UNIVERSITY OF ALBERTA

MODELS IN CONDENSED MATTER NEAR CRITICALITY

BY

KENNETH JOHN ERNEST VOS

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY.**

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ABSTRACT

The Landau - Ginzburg free energy expansion of the superconducting order parameter in the presence of a magnetic vector potential has been used as a basis for the analysis of magnetic field penetration in superconductors. Several specific cases have been examined in one and two dimensions in order to solve the complicated system of coupled nonlinear partial differential equations that describe interactions between the magnetic field and superconducting charge density. Exact solutions are found at the critical temperature, while accurate series expansions are used close to it. Oscillatory damped profiles of the magnetic field penetration are found. In addition, periodic patterns of magnetic induction and a phase - shifted superconducting charge density are obtained. Other approximate methods were used to examine the behavior of the superconducting system for arbitrary temperatures below the critical temperature. For two dimensions, two types of solutions were obtained; vortices which exist below the critical temperature and spirals which exist at the critical temperature. In two dimensions, the symmetries that were considered so as to reduced the equations to nonlinear ordinary differential equations gave us in some cases, equations which have no known analytical solutions. A method was developed to find analytical solutions to some of these equations which are of the particular form; $f'' + A(f)f' + B(f)(f')^2 + C(f) = 0$. The method consists of converting the ordinary differential equation into a system of coupled algebraic equations which in principal can be solved using a symbolic solver program on a computer. Also, solutions were found for seven special cases, one of which is the solutions to the nonlinear ordinary differential equation; $f'' + B(f)(f')^2 + C(f) = 0$. Several examples of the method are given as a demonstration of the power of the method. The last part consists of investigating a generalized Hamiltonian of two interacting quasi - particles. The method of coherent structures is developed for systems with two different types of interacting particles. The general formalism is worked out leading to coupled field equations relevant for all three combinations of quantum statistics. The Fröhlich Hamiltonian for electron - phonon coupling in metals is analyzed as an example.

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a	Landau - Ginzburg parameter ($= a_o(T - T_c)$)
a_o	Landau - Ginzburg Parameter
$\mathbf{A}$	Magnetic vector potential
α	Parameter ($= 2m^*a/\hbar^2$)
am	Jacobi elliptic amplitude function
b	Landau - Ginzburg parameter
β	Parameter ($= 2m^*b/\hbar^2$)
c	Speed of light
cos	trigonometric cosine function
cosh	hyperbolic cosine function
cn	Jacobi elliptic function
dn	Jacobi elliptic function
e	electron charge
e^*	Effective charge
$E[k]$	Complete elliptic integral of the second kind
$E[\phi,k]$	Elliptic integral of the second kind
f	Landau - Ginzburg free energy density
f_o	Free energy density in the normal phase
${}_pF_q$	Hypergeometric function
$F[\phi,k]$	Elliptic integral of the first kind

γ	Parameter (= $4\pi(e^*)^2/(m^*c^2)$)
∇	Gradient operator
$\mathbf{h}$	Magnetic field vector
H	Hamiltonian
H_{LG}	Landau - Ginzburg Hamiltonian density
H_c	Critical magnetic field
$\hbar$	Planck's constant
i	$\sqrt{-1}$
I_μ	Modified Bessel function of the first kind
$\mathbf{j}$	current density vector
j_c	Critical current density
j_s	Superconducting current density
k	Modulus for Jacobi elliptic functions and elliptic integrals
K[k]	Complete elliptic integral of the first kind
K_μ	Modified Bessel function of the second kind
κ_{LG}	Landau - Ginzburg ratio of characteristic lengths (λ_{LG}/ξ_{LG})
λ_{LG}	Landau - Ginzburg penetration depth
LG	Landau - Ginzburg
m	electron mass
m*	Effective mass
n	Total number of particle density
n_s	Superconducting charge density

ODE	Ordinary Differential Equation
P_ν^μ	Associated Legendre function of the first kind
π	Constant pi
$\Pi[\phi,n,k]$	Elliptic integral of the third kind
PDE	Partial Differential Equation
Q_ν^μ	Associated Legendre function of the second kind
sin	trigonometric sine function
sinh	hyperbolic sine function
sech	hyperbolic secant function
sn	Jacobi elliptic function
T	Temperature (in chapter 2), a function (in chapter 3), and kinetic energy (in chapter 4)
T_c	Critical temperature
T_c^*	Modified critical temperature due to a nonzero magnetic field
tan	trigonometric tangent function
tanh	hyperbolic tangent function
tn	Jacobi elliptic function
$\mathbf{u}$	Superconducting scaled velocity vector
$\mathbf{v}_s$	Superconducting drift velocity vector
ξ_{LG}	Landau - Ginzburg coherence length

CHAPTER 1: INTRODUCTION

In the past decade, there has been much advancement in the study of nonlinear systems. This work will focus on applying this to a couple of models that have been of importance in physics in the past. The first model which will be analyzed is the Landau-Ginzburg phenomenological model for superconductors with a magnetic field. The model was very successful in describing the change in thermodynamical quantities at the phase transition in conventional superconductors. The second model that will be analyzed is the Fröhlich Hamiltonian which also was important in the understanding of low temperature superconductors. The Fröhlich Hamiltonian is a simplified model of free electrons interacting with phonons in a metal. Cooper pairs are the singlet state formed by electrons that are bound together by phonons and so this critical state should be present in the Fröhlich Hamiltonian. If one uses a unitary transformation to decouple the electrons and phonons to first order in the Fröhlich Hamiltonian, then one obtains a simplified version of the BCS model. However, one can show that the superconducting state is obtainable from the Fröhlich Hamiltonian from a different approach, which shall be done in chapter four. The third chapter is mathematical and deals with a method for solving autonomous nonlinear ordinary differential equations. This chapter is necessary, since the work that will be done with both the Landau - Ginzburg model and the Fröhlich Hamiltonian, will lead to nonlinear differential equations which need to be solved. The Landau - Ginzburg model shall be analyzed in chapter two.

Chapter two is a nonlinear study of the Landau - Ginzburg phenomenological model for superconductors with the presence of a magnetic field. The introduction section of this chapter gives a brief review of what has been done in the past using this model. Section II gives the equations of state that result from the minimization of the Landau - Ginzburg free energy functional. The section also shows the relationships between the physical quantities that are of interest such as the magnetic field, the superconducting current density and the superconducting drift velocity. It should be noted that the Landau - Ginzburg model basically consists of expanding the free energy functional in terms of an order parameter which in this case is the modulus squared of a complex field, and also terms consisting of the gradient of the order parameter. The order parameter is a quantity that is introduced to explain the change in thermodynamical quantities at a phase transition and as such, this means that our model is valid only near the critical temperature. We use this fact to realize that we can find solutions at the critical temperature and then perturb around the critical temperature. In this model one of the parameters vanishes at the critical temperature and so it is used as the expansion coefficient of the solutions about the critical temperature. In the third section, three different sets of solutions are found in one dimension and then expanded below the critical temperature. Also, in section III, a numerical plot of the full equations of state is presented so as to test the validity of this type of expansion. Section IV is basically an extension of the work which has been done in the past, where the effects of the magnetic field upon the superconducting charge are neglected. For type I superconductors where the magnetic field can not penetrate into the superconductor except for a very small distance or in type II superconductors where the superconducting current is basically constant, then this approximation is reasonably good. The fourth section is divided into three parts, the first of which discusses the approximation and the trivial solutions for the equations of state. The second part deals with a wealth of solutions; however, more approximations are necessary. Even with these approximations, the solutions which will be obtained are in good agreement with what one would expect. There is however, one set of solutions that will be obtained without further approximations. In the third part , some exact solutions are found using only the initial approximation of neglecting the effects of the magnetic field upon the superconducting charge.

Up to this point, we have been dealing exclusively with one dimensional solutions for the equations of state and now in section five of chapter two, the focus will be on quasi-two dimensional solutions. The method used to solve the equations of state in this section will be the same as that in section III, an expansion about the critical temperature. The section is broken into two parts, the first of which is called vortices. In this part, only radial symmetry in cylindrical coordinates will be considered so as to obtain nonlinear ordinary differential equations for the equations of state. Magnetic vortex solutions for small radial distances will be found in contrast to past attempts which only gave asymptotic solutions or solutions obtained from linearization of the equations. Some very good results for two dimensional solutions will be obtained, showing the interaction between the magnetic field and the superconducting charge in a more realistic system than the one dimensional solutions. The second part will use the spiral symmetry to reduce the equations of state to coupled nonlinear ordinary differential equations, however, it is necessary to introduce the approximation that small radial distances from the core (origin) are required. Again, some very interesting two dimensional solutions will be found.

The sixth section of chapter two deals with a unique approximation to simplify the coupled nonlinear partial differential equations. Essentially, the dependent variables will be rescaled so as to be able to solve one of dependent functions in terms of the other dependent function. This will leave only one equation to solve, which will turn out to be the cubic nonlinear Klein - Gordon equation in three dimensional space. Analytical solutions to this equation are discussed and their meaning for the physical parameters. Section VII discusses and reviews the results that will be obtained in chapter two. Three appendices are enclosed: the first looks at decoupling the two equations of state in such a way as to give more solutions for section three. The second appendix simply lists the differential equations that the Jacobi elliptic functions satisfy. The final appendix shows a couple of ways in which the two coupled equations of state in one dimension can be decoupled exactly.

Chapter three is a continuation of chapter two. In section V of chapter two when looking for vortices, autonomous nonlinear ordinary differential equations were obtained that have no known analytical solutions. In an effort to obtain analytical solutions to these equations, a method is developed for finding some analytical solutions to these equations. The introduction in section I discusses some of the previous work along these lines. The second section shows seven special cases of solutions and also discusses the method of reducing the differential equation into a system of algebraic equations. With the improvement of symbolic solvers for computers over recent years, this becomes very desirable. It is also discovered that for one of the special cases, elliptic integrals are a subclass of the resulting integral solution. In section three, further detail on the reduction of the nonlinear ordinary differential equation to a system of algebraic equations is discussed. In this section, it is shown that by one reduction, exponential solutions are obtained and by another reduction, Jacobi elliptic functions are obtained as solutions. Section IV of chapter three looks at a few physical examples where the algebraic method and the special cases are applicable. Section V summarizes the results of this chapter.

Chapter four deals with coupled systems at the microscopic level. The introduction in section I discusses some of the previous work done in this area, giving the historical background. Section II starts by looking at a generic second quantized Hamiltonian for two interacting types of particles. The section goes on to find the Heisenberg equations of motion for each possible statistics that the particles can obey: Boson - Boson interactions, indistinguishable Fermion - Fermion interactions, distinguishable Fermion - Fermion interactions, and Fermion - Boson interactions. The section also discusses the methodology where the coefficients are expanded in a Taylor series near some critical point in momentum space and also field operators are introduced which have a plane wave basis. This leads to two coupled generalized nonlinear Schrödinger equations and which are assumed to be classical to zeroth order. Also, in section II it will be shown that the equations of motion for the classical fields are equivalent to those obtained from a Landau - Ginzburg type Hamiltonian density. In section III, the classical field equations for the distinguishable Fermion - Fermion case will be discussed and some solutions to these equations are obtained. Section IV is a second example, but instead deals with the case of interacting Bosons and Fermions. The specific system in this case is that of interacting electrons and phonons in a metal as described by the Fröhlich Hamiltonian. The fourth section is divided into three parts, the first of which is a discussion on the general background information of the Fröhlich Hamiltonian and also the resulting classical field equations that will be obtained. In the second part, the equations of motion when the coupling coefficient is a constant will be discussed. In this part, the solutions to the classical equations of motion are found which permit the formation of Cooper pairs and thus lead to superconductivity. The third part is an extension of the second part, except that now the coupling coefficient between the electrons and the phonons is considered to have a q dependence which will lead to extra terms in the equations of motion. This part however, leads to implicit solutions in the form of an indefinite integral which in general can not be solved, but for special cases the integral can be done and then the solution can be inverted. However, there is much information that can be gained from the integral without actually solving it, which will be discussed. In this part solutions for free electrons scattering off the phonons and solutions for Cooper pair formation are found. The fifth section closes chapter four by summarizing the results and discussing further applications for the method discussed.

The last chapter, namely, chapter five gives some concluding remarks and a summary of the results found in chapters two through four. It also points out possible avenues where this work can be extended.

CHAPTER 2: MAGNETIC FIELD PENETRATION IN SUPERCONDUCTORS USING THE LANDAU - GINZBURG MODEL*

SECTION I: INTRODUCTION

Models of a Landau-Ginzburg (LG) type have played a central role in modern theories of phase transitions[1)] and, in particular, in theories of superconductivity[2)]. Despite the powerful predictive properties of these models, they were often considered to be of little relevance due to their phenomenological development. This was despite the fact that in 1959, Gor'kov[3)] established a connection between the microscopic BCS theory of superconductivity and the LG model. Recently, using an entirely different approach, a direct link has been established between an effective microscopic second quantized Hamiltonian describing interacting electrons in a phonon field and the order parameter picture in the LG theory[4)]. Thus, although apparently phenomenological, such models do contain the basic physical notions required to understand and adequately describe superconductivity. The equations of state, obtained from a LG model, are highly nonlinear, and when applied to standard superconductors, with long range order in the superconducting state and a fairly long coherence length, it is conventional to rely heavily on solutions in which the modulus squared of the order parameter is a constant. For low temperature superconductors, this approximation in many instances is not a bad one, and such phenomena as penetration depth and the critical current as a function of temperature are correctly predicted. However, this type of approximation predicts, for example, critical exponents which may differ significantly from those resulting from renormalization group studies[5)], where fluctuations are incorporated correctly. In the high temperature superconductors, where coherence lengths on the order of 20 - 30 Å in the a-b plane are much shorter than those of conventional superconductors, magnetic

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fields penetrate much farther into the superconductor, and furthermore, penetration depth measurements[6)] appear to show a temperature dependence which is inconsistent with the standard form for ordinary superconductors. Thus models relying on constant order parameter modulus solutions are inadequate.

In the presence of an external magnetic field, the LG free energy density is given by[1),2)]

$$f = f_o + a|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{\hbar^2}{2m^*}\left|\left\{\nabla - \frac{ie^*}{\hbar c}\mathbf{A}\right\}\Psi\right|^2 + \frac{1}{8\pi}(\nabla \times \mathbf{A})^2, \tag{2.1.1}$$

where f_0 is the free energy density of the normal phase, e^* is the effective charge, m^* is the effective mass, and $\mathbf{A}$ is the vector potential. Usually, the order parameter ψ is identified with the (complex) wave function of a Cooper pair, so that $n_s(\mathbf{r}) = |\Psi|^2$ is the local density of superconducting charges. Conventionally, all the parameters of the theory, apart from a, for which $a = a_o(T - T_c)$ following Landau, are temperature independent. Both the complex order parameter Ψ and the vector potential $\mathbf{A}$ are a priori independent quantities, so that a subsequent minimization of the free energy functional

$$F = \int d^3x\, f \tag{2.1.2}$$

should be carried out using the variational principle applied with respect to both these quantities separately. The result of such a procedure is the following set of two coupled nonlinear differential equations[1),2)], where $\mathbf{A}$ and Ψ are the dependent variables and the spatial coordinates are the independent variables:

$$\nabla \times (\nabla \times \mathbf{A}) + 2\pi i \frac{\hbar e^*}{m^* c}(\Psi^* \nabla\Psi - \Psi \nabla\Psi^*) + \frac{4\pi(e^*)^2}{m^* c^2}|\Psi|^2 \mathbf{A} = 0, \tag{2.1.3}$$

and

$$a\Psi + b|\Psi|^2 \Psi - \frac{\hbar^2}{2m^*}\left\{\nabla - \frac{ie^*}{\hbar c}\mathbf{A}\right\}^2 \Psi = 0. \tag{2.1.4}$$

The constant solutions of equation (2.1.4) are easiest to find and they are, for $T > T_c$,

$$|\Psi|^2 = 0, \tag{2.1.5}$$

while, for $T < T_c$,

$$|\Psi|^2 = |\Psi_\infty|^2 = -\frac{a}{b}, \tag{2.1.6}$$

when the vector potential is ignored in equation (2.1.4). This yields the scaling of n_s as $n_s \propto (T - T_c)$. It then follows from equation (2.1.3) that

$$\nabla^2 \mathbf{h} = \frac{\mathbf{h}}{\lambda_{LG}^2}, \tag{2.1.7}$$

where $\lambda_{LG} = (m^* c^2 / 4\pi(e^*)^2 n_s)^{1/2}$ is the characteristic *LG penetration depth*, which, because of its dependence on $n_s = |\Psi|^2$, varies with temperature in a standard way1) as $\lambda_{LG} \propto |T - T_c|^{-1/2}$. Equation (2.1.7) describes an isotropic penetration depth, so that the magnetic field **h** is exponentially damped to zero as it penetrates the sample.

Another easily tractable case is that of zero magnetic field. Defining a reduced superconducting order parameter

$$\hat{\Psi} = \frac{\Psi}{\Psi_\infty}, \tag{2.1.8}$$

reduces the problem in one dimension to solving the equation

$$\frac{\hbar^2}{2m^* |a|} \frac{d^2 \hat{\Psi}}{dx^2} + \hat{\Psi} - \left(\hat{\Psi}\right)^3 = 0. \tag{2.1.9}$$

This form introduces, in a natural way, the *LG coherence length*, ξ_{LG} describing the length over which spatial variation of the order parameter exists and is defined by

$$\xi_{LG}^2(T) = \frac{\hbar^2}{2m^*|a|} \propto (T - T_c)^{-1}. \tag{2.1.10}$$

The combined presence of the order parameter $\Psi(x)$ and the vector potential $A(x)$ makes the problem much more difficult to treat, but approximate calculations have been done in the past[1),8)]. An important characteristic quantity is the ratio of the two characteristic lengths, i.e.,

$$\kappa_{LG} = \frac{\lambda_{LG}}{\xi_{LG}}. \tag{2.1.11}$$

Depending on the value of κ_{LG}, superconductivity can be divided into two distinct classes. For $\kappa_{LG} < 1/\sqrt{2}$ pure superconductors are designated type I, otherwise they are designated type II ("dirty" superconductors).

It was shown long ago by Abrikosov[9)] that the LG equations of state for superconductivity possess vortex solutions. Conservation laws governing the topological charges, which are a measure of magnetic flux quantization in type II superconductors, ensure their stability. Such vortices are two dimensional examples of solitons, which in turn are often the solutions of nonlinear equations. All these factors strongly suggest that an attempt should be made to solve the nonlinear LG equations of state in the presence of a magnetic field without using a constant modulus and making use of the powerful techniques which are now available for solving nonlinear equations.

In this chapter, we examine the LG equations for superconductors subjected to external magnetic fields and/or trapped magnetic fields, going significantly beyond previous linearization attempts. Under special conditions (e.g., the immediate vicinity of $T = T_c$) exact solutions of these nonlinear equations will be found. These will be subsequently used as a starting point for approximate calculations involving series expansions. Another approach will be to postulate a special ansätze whereby nonlinear coupled equations may be effectively separated and solved independently.

of state that result from the minimization of the Landau - Ginzburg free energy functional. The section also shows the relationships between the physical quantities that are of interest such as the magnetic field, the superconducting current density and the superconducting drift velocity. In the third section, three different sets of solutions are found in one dimension at the critical temperature and then expanded below the critical temperature. Also, in section III, a numerical plot of the full equations of state is presented so as to test the validity of this type of expansion. Section IV is basically an extension of the work which has been done in the past, where the effects of the magnetic field upon the superconducting charge are neglected. The fourth section is divided into three parts, the first of which discusses the approximation and the trivial solutions for the equations of state. The second part deals with a wealth of solutions, however, more approximations are necessary except for one of the nontrivial solutions which can be solved without further approximations. Even with these approximations, the solutions which will be obtained are in good agreement with what one would expect. In the third part, some exact solutions are found using only the initial approximation of neglecting the effects of the magnetic field upon the superconducting charge. In section five, the focus will be on quasi-two dimensional solutions. The method used to solve the equations of state in this section will be the same as that in section III, an expansion about the critical temperature. The section is broken into two parts, the first of which is called vortices. In this part, only radial symmetry in cylindrical coordinates will be considered so as to obtain nonlinear ordinary differential equations for the equations of state. Magnetic vortex solutions for small radial distances will be found in contrast to past attempts which only gave asymptotic solutions or solutions obtained from linearization of the equations of state. Some very good results for two dimensional solutions will be obtained, showing the interaction between the magnetic field and the superconducting charge density in a more realistic system than the one dimensional solutions. The second part will use the equations, however, it is necessary to introduce the approximation that small radial distances from the core (origin) are required. Again, some very interesting two dimensional solutions will be found. The sixth section deals with an unique approximation to simplify the coupled nonlinear partial differential equations. Essentially, the dependent variables will be rescaled so as to be able to solve one of dependent functions in terms of the other dependent function. This will leave only one equation to solve, which will turn out to be the cubic nonlinear Klein - Gordon equation in three dimensional space. Analytical solutions to this equation are discussed and their meaning for the physical parameters. There are also three appendices: the first looks at decoupling the two equations of state in such a way as to give more solutions for section three. The second appendix lists the differential equations that the Jacobi elliptic functions satisfy. The final appendix shows a couple of ways in which the two coupled equations of state in one dimension can be decoupled exactly.

SECTION II: LG FREE ENERGY AND ITS MINIMIZATION

In the presence of a vector potential **A**, minimization of the free energy functional with respect to both **A** and Ψ leads to the coupled equations (2.1.3) and (2.1.4), which we now intend to simplify. First, we represent the complex order parameter in the modulus argument form:

$$\Psi = \eta \exp(i\chi).$$

Equations (2.1.3) and (2.1.4) then become the following three coupled equations for the real functions, the vector potential **A**, the order parameter envelope η, and the order parameter phase χ:

$$\nabla \times (\nabla \times \mathbf{A}) + 4\pi\eta^2 \frac{\hbar e^*}{m^* c}\left\{\frac{e^*}{\hbar c}\mathbf{A} - \nabla\chi\right\} = 0, \tag{2.2.1}$$

$$\nabla^2\eta - \eta\left\{\frac{2m^* a}{\hbar^2} + \frac{2m^* b}{\hbar^2}\eta^2 + \left\{\frac{e^*}{\hbar c}\mathbf{A} - \nabla\chi\right\}\right\} = 0, \tag{2.2.2}$$

and

$$\nabla \bullet \left\{\eta^2\left(\frac{e^*}{\hbar c}\mathbf{A} - \nabla\chi\right)\right\} = 0. \tag{2.2.3}$$

From equations (2.2.1) through (2.2.3) we can relate the vector potential **A** and the phase χ to the physical quantities that we are interested in, namely, the magnetic field **h**, the superconducting current density $\mathbf{j}_s$, and the drift velocity of superconducting charges $\mathbf{v}_s$, where we have used equation (2.2.1) to obtain the following relationship:

$$\frac{e^*}{\hbar c}\mathbf{A} - \nabla\chi = \frac{-m^* c}{4\pi\hbar e^*}\frac{\nabla \times \mathbf{h}}{n_s} = \frac{-m^*}{\hbar e^*}\frac{\mathbf{j}_s}{n_s} = \frac{-m^*}{\hbar}\mathbf{v}_s. \tag{2.2.4}$$

Now, if we suppose that $[(e^*/\hbar c)\mathbf{A} - \nabla\chi] = \mathbf{u}$, which represents a scaled velocity field (see equation (2.2.4)) of the superconducting charge density, then equations (2.2.1) through (2.2.3) become

$$\nabla \times (\nabla \times \mathbf{u}) + \frac{4\pi(e^*)^2}{m^* c^2}\eta^2\mathbf{u} = \mathbf{0}, \tag{2.2.5}$$

$$\nabla^2\eta - \eta\left\{\frac{2m^* a}{\hbar^2} + \frac{2m^* b}{\hbar^2}\eta^2 + |\mathbf{u}|^2\right\} = 0, \tag{2.2.6}$$

and

$$\nabla \bullet \left[\eta^2\,\mathbf{u}\right] = 0. \tag{2.2.7}$$

Note that equation (2.2.7) represents a continuity equation for the superconducting current density, and it is automatically satisfied by equation (2.2.5). To solve the remaining two equations, (i.e., equations (2.2.5) and (2.2.6)) exactly is extremely good approximation.

SECTION III: PROXIMITY OF THE CRITICAL TEMPERATURE

In this section we shall approach the magnetic field penetration problem described by equations (2.2.5) and (2.2.6) by considering the temperature to be close to the critical temperature T_c, so that the coefficient a is negligibly small. Thus our objective here will be to establish whether there exist special types of charge distribution and magnetic field patterns which are characteristic of the vicinity of the critical point. Since the relationships under study may possess scale invariance at $T = T_c$, where $a = 0$, different types of solutions are expected to occur at criticality. First, let us simplify the equations by setting

$$\alpha = \frac{2m^* a}{\hbar^2}, \beta = \frac{2m^* b}{\hbar^2}, \text{ and } \gamma = \frac{4\pi(e^*)^2}{m^* c^2}, \tag{2.3.1}$$

so that equations (2.2.5) and (2.2.6) may be rewritten more compactly as

$$\nabla \times (\nabla \times \mathbf{u}) + \gamma\, \eta^2\, \mathbf{u} = \mathbf{0}, \tag{2.3.2}$$

$$\nabla^2 \eta - \eta\left(\alpha + \beta \eta^2 + |\mathbf{u}|^2\right) = 0. \tag{2.3.3}$$

Note that the LG coherence length $\xi_{LG} = |\alpha|^{-1/2}$, so that in this section we are dealing with a very large LG coherence length. Concentrating on the one dimensional case where $\eta = \eta(x)$ and $\mathbf{u} = \mathbf{u}(x)$, we immediately obtain that the x component of the scaled velocity $u_x = 0$ and that the remaining two components u_y and u_z are both proportional to some function u. This then reduces the problem to the following system of coupled ordinary differential equations (ODE's)

$$\frac{d^2u}{dx^2} = \gamma\, \eta^2\, u \tag{2.3.4}$$

$$\frac{d^2\eta}{dx^2} = \eta(\alpha + \beta\,\eta^2 + u^2). \qquad (2.3.5)$$

It should be noted that when $\gamma = 2\beta > 0$ and $\alpha < 0$ or when $\gamma = \beta$, then equations (2.3.4) and (2.3.5) are of the Painlevé-type. For a brief discussion of the Painlevé test, see section I of chapter three and also the properties of Painlevé-type equations. Also, references which give a more detailed discussion can be found there. As can been seen by the constraints on the parameters, these cases are very specialized and so we will not look for their solutions here.

We have made numerous attempts at solving these two coupled equations by postulating that u be some a priori unknown function of η (see Appendix A). All these attempts led to inconsistencies except when we assume a linear relationship between the order parameter's envelope η and the scaled velocity u:

$$\eta = \frac{|u|}{\sqrt{\gamma - \beta}}. \qquad (2.3.6)$$

However, this demands that $\alpha = 0$, i.e., it is valid only at the critical point $T = T_c$. Furthermore, for the order parameter η and scaled velocity u to be real, we have to have $\gamma > \beta$, which implies the following constraint on b, namely,

$$b \lesssim 2\pi \left\{\frac{e^* \hbar}{m^* c}\right\}^2, \text{ if } m^* \gtrless 0. \qquad (2.3.7)$$

Looking for solutions at the critical temperature, we shall find three different sets of solutions for equations (2.3.4) and (2.3.5) when a = 0. The first set of solutions that was found is for the assumption given in equation (2.3.6). The other two sets were found when one of the dependent variables was set to zero. It should be noted that these three sets of solutions do not cover all possible solutions to equations (2.3.4) and (2.3.5) since gives two. At this point we wish to digress and comment on the physical possibility of a negative effective mass. In conventional studies of LG free energies, a negative coefficient of the gradient term has been customarily disallowed because of a seemingly unbounded energy of the corresponding order parameter solutions. However, there exist indications that such a conclusion may be premature. A number of arguments can be put forward, indeed, to justify a physical relevance of the negative mass situation. First, as outlined in reference 7, the elliptic cn solutions that arise from solving the corresponding LG equations are themselves not divergent. Second[10)], a linear stability analysis shows that these solutions are stable with respect to perturbations. The question of the lower limit of the energy spectrum can be answered in a positive way by introducing a cutoff on the wavelength of these periodic solutions due to inherent lattice periodicity. As a continuum model, LG theory does not exhibit discrete translational symmetries typical of the underlying crystal structure. Thus a reintroduction of a natural period is fully justified even if it is done by hand. The net result of such a procedure will be a restoration of the lower bound of energy. Furthermore, a more elegant way of accomplishing the same result can be obtained by continuing the free energy expansion in gradient terms beyond the Ginzburg term $|\nabla\Psi|^2$. This procedure has been attempted before[11)], and the inclusion of $|\nabla\Psi|^4$ or $(\nabla^2\Psi)^2$, in the free energy density, leads invariably to a desired stabilization with the selection of a particular wave vector (and thus periodicity) corresponding to a functional minimum. It is interesting to note, in this context, that such terms arise naturally in a first principles approach from a fundamental microscopic basis[4)]. This, therefore, leads us to admit both $m^* > 0$ and $m^* < 0$ as physically viable options. With the requirements of equation (2.3.7) satisfied, the resultant equation for u takes the elliptic form of;

$$\frac{d^2u}{dx^2} = \frac{\gamma\, u^3}{\gamma - \beta}. \tag{2.3.8}$$

given by

$$u_o = \bar{u}_o \, \mathrm{cn}\left(\bar{u}_0(x - x_0)\left|\frac{\gamma}{\gamma - \beta}\right|^{1/2}, \frac{1}{\sqrt{2}} \right), \tag{2.3.9}$$

where $\bar{u}_o$ and x_o are arbitrary constants of integration. The period of this solution is

$$\frac{4K(1/\sqrt{2})}{\bar{u}_o}\left|\frac{\gamma}{\gamma - \beta}\right|^{1/2},$$

where $K(1/\sqrt{2})$ is the complete elliptic integral of the first kind.

In order to extend the validity of the solution below $T = T_c$, we have carried out a perturbation expansion in powers of α, which, when written in matrix notation, takes the form

$$\begin{pmatrix} u \\ \eta \end{pmatrix} = \sum_{n=0}^{\infty} \alpha^n \begin{pmatrix} u_n \\ \eta_n \end{pmatrix}, \tag{2.3.10}$$

where u_n and η_n are nth-order corrections to the scaled velocity u and the order parameter envelope η, which appears in equations (2.3.4) and (2.3.5). The zeroth, first, and second order equations for this type of expansion are as follows:

$$\frac{d^2u_o}{dx^2} = \gamma \eta_o^2 u_o, \tag{2.3.11}$$

$$\frac{d^2\eta_o}{dx^2} = \eta_o(u_o^2 + \beta \eta_o^2), \tag{2.3.12}$$

$$\frac{d^2u_1}{dx^2} = \gamma \eta_o(\eta_o u_1 + 2u_o\eta_1), \tag{2.3.13}$$

$$\frac{d^2\eta_1}{dx^2} = \eta_o + 2\eta_o u_o u_1 + (u_o^2 + 3\beta \eta_o^2)\eta_1, \tag{2.3.14}$$

$$\frac{d^2u_2}{dx^2} = \gamma\left[u_0\eta_1^2 + 2\eta_0\eta_1 u_1 + 2\eta_0\eta_2 u_0 + \eta_0^2 u_2\right], \tag{2.3.15}$$

$$\frac{d^2\eta_2}{dx^2} = \eta_1\left[1 + 3\beta\eta_0\eta_1 + 2u_0u_1\right] + \eta_0 u_1^2 + 2\eta_0 u_0 u_2 + \eta_2\left[3\beta\eta_0^2 + u_0^2\right]. \tag{2.3.16}$$

This perturbative procedure can be extended to higher orders and is very tedious but algorithmic. It should be pointed out that all the nonlinearity is contained in the zeroth order equations (2.3.11) and (2.3.12) and in higher orders, for example, equations (2.3.13) and (2.3.14), they are all linear and their coefficients are dependent only on the solutions of the lower order equations. The first order corrections for the solution in equation (2.3.9) can be conveniently written as the following rapidly convergent series:

$$\begin{pmatrix} u_1 \\ \eta_1 \end{pmatrix} = \sum_{n=0}^{\infty} \eta_o^{4n} \left\{ \eta_0^3 \begin{pmatrix} c_{0,n} \\ c_{1,n} \end{pmatrix} + \begin{pmatrix} c_{2,n} \\ c_{3,n} \end{pmatrix} + \eta_0 \begin{pmatrix} c_{4,n} \\ c_{5,n} \end{pmatrix} \right\}, \tag{2.3.17}$$

where η_o is defined by

$$\eta_o = \frac{|u_o|}{\sqrt{\gamma - \beta}}, \tag{2.3.18}$$

and u_o is given in equation (2.3.9). In equation (2.3.17) the constants $c_{0,n}$ and $c_{1,n}$ satisfy the recurrence relationship for the particular solution, which can be found in a straightforward manner and the other set of four constants satisfy the recurrence relationship for the homogeneous coupled equations. However, an easier way of solving the equations for first order corrections is by changing the independent and dependent variables, thus by letting:

$$y = cn^4\left[\bar{u}_0(x - x_0)\sqrt{\frac{|\gamma|}{\gamma - \beta}}, \frac{1}{\sqrt{2}}\right], \tag{2.3.19}$$

$$u_1 = u_1(y), \tag{2.3.20}$$

and

$$\eta_1 = (\gamma - \beta)^{-1/2} R(y), \qquad (2.3.21)$$

along with equations (2.3.9) and (2.3.18), then our first order corrections (i.e. equations (2.3.13) and (2.3.14)) become;

$$y(1-y)u_1'' + \frac{3-5y}{4}u_1' + \frac{u_1}{8} + \frac{R}{4} = 0, \qquad (2.3.22)$$

and

$$y(1-y)R'' + \frac{(3-5y)}{4}R' + \frac{\gamma+2\beta}{8\gamma}R + \frac{\gamma-\beta}{4\gamma}u_1 + \frac{\gamma-\beta}{8\gamma\bar{u}_0 y^{1/4}} = 0. \qquad (2.3.23)$$

These two linear ODE's can be decoupled by the following transformation; $u_1 = T + S$ and $R = T - (1 - \beta/\gamma)S$, which gives us;

$$y(1-y)S'' + \frac{3-5y}{4}S' + \frac{2\beta-\gamma}{8\gamma}S + \frac{\gamma-\beta}{8\bar{u}_0(\beta-2\gamma)y^{1/4}} = 0 \qquad (2.3.24)$$

and

$$y(1-y)T'' + \frac{3-5y}{4}T' + \frac{3}{8}T - \frac{\gamma-\beta}{8\bar{u}_0(\beta-2\gamma)y^{1/4}} = 0. \qquad (2.3.25)$$

Equations (2.3.24) and (2.3.25) are hypergeometric equations and their solutions are found to be;

$$S = \frac{-(\gamma-\beta)y^{3/4}}{3\bar{u}_0(\beta-2\gamma)}\,{}_3F_2\left[1,\frac{7-\sqrt{16\beta/\gamma-7}}{8},\frac{7+\sqrt{16\beta/\gamma-7}}{8};\frac{3}{2},\frac{7}{4};y\right] +$$
$$c_1\,{}_2F_1\left[\frac{1-\sqrt{16\beta/\gamma-7}}{8},\frac{1+\sqrt{16\beta/\gamma-7}}{8};\frac{3}{4};y\right] + c_2 y^{1/4} \qquad (2.3.26)$$
$${}_2F_1\left[\frac{3-\sqrt{16\beta/\gamma-7}}{8},\frac{3+\sqrt{16\beta/\gamma-7}}{8};\frac{5}{4};y\right]$$

and

$$T = \frac{(\gamma-\beta)y^{3/4}}{3\bar{u}_0(\beta-2\gamma)}\,{}_2F_1\left[1,\frac{1}{4};\frac{7}{4};y\right] + c_3\sqrt{1-y} + c_4 y^{1/4}\,{}_2F_1\left[\frac{-1}{4},1;\frac{5}{4};y\right]. \qquad (2.3.27)$$

For the particular solutions given in equations (2.3.26) and (2.3.27), we had to let $T \rightarrow y^{3/4}T'$ and $S \rightarrow y^{3/4}S'$, and then we introduced a power series which leads to hypergeometric recurrence relationships. Now combining equations (2.3.9), (2.3.26), and (2.3.27) gives us the solutions for u and η to first order. Thus, the solutions for the superconducting charge density ($n_s = |\Psi|^2$) and the magnitude of the magnetic field h are;

$$n = \frac{\bar{u}_0^2 y^{1/2}}{\gamma-\beta}\left\{1+\frac{2\alpha}{|\bar{u}_0|y^{1/4}}\left[T+\left(\frac{\beta}{\gamma}-1\right)S\right]+O(\alpha^2)\right\} \qquad (2.3.28)$$

and

$$h = \frac{\hbar c\bar{u}_0^2}{e^*}\sqrt{\frac{|\gamma|(1-y)}{2(\gamma-\beta)}}\left\{1+\frac{4\alpha y^{3/4}}{\bar{u}_0}\left[\frac{dT}{dy}+\frac{dS}{dy}\right]+O(\alpha^2)\right\}, \qquad (2.3.29)$$

where y is given by equation (2.3.19). If we ignore the homogeneous solutions to the first order corrections (i.e. $c_1 = c_2 = c_3 = c_4 = 0$) and we let $x = x_0 + \frac{\hat{x}}{\bar{u}_0}\sqrt{\frac{\gamma-\beta}{|\gamma|}}$, $n = \frac{\hat{n}\bar{u}_0^2}{\gamma-\beta}$, and $h = \frac{\hat{h}\hbar c\bar{u}_0^2}{e^*}\sqrt{\frac{|\gamma|}{2(\gamma-\beta)}}$, then equations (2.3.28) and (2.3.29) become;

$$n = y^{1/2}\left\{1+\frac{2\alpha(\gamma-\beta)y^{1/2}}{3\bar{u}_0^2(\beta-2\gamma)}\left\{{}_2F_1\left(1,\frac{1}{4};\frac{7}{4};y\right)-\left(\frac{\beta}{\gamma}-1\right)\right.\right.$$
$$\left.\left.{}_3F_2\left(1,\frac{7-\sqrt{-7+16\beta/\gamma}}{8},\frac{7+\sqrt{-7+16\beta/\gamma}}{8};\frac{3}{2},\frac{7}{4};y\right)\right\}\right\} \qquad (2.3.30)$$

and

$$h = \sqrt{(1-y)}\left\{1+\frac{\alpha(\gamma-\beta)y^{3/4}}{3\bar{u}_0^2(\beta-2\gamma)}\left\{{}_2F_1\left(1,\frac{5}{4};\frac{7}{4};y\right)-\frac{7\gamma-2\beta}{3\gamma}\right.\right.$$
$$\left.\left.{}_3F_2\left(1,\frac{15+\sqrt{-7+16\beta/\gamma}}{8},\frac{15-\sqrt{-7+16\beta/\gamma}}{8};\frac{5}{2},\frac{7}{4};y\right)\right\}\right\}, \qquad (2.3.31)$$

and (2.3.31) are plotted in figures II-1 through II-3. Figure II-1 was plotted with $\alpha = 0$ and over half a period (i.e. $\hat{x}$ going from 0 to $2\,K[1/\sqrt{2}]$). This figure shows the physically correct picture of the magnetic field being phase shifted from the condensate density by a quarter period. It is also clear from figure II-1 that the magnetic field has a larger domain at which it is a maximum as compared to the condensate density. Figure II-2 is a plot of equation (2.3.30) with $\alpha = 0$ (the thin line) and the thick line corresponds to $\hat{n}$ to first order. We see that with the dropping of the temperature, that the magnitude increases at the centers of nucleation, and a pushing out into the regions that were in the normal state. From figure II-3 we have plotted equation (2.3.31) with the thin line being the zeroth order magnetic field and the thick line being the magnetic field to first order. It should be noted, that for figures II-1 through II-3, the choice of α and β were arbitrary and do not represent any real physical system.

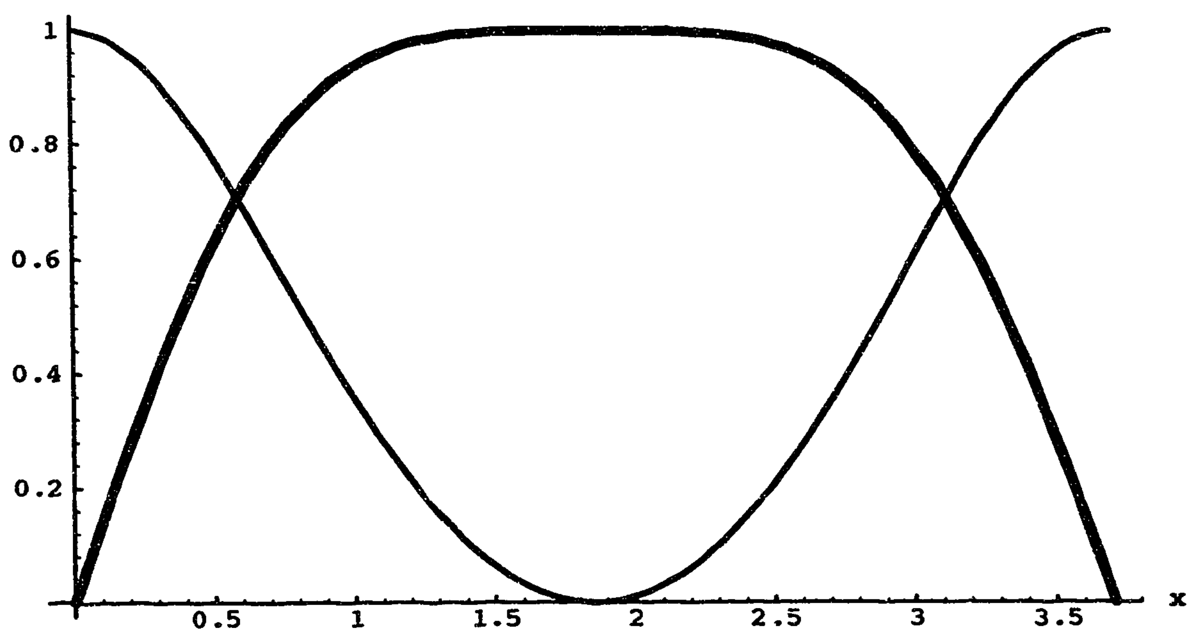

Figure II-1: Plot of equations (2.3.30) and (2.3.31) to zeroth order in α with the thin line corresponding to superconducting charge density and the thick line corresponding to the magnetic field.

The thin lines in figures II-2 and II-3 are the same as the respective curve in figure II-1. Thus we obtain a physically correct picture of the magnetic field penetration into the superconducting sample in the regions of vanishing condensate density n_s, while complete expulsion of the magnetic field $\tilde{h}$ occurs in regions saturated with the superconducting order parameter. We see that the region over which the magnetic field is a maximum stays unchanged as we start to drop the temperature, but the drop off to zero becomes steeper giving the plot more of a rectangular form. On the other hand, the change in the superconducting charge density is more dramatic than that of the magnetic field. However, we do expect that the farther we are from the critical temperature, the more the magnetic field will become a spike. Finally, it should be emphasized that the correction terms revealed by equation (2.3.17) are premultiplied by a factor proportional to $(T - T_c)$ and therefore are very accurate in the immediate vicinity of the critical temperature.

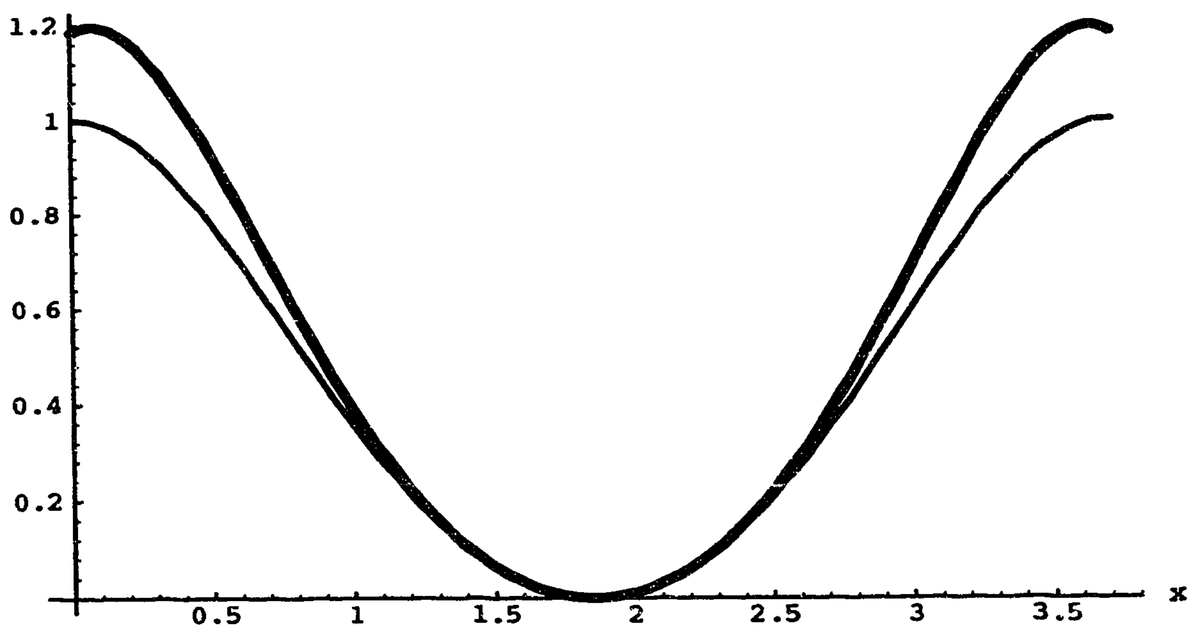

Figure II-2: Plot of the superconducting charge density as given by equation (2.3.30) where the thin line corresponds to the zeroth order solution and the thick line corresponds to the solution to first order in α.

It can be seen from equations (2.3.11) and (2.3.12) that the previous solution is not unique and that other relationships do exist. We can, however, find limiting solutions to the above equations which simplify the two nonlinear equations (2.3.11) and (2.3.12). The first one applies to the case of weak effects due to the magnetic field penetration, and hence we set $u_o = 0$. Therefore, equations (2.3.12) through (2.3.14) become

$$\frac{d^2\eta_o}{dx^2} = \beta\eta_o^3, \tag{2.3.32}$$

$$\frac{d^2u_1}{dx^2} = \gamma\eta_o^2 u_1, \tag{2.3.33}$$

and

$$\frac{d^2\eta_1}{dx^2} = \eta_o + 3\beta\eta_o^2\eta_1. \tag{2.3.34}$$

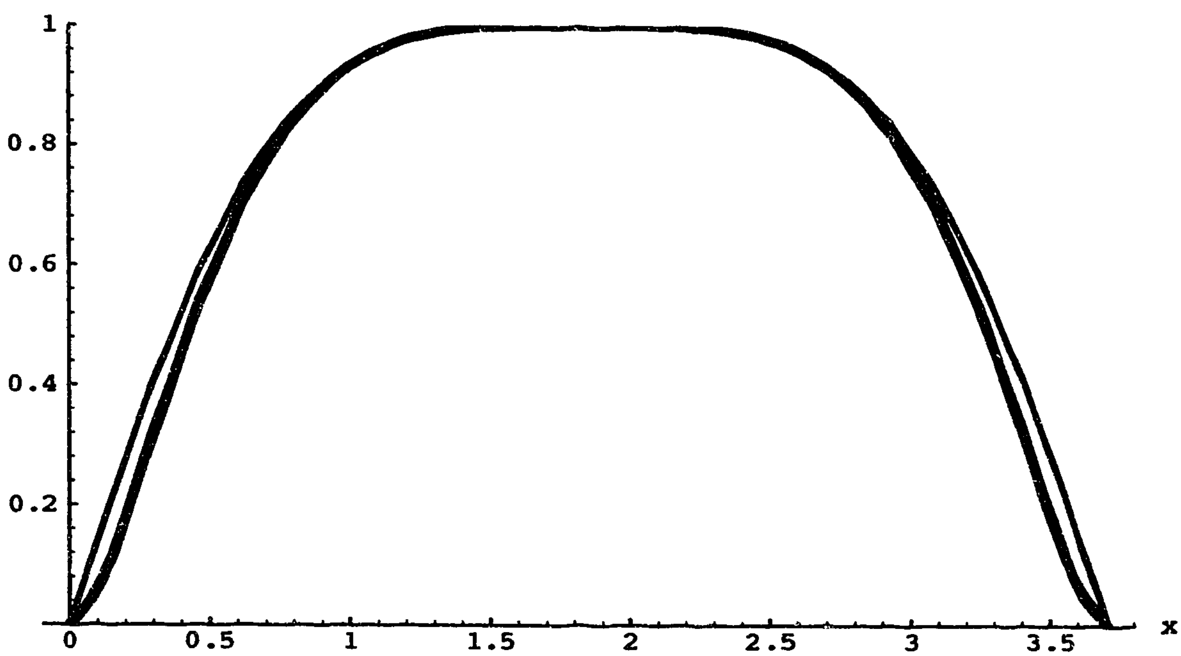

Figure II-3: Plot of the magnetic field as given by equation (2.3.31) where the thin line corresponds to the zeroth order solution and the thick line corresponds to the solution to first order in α.

It is clear that the only nonsingular solution for equation (2.3.32) exists for $\beta < 0$ and is given by equation (2.3.9) except with different coefficients;

$$\eta_o = \bar{\eta}_o \,\mathrm{cn}\left(\bar{\eta}_o(x - x_0)\sqrt{|\beta|}, \frac{1}{\sqrt{2}}\right). \tag{2.3.35}$$

If we let

$$y = \mathrm{cn}^4\left(\bar{\eta}_o(x - x_0)\sqrt{|\beta|}, \frac{1}{\sqrt{2}}\right) \tag{2.3.36}$$

in a similar manner to equation (2.3.19) then our first order equations (i.e. equations (2.3.33) and (2.3.34)) become;

$$y(1-y)u_1'' + \frac{3-5y}{4}u_1' + \frac{\gamma}{8\beta}u_1 = 0 \tag{2.3.37}$$

and

$$y(1-y)\eta_1'' + \frac{3-5y}{4}\eta_1' + \frac{3}{8}\eta_1 + \frac{1}{8\beta\bar{\eta}_o y^{1/4}} = 0. \tag{2.3.38}$$

The solutions to equations (2.3.37) and (2.3.38) are easily found to be (see equations (2.3.24) and (2.3.25)):

$$u_1 = c_1 \, {}_2F_1\left(\frac{1+\sqrt{1+8\gamma/\beta}}{8}, \frac{1-\sqrt{1+8\gamma/\beta}}{8}; \frac{3}{4}; y\right) + \tag{2.3.39}$$
$$c_2 y^{1/4} \, {}_2F_1\left(\frac{3+\sqrt{1+8\gamma/\beta}}{8}, \frac{3-\sqrt{1+8\gamma/\beta}}{8}; \frac{5}{4}; y\right)$$

and

$$\eta_1 = \frac{y^{3/4}}{3\beta\bar{\eta}_o} \, {}_2F_1\left(1, \frac{1}{4}; \frac{7}{4}; y\right) + c_3\sqrt{1-y} + c_4 y^{1/4} \, {}_2F_1\left(1, -\frac{1}{4}; \frac{5}{4}; y\right). \tag{2.3.40}$$

Thus, our solutions for the superconducting charge density n_s and the magnetic field h to first order are;

$$h = \frac{\alpha\hbar c\bar{\eta}_0}{e^*}\sqrt{\frac{|\beta|(1-y)}{2}}\left\{\frac{-2c_1\gamma y^{3/4}}{3\beta}{}_2F_1\left(\frac{9+\sqrt{1+\frac{8\gamma}{\beta}}}{8},\frac{9-\sqrt{1+\frac{8\gamma}{\beta}}}{8};\frac{7}{4};y\right)+\right.$$

$$c_2\left\{{}_2F_1\left(\frac{3+\sqrt{1+\frac{8\gamma}{\beta}}}{8},\frac{3-\sqrt{1+\frac{8\gamma}{\beta}}}{8};\frac{5}{4};y\right)+\frac{2\left(1-\frac{\gamma}{\beta}\right)y}{5}\right. \tag{2.3.41}$$

$$\left.\left.{}_2F_1\left(\frac{11+\sqrt{1+\frac{8\gamma}{\beta}}}{8},\frac{11-\sqrt{1+\frac{8\gamma}{\beta}}}{8};\frac{9}{4};y\right)\right\}\right\} ,$$

and

$$n_s = \bar{\eta}_0^2\sqrt{y} + \frac{2\alpha y}{3\beta}{}_2F_1\left(1,\frac{1}{4};\frac{7}{4};y\right) + 2c_3\alpha\bar{\eta}_0 y^{1/4}\sqrt{1-y} +$$

$$2c_4\alpha\bar{\eta}_0 y^{1/2}{}_2F_1\left(1,-\frac{1}{4};\frac{5}{4};y\right) \tag{2.3.42}$$

where y is given by equation (2.3.36). If we make a change to the dependent and independent variables in a similar manner as for the previous solutions, then we let $n = \hat{n}\bar{\eta}_0^2$ and $x = x_0 + \frac{\hat{x}}{\bar{\eta}_0\sqrt{|\beta|}}$. In figure II-4, we have plotted equations (2.3.42) and (2.3.41) with these substitutions for $\hat{n}$ and $\hat{x}$. The value of β was taken to be the same as that used in figures II-1 through II-3. In figure II-4, we have chosen c_1 through c_4 so that $\hat{n}$ vanishes when h is a maximum and h vanishes when $\hat{n}$ is a maximum.

The second limiting case is for large magnetic field penetration into the sample, and hence in this case we take $\eta_o = 0$ (normal phase) as the starting point, so that equations (2.3.11), (2.3.13), (2.3.14), (2.3.15), and (2.3.16) become

$$\frac{d^2u_o}{dx^2} = 0, \tag{2.3.43}$$

$$\frac{d^2u_1}{dx^2} = 0, \tag{2.3.44}$$

$$\frac{d^2\eta_1}{dx^2} = u_o^2\eta_1, \tag{2.3.45}$$

$$\frac{d^2u_2}{dx^2} = \gamma u_0\eta_1^2, \tag{2.3.46}$$

and

$$\frac{d^2\eta_2}{dx^2} = \eta_1\left[1 + 2u_0u_1\right] + u_0^2\eta_2. \tag{2.3.47}$$

The solutions to these equations are straightforward and can be written down immediately as

$$u_o = c_o x + c_1, \tag{2.3.48}$$

$$u_1 = c_2 x + c_3, \tag{2.3.49}$$

$$\eta_1 = \sqrt{|u_0|}\left\{c_4 I_{\frac{1}{4}}\left(\frac{u_0^2}{2c_0}\right) + c_5 K_{\frac{1}{4}}\left(\frac{u_0^2}{2c_0}\right)\right\}, \tag{2.3.50}$$

and

$$u_2 = c_6 x + c_7 + \gamma\int^x dy\int^y dz\, u_0(z)\eta_1^2(z). \tag{2.3.51}$$

We have not bothered to write down the solution for η_2 since we are only interested in the solution for the magnetic field h and the superconducting charge density n_s to second order in α and η_2 will not contribute to the solution until the α^3 term in the solution for the superconducting charge density. These results clearly show that the magnetic field is constant up to first order and does not start to fluctuate spatially until second order. This follows from the fact that h is proportional to a spatial derivative of u and u is a linear function of x. Therefore, c_o is proportional to the critical magnetic field h_c. Since the modified Bessel functions are monotonically decreasing and increasing functions this leads to either a clustering of electrons at the boundaries (i.e. surface superconductivity) or a clustering of electrons at the core. Thus we can write the solutions for the superconducting charge density and the magnetic field as;

$$n = \alpha^2 |u_0| \left\{ c_4 I_{\frac{1}{4}}\left(\frac{u_0^2}{2c_0}\right) + c_5 K_{\frac{1}{4}}\left(\frac{u_0^2}{2c_0}\right) \right\}^2 \tag{2.3.52}$$

and

$$h = h_c + \alpha^2 \gamma \frac{\hbar c}{e^*} \int^x dy\, u_0(y) \eta_1^2(y), \tag{2.3.53}$$

where h_c is defined as; $h_c = \frac{\hbar c}{e^*}\left\{c_0 + \alpha c_2 + \alpha^2 c_6\right\}$.

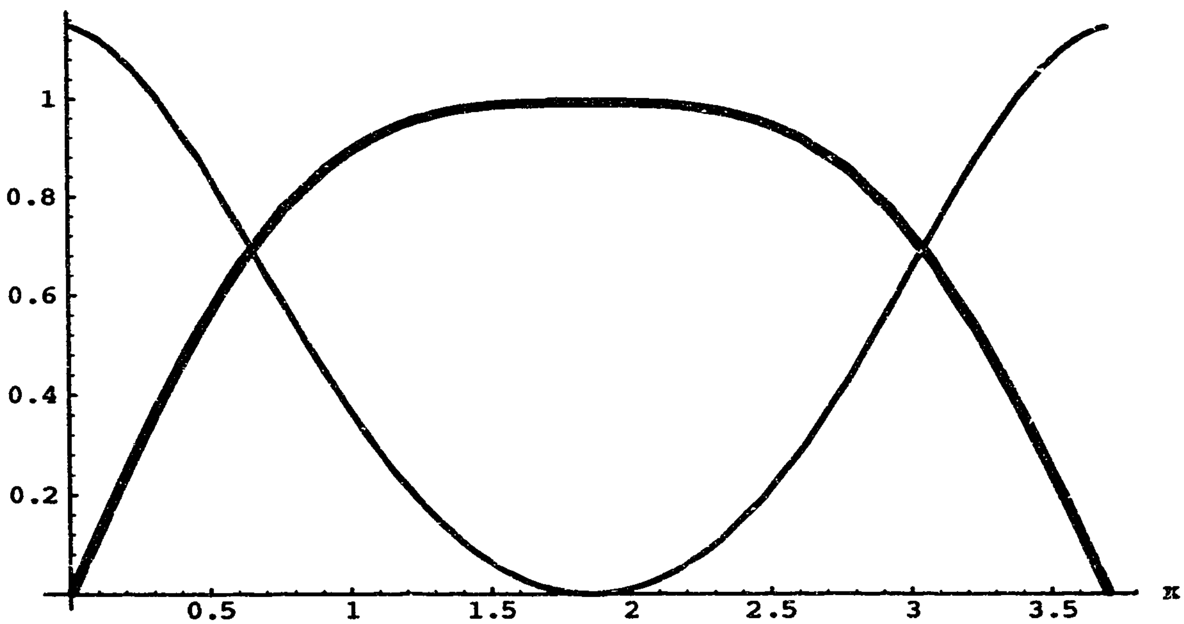

Figure II-4: Plot of the superconducting charge density represented by the thin line and the magnetic field represented by the thick line, as given by equations (2.3.42) and (2.3.41) respectively.

The solutions given by equations (2.3.52) and (2.3.53) are plotted in figure II-5 for an arbitrary choice of the parameters and the boundary conditions. For figure II-5, we define

$\hat{h} = \frac{h}{h_c}$ and $\hat{x} = \frac{c_0 x + c_1}{\sqrt{2c_0}}$. As can be seen from figure II-5, the superconducting charge is along the boundaries with a small dip in the magnetic field also occurring at the boundaries.

To summarize the results of this section, close to the critical temperature T_c, perturbative approaches starting from particular nonlinear solutions (and not linearization schemes) appear to be quite successful as they lead to physically plausible behavior of the magnetic field penetration into the regions free of the superconducting charges and, conversely, field expulsion in the regions occupied by superconducting charges. We were able to obtain three exact solutions valid at the critical temperature, two of which were in the form of periodically modulated elliptic functions with the magnetic field and the superconducting charge density phase shifted. The third solution describes surface superconductivity with the bulk of the sample residing in the normal phase.

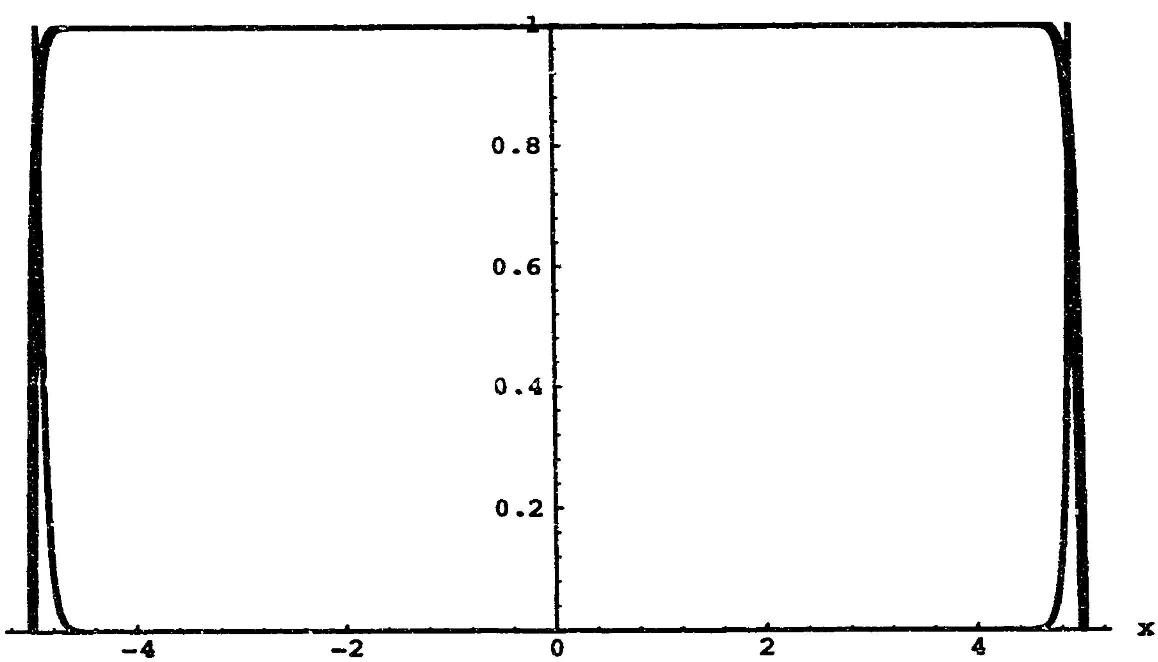

Figure II-5: Plot of the superconducting charge density represented by the thin line and the magnetic field represented by the thick line, as given by equations (2.3.52) and (2.3.53) respectively.

The first two periodic solutions appear, on a macroscopic scale, to be in a normal phase if the net effect was observed at the critical temperature ($\alpha = 0$). As the temperature is lowered below the critical temperature, a gradual overlap between neighboring maxima develops as a result of first and higher order corrections. This may not be clear from figures II-2 through II-4, since for these figures we either ignored the homogeneous solution for the first order corrections (see figures II-2 and II-3) or we had chosen the boundary condition so that the superconducting charge density vanished when the magnetic field was a maximum. If we had not done this we would obtain an overlap between the nucleation centers of the superconducting charge. For example, in equation (2.3.27), we have the term $c_3\sqrt{1-y}$ which for $c_3 \neq 0$ and $\alpha \neq 0$, then the superconducting charge density would be nonzero for all space (within the boundaries). Another interesting effect caused by the homogeneous solutions for the first order corrections in equations (2.3.26) and (2.3.27) is that depending on the boundary conditions, the magnetic field and the superconducting charge density can be asymmetrical. To illustrate this, we shall show a numerical plot of this effect as shown in figure II-6 and also, show the individual contributions of each of the homogeneous solutions as given in equations (2.3.26) and (2.3.27) by figures II-7 and II-8. For figure II-6, we started with equations (2.3.4) and (2.3.5) and then rewrote them in terms of the magnetic field h and the superconducting charge density n_s, (see equations (2.4.1) and (2.4.2) in the next section,) and after rescaling the dependent and independent variables we numerically plotted the full equations. In figure II-6, we see the asymmetrical peaks in the magnetic field and the superconducting charge density, but periodic symmetry is also evident in the figure. In figures II-7 and II-8, we see that the c_1 term and the c_3 term are the ones that contribute to making the superconducting charge density nonzero throughout the sample, but it is the c_2 and c_4 terms that generate the asymmetrical features as shown by the numerical plot in figure II-6. Considering the fact that equations (2.3.26) and (2.3.27) are only to first order and the fact that they can pr' 'uce figure II-6 quite closely, which is the numerical plot of the full equations, shows the accuracy of our expansion.

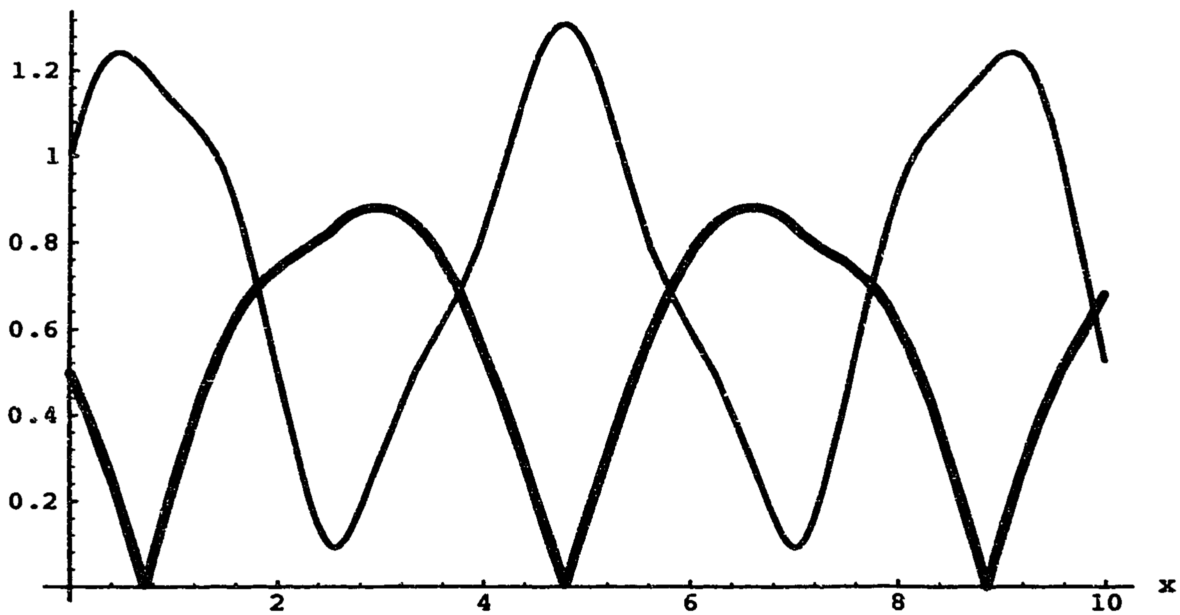

Figure II-6: Numerical plot of equations (2.4.1) and (2.4.2) for the magnetic field (thick line) and the charge density (thin line).

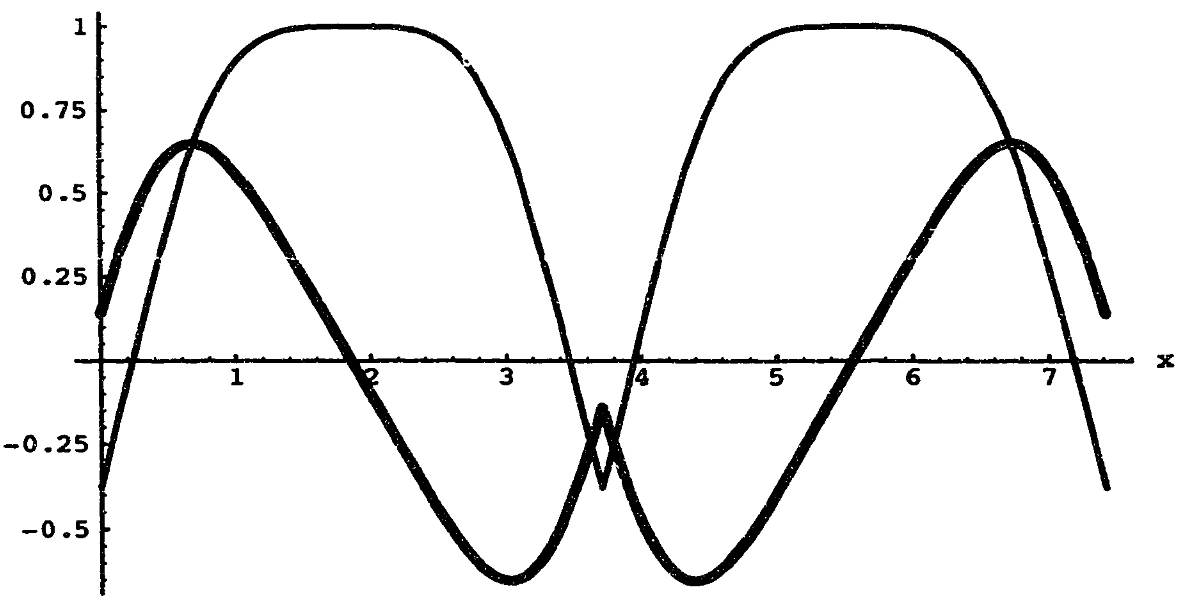

Figure II-7: Plot of equation (2.3.26), where the thick line represents the c_2 term and the thin line represents the c_1.

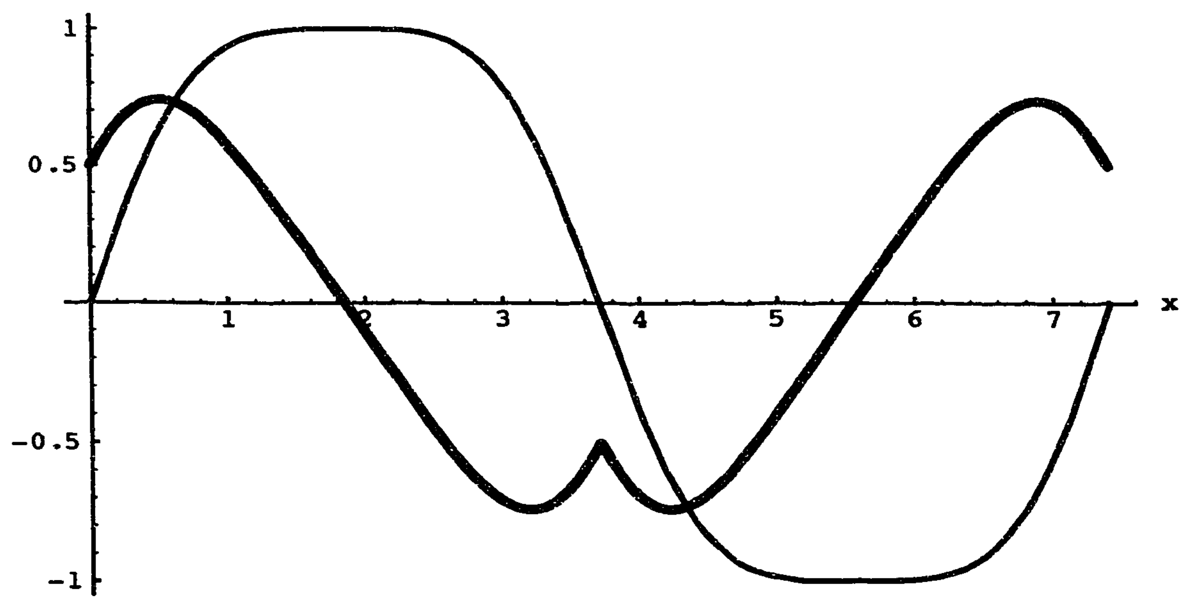

Figure II-8: Plot of equation (2.3.27), where the thick line represents the c_4 term and the thin line represents the c_3 term, while we have ignored the nonhomogeneous solution.

SECTION IV: NEGLECTING THE EFFECTS OF THE MAGNETIC FIELD UPON THE SUPERCONDUCTING CHARGE DENSITY

A. General Comments

The approximate treatment of the penetration depth problem in this section consists in making use of the exact expressions for the order parameter envelope η obtained in the absence of the magnetic field. This means, physically, that either the current (magnetic field) is very weak or that the superconducting phase is very strongly developed so that it is only slightly affected by the presence of a moderate strength current (field). For this we shall write the equations (2.2.1) and (2.2.2) in terms of the magnetic field **h** and superconducting charge density n_s:

$$\frac{\nabla^2 \mathbf{h}}{n_s} + \frac{\nabla n_s \times (\nabla \times \mathbf{h})}{n_s^2} - \gamma \mathbf{h} = 0 \qquad (2.4.1)$$

and

$$\nabla^2 n_s - \frac{(\nabla n_s)^2}{2n_s} - 2n_s\left\{\beta n_s + \alpha + \left(\frac{m^* c(\nabla \times h)}{4\pi e^* \hbar n_s}\right)^2\right\} = 0, \tag{2.4.2}$$

where $\alpha = 2m^*a/\hbar^2$, $\beta = 2m^*b/\hbar^2$, and $\gamma = 4\pi(e^*)^2/(m^*c^2)$. The following Maxwell's equation must be satisfied as well:

$$\nabla \bullet \mathbf{h} = 0. \tag{2.4.3}$$

Under the above approximation, equation (2.3.3) becomes

$$\nabla^2 n_s - \frac{(\nabla n_s)^2}{2n_s} - 2n_s\left\{\alpha + \beta n_s + \left\{\frac{m^*}{e^* \hbar}\frac{\mathbf{j}_s}{n_s}\right\}^2\right\} = 0 \tag{2.4.4}$$

and is independent of the magnetic field with the superconducting current density $\mathbf{j}_s$ considered to be either zero or a nonzero constant to zeroth order. In the first instance we shall discuss the influence of the constant current density $\mathbf{j}_s$ on the constant solutions. This means that the superconducting charge density $n_s = \eta^2$ satisfies the cubic equation

$$\beta n_s^3 + \alpha n_s^2 + \left\{\frac{m^*}{e^* \hbar}\mathbf{j}_s\right\}^2 = 0. \tag{2.4.5}$$

Note that if $j_s^2 < j_c^2 = -8(e^*)^2 a^3 / (27m^* b^2)$, then we have three real and unequal roots in equation (2.4.5), whereas if $j_s^2 = j_c^2$, then all the roots are real and two of the roots are equal. Otherwise, if $j_s^2 > j_c^2$, then there is only one real root. The equality given above gives a possible criterion for the critical current and shows that the square of the critical current is proportional to the cube power of the critical temperature; i.e., j_c scales as $(T - T_c)^{3/2}$, which is a standard relationship. For a more detailed discussion of this derivation, the reader is referred to reference 7. In the limit of $j_s \rightarrow 0$, we recover the standard result for $n_s(T)$ as outlined in the Introduction of this chapter.

B. Approximate Treatment

When constant solutions are physically inappropriate, for example, because of the very short coherence length, as is the case with high temperature superconductors, we need to look at the nonconstant solutions and to do this we shall assume that $\mathbf{h} = h(x)\hat{y}$ and $n_s = n_s(x)$ for convenience, where the superconducting charge density n_s is taken to be the solution of the equation[7] (2.4.4), so that equation (2.4.1) becomes

$$\frac{d^2h}{dx^2} = \frac{1}{n_s}\frac{dn_s}{dx}\frac{dh}{dx} + \gamma h n_s. \tag{2.4.6}$$

Equation (2.4.6) can be rewritten as

$$\frac{d^2h}{ds^2} = \frac{\gamma h}{n_s}, \tag{2.4.7}$$

where an independent variable has been introduced as $s = \int dx\, n_s$ and n_s is one of the solutions of equation (2.4.4) as given in the paper by Tuszyński and Dixon[7]. Looking at equation (2.4.7) we see that the effective mass m^* under these approximations must be greater than zero since $-\gamma/n_s$ acts like an effective potential in a classical Hamiltonian with zero energy and we want the magnetic field and superconducting charge to be mutually repulsive so that we need $\gamma > 0$, which implies that $m^* > 0$. This means that for some of the solutions presented later in this section the nonlinearity coefficient b will necessarily be taken to be negative. There are two cases that have been studied in that paper[7], and they are: (i) when the current is zero and (ii) when it is a nonzero constant. However, we shall treat these conditions on the current as being approximate, which can be further improved by perturbative methods. In the following we use am, sn, cn, and dn to denote the Jacobian elliptic functions. Appendix B lists the differential equations satisfied by these functions. Also, we denote the elliptic integral of the first kind by K and the elliptic integral of the second kind by E[14]. The nonsingular solutions that were found[7] (excluding the constant solutions) will be discussed in conjunction with Table II-1. We have represented the charge density as

$$n_s = A + B\,G(Z,k), \tag{2.4.8}$$

where the constants A and B depend on the particular case given in Table II-1 and the reduced spatial variable Z is also given in Table II-1. The Jacobi modulus k is related to the form of the quartic polynomial

$$(\eta^2 - \eta_1^2)(\eta^2 - \eta_2^2) = \eta^4 + \frac{2a}{b}\eta^2 + k_2, \tag{2.4.9}$$

where k_2 is an integration constant.

TABLE II-1. Superconducting charge density forms corresponding to the nonsingular solutions of equation (2.4.4) for a variety of conditions.

Case	A	B	G	Z	k^2	β	j_s	Conditions
1	0	η_1^2	sn^2	$\eta_2(x - x_o)\sqrt{\frac{\beta}{2}}$	$\frac{\eta_1}{\eta_2}$	>0	0	$\eta_1 < \eta_2$
2	0	η_1^2	cn^2	$(x - x_o)\sqrt{\left\lvert\frac{\beta}{2}\right\rvert(\eta_1^2 + \lvert\eta_2\rvert^2)}$	$\frac{\eta_1^2}{\eta_1^2 + \lvert\eta_2\rvert^2}$	<0	0	$\eta_2 \in C$
3	0	η_1^2	dn^2	$(x - x_o)\eta_1\sqrt{\left\lvert\frac{\beta}{2}\right\rvert}$	$1 - \left\{\frac{\eta_2}{\eta_1}\right\}^2$	<0	0	$\eta_1 > \eta_2$
4	c_3	$c_2 - c_3$	sn^2	$(x - x_o)\sqrt{\frac{\beta}{2}(c_1 - c_3)}$	$\frac{c_2 - c_3}{c_1 - c_3}$	>0	$\neq 0$	$c_1 > c_2 \geq n > c_3$
5	c_1	$c_2 - c_1$	sn^2	$(x - x_o)\sqrt{\frac{\beta}{2}(c_3 - c_1)}$	$\frac{c_1 - c_2}{c_1 - c_3}$	<0	$\neq 0$	$c_1 > n \geq c_2 > c_3$
6	c_1	$c_2 - c_1$	$\tanh^2$	$(x - x_o)\sqrt{\frac{\beta}{2}(c_2 - c_1)}$		>0	$\neq 0$	$c_3 = c_2 \geq n > c_1$

The coefficient $c_1, c_2,$ and c_3 appearing in Table II-1 are defined through the cubic polynomial;

$$(n_s - c_1)(n_s - c_2)(n_s - c_3) = n_s^3 + \frac{2a}{b} n_s^2 + k_2 n_s - \frac{m^* j_s^2}{b(e^*)^2}. \qquad (2.4.10)$$

The first three solutions in Table II-1 are valid for the case with no superconducting currents and the rest have a nonzero constant current. The first five solutions in Table II-1 are oscillatory, and their period of spatial oscillation is uniquely determined by the value k. In the limit when $k \to 1$, the period tends to infinity and cases 2 and 3 become sech^2 whereas cases 1, 4, and 5 go into case 6 with an appropriate change of constants.

In equation (2.4.7) we have defined an independent variable s as an indefinite space integral over the superconducting charge density n_s. Table II-2 lists the results of this type of integration for the cases given in Table II-1. Note that the variable Z is expressed in terms of n_s, where A_o, A_1, and A_2 are given in columns for each case.

TABLE II-2. Expressions for $s = \int dx\, n_s$ which were obtained using the results for n_s from Table II-1.

Case	s	A_0	A_1	A_2	Z	k^2
1	$\frac{A_1}{A_2 k^2}(F[Z;k] - E[Z;k])$	0	η_1^2	$\eta_2\sqrt{\frac{\beta}{2}}$	$\sin^{-1}\sqrt{\frac{n_s}{A_1}}$	$\frac{\eta_1}{\eta_2}$
2	$\frac{A_1}{A_2 k^2}(E[Z;k] - (1-k^2)F[Z;k])$	0	η_1^2	$\sqrt{\left\|\frac{\beta}{2}\right\|\left[\eta_1^2 + \|\eta_2\|^2\right]}$	$\cos^{-1}\sqrt{\frac{n_s}{A_1}}$	$\frac{\eta_1^2}{\eta_1^2 + \|\eta_2\|^2}$
3	$\frac{A_1}{A_2}E(Z;k)$	0	η_1^2	$\eta_1\sqrt{\left\|\frac{\beta}{2}\right\|}$	$\sin^{-1}\sqrt{\frac{1-n_s A_1}{k^2}}$	$1 - \frac{\eta_2^2}{\eta_1^2}$
4	$\frac{A_1}{A_2 k^2}\left\{\left(1+\frac{k^2 A_o}{A_1}\right)F[Z;k] - E[Z;k]\right\}$	c_3	$c_2 - c_3$	$\sqrt{\frac{\beta}{2}(c_1 - c_3)}$	$\sin^{-1}\sqrt{\frac{n_s - A_o}{A_1}}$	$\frac{c_2 - c_3}{c_1 - c_3}$
5	same as 4	c_1	$c_2 - c_1$	$\sqrt{\left\|\frac{\beta}{2}\right\|(c_1 - c_3)}$	$\sin^{-1}\sqrt{\frac{n_s - A_o}{A_1}}$	$\frac{c_1 - c_2}{c_1 - c_3}$
6	$\frac{A_o + A_1}{2A_2}\ln\left\|\frac{1+\sqrt{Z}}{1-\sqrt{Z}}\right\| - \frac{A_1}{A_2}\sqrt{Z}$	c_1	$c_2 - c_1$	$\sqrt{\frac{\beta}{2}(c_2 - c_1)}$	$\frac{n_s - A_o}{A_1}$	
7	$\frac{A_1}{A_2}\sqrt{1-Z}$	0	η_1^2	$\sqrt{\left\|\frac{\beta}{2}\right\|}\eta_1$	$\frac{n_s}{A_1}$	1

The squared elliptic modulus k^2 is also provided for each solution in Table II-2 except in case 6, where k is 1 when taking the limit in cases 1, 4, and 5 given in Table II-1. The expressions for s in Table II-2 give us an implicit form for n_s in terms of s which we need to substitute into equation (2.4.7). It should be noted that in case 3 in Table II-2 when k $\rightarrow$ 0, we obtain the special case of n_s equal to -a/b, which is our standard solution as outlined in the introduction of this chapter. For cases 1, 2, 4, and 5 in Table II-1 when k $\rightarrow$ 0, we obtain trigonometric functions instead of elliptic functions, however, the coefficient vanishes, which means that n_s goes to a constant or to zero. When k $\rightarrow$ 1, cases 1, 4, and 5 degenerate into the form given in case 6 with appropriate changes in the constants A_o, A_1, and A_2. Similarly, examples 2 and 3, in the limit k $\rightarrow$ 1, transform into the situation as given in case 7.

The only nontrivial solution that can be inverted to obtain n_s as a function of s, in equation (2.4.7), without approximations, is given by case 7 in Table II-2, which results in;

$$n_s = A_1 - \frac{(A_2)^2 s^2}{A_1}, \tag{2.4.11}$$

so that equation (2.4.7) becomes

$$\frac{d^2h}{ds^2} = \frac{\gamma A_1 h}{A_1^2 - A_2^2 s^2}. \tag{2.4.12}$$

Equation (2.4.12) is solved in terms of the variable s, which we then convert back to our original independent variablex, so that we obtain;

$$h = \mathrm{sech}(A_2 x)\left\{A P_\mu^1 |\tanh(A_2 x)| + B Q_\mu^1 |\tanh(A_2 x)|\right\}, \tag{2.4.13}$$

where $\mu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \gamma\left\{\frac{A_1}{A_2}\right\}^2}$. Also, A and B are constants satisfying the boundary conditions. In equation (2.4.13), P_μ^ν and Q_μ^ν denote the associated Legendre functions of the first and second kind respectively. This solution corresponds to the limit of $k \to 1$ in both cases 2 and 3 of Table II-1. In cases 2 and 3 when $k \to 1$, we have $\eta_2 = 0$ and the other constant from equation (2.4.9) is $\eta_1 = \sqrt{-2a/b}$ along with $k_2 = 0$ in equation (2.4.9). The signs of our three parameters in this case are: $a > 0$, $b < 0$, and $m^* > 0$. Writing our solutions for the superconducting charge density n_s and the magnetic field h in terms of our original parameters, we obtain;

$$n_s = \frac{-2a}{b}\operatorname{sech}^2\left((x - x_0)\frac{\sqrt{2am^*}}{\hbar}\right) \tag{2.4.14}$$

and

$$h = \operatorname{sech}\left((x - x_0)\frac{\sqrt{2am^*}}{\hbar}\right)\left\{A\, P_\mu^1\left|\tanh\left((x - x_0)\frac{\sqrt{2am^*}}{\hbar}\right)\right| + B\, Q_\mu^1\left|\tanh\left((x - x_0)\frac{\sqrt{2am^*}}{\hbar}\right)\right|\right\}, \tag{2.4.15}$$

where $\mu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\pi a}{2}\left(\frac{4e^*\hbar}{bm^*c}\right)^2}$. Physically, we can interpret this sech like solution as indicating a localized cluster of superconducting electrons along the x axis. It can be noted that this may be a picture describing a two dimensional superconductor. The effect of the Legendre function modulation on the sech "bump" is to decrease markedly its amplitude in the region where the superconducting order parameter is a maximum. If we let $n_s = \frac{-2a\hat{n}}{b}$ and $x = x_0 + \frac{\hat{x}\hbar}{\sqrt{2am^*}}$ then in figure II-9 we have plotted equations (2.4.14) and (2.4.15) using $\hat{x}$, $\hat{n}$, and h. Figure II-9 clearly shows the nucleation of the superconducting charge density. The constants A and B were chosen so that the magnetic field h vanishes at $\hat{x} = 0$ for a specific μ which we have chosen to be -1/4.

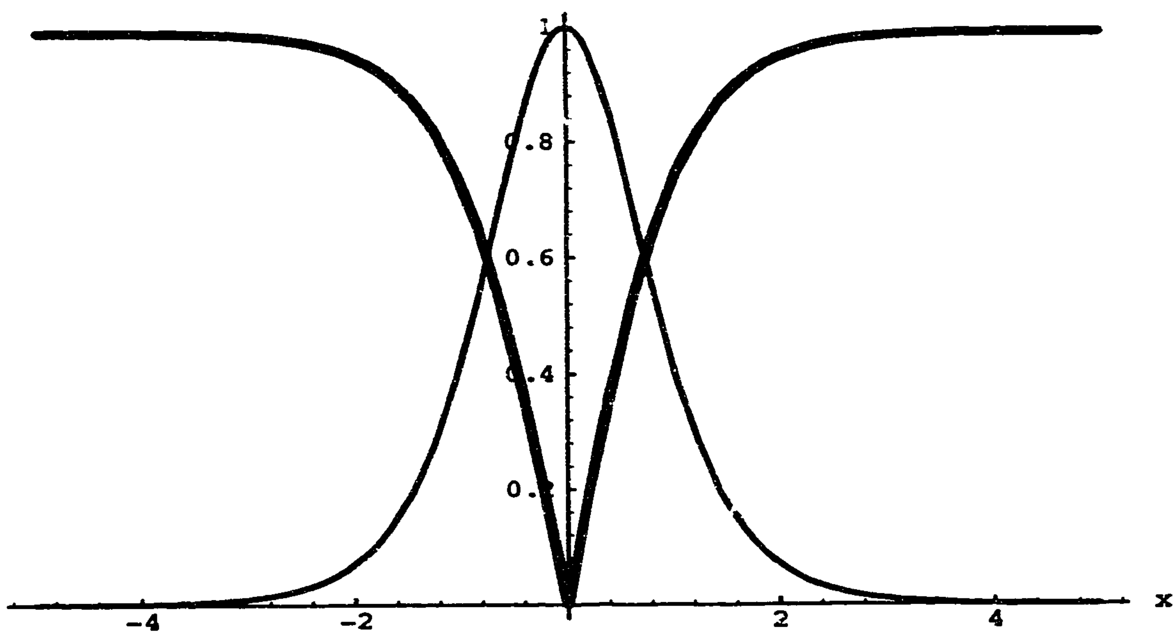

Figure II-9: Plot of the superconducting charge density represented by the thin line and the magnetic field represented by the thick line, as given by equations (2.4.14) and (2.4.15) respectively.

Next we wish to look at the other cases and since they can not be inverted exactly, we will approximate the relationship between n_s and s for small values of the superconducting charge density n_s. The first case we consider is case 6 from Table II-2, here we obtain, approximately,

$$n_s \approx \begin{cases} A_0 + A_1\left(\dfrac{A_2 s}{A_0}\right)^2 & \text{, if } A_0 \neq 0, \\ A_1\left(\dfrac{3A_2 s}{A_1}\right)^{2/3} & \text{, if } A_0 = 0. \end{cases} \tag{2.4.16}$$

The solution of equation (2.4.7) in this instance takes the form

$$h = \begin{cases} \sqrt{1+\dfrac{A_1 q^2}{A_0}}\left\{b_1 P_\mu^1\left(|q|\sqrt{\dfrac{-A_1}{A_0}}\right) + b_2 Q_\mu^1\left(|q|\sqrt{\dfrac{-A_1}{A_0}}\right)\right\} & \text{, if } A_0 \neq 0, \\ |q|^{3/2}\left[b_3 I_{3/4}(\delta q^2) + b_4 K_{3/4}(\delta q^2)\right] & \text{, if } A_0 = 0, \end{cases} \tag{2.4.17}$$

where $I_{3/4}$ and $K_{3/4}$ refer to modified Bessel functions and

$$q = \tanh(A_2 x), \text{ and } \delta^2 = \frac{\gamma A_1}{4A_2^2}, \tag{2.4.18}$$

with $\mu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \left(\frac{2\delta A_o}{A_1}\right)^2}$.

Now, we wish to write the solution in terms of our original parameters and to do this we must look at each case that is applicable separately. That is we need to look at cases 1, 4, 5, and 6 in Table II-1 with $k \rightarrow 1$ in cases 1, 4, and 5. Starting with case 1 in Table II-1 (with $k \rightarrow 1$), we obtain the following constraints on the parameters: $a < 0$, $b > 0$, $m^* > 0$, and $j_s = 0$, so that the superconducting charge density n_s and the magnetic field h become;

$$n_s = \frac{-a}{b} q^2 \tag{2.4.19}$$

and

$$h = |q|^{3/2}\left[b_3 I_{\frac{3}{4}}\left(\delta q^2\right) + b_4 K_{\frac{3}{4}}\left(\delta q^2\right)\right], \tag{2.4.20}$$

where $q = \tanh\left\{(x - x_0)\frac{\sqrt{-am^*}}{\hbar}\right\}$ and $\delta^2 = \frac{\pi}{b}\left\{\frac{e^* \hbar}{m^* c}\right\}^2$. By introducing $\hat{x} = \frac{\sqrt{-am^*}}{\hbar}(x-x_0)$ and $\hat{n} = -bn_s/a$, we can plot equations (2.4.19) and (2.4.20) in figure II-10 for an arbitrary choice of boundary conditions. The solutions given by equations (2.4.19) and (2.4.20) show the coherence length and penetration depth quite clearly. We see that the coherence length is given by $\xi = \hbar\,(-am^*)^{-1/2}$ which is $\sqrt{2}$ larger than the standard definition for the coherence length as discussed in section I. The penetration depth on the other hand is proportional to the inverse of δ. The larger δ becomes the faster the magnetic field drops to zero (see figure II-10), but since δ is a dimensionless parameter, it is probably proportional to the ratio of the coherence length and the penetration depth. If this ratio is exactly equal to δ then the penetration depth λ is ξ/δ or $(bm^*c^2/(2\pi(e^*)^2|a|))^{1/2}$, which is $\sqrt{2}$ times larger than the penetration depth that was defined in section I and corresponds to the $\sqrt{2}$ that the coherence length is larger.

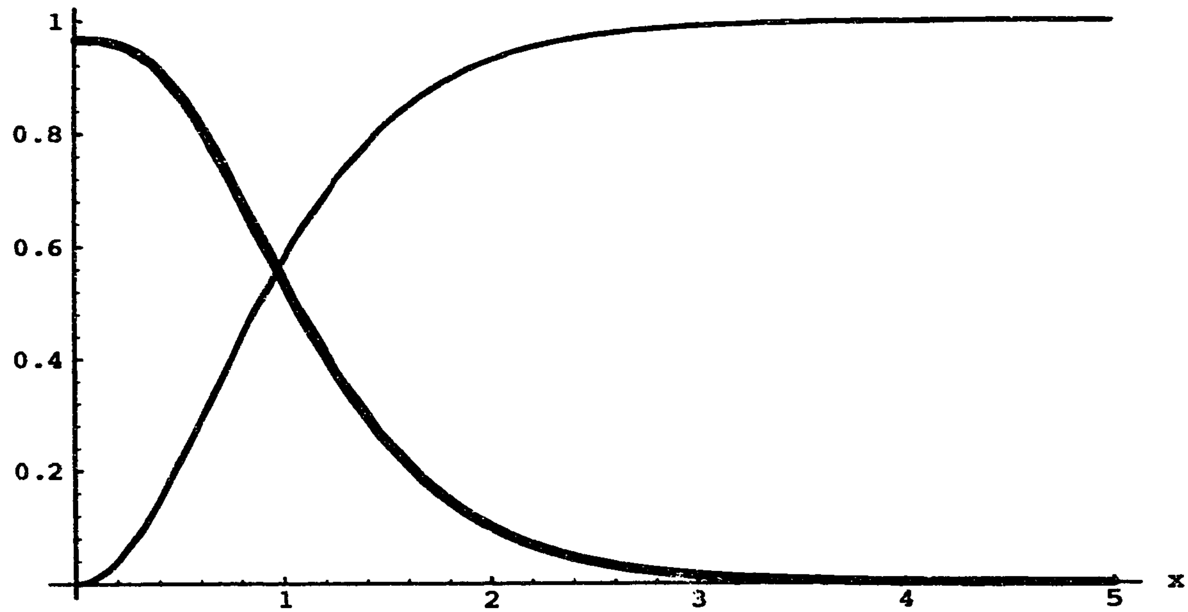

Figure II-10: Plot of the superconducting charge density represented by the thin line and the magnetic field represented by the thick line, as given by equations (2.4.19) and (2.4.20) respectively.

For case 4 from Table II-1, it is equivalent to case 6 from Table II-1 when $k \rightarrow 1$ and with the change $c_1 \leftrightarrow c_3$. In this situation the parameters satisfy the following inequalities; $a < 0$, $b > 0$, and $m^* > 0$. We also obtain a relationship a relationship between the current and the integration constant k_2 which is;

$$j_s^2 = \frac{2(e^*)^2}{27b^2m^*}\left[8a^3 - 9ab^2k_2 - \left(4a^2 - 3b^2k_2\right)^{3/2}\right]$$

By demanding j_s real and $n_s > 0$, we obtain the inequality $1 \leq (b/a)^2k_2 \leq 4/3$. The equality 4/3 leads to n_s being a constant which is not of much interest and the equality of 1 leads to the same solutions as given by case 1 of Table II-1 and thus are given by equations (2.4.19) and (2.4.20). Recall also figure II-10 which plots this set of solutions and the current j_s is zero. For k_2 between these values, the current j_s is real and n_s is positive and real, however, the solution for the magnitude of the magnetic field, i.e. the associated Legendre polynomials have pure imaginary arguments and thus are not allowed. Thus, the only solutions of interest is when $k_2 = (a/b)^2$ and the solutions are given by equations (2.4.19) and (2.4.20).

We have one last case to look at and that is case 5 when $k \to 1$. For this case, our parameters satisfy the following inequalities; $a > 0$, $b < 0$, and $m^* > 0$. The superconducting current j_s has the same form as given just above, which means that it is real only for $k_2 = 0$ which gives $j_s = 0$. Upon substitution of these values into the solutions for n_s and h, we obtain equations (2.4.14) and (2.4.15) as our solutions with the only change being the value of $\mu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{\pi}{b}\left\{\frac{2e^* \hbar}{m^* c}\right\}^2}$, with the corresponding plot of the solutions given by figure II-9. This concludes the limiting cases for $k \to 0$ and 1, however, as of yet we have not done an analysis for arbitrary k.

C. Exact solutions of equation (2.4.7)

In order to generate solutions of equation (2.4.7), we first substitute $n_s = \eta^2$ in equation (2.4.4). When this equation describes a one dimensional situation, it can then be integrated once and rewritten in terms of n_s again to yield

$$\left(\frac{dn_s}{dx}\right)^2 = 4n_s^2\left\{\alpha + \frac{\beta n_s}{2} - \frac{j_o^2}{n_s^2}\right\} + 4b_o n_s, \tag{2.4.21}$$

where j_o^2 is defined by $j_o^2 = (m^* j_s / \hbar e^*)^2$. Differentiating equation (2.4.21) once produces

$$\frac{d^2 n_s}{dx^2} = n_s(4\alpha + 3\beta n_s) + 2b_o. \tag{2.4.22}$$

We will consider the case where $b_o = 0$ for convenience and the magnetic field **h** is along the z axis, so that $\mathbf{h} = h(n_s)\hat{\mathbf{z}}$. Under these circumstances, equation (2.4.1) then becomes:

$$\left\{n_s^3 + \frac{2\alpha}{\beta}n_s^2 - \frac{2j_o^2}{\beta}\right\}\frac{d^2h}{dn_s^2} + \left\{\frac{n_s}{2} + \frac{2j_o^2}{\beta n_s}\right\}\frac{dh}{dn_s} - \frac{\gamma n_s}{2\beta}h = 0. \tag{2.4.23}$$

Equation (2.4.23) can be solved exactly for two special cases and also the recurrence relationships for the general case will be given below.

1: $\beta = j_0 = 0$; no currents or cubic nonlinearity

The magnetic field for this situation takes the following form when $\alpha > 0$:

$$h = h_o\sqrt{n_s}\left\{d_o I_1\sqrt{\frac{\gamma n_s}{\alpha}} + d_1 K_1\sqrt{\frac{\gamma n_s}{\alpha}}\right\}. \tag{2.4.24}$$

Here, $h_o, d_o,$ and d_1 are constants and I_1 and K_1 are modified Bessel functions. Although K_1 diverges near the origin, the $\sqrt{n_s}$ prefactor ensures that the solutions are nonsingular. The case when $\alpha < 0$ leads to imaginary values of n_s and thus was dropped as unphysical. The solution for the superconducting charge density n_s, from equation (2.4.21) is;

$$n_s = \exp\left[\pm 2\sqrt{\alpha}(x - x_0)\right]. \tag{2.4.25}$$

Figure II-11 illustrates the form of the solutions as given by equations (2.4.24) and (2.4.25). Again, we see that $(\alpha)^{-1/2}$ is proportional to the coherence length in agreement with the definition in section I and from equation (2.4.24) we see that the penetration depth is proportional to $(\gamma/\alpha)^{-1/2}$ which has the same temperature dependence as the penetration depth as given in section I.

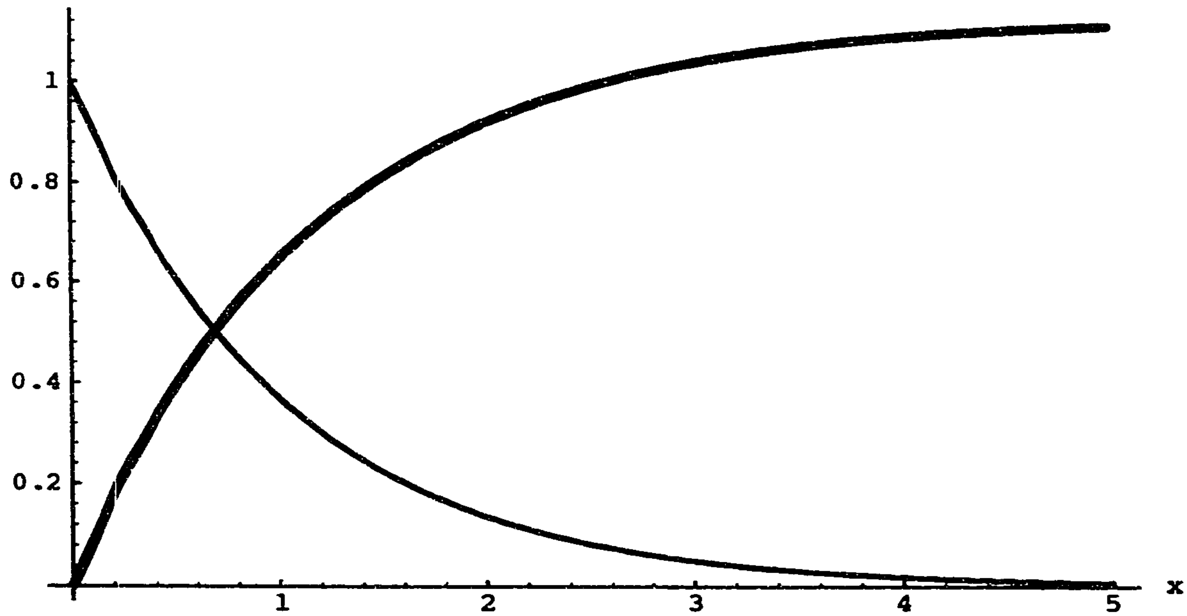

Figure II-11: Plot of the magnetic field represented by the thick line and the superconducting charge density represented by the thin line, as given by equations (2.4.24) and (2.4.25) respectively.

2: $j_0 = 0$; no currents

For this case, equation (2.4.23) becomes

$$n_s\left\{n_s + \frac{2\alpha}{\beta}\right\}\frac{d^2h}{dn_s^2} + \frac{n_s}{2}\frac{dh}{dn_s} - \frac{\gamma h}{2} = 0. \tag{2.4.26}$$

On making the substitutions $h = -(2\alpha/\beta)q\,\omega$ and $n_s = -(2\alpha/\beta)q$ in equation (2.4.26), we find

$$q(1-q)\frac{d^2\omega}{dq^2} + \left(2 - \frac{5q}{2}\right)\frac{d\omega}{dq} - \omega\left\{\frac{1}{2} - \frac{\gamma}{2\beta}\right\} = 0. \tag{2.4.27}$$

Equation (2.4.27) is an example of Gauss' hypergeometric equation, the solution of which is

$$\omega = \left\{c_0 + \frac{\gamma c_1}{2\beta}\left(1 - \ln|q| - \frac{\beta}{\gamma}\right)\right\}{}_2F_1\left(\frac{3}{4} + \sqrt{\frac{\gamma}{2\beta} + \frac{1}{16}}, \frac{3}{4} - \sqrt{\frac{\gamma}{2\beta} + \frac{1}{16}}; 2; q\right) + \frac{c_1}{q} - \frac{\gamma c_1}{2\beta}\lim_{s \to -1}\frac{\partial}{\partial s}{}_3F_2\left(s + \frac{7}{4} + \sqrt{\frac{\gamma}{2\beta} + \frac{1}{16}}, s + \frac{7}{4} - \sqrt{\frac{\gamma}{2\beta} + \frac{1}{16}}, 1; 2 + s, 3 + s; q\right). \tag{2.4.28}$$

Thus, our solutions for the superconducting charge density and the magnetic field are as follows:

$$n_s = \frac{-2\alpha q}{\beta} \tag{2.4.29}$$

and

$$h = \frac{-2\alpha q}{\beta}\left\{c_0\,{}_2F_1\left(\frac{3}{4} + \sqrt{\frac{\gamma}{2\beta} + \frac{1}{16}}, \frac{3}{4} - \sqrt{\frac{\gamma}{2\beta} + \frac{1}{16}}; 2; q\right) + c_1 \lim_{s \to -1}\frac{\partial}{\partial s}\left\{(s+1)q^s\,{}_3F_2\left(1, s + \frac{3}{4} + \sqrt{\frac{\gamma}{2\beta} + \frac{1}{16}}, s + \frac{3}{4} - \sqrt{\frac{\gamma}{2\beta} + \frac{1}{16}}; s+1, s+2; q\right)\right\}\right\}, \tag{2.4.30}$$

where

$$q = \begin{cases} \operatorname{sech}^2\left[(x - x_0)\sqrt{\alpha}\right] & \text{if } \alpha > 0 \text{ and } \beta < 0 \\ \sec^2\left[(x - x_0)\sqrt{-\alpha}\right] & \text{if } \alpha < 0 \text{ and } \beta > 0 \end{cases}. \tag{2.4.31}$$

In the special case where the curly bracket in equation (2.4.27) vanishes or

$$b = 2\pi\left\{\frac{\hbar e^*}{m^* c}\right\}^2, \tag{2.4.32}$$

the solution for the magnetic field h, as given by equation (2.4.30), reduces to the form;

$$h = \frac{-2\alpha q}{\beta}\begin{cases} d_0 - d_1\left(\sqrt{\alpha}(x - x_0) + \dfrac{\sinh\left[2\sqrt{\alpha}(x - x_0)\right]}{2}\right), & \text{if } \alpha > 0 \\ d_0 + d_1\left(\sqrt{-\alpha}(x - x_0) + \dfrac{\sin\left[2\sqrt{-\alpha}(x - x_0)\right]}{2}\right), & \text{if } \alpha < 0 \end{cases}. \tag{2.4.33}$$

If we introduce $\hat{x} = \sqrt{\alpha}(x-x_0)$ and $\hat{n} = q$, then plotting equations (2.4.29) and (2.4.33) for $\alpha > 0$, we obtain figure II-12. We have plotted equation (2.4.33) instead of equation (2.4.30) since the important term has the coefficient c_1 and is very difficult to obtain a closed form expression. However as can be seen from figure II-12, equation (2.4.33) is a good substitute even though it is for a specific value of b. It is interesting to note that the magnetic field has an initial hump at $\hat{x} \sim 1.25$ and then decays to its asymptotic value as $\hat{x} \rightarrow \infty$. The hump is due to the expulsion of the field at the origin and the field is circumventing the region at the core.

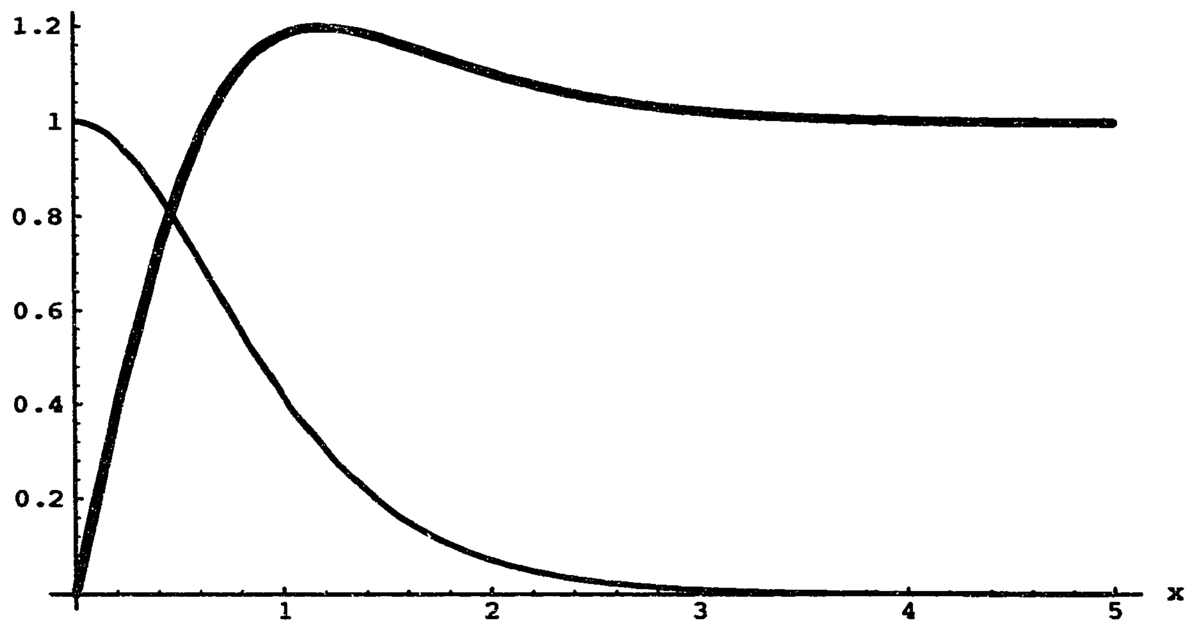

Figure II-12: Plot of the superconducting charge density represented by the thin line and the magnetic field represented by the thick line, as given by equations (2.4.29) and (2.4.33) respectively, for $\alpha > 0$ and $\beta < 0$.

3: The general case

For this situation we use a power series solution for the magnetic field h, given by;

$$h = A_0q^2 + A_1q^3 + \sum_{k=2}^{\infty}\left(A_kq^2 + B_k\right)q^k, \tag{2.4.34}$$

where the recurrence formulas for the A_k's and B_k's are given by;

$$A_k = \frac{\varepsilon_\alpha\left\{\delta - 3/2 + 5k/2 - k^2\right\}}{j_1^2k(k+2)}A_{k-3} + \frac{\varepsilon_\alpha(k-1)}{j_1^2(k+2)}A_{k-2} + \frac{b_1(k+1)(k-1/2)}{4j_1^2k(k+2)}A_{k-1}, \quad k \geq 3 \tag{2.4.35}$$

and

$$B_k = \frac{\varepsilon_\alpha\left\{\delta - 21/2 + 13k/2 - k^2\right\}}{j_1^2k(k-2)}B_{k-3} + \frac{\varepsilon_\alpha(k-3)}{j_1^2k}B_{k-2} + \frac{b_1(k-1)(k-5/2)}{4j_1^2k(k-2)}B_{k-1}, \quad k \geq 3 \quad , \tag{2.4.36}$$

where A_0 and B_2 are arbitrary, $A_1 = \dfrac{b_1A_0}{12j_1^2}$, $A_2 = \dfrac{A_0}{j_1^2}\left\{\varepsilon_\alpha + \dfrac{3b_1^2}{256j_1^2}\right\}$, and $B_0 = B_1 = 0$.

The parameters and the independent variable are related to those given at the beginning of the section as follows: $\varepsilon_\alpha = 1$ if $\alpha > 0$ and $\varepsilon_\alpha = -1$ if $\alpha < 0$, $\delta = \gamma/(2\beta)$, $b_1 = -2\beta b_0\varepsilon_\alpha(\alpha)^{-2}$, $y = \sqrt{|\alpha|}\,x$, and $n_s = -2\alpha q(y)/\beta$ where q(y) satisfies the following first order differential equation;

$$(q')^2 = 4\varepsilon_\alpha q^2(1-q) + b_1q - 4j_1^2. \tag{2.4.37}$$

As can be seen, the recursion relationships given by equations (2.4.35) and (2.4.36) do not apply when the current is zero and so when the current is zero, the power series solution and the recursion relationships become;

$$h = B_0 + A_0q^{3/2} - \frac{6\varepsilon_\alpha A_0}{5b_1}q^{5/2} + \sum_{k=2}^{\infty}\left(A_kq^{3/2} + B_k\right)q^k, \tag{2.4.38}$$

$$A_k = \frac{4\varepsilon_\alpha\left\{\left(k^2 - 3k/2 + 1/2 - \delta\right)A_{k-2} - \left(k^2 - 1/4\right)A_{k-1}\right\}}{b_1 k(k + 3/2)}, \; k \geq 2, \tag{2.4.39}$$

and

$$B_k = \frac{4\varepsilon_\alpha\left\{\left(k^2 - 9k/2 + 5 - \delta\right)B_{k-2} - (k-1)(k-2)B_{k-1}\right\}}{b_1 k(k - 3/2)}, \; k \geq 2. \tag{2.4.40}$$

Comparing equations (2.4.35) and (2.4.36) with equations (2.4.39) and (2.4.40), we see that there is no continuous way in which to go from a constant current to a zero current. This is also true for b_1 when the current is zero but not when the current is a nonzero constant.

In all three cases, the region of validity of the obtained solutions must be determined by first checking that they indeed satisfy equation (2.4.21) and second by finding the parameter ranges for which the Maxwell equation $\mathbf{j} = (c/4\pi)\nabla \times \mathbf{h}$ holds, i.e., where $h' = (4\pi/c)j_0 =$ constant and in cases (i) and (ii) the constant is identically zero. This is also true for part B.

To summarize the results of this section, a number of interesting results were obtained based on the assumption that the magnetic field may have a relatively small influence on the order parameter, provided h is small enough. An approximate treatment was presented where a linear ODE for the magnetic field was derived with a new independent variable representing an integral transform of the superconducting charge density. Using order parameter patterns found earlier in reference 7, in the absence of magnetic fields, analytical forms for the reduced variable s and the effective potential in the equation for the magnetic field were obtained. In several special cases, closed form results for the magnetic field were derived while the remaining cases are tractable numerically. In section IV C an analysis was provided which relies on particular exact solutions for superconducting charge density n_s, and the resultant forms for the magnetic field h were listed for two special cases. A solution in the general case was also given in the form of a power series and its recursion relationships. Also, in this section, the penetration depth and the coherence length that were given in section I come out quite naturally. This is not surprising since previous work also ignored the effects of the magnetic field upon the superconducting charge density, but they then went on to linearize or consider the superconducting charge density a constant.

SECTION V: TWO DIMENSIONS; VORTICES AND SPIRALS

A. Vortices

In part A of this section we shall look at the equations in cylindrical coordinates, but shall assume that all dependent variables are functions of only the radial component and that the scaled velocity is given by $\mathbf{u} = u(r)\hat{\phi}$, so that equations (2.2.5) and (2.2.6) become

$$\frac{d}{dr}\left\{\frac{1}{r}\frac{d(ru)}{dr}\right\} = \gamma\eta^2 u, \tag{2.5.1}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\eta}{dr}\right) - \eta(\alpha + \beta\eta^2 + u^2) = 0. \tag{2.5.2}$$

If we let $u = f/r$, $\eta = g/r$, and $r^2 = R$, then our equations become;

$$\frac{d^2f}{dR^2} = \frac{\gamma g^2 f}{4R^2}, \tag{2.5.3}$$

$$\frac{d^2g}{dR^2} = \frac{g}{4R^2}\left[f^2 + \alpha R - 1 + \beta g^2\right]. \tag{2.5.4}$$

Now, if we consider the temperature to be close to the critical temperature $T = T_c$, so that we can again use the procedure outlined in section III and assume that the function f and g can be expanded in the power series

$$\begin{pmatrix} f \\ g \end{pmatrix} = \sum_{n=0}^{\infty} \alpha^n \begin{pmatrix} f_n \\ g_n \end{pmatrix}, \tag{2.5.5}$$

then equations (2.5.3) and (2.5.4) become, up to second order,

$$\frac{d^2 f_o}{dR^2} = \frac{\gamma g_o^2 f_o}{4R^2}, \tag{2.5.6}$$

$$\frac{d^2 g_o}{dR^2} = \frac{g_o}{4R^2}(f_o^2 - 1 + \beta g_o^2), \tag{2.5.7}$$

$$\frac{d^2 f_1}{dR^2} = \frac{\gamma}{4R^2}\left[g_o^2 f_1 + 2f_0 g_0 g_1\right], \tag{2.5.8}$$

$$\frac{d^2 g_1}{dR^2} = \frac{g_o}{4R^2}(2f_o f_1 + R) + \frac{g_1}{4R^2}(f_o^2 - 1 + 3\beta g_o^2), \tag{2.5.9}$$

$$\frac{d^2 f_2}{dR^2} = \frac{\gamma}{4R^2}\left\{f_0\left[2g_0 g_2 + g_1^2\right] + 2f_1 g_0 g_1 + f_2 g_0^2\right\}, \tag{2.5.10}$$

and

$$\frac{d^2 g_2}{dR^2} = \frac{g_o}{4R^2}(2f_o f_2 + f_1^2) + \frac{g_1}{4R^2}(2f_o f_1 + R + 3\beta g_o g_1) + \frac{g_2}{4R^2}(f_o^2 - 1 + 3\beta g_o^2). \tag{2.5.11}$$

There are two simple sets of solutions which can be extracted from equations (2.5.6) and (2.5.7). The first set is obtained by setting $f_o = 0$ and then calculating the corresponding g_o and subsequent higher order corrections in f and g. The second set is found by assuming that $g_o = 0$ and repeating the above procedure for f_o and higher order corrections in f and g, i.e., in equations (2.5.7) through (2.5.11).

In the first case, i.e. setting $f_o = 0$, equations (2.5.7) through (2.5.11) become

$$\frac{d^2 g_o}{dR^2} = \frac{g_o}{4R^2}(\beta g_o^2 - 1), \tag{2.5.12}$$

$$\frac{d^2 f_1}{dR^2} = \frac{\gamma}{4R^2} g_o^2 f_1, \tag{2.5.13}$$

$$\frac{d^2g_1}{dR^2} = \frac{g_o}{4R} + \frac{g_1}{4R^2}(3\beta g_o^2 - 1), \tag{2.5.14}$$

$$\frac{d^2f_2}{dR^2} = \frac{\gamma}{4R^2}(2f_1g_0g_1 + f_2g_0^2), \tag{2.5.15}$$

and

$$\frac{d^2g_2}{dR^2} = \frac{g_o}{4R^2}f_1^2 + \frac{g_1}{4R^2}(R + 3\beta g_o g_1) + \frac{g_2}{4R^2}(3\beta g_o^2 - 1). \tag{2.5.16}$$

To satisfy equation (2.5.12) we can use two constant solutions for g_o, as follows: $g_o = 0$ and $g_o = 1/\sqrt{\beta}$. Choosing $g_o = 0$, we obtain the solutions for the first order corrections, since $f_1'' = 0$ and $g_1'' = \frac{-g_1}{4R^2}$, in a straightforward manner as follows:

$$f_1 = c_oR + c_1 \tag{2.5.17}$$

and

$$g_1 = \sqrt{R}[c_2 \ln(R) + c_3]. \tag{2.5.18}$$

As a result, we can calculate the corresponding order parameter η and the scaled velocity u, recalling that $R=r^2$, as follows;

$$\eta = \frac{g}{r} = \alpha[2c_2 \ln(r) + c_3] \tag{2.5.19}$$

and

$$u = \frac{f}{r} = \frac{\alpha}{r}\left[c_0r^2 + c_1\right]. \tag{2.5.20}$$

Thus, our solutions in terms of the magnetic field h and the superconducting charge density n_s are;

$$h = \frac{e^*}{\hbar c r}\frac{df}{dr} = \frac{2e^* c_o\alpha}{\hbar c} \tag{2.5.21}$$

and

$$n_s = \left(\frac{g}{r}\right)^2 \alpha^2\left[2c_2 \ln(r) + c_3\right]^2. \tag{2.5.22}$$

In order to justify physically the coexistence of a constant magnetic field with the order parameter field, η in equation (2.5.19), it is necessary to demand that the radius is near r $\sim \exp[-c_3/(2c_2)]$ or else we need to set $c_2 = 0$ and $c_0 \sim 0$ so that we obtain magnetic expulsion by the superconducting charge.

The other constant solution for g_0 is $(\beta)^{-1/2}$ and substituting this into equations (2.5.13) and (2.5.14) we obtain;

$$\frac{d^2f_1}{dR^2} = \frac{\gamma}{4\beta R^2} f_1 \tag{2.5.23}$$

and

$$\frac{d^2g_1}{dR^2} = \frac{1}{4\sqrt{\beta}R} + \frac{g_1}{2R^2}. \tag{2.5.24}$$

These equations have the solutions;

$$f_1 = c_0 R^{\left(1+\sqrt{1+\gamma/\beta}\right)/2} + c_1 R^{\left(1-\sqrt{1+\gamma/\beta}\right)/2} \tag{2.5.25}$$

and

$$g_1 = c_2 R^{\left(1+\sqrt{3}\right)/2} + c_3 R^{\left(1-\sqrt{3}\right)/2} - \frac{R}{2\sqrt{\beta}}. \tag{2.5.26}$$

Thus, the magnetic field h and the superconducting charge density n_s to first order are;

$$h = \frac{e^*}{\hbar c r}\frac{df}{dr} = \frac{\alpha e^*}{\hbar c}\left\{c_0\left(1+\sqrt{1+\gamma/\beta}\right) r^{\sqrt{1+\gamma/\beta}-1} + \frac{c_1\left(1-\sqrt{1+\gamma/\beta}\right)}{r^{1+\sqrt{1+\gamma/\beta}}}\right\} \tag{2.5.27}$$

and

$$n_s = \left(\frac{g}{r}\right)^2 = \frac{\left(1-\alpha r^2\right)}{\beta r^2} + \frac{2\alpha}{\sqrt{\beta}}\left\{c_2 r^{\sqrt{3}-1} + \frac{c_3}{r^{1+\sqrt{3}}}\right\}. \tag{2.5.28}$$

It can be seen from equations (2.5.27) and (2.5.28) that the general solution is divergent at both the origin and at infinity and thus we need specific boundary conditions so as to eliminate the divergent terms. The singularities at the origin do not really exist and would be eliminated by taking the full solutions for the superconducting charge density and the magnetic field into account and just not to order α. The solutions given by equations (2.5.27) and (2.5.28) are plotted in figure II-13, where we have chosen an arbitrary set of boundary conditions.

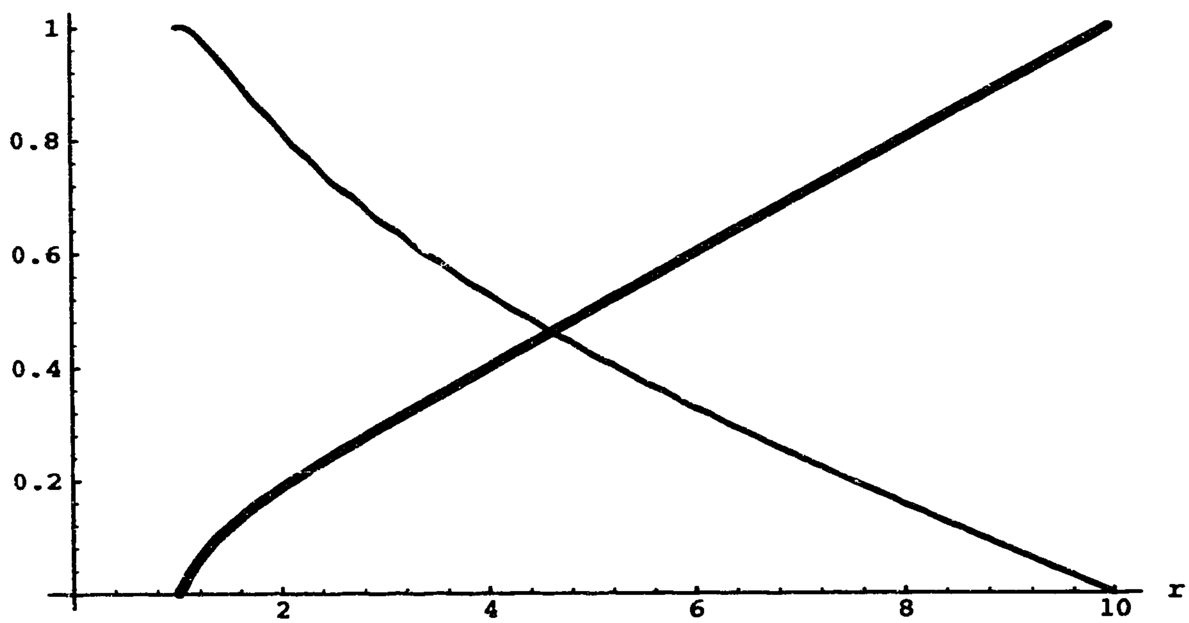

Figure II-13: Plot of the magnetic field represented by the thick line and the superconducting charge density represented by the thin line, as given by equations (2.5.27) and (2.5.28) respectively.

In order to obtain nonconstant charge and field distributions, f_o remaining zero, we can reanalyze the system of equations (2.5.12) through (2.5.16) by substituting $g_o = \omega / \sqrt{\beta}$ and $R = \exp(\chi)$, whereby equation (2.5.12) becomes

$$\frac{d^2\omega}{d\chi^2} - \frac{d\omega}{d\chi} + \frac{\omega}{4}(1-\omega^2) = 0. \tag{2.5.29}$$

Equation (2.5.29) is in the form of the unforced Duffing equation[15]. Unfortunately, to the best of our knowledge, no analytical solutions to this equation are known except the constant ones $\omega = 0$, $\omega = 1$, and $\omega = -1$.

If equation (2.5.29) had the term $2\omega/9$ instead of $\omega/4$ then it would be of the Painlevé-type which can be solved exactly. However, there are methods of approximation that give analytical solutions which are good in the region of validity applicable for that approximation. We now will solve equation (2.5.29) using two different approximation methods. The first approach is to linearize equation (2.5.29) around the constant solutions. This gives us asymptotic solutions which are valid for $\chi \to \pm\infty$. The second approach is to use a multiple-scale expansion around $\chi = 0$ and since we shall only do the expansion to first order, the range of validity is up to $\chi \sim O(36)$. In terms of our original independent variable, this range is $e^{-18} < r < e^{18}$, where we have recalled the fact that $r^2 = R = e^{\chi}$. Thus, the first approach of linearizing equation (2.5.29) about a constant solution leads to;

$$\left\{\frac{d^2}{d\chi^2} - \frac{d}{d\chi} + \frac{\left(1-3\omega_0^2\right)}{4}\right\}\omega_1 = 0, \tag{2.5.30}$$

where $\omega = \omega_0 + \omega_1$ and ω_0 is one of the constant solutions mentioned above. Equation (2.5.30) is easily solved to give us;

$$\omega = \begin{cases} \pm 1 + c_1 e^{\frac{\chi}{2}(1+\sqrt{3})} + c_2 e^{\frac{\chi}{2}(1-\sqrt{3})}, \ \omega_0 = \pm 1, \\ e^{\frac{\chi}{2}}\left(c_1 + c_2\chi\right), \quad \omega_0 = 0. \end{cases} \tag{2.5.31}$$

We see that the linearization of equation (2.5.29) is not bounded in general for both $\chi \to \pm\infty$, but can be may bounded in one of the limits. The solution for g_0 is;

$$g_0 = \frac{1}{\sqrt{\beta}}\begin{cases} \pm 1 + c_1 e^{\frac{\chi}{2}(1+\sqrt{3})} + c_2 e^{\frac{\chi}{2}(1-\sqrt{3})}, \ \omega_0 = \pm 1, \\ e^{\frac{\chi}{2}}(c_1 + c_2\chi), \quad \omega_0 = 0. \end{cases} \tag{2.5.32}$$

and consequently the order parameter envelope behaves, asymptotically, as

$$n_s = \frac{1}{\beta}\begin{cases} \left(\pm\frac{1}{r} + c_1 r^{\sqrt{3}} + \frac{c_2}{r^{\sqrt{3}}}\right)^2, \ \omega_0 = \pm 1, \\ (c_1 + 2c_2 \ln|r|)^2, \quad \omega_0 = 0. \end{cases} . \tag{2.5.33}$$

From equation (2.5.33), we see that the superconducting charge density is singular at the origin for $\omega_0 = \pm 1$ and for $\omega_0 = 0$ if $c_2 \neq 0$. Also, for $\omega_0 = 0$ when $c_2 = 0$, the solutions for η_0 are constants. When $r \to \infty$, the solution for η_0 will go either to zero or "blow up", except for $\omega_0 = 0 = c_2$ which remains a constant that can be nonzero. Furthermore, from equation (2.5.8) we deduce that the form of the first order correction for f is as follows:

$$f_1 = \begin{cases} c_3 r^{1+\sqrt{1+\frac{1}{\beta}}} + c_4 r^{1-\sqrt{1+\frac{1}{\beta}}}, & \omega_0 = \pm 1, c_1 = c_2 = 0 \\ r\left\{c_3 I_p\left(\frac{2r^{(1+\sqrt{3})/2}}{1+\sqrt{3}}\sqrt{\frac{c_1\gamma}{\beta}}\right) + c_4 K_p\left(\frac{2r^{(1+\sqrt{3})/2}}{1+\sqrt{3}}\sqrt{\frac{c_1\gamma}{\beta}}\right)\right\}, & \omega_0 = \pm 1, c_1 = 0 \\ r\left\{c_3 I_q\left(\frac{2r^{(1-\sqrt{3})/2}}{1-\sqrt{3}}\sqrt{\frac{c_2\gamma}{\beta}}\right) + c_4 K_q\left(\frac{2r^{(1-\sqrt{3})/2}}{1-\sqrt{3}}\sqrt{\frac{c_2\gamma}{\beta}}\right)\right\}, & \omega_0 = \pm 1, c_2 = 0 \\ r\left\{c_3 I_2\sqrt{\frac{4\gamma c_1 r}{\beta}} + c_4 K_2\sqrt{\frac{4\gamma c_1 r}{\beta}}\right\}, & \omega_0 = c_2 = 0, \end{cases} \tag{2.5.34}$$

where $p = \frac{2}{1+\sqrt{3}}\sqrt{1 \pm \frac{\gamma}{\beta}}$ and $q = \frac{2}{1-\sqrt{3}}\sqrt{1 \pm \frac{\gamma}{\beta}}$. This gives for the magnetic field;

$$h = \frac{e^* \alpha}{\hbar c}\begin{cases} \dfrac{\left(1+\sqrt{1+\frac{\gamma}{\beta}}\right)c_3}{r^{1-\sqrt{1+\frac{\gamma}{\beta}}}} + \dfrac{\left(1-\sqrt{1+\frac{\gamma}{\beta}}\right)c_4}{r^{1+\sqrt{1+\frac{\gamma}{\beta}}}}, & c_1 = c_2 = 0 \\ \dfrac{r^{(\sqrt{3}-1)/2}}{2}\sqrt{\dfrac{c_1\gamma}{\beta}}\left\{\left(1-\dfrac{1-\sqrt{3}}{p}\right)\left\{c_3 I_{p-1}\left(Ar^{(1+\sqrt{3})/2}\right) - c_4 K_{p-1}\left(Ar^{(1+\sqrt{3})/2}\right)\right\}\right. & \\ \quad \left. + \left(1+\dfrac{1-\sqrt{3}}{p}\right)\left\{c_3 I_{p+1}\left(Ar^{(1+\sqrt{3})/2}\right) - c_4 K_{p+1}\left(Ar^{(1+\sqrt{3})/2}\right)\right\}\right\}, & c_1 = 0 \\ \dfrac{1}{2r^{(\sqrt{3}+1)/2}}\sqrt{\dfrac{c_2\gamma}{\beta}}\left\{\left(1-\dfrac{1+\sqrt{3}}{q}\right)\left\{c_3 I_{q-1}\left(Br^{(1-\sqrt{3})/2}\right) - c_4 K_{p-1}\left(Br^{(1-\sqrt{3})/2}\right)\right\}\right. & \\ \quad \left. + \left(1+\dfrac{1+\sqrt{3}}{q}\right)\left\{c_3 I_{q+1}\left(Br^{(1-\sqrt{3})/2}\right) - c_4 K_{q+1}\left(Br^{(1-\sqrt{3})/2}\right)\right\}\right\}, & c_2 = 0 \\ \sqrt{\dfrac{\gamma c_1}{\beta r}}\left\{c_3 I_1\sqrt{\dfrac{4\gamma c_1 r}{\beta}} + c_4 K_1\sqrt{\dfrac{4\gamma c_1 r}{\beta}}\right\}, & \omega_0 = c_2 = 0, \end{cases} \tag{2.5.35}$$

where p and q are given above. Also, $A = \frac{2}{1+\sqrt{3}}\sqrt{\frac{c_1\gamma}{\beta}}$ and $B = \frac{2}{1-\sqrt{3}}\sqrt{\frac{c_2\gamma}{\beta}}$. It is clear from the solution for the magnetic field given by equation (2.5.35) that the c_4 term gives a magnetic vortex whereas, the c_3 term gives a nucleation center of superconducting charge surrounded by a magnetic field. Also, if we compare the solution for the magnetic field in equation (2.5.35) when $\omega_0 = 0$ with the standard single vortex solution to the London equations[16], i.e., $h(r) = (\Phi_o / (2\pi\lambda^2))K_o(r/\lambda)$, we see that the solutions are basically identical when $c_3 = 0$ except that the divergence at the origin is at a slightly different rate with K_0 going as a logarithm.

The second approach that we shall use to solve equation (2.5.29) is the multiple-scale expansion and noting that the term ω with a coefficient of 2/9 instead of 1/4 gives an exactly solvable equation and so by letting $\omega = 2F$, then equation (2.5.29) becomes;

$$\frac{d^2F}{d\chi^2} - \frac{dF}{d\chi} + \frac{2F}{9} - F^3 + \varepsilon F = 0, \tag{2.5.36}$$

where $\varepsilon = 1/36$. Thus, introducing an expansion in $F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \ldots$ and also $\frac{d}{d\chi} \rightarrow \frac{\partial}{\partial\tau_0} + \varepsilon\frac{\partial}{\partial\tau_1} + \ldots$ gives us to first order;

$$\frac{d^2F_0}{d\tau_0^2} - \frac{dF_0}{d\tau_0} + \frac{2F_0}{9} - F_0^3 = 0 \qquad (2.5.37)$$

and

$$\frac{d^2F_1}{d\tau_0^2} - \frac{dF_1}{d\tau_0} + \left\{\frac{2}{9} - 3F_0^2\right\}F_1 = \frac{\partial}{\partial\tau_1}\left\{1 - \frac{\partial}{\partial\tau_0}\right\}F_0 - F_0. \qquad (2.5.38)$$

Note that the τ_1 solution for F_0 is not really needed in this case since there is no secular terms in equation (2.5.38), however, it can be used to eliminate the right hand side of equation (2.5.38) so that we only have a linear homogeneous equation for the first order corrections. The solution to equation (2.5.37) has to constants of integration which are functions of τ_1 and then we set the right hand side of equation (2.5.38) to zero to solve for τ_1. Note that when we convert back to χ, we have $\tau_0 = \chi$ and $\tau_1 = \varepsilon\chi$. We have not solved for τ_1 yet but will now give the solution for equation (2.5.37) and will ignore equation (2.5.38). Equation (2.5.37) can be transformed into an elliptic equation by the transformation $F_0 = e^{\chi/3}G(e^{\chi/3})/3$ so that the G satisfies $G'' = G^3$ or $(G')^2 = G^4/2 - c_0$ and gives the following real solutions for the order parameter envelope η;

$$n_s = \frac{4}{9\beta r^{2/3}}\begin{cases} \dfrac{\sqrt{2c_0}}{cn^2\left[(2c_0)^{1/4}\left(r^{2/3} + c_1\sqrt{2}\right);(2)^{-1/2}\right]}; & c_0 > 0 \\[2ex] \dfrac{2}{\left(\sqrt{2}c_1 \pm r^{2/3}\right)^2}; & c_0 = 0, \\[2ex] \dfrac{\sqrt{-2c_0}\,sn^2\left[2(-2c_0)^{1/4}\left(c_1 \pm r^{2/3}(2)^{-1/2}\right);(2)^{-1/2}\right]}{1 + cn^2\left[2(-2c_0)^{1/4}\left(c_1 \pm r^{2/3}(2)^{-1/2}\right);(2)^{-1/2}\right]}; & c_0 < 0 \end{cases} \qquad (2.5.39)$$

where we have used the relationship $e^{x/2} = r$, so as to write equation (2.5.39) in terms of our original independent variable.

The solutions given in equation (2.5.39) shows a damped oscillatory behavior which are illustrated in figures II-14 through II-16. This is in agreement with numerical studies[17)] of equation (2.5.29) which shows that damped oscillatory functions $\omega(R)$ can be found as solutions, and depending on the initial conditions, they may asymptotically tend to any of the constant solutions above. Note that by 'damped oscillatory behavior', we mean a periodic function which decays to some constant value asymptotically and becomes no longer periodic. Also, as was mentioned above, the range of validity for this solution is $e^{-18} < r < e^{18}$, however, the closer one is to $r = 1$, the more accurate the solution is for equation (2.5.29).

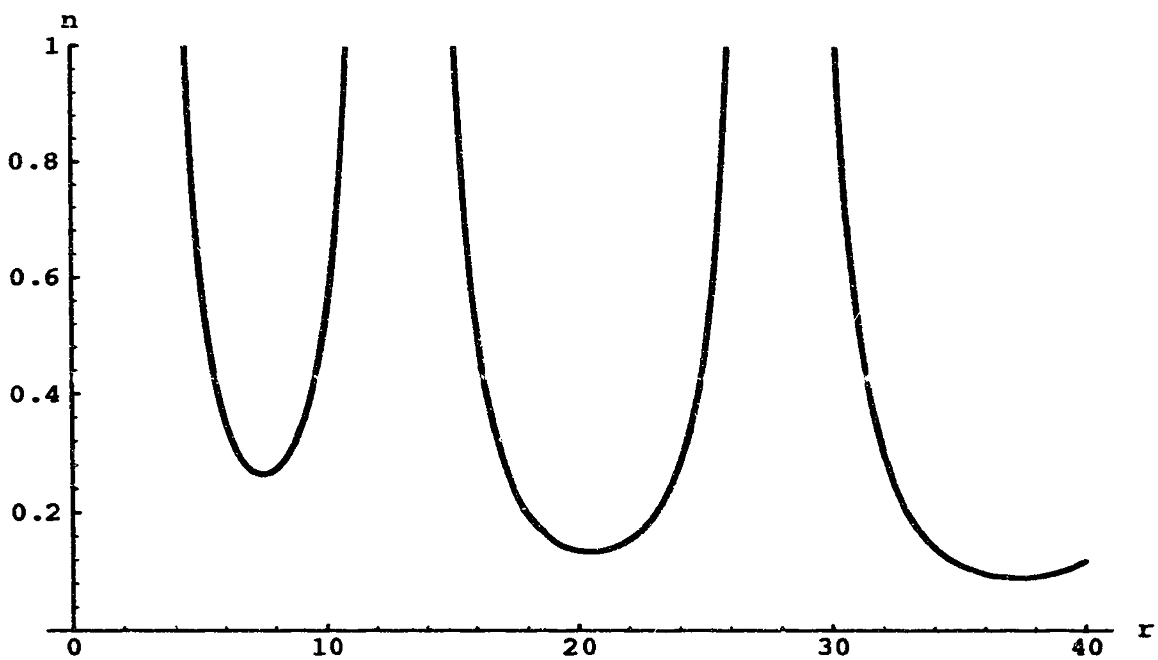

Figure II-14: The superconducting charge density as given by equation (2.5.39) with $c_0 > 0$.

Figure II-14 is from equation (2.5.39) with $c_0 > 0$ and clearly shows the multiple singularities. If this was the complete solution, these singularities would have a finite limit. However, it does show some interesting results. First we see that the superconducting charge density is asymptotically going to zero with a periodic oscillation and with a period that grows with the distance from the core. Second we can also state that the magnitude of the magnetic field is periodic and grows asymptotically to some value with the peaks in the magnitude occurring at the minimums for n_s.

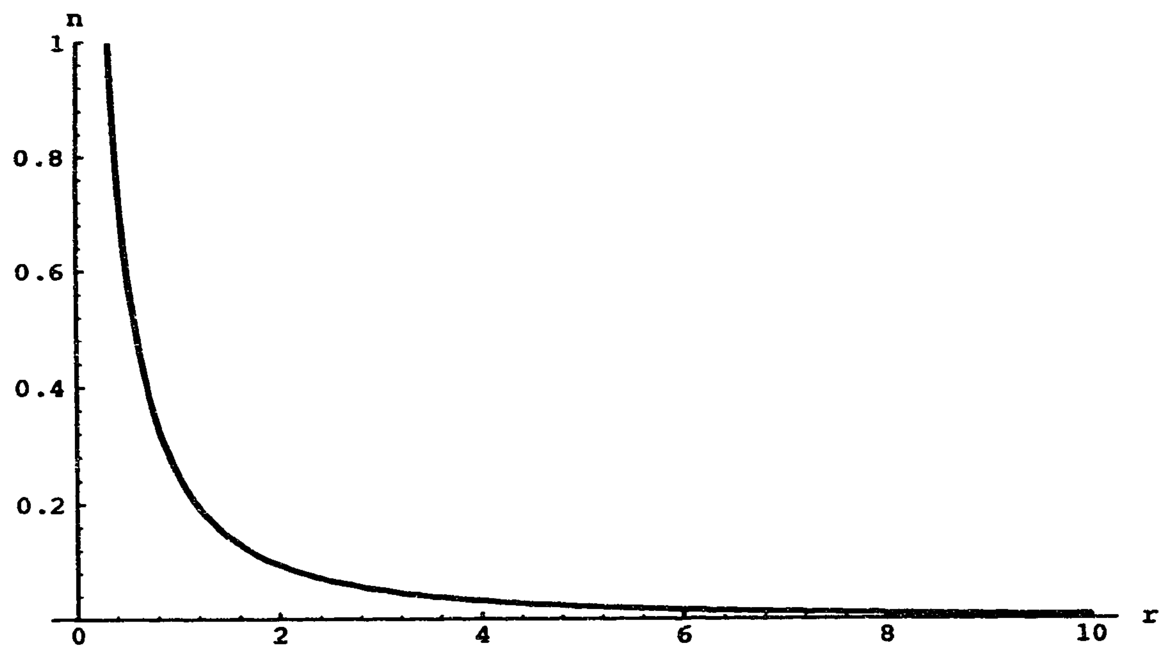

Figure II-15: The superconducting charge density as given by equation (2.5.39) with $c_0 = 0$.

Figure II-15 corresponds to the solution in equation (2.5.39) with $c_0 = 0$ and is a damped solution with the nucleation of the superconducting charge at the origin but with no periodicity.

The only solution in equation (2.5.39) that can be nonsingular at the origin is for $c_0 < 0$, which we have plotted in figure II-16. This figure clearly shows the damped oscillatory behavior of the solution. Again, we can see from figure II-16 that the period of oscillation grows with the distance from the core. At this stage we have not yet analyzed the correction terms for the superconducting charge density or the magnetic field. However, from the solution for the condensate density as illustrated by figure II-16 we can infer that the magnitude of the magnetic field has a peak at the origin and is oscillatory with each peak growing to some asymptotic value away from the core.

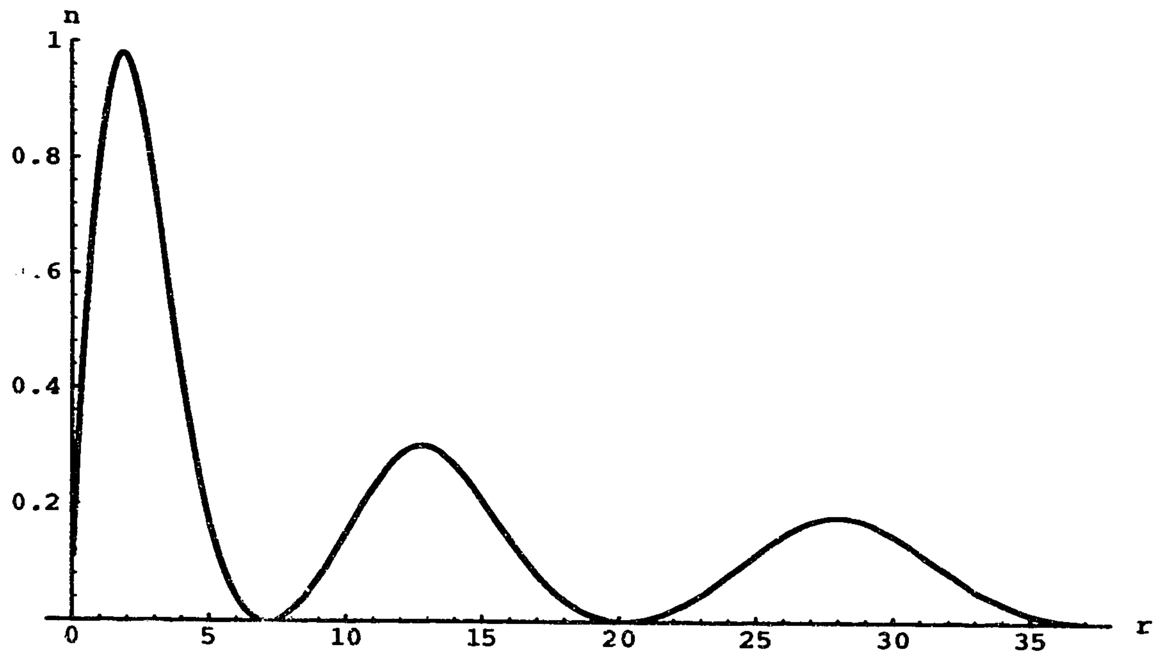

Figure II-16: The superconducting charge density as given by equation (2.5.39) with $c_0 < 0$.

In the second case we start by putting $g_0 = 0$, so that equation (2.5.6) and equations (2.5.8) through (2.5.11) become:

$$\frac{d^2 f_o}{dR^2} = 0, \tag{2.5.40}$$

$$\frac{d^2 f_1}{dR^2} = 0, \tag{2.5.41}$$

$$\frac{d^2 g_1}{dR^2} = \frac{g_1}{4R^2}(f_o^2 - 1), \tag{2.5.42}$$

$$\frac{d^2 f_2}{dR^2} = \frac{\gamma}{4R^2} f_0 g_1^2, \tag{2.5.43}$$

and

$$\frac{d^2g_2}{dR^2} = \frac{g_1}{4R^2}(2f_of_1 + R) + \frac{g_2}{4R^2}(f_o^2 - 1). \tag{2.5.44}$$

Equations (2.5.40) and (2.5.41) are straight forward to solve and thus we obtain;

$$f_0 = c_0R + c_1 \tag{2.5.45}$$

and

$$f_1 = c_2R + c_3 \tag{2.5.46}$$

which can be substituted into equations (2.5.42) through (2.5.44). To avoid singularities at the origin for the magnetic field, we assume $c_1 = c_3 = 0$. For convergence, it is required that $c_0 > \alpha\, c_2$. Hence equations (2.5.42) through (2.5.44) become

$$\frac{d^2g_1}{dR^2} = \frac{g_1}{4}\left\{c_o^2 - \frac{1}{R^2}\right\}, \tag{2.5.47}$$

$$\frac{d^2f_2}{dR^2} = \frac{\gamma c_0}{4R}g_1^2, \tag{2.5.48}$$

and

$$\frac{d^2g_2}{dR^2} = g_1\left\{\frac{c_oc_1}{2} + \frac{1}{4R}\right\} + \frac{g_2}{4}\left\{c_o^2 - \frac{1}{R^2}\right\}. \tag{2.5.49}$$

Equation (2.5.47) can be solved to give the analytical form for g_1, namely,

$$g_1 = \sqrt{R}\left\{c_4I_0\left(\frac{c_0R}{2}\right) + c_5K_0\left(\frac{c_0R}{2}\right)\right\}, \tag{2.5.50}$$

where I_0 and K_0 are modified Bessel functions. As a consequence, the order parameter envelope η, for small r, can be approximated as;

$$\eta = \alpha c_2I_0\left(\frac{c_0R}{2}\right) = \alpha c_2\left\{1 + \frac{c_0^2}{16}r^4 + \ldots\right\}. \tag{2.5.51}$$

Hence the corresponding correlation length is $\xi = \sqrt{2/c_0}$. On the other hand, the magnetic field to first order does not depend on the radius r, however, solving equation (2.5.48) and then calculating the corresponding magnetic field h yields

$$h \approx h_0 - h_2 r^4 + \ldots, \tag{2.5.52}$$

with the corresponding penetration length given by $\lambda \approx 1/\sqrt{|h_2|}$.

The full solutions for both the magnetic field and the superconducting charge density to second order (α^2) is;

$$h = \frac{e^*}{\hbar c r}\frac{df}{dr} = \frac{2e^*}{\hbar c}\left\{c_0 + \alpha c_2 + \alpha^2 c_6 + \frac{\alpha^2 \gamma c_0}{4}\int^{r^2} dR \left\{c_4 I_0\left(\frac{c_0 R}{2}\right) + c_5 K_0\left(\frac{c_0 R}{2}\right)\right\}^2\right\} \tag{2.5.53}$$

and

$$n_s = \left(\frac{g}{r}\right)^2 = \alpha^2 \left\{c_4 I_0\left(\frac{c_0 R}{2}\right) + c_5 K_0\left(\frac{c_0 R}{2}\right)\right\}^2. \tag{2.5.54}$$

We have plotted equations (2.5.53) and (2.5.54) in figures II-17, II-18, and II-19 with different boundary conditions. For figure II-17, we have at the origin a nucleation of superconducting charge with a surrounding magnetic field. Figure II-18 illustrates a magnetic vortex, where the structure of the vortex is highly dependent upon the boundary conditions. The shape of the magnitude of the vortex can vary from being a semi-hemisphere to cylindrical in shape. Figure II-19 is the equivalent of figure II-18 with the boundary conditions changed slightly to give a more rounded top to the vortex. It should be noted that the larger the radius of the vortex, the more cylindrical its shape which is expected since the surrounding superconducting charge penetrates into less of the magnetic fields' vortex.

This case, therefore, indicates a much slower decay of the magnetic field as it penetrates the sample associated with a greatly reduced superconducting charge density, in striking contrast to the previously discussed case. Thus we have shown in this subsection that quasi two dimensional cylindrically symmetric solutions to the magnetic field penetration problem in a superconductor can be found.

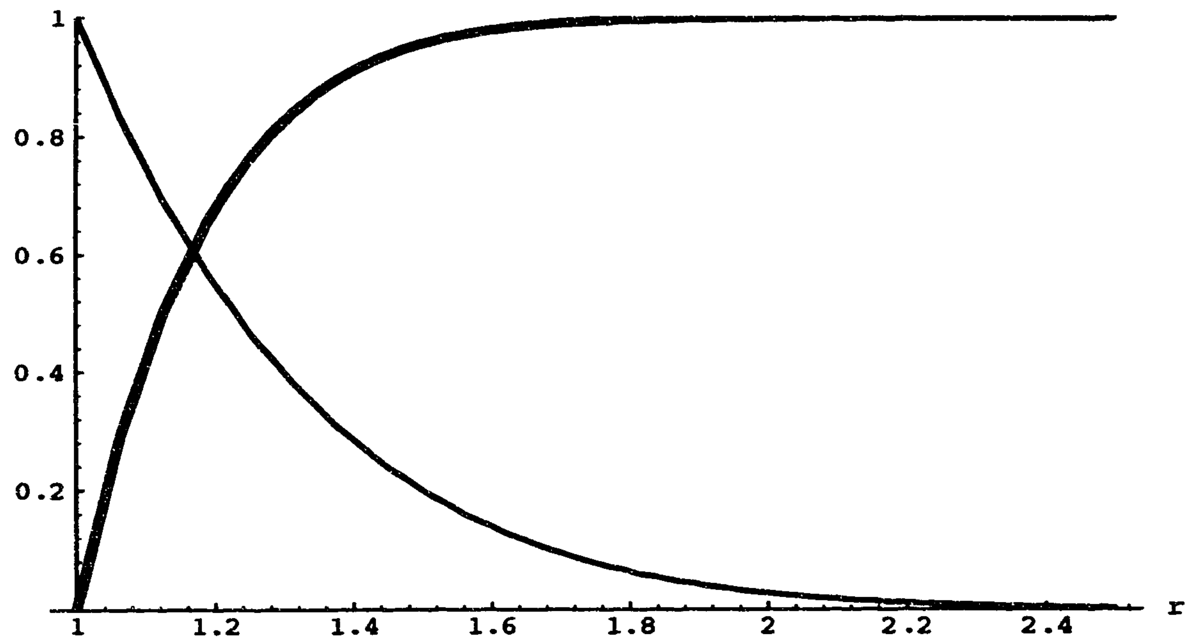

Figure II-17: Plot of the magnetic field represented by the thick line and the superconducting charge density represented by the thin line, as given by equations (2.5.53) and (2.5.54) respectively, with $c_4 = 0$.

Of course, the existence of vortices was known long time ago[9),18)], but virtually the only information about their structure was related to the asymptotic behavior as $r \to \infty$. Here we have shown that close to the critical temperature some closed form approximate solutions can be found which may be nonsingular and accurate perturbation expansions can be performed away from the critical temperature. Valuable information about the dominant behavior of both the order parameter envelope η and the magnetic field h in the vicinity of the vortex core has been obtained in two particular cases.

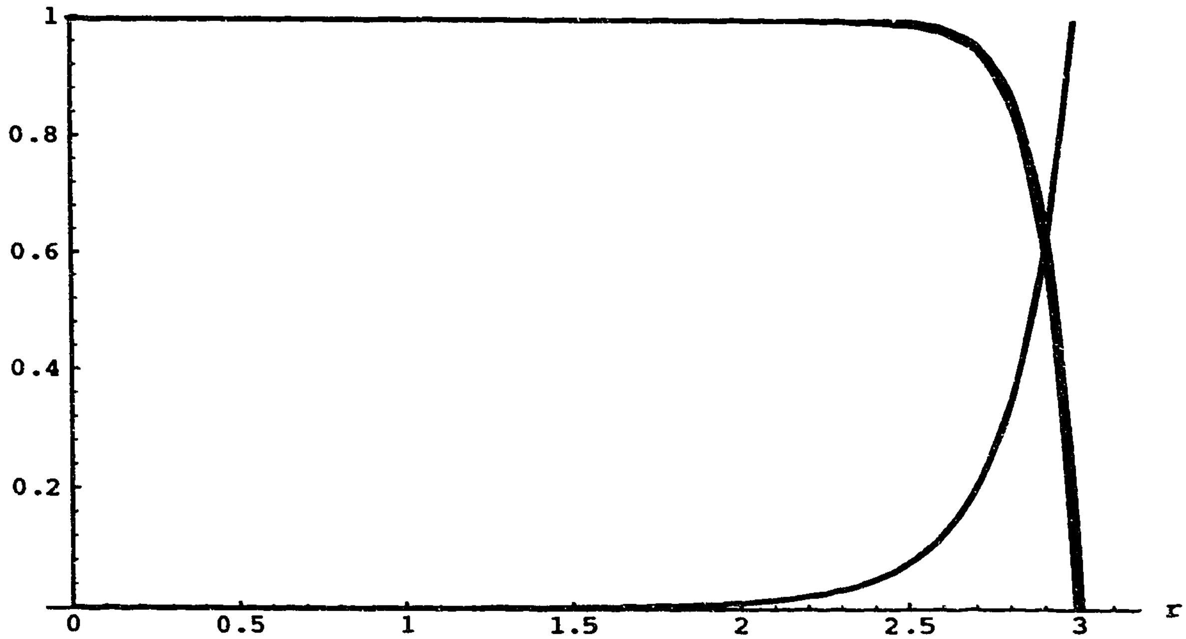

Figure II-18: Plot of the magnetic field represented by the thick line and the superconducting charge density represented by the thin line, as given by equations (2.5.53) and (2.5.54) respectively, with $c_5 = 0$.

B. Spirals

Next, we shall look at the possibility of the existence of spiral solutions. Based on the fact that certain nonlinear partial equations[12)] (PDE's) possess spiral solutions when scaling symmetry is present, we have anticipated such an eventuality in our case of the system of coupled PDE's, equations (2.2.5) through (2.2.7). We expect this type of solution to occur only under very specific circumstances, and as we shall see, this corresponds physically to either creation or destruction of vortex like structures. This therefore suggests that spiral solutions may be metastable. We now proceed to demonstrate the form of these solutions and the conditions required for their existence.

Beginning with equation (2.2.7), we intend to find the order parameter envelope η and the scaled velocity vector $\mathbf{u}$ which would satisfy equations (2.2.5) through (2.2.7) and would depend on a spiral type symmetry variable, which is postulated in the form

$$Q = B_o \ln(r) + \phi, \tag{2.5.55}$$

where simultaneously the order parameter is sought as

$$\eta = \frac{f(Q)}{r}. \tag{2.5.56}$$

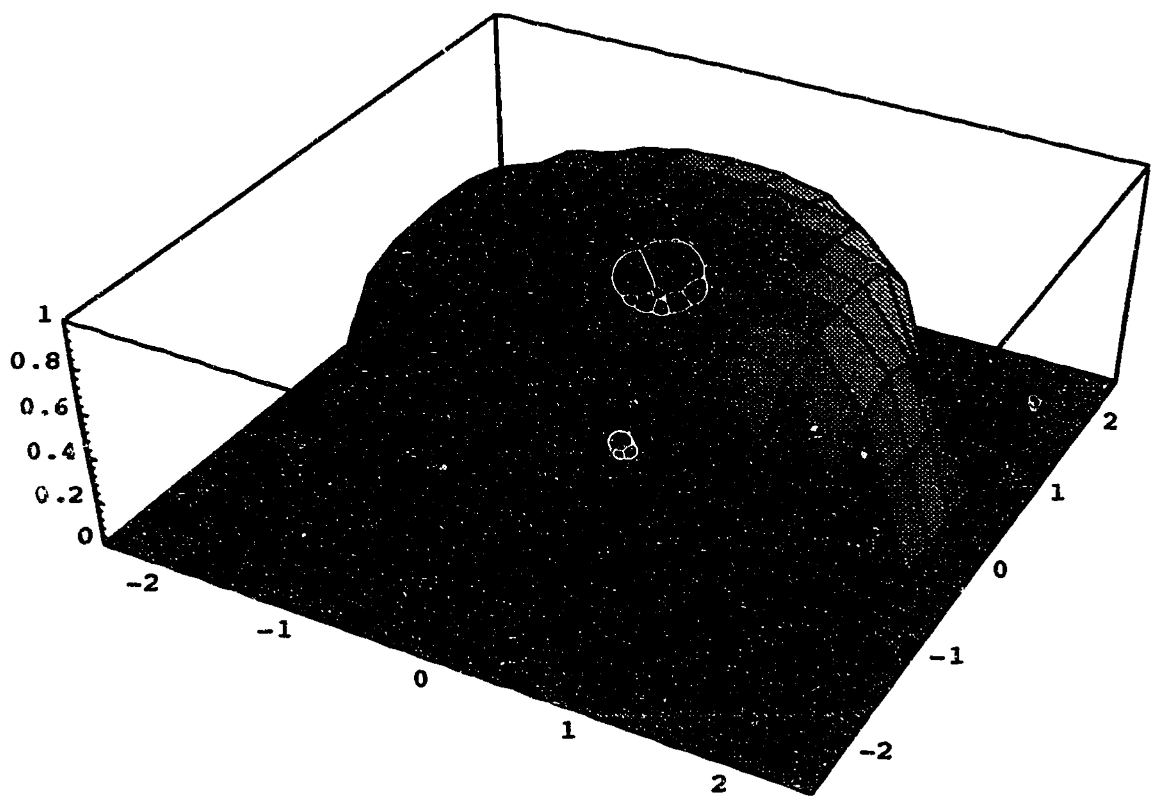

Figure II-19: Three dimensional representation of the magnetic field as given by equation (2.5.53) with $c_5 = 0$.

The scaled velocity vector **u** is expected to have both radial and azimuthal components, i.e.,

$$\mathbf{u} = \hat{\mathbf{r}}\, u_r + \hat{\boldsymbol{\phi}}\, u_\phi. \tag{2.5.57}$$

Thus, equation (2.2.7) becomes

$$\frac{2}{f}(B_0 u_r + u_\phi)\frac{df}{dQ} + r\frac{\partial u_r}{\partial r} + \frac{\partial u_\phi}{\partial \phi} - u_r = 0. \qquad (2.5.58)$$

A simple way of satisfying equation (2.5.58) without in any way affecting the form of f is to put

$$u_r = -\frac{u_\phi}{B_0} \qquad (2.5.59)$$

and

$$r\frac{\partial u_r}{\partial r} + \frac{\partial u_\phi}{\partial \phi} - u_r = 0. \qquad (2.5.60)$$

Using the method of Lagrange[19)], the solution to equations (2.5.59) and (2.5.60) is straightforward and found to be $\chi(Q, u_\phi / r) = 0$, where χ is an arbitrary function. In particular, this could be written as $u_\phi = rF(Q)$, where at this point F is an arbitrary function. It is easy to demonstrate that the associated magnetic field **h** can be expressed as:

$$\mathbf{h} = \frac{\hbar c}{e^*}(\nabla \times \mathbf{u}) = \hat{\mathbf{z}}\frac{\hbar c}{e^*}\left\{2F + \left\{B_0 + \frac{1}{B_0}\right\}\frac{dF}{dQ}\right\}. \qquad (2.5.61)$$

Now we must satisfy equations (2.2.9) and (2.2.10), which may be written in the form of two coupled ODE's:

$$(B_0^2 + 1)\frac{d^2 f}{dQ^2} - 2B_0\frac{df}{dQ} = f\left\{\alpha r^2 - 1 + \beta f^2 + (B_0^2 + 1)\frac{r^4 F^2}{B_0^2}\right\} \qquad (2.5.62)$$

and

$$(B_0^2 + 1)\frac{d^2 F}{dQ^2} + 2B_0\frac{dF}{dQ} = \gamma f^2 F. \qquad (2.5.63)$$

nonlinear PDE's to ODE's. However, as can be seen in equation (2.5.62), there still remains one of the old independent variables, namely r, which is not allowed for a true reduction. This problem however, can be circumvented by doing the analysis close to the origin so that $r^4 \approx 0$. Also, we need to eliminate the term αr^2 which can be done either by constraining ourselves even closer to the origin so that $r^2 \approx 0$ or we can look for solutions at the critical temperature and then do an expansion in α similar to section three. Once this is done, we can then perturb about the solutions to add corrections to minimize the approximations. Under this approximation, equation (2.5.63) is unchanged, but equation (2.5.62) becomes

$$(B_0^2+1)\frac{d^2f}{dQ^2} - 2B_0\frac{df}{dQ} + f\left[1-\beta f^2\right] = 0. \tag{2.5.64}$$

Now, if we rescale the independent and dependent variables according to $Q \rightarrow \tau Q$ and $f \rightarrow \nu f$, then equation (2.5.64) takes the form

$$\frac{d^2f}{dQ^2} + \frac{df}{dQ} + \delta^2 f + \varepsilon f^3 = 0. \tag{2.5.65}$$

where ε = -1 for positive β and ε = 1 for negative β and also $\delta^2 = (B_o^2+1)/(4B_o^2)$, $\tau = \left[B_0^2+1\right]/(-2B_0)$, and $\nu = \pm 1/\sqrt{\delta^2|\beta|}$. This is again a standard classical anharmonic oscillator equation with dissipation (the unforced Duffing equation) and is of the same form as equation (2.5.37) except that the dissipation term has the opposite sign and the coefficient of f is arbitrary. Note that equation (2.5.65) is of the Painlevé-type when δ^2 = 2/9. For this value of δ, equation (2.5.65) can be transformed into an elliptic equation which can be solved exactly. These solutions are given below under the subheadings: kink solutions and elliptic like solutions. Thus, the solutions are found in the same way as those for equation (2.5.37). Although a complete set of solutions to this equation for arbitrary coefficients is, to the best of our knowledge, not available at present, some exact value of B_o, it can thus be chosen to correspond to those values for which we know the exact solution. Also, there are some special solutions for unspecified values of B_o. In the following, we list some specific solutions of equation (2.5.65).

1. Constant Solutions

Constant solutions are $f = 0$, and if $\varepsilon = -1$, $f = \pm|\delta|$.

2. Kink Solutions

Kink solutions exist when $\varepsilon = -1$ and $\delta^2 = 2/9$ and the solutions take the form

$$f = \left\{c_1 \exp\left(\frac{Q}{3}\right) \pm \frac{3}{\sqrt{2}}\right\}^{-1}, \tag{2.5.66}$$

which interpolates between $f = 0$ and $f = \pm\sqrt{2}/3$ as $Q \to \infty$ and $Q \to -\infty$, respectively. If $\varepsilon = 1$ then the solution becomes complex and is not allowed.

Now, from the approximation that the radius is very small, the solution to equation (2.5.62) implies that Q is essentially given by $B_0 \ln|r|$, which dominates the ϕ term. Taking this into account and dropping ϕ along with the fact that for $\delta^2 = 2/9$ implies that $B_0 = \pm i3$, gives us the following solution for the superconductin density;

$$n_s = \frac{f^2(Q = B_0 \ln r)}{r^2} = \frac{9}{2|\beta|\left\{c_1 r^{1/4} \pm \frac{3r}{\sqrt{2}}\right\}^2}. \tag{2.5.67}$$

The solution given by equation (2.5.67) is represented by figure II-20 below. We see that we have a singularity at the origin and going to zero asymptotically. Comparing the solutions under the spiral symmetry with those in the vortex subsection, the solutions are similar but the powers of r are different. This is also true for the next case.

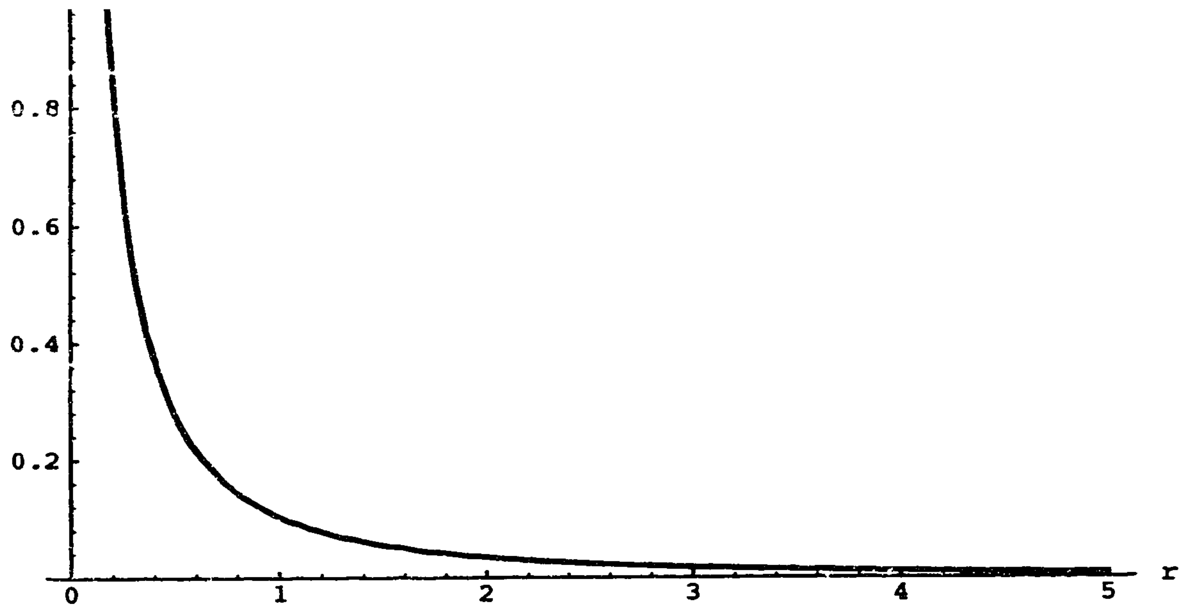

Figure II-20: Plot of the superconducting charge density as given by equation (2.5.67).

3. Elliptic like Solutions

For $\delta^2 = 2/9$, equation (2.5.65) has been recently analyzed[20] and a set of solutions in terms of elliptic functions has been found similar to those in equation (2.5.39), namely for $\varepsilon = -1$;

$$f = \frac{e^{-Q/3}}{3}\begin{cases} \dfrac{(2c_0)^{1/4}}{\left|\mathrm{cn}\left[(2c_0)^{1/4}\left(\sqrt{2}c_1 + e^{-Q/3}\right);(2)^{-1/2}\right]\right|}, & c_0 > 0 \\ \dfrac{(-2c_0)^{1/4}\left|\mathrm{sn}\left[2(-2c_0)^{1/4}\left(\sqrt{2}c_1 + e^{-Q/3}\right);(2)^{-1/2}\right]\right|}{\sqrt{1+\mathrm{cn}^2\left[2(-2c_0)^{1/4}\left(\sqrt{2}c_1 + e^{-Q/3}\right);(2)^{-1/2}\right]}}, & c_0 < 0 \end{cases} \qquad (2.5.68)$$

Before finding the superconducting charge density, it should be noted that for $\delta^2 = 2/9$ implies that $B_0 = \pm i3$, $Q = -3(3 \ln r \pm i\phi)/4 \approx \ln(r^{-9/4})$ since $r \approx 0$, and $n_s = 3f^2(Q)\left(r\sqrt{2|\beta|}\right)^{-1}$. Thus, the superconducting charge density is as follows;

$$n_s = \frac{1}{2|\beta|\sqrt{r}} \begin{cases} \dfrac{\sqrt{2c_0}}{\mathrm{cn}^2\left[(2c_0)^{1/4}\left(\sqrt{2}c_1 + r^{3/4}\right); (2)^{-1/2}\right]}, & c_0 > 0 \\ \dfrac{\sqrt{-2c_0}\,\mathrm{sn}^2\left[(-2c_0)^{1/4}\left(\sqrt{2}c_1 + r^{3/4}\right); (2)^{-1/2}\right]}{1 + \mathrm{cn}^2\left[2(-2c_0)^{1/4}\left(\sqrt{2}c_1 + r^{3/4}\right); (2)^{-1/2}\right]}, & c_0 < 0 \end{cases} \quad (2.5.69)$$

The solutions are similar to those given in the vortex subsection, but the power of r is different. The two solutions given by equation (2.5.69) are represented in figures II-21 and II-22. Compared with figures II-15 and II-17 respectively, we see that the oscillations are more closely packed together in figures II-21 and II-22, which is due to the radius r having a power of 3/4 instead of 2/3 as is the case in equation (2.5.37).

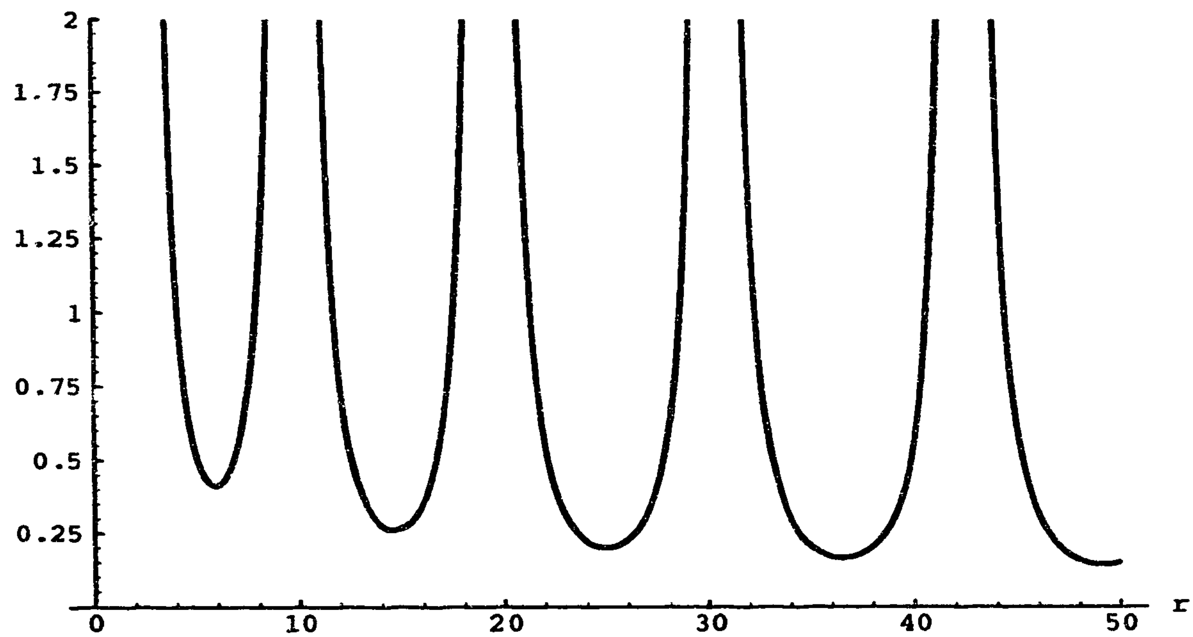

Figure II-21: Plot of the superconducting charge density as given by equation (2.5.69) with $c_0 > 0$.

When $\varepsilon = 1$, the $c_0 < 0$ solution is pure imaginary, but for $c_0 > 0$ we obtain the following solution for f;

$$f = \frac{(2c_0)^{1/4} e^{-Q/3}}{3} \sqrt{\frac{sn^2\left[c_1 + e^{-Q/3}\sqrt{c_0}/2;(2)^{-1/2}\right]}{1 + cn^2\left[c_1 + e^{-Q/3}\sqrt{c_0}/2;(2)^{-1/2}\right]}}. \tag{2.5.70}$$

This gives us the following solution for the superconducting charge density;

$$n_s = \frac{1}{2|\beta|}\sqrt{\frac{2c_0}{r}} \frac{sn^2\left[c_1 + r^{3/4}\sqrt{c_0}/2;(2)^{-1/2}\right]}{1 + cn^2\left[c_1 + r^{3/4}\sqrt{c_0}/2;(2)^{-1/2}\right]}. \tag{2.5.71}$$

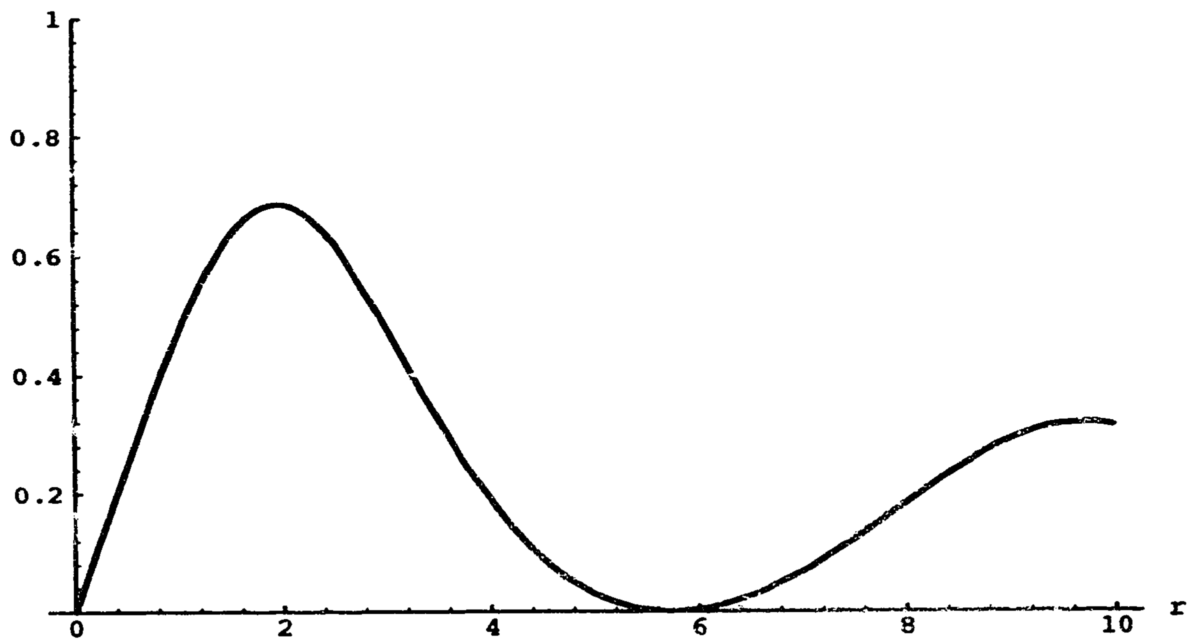

Figure II-22: Plot of the superconducting charge density as given by equation (2.5.69) with $c_0 < 0$.

The solution given in equation (2.5.71) is identical to the one given in equation (2.5.71) when $c_0 < 0$ and is also represented by the solution plotted into figure II-22. Since we had to approximate our equations for small r to find solutions, we see that they are very similar to those found in the vortex solution, supporting the idea that vortices break up at the critical temperature through a type of spiral formation. This hypothesis can probably not be verified since it occurs right at the critical temperature and thermal fluctuations would destroy any uniform constant temperature.

Finally, when f is close to a constant solution, say, $f = f_o$, we can put $f = f_o + \hat{x}$ with $\hat{x}$ small. Equation (2.5.65) then becomes, linearizing around f_0;

$$\frac{d^2\hat{x}}{dQ^2} + \frac{d\hat{x}}{dQ} + (\delta^2 + 3\varepsilon f_0^2)\hat{x} = 0. \tag{2.5.72}$$

where $f_0 = 0, \pm\delta$ if $\varepsilon = -1$ or just 0 if $\varepsilon = 1$. The solution of this equation are well known and take the form

$$\hat{x}(Q) = \exp\left(-\frac{Q}{2}\right)\begin{cases} c_1 \cos\left(\frac{Q}{2}\sqrt{4\delta^2 - 1}\right) + c_2 \sin\left(\frac{Q}{2}\sqrt{4\delta^2 - 1}\right), \text{ if } f_0 = 0, \\ c_1 \exp\left(\frac{Q}{2}\sqrt{8\delta^2 + 1}\right) + c_2 \exp\left(\frac{-Q}{2}\sqrt{8\delta^2 + 1}\right), \text{ if } f_0 = \pm\delta. \end{cases} \tag{2.5.73}$$

The superconducting charge density that corresponds to the solution given in equation (2.5.73) is

$$n_s = \frac{1}{|\beta|(\delta r)^2}\begin{cases} \Bigg\{ r^{1/2\delta^2} \exp\left(\frac{\phi}{2B_0\delta^2}\right)\left\{c_1 \cos\left(\frac{\sqrt{4\delta^2 - 1}}{4\delta^2}\left(\ln r + \frac{\phi}{B_0}\right)\right) - c_2 \right. \\ \qquad \left. \sin\left(\frac{\sqrt{4\delta^2 - 1}}{4\delta^2}\left(\ln r + \frac{\phi}{B_0}\right)\right)\right\}\Bigg\}^2, \; f_0 = 0, \\ \Bigg\{ f_0 + r^{1/4\delta^2} \exp\left(\frac{\phi}{4B_0\delta^2}\right)\left\{c_1 \exp\left(\frac{\sqrt{8\delta^2 + 1}}{4\delta^2}\left(\ln r + \frac{\phi}{B_0}\right)\right)\right. \\ \qquad \left. + c_2 \exp\left(\frac{-\sqrt{8\delta^2 + 1}}{4\delta^2}\left(\ln r + \frac{\phi}{B_0}\right)\right)\right\}\Bigg\}^2, \; f_0 = \pm\delta. \end{cases} \tag{2.5.74}$$

It should be noted that in equation (2.5.74), we have included ϕ in the expression for the superconducting charge density even though it must be remembered that this is a small correction to the ln r which is the more dominate.

Recent experiments[illegible] on magnetic penetration and related effects in high temperature 1-2-3 compounds provide magnetic field profiles as a function of depth which are reminiscent of our elliptic solutions in two dimensions. Among them we find profiles with a simple peak in the center as well as some with two peaks of slightly different heights, as might be expected from the generalized Emden equation which would be applicable in this case[22].

In this section we have concentrated on two dimensional structures involving the coupled superconducting charge density and magnetic vector potential fields. The first type of pattern studied, i.e., vortices, were known to exist for a long time, but we have gone significantly beyond the asymptotic behavior known earlier and found functional forms which [illegible]ally exact at the critical temperature and can be accurately extended below the critical temperature in a recursive manner. A less well known type of solution to the magnetic field penetration problem in superconductors is spirals. It appears that spirals may exist only in the immediate vicinity of the critical point, and they seem to be related to the creation and destruction of cylindrical vortices.

SECTION VI: MODERATE FIELDS AND CHARGE DENSITIES

In this section we intend to present a different approach to solving equations (2.4.1) and (2.4.2) which applies to moderate values of the magnetic field $\mathbf{h}$ and the superconducting charge density n_s. In this section we take the magnetic field to be;

$$\mathbf{h} = H\mathbf{w}\,, \tag{2.6.1}$$

where H is a constant and $\mathbf{w}$ is a vector function with its norm bounded through, $|\mathbf{w}| \leq 1$. The superconducting charge density is to be taken as;

$$n_s = Ns = N t^2 = \eta^2, \tag{2.6.2}$$

where N is another constant representing a maximum amplitude for the superconducting charge density and $|s| \leq 1$. In order to simplify the form of the equations, the following scaling is introduced for the independent variable:

$$x = \frac{x'}{\sqrt{\gamma N}}. \tag{2.6.3}$$

As a result, the following set of coupled differential equations is obtained for equations (2.4.1) and (2.4.2):

$$\mathbf{w} + \nabla' \times \left\{ \frac{\nabla' \times \mathbf{w}}{s} \right\} = 0, \tag{2.6.4}$$

and

$$\nabla'^2 t - \frac{m^{*2} c^2 t}{2\pi\hbar^2 e^{*2} N} \left\{ a + \frac{H^2}{8\pi N} \frac{(\nabla' \times \mathbf{w})^2}{t^2} + bNt^2 \right\} = 0, \tag{2.6.5}$$

where ∇' denotes the gradient operator with respect to the primed coordinates. From the fact that the magnetic field and superconducting charge density are mutually exclusive and that s and **w** are bounded by unity, we have that $|s\,\mathbf{w}| << 1$, so that equation (2.6.4) becomes

$$\nabla' \times (\nabla' \times \mathbf{w}) - \frac{\nabla' s}{s} \times (\nabla' \times \mathbf{w}) = -s\mathbf{w}. \tag{2.6.6}$$

Since in most applications, the magnetic field is taken to be along only one spatial coordinate, then sw becomes a scalar which is equivalent to the norm |sw| and thus we can approximate the right hand side of equation (2.6.6) as zero. It is easy to see that the solution to equation (2.6.6) is

$$\nabla' \times \mathbf{w} = \Lambda s, \tag{2.6.7}$$

where Λ is a constant vector. Substituting equation (2.6.7) into equation (2.6.5) gives an equation strictly in terms of t as the only dependent variable. The resultant equation takes the form of the well known cubic nonlinear Klein-Gordon equation in three dimensional space, i.e.,

$$\nabla'^2\iota = \iota(A + B\iota^2), \tag{2.6.8}$$

where the constant coefficients A and B are

$$A = \left\{\frac{m^* c}{4\pi N\hbar e^*}\right\}^2\left[8\pi Na + \Lambda^2H^2\right] \tag{2.6.9}$$

and

$$B = \frac{b}{2\pi}\left\{\frac{m^* c}{\hbar e^*}\right\}^2 > 0. \tag{2.6.10}$$

One can therefore define a modified LG coherence length by analogy with ξ_{LG} of equation (2.1.10) as

$$\xi_{LG}(T,H) = |A|^{-1/2}.$$

Note that the values of the amplitudes of the field H and charge distribution N can be adjusted arbitrarily to the boundary condition of the problem. The sign of the coefficient A depends in a direct way on the temperature through $a = a'(T - T_c)$, so that $A = 0$ when $T = T_c^* \equiv T_c - \Lambda^2H^2/(8\pi Na')$ and $A > 0$ for $T > T_c^*$, while $A < 0$ for $T < T_c^*$. From these relationships we can deduce that an approximate form of the phase coexistence line is

$$T_c(H) = T_c(0) - \frac{\Lambda^2H_c^2}{8\pi Na'}. \tag{2.6.11}$$

If Λ and N are only are only weakly temperature dependent, this expression represents the well known parabolic down turned plot for $T_c(H)$ vs H. Whereas the weak temperature dependence of Λ and N may be justified for standard superconductors, it does not appear to be valid for high T_c materials. The value of N is expected to be sensitive to the doping rate as well as to depend strongly on $(T - T_c)$. At this stage we can only speculate that the temperature dependence of Λ and N might provide an explanation of the two interesting features occurring in $H_{c1}(T)$ plots for high T_c superconductors, i.e., for a gradual drop off close to the critical temperature and a steep upturn close to zero Kelvin[22].

Inverting the relationship in equation (2.6.11) to find H_c as a function of temperature T such that the point trace the phase boundary, we obtain

$$H_c = \sqrt{8\pi a'}\,\frac{\sqrt{N(T)\left(T_c(0) - T\right)}}{\Lambda(T)} \tag{2.6.12}$$

We now postulate an empirical dependence of N and $|\Lambda|$ on temperature to be of the general forms

$$N(T) = N_0 (T_c(0) - T)^{2\mu} \tag{2.6.13}$$

and

$$\Lambda(T) = \Lambda_0 + \sum_{n=1}^{k} \Lambda_n T^n, \tag{2.6.14}$$

where N_o, μ, and Λ_k are parameters to be adjusted to experiment. The form of N is consistent with its expected behavior close to criticality. It is also expected that, as $T \rightarrow 0$ K, the value of Λ decreases to Λ_o, ensuring an upturn in $H_c(T)$ in agreement with experimental results. The slope of $H_c(T)$ close to absolute zero now depends crucially on the value of μ and, under normal circumstances, is always negative. In the other extreme limit, i.e., $T \rightarrow T_c(0)$, we obtain a predominantly linear approach to the temperature axis provided $\mu \geq 1/2$. In order to recover the standard form

$$H = H_0\left\{1 - \left(\frac{T}{T_c}\right)^2\right\},$$

it is sufficient to assume that

$$N \propto |T - T_c| \text{ and } A \propto (T + T_c)^{-1}.$$

It is very interesting to note that equation (2.6.8) has been the subject of a recent extensive investigation[12] using a sophisticated mathematical technique, namely, the symmetry reduction method. Here we only briefly summarize the results of this analysis. If $A \neq 0$, there can only be three general types of solutions to equation (2.6.8): (i) quasi-linear, where $t = t(x_1)$ and x_1 may be x, y, or z; (ii) cylindrical, where $t = t(r)$, with $r = \sqrt{x_i^2 + x_j^2}$, where x_i and x_j are any two of (x, y, z); and (iii) spherical, where $t = t(r)$ and $r = \sqrt{x^2 + y^2 + z^2}$.

In cases (ii) and (iii), equation (2.6.8) reduces to an Emden equation[23] with its localized and damped oscillatory solutions. On the other hand, when $A = 0$ identically, a scale invariance symmetry is present and new types of solutions are admitted in addition to cases (i) through (iii), including spiral structures with $t = t(\xi_1)$, where $\xi_1 = \arctan(y/x) + \alpha \ln(r)$ and hyperboloidal waves with $t = t(\xi_2)$, where $\xi_2^2 = x^2 + y^2 - z^2$. The most common solutions, however, are quasi-linear ones, which take the form of elliptic waves. In this case equation (2.6.8) can be integrated once to give

$$\frac{1}{2}\left(\frac{dt}{dx'}\right)^2 = \frac{At^2}{2} + \frac{Bt^4}{4} + C_0 = \frac{1}{2}p(t), \qquad (2.6.15)$$

where C_o is an integration constant. All of the solutions to equation (2.6.15) are listed and discussed in reference 12, but we shall just draw the attention of the reader to the following classes of nonsingular solutions.

(a) Constant solutions $t = 0$ or $t = (-A/B)^{1/2}$, representing the normal and superconducting phases, respectively.

(b) Localized solutions occur when $A < 0$ and $C_o = A^2/(4B)$. They are given by

$$t = \sqrt{\frac{-A}{B}}\left|\tanh\left(x'\sqrt{\frac{-A}{2}}\right)\right|. \tag{2.6.16}$$

They represent an expulsion of the superconducting charge from a region near the center of a magnetic vortex at $x' = 0$. Thus, the solutions for the magnetic field and the superconducting charge density are;

$$h = \frac{HA\sqrt{-2A}}{B}\left\{x'\sqrt{\frac{-A}{2}} - \tanh\left(x'\sqrt{\frac{-A}{2}}\right)\right\} + c_1 \tag{2.6.17}$$

and

$$n_s = \frac{-AN}{B}\tanh^2\left(x'\sqrt{\frac{-A}{2}}\right). \tag{2.6.18}$$

Equations (2.6.17) and (2.6.18) are represented by the curves plotted in figure II-23. In figure II-23, we obtain the expected behavior between the superconducting charge density and the magnetic field with the magnetic field expelled from the superconducting region. There is however a problem with the magnetic field in that it diverges as $x \to \pm\infty$ and thus we expect the solutions to become less accurate the farther we are from the origin, depending of course on the boundary conditions of the system.

(c) Periodic solutions may only occur when $A < 0$, and they are given by

$$t = \left|t_1 \,\mathrm{sn}\left(t_2\, x'\sqrt{\frac{B}{2}}, k\right)\right|, \tag{2.6.19}$$

where the Jacobi modulus $k = |t_1 / t_2|$ and the symbols t_1 and t_2 denote the two real roots of the polynomial $p(t) = (B/2)(t^2 - t_1^2)(t^2 - t_2^2)$ such that $|t_2| > |t_1|$. Note that $0 \le k \le 1$, and as $k \to 0$, $sn(\bullet) \to \sin(\bullet)$, while as $k \to 1$, $sn(\bullet) \to \tanh(\bullet)$. These solutions form periodic arrangements of "downward spikes" in the superconducting charge density along the x-axis. Thus, from equation (2.6.19), the magnetic field h and the superconducting charge density n_S are;

$$h = \frac{H\Lambda t_1^2}{t_2 k^2}\sqrt{\frac{2}{B}}\left\{x' t_2\sqrt{\frac{B}{2}} - E\left(am\left(x' t_2\sqrt{\frac{B}{2}}\right), k\right)\right\} + c_1 \qquad (2.6.20)$$

and

$$n_s = N t_1^2 sn^2\left(t_2\, x'\sqrt{\frac{B}{2}}, k\right). \qquad (2.6.21)$$

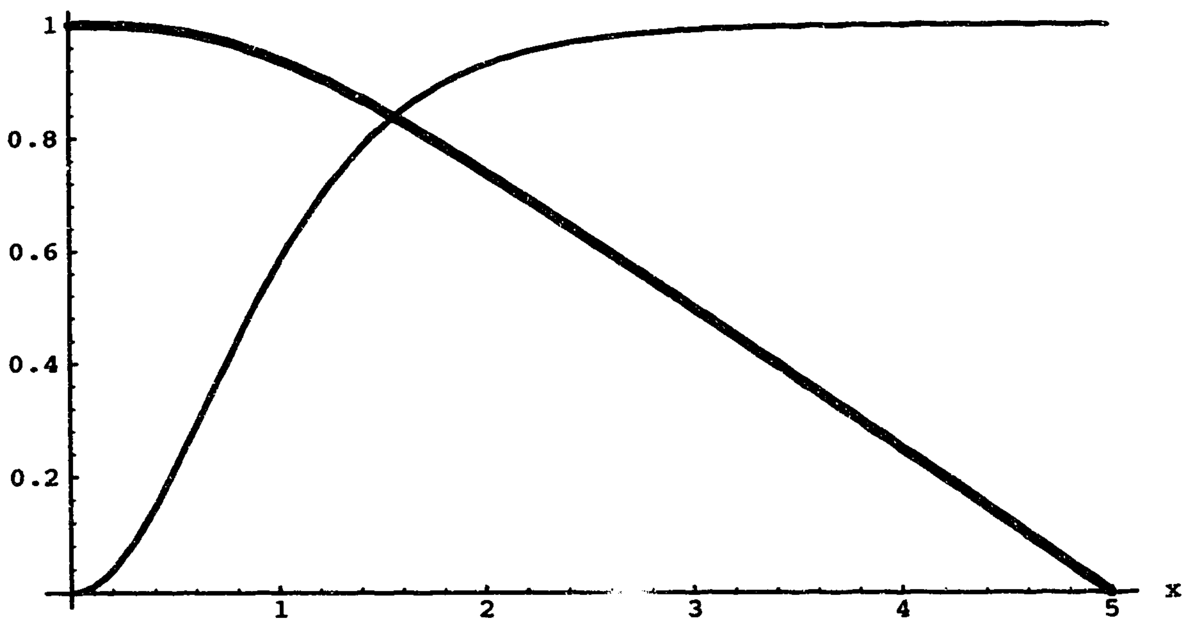

Figure II-23: Plot of the magnetic field represented by the thick line and the superconducting charge density represented by the thin line, as given by equations (2.6.17) and (2.6.18) respectively.

The solutions given in equations (2.6.20) and (2.6.21) are represented in figure II-24 and again show the proper behavior between the magnetic field and the superconducting charge density. However, as with the previous case the magnetic field diverges as $x \to \pm\infty$ and thus boundary conditions are needed.

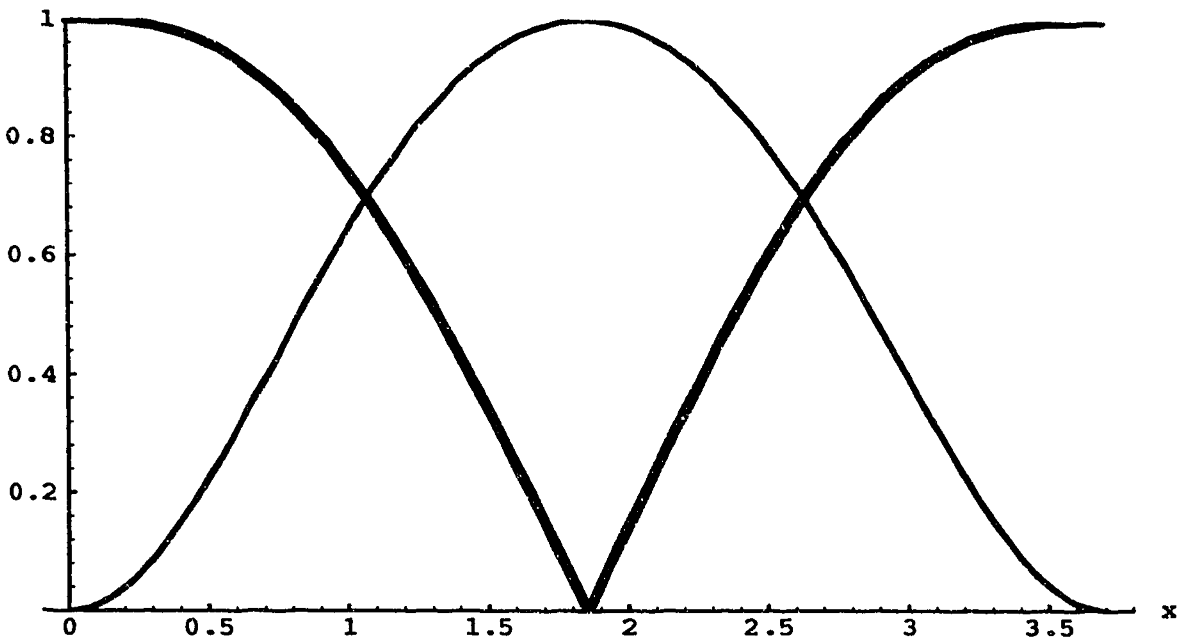

Figure II-24: Plot of the magnetic field represented by the thick line and the superconducting charge density represented by the thin line, as given by equations (2.6.20) and (2.6.21) respectively.

If we allow the coefficient B to be negative, then two additional types of elliptic waves and one solitary wave will be added to this list, i.e., we have the following.

(d) Cnoidal waves:

$$t = \left| t_1 \, \mathrm{cn}\left(x' \sqrt{\frac{-B\left[t_1^2 - t_2^2\right]}{2}}, k \right) \right|, \tag{2.6.22}$$

where t_1 is real, t_2 is imaginary, and $k^2 = t_1^2 / (t_1^2 - t_2^2)$. These solutions appear to represent a modulated superconducting phase with intermittent gaps in the superconducting order parameter which are much broader than the "spikes" in (c). For this case, the solutions for the magnetic field h and the superconducting charge density n_s are;

$$h = \frac{H\Lambda t_1^2}{k^2}\left\{\sqrt{\frac{2}{-B(t_1^2 - t_2^2)}}E\left(\mathrm{am}\left(x'\sqrt{\frac{-B(t_1^2 - t_2^2)}{2}}\right), k\right) - (1 - k^2)x'\right\} + c_1 \qquad (2.6.23)$$

and

$$n_s = Nt_1^2\mathrm{cn}^2\left(x'\sqrt{\frac{-B[t_1^2 - t_2^2]}{2}}, k\right). \qquad (2.6.24)$$

The solutions given by equations (2.6.23) and (2.6.24) have been plotted in figure II-25. Even though figure II-25 shows an agreement with what we expect, for generic boundary conditions it does not, since the magnetic field is divergent when $x' \to \pm\infty$ and thus special boundary conditions are needed.

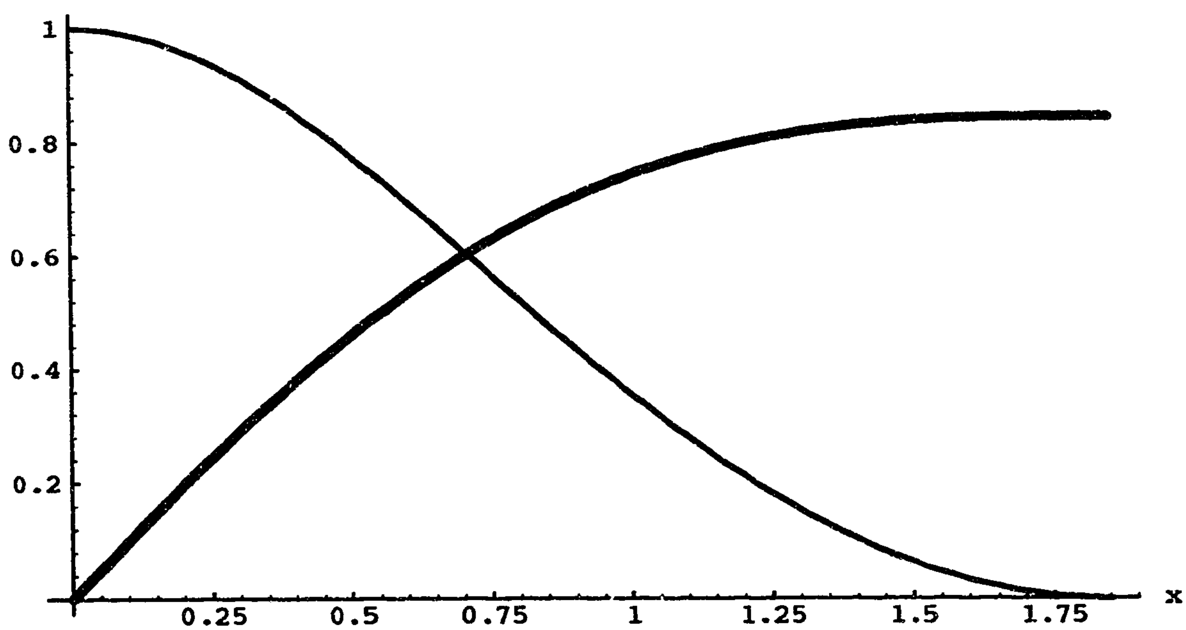

Figure II-25: Plot of the magnetic field represented by the thick line and the superconducting charge density represented by the thin line, as given by equations (2.6.23) and (2.6.24) respectively.

(e) Dnoidal waves:

$$t = \left| t_2 \mathrm{dn}\left(t_2 x' \sqrt{\frac{-B}{2}}, k \right) \right|, \tag{2.6.25}$$

where $k = (1 - t_1^2 / t_2^2)^{1/2}$, and these waves can be interpreted as small spatially periodic fluctuations of the superconducting order parameter envelope. For this case, the magnetic field h and the superconducting charge density n_S become;

$$h = H\Lambda t_2 \sqrt{\frac{2}{-B}} E\left\{ \mathrm{am}\left(x' t_2 \sqrt{\frac{-B}{2}} \right), k \right\} + c_1 \tag{2.6.26}$$

and

$$n_s = N t_2^2 \mathrm{dn}^2\left(t_2 x' \sqrt{\frac{-B}{2}}, k \right). \tag{2.6.27}$$

The solutions for the magnetic field and the superconducting charge density, given by equations (2.6.26) and (2.6.27) are represented in figure II-26. The superconducting charge density is nonzero for all x, except when $k \to 1$ and dn becomes a sech function which vanishes only at infinity. Thus the superconducting charge density is a well behaved periodic function, however, the magnetic field is a divergent function as can be seen in figure II-26 and does not seem to be dependent upon the form or value of the superconducting charge density. The only reasonable value for the magnetic field in this case seems to be zero.

(f) Solitary waves in the form of a bump:

$$t = \left| t_1 \mathrm{sech}\left(t_1 x' \sqrt{\frac{-B}{2}} \right) \right|, \tag{2.6.28}$$

This solution can be found as a limiting case of (d) with only a single domain of superconductivity nucleating out of a normal background. This gives a simplified version for the magnetic field h and the superconducting charge density n_S as given by equations (2.6.23) and (2.6.24) in the following form;

$$h = H\Lambda t_1\sqrt{\frac{-2}{B}}\tanh\left\{x' t_1\sqrt{\frac{-B}{2}}\right\} + c_1 \tag{2.6.29}$$

and

$$n_s = N t_1^2 \operatorname{sech}^2\left(t_1 x'\sqrt{\frac{-B}{2}}\right). \tag{2.6.30}$$

The solutions for the magnetic field and the superconducting charge density as given by equations (2.6.29) and (2.6.30) are represented in figure II-27. We see from figure II-27 that the solutions are well behaved for all x and give results as we have expected with a nucleation center of superconducting charge and surrounded by a magnetic field.

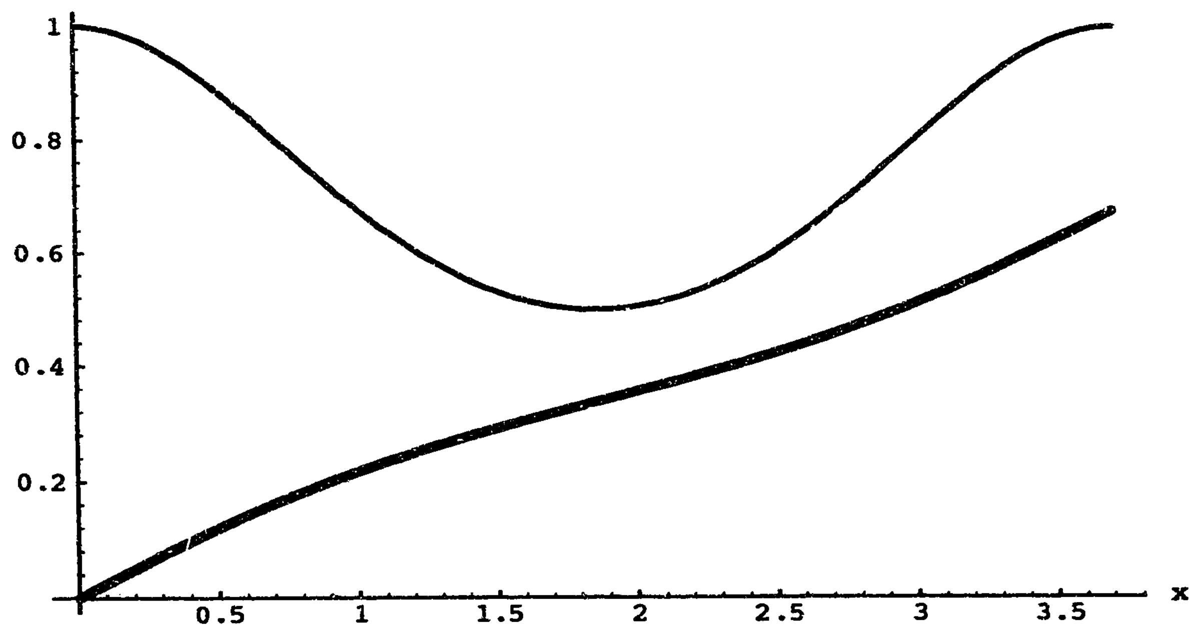

Figure II-26: Plot of the magnetic field represented by the thick line and the superconducting charge density represented by the thin line, as given by equations (2.6.26) and (2.6.27) respectively.

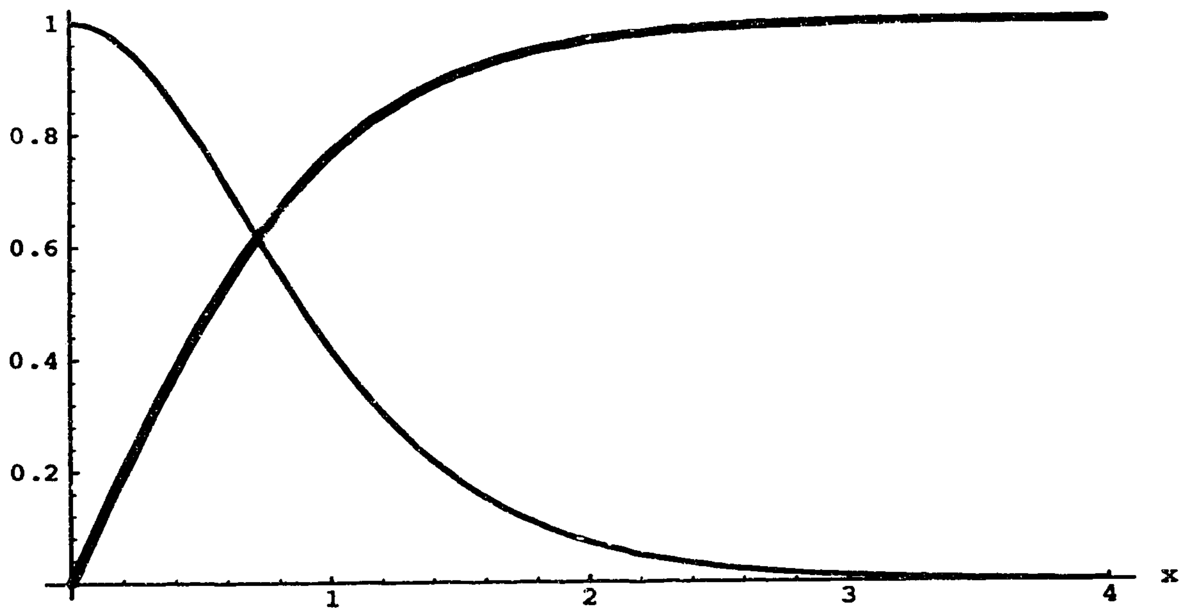

Figure II-27: Plot of the magnetic field represented by the thick line and the superconducting charge density represented by the thin line, as given by equations (2.6.29) and (2.6.30) respectively.

SECTION VII: SUMMARY AND CONCLUSION

In this chapter, we have investigated solutions to the equations of state describing magnetic field penetration in a LG superconductor. Following the minimization procedure, three approximate but very accurate approaches have been presented. In the third section, three different sets of solutions in one dimension at the critical temperature were found and then these solutions were expanded below the critical temperature. The first set of solutions obtained gave a periodically fluctuating solution for the superconducting charge density and also a periodic solution for the magnetic field but shifted by a quarter of a period relative to the superconducting charge density. The first order corrections for just below the critical temperature were also found. The solutions for the homogeneous first order equations provide a wealth of possibilities for change in the solutions as the temperature is lowered. The particular solution for the first order corrections without the homogeneous solutions provided us with the expected behavior in the magnetic field and the superconducting charge density by increasing the range over which superconducting charge density is dominant and decreasing the range over which the magnetic field is dominant. The inclusion of the homogeneous solutions would be necessary to fit the solutions to the boundary conditions. The second set of solutions are similar to the first set. Both the magnetic field and the superconducting charge density are periodic and shifted by a quarter of a period relative to each other. The main difference between this set of solutions and the first set is that the magnetic field vanishes at the critical temperature and thus represents trapped flux with no external magnetic field present. The third set of solutions found in section three represent an extreme case in which the magnetic field is so strong it has just about driven the superconductor into the normal phase. This leaves only the surface of the material superconducting and the rest of the material in the normal phase. At the end of section three a numerical plot of the full equations was presented so as to test how accurate the expansion was for this model. Even though it was not explicitly shown to be true, the first set of solutions and to a slightly lesser degree, the second set of solutions can replicate the numerical plot to very good agreement. This means that even though the solutions are only to first order in the parameter α, the series converges very fast giving us good results to work with. In the fourth section, the effects of the magnetic field upon the superconducting charge is neglected. By treating the superconducting current density as a constant, the effects of the magnetic field upon the superconducting charge density are ignored. The equation describing the superconducting charge density under this approximation has been previously analyzed. By simplifying the equation describing the magnetic field, a new independent variable was obtained which is the indefinite integral of the superconducting charge density. In the second part of this section, besides the trivial solution and the standard solution where the superconducting charge density is taken to be a constant, one further solution was obtained where no further approximations were needed. The solution corresponds to the superconducting charge density being a sech^2 solution, which would correspond to a superconducting plane. The rest of the solutions for the superconducting charge density led to very complicated indefinite integrals which needed to be approximated further in order to solve the equation for the magnetic field. A few of the limiting cases were considered and even though these were limiting cases and required further approximations, they did provide solutions that were physically correct. The last part of section four dealt with looking at the equation for the magnetic field from a different point of view. This method gave solutions for the magnetic field and the superconducting charge density for two special cases and also we obtained recursion relationships for the general solution The first special case was when there was no cubic nonlinearity and no superconducting current density and the second special case was when there was just no superconducting current density. In sections three and four, only one dimensional symmetry was discussed and so in section five the equations of state with a cylindrical symmetry was considered. The method used to solve the equations of state in this section was the same as that in section III, an expansion about the critical temperature. Section five contained two parts, the first was called vortices and the second spirals. The first part of section five considered the magnetic field and the superconducting charge density to have only radial dependence. The vortex part of this section led to many interesting solutions some of which were very singular and others that appear to have applications to magnetic field penetration in high temperature superconductors. Some of the obtained solutions included a solution for the magnetic field close to the core of a magnetic vortex and a solution of damped periodic rings of superconducting charge density to mention a couple. The spiral part of section five gave solutions that were similar to those obtained in the vortex part. For the spiral part of the section, it was assumed that the magnetic field and the superconducting charge density were both functions of an independent variable that leads to spiral solutions. For this part, to simplify the equations, the radial distance from the core was assumed to be small so that the ln(r) in the spiral symmetry is dominate. As in the vortex part of section five, many solutions were obtained, some of which were kink solutions and damped periodic solutions. The sixth section dealt with rescaling the magnetic field and the superconducting charge density in order to validate an approximation where the magnetic field is written as a function of the superconducting charge density and thus decoupled the two equations. The decoupling gave a single equation which was the cubic nonlinear Klein - Gordon equation which has been studied in the past. These solutions were reviewed in section six as were the corresponding magnetic field and superconducting charge density that resulted from these solutions. For nontrivial solutions, there were five sets of solutions found for the magnetic field and the superconducting charge density. Even though the solutions found were physically acceptable for specific boundary conditions, the generic solutions appeared to be unphysical, with the magnetic field being unbounded in most of the cases. There were a few cases that gave well behaved solutions for the generic case.

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CHAPTER 3: A METHOD FOR FINDING ANALYTICAL SOLUTIONS TO SOME NONLINEAR DIFFERENTIAL EQUATIONS OF DISSIPATIVE CRITICAL DYNAMICS*

SECTION I: INTRODUCTION

Much of the mathematical difficulty related to solving the problems of modern many-body physics is due to the fact that the resultant nonlinear differential equations are often very difficult to treat, especially when dissipative effects are present. Unlike the linear formalism of classical electrodynamics and quantum mechanics, nonlinear physics as yet has not developed a systematic and consistent mathematical approach. The nonlinear differential equations that are often found in the descriptions (for example in the kinetics of critical systems) possess many interesting properties (e.g. solitons, limit cycles, instabilities, chaotic behavior, etc.) but are also much more poorly understood than their linear counterparts. For example, unless an ordinary differential equation (ODE) satisfies a so-called Painlevé property[1)], the hope of finding analytical solutions to it is virtually nonexistent[2-5,6,78)].

At this point, we shall briefly look at the Painlevé property and the Painlevé test[4,9,10)]. There are different types of singularities that the solutions to differential equations may possess, namely; poles, branch points, and essential singularities. The equations that are of the Painlevé-type are those equations which have nonmovable singularities except the poles. Note that a movable singularity is one which depends upon the constants of integration and thus change when the boundary and/or initial conditions change. For second order ordinary differential equations, there are fifty different types of equations that are of the Painlevé-type and are listed by Ince[4)]. The solutions to Painlevé-type equations can be written in terms of elementary functions, elliptic functions or in

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* A version of this chapter has been accepted for publication. K. Vos and J. A. Tuszyński. 1992. Physical Review A.

terms of the six Painlevé transcendentals. The Painlevé test for second order differential equations of degree one (i.e. $f'' = F(f', f, x)$) need the function F to be rational in f' and f, and also we need the independent variable x to be analytical. If the ordinary differential equation is of this form then there are three steps involved in the Painlevé test. The first part is to obtain the *leading term* in a Laurent series and find the lowest power possible for $(x - x_0)$. For this part we want the power to be a negative integer so that we obtain a pole. The second part of the test is to look at the *resonance,* which is the power of the first term after the leading term. For example, if p is the power of the leading term and is a negative integer then in the second term we have $(x - x_0)^{p+r}$ where r is our resonance. The second part of the test is past if r is a positive integer. The last part of the test with finding the expansion coefficients in terms of the integration constants and is achieved by introducing the expansion $a(x - x_0)^{p+r} + \sum_{k=1}^{r} a_k(x - x_0)^{p+k}$. It should be pointed out that some equations with essential singularities will pass this test while not being of the Painlevé-type since this is a necessary condition, it is not a sufficient condition.

Even when we fix our attention on equations with a single dependent variable (for single component critical systems, for example), the situation looks rather grim since most of the multidimensional (in terms of independent variables, i.e. space-time) symmetry reduction analyses for physically relevant partial differential equations (PDE's) result in non-Painlevé ODE's. However, many of those reductions lead to autonomous ODE's which quite often take the form

$$\frac{d^2f}{dt^2} + A(f)\frac{df}{dt} + B(f)\left(\frac{df}{dt}\right)^2 + C(f) = 0. \tag{3.1.1}$$

This could be broadly classified as an anharmonic oscillator equation with both linear and quadratic friction (or dissipation) terms. Thus, it could also apply to a number of physically interesting cases of multistable systems with velocity-dependent friction terms. In particular, kink and nucleation center dynamics[11)] falls in this category. In terms of possible physical applications a recent example involves single-mode wave propagation in a Kerr dielectric guide when higher-order nonlinearities are accounted for[12] and thus applies to optical solitons in fibers. It is interesting to note that reductions of particular PDE's to this form can be found in the multidimensional time-dependent Landau-Ginzburg equation[6] (for spatial-only, traveling-wave and spiral-type solutions), the nonlinear Schrödinger equation[7] (for traveling waves), and the nonlinear Klein-Gordon equation[8] (for a host of symmetry variables, including degenerate ones). It has also been recently demonstrated[13] that the celebrated Emden equation (for spherical and cylindrical patterns in critical dynamics) can be transformed to

$$f'' + f'(b_0 + b_1 f) + b_3 f(f - f_1)(f - f_2) = 0 \qquad (3.1.2)$$

which is also a special case of equation (3.1.1). A very interesting result has been published[14] not long ago in connection with a transition from discrete lattice equations to continuum. This procedure has been shown to generate higher order friction terms yielding an equation of the form

$$f'' + \lambda_1 f' + \lambda_2 (f')^2 + \mu(f) = 0, \qquad (3.1.3)$$

which is also a special case of equation (3.1.1). Thus, we conclude that this class of ODE's appears quite prominently in both dissipative critical dynamics and in conservative multidimensional critical systems possessing special symmetry properties.

Our motivation in the present chapter is to study equation (3.1.1) and seek analytical ways of finding its solutions even if they turn out to be special ones only. The approach that will be presented is a systematic one and is a significant generalization of the one proposed in a recent paper by Otwinowski et al[15] who developed an ansatze approach to solving a generalized anharmonic oscillator equation with linear dissipation, i.e.

$$f'' + \lambda f' + C(f) = 0, \qquad (3.1.4)$$

where C(f) is a polynomial in f. Their basic assumption was to seek solutions to equation (3.1.4) and which also satisfy simultaneously the first order ODE

$$f' = D(f), \qquad (3.1.5)$$

where D(f) is á priori unknown. Our method goes a significant step beyond the one given by Otwinowski et al[15] in both the scope of applications and the form of solutions. Theirs succeeds in finding only localized kink and bump solutions while ours will not be limited this way.

In what is to follow, we shall present a technique by which analytical solutions to equation (3.1.1) can be obtained. First, we shall look at some special cases when two of the functions in equation (3.1.1) A(f), B(f), and C(f) are arbitrary functions and the third one is a specific function of f and shall give the solutions to these special cases. Second, we will look at the case when A(f), B(f), and C(f) are polynomials in f with constant coefficients. In Section II we define a new function D(f) as the indefinite integral of B(f) (i.e. $D(f) = \int df\, B(f)$). Throughout the following discussion, we shall consider $A(f) \neq 0$ in equation (3.1.1) except for special case 1 in section II which gives the general solution to equation (3.1.1) in principle, when $A(f) = 0$.

This chapter is organized as follows. The second section shows seven special cases of solutions and also discusses the method of reducing the differential equation into a system of algebraic equations. It is also discovered that for one of the special cases, elliptic integrals are a subclass of the resulting integral solution. In section three, further details on the reduction of the nonlinear ordinary differential equation to a system of algebraic equations is discussed. In this section, it is shown that by one reduction, exponential solutions are obtained and by another reduction, Jacobi elliptic functions are obtained as solutions. Section IV looks at a few physical examples where the algebraic method and the special cases are applicable. Conclusions and an outlook for future generalizations close the chapter.

SECTION II: THE METHOD AND ITS SPECIAL CASES

In order to find special analytical solutions of equation (3.1.1) we make the following transformation

$$f = \frac{R(z)}{S(z)}. \tag{3.2.1}$$

Also, we define the new independent variable z through

$$z' = \frac{T(z)}{U(z)}. \tag{3.2.2}$$

The primary motivation for this type of transformation comes from equation (2.5.29) which is not of the Painlevé-type and we had originally tried the anatze $\omega' = A + B\omega + C\omega^2$ which lead to inconsistancies in the three parameters. Thus, I felt that the introduction of more parameters might circumvent this difficulty. The introduction of a new dependent variable brought in a significantly larger number parameters where it was hoped that a solution for the parameters without inconsistencies would be found.

From equations (3.2.1) and (3.2.2), we find that:

$$f' = \frac{TV}{US^2}, \tag{3.2.3}$$

and

$$f'' = \frac{T}{U^3S^3}\left\{USVT' + T\left[USV' - (SU' + 2S'U)V\right]\right\}, \tag{3.2.4}$$

where V = R'S - RS'. Since the independent variable is transformed through equation (3.2.2) and the dependent one is assumed quite general in equation (3.2.3) our ansatze is much more general than that of equation (3.1.5) as postulated by Otwinowski et al[13]. Substitution into equation (3.1.1) leads to

$$\frac{T}{U^3S^3}\left\{T'USV + T\left[USV' - (SU' + 2S'U)V\right]\right\} + A\left(\frac{R}{S}\right)\frac{TV}{US^2} + B\left(\frac{R}{S}\right)\frac{T^2V^2}{U^2S^4} + C\left(\frac{R}{S}\right) = 0. \tag{3.2.5}$$

The method we propose to consider relies on analyzing and solving equation (3.2.5) instead of equation (3.1.1). This can be done quite readily for some special cases listed below. In these <u>special cases,</u> we let

$$T = \frac{dz}{df}\exp\left[-\int^f dx\, B(x)\right] = \frac{dz}{df}\exp[-D(f)], \tag{3.2.6}$$

where

$$D(f) = \int^f dx\, B(x). \tag{3.2.7}$$

is the indefinite integral of B(f). Thus, equations (3.2.5) and (3.2.2) become respectively;

$$\frac{dU}{df} = \left\{A(f) + C(f)e^D U\right\}e^D U^2 \tag{3.2.8}$$

and

$$t - t_0 = \int^f dx\, e^{D(x)}U(x), \tag{3.2.9}$$

where the variable t is our original independent variable. Clearly, solving equations (3.2.8) and (3.2.9) is equivalent to solving equation (3.1.1). There are seven special cases that solve equations (3.2.8) and (3.2.9), which are as follows:

Special Case I

Here A(f) = 0. In this case, equation (3.1.1) takes the form

$$f'' + B(f)(f')^2 + C(f) = 0. \qquad (3.2.10)$$

Clearly, the corresponding solution given in Table III-1 is only formal since the integrals in general cannot be done. In fact the elliptic integrals satisfy an equation which is a special subcase of equation (3.2.10).

There is some information that can be gained without specifying what B(f) and C(f) are. To do this we consider the classical Hamiltonian $H = (df/dt)^2 + V(f)$ where in this special case the corresponding potential is given by

$$V(f) = 2e^{-2D(f)}\int^{f} dx\, C(x)e^{2D(x)}. \qquad (3.2.11)$$

For periodic solutions to exist we need bounded solutions which means that V(f) has to have at least one local minimum. Also, for the system to have a separatrix, V(f) needs to have a local maximum along with the local minimum. Thus,

$$\left.\frac{dV}{df}\right|_{f=f_1} = 0 = C(f_1) - 2B(f_1)e^{-2D(f_1)}\int^{f_1} dx\, C(x)e^{2D(x)} \qquad (3.2.12)$$

and

$$\left.\frac{d^2V}{df^2}\right|_{f=f_1} = 2\left.\frac{dC}{df}\right|_{f=f_1} - \frac{2C(f_1)}{B(f_1)}\left.\frac{dB}{df}\right|_{f=f_1}. \qquad (3.2.13)$$

Clearly, the first derivative of the coefficient B plays a very important role in shifting a local maximum to a local minimum and vice versa. Also, B itself causes a shifting in the location of the local extrema. We shall now consider a simple example by noting that when B = 0 and C = f, then we have periodic solutions so that when B = 0 and C= -f, then there are no periodic solutions. If we now consider the case B = b/f and C = -f, then we do obtain periodic solutions when $b < -1$ and no periodic solutions for $b \geq -1$. Since Special Case 1 covers all cases when A(f) = 0, we will consider A(f) ≠ 0 in the rest of the chapter.

Special Case II

Here, C(f) = 0 and consequently, equation (3.1.1) takes the form

$$f'' + A(f)f' + B(f)(f')^2 = 0. \tag{3.2.14}$$

Also, the corresponding potential here is

$$V(f) = e^{-2D(f)}\left\{\int^f dx\, A(x)e^{D(x)}\right\}^2. \tag{3.2.15}$$

Special Case III

In this case we assume that $A(f) = C(f)\, e^{D(f)}$, and thus equation (3.1.1) becomes

$$f'' + A(f)f' + B(f)(f')^2 + A(f)\exp[-D(f)] = 0. \tag{3.2.16}$$

In this case, the formula for its solution shown in Table III-1 needs to be inverted before we can substitute for U in equation (3.2.9). However, depending on what the integral on the left hand side of this equation is, it may be easier to take f as a function of U and then replace the integrand in equation (3.2.9) with the appropriate integral in U. Once this integral is solved, one may be able to invert equation (3.2.9) so as to obtain U as a function of t and thus obtain f as a function of t.

A simple approximation is to assume $1/U \approx 0$ so that we can expand the logarithm to obtain

$$t - t_0 = \int^f dx \frac{e^{D(x)}}{\sqrt{-2\int^x dy\, A(y)e^{D(y)}}}, \tag{3.2.17}$$

which has the potential

$$V(f) = -2e^{-2D(f)}\int^f dy\, A(y)e^{D(y)}. \tag{3.2.18}$$

Special Case IV

Here, we take

$$A(f) = U(f) = \left\{-2\int^f dy\,[1 + C(y)e^{D(y)}]e^{D(y)}\right\}^{-1/2}.$$

In this case, equation (3.1.1) becomes an integro-differential equation of the form

$$f'' + \frac{f'}{\sqrt{-2\int^f dy\,[1 + C(y)e^{D(y)}]e^{D(y)}}} + B(f)(f')^2 + C(f) = 0. \tag{3.2.19}$$

Also, the corresponding potential is

$$V(f) = 2e^{-2D(f)}\int^f dx[1 + C(x)e^{D(x)}]e^{D(x)}. \tag{3.2.20}$$

Special Case V

Here,

$$C(f) = \frac{1}{U(f)} = -\int^f dy[A(y) + e^{D(y)}]e^{D(y)}.$$

In this case, equation (3.1.1) is

$$f'' + A(f)f' + B(f)(f')^2 - \int^f dy[A(y) + e^{D(y)}]e^{D(y)} = 0. \tag{3.2.21}$$

The potential in this case is

$$V(f) = -e^{-2D(f)}\left\{\int^{f} dx[A(x) + e^{D(x)}]e^{D(x)}\right\}^2. \tag{3.2.22}$$

Special Case VI

Here, we assume that

$$e^{D(f)} = \frac{1}{U(f)} \quad \text{or} \quad B(f) = \frac{-1}{U(f)}\frac{dU}{df} = -A(f) - C(f),$$

and, as a result we can generalize this condition some what so that we obtain

$$f = c_0 t + f_0. \tag{3.2.23}$$

In this case, equation (3.1.1) becomes

$$f'' + A(f)f' + B(f)(f')^2 - c_0 A(f) - c_0^2 B(f) = 0. \tag{3.2.24}$$

This is a trivial situation and is not of much interest.

Special Case VII

Here, $A(f)$ = constant (a) and $C(f) = e^{-D(f)}$. Also, equation (3.1.1) becomes

$$f'' + af' + B(f)(f')^2 + \exp[-D(f)] = 0. \tag{3.2.25}$$

In this case, we see from Table III-1 that the indefinite integral must be determined before we can write f as an explicit function of the independent variable t. This can not always be done since for a general D(f), the integral can not be solved.

For this special case, we have two simple examples;

(i) Let $B(f) = b/f$ so that $e^{D(f)} = f^b$ and thus our differential equation becomes;

$$f'' + af' + b(f')^2/f + f^{-b} = 0$$

and the solution is found from Table III-1 to be;

$$f = \begin{cases} \exp\left(c_0 + \dfrac{1 - a(t - t_0) - e^{-a(t-t_0)}}{a^2} \right), & b = -1, \\ (b+1)^{1/(b+1)} \left\{ c_0 + \dfrac{1 - a(t - t_0) - e^{-a(t-t_0)}}{a^2} \right\}^{1/(b+1)}, & b \neq -1. \end{cases}$$

(ii) For this example, our differential equation becomes;

$$f'' + af' + b\tanh(f)\,(f')^2 + (\operatorname{sech} f)^b = 0$$

and the solution for is found from Table III-1 to be;

$$f = \begin{cases} \ln\left|F(t) + \sqrt{F^2(t) + 1}\right|, & b = 1 \\ \ln\left|\dfrac{1 + \sqrt{2}\,|sn(u,k)dn(u,k)|}{cn^2(u,k)}\right|, & b = -\dfrac{1}{2} \\ \ln\left|\tan\left(\dfrac{F(t)}{2}\right)\right|, & b = -1 \\ \dfrac{1}{2}\ln\left|\dfrac{1 + F(t)}{1 - F(t)}\right|, & b = -2 \end{cases},$$

where we have chosen four specific values for b. Also, $F(t) = c_0 + a^{-2}(1 - e^{-a(t-t_0)}) - a^{-1}(t - t_0)$, $u = F(t)/\sqrt{2}$, and $k^2 = 1/2$.

Table III-1 gives the form of the equation and its corresponding solution. The solutions are written as indefinite integrals and thus the constants of integration must not be forgotten since this significantly changes the form of the solution. These integrals in general cannot be solved analytically as can be seen in case I which contains the elliptic equations as a special subcase. Cases II through VII are very particular cases but may be of interest in special applications in physics. In case VII, we have only one general function B(f), while C(f) is related to B(f) and A(f) is a constant. In the next section we develop a general approach to analyzing and solving equation (3.1.1).

The results of this section are summarized in table III-1 below:

TABLE III-1: Special cases of Equation (3.1.1).

Case	Equation	Solution
I	$f'' + B(f)\,(f')^2 + C(f) = 0$	$t - t_0 = \int^f dx \frac{e^{D(x)}}{\sqrt{-2\int^x dy\, C(y) e^{2D(y)}}}$
II	$f'' + A(f)\, f' + B(f)\,(f')^2 = 0$	$t - t_0 = -\int^f dx \frac{e^{D(x)}}{\int^x dy\, A(y) e^{D(y)}}$
III	$f'' + A(f)f' + B(f)(f')^2 + A(f)\exp[-D(f)] = 0$	$t - t_0 = \int^f dx\, e^{D(x)} U(x)$, where $\int^f dx\, e^{D(x)} A(x) = \ln\left\lvert 1 + \frac{1}{U}\right\rvert - \frac{1}{U}$
IV	$f'' + \frac{f'}{\sqrt{-2\int^f dx [1 + C(x) e^{D(x)}] e^{D(x)}}} + B(f)(f')^2 = -C(f)$	$t - t_0 = \int^f dx \frac{e^{D(x)}}{\sqrt{-2\int^x dy [1 + C(y) e^{D(y)}] e^{D(y)}}}$
V	$f'' + A(f)f' + B(f)(f')^2 - \int^f dx \left[A(x) + e^{D(x)}\right] e^{D(x)} = 0$	$t - t_0 = -\int^f dx \frac{e^{D(x)}}{\int^x dy \left[A(y) + e^{D(y)}\right] e^{D(y)}}$
VI	$f'' + A(f)f' + B(f)(f')^2 - c_0 A(f) - c_0^2 B(f) = 0$	$f = c_0 t + f_0$
VII	$f'' + a f' + B(f)(f')^2 + e^{-D(f)} = 0$	$\int^f dx\, e^{D(x)} = c_0 + \frac{1 - a(t - t_0) - e^{-a(t-t_0)}}{a^2}$

SECTION III: A GENERAL ANALYSIS

This section deals with cases that do not fall in Section II and where A(f), B(f), and C(f) are ratios of polynomials in f with no explicit dependence on the independent variable t. We shall denote P(t; k) as representing a polynomial in the variable t to order k (i.e. $p_o + p_1 t + p_2 t^2 + \ldots + p_k t^k$). Then, A(f), B(f) and C(f) can be represented as

$$A(f) = \frac{A_1(f)}{A_2(f)} = \frac{P(f;\alpha_1)}{P(f;\alpha_2)}, \tag{3.3.1}$$

$$B(f) = \frac{B_1(f)}{B_2(f)} = \frac{P(f;\beta_1)}{P(f;\beta_2)}, \tag{3.3.2}$$

and

$$C(f) = \frac{C_1(f)}{C_2(f)} = \frac{P(f;\chi_1)}{P(f;\chi_2)}. \tag{3.3.3}$$

Thus, equation (3.1.1) can be written as follows

$$E_0(f)\frac{d^2f}{dt^2} + A_0(f)\frac{df}{dt} + B_0(f)\left(\frac{df}{dt}\right)^2 + C_0(f) = 0, \tag{3.3.4}$$

where $E_0(f) = P(f;\, \varepsilon_0)$, $A_0(f) = P(f;\, \alpha_0)$, $B_0(f) = P(f;\, \beta_0)$, $C_0(f) = P(f;\, \chi_0)$.

The purpose of this section is to reduce the problem of solving the nonlinear ordinary differential equation into an equivalent form of solving a known differential equation and a system of nonlinear algebraic equations. This is more desirable since the algebraic system is easier to solve using symbolic solver programs that are available on computers. We can accomplish this objective by introducing a new dependent variable and transforming equation (3.1.1) into a polynomial in the new dependent variable. Thus, for the polynomial to be zero, we must demand that each coefficient of each power of the new dependent variable is zero. This will produce a certain number of parameters and a certain number of coefficients which become our algebraic equations. Now, if we assume that

$$f = \frac{R(z)}{S(z)} = \frac{P(z;\kappa)}{P(z;\kappa)} \text{ and } \frac{dz}{dt} = T(z) = P(z;n),$$

then equation (3.3.4) becomes

$$\frac{F_1(z)+F_2(z)}{S^{\varepsilon_0+3}} + \frac{F_3(z)}{S^{\alpha_0+2}} + \frac{F_4(z)}{S^{\beta_0+4}} + \frac{F_5(z)}{S^{\chi_0}} = 0, \tag{3.3.5}$$

where

$$F_1(z) = E_0\left(\frac{R}{S}\right)S^{\varepsilon_0}T^2(SV' - S'V) = P[z;(\varepsilon_0+3)\kappa + 2n - 4], \tag{3.3.6}$$

$$F_2(z) = E_0\left(\frac{R}{S}\right)S^{\varepsilon_0}TV(ST' - S'T) = P[z;(\varepsilon_0 + 3)\kappa + 2n - 4], \tag{3.3.7}$$

$$F_3(z) = A_0\left(\frac{R}{S}\right)S^{\alpha_0}TV = P[z;(\alpha_0 + 2)\kappa + n - 2], \tag{3.3.8}$$

$$F_4(z) = B_0\left(\frac{R}{S}\right)S^{\beta_0}T^2V^2 = P[z;(\beta_0 + 4)\kappa + 2n - 4], \tag{3.3.9}$$

$$F_5(z) = C_0\left(\frac{R}{S}\right)S^{\chi_0} = P[z;\chi_0\kappa], \tag{3.3.10}$$

and

$$V(z) = \frac{dR}{dz}S - R\frac{dS}{dz}. \tag{3.3.11}$$

The primes in equations (3.3.6) and (3.3.7) represent differentiation with respect to z. The $F_1(z)$ and $F_2(z)$ terms in equation (3.3.5) arise from f" in equation (3.3.4), the $F_3(z)$ term comes from $A_0(f)f'$, the $F_4(z)$ term comes from $B_0(f)(f')^2$, and the $F_5(z)$ term comes from $C_0(f)$. As can be seen, the power n in each term of equation (3.3.5) appears as n-2 or 2n-4 and thus can be eliminated by choosing n=2. This corresponds to T(z) being a quadratic which means that along with solving the system of algebraic equations, we have to solve the following first order ordinary differential equation on the independent variable

$$\frac{dz}{dt} = \tau_0 + \tau_1 z + \tau_2 z^2, \tag{3.3.12}$$

and since the coefficients are complex constants in general, this leads to hyperbolic and/or trigonometric solutions. It should be noted that the definitions of $F_1(z)$ and $F_2(z)$ are not completely true and that they are polynomials $P[z; (3+\varepsilon_0)\kappa + 2n - 3]$. However, the coefficient of $z^{(3+\varepsilon_0)\kappa+2n-3}$ contains n-2 as a factor after $F_1(z)$ and $F_2(z)$ have been added together and thus cancel when n=2. The n=2 requirement can also be interpreted as the square root of a quartic polynomial which changes equation (3.3.12) into an elliptic differential equation. However, $F_3(z)$ gives rise to an extra square root, which would have to be eliminated. To eliminate the square root, we have two choices, one is to have $A(f) = 0$ or else to place $F_3(z)$ on the right hand side of equation (3.3.5) and square both sides. For $A(f) = 0$, all we need to do is use case I in table III-1 and thus this procedure is not necessary. The alternative of squaring both sides to eliminate the square root shall be considered later on and at present we will consider T(z) as given in equation (3.3.12). It should be noted that one could extend this argument to a polynomial of order 2n taken to the (1/n)th power. However, this is impractical at this stage since the reduction is intended to obtain solvable first order differential equations. This extension also includes the $z^{(3+\varepsilon_o)\kappa+2n-3}$ term which in this case has the coefficient n/l - 2 where n is the order of the polynomial and 1/l is the power of the polynomial which is again eliminated by choosing n=2l. Multiplying equation (3.3.5) by S(z) to the highest power in the denominator will make each of the five terms a polynomial to the same order as the other four. Of course, we want to have at least as many parameters as there are algebraic equations. This is obtained by restricting what κ is since from R we have $\kappa+1$ parameters and the same from S. We also obtain 3 parameters from T(z) for a total of $2\kappa+5$ parameters.

Table III-2 gives for each case the maximum order for each of the polynomials of A_o, B_o, C_o, and E_o. This table also gives κ which is the maximum order that R and S can assume. Also, table III-2 contains two distinct classes, those when $B(f) = 0$ (cases 1 through 9) and the cases when $B(f) \neq 0$. It is clear from the conditions on κ that there is always a solution even if it is just the trivial solution $\kappa = 0$. This means that by equating parameters to zero, the number of equations are reduced faster than the number of parameters. It should also be noted that since f is the ratio of R and S, that for κ at its maximum value, one of the parameters must necessarily be arbitrary and therefore this implies that we have one equation too many. This problem can be circumvented either by the fact that not all of the equations may be independent or if they are, we will need at least one of the parameters set to zero.

TABLE III-2: Reduction to algebraic equations for exponential solutions.

Case	κ	α_0	β_0	χ_0	ε_0	# of algebraic equations	B(f)
I	≤ 4	≤ 1	$= 0$	≤ 3	$= 0$	$3\kappa + 1$	$= 0$
II	≤ 2	≤ 2	$= 0$	≤ 4	$= 1$	$4\kappa + 1$	$= 0$
III	≤ 1	≤ 4	$= 0$	≤ 6	$= 3$	$6\kappa + 1$	$= 0$
IV	≤ 4	$= 1$	$= 0$	≤ 3	$= 0$	$3\kappa + 1$	$= 0$
V	≤ 2	$= 2$	$= 0$	≤ 4	≤ 1	$4\kappa + 1$	$= 0$
VI	≤ 1	$= 4$	$= 0$	≤ 6	≤ 3	$6\kappa + 1$	$= 0$
VII	≤ 4	≤ 1	$= 0$	$= 3$	$= 0$	$3\kappa + 1$	$= 0$
VIII	≤ 2	≤ 2	$= 0$	$= 4$	≤ 1	$4\kappa + 1$	$= 0$
IX	≤ 1	≤ 4	$= 0$	$= 6$	≤ 3	$6\kappa + 1$	$= 0$
X	≤ 2	≤ 2	$= 0$	≤ 4	$= 1$	$4\kappa + 1$	$\neq 0$
XI	≤ 1	≤ 4	≤ 2	≤ 6	$= 3$	$6\kappa + 1$	$\neq 0$
XII	≤ 2	≤ 2	$= 0$	≤ 4	≤ 1	$4\kappa + 1$	$\neq 0$
XIII	≤ 1	≤ 4	≤ 2	≤ 6	≤ 3	$6\kappa + 1$	$\neq 0$
XIV	≤ 2	≤ 2	$= 0$	≤ 4	≤ 1	$4\kappa + 1$	$\neq 0$
XV	≤ 1	≤ 4	≤ 2	≤ 6	≤ 3	$6\kappa + 1$	$\neq 0$
XVI	≤ 2	≤ 2	$= 0$	≤ 4	≤ 1	$4\kappa + 1$	$\neq 0$
XVII	≤ 1	≤ 4	≤ 2	≤ 6	≤ 3	$6\kappa + 1$	$\neq 0$

As mentioned above, we shall now discuss the case when n = 4. The reduction leads to elliptic differential equation of the form

$$\frac{dz}{dt} = T(z) = \sqrt{\tau_0 + \tau_1 z + \tau_2 z^2 + \tau_3 z^3 + \tau_4 z^4}\,, \tag{3.3.13}$$

instead of equation (3.3.12). The cases that are possible appear in Table III-3.

Clearly, all seven cases reduce to the same differential equation, which is

$$\frac{d^2f}{dt^2} + \frac{a_0 + a_1 f + a_2 f^2}{e_0 + e_1 f}\frac{df}{dt} + \frac{b_0}{e_0 + e_1 f}\left(\frac{df}{dt}\right)^2 + \frac{c_0 + c_1 f + c_2 f^2 + c_3 f^3 + c_4 f^4}{e_0 + e_1 f} = 0. \tag{3.3.14}$$

Even though we have not investigated this equation, it would be interesting to see what type of solutions do arise.

TABLE III-3: Reduction to algebraic equations for elliptic solutions.

Case	κ	α_0	β_0	χ_0	ε_0	# of algebraic equations	B(f)
I	≤ 1	≤ 2	$= 0$	≤ 4	$= 1$	$8\kappa + 1$	$= 0$
II	≤ 1	$= 2$	$= 0$	≤ 4	≤ 1	$8\kappa + 1$	$= 0$
III	≤ 1	≤ 2	$= 0$	$= 4$	≤ 1	$8\kappa + 1$	$= 0$
IV	≤ 1	≤ 2	$= 0$	≤ 4	$= 1$	$8\kappa + 1$	$\neq 0$
V	≤ 1	$= 2$	$= 0$	≤ 4	≤ 1	$8\kappa + 1$	$\neq 0$
VI	≤ 1	≤ 2	$= 0$	$= 4$	≤ 1	$8\kappa + 1$	$\neq 0$
VII	≤ 1	≤ 2	$= 0$	≤ 4	≤ 1	$8\kappa + 1$	$\neq 0$

The above analysis covers a large class of nonlinear ordinary differential equations, however, in most cases when transforming to the type of differential equation as given in equation (3.1.1) there is usually some explicit dependence on the independent variable still in the equation. Thus, if we were to assume that the coefficient $A_0(f,t) = P(f; \alpha_0)$ and similarly for $B_0(f,t)$, $C_0(f,t)$, and $E_0(f,t)$, i.e. the equation would become nonautonomous, then we can use the method above with the parameters now functions of t. This, however, will lead to a system of coupled second order differential equations which are probably not solvable for general functions of t but may be solvable when the coefficients have a simple dependence on t. Also, equation (3.3.12) now becomes the generalized Ricatti equation.

SECTION IV: EXAMPLES

In this section we wish to demonstrate the power and usefulness of the presented method on several important physical examples of recent interest. The first example is that of a damped-driven anharmonic oscillator equation and it falls into the general type of analysis described in Section III. The second example is concerned with an anharmonic oscillator equation with quadratic dissipation and it can be analyzed using a special case given in Section III. The third example is that of a damped-driven Morse oscillator and it calls for the use of various special cases. It can also be analyzed using the algebraic method but this has not been done yet. The fourth example studied involves the equation of motion for interacting q-boson system and it can be integrated implicitly and analyzed qualitatively. The final part of this section lists a number of other equations that can be treated with the help of our method and discusses the physical contexts in which they emerge.

Example 1

As our first illustration of the usefulness of the method, we consider the equation

$$f'' + c_0 f' + c_1 + c_2 f + c_3 f^2 + c_4 f^3 = 0, \tag{3.4.1}$$

which describes the motion of a classical cubic anharmonic oscillator with dissipation and in the presence of a bias field (due to c_1). A particularly important application of this equation occurs in the studies of the kinetics of phase transitions where, for nonconserved order parameters η, the governing equation is the celebrated time-dependent Landau-Ginzburg equation (TDLG)[16)]

$$\frac{1}{\Gamma}\frac{\partial \eta}{\partial t} = D\nabla^2 \eta + P(\eta), \tag{3.4.2}$$

where P is a polynomial in η. Examples of the uses of the TDLG abound in many areas of physics, chemistry, and biology, with specific applications to liquid crystals[17], ferroelectrics[18] and structural phase transitions[19]. Traveling wave solutions of equation (3.4.2): $\eta = \eta(x\text{-}vt)$ obey an equation as given in equation (3.4.1). A very recent group of interesting physical examples governed by this type of equation arise in the context of propagating pattern selection[11] with Rayleigh-Bénard convection in fluids, solidification fronts[20] and cellular flame fronts being particular realizations in liquid, solid and gas systems. In addition, chemical kinetics provides additional examples in the case of reaction-diffusion systems[21].

We should also point out that equation (3.4.1) represents the case of a Duffing oscillator with a constant forcing term and reference 22 maintains that no exact solutions for it were reported in the literature unless the force is set to zero. At the end of this subsection we give an explicit analytical formula for a special solution of this equation with $c_1 \neq 0$. Also, note that important applications of the Duffing oscillator equation include the formation of laser instabilities[23].

Having discussed its importance, we now return to the study of equation (3.4.1). Using a linear shift operation for f and a scaling of variables we can reduce this equation to the form

$$F'' + F' + a + bF \pm F^3 = 0. \tag{3.4.3}$$

A recent paper[24] has dealt extensively with this type of equation (for a = 0) including an up-to-date review of existing literature and we refer the interested reader for details. It turns out that except for the Painlevé case of b = 2/9, only trivial (constant) analytical solutions have been known. By using the method outlined in previous section for exponential solutions we find that $\kappa \leq 4$. This means that we can have up to 13 coupled algebraic equations on the polynomial coefficients. It is worth pointing out that for $\kappa = 0$, we recover all the trivial solutions. For $\kappa = 1$, on the other hand, we again obtain the trivial solutions and, in addition, a nontrivial solution for the following special case of equation (3.4.3);

$$F'' + F' \pm \frac{\sqrt{2}}{54}\left(1 - 36g^2\right) + \frac{\left(1 + 12g^2\right)}{6} F - F^3 = 0. \tag{3.4.4}$$

The obtained solution is given by

$$F = \pm\frac{\sqrt{2}}{6} \pm \sqrt{2}\, g\left\{\frac{1 + C_0 \tanh\left[g(t - t_0)\right]}{C_0 + \tanh\left[g(t - t_0)\right]}\right\}, \tag{3.4.5}$$

where C_0 and t_0 are integration constants and the $\pm$ sign above corresponds to the one in equation (3.4.4). For $|C_0| \le 1$, this solution is singular, while for $|C_0| > 1$ it represents a kink. To the best of our knowledge this solution has not been listed earlier in the literature. The situation for $2 \le \kappa \le 4$ has not been fully investigated yet but calculations are underway and will be reported separately. However, what we do in principle in the cases when $\kappa \le 4$ is to let

$$F = \frac{r_0 + r_1 z + r_2 z^2 + r_3 z^3 + r_4 z^4}{s_0 + s_1 z + s_2 z^2 + s_3 z^3 + s_4 z^4}, \tag{3.4.6}$$

and

$$\frac{dz}{dt} = T(z) = \tau_0 + \tau_1 z + \tau_2 z^2. \tag{3.4.7}$$

This gives rise to 13 adjustable parameters subject to 13 coupled algebraic equations. For $\kappa = 1$, we have set $r_2 = r_3 = r_4 = s_2 = s_3 = s_4 = 0$.

Example 2

A recent paper[12)] demonstrated a very interesting property arising as a result of applying the continuum limit to the dynamics of anharmonic lattices. Due to kinkfluctuation coupling and trapping which are characteristic for discrete lattices, dissipative terms appear in the equation of motion which are powers of the field's time derivative. Compared to the previous example then the lowest nontrivial term that appears in the equation of motion due to discreteness is proportional to $(\eta_t)^2$. In order to compare the two distinct situations, we now consider ***an anharmonic oscillator equation with quadratic dissipation*** given by

$$f'' + B(f')^2 + Cf + f^3 = 0, \tag{3.4.8}$$

where we have neglected a constant forcing term for simplicity. This is clearly in the form discussed as Special Case I in Section II. We then readily find that

$$t - t_0 = \int^f dx \exp[D(x)]\left\{-2\int^x dy\, C(y) \exp[2D(y)]\right\}^{-1/2}, \tag{3.4.9}$$

where

$$C(y) = Cy + y^3, \tag{3.4.10}$$

and

$$D(y) = B(y - y_0). \tag{3.4.11}$$

We therefore obtain an implicitly given solution in the form

$$t - t_0 = \int^f dx\left\{C_0 e^{-2Bx} - \frac{Cx}{B} + \frac{C}{2B^2} - \frac{x^3}{B} + \frac{3x^2}{2B^2} + \frac{3x}{2B^3} + \frac{3}{4B^4}\right\}^{-1/2}, \tag{3.4.12}$$

with C_0 denoting an integration constant. When $C_0 = 0$, the integral in equation (3.4.12) is an elliptic integral.

This can be readily extended to a more general case where

$$f'' + B(f')^2 + C(f) = 0,$$

with C(f) being an arbitrary function of f. Examples could include C being a trigonometric function (sin f or a sin f + b sin 2f for sine-Gordon and double-sine Gordon applications, respectively) or an exponential function for studying a damped Liouville equation. An implicit form of the solution is then found to be

$$t - t_0 = \int^{f} dx \left\{ e^{-2Bx} \left[c_0 - 2\int^{x} dy\, e^{2By} C(y) \right] \right\}^{-1/2}. \tag{3.4.13}$$

It is interesting to note that the equation for local maximum and minimum in the (f', f) phase space is

$$(f')^2 = \frac{-C(f_1)}{B} = e^{-2Bf_1} \left\{ c_0 - 2\int^{f_1} dx\, e^{2Bx} C(x) \right\}, \tag{3.4.14}$$

while in the nondissipative case (B = 0), one obtains;

$$(f')^2 = c_0 - 2\int^{f} dx\, C(x), \tag{3.4.15}$$

with c_0 corresponding to a local minimum or maximum when $C(f_1) = 0$. Comparing the solutions of equation (3.4.13) with those in Example 1 for linear dissipation, we find an important qualitative difference. Whereas linear dissipation caused all the solutions to be damped and hence nonperiodic, quadratic dissipation appears to allow periodic solutions which, in the extreme limit of their period tending to infinity and converge to the separatrix orbit given by equation (3.4.14).

Example 3

In this subsection we study the so called *damped-driven Morse oscillator equation*

$$\ddot{x} + \alpha\dot{x} + \beta e^{-x}\left(1 - e^{-x}\right) - Q = 0, \tag{3.4.16}$$

where Q represents a constant force term and $e^{-x}(1-e^{-x})$ is the Morse potential term. This equation has recently gained much interest[25)] in applications to the modeling of vibrational spectra of diatomic molecules. In particular, infrared multiphoton excitation and dissociation dynamics was studied with its use[26] as well as an investigation into the dissociation of van der Waals complexes undertaken[27]. Other applications include laser isotope separation and stimulated Raman emission modeling.

To solve equation (3.4.16), we first substitute

$$x = -\ln f; \quad \dot{x} = -\frac{f'}{f}; \quad \text{and} \quad \ddot{x} = -\frac{f''}{f} + \frac{(f')^2}{f^2}, \tag{3.4.17}$$

so that the equation takes the form

$$f'' - \frac{(f')^2}{f} + \alpha f' + Qf - \beta f^2(1-f) = 0. \tag{3.4.18}$$

This equation can be solved using the algebraic method in the previous section, but this has not been completed as of yet. However, for certain values of α, β, and Q, equation (3.4.18) does satisfy some of the special cases as discussed below.

<u>Special case a)</u>

We assume that $\alpha = 0$ (no dissipation) and integrate equation (3.4.16) to obtain

$$t - t_0 = \int^x \frac{dy}{\sqrt{C_0 - \int^y dz\left[\beta e^{-z}\left(1 - e^{-z}\right) - Q\right]}}, \tag{3.4.19}$$

which is equivalent to an elliptic integral when $Q = 0$ (see equation (3.4.18)). If in addition to $\alpha = 0$, $Q = 0$ then we find

$$x = \ln\left|1 - C_0 \exp\left[-\sqrt{-\beta}\left(t - t_0\right)\right]\right|, \tag{3.4.20}$$

which is an explicit analytical solution.

Special case b)

Assuming that $Q = 0$ but $\alpha \neq 0$ and $\beta \neq 0$, we find for $\beta = \alpha^2$

$$x = \ln\left|C_0 - \alpha(t - t_0)\right|. \tag{3.4.21}$$

Special case c)

Here, we take $\alpha \neq 0 \neq \beta$ but Q has a specific value, namely $Q = \pm\alpha\sqrt{-\beta}$. Then

$$x = \ln\left|\frac{2\sinh\left[\Delta\sqrt{-\beta}(t - t_0)\right]}{\sinh\left[\Delta\sqrt{-\beta}(t - t_0)\right] \pm \Delta\cosh\left[\Delta\sqrt{-\beta}(t - t_0)\right]}\right|, \tag{3.4.22}$$

where $\Delta = (1 \pm 4\alpha/\sqrt{-\beta})^{1/2}$ and $\beta < 0$ is required. These three solutions were obtained using the special cases given in section II. From these three special cases, we would expect exponential solutions for equation (3.4.18) in conjunction with the algebraic method presented in section III.

Example 4

In a recent paper[28)], a system of strongly interacting q-bosons was considered with the characteristic commutation relation for each quasi-particle labeled by k given below

$$a_k a_k^+ - q^{-1} a_k^+ a_k = q^N, \tag{3.4.23}$$

where q is in general a complex number. For the generic Hamiltonian

$$H_{eff} = \sum_{k,l} \omega_{k,l} a_k^+ a_l + \sum_{k,l,m} \Delta_{k,l,m} a_k^+ a_l^+ a_m a_{k+l-m} \tag{3.4.24}$$

describing quasi-particle interactions, the Heisenberg equation of motion can easily be found for each annihilator. This is then converted to a field equation for

$$\Phi = \frac{1}{\sqrt{\Omega}} \sum_{k} e^{-ik\cdot r} a_k = \eta\, e^{i\psi} \tag{3.4.25}$$

represented above in the modulus-argument form. It is subsequently shown that, for a phase ψ which is spatially independent, i.e. $\nabla\psi = \nabla^2\psi = 0$, and time modulated through $\psi = vt$, the effective equation of motion for the amplitude η is

$$\begin{aligned}\Omega_2\nabla^2\eta + (\Omega_4 + \Omega_5\eta)(\nabla\eta)^2 = -\Omega_1\eta - \Omega_3\eta^3 - \hbar v\eta\{(1+q)\exp[(\ln q)\eta^2] \\ +(q^{-1} - q)\eta^2\}^{-1},\end{aligned} \tag{3.4.26}$$

which is a *highly* nonlinear partial differential equation. However, since in one dimensional space it is essentially of the form

$$\frac{d^2y}{dx^2} + B(y)\left(\frac{dy}{dx}\right)^2 + C(y) = 0 \tag{3.4.27}$$

it can be integrated using our method (see Special Case 1) with

$$B(y) = \frac{\Omega_4 + \Omega_5\eta}{\Omega_2}$$

$$-\Omega_2 C(y) = \text{RHS of equation (3.4.26)}.$$

We find its formal solution as

$$x - x_0 = \int^{\eta} \frac{dy \quad e^{\left(\frac{\Omega_4 y + \Omega_5 y^2}{2\Omega_2}\right)}}{\sqrt{c_0 - \frac{2}{\Omega_2}\int^{y} dz \left\{\Omega_1 + \Omega_3 z^2 + \frac{\hbar v}{(1+q)|q|^{z^2} + (q^{-1} - q)z^2}\right\} z e^{z\left[\frac{2\Omega_4 + \Omega_5 z}{\Omega_2}\right]}}}. \tag{3.4.28}$$

It can be demonstrated[28)] that this type of solution admits solitons.

Other Examples

We conclude this section by listing several other equations that have recently appeared in the physical literature and which can be investigated using the method proposed in this article.

(a) A higher order nonlinear Schrödinger equation of the type

$$i\psi_t + \psi_{xx} = a_1|\psi|^2\psi + a_2|\psi|^4\psi + ia_3\left(\psi|\psi|^2\right)_x + \left(a_4 + ia_5\right)\psi\left(|\psi|^2\right)_x \tag{3.4.29}$$

appears in the study of single mode wave propagation in a Kerr dielectric guide with nonlinear properties[12]. Traveling wave solutions will have amplitude equations that conform with the general type in equation (3.1.1)[29]. This illustrated by taking $\psi = \eta(\zeta)e^{i\phi(\zeta)}$, where $\zeta = x + vt$, then equation (3.4.29) becomes upon splitting real and imaginary parts;

$$\text{Im.:}\quad v\eta' + 2\eta'\phi' + \eta\phi'' = \left(3a_3 + 2a_5\right)\eta^2\eta' \tag{3.4.30}$$

and

$$\text{Re.:}\quad -v\eta\phi' + \eta'' - \eta(\phi')^2 = a_1\eta^3 + a_2\eta^5 - a_3\eta^3\phi' + 2a_4\eta^2\eta'. \tag{3.4.31}$$

Equation (3.4.30) can be readly solved for ϕ' as follows;

$$\phi' = \frac{c_0}{\eta^2} - \frac{v}{2} + \left(\frac{3a_3}{4} + \frac{a_5}{2}\right)\eta^2, \tag{3.4.32}$$

Upon substitution of equation (3.4.32) into equation (3.4.31) leads to an equation of the form given in equation (3.1.1).

(b) The motion of a particle under a potential force and generalized friction was recently discussed which satisfies the equation[30]

$$\ddot{x} + \dot{x}|x|^p + \beta x|x|^{2p} = 0, \tag{3.4.33}$$

which, again, is of the general form in equation (3.1.1).

(c) Crystal growth dynamics[31]) has been demonstrated to satisfy two equations, depending on the regime, one of which falls into our general class (when traveling wave solutions are sought), i.e. the Kardar-Parisi-Zhang equation

$$\frac{\partial h}{\partial t} + V\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta = 0, \tag{3.4.34}$$

where h denotes the surface's height and η is a forcing (or noise) term. If we assume that h(x + vt) then equation (3.4.34) becomes;

$$vh' + Vh'' + \frac{\lambda}{2}(h')^2 + \eta = 0, \tag{3.4.35}$$

which is a Riccati equation if the forcing term η is either a constant or a function of x + vt. In which case we can linearize equation (3.4.35) by letting $h' = 2Vf'(\lambda f)^{-1}$;

$$f'' + \frac{v}{V}f' + \frac{\lambda\eta f}{2V^2} = 0. \tag{3.4.36}$$

If on the other hand, η is a function of h only, then equation (3.4.35) is of the form given in equation (3.1.1).

We therefore conclude that a very broad range of physical systems satisfy equations of motion that can be successfully analyzed using the method outlined in this chapter.

SECTION V: CONCLUSION

In this chapter we have focused on a class of autonomous nonlinear differential equations which can be described as generalizations of the anharmonic oscillator equation with linear and quadratic dissipation. These types of equations appear frequently in the descriptions of dynamics of multistable dissipative systems and they also occur as reduced ODE's for multidimensional PDE's in conservative multistable systems. Physical examples of relevant systems were given. The method that was proposed in the present chapter relies on representing the solution as a ratio of polynomials. However, the independent variable is simultaneously transformed as well, so that it can take a linear, exponential, trigonometric or hyperbolic form, depending on the case. Several special types of the investigated class of equations were solved analytically to either explicit or implicit forms but a general approach has been presented as an algorithm that results in a system of coupled algebraic equations on the polynomial coefficients in the ansatze. Finally, several physically interesting particular examples of both the special cases and the general method were described showing the power of this way of solving this class of ordinary differential equations.

A final comment should be made in regard to the question of invertibility for the solutions derived. The algebraic method gives solutions that are always invertible, the exponential case leads to hyperbolic functions and/or trigonometric functions while the elliptic case leads to Jacobi elliptic functions. However, in general, we cannot expect invertibility to exist for solutions of the special cases.

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CHAPTER 4 : FORMATION OF COHERENT STRUCTURES IN SYSTEMS WITH TWO STRONGLY INTERACTING TYPES OF QUASI - PARTICLES*

SECTION I: INTRODUCTION

A large variety of condensed matter systems exhibit properties which manifest underlying interactions between two distinct order parameters. Examples of such behavior include metamagnets[1)], ferroelectric-ferromagnetic systems[2)], ferroelectric-piezoelectric crystals[3)], crystalline-superfluid systems[4)] as well as orientation-position ordering phenomena in molecular liquid crystals[4)]. It is well known that an interplay between two distinct orders may result in critical temperature shifts as well as crossover phenomena. This can be readily analyzed using the mean field approximation[5)]. However, a more fundamental microscopic approach to the problem poses a serious difficulty due to inherent nonlinearities in the description. The objective of the present paper is to provide an insight into the problem of coupled degrees of freedom using a recently developed technique[6,7)], which applies to systems composed of a large number of strongly interacting particles. The method, henceforth referred to as the Method of Coherent Structures (MCS), is based on a nonlinear analysis of collective modes of behavior in these many-body systems. The starting point is a generic second-quantized Hamiltonian which includes both one- and two-body interaction terms. Then, Heisenberg's equations of motion are calculated for the creation and annihilation operators for each quasi-particle. This technique relies on an expansion of the coupling coefficients in both one- and two-body terms in a power series about a specifically chosen point in momentum space (or the space of quantum numbers). Also, quantum fields are defined, in the standard manner, using a plane wave basis. The definition of the fields along with the coefficient expansions are then used to convert Heisenberg's equations of

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motion into a coupled system of nonlinear partial differential equations (PDE's) for the quantum fields. In order to make the problem tractable, a standard approach in quantum field theory[8,9] is then implemented whereby the quantum field is decomposed into its *classical part* and a *quantum correction.* Close to criticality, which is of particular interest to us, the classical part predominates to a very large degree so that the equations, although nonlinear, may be first solved for the classical component to a very good degree of approximation (corrections are of the order of $\hbar$). This classical envelope equation is usually solved using recent mathematical discoveries in the area of nonlinear PDE's[10-12]. The new powerful mathematical techniques at our disposal allow us, at the very least, to extract important physical information about the geometry of the system and sometimes the analytical form of these classical nonlinear fields existing in multi-dimensional space-time can be determined. Once the stable classical solutions are found they provide an effective potential in a linear Schrödinger equation for the quantum oscillations. The presence of bound states is an additional criterion determining the stability of the envelope. It should be pointed out in this connection that approaches very similar in spirit were proposed almost simultaneously by other authors (see e.g. Ref. 13) and a precise relationship between these methods is to be established in the near future.

Apart from the general framework, several specific physical applications have been recently worked out with the aim of testing the validity of the method. First, the BCS Hamiltonian for superconductivity has been used and a careful analysis resulted in a remarkable confirmation of earlier standard scaling laws for the superconducting current and energy gap[14]. Another application was concerned with the equilibrium phases of metamagnets, i.e. spin systems with two or more sublattices. Again, an independent analytical support was provided[15] for a numerical form of the phase boundaries between the three possible ground states[16]. Further applications are being currently investigated. In particular, the Haldane gap problem in quantum Heisenberg spin chains and also the bound states in multielectron atoms are being studied. Another important observation has been recently made[17] in connection with the presence of spin degrees of freedom. It was rigorously demonstrated that the inclusion of spin does not alter the *form* of the quantum equations of motion and hence the basic results of the calculations remain valid. What is affected, however, by the presence of spin is the magnitude of the nonlinear coupling coefficient when the spin $S > \frac{1}{2}$, in which case it becomes multiplied by the spin degeneracy $(2S + 1)$ (for integer spins only). This may also lend support to the approach described since the critical temperature will be spin dependent and, for example, for superfluid ^3He and ^4He the transition temperature is much higher in the latter case than in the former[18]. Finally, it may be demonstrated that the form of the nonlinear field equations, provided the field is suitably defined, is exactly the same in any orthonormal basis of states and not just in a plane-wave basis as originally used.

The motivation for this chapter is to extend the MCS beyond a single quasi-particle type and include another set of degrees of freedom, which may represent either another subsystem (e.g. heat reservoir) or a different type of quasi-particle (electrons and phonons, for instance), when the two subsystems are coupled together. This is not an easy exercise but an <u>important step</u> in the direction towards realistic modeling of complex many-body systems. In fact, under close scrutiny nearly all many-body structures are comprised of several distinct subsystems or components. Low temperature superconductivity, for example, involves electrons and phonons dynamically coupled via an electron-phonon interaction[19]. This particular case will be discussed in greater detail in this chapter in Section IV using the Fröhlich Hamiltonian and the MCS as a basis. The general situation described here is indeed ubiquitous in condensed matter physics with, for example, various phonon modes interacting amongst themselves and leading to structural instabilities[20]. Practically speaking, any combination of two types of excitations present in a solid may be described in this way (electron-plasmon, magnon-phonon, electron-polaron, photon-electron, etc.). This chapter, therefore, addresses the very important question of the physical behavior of cross-coupled quasi-particles or subsystems.

The rest of the chapter is divided as follows. Section II starts by looking at a generic second quantized Hamiltonian for two interacting types of particles. The section goes on to find the Heisenberg equations of motion for each possible statistics that the particles can obey: Boson - Boson interactions, indistinguishable Fermion - Fermion interactions, distinguishable Fermion - Fermion interactions, and Fermion - Boson interactions. The section also discusses the methodology where the coefficients are expanded in a Taylor series near some critical point in momentum space and also field operators are introduced using a plane wave basis. This leads to two coupled generalized nonlinear Schrödinger equations and which are assumed to be classical to zeroth order. Also, in section II it will be shown that the equations of motion for the classical fields are equivalent to those obtained from a Landau - Ginzburg type Hamiltonian density. Section II closes with a brief discussion of the physical interpretation of these classical solutions. Section III is our first example, the classical field equations for the distinguishable Fermion - Fermion case will be discussed and some solutions to these equations are obtained. Section IV is our second example which deals with the case of interacting Bosons and Fermions. The specific system in this case is that of interacting electrons and phonons in a metal as described by the Fröhlich Hamiltonian. The fourth section is divided into three parts, the first of which is a discussion of the general background information of the Fröhlich Hamiltonian and also the resulting classical field equations that will be obtained. In the second part, the equations of motion when the coupling coefficient is taken only to zeroth order will be discussed. In this part, the solutions to the classical equations of motion are found which permit the formation of Cooper pairs and thus leads to superconductivity. The third part is an extension of the second part, except that now the coupling coefficient between the electrons and the phonons is considered to have a q dependence and is expanded to second order, which will lead to extra terms in the equations of motion. This part however, leads to implicit solutions in the form of an indefinite integral which in general can not be solved, but for special cases the integral can be done and then the solution can be inverted. However, there is much information that can be gained from the integral without actually solving it, which will be discussed. In this part, solutions for free electrons scattering off the phonons and solutions for Cooper pair formation are found. The fifth section closes the chapter by summarizing the results and discussing further applications for the method discussed.

SECTION II: METHODOLOGY

In this section, the MCS is applied to a general Hamiltonian describing many-particle systems. Here, the entire physical system is divided into two parts, denoted A and B, with strong interactions within each of the systems and coupling terms between the two subsystems. The form of the coupling terms in this study is quite general and includes one- and two-quantum exchanges between systems A and B with associated second quantized operators (a,a^+) and (b,b^+), respectively. The model Hamiltonian will be taken generically as

$$H = H_a + H_b + H_{ab}, \tag{4.2.1}$$

where

$$\begin{aligned} H_{ab} = \sum_k \left\{\xi_k a_k^+ b_k + \xi_k^* b_k^+ a_k\right\} + \sum_{k,l} \Big\{\eta_{k,l} a_k^+ a_l b_{k-l} + \eta_{k,l}^* b_{k-l}^+ a_l^+ a_k + \\ \lambda_{k,l} b_k^+ b_l a_{k-l} + \lambda_{k,l}^* a_{k-l}^+ b_l^+ b_k\Big\} + \sum_{k,l,m} \Big\{\mu_{k,l,m} b_k^+ a_l^+ a_m a_{k+l-m} + \\ \mu_{k,l,m}^* a_{k+l-m}^+ a_m^+ a_l b_k + \gamma_{k,l,m} a_k^+ b_l^+ b_m b_{k+l-m} + \gamma_{k,l,m}^* b_{k+l-m}^+ b_m^+ b_l a_k + \\ \alpha_{k,l,m} a_k^+ b_l^+ b_m a_{k+l-m} + \alpha_{k,l,m}^* a_{k+l-m}^+ b_m^+ b_l a_k\Big\}, \end{aligned} \tag{4.2.2}$$

$$H_a = \sum_k \omega_k a_k^+ a_k + \sum_{k,l,m} \delta_{k,l,m}^{(0)} a_k^+ a_l^+ a_m a_{k+l-m} + \sum_{k,l} \left\{\varepsilon_{k,l} a_k^+ a_l a_{k-l} + \varepsilon_{k,l}^* a_{k-l}^+ a_l^+ a_k\right\}, \tag{4.2.3}$$

and

$$H_b = \sum_{k} \Omega_k b_k^+ b_k + \sum_{k,l,m} \Delta_{k,l,m} b_k^+ b_l^+ b_m b_{k+l-m} + \sum_{k,l} \left\{ \Lambda_{k,l} b_k^+ b_l b_{k-l} + \Lambda_{k,l}^* b_{k-l}^+ b_l^+ b_k \right\}, \qquad (4.2.4)$$

where H_a and H_b are the associated Hamiltonian for the two separate subsystems A and B, and H_{ab} is an operator providing interactions between them. Each of the labels $\underline{k}$, $\underline{l}$ and $\underline{m}$ should denote a set of quantum numbers for a complete set of states. In view of the fact that spin labels do not alter the form of the MCS equations of motion, provided the interactions themselves are spin independent, we drop spin labels and use a plane-wave basis. The interaction terms in H_a and H_b are such that linear momentum is a conserved quantity so indeed an assumption about the form of interaction has been made, but this would cover most examples in physics. Note that the interaction H_{ab} has to be Hermitian and this is reflected in the form of equation (4.2.2). Three-body interchanges have been specifically excluded, i.e. six-legged operators will not appear in the Hamiltonian. The reader should not presuppose that in all four cases the form of H as given in equation (4.2.1) is the same but each particular case will never include terms which do not appear there. Thus, when we consider individual cases we can drop terms from equation (4.2.1) as appropriate and do away with the need to have a separate Hamiltonian in each section.

The first step in this procedure is to calculate the Heisenberg equations of motion for the ladder operators of the two systems. In general, four cases can be considered depending on the types of statistics obeyed by the quasi-particles, i.e. 1) Boson-Boson, 2) indistinguishable Fermion-Fermion, 3) distinguishable Fermion-Fermion and 4) Boson-Fermion cases. We consider each of these cases in turn as, in general, they lead to different equations of motion, case three being the simplest and case one with the Boson-Boson situation is the most complex.

Case 1: The Boson-Boson System

For this situation all terms in equations (4.2.1), (4.2.2), (4.2.4) and (4.2.2) are included. The a and b operators satisfy Bose-Einstein commutation rules, namely

$$\left[a_l, a_k^+\right]_- = \delta_{k,l} \quad , \quad \left[b_l, b_k^+\right]_- = \delta_{k,l} \quad , \tag{4.2.5}$$

$$\left[a_k^+, a_l^+\right]_- = \left[a_k, a_l\right]_- = \left[b_k^+, b_l^+\right]_- = \left[b_k, b_l\right]_- = 0 \quad , \tag{4.2.6}$$

and

$$\left[a_k^+, b_l^+\right]_- = \left[a_k^+, b_l\right]_- = \left[a_k, b_l^+\right]_- = \left[a_k, b_l\right]_- = 0 \quad . \tag{4.2.7}$$

We have specifically excluded interactions or terms internal to a subsystem which create one particle from the vacuum but have allowed transformation of one particle to another and two of one type to scatter off another but in the process linear momentum is conserved. Since there are two-body terms in equation (4.2.2) we include not only interactions which preserve different particles separately but incorporate terms where, for example, two B-particles are destroyed, a B-particle is created but an A-particle also appears to conserve momentum. No processes of the type in which a B-particle is destroyed at the expense of creating three A-particles have been included, etc. Employing Heisenberg's equation of motion in the form

$$i\hbar\, \partial_t a_n = \left[a_n, H\right]_- , \tag{4.2.8}$$

we find

$$\begin{aligned} i\hbar\partial_t a_n = {} & \omega_n a_n + \sum_{l,m}\left(\delta^{(0)}_{n,l,m} + \delta^{(0)}_{l,n,m}\right) a_l^+ a_m a_{n+l-m} + \xi_n b_n + \sum_l \Big\{ \eta_{n,l} a_l b_{n-l} \\ & + \eta^*_{l,n} b^+_{l-n} a_l + \lambda^*_{n+l,l} b_l^+ b_{n+l} + \varepsilon_{n,l} a_l a_{n-l} + \varepsilon^*_{n+l,l} a_l^+ a_{n+l} + \varepsilon^*_{l,n} a^+_{l-n} a_l \Big\} \\ & + \sum_{l,m} \Big\{ \mu_{l,n,m} b_l^+ a_m a_{n+l-m} + \mu^*_{n+m-l,l,m} a_m^+ a_l b_{n+m-l} + \mu^*_{l,m,n} a^+_{l+m-n} a_m b_l \\ & + \gamma_{n,l,m} b_l^+ b_m b_{n+l-m} + \alpha_{n,l,m} b_l^+ b_m a_{n+l-m} + \alpha^*_{n+m-l,l,m} b_m^+ b_l a_{n+m-l} \Big\} \end{aligned} \tag{4.2.9}$$

and

$$\begin{aligned} i\hbar\partial_t b_n = \Omega_n b_n + \sum_{l,m}\left(\Delta_{n,l,m} + \Delta_{l,n,m}\right) b_l^+ b_m b_{n+l-m} + \xi_n^* a_n + \sum_l \Big\{ \lambda_{n,l} b_l a_{n-l} + \\ \lambda_{l,n}^* a_{l-n}^+ b_l + \eta_{n+l,l}^* a_l^+ a_{n+l} + \Lambda_{n,l} b_l b_{n-l} + \Lambda_{n+l,l}^* b_l^+ b_{n+l} + \Lambda_{l,n}^* b_{l-n}^+ b_l \Big\} \\ + \sum_{l,m} \Big\{ \gamma_{l,n,m} a_l^+ b_m b_{n+l-m} + \gamma_{n+m-l,l,m}^* b_m^+ b_l a_{n+m-l} + \gamma_{l,m,n}^* b_{l+m-n}^+ b_m a_l \\ + \mu_{n,l,m} a_l^+ a_m b_{n+l-m} + \alpha_{l,n,m} a_l^+ b_m a_{n+l-m} + \alpha_{l,m,n}^* a_{m-n+l}^+ b_m a_l \Big\}. \end{aligned} \tag{4.2.10}$$

In order to obtain quantum field equations, two quantum fields are defined corresponding to the two sets of operators as follows

$$\Phi(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_k b_k \exp[-i\mathbf{k} \bullet \mathbf{x}] \tag{4.2.11}$$

and

$$\Psi(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_k a_k \exp[-i\mathbf{k} \bullet \mathbf{x}], \tag{4.2.12}$$

where V is a volume over which the plane waves are normalized. For completeness we give the well-known relations

$$\int_V \exp[i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{x}] d^3x = V\,\delta_{\mathbf{k}_1,\mathbf{k}_2} \tag{4.2.13}$$

$$\sum_k \exp[i(\mathbf{r} - \mathbf{r}') \cdot \mathbf{k}] = V\,\delta(\mathbf{r} - \mathbf{r}'). \tag{4.2.14}$$

The procedure outlined earlier[6,7] for a single field is now followed and the coefficients in each equation of motion are Taylor expanded about a particular point in reciprocal or quantum number space. It is assumed that there is at least one such "point" which is common to both subsystems. This point may be chosen as an actual critical point of a subsystem with coordinates ($\mathbf{n}_o$, $\mathbf{k}_o$, $\mathbf{m}_o$) in "reciprocal" space or it may be a point in a regime where the system is close to classical from the Correspondence Principle[21]. As an example, expanding ω we find

$$\omega_{\mathbf{n}} = \omega_{\mathbf{n}_o} + \sum_{s=1}^{\infty} \frac{\left[(\mathbf{n}-\mathbf{n}_o)\cdot\nabla_{\mathbf{n}}\right]^s \omega_o}{s!}. \tag{4.2.15}$$

We expand both ω and Ω to second order in deviations from the critical point and assume the remaining model parameters to be "momentum" independent. For more details of these expansions, see reference 6. Having made the appropriate Taylor expansions and defined fields for each subsystem we multiply both sides of equations (4.2.9) and (4.2.10) by $V^{-1/2}\exp[-i\mathbf{n}\cdot\mathbf{r}]$ and then sum over $\mathbf{n}$. We find that

$$\begin{aligned} i\hbar\partial_t\Psi = {} & \omega_o\Psi + i\omega_1\cdot\nabla\Psi + \omega_2\nabla^2\Psi + 2\delta_o\Psi^+\Psi\Psi + \xi_o^*\Phi + \eta_o\Psi\Phi + \\ & \eta_o^*\Phi^+\Psi + \lambda_o^*\Phi^+\Phi + \varepsilon_o\Psi\Psi + 2\varepsilon_o^*\Psi^+\Psi + \mu_o\Phi^+\Psi\Psi + \\ & 2\mu_o^*\Psi^+\Psi\Phi + \gamma_o\Phi^+\Phi\Phi + \left[\alpha_o + \alpha_o^*\right]\Phi^+\Phi\Psi \quad , \end{aligned} \tag{4.2.16}$$

where the new parameters are related in an obvious fashion to those in the original equation, i.e. equation (4.2.9).

In a similar manner we obtain, for the other field Φ,

$$\begin{aligned} i\hbar\partial_t\Phi = {} & \Omega_o\Phi + i\Omega_1\cdot\nabla\Phi + \Omega_2\nabla^2\Phi + 2\Delta_o\Phi^+\Phi\Phi + \xi_o^*\Psi + \lambda_o\Phi\Psi + \\ & \lambda_o^*\Psi^+\Phi + \eta_o^*\Psi^+\Psi + \Lambda_o\Phi\Phi + 2\Lambda_o^*\Phi^+\Phi + \mu_o\Psi^+\Psi\Psi + \\ & 2\gamma_o^*\Phi^+\Phi\Psi + \gamma_o\Psi^+\Phi\Phi + \left[\alpha_o + \alpha_o^*\right]\Psi^+\Phi\Psi \quad . \end{aligned} \tag{4.2.17}$$

It should be noted that ω_2 and Ω_2 in equations (4.2.16) and (4.2.17), respectively, are written in this form for convenience since in general they are second order tensors and the Laplacian is replaced by mixed second order derivatives. However, for numerous physical systems they will reduce to a Laplacian operator due to their inherent spatial symmetries. These equations, even when treated classically, are extremely difficult to solve but can be useful when considering specific physical systems. At present we have not analyzed any physical system which involves a Boson-Boson interaction but hope to do so in the near future.

Case 2: Fermion-Fermion, Indistinguishable Particles

As an example of such systems we might imagine two atoms in a solid each with their own electrons[22]). As these electrons are indistinguishable, the associated annihilators and creators satisfy the following commutation rules

$$\left[a_k, a_l^+\right]_+ = \delta_{k,l}\,, \left[a_k, a_l\right]_+ = \left[a_k^+, a_l^+\right]_+ = 0, \tag{4.2.18}$$

$$\left[b_k, b_l^+\right]_+ = \delta_{k,l}\,, \left[b_k, b_l\right]_+ = \left[b_k^+, b_l^+\right]_+ = 0, \tag{4.2.19}$$

and

$$\left[a_k, b_l^+\right]_+ = \left[a_k, b_l\right]_+ = \left[a_k^+, b_l\right]_+ = \left[a_k^+, b_l^+\right]_+ = 0. \tag{4.2.20}$$

Going through the same procedure as before and noting that ε, Λ, η, λ terms vanish since charge must be conserved, we find for the equations of motion the following

$$\begin{aligned} i\hbar\,\partial_t a_n = \omega_n a_n + \xi_n b_n + \sum_{l,m}\Big\{&\left(\delta^{(0)}_{n,l,m} - \delta^{(0)}_{l,n,m}\right)a_l^+ a_m a_{n+l-m} + \\ &\alpha_{n,l,m} b_l^+ b_m a_{n+l-m} - \alpha^*_{n+l-m,m,l} b_l^+ b_m a_{n+l-m} + \gamma_{n,l,m} b_l^+ b_m b_{n+l-m} \\ &-\mu_{l,n,m} b_l^+ a_m a_{n+l-m} + \mu^*_{n-l+m,l,m} a_m^+ a_l b_{n-l+m} - \mu^*_{l,m,n} a^+_{l-n+m} a_m b_l\Big\}. \end{aligned} \tag{4.2.21}$$

For the other field annihilator we obtain

$$\begin{aligned} i\hbar\,\partial_t b_n = \Omega_n b_n + \xi_n^* a_n + \sum_{l,m}\Big\{&\left(\Delta_{n,l,m} - \Delta_{l,n,m}\right)b_l^+ b_m b_{n+l-m} - \\ &\alpha_{l,n,m} a_l^+ b_m a_{n+l-m} - \alpha^*_{l,m,n} a^+_{l+m-n} b_m a_l + \mu_{n,l,m} a_l^+ a_m a_{n+l-m} - \\ &\gamma_{l,n,m} a_l^+ b_m b_{n+l-m} - \gamma^*_{n-l+m,l,m} b_m^+ b_l a_{n-l+m} + \gamma^*_{l,m,n} b^+_{l-n+m} b_m a_l\Big\}. \end{aligned} \tag{4.2.22}$$

To convert equations (4.2.21) and (4.2.22) to field equations of motion we again divide by $\sqrt{V}$ and an appropriate exponential (see definition of the field operators as given by equations (4.2.9) and (4.2.10)) to obtain

$$\begin{aligned} i\hbar\partial_t\Psi = \omega_o\Psi + i\omega_1\cdot\nabla\Psi + \omega_2\nabla^2\Psi + 2\delta_o\Psi^+\Psi\Psi + \xi_o\Phi - \mu_o\Phi^+\Psi\Psi + \\ \gamma_o\Phi^+\Phi\Phi + \left[\alpha_o + \alpha_o^*\right]\Phi^+\Phi\Psi \quad , \end{aligned} \tag{4.2.23}$$

since for Fermions $\delta^{(0)}_{n,l,m} = -\delta^{(0)}_{l,n,m}$ so that we obtain $2\delta_0|\Psi|^2\Psi$, however, there are two terms in μ^* that do cancel out. This is obvious since otherwise we would either create or destroy a particle. For the other field we find

$$i\hbar\partial_t\Phi = \Omega_o\Phi + i\Omega_1\cdot\nabla\Phi + \Omega_2\nabla^2\Phi + 2\Delta_o\Phi^+\Phi\Phi + \xi_o^*\Psi + \mu_o\Psi^+\Psi\Psi - \gamma_o\Psi^+\Phi\Phi - \left[\alpha_o + \alpha_o^*\right]\Psi^+\Phi\Psi \quad . \tag{4.2.24}$$

In equations (4.2.23) and (4.2.24), we have again used the notations ω_2 and Ω_2 respectively, to simplify the notation of the second order derivatives as a Laplacian but can be generalized if need be to give mixed second order derivatives. One of the examples discussed in the introduction was the interaction between valence electrons of two or more atoms or molecules and would fall into this category of indistinguishable Fermion-Fermion particles. Specific examples of this type of interaction are not being considered here but will be examined in the future.

Case 3: Fermion-Fermion, Distinguishable Particles

Here, as an example, we might have a plasma made up of electrons and protons. Unlike Case 2 here we must preserve each particle type separately so single annihilators or creators for either subsystem are not allowed. The second quantized operators here satisfy

$$\left[a_k, a_l^+\right]_+ = \delta_{k,l}, \left[a_k, a_l\right]_+ = \left[a_k^+, a_l^+\right]_+ = 0, \tag{4.2.25}$$

and

$$\left[b_k, b_l^+\right]_+ = \delta_{k,l}, \left[b_k, b_l\right]_+ = \left[b_k^+, b_l^+\right]_+ = 0, \tag{4.2.26}$$

but between subsystems they commute (not anti-commute as in Case 2), i.e.

$$\left[a_k, b_l^+\right]_- = \left[a_k, b_l\right]_- = \left[a_k^+, b_l\right]_- = \left[a_k^+, b_l^+\right]_- = 0, \tag{4.2.27}$$

For this case ε, Λ, ξ, η, γ, μ, γ-type interactions are not present. The equations of motion become;

$$i\hbar\,\partial_t a_n = \omega_n a_n + \xi_n b_n + \sum_{l,m}\Big\{\Big(\delta^{(0)}_{n,l,m} - \delta^{(0)}_{l,n,m}\Big)a_l^+ a_m a_{n+l-m} + \Big(\alpha_{n,l,m} + \alpha^*_{n+l-m,m,l}\Big)b_l^+ b_m a_{n+l-m}\Big\} \tag{4.2.28}$$

and

$$i\hbar\,\partial_t b_n = \Omega_n b_n + \xi_n^* a_n + \sum_{l,m}\Big\{\Big(\Delta_{n,l,m} - \Delta_{l,n,m}\Big)b_l^+ b_m b_{n+l-m} + \alpha_{l,n,m} a_l^+ b_m a_{n+l-m} + \alpha^*_{l,m,n} a^+_{l+m-n} b_m a_l\Big\} \quad . \tag{4.2.29}$$

Thus the field equations for this case become, using equations (4.2.11) and (4.2.12), as follows:

$$i\hbar\partial_t\Psi = \omega_o\Psi + i\omega_1\cdot\nabla\Psi + \omega_2\nabla^2\Psi + 2\delta_o\Psi^+\Psi\Psi + \left[\alpha_o + \alpha_o^*\right]\Phi^+\Phi\Psi \tag{4.2.30}$$

and

$$i\hbar\partial_t\Phi = \Omega_o\Phi + i\Omega_1\cdot\nabla\Phi + \Omega_2\nabla^2\Phi + 2\Delta_o\Phi^+\Phi\Phi + \left[\alpha_o + \alpha_o^*\right]\Psi^+\Psi\Phi. \tag{4.2.31}$$

Equations (4.2.30) and (4.2.31) may, at first, appear complicated but some important information can be gained from them and will be analyzed in Section III.

Case 4: Fermion (a) - Boson (b)

For this particular situation we have in mind the important example of interacting electrons and phonons. Thus, if we let the a operators represent the Fermions which satisfy Fermi-Dirac commutation rules

$$\left[a_k, a_l^+\right]_+ = \delta_{k,l}\,, \left[a_k, a_l\right]_+ = \left[a_k^+, a_l^+\right]_+ = 0, \tag{4.2.32}$$

whereas the Boson operators satisfy

$$\left[b_k, b_l^+\right]_- = \delta_{k,l}, \left[b_k, b_l\right]_- = \left[b_k^+, b_l^+\right]_- = 0, \tag{4.2.33}$$

and between subsystems the operators commute so that

$$\left[a_k, b_l^+\right]_- = \left[a_k, b_l\right]_- = \left[a_k^+, b_l^+\right]_- = \left[a_k^+, b_l\right]_- = 0. \tag{4.2.34}$$

In this case terms of type ε, ξ, μ and λ must be omitted from the Hamiltonian. This case is relatively straight forward to derive the equations of motion for the annihilators to give

$$\begin{aligned} i\hbar\partial_t a_n = \omega_n a_n + \sum_{l,m}\left(\delta^{(0)}_{n,l,m} - \delta^{(0)}_{l,n,m}\right)a_l^+ a_m a_{n+l-m} + \sum_l\left\{\eta_{n,l} a_l b_{n-l} + \right. \\ \left. \eta^*_{l,n} b^+_{l-n} a_l\right\} + \sum_{l,m}\left\{\alpha_{n,l,m} b_l^+ b_m a_{n+l-m} + \alpha^*_{n+m-l,l,m} b_m^+ b_l a_{n+m-l}\right\} \end{aligned} \tag{4.2.35}$$

and

$$\begin{aligned} i\hbar\partial_t b_n = \Omega_n b_n + \sum_{l,m}\left(\Delta_{n,l,m} + \Delta_{l,n,m}\right)b_l^+ b_m b_{n+l-m} + \sum_l\left\{\Lambda_{n,l} b_l b_{n-l} + \right. \\ \Lambda^*_{n+l,l} b_l^+ b_{n+l} + \Lambda^*_{l,n} b^+_{l-n} b_l + \eta^*_{n+l,l} a_l^+ a_{n+l}\Big\} + \sum_{l,m}\Big\{\alpha_{l,n,m} a_l^+ b_m a_{n+l-m} \\ \left. + \alpha^*_{l,m,n} a^+_{m-n+l} b_m a_l\right\}. \end{aligned} \tag{4.2.36}$$

The corresponding two field equations are found in the same way as the first three cases and they are;

$$\begin{aligned} i\hbar\partial_t\Psi = \omega_o\Psi + i\omega_1\cdot\nabla\Psi + \omega_2\nabla^2\Psi + 2\delta_o\Psi^+\Psi\Psi + \eta_o\Psi\Phi + \eta_o^*\Phi^+\Psi + \\ \left[\alpha_o + \alpha_o^*\right]\Phi^+\Phi\Psi \end{aligned} \tag{4.2.37}$$

and

$$\begin{aligned} i\hbar\partial_t\Phi = \Omega_o\Phi + i\Omega_1\cdot\nabla\Phi + \Omega_2\nabla^2\Phi + 2\Delta_o\Phi^+\Phi\Phi + \eta_o^*\Psi^+\Psi + \Lambda_o\Phi\Phi + \\ 2\Lambda_o^*\Phi^+\Phi + \left[\alpha_o + \alpha_o^*\right]\Psi^+\Phi\Psi. \end{aligned} \tag{4.2.38}$$

In all the cases above the equations of motion are of the nonlinear Schrödinger (NLS) type with additional terms due to the mutual interactions between the fields. First, there are cross-terms proportional to $\Phi\,\Psi$ as well $|\Phi|^2\Psi$ and $|\Psi|^2\Phi$. Secondly, "source" terms appear proportional to the other field and its squared modulus. The method for solving these nonlinear coupled differential equations is to first treat the fields as classical and effectively decouple them by assuming a particular type of analytic orbit in the phase space given by the general formula

$$G(\Phi, \Psi) = 0 \; . \tag{4.2.39}$$

Some commonly used examples of this function involve straight lines, parabolas, ellipses and hyperbolae but it is not at all guaranteed that any of these approaches will actually produce analytical results in all the cases considered. Each time a particular orbit is investigated, compatibility conditions have to be satisfied and for stationary orbits the condition

$$\left|\frac{\partial G}{\partial \Phi}\right|^2 |\nabla\Phi|^2 = \left|\frac{\partial G}{\partial \Psi}\right|^2 |\nabla\Psi|^2, \tag{4.2.40}$$

which follows from equation (4.2.39), can be used to effectively decouple the two equations. Rather than provide an extensive general discussion on the applications of this general technique we shall refer to reader to the monograph by Rajaraman[9)] where several cases have been worked out in detail. A specific example of the Fröhlich Hamiltonian is provided as an illustration in Section IV. It should also be mentioned that, in addition to these analytical solutions, it is well known that large segments of the phase space for coupled systems, like the ones discussed in this section, can be filled with chaotic behavior.

We now wish to note the relationship between the MCS and the Landau-Ginzburg Theory, where chapter two is an example of the Landau-Ginzburg theory. In the previous paper on the MCS[6)] it has been observed that for a single field, the equations of motion can be derived via the Euler-Lagrange equations, from a Hamiltonian density of the Landau-Ginzburg form

$$H^a_{LG} = \alpha_a \Psi^+\Psi + \beta_a \Psi^+\Psi^+\Psi\Psi + \gamma_a\left(\nabla\Psi^+\right)\cdot\left(\nabla\Psi\right). \qquad (4.2.41)$$

It can be readily shown that the corresponding Hamiltonian density in our case of two coupled fields can also be written within a Landau-Ginzburg formalism as

$$H = H^a_{LG}(\Psi, \nabla\Psi) + H^b_{LG}(\Phi, \nabla\Phi) + H^{ab}_{LG}(\Phi, \Psi, \nabla\Phi, \nabla\Psi), \qquad (4.2.42)$$

where H^a_{LG} is that of equation (4.2.41), H^b_{LG} is obtained from equation (4.2.41) by changing Ψ to Φ and replacing a by b. The last term in equation (4.2.42) takes the form (in general)

$$\begin{aligned} H^{ab}_{LG} = \mu_1|\Phi|^2|\Psi|^2 + \mu_2\left(\Phi + \Phi^*\right)|\Psi|^2 + \mu_3|\Phi|^2\left(\Psi + \Psi^*\right) + \mu_4\left(\Phi^*\Psi + \right. \\ \left. \Phi\Psi^*\right) + \mu_5\left(\Phi^*\Psi + \Phi\Psi^*\right)|\Psi|^2 + \mu_6|\Phi|^2\left(\Phi^*\Psi + \Phi\Psi^*\right) + \ldots \quad . \end{aligned} \qquad (4.2.43)$$

On going to first order in the interaction (the reader should consult the original MCS papers, references 6 and 7, where the form of H^{ab}_{LG} would become modified to include terms which involve gradient of one field and are proportional to the square or modulus squared of the other field. Second order corrections will bring in Laplacians of one field and squares of the other. In specific systems, cubic terms like $\left(\Phi + \Phi^*\right)|\Psi|^2$ may be excluded on symmetry grounds but often are important and, for example, in an electron-phonon system, the $\Phi + \Phi^*$ would denote a phonon field representation of the lattice displacement and $|\Psi|^2$ an electron number density. In Section IV, the example of the electron-phonon interaction will be considered in more detail making use of the Fröhlich Hamiltonian. The next section, however, deals with the example of coupled distinguishable Fermion systems.

Before going on to the examples in section III and IV, we need to look at the meaning of treating these equations as classical. If we recall that the total number of particle operator is;

$$N_{OP.} = \sum_k a_k^\dagger a_k = \int_V d^3x\, \Psi^\dagger(x)\Psi(x) \tag{4.2.44}$$

so that $N_{OP.}$ acting on a state will give the eigenvalue which is the total number of particles in that energy level. Therefore, when we do an expansion of the coefficients in **k** space around a particular point $\mathbf{k_0}$ and taking the field Ψ as classical means that the modulus squared of the classical field is the classical number density for all the particles that are near $\mathbf{k_0}$.

SECTION III: SOLUTIONS FOR THE DISTINGUISHABLE FERMION - FERMION CASE

We shall now examine case 3 of section II so as to obtain exact solutions to the system of equations of motion for Ψ and Φ, namely equations (4.2.30) and (4.2.31), under a certain assumption given later on. It is assumed for simplicity that the coordinate system is chosen in such a way that $\omega_1 \cdot \nabla\Psi = \Omega_1 \cdot \nabla\Phi = 0$. This can be achieved by either a convenient rotation of the coordinates and/or the choice of a moving frame of reference. Before continuing on, it should be pointed out that others have done work on coupled nonlinear Schrödinger equations[23-28]. The time dependence of the two fields is taken in the form

$$\Phi = u\, e^{iE_1 t/\hbar} \quad \text{and} \quad \Psi = w\, e^{iE_2 t/\hbar}. \tag{4.3.1}$$

Then, the two fields are assumed to be linearly dependent, or namely

$$\Phi = \lambda\, \Psi, \tag{4.3.2}$$

which transforms equations (4.2.30) and (4.2.31) into two similar equations. This means that we are assuming that the number density of the two fields are proportional. It should be noted that for certain values of the parameters, solutions have been found without making the assumption given in equation (4.3.2) with both one dimensional space dependence[25-27] and two dimensional space dependence[28]. Since equations (4.2.30) and (4.2.31) now refer to the same field, compatibility is then required. The equations take the form

$$0 = (\omega_0 + E_1)\Psi + \omega_2\nabla^2\Psi + \left[2\delta_0 + \lambda^2(\alpha_0 + \alpha_0^*)\right]\Psi^+\Psi\Psi \tag{4.3.3}$$

and

$$0 = (\Omega_0 + E_2)\Psi + \Omega_2\nabla^2\Psi + \left[2\Delta_0\lambda^2 + (\alpha_0 + \alpha_0^*)\right]\Psi^+\Psi\Psi. \tag{4.3.4}$$

Dividing equation (4.3.3) by ω_2 and equation (4.3.4) by Ω_2 and comparing corresponding terms in the two equations, we obtain the relationships;

$$\Omega_2(\omega_0 + E_1) = \omega_2(\Omega_0 + E_2), \tag{4.3.5}$$

and

$$\Omega_2\left[2\delta_0 + \lambda^2(\alpha_0 + \alpha_0^*)\right] = \omega_2\left[2\Delta_0\lambda^2 + \alpha_0 + \alpha_0^*\right]. \tag{4.3.6}$$

We may solve equation (4.3.5) for E_2 to give us

$$E_2 = \frac{\Omega_2}{\omega_2}(\omega_0 + E_1) - \Omega_0, \tag{4.3.7}$$

where of course, E_1 is still arbitrary. Solving for λ in equation (4.3.6), we obtain

$$\lambda^2 = \frac{\frac{\omega_2}{\Omega_2}(\alpha_0 + \alpha_0^*) - 2\delta_0}{\alpha_0 + \alpha_0^* - 2\Delta_0\frac{\omega_2}{\Omega_2}}. \tag{4.3.8}$$

The net result is that the two equations in (4.3.3) and (4.3.4) are made compatible and both become a stationary nonlinear Schrödinger equation with constant coefficients. There are a number of solutions of this equation which have been thoroughly investigated in the past. Suffice it to say that, among the spatially inhomogeneous solutions one finds elliptic waves of several kinds as well as hyperbolic localized solutions and there exist critical currents above which nonsingular solutions cease to exist[29].

SECTION IV: INTERACTING ELECTRONS AND PHONONS

A. The Fröhlich Hamiltonian

To illustrate the general method described in Section II, we will use another specific example in which Bosons will be represented by phonons and Fermions by electrons. Interactions between electrons and phonons in a metal were first described in a quantum mechanical formalism[19] using the Fröhlich Hamiltonian below

$$H = \sum_{k} \frac{\hbar^2}{2m} k^2 a_k^+ a_k + \sum_{q} \hbar\omega_q b_q^+ b_q + \sum_{k,l} M_{k,l}\left[b_q + b_{-q}^+\right] a_k^+ a_l, \tag{4.4.1}$$

where $\mathbf{q} = \mathbf{k} - \mathbf{l}$. Here, the operators a_k^+ and a_k refer to the electrons while b_k^+ and b_k refer to the phonons. This Hamiltonian is one of the simplest possible examples of the general Fermion-Boson type since only ω, η and Ω terms are retained. The effective coupling constant $M_{\mathbf{k},\mathbf{l}}$ is usually[18] written as

$$M_{k,l} = i\sqrt{\frac{N\hbar}{2M\omega_q}}(\mathbf{l} - \mathbf{k})\cdot \mathbf{s} V_{k-l}, \tag{4.4.2}$$

where $V_{\mathbf{k}-\mathbf{l}}$ represents the screened Coulomb potential between two electrons in a metal, and N is the number of electrons. Typically $V_{\mathbf{k}-\mathbf{l}}$ takes the form

$$V_{k-l} = \frac{2\pi e^2}{|\mathbf{k} - \mathbf{l}|^2 + q_c^2}, \tag{4.4.3}$$

where q_c^{-1} is of the order of the interparticle distance and plasma waves only exist when their wavelength is greater than this. Also, s is the polarization vector for the phonons and the acoustical phonons are taken to be parallel to the vector **l** - **k**.

In this simple model the ions of the metal are assumed to interact with one another and with conduction electrons via a short-range screened potential. The conduction electrons, on the other hand, are considered to be essentially independent Fermions. Note that the "bare" Coulomb interaction between the ions and the conduction electrons is <u>not</u> used, and, to incorporate screening repulsive terms have been built in to some extent via the short-range nature of the effective interaction remaining. The Fröhlich Hamiltonian, in spite of its approximate nature, played an important role in the development of the theory of superconductivity by leading directly to the BCS model, which worked quite well for describing low temperature superconductors.

Our approach to the problem of the electron-phonon interaction will be based on the Fröhlich Hamiltonian but will differ from the standard perturbative approach. Instead, we shall employ the formalism of the MCS developed in Section II and obtain exact solutions to the equations of motion for the classical order parameter fields.

Using the results of Section II the following coupled equations for the electron field Ψ and phonon field Φ are found to be

$$i\hbar\, \partial_t \Psi = -\omega_2 \nabla^2 \Psi + 2\, \mathrm{Re}\left[\eta_0 \Phi\right] \Psi \tag{4.4.4}$$

and

$$i\hbar\, \partial_t \Phi = -\Omega_2 \nabla^2 \Phi + \eta_0^* |\Psi|^2 \tag{4.4.5}$$

where the coefficient $\omega_2 = \dfrac{\hbar^2}{2m}$ and is due to the dispersion relation for free electrons.

B. The q-Independent Coupling Constant Case

(i) Fully NonLinear Calculation

The expansion in terms of k has to be around $q_F = 2k_F$ since the conduction electrons are close to the Fermi level[18]. Consequently, to zeroth order

$$\eta_0 = i\sqrt{\frac{N\hbar}{2M\omega(q_F)}\frac{2\pi e^2 q_F}{q_F^2 + q_c^2}}. \tag{4.4.6}$$

Note also that we have conveniently removed gradient terms which can be readily absorbed into the time dependence of Ψ by shifting to a moving frame of reference.

To begin we adopt the simplest ansatze and write

$$\Psi = e^{i\theta t}\eta_1 e^{i\phi_1} \tag{4.4.7}$$

and

$$\Phi = \eta_2 e^{i\phi_2}. \tag{4.4.8}$$

In equation (4.4.7) θ is a constant and η_1, ϕ_1, η_2 and ϕ_2 are spatially dependent only. Inserting equations (4.4.7) and (4.4.8) into equations (4.4.4) and (4.4.5) and splitting each equation up into real and imaginary parts we obtain,

$$\omega_2\nabla^2\eta_1 - \left[\omega_2(\nabla\phi_1)^2 + \hbar\theta\right]\eta_1 + 2|\eta_0|\eta_1\eta_2\sin\phi_2 = 0, \tag{4.4.9}$$

$$\nabla\cdot\left[\eta_1^2\nabla\phi_1\right] = 0, \tag{4.4.10}$$

$$\nabla^2\eta_2 = \eta_2(\nabla\phi_2)^2 - \frac{|\eta_0|}{\Omega_2}\eta_1^2\sin\phi_2, \tag{4.4.11}$$

and

$$\Omega_2\nabla\cdot\left[\eta_2^2(\nabla\phi_2)\right] + |\eta_0|\eta_1^2\eta_2\cos\phi_2 = 0. \tag{4.4.12}$$

It is possible to find solutions to this system of nonlinear equations by assuming the relationships below. We shall assume that $\phi_2 = (n+1/2)\pi$ with n an integer, which automatically satisfies equation (4.4.12). There are then four possible cases, which will satisfy equation (4.4.10), namely:

(i) $\nabla\phi_1 = \text{constant}, \quad \eta_1 = \text{constant},$

(ii) $\nabla\phi_1 = \dfrac{j_1}{\eta_1^2},$

(iii) $\nabla\phi_1 = 0, \quad \eta_1$ arbitrary,

and

(iv) $\eta_1^2\nabla\phi_1 = \nabla \times \mathbf{f}.$

The first case, i.e. (i), leads to $\eta_1 = 0$ and η_2 being an arbitrary constant. Such a condition exhibits no free electrons and thus is not valid. The second case leads to two nonlinear ODE's for η_1 and η_2 which are very difficult to solve but may be made compatible. The second and fourth case will not be discussed here since a more general form will be looked at in part C where we will find some analytical solutions. The third case, however, results in two coupled equations in the form

$$\omega_2\nabla^2\eta_1 = \hbar\theta\eta_1 \pm 2|\eta_0|\eta_1\eta_2 \tag{4.4.13}$$

and

$$\nabla^2\eta_2 = \pm\frac{|\eta_0|}{\Omega_2}\eta_1^2. \tag{4.4.14}$$

Equations (4.4.13) and (4.4.14) can be made compatible by assuming a linear relationship between η_2 and η_1. For example,

$$\eta_2 = \mu\eta_1 + \nu, \tag{4.4.15}$$

where μ and ν are constants to be determined. Compatibility conditions imposed on the system yield

$$\mu = \pm\sqrt{\frac{\omega_2}{2\Omega_2}} \tag{4.4.16}$$

and

$$\nu = \mp\frac{\hbar\Theta}{2|\eta_0|}. \tag{4.4.17}$$

We now have one equation which can be integrated and so on integrating equation (4.4.14) once, we obtain;

$$(\nabla\eta_1)^2 = \mp\frac{2|\eta_0|}{3}\sqrt{\frac{2}{\omega_2\Omega_2}}\,\eta_1^3 + c, \tag{4.4.18}$$

where we have assumed $\nabla\eta_1 \neq 0$. If we plot the left-hand side as a function of η_1 it is easy to see that all the real solutions are singular since the plot is monotonic and, no matter what the integration constant c, there will not be two turning points for the same value of c. The conclusion must, therefore, be drawn that the adopted approach is, in some sense, inadequate because localized nonsingular solutions, which would manifest the onset of superconductivity do not exist. The reason for the failure to obtain valid physical solutions is that in equation (4.4.15), we have assumed a linear relationship between the phonon density and the electron density, which is not the case. The onset of superconductivity however, can be seen by tackling the problem from another point of view, which relies on a small coupling energy compared to the phonon energy and is discussed in the following subsection.

(ii) Weakly Coupled Elections and Phonons

In this subsection we will again consider the interaction between conduction electrons and phonons to zeroth order and, for convenience, will consider the effects of the electron field on the phonon field to be weak. Also, we not only retain the gradient in the phonon field but do not assume that the second order terms have already been reduced to a diagonal form of a Laplacian operator. Since the electrons are initially free, we have for the energy $-\hbar^2k^2/(2m)$ which is a Laplacian, so that our starting equations are;

$$i\hbar\partial_t\Psi = -\omega_2\nabla^2\Psi + 2\mathrm{Re}\left[\eta_0\Phi\right]\Psi \tag{4.4.19}$$

and

$$i\hbar\partial_t\Phi = \Omega_0\Phi + i\Omega_1\cdot\nabla\Phi - \Omega_{2,ij}\frac{\partial^2\Phi}{\partial x_i\partial x_j} + \eta_0^*|\Psi|^2, \tag{4.4.20}$$

where $\omega_2 = \hbar^2/(2m)$. We first note that equation (4.4.20) is essentially linear and inhomogeneous with a driving term in η_0^*. The independent variables are now changed to one which is a function of time and space in a traveling-wave form. Thus, we define

$$w = c_0 + \sqrt{\frac{-2\Omega_0}{\Omega_{2,ij}\Omega_{1,i}\Omega_{1,j}}}\left[|\Omega_1|^2 t + \Omega_1\cdot\mathbf{r}\right], \tag{4.4.21}$$

where, with $\Omega_0 > 0$, we assume $\Omega_{2ij}\ \Omega_{1i}\ \Omega_{1j}$ is negative, Ω_{1i} being the ith component of the vector Ω_1. The parameter c_0 is an arbitrary constant which may be fixed by boundary conditions. We are, of course, implicitly assuming in equation (4.4.21) summation over repeated indices. This reduces equation (4.4.20) into a linear ordinary differential equation (ODE) which is easy to solve. Without loss of generality we can take

$$\Phi = i\,F(w), \tag{4.4.22}$$

to be purely imaginary and equation (4.4.20) becomes

$$F'' + F - \left(\hbar\Omega_0\right)^{-1}|\eta_0|n(w) = 0, \tag{4.4.23}$$

where $n(w) = |\Psi|^2$ is the electron number density and the primes denote differentiation with respect to w. The independent solutions to the homogeneous part of equation (4.4.23) are cos w and sin w and it is easy to demonstrate that the general solution of equation (4.4.23) is

$$F = c_1 \cos w + c_2 \sin w + \frac{|\eta_0|}{\hbar\Omega_0} \cos w \int_0^w \frac{dw'}{\cos^2 w'} \int_0^{w'} dw'' n(w'') \cos w''. \tag{4.4.24}$$

Having solved equation (4.4.20) via equation (4.4.24) we can now attempt to solve equation (4.4.19) and we do this by assuming that the electron field may be written as

$$\Psi = f(w) \exp\left[-\frac{iEt}{\hbar} + i g(w)\right], \tag{4.4.25}$$

where the functions f(w) and g(w) are to be determined from equation (4.4.19). To this end we insert equation (4.4.25) into equation (4.4.19) and separate real and imaginary parts. We find that the imaginary component gives us a form of continuity equation, namely,

$$\partial_t f^2 + \frac{2\omega_2}{\hbar} \nabla \cdot \left[f^2 \nabla g\right] = 0. \tag{4.4.26}$$

Defining Δ by $\Delta = \sqrt{\frac{-2\Omega_0}{\Omega_{2,ij}\Omega_{1,i}\Omega_{1,j}}}$, we can rearrange equation (4.4.26) to be in terms of an ODE with the independent variable w as follows;

$$\left(f^2\right)' + \frac{2\omega_2}{\hbar} \Delta \left[f^2 g'\right]' = 0. \tag{4.4.27}$$

Integrating equation (4.4.27) once and solving for g' we find

$$g' = \left\{\frac{c_3}{f^2} - 1\right\} \frac{\hbar}{2\omega_2 \Delta}. \tag{4.4.28}$$

In equation (4.4.28) c_3 is an arbitrary constant whose physical meaning will be shown below. Now, the real part of equation (4.4.19), on substituting equation (4.4.25), is;

$$E f - \hbar(\partial_t g) f = -\omega_2 \nabla^2 f + \omega_2 f \nabla^2 g - |\eta_0| f F. \tag{4.4.29}$$

Equation (4.4.29) can be converted into an ODE since g and f are functions only of w to give

$$\left[E - \hbar|\Omega_1|^2 \Delta g'\right] f = -\omega_2 |\Omega_1|^2 \Delta^2 \left[f'' - f(g')^2\right] - |\eta_0| f F. \tag{4.4.30}$$

Substituting for g' from equation (4.4.28) into (4.4.30) decouples the two functions and yields

$$\left\{E - \frac{\hbar^2 |\Omega_1|^2}{4\omega_2}\left(\frac{c_3^2}{f^4} - 1\right)\right\} f = -\omega_2 |\Omega_1|^2 \Delta^2 f'' - |\eta_0| f F. \tag{4.4.31}$$

The expression for F(w) from equation (4.4.24) may now be inserted into equation (4.4.31) which, of course, at this stage, is an integro-differential equation and it is now that we make approximations. First, if we assume that the electron field has a weak effect on the phonon field, then $|\eta_0|(\hbar\Omega_0)^{-1}$ in equation (4.4.24) is very small so we drop this term. The second point which we note is that the electron current density is proportional to

$$j \propto f^2 \nabla g = \frac{\Omega_1}{2\omega_2}\left[c_3 - f^2\right]. \tag{4.4.32}$$

It is clear from equation (4.4.32) that c_3 is a constant current density which, if considered to be an external current (remember it arose as an integration constant), destroys superconductivity. Thus, at this stage we take $c_3 = 0$. Taking these two effects into account, equation (4.4.31) becomes Mathieu's equation.

Thus, equation (4.4.31) becomes

$$f'' + \frac{|\eta_0| f}{\omega_2 |\Omega_1|^2 \Delta^2}[c_1 \cos w + c_2 \sin w] = \left\{E + \frac{\hbar^2 |\Omega_1|^2}{4\omega_2}\right\} \frac{f}{-\omega_2 |\Omega_1|^2 \Delta^2}. \tag{4.4.33}$$

It is through a simple rescaling of the independent variable w that we can transform this into the standard form of Mathieu's equation, with cos 2u, by letting w = 2u + arctan(c_2/c_1). There are two types of solutions represented by equation (4.4.33); those where electrons are scattered by the phonons and those where the electrons are bound by the phonons (i.e. manifesting superconductivity). The bounded solutions occur whenever

$$E < |\eta_0| \sqrt{c_1^2 + c_2^2} - \frac{\hbar^2 |\Omega_1|^2}{4\omega_2}, \tag{4.4.34}$$

otherwise, we obtain free electrons scattering off the phonons. The above puts a constraint on how weak the coupling between the phonons and the electrons ($|\eta_0|$) can be for superconductivity to occur or, conversely, how strong $|\Omega_1|$ can be.

As a final note to this subsection, equation (4.4.25) can be easily generalized by including in the exponential, $i\mathbf{k_0}\cdot\mathbf{x}$ and if $\mathbf{k_0}\cdot\mathbf{\Omega_1} = 0$ then the only change to our equations is a shift in E to E-$\omega_2 k_0^2$. Equation (4.4.24) remains unchanged but has a profound effect upon equations (4.4.31), (4.4.33) and (4.4.34). Basically, from equation (4.4.34), we see that this inclusion expands the range that E can be within and still have bound states. As an example, consider w lying in a two-dimensional plane so that we wish the electrons to be bounded within the potential wells of the phonons on the plane but free in the perpendicular direction (say, the z-direction). Then, we have equation (4.4.25) but multiplied by e^{ikz} so that the conditions for Cooper pairs to form becomes;

$$E < |\eta_0| \sqrt{c_1^2 + c_2^2} + \omega_2 k^2 - \frac{\hbar^2 |\Omega_1|^2}{4\omega_2}. \tag{4.4.35}$$

Thus, by having superconductivity only ou a plane and free electrons in the perpendicular direction we have raised the critical temperature T_c.

C. Introducing q-Dependence in the Interactions

With the full knowledge that superconductivity does arise from the Fröhlich Hamiltonian we re-examine the approximation in equation (4.4.6) in order to find out if q-dependence of the coupling term may provide further insight. We proceed by expanding linearly in q^2 to obtain a q-dependent correction to the coupling constant. Thus, we obtain

$$\eta_q = \eta_0 - \eta_0 \Lambda q^2, \tag{4.4.36}$$

where $\Lambda = 1/q_c^2$. This will introduce extra terms into the equations of motion for the Ψ and Φ fields of equations (4.4.4) and (4.4.5), namely

$$i\hbar\partial_t\Psi = -\omega_2\nabla^2\Psi + 2\,\mathrm{Re}\left[\eta_0\Phi\right]\Psi + 2\Lambda\nabla^2\left[\mathrm{Re}\left[\eta_0\Phi\right]\Psi\right] \tag{4.4.37}$$

and

$$i\hbar\partial_t\Phi = -\Omega_2\nabla^2\Phi + \eta_0^*|\Psi|^2 + \Lambda\eta_0^*\nabla^2\left[|\Psi|^2\right]^* \tag{4.4.38}$$

To analyze equation (4.4.38) we write

$$\Phi = \Phi_R + i\Phi_I, \tag{4.4.39}$$

$$\eta_0 = i\,\chi, \tag{4.4.40}$$

and

$$\Psi = \eta e^{i\phi_0} e^{-iEt/\hbar}, \tag{4.4.41}$$

so that the real and imaginary parts of equation (4.4.38) become

$$\text{Re:}\quad \nabla^2\Phi_R = \frac{\hbar}{\Omega_2}\dot{\Phi}_I\,, \tag{4.4.42}$$

$$\text{Im:}\quad \nabla^2\left[\Omega_2\Phi_I + \Lambda\chi\eta^2\right] = -\hbar\dot{\Phi}_R - \chi\eta^2\,. \tag{4.4.43}$$

We then assume that the argument of the left hand side of equation (4.4.43) is zero so that

$$\Phi_I = -\frac{\Lambda\chi\eta^2}{\Omega_2} \tag{4.4.44}$$

and thus we also need

$$\dot{\Phi}_R = -\frac{\chi\eta^2}{\hbar} \tag{4.4.45}$$

so as to satisfy equation (4.4.43). In principle, we could add any harmonic function of spaceand an arbitrary function of time to the right-hand side of equation (4.4.44) so that the left hand side of equation (4.4.43) is still zero but we draw back at such, possibly not necessary, complexity. Next we differentiate equation (4.4.42) with respect to time once and substitute equations (4.4.44) and (4.4.45) into equation (4.4.42) to obtain a wave-equation in the form

$$\nabla^2\eta^2 = \frac{\hbar^2\Lambda}{\Omega_2^2}\partial_{tt}\eta^2, \tag{4.4.46}$$

the solution of which in one dimension may be written as

$$\eta^2 = \eta^2(\xi), \tag{4.4.47}$$

where $\xi = x \pm vt$, with $v = \Omega_2 / (\hbar\sqrt{\Lambda})$. Note that the function in equation (4.4.47) is an <u>arbitrary</u> function of ξ and this does not impose any constraints on the procedure to follow, except of course that η has the independent variable ξ.

We now return to equation (4.4.37) with equation (4.4.44) providing a link and making use of equation (4.4.39), to obtain

$$i\hbar\partial_t\Psi = -\omega_2\nabla^2\Psi + 2\frac{\Lambda}{\Omega_2}\chi^2|\Psi|^2\Psi + 2\frac{\Lambda^2}{\Omega_2}\chi^2\nabla^2\left(|\Psi|^2\Psi\right). \tag{4.4.48}$$

In order to interpret the meaning of E in equation (4.4.41), we consider the kinetic energy part of the Lagrangian density L = T - V which takes the form

$$T = \frac{i}{2}\left[\Psi^*\Psi_t - \Psi_t^*\Psi\right]. \tag{4.4.49}$$

Thus, inserting equation (4.4.41) into equation (4.4.49) gives the kinetic energy T as

$$T = \frac{E\eta^2}{\hbar}. \tag{4.4.50}$$

From equation (4.4.50) it is clear that E must be positive i.e. the kinetic energy is proportional to E and the amplitude squared of the field.

Using equation (4.4.41) and separating real and imaginary parts yields for equation (4.4.48);

$$\begin{aligned}\text{Re:}\quad -\hbar\eta\frac{\partial\phi_0}{\partial t} + E\eta = -\omega_2\left[\nabla^2\eta - \eta(\nabla\phi_0)^2\right] + \lambda\eta^3 + \lambda\Lambda\Big[6\eta(\nabla\eta)^2 + \\ 3\eta^2\nabla^2\eta - \eta^3(\nabla\phi_0)^2\Big]\end{aligned} \tag{4.4.51}$$

and

$$\text{Im:}\quad \hbar\frac{\partial\eta}{\partial t} = -\omega_2\left[2\nabla\eta\cdot\nabla\phi_0 + \eta\nabla^2\phi_0\right] + \lambda\Lambda\eta^2\left[6\nabla\eta\cdot\nabla\phi_0 + \eta\nabla^2\phi_0\right], \tag{4.4.52}$$

where λ is defined by $\lambda = 2\chi^2\Lambda/\Omega_2$. Equation (4.4.52) can be understood as a form of continuity equation. Defining charge density as $n = \eta^2$ and current density as $\mathbf{j} = \eta^2\nabla\phi_0$, equation (4.4.52) can be written as

$$\nabla\cdot\left\{\frac{4}{\hbar}(\omega_2-\lambda\Lambda n)^2\mathbf{j}\right\}-\frac{1}{\lambda\Lambda}\frac{\partial}{\partial t}[\omega_2-\lambda\Lambda n]^2=0, \tag{4.4.53}$$

where we see that equation (4.4.53) defines a new current density and number density which are respectively $\mathbf{J} = -4(\omega_2 - \lambda\Lambda n)^2 n\nabla\phi_0/(\lambda\Lambda\hbar)$ and $\rho = (n - \omega_2/(\lambda\Lambda))^2$ and are due to the interaction between the fields.

If we now recall the one dimensional solution for η (see equation (4.4.46) and (4.4.47)), then equation (4.4.53) can be written as a first order ordinary differential equation as follows, using $\xi = x - vt$;

$$\frac{d}{d\xi}\left\{(\omega_2-\lambda\Lambda n)^2\left(\frac{4j_x}{\hbar}+\frac{v}{\lambda\Lambda}\right)\right\}=0. \tag{4.4.54}$$

We have chosen $j_y = j_z = 0$ for convenience. Equation (4.4.54) can be solved for the current by integrating once to obtain, where we have dropped the subscript x on the current;

$$j=\frac{c_0}{(\lambda\Lambda n-\omega_2)^2}-\frac{\hbar v}{4\lambda\Lambda}. \tag{4.4.55}$$

Now, we return to the real part as given in equation (4.4.51) and insert $\nabla\phi_0 = \mathbf{j}/\eta^2$ to obtain;

$$\frac{\hbar v j}{\eta}+E\eta=-\omega_2\left\{\eta''-\frac{j^2}{\eta^3}\right\}+\Lambda\lambda\left\{6\eta(\eta')^2+3\eta^2\eta''-\frac{j^2}{\eta}\right\}+\lambda\eta^3. \tag{4.4.56}$$

Since equation (4.4.56) does not have an explicit dependence on the independent variable, we can use the method outlined in chapter 3 or make the following substitution to obtain a first order ordinary differential equation;

$$(\eta')^2=P(n) \tag{4.4.57}$$

so that equation (4.4.56) becomes;

$$[3\lambda\Lambda n-\omega_2]P'+6P\lambda\Lambda=\frac{\hbar vj}{n}+E-\lambda n-\frac{j^2(\omega_2-\lambda\Lambda n)}{n^2}. \tag{4.4.58}$$

This is solved using the integrating factor method as

$$P=\frac{1}{[3\lambda\Lambda n-\omega_2]^2}\left\{c_1+\int^n dz[3\lambda\Lambda z-\omega_2]\left\{\frac{j^2(\lambda\Lambda z-\omega_2)}{z^2}+\frac{\hbar vj}{z}+E-\lambda z\right\}\right\}. \tag{4.4.59}$$

Thus, we can formally write the solution for η in an implicit form as

$$\xi-\xi_0=\int\frac{d\eta}{\sqrt{P(\eta)}}. \tag{4.4.60}$$

Upon substituting for the current from equation (4.4.55) into equation (4.4.59), we can solve the first integral to obtain;

$$P=\frac{1}{[3\lambda\Lambda n-\omega_2]^2}\left\{c_1-\lambda^2\Lambda n^3+\frac{\lambda n^2}{2}(3E\Lambda+\omega_2)-\frac{c_0^2\lambda\Lambda}{\omega_2(\omega_2-\lambda\Lambda n)^2}-\right.$$
$$\left.\frac{n(9\hbar^2v^2+16E\omega_2)}{16}-\frac{(4c_0\lambda\Lambda-\hbar v\omega_2^2)^2}{(4\lambda\Lambda\omega_2)^2 n}+\frac{c_0(2\hbar v\omega_2^2-c_0\lambda\Lambda)}{\omega_2^2(\omega_2-\lambda\Lambda n)}\right\}. \tag{4.4.61}$$

As was mentioned earlier, superconductivity corresponds to a localized or periodic solution which would describe bound states for the electrons formed by the phonons. An analysis of equation (4.4.61) provides a quantitative criterion for superconductivity. Note here that localized solutions correspond to the existence of at least two different real roots of $P(\eta)$, one of which must be a multiple root.

From equation (4.4.61) the resultant integration constant will determine the values of the corresponding roots of $P(\eta)$. Analyzing the asymptotic behavior of $P(\eta)$ for $\eta\to\infty$ we see that $P(\eta)\to-\infty$ and for $\eta\to 0$, we find that $P(\eta)\to-\infty$ unless $c_0=\hbar v\omega_2^2/(4\lambda\Lambda)$, in which case we have $P(\eta)\to(8\lambda\Lambda c_1+3\hbar^2v^2\omega_2)(8\lambda\Lambda\omega_2^2)^{-1}$. This value for c_0 corresponds to a special value for the current density j, which may possibly represent j_c.

In searching for localized solutions we must determine whether there exists at least <u>one</u> local maximum for $P(\eta)$ where $P(\eta) \geq 0$.

Before examining equation (4.4.61) in more detail, we will simplify the equation by introducing new parameters and rescaling n and P as follows:

$$P(n) = \frac{\omega_2 Q(x)}{\lambda\Lambda^2},\ n = \frac{\omega_2 x}{\lambda\Lambda},\ E = \frac{\omega_2}{3\Lambda}(2\sigma - 1),\ \omega_2 = \frac{\hbar v\sqrt{\Lambda}}{4\varpi},\ c_0 = \frac{\omega_2^3 d_0}{\lambda\Lambda^{3/2}},\ \text{and}\ c_1 = \frac{\omega_2^3 d_1}{\lambda\Lambda^2}\quad \text{so}$$

that equation (4.4.61) becomes;

$$Q = \frac{1}{[3x-1]^2}\left\{d_1 - x^3 + \sigma x^2 - \frac{d_0^2}{(1-x)^2} - \frac{x(27\varpi + 2\sigma - 1)}{3} - \frac{(\varpi - d_0)^2}{x} + \frac{d_0(2\varpi - d_0)}{(1-x)}\right\}. \tag{4.4.62}$$

Thus, we see from equation (4.4.62) that we essentially have only two parameters and that Q is singular at x = 0, 1/3, 1, and ∞, unless we have specific boundary conditions. Note that the singularities at x = 0 and 1 can not both be eliminated at the same time unless $\varpi = d_0 = 0$, and to eliminate the singularity also at x = 1/3, we need $d_1 = (3\sigma-2)/27$ so that $Q = (3\sigma - 3x - 2)/27$. Also, the reader should note that we are assuming that all the parameters at this point are positive, however, if some of them are negative then the change would have to be incorporated into x, Q, and the parameters σ and ϖ. As an example, if $\lambda < 0$ say, then we need to look at $Q \leq 0$ and $x \leq 0$ instead of both of them being nonnegative. If we let $j_x = -\hbar vJ(4\lambda\Lambda)^{-1}$ and substitute this along with the new parameters and constants defined just above equation (4.4.62), then equation (4.4.55) becomes;

$$J = 1 - \frac{d_0}{\varpi(x-1)^2}. \tag{4.4.63}$$

There is much that can be gained from equations (4.4.62) and (4.4.63), the first of which is that if we solve for d_1 such that $([3x-1]^2 Q(x)) \to 0$ when $x \to 1/3$ then for any

d_1 less than this value, $Q \to -\infty$ and $Q \to \infty$ for d_1 greater than this value. Thus, we have an upper limit on the value of d_1 and still obtain nonsingular solutions for $x < 1/3$. This is also true for $1/3 < x < 1$ if $d_0 \neq 0$ or $1/3 < x < \infty$ if $d_0 = 0$. The lower limit on d_1 is determined by the fact that $Q \geq 0$ and the lower limit of d_1 will occur when $Q \leq 0$ for all $x \geq 0$. Now d_0 is interesting in that besides setting the range of values for the currents, it has a strong effect upon Q. For $d_0 = 0$ or for a constant current we see from equation (4.4.62) that Q is no longer singular at $x = 1$ which means that we have only two regimes for x, namely, $0 < x < 1/3$ and $1/3 < x < \infty$. We also have only two singularities when $d_0 = \varpi$, which are at $x = 1/3$ and 1. This means that at $x = 0$, the current is zero and Q in no longer singular which makes since we can not have a current where there are no electrons and this is the only value of d_0 that allows $x = 0$ to be part of the solution. For any value of d_0 besides the two given above, we have three singularities and so our solutions can be periodic functions with the range in values between: $0 < x < 1/3$, $1/3 < x < 1$, or $1 < x < \infty$. It is our opinion that for the range $0 < x < 1/3$, we have the electrons being bounded by the phonons whereas for the higher ranges we look at them as being electrons which are scattering of the phonon field. The reason we think the lower range is the superconducting solutions is first; for the lower range, our solutions are highly dependent upon the values of the d's. The larger d_0 becomes the lower the upper bound for d_1 is and thus in essence shows that there is a value for a critical current. Second, it is possible to have x vanish in a periodic fashion when $d_0 = \varpi$ and for free electrons scattering off a back ground phonon field, there would not be any locations in space where there are no electrons. The third reason is that the upper range $1 < x < \infty$, is not restricted by an upper limit for d_1, as d_1 grows so does the range in which x oscillates over and represents a build of the electron density. This of course is contrary to Cooper pairs which require a low concentration so that the Coulomb repulsion does not destroy the pairing. For the mid range values of x (i.e. $1/3 < x < 1$), there does not seem to be any solutions since it seems to remain below the axis and as one tries to get Q positive in this region, the maximum d_1 is reached and Q jumps up to $+\infty$ at $x = 1/3$ and thus the solutions become singular. In figures IV-1 through IV-4, we have plotted Q as a function of x with the choice $\varpi = 1/4$ and $\sigma = 4$. Also, d_1 was chosen just below its critical value for all four figures. For figure IV-1, we have chosen $d_0 = 0$ so that the current is a constant $J = 1$. The only valid solution is for $0 < x < 1/3$ since for larger x the curve is just below the axis. The solution shown in figure IV-1 is "blown up" in figure IV-2 showing in more detail the curvature of the solution between $0 < x < 1/3$. Thus we see that we can lower d_1 down till Q only touches the axis at one point between $0 < x < 1/3$ which can be determined numerically and as we raise d_1 above the critical value, the solutions for $x < 1/3$ become singular and the solutions for $x > 1/3$ become allowed and nonsingular thus demonstrating the break up of the Cooper pairs and the formation of free electrons scattering off the phonon field. This is also shown by the sudden jump in the density of the electrons. For figures IV-3 and IV-4, we have chosen $d_0 = \varpi$, which gives for x the range $0 \le x < 1/3$. Again, we see the superconducting state and if we increase d_1, we would get the free electron state. Figure IV-4 just shows figure IV-3 between $0 < x < 1/3$. Figures IV-5 and IV-6 confirm the existence of a critical current since for all six figures we have chosen d_1 just below its critical value, at which there is a switch in the curvature at the $x = 1/3$ singularity from $-\infty$ to ∞. In figure IV-6, we have chosen the current density very large and as can be seen, the nonsingular solutions for the superconducting phase do not exist.

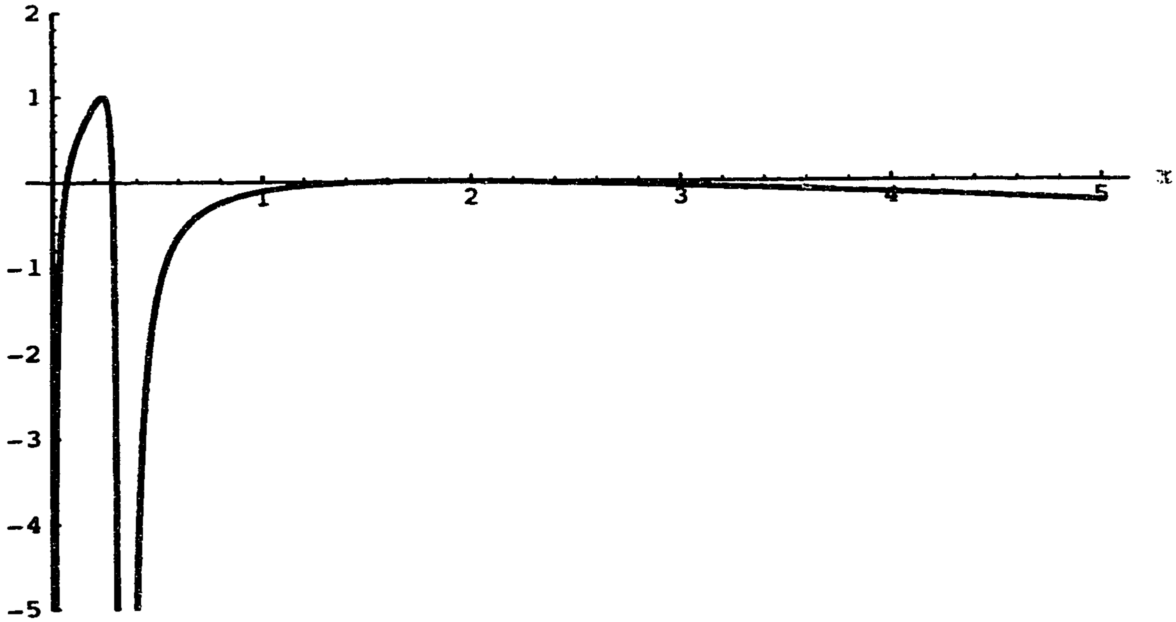

Figure IV-1: Plot of Q(x) as given by equation (4.4.62) with $d_0 = 0$.

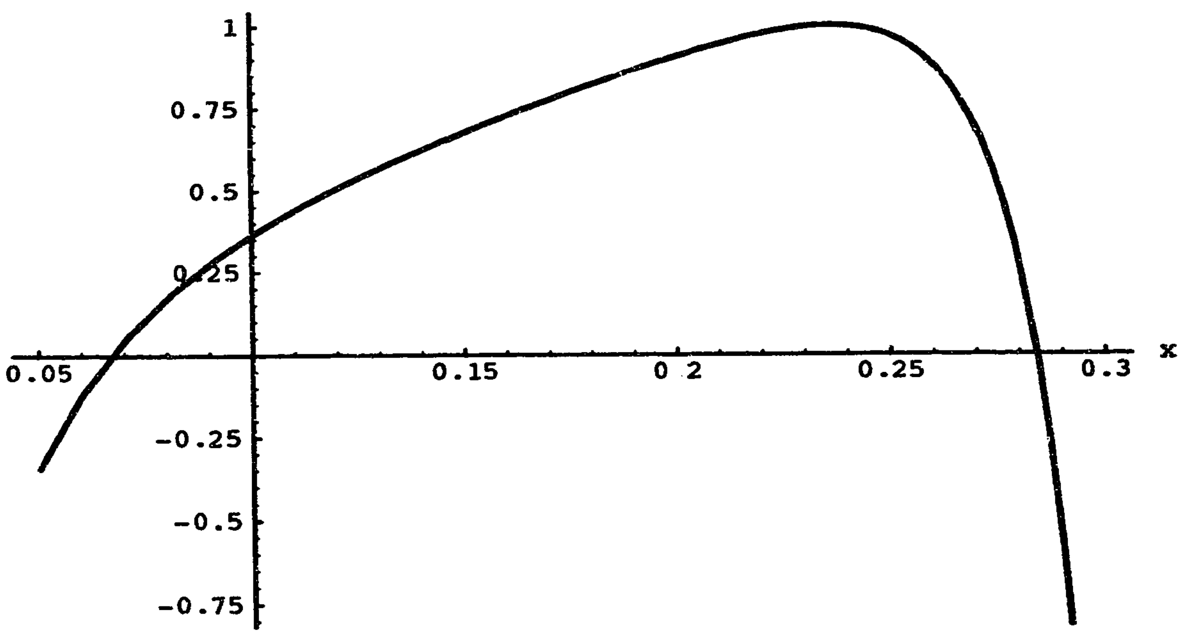

Figure IV-2: Plot of figure IV-1 for small x.

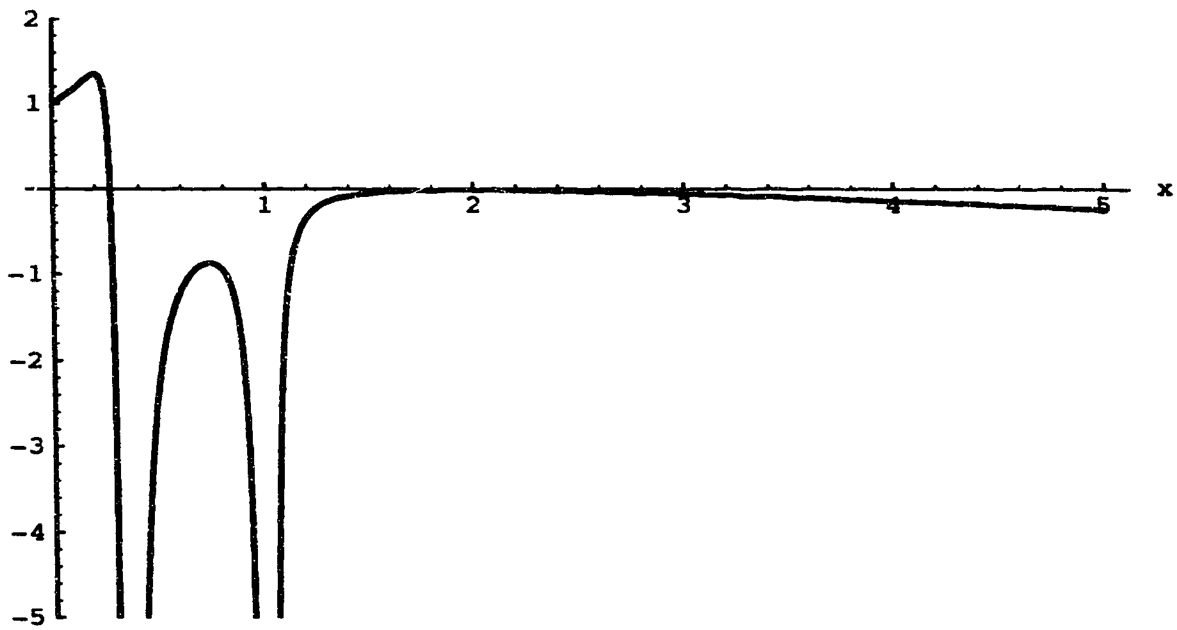

Figure IV-3: Plot of Q(x) as given by equation (4.4.62) with $d_0 = \varpi$.

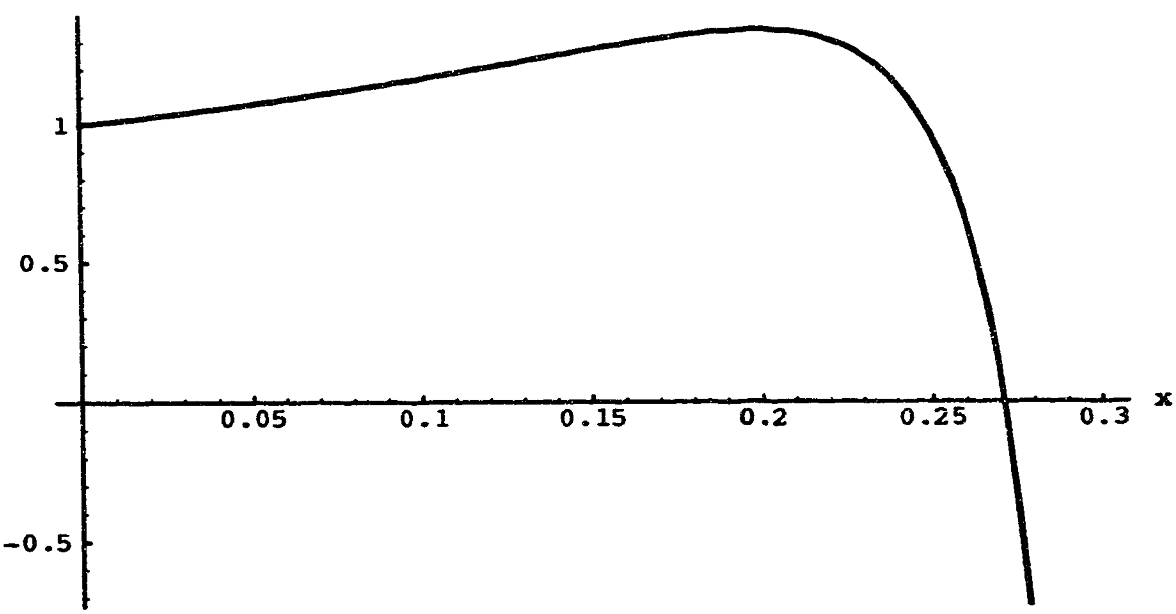

Figure IV-4: Plot of figure IV-3 for small x.

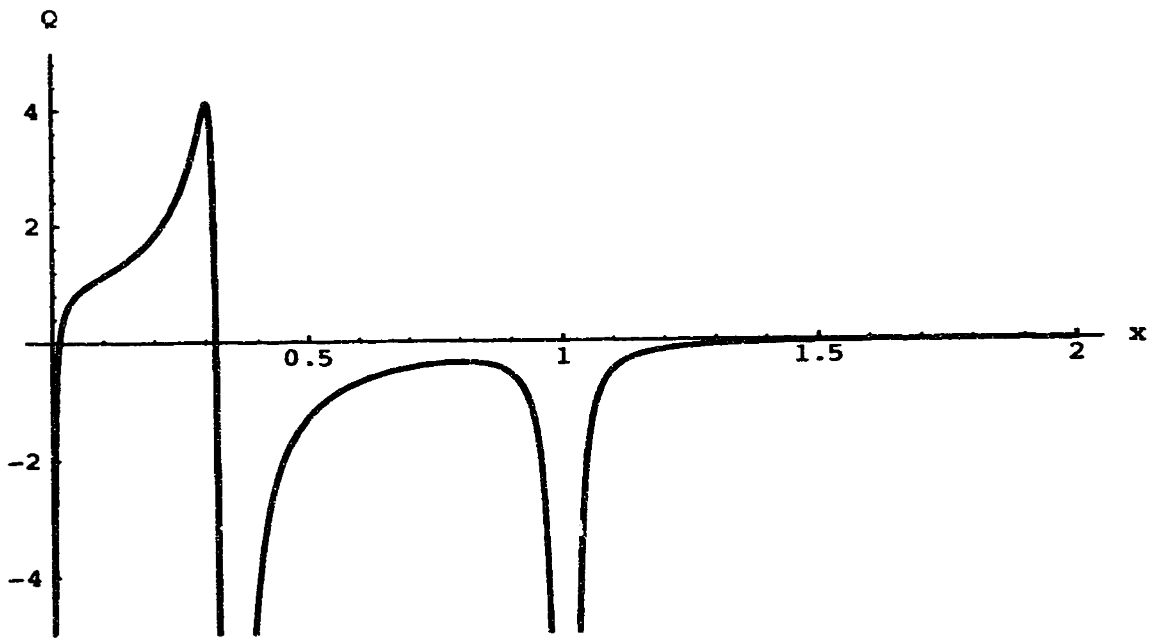

Figure IV-5: Plot of Q(x) as given by equation (4.4.62) with $d_0 = \varpi/2$.

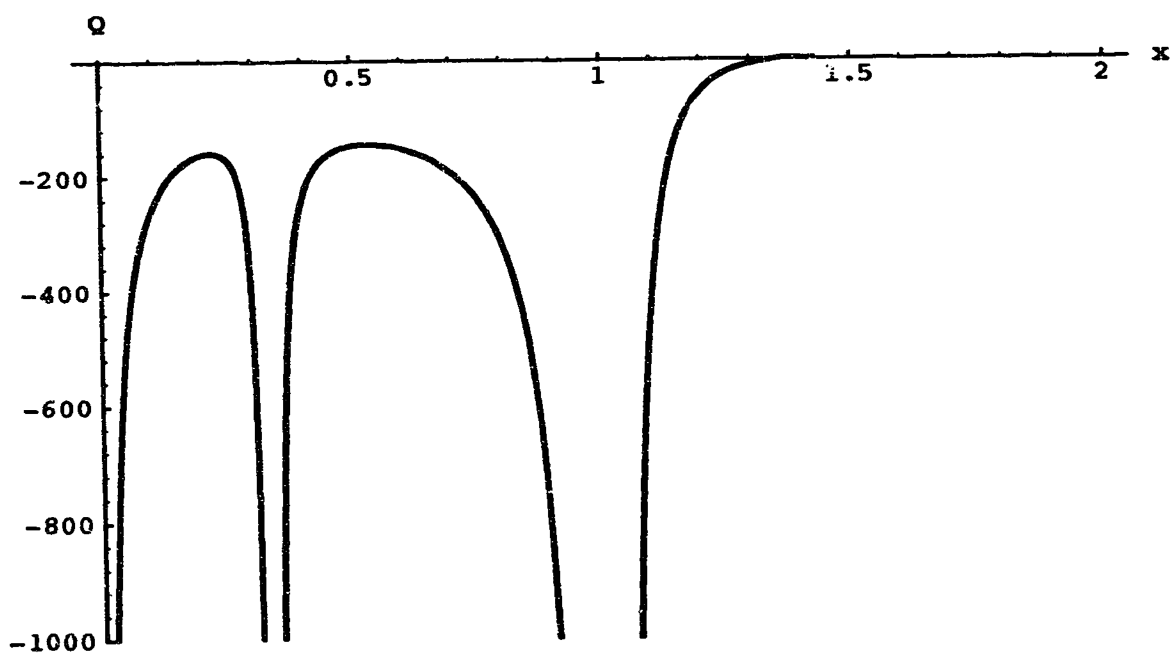

Figure IV-6: Plot of Q(x) as given by equation (4.4.62) with $d_0 = 20\varpi$.

SECTION V: CONCLUSION

In this chapter, we have provided a method of analyzing coupled many-body systems of two degrees of freedom which occur in condensed matter physics and also other branches of physics relying on many-body formalism. The approach to the problem is an important new application to a very general class of problems using the method of coherent structures[6,7]. Since a growing number of important physical phenomena (superconductivity, metamagnetism, structural phase transitions, Mott insulators, etc.) can be adequately described only in terms of two or more interacting degrees of freedom, we feel that our work represents a significant step forward in unifying these diverse systems within a single theory. In our paper, a general calculation of the field equations has been performed that can be applied to an arbitrary system with two field degrees of freedom provided its energy dispersion relations and coupling constants are known or can be estimated. All four types of system, i.e. Boson-Boson, Fermion-Fermion distinguishable as well as indistinguishable and Fermion-Boson have been investigated and their equations of motion provided.

In addition, we have provided two application sections. The first one, being the simplest, deals with the distinguishable Fermion-Fermion system and demonstrates a direct connection between the equations of motion for the fields and the celebrated nonlinear Schrödinger equation, leading to a wealth of analytical solutions including solitons.

In the second application section of this paper we have given a detailed analysis of the Fröhlich Hamiltonian for electron-phonon interactions in a metal which was so crucial in building a modern theory of superconductivity. Our approach has been fully nonlinear and analytical solutions have been provided for both the phonon and electron fields under the general conditions of arbitrary dimensionality and the strength of coupling. The conclusion reached here is that in order to demonstrate Cooper pair formation in a Fröhlich-like Hamiltonian the electron-phonon coupling coefficient must be sufficiently strong for q-independent interactions or must not exceed a critical value for q-dependent couplings. Precise formulae linking the model parameters have been obtained as quantitative criteria for superconductivity. If they are not satisfied the only solutions which arise are either identically zero or singular.

We intend to carry out further applications to this and other important systems (structurally unstable crystal lattices, Jahn-Teller compounds, magnetoelastically coupled spin systems, etc.) in our future work.

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CHAPTER 5: CONCLUSIONS

This work has examined in detail, two models near criticality. The first was the Landau - Ginzburg phenomenological model for superconductors and the second was the Fröhlich Hamiltonian which was presented as an example in chapter four. Both the models were instrumental in the theoretical understanding of low temperature superconductors. Chapter two dealt exclusively with the Landau - Ginzburg model. Chapter four dealt mainly with the method for finding the classical number of particle density from a quantum picture and then the Fröhlich Hamiltonian for describing the electron - phonon interaction in metals was used as an example. The third chapter was mathematical in nature and was to developed so as to help in the analysis of the models studied in the other two chapters. Now, we will briefly summarize the results and conclusions from the previous chapters.

Chapter two was a nonlinear study of the Landau - Ginzburg phenomenological model for superconductors with the presence of a magnetic field. In the third section, three different sets of solutions in one dimension at the critical temperature were found and then these solutions were expanded below the critical temperature. The first set of solutions obtained gave a periodically fluctuating solution for the superconducting charge density and also a periodic solution for the magnetic field but shifted by a quarter of a period relative to the superconducting charge density. The first order corrections for just below the critical temperature were also found. The solutions for the homogeneous equations provide a wealth of possibilities for change in the solutions as the temperature is lowered. The particular solution for the first order corrections without the homogeneous solutions provided us with the expected behavior in the magnetic field and the superconducting charge density by increasing the range over which superconducting charge density is dominate and decreasing the range over which the magnetic field is dominate. The inclusion of the homogeneous solutions would be necessary to fit the solutions to the boundary conditions. The second set of solutions are similar to the first set. Both the magnetic field and the superconducting charge density are periodic and shifted by a quarter of a period relative to each other. The main difference between this set of solutions and the first set is that the magnetic field vanishes at the critical temperature and thus represents trapped flux with no external magnetic field present. The third set of solutions found in section three represent an extreme case in which the magnetic field is so strong it has just about driven the superconductor into the normal phase. This leaves only the surface of the material superconducting and the rest of the material in the normal phase. At the end of section three a numerical plot of the full equations was presented so as to test how accurate the expansion was for this model. Even though it was not explicitly shown to be true, the first set of solutions and to a slightly lesser degree, the second set of solutions can replicate the numerical plot to very good agreement. This means that even though the solutions are only to first order in the parameter α, the series converges very fast giving us good results to work with.

The fourth section of chapter two is basically an extension of the work which has been done in the past, where the effects of the magnetic field upon the superconducting charge is neglected. This is an adequate approximation for type I superconductors where the magnetic field can not penetrate into the superconductor except for the skin, depending of course on the macroscopic shape of the superconductor. These solutions are also valid for type II superconductors in regions were the superconducting current density is essentially a constant. By treating the superconducting current density as a constant, the effects of the magnetic field upon the superconducting charge density are ignored. The equation describing the superconducting charge density under this approximation has been previously analyzed. By simplifying the equation describing the magnetic field, a new independent variable was obtained which is the indefinite integral of the superconducting charge density. In the second part of this section, besides the trivial solution and the standard solution where the superconducting charge density is taken to be a constant, one further solution was obtained where no more approximations were needed. The solution corresponds to the superconducting charge density being a sech^2 solution, which would correspond to a superconducting plane. The rest of the solutions for the superconducting charge density led to very complicated indefinite integrals which needed to be approximated further in order to solve the equation for the magnetic field. A few of the limiting cases were considered and even though these were limiting cases and required further approximations, they did provide solutions that were physically correct. The last part of section four dealt with looking at the equation for the magnetic field from a different point of view. This method gave solutions for the magnetic field and the superconducting charge density for two special cases and also obtained recursion relationships for the general solution The first special case was when there was no cubic nonlinearity and no superconducting current density and the second special case was when there was just no superconducting current density.

In sections three and four of chapter two, only one dimensional symmetry was discussed and so in section five the equations of state with a cylindrical symmetry were considered. The method used to solve the equations of state in this section was the same as that in section III, an expansion about the critical temperature. Section five contained two parts, the first was called vortices and the second spirals. The first part of section five considered the magnetic field and the superconducting charge density to have only radial dependence. The vortex part of this section led to many interesting solutions some of which were very singular and others that appear to have applications to magnetic field penetration in high temperature superconductors. Some of the obtained solutions included a solution for the magnetic field close to the core of a magnetic vortex and a solution of damped periodic rings of superconducting charge density to mention a couple. The spiral part of section four also gave many solutions that were similar to those obtained in the vortex part. For the spiral part of the section, it was assumed that the magnetic field and the superconducting charge density were both functions of an independent variable that leads to spiral solutions. For this part, to simplify the equations, the radial distance from the core was assumed to be small so that the ln(r) in the spiral symmetry is dominate. As in the vortex part of section five, many solutions were obtained, some of which were kink solutions and damped periodic solutions.

The sixth section of chapter two dealt with rescaling the magnetic field and the superconducting charge density in order to validate an approximation where the magnetic field is written as a function of the superconducting charge density and thus decouple the two equations. The decoupling gave a single equation which was the cubic nonlinear Klein - Gordon equation which was studied in the past. These solutions were reviewed in section six as were the corresponding magnetic field and superconducting charge density that resulted from these solutions. For nontrivial solutions, there were five sets of solutions found for the magnetic field and the superconducting charge density. Even though the solutions found were physically acceptable for specific boundary conditions, the generic solutions appeared to be unphysical, with the magnetic field being unbounded in most of the cases. There were a few cases that gave well behaved solutions for the generic case.

Chapter three was a purely mathematical chapter with no physics involved except in the examples where some important physical models were discussed and the applicability of this chapter to those models. The chapter looks at a class of autonomous nonlinear ordinary differential equations and seeks solutions for these equations. The second section of chapter three looked at seven special cases in which the general equation could be solved. For six of the seven cases, the restriction on the equation was that two of the three initially unknown functions remained unknown and the third function was a specific function of the other two arbitrary functions. The seventh special case however, was for only one arbitrary function and for the other two, one was a constant and the other was a function of the arbitrary function. In the third section, the algebraic method was introduced. The section showed how the nonlinear ordinary differential equation could be reduced to a system of algebraic equations that could be solved in principle by a symbolic solver. The section also showed that depending on the type of reduction, one can obtain either exponential solutions or solutions of Jacobi elliptic functions. The fourth section dealt with several examples, some of which are physically important. The examples showed how to approach the problem using either the special cases and/or the algebraic method. There were four examples analyzed and another three were briefly discussed.

Chapter four starts with a very general and complicated generic second quantized Hamiltonian for a system of two types of interacting particles. From this Hamiltonian, the Heisenberg equations of motion are obtained for four cases; the Boson - Boson case, the indistinguishable Fermion - Fermion case, the distinguishable Fermion - Fermion case, and the Boson - Fermion case. In each case, the creation and annihilation operators are written in terms of field operators for both types of particles. A plane wave basis was used and all the coefficients were expanded in a Taylor series, with the single particle energies expanded to second order and all other coefficients expanded only to [illegible] order. For each case, the resulting equations of motion were of the nonlinear Schrödinger type coupled together. This was the end for each case since the obtained equations were too complicated to analyze and only when considering a specific model could the equations be reduced to something more manageable and even then not much could be done with them unless treated classically. Once the equations were treated classically and their classical solutions obtained, the modulus squared of these solutions would correspond to the classical particle number density. In the third section, an attempt is made to find classical solutions for the general equations of motion for the distinguishable Fermion - Fermion case. Under the assumption that the two fields were linearly dependent, the equations of motion reduce to a single nonlinear Schrödinger equation. A great deal of work has already been done in finding analytical solutions for this equation with the solutions including elliptic waves and hyperbolic localized solutions. The fourth section dealt with the Fröhlich Hamiltonian, which is the electron - phonon interaction in metals and thus falls into the Fermion - Boson case. Only three of the coefficients in the generic Hamiltonian remained in the example and thus simplified the equations of motion for the classical fields significantly. The first part was a general discussion of the Fröhlich Hamiltonian and the derivation of the equations of motion for this case. The second part dealt with the interaction being q independent or equivalently, only to zeroth order in the Taylor expansion of the coefficient was considered. After the two equations were decoupled, a single integro - differential equation was obtained. This was not possible to solve but by assuming that the effects of the electrons upon the phonon field were weak, then the term with the integral could be dropped. This gave a second order nonlinear ordinary differential equation to solve, however, by assuming no external current was present, then the equation reduced down to Mathieu's equation. For Mathieu's equation, there are bounded periodic solutions and also unbounded solutions depending on the energy of the electrons. Thus, even with the assumptions made to simplify, the solutions still showed superconductivity through the formation of Cooper pairs by the bounded solutions and also solutions for free electrons scattering off the phonon field were present. In the third part, higher order terms in the Taylor expansion of the interaction coefficient were considered. This lead to a pair of complicated coupled nonlinear partial differential equations to solve. However, again it was possible to find a relationship between the phonon field and the electron field which gave the constraint that the envelop of the electron field must satisfy the wave equation. Using the relationship between the phonon field and the electron field and also the constraint, one is able to reduce the second equation down to a nonlinear ordinary differential equation and also a continuity equation. In one dimension, the continuity equation was easily solved for the current density. The final solution for the electron envelop was written as indefinite integral, giving an implicit solution for the envelop. However, it was possible to be gain some information without actually solving the integral. It was found that there were only two independent parameters which is an acceptable number of parameters. Also, it was shown that there exists solutions for superconductivity and solutions for free electrons scattering off the phonon field as was illustrated in the figures IV-1 through IV-4.

Thus, this work has analyzed two models that were fundamental in the understanding of low temperature superconductors. The work presented here expanded on the previous work done on these models to a great extent. Also, new techniques for solving nonlinear ordinary differential equations was introduced. Finally, a method was developed in chapter four to find the classical number density for large systems of interacting particles.

APPENDIX A: POLYNOMIAL ANSATZE FOR CHAPTER TWO SECTION III

Consider the system of equations (2.3.4) and (2.3.5). Assume first a linear relationship between u and η, i.e.,

$$u = a_1\eta + a_2. \tag{A1}$$

Then we substitute equation (A1) back into equations (2.3.4) and (2.3.5), both of which can now be compared as to their form. Compatibility conditions that arise read

$$\alpha + a_2 = 0, \tag{A2}$$

$$\frac{\gamma a_2}{a_1} = 2a_1a_2, \tag{A3}$$

and

$$\gamma = \beta + a_1^2. \tag{A4}$$

Equation (A3) can be satisfied if either $a_2 = 0$ or $a_1^2 = \gamma/2$. The first possibility, i.e., $a_2 = 0$, leads immediately to $\alpha = 0$, which means that the temperature must be at the critical temperature. This must also be accompanied by $a_1^2 = \gamma - \beta > 0$. On the other hand, the requirement that $a_1^2 = \gamma/2$ also implies that $a_2^2 = -\alpha$ and $\gamma = 2\beta$. Therefore, a very specific (and highly unlikely) set of coupling parameters is needed in this second case to satisfy compatibility.

A more general Ansätze than that of equation (A1) was also contemplated, namely,

$$u = a_1\eta + a_2 + a_p\eta^p, \tag{A5}$$

with an a priori arbitrary value of p. Substituting equation (A5) into equations (2.3.4) and (2.3.5) and comparing terms of the same order in η leads to compatibility conditions very similar to those stated in equations (A2) through (A4). This time, however, there are many more possibilities than before. We have examined all such cases and found that only one additional situation is remotely possible. This is when p = 1/2 with a host of restrictions, namely,

(i) $a_2^2 + \alpha = 0$,

(ii) $32a_1a_2 = -5a_1^2 / 2$,

(iii) $19a_1^2 + 4a_1 = 8\gamma\beta$,

and

(iv) $-11\gamma + 20\beta = 4a_1$.

The last two of these equations may be satisfied only if γ and β are specifically related, requiring a particular temperature value for a given nonlinearity parameter in the model.

APPENDIX B: DIFFERENTIAL EQUATIONS FOR JACOBI ELLIPTIC FUNCTIONS

$$\left\{\frac{d}{du}\mathrm{sn}(u)\right\}^2 = \left[1-\mathrm{sn}^2u\right]\left[1-k^2\mathrm{sn}^2u\right], \tag{B1}$$

$$\left\{\frac{d}{du}\mathrm{cn}(u)\right\}^2 = \left[1-\mathrm{cn}^2u\right]\left[(k')^2+k^2\mathrm{cn}^2u\right], \tag{B2}$$

$$\left\{\frac{d}{du}\mathrm{dn}(u)\right\}^2 = (1-\mathrm{dn}^2u)(\mathrm{dn}^2u-(k')^2), \tag{B3}$$

$$\left\{\frac{d}{du}\mathrm{tn}(u)\right\}^2 = (1+\mathrm{tn}^2u)(1+(k')^2\mathrm{tn}^2u), \tag{B4}$$

where k is the Jacobi modulus and k' is the complementary modulus with $(k')^2 + k^2 = 1$.

APPENDIX C: REDUCTION OF THE SYSTEM OF EQUATIONS IN CHAPTER TWO TO A SINGLE EQUATION

It is an interesting observation that in one dimension it is possible to reduce our system of equations to a single nonlinear ODE. We start by considering the reduced equations (2.3.4) and (2.3.5) where for convenience we set $u_y = u$ and $u_z = 0$. We now write η in terms of n_s (which we simply [illegible] henceforth as n), so that equations (2.3.4) and (2.3.5) become

$$\frac{d^2u}{dx^2} = \gamma n u, \tag{C1}$$

and

$$\frac{d^2n}{dx^2} - \frac{1}{2n}\left\{\frac{dn}{dx}\right\}^2 = 2n(\alpha + \beta n + u^2). \tag{C2}$$

Equations (C1) and (C2) are two autonomous equations which we will reduce down to a single nonautonomous equation by assuming that $n = n(u)$, so that we obtain the following consistency equation along with the physical parameters in terms of u, which are;

$$\gamma n u = \frac{d}{du}\left\{\frac{2n^2\left[2\alpha + 2\beta n + 2u^2 - \gamma u n'\right]}{2nn'' - (n')^2}\right\}, \tag{C3}$$

$$x + x_o = \int_{u^*}^{u} dv \left\{\int_{v^*}^{v} d\omega\, 2\gamma\omega\, n(\omega)\right\}^{-1/2}, \tag{C4}$$

$$\mathbf{v}(u) = -\frac{\hbar u \hat{\mathbf{y}}}{m^*}, \tag{C5}$$

$$\mathbf{j}_s(u) = -\frac{\hbar e^*}{m^*} u n \hat{\mathbf{y}}, \tag{C6}$$

$$\mathbf{h}(u) = \frac{\hbar c \hat{\mathbf{z}}}{e^*}\sqrt{2\gamma \int_{u^*}^{u} dv\, v n(v)}. \tag{C7}$$

Equation (C3) is a third order nonlinear ODE which is very difficult to solve analytically, but can be reduced to a second order ODE by the transformations $u^2 = y$ and $n = f'/\gamma$, so that equation (C3) becomes

$$\left\{\frac{d^2f}{dy^2}\right\}^2 = \frac{1}{y(f+c_0)}\frac{df}{dy}\left\{y\frac{df}{dy} - f + c_1 + \frac{1}{2\delta}\left\{\alpha + \delta\frac{df}{dy}\right\}^2\right\}, \tag{C8}$$

where c_o and c_1 are constants of integration and $\delta = \beta/\gamma = (b/2\pi)(m^*c/\hbar e^*)^2$. Thus solving equation (C8) is essentially equivalent to solving equations (C1) and (C2), however, once (C8) is solved, we need to analyize equations (C4) through (C7).

An alternative method for the reduction of equations (2.3.4) and (2.3.5) is to first note that if we multiply equation (2.3.4) by u' and multiply equation (2.3.5) by η', then we add them together to obtain a complete differential, which can be integrated once to yield

$$\frac{1}{\gamma}\left\{\frac{du}{dx}\right\}^2 + \left\{\frac{d\eta}{dx}\right\}^2 = c_0 + \eta^2\left(\alpha + \frac{\beta\eta^2}{2} + u^2\right). \tag{C9}$$

Now, if we take $u' = F(u,\eta)$ and $\eta' = G(u,\eta)$, then we obtain the relationship $F_\eta = \gamma G_u$. So we now take $R^2 = G^2 + F^2/\gamma$ and $S = G/F$, which reduces equations (2.3.4) and (2.3.5) to the following set of equations:

$$\frac{du}{dx} = F(u,\eta(u)), \tag{C10}$$

$$\frac{d\eta}{dx} = G(u(\eta),\eta), \tag{C11}$$

$$\frac{d\eta}{du} = S(u,\eta), \tag{C12}$$

and

$$\gamma S^3\frac{\partial S}{\partial u} + S\frac{\partial S}{\partial \eta} = \frac{\eta(1/\gamma + S^2)(\alpha + u^2 + \beta\eta^2 - \gamma\eta u S)}{c_0 + \eta^2(\alpha + u^2 + \beta\eta^2/2)}. \tag{C13}$$

Thus solving equations (2.3.4) and (2.3.5) is equivalent to solving equations (C12) and (C13) along with either equation (C10) or (C11). This means that we now have two first order ODE's and one first order PDE to solve.

For the special case $\alpha = c_o = 0$ (i.e. at the critical temperature), this becomes four first order ODE's by letting $u = v\,\omega$ and $\eta = v$, and hence equation (C13) becomes

$$\frac{dv}{v} = \frac{d\omega}{\gamma S^2 - \omega} = \frac{(\omega^2 + \beta/2)SdS}{(\omega^2 + \beta - \gamma\omega S)(1/\gamma + S^2)} \tag{C14}$$

or

$$\frac{d\omega}{\gamma S^2 - \omega} - \frac{(\omega^2 + \beta/2)SdS}{(\omega^2 + \beta - \gamma\omega S)(1/\gamma + S^2)} = 0 \tag{C15}$$

and

$$\ln|v| - \int d\omega \frac{1}{\gamma S^2(\omega) - \omega} = c_2. \tag{C16}$$

From equation (C15) we obtain $f_1(S,\omega) = c_1$, and from equation (C16) we obtain $f_2(v,\omega) = c_2$, so that our general solution is

$$f_1\left(\frac{d\eta}{du}, \frac{u}{\eta}\right) = F\left(f_2\left(\eta, \frac{u}{\eta}\right)\right), \tag{C17}$$

where F is an arbitrary function. It remains to be seen, however, whether the results of this appendix lead to any practical applications or additional physical insight.