# University of Alberta

Irreducible Characters of GL(n, Z/p^l Z)

by

Qianglong Wen

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Department of Mathematical and Statistical Sciences


Fall 2011
Edmonton, Alberta

# ABSTRACT

We first find all the irreducible complex characters of the general linear group $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$ over the ring $\mathbb{Z}/p^\ell\mathbb{Z}$, where $\ell$ is an integer $> 1$ and $p$ is an odd prime, and determine all the character values. Our methods rely on Clifford Theory and can be modified easily to get all the irreducible complex characters of $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$ when $p = 2$.

We deal with irreducible characters which are not inflated from $\mathrm{GL}(2, \mathbb{Z}/p^{\ell-1}\mathbb{Z})$. These have three possible degrees. There are characters induced from a Borel subgroup, which have degree $(p+1)p^{\ell-1}$; and there are two other families of characters, of degrees $(p-1)p^{\ell-1}$ and $(p^2-1)p^{\ell-2}$.

Many results can be extended to the group $G = \mathrm{GL}(2, R)$ with $R = S/P^\ell$ where $S$ is the ring of integers in a local or global field and $P$ is a maximal ideal. If $S/P$ has $q$ elements, we can replace $p$ by $q$ in the degree and number of each degree formulas we find. We study $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$ in our work not only because it can give us some general results, but also it is simpler when we deal with character values.

We also construct irreducible characters of $\mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$ and $\mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$. There are 7 kinds of irreducible characters for each group, and these 7 kinds of irreducible characters also show up for group $\mathrm{GL}(3, \mathbb{Z}/p^\ell\mathbb{Z})$ for any $\ell > 1$. We have all the degrees and the number of characters of each degree for the

$\mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$. Moreover, we find all the irreducible constituents of character $\mathrm{Ind}_B^G(1)$ for the two groups $\mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$ and $\mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$, where $B$ is the corresponding Borel subgroup.

## ACKNOWLEDGEMENTS

I sincerely thank my supervisor, Dr. Gerald Cliff, for his kind guidance, supervision and financial support in the course of this research programme. I would like to thank him for bringing me into this area in my master program and continuing help me with it during the PhD program. This experience is undoubtedly my lifetime benefit. I also wish to thank my parents for giving me their constant love, encouragement and support. Lastly, I also wish to thank the University of Alberta for the financial support provided during this period.

# Table of Contents


**1 Introduction** **1**

**2 Some Preliminaries** **5**
- 2.1 Character Theory . . . . . 5
- 2.2 Clifford Theory . . . . . 6
- 2.3 Useful results . . . . . 9
- 2.4 Conjugacy Classes . . . . . 12

**3 Irreducible character degrees of** $\mathrm{GL}(2,\mathbb{Z}/p^\ell\mathbb{Z})$ **15**
- 3.1 Construction of Irreducible Characters of $\mathrm{GL}(2,\mathbb{Z}/p^\ell\mathbb{Z})$ . . . . 15
- 3.2 The number of characters . . . . . 18
  - 3.2.1 $G=\mathrm{GL}(2,\mathbb{Z}/p^{2m}\mathbb{Z})$, i.e. $\ell=2m$ . . . . . 19
  - 3.2.2 $G=\mathrm{GL}(2,\mathbb{Z}/p^{2m+1}\mathbb{Z})$, i.e. $\ell=2m+1$ . . . . . 20
- 3.3 About Character Values . . . . . 22
- 3.4 When $p=2$ . . . . . 23
- 3.5 Replacing $\mathbb{Z}/p^\ell\mathbb{Z}$ by $R/P^\ell$ . . . . . 24

**4 Character Values** **26**
- 4.1 Values for Characters with Degree $(p+1)p^{\ell-1}$ . . . . . 26
  - 4.1.1 Character Values by Parabolic Induction . . . . . 26
  - 4.1.2 Character Values by Clifford's Theorem . . . . . 27
- 4.2 Values for $(p-1)p^{\ell-1}$-Degree Characters . . . . . 30
  - 4.2.1 Character Values of Elements in $K_{\ell-i}, 1\le i\le\frac{\ell}{2}$ . . . . 30
  - 4.2.2 Remaining Values when $\ell=2m$ is even . . . . . 32
  - 4.2.3 Remaining Values when $\ell=2m+1$ is odd . . . . . 35
  - 4.2.4 Character Values of $\psi$ on $T$ . . . . . 39
- 4.3 Characters of degree $(p^2-1)p^{\ell-2}$ and the character values . . 41
  - 4.3.1 Character values of elements in $K_{\ell-i}, 1\le i\le\frac{\ell}{2}$ . . . . 41
  - 4.3.2 Remaining Values for Characters of Degree $(p^2-1)p^{\ell-2}$ 43

**5 Irreducible Characters of** $\mathrm{GL}(3,\mathbb{Z}/p^\ell\mathbb{Z})$ **50**
- 5.1 The Irreducible Characters of $\mathrm{GL}(3,\mathbb{Z}/p^2\mathbb{Z})$ . . . . . 50
  - 5.1.1 Irreducible Characters of Degree $p^2(p^2+p+1)$ . . . . 52

5.1.2 Irreducible Characters of Degree $p^3(p+1)(p^2+p+1)$ . 53
5.1.3 Irreducible Characters of Degree $p^3(p-1)^2(p+1)$ . . . 54
5.1.4 Irreducible Characters of Degree $(p^3-1)(p+1)$ . . . . 55
5.1.5 Irreducible Characters of Degree $p(p^3-1)(p^2-1)$ . . . 56
5.1.6 Irreducible Characters of Degree $p^3(p^3-1)$ . . . . . . . 57
5.1.7 Irreducible Characters of Degree $p^2(p+1)(p^3-1)$ . . . 58
5.1.8 All the irreducible characters of $\mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$ . . . . . 59
5.2 Irreducible Characters of $\mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$ . . . . . . . . . . . . . 60
5.2.1 Irreducible Characters of Degree $p^6(p-1)^2(p+1)$ . . . 61
5.2.2 Irreducible Characters of Degree $p^4(p^3-1)(p^2-1)$ . . 62
5.2.3 Irreducible Characters of Degree $p^6(p^3-1)$ . . . . . . . 63
5.2.4 Irreducible Characters of Degree $p^5(p+1)(p^3-1)$ . . . 64
5.2.5 Irreducible Characters of Degree $p^6(p+1)(p^2+p+1)$ . 66
5.2.6 Irreducible Characters of Degree $p^4(p^2+p+1)$ . . . . 68
5.2.7 Irreducible Characters of Degree $p^2(p^3-1)(p+1)$ . . . 72
5.3 Decomposition of $\mathrm{Ind}_B^G(1_B)$ . . . . . . . . . . . . . . . . . . . . . 73
5.3.1 Double Cosets of the Borel subgroup . . . . . . . . . . 73
5.3.2 $G = \mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$ . . . . . . . . . . . . . . . . . . . . 76
5.3.3 $G = \mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$ . . . . . . . . . . . . . . . . . . . . 80

**6 Parabolic Induction of $\mathrm{GL}(n, \mathbb{Z}/p^\ell\mathbb{Z})$** **84**
6.1 $G = \mathrm{GL}(n, \mathbb{Z}/p^{2m}\mathbb{Z})$ . . . . . . . . . . . . . . . . . . . . . . . 85
6.2 $G = \mathrm{GL}(n, \mathbb{Z}/p^{2m+1}\mathbb{Z})$ . . . . . . . . . . . . . . . . . . . . . 86
6.3 More general result . . . . . . . . . . . . . . . . . . . . . . . . 87

**Bibliography** **89**

# Chapter 1

# Introduction

In this thesis, we apply Clifford Theory to construct irreducible characters of the groups $\mathrm{GL}(2, \mathbb{Z}/p^n\mathbb{Z})$ and $\mathrm{GL}(3, \mathbb{Z}/p^n\mathbb{Z})$. The main idea of Clifford's Theorem is as follows.

Let $N \lhd G$ be a normal subgroup of $G$. For any character $\phi$ of $N$, we can define

$$\phi^g : N \to \mathbb{C}; \phi^g(n) = \phi(gng^{-1}), \forall g \in G, n \in N.$$

$\phi^g$ is also a character of $N$. Let $\phi \in Irr(N)$, denote $I_G(\phi) = \{g \in G \mid \phi^g = \phi\}$. If $\psi \in \mathrm{Irr}(I_G(\phi))$, such that $[\psi_N, \phi] \neq 0$, then Clifford's Theorem tells us that $\psi^G$ is an irreducible character of $G$.

In Chapter 2 and 3, we determine the values of the irreducible complex characters of the general linear group $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$ over the ring $\mathbb{Z}/p^\ell\mathbb{Z}$, where $\ell$ is an integer $> 1$ and $p$ is an odd prime. The degrees of these characters, and the number of characters of each degree, follow from work of Nobs [1]. However Nobs did not consider the problem of finding the character values.

Our methods can be modified easily to get all the irreducible characters when $p = 2$, which are quite different than those of Nobs. Our methods of constructing the irreducible characters are somewhat similar to those of Kutzko [8], who was interested in representations of $\mathrm{GL}(2, F)$ where $F$ is a $p$-adic field; Kutzko did not find character values in [8]. Indeed, one of the reasons for our interest in this problem is that smooth, irreducible super-cuspidal representations of $\mathrm{GL}(2, F)$ are induced from those of $\mathrm{GL}(2, \mathcal{O})$ where $\mathcal{O}$ is the ring of integers of $F$, and these in turn arise from representations of $\mathrm{GL}(2, k)$ where $k$ is a finite factor ring of $\mathcal{O}$.

We deal with irreducible characters which are not inflated from $\mathrm{GL}(2, \mathbb{Z}/p^{\ell-1}\mathbb{Z})$. These have three possible degrees. There are characters induced from a Borel subgroup, which have degree $(p + 1)p^{\ell-1}$; and there are two other families of characters, of degrees $(p - 1)p^{\ell-1}$ and $(p^2 - 1)p^{\ell-2}$.

Many results can be extended to the group $G = \mathrm{GL}(2, R)$ with $R = S/P^\ell$ where $S$ is the ring of integers in a local or global field and $P$ is a maximal

ideal. If $S/P$ has $q$ elements, we can replace $p$ by $q$ in the degree and number of each degree formulas. We will see one example in section 3.5. We study $\mathrm{GL}(2,\mathbb{Z}/p^\ell\mathbb{Z})$ in our work not only because it can give us some general results, but also it is simpler when we deal with character values.
In Chapter 5, we construct irreducible characters of $\mathrm{GL}(3,\mathbb{Z}/p^2\mathbb{Z})$ and $\mathrm{GL}(3,\mathbb{Z}/p^3\mathbb{Z})$. There are 7 kinds of irreducible characters for each group, and these 7 kinds of irreducible characters also show up for group $\mathrm{GL}(3,\mathbb{Z}/p^\ell\mathbb{Z})$ for any $\ell > 1$. For $\mathrm{GL}(3,\mathbb{Z}/p^2\mathbb{Z})$, we define one-dimensional character $\phi_A$ on its normal subgroup $K_1 = \{I + pB, B \in M(3,\mathbb{Z}/p\mathbb{Z})\}$, find the stabilizer $T$ of $\phi_A$ and then extend $\phi_A$ to $\psi_A$ of $T$ such that $\psi_A \mid_{K_1} = \phi_A$. By Clifford Theory we know $\chi_A = \mathrm{Ind}_T^G(\psi_A) \in \mathrm{Irr}(G)$. The process is as follows:

$$\begin{array}{ccccc} K_1 & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \psi_A & \xrightarrow{\text{ind}} & \chi_A \end{array}.$$

We have all the degrees and the number of characters of each degree for the $\mathrm{GL}(3,\mathbb{Z}/p^2\mathbb{Z})$ as follows.

| Degrees | Number of this degree |
|---|---|
| $p^2(p^2+p+1)$ | $p(p-1)^4$ |
| $p^3(p^2+p+1)$ | $p(p-1)^4$ |
| $p^2(p^2+p+1)(p+1)$ | $\frac{p(p-1)^4(p-1)}{2}$ |
| $p^2(p^2+p+1)(p-1)$ | $\frac{p^2(p-1)^4}{2}$ |
| $p^3(p+1)(p^2+p+1)$ | $\frac{p(p-1)(p-2)(p-1)^3}{6}$ |
| $p^2(p+1)(p^3-1)$ | $p^2(p-1)^3$ |
| $p^3(p^3-1)$ | $\frac{p^2(p-1)^3(p+1)}{2}$ |
| $p(p^3-1)(p^2-1)$ | $p^3(p-1)$ |
| $(p^3-1)(p+1)$ | $p(p-1)^2$ |
| $(p^3-1)(p+1)(p-1)$ | $p(p^2-1)$ |
| $p(p^3-1)(p+1)$ | $p(p-1)^3$ |
| $p^3(p-1)^2(p+1)$ | $\frac{p(p-1)(p+1)(p^3-1)}{3}$ |

For $\mathrm{GL}(3,\mathbb{Z}/p^3\mathbb{Z})$, we define one-dimensional characters $\phi$ on the normal subgroup $K_2 = \{I + p^2B\}$, find the stabilizer $T$ of $\phi$, and then get $\psi \in \mathrm{Irr}(T)$ such that $[\psi \mid_{K_2}, \phi_A] \neq 0$. Eventually by Clifford Theory, we have $\chi = \mathrm{Ind}_T^G(\psi) \in \mathrm{Irr}(G)$. Depending on the definition of $\phi$ on $K_2$, we have two different construction processes. The first one is as follows.

$$(1)\quad \begin{array}{ccccccccccc} K_2 & \longrightarrow & K_2(K_1\cap S) & \longrightarrow & H & \longrightarrow & K_1 & \longrightarrow & K_1S & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ext}} & \phi'' & \xrightarrow{\text{ind}} & \theta & \xrightarrow{\text{ext}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array},$$

where $H \lhd K_1$ with index $p^3$.

The second one is

$$
(2) \qquad \begin{array}{ccccccccc} K_2 & \longrightarrow & N & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi'_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ind}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array},
$$

for which we choose a normal subgroup $N$ of $T$ such that $\frac{|N|}{|K_2|}$ is as big as possible while we can still extend $\phi_A$ to $\phi'_A$ of $N$. $T'$ is the stabilizer of $\phi'_A$ in $T$. These two construction processes will give us 7 kinds of irreducible characters of $\mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$.

In the last section of chapter 5, we find all the irreducible constituents of the permutation character $\mathrm{Ind}_B^G(1)$ for the two groups $\mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$ and $\mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$, where $B$ is the corresponding Borel subgroup. For $\mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$, the decomposition is

$$
\mathrm{Ind}_B^G(1_B) = \mathrm{Ind}_{B'}^{G'}(1_{B'}) \bigoplus \chi_1 \bigoplus 3\chi_2 \bigoplus \chi_3 \bigoplus \chi_4.
$$

In the case of $\mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$, the complete decomposition is as follows.

$$
\mathrm{Ind}_B^G(1_B) = \mathrm{Ind}_{B''}^{G''}(1_{B''}) \bigoplus 4\chi_1 \bigoplus_{i=1}^{p-2} \chi_{1i} \bigoplus_{j=2}^{6} \chi_j.
$$

$B'$ and $B''$ are the corresponding Borel subgroups of $G'$ and $G''$. Details will be given in section 5.3.

In Chapter 6, we generalize the parabolic induction to construct irreducible characters of group $\mathrm{GL}(n, \mathbb{Z}/p^\ell\mathbb{Z})$. In the $2\times 2$ case, let $B \subset G = GL(2, \mathbb{Z}/p^\ell\mathbb{Z})$ be the Borel subgroup $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}$. Let $\lambda$ be an injective character of $(\mathbb{Z}/p^\ell\mathbb{Z})^\times$ and define $\phi$ by $\phi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \lambda(a)$. $\chi = \mathrm{Ind}_B^G \phi$ is irreducible.

The most general version for $G = \mathrm{GL}(n, \mathbb{Z}/p^\ell\mathbb{Z})$ is as follows. Let

$$
B = \left\{ \begin{pmatrix} A_1 & * & * & \cdots & * \\ 0 & A_2 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{k-1} & * \\ 0 & 0 & 0 & \cdots & A_k \end{pmatrix} \right\} \subset G
$$

where each $A_i$ is a $n_i \times n_i$ matrix. Let $\lambda_1, \lambda_2, ..., \lambda_{k-1} : (\mathbb{Z}/p^\ell\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ be homomorphisms such that $\left\{1 + p^{[\ell/2]}x\right\} \nsubseteq \ker \lambda_i\lambda_j^{-1}, i \neq j, 1 \leq i, j \leq k-1$. Let

$$
\phi_i \in \mathrm{Irr}(\mathrm{GL}(n_i, \mathbb{Z}/p^\ell\mathbb{Z})), \quad 1 \leq i \leq k
$$

be inflated from $\mathrm{GL}(n_i, \mathbb{Z}/p^{\ell-[\frac{\ell}{2}]}\mathbb{Z})$. Define $\Phi : B \longrightarrow \mathbb{C}^\times$,

$$\Phi\begin{pmatrix} A_1 & * & * & \cdots & * \\ 0 & A_2 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{k-1} & * \\ 0 & 0 & 0 & \cdots & A_k \end{pmatrix} = \lambda_1[\det(A_1)]\lambda_2[\det(A_2)]\cdots\lambda_{k-1}[\det(A_{k-1})]$$

$$\times\phi_1(A_1)\phi_2(A_2)\cdots\phi_k(A_k),$$

then

$$\mathrm{Ind}_B^G(\Phi) \in \mathrm{Irr}(G).$$

# Chapter 2

# Some Preliminaries

## 2.1 Character Theory

**Definition 2.1.1.** Let $V$ be a finite-dimensional vector space over $\mathbb{C}$. A *representation* $\rho$ of a group $G$ is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. $\dim(V)$ is also called the *dimension* of $\rho$, denoted by $\dim(\rho)$.

We know that if we choose a basis of $V$, then $\mathrm{GL}(V) \cong \mathrm{GL}(n, \mathbb{C})$, where $n = \dim(V)$. So it is equivalent to say that a group homomorphism $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ is also a representation. In particular, a group homomorphism $\lambda : G \to \mathbb{C}^{\times}$ is a representation.

**Definition 2.1.2.** A subspace $W$ of $V$ is invariant under $\rho$ if for each $w \in W$ and for all $g \in G, \rho(g)(w) \in W$. A representation $\rho : G \to \mathrm{GL}(V)$ is *irreducible* if there is no proper nonzero invariant subspace $W$ of $V$ under $\rho$.

We usually use character theory to determine whether a representation is irreducible.

**Definition 2.1.3.** Let $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ be a representation of $G$. The character $\chi$ of $G$ afforded by $\rho$ is the function given by $\chi(g) = \mathrm{tr}(\rho(g))$. $\chi$ is called *irreduciblee* if $\rho$ is irreducible. The degree of $\chi$ is defined by $\deg(\chi) = \dim(\rho) = \chi(1)$.

From now on, let $\mathrm{Irr}(G)$ represent the set of all irreducible characters of the group $G$.

**Proposition 2.1.4.** Let $\chi$ and $\psi$ be characters of $G$. Define $\chi\psi$ on $G$ by setting $(\chi\psi)(g) = \chi(g)\psi(g)$. $\chi\psi$ is also a character of $G$.

From the definitions above, it is clear that a $1-$*dimensional* representation $\rho$ is irreducible. Moreover, if $\chi$ is the character afforded by $\rho$, we have $\rho = \chi$. Namely, a $1-$*dimensional* character is also a representation. We will use this fact in the next two chapters very often.

**Definition 2.1.5.** Let $N < G$ be a subgroup and suppose that $\phi$ is a character of $N$. We say $\phi$ is extendible to $G$ if $\exists\psi$, a character of $G$, such that $\psi_N = \phi$. We call $\psi$ an extension of $\phi$ to $G$.

**Definition 2.1.6.** Let $\phi$ and $\theta$ be characters of a group $G$.

$$[\phi, \theta] = \frac{1}{\mid G \mid} \sum_{g \in G} \phi(g)\overline{\theta(g)}$$

is the inner product of $\phi$ and $\theta$.

**Theorem 2.1.7.** Let $\lambda$ and $\psi$ be characters of G. $[\lambda, \psi] = [\psi, \lambda]$ is a non-negative integer. Also $\lambda$ is irreducible if and only if $[\lambda, \lambda] = 1$.

**Definition 2.1.8.** Let $H < G$ be a subgroup and let $\phi$ be a character of $H$. $\phi^G$, the induced character on $G$, is given by

$$\phi^G(g) = \frac{1}{\mid H \mid} \sum_{x \in G} \phi^\circ(xgx^{-1}),$$

where $\phi^\circ$ is defined by $\phi^\circ(h) = \phi(h)$ if $h \in H$ and $\phi^\circ(y) = 0$ if $y \notin H$.

By the definition above, it is easy to calculate that

$$\deg(\phi^G) = \deg(\phi)\frac{\mid G \mid}{\mid H \mid}.$$

Also from the definition of induced character, we have the following proposition.

**Proposition 2.1.9.** Let $H < K < G$ and suppose that $\phi$ is a character of $H$, then $(\phi^K)^G = \phi^G$.

**Lemma 2.1.10.** (*Frobenius Reciprocity*) Let $H < G$ and suppose that $\phi$ is a character of $H$ and that $\theta$ is a character on G, then

$$[\phi, \theta_H] = [\phi^G, \theta].$$

## 2.2 Clifford Theory

Let $H \lhd G$. If $\theta$ is a character of $H$ and $g \in G$, we define $\theta^g : H \to \mathbb{C}$ by $\theta^g(h) = \theta(ghg^{-1})$. We say that $\theta^g$ is *conjugate* to $\theta$ in $G$.

**Lemma 2.2.1.** Let $H \lhd G$ and let $\phi$, $\theta$ be characters of $H$ and $x, y \in G$.
(a) $\phi^x$ is a character;
(b) $(\phi^x)^y = \phi^{xy}$;
(c) $[\phi^x, \theta^x] = [\phi, \theta]$;
(d) $[\chi_H, \phi^x] = [\chi_H, \phi]$ for characters $\chi$ of $G$.

The Lemma follows from direct calculation.

**Definition 2.2.2.** Let $H \lhd G$ and let $\theta \in \mathrm{Irr}(H)$.

$$I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$$

is the inertia group of $\theta$ in $G$.

We also call $I_G(\theta)$ the stabilizer of $\theta$ in $G$ and use notation $\mathrm{Stab}_G(\theta)$ in later sections. When $I_G(\theta) = G$, we say $\theta$ is stable under $G$, or invariant in $G$.

**Theorem 2.2.3.** (*Clifford*, [1]) Let $H \lhd G$, $\theta \in \mathrm{Irr}(H)$, and $T = I_G(\theta)$. Let

$$A = \{\psi \in \mathrm{Irr}(T) \mid [\psi_H, \theta] \neq 0\}, B = \{\chi \in \mathrm{Irr}(G) \mid [\chi_H, \theta] \neq 0\}.$$

(a) If $\psi \in A$, then $\psi^G$ is irreducible;
(b) The map $\psi \mapsto \psi^G$ is a bijection of $A$ onto $B$;
(c) If $\psi^G = \chi$, with $\psi \in A$, then $\psi$ is the unique irreducible constituent of $\chi_T$ which lies in $A$;
(d) If $\psi^G = \chi$, with $\psi \in A$, then $[\psi_H, \theta] = [\chi_H, \theta]$.

In general, it is hard to tell whether the character of $G$ induced from an irreducible character of $H < G$ is still irreducible. This Theorem tells us when the induced character is irreducible . So we can apply this theorem to construct some irreducible characters of $G$, from certain irreducible characters of the normal subgroup $H$. Part $(a)$ of this theorem is used throughout the following two chapters.

**Corollary 2.2.4.** Let $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$. $\theta^G \in \mathrm{Irr}(G)$ if and only if $I_G(\theta) = N$.

The $I_G(\theta) = N \Rightarrow \theta^G \in \mathrm{Irr}(G)$ direction follows immediately from (a) of last theorem and we will use this result very often in the next two chapters.

**Corollary 2.2.5.** Let $N \lhd G$ and let $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(N)$ with $[\chi_N, \theta] \neq 0$. The following are equivalent:
(a) $\chi_N = e\theta$, with $e^2 = \mid G : N \mid$;
(b) $\chi$ vanishes on $G - N$ and $\theta$ is invariant in $G$;
(c) $\chi$ is the unique irreducible constituent of $\theta^G$ and $\theta$ is invariant in $G$.

**Theorem 2.2.6.** (*Gallagher*, [1]) Let $N \lhd G, \chi \in \mathrm{Irr}(G)$ be such that $\chi_N = \theta \in \mathrm{Irr}(N)$. The characters $\beta\chi$ for $\beta \in \mathrm{Irr}(G/N)$ are irreducible, distinct for distinct $\beta$, and are all of the irreducible constituents of $\theta^G$.

Note that there is a projection $\pi : G \to G/N$. Thus, for any group representation $\rho$ of $G/N$, $\rho \circ \pi$ is a representation of $G$. If $\rho$ is irreducible, $\rho \circ \pi$ is also irreducible. As a result, we can consider the character $\beta \in \mathrm{Irr}(G/N)$ as an irreducible character of $G$.Therefore, $\beta\chi$ above is well defined.
Considering the set $A$ in Theorem 2.2.3, we have

$$A = \{\psi \in \mathrm{Irr}(T) \mid [\psi_H, \theta] \neq 0\} = \{\psi \in \mathrm{Irr}(T) \mid [\psi, \theta^T] \neq 0\}.$$

In order to apply theorem 2.2.3 to construct irreducible characters of $G$, we need to induce up the characters in $A$. Theorem 2.2.6 tells us that, if we can actually extend $\theta$ to $T$, then by finding out all the irreducible characters of $T/H$, we can construct all the irreducible characters in $A$ and, as a result, we will find more irreducible characters of $G$.
We will apply Theorem 2.2.6 in chapter 4.

**Theorem 2.2.7.** Let $N \lhd G$ with $G/N$ cyclic and let $\theta \in \mathrm{Irr}(N)$ be invariant in $G$, then $\theta$ is extendible to $G$.

By applying this theorem, we will come up with some crucial results. The following three lemmas are useful in the following two chapters to construct certain extensions of some characters of degree one.

**Lemma 2.2.8.** Let $G$ be a group, $N \lhd G, H < G$ and $G = NH$. Let $\phi \in \mathrm{Irr}(N), \psi \in \mathrm{Irr}(H)$ be such that $\deg(\phi) = \deg(\psi) = 1$. Assume $\phi_{N\cap H} = \psi_{N\cap H}$ and $\forall h \in H, \phi^h = \phi$. $\exists \theta \in \mathrm{Irr}(G)$ such that $\deg(\theta) = 1$ and $\theta_N = \phi$.

*Proof.* Define

$$\theta : G \to \mathbb{C}^{\times}; \theta(nh) = \phi(n)\psi(h), \forall n \in N, h \in H.$$

Since $\phi$ and $\psi$ are of degree one, they are also group homomorphisms. Since $\phi_{N\cap H} = \psi_{N\cap H}$, we know that $\theta$ is well-defined. In addition, $\forall n_1, n_2 \in N, h_1, h_2 \in H$, we have

$$\begin{aligned}
\theta(n_1h_1n_2h_2) &= \theta(n_1h_1n_2h_1^{-1}h_1h_2) \\
&= \phi(n_1h_1n_2h_1^{-1})\psi(h_1h_2) \\
&= \phi(n_1)\phi(h_1n_2h_1^{-1})\psi(h_1)\psi(h_2) \\
&= \phi(n_1)\phi^{h_1}(n_2)\theta(h_1)\theta(h_2) \\
&= \phi(n_1)\phi(n_2)\theta(h_1)\theta(h_2) \\
&= \theta(n_1h_1)\theta(n_2h_2).
\end{aligned}$$

Thus, $\theta$ is of degree one. It is clear that $\theta_N = \phi$. □

**Lemma 2.2.9.** Let $G$ be a finite abelian group, let $N \lhd G$ and $\lambda \in \mathrm{Irr}(N)$, then $\lambda$ is extendible to $G$.

*Proof.* Since $G$ is a finite abelian group, it is a direct product of cyclic groups. Thus, we can find the subgroups $N_1, N_2, ..., N_m$ of $G$, such that $N_1/N, N_2/N_1$, ..., $N_m/N_{m-1}$ and $G/N_m$ are all cyclic. Thus, by Theorem 2.2.7, $\lambda$ can be extended to $N_1$. Call the extension $\lambda_1$. Since $G$ is abelian, we have that any character of any subgroup of $G$ is stable under $G$. Therefore $\lambda_1$ is stable under G, so is stable under $N_2$. Hence it is extended to $N_2$. So $\lambda$ is extended to $N_2$. Keeping doing this, we know that finally $\lambda$ will be extended to $G$. □

**Lemma 2.2.10.** Let $G$ be a group, $N \lhd G, S < G, S$ is abelian, and $G = NS$. Let $\phi \in \mathrm{Irr}(N)$ be such that $\deg(\phi) = 1$. Assume $\phi$ is stable under $G$, then $\phi$ is extendible to $G$.

*Proof.* Let $\psi = \phi_{S\cap N}$, then $\psi \in \mathrm{Irr}(S \cap N)$. Since $S$ is abelian, we know $S \cap N \lhd S$. By Lemma 2.2.9, $\exists\theta \in \mathrm{Irr}(S)$ such that $\theta_{S\cap N} = \psi = \phi_{S\cap N}$. Apply Lemma 2.2.8, we know that $\phi$ is extendible to $G$. □

Lemma 2.2.10 will be used a lot.

## 2.3 Useful results

In this section, we will calculate the orders of groups $\mathrm{GL}(2, \mathbb{Z}/p^n\mathbb{Z}), \mathrm{GL}(3, \mathbb{Z}/p^n\mathbb{Z})$ and some of their important subgroups.

In $\mathrm{GL}(k, \mathbb{Z}/p^n\mathbb{Z})$, define $K = \{I + pA \mid A \in M_{k\times k}(\mathbb{Z}/p^{n-1}\mathbb{Z})\}$. $\mid K \mid=\mid M_{k\times k}(\mathbb{Z}/p^{n-1}\mathbb{Z}) \mid= p^{k^2(n-1)}$. Formally speaking, matrix $A \in M_{k\times k}(\mathbb{Z}/p^{n-1}\mathbb{Z})$ doesn't belong to $M_{k\times k}(\mathbb{Z}/p^n\mathbb{Z})\}$ since $\mathbb{Z}/p^{n-1}\mathbb{Z}$ is not a subset of $\mathbb{Z}/p^n\mathbb{Z}$. In this thesis, we treat $\mathbb{Z}/p^i\mathbb{Z}$ as a subset of $\mathbb{Z}/p^j\mathbb{Z}$ for $i < j$ to simplify notations and this does not cause confusion.

**Proposition 2.3.1.** $\mid \mathrm{GL}(k, \mathbb{Z}/p^n\mathbb{Z}) \mid= p^{k^2(n-1)} \prod_{t=1}^{k} (p^k - p^{t-1})$.

*Proof.* Recall that there is a group homomorphism

$$\phi : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}; \quad \phi(a) = \overline{a}, \quad \forall a \in \mathbb{Z}/p^n\mathbb{Z}.$$

Thus, we can define

$$\psi : \mathrm{GL}(k, \mathbb{Z}/p^n\mathbb{Z}) \to \mathrm{GL}(k, \mathbb{Z}/p\mathbb{Z}); \psi(A) = \overline{A},$$

where $A \in \mathrm{GL}(k, \mathbb{Z}/p^n\mathbb{Z})$ and $\overline{A}_{ij} = \phi(A_{ij})$. It is easy to check that $\psi$ is a surjective group homomorphism. Moreover, $\ker(\psi) = K$. Hence, we have

$$\mathrm{GL}(k, \mathbb{Z}/p^n\mathbb{Z})/K \cong \mathrm{GL}(k, \mathbb{Z}/p\mathbb{Z})$$

$$\Rightarrow\mid \mathrm{GL}(k, \mathbb{Z}/p^n\mathbb{Z}) \mid=\mid \mathrm{GL}(k, \mathbb{Z}/p\mathbb{Z}) \mid\mid K \mid .$$

Since it is known that $\mid \mathrm{GL}(k, \mathbb{Z}/p\mathbb{Z}) \mid = \prod_{t=1}^{k}(p^k - p^{t-1})$[6], the proposition follows. $\square$

**Corollary 2.3.2.** $\mid \mathrm{GL}(2, \mathbb{Z}/p^n\mathbb{Z}) \mid = (p^2-p)(p^2-1)p^{4n-4}$, $\mid \mathrm{GL}(3, \mathbb{Z}/p^n\mathbb{Z}) \mid = (p^3-1)(p^3-p)(p^3-p^2)p^{9n-9}$.

Next, we will calculate the orders of two important subgroups of $\mathrm{GL}(2, \mathbb{Z}/p^n\mathbb{Z})$ and $\mathrm{GL}(3, \mathbb{Z}/p^n\mathbb{Z})$.
Let $\varepsilon \in \mathbb{Z}/p\mathbb{Z}$ be such that $\sqrt{\varepsilon} \notin \mathbb{Z}/p\mathbb{Z}$, *i.e.* there is no $a \in \mathbb{Z}/p\mathbb{Z}$ such that $a^2 = \varepsilon$.
Let

$$S' = \left\{ s' = \begin{pmatrix} x & y\varepsilon \\ y & x \end{pmatrix} \mid s' \in \mathrm{GL}(2, \mathbb{Z}/p\mathbb{Z}) \right\}.$$

$S' < \mathrm{GL}(2, \mathbb{Z}/p\mathbb{Z})$. Moreover, we can prove that

$$S' \cong (\mathbb{Z}/p\mathbb{Z}[\sqrt{\varepsilon}])^{\times},$$

therefore $\mid S' \mid = p^2 - 1$. The proof is exactly the same as the one in the $\mathrm{GL}(3, \mathbb{Z}/p^n\mathbb{Z})$ case and we will talk about it later. Consider $\varepsilon$ as an element of $\mathbb{Z}/p^n\mathbb{Z}$, *e.g.* $3 \in \mathbb{Z}/5\mathbb{Z}$, 3 is also an element of $\mathbb{Z}/25\mathbb{Z}$.
Define

$$S = \left\{ s = \begin{pmatrix} x & y\varepsilon \\ y & x \end{pmatrix} \mid s \in \mathrm{GL}(2, \mathbb{Z}/p^n\mathbb{Z}) \right\}.$$

**Proposition 2.3.3.** $\mid S \mid = (p^2-1)p^{2n-2}$.

*Proof.* Let

$$\psi : \mathrm{GL}(2, \mathbb{Z}/p^n\mathbb{Z}) \to \mathrm{GL}(2, \mathbb{Z}/p\mathbb{Z})$$

be the surjective group homomorphism defined in the proof of Proposition 2.3.1 in the case $k = 2$. Consider the restriction of $\psi$ to $S$, then it is clear that $\psi$ maps $S$ onto $S'$ and

$$\ker \psi = \left\{ t = \begin{pmatrix} 1+px & py\varepsilon \\ py & 1+px \end{pmatrix} \mid t \in \mathrm{GL}(2, \mathbb{Z}/p^n\mathbb{Z}) \right\}.$$

Clearly, $\mid \ker \psi \mid = p^{2n-2}$. Since $S/\ker\psi \cong S'$, we have that

$$\mid S \mid = \mid S' \mid \mid \ker \psi \mid = (p^2-1)p^{2n-2}.$$

$\square$

**Corollary 2.3.4.** Suppose $n > m$. Let $G = \mathrm{GL}(2, \mathbb{Z}/p^n\mathbb{Z}), K_m = \{I + p^m A \mid A \in M_{2\times 2}(\mathbb{Z}/p^{n-m}\mathbb{Z})\}$ and $S = \left\{ s = \begin{pmatrix} x & y\varepsilon \\ y & x \end{pmatrix} \mid s \in G \right\}$, then $\mid K_m S \mid = p^{4n-2m-2}(p^2-1)$.

*Proof.* It is clear that

$$K_m \cap S = \left\{ \begin{pmatrix} 1+p^m a & p^m b\varepsilon \\ p^m b & 1+p^m a \end{pmatrix} \mid a, b \in \mathbb{Z}/p^{n-m}\mathbb{Z} \right\},$$

so $\mid K_m \cap S \mid = (p^{n-m})^2$. Since we also have $\mid K_m \mid = (p^{n-m})^4$ and $\mid S \mid = (p^2-1)p^{2n-2}$, we can conclude that $\mid K_m S \mid = \frac{|K_m||S|}{|K_m \cap S|} = p^{4n-2m-2}(p^2-1)$. $\square$

In particular, when $G = \mathrm{GL}(2, \mathbb{Z}/p^{2m}\mathbb{Z})$, we have $\mid K_m S \mid = p^{6m-2}(p^2-1)$; and if $G = \mathrm{GL}(2, \mathbb{Z}/p^{2m+1}\mathbb{Z})$, we have $\mid K_m S \mid = p^{6m+2}(p^2-1)$. The subgroups $S$ and $K_m S$ above play an important role in Chapter 3.
In the $\mathrm{GL}(3, \mathbb{Z}/p^n\mathbb{Z})$ case, there is a similar subgroup and we will now talk about it.
Let $t^3 - ct^2 - bt - a$ be an irreducible polynomial in $\mathbb{Z}/p\mathbb{Z}[t]$. We have a field extension of $\mathbb{Z}/p\mathbb{Z}$ corresponding to the polynomial $t^3 - ct^2 - bt - a$. Call the field extension $\mathbb{Z}/p\mathbb{Z}[\alpha]$, then $\alpha^3 - c\alpha^2 - b\alpha - a = 0$. We know that $\mathbb{Z}/p\mathbb{Z}[\alpha]$ is a 3 - dimensional linear space over $\mathbb{Z}/p\mathbb{Z}$, the basis is $\{1, \alpha, \alpha^2\}$.
Consider

$$1 \to \alpha, \alpha \to \alpha^2, \alpha^2 \to \alpha^3$$

as a linear transformation from $\mathbb{Z}/p\mathbb{Z}[\alpha]$ to $\mathbb{Z}/p\mathbb{Z}[\alpha]$. The corresponding matrix is

$$B = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}.$$

Thus,

$$S' = \{s' = xI + yB + zB^2 \mid s' \in \mathrm{GL}(3, \mathbb{Z}/p\mathbb{Z})\} \cong (\mathbb{Z}/p\mathbb{Z}[\alpha])^\times \Rightarrow \mid S' \mid = p^3 - 1.$$

Consider $B$ above as a matrix in $\mathrm{GL}(3, \mathbb{Z}/p^n\mathbb{Z})$, then

$$S = \{s = xI + yB + zB^2 \mid x, y, z \in \mathbb{Z}/p^n\mathbb{Z}, s \in \mathrm{GL}(3, \mathbb{Z}/p^n\mathbb{Z})\} < \mathrm{GL}(3, \mathbb{Z}/p^n\mathbb{Z}).$$

**Proposition 2.3.5.** Let $S$ be the same as above, then $\mid S \mid = (p^3-1)p^{3n-3}$.

*Proof.* By the same argument as in (2.3.3), we know that $\mid S \mid = \mid S' \mid\mid \ker\psi \mid$. In this case, $\ker\psi = \{s = (1+px)I + pyB + pzB^2 \mid s \in \mathrm{GL}(3, \mathbb{Z}/p^n\mathbb{Z})\}$. Clearly, $\mid \ker\psi \mid = p^{3(n-1)}$, and the proposition follows. $\square$

**Corollary 2.3.6.** Suppose $n > m$. Let $G = \mathrm{GL}(3, \mathbb{Z}/p^n\mathbb{Z}), K_m = \{I + p^m A \mid A \in M_{3\times 3}(\mathbb{Z}/p^{n-m}\mathbb{Z})\}$, and $S$ be the same as above, then $\mid K_m S \mid = (p^3 - 1)p^{9n-6m-3}$.

*Proof.* By the same argument as in Corollary (2.3.4), note that in this case,

$$K_m \cap S = \{s = (1 + p^m x)I + p^m yB + p^m zB^2 \mid x, y, z \in \mathbb{Z}/p^{n-m}\mathbb{Z}\}$$

$$\Rightarrow \mid K_m \cap S \mid = (p^{n-m})^3.$$

The corollary follows. $\square$

Again from the above corollary, when $G = \mathrm{GL}(3, \mathbb{Z}/p^{2m}\mathbb{Z}), \mid K_m S \mid = p^{12m-3}(p^3 - 1)$; and if $G = \mathrm{GL}(3, \mathbb{Z}/p^{2m+1}\mathbb{Z}), \mid K_m S \mid = p^{12m+6}(p^3 - 1)$. As we will see in Chapter 4, the above two subgroups are the stabilizers of the characters of $K_m$ in $\mathrm{GL}(3, \mathbb{Z}/p^{2m}\mathbb{Z})$ and $K_{m+1}$ in $\mathrm{GL}(3, \mathbb{Z}/p^{2m+1}\mathbb{Z})$ respectively.

## 2.4 Conjugacy Classes

Let $R$ be a finite commutative principal local ring of odd characteristic. This means that there is a nilpotent element $\pi \in R$ and a positive integer $\ell$ such that $\pi^\ell = 0$ and every nonzero $x \in R$ can be written as

$$x = u\pi^k$$

for some $u \in R^\times$ and a unique $k, 0 \leq k < \ell$. In particular, $R = R^\times \dot{\cup} \pi R$, so that $R/\pi R = \mathbb{F}_q$, where $q = \mid R/\pi R \mid$ .
Let

$$M = M_2(R) = \{\, 2 \times 2 \text{ matrices over } R \,\}, M_0 = \{\, A \in M : \mathrm{tr}(A) = 0 \,\},$$

$$G = \mathrm{GL}(2, R) = M^\times = \{A \in M : \det(A) \in R^\times\}.$$

Let $I$ denote the identity matrix. Now $G$ acts on $M$ by conjugation, preserving both trace and determinant. It follows that the conjugacy class of an element $g$ in $G$ is equal to the orbit of the same matrix $g \in M$ under this action. Moreover, the action restricts nontrivially to $M_0$ and trivially to $\{\alpha I : \alpha \in R\}$. Since these two subgroups generate $M^+$ additively (provided that $R$ has odd characteristic), it is sufficient to describe the orbits in $M_0$.
We identify some invariant subgroups of $M_0$. Let $L_i = \pi^i M_0, 0 \leq i \leq l$, the subset of matrices all of whose entries are multiples of $\pi^i$. This is invariant under the action of $G$ because the constant $\pi^i$ factors through the conjugation. We have $\{0\} \subseteq L_\ell \subseteq L_{\ell-1} \subseteq \cdots \subseteq L_0 = M_0$.
Let $A \in M_0$ be a matrix that is not in $L_1$. We will find a canonical representative for the similarity class (orbit) of $A$. Form the $2 \times 2$ matrix $B$ over $\mathbb{F}_q = R/\pi R$ by reducing the entries of $A$ modulo $\pi R$. $B$ has trace 0 but is not the zero matrix, hence it is not a multiple of the identity matrix. We therefore know that $B$ is conjugate by some element of $\mathrm{GL}(2, \mathbb{F}_q)$ to the matrix

$\begin{pmatrix} & -\det(B) \\ 1 & \end{pmatrix}$. It follows that $A$ is conjugate by some element of $G$ to the matrix

$$A' = \begin{pmatrix} \pi\alpha & \beta \\ 1+\pi\gamma & -\pi\alpha \end{pmatrix}$$

for some $\alpha, \beta, \gamma \in R$, $\beta \equiv -\det(B) \pmod{\pi R}$. Since

$$\begin{pmatrix} 1 & \pi\alpha \\ & 1+\pi\gamma \end{pmatrix}^{-1} A' \begin{pmatrix} 1 & \pi\alpha \\ & 1+\pi\gamma \end{pmatrix} = \begin{pmatrix} & \pi^2\alpha^2 + (1+\pi\gamma)\beta \\ 1 & \end{pmatrix} = \begin{pmatrix} & -\det(A') \\ 1 & \end{pmatrix},$$

we have shown that the orbit of $A$ contains a unique representative of the form $\begin{pmatrix} & \beta \\ 1 & \end{pmatrix}$. The set of all such matrices, with $\beta \in R$, contains one representative from each orbit in $M_0 \setminus L_1$.

Now consider any matrix $C \in L_i \setminus L_{i+1}, 0 \leq i < \ell$. Thus $C = \pi^i A$ for some matrix $A \in M_0 \setminus L_1$. We will reduce the problem of finding a representative for the orbit of $C$ to the special case we have already solved, but over a different ring. Let $R_i = R/\pi^{\ell-i}R$. There is an additive isomorphism $\theta : R_i \to \pi^i R$ given by $\theta(x) = \pi^i x$. Write $\theta$ also for the corresponding map of $2 \times 2$ matrices, and use bars to denote reduction modulo $\pi^{l-i}R$. Hence, for $g \in G$,

$$gCg^{-1} = g(\pi^i A)g^{-1} = \pi^i gAg^{-1} = \theta(\bar{g}\bar{A}\bar{g}^{-1}).$$

That is, the orbit of $C = \theta(\bar{A})$ under the action of $\mathrm{GL}(2, R)$ is $\theta(O)$ where $O$ is the orbit of $\bar{A}$ under the action of $\mathrm{GL}(2, R_i)$. We know the orbit representatives for this action, because it is the special case considered before. Thus holding $i$ fixed, $C$ must be in the orbit of exactly one of

$$\begin{pmatrix} & \beta \\ \pi^i & \end{pmatrix}_{\beta \in \pi^i R.}$$

There is another special case, $i = \ell$, but it includes only the 0 matrix.

As remarked before, knowledge of the similarity classes of $M_0$ implies knowledge of the similarity classes of $M$ and the conjugacy classes of $G$. The following is a set of similarity class representatives for $M$ : the representatives of $M_0$, plus arbitrary multiples of the identity.

$$\begin{pmatrix} \alpha & \beta \\ \pi^i & \alpha \end{pmatrix}_{0 \leq i \leq l, \alpha \in R, \beta \in \pi^i R.}$$

For conjugacy class representatives of $G$, it suffices to discard singular matrices from the above list. We can make a more useful list at the cost of distinguishing a few cases. First we have the case $i = \ell$, for which the matrix is a multiple of the identity. Otherwise, fix $i < \ell$, and let $\beta = \theta(\gamma), \gamma \in R_i$.

If $\gamma$ is the square of a unit in $R_i$, say $\gamma = \delta^2 \in R_i^\times$, then

$$\begin{pmatrix} \alpha & \beta \\ \pi^i & \alpha \end{pmatrix} \sim \begin{pmatrix} \alpha + \pi^i \delta & \\ & \alpha - \pi^i \delta \end{pmatrix}.$$

Note that $\delta$ is only defined up to sign.

Fix a nonsquare unit $\epsilon$ in $R$. If $\gamma$ is a nonsquare unit in $R_i$, then $\gamma\bar{\epsilon}^{-1}$ is a square, say $\gamma = \delta^2\bar{\epsilon}, \delta \in R_i^\times$. In this case,

$$\begin{pmatrix} \alpha & \beta \\ \pi^i & \alpha \end{pmatrix} \sim \begin{pmatrix} \alpha & \pi^i\delta\epsilon \\ \pi^i\delta & \alpha \end{pmatrix}.$$

Once again, $\delta$ is only defined up to sign.

For the remaining case when $\gamma$ is not a unit, we have $\beta = \pi^{i+1}\beta'$ and therefore

$$\begin{pmatrix} \alpha & \beta \\ \pi^i & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & \pi^{i+1}\beta' \\ \pi^i & \alpha \end{pmatrix}$$

is another class type.

**Summary: Conjugacy Classes of $\mathrm{GL}(2,R)$, where $R = \mathbb{Z}/p^\ell\mathbb{Z}$**

For $0 \le i < \ell$ and non square unit $\epsilon \in R$, we have the following summarization:

| Name of class type | Parameters | Representatives |
|---|---|---|
| $I_\alpha$ | $\alpha \in R^\times$ | $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ |
| $B_{i\alpha\beta}$ | $\alpha \in R^\times, \beta \in R/p^\ell R$ | $\begin{pmatrix} \alpha & p^{i+1}\beta \\ p^i & \alpha \end{pmatrix}$ |
| $C_{i\alpha\beta}$ | $\alpha \in R, \beta \in R^\times, \alpha^2 - \epsilon\beta^2 p^{2i} \in R^\times$ | $\begin{pmatrix} \alpha & p^i\epsilon\beta \\ p^i\beta & \alpha \end{pmatrix}$ |
| $D_{i\alpha\delta}$ | $\alpha, \delta \in R^\times, \alpha - \delta \in p^i R^\times$ | $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ |

| Name | # of classes if $i = 0$ | # of classes if $i > 0$ | Size of class |
|---|---|---|---|
| $I_\alpha$ | $-$ | $(p-1)p^{\ell-1}$ | $1$ |
| $B_{i\alpha\beta}$ | $(p-1)p^{2\ell-2}$ | $(p-1)p^{2\ell-i-2}$ | $(p-1)(p+1)p^{2\ell-2i-2}$ |
| $C_{i\alpha\beta}$ | $\frac{1}{2}(p-1)p^{2\ell-1}$ | $\frac{1}{2}(p-1)^2p^{2\ell-i-2}$ | $(p-1)p^{2\ell-2i-1}$ |
| $D_{i\alpha\delta}$ | $\frac{1}{2}(p-1)(p-2)p^{2\ell-2}$ | $\frac{1}{2}(p-1)^2p^{2\ell-i-2}$ | $(p+1)p^{2\ell-2i-1}$ |

# Chapter 3

# Irreducible character degrees of $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$

In this chapter, we first construct 3 types of irreducible characters of $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$, then count the number to see that we do have all of them. We will also see how to modify the case when $p = 2$.

## 3.1 Construction of Irreducible Characters of $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$

In this section, we will apply Clifford Theory to construct 3 kinds of irreducible characters of $G = \mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$. The main idea of Clifford Theory [5] is as follows.

Let $N \lhd G$ be a normal subgroup of $G$. For any character $\phi$ of $N$, we can define

$$\phi^g : N \to \mathbb{C}; \phi^g(n) = \phi(gng^{-1}), \forall g \in G, n \in N.$$

$\phi^g$, the conjugate to $\phi$, is also a character of $N$. Let $\mathrm{Irr}(N)$ be the set of irreducible characters of N, denote $T = \mathrm{Stab}_G(\phi) = \{g \in G \mid \phi^g = \phi\}$. Let $\phi \in \mathrm{Irr}(N)$, Clifford's Theorem indicates that there exists $\psi \in \mathrm{Irr}(T)$, such that $\psi\mid_N$ is a multiple of $\phi$, namely the inner product $[\psi\mid_N, \phi] \neq 0$, and then the induced character $\mathrm{Ind}_T^G \psi$ is in $\mathrm{Irr}(G)$. Also, the map $\psi \to \psi^G$ is a bijection of $\{\psi \in \mathrm{Irr}(T) \mid [\psi\mid_N, \phi] \neq 0\}$ onto $\{\chi \in \mathrm{Irr}(G) \mid [\chi_N, \phi] \neq 0\}$.

Let $p > 2$ be prime, $\ell \geq 2$ be a positive integer, $R = \mathbb{Z}/p^\ell\mathbb{Z}, m = \lfloor l/2 \rfloor, G = \mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$, and $K_i = \{I + p^iB : B \in M_{2\times2}(R)\}$ for $1 \leq i < \ell$. Note that $K_i \trianglelefteq G$, for all $i$, and that $K_i$ is abelian if $i \geq \ell/2$, because $(I+p^iB)(I+p^iC) = I + p^i(B + C)$. Since

$$K_i = \left\{ \begin{pmatrix} 1 + p^ia & p^ib \\ p^ic & 1 + p^id \end{pmatrix} : 0 \leq a, b, c, d < p^{\ell-i} \right\} \text{ then } |K_i| = p^{4(\ell-i)}.$$

Characters with kernel containing $K_{\ell-1}$ are lifted from the quotient group $G/K_{\ell-1} \cong \mathrm{GL}(2, Z/p^{\ell-1}\mathbb{Z})$; we assume that these are already known inductively.

We first describe the characters of the abelian group $K_i$, $i \geq l/2$. Assign a fixed injective homomorphism $\lambda : \mathbb{Z}/p^\ell\mathbb{Z} \to \mathbb{C}^\times$ and let $A \in M_{2\times 2}(\mathbb{Z}/p^\ell\mathbb{Z})$, then $\phi_A$ defined as

$$\phi_A(I + p^i B) = \lambda(\mathrm{tr}(p^i AB))$$

is clearly a character on $K_i$ of degree 1. Notice that $\phi_A = \phi_{A+p^{\ell-i}A'}$, we only need to consider those matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $0 \leq a, b, c, d < p^{\ell-i}$. Let

$$A \to \bar{A}, \quad \mathrm{GL}(2, \mathbb{Z}/p^\ell) \to \mathrm{GL}(2, \mathbb{Z}/p^{\ell-i})$$

be the natural map, then $\phi_A$ is determined by $\bar{A} \in \mathrm{GL}(2, \mathbb{Z}/p^{\ell-i})$. To simplify notations, we can consider $A$ to be a matrix over $\mathbb{Z}/p^{\ell-i}\mathbb{Z}$ and then we have that the irreducible characters $\phi_A$ of $K_i$ are in correspondence with $2 \times 2$ matrices $A$ over $\mathbb{Z}/p^{\ell-i}\mathbb{Z}$. By the definition of $K_i$, we can also treat $B$ as a matrix over $\mathbb{Z}/p^{\ell-i}\mathbb{Z}$. We can also find injective homomorphism $\lambda' : \mathbb{Z}/p^{\ell-i}\mathbb{Z} \to \mathbb{C}^\times$ such that $\lambda(\mathrm{tr}(p^i AB)) = \lambda'(\mathrm{tr}(AB))$. Thus, when there is no confusion, we can also define $\phi_A$ this way:

$$\phi_A(I + p^i B) = \lambda(\mathrm{tr}(AB))$$

where $\lambda : \mathbb{Z}/p^{\ell-i}\mathbb{Z} \to \mathbb{C}^\times$ is an injective homomorphism. In the following sections, we may use different versions of $\phi_A$ for different purposes and this does not cause any confusion once we use the above identification.

An element $g \in G$ acts on $K_i$ by conjugation via $(I + p^i B)^g = I + p^i gBg^{-1}$, and thus $g$ also acts on the characters of $K_i$ via

$$\begin{aligned}(\phi_A)^g(I + p^i B) &= \phi_A(I + p^i gBg^{-1}) \\ &= \lambda(\mathrm{tr}(p^i AgBg^{-1})) \\ &= \lambda(\mathrm{tr}(p^i g^{-1}AgB)) \\ &= \phi_{A^{g^{-1}}}(I + p^i B).\end{aligned}$$

The stabilizer of $\phi_A$ is

$$T = \mathrm{Stab}_G(\phi_A) = \{g \in G : gA = Ag\}.$$

Clifford's theorems imply that all the characters of $G$ can be obtained by inducing from $T$ to $G$ all possible characters $\psi$ of $T$ that restrict to multiples of $\phi_A$ on $K_i$.

When $\ell = 2m$ is even, the existence of the abelian normal subgroup $K_m$ (with $i = m$) allows characters to be constructed easily. The process is as follows:

$$\begin{array}{ccccc} K_m & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array}$$

where $\phi_A(I+p^m B) = \lambda(\mathrm{tr}(p^m AB))$, $T = \mathrm{Stab}_G(\phi_A) = K_m S$ for some subgroup $S$ of $G$, depending on the choices of $A$. One nice thing in the even case is that we can always extend $\phi_A$ to $\psi$. Moreover, we also have an explicit formula for $\psi$, which helps us a lot to find values of $\chi$ later. By Clifford's theorem, we know $\chi = \mathrm{Ind}_T^G \psi$ is irreducible.

Starting with different matrices $A$ to define $\phi_A$ on $K_m$ will give us different $S$ for the stabilizer $T$, and therefore will end up giving us irreducible character $\chi$ of $G$ with different degrees. We have three cases and this gives us all the degrees we need.

i) $A = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$, $k \in (\mathbb{Z}/p^m\mathbb{Z})^\times$. In this case, we have $S = \left\{ \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix} \right\}$. The construction will give us irreducible characters of $G$ with degree $(p+1)p^{\ell-1}$.

ii) $A = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}$, where $\epsilon$ is a non-square unit. We have $S = \left\{ \begin{pmatrix} w & \epsilon y \\ y & w \end{pmatrix} \right\}$ and we can get irreducible characters of $G$ with degree $(p-1)p^{\ell-1}$.

iii) $A = \begin{pmatrix} 0 & p\beta \\ 1 & 0 \end{pmatrix}$, $\beta \in \mathbb{Z}/p^m\mathbb{Z}$. We get $S = \left\{ \begin{pmatrix} w & p\beta y \\ y & w \end{pmatrix} \right\}$ and we will have irreducible characters of $G$ with degree $(p^2-1)p^{\ell-2}$.

If $\ell = 2m+1$ is odd, the construction is a little more complicated. Notice that $K_m$ in this case is not abelian, we start with the normal subgroup $K_{m+1}$ and define $\phi_A$ on $K_{m+1}$ using the same formula as before. We also have $T = \mathrm{Stab}_G(\phi_A) = K_m S$ for some subgroup $S$, depending the choices of $A$. Unlike the even case, we cannot extend $\phi_A$ to $T$ directly. Instead, we can construct irreducible characters $\psi \in \mathrm{Irr}(T)$, with degree $p$, such that $[\psi\,|_{K_{m+1}}, \phi_A] \neq 0$. By Clifford's theorem, we know $\mathrm{Ind}_T^G \psi \in \mathrm{Irr}(G)$.
When we use $A = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$ or $A = \begin{pmatrix} 0 & p\beta \\ 1 & 0 \end{pmatrix}$ to define $\phi_A$ on $K_{m+1}$, we will end up with finding irreducible characters of $G$ with degree $(p+1)p^{\ell-1}$ and $(p^2-1)p^{\ell-2}$ respectively. The construction process is as follows.

$$\begin{array}{ccccccccc} K_{m+1} & \longrightarrow & N & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi'_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ind}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array}.$$

$T = \mathrm{Stab}_G(\phi_A)$. We pick a proper normal subgroup $N \lhd T$ so that we extend $\phi_A$ to $\phi'_A$ naturally. $T' = \mathrm{Stab}_T(\phi'_A)$ and we can also extend $\phi'_A$ to $\phi'$ of $T'$. By Clifford's theorem, we know that $\psi = \mathrm{Ind}_{T'}^T(\phi')$ is irreducible and clearly $[\psi\,|_{K_{m+1}}, \phi_A] \neq 0$. Therefore, $\chi = \mathrm{Ind}_T^G \psi$ is irreducible.

If we start with $A = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}$ to define $\phi_A$, we will get irreducible characters of $G$ with degree $(p-1)p^{\ell-1}$ and here is the construction process.

$$\begin{array}{ccccccccccc} K_{m+1} & \longrightarrow & N_{m+1} & \longrightarrow & H & \longrightarrow & N_m & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ext}} & \phi'' & \xrightarrow{\text{ind}} & \theta & \xrightarrow{\text{ext}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array},$$

where

$$N_i = K_i(K_1 \cap S)\left\{\begin{pmatrix} a & \\ & a \end{pmatrix}\right\}, H = N_{m+1}\left\langle\begin{pmatrix} 1+p^m & \\ & 1 \end{pmatrix}\right\rangle.$$

The extensions $\phi'$ and $\phi''$ are constructible. $\theta$ is irreducible because $\mathrm{Stab}_{N_m}(\phi'') = H$. $\theta$ is extendible to $\psi$ because $\theta$ is stable under $T$ and $T/N_m$ is cyclic. Eventually, we have irreducible character $\chi$ of $G$ with degree $(p-1)p^{\ell-1}$. More details of the constructions will be given in later sections when we need to evaluate irreducible character of $\chi$ in each case.

## 3.2 The number of characters

In the last section, we know we can construct 3 kinds of irreducible characters of $G$. Now, we want to count the number of each kind and see that we actually have all the irreducible characters of each degree. From the constructions we had before and the work of Nobs [1], we have the following table(*).

| $A$ | $S$ | $\deg\chi$ | number $\chi$ of this degree |
|---|---|---|---|
| $\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}$ | $(p+1)p^{\ell-1}$ | $\frac{1}{2}(p-1)^3p^{2\ell-3}$ |
| $\begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} w & \epsilon y \\ y & w \end{pmatrix}$ | $(p-1)p^{\ell-1}$ | $\frac{1}{2}(p-1)(p^2-1)p^{2\ell-3}$ |
| $\begin{pmatrix} 0 & p\beta \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} w & p\beta y \\ y & w \end{pmatrix}$ | $(p^2-1)p^{\ell-2}$ | $(p-1)p^{2\ell-2}$ |

Recall that we constructed 3 kinds of irreducible characters using the simple matrices $A$ as in the above table, but one single $A$ does not give us the complete corresponding irreducible characters $\chi$ of each degree. Fix one injective $\lambda : \mathbb{Z}/p^\ell\mathbb{Z} \to \mathbb{C}^\times$, in order to get all the irreducible characters of the above three degrees, we start with more general matrices $A'$, such that the stabilizer $T$ stays the same in each case. This will give us more irreducible characters of $G$ with the same corresponding degree. Since starting with conjugate characters $\phi_A$ will give us the same irreducible character $\chi$ of $G$ and $\phi_{A_1}$ is conjugate to $\phi_{A_2}$ if and only if $A_1$ is conjugate to $A_2$, counting the number of non-conjugate matrices of $A'$ in each case can give us the number of the irreducible characters of $G$ of each degree.

Since our constructions in the even and odd cases are different, we will first count the number in the even case and then do it similarly in the odd case.

### 3.2.1 $G = \mathrm{GL}(2, \mathbb{Z}/p^{2m}\mathbb{Z})$, i.e. $\ell = 2m$

Recall that in this case the general process to construct irreducible characters of $G$ is as follows:

$$\begin{array}{ccccc} K_m & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array}$$

where $\phi_A(I + p^m B) = \lambda(\mathrm{tr}(p^m AB))$, $T = \mathrm{Stab}_G(\phi_A)$, $\psi$ is an extension of $\phi_A$ and $\chi = \mathrm{Ind}_T^G \psi$. By Clifford theory, we know $\chi$ is irreducible and we have three kinds of irreducible characters of $G$. In each case, we have

$$\# \text{ of irreducible characters of } G = \# \text{ of non-conjugate } \phi_A \times \# \text{ of extensions },$$

so we count how many non-conjugate $\phi_A$ we can get in each case.

In the first case where $A = \begin{pmatrix} k & \\ & \end{pmatrix}$, we can start with more general matrices

$$A_{\alpha,k} = \alpha I + A = \begin{pmatrix} \alpha + k & \\ & \alpha \end{pmatrix}, \text{ where } \alpha \in \mathbb{Z}/p^m\mathbb{Z}, k \in (\mathbb{Z}/p^m\mathbb{Z})^\times.$$

We can define $\phi_{A_{\alpha,k}}$ on $K_m$ using the same formula as $\phi_A$. Later we will see

$$\mathrm{Stab}_G(\phi_{A_{\alpha,k}}) = \mathrm{Stab}_G(\phi_A) = T$$

and we can also extend $\phi_{A_{\alpha,k}}$ to $T$. Therefore, we will get irreducible characters of $G$ with degree $(p+1)p^{2m-1}$. Notice that $\begin{pmatrix} \alpha + k & \\ & \alpha \end{pmatrix}$ is conjugate to $\begin{pmatrix} \alpha & \\ & \alpha + k \end{pmatrix}$, hence the total number of non-conjugate $A_{\alpha,k}$, and so of non-conjugate $\phi_{A_{\alpha,k}}$ is

$$\frac{1}{2}p^m(p^m - p^{m-1}) = \frac{1}{2}p^{2m-1}(p-1) = \frac{1}{2}(p-1)p^{\ell-1}.$$

Multiplying it by the number of extensions from $K_m$ to $T$, which is $(p-1)^2p^{\ell-2}$in this case, gives us all the irreducible characters of degree $(p+1)p^{\ell-1}$.

In the second case when $A = \begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix}$, we can start with general matrices

$$A_{\alpha,\epsilon} = \alpha I + A = \begin{pmatrix} \alpha & \epsilon \\ 1 & \alpha \end{pmatrix}$$

where $\alpha \in \mathbb{Z}/p^m\mathbb{Z}$ and $\epsilon$ is a non-square unit in $\mathbb{Z}/p^m\mathbb{Z}$. Notice that $A_{\alpha,\epsilon}$ is conjugate to $A_{\alpha',\epsilon'}$ if and only if $\alpha = \alpha', \epsilon = \epsilon'$. The number of non-square unit

$\epsilon$ is $\frac{p^m - p^{m-1}}{2}$, therefore we have $\frac{1}{2}(p^m - p^{m-1})p^m = \frac{1}{2}(p-1)p^{\ell-1}$ non conjugate characters $\phi_{A_{\alpha,\epsilon}}$ on $K_m$, which is exactly what we need to complete this case.

The last case when $A = \begin{pmatrix} & p\beta \\ 1 & \end{pmatrix}$, we can use general matrices

$$A_{\alpha,\beta} = \alpha I + A = \begin{pmatrix} \alpha & p\beta \\ 1 & \alpha \end{pmatrix} \text{ where } \alpha \in \mathbb{Z}/p^m\mathbb{Z} \text{ and } \beta \in (\mathbb{Z}/p^{m-1}\mathbb{Z})^{\times}.$$

By counting the number of $\alpha$ and $\beta$, we have $p^m p^{m-1} = p^{\ell-1}$ non conjugate $\phi_{\alpha,\beta}$ on $K_m$ and this is exactly what we need.

### 3.2.2 $G = \mathrm{GL}(2, \mathbb{Z}/p^{2m+1}\mathbb{Z})$, i.e. $\ell = 2m+1$

Now the construction of irreducible characters of each kind is different from the even case. We will count the number case by case.

(i) $A = \begin{pmatrix} k & \\ & \end{pmatrix}, k \in (\mathbb{Z}/p^m\mathbb{Z})^{\times}$. Starting with $\phi_A$ on $K_{m+1} \lhd G$, we end up with irreducible characters of $G$ with degree $(p+1)p^{\ell-1}$. The process is as follows:

$$\begin{array}{ccccccccc} K_{m+1} & \longrightarrow & N & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi'_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ind}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array}$$

where $N = \left\{ \begin{pmatrix} 1+p^m a & p^{m+1}b \\ p^m c & 1+p^m d \end{pmatrix} \right\}, T' = \left\{ \begin{pmatrix} a & p^{m+1}b \\ p^m c & d \end{pmatrix} \right\}$, and $T = \left\{ \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix} \right\}$. $N \lhd T$ and $\phi'_A(n) = \lambda(\mathrm{tr}(A(n-I))), \forall n \in N$. It is easy to check that $\phi'_A$ is a one-dimensional character and an extension of $\phi_A$. We also have $\mathrm{Stab}_T(\phi'_A) = T'$ and we can extend $\phi'_A$ to $\phi'$. By Clifford theory, we know $\psi = \mathrm{Ind}_{T'}^{T} \phi' \in \mathrm{Irr}(T)$, and therefore, $\chi = \mathrm{Ind}_T^G \psi = \mathrm{Ind}_{T'}^{T} \phi' \in \mathrm{Irr}(G)$. To count how many irreducible characters of $G$ we can get in this case, we need to look at the following piece of the construction:

$$\begin{array}{ccccc} N & \longrightarrow & T' & \longrightarrow & T \\ \phi'_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ind}} & \psi \end{array}.$$

Similar to the argument in the even case, we know that non-conjugate $\phi'_A$ on $T'$ can give us different $\psi$ on $T$, and eventually will give us different $\chi \in \mathrm{Irr}(G)$. Therefore, we want to count how many non-conjugate $\phi'_A$ we have on $T'$. By making $A$ into more general matrices

$$A_{\alpha,k} = \alpha I + kA = \begin{pmatrix} \alpha + k & \\ & \alpha \end{pmatrix}, \text{ where } \alpha \in \mathbb{Z}/p^{m+1}\mathbb{Z}, k \in (\mathbb{Z}/p^{m+1}\mathbb{Z})^{\times},$$

we have $\frac{1}{2}(p^{m+1}-p^m)p^{m+1} = \frac{1}{2}p^\ell(p-1)$ non-conjugate $\phi'_{A_{\alpha,k}}$ on $N$. Notice that the number of extensions from $N$ to $T'$ is $\frac{|T'|}{|N|} = p^{\ell-3}(p-1)^2$, we have $\frac{1}{2}p^\ell(p-1)p^{\ell-3}(p-1)^2 = \frac{1}{2}(p-1)^3p^{2\ell-3}$ irreducible characters of $G$ with degree $(p+1)p^{\ell-1}$.

(ii) Now let us consider the case when $A = \begin{pmatrix} & p^j\beta \\ 1 & \end{pmatrix}$. We can find irreducible characters of $G$ with degree $(p-1)p^{2\ell-2}$ and the process to construct these characters is very similar to the above case. We have

$$\begin{array}{ccccccccc} K_{m+1} & \longrightarrow & N & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi'_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ind}} & \psi & \xrightarrow{\text{ind}} & \psi \end{array}$$

where

$$N = \left\{ \begin{pmatrix} 1+p^m a & p^{m+1}b \\ p^m c & 1+p^m d \end{pmatrix} \right\}, T' = \left\{ \begin{pmatrix} a & p^j\beta b + p^{m+1}c \\ b & a+p^m d \end{pmatrix} \right\},$$

and

$$T = \left\{ \begin{pmatrix} a & p^j\beta b + p^m c \\ b & a+p^m d \end{pmatrix} \right\}.$$

By a similar argument, we only need to look at the following piece

$$\begin{array}{ccccc} N & \longrightarrow & T' & \longrightarrow & T \\ \phi'_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ind}} & \psi \end{array}$$

and we can start with general matrices

$$A_{\alpha,\beta} = \alpha I + A = \begin{pmatrix} \alpha & p\beta \\ 1 & \alpha \end{pmatrix}$$

where $\alpha \in \mathbb{Z}/p^{m+1}\mathbb{Z}$ and $\beta \in \mathbb{Z}/p^m\mathbb{Z}$ to define $\phi_{A_{\alpha,\beta}}$ on $N$. The number of such $\phi_{\alpha,\beta}$ is $p^m p^{m+1} = p^\ell$ and the number of extensions in this case is $\frac{|T'|}{|N|} = (p-1)p^{\ell-2}$. Therefore, we have $(p-1)p^{2\ell-2}$ irreducible characters of $G$ with degree $(p-1)p^{\ell-1}$.

(iii) Now consider the case $A = \begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix}$. The process to construct irreducible characters of $G$ with degree $(p^2-1)p^{\ell-2}$ is as follows:

$$\begin{array}{ccccccccccc} K_{m+1} & \longrightarrow & N_{m+1} & \longrightarrow & H & \longrightarrow & N_m & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ext}} & \phi'' & \xrightarrow{\text{ind}} & \theta & \xrightarrow{\text{ext}} & \psi & \xrightarrow{\text{ind}} & \operatorname{Ind}_T^G \psi \end{array}$$

where

$$S = \left\{ \begin{pmatrix} x & y\epsilon \\ y & x \end{pmatrix} \right\}, N_i = K_i(K_1 \cap S) \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \right\}, H = N_{m+1} \left\langle \begin{pmatrix} 1+p^m & \\ & 1 \end{pmatrix} \right\rangle.$$

To count how many $\mathrm{Ind}_T^G \psi$ we can get, it is equivalent to count the number of $\psi$ on $T$. With this construction process, we always have the following properties:

$$\theta\,|_{N_{m+1}} = p\phi', \theta\,|_{N_m - N_{m+1}} = 0.$$

Therefore, later we only need to consider the extensions from $K_{m+1}$ to $N_{m+1}$ and $N_m$ to $T$.
Generalize matrix $A$ with

$$A_{\alpha,\epsilon} = \alpha I + A = \begin{pmatrix} \alpha & \epsilon \\ 1 & \alpha \end{pmatrix}$$

where $\alpha \in \mathbb{Z}/p^m\mathbb{Z}$ and $\epsilon$ is a non-square unit in $\mathbb{Z}/p^m\mathbb{Z}$, we have $\frac{1}{2}(p^m - p^{m-1})p^m = \frac{1}{2}(p-1)p^{\ell-2}$ non-conjugate $\phi_{A_{\alpha,\epsilon}}$ on $K_{m+1}$. The number of extensions from $\phi_{A_{\alpha,\epsilon}}$ to $\phi'$ is $\frac{|N_{m+1}|}{|K_{m+1}|} = (p-1)p^{\ell-1}$, the number of extensions from $\theta$ to $\psi$ is $\frac{|T|}{|N_m|} = p+1$. Therefore, we have total number of $\frac{1}{2}(p-1)p^{\ell-2}(p-1)p^{\ell-1}(p+1) = \frac{1}{2}(p-1)(p^2-1)p^{2\ell-3}$ irreducible characters of $G$ with degree $(p^2-1)p^{\ell-2}$.

## 3.3 About Character Values

Now we will show that as long as we have the values of the irreducible characters constructed using the simple matrix $A$ in the table (*), we can also deduce the character values of the remaining ones easily. Let $\chi_A$ be an irreducible character of $G$ constructed using simple matrix $A$ in the table at the beginning of this section, and let $\chi_{A'}$ be an irreducible character of $G$ constructed using corresponding general matrix $A'$. We will give a formula that relates $\chi_{A'}$ and $\chi_A$.

Let us use the notations in the case when $G = \mathrm{GL}(2, \mathbb{Z}/p^{2m}\mathbb{Z})$, $A = \begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix}$, and the general $A' = A_{\alpha,\epsilon} = \alpha I + A$. The construction process is as follows:

$$\begin{array}{ccccc} K_m & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array}.$$

We will prove the formula in this case and all the other cases follow exactly the same way.
Given $\lambda : \mathbb{Z}/p^\ell\mathbb{Z} \longrightarrow C^\times$, recall the definition of $\phi_A$ and $\phi_{A_{\alpha,\epsilon}}$, we have

$$\begin{aligned} \phi_{A_{\alpha,\epsilon}}(I + p^m B) &= \lambda(\mathrm{tr}(p^m A_{\alpha,\epsilon} B)) \\ &= \lambda(\mathrm{tr}(p^m \alpha B + p^m AB)) \\ &= \lambda^\alpha(\mathrm{tr}(p^m B)\lambda(p^m AB) \\ &= \lambda^\alpha(\mathrm{tr}(p^m B))\phi_A(I + p^m B) \end{aligned}$$

where $\lambda^\alpha(g) = \lambda(\alpha g)$.
Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $\mathrm{tr}(B) = (a+d)$ and $\det(I + p^m B) = 1 + p^m(a+d)$.
Note that $\{1 + p^m x\}^\times \cong \{p^m x\}^+$ in $\mathbb{Z}/p^\ell\mathbb{Z}$, we can find a character $\mu_\alpha = \mu : (\mathbb{Z}/p^\ell\mathbb{Z})^\times \to \mathbb{C}^\times$ such that

$$\lambda^\alpha(\mathrm{tr}(p^m B)) = \mu(\det(I + p^m B)).$$

Therefore,

$$\phi_{A_{\alpha,\epsilon}} = (\mu \circ \det) \times \phi_A.$$

Notice that $\mu \circ \det$ is a linear character of $G$, and hence it is stable under $G$, we have $\mathrm{Stab}_G(\phi_{A_{\alpha,\epsilon}}) = \mathrm{Stab}_G(\phi_A)$ and $(\mu \circ \det) \times \psi$ is an extension of $\phi_{A_{\alpha,\epsilon}}$ provided that $\psi$ is an extension of $\phi_A$. It is clear that

$$\chi_{A_{\alpha,\epsilon}} = \mathrm{Ind}_T^G[(\mu \circ \det) \times \psi] = (\mu \circ \det) \times \mathrm{Ind}_T^G \psi = (\mu \circ \det) \times \chi.$$

From the above formula, we know that as long as we have the character values of $\chi_A$, we can have all the remaining character values easily. We will start to evaluate $\phi_A$ in the next chapter.

## 3.4 When $p = 2$

In section 3.1, we used 3 types of matrices $A$ to define $\phi_A$ and eventually got 3 types of irreducible characters of $G$. Now if $p = 2$, two types of construction work exactly the same way as before and the only one that needs to be modified is the case where we used $A = \begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix}$. The reason we chose matrix $\begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix}$ before was that this matrix can generate the group $\left\langle \begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix} \right\rangle$ of order $p^2 - 1$ in the field case. Now if $p = 2$, the matrix that plays this role is $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and we can use this type of matrix to define $\phi_A$ and eventually get all the irreducible characters of degree $(p-1)p^{\ell-1}$.
In the even case when $\ell = 2m$, we still have the same process

$$\begin{array}{ccccc} K_m & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array}$$

where $\phi_A$ and $K_m$ are defined the say way as before and $T = K_m S$ with $S$ abelian. The general matrices we can use are $A_{\alpha,\beta} = \begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix}$ where $\alpha$ and $\beta$ are units in $\mathbb{Z}/p^m\mathbb{Z}$, so that $A_{\alpha,\beta}$ is mapped to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ by the natural map from $\mathrm{GL}(2,\mathbb{Z}/p^{2m}\mathbb{Z})$ to $\mathrm{GL}(2,\mathbb{Z}/p\mathbb{Z})$. By looking at the trace and determinant, we can see that $A_{\alpha,\beta}$ is conjugate to $A_{\alpha',\beta'}$ if and only if $\alpha=\alpha',\beta=\beta'$. Therefore, the number of such non-conjugate matrices $A_{\alpha,\beta}$ are

$$(p^m-p^{m-1})^2=p^{2m-2}(p-1)^2=2^{\ell-2},$$

which is the same as $\frac{1}{2}(p-1)p^{\ell-1}$ for $p=2$. The number of extensions from $\phi_A$ to $\psi$ is $\mid T:K_m \mid = p^{\ell-2}(p^2-1)$, which is also the same as before. Therefore, we can have all the $(p-1)p^{\ell-1}$-dimensional irreducible characters of $\mathrm{GL}(2,\mathbb{Z}/2^\ell\mathbb{Z})$. As in the odd case before, we will have the process

$$\begin{array}{ccccccccc} K_{m+1} & \longrightarrow & N & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi'_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ind}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array}$$

when $\ell=2m+1$. Just making the general matrices $A_{\alpha,\beta}=\begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix}$, we have the similar way to count the total number of irreducible characters in this case as well.

To summarize, the formulas we found before about irreducible degrees and number of each degree also work when $p=2$.

## 3.5 Replacing $\mathbb{Z}/p^\ell\mathbb{Z}$ by $R/P^\ell$

We have seen formulas of the irreducible degrees and number of each degree for group $\mathrm{GL}(2,\mathbb{Z}/p^\ell\mathbb{Z})$. Now let $R$ be the ring of integers of a local or global field, and $P$ be a prime ideal of $R$ such that $R/P$ is a field of $q$ elements, $q$ odd. We will see that, by replacing $p$ by $q$ in the previous sections, the formulas about character degrees and the number of each degree also work for group $G=\mathrm{GL}(2,R/P^\ell)$. We will look at one construction example when $\ell=2$ and the other cases are similar.

Now $R/P$ is a field of $q$ elements, then we can get that $\mid \mathrm{GL}(2,R/P^\ell)\mid = q^{4\ell-4}(q^2-1)(q^2-q)$. In particular, $\mid \mathrm{GL}(2,R/P^2)\mid = q^4(q^2-1)(q^2-q)$. Let $\epsilon$ be a non-square unit in $R/P$ and let $A=\begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}$. Let $K_1 = \left\{\begin{pmatrix} 1+p_1 & p_2 \\ p_3 & 1+p_4 \end{pmatrix}, p_i\in P/P^2\right\} = \{I+B, B\in M(2,P/P^2)\}$, then $\mid K_1\mid = q^4$.

Define $\phi_A : K_1\to\mathbb{C}^\times, \phi_A(I+B)=\lambda(\mathrm{tr}(AB))$, that is

$$\phi_A\begin{pmatrix} 1+p_1 & p_2 \\ p_3 & 1+p_4 \end{pmatrix} = \lambda(\epsilon p_3+p_2),$$

where $\lambda : R/P^2 \to \mathbb{C}^\times$ is a homomorphism such that $P \notin \ker(\lambda)$. The stabilizer of $\phi$ is

$$T = \mathrm{Stab}_G(\phi_A) = K_1 S \text{ where } S = \left\{ \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix} \right\}.$$

By similar arguments as before, we have $\mid S \mid = (q^2-1)q^2, \mid T \mid = (q^2-1)q^4$. We also have that $T/K_1$ is cyclic, therefore we can extend $\phi_A$ to $\psi$ of $T$ and the following construction process:

$$\begin{array}{ccccc} K_1 & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array}.$$

Now $\chi \in \mathrm{Irr}(G)$ and $\deg(\chi) = \frac{|G|}{|T|} = q^2 - q = q(q-1)$.
We can also count the number of this degree by the same method as before. The general matrices we can use to replace $A$ before are

$$A_{\alpha,\epsilon} = \begin{pmatrix} \alpha & \epsilon \\ 1 & \alpha \end{pmatrix}, \alpha \in R/P.$$

Since the number of non-square unit $\epsilon$ is $\frac{q-1}{2}$, we have $\frac{1}{2}q(q-1)$ non-conjugate $A_{\alpha,\epsilon}$. The number of extensions from $\phi_A$ to $\psi$ is equal to

$$\frac{\mid T \mid}{\mid K_1 \mid} = q^2 - 1.$$

Therefore, there are

$$\frac{1}{2}q(q-1)(q^2-1)$$

irreducible characters of degree $q(q-1)$.
Notice that the corresponding formulas of this degree and number of this degree formula in previous sections are $p(p-1)$ and $\frac{1}{2}p(p-1)(p^2-1)$ respectively, the only difference here is that we replace $p$ by $q$.

# Chapter 4

# Character Values

We will evaluate the values of irreducible characters we constructed in the last chapter. Some proofs are skipped and can be found in [10].

## 4.1 Values for Characters with Degree $(p+1)p^{\ell-1}$

In this section, we will first find values for characters of degree $(p+1)p^{\ell-1}$ by parabolic induction. We will use Clifford Theory to construct irreducible characters with the same degree and show that these two kinds of irreducible characters are the same.

### 4.1.1 Character Values by Parabolic Induction

Here $\ell$ can be any positive integer. Let $B \subset G = \mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$ be the Borel subgroup $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}$. Let $\lambda$ be injective character of $R^\times$, let $\phi$ be the character of $B$ given by $\phi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \lambda(a)$, and let $\chi = \mathrm{Ind}_B^G \phi$.
Claim: $\chi$ is irreducible.

*Proof.* It suffices to show $[\chi, \chi] = [\phi, \chi \mid_B] = 1$. By Mackey's Theorem, we have

$$\chi \mid_B = \sum_{G = \cup BgB} \mathrm{Ind}^B_{gBg^{-1} \cap B}(\phi_g), \text{where } \phi_g(gXg^{-1}) = \phi(X), X \in B.$$

In order to calculate $[\phi, \chi \mid_B]$, we want to look at $[\phi, \mathrm{Ind}^B_{gBg^{-1}\cap B}(\phi_g)] = [\phi \mid_{gBg^{-1}\cap B}, \phi_g]$ for each double coset representative $g$ of $B$. Pick the double coset representatives of $B$ to be

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, g_i = \begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix}, 1 \le i \le \ell.$$

Let $X = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, we have $gXg^{-1} = \begin{pmatrix} c & 0 \\ b & a \end{pmatrix}$. Therefore,

$$gBg^{-1} \bigcap B = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \right\}, \phi \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} = \lambda(a), \phi_g \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} = \lambda(c).$$

Since $\phi \neq \phi_g$ and they both have degree 1, we know $[\phi \mid_{gBg^{-1} \cap B}, \phi_g] = 0$. Similarly we have

$$g_i B g_i^{-1} \bigcap B = \left\{ \begin{pmatrix} p^{\ell-i}a + c & b \\ 0 & p^i b + c \end{pmatrix} \right\}, \text{ and } [\phi \mid_{g_i B g_i^{-1} \cap B}, \phi_{g_i}] = 0,$$

when $1 \le i < \ell$. It is clear that $[\phi \mid_{g_\ell B g_\ell^{-1} \cap B}, \phi_{g_\ell}] = 1$ and hence, $[\phi, \chi_B] = [\chi, \chi] = 1$.

□

To find the character value on an arbitrary conjugacy Class $C$, we use the following formula:

$$\chi(C) = \frac{[G:B]}{\mid C \mid} \sum_{a \in R^\times} \lambda(a) \left| \left\{ (b,c) \in R^2 : \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in B \cap C \right\} \right|. \quad (*)$$

The result is as follows:

$$\chi(D_{iac}) = p^i(\lambda(a) + \lambda(c)), \chi(B_{(\ell-1)\alpha 0}) = p^{\ell-1}\lambda(a),$$

$$\chi(I_\alpha) = \lambda(\alpha) \deg(\chi) = (p+1)p^{\ell-1}\lambda(\alpha)$$

and $\chi$ is 0 on all other conjugacy classes.

### 4.1.2 Character Values by Clifford's Theorem

Now we will use Clifford's theorem to construct irreducible characters of $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$ with degree $p^{\ell-1}(p+1)$ and we will see that they have the same character values as $\chi$ in the last section. We first assume $\ell = 2m$ and will talk about the odd case later.
Let $\lambda : (\mathbb{Z}/p^{2m}\mathbb{Z})^\times \to \mathbb{C}^\times$ be injective. Define

$$\phi' : K_m \to \mathbb{C}^\times, \phi' \begin{pmatrix} 1 + p^m a & p^m b \\ p^m c & 1 + p^m d \end{pmatrix} = \lambda(1 + p^m a),$$

then we have $\mathrm{Stab}_G(\phi') = T = \left\{ \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix} \right\}$. We also have

$$\phi : T \to \mathbb{C}^\times, \phi \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix} = \lambda(a)$$

is an extension of $\phi'$. By Clifford Theory, $\psi = \mathrm{Ind}_T^G \phi \in \mathrm{Irr}(G)$, and $\deg(\psi) = p^{\ell-1}(p+1)$.

Next, we want to find the character values of $\psi$.

**Lemma 4.1.** *If $u, v \in R^\times$ and $m \leq k < l$, then* $\sum_{0 \leq t < p^m} \lambda(u + p^k tv) = 0$.

Proof.

$$\begin{aligned} \sum_{0 \leq t < p^m} \lambda(u + p^k tv) &= \sum_{0 \leq t < p^m} \lambda(u)\lambda(1 + p^k tvu^{-1}) \\ &= \lambda(u) \sum_{0 \leq t < p^m} \lambda(1 + p^k t) \\ &= 0. \end{aligned}$$

Pick the coset representatives of $T$ to be

$$E_{xy} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, F_{xz} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} pz & 1 \\ 1 & pz \end{pmatrix}, 0 \leq x, y, pz < p^m.$$

We first evaluate $\psi(C_{i\alpha\beta})$. Since $T \cap C_{i\alpha\beta} = \varnothing$ if $i < m$, we assume $l > i \geq m$. Notice that

$$E_{xy} \begin{pmatrix} \alpha & p^i \epsilon\beta \\ p^i\beta & \alpha \end{pmatrix} E_{xy}^{-1} = \begin{pmatrix} \alpha - p^i\epsilon\beta y + p^i\beta x(1 - \epsilon y^2) & p^m * \\ p^m * & * \end{pmatrix},$$

$$F_{xz} \begin{pmatrix} \alpha & p^i \epsilon\beta \\ p^i\beta & \alpha \end{pmatrix} F_{xz}^{-1}$$

$$= \begin{pmatrix} [\alpha + p^{i+1}\beta z(\epsilon - 1)(1 - p^2z^2)^{-1}] + p^i x\beta(\epsilon - p^2z^2)(1 - p^2z^2)^{-1} & * \\ p^m * & * \end{pmatrix},$$

$$\psi(C_{i\alpha\beta}) = \sum_{0 \leq x,y < p^m} \lambda[(\alpha - p^i\epsilon\beta y) + p^i x\beta(1 - \epsilon y^2)] +$$

$$\sum_{0 \leq pz, x < p^m} \lambda\{[\alpha + p^{i+1}\beta z(\epsilon - 1)(1 - p^2z^2)^{-1}] + p^i x\beta(\epsilon - p^2z^2)(1 - p^2z^2)^{-1}\}.$$

By lemma 4.1,

$$\sum_{0 \leq x < p^m} \lambda[(\alpha - p^i\epsilon\beta y) + p^i x\beta(1 - \epsilon y^2)] = 0,$$

$$\sum_{0\le x<p^m} \lambda\{[\alpha + p^{i+1}\beta z(\epsilon-1)(1-p^2z^2)^{-1}] + p^i x\beta(\epsilon - p^2z^2)(1-p^2z^2)^{-1}\} = 0.$$

Therefore,

$$\psi(C_{i\alpha\beta}) = 0.$$

Using the similar method and applying lemma 4.1 throughout the calculation, we have

$$\psi(B_{(\ell-1)\alpha\beta}) = p^{\ell-1}\lambda(\alpha), \psi(D_{i\alpha\beta}) = p^i[\lambda(\alpha)+\lambda(\delta)], \psi(I_\alpha) = \lambda(\alpha)\deg(\psi).$$

Comparing the $\psi$ we constructed here with the $\chi$ by parabolic induction in the last section, we notice that $\psi$ and $\chi$ take the same non zero values. Since $\deg(\psi) = \deg(\chi)$ and $\psi$ is irreducible, we must have

$$[\psi, \chi] = [\chi, \chi] = 1.$$

Therefore, $\psi = \chi$. and this is another way to show that $\chi$ in the last section is irreducible.

Now we talk about the odd case. Although the way we construct $\psi$ is a little different when $\ell = 2m+1$, we still have the same result. Suppose $\ell = 2m+1$ now. Let $\lambda : (\mathbb{Z}/p^\ell\mathbb{Z})^\times \to \mathbb{C}^\times$ be injective. Denote

$$N = \left\{ \begin{pmatrix} 1+p^m a & p^{m+1}b \\ p^m c & 1+p^m d \end{pmatrix} \right\}, T' = \left\{ \begin{pmatrix} a & p^{m+1}b \\ p^m c & d \end{pmatrix} \right\}, T = \left\{ \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix} \right\}.$$

We will have the following construction process:

$$\begin{array}{ccccccccc} K_{m+1} & \longrightarrow & N & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & G \\ \phi & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ext}} & \psi' & \xrightarrow{\text{ind}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array},$$

where

$$\phi \begin{pmatrix} 1+p^{m+1}a & p^{m+1}b \\ p^{m+1}c & 1+p^{m+1}d \end{pmatrix} = \lambda(1+p^{m+1}a), \quad \mathrm{Stab}_G(\phi) = T.$$

Also, $N \lhd T$ and $\phi'$ on $N$ is an extension of $\phi$ such that $\phi'(n) = \lambda(n_{11})$ for all $n \in N$. Moreover, $\mathrm{Stab}_T(\phi') = T'$ and we can extend $\phi'$ to $\psi'$ of $T'$ with $\phi(t') = \lambda(t'_{11}), \forall t' \in T'$. Therefore,

$$\psi = \mathrm{Ind}_{T'}^T \psi' \in \mathrm{Irr}(T) \text{ and } (\psi_{K_{m+1}}, \phi) \neq 0.$$

By Clifford Theory,

$$\chi = \mathrm{Ind}_T^G \psi = \mathrm{Ind}_{T'}^G \psi' \in \mathrm{Irr}(G) \text{ and } \deg(\chi) = p^{\ell-1}(p+1).$$

Consider $\chi$ as the induction of $\psi'$ from $T'$, the evaluation process is the same as in the even case and the same result follows. Namely, the irreducible characters of degree $p^{\ell-1}(p+1)$ by Clifford's theorem are the same as the ones constructed by Parabolic induction and we know the character values.

## 4.2 Values for $(p-1)p^{\ell-1}$-Degree Characters

In this section, we will construct the irreducible characters of $G$ with degree $(p-1)p^{\ell-1}$ and find the character values. We first find character values on $K_{\ell-i}, 1 \le i \le \frac{\ell}{2}$, and then work on the remaining character values in two cases depending on whether $\ell$ is even or odd.

### 4.2.1 Character Values of Elements in $K_{\ell-i}, 1 \le i \le \frac{\ell}{2}$

Let $S$ denote the subgroup of $\mathrm{GL}(2,R), R = \mathbb{Z}/p^\ell\mathbb{Z}$ consisting of matrices of the form $\begin{pmatrix} a & b\epsilon \\ b & a \end{pmatrix}$, where $\epsilon$ is a non square unit in $R$.

**Lemma 4.2.** *The group $S$ has order $(p^2-1)p^{2\ell-2}$. Moreover $S$ is the semi-direct product $S = (K_1 \cap S)\langle s_o \rangle$ where $s_0$ has order $p^2-1$; $s_0^{p+1}$ has the form $\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$.*

Let $A$ be the matrix $\begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}$ over the ring $R_i = \mathbb{Z}/p^i\mathbb{Z}$, and let $\phi_A$ be the corresponding character of $K_{\ell-i}$ :

$$\phi_A \begin{pmatrix} 1+p^{\ell-i}a & p^{\ell-i}b \\ p^{\ell-i} & 1+p^{\ell-i}a \end{pmatrix} = \lambda\left(\operatorname{tr}\left(\begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 1 & a \end{pmatrix}\right)\right) = \lambda(b+\epsilon).$$

Let $S_i$ denote the subgroup of $\mathrm{GL}(2,R_i)$ consisting of matrices of the form $\begin{pmatrix} a & b\epsilon \\ b & a \end{pmatrix}$, so $|S_i| = (p^2-1)p^{2i-2}$

**Lemma 4.3.** *The following set of cardinality $(p-1)p^{2i-1}$ includes exactly one representative from each right coset of $S_i$ in $\mathrm{GL}(2,R_i)$.*

$$\left\{\begin{pmatrix} 1 & c \\ & d \end{pmatrix} : c \in R_i, d \in R_i^\times\right\}.$$

*Proof.* It is easy to check that the above set actually forms a subgroup of $\mathrm{GL}(2,R_i)$ and the only element that lies in $S_i$ is the identity. The number of elements in this subgroup is $(p^i - p^{i-1})p^i = p^{2i-1}(p-1)$. On the other hand, from proposition 2.3.1,

$$|\operatorname{GL}(2,\mathbb{Z}/p^i\mathbb{Z})| = p^{4(i-1)}(p^2-1)(p^2-p),$$

so the index $[\mathrm{GL}(2,\mathbb{Z}/p^i\mathbb{Z}) : S_i] = p^{2i-1}(p-1)$. We have a complete list of coset representatives. □

**Lemma 4.4.** . *Let $z \in R_i^\times$. The number of solutions $(x, y) \in R_i^2$ of the equation $x^2 - \epsilon y^2 = z$ is $(p+1)p^{i-1}$.*

*Proof.* We claim that the map $\det : S_i \to R_i^\times$ is surjective. This is easily seen if $i = 1$. In general, the claim follows using the commutative diagram

$$\begin{array}{ccc} S_i & \longrightarrow & S_1 \\ \Big\downarrow & & \Big\downarrow \\ (\mathbb{Z}/p^i\mathbb{Z})^\times & \longrightarrow & (\mathbb{Z}/p\mathbb{Z})^\times \end{array}$$

where the horizontal maps are "mod $p$" and the vertical maps are det. The number of solutions to $x^2 - \varepsilon y^2 = z$ is the number of matrices $\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$ in $S_i$ whose determinant is $z$. This number is $|S_i|/|R_i^\times| = (p+1)p^{i-1}$. $\square$

**Lemma 4.5.** *If $\lambda : R_i^+ \to \mathbb{C}^\times$ is injective and $0 \leq j \leq i$ then*

$$\sum\{\lambda(y) : y \in p^j R_i^\times\} = \begin{cases} 0, & \text{if } j < i-1, \\ -1, & \text{if } j = i-1, \\ 1, & \text{if } j = i. \end{cases}$$

The proof uses the fact that for any $y_0 \in R_i$ and $j < i$,

$$\sum\{\lambda(y) : y \equiv y_0(\bmod p^j)\} = 0.$$

Suppose $\chi$ is any irreducible character of $G$ whose restriction to $K_{\ell-i}$ contains copies of $\phi_A$. For any $X \in K_{\ell-i}$, by Clifford's theorem,

$$\chi(X) = e\sum_{k=1}^{t} \phi_k(X),$$

where $\phi_1, \phi_2, ..., \phi_t$ are the distinct conjugates of $\phi_A$ in $G$.
Choose

$$E_{cd} = \begin{pmatrix} 1 & c \\ & d \end{pmatrix} : c \in R_i, d \in R_i^\times$$

from lemma 4.3, we have each $\phi_k = \phi_{E_{cd}^{-1} A E_{cd}}$ for some $c, d$, which implies $t = (p-1)p^{2i-1}$. Notice that each $\phi_i$ has degree 1, we have $e = \frac{\deg(\chi)}{(p-1)p^{2i-1}}$. Therefore,

$$\chi(X) = \frac{\deg(\chi)}{(p-1)p^{2i-1}} \sum_{c \in R_i} \sum_{d \in R_i^\times} \lambda\left(\operatorname{tr}\left(A \frac{E_{cd} X E_{cd}^{-1}}{p^{\ell-i}}\right)\right).$$

Now when $X = I + p^{\ell-i}\begin{pmatrix} a & b \\ 1 & a \end{pmatrix}$,

$$\begin{aligned} E_{cd}XE_{cd}^{-1} &= 1 + p^{\ell-i}\begin{pmatrix} 1 & \\ & d \end{pmatrix}\begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 1 & a \end{pmatrix}\begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & d^{-1} \end{pmatrix} \\ &= 1 + p^{\ell-i}\begin{pmatrix} a+c & d^{-1}(b-c^c) \\ d & a-c \end{pmatrix}. \end{aligned}$$

Therefore $\operatorname{tr}\left(A\frac{E_{cd}XE_{cd}^{-1}}{p^{\ell-i}}\right) = \epsilon d + d^{-1}(b-c^2)$, so that

$$\chi(X) = p^{\ell-2i}\sum_{c\in R_i}\sum_{d\in R_i^\times}\lambda(\epsilon d + d^{-1}(b-c^2)).$$

Define

$$P = \sum_{c\in R_i}\sum_{d\in R_i^\times}\lambda(\epsilon d + d^{-1}(b-c^2))$$

so that $\chi(X) = p^{\ell-2i}P$. We can find that [10]

$$P = (-p)^i(\lambda(2u) + \lambda(-2u)) \text{ if } u^2 = \epsilon b \in R_i^\times$$

$$P = -p \text{ if } i = 1 \text{ and } b = 0$$

$$P = 0 \text{ otherwise .}$$

Therefore, we have

$$\begin{gathered} \chi(I_\alpha) = \deg(\chi) = (p-1)p^{\ell-1}, \\ \chi(C_{i\alpha\beta}) = (-1)^i p^{\ell-i}(\lambda(2\epsilon\beta) + \lambda(-2\epsilon\beta)) \\ = (-1)^i p^{\ell-i}\left(\phi_A\begin{pmatrix} \alpha & p^i\epsilon\beta \\ p^i\beta & \alpha \end{pmatrix} + \phi_A\begin{pmatrix} \alpha & -p^i\epsilon\beta \\ -p^i\beta & \alpha \end{pmatrix}\right), \\ \chi(B_{i\alpha\beta}) = (-p)^{\ell-1} \text{ if } i = \ell-1 \text{ and } 0 \text{ otherwise }, \\ \chi(D_{i\alpha\delta}) = 0 \end{gathered}$$

valid when $i \geq \frac{\ell}{2}$.

### 4.2.2 Remaining Values when $\ell = 2m$ is even

**Lemma 4.6.** *If $i$ and $j$ are positive integers and $\lambda' : (\mathbb{Z}/p^i\mathbb{Z})^+ \to \mathbb{C}^\times$ is an injective homomorphism then*

$$\sum_{e,f\in\mathbb{Z}/p^i\mathbb{Z}}\lambda'\left(\frac{\epsilon f^2 - e^2}{1+p^j f}\right) = (-p)^i.$$

*Proof.* Change variables:

$$f' = \frac{f}{\sqrt{1+p^j f}}, \quad e' = \frac{e}{\sqrt{1+p^j f}},$$

where the square root having remainder $+1 \pmod p$ is taken. It can be shown that the map

$$f \to \frac{f}{\sqrt{1+p^j f}}$$

is injective. Therefore, the desired sum is equal to

$$\sum_{e', f' \in \mathbb{Z}/p^i\mathbb{Z}} \lambda'(\epsilon f'^2 - e'^2).$$

By counting the number of solutions to the equation $x^2 - \epsilon y^2 = p^k z$, [10] where $k < i, z \in (\mathbb{Z}/p^{i-k}\mathbb{Z})^\times$, this sum can be evaluated using lemma 4.5 of the previous section.
Now we briefly show that the map

$$f \to \frac{f}{\sqrt{1+p^j f}}$$

is injective. There exists a polynomial $S(X)$ with coefficients in $R$ such that

$$\frac{1}{\sqrt{1+p^j f}} = 1 + p^j S(f).$$

Indeed, the Taylor series terminates since $p^j f$ is nilpotent. If $f_1$ and $f_2$ map to the same element, we deduce

$$0 = f_1 - f_2 + p^j(S(f_1) - S(f_2)).$$

One can show that the coefficient of $p^j$ contains a factor $f_1 - f_2$, so

$$0 = (f_1 - f_2)(1 + p^j Q(f_1, f_2))$$

for some polynomial Q(X,Y). Since $j > 1$, the second factor is a unit, so $f_1 = f_2$, as required.

□

Let $\ell = 2m$. We found that every character $\chi$ of $G$ with degree $(p-1)p^{\ell-1}$ is induced from a linear character $\psi$ of the subgroup

$$T = \begin{pmatrix} a & \epsilon b + p^m c \\ b & a + p^m d \end{pmatrix} \subseteq G.$$

The following is a list of left coset representatives of $T$ :

$$E_{cd} = \begin{pmatrix} 1 & c \\ & d \end{pmatrix}, 0 \leq c < p^m, 0 < d < p^m, p \nmid d.$$

For $X \in T$, we have

$$\chi(X) = \sum_{c=0}^{p^m-1} \sum_{0<d<p^m, p\nmid d} \dot{\psi}(E_{cd} X E_{cd}^{-1})$$

where as usual, $\dot{\psi}$ is the extension of the function $\psi$ which is 0 off $T$. Assume that $X \notin K_m$, because we have calculated character values on $K_m$ in the previous section. The only conjugacy class type that intersects $T$ is $C_{i\alpha\beta}$. Thus, let $X = \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix}, b \in R^\times, 0 \leq i < m$. We have

$$E_{cd} X E_{cd}^{-1} = \begin{pmatrix} a + p^i bc & p^i b d^{-1}(\epsilon - c^2) \\ p^i bd & a - p^i bc \end{pmatrix}$$

which is in $T$ if and only if $p^m \mid p^i bc$ and $p^m \mid p^i b(d+1)(d-1)$. This is the condition for $\dot{\psi} \neq 0$.
First consider the case $i = 0$. $\dot{\psi} = 0$ unless $c = 0, d = 1$ or $c = 0, d = p^m - 1$, so that

$$\begin{aligned}
\chi(X) &= \psi(X) + \psi(E_{0(p^m-1)} X E_{0(p^m-1)}^{-1}) \\
&= \psi(X) + \psi\left(\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} X \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right) \\
&= \psi \begin{pmatrix} a & \epsilon b \\ b & a \end{pmatrix} + \psi \begin{pmatrix} a & -\epsilon b \\ -b & a \end{pmatrix}.
\end{aligned}$$

The second-last equality uses the fact that $\psi$ is a class function on $T$.
Henceforth assume that $i > 0$. The values of $d$ such that $\dot{\psi} \neq 0$ are $p^{m-i} f \pm 1$ for $0 \leq f < p^i$. The $+$ and $-$ alternatives are interchanged when $\begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix}$ is replaced with $\begin{pmatrix} a & -\epsilon p^i b \\ -p^i b & a \end{pmatrix}$; therefore

$$\chi(X) = \sum_{e,f=0}^{p^i-1} \psi\left(E_{(p^{m-i}e)(1+p^{m-i}f)} \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix} E_{(p^{m-i}e)(1+p^{m-i})}^{-1}\right)$$

$$+ \sum_{e,f=0}^{p^i-1} \psi\left(E_{(p^{m-i}e)(1+p^{m-i}f)} \begin{pmatrix} a & -\epsilon p^i b \\ -p^i b & a \end{pmatrix} E_{(p^{m-i}e)(1+p^{m-i})}^{-1}\right).$$

It suffices to compute the first sum because the second is symmetrical. The first is equal to

$$\sum_{e,f=0}^{p^i} \psi\left[\begin{pmatrix}1 & \\ & 1+p^{m-i}f\end{pmatrix}\begin{pmatrix}1 & p^{m-i}e \\ & 1\end{pmatrix}, \begin{pmatrix}a & \epsilon p^i b \\ p^i b & a\end{pmatrix}\right] \psi\begin{pmatrix}a & \epsilon p^i b \\ p^i b & a\end{pmatrix},$$

using the convention

$$[x,y] = xyx^{-1}y^{-1}.$$

We find that (modulo $p^{2m}$),

$$\begin{aligned}[E_{cd}, X] &= \left[\begin{pmatrix}1 & \\ & 1+p^{m-i}f\end{pmatrix}\begin{pmatrix}1 & p^{m-i}e \\ & 1\end{pmatrix}, \begin{pmatrix}a & \epsilon p^i b \\ p^i b & a\end{pmatrix}\right] \\ &= \begin{pmatrix}1 & \\ & 1\end{pmatrix} + \frac{p^m}{a^2 - \epsilon p^{2i} b^2}\begin{pmatrix} abe + p^i \epsilon b^2 f & -p^i \epsilon b^2 e - \frac{b(\epsilon a f + p^{m-i} a e^2)}{1+p^{m-i}f} \\ abf + p^i b^2 e & -p^i \epsilon b^2 f - abe\end{pmatrix}.\end{aligned}$$

This commutator is in $K_m$, so we can describe its image under $\psi$ in terms of an appropriate matrix $A = \begin{pmatrix}\alpha & \epsilon\beta \\ \beta & \alpha\end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}/p^m\mathbb{Z})$ and linear character $\lambda$ of $\mathbb{Z}/p^m\mathbb{Z}$ as

$$\begin{aligned}\psi([E_{cd}, X]) &= \lambda(\mathrm{tr}(p^{-m} A([E_{cd}, X] - 1))) \\ &= \lambda\left(\frac{\beta}{a^2 - \epsilon p^{2i} b^2}(-p^i \epsilon b^2 e - \frac{b(\epsilon a f + p^{m-i} a e^2)}{1+p^{m-i}f}) + \frac{\epsilon\beta(abf + p^i b^2 e)}{a^2 - \epsilon p^{2i} b^2}\right) \\ &= \lambda\left(\frac{\beta ab}{a^2 + \epsilon p^{2i} b^2} p^{m-i} \frac{\epsilon f^2 - e^2}{1+p^{m-i}f}\right) \\ &= \lambda'\left(\frac{\epsilon f^2 - e^2}{1+p^{m-i}f}\right)\end{aligned}$$

if we define

$$\lambda' = \lambda^{p^{m-i} ab\beta/(a^2 - \epsilon p^{2i} b^2)},$$

an injective linear character of $\mathbb{Z}/p^i\mathbb{Z}$. The lemma 4.6 now applies and we have

$$\chi(C_{iab}) = (-p)^i\left(\psi\begin{pmatrix}a & \epsilon p^i b \\ p^i b & a\end{pmatrix} + \psi\begin{pmatrix}a & -\epsilon p^i b \\ -p^i b & a\end{pmatrix}\right).$$

### 4.2.3 Remaining Values when $\ell = 2m+1$ is odd

In this section $\ell = 2m+1$. Let $A = \begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix}$, and let $\lambda : \mathbb{Z}/p^{2m+1}\mathbb{Z} \longrightarrow C^\times$ be injective. Define $\phi_A$ on $K_{m+1}$ by

$$\phi_A\begin{pmatrix}1+p^{m+1}a & p^{m+1}b \\ p^{m+1}c & 1+p^{m+1}d\end{pmatrix} = \lambda(p^{m+1}b + \epsilon p^{m+1}c).$$

The stabilizer of $\phi_A$ is $T = K_m S$ where $S = \left\{ \begin{pmatrix} x & y\epsilon \\ y & x \end{pmatrix} \right\}$. We will construct an irreducible character $\psi$ of $T$ such that $[\psi|_{K_{m+1}}, \phi_A] \neq 0$; then by Clifford's Theorem, $\chi = \mathrm{Ind}_T^G \in \mathrm{Irr}(G)$.

Denote

$$N_i = K_i(K_1 \cap S) \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \right\}.$$

The process to construct $\psi$ is as follows. Let $H = N_{m+1} \left\langle \begin{pmatrix} 1+p^m & \\ & 1 \end{pmatrix} \right\rangle$. We extend $\phi_A$ to a character $\phi'$ of $N_{m+1}$, then to a character $\phi''$ of $H$, then induce to $N_m$, then extend to $T$. Pictorially,

$$\begin{array}{ccccccccc} K_{m+1} & \longrightarrow & N_{m+1} & \longrightarrow & H & \longrightarrow & N_m & \longrightarrow & T \\ \phi_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ext}} & \phi'' & \xrightarrow{\text{ind}} & \theta & \xrightarrow{\text{ext}} & \psi \end{array}$$

Since $N_m/N_{m+1}$ is abelian, any subgroup of $N_m$ containing $N_{m+1}$ is normal. Thus $H \lhd N_m$ and the index is $p$. Here are the details of this process:

(i) Extend $\phi_A$ to $\phi'$.
Our first attempt is to define $\phi'$ on $K_1 \cap S$ using the same formula as we used for $\phi_A$, namely

$$\phi'(I + p(xI + yA)) = \lambda(\mathrm{tr}(p(xI + yA)A) = \lambda(2py\varepsilon).$$

However this does not preserve multiplication, since

$$\begin{aligned} &\phi'(I + p(x_1 I + y_1 A)(I + p(x_2 I + y_2 A)) \\ &= \phi'(I + p(x_1 + x_2)I + p(x_1 y_2 + x_2 y_1)A + p(y_1 + y_2)A^2) \\ &= \lambda(2p(x_1 y_2 + x_2 y_1)\varepsilon + 2p(y_1 + y_2)\varepsilon). \end{aligned}$$

Note that multiplication would be preserved if either $x_1$ and $x_2$ are both divisible by $p^{2m}$ or $y_1$ and $y_2$ are both divisible by $p^{2m}$. So we define $\phi'$ on $K_a = \left\{ \begin{pmatrix} 1+p^{2m}x & py\epsilon \\ py & 1+p^{2m}x \end{pmatrix} \right\}$ and $K_b = \left\{ \begin{pmatrix} 1+px & p^{2m}y\epsilon \\ p^{2m}y & 1+px \end{pmatrix} \right\}$ using the same formula as $\phi_A$, namely,

$$\phi' \begin{pmatrix} 1+p^{2m}x & py\epsilon \\ py & 1+p^{2m}x \end{pmatrix} = \lambda(2py\epsilon), \quad \phi' \begin{pmatrix} 1+px & p^{2m}y\epsilon \\ p^{2m}y & 1+px \end{pmatrix} = \lambda(2p^{2m}y\epsilon).$$

Since $K_1 \cap S = K_a K_b$ and $K_a \cap K_b \subset K_{2m} \subset K_{m+1}$, we can define the homomorphism $\phi'$ on $K_1 \cap S$ by

$$\phi'(gh) = \phi'(g)\phi'(h), \qquad g \in K_a,\ h \in K_b.$$

Note that $\phi'\begin{pmatrix}1+px & \\ & 1+px\end{pmatrix} = \lambda(0) = 1$; so we can define $\phi'$ to be trivial on all central elements $\begin{pmatrix}a & \\ & a\end{pmatrix}$ in $N_{m+1}$, and realize that we can construct a new extension by multiplying the above $\phi'$ by a root of unity on a central element. It is clear that $\phi'$ is an extension of $\phi_A$.

(ii) Extend $\phi'$ of $N_{m+1}$ to $\phi''$ of $H$.

We only need to define $\phi''$ on $\left\langle\begin{pmatrix}1+p^m & \\ & 1\end{pmatrix}\right\rangle$. This can be done by first defining $\phi''$ to be trivial on $\begin{pmatrix}1+p^m & \\ & 1\end{pmatrix}$ and then multiplying $\phi''$ by a $p$th root of unity to get different extensions.

(iii) Induction from $\phi''$ to $\theta$

It is easy to find an element in $N_m$ that does not stabilize $\phi''$ and since the index of $H$ in $N_m$ is $p$, we have $\mathrm{Stab}_{N_m}(\phi'') = H$. Clifford's Theorem tells us that $\theta = \mathrm{Ind}_H^{N_m}\phi''$ is irreducible.

Let $\Theta$ denote the induced representation affording $\theta$. The following result will help us to find the character values of $\psi$ on $T$ in section 4.4.2.

**Lemma 4.7.** *For $s \in S \cap K_1$, $s = \begin{pmatrix}1+px & py\varepsilon \\ py & 1+px\end{pmatrix}$ then $\Theta(s) = \lambda(2py\varepsilon)I$ where $I$ is the identity matrix.*

*Proof.* Coset representatives of $H$ in $N_m$ are given by

$$\left\{n(k) = \begin{pmatrix}1 & p^m k \\ 0 & 1\end{pmatrix} : 0 \le k < p\right\}.$$

Suppose that $s = \begin{pmatrix}1+px & py\varepsilon \\ py & 1+px\end{pmatrix} = \begin{pmatrix}1+px & 0 \\ 0 & 1+px\end{pmatrix} + \begin{pmatrix}0 & py\varepsilon \\ py & 0\end{pmatrix}$.

$$\begin{aligned} n(k)^{-1}sn(k) &= \begin{pmatrix}1+px & 0 \\ 0 & 1+px\end{pmatrix} + \begin{pmatrix}1 & -p^m k \\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & py\varepsilon \\ py & 0\end{pmatrix}\begin{pmatrix}1 & p^m k \\ 0 & 1\end{pmatrix} \\ &= \begin{pmatrix}1+px & 0 \\ 0 & 1+px\end{pmatrix} + \begin{pmatrix}-p^{m+1}yk & py\varepsilon \\ py & p^{m+1}yk\end{pmatrix}. \end{aligned}$$

This belongs to $N_{m+1} \subset H$, and $\lambda\big(n(k)^{-1}sn(k)\big) = \lambda(2py\varepsilon)$. $\Theta(s) = \lambda(2py\varepsilon)I_p$. □

(iv) Extend $\theta$ to $\psi$.

Since $\theta$ is stable under $T$ and $T/N_m$ is cyclic, we know that $\psi$ exists. Other extensions of $\phi'$ to a character of $H$ have the form $\phi''\alpha$ where $\alpha$ is a character of

$H/N_m$. Each $\phi''\alpha$ is a component of the restriction of $\theta$ to $H$. So $\theta|_H = \sum_\alpha \phi''\alpha$ where the sum is over the $p$ irreducible characters $\alpha$ of $H/N_m$. It follows that

$$\psi\mid_{N_{m+1}} = p\phi'_A \text{ and } \psi\mid_{N_m - N_{m+1}} = 0.$$

Now we will calculate the character values of $\chi = \mathrm{Ind}_T^G \psi \in \mathrm{Irr}(G)$. Similar to the even case, we have the same left coset representatives $E_{cd}$ and we only need to calculate $\chi(X)$ where

$$X = \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix}, b \in R^\times, 0 \le i < m+1.$$

We have

$$E_{cd} X E_{cd}^{-1} = \begin{pmatrix} a + p^i bc & p^i b d^{-1}(\epsilon - c^2) \\ p^i bd & a - p^i bc \end{pmatrix}.$$

This time, the condition for $\dot{\phi} \neq 0$ yields that $p^{m+1} \mid p^i bc$ and $p^{m+1} \mid p^i b(d+1)(d-1)$.

For the case $i = 0$, we have the same argument as in the even case and we find that

$$\chi(X) = \psi \begin{pmatrix} a & \epsilon b \\ b & a \end{pmatrix} + \psi \begin{pmatrix} a & -\epsilon b \\ -b & a \end{pmatrix}.$$

Assume that $i > 0$. We have

$$\chi(X) = \sum_{e,f=0}^{p^i-1} \psi \left( E_{(p^{m+1-i}e)(1+p^{m+1-i}f)} \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix} E^{-1}_{(p^{m+1-i}e)(1+p^{m+1-i}f)} \right)$$

$$+ \sum_{e,f=0}^{p^i-1} \psi(E_{(p^{m+1-i}e)(1+p^{m+1-i}f)} \begin{pmatrix} a & -\epsilon p^i b \\ -p^i b & a \end{pmatrix} E^{-1}_{(p^{m+1-i}e)(1+p^{m+1-i}f)}).$$

To evaluate the first sum, notice that

$$E_{(p^{m+1-i}e)(1+p^{m+1-i}f)} \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix} E^{-1}_{(p^{m+1-i}e)(1+p^{m+1-i})} \in N_{m+1} \text{ and } \psi|_{N_{m+1}} = p\psi',$$

we factor out $p$ and use the same method as in the even case, because $\phi'$ is a homomorphism. Finally, we will get

$$\chi(C_{iab}) = (-p)^i \left( \psi \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix} + \psi \begin{pmatrix} a & -\epsilon p^i b \\ -p^i b & a \end{pmatrix} \right),$$

which is the same formula as the even case.

### 4.2.4 Character Values of $\psi$ on $T$

In the last section, we have the formula for $\chi(C_{iab})$ and notice that it depends on $\psi$ where $\psi$ is the corresponding character on the stabilizer $T$. In this section, we will consider the character values of $\psi$.

**Even Case, i.e.** $\ell = 2m$

Denote

$$K_m = \{I + p^m B\}, K_1 = \{I + pB\}, S = \left\{ \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix} \right\}, A = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}.$$

Let $\lambda : (\mathbb{Z}/p^{2m}\mathbb{Z})^+ \to \mathbb{C}^\times$ be an injective homomorphism.
Define

$$\phi_A : K_m \to \mathbb{C}^\times, \phi_A(I + p^m B) = \lambda(\mathrm{tr}(p^m AB)).$$

$\mathrm{Stab}_G(\phi_A) = K_m S$, and hence $\phi_A \mid_{K_m \cap S}$ is stable under $K_1 \cap S$. We want to extend $\phi_A$ to $\psi$ of $T = K_m S$ and we will approach it in the following two steps.

Firstly, we extend $\phi_A$ to $\phi'_A$ of $K_m(K_1 \cap S)$. In order to do this, we only need to extend $\phi_A \mid_{K_m \cap S}$ to $\phi'$ of $K_1 \cap S$. Since $\frac{|K_1 \cap S|}{|K_m \cap S|} = p^{2m-2}$, there are $p^{2m-2}$ extensions. Notice that

$$K_1 \cap S = (K_m \cap S) \left\langle \begin{pmatrix} 1 + pa & \\ & 1 + pa \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 1 & p\epsilon \\ p & 1 \end{pmatrix} \right\rangle,$$

we only need to define $\phi'$ on $\left\langle \begin{pmatrix} 1 + pa & \\ & 1 + pa \end{pmatrix} \right\rangle$ and $\left\langle \begin{pmatrix} 1 & p\epsilon \\ p & 1 \end{pmatrix} \right\rangle$.

Since

$$C = \begin{pmatrix} 1 + pa & \\ & 1 + pa \end{pmatrix}^{p^{m-1}} \in K_m \cap S,$$

we can define $\phi'$ such that

$$\phi' \begin{pmatrix} 1 + pa & \\ & 1 + pa \end{pmatrix} = \sqrt[p^{m-1}]{\phi_A(C)}.$$

Similarly,

$$D = \begin{pmatrix} 1 & p\epsilon \\ p & 1 \end{pmatrix}^{p^{m-1}} \in K_m \cap S \Rightarrow \phi' \begin{pmatrix} 1 & p\epsilon \\ p & 1 \end{pmatrix} = \sqrt[p^{m-1}]{\phi_A(D)}.$$

This way, we essentially extend $\phi_A$ to $K_m(K_1 \cap S)$ and there are indeed $p^{2m-2}$ extensions.

Secondly, we want to extend $\phi'$ to $\psi$ of $K_m S$. From Lemma 4.2,

$$K_m S = K_m(K_1 \cap S) \langle s_0 \rangle, \text{ and } K_m(K_1 \cap S) \cap \langle s_0 \rangle = 1.$$

Define $\psi \mid_{K_m(K_1 \cap S)} = \phi'$ and $\psi(s_0^i) = \zeta^i$ where $\zeta$ is a $(p^2 - 1)$ root of unity; then $\psi$ is an extension of $\phi_A$ and we know the values of $\psi$.

**Odd Case, i.e.** $\ell = 2m+1$

Now we want consider the character values of $\psi$ on $T$ in the odd case. Let $h = \begin{pmatrix} a & \varepsilon p^i b \\ p^i b & a \end{pmatrix}$ in the class $C_{iab}$. If $i > 0$, then $h \in N_m$. Since

$$\psi|_{N_{m+1}} = p\phi', \qquad \psi(n) = 0 \text{ if } n \in N_m, n \notin N_{m+1}$$

we know the character value $\phi(h)$ if $h \in N_m$. So suppose that $h \notin N_m$, that is, $i = 0$.
In section 4.2.3 we constructed $\psi$ this way:

$$\begin{array}{ccccccccc} K_{m+1} & \longrightarrow & N_{m+1} & \longrightarrow & H & \longrightarrow & N_m & \longrightarrow & T \\ \phi_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ext}} & \phi'' & \xrightarrow{\text{ind}} & \theta & \xrightarrow{\text{ext}} & \psi \end{array}$$

where

$$N_i = K_i(K_1 \cap S)\left\{\begin{pmatrix} a & \\ & a \end{pmatrix}\right\}, H = N_{m+1}\left\langle \begin{pmatrix} 1+p^m & \\ & 1 \end{pmatrix} \right\rangle.$$

Note that

$$N_m = \left\{\begin{pmatrix} t+p^m x + pa & p^m y + pb\epsilon \\ pb & t+pa \end{pmatrix}\right\}.$$

Coset representatives of $N_{m+1}$ in $N_m$ are given by

$$\left\{ g(x,y) = \begin{pmatrix} 1+p^m x & p^m y \\ 0 & 1 \end{pmatrix} : 0 \le x, y < p \right\}$$

and the coset $g(x,y)N_{m+1}$ is equal to

$$\left\{\begin{pmatrix} t+p^m x + p^{m+1}d + pa & p^m y + p^{m+1}e + pb\epsilon \\ pb & t+pa \end{pmatrix}\right\}.$$

It follows that an element $\begin{pmatrix} r & s \\ u & v \end{pmatrix}$ of $N_m$ is in the coset $g(x,y)N_{m+1}$ precisely when

$$(r-v)/p^m \equiv x \mod p \text{ and } (s-\epsilon u)/p^m \equiv y \mod p.$$

**Lemma 4.8.** *Suppose that* $h \in T$ *and* $h \notin N_m$. $|\psi(h)| = 1$.

Since $h = \begin{pmatrix} a & \varepsilon b \\ b & a \end{pmatrix}$, then $h \in S$. We know that $S = (K_1 \cap S)\langle s_0 \rangle$, and that for $s \in K_1 \cap S$ we have $\Theta(s) = \alpha I$ for some $\alpha \in \mathbb{C}^\times$. So it suffices to calculate $\psi(s)$ for $s \in \langle s_0 \rangle = S_0$.

**Lemma 4.9.** *For* $s \in S_0$*, the value of* $\psi(s)$ *is a* $p^2-1$ *root of unity.*

Above all, we know that we can find an extension $\phi$ on $T$ such that $\phi(s_0) = \pm 1$. We can construct different extensions by multiplying $\phi$ above by a $(p+1)$ root of unity.

## 4.3 Characters of degree $(p^2-1)p^{\ell-2}$ and the character values

In this section, we will construct the irreducible characters of $G$ with degree $(p^2-1)p^{\ell-2}$ and find the character values. Similarly to the last section, we first find character values on $K_{\ell-i}, 1 \le i \le \frac{\ell}{2}$. We will then work on the remaining character values in two cases depending on whether $\ell$ is even or odd.

### 4.3.1 Character values of elements in $K_{\ell-i}, 1 \le i \le \frac{\ell}{2}$

Let $A$ be the matrix $\begin{pmatrix} & p^j\beta \\ 1 & \end{pmatrix}$ over the ring $R_i = \mathbb{Z}/p^i\mathbb{Z}$ where $\beta \in R^\times, 1 \le j \le i$. Let $\phi_A$ be the corresponding character of $K_{\ell-i}$ :

$$\phi_A \begin{pmatrix} 1+p^{\ell-i}a & p^{\ell-i}b \\ p^{\ell-i} & 1+p^{\ell-i}a \end{pmatrix} = \lambda\left(\operatorname{tr}\begin{pmatrix} 0 & p^j\beta \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 1 & a \end{pmatrix}\right) = \lambda(b+p^j\beta).$$

**Lemma 4.10.** *The following list of cardinality $p^{2i-2}(p^2-1)$ includes exactly one representative from each right coset of* $\left\{\begin{pmatrix} w & p^j\beta y \\ y & w \end{pmatrix}\right\} \subset \mathrm{GL}(2,R_i)$ :

$$\left\{\begin{pmatrix} 1 & \\ & d \end{pmatrix}\begin{pmatrix} 1 & c \\ & 1 \end{pmatrix} : c \in R_i, d \in R_i^\times\right\},$$

$$\left\{\begin{pmatrix} 1 & \\ & d \end{pmatrix}\begin{pmatrix} 1 & pc \\ & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : pc \in R_i, d \in R_i^\times\right\}.$$

*Proof.* It is easy to check that, for any two matrices $B, C$ from the list, $BC^{-1} \notin \left\{\begin{pmatrix} w & p^j\beta y \\ y & w \end{pmatrix}\right\}$. Since the index of $\left\{\begin{pmatrix} w & p^j\beta y \\ y & w \end{pmatrix}\right\}$ in $\mathrm{GL}(2,R_i)$ is $p^{2i-2}(p^2-1)$, we know we have all the coset representatives. $\square$

Suppose $\chi$ is any irreducible character of $G$ whose restriction to $K_{l-i}$ contains copies of $\phi_A$. Similarly to section 4.2.1, let $X = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + p^{\ell-i}\begin{pmatrix} a & b \\ 1 & a \end{pmatrix}$, by Clifford's theorem, we have

$$\chi(X) = \frac{\deg\chi}{p^{2i-2}(p^2-1)} \sum_{c\in R_i, d\in R_i^\times} \lambda\left(\operatorname{tr}\left(A\frac{E_{cd}(X-I)E_{cd}^{-1}}{p^{\ell-i}}\right)\right)$$

$$+ \sum_{pc\in R_i, d\in R_i^\times} \lambda\left(\operatorname{tr}\left(A\frac{F_{cd}(X-I)F_{cd}^{-1}}{p^{\ell-i}}\right)\right),$$

where

$$E_{cd} = \begin{pmatrix} 1 & \\ & d \end{pmatrix} \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}, F_{cd} = \begin{pmatrix} 1 & \\ & d \end{pmatrix} \begin{pmatrix} 1 & pc \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By calculation, we have

$$\operatorname{tr}\left(A\left(\frac{E_{cd}(X-I)E_{cd}^{-1}}{p^{\ell-i}}\right)\right) = p^j\beta d + d^{-1}(b-c^2),$$

$$\operatorname{tr}\left(A\frac{F_{cd}(X-I)F_{cd}^{-1}}{p^{\ell-i}}\right) = p^j\beta bd + d^{-1}(1-p^2c^2b).$$

Therefore

$$\chi(X) = p^{\ell-2i}[\sum_{c\in R_i, d\in R_i^\times} \lambda(p^j\beta d+d^{-1}(b-c^2))+\sum_{pc\in R_i, d\in R_i^\times} \lambda(p^j\beta bd+d^{-1}(1-p^2c^2b)].$$

We want to evaluate the above two sums. For the second one, we have

$$\forall b, \sum_{pc\in R_i, d\in R_i^\times} \lambda(p^j\beta bd + d^{-1}(1-p^2c^2b)) = \begin{cases} 0, & \text{if } i > j, \\ -p^{i-1}, & \text{if } i = j. \end{cases}$$

For the first sum, denote $P = \sum_{c\in R_i, d\in R_i^\times} \lambda(p^j\beta d + d^{-1}(b-c^2))$ and $x = p^j\beta d + d^{-1}(b-c^2)$, to evaluate $P$, we need to consider the values that $x$ can take. From Lemma 4.5, we know that only care about the cases when $p^{i-1} \mid x$ or $x = 0$, because the rest $\lambda(x)$ will sum to 0. We will deal with the case when $i = j$ first. The case when $j < i$ is similar but more complicated. When $i = j, x = d^{-1}(b-c^2)$. We have the following cases.

(i) If $b$ is non square unit, then $b - c^2$ is always a unit and therefore $x = d^{-1}(b-c^2)$ can only take units. By lemma 4.5, we have

$$P = \begin{cases} 0, & \text{if } i > 1, \\ -p^i, & \text{if } i = 1. \end{cases}$$

(ii) $b = u^2$, for some unit $u$. In this case,

$$p^{i-1} \mid x \Leftrightarrow p^{i-1} \mid u^2 - c^2 \Leftrightarrow c = \pm u + p^{i-1}v.$$

There are $2p$ such $c$. For $b - c^2$, it equals 0 twice (when $c = \pm u$); and equals $p^{i-1} * unit$ $2(p-1)$ times. From lemma 4.5 again, we have

$$P = \begin{cases} 2(p-1)(p^{i-1}-1), & \text{if } i > 1, \\ p, & \text{if } i = 1. \end{cases}$$

(iii) $b = p^{i-1}u, u \in R_i^\times$. The implicit condition for this case is $i > 1$, otherwise it is done already. Now $p^{i-1} \mid b - c^2 \Leftrightarrow p^{i-1} \mid c^2$ and we have the following cases.
(a) $u$ is non square and $i - 1 = 2k$ for some $k$. In this case, $p^{i-1} \mid c^2 \rightarrow c = p^k v, v \in R_i$, and $b - c^2 = p^{2k}(u - v^2)$ where $u - v^2$ is always a unit. The number of $v$ is $p^{i-k}$. By Lemma 4.5, $P = -p^{i-k} = -p^{\frac{i+1}{2}}$.
(b) $u = w^2, i - 1 = 2k$. Like in (a), $c = p^k v$ in order for $p^{i-1} \mid x$. Now $b - c^2 = p^{2k}(w^2 - v^2)$ where $w^2 - v^2$ is not always a unit. When $v = \pm w + p*, p \mid w^2 - v^2$ and $p^{2k}(w^2 - v^2) = 0$. There are $2p^{i-k-1}$ such $v$. For remaining $v$, the number of which is $p^{i-k} - 2p^{i-k-1} = p^{i-k-1}(p-2)$, $w^2 - v^2$ is a unit. Applying Lemma 4.5, we have

$$P = 2p^{i-k-1}(p^i - p^{i-1}) - p^{i-k-1}(p-2).$$

(c) $i - 1 = 2k + 1$. Now $p^{i-1} \mid b - c^2 \Leftrightarrow c^2 = 0 \Rightarrow c = p^{k+1} *$. The number of such $c$ is $p^{i-k-1}$ and we have

$$P = -p^{i-k-1}.$$

(iv) $b = p^t u, 1 \le t < i - 1$. This case can be dealt with similarly to case (iii). The results can be summarized as follows.
(a) $t$ is odd, then there is no $c$ making $p^{i-1} \mid x$, hence, $P = 0$.
(b) $t$ is even and $u$ is non square unit, $P = 0$.
(c) $t$ is even and $u = v^2, P = 2p^{\frac{t}{2}+1}(p^i - p^{i-1} - p^{\frac{t}{2}} + 1)$.
(v) $b = 0$. Now $x = -d^{-1}c^2$ and $P = \sum\limits_{c \in R_i, d \in R_i^\times} \lambda(-d^{-1}c^2) = (p^i - p^{i-1}) \sum\limits_{c \in R_i} \lambda(c^2)$.
We have the evaluation of the above sum in 4.3.2, so this case is done as well.
When $j < i$, the sum can be discussed case by case like before. There are cases where we have to consider the possible values for $\beta$ as well, making the discussion more complicated. Some cases are very simple and similar to what we just did. For example, when $b$ is a non square unit, $x$ can only take units and therefore $P = 0$.
From the above results, we can figure out the character values of $\chi$ on each kind of conjugacy classes.

### 4.3.2 Remaining Values for Characters of Degree $(p^2 - 1)p^{\ell-2}$

In this section, we want to evaluate the remaining character values. We will first work on the even case, then the odd case will follow similarly.
Denote

$$G = \mathrm{GL}(2, \mathbb{Z}/p^{2m}\mathbb{Z}), K_m = \{I + p^m B\},$$

$$A = \begin{pmatrix} & p^j\beta \\ 1 & \end{pmatrix}, 1 \le j \le m, \beta \in R^\times, R = \mathbb{Z}/p^{2m}\mathbb{Z}.$$

Now let $\lambda : \mathbb{Z}/p^{2m}\mathbb{Z} \to \mathbb{C}^\times$, injective; $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Define

$$\phi_A : K_m \to \mathbb{C}^\times, \phi_A(I + p^m B) = \lambda(\mathrm{tr}(p^m AB)) = \lambda(p^m b + p^{m+j} c\beta).$$

$$\mathrm{Stab}_G(\phi_A) = T = \left\{ \begin{pmatrix} a & p^j\beta b + p^m c \\ b & a + p^m d \end{pmatrix} \right\}.$$

We can extend $\phi_A$ to $\psi$ of $T$ then we know $\chi = \mathrm{Ind}_T^G \psi \in \mathrm{Irr}(G)$. We can first define $\psi$ satisfying

$$\psi \begin{pmatrix} 1 + p^m a & p^m b \\ p^{m-j} c & 1 + p^m d \end{pmatrix} = \lambda(p^m b + p^m c\beta)$$

and get different extensions by multiplying by roots of unity.

Pick coset representatives of $T$ in $G$ to be

$$E_{cd} = \begin{pmatrix} 1 & \\ & d \end{pmatrix} \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}, 0 \le c, d < p^m, d \in R^\times,$$

and

$$F_{cd} = \begin{pmatrix} 1 & \\ & d \end{pmatrix} \begin{pmatrix} 1 & pc \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, 0 \le pc, d < p^m, d \in R^\times.$$

Notice that the only conjugacy class type that intersects $T$ is $B_{i\alpha\beta}$, so we only need to evaluate the character values of $\chi$ on $X = \begin{pmatrix} a & p^{i+j}\beta \\ p^i & a \end{pmatrix}, 0 \le i < m$. By calculation, we have

$$F_{cd} X F_{cd}^{-1} \notin T, \forall c, d,$$

and

$$E_{cd} X E_{cd}^{-1} = \begin{pmatrix} a + p^i c & p^i d^{-1}(p^j\beta - c^2) \\ p^i d & a - p^i c \end{pmatrix}.$$

First, we assume $m > i + j$. In order for $E_{cd} X E_{cd}^{-1} \in T$, we must have

$$c = p^{m-i} e, d = p^{m-i-j} f \pm 1, 0 \le e < p^i, 0 \le f < p^{i+j}.$$

Since

$$\begin{aligned} Y &= E_{(p^{m-i}e)(1+p^{m-i-j}f)} X E^{-1}_{(p^{m-i}e)(1+p^{m-i-j}f)} \begin{pmatrix} a & -p^{i+j}\beta \\ -p^i & a \end{pmatrix} \\ &= (a^2 - p^{2i+1}\beta) \begin{pmatrix} 1 + p^m * & d^{-1} w \\ p^{m-j} f a^{-1} + p^{m+i} e a^{-2} & 1 + p^m * \end{pmatrix}, \end{aligned},$$

where $w = (-p^{2m-i}a^{-1}e^2 - p^{m+i+j}\beta e a^{-2} - p^m \beta^{-1} f)$, we have

$$\psi(Y) = \psi\begin{pmatrix} a^2 - p^{2i+1}\beta & \\ & a^2 - p^{2i+1}\beta \end{pmatrix} \lambda[p^{2m-i-j}a^{-1}(1+p^{m-i-j}f)^{-1}(\beta f^2 - p^j e^2)].$$

Notice that

$$\psi(X)\psi\begin{pmatrix} a & -p^{i+j}\beta \\ -p^i & a \end{pmatrix} = \psi\begin{pmatrix} a^2 - p^{2i+1}\beta & \\ & a^2 - p^{2i+1}\beta \end{pmatrix},$$

we know

$$\begin{aligned} &\psi(E_{(p^{m-i}e)(1+p^{m-i-j}f)} X E^{-1}_{(p^{m-i}e)(1+p^{m-i-j}f)}) \\ = &\psi\begin{pmatrix} a & p^{i+j}\beta \\ p^i & a \end{pmatrix} \lambda[p^{2m-i-j}a^{-1}(1+p^{m-i-j}f)^{-1}(\beta f^2 - p^j e^2)]. \end{aligned}$$

Similarly,

$$\begin{aligned} &\psi(E_{(p^{m-i}e)(p^{m-i-j}f-1)} X E^{-1}_{(p^{m-i}e)(p^{m-i-j}f-1)}) \\ = &\psi\begin{pmatrix} a & -p^{i+j}\beta \\ -p^i & a \end{pmatrix} \lambda[p^{2m-i-j}a^{-1}(p^{m-i-j}f-1)^{-1}(\beta f^2 - p^j e^2)]. \end{aligned}$$

Note that $1 + p^{m-i-j}f$ and $1 - p^{m-i-j}f$ are two square units, we can make a substitution,

$$f' = \frac{f}{\sqrt{1+p^{m-i-j}f}}, e' = \frac{e}{\sqrt{1+p^{m-i-j}f}},$$

to get

$$\begin{aligned} \chi\begin{pmatrix} a & p^{i+j}\beta \\ p^i & a \end{pmatrix} = &\sum_{e=0}^{p^i-1}\sum_{f=0}^{p^{i+j}-1}\left\{\psi\begin{pmatrix} a & p^{i+j}\beta \\ p^i & a \end{pmatrix}\lambda[p^{2m-i-j}a^{-1}(\beta f^2 - p^j e^2)]\right\} \\ &+\sum_{e=0}^{p^i-1}\sum_{f=0}^{p^{i+j}-1}\left\{\psi\begin{pmatrix} a & -p^{i+j}\beta \\ -p^i & a \end{pmatrix}\lambda[p^{2m-i-j}a^{-1}(p^j e^2 - \beta f^2)]\right\}. \end{aligned}$$

The above two summations can be calculated because

$$\begin{aligned} &\sum_{e=0}^{p^i-1}\sum_{f=0}^{p^{i+j}-1}\lambda[p^{2m-i-j}a^{-1}(\beta f^2 - p^j e^2)] \\ = &\sum_{e=0}^{p^i-1}\lambda(-p^{2m-i}a^{-1}e^2)\sum_{f=0}^{p^{i+j}-1}\lambda(p^{2m-i-j}a^{-1}\beta f^2) \\ = &\sum_{e=0}^{p^i-1}\lambda_1(e^2)\sum_{f=0}^{p^{i+j}-1}\lambda_2(f^2), \end{aligned}$$

for some injective homomorphisms

$$\lambda_1 : \mathbb{Z}/p^i\mathbb{Z} \longrightarrow C^\times, \lambda_2 : \mathbb{Z}/p^{i+j}\mathbb{Z} \longrightarrow C^\times.$$

Since we have the results that

$$\sum_{e=0}^{p^{2k}-1} \lambda(e^2) = p^k, \sum_{e=0}^{p^{2k+1}-1} \lambda(e^2) = p^k G(\lambda),$$

where

$$G(\lambda) = \sum_{r=0}^{p-1} \lambda(r^2)$$

is the quadratic Gauss sum [2] which has a formula depending on $\lambda$ and $p$, we can evaluate the summations in the formula of $\chi \begin{pmatrix} a & p^{i+j}\beta \\ p^i & a \end{pmatrix}$.

Now if $i + j \geq m$, we have $E_{cd} X E_{cd}^{-1} \in T \Rightarrow c = p^{m-i}e, d \in R^\times$. Therefore,

$$\psi(E_{cd} X E_{cd}^{-1}) = \psi \begin{pmatrix} a & \\ & a \end{pmatrix} \lambda[a^{-1}(p^{i+j}\beta d + p^{i+j}\beta d^{-1} - p^{2m-i}e^2 d^{-1})],$$

and we have

$$\chi(X) = \psi \begin{pmatrix} a & \\ & a \end{pmatrix} \sum_{0 \leq e < p^i, d \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \lambda[a^{-1}(p^{i+j}\beta d + p^{i+j}\beta d^{-1} - p^{2m-i}e^2 d^{-1})].$$

Denote $P = \sum_{0 \leq e < p^i, d \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \lambda[a^{-1}(p^{i+j}\beta d + p^{i+j}\beta d^{-1} - p^{2m-i}e^2 d^{-1})]$, in order to evaluate $P$ we have the following 3 cases.

(i) $i + j > 2m - i$.

In this case, we can find $\lambda' : \mathbb{Z}/p^i\mathbb{Z} \longrightarrow \mathbb{C}^\times$ such that

$$\begin{aligned} P &= \sum_{e \in \mathbb{Z}/p^i\mathbb{Z}, d \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \lambda'[p^{2i+j-m}\beta(d + d^{-1}) - e^2 d^{-1}] \\ &= p^{m-i} \sum_{e \in \mathbb{Z}/p^i\mathbb{Z}, d \in (\mathbb{Z}/p^i\mathbb{Z})^\times} \lambda'[p^{2i+j-m}\beta d + d^{-1}(p^{2i+j-m}\beta - e^2)]. \end{aligned}$$

Now $P$ can be evaluated because the above summation has been done in 4.3.1.

(ii) $i + j = 2m - i$.

In this case

$$P = p^{m-i} \sum_{e \in \mathbb{Z}/p^i\mathbb{Z}, d \in (\mathbb{Z}/p^i\mathbb{Z})^\times} \lambda'[\beta d + d^{-1}(\beta - e^2)].$$

Compare the above summation with the one in 4.3.1, we know it can be evaluated using the same argument.

(iii) $i + j < 2m - i$.

Now we can find injective $\lambda_1 : \mathbb{Z}/p^{2m-i-j}\mathbb{Z} \longrightarrow \mathbb{C}^\times, \lambda_2 : \mathbb{Z}/p^i\mathbb{Z} \longrightarrow \mathbb{C}^\times$ to simplify $P$ such that

$$\begin{aligned} P &= \sum_{d\in(\mathbb{Z}/p^m\mathbb{Z})^\times} \lambda_1(d+d^{-1}) \sum_{e\in\mathbb{Z}/p^i\mathbb{Z}} \lambda_2(e^2d^{-1}) \\ &= p^{i+j-m} \sum_{d\in(\mathbb{Z}/p^{2m-i-j}\mathbb{Z})^\times} \lambda_1(d+d^{-1}) \sum_{e\in\mathbb{Z}/p^i\mathbb{Z}} \lambda_2(e^2d^{-1}) \end{aligned}.$$

Note that the second summation above is the quadratic Gauss sum and we have

$$\sum_{e\in\mathbb{Z}/p^{2k}\mathbb{Z}} \lambda(e^2) = p^k$$

for any injective $\lambda : \mathbb{Z}/p^{2k} \longrightarrow \mathbb{C}^\times$, thus when $i$ is even, we have

$$P = p^{\frac{i}{2}} p^{i+j-m} \sum_{d\in(\mathbb{Z}/p^{2m-i-j}\mathbb{Z})^\times} \lambda_1(d+d^{-1})$$

which involves an unknown summation and we will stop here.

**Odd Case**

Now denote

$$G = \mathrm{GL}(2, \mathbb{Z}/p^{2m+1}\mathbb{Z}), K_m = \{I + p^m B\}, K_{m+1} = \{I + p^{m+1}B\},$$

$$A = \begin{pmatrix} & p^j\beta \\ 1 & \end{pmatrix}, 1 \le j \le m, \beta \in R^\times.$$

Let $\lambda : \mathbb{Z}/p^{2m+1}\mathbb{Z} \to \mathbb{C}^\times$, injective; $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Define

$$\phi_A : K_{m+1} \to \mathbb{C}^\times, \phi_A(I + p^{m+1}B) = \lambda(\mathrm{tr}(p^{m+1}AB)) = \lambda(p^{m+1}b + p^{m+j+1}c\beta).$$

$$T = \mathrm{Stab}_G(\phi_A) = \left\{ \begin{pmatrix} a & p^j\beta b + p^m c \\ b & a + p^m d \end{pmatrix} \right\} = K_m S, \text{ where } S = \left\{ \begin{pmatrix} w & p^j\beta y \\ y & w \end{pmatrix} \right\}.$$

Recall that we have the following picture

$$\begin{array}{ccccccccc} K_{m+1} & \longrightarrow & N & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi'_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ind}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array},$$

where $N = \left\{ \begin{pmatrix} 1+p^m a & p^{m+1}b \\ p^{m+1-j}c & 1+p^m d \end{pmatrix} \right\}$, and we can extend $\phi_A$ to $\phi'_A$ of $N$ such that

$$\phi'_A \begin{pmatrix} 1+p^m a & p^{m+1}b \\ p^{m+1-j}c & 1+p^m d \end{pmatrix} = \lambda(p^{m+1}b + p^{m+1}c\beta).$$

Similarly to the even case, we can get different extensions $\phi'_A$ by multiplying by roots of unity.

Let $T' = \left\{ \begin{pmatrix} a & p^j\beta b + p^{m+1}c \\ b & a+p^m d \end{pmatrix} \right\} = NS$, then $N \lhd T'$ and $\phi'_A$ is stable under $T'$. Thus, we can extend $\phi'_A$ of $N$ to $\phi'$ of $T'$ such that $\phi'$ is trivial on the center of $G$. Since $T' \lhd T$, and $\mathrm{Stab}_T(\phi') = T'$, we have $\psi = \mathrm{Ind}_{T'}^T \phi' \in \mathrm{Irr}(T)$. Therefore, $\chi = \mathrm{Ind}_T^G \psi = \mathrm{Ind}_{T'}^G \phi' \in \mathrm{Irr}(G)$. In order to evaluate the character values of $\chi$, we can consider $\chi$ as induced from $\phi'$ of $T'$. The coset representatives of $T'$ are

$$E_{cd} = \begin{pmatrix} 1 & \\ & d \end{pmatrix} \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}, 0 \le d < p^m, d \in R^\times, 0 \le c < p^{m+1}$$

and

$$F_{cd} = \begin{pmatrix} 1 & \\ & d \end{pmatrix} \begin{pmatrix} 1 & pc \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, 0 \le c, d < p^m, d \in R^\times.$$

Let $X = \begin{pmatrix} a & p^{i+j}\beta \\ p^i & a \end{pmatrix}, 0 \le i \le m$, then

$$F_{cd} X F_{cd}^{-1} \notin T', \forall c, d, \text{ and } E_{cd} X E_{cd}^{-1} = \begin{pmatrix} a + p^i c & p^i d^{-1}(p^j\beta - c^2) \\ p^i d & a - p^i c \end{pmatrix}.$$

We first assume that $m+1 > i+j$, then in order for $E_{cd} X E_{cd}^{-1} \in T'$, we must have

$$c = p^{m-i}e, d = p^{m+1-i-j}f \pm 1, 0 \le e < p^{i+1}, 0 \le f < p^{i+j-1}.$$

By the same method as in the even case, we first calculate

$$Y = E_{(p^{m-i}e)(1+p^{m+1-i-j}f)} X E^{-1}_{(p^{m-i}e)(1+p^{m+1-i-j}f)} \begin{pmatrix} a & -p^{i+j}\beta \\ -p^i & a \end{pmatrix}$$

and deduce that

$$\phi'(Y) = \phi' \begin{pmatrix} a^2 - p^{2i+1}\beta & \\ & a^2 - p^{2i+1}\beta \end{pmatrix}$$
$$\times \lambda[p^{\ell-i-j}a^{-1}(1+p^{m+1-i-j}f)^{-1}(p^j\beta f^2 - e^2)].$$

Denote $\phi' \begin{pmatrix} a & p^{i+j}\beta \\ p^i & a \end{pmatrix} = P^+, \phi' \begin{pmatrix} a & -p^{i+j}\beta \\ -p^i & a \end{pmatrix} = P^-$, we have

$$\phi'(E_{(p^{m-i}e)(1+p^{m+1-i-j}f)} X E^{-1}_{(p^{m-i}e)(1+p^{m+1-i-j}f)})$$

$$= P^+ * \lambda[p^{\ell-i-j}a^{-1}(1+p^{m+1-i-j}f)^{-1}(p^j\beta f^2 - e^2)].$$

Similarly,

$$\phi'(E_{(p^{m-i}e)(p^{m+1-i-j}f-1)}XE^{-1}_{(p^{m-i}e)(p^{m+1-i-j}f-1)})$$
$$= P^- * \lambda[p^{\ell-i-j}a^{-1}(p^{m+1-i-j}f-1)^{-1}(p^j\beta f^2 - e^2)].$$

Making a substitution gives us

$$\chi\begin{pmatrix} a & p^{i+j}\beta \\ p^i & a \end{pmatrix} = \sum_{e=0}^{p^{i+1}-1}\sum_{f=0}^{p^{i+j-1}} \left\{P^+ * \lambda[p^{\ell-i-j}a^{-1}(p^j\beta f^2 - e^2)]\right\}$$
$$+ \sum_{e=0}^{p^{i+1}-1}\sum_{f=0}^{p^{i+j-1}} \left\{P^- * \lambda[p^{\ell-i-j}a^{-1}(e^2 - p^j\beta f^2)]\right\}.$$

Once again, the above summations can be written as a product of two Gauss sums, hence can be calculated.

Like the even case, if $i + j \geq m + 1$, we have

$$\chi(X) = \phi\begin{pmatrix} a & \\ & a \end{pmatrix} \sum_{0\leq e<p^{i+1}, d\in(\mathbb{Z}/p^m\mathbb{Z})^\times} \lambda[a^{-1}(p^{i+j}\beta d + p^{i+j}\beta d^{-1} - p^{2m-i}e^2d^{-1})].$$

Similarly, the above summation can be discussed in the same way as the even case.

# Chapter 5

# Irreducible Characters of $\mathrm{GL}(3, \mathbb{Z}/p^\ell\mathbb{Z})$

In this section, we first work on $G = \mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$ and find all the irreducible characters of it and the number of characters of each degree. See also [9]. We move on to find irreducible characters of $\mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$. There are 7 kinds of conjugacy classes for the group $\mathrm{GL}(3, \mathbb{Z}/p\mathbb{Z})$ and each conjugacy class gives us one kind of irreducible characters of $G$. Since these 7 kinds of conjugacy classes show up for the group $\mathrm{GL}(3, \mathbb{Z}/p^\ell\mathbb{Z})$ for any $\ell$, the 7 kinds of irreducible characters also show up for any group $\mathrm{GL}(3, \mathbb{Z}/p^\ell\mathbb{Z})$.

## 5.1 The Irreducible Characters of $\mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$

Denote $G = \mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z}), K_1 = \{I + pB, B \in M(3, \mathbb{Z}/p\mathbb{Z})\}$. It's easy to see that $K_1$ is abelian and $K_1 \lhd G$. Fix an injective $\lambda : \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{C}^\times$, we first define one-dimensional character $\phi_A$ on $K_1$ using the formula

$$\phi_A(I + pB) = \lambda(\mathrm{tr}(pAB))$$

for some $A \in M_3(\mathbb{Z}/p\mathbb{Z})$, the set of $3 \times 3$ matrices over $\mathbb{Z}/p\mathbb{Z}$. Here we use $A \in M_3(\mathbb{Z}/p\mathbb{Z})$ because $(A + pC)(pB) = pAB$; but we treat $A$ as a matrix in $\mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$, so the matrix multiplication $pAB$ makes sense.
Like in the $2 \times 2$ case before, we have

$$(\phi_A)^g(I + pB) = \phi_{g^{-1}Ag}(I + pB)$$

and the stabilizer of $\phi_A$ is

$$T = \mathrm{Stab}_G(\phi_A) = \{g \in G : pgA = pAg\}.$$

As it turns out, $\mathrm{Stab}_G(\phi_A) = T = K_1S$,where $S$ is a subgroup of $G$ and depends on the choice of $A$. Next, we extend $\phi_A$ to $\psi_A$ of $T$ such that $\psi_A\mid_{K_1}=$

$\phi_A$. By Clifford Theory we know $\chi_A = \mathrm{Ind}_T^G(\psi_A) \in \mathrm{Irr}(G)$. The process is as follows:

$$\begin{array}{ccccc} K_1 & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \psi_A & \xrightarrow{\text{ind}} & \chi_A \end{array}$$

Since $\chi_A = \chi_{A'}$ if $A$ is conjugate to $A'$, we want to use non-conjugate matrices $A$ and have 7 cases. The 7 cases will give us 7 kinds of irreducible characters. Clifford theory tells us that $\mathrm{Ind}_T^G(\psi_A\beta) \in \mathrm{Irr}(G)$ for any $\beta \in \mathrm{Irr}(T/K_1)$ and we know

$$\deg(\mathrm{Ind}_T^G(\psi_A\beta)) = \deg(\beta)\deg(\mathrm{Ind}_T^G(\psi_A)).$$

If $T/K_1$ is abelian, then $\deg(\beta) = 1$ and we have $\deg(\mathrm{Ind}_T^G(\psi_A\beta)) = \deg(\chi_A)$. For other cases, $T/K_1$ is not abelian, we can get new irreducible characters of $G$ with different degrees by finding all the irreducible characters of $T/K_1$.
Denote $A_\alpha = \alpha I + A$. For the same $\lambda : \mathbb{Z}/p^2\mathbb{Z} \longrightarrow C^\times$, we can define $\phi_{A_\alpha}$ on $K_1$ in the same way as defining $\phi_A$. Let $\lambda^\alpha(g) = \lambda(\alpha g)$, then we have

$$\phi_{A_\alpha}(I + pB) = \lambda^\alpha(\mathrm{tr}(pB))\phi_A(I + pB).$$

Let $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, we have $\mathrm{tr}(B) = (a + e + i)$ and $\det(I + pB) = 1 + p(a + e + i)$. Note that $\{1 + px\}^\times \cong \{px\}^+$ in $\mathbb{Z}/p^2\mathbb{Z}$, we can find a character $\mu_\alpha = \mu : (\mathbb{Z}/p^2\mathbb{Z})^\times \to \mathbb{C}^\times$ such that $\lambda(\alpha pg) = \mu(1 + pg)$.

$$\lambda^\alpha(\mathrm{tr}(pB)) = \mu(\det(I + pB)).$$

Therefore,

$$\phi_{A_\alpha} = (\mu \circ \det) \times \phi_A.$$

Since $\mu \circ \det$ is a linear character of $G$, it is stable under $G$. We also have $\mathrm{Stab}_G(\phi_{A_\alpha}) = \mathrm{Stab}_G(\phi_A)$ and $(\mu \circ \det) \times \psi_A$ is an extension of $\phi_{A_\alpha}$ provided that $\psi_A$ is an extension of $\phi_A$. It is clear that

$$\chi_{A_\alpha} = \mathrm{Ind}_T^G[(\mu \circ \det) \times \psi_A] = (\mu \circ \det) \times \mathrm{Ind}_T^G \psi_A = (\mu \circ \det) \times \chi,$$

therefore

$$\deg(\chi_A) = \deg(\chi_{A_\alpha}).$$

In order to show that we have all the irreducible characters for $G$, we need to count how many irreducible characters of each degree. In each of the 7 cases, we can get the same construction process if we replace $A$ by $\alpha I + A$, because the stabilizer does not change and we can always find the corresponding extension $\psi_A$ of $\phi_A$. The way to count how many irreducible characters of each degree is similar to the $2 \times 2$ case before, that is,

$$\#\text{ of each degree } = \#\text{ of nonconjugate } \alpha I + A$$

$$\times\#\text{ of irreducible characters of } T/K_1.$$

In the end, we will verify that the orthogonality relation holds. Like in the $2\times 2$ case, we give details for a specific choice of $A$ and then the argument can be generalized for matrices $\alpha I + A$.
Since $(I+pA)(I+pB) = I + p(A+B)$, in some cases it is more convenient to use a multiplicative $\lambda' : (\mathbb{Z}/p^2\mathbb{Z})^\times \to \mathbb{C}^\times$ to define $\phi_A$ such that

$$\phi_A(I+pB) = \lambda'[\mathrm{tr}((I+pB)A)].$$

We will specify the choice of multiplicative $\lambda'$ when we use it later, otherwise we use the additive one as before.

### 5.1.1 Irreducible Characters of Degree $p^2(p^2+p+1)$

We pick $A_k = \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $k \in (\mathbb{Z}/p\mathbb{Z})^\times$. Let $\lambda' : (\mathbb{Z}/p^2\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ be such that

$$\lambda(kpx) = \lambda'(1+px).$$

We can define $\phi_A$ as

$$\phi_A \begin{pmatrix} 1+pa_{11} & p* & p* \\ p* & 1+p* & p* \\ p* & p* & 1+p* \end{pmatrix} = \lambda'(1+pa_{11}).$$

The stabilizer is $\mathrm{Stab}_G(\phi_A) = T = \left\{ \begin{pmatrix} a & p* & p* \\ p* & x & y \\ p* & z & w \end{pmatrix} \right\}$. We can extend $\phi_A$ to $\psi_A$ by defining

$$\psi_A \begin{pmatrix} a & p* & p* \\ p* & x & y \\ p* & z & w \end{pmatrix} = \lambda'(a).$$

We have

$$\chi_A = \mathrm{Ind}_T^G(\psi_A) \in \mathrm{Irr}(G) \text{ and } \deg(\chi_A) = \frac{|\,G\,|}{|\,T\,|} = p^2(p^2+p+1).$$

Notice that $T/K_1 \cong (\mathbb{Z}/p\mathbb{Z})^\times \times \mathrm{GL}(2,\mathbb{Z}/p\mathbb{Z})$ and the degrees of irreducible characters of $(\mathbb{Z}/p\mathbb{Z})^\times \times \mathrm{GL}(2,\mathbb{Z}/p\mathbb{Z})$ are

$$1, p, p+1, p-1 \quad (*)$$

with numbers

$$(p-1)^3, (p-1)^3, \frac{(p-1)^3(p-2)}{2}, \frac{p(p-1)^3}{2} \quad (**)$$

respectively, we can get more irreducible characters of $G$ with degrees

$$p^2(p^2+p+1), p^3(p^2+p+1), p^2(p^2+p+1)(p+1), p^2(p^2+p+1)(p-1).$$

A general choice of $A$ is

$$A_{k,\alpha} = A_k + \alpha I = \begin{pmatrix} k+\alpha & & \\ & \alpha & \\ & & \alpha \end{pmatrix}, k \in (\mathbb{Z}/p\mathbb{Z})^\times, \alpha \in \mathbb{Z}/p\mathbb{Z}.$$

Since $A_{k,\alpha}$ is conjugate to $A_{k',\alpha'}$ if and only if $k = k', \alpha = \alpha'$, the number of non-conjugate such $A_{k,\alpha}$ is $p(p-1)$. To count the number of irreducible characters with degrees in $(*)$, we only need to multiply the numbers in $(**)$ by $p(p-1)$. All the degrees and corresponding number of each degree are summarized in the following table.

| Degrees | Number of this degree |
|---|---|
| $p^2(p^2+p+1)$ | $p(p-1)^4$ |
| $p^3(p^2+p+1)$ | $p(p-1)^4$ |
| $p^2(p^2+p+1)(p+1)$ | $\frac{p(p-1)^4(p-1)}{2}$ |
| $p^2(p^2+p+1)(p-1)$ | $\frac{p^2(p-1)^4}{2}$ |

### 5.1.2 Irreducible Characters of Degree $p^3(p+1)(p^2+p+1)$

Let $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $a, b \in \mathbb{Z}/p\mathbb{Z}, a \neq b, a \neq 0, b \neq 0$. Let

$$\lambda_1, \lambda_2 : (\mathbb{Z}/p^2\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

be such that

$$\lambda_1(1+p*) = \lambda[a(1+p*)], \lambda_2(1+p*) = \lambda[b(1+p*)],$$

then $\phi_A$ can be defined as

$$\phi_A \begin{pmatrix} 1+pa_{11} & p* & p* \\ p* & 1+pa_{22} & p* \\ p* & p* & 1+p* \end{pmatrix} = \lambda_1(1+pa_{11})\lambda_2(1+pa_{22}).$$

The stabilizer is $T = \left\{ \begin{pmatrix} a & p* & p* \\ p* & y & p* \\ p* & p* & z \end{pmatrix} \right\}$ and one extension $\psi_A$ on $T$ can be defined as

$$\psi_A \begin{pmatrix} a & p* & p* \\ p* & y & p* \\ p* & p* & z \end{pmatrix} = \lambda_1(a)\lambda_2(y).$$

Since

$$T/K_1 \cong \left\{ \begin{pmatrix} a & & \\ & y & \\ & & z \end{pmatrix} \right\}$$

is abelian, the only degree we get in this case is

$$\deg(\chi_A) = \frac{\mid G \mid}{\mid T \mid} = p^3(p+1)(p^2+p+1).$$

The number of different extensions $\psi_A$ is

$$\mid T/K_1 \mid = (p-1)^3.$$

The general matrix $A$ we can use is

$$A_\alpha = \alpha I + A = \begin{pmatrix} a+\alpha & 0 & 0 \\ 0 & b+\alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \alpha \in \mathbb{Z}/p\mathbb{Z}.$$

Since we get conjugate matrices if we permutate the diagonal entries of $A_\alpha$, the number of non-conjugate such matrices is $\frac{p(p-1)(p-2)}{6}$. To summarize, we have

| Degrees | Number of this degree |
|---|---|
| $p^3(p+1)(p^2+p+1)$ | $\frac{p(p-1)(p-2)(p-1)^3}{6}$ |

### 5.1.3 Irreducible Characters of Degree $p^3(p-1)^2(p+1)$

Now $A = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}$ such that the polynomial $t^3 - ct^2 - bt - a$ is irreducible in $\mathbb{Z}/p\mathbb{Z}[t]$. By subtracting the number of reducible polynomials from the total number of polynomials over $\mathbb{Z}/p\mathbb{Z}$, we can get that the number of irreducible polynomials, and so is the number of matrices $A$, is $\frac{p(p-1)(p+1)}{3}$. Define $\phi_A$ on $K_1$ as

$$\phi_A(I + pB) = \lambda(\mathrm{tr}(pAB)).$$

The stabilizer in this case is

$$T = K_1 S, S = \{xI + yA + zA^2\} < \mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z}).$$

Since $S$ is abelian,we do not need to find an explicit formula for $\psi_A$. The existence of $\psi_A$ on $T$ is guaranteed by Lemma 2.2.10 and the number of such $\psi_A$ is

$$\frac{\mid K_1 S \mid}{\mid K_1 \mid} = p^3 - 1.$$

Since $K_1S/K_1$ is abelian, the only degree we get in this case is

$$\deg(\chi_A) = \frac{| G |}{| T |} = p^3(p-1)^2(p+1).$$

We do not need to count

$$A_\alpha = \begin{pmatrix} \alpha & 0 & a \\ 1 & \alpha & b \\ 0 & 1 & \alpha + c \end{pmatrix}$$

in this case because $A_\alpha$ is actually conjugate to $A = \begin{pmatrix} 0 & 0 & a' \\ 1 & 0 & b' \\ 0 & 1 & c' \end{pmatrix}$ for certain $a', b', c'$. Therefore the number of non-conjugate $A$ is $\frac{p(p-1)(p+1)}{3}$. We have

| Degrees | Number of this degree |
|---|---|
| $p^3(p-1)^2(p+1)$ | $\frac{p(p-1)(p+1)(p^3-1)}{3}$ |

### 5.1.4 Irreducible Characters of Degree $(p^3 - 1)(p + 1)$

Now let $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then

$$\phi_A \begin{pmatrix} 1+p* & p* & p* \\ p* & 1+p* & p* \\ p*a_{31} & p* & 1+p* \end{pmatrix} = \lambda(pa_{31})$$

and the stabilizer is $T = \left\{ \begin{pmatrix} a & w & y \\ p* & x & z \\ p* & p* & a+p* \end{pmatrix} \right\}$. One extension $\psi_A$ can be defined as

$$\psi_A \begin{pmatrix} a & w & y \\ p* & x & z \\ pa_{31} & p* & a+p* \end{pmatrix} = \lambda(pa_{31}a^{-1}).$$

We can get the degree of $\chi_A$ as

$$\deg(\chi_A) = \frac{| G |}{| T |} = (p^3-1)(p+1)$$

right away by inducing up $\psi_A$ to $G$. However, since

$$T/K_1 \cong \left\{ \begin{pmatrix} a & w & y \\ 0 & x & z \\ 0 & 0 & a \end{pmatrix} \right\} \subset \mathrm{GL}(3, \mathbb{Z}/p\mathbb{Z})$$

is not abelian and all the irreducible characters of $\left\{\begin{pmatrix} a & w & y \\ 0 & x & z \\ 0 & 0 & a \end{pmatrix}\right\}$ have degrees

$$1, p, p-1$$

with number

$$(p-1)^2, p^2-1, (p-1)^3$$

respectively, we can also have new irreducible $\chi \in \mathrm{Irr}(G)$ with degrees

$$(p^3-1)(p+1)(p-1), (p^3-1)(p+1)p.$$

The general matrix we can use to replace $A$ is

$$A_\alpha = \begin{pmatrix} \alpha & 0 & 1 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \text{ with } \alpha \in \mathbb{Z}/p\mathbb{Z}$$

and we have $p$ such matrices. Clearly they are not conjugate to each other. To summarize, we get

| Degrees | Number of this degree |
|---|---|
| $(p^3-1)(p+1)$ | $p(p-1)^2$ |
| $(p^3-1)(p+1)(p-1)$ | $p(p^2-1)$ |
| $p(p^3-1)(p+1)$ | $p(p-1)^3$ |

We can see some details of constructing irreducible characters of the group $\left\{\begin{pmatrix} a & w & y \\ 0 & x & z \\ 0 & 0 & a \end{pmatrix}\right\}$ in a later section.

### 5.1.5 Irreducible Characters of Degree $p(p^3-1)(p^2-1)$

Pick $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then

$$\phi_A \begin{pmatrix} 1+p* & p* & p* \\ pa_{21} & 1+p* & p* \\ p* & pa_{32} & 1+p* \end{pmatrix} = \lambda(pa_{21}+pa_{32})$$

and we have the stabilizer

$$T = \left\{\begin{pmatrix} a & b & c \\ p* & a+p* & b+p* \\ p* & p* & a+p* \end{pmatrix}\right\} = K_1 S \text{ where } S = \left\{\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}\right\}.$$

Therefore

$$\deg(\chi_A) = \frac{| G |}{| T |} = p(p^3 - 1)(p^2 - 1).$$

The existence of $\psi_A$ is guaranteed by Lemma 2.2.10 again, but here we give some details because we will use it in later sections. Notice that

$$\phi_A \mid_{K_1 \cap S} = 1,$$

we can define $\psi_A$ on $T$ this way:

$$\psi_A \mid_{K_1} = \phi_A, \psi_A \mid_S = 1, \psi_A(k_1 s) = \phi_A(k_1), \forall k_1 \in K_1, s \in S.$$

$\psi_A$ is well-defined and clearly an extension of $\phi_A$. To get more extensions, we can multiply $\psi_A$ by any irreducible character of $T/K_1$. Since $T/K_1$ is abelian, the total number of extensions is

$$\frac{| T |}{| K_1 |} = p^2(p - 1).$$

The more general matrix to start with is $A_\alpha = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}$ and we have $p$ non-conjugate ones. The degree and number of this degree we get is as follows.

| Degrees | Number of this degree |
|---|---|
| $p(p^3 - 1)(p^2 - 1)$ | $p^3(p - 1)$ |

### 5.1.6 Irreducible Characters of Degree $p^3(p^3 - 1)$

We will use $A = \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \epsilon & 0 \end{pmatrix}$ in this case, where $\epsilon$ is a non-square unit in $\mathbb{Z}/p\mathbb{Z}$. Let $\lambda_1 : (\mathbb{Z}/p^2\mathbb{Z})^\times \to \mathbb{C}^\times$ be injective such that $\lambda(kpx) = \lambda_1(1 + px)$. An explicit formula for $\phi_A$ on $K_1$ is

$$\phi_A \begin{pmatrix} 1 + pa_{11} & p* & p* \\ p* & 1 + p* & pa_{23} \\ p* & pa_{32} & 1 + p* \end{pmatrix} = \lambda_1(1 + pa_{11})\lambda(pa_{32} + \epsilon pa_{23}).$$

The stabilizer is

$$T = \left\{ \begin{pmatrix} a & p* & p* \\ p* & b & c \\ p* & c\epsilon + p* & b + p* \end{pmatrix} \right\} = K_1 S \text{ where } S = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c\epsilon & b \end{pmatrix} \right\}.$$

Since $S$ is abelian, the existence of extension $\psi_A$ is guaranteed by Lemma 2.2.10 and the number of extensions here is

$$\frac{\mid T \mid}{\mid K_1 \mid} = (p-1)^2(p+1).$$

The general matrix we can use is $A_{x,y} = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 1 \\ 0 & \epsilon & y \end{pmatrix}$. Since $\begin{pmatrix} x & 0 & 0 \\ 0 & y & 1 \\ 0 & \epsilon & y \end{pmatrix}$ is conjugate to $\begin{pmatrix} x' & 0 & 0 \\ 0 & y' & 1 \\ 0 & \epsilon' & y' \end{pmatrix}$ if and only if $x = x'$ and $\begin{pmatrix} y & 1 \\ \epsilon & y \end{pmatrix}$ is conjugate to $\begin{pmatrix} y' & 1 \\ \epsilon' & y' \end{pmatrix}$, the number of non-conjugate $A_{x,y}$ is $\frac{p^2(p-1)}{2}$. The following table summarizes this case.

| Degrees | Number of this degree |
|---|---|
| $p^3(p^3-1)$ | $\frac{p^2(p-1)^3(p+1)}{2}$ |

### 5.1.7 Irreducible Characters of Degree $p^2(p+1)(p^3-1)$

The last case we let $A = \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $k \in (\mathbb{Z}/p\mathbb{Z})^\times$ and let $\lambda_1 : (\mathbb{Z}/p^2\mathbb{Z})^\times \to \mathbb{C}^\times$ be injective such that $\lambda(kpx) = \lambda_1(1+px)$. $\phi_A$ on $K_1$ can be defined as

$$\phi_A \begin{pmatrix} 1+pa_{11}* & p* & p* \\ p* & 1+p* & pa_{23} \\ p* & p* & 1+p* \end{pmatrix} = \lambda_1(1+pa_{11})\lambda_2(pa_{23}).$$

This will give us the stabilizer

$$T = \left\{ \begin{pmatrix} a & p* & p* \\ p* & b & c \\ p* & p* & b+p* \end{pmatrix} \right\} = K_1 S \text{ where } S = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & b \end{pmatrix} \right\}.$$

$S$ being abelian guarantees the existence of extension $\psi_A$ and the number of such extensions is

$$\frac{\mid T \mid}{\mid K_1 \mid} = p(p-1)^2.$$

Also the degree of $\chi_A$ we get in this case is

$$\deg(\chi_A) = \frac{\mid G \mid}{\mid T \mid} = p^2(p+1)(p^3-1).$$

More general choices of matrix $A$ are

$$A_{x,y} = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 1 \\ 0 & 0 & y \end{pmatrix}, x \neq 0.$$

Since $A_{x,y}$ is conjugate to $A_{x',y'}$ if and only if $x = x', y = y'$, we have $p(p-1)$ non-conjugate ones.We will have

| Degrees | Number of this degree |
|---|---|
| $p^2(p+1)(p^3-1)$ | $p^2(p-1)^3$ |

### 5.1.8 All the irreducible characters of $\mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$

We can summarize all the irreducible characters found in the previous 7 sections in the following table

| Degrees | Number of this degree |
|---|---|
| $p^2(p^2+p+1)$ | $p(p-1)^4$ |
| $p^3(p^2+p+1)$ | $p(p-1)^4$ |
| $p^2(p^2+p+1)(p+1)$ | $\frac{p(p-1)^4(p-1)}{2}$ |
| $p^2(p^2+p+1)(p-1)$ | $\frac{p^2(p-1)^4}{2}$ |
| $p^3(p+1)(p^2+p+1)$ | $\frac{p(p-1)(p-2)(p-1)^3}{6}$ |
| $p^2(p+1)(p^3-1)$ | $p^2(p-1)^3$ |
| $p^3(p^3-1)$ | $\frac{p^2(p-1)^3(p+1)}{2}$ |
| $p(p^3-1)(p^2-1)$ | $p^3(p-1)$ |
| $(p^3-1)(p+1)$ | $p(p-1)^2$ |
| $(p^3-1)(p+1)(p-1)$ | $p(p^2-1)$ |
| $p(p^3-1)(p+1)$ | $p(p-1)^3$ |
| $p^3(p-1)^2(p+1)$ | $\frac{p(p-1)(p+1)(p^3-1)}{3}$ |

To verify that we have all the irreducible characters of $G$, we first find the sum of square of the degrees. We have

$$(i) \quad \sum \text{number of degree} \times \deg^2 = p^4(p^3-1)(p^2-1)(p-1)(p^8-1).$$

Notice that we also have

$$(ii) \quad \sum_{\chi \text{ from } \mathrm{Irr}(\mathrm{GL}(3,\mathbb{Z}/p\mathbb{Z}))} \deg(\chi)^2 = p^3 \mid \mathrm{GL}(3, \mathbb{Z}/p\mathbb{Z}) \mid = p^4(p^3-1)(p^2-1)(p-1).$$

By the second orthogonality relationship, we know that

$$\sum_{\chi \in \mathrm{Irr}(G)} \deg(\lambda)^2 = \mid G \mid, \text{ for any finite group } G\ .$$

Since $(i) + (ii) = \mid \mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z}) \mid$, we do have all the irreducible characters of $G$.

## 5.2 Irreducible Characters of $\mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$

Now let $G = \mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z}), K_1 = \{I + pB\}, K_2 = \{I + p^2B\}$. To construct irreducible characters of $G$, we start with one-dimensional character $\phi_A$ of $K_2$, where $\phi_A$ is defined similarly to the even case. That is, fix an injective $\lambda : \mathbb{Z}/p^3\mathbb{Z} \to \mathbb{C}^\times$ and define

$$\phi_A(I + p^2B) = \lambda(\mathrm{tr}(p^2AB)),$$

for some $A \in \mathrm{GL}(3, \mathbb{Z}/p\mathbb{Z})$. We pick $A \in \mathrm{GL}(3, \mathbb{Z}/p\mathbb{Z})$ and treat it as a matrix in $\mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$ for the same reason as in the even case. For some cases, it is more convenient to pick an injective multiplicative $\lambda' : (\mathbb{Z}/p^3\mathbb{Z})^\times \to \mathbb{C}^\times$ and use it to define $\phi_A$. We will specify in those cases, otherwise $\phi_A$ is defined by the above formula.

Next we calculate the stabilizer of $\phi_A$ and will see $T = \mathrm{Stab}_G(\phi_A) = K_1S$ for some subgroup $S < G$ which depends on $A$. Unlike the even case, we can not extend $\phi_A$ to $T$. Instead, we will try to find $\psi \in \mathrm{Irr}(T)$ such that $[\psi\mid_{K_2}, \phi_A] \neq 0$. From Clifford Theory, we have $\chi = \mathrm{Ind}_T^G(\psi) \in \mathrm{Irr}(G)$.

We have two different construction processes. The first one is

$$(1) \qquad \begin{array}{ccccccccccc} K_2 & \longrightarrow & K_2(K_1 \cap S) & \longrightarrow & H & \longrightarrow & K_1 & \longrightarrow & K_1S & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ext}} & \phi'' & \xrightarrow{\text{ind}} & \theta & \xrightarrow{\text{ext}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array},$$

where $H \lhd K_1$ with index $p^3$. When using the above process, $S$ is abelian in each case. Therefore, the existence of $\phi'$ on $K_2(K_1 \cap S)$ is guaranteed by Lemma 2.2.10 because $K_1 \cap S$ is abelian. We can define $\phi''$ depending the choice of $H$ and have the stabilizer $\mathrm{Stab}_{K_1}(\phi'') = H$. By Clifford Theory, we have $\theta \in \mathrm{Irr}(H)$. Moreover,

$$\theta\mid_{K_2(K_1\cap S)} = p^3\phi', \theta\mid_{K_1 - K_2(K_1\cap S)} = 0.$$

So $\theta$ is invariant under $K_1S$ and $K_1S/K_1$ is cyclic, we can extend $\theta$ to $\psi$. Notice that $\deg(\psi) = p^3$, so

$$\deg(\chi) = p^3 \frac{\mid G \mid}{\mid K_1S \mid}.$$

This construction process can give us irreducible characters of one degree only. We can also have the number of each degree when using this process. As in the even case before, we can replace $A$ by $\alpha I + A$ to get more irreducible characters of $G$. We also want to count how many extensions we can get in each case. As we can see from the construction process and the property of $\theta$ in the picture, $\theta$ only depends on $\phi'$. Therefore, we only need to count the number extensions from $\phi_A$ to $\phi'$, which is equal to $\frac{|K_2(K_1\cap S)|}{|K_2|}$, and the number of extensions from $K_1$ to $K_1S$ which is equal to $\frac{|K_1S|}{|K_1|}$. To summarize, we have

$$\#\text{ of each degree } = \#\text{ of nonconjugate } (\alpha I + A)\ \times \frac{\mid K_2(K_1\cap S)\mid}{\mid K_2\mid} \times \frac{\mid K_1S\mid}{\mid K_1\mid}.$$

The degrees and numbers are both $p^3$ times the corresponding ones in the even case. There are 4 kinds of irreducible characters constructed using the above picture and we will give details of each construction.

### 5.2.1 Irreducible Characters of Degree $p^6(p-1)^2(p+1)$

Let $A = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}$ such that the polynomial $t^3 - ct^2 - bt - a$ is irreducible in $\mathrm{GL}(3,\mathbb{Z}/p\mathbb{Z})$. The stabilizer of $\phi_A$ is

$$T = K_1S \text{ where } S = \{xI + yA + zA^2\} < \mathrm{GL}(3,\mathbb{Z}/p^3\mathbb{Z}).$$

Choose

$$H = \left\{\begin{pmatrix} 1+p* & p^2* & p^2* \\ p* & 1+p* & p^2* \\ p^2* & p^2* & 1+p^2* \end{pmatrix}\right\} K_2(K_1\cap S),$$

then $H$ is a normal subgroup of $K_1$ with index $p^3$. Since

$$\left\{\begin{pmatrix} 1+p* & p^2* & p^2* \\ p* & 1+p* & p^2* \\ p^2* & p^2* & 1+p^2* \end{pmatrix}\right\} \cap (K_2(K_1\cap S)) \subset K_2,$$

we can define $\phi''$ of $H$ on $\left\{\begin{pmatrix} 1+p* & p^2* & p^2* \\ p* & 1+p* & p^2* \\ p^2* & p^2* & 1+p^2* \end{pmatrix}\right\}$ this way:

$$\phi''\begin{pmatrix} 1+p* & p^2a_{12} & p^2* \\ p* & 1+p* & p^2a_{23} \\ p^2a_{31} & p^2a_{32} & 1+p^2a_{33} \end{pmatrix} = \lambda\left[\mathrm{tr}\left(A\begin{pmatrix} p* & p^2a_{12} & p^2* \\ p* & p* & p^2a_{23} \\ p^2a_{31} & p^2a_{32} & p^2a_{33} \end{pmatrix}\right)\right],$$

and on $K_2(K_1\cap S)$ as

$$\phi'' \mid_{K_2(K_1\cap S)} = \phi' \mid_{K_2(K_1\cap S)}.$$

$$\phi''(h_1h_2) = \phi''(h_1)\phi''(h_2)$$

for

$$h_1 \in \left\{ \begin{pmatrix} 1+p* & p^2* & p^2* \\ p* & 1+p* & p^2* \\ p^2* & p^2* & 1+p^2* \end{pmatrix} \right\} \text{ and } h_2 \in K_2(K_1 \cap S).$$

$\phi''$ is a well defined extension of $\phi'$. Since $\mid T \mid = p^{18}(p^3-1)$, we have

$$\deg(\chi) = p^3 \frac{\mid G \mid}{\mid T \mid} = p^6(p-1)^2(p+1).$$

Notice that the number of non-conjugate matrices $\alpha I + A$ is $\frac{p(p-1)(p+1)}{3}$, extensions from $\phi_A$ to $\phi'$ is $p^3$ and from $\theta$ to $\psi$ is $p^3 - 1$, multiplying the above 3 will give us the number of irreducible characters in this case. To summarize,

| Degrees | Number of this degree |
|---|---|
| $p^6(p-1)^2(p+1)$ | $\frac{p^4(p-1)(p+1)(p^3-1)}{3}$ |

### 5.2.2 Irreducible Characters of Degree $p^4(p^3-1)(p^2-1)$

Now pick $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then the explicit formula for $\phi_A$ is

$$\phi_A \begin{pmatrix} 1+p^2* & p^2* & p^2* \\ p^2 a_{21} & 1+p^2* & p^2* \\ p^2* & p^2 a_{32} & 1+p^2 \end{pmatrix} = \lambda(p^2 a_{21} + p^2 a_{32}).$$

The stabilizer is

$$T = K_1 S = \left\{ \begin{pmatrix} a & b & c \\ p* & a+p* & b+p* \\ p* & p* & a+p* \end{pmatrix} \right\} \text{ where } S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \right\}.$$

$\phi'$ on $K_2(K_1 \cap S)$ can be defined as

$$\phi' \mid_{K_2} = \phi_A, \phi' \mid_{K_1 \cap S} = 1, \phi'(k_2 s) = \phi_A(k_2) \quad \forall k_2 \in K_2, s \in S.$$

$\phi'$ is clearly an extension of $\phi_A$. Choose

$$H = \left\{ \begin{pmatrix} 1+p* & p* & p* \\ p^2* & 1+p* & p* \\ p^2* & p^2* & 1+p* \end{pmatrix} \right\},$$

then $\phi''$ on $H$ can be defined as

$$\phi'' \begin{pmatrix} 1+p* & p* & p* \\ p^2 a_{21} & 1+p* & p* \\ p^2* & p^2 a_{32} & 1+p* \end{pmatrix} = \lambda(p^2 a_{21} + p^2 a_{32}).$$

$| T |= p^{20}(p-1)$ implies that

$$\deg(\chi) = p^3 \frac{| G |}{| T |} = p^4(p^3-1)(p^2-1).$$

The above argument works similarly if we replace $A$ by $A_\alpha = \alpha I + A$. There are $p$ non-conjugate such matrices. We can also get

$$\frac{| K_2(K_1 \cap S) |}{| K_2 |} \times \frac{| K_1 S |}{| K_1 |} = p^5(p-1)$$

total extensions. Therefore, the total number of irreducible characters of this degree can be summarized as follows:

| Degrees | Number of this degree |
|---|---|
| $p^4(p^3-1)(p^2-1)$ | $p^6(p-1)$ |

### 5.2.3 Irreducible Characters of Degree $p^6(p^3-1)$

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \epsilon & 0 \end{pmatrix}$ with $\epsilon$ to be a non-square unit in $\mathbb{Z}/p\mathbb{Z}$. The stabilizer is of $\phi_A$ in this case is

$$T = \left\{ \begin{pmatrix} a & p* & p* \\ p* & b & c \\ p* & \epsilon + p* & b + p* \end{pmatrix} \right\} = K_1 S, \text{ where } S = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c\epsilon & b \end{pmatrix} \right\}.$$

We can choose

$$H = H' K_2(K_1 \cap S) \text{ where } H' = \left\{ \begin{pmatrix} 1 & p* & p* \\ 0 & 1 + p* & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

$\phi''$ on $H$ can be defined as

$$\phi'' \mid_{K_2(K_1 \cap S)} = \phi', \phi'' \mid_{H'} = 1$$

and

$$\phi''(h's) = \phi''(h')\phi''(s) \text{ for } h' \in H' \text{ and } s \in K_2(K_1 \cap S).$$

Since $| T |= p^{18}(p-1)^2(p+1)$, we have

$$\deg(\chi) = p^3 \frac{| G |}{| T |} = p^6(p^3-1).$$

The general matrices we can use to replace $A$ are

$$A_{x,y} = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 1 \\ 0 & \epsilon & y \end{pmatrix} \text{ with } x, y \in \mathbb{Z}/p\mathbb{Z} \text{ and } \epsilon \text{ a non-square unit in } \mathbb{Z}/p\mathbb{Z},$$

the total number of such non-conjugate $A_{x,y}$ is $\frac{p^2(p-1)}{2}$. Also, the number of $\psi's$ is

$$\frac{\mid K_2(K_1 \cap S) \mid}{\mid K_2 \mid} \times \frac{\mid K_1 S \mid}{\mid K_1 \mid} = p^3(p-1)^2(p+1).$$

The following table summarizes this case.

| Degrees | Number of this degree |
|---|---|
| $p^6(p^3-1)$ | $\frac{p^5(p-1)^3(p+1)}{2}$ |

### 5.2.4 Irreducible Characters of Degree $p^5(p+1)(p^3-1)$

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\lambda_1 : (\mathbb{Z}/p^3\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ be injective, then an explicit formula of $\phi_A$ can be

$$\phi_A \begin{pmatrix} 1+pa_{11} & p* & p* \\ p^2* & 1+p* & p* \\ p^2* & p^2a_{32} & 1+p* \end{pmatrix} = \lambda_1(1+p^2a_{11})\lambda(p^2a_{32}).$$

This will give us the stabilizer

$$T = \left\{ \begin{pmatrix} a & p* & p* \\ p* & b & c \\ p* & p* & b+p* \end{pmatrix} \right\} = K_1 S, \text{ where } S = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & b \end{pmatrix} \right\}.$$

$\phi'$ on $K_2(K_1 \cap S)$ can be defined as

$$\phi' \mid_{K_2} = \phi_A, \phi' \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & b \end{pmatrix} = \lambda_1(a)$$

and

$$\phi'(k_2 s) = \phi'(k_2)\phi'(s) \text{ for } k_2 \in K_2 \text{ and } s \in K_1 \cap S.$$

We can choose

$$H = \left\{ \begin{pmatrix} 1+p* & p* & p* \\ p^2* & 1+p* & p* \\ p^2* & p^2* & 1+p* \end{pmatrix} \right\}$$

and have $\phi''$ on $H$ as

$$\phi''\begin{pmatrix} 1+pa_{11} & p* & p* \\ p^2* & 1+p* & p* \\ p^2* & p^2a_{32} & 1+p* \end{pmatrix} = \lambda_1(1+pa_{11})\lambda(p^2a_{32}).$$

$\mid T \mid= p^{19}(p-1)^2$ in this case, so we have

$$\deg(\chi) = p^3\frac{\mid G \mid}{\mid T \mid} = p^5(p+1)(p^3-1).$$

We can use more general matrices

$$A_{x,y} = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 1 \\ 0 & 0 & y \end{pmatrix}, x, y \in \mathbb{Z}/p\mathbb{Z}, x \neq 0$$

to replace $A$ and we can get the same degree by similar argument. The number of such non-conjugate $A_{x,y}$ is $p(p-1)$. Also, the total number of $\psi's$ in this case is

$$\frac{\mid K_2(K_1\cap S)\mid}{\mid K_2 \mid} \times \frac{\mid K_1S \mid}{\mid K_1 \mid} = p^4(p-1)^2.$$

Therefore, we will have

| Degrees | Number of this degree |
|---|---|
| $p^5(p+1)(p^3-1)$ | $p^5(p-1)^3$ |

Now we will work on the second construction process:

$$(2) \qquad \begin{array}{ccccccccc} K_2 & \longrightarrow & N & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi'_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ind}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array}.$$

Again, we fix an injective $\lambda : \mathbb{Z}/p^3\mathbb{Z} \to \mathbb{C}^\times$ and define $\phi_A$ as

$$\phi_A(I + p^2B) = \lambda(\mathrm{tr}(p^2AB)).$$

We choose a normal subgroup $N$ of $T$ such that $\frac{|N|}{|K_2|}$ is as big as possible while we can still extend $\phi_A$ to $\phi'_A$ of $N$. $T'$ is the stabilizer of $\phi'_A$ in $T$. From Clifford theory we know that

$$\psi \in \mathrm{Irr}(T) \text{ and } [\psi\mid_{K_2}, \phi_A] \neq 0,$$

which implies that

$$\chi \in \mathrm{Irr}(G).$$

Since

$$\chi = \mathrm{Ind}_T^G(\psi) = \mathrm{Ind}_T^G(\mathrm{Ind}_{T'}^T(\phi')) = \mathrm{Ind}_{T'}^G(\phi'),$$

we have

$$\deg(\chi) = \frac{\mid G \mid}{\mid T' \mid}.$$

Look at the

$$\begin{array}{ccccc} N & \longrightarrow & T' & \longrightarrow & T \\ \phi'_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ind}} & \psi \end{array}$$

piece of the construction process, we know that non-conjugate $\phi'_A$ on $N$ will give us different $\psi$ on $T$, and hence different $\chi \in \mathrm{Irr}(G)$. Therefore, in order to count how many irreducible characters we can get using the second construction process, we need to count how many non-conjugate $\phi'_A$ we have on $N$. One natural way to define $\phi'_A$ is to use the same formula as $\phi_A$, that is,

$$\phi'_A(I + pB) = \lambda(\mathrm{tr}(pAB)), \text{ for } I + pB \in N.$$

But we can easily get more extension by using $A + pC$ to extend $\phi_A$, which gives us

$$\phi'_{A+pC}(I + pB) = \lambda(\mathrm{tr}[(A + pC)pB]), \text{ for } I + pB \in N.$$

Notice that

$$\phi_{A+pC}(I + p^2B) = \lambda(\mathrm{tr}[p^2(A + pC)B]) = \lambda(\mathrm{tr}(p^2AB)) = \phi_A(I + p^2B),$$

so $\phi'_{A+pC}$ is indeed another extension of $\phi_A$. Using non-conjugate matrices $A + pC$ in $T$ to define $\phi'_{A+pC}$ will give us different stabilizers $T'$ with different indexes, therefore we can have irreducible characters with different degrees. To get the number in each case, we first need to find all types of non-conjugate matrices $A + pC$, then like the cases before, to count how many non-conjugate matrices $\alpha I + A + pC$ in $T$. We also need the number of extensions from $\phi'_A$ to $\phi'$, which is $\frac{|T'|}{|N|}$. The following 3 kinds of irreducible characters use this construction process.

### 5.2.5 Irreducible Characters of Degree $p^6(p+1)(p^2+p+1)$

Let $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $a, b \in \mathbb{Z}/p\mathbb{Z}, a \neq b, a \neq 0, b \neq 0$. Let

$$\lambda_1, \lambda_2 : (\mathbb{Z}/p^3\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

be such that

$$\lambda_1(1 + p^2x) = \lambda[a(1 + p^2x)], \lambda_2(1 + p^2x) = \lambda[b(1 + p^2x)],$$

then $\phi_A$ on $K_2$ can be defined as

$$\phi_A \begin{pmatrix} 1+p^2a_{11} & p^2* & p^2* \\ p^2* & 1+pa_{22}^2 & p^2* \\ p^2* & p^2* & 1+p^2* \end{pmatrix} = \lambda_1(1+p^2a_{11})\lambda_2(1+p^2a_{22}).$$

The stabilizer is

$$T = \left\{ \begin{pmatrix} x & p* & p* \\ p* & y & p* \\ p* & p* & z \end{pmatrix} \right\}.$$

Let

$$N = \left\{ \begin{pmatrix} 1+p* & p^2* & p^2* \\ p* & 1+p* & p^2* \\ p* & p* & 1+p* \end{pmatrix} \right\}$$

then $N \lhd T$. For this $N$, one explicit formula for $\phi'_A$ could be

$$\phi'_A \begin{pmatrix} 1+pa_{11} & p^2* & p^2* \\ p* & 1+pa_{22} & p^2* \\ p* & p* & 1+p* \end{pmatrix} = \lambda_1(1+pa_{11})\lambda_2(1+pa_{22}).$$

The stabilizer of $\phi'_A$ under $T$ is

$$\mathrm{Stab}_{\phi'_A}(T) = T' = \left\{ \begin{pmatrix} x & p^2* & p^2* \\ p* & y & p^2* \\ p* & p* & z \end{pmatrix} \right\}$$

and we can define $\phi'$ on $T'$ as

$$\phi' \begin{pmatrix} x & p^2* & p^2* \\ p* & y & p^2* \\ p* & p* & z \end{pmatrix} = \lambda_1(x)\lambda_2(y).$$

The degree of $\chi$ we get in this case is

$$\deg(\chi) = \frac{\mid G \mid}{\mid T' \mid} = p^6(p+1)(p^2+p+1).$$

As discussed before, we can use $A+pC$ to define an extension of $\phi_A$ this way:

$$\phi'_{A+pC}(I+pB) = \lambda(\mathrm{tr}[(A+pC)pB]), \text{ where } I+pB \in N.$$

It turns out that all non-conjugate matrices $\alpha I + A + pC$ are in the form of

$$A_{a,b,c} = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}, \text{ with } a,b,c \in \mathbb{Z}/p^2\mathbb{Z} \text{ and } p \nmid a-b, a-c, b-c$$

and we have $\frac{p^4(p-1)(p-2)}{6}$ non-conjugate ones. The number of extensions from $N$ to $T'$ is

$$\frac{\mid T' \mid}{\mid N \mid} = (p-1)^3.$$

Since $T'/N$ is abelian, we don't have new degrees of $\psi$, therefore no new degrees of $\chi$. To summarize, we have

| Degrees | Number of this degree |
|---|---|
| $p^6(p+1)(p^2+p+1)$ | $\frac{p^4(p-1)(p-2)(p-1)^3}{6}$ |

### 5.2.6 Irreducible Characters of Degree $p^4(p^2+p+1)$

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ pick some $\lambda' : (\mathbb{Z}/p^3\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ be injective such that $\lambda'(1+p^2x) = \lambda(p^2x)$. An explicit formula of $\phi_A$ ca be

$$\phi_A \begin{pmatrix} 1+p^2a_{11} & p^2* & p^2* \\ p^2* & 1+p^2* & p^2* \\ p^2* & p^2* & 1+p^2* \end{pmatrix} = \lambda'(1+p^2a_{11}).$$

The stabilizer $T = \left\{ \begin{pmatrix} a & p* & p* \\ p* & x & y \\ p* & z & w \end{pmatrix} \right\}$. We can pick

$$N = \left\{ \begin{pmatrix} 1+p* & p^2* & p^2* \\ p* & 1+p* & p* \\ p* & p* & 1+p* \end{pmatrix} \right\}$$

then $N$ is a normal subgroup of $T$. Now we want to use $A+pC$ to define an extension of $\phi_A$ on $N$. As discussed before, we only want to consider the non-conjugate ones under $T$ and we have the following 4 cases.
(i) $C = 0$. In this case, we can define $\phi'_A$ on $N$ by

$$\phi'_A \begin{pmatrix} 1+pa_{11} & p^2* & p^2* \\ p* & 1+p* & p* \\ p* & p* & 1+p* \end{pmatrix} = \lambda'(1+pa_{11}).$$

$$\mathrm{Stab}_T(\phi'_A) = T' = \left\{ \begin{pmatrix} a & p^2* & p^2* \\ p* & x & y \\ p* & z & w \end{pmatrix} \right\}.$$

Define $\phi'$ on $T'$ as

$$\phi' \begin{pmatrix} a & p^2* & p^2* \\ p* & x & y \\ p* & z & w \end{pmatrix} = \lambda'(a),$$

which is clearly an extension of $\phi'_A$. By inducing $\phi'$ of $T'$ to $G$, we have $\chi \in Irr(G)$ and

$$\deg(\chi) = \frac{|\, G \,|}{|\, T' \,|} = p^4(p^2+p+1).$$

Like in the even case,

$$T'/N \cong (\mathbb{Z}/p\mathbb{Z})^\times \times \mathrm{GL}(2, \mathbb{Z}/p\mathbb{Z})$$

and we already know the irreducible characters of $(\mathbb{Z}/p\mathbb{Z})^\times \times \mathrm{GL}(2, \mathbb{Z}/p\mathbb{Z})$ have degrees

$$1, p, p+1, (p-1)$$

with numbers

$$(p-1)^3, (p-1)^3, \frac{(p-1)^3(p-2)}{2}, \frac{p(p-1)^3}{2}$$

respectively. Also, non-conjugate matrices $\alpha I + kA + pC$ in this case are in the form of

$$\begin{pmatrix} x & & \\ & y & \\ & & y \end{pmatrix}, x, y \in \mathbb{Z}/p^2\mathbb{Z}, p \nmid x$$

and we have $p^3(p-1)$ non-conjugate ones. To summarize this case, we have

| Degrees | Number of this degree |
|---|---|
| $p^4(p^2+p+1)$ | $p^3(p-1)^4$ |
| $p^5(p^2+p+1)$ | $p^3(p-1)^4$ |
| $p^4(p^2+p+1)(p+1)$ | $\frac{p^3(p-1)^4(p-1)}{2}$ |
| $p^4(p^2+p+1)(p-1)$ | $\frac{p^4(p-1)^4}{2}$ |

(ii) Let $C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ then $A + pC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix}$ and the explicit formula for $\phi'_{A+pC}$ is

$$\phi'_{A+pC} \begin{pmatrix} 1+pa_{11} & p^2* & p^2* \\ p* & 1+p* & p* \\ p* & p*a_{32} & 1+p* \end{pmatrix} = \lambda'(1+pa_{11})\lambda(p^2a_{32}).$$

The stabilizer of $\phi'_{A+pC}$ under $T$ is

$$T' = \left\{ \begin{pmatrix} a & p^2* & p^2* \\ p* & x & y \\ p* & p* & x+p* \end{pmatrix} \right\}.$$

We can define $\phi'$ on $T'$ as

$$\phi'\begin{pmatrix} a & p^2* & p^2* \\ p* & x & y \\ p* & pz & x+p* \end{pmatrix} = \lambda'(a)\lambda(p^2zx^{-1}),$$

then $\phi'$ is an extension of $\phi'_{A+pC}$ that satisfies the conditions in our second construction process. Since $\mid T' \mid = p^{17}(p-1)^2$, the degree of $\chi \in \mathrm{Irr}(G)$ in this case is

$$\deg(\chi) = \frac{\mid G \mid}{\mid T' \mid} = p^4(p+1)(p^3-1).$$

Non-conjugate matrices $\alpha I + kA + pC$ are all in the form of

$$\begin{pmatrix} x & & \\ & y & \\ & p & y \end{pmatrix}, x, y \in \mathbb{Z}/p^2\mathbb{Z}, p \nmid x$$

and the number is $p^3(p-1)$. Notice that $T'/N$ is abelian in this case, we can only get one degree of $\phi'$ and therefore only one degree of $\chi \in \mathrm{Irr}(G)$. Also, the number of extensions from $\phi'_A$ to $\phi'$ is $\frac{|T'|}{|N|} = p(p-1)^2$, so we can get total $p^4(p-1)^3$ irreducible characters of $G$ in this case. To summarize, we have

| Degrees | Number of this degree |
|---|---|
| $p^4(p+1)(p^3-1)$ | $p^4(p-1)^3$ |

(iii) Now $C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\beta \in \mathbb{Z}/p\mathbb{Z}$, then $A + pC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p\beta & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and the explicit formula for $\phi'_{A+pC}$ is

$$\phi'_{A+pC}\begin{pmatrix} 1+pa_{11} & p^2* & p^2* \\ p* & 1+pa_{22}* & p* \\ p* & p* & 1+p* \end{pmatrix} = \lambda'(1+pa_{11})\lambda''(p^2a_{32})$$

where $\lambda'' : (\mathbb{Z}/p^3\mathbb{Z})^\times \to \mathbb{C}^\times$ satisfies $\lambda''(1+p^2x) = \lambda(p^2\beta x)$. The stabilizer of $\phi'_{A+pC}$ under $T$ is

$$T' = \left\{ \begin{pmatrix} x & p^2* & p^2* \\ p* & y & p* \\ p* & p* & z \end{pmatrix} \right\}.$$

We can define $\phi'$ on $T'$ as

$$\phi'\begin{pmatrix} x & p^2* & p^2* \\ p* & y & p* \\ p* & p* & z \end{pmatrix} = \lambda'(a)\lambda''(y),$$

then $\phi'$ is an extension of $\phi'_{A+pC}$ that satisfies the conditions in our second construction process. Since $\mid T' \mid= p^{16}(p-1)^3$, the degree of $\chi \in \mathrm{Irr}(G)$ in this case is

$$\deg(\chi) = \frac{\mid G \mid}{\mid T' \mid} = p^5(p+1)(p^2+p+1).$$

Now non-conjugate matrices $\alpha I + kA + pC$ are all in the form of

$$\begin{pmatrix} x & & \\ & y+p\beta & \\ & & y \end{pmatrix}, x, y \in \mathbb{Z}/p^2\mathbb{Z}, p \nmid x$$

and the number is $\frac{p^3(p-1)}{2}$. Since $T'/N$ is abelian, we can have only one degree of $\chi \in \mathrm{Irr}(G)$. The following table summarizes this case.

| Degrees | Number of this degree |
|---|---|
| $p^5(p+1)(p^2+p+1)$ | $\frac{p^3(p-1)^5}{2}$ |

(iv) The last case we let $C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & 1 & 0 \end{pmatrix}$ where $\epsilon$ is a non-square unit in $\mathbb{Z}/p\mathbb{Z}$, then $A + pC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & p\epsilon \\ 0 & 1 & 0 \end{pmatrix}$ and the explicit formula for $\phi'_{A+pC}$ is

$$\phi'_{A+pC}\begin{pmatrix} 1+pa_{11} & p^2* & p^2* \\ p* & 1+p* & pa_{23} \\ p* & pa_{32} & 1+p* \end{pmatrix} = \lambda'(1+pa_{11})\lambda(p^2a_{23}+p^2\epsilon a_{32}).$$

The stabilizer of $\phi'_{A+pC}$ under $T$ is

$$T' = \left\{ \begin{pmatrix} x & p^2* & p^2* \\ p* & y & w \\ p* & \epsilon w + p* & y + p* \end{pmatrix} \right\}.$$

Notice that we actually have a $2 \times 2$ block here which is one of the $2 \times 2$ cases before. Therefore, the existence of extension $\phi'$ on $T'$ is guaranteed. Since $\mid T' \mid= p^{16}(p-1)^2(p+1)$, the degree of $\chi \in \mathrm{Irr}(G)$ in this case is

$$\deg(\chi) = \frac{\mid G \mid}{\mid T' \mid} = p^5(p^3-1).$$

Look at $\alpha I + kA + pC$ and we have that non-conjugate ones are in the form

$$\begin{pmatrix} x & & \\ & y & p\epsilon \\ & py & \end{pmatrix}, x, y \in \mathbb{Z}/p^2\mathbb{Z}, p \nmid x \text{ and } \epsilon \text{ is a non-square unit in } \mathbb{Z}/p\mathbb{Z}\ .$$

The number is $\frac{p^3(p-1)}{2}$. Since $T'/N$ is abelian, we can have only one degree of $\chi \in \mathrm{Irr}(G)$. The following table summarizes this case.

| Degrees | Number of this degree |
|---|---|
| $p^5(p^3-1)$ | $\frac{p^3(p-1)^5}{2}$ |

### 5.2.7 Irreducible Characters of Degree $p^2(p^3-1)(p+1)$

Now let $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then

$$\phi_A \begin{pmatrix} 1+p^2* & p^2* & p^2* \\ p^2* & 1+p^2* & p^2* \\ p^2a_{31} & p^2* & 1+p^2* \end{pmatrix} = \lambda(p^2 a_{31}).$$

The stabilizer is $T = \left\{ \begin{pmatrix} a & w & y \\ p* & x & z \\ p* & p* & a+p* \end{pmatrix} \right\}$. We can choose a normal subgroup of $T$ as

$$N = \left\{ \begin{pmatrix} 1+p* & p* & p* \\ p* & 1+p* & p* \\ p^2* & p^2* & 1+p* \end{pmatrix} \right\}$$

and define $\phi_{A'}$ on $N$ by

$$\phi'_A \begin{pmatrix} 1+p* & p* & p* \\ p* & 1+p* & p* \\ p^2a_{31} & p^2* & 1+p* \end{pmatrix} = \lambda(p^2 a_{31}).$$

The stabilizer of $\phi'_A$ is

$$\mathrm{Stab}_T(\phi'_A) = T' = \left\{ \begin{pmatrix} a & w & y \\ p* & x & z \\ p^2* & p^2* & a+p* \end{pmatrix} \right\}.$$

Define $\phi'$ on $T'$ as

$$\phi' \begin{pmatrix} a & w & y \\ p* & x & z \\ p^2a_{31}* & p^2* & a+p* \end{pmatrix} = \lambda(p^2 a_{31} a^{-1}),$$

then $\phi'$ is clearly an extension of $\phi'_A$ and the degree we can get by inducing up $\phi'$ of $T'$ is

$$\deg(\chi) = \frac{|G|}{|T'|} = p^2(p^3-1)(p+1).$$

Since

$$T/K_1 \cong \left\{ \begin{pmatrix} a & w & y \\ 0 & x & z \\ 0 & 0 & a \end{pmatrix} \right\}$$

and we already know that all the irreducible characters of $\left\{ \begin{pmatrix} a & w & y \\ 0 & x & z \\ 0 & 0 & a \end{pmatrix} \right\}$ have degrees

$$1, p, p-1,$$

therefore we can have more irreducible characters of $G$ with degrees

$$p^2(p^3-1)(p+1)(p-1) \text{ and } p^3(p^3-1)(p+1).$$

As discussed before, we can still define $\phi'_{A+pC}$ using $A+pC$. However, every matrix of the following from

$$\begin{pmatrix} & & 1 \\ p & & \\ pa & pb & pc \end{pmatrix}, a, b, c \in \mathbb{Z}/p\mathbb{Z}$$

are not conjugate to each other for different $a, b, c$, which is quite complicated to deal with and we don't have every case.
We will see a new way of defining $\phi'_A$ on $N$ in a later section and see that we indeed have different irreducible character degrees of $G$.

## 5.3 Decomposition of $\mathrm{Ind}_B^G(1_B)$

Let $B$ be the Borel subgroup of $G = \mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})$, to determine irreducible constituents of $\mathrm{Ind}_B^G(1_B)$, we study $End_{\mathbb{C}G}(\mathrm{Ind}_B^G(1_B))$ which has basis $\phi_w$, where $w's$ are double coset representatives of $B$ such that $G = \mathop{\mathring{\bigcup}}_{w \in G} BwB$. In the field case, that is, when $G = \mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})$, $w$ corresponds to elements of $S_n$ and $\dim(End_{\mathbb{C}G}(\mathrm{Ind}_B^G(1_B))) = n!$ which does not depend on $p$. As we will see in the next section when $\ell > 1$, the number of double cosets depends on $n, p$ and $\ell$. In sections 5.3.2 and 5.3.3, we will see the complete decompositions of $\mathrm{Ind}_B^G(1_B)$ when $G = \mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$ and $G = \mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$.

### 5.3.1 Double Cosets of the Borel subgroup

Let $S$ be a subgroup of $G$, then we have

$$G = \mathop{\mathring{\bigcup}}_{w \in G} SgS$$

where $SgS$ are disjoint double cosets of $S$ and $g's$ are called double coset representatives.
In this section, we want to see how many double cosets of the Borel subgroup $B$ of groups $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$ and $\mathrm{GL}(3, \mathbb{Z}/p^\ell\mathbb{Z})$ there are. It is known that in the field case, that is $G = \mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})$, one choice of double coset representatives for the Borel subgroup of $\mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})$ are the permutation matrices, so the number of double cosets is $n!$, which does not depend on $p$ or $\ell$.
For $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$, one choice for the double coset representatives of the Borel subgroup are the following matrices:

$$\begin{pmatrix} 1 & \\ p^k & 1 \end{pmatrix}, 1 \leq k \leq \ell, \text{ and } \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

Therefore, the number of double cosets in this case is $\ell + 1$.
Next, let us look at an idea to find double coset representatives for the Borel subgroup $B$ of $G = \mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$. We want to write $G$ as disjoint unions of $BgB$, therefore for any matrix $A = \begin{pmatrix} a & k & c \\ d & e & f \\ g & h & i \end{pmatrix} \in G$, it must be in one double coset $BgB$ for some $w$; so we must have $g^{-1}bAb' \in B$ for some matrices $b, b' \in B$.
Notice that $B$ contains upper triangular elementary matrices, multiplying $A$ by those matrices on the left and right is equivalent to performing the corresponding elementary row and column operations on $A$. Thus, in order to make $g^{-1}bAb' \in B$, we can think of applying some elementary row and column operations on $A$ first, and then at certain stage, we will need to multiply the resulting matrix by $w^{-1}$ to get an upper triangular matrix. We discuss all the possible forms of matrix $A$ and each form can give us a double coset representative $g$, thus we will have all the double coset representatives.
For example, let $G = \mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$ and $A = \begin{pmatrix} a & k & c \\ d & e & f \\ g & h & i \end{pmatrix} \in G$ be such that $p|d, g, h$ and $g \neq 0$. To simplify notation, write $A = \begin{pmatrix} a & k & c \\ pd & e & f \\ pg & ph & i \end{pmatrix}$ with $g$ being a unit in $\mathbb{Z}/p^2\mathbb{Z}$. Let $b = \begin{pmatrix} 1 & & \\ & 1 & -dg^{-1} \\ & & g^{-1} \end{pmatrix}$ and $b' = \begin{pmatrix} 1 & -hg^{-1} & \\ & 1 & \\ & & 1 \end{pmatrix}$, then $bAb' = \begin{pmatrix} * & * & * \\ 0 & * & * \\ p & 0 & * \end{pmatrix}$. To reduce the above matrix into $B$, we will pick

$g = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ p & 0 & 1 \end{pmatrix}$ so that $g^{-1}bAb' \in B$ and we have one double coset representative $\begin{pmatrix} 1 & & \\ & 1 & \\ p & & 1 \end{pmatrix}$.

By similar arguments to run all the possible forms of matrix $A \in G$, we have the following double coset representatives for $B < \mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$:

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}, \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix},$$

$$\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ p & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & p & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ p & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ p & 1 & \\ & p & 1 \end{pmatrix},$$

$$\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & p & 1 \end{pmatrix}, \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ p & & 1 \end{pmatrix},$$

$$\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ p & p & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}\begin{pmatrix} 1 & & \\ p & 1 & \\ p & & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}\begin{pmatrix} 1 & & \\ p & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ p & & 1 \end{pmatrix},$$

$$\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}\begin{pmatrix} 1 & & \\ p & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & p & 1 \end{pmatrix}.$$

The number of double cosets in this case is 18.

We can also find a choice of double coset representatives for the Borel subgroup of $\mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$ as follows:

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix},$$

$$\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ p & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ p^2 & & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & & \\ p & 1 & \\ p^2 & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ p^2 & p & 1 \end{pmatrix},$$

$\begin{pmatrix} 1 & & \\ p & 1 & \\ p^2g & p & 1 \end{pmatrix}$ with $g$ being units in $\mathbb{Z}/p^3\mathbb{Z}$, the number of distinct $p^2g$ is $p-1$,

$$\begin{pmatrix} 1 & & \\ p^i & 1 & \\ & p^j & 1 \end{pmatrix}, \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ p^i & p^j & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \begin{pmatrix} 1 & & \\ p^i & 1 & \\ p^j & & 1 \end{pmatrix}, 1 \leq i, j \leq 3, \text{ not both } 3$$

$$\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \begin{pmatrix} 1 & & \\ p^k & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & p^k & 1 \end{pmatrix}, 1 \leq k \leq 2.$$

The number of double cosets in this case is $p + 37$.
We will use some of the double cosets in the following two sections. From Mackey's Theorem, we have

$$[\mathrm{Ind}_B^G(1_B), \mathrm{Ind}_B^G(1_B)] = \text{ number of double cosets of } B,$$

after we find all the irreducible constituents of $\mathrm{Ind}_B^G(1_B)$ in the next two sections, we will verify that the number of double cosets we found for $\mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$ and $\mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$ are indeed correct.

Let $B = \left\{ \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} \right\} \subset G$ be the Borel subgroup of $G$ and let $1_B$ be the identity character on $B$. We know $\mathrm{Ind}_B^G(1_B)$ is not irreducible. We want to see the decomposition of $\mathrm{Ind}_B^G(1_B)$ when $G = \mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$ and $G = \mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$.

### 5.3.2 $G = \mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$

In section 5.1, we have seen all the irreducible characters of $G$. Now we want to see how to decompose $\mathrm{Ind}_B^G(1_B)$. It's easy to see that $\mid B \mid = p^9(p-1)^3$ so

$$\deg(\mathrm{Ind}_B^G(1_B)) = \frac{\mid G \mid}{\mid B \mid} = p^3(p+1)(p^2+p+1).$$

We will give the construction of each irreducible constituent, show that it is indeed an irreducible constituent and also give the multiplicity.

(1) Let $\chi_1$ be the irreducible constructed in 5.1.5. That is, $\chi_1 = \mathrm{Ind}_{T_1}^G(\psi_1)$ where

$$T_1 = \left\{ \begin{pmatrix} a & b & c \\ p* & a+p* & b+p* \\ p* & p* & a+p* \end{pmatrix} \right\} = K_1 S, S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \right\}.$$

Recall that $\psi_1$ on $T_1$ is defined as

$$\psi_1 \begin{pmatrix} 1+p* & p* & p* \\ pa_{21} & 1+p* & p* \\ p* & pa_{32} & 1+p \end{pmatrix} = \lambda(pa_{21} + pa_{32}),$$

$$\psi_1 \mid_S = 1, \psi_1(k_1 s) = \psi_1(k_1) \forall k_1 \in K_1, s \in S.$$

By Mackey's Theorem, we have

$$\mathrm{Ind}_B^G(1_B) \mid_{T_1} = \sum_{G = \cup T_1 g B} \mathrm{Ind}_{gBg^{-1} \cap T}^B(1).$$

Notice that $\psi_1 \mid_{T \cap B} = 1$, we have

$$\begin{aligned} [\mathrm{Ind}_B^G(1_B), \mathrm{Ind}_{T_1}^G(\psi_1)] &= [\psi_1, \mathrm{Ind}_B^G(1_B) \mid_T] \\ &= \sum_{G = \cup T_1 g B} [\psi_1, \mathrm{Ind}_{gBg^{-1} \cap T}^B(1)] \\ &= \sum_{G = \cup T_1 g B} [\psi_1 \mid_{gBg^{-1} \cap T_1}, 1 \mid_{gBg^{-1} \cap T_1}] \geq 1. \end{aligned}$$

Therefore, $\chi_1$ is an irreducible constituent of $\mathrm{Ind}_B^G(1_B)$.
(2) Let $\chi_2 = \mathrm{Ind}_{T_2}^G(\psi_2)$ denote the irreducible character constructed in 5.1.4 where

$$T_2 = \left\{ \begin{pmatrix} a & w & y \\ p* & x & z \\ p* & p* & a+p* \end{pmatrix} \right\}$$

and

$$\psi_2 \begin{pmatrix} a & w & y \\ p* & x & z \\ pa_{31}* & p* & a+p* \end{pmatrix} = \lambda(pa_{31} a^{-1}).$$

Let $g_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, then

$$B_1 = g_1 B g_1^{-1} = \left\{ \begin{pmatrix} b & 0 & f \\ d & a & e \\ 0 & 0 & c \end{pmatrix} \right\}, \quad B_2 = g_2 B g_2^{-1} = \left\{ \begin{pmatrix} a & c & b \\ 0 & f & 0 \\ 0 & e & d \end{pmatrix} \right\}$$

are two different conjugates of $B$. From the definition of $\psi_2$, it is clear that

$$\psi_2 \mid_{B_1 \cap T_2} = \psi_2 \mid_{B_2 \cap T_2} = \psi_2 \mid_{B \cap T_2} = 1.$$

Therefore,

$$[\chi_2, \mathrm{Ind}_B^G(1_B)] = \sum_{G = \cup T_2 g B} [\psi_2 \mid_{gBg^{-1} \cap T_2}, 1 \mid_{gBg^{-1} \cap T_2}] \geq 3,$$

which shows that $\chi_2$ is an irreducible constituent of $\mathrm{Ind}_B^G(1_B)$ with multiplicity at least 3.

(3) Using characters and groups from case (2), we want to construct an irreducible character $\chi_3 \in \mathrm{Irr}(G)$ such that $\chi_3 = \mathrm{Ind}_{T_2}^G(\psi_2 \beta_3)$ where $\beta_3 \in \mathrm{Irr}(T_2/K_1)$. Since

$$T_2/K_1 \cong P = \left\{ \begin{pmatrix} a & w & y \\ 0 & x & z \\ 0 & 0 & a \end{pmatrix} \right\} \subset \mathrm{GL}(3, \mathbb{Z}/p\mathbb{Z}),$$

we work on group $P$ to construct $\beta_3 \in \mathrm{Irr}(P)$. Let $N = \left\{ \begin{pmatrix} 1 & w & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset$ $\mathrm{GL}(3, \mathbb{Z}/p\mathbb{Z})$, then $N \lhd P$. Define $\alpha_3$ on $N$ to be

$$\alpha_3 \begin{pmatrix} 1 & w & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \mu(w) \text{ where } \mu : (\mathbb{Z}/p\mathbb{Z})^+ \longrightarrow \mathbb{C}^\times.$$

$\mathrm{Stab}_P(\alpha_3) = M = \left\{ \begin{pmatrix} a & w & y \\ 0 & a & z \\ 0 & 0 & a \end{pmatrix} \right\}$ and we can have an extension of $\alpha_3$ by defining $\alpha_3'$ on $M$ by

$$\alpha_3' \begin{pmatrix} a & w & y \\ 0 & a & z \\ 0 & 0 & a \end{pmatrix} = \mu(wa^{-1}).$$

Let $\beta_3 = \mathrm{Ind}_M^P(\alpha_3')$, then $\beta_3 \in \mathrm{Irr}(P)$. Notice that $M \lhd P$ and pick

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}, t \in (\mathbb{Z}/p\mathbb{Z})^\times$$

to be coset representatives of $M$ in $P$, we have

$$\beta_3 \begin{pmatrix} a & w & y \\ 0 & a & z \\ 0 & 0 & a \end{pmatrix} = \sum_{t \in (\mathbb{Z}/p\mathbb{Z})^\times} \mu(wt^{-1}a^{-1}), \beta_3 \begin{pmatrix} a & w & y \\ 0 & x & z \\ 0 & 0 & a \end{pmatrix} = 0 \text{ if } x \neq a.$$

Notice that

$$\psi_2\beta_3\begin{pmatrix} a & w & y \\ p* & x & z \\ pa_{31} & p* & a+p* \end{pmatrix} = \lambda(pa_{31}a^{-1})\beta_3\begin{pmatrix} \bar{a} & \bar{w} & \bar{y} \\ 0 & \bar{a} & \bar{z} \\ 0 & 0 & \bar{a} \end{pmatrix},$$

we have

$$\psi_2\beta_3\mid_{T_2\cap B_1} = \begin{cases} p, & \text{if } x = a, \\ 0, & \text{if } x \neq a. \end{cases}$$

Therefore,

$$[\psi_2\beta_3\mid_{B_1\cap T_2}, 1] = 1,$$

showing that $\chi_3$ is an irreducible constituent of $\mathrm{Ind}_B^G(1_B)$.
(4) Here we want to construct $\chi_4 = \mathrm{Ind}_{T_2}^G(\psi_2\beta_4)$ where $\beta_4 \in \mathrm{Irr}(P)$ and is constructed similarly to $\beta_3$ above. Just define $\alpha_4$ on $M$ as

$$\alpha_4\begin{pmatrix} 1 & w & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \mu(z) \text{ where } \mu : (\mathbb{Z}/p\mathbb{Z})^+ \longrightarrow \mathbb{C}^\times,$$

we can have a corresponding $\beta_4$ and we can show that

$$[\psi_2\beta_4\mid_{B_2\cap T_2}, 1] = 1,$$

hence $\chi_4$ is also an irreducible constituent of $\mathrm{Ind}_B^G(1_B)$.
Let $G' = \mathrm{GL}(3, \mathbb{Z}/p\mathbb{Z})$ and $B'$ be the corresponding Borel subgroup of $G'$, then

$$\deg(\mathrm{Ind}_{B'}^{G'}(1_{B'})) = (p+1)(p^2+p+1).$$

Since

$$\deg(\mathrm{Ind}_B^G(1_B)) = \deg(\mathrm{Ind}_{B'}^{G'}(1_{B'})) + \deg(\chi_1) + 3\deg(\chi_2) + \deg(\chi_3) + \deg(\chi_4),$$

we have the complete decomposition

$$\mathrm{Ind}_B^G(1_B) = \mathrm{Ind}_{B'}^{G'}(1_{B'}) + \chi_1 + 3\chi_2 + \chi_3 + \chi_4.$$

We also have

$$[\mathrm{Ind}_B^G(1_B), \mathrm{Ind}_B^G(1_B)] = 6 + 1^2 + 3^2 + 1^2 + 1^2 = 18,$$

which is equal to the number of double cosets for the Borel subgroup $B$ of $\mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$.

### 5.3.3 $G = \mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$

Although we did not have all the irreducible characters of $G$ in this case, we can still completely decompose $\mathrm{Ind}_B^G(1_B)$. The following gives the construction of each constituent.
(1) Let $\chi_1 = \mathrm{Ind}_{T_1}^G(\phi_1)$ be the irreducible character constructed in 5.2.7, where where

$$T_1 = \left\{ \begin{pmatrix} a & w & y \\ p* & x & z \\ p^2* & p^2* & a+p* \end{pmatrix} \right\}$$

and

$$\phi_1 \begin{pmatrix} a & w & y \\ p* & x & z \\ p^2 a_{31}* & p^2* & a+p* \end{pmatrix} = \lambda(p^2 a_{31} a^{-1}).$$

Let $g_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $g_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & p & 1 \end{pmatrix}$, then $T_1 g_i B$ are three different double cosets and

$$B_1 = g_1 B g_1^{-1} = \left\{ \begin{pmatrix} b & 0 & f \\ d & a & e \\ 0 & 0 & c \end{pmatrix} \right\}, \quad B_2 = g_2 B g_2^{-1} = \left\{ \begin{pmatrix} a & c & b \\ 0 & f & 0 \\ 0 & e & d \end{pmatrix} \right\}$$

$$\text{and } B_3 = g_3 B g_3^{-1} = \left\{ \begin{pmatrix} a & d-pe & e \\ 0 & b-pf & f \\ 0 & p(b-c)-p^f & c+pf \end{pmatrix} \right\}.$$

From the definition of $\phi_1$, it is clear that

$$\phi_1 \mid_{B_1 \cap T_1} = \psi_1 \mid_{B_2 \cap T_1} = \phi_1 \mid_{B_3 \cap T_1} = \phi_2 \mid_{B \cap T_1} = 1.$$

Therefore,

$$[\chi_1, \mathrm{Ind}_B^G(1_B)] = \sum_{G = \cup T_1 g B} [\phi_1 \mid_{gBg^{-1} \cap T_1}, 1 \mid_{gBg^{-1} \cap T_1}] \geq 4,$$

which shows that $\chi_1$ is an irreducible constituent of $\mathrm{Ind}_B^G(1_B)$ with multiplicity at least 4.

(2) Recall that in section 5.2.7, $N = \left\{ \begin{pmatrix} 1+p* & p* & p* \\ p* & 1+p* & p* \\ p^2* & p^2* & 1+p* \end{pmatrix} \right\}$ and we have $\mathrm{Ind}_{T_1}^G(\phi_1 \beta) \in \mathrm{Irr}(G)$ for any $\beta \in \mathrm{Irr}(T_1/N)$ where $T_1/N \cong P =$

$\left\{\begin{pmatrix} a & w & y \\ 0 & x & z \\ 0 & 0 & a \end{pmatrix}\right\}$. Let $\phi_{1i} = \phi_1\zeta_i$ where

$$\zeta_i \begin{pmatrix} a & w & y \\ 0 & x & z \\ 0 & 0 & a \end{pmatrix} = \mu_i(ax^{-1}), \mu_i : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times, \mu_i \neq 1.$$

$$\phi_{1i} \begin{pmatrix} a & w & y \\ p* & x & z \\ p^2a_{31}* & p^2* & a+p* \end{pmatrix} = \lambda(p^2 a_{31} a^{-1})\mu_i(\bar{a}\bar{x}^{-1}).$$

Notice that

$$B_3 \cap T_1 \subset \left\{\begin{pmatrix} a & * & * \\ 0 & a+p* & * \\ 0 & p^2* & a+p* \end{pmatrix}\right\},$$

so $\phi_{1i} \mid_{B_3\cap T_1} = 1$. Denote $\chi_{1i} = \mathrm{Ind}_{T_1}^G(\phi_{1i})$, then $[\chi_{1i}, \mathrm{Ind}_B^G(1_B)] \geq 1$, showing that each $\chi_{1i}$ above is an irreducible constituent of $\mathrm{Ind}_B^G(1_B)$. Since there are $p-2$ such $\mu_i$, we have $p-2$ $\zeta_i$ and $\chi_{1i}$.

(3) Let$\beta_3$ and $\beta_4$ be the two irreducible characters of $P$ constructed in 5.3.1. Let $\chi_2 = \mathrm{Ind}_{T_1}^G(\phi_1\beta_3), \chi_3 = \mathrm{Ind}_{T_1}^G(\phi_1\beta_4)$, then $\chi_2, \chi_3 \in \mathrm{Irr}(G)$ by Clifford Theory and $\deg(\chi_2) = \deg(\chi_3) = p^2(p^2-1)(p^3-1)$. By the same argument as in 5.3.1. (3), we can show that

$$[\phi_1\beta_3 \mid_{B_1\cap T_1}, 1] = 1, [\phi_1\beta_4 \mid_{B_2\cap T_1}, 1] = 1,$$

where $B_1$ and $B_2$ are in (1). Therefore, we know that $\chi_2$ and $\chi_3$ are also irreducible constituent of $\mathrm{Ind}_B^G(1_B)$.

(4) In 5.2.7 when we use the construction process

$$\begin{array}{ccccccccc} K_2 & \longrightarrow & N & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi'_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ind}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array},$$

we mentioned that if we define $\phi'_A$ on $N$ differently, we may have a different stabilizer $T'$ and eventually a different $\chi \in \mathrm{Irr}(G)$ with a new degree. We will see a new construction here. Define $\phi'_2$ on $N$ as

$$\phi'_2 \begin{pmatrix} 1+p* & pa_{12} & p* \\ p* & 1+p* & p* \\ p^2a_{31} & p^2* & 1+p* \end{pmatrix} = \lambda(p^2a_{31} + p^2a_{12}),$$

then $T_2 = \mathrm{Stab}_T(\phi'_2) = \left\{\begin{pmatrix} a & pb & c \\ p* & a+p* & f \\ p^2* & p^2* & a+p* \end{pmatrix}\right\}$. Define $\phi_2$ on $T_2$ by

$$\phi_2 \begin{pmatrix} a & pb & c \\ p* & a+p* & f \\ p^2d & p^2* & a+p* \end{pmatrix} = \lambda(p^2(b+d)a^{-1}),$$

then $\phi_2$ is an extension of $\phi_2'$ and we have $\chi_4 = \mathrm{Ind}_{T_2}^G(\phi_2) \in \mathrm{Irr}(G)$ with

$$\deg(\chi_4) = p^3(p^2-1)(p^3-1).$$

Moreover, we can see

$$\phi_2 \mid_{T_2 \cap B_2} = 1,$$

therefore $\chi_4$ is an irreducible constituent of $\mathrm{Ind}_B^G(1_B)$.
Similarly, we can define $\phi_3'$ on $N$ by

$$\phi_3' \begin{pmatrix} 1+p* & p* & p* \\ p* & 1+p* & pa_{23} \\ p^2a_{31} & p^2* & 1+p* \end{pmatrix} = \lambda(p^2a_{31} + p^2a_{23})$$

and get the stabilizer $T_3$ and define $\phi_3$ on $T_3$. We have $\chi_5 = \mathrm{Ind}_{T_3}^G(\phi_3) \in \mathrm{Irr}(G)$ with $\deg(\chi_5) = p^3(p^2-1)(p^3-1)$. We can also see that $\phi_3 \mid_{B_3 \cap T_3} = 1$, implying $\chi_5$ is an irreducible constituent of $\mathrm{Ind}_B^G(1_B)$ as well.
(5) Let $\chi_6 = \mathrm{Ind}_{T_6}^G(\psi_6)$ be the irreducible constructed in 5.2.2 where $T_6 = K_1S$ with $S = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$. We used the following construction process

$$\begin{array}{ccccccccccc} K_2 & \longrightarrow & K_2(K_1 \cap S) & \longrightarrow & H & \longrightarrow & K_1 & \longrightarrow & K_1S & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ext}} & \phi'' & \xrightarrow{\text{ind}} & \theta & \xrightarrow{\text{ext}} & \psi_6 & \xrightarrow{\text{ind}} & \chi \end{array}$$

and chose $H = \left\{ \begin{pmatrix} 1+p* & p* & p* \\ p^2* & 1+p* & p* \\ p^2* & p^2* & 1+p* \end{pmatrix} \right\}$. We already had

$$\phi' \mid_{K_2} = \phi_A, \phi' \mid_{K_1 \cap S} = 1 \text{ and } \theta \mid_{K_2(K_1 \cap S)} = p^3\phi', \theta \mid_{K_1 - K_2(K_1 \cap S)} = 0.$$

We choose $\psi_6$ be the extension such that $\psi_6 \mid_S = p^3$. We can get

$$\psi_6 \mid_{K_1S \cap B - K_2S \cap B} = 0, \psi_6 \mid_{K_2 \cap S} = p^3.$$

This will give us

$$[\psi_6 \mid_{K_1S \cap B}, 1] = 1,$$

showing that $\chi_6$ is an irreducible constituent of $\mathrm{Ind}_B^G(1_B)$.
Let $G'' = \mathrm{GL}(3, \mathbb{Z}/p^2\mathbb{Z})$ and $B''$ be the Borel subgroup of $G''$, we have

$$\deg(\mathrm{Ind}_B^G(1_B)) = \deg(\mathrm{Ind}_{B''}^{G''}(1_{B''})) + 4\deg(\chi_1) + \sum_{i=1}^{p-2} \deg(\chi_{1i}) + \sum_{j=2}^{6} \deg(\chi_j).$$

Therefore, the complete decomposition in this case is

$$\mathrm{Ind}_B^G(1_B) = \mathrm{Ind}_{B''}^{G''}(1_{B''}) + 4\chi_1 + \sum_{i=1}^{p-2} \chi_{1i} + \sum_{j=2}^{6} \chi_j.$$

We can see now

$$[\mathrm{Ind}_B^G(1_B), \mathrm{Ind}_B^G(1_B)] = 18 + 4^2 + (p-2) + 5 = p + 37,$$

which is the number of double cosets for the Borel subgroup of $\mathrm{GL}(3, \mathbb{Z}/p^3\mathbb{Z})$.

# Chapter 6

# Parabolic Induction of $\mathrm{GL}(n, \mathbb{Z}/p^\ell\mathbb{Z})$

In Chapter 4, we had Parabolic Induction for the $2 \times 2$ case, now we want to generalize that result.

Let $G = \mathrm{GL}(n, \mathbb{Z}/p^\ell\mathbb{Z})$ and $B = \left\{ \begin{pmatrix} a_1 & * & * & \cdots & * \\ 0 & a_2 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & * \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix} \right\} \subset G$. Let

$$\lambda_1, \lambda_2, ..., \lambda_{n-1} : (\mathbb{Z}/p^\ell\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

be homomorphisms. Define

$$\phi : B \longrightarrow \mathbb{C}^\times, \phi \begin{pmatrix} a_1 & * & * & \cdots & * \\ 0 & a_2 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & * \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix} = \lambda_1(a_1)\lambda_2(a_2)\cdots\lambda_{n-1}(a_{n-1}).$$

In the field case when $\ell = 1$, it is known that $\mathrm{Ind}_B^G(\phi) \in \mathrm{Irr}(G)$ if and only if $\lambda_i \neq \lambda_j$ for $i \neq j$. In our case, we want to prove that $\chi = \mathrm{Ind}_B^G(\phi) \in \mathrm{Irr}(G)$ when $\left\{1 + p^{[\ell/2]}x\right\} \nsubseteq \ker \lambda_i\lambda_j^{-1}, i \neq j, 1 \leq i, j \leq n-1$. One implicit condition here is that $\ell$ is large enough so that we do have those $\lambda's$. More generally, if all the $a_i$ are block matrices, we can still have a similar result.

## 6.1 $G = \mathrm{GL}(n, \mathbb{Z}/p^{2m}\mathbb{Z})$

We deal with the even case first. Denote $K_m = \{I + p^m C\}$. Let $A = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$ with $a_i$ and $a_i - a_j \in (\mathbb{Z}/p^m\mathbb{Z})^\times$ for $i \neq j$. Let $\lambda : (\mathbb{Z}/p^{2m}\mathbb{Z})^+ \longrightarrow \mathbb{C}^\times$ be a homomorphism such that $\lambda \mid_{\{p^m *\}}$ is injective. Define

$$\phi_A : K_m \longrightarrow \mathbb{C}^\times, \phi_A(I + p^m C) = \lambda[\mathrm{tr}(p^m AC)],$$

then $\mathrm{Stab}_G(\phi_A) = T = \left\{ \begin{pmatrix} a_1 & p^m* & p^m* & \cdots & p^m* \\ p^m* & a_2 & p^m* & \cdots & p^m* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p^m* & p^m* & \cdots & a_{n-1} & p^m* \\ p^m* & p^m* & p^m* & \cdots & a_n \end{pmatrix} \right\}$. Notice that $\{1 + p^m x\}^\times \cong \{p^m x\}^+$, we can find $\lambda_1, \lambda_2, ..., \lambda_{n-1} : (\mathbb{Z}/p^{2m}\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ such that $\lambda_i(1 + p^m x) = \lambda(a_i p^m x)$. Define

$$\psi : T \longrightarrow \mathbb{C}^\times, \psi \begin{pmatrix} a_1 & p^m* & p^m* & \cdots & p^m* \\ p^m* & a_2 & p^m* & \cdots & p^m* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p^m* & p^m* & \cdots & a_{n-1} & p^m* \\ p^m* & p^m* & p^m* & \cdots & a_n \end{pmatrix} = \lambda_1(a_1) \cdots \lambda_{n-1}(a_{n-1}),$$

then $\psi$ is an extension of $\phi_A$, that is, $\psi \mid_{K_m} = \phi_A$. By Clifford Theory, we have $\psi^G \in \mathrm{Irr}(G)$.

Next we want to show that $\psi^G = \phi^G$. Since $\mid B \mid = \mid T \mid$ and we already know $\psi^G \in \mathrm{Irr}(G)$, it suffices to show that $[\psi^G, \phi^G] = [\psi, \phi^G \mid_T] = 1$. By Mackey's Theorem, we have

$$\phi^G \mid_T = \bigoplus_{G = \cup TgB} \mathrm{Ind}^B_{gBg^{-1} \cap T}(\phi_g), \text{where } \phi_g(gXg^{-1}) = \phi(X), X \in B.$$

Therefore

$$[\psi, \phi^G \mid_T] = \sum_{G = \cup TgB} [\psi, \mathrm{Ind}^B_{gBg^{-1} \cap T}(\phi_g)] = \sum_{G = \cup TgB} [\psi \mid_{gBg^{-1} \cap T}, \phi_g \mid_{gBg^{-1} \cap T}].$$

It is clear that $\psi \mid_{B \cap T} = \phi \mid_{B \cap T}$, thus the above sum must be 1, showing that $\psi^G = \phi^G \in \mathrm{Irr}(G)$.

## 6.2 $G = \mathrm{GL}(n, \mathbb{Z}/p^{2m+1}\mathbb{Z})$

Now let us work on the odd case. Similar to the even case, we want to construct $\psi$ on $T'$ such that $\psi^G \in \mathrm{Irr}(G)$ and $\mid T' \mid = \mid B \mid$ and then we will show $\psi^G = \phi^G$. We will use the following construction process:

$$\begin{array}{ccccccccc} K_{m+1} & \longrightarrow & N & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi'_A & \xrightarrow{\text{ext}} & \psi & \xrightarrow{\text{ind}} & \psi' & \xrightarrow{\text{ind}} & \chi \end{array}.$$

Here $K_{m+1} = \{I + p^{m+1}C\}$, $A = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$ with $a_i$ and $a_i - a_j \in (\mathbb{Z}/p^m\mathbb{Z})^\times$. Pick $\lambda : \mathbb{Z}/p^{2m+1}\mathbb{Z} \longrightarrow \mathbb{C}^\times$ to be a homomorphism such that $\lambda \mid_{\{p^{m+1}*\}}$ is injective. We can find $\lambda_1, \lambda_2, ..., \lambda_{n-1} : (\mathbb{Z}/p^{2m+1}\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ such that $\lambda_i(1 + p^{m+1}x) = \lambda(a_i p^{m+1}x)$. We can define $\phi_A$ on $K_{m+1}$ using the same formula as in the even case, that is,

$$\phi_A(I + p^{m+1}C) = \lambda(\mathrm{tr}(p^{m+1}CA)).$$

The stabilizer is

$$T = \left\{ \begin{pmatrix} a_1 & p^m* & p^m* & \cdots & p^m* \\ p^m* & a_2 & p^m* & \cdots & p^m* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p^m* & p^m* & \cdots & a_{n-1} & p^m* \\ p^m* & p^m* & p^m* & \cdots & a_n \end{pmatrix} \right\}.$$

Let $N = \left\{ \begin{pmatrix} 1+p^m* & p^{m+1}* & p^{m+1}* & \cdots & p^{m+1}* \\ p^m* & 1+p^m* & p^{m+1}* & \cdots & p^{m+1}* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p^m* & p^m* & \cdots & 1+p^m & p^{m+1}* \\ p^m* & p^m* & p^m* & \cdots & 1+p^m* \end{pmatrix} \right\}$, then $N \lhd T$ and we can define $\phi'_A$ on $N$ by

$$\phi'_A \begin{pmatrix} 1+p^m a_1 & p^{m+1}* & p^{m+1}* & \cdots & p^{m+1}* \\ p^m* & 1+p^m a_2 & p^{m+1}* & \cdots & p^{m+1}* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p^m* & p^m* & \cdots & 1+p^m a_{n-1} & p^{m+1}* \\ p^m* & p^m* & p^m* & \cdots & 1+p^m a_n \end{pmatrix}$$

$$= \lambda_1(1 + p^m a_1) \cdots \lambda_{n-1}(1 + p^m a_{n-1})$$

and we can get

$$\mathrm{Stab}_{\phi'_A}(T) = T' = \left\{ \begin{pmatrix} a_1 & p^{m+1}* & p^{m+1}* & \cdots & p^{m+1}* \\ p^m* & a_2 & p^m* & \cdots & p^{m+1}* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p^m* & p^m* & \cdots & a_{n-1} & p^{m+1}* \\ p^m* & p^m* & p^m* & \cdots & a_n \end{pmatrix} \right\}.$$

By the same argument as in the even case, we can define $\psi$ on $T'$, an extension of $\phi'_A$ with

$$\psi \begin{pmatrix} a_1 & p^{m+1}* & p^{m+1}* & \cdots & p^{m+1}* \\ p^m* & a_2 & p^m* & \cdots & p^{m+1}* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p^m* & p^m* & \cdots & a_{n-1} & p^{m+1}* \\ p^m* & p^m* & p^m* & \cdots & a_n \end{pmatrix} = \lambda_1(a_1)\lambda_2(a_2)\cdots\lambda_{n-1}(a_{n-1}),$$

then $\psi^G \in \mathrm{Irr}(G)$. Notice that $\mid T' \mid = \mid B \mid$, to show that $\psi^G = \phi^G$ is exactly the same as in the even case——just replace $T$ by $T'$.

## 6.3 More general result

Now let $B = \left\{ \begin{pmatrix} A_1 & * & * & \cdots & * \\ 0 & A_2 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{k-1} & * \\ 0 & 0 & 0 & \cdots & A_k \end{pmatrix} \right\} \subset G$ where each $A_i$ is a $n_i \times n_i$ matrix. Let $\lambda_1, \lambda_2, ..., \lambda_{k-1} : (\mathbb{Z}/p^\ell\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ be homomorphisms such that $\left\{1 + p^{[\ell/2]}x\right\} \nsubseteq \ker \lambda_i\lambda_j^{-1}, i \neq j, 1 \leq i, j \leq k-1$. Let

$$\phi_i \in \mathrm{Irr}(\mathrm{GL}(n_i, \mathbb{Z}/p^\ell\mathbb{Z})), 1 \leq i \leq k \text{ be inflated from } \mathrm{GL}(n_i, \mathbb{Z}/p^{\ell-[\frac{\ell}{2}]}\mathbb{Z}).$$

Define $\Phi : B \longrightarrow \mathbb{C}^\times$,

$$\Phi \begin{pmatrix} A_1 & * & * & \cdots & * \\ 0 & A_2 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{k-1} & * \\ 0 & 0 & 0 & \cdots & A_k \end{pmatrix} = \lambda_1[\det(A_1)]\lambda_2[\det(A_2)]\cdots$$

$$\lambda_{k-1}[\det(A_{k-1})]\phi_1(A_1)\phi_2(A_2)\cdots\phi_k(A_k).$$

$$\mathrm{Ind}_B^G(\Phi) \in \mathrm{Irr}(G).$$

To prove this result, we still need to discuss even and odd cases separately, but the arguments are similar to the cases we had before. I will sketch the proof for the even case and the odd case will follow similarly.
Suppose now $\ell = 2m$. Let $K_m$ be the same as before. Let

$$A = \begin{pmatrix} a_1 I_{n_1} & 0 & 0 & \cdots & 0 \\ 0 & a_2 I_{n_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{k-1} I_{k-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \text{ with } a_i \text{ and } a_i - a_j \in (\mathbb{Z}/p^m\mathbb{Z})^\times.$$

Let $\lambda : (\mathbb{Z}/p^{2m}\mathbb{Z})^+ \longrightarrow C^\times$ be a homomorphism such that $\lambda \mid_{\{p^m *\}}$ is injective.
Define

$$\Phi_A : K_m \longrightarrow C^\times, \Phi_A(I + p^m C) = \lambda[\mathrm{tr}(p^m AC)],$$

then $\mathrm{Stab}_G(\phi_A) = T = \left\{ \begin{pmatrix} A_1 & p^m * & p^m * & \cdots & p^m * \\ p^m * & A_2 & p^m * & \cdots & p^m * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p^m * & p^m * & \cdots & A_{k-1} & p^m * \\ p^m * & p^m * & p^m * & \cdots & A_k \end{pmatrix} \right\}$. Let

$$\lambda_i : (\mathbb{Z}/p^{2m}\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

be such that $\lambda_i(1 + p^m x) = \lambda(a_i p^m x)$, then

$$\Psi' : T \longrightarrow \mathbb{C}^\times, \Psi' \begin{pmatrix} A_1 & p^m * & p^m * & \cdots & p^m * \\ p^m * & A_2 & p^m * & \cdots & p^m * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p^m * & p^m * & \cdots & A_{k-1} & p^m * \\ p^m * & p^m * & p^m * & \cdots & A_k \end{pmatrix}$$

$$= \lambda_1[\det(A_1)] \cdots \lambda_{k-1}[\det(A_{k-1})]$$

is an extension of $\Phi_A$ and $\mathrm{Ind}_T^G(\Psi') \in \mathrm{Irr}(G)$.
Let $\phi_i \in \mathrm{Irr}(\mathrm{GL}(n_i, \mathbb{Z}/p^m\mathbb{Z})), 1 \leq i \leq k$ and let $\Psi = \Psi' \phi_1 \phi_2 \cdots \phi_k$. Since

$$T/K_m \cong \mathrm{GL}(n_1, \mathbb{Z}/p^m\mathbb{Z}) \times \mathrm{GL}(n_2, \mathbb{Z}/p^m\mathbb{Z}) \times \cdots \times \mathrm{GL}(n_k, \mathbb{Z}/p^m\mathbb{Z}),$$

by Clifford Theory, we still have $\mathrm{Ind}_T^G(\Psi) \in \mathrm{Irr}(G)$.
To show that $\mathrm{Ind}_T^G(\Psi) = \mathrm{Ind}_T^G(\Phi)$, we can apply the same method by showing that $[\mathrm{Ind}_T^G(\Psi), \mathrm{Ind}_T^G(\Phi)] = 1$ because $\deg(\mathrm{Ind}_T^G(\Psi)) = \deg(\mathrm{Ind}_T^G(\Phi))$. Notice that

$$\mathrm{Ind}_T^G(\Psi) \mid_{T \cap B} = \mathrm{Ind}_T^G(\Phi) \mid_{T \cap B},$$

the statement follows just as we had before.

# Bibliography

[1] A. Nobs, Die irreduziblen Darstellungen von $\mathrm{GL}_2(\mathbb{Z}_p)$ insbesondere $\mathrm{GL}_2(\mathbb{Z}_2)$, Math. Ann. 229(1974), 113–133.

[2] Fernando Szechtman. Quadratic Gauss Sums over Finite Commutative Rings. Journal of Number Theory 95, 1-13(2002).

[3] G. Hill. Semisimple and cupsidal characters of $\mathrm{GL}_n(\mathcal{O})$. Comm. Algebra, 23(1995), 7–25.

[4] I. Martin Isaacs, Characters of Solvable and Symplectic Groups, American Journal of Mathematics, 95(1973), 594–635.

[5] I. Martin Isaacs, Character theory of finite groups, Dover Publictions, Inc. New York, 1976.

[6] J.L. Alperin, Rowen B.Bell, Groups and Representations, Springer, 1995.

[7] N. Avni, U. Onn, A. Prasad, and L. Vaserstein. Similarity classes of $3\times 3$ matrices over a local principal ideal ring. Comm. Algebra, 37(2009), 2601–2615.

[8] P. C. Kutzko, On the supercuspidal representations of $\mathrm{Gl}_2$ II, American Journal of Mathematics, 100(1978), 705–716.

[9] P. Singla. On representations of general linear groups over principal ideal local rings of length two. Journal of Algebra, 324(2010), 2543-2563.

[10] R. Barrington Leigh, G. Cliff, and Q. Wen. Character values for $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$. Journal of Algebra, 323(2010), 1288-2320.

[11] U. Onn. Representations of automorphism groups of finite o-modules of rank two. Adv. Math., 219(2008), 2058–2085.