}_1}(\mathbf{Cat}^{\times}_\infty)$.

Recall from [28, Remark 4.8.1.6] that $\mathbf{Pres}^{L}_\infty$ admits a symmetric monoidal tensor product with the universal property tht cocontinuous functors $\mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$ correspond to ordinary functors $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ that are cocontinuous in each variable. Using this monoidal structure, we can define locally presentable analogues of monoidal ∞-categories:

Definition 5.1.1.13. For a symmetric ∞-operad $\mathcal{O}$, define $\mathbf{Mon}^{\mathbf{Pres}}_{\mathcal{O}} := \mathbf{Alg}_{\mathcal{O}}(\mathbf{Pres}^{L,\otimes}_\infty)$ to be the ∞-category of *presentably $\mathcal{O}$-monoidal ∞-categories*. In particular, define the ∞-category of *presentably monoidal ∞-categories* to be $\mathbf{Mon}^{\mathbf{Pres}}_\infty := \mathbf{Mon}^{\mathbf{Pres}}_{\mathbb{E}_1}$, and the ∞-category of *presentably symmetric monoidal ∞-categories* to be $\mathbf{SymMon}^{\mathbf{Pres}}_\infty := \mathbf{Mon}^{\mathbf{Pres}}_{\mathbb{E}_\infty}$.

Lemma 5.1.1.14. *All of the forgetful functors in the diagram*

$$\begin{tikzcd}
\mathbf{SymMon}^{\mathbf{Pres}}_\infty \arrow[r] \arrow[d] & \mathbf{Mon}^{\mathbf{Pres}}_\infty \arrow[r] \arrow[d] & \mathbf{Pres}^{L}_\infty \arrow[d] \\
\mathbf{SymMon}_\infty \arrow[r] & \mathbf{Mon}_\infty \arrow[r] & \mathbf{Cat}_\infty
\end{tikzcd}$$

create limits.

Proof. Note that [28, Proposition 3.2.2.1] implies for any symmetric monoidal category $\mathcal{V}^{\otimes}$ and any symmetric operad $\mathcal{O}$ that the forgetful functor $\mathbf{Alg}^{\Sigma}_{\mathcal{O}}(\mathcal{V}) \to \mathcal{V}$ creates limits. Therefore, all of the functors on the bottom row of the above diagram create limits by noting that $\mathbf{Mon}_{\infty} \simeq \mathbf{Alg}^{\Sigma}_{\mathbb{E}_1}(\mathbf{Cat}^{\times}_{\infty})$ and

$$\begin{aligned}\mathbf{SymMon}_{\infty} &\simeq \mathbf{Alg}^{\Sigma}_{\mathbb{E}_{\infty}}(\mathbf{Cat}^{\times}_{\infty}) \simeq \mathbf{Alg}^{\Sigma}_{\mathbb{E}_{\infty}\otimes\mathbb{E}_1}(\mathbf{Cat}^{\times}_{\infty}) \\ &\simeq \mathbf{Alg}^{\Sigma}_{\mathbb{E}_{\infty}}(\mathbf{Alg}^{\Sigma}_{\mathbb{E}_1}(\mathbf{Cat}^{\times}_{\infty})^{\times}) \simeq \mathbf{Alg}^{\Sigma}_{\mathbb{E}_{\infty}}(\mathbf{Mon}^{\times}_{\infty})\end{aligned}$$

using the closed symmetric monoidal structure on symmetric operads described in [28, §2.2.5, §3.2.4], and the Dunn Additivity Theorem $\mathbb{E}_{\infty} \otimes \mathbb{E}_1 \simeq \mathbb{E}_1$. The fact that the functors on the top row create limits follows similarly.

That the vertical functors create limits follows by [26, Proposition 5.5.3.13]. □

5.1.2 Continuity of enrichment

The purpose of this subsection is to describe the construction of the category $\mathcal{V}\mathbf{Cat}$ of $\mathcal{V}$-enriched categories from a monoidal category $\mathcal{V}$, and prove that this functorial construction preserves weakly contractible limits.

Definition 5.1.2.1. For a space S, let $\mathbf{\Delta}^{\mathrm{op}}_S \to \mathbf{\Delta}^{\mathrm{op}}$ be the cocartesian fibration associated to the functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}_{\infty}$ mapping $[n] \mapsto S^{\times(n+1)}$, where the degeneracies are given by diagonal functors, and faces by projections. As described in [10, §4.1], this construction defines a functor $\mathbf{\Delta}^{\mathrm{op}}_{(-)} : \mathcal{S} \to \mathbf{Opd}^{\mathrm{gen}}_{\infty}$.

For a monoidal category $\mathcal{V}^{\otimes}$, define the category $\mathbf{Alg}_{\mathrm{cat}}(\mathcal{V})$ of $\mathcal{V}$-*categorical algebras* as the pullback

$$\begin{tikzcd} \mathbf{Alg}_{\mathrm{cat}}(\mathcal{V}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Alg}(\mathcal{V}) \arrow[d] \\ \mathcal{S} \arrow[r, "L_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_{(-)}"'] & \mathbf{Opd}_{\infty} \end{tikzcd}$$

where the vertical map on the right is the algebra fibration of Definition 5.1.1.9. In particular, a $\mathcal{V}$-categorical algebra with space of objects S is precisely a map of generalised operads $\mathcal{C} : \mathbf{\Delta}^{\mathrm{op}}_S \to \mathcal{V}^{\otimes}$.

Definition 5.1.2.2. Fix a monoidal category $\mathcal{V}^{\otimes}$. For a space S, define the *trivial* $\mathcal{V}$-*category* $E^{\mathcal{V}}_S$ on S to be the composite $\mathbf{\Delta}^{\mathrm{op}}_S \to \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathbf{B}\mathbb{1}} \mathcal{V}^{\otimes}$, where $\mathbf{B}\mathbb{1}$ is the delooping of the tensor unit of $\mathcal{V}^{\otimes}$ viewed as a monoid (see [10, Proposition 3.1.18]). In particular, let $E^1 := E^{\mathcal{V}}_{\{0,1\}}$ be the walking $\mathcal{V}$-enriched equivalence, and $E^0 := E^{\mathcal{V}}_{\{0\}} = \mathbf{B}\mathbb{1}$.

Say that a $\mathcal{V}$-categorical algebra $\mathcal{C}$ is a *$\mathcal{V}$-category* if it is local with respect to the canonical map $E^1 \to E^0$. Then, define the category $\mathcal{V}\mathbf{Cat}$ of $\mathcal{V}$-categories to be the full subcategory of $\mathbf{Alg}_{\mathrm{cat}}(\mathcal{V})$ spanned by the $\mathcal{V}$-categories. By [10, Corollary 5.7.6], this construction defines a functor $(-)\mathbf{Cat} : \mathbf{Mon}_\infty^{\mathrm{lax}} \to \mathbf{Cat}_\infty$.

Remark 5.1.2.3. By [10, Theorem 5.6.6], the category $\mathcal{V}\mathbf{Cat}$ is precisely the localisation of $\mathbf{Alg}_{\mathrm{cat}}(\mathcal{V})$ at the $\mathcal{V}$-functors that are fully faithful and essentially surjective.

Remark 5.1.2.4. The locality with respect to $E^1 \to E^0$ asserts that the $\mathcal{V}$-categorical algebra is *Rezk-complete*, completely analogous to Lemma 3.3.4.1. In fact, this analogy is made precise in [18, Proposition 7.16].

Proposition 5.1.2.5. [10, Corollary 5.7.12, Proposition 5.7.16] *If $\mathcal{V}$ is a symmetric monoidal category, then $\mathcal{V}\mathbf{Cat}$ admits a symmetric tensor product as well. If $\mathcal{V}$ is moreover presentably symmetric monoidal, then so is $\mathcal{V}\mathbf{Cat}$. In particular, the enrichment functor canonically restricts to an endofunctor on $\mathbf{SymMon}_\infty$, and restricts further to an endofunctor on $\mathbf{SymMon}_\infty^{\mathbf{Pres}}$ also.*

Remark 5.1.2.6. The tensor product above is a homotopy-coherent generalisation of the classical tensor product of enriched 1-categories described in [24, §1.4].

The remainder of this section is dedicated to proving the following result:

Theorem 5.1.2.7. *Suppose $(\mathcal{V}_i)_{i\in K}$ is a diagram in $\mathbf{Mon}_\infty$ indexed by a weakly contractible simplicial set K. If each of the functors $F_{i,j} : \mathcal{V}_i \to \mathcal{V}_j$ induces natural equivalences*

$$\mathrm{Hom}_{\mathcal{V}_i}(\mathbb{1}_{\mathcal{V}_i}, -) \Rightarrow \mathrm{Hom}_{\mathcal{V}_j}(\mathbb{1}_{\mathcal{V}_j}, F(-))$$

for all edges $i \to j$ in K, then the induced map

$$\left(\varprojlim_{i\in K} \mathcal{V}_i\right)\mathbf{Cat} \to \varprojlim_{i\in K} (\mathcal{V}_i\mathbf{Cat})$$

is an equivalence.

In order to prove Theorem 5.1.2.7, we will piece through the construction of the enrichment functor, and study the limits preserved at each step.

Lemma 5.1.2.8. *Let $\mathcal{C}, \mathcal{D}$ be categories, and $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Cat}_\infty$ a functor. For a simplicial set K, suppose $\mathcal{D}$ has all K-indexed limits, and that $F_c : \mathcal{D} \to \mathbf{Cat}_\infty$ preserves these limits for all $c \in \mathcal{C}_0$. Then, the corresponding functor $\mathcal{D} \to \mathbf{Cat}_{\infty,/\mathcal{C}}$ preserves K-indexed limits.*

Proof. Consider the adjunct functor $\mathcal{D} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_\infty)$. Since limits in functor categories are computed pointwise by [26, Corollary 5.1.2.3], the assumptions of the lemma imply that this adjunct functor preserves K-indexed limits. The desired functor is the composite $\mathcal{D} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_\infty) \to \mathbf{Cat}_{\infty,/\mathcal{C}}$ of this adjunct with unstraightening, the latter of which is a right adjoint by [26, Theorem 3.1.5.1(A0)]. □

Corollary 5.1.2.9. *The functor* $\mathbf{Alg}_{\mathrm{cat}} : \mathbf{Mon}_\infty \to \mathbf{Cat}_{\infty,/\mathcal{S}}$ *preserves all limits.*

Proof. Applying Lemma 5.1.2.8 to $\mathrm{Hom}_{\mathbf{Opd}_\infty}(-,-) : \mathbf{Opd}_\infty^{\mathrm{op}} \times \mathbf{Opd}_\infty \to \mathbf{Cat}_\infty$, we see that the corresponding functor $\mathbf{Opd}_\infty \to \mathbf{Cat}_{\infty,/\mathbf{Opd}_\infty}$ preserves all limits. Note that the restriction of this functor to $\mathbf{Mon}_\infty^{\mathrm{lax}}$ is precisely $\mathbf{Alg}(-) : \mathbf{Mon}_\infty^{\mathrm{lax}} \to \mathbf{Cat}_{\infty,/\mathbf{Opd}_\infty}$. Since $\mathbf{Mon}_\infty \to \mathbf{Opd}_\infty$ is a right adjoint, it follows that $\mathbf{Alg} : \mathbf{Mon}_\infty \to \mathbf{Opd}_\infty \to \mathbf{Cat}_{\infty,/\mathbf{Opd}_\infty}$ is continuous. Observing that $\mathbf{Alg}_{\mathrm{cat}}$ is recovered as the composite

$$\mathbf{Mon}_\infty \xrightarrow{\mathbf{Alg}} \mathbf{Cat}_{\infty,/\mathbf{Opd}_\infty} \xrightarrow{(L_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_{(-)})^*} \mathbf{Cat}_{\infty,/\mathcal{S}}$$

and base change is a right adjoint, the result follows. □

Corollary 5.1.2.10. *The functor* $\mathbf{Alg}_{\mathrm{cat}} : \mathbf{Mon}_\infty \to \mathbf{Cat}_\infty$ *preserves limits of diagrams indexed by weakly contractible simplicial sets.*

Proof. Follows by combining Corollary 5.1.2.9 with the dual of Lemma 3.3.2.4. □

Proof of Theorem 5.1.2.7. Let $(\mathcal{V}_i)_{i\in K}$ be a weakly contractible diagram of monoidal categories where each $F_{i,j} : \mathcal{V}_i \to \mathcal{V}_j$ induces a natural equivalence

$$\mathrm{Hom}_{\mathcal{V}_i}(\mathbb{1}_{\mathcal{V}_i}, -) \Rightarrow \mathrm{Hom}_{\mathcal{V}_j}(\mathbb{1}_{\mathcal{V}_j}, F(-))$$

of spaces.

Let $\mathcal{V} := \varprojlim_i \mathcal{V}_i$ in $\mathbf{Mon}_\infty$ with strongly monoidal projections $F_i : \mathcal{V} \to \mathcal{V}_i$. Then, $\mathrm{Hom}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, -) \simeq \varprojlim_i \mathrm{Hom}_{\mathcal{V}_i}(\mathbb{1}_{\mathcal{V}_i}, F_i(-))$ is a limit of an essentially constant diagram, by assumption, and therefore each $F_i : \mathcal{V} \to \mathcal{V}_i$ induces a natural equivalence $\mathrm{Hom}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, -) \Rightarrow \mathrm{Hom}_{\mathcal{V}_i}(\mathbb{1}_{\mathcal{V}_i}, F_i(-))$.

Recall that $\mathcal{V}\mathbf{Cat}$ is the full subcategory of $\mathbf{Alg}_{\mathrm{cat}}(\mathcal{V})$ spanned by the objects that are local with respect to the morphism $s_{\mathcal{V}} : E^1 \to E^0$. Note for any space S that the image of the trivial $\mathcal{V}$-category $E_S^{\mathcal{V}}$ under the projection $\mathcal{V} \to \mathcal{V}_i$ is precisely $E_S^{\mathcal{V}_i}$. Indeed, since $\mathcal{V} \to \mathcal{V}_i$ is strongly monoidal, it preserves the tensor unit of $\mathcal{V}$, and so the composite $\mathbf{\Delta}_S^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathbf{B}\mathbb{1}_{\mathcal{V}}} \mathcal{V}^{\otimes} \to \mathcal{V}_i^{\otimes}$ is equivalent to $E_S^{\mathcal{V}_i}$.

Moreover, since F_i induces a natural equivalence $\mathrm{Hom}_{\mathcal{V}_i}(\mathbb{1}_{\mathcal{V}_i}, -) \Rightarrow \mathrm{Hom}_{\mathcal{V}_j}(\mathbb{1}_{\mathcal{V}_j}, F(-))$, it follows that the induced functor $F_{i,*} : \mathbf{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathbf{Alg}_{\mathrm{cat}}(\mathcal{V}_i)$ induces natural equivalences $\mathrm{Hom}_{\mathrm{cat}}(E_S^{\mathcal{V}}, -) \Rightarrow \mathrm{Hom}_{\mathrm{cat}}(E_S^{\mathcal{V}_i}, F_{i,*}(-))$ for every S. In particular, each $F_{i,*}$ sends $\mathcal{V}$-categories to $\mathcal{V}_i$-categories.

If $\mathcal{C}$ is a categorical $\mathcal{V}$-algebra, denote by $\mathcal{C}_i := F_i(\mathcal{C})$ the categorical $\mathcal{V}_i$-algebra induced by the canonical projection $F_i : \mathcal{V} \to \mathcal{V}_i$. We have an equivalence $\mathbf{Alg}_{\text{cat}}(\mathcal{V}) \simeq \varprojlim_i \mathbf{Alg}_{\text{cat}}(\mathcal{V}_i)$ by Corollary 5.1.2.10. Therefore, we have for all $\mathcal{C}, \mathcal{D}$ in $\mathbf{Alg}_{\text{cat}}(\mathcal{V})$ that $\text{Hom}_{\text{cat}}(\mathcal{D}, \mathcal{C}) \simeq \varprojlim_i \text{Hom}_{\text{cat}}(\mathcal{D}_i, \mathcal{C}_i)$. In particular, if $\mathcal{C}$ is a categorical $\mathcal{V}$-algebra such that $\mathcal{C}_i$ is a $\mathcal{V}_i$-category for every i, then the homotopy equivalences $\text{Hom}_{\text{cat}}(E_i^0, \mathcal{C}_i) \to \text{Hom}_{\text{cat}}(E_i^1, \mathcal{C}_i)$ induce a homotopy equivalence $\varprojlim_i \text{Hom}_{\text{cat}}(E_i^0, \mathcal{C}_i) \to \varprojlim_i \text{Hom}_{\text{cat}}(E_i^1, \mathcal{C}_i)$. From the above discussion, this is precisely the map $\text{Hom}_{\text{cat}}(E^0, \mathcal{C}) \to \text{Hom}_{\text{cat}}(E^1, \mathcal{C})$ induced by $s_{\mathcal{V}}$. Therefore, $\mathcal{C}$ is a $\mathcal{V}$-category.

Altogether, this proves that $\mathcal{C} \in \mathbf{Alg}_{\text{cat}}(\mathcal{V})$ is a $\mathcal{V}$-category if and only if every projection $\mathcal{C}_i \in \mathbf{Alg}_{\text{cat}}(\mathcal{V}_i)$ is a $\mathcal{V}_i$-category. In other words, the equivalence of categories $\mathbf{Alg}_{\text{cat}}(\mathcal{V}) \simeq \varprojlim_i \mathbf{Alg}_{\text{cat}}(\mathcal{V}_i)$ restricts to an equivalence $\mathcal{V}\mathbf{Cat} \simeq \varprojlim_i (\mathcal{V}_i\mathbf{Cat})$. $\square$

5.2 Universal fixed points

Let $\mathbf{Alg}_{\mathbf{Enr}}$ denote the category of algebras in $\mathbf{SymMon}_\infty$ for the endofunctor $(-)\mathbf{Cat}$, and define $\mathbf{Fix}_{\mathbf{Enr}}$ to be the full subcategory of $\mathbf{Alg}_{\mathbf{Enr}}$ spanned by those enrichment algebras $(\mathcal{V}, \tau)$ where the action $\tau : \mathcal{V}\mathbf{Cat} \to \mathcal{V}$ is an equivalence. Analogously, let $\mathbf{Alg}_{\mathbf{Enr}}^{\mathbf{Pres}}$ denote the category of enrichment algebras in $\mathbf{SymMon}_\infty^{\mathbf{Pres}}$, and $\mathbf{Fix}_{\mathbf{Enr}}^{\mathbf{Pres}}$ the full subcategory of $\mathbf{Alg}_{\mathbf{Enr}}^{\mathbf{Pres}}$ spanned by the locally presentable enrichment algebras with trivial action.

The goal of this section is to prove our main result:

Theorem 5.2.0.1. $\mathbf{Cat}_{(\infty,\infty)}$ *underlies an initial object of* $\mathbf{Fix}_{\mathbf{Enr}}^{\mathbf{Pres}}$.

Proof. Follows from Proposition 5.2.1.4. $\square$

Remark 5.2.0.2. Note that the fixed point structure of $\mathbf{Cat}_{(\infty,\infty)}$ in Theorem 5.2.0.1 coincides with the equivalence provided in the proof of Proposition 5.0.0.1, which is induced by Theorem 3.3.4.5.

To prove this result, we use the machinery developed in Chapter 4.

5.2.1 The initial presentable enrichment algebra

Proposition 2.1.1.13 suggests that $\mathbf{Cat}_{(\infty,\infty)}$ is analogous to the category $\omega\mathbf{Cat}^+$ of marked strict ω-categories from Definition 2.1.1.10. To strengthen this analogy, we need an appropriate truncation map $\tau_{\leq(n,r)} : \mathbf{Cat}_{(n+1,r+1)} \to \mathbf{Cat}_{(n,r)}$ analogous to that of Definition 2.1.1.7. By Remark 3.2.3.2, however, the inclusion $\mathbf{Cat}_{(n,r)} \hookrightarrow \mathbf{Cat}_{(n+1,r+1)}$ does not admit a right adjoint in general. Therefore, we construct $\tau_{\leq(n,r)}$ more directly.

Notation 5.2.1.1. For $-2 \leq n \leq \infty$ and $0 \leq r \leq n+2$, let

$$\tau_{\leq(n,r)} : \mathbf{Cat}_{(n+1,r+1)} \xrightarrow{\kappa_{\leq r}} \mathbf{Cat}_{(n+1,r)} \xrightarrow{\pi_{\leq n}} \mathbf{Cat}_{(n,r)}$$

Lemma 5.2.1.2. *For $-2 \leq n \leq \infty$ and $0 \leq r < n+2$, we have a commutative square*

$$\begin{tikzcd} \mathbf{Cat}_{(n+1,r+1)} \arrow[r, "\kappa_{\leq r}"] \arrow[d, "\pi_{\leq n}"'] & \mathbf{Cat}_{(n+1,r)} \arrow[d, "\pi_{\leq n}"] \\ \mathbf{Cat}_{(n,r+1)} \arrow[r, "\kappa_{\leq r}"'] & \mathbf{Cat}_{(n,r)} \end{tikzcd}$$

Proof. By iterating $(-)\mathbf{Cat}$, it suffices to prove that the square

$$\begin{tikzcd} \mathbf{Cat}_{(n+1,1)} \arrow[r, "\kappa_{\leq 0}"] \arrow[d, "\pi_{\leq n}"'] & \mathbf{Grpd}_{n+1} \arrow[d, "\pi_{\leq n}"] \\ \mathbf{Cat}_{(n,1)} \arrow[r, "\kappa_{\leq 0}"'] & \mathbf{Grpd}_{n} \end{tikzcd}$$

commutes for $-2 \leq n \leq \infty$. Note that the localisations are trivial when $n = \infty$, so assume $n < \infty$.

The $\pi_{\leq n}$ in this case can be described using the functors $h_n : \mathbf{Cat}_\infty \to \mathbf{Cat}_{(n,1)}$ defined in [26, Proposition 2.3.4.12]. Explicitly, for a simplicial set K, let $[K, \mathcal{C}]_n$ be the subset of $\mathrm{Map}(\mathrm{sk}^n K, \mathcal{C})$ consisting of restrictions of maps $\mathrm{sk}^{n+1} K \to \mathcal{C}$. Then, the k-cells of $h_n\mathcal{C}$ are homotopy classes of maps in $[\Delta[k], \mathcal{C}]_n$ relative to $\mathrm{sk}^{n-1}(\Delta[k])$.

For an $(n+1,1)$-category $\mathcal{C}$, then we have $\pi_{\leq n}\mathcal{C} = h_n\mathcal{C}$. Note that the k-cells of $\kappa_{\leq 0} h_n \mathcal{C}$ are given by the k-cells of $h_n\mathcal{C}$ whose edges are all invertible in $h_n\mathcal{C}$. On the other hand, the k-cells of $h_n\kappa_{\leq 0}\mathcal{C}$ are given by homotopy equivalence classes of maps in $[\Delta[k], \kappa\mathcal{C}]_n$ relative to $\mathrm{sk}^{n-1}(\Delta[k])$, where k-cells of $\kappa_{\leq 0}\mathcal{C}$ are the k-cells of $\mathcal{C}$ whose edges are all invertible in $\mathcal{C}$. If $\mathcal{C}$ is an $(n+1,1)$-category, then all higher morphisms are invertible, which implies that an edge of $\mathcal{C}$ is invertible if and only if its image in $h_n\mathcal{C}$ is invertible. Therefore, both $\kappa_{\leq 0} h_n\mathcal{C}$ and $h_n\kappa_{\leq 0}\mathcal{C}$ describe the same simplicial set: the k-cells are homotopy classes of maps in $[\Delta[k], \mathcal{C}]_n$ whose edges are all invertible in $\mathcal{C}$. ∎

Proposition 5.2.1.3. *For all $0 \leq n \leq \infty$ and finite $0 \leq r \leq n$, the limit*

$$\mathbf{Enr}^\infty(\mathbf{Cat}_{(n,r)}, \tau_{\leq (n,r)}) := \varprojlim \left(\cdots \to \mathbf{Cat}_{(n+2,r+2)} \xrightarrow{\tau_{\leq(n+1,r+1)}} \mathbf{Cat}_{(n+1,r+1)} \xrightarrow{\tau_{\leq(n,r)}} \mathbf{Cat}_{(n,r)} \right)$$

is equivalent to $\mathbf{Cat}_{(\infty,\infty)}$.

Proof. Note first that $\mathbf{Enr}^\infty(\mathbf{Cat}_{(n,r)}, \tau_{\leq(n,r)}) \simeq \mathbf{Enr}^\infty(\mathbf{Grpd}_{n-r}, \tau_{\leq(n-r,0)})$, so we may assume without loss of generality that $r = 0$. By Lemma 5.2.1.2, the diagram

$$\begin{tikzcd} \mathbf{Cat}_{(n+2,1)} \arrow[r, "\pi_{\leq n+1}"] \arrow[d, "\kappa_{\leq 0}"] \arrow[dd, bend right=60, "\tau_{\leq(n+1,0)}"'] & \mathbf{Cat}_{(n+1,1)} \arrow[d, "\kappa_{\leq 0}"] \arrow[ddl, "\kappa_{\leq 0}"] \arrow[dd, bend left=60, "\tau_{\leq(n,0)}"] \\ \mathbf{Grpd}_{n+2} \arrow[d, "\pi_{\leq n+1}"] & \mathbf{Grpd}_{n+1} \arrow[d, "\pi_{\leq n}"] \\ \mathbf{Grpd}_{n+1} \arrow[r, "\pi_{\leq n}"'] & \mathbf{Grpd}_{n} \end{tikzcd}$$

commutes for every $0 \leq n < \infty$. Although the pair $(\mathbf{SymMon}_\infty, (-)\mathbf{Cat})$ is not necessarily compatible with inverse ω-sequences in the sense of Remark 4.3.2.10, the commutativity of the above square implies through the argument proving Proposition 4.3.2.8 that $\mathbf{Enr}^\infty(\mathbf{Grpd}_n, \tau_{\leq(n,0)}) \simeq \mathbf{Enr}^\infty(\mathbf{Grpd}_{n+1}, \tau_{\leq(n+1,0)})$ for every $0 \leq n < \infty$. The proposition now follows from the fact that $\mathbf{Grpd}_\infty \simeq \varprojlim_n \mathbf{Grpd}_n$, noting that the truncation map $\tau_{\leq 0} : \mathbf{Cat}_{(n,1)} \to \mathbf{Grpd}_n$ reduces to $\kappa_{\leq 0}$ when $n = \infty$. Indeed, the equivalence follows from the fact that Postnikov towers converge in $\mathbf{Grpd}_\infty$; see [26, Remark 5.5.6.24]. $\square$

We now prove that $\mathbf{Cat}_{(\infty,\infty)}$ enjoys a universal property with respect to enrichment in the presentable setting.

Proposition 5.2.1.4. *The category $\mathbf{Cat}_{(\infty,\infty)}$ defines an initial object in $\mathbf{Alg}^{\mathbf{Pres}}_{\mathbf{Enr}}$, the category of algebras for the endofunctor $(-)\mathbf{Cat}$ over $\mathbf{SymMon}^{\mathbf{Pres}}_\infty$.*

Proof. By definition, $\mathbf{Cat}_{(\infty,\infty)}$ is given as the limit of categories

$$\mathbf{Cat}_{(\infty,\infty)} := \varprojlim \left(\cdots \to \mathbf{Cat}_{(\infty,3)} \xrightarrow{\kappa_{\leq 2}} \mathbf{Cat}_{(\infty,2)} \xrightarrow{\kappa_{\leq 1}} \mathbf{Cat}_{(\infty,1)} \xrightarrow{\kappa_{\leq 0}} \mathbf{Cat}_{(\infty,0)} \right)$$

where every $\kappa_{\leq n}$ is a right adjoint. By [26, Theorem 5.5.3.18], this limit can be computed in $\mathbf{Pres}^R_\infty$. With [26, Corollary 5.5.3.4], we can therefore equivalently calculate $\mathbf{Cat}_{(\infty,\infty)}$ as the colimit of left adjoints

$$\mathbf{Cat}_{(\infty,\infty)} \simeq \varinjlim \left(\mathbf{Cat}_{(\infty,0)} \subseteq \mathbf{Cat}_{(\infty,1)} \subseteq \mathbf{Cat}_{(\infty,2)} \subseteq \cdots \right) \tag{5.1}$$

in $\mathbf{Pres}^L_\infty$.

Now, the forgetful functor $\mathbf{SymMon}^{\mathbf{Pres}}_\infty := \mathbf{Alg}^{\Sigma}_{\mathbb{E}_\infty}(\mathbf{Pres}^{L,\otimes}_\infty) \to \mathbf{Pres}_\infty$ creates sifted colimits by [28, Corollary 3.2.3.2]. Therefore, we obtain the presentably symmetric monoidal category $\mathbf{Cat}_{(\infty,\infty)}$ as the colimit (5.1) computed in $\mathbf{SymMon}^{\mathbf{Pres}}_\infty$.

To see that this colimit of left adjoints is preserved by $(-)\mathbf{Cat}$, it suffices to show the preservation of the corresponding limit of right adjoints. For (∞,r)-categories $\mathcal{C}, \mathcal{D}$, the space $\mathrm{Hom}_{\mathbf{Cat}_{(\infty,r)}}(\mathcal{C}, \mathcal{D})$ is the underlying groupoid of functors $\mathcal{C} \to \mathcal{D}$. In particular, $\mathrm{Hom}_{\mathbf{Cat}_{(\infty,r)}}(*, \mathcal{C}) \simeq \kappa_{\leq 0}\mathcal{C}$. This implies that $\kappa_{\leq r} : \mathbf{Cat}_{(\infty,r+1)} \to \mathbf{Cat}_{(\infty,r)}$ induces a natural equivalence $\mathrm{Hom}_{\mathbf{Cat}_{(\infty,r+1)}}(*, -) \Rightarrow \mathrm{Hom}_{\mathbf{Cat}_{(\infty,r)}}(*, \kappa_{\leq r}(-))$ of spaces. Therefore, Theorem 5.1.2.7 ensures that the limit of right adjoints is preserved by $(-)\mathbf{Cat}$, as desired.

By [10, Remark 3.1.25], the initial object of $\mathbf{SymMon}^{\mathbf{Pres}}_\infty$ is $\mathbf{Cat}_{(\infty,0)}$ with its cartesian monoidal structure. Therefore, the colimit (5.1) in $\mathbf{SymMon}^{\mathbf{Pres}}_\infty$ is precisely Adámek's construction of an initial algebra for $(-)\mathbf{Cat}$, and the construction terminates after countably many steps, so the proposition follows by Corollary 4.3.1.8. $\square$

5.2.2 Noetherian (∞, ∞)-categories

Proposition 5.2.1.4 establishes that $\mathbf{Cat}_{(\infty,\infty)}$ is only the initial fixed point of enrichment among the presentably monoidal categories. It is natural, then, to wonder what the initial algebra for the endofunctor $(-)\mathbf{Cat}$ is over arbitrary symmetric monoidal categories. The purpose of this subsection is to address this curiosity, which in turn resolves [11, Conjecture 3.4.3].

Proposition 5.2.2.1. *Adámek's construction of an initial algebra for* $(-)\mathbf{Cat}$ *over the category* $\mathbf{SymMon}_\infty$ *does not terminate after* ω *steps.*

Proof. The initial object of $\mathbf{SymMon}_\infty$ is the one-object category with its unique symmetric tensor product, which is equivalent to $\mathbf{Grpd}_{-2}$. In particular, after ω steps, Adámek's construction produces the colimit

$$\mathbf{Cat}_{<\omega} := \varinjlim \left(\mathbf{Cat}_{(-2,0)} \subseteq \mathbf{Cat}_{(-1,1)} \subseteq \mathbf{Cat}_{(0,2)} \subseteq \mathbf{Cat}_{(1,3)} \subseteq \cdots\right) = \bigcup_{\substack{0\le n<\infty \\ r\ge 0}} \mathbf{Cat}_{(n,r)}$$

which is the category of finite-dimensional higher categories.

On the other hand, the objects of $(\mathbf{Cat}_{<\omega})\mathbf{Cat}$ are the (∞,∞)-categories $\mathcal{C}$ that are locally finite-dimensional, in the sense that $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is finite-dimensional for all $x, y \in \mathcal{C}$. In particular, the induced map $\mathbf{Cat}_{<\omega} \subseteq (\mathbf{Cat}_{<\omega})\mathbf{Cat}$ is not an equivalence. $\square$

To better understand Adámek's construction in this setting, we introduce the following measure of finiteness to (∞,∞)-categories:

Definition 5.2.2.2. We define the *rank* of an (∞,∞)-category $\mathcal{C}$ by transfinite induction.

- Say that $\operatorname{rank} \mathcal{C} < 0$ if and only if $\mathcal{C} \simeq *$.
- For an ordinal θ, say that $\operatorname{rank} \mathcal{C} < \theta + 1$ if $\operatorname{rank} \mathrm{Hom}_{\mathcal{C}}(x, y) < \theta$ for all $x, y \in \mathcal{C}$.
- For a limit ordinal λ, say that $\operatorname{rank} \mathcal{C} < \lambda$ if $\operatorname{rank} \mathcal{C} < \theta$ for some $\theta < \lambda$.

Say $\operatorname{rank} \mathcal{C} = \theta$ if $\operatorname{rank} \mathcal{C} < \theta + 1$ but $\operatorname{rank} \mathcal{C} \not< \theta$. Note that the rank of $\mathcal{C}$ is invariant under equivalence.

For an ordinal θ, let $\mathbf{Cat}_{<\theta}$ denote the full subcategory of $\mathbf{Cat}_{(\infty,\infty)}$ spanned by those $\mathcal{C}$ with $\operatorname{rank} \mathcal{C} < \theta$.

Remark 5.2.2.3. By Lemma 5.2.2.5 below, if $\operatorname{rank} \mathcal{C} < \theta$ and $\theta < \theta'$, then also $\operatorname{rank} \mathcal{C} < \theta'$.

Example 5.2.2.4. As in Proposition 5.2.2.1, the category $\mathbf{Cat}_{<\omega}$ consists of the finite-dimensional higher categories, and $\mathbf{Cat}_{<\omega+1}$ consists of the locally finite-dimensional higher categories.

Lemma 5.2.2.5. *The categories $\mathbf{Cat}_{<\theta}$ can be constructed through transfinite induction:*

- $\mathbf{Cat}_{<0} \simeq \mathbf{Grpd}_{-2} \simeq \{*\}$,
- $\mathbf{Cat}_{<\theta+1} \simeq (\mathbf{Cat}_{<\theta})\mathbf{Cat}$*; in particular,* $\mathbf{Cat}_{<\theta}$ *is a full subcategory of* $\mathbf{Cat}_{<\theta+1}$*,*
- *For a limit ordinal* λ*,*
$$\mathbf{Cat}_{<\lambda} \simeq \varinjlim_{\theta<\lambda} \mathbf{Cat}_{<\theta}$$

Proof. That $\mathbf{Cat}_{<0} \simeq \{*\}$ and $\mathbf{Cat}_{<\theta+1} \simeq (\mathbf{Cat}_{<\theta})\mathbf{Cat}$ follow by definition. For the limit case, suppose by transfinite induction that $\mathbf{Cat}_{<\theta} \subseteq \mathbf{Cat}_{<\theta'}$ for all $\theta < \theta' < \lambda$. Then,

$$\varinjlim_{\theta<\lambda} \mathbf{Cat}_{<\theta} \simeq \bigcup_{\theta<\lambda} \mathbf{Cat}_{<\theta} = \mathbf{Cat}_{<\lambda}$$

as desired. □

Lemma 5.2.2.6. *For every ordinal θ, there is an (∞,∞)-category $\mathcal{C}$ such that* $\operatorname{rank} \mathcal{C} = \theta$*; that is,* $\operatorname{rank} \mathcal{C} < \theta + 1$ *but* $\operatorname{rank} \mathcal{C} \not< \theta$*.*

Proof. We prove this by transfinite induction. For $\theta = 0$, we take $\mathcal{C} = \varnothing$. Indeed, $\operatorname{rank} \mathcal{C} < 1$ is vacuous, and $\operatorname{rank} \mathcal{C} \not< 0$ because $\mathcal{C} \not\simeq *$.

Suppose we have an (∞,∞)-category $\mathcal{D}$ such that $\operatorname{rank} \mathcal{D} = \theta$. Then, $\operatorname{rank} \mathcal{C} = \theta + 1$ for $\mathcal{C} := \Sigma\mathcal{D}$.

Finally, suppose λ is a limit ordinal such that for every $\theta < \lambda$, there exists an (∞,∞)-category $\mathcal{D}^\theta$ such that $\operatorname{rank} \mathcal{D}^\theta = \theta$. Then, take $\mathcal{C} := \coprod_{\theta<\lambda} \mathcal{D}^\theta$.

Let $x, y \in \mathcal{C}$. If $x \in \mathcal{D}^\theta$ and $y \in \mathcal{D}^{\theta'}$ with $\theta \neq \theta'$, then $\operatorname{rank} \operatorname{Hom}_{\mathcal{C}}(x,y) = \operatorname{rank} \varnothing = 0 < \lambda$. Otherwise, $\operatorname{rank} \operatorname{Hom}_{\mathcal{C}}(x,y) = \operatorname{rank} \operatorname{Hom}_{\mathcal{D}^\theta}(x,y) < \lambda$. In particular, $\operatorname{rank} \mathcal{C} < \lambda + 1$. On the other hand, $\operatorname{rank} \mathcal{C} \not< \theta$ for all $\theta < \lambda$ since $\mathcal{D}^\theta$ is a (full) subcategory of $\mathcal{C}$, and $\operatorname{rank} \mathcal{D}^\theta \not< \theta$. Therefore, $\operatorname{rank} \mathcal{C} \not< \lambda$, proving that $\operatorname{rank} \mathcal{C} = \lambda$, as desired. □

Proposition 5.2.2.7. *Adámek's construction of an initial algebra for* $(-)\mathbf{Cat}$ *over the category* $\mathbf{SymMon}_\infty$ *does not terminate.*

Proof. The θth stage of Adámek's construction yields $\mathbf{Cat}_{<\theta}$ by Lemma 5.2.2.5. Therefore, the proposition follows from Lemma 5.2.2.6. □

The failure of Adámek's construction to terminate is purely a size issue. For instance, let $(-)\mathbf{Cat}^{<\omega}$ denote the subfunctor of $(-)\mathbf{Cat}$ that sends $\mathcal{V}$ to the full subcategory $\mathcal{V}\mathbf{Cat}^{<\omega}$ of $\mathcal{V}\mathbf{Cat}$ spanned by those $\mathcal{V}$-enriched categories with finitely many equivalence classes of objects (that is, the underlying space of objects has finitely many path-connected components). Then, Adámek's construction for $(-)\mathbf{Cat}^{<\omega}$ terminates after ω steps, and the initial algebra consists of those finite-dimensional higher categories with finitely many equivalence classes of k-morphisms for each $k \geq 0$.

This phenomenon can be shown more generally:

Lemma 5.2.2.8. *Fix a regular cardinal λ. Let $\mathcal{C}$ be an (∞,∞)-category such that the set of equivalence classes of objects of $\mathcal{C}$ is λ-small, and* $\operatorname{rank}\operatorname{Hom}_{\mathcal{C}}(x,y) < \lambda$ *for all $x, y \in \mathcal{C}$. Then,* $\operatorname{rank}\mathcal{C} < \lambda$.

Proof. For $x, y \in \mathcal{C}$, let $\theta_{x,y} < \lambda$ such that $\operatorname{rank}\operatorname{Hom}_{\mathcal{C}}(x,y) < \theta_{x,y}$; such an ordinal exists because a regular cardinal is necessarily a limit ordinal. Then, let $\theta := \sup_{x,y\in\mathcal{C}} \theta_{x,y}$. Note that if $x \simeq x'$ and $y \simeq y'$, then $\theta_{x,y} = \theta_{x',y'}$. Since $\mathcal{C}$ has fewer than λ objects up to equivalence, it follows from the fact that λ is a regular cardinal that $\theta < \lambda$, and therefore also that $\theta + 1 < \lambda$. Therefore, $\operatorname{rank}\mathcal{C} < \theta + 1 < \lambda$, as desired. □

Proposition 5.2.2.9. *For a regular cardinal λ, let $(-)\mathbf{Cat}^{<\lambda}$ denote the subfunctor of $(-)\mathbf{Cat} : \mathbf{SymMon}_\infty \to \mathbf{SymMon}_\infty$ that associates to a symmetric monoidal category $\mathcal{V}$ the full subcategory $\mathcal{V}\mathbf{Cat}^{<\lambda}$ of $\mathcal{V}\mathbf{Cat}$ spanned by those $\mathcal{V}$-enriched categories such that the set of path-connected components of its underlying space of objects is λ-small. Then, Adámek's construction of an initial algebra for $(-)\mathbf{Cat}^{<\lambda}$ over $\mathbf{SymMon}_\infty$ terminates after no fewer than λ steps.*

Proof. For an ordinal θ, let $\mathbf{Cat}^{<\lambda}_{<\theta}$ denote the full subcategory of $\mathbf{Cat}_{<\theta}$ on those (∞,∞)-categories $\mathcal{C}$ such that the set of equivalence classes of k-morphisms is λ-small for every $k \geq 0$. Then, $\mathbf{Cat}^{<\lambda}_{<\theta}$ can be constructed by transfinite induction, analogous to Lemma 5.2.2.5:

- $\mathbf{Cat}^{<\lambda}_{<0} \simeq \mathbf{Grpd}_{-2} \simeq \{*\}$, which is the initial object in $\mathbf{SymMon}_\infty$,
- $\mathbf{Cat}^{<\lambda}_{<\theta+1} \simeq (\mathbf{Cat}^{<\lambda}_{<\theta})\mathbf{Cat}^{<\lambda}$,
- For a limit ordinal μ,
$$\mathbf{Cat}^{<\lambda}_{<\mu} \simeq \varinjlim_{\theta<\mu} \mathbf{Cat}^{<\lambda}_{<\theta}$$

Following the proof of Lemma 5.2.2.6, there still exists $\mathcal{C} \in \mathbf{Cat}^{<\lambda}_{<\theta+1}$ such that $\mathcal{C} \notin \mathbf{Cat}^{<\lambda}_{<\theta}$, so long as $\theta < \lambda$. However, Lemma 5.2.2.8 shows that $\mathbf{Cat}^{<\lambda}_{<\lambda} \subseteq \mathbf{Cat}^{<\lambda}_{<\theta}$ is an equivalence for all $\theta > \lambda$.

Therefore, Adámek's construction terminates in exactly λ steps, as desired, and $\mathbf{Cat}^{<\lambda}_{<\lambda}$ carries the structure of an initial algebra for $(-)\mathbf{Cat}^{<\lambda}$ over $\mathbf{SymMon}_\infty$. □

We conclude this subsection by proving that $(-)\mathbf{Cat}$ has an initial algerba over $\mathbf{SymMon}_\infty$.

Definition 5.2.2.10. A *parallel morphism tower* $(\vec{\alpha}, \vec{\beta})$ in an (∞,∞)-category $\mathcal{C}$ is a (countable) sequence of pairs

$$(\alpha_0, \beta_0), (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots$$

where α_0, β_0 are objects of $\mathcal{C}$, and α_{n+1} and β_{n+1} are parallel $(n+1)$-morphisms $\alpha_n \to \beta_n$ in $\mathcal{C}$ for all $n \geq 0$.

Say that an (∞,∞)-category $\mathcal{C}$ is *Noetherian* if for any parallel morphism tower $(\vec{\alpha}, \vec{\beta})$, there exists $N \gg 0$ such that $\mathrm{Hom}_{\mathcal{C}}(\alpha_N, \beta_N) \simeq *$.

Denote by $\mathbf{Cat}^{\mathrm{Noeth}}_{(\infty,\infty)}$ the full subcategory of $\mathbf{Cat}_{(\infty,\infty)}$ spanned by the Noetherian (∞,∞)-categories.

Remark 5.2.2.11. The above definition of Noetherian is stronger than that proposed in [11, Definition 3.4.2], which only requires that any parallel morphism tower $(\vec{\alpha}, \vec{\beta})$ admits $N \gg 0$ such that α_n and β_n are equivalences for all $n \geq N$. Indeed, any ∞-groupoid satisfies this weaker property, but not every ∞-groupoid is Noetherian in the sense of Definition 5.2.2.10.

Remark 5.2.2.12. In some sense, the Noetherian (∞,∞)-categories are the (∞,∞)-categories for which (EQ1) and (EQ2) from Section 2.1.1 uniquely determine its equivalences.

Lemma 5.2.2.13. *For an (∞,∞)-category $\mathcal{C}$, the following are equivalent:*

(i) $\mathcal{C}$ is Noetherian,

(ii) $\mathcal{C}$ is locally Noetherian, in the sense that $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is Noetherian for all $x, y \in \mathcal{C}$,

(iii) $\mathcal{C}$ has small rank, in that $\operatorname{rank} \mathcal{C} < \theta$ for some ordinal θ,

(iv) $\mathcal{C}$ locally has small rank, in that $\mathrm{Hom}_{\mathcal{C}}(x, y)$ has small rank for all $x, y \in \mathcal{C}$.

Proof. The equivalence between (i) and (ii) follows by definition.

Note that (iii) certainly implies (iv): if $\operatorname{rank} \mathcal{C} < \theta$, then $\operatorname{rank} \mathcal{C} < \theta + 1$, and therefore $\operatorname{rank} \mathrm{Hom}_{\mathcal{C}}(x, y) < \theta$ for all $x, y \in \mathcal{C}$. Conversely, if for all $x, y \in \mathcal{C}$ there exists an ordinal $\theta_{x,y} \gg 0$ such that $\operatorname{rank} \mathrm{Hom}_{\mathcal{C}}(x, y) < \theta_{x,y}$, choose $\lambda \gg 0$ such that the set of equivalence classes of objects in $\mathcal{C}$ is λ-small, and such that $\lambda \geq \theta_{x,y}$ for all $x, y \in \mathcal{C}$. Then, $\operatorname{rank} \mathcal{C} < \lambda$ by Lemma 5.2.2.8. This proves that (iii) is equivalent to (iv).

Since the singleton $*$ is certainly Noetherian, and locally Noetherian (∞,∞)-categories are Noetherian, it follows by transfinite induction on the rank that every (∞,∞)-category $\mathcal{C}$ with small rank is Noetherian. This shows that (iii) implies (i).

To prove the converse, suppose $\mathcal{C}$ does not have small rank. Then, $\mathcal{C}$ does not locally have small rank, so there must exist $\alpha_0, \beta_0 \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(\alpha_0, \beta_0)$ does not have small rank. Proceeding recursively, we obtain a parallel morphism tower $(\vec{\alpha}, \vec{\beta})$ where each $\mathrm{Hom}_{\mathcal{C}}(\alpha_n, \beta_n)$ does not have small rank. In particular, $\mathrm{Hom}_{\mathcal{C}}(\alpha_n, \beta_n) \not\simeq *$ for every $n \geq 0$. Therefore, if $\mathcal{C}$ does not have small rank, then $\mathcal{C}$ is not Noetherian, completing the proof. $\square$

Theorem 5.2.2.14. $\mathbf{Cat}^{\mathrm{Noeth}}_{(\infty,\infty)}$ *carries the structure of an initial algebra for* $(-)\mathbf{Cat}$ *over* $\mathbf{SymMon}_\infty$.

Proof. By Lemma 5.2.2.13, the canonical inclusion $\mathbf{Cat}^{\mathrm{Noeth}}_{(\infty,\infty)} \subseteq (\mathbf{Cat}^{\mathrm{Noeth}}_{(\infty,\infty)})\mathbf{Cat}$ is an equivalence.

By expanding universes, let Λ denote the large ordinal of all (small) ordinals. Then, Lemma 5.2.2.13 implies that $\mathbf{Cat}^{\mathrm{Noeth}}_{(\infty,\infty)}$ is the Λ-filtered colimit

$$\mathbf{Cat}^{\mathrm{Noeth}}_{(\infty,\infty)} = \bigcup_{\theta} \mathbf{Cat}_{<\theta} \simeq \varinjlim_{\theta<\Lambda} \mathbf{Cat}_{<\theta}$$

in $\mathbf{SymMon}_\infty$, which by Lemma 5.2.2.5 is precisely Λ stages of Adámek's initial algebra construction. Since the construction terminates after Λ steps by the previous discussion, the theorem follows from Corollary 4.3.1.8. $\square$

Remark 5.2.2.15. Although Adámek's construction in this case requires a large colimit, this colimit is small relative to an expanded universe, and Corollary 4.3.1.8 applies also to categories that are small (relative to the universe of discourse).

Chapter 6

Conclusion

The purpose of this thesis was to study fully-general (∞, ∞)-categories. By the Unicity Theorem of [6], the theory of (∞, r)-categories is uniquely determined for all finite $r \geq 0$. However, the theory of (∞, r)-categories is ambiguous when $r = \infty$, as illustrated in the strict case in Section 2.1.1, with the source of discrepancy being the notion of equivalence in an (∞, ∞)-category.

In Chapter 3, we provided an explicit model $\mathbf{Cat}_{(n,r)}$ of (n, r)-categories for $-2 \leq n \leq \infty$ and $0 \leq r \leq n + 2$. In this model, the (∞, ∞)-categories have the weakest constraints on their equivalences: they are generated inductively. Any theory of (∞, ∞)-categories should be invariant under enrichment. In Chapter 4, we extended the theory of endofunctor algebras and endofunctor fixed points to the $(\infty, 1)$-categorical setting, which enables us to study the behaviour of $\mathbf{Cat}_{(\infty,\infty)}$ under enrichment. We then employed the theory in Chapter 5 to prove that $\mathbf{Cat}_{(\infty,\infty)}$ is an initial locally presentable fixed point of enrichment. In particular, this uniquely characterises $\mathbf{Cat}_{(\infty,\infty)}$.

Note that $\mathbf{Cat}_{(\infty,\infty)}$ is not initial among the fixed points of enrichment that are possibly not locally presentable. In Section 5.2.2, we completed this picture by showing that the full subcategory of $\mathbf{Cat}_{(\infty,\infty)}$ spanned by the Noetherian (∞, ∞)-categories form an initial object among arbitrary fixed points.

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