Illumination of Convex Bodies with Symmetries in Dimensions 3 and 4

by

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Abstract

Let $n \geq 3$ and $\mathbf{B} \subset \mathbb{R}^n$. The Illumination Conjecture states that the minimal number $\mathcal{I}(\mathbf{B})$ of directions/'light sources' that illuminate the boundary of a convex body $\mathbf{B}$, which is not the affine image of a cube, is strictly less than 2^n. The conjecture in most cases is widely open, and it has only been verified for certain special classes of convex bodies. For instance, significant progress in dimension 3 has been made for convex bodies with certain symmetries. Moreover, in large dimensions with dimension greater than some universal constant C, Konstantin Tikhomirov [22] showed the conjecture in the setting of 1-symmetric bodies, but unfortunately there is a gap in his proof which still leaves a case to be handled. In addition, an explicit value for this constant C was not computed. The natural question which follows from Tikhomirov's paper is whether we can use Tikhomirov's method for 1-symmetric bodies or 1-unconditional bodies in low dimensions. In this thesis, we will first fill the gap in Tikhomirov's results. Through this process, we are also able to prove the Illumination Conjecture, with bound 2^n (and not $< 2^n$), for any 1-symmetric convex body regardless of dimension. Additionally, we are able to show that for 1-symmetric convex bodies in dimensions 3 and 4, $\mathcal{I}(\mathbf{B}) \leq 7$ and $\mathcal{I}(\mathbf{B}) \leq 15$, respectively. Finally, we are also able to show that for 1-unconditional polytopes in dimensions 3 and 4, $\mathcal{I}(\mathbf{B}) \leq 6$ and $\mathcal{I}(\mathbf{B}) \leq 16$, respectively.

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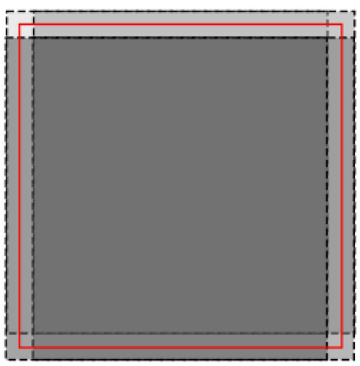
1 Introduction

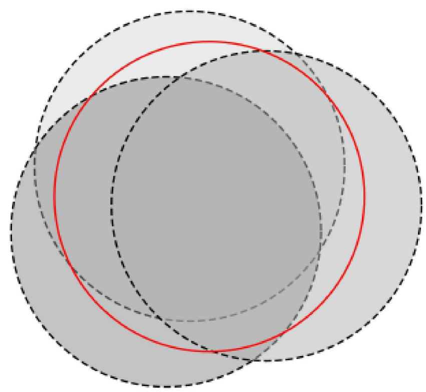
1.1 Some Brief History

The conjecture of interest is a question of Combinatorial Geometry and Discrete Geometry. It is motivated by the idea of covering a *convex body*, a compact convex set with nonempty interior, with slightly smaller copies of itself. The notion of covering which I refer to is "to set theoretically contain". The origin of our question is:

Let L_n denote the smallest natural number such that any n-dimensional convex body can be covered by the interior of a union of at the most L_n of its translates.
What is L_n for $n \geq 3$?



(a) Covering the cube with 4 copies of its interior.



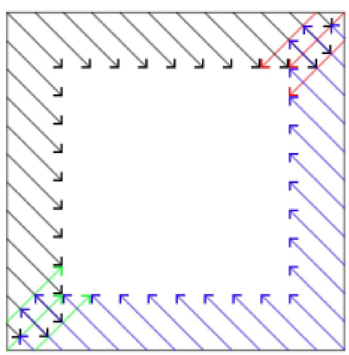
(b) Covering the ball with 3 copies of its interior.

Figure 1: Examples of covering by copies of the interior in dimension 2.

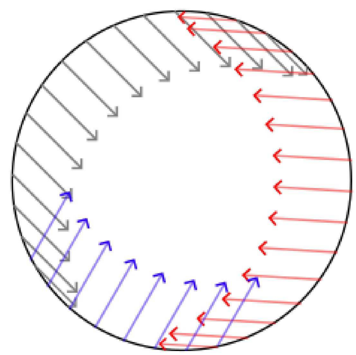
Hadwiger was well known for coming up with such unsolved problems [9, pp. 389-390], including the above conjecture, known as the Hadwiger Conjecture [11]. However, this problem was previously studied by Levi who solved the question for the 2-dimensional case in 1955 [16]. In 1960, Gohberg and Markus restated the question in terms of covering by homothetic copies [10], which, roughly speaking, are translations of smaller copies of the original object. The Covering Conjecture stated by Gohberg and Markus [10] is the topic of interest.

We can cover any n-dimensional convex body by 2^n or fewer of its smaller homothetic copies in Euclidean n-space, $n \geq 3$. Furthermore, 2^n homothetic copies are required only if the body is an affine n-cube.

In this thesis, we will indirectly work on this conjecture via illumination. To introduce the idea of illumination, picture an object suspended and centered in a dark room. Without considering reflections, in how many different directions we would have to shine a light on the object for each point on the surface of the object to be slightly penetrated by light? Formally, let $\mathbf{B} \subset \mathbb{R}^n$ be a convex body. A boundary point $x \in \partial\mathbf{B}$ is *illuminated* by a direction $d \in \mathbb{R}^n \backslash \{0\}$ if there exists an $\varepsilon > 0$ such that $x + \varepsilon d$ belongs to the interior of $\mathbf{B}$. The *illumination number* of a convex body, $\mathcal{I}(\mathbf{B})$, is the smallest number of directions which can illuminate the boundary of a convex body.



(a) Illuminating the cube with 4 directions.



(b) Illuminating the ball with 3 directions.

Figure 2: Examples of illuminating in dimension 2.

The following conjecture by Hadwiger and Boltyanski is the Illumination Conjecture and it is equivalent to the Covering Conjecture [3, 12]. This is because the illumination number $\mathcal{I}(\mathbf{B})$ of a convex body $\mathbf{B}$ and its covering number are always equal.

The illumination number, $\mathcal{I}(\mathbf{B})$, of any n-dimensional convex body $\mathbf{B}$, $n \geq 3$, is at most 2^n and $\mathcal{I}(\mathbf{B}) = 2^n$ only if $\mathbf{B}$ is an affine n-cube.

Since the problem is still unsolved in full generality, even for low dimensions greater than $n = 2$, a proof for this conjecture will have a lead to a better understanding of the boundaries of convex bodies in different dimensions.

1.2 Known Results

Recall that Levi already solved the conjecture in dimension 2. He also observed that for smooth convex bodies of any dimension $\mathcal{I}(\mathbf{B}) \leq n + 1$ [16] (smooth here means that every boundary point of the body has a unique tangent/supporting hyperplane).

Definition 1.1.

1. A zonotope is a set which can be expressed as the Minkowski sum of line segements. A zonotope will always be a convex body.

2. A zonoid is a convex body that is the limit of zonotopes in the Hausdorff metric.

For zonoids which are not the parallelepipeds, Boltyanski and P.S. Soltan have shown that $\mathcal{I}(\mathbf{B}) \leq 3 \cdot 2^{n-2}$ [4, 5].

So far, the best result for general convex bodies in dimension 3 is due to Papadoperakis, who was able to show that $\mathcal{I}(\mathbf{B}) \leq 16$ [17]. There are also some results due to Prymak and Shepelska who gave upper bounds for $\mathcal{I}(\mathbf{B})$ of 96, 1091 and 15373 in dimensions 4, 5, 6, respectively [18]. As we can see, the results for general convex bodies are still rather crude when compared to the conjecture. There are many partial results in $\mathbb{R}^3$ which are not stronger than each other and do not contradict the conjecture. For many of such results there is some underlying symmetry.

Definition 1.2. We say that a convex polyhedron $\mathbf{B}$ has *affine symmetry* if the affine symmetry group of $\mathbf{B}$ is non trivial.

Bezdek has shown that if $\mathbf{B} \subset \mathbb{R}^3$ is a convex polyhedron with affine symmetry, then $\mathcal{I}(\mathbf{B}) \leq 8$ [1].

Definition 1.3.

1. We say that a convex body $\mathbf{B} \subset \mathbb{R}^n$ is *o-symmetric (origin symmetric)* if $\mathbf{B} = -\mathbf{B}$.
2. We say that a convex body $\mathbf{B} \subset \mathbb{R}^n$ is *centrally symmetric* if some translate of $\mathbf{B}$ is o-symmetric.

Through collaboration, Erdös, Rogers and Shepard has shown the following. [8, 19, 20]

$$\mathcal{I}(\mathbf{B}) = \text{covering number} \leq \frac{\text{vol}_n(\mathbf{B} - \mathbf{B})}{\text{vol}_n(\mathbf{B})} n(\ln n + \ln\ln n + 5) \leq O\left(4^n \sqrt{n} \ln n\right).$$

This bound was the best estimate for general convex bodies for a long time. It was proven using covering rather than illumination. If $\mathbf{B}$ is centrally symmetric, we get

$$\mathcal{I}(\mathbf{B}) = \text{covering number} \leq \frac{\text{vol}_n(\mathbf{B} - \mathbf{B})}{\text{vol}_n(\mathbf{B})} n(\ln n + \ln\ln n + 5) \leq O\left(2^n n \ln n\right).$$

Lassak has shown that the boundary of any centrally symmetric convex body $\mathbf{B} \subset \mathbb{R}^3$ can be illuminated by four pairs of opposite directions, and thus $\mathcal{I}(\mathbf{B}) \leq 8$ [14]. Dekster has given the following complementary result: if $\mathbf{B} \subset \mathbb{R}^3$ is a convex body symmetric about a plane then $\mathcal{I}(\mathbf{B}) \leq 8$ [7].

Regarding Lassak's result, in his paper [14], it was conjectured that for any centrally symmetric convex body $\mathbf{B} \subset \mathbb{R}^n$, where $n \geq 3$, which is not a parallelepiped, there are $2^{n-1} - 1$ pairs of opposite directions which form an illuminating set. Although our results are focused on more restricted classes, the 1-symmetric and 1-unconditional cases, it turns out that at least many of our results confirm his conjecture. In fact in the case where $\mathbf{B} \subset \mathbb{R}^3$ is a 1-unconditional polytope, we are able to show that this conjecture is true (Theorem 6.9).

Definition 1.4.

1. The *width* of a convex body $\mathbf{B} \subset \mathbb{R}^n$ is given by

$$\text{Width}(\mathbf{B}) = \min_{\substack{H_1 \| H_2, \\ H_1, H_2 \text{ support } \mathbf{B}}} \text{d}(H_1, H_2)$$

2. We say that a convex body $\mathbf{B} \subset \mathbb{R}^n$ is of *constant width* if for any hyperplanes H_1, H_2 which are parallel and support $\mathbf{B}$,

$$\mathrm{Width}(\mathbf{B}) = \mathrm{d}(H_1, H_2)$$

For convex bodies of constant width in dimension 3, it has been shown that $\mathcal{I}(\mathbf{B}) \leq 6$; this was also shown by Lassak [15]. In $\mathbb{R}^n$, for convex bodies of constant width, Schramm has shown that

$$\mathcal{I}(\mathbf{B}) < 5n\sqrt{n}(4 + \log(n)) \left(\frac{3}{2}\right)^{\frac{1}{2}} \quad [21],$$

which gives $\mathcal{I}(\mathbf{B}) < 2^n$ for $n > 15$.

Definition 1.5. We say that $\mathbf{B}$ is a *cap body of a ball* if $\mathbf{B}$ is the convex hull of a Euclidean ball and a countable set of points outside the ball under the condition that each segment connecting two of these points intersects the ball.

Definition 1.6. We say that a convex body $\mathbf{B} \subset \mathbb{R}^n$ is 1-*unconditional* if

$$(x_1, ..., x_n) \in \mathbf{B} \implies (\pm x_1, ..., \pm x_n) \in \mathbf{B}.$$

Ivanov and Strachan have shown that for if $\mathbf{B} \subset \mathbb{R}^n$ is a centrally symmetric cap body of the ball, then $I(\mathbf{B}) \leq 6$ [13]. Moreover, in $\mathbb{R}^4$, for 1-unconditional cap bodies $\mathbf{B}$ of the ball, they have shown that $I(\mathbf{B}) \leq 8$ [13].

There are many results regarding illumination. If the reader would like to know more on this topic, we recommend reading the survey paper by Bezdek and Khan [2]. Another good reference is the textbook by Brass, Moser and Pach [6]. We finish with one more result, the most relevant for this thesis.

Definition 1.7. We say that a convex body $\mathbf{B} \subset \mathbb{R}^n$ is 1-*symmetric* if $\mathbf{B}$ satisfies the 1-unconditional condition, and

$$(x_1, ..., x_n) \in \mathbf{B} \implies \left(x_{\sigma(1)}, ..., x_{\sigma(n)}\right) \in \mathbf{B}$$

for any permutation σ on n elements.

Tikhomirov has shown that there exists a universal constant C, so that when $n > C$, if $\mathbf{B} \subset \mathbb{R}^n$ is a 1-symmetric convex body, then $\mathcal{I}(\mathbf{B}) < 2^n$, except in the following subcase (for which we discovered his proof has a gap): if $1 \neq \mathrm{d}(\mathbf{B}, [-1,1])^n < 2$ and $\|e_i + e_j\|_{\mathbf{B}} > \|e_i\|_{\mathbf{B}}$ for every $i, j \in [n]$.

In this thesis, we will study Tikhomirov's paper in depth in sections 3 and 4. In section 5 we are able to fill in the gap in Tikhomirov's results and also obtain some of our main results. Then we will stem off and discuss our findings in sections 6 and 7 for 1-unconditional convex polytopes. Let's take closer a look at Tikhomirov's method and our results in the next subsection.

1.3 Tikhomirov's Method and Our Main Results

Tikhomirov's idea is to start with a common illuminating set with $3^n - 1$ illuminating directions

$$\mathcal{I}_T^n := \{-1, 0, 1\}^n \backslash \{\vec{0}\}.$$

Then he shows that one can find sufficiently smaller subsets of it that are still illuminating. We can see that for any boundary point $x \in \mathbf{B}$, if we set $d \in \mathbb{R}^n$ so that

$$d_i = \begin{cases} -\operatorname{sign}(x_i), & x_i \neq 0 \\ 0, & x_i = 0 \end{cases},$$

then $d \in \mathcal{I}_T^n$ and d illuminates x (lemma 2.42).

Recall that for a direction $d \in \mathbb{R}^n \backslash \{0\}$ to illuminate a boundary point $x \in \partial\mathbf{B}$, it means there exists $\varepsilon > 0$ so that $x + \varepsilon d \in \operatorname{int} \mathbf{B}$. Thus, arbitrary scalar multiples of d will illuminate (parts of the boundary of) $\mathbf{B}$ as long as d illuminates (parts of the boundary of) $\mathbf{B}$. Because of this, just as Tikhomirov does, we will only consider d which satisfies $\|d\|_\infty = 1$. Finally, it is enough to consider affine transformations of convex bodies (because both the illumination number and the covering number of a convex body are affine invariants), so WLOG we can require that $\|e_i\|_{\mathbf{B}} = 1$. We will give justification for this choice of normalization of $\mathbf{B}$ at the beginning of section 5. Alongside convexity, this condition implies that $\|x\|_\infty \leq 1$ and $x \in \mathbf{B}$. In layman's terms, we will shrink or enlarge our 1-symmetric convex body so that it will fit tightly into the unit cube.

For Tikhomirov's proof to work, we require the dimension of the smaller Euclidean space which contains $\mathbf{B}$ to be sufficiently large. So, a natural approach would be to try to estimate the maximum *'low'* dimension for which Tikhomirov's proof stops working. In this paper, we find that if $\mathbf{B}$ lives in dimensions 3 and 4, his method does not go through, and perhaps it is too 'rigid' to work (see lemmas 4.1, 4.2, and 5.5, and remark 5.6). In fact, we also found an error in his method, which we will discuss in section 4, which is why we have only stated his result as partial for now. On the other hand, Tikhomirov uses certain norm conditions to distinguish between interesting cases which is very important for our methods too in sections 6 and 7. In the 1-symmetric case, we look at the cases:

$$\forall i,j \in [n],\ i \neq j,\ \|e_i + e_j\|_{\mathbf{B}} = \|e_i\|_{\mathbf{B}} \quad \text{or} \quad \forall i,j \in [n],\ \|e_i + e_j\|_{\mathbf{B}} > \|e_i\|_{\mathbf{B}}.$$

We will also look at an analogue of these cases in the 1-unconditional setting. At the same time, our methods to prove illumination will differ significantly at certain points from those of Tikhromirov's. We will be carefully choosing sections of our convex bodies in order to see that the boundary points belonging to those sections are illuminated by certain directions. While imitating the cases/subcases Tikhomirov distinguishes, and while playing with directions similar to his, we successfully develop some core combinatorial and geometric intuition for choosing appropriate sections of our convex body to look at and construct small illuminating sets.

With our methods, we are able to strengthen Tikhomirov's result: when $n > C$, if $\mathbf{B} \subset \mathbb{R}^n$ is a 1-symmetric convex body, then $\mathcal{I}(\mathbf{B}) < 2^n$. This is because we are able to find a replacement for the remaining, problematic case (corollary 5.20). Alongside this, we are able to show that for every $n \geq 3$, if $\mathbf{B} \subset \mathbb{R}^n$ is a 1-symmetric convex body, then $\mathcal{I}(\mathbf{B}) \leq 2^n$ (Theorem 5.18). When combining our results with Tikhomirov's results, we are even able to show that in dimensions 3 and 4, if $\mathbf{B}$ is a 1-symmetric convex body, then $\mathcal{I}(\mathbf{B}) \leq 7$ (corollary 5.21) and $\mathcal{I}(\mathbf{B}) \leq 15$ (corollary 5.22), respectively.

Sections 7 and 8 are dedicated to 1-unconditional polytopes. In the case of 3-dimensional 1-unconditional polytopes, we are able to construct sets to show that $\mathcal{I}(\mathbf{B}) \leq 6 < 2^3$ (Theorem 6.9); in the case of dimension 4, we are able to show that for many cases, $\mathcal{I}(\mathbf{B}) \leq 14$, and $\mathcal{I}(\mathbf{B}) \leq 16$ for the remaining cases (see Theorem 7.15 and table 9).

Our method of proof relies on looking at specific sections of the convex body, therefore the proofs that we give already come with a concrete construction of an illuminating set, similar to that of Tikhomirov's and Lassak's but different from Levi's.

For the sake of transparency, Prof. Vritsiou suggested several crucial technical details and changes which led to a simplified proof of proposition 5.13, and strengthenings of previous versions of Theorem 5.18, and corollary 5.20. Additionally, she has also pushed some results further, such as in propositions 7.4, 7.6 and 7.12.

2 Preliminaries and Notations

We begin here with some basic definitions and lemmas about convex geometry and illumination. We attempt to keep the notation as consistent as possible to the notation from Tikhomirov's paper.

2.1 Basic Notations

Notation 2.1. We use the following notation to denote the set of naturals less than or equal to n:

$$[n] := \{1, 2, ..., n\}.$$

Notation 2.2. Let $n \in \mathbb{N}$, $c \in \mathbb{R}$. For a vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$, let

$$I_c^x := \{i \in [n] : x_i = c\}.$$

In particular, by setting $c = 0$, this set keeps track of how many zero entries there are in a vector, and also where the zero entries are located.

Notation 2.3. Let $n \in \mathbb{N}$ and $i \in [n]$. We use the following notation for the standard basis coordinates of $\mathbb{R}^n$:

$$e_i = (0, ..., 0, \underset{i^{th} \text{ entry}}{1}, 0, ..., 0).$$

2.2 Convex Geometry

Definition 2.4. A *convex body* $\mathbf{B} \subset \mathbb{R}^n$ is a compact convex set with non-empty interior. The polytopes in this thesis will all be convex bodies.

Definition 2.5.

1. We say $\mathbf{B} \in \mathcal{S}^n$ if $\mathbf{B} \subset \mathbb{R}^n$ is a 1-symmetric convex body that satisfies $\|e_i\|_{\mathbf{B}} = 1$ for every $i \in [n]$.

2. We say $\mathbf{B} \in \mathcal{U}^n$ if $\mathbf{B} \subset \mathbb{R}^n$ is a 1-unconditional convex body that satisfies $\|e_i\|_{\mathbf{B}} = 1$ for every $i \in [n]$.

We will justify why we impose $\|e_i\|_{\mathbf{B}} = 1$ for every $i \in [n]$ in subsection 2.3.

Definition 2.6. Let $\mathbf{B} \subset \mathbb{R}^n$ be a convex body. A *supporting hyperplane* H is a hyperplane in $\mathbb{R}^n$ which satisfies $H \cap \mathbf{B} = H \cap \partial\mathbf{B} \neq \varnothing$.

Notation 2.7. Let $x, \mathbf{n} \in \mathbb{R}^n$. We can denote

$$H_x(\mathbf{n}) := \{\xi \in \mathbb{R}^n : \langle \xi, \mathbf{n} \rangle = \langle x, \mathbf{n} \rangle\}.$$

So $\mathbf{n}$ is a normal unit vector to $H_x(\mathbf{n})$.

Definition 2.8. Let $\mathbf{B} \subset \mathbb{R}^n$ be a convex body and $x \in \partial\mathbf{B}$. An *outer normal* is a unit vector $\mathbf{n} \in S^{n-1}$ so that

$$\mathbf{B} \subset \{\xi \in \mathbb{R}^n : \langle \xi, \mathbf{n} \rangle \leq \langle x, \mathbf{n} \rangle\}.$$

Moreover, then $H_x(\mathbf{n})$ is a supporting hyperplane of $\mathbf{B}$. The set of outer normals of $\mathbf{B}$ at x is denoted by $\nu(\mathbf{B}, x)$.

Definition 2.9. Let $x \in \mathbb{R}^n$, $\mathbf{n} \in S^{n-1}$. We call the set

$$\{\xi \in \mathbb{R}^n : \langle \xi, \mathbf{n} \rangle \leq \langle x, \mathbf{n} \rangle\}$$

the *negative half-space* of $\mathbb{R}^n$ out of the two half-spaces with boundary $H_x(\mathbf{n})$, and

$$\{\xi \in \mathbb{R}^n : \langle \xi, \mathbf{n} \rangle > \langle x, \mathbf{n} \rangle\}$$

the *positive half-space* of $\mathbb{R}^n$ with boundary $H_x(\mathbf{n})$.

Fact 2.10. A convex body is the intersection of the negative half-spaces of all of its supporting hyperplanes.

Notation 2.11. Let $r > 0$. We denote the following convex bodies in dimension n centered at the origin as follows.

1. The cube (with sidelength $2r$) is denoted by:

$$C_r^n := \{x \in \mathbb{R}^n : \|x\|_\infty \leq r\}.$$

2. The cross-polytope (with diameter $2r$) is denoted by:

$$CP_r^n := \{x \in \mathbb{R}^n : \|x\|_1 \leq r\}.$$

Definition 2.12. Let $x \in \mathbb{R}^n$ and $\mathbf{B}$ be an origin symmetric convex body. The norm induced on $\mathbb{R}^n$ by the convex body $\mathbf{B}$ is defined by

$$\|x\|_{\mathbf{B}} := \min\{t \geq 0 : x \in t\mathbf{B}\}.$$

Notation 2.13. Let $x \in \mathbb{R}^n$. Then according to our notation

$$\|x\|_\infty = \|x\|_{C_1^n}.$$

Definition 2.14. Let $S \subset \mathbb{R}^n$ be a set, the *convex hull of* S is the smallest convex set containing S, or equivalently, the intersection of all convex sets containing S. It can be shown that

$$\operatorname{conv} S = \left\{ \sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N},\ x_1, ..., x_m \in S,\ \sum_{i=1}^{m} \lambda_i = 1,\ 0 \leq \lambda_1, ..., \lambda_m \leq 1 \right\},$$

which is the set of all possible convex combinations of points from S.

Definition 2.15. We define the distance between two centrally symmetric convex bodies, $\mathbf{B}_1, \mathbf{B}_2 \subset \mathbb{R}^n$ to be

$$\mathrm{d}(\mathbf{B}_1, \mathbf{B}_2) = \min\{\lambda \geq 1 : \mathbf{B}_1 \subset r\mathbf{B}_2 \subset \lambda \mathbf{B}_1 \text{ for some } r > 0\}.$$

Remark 2.16. This looks quite similar to the Banach-Mazur distance; however, since we do not allow arbitrary linear transformations of $\mathbf{B}_1$ and $\mathbf{B}_2$, this means that $\mathrm{d}(CP_1^2, C_1^2) \neq 1$.

Lemma 2.17. Let $\mathbf{B} \in \mathcal{U}^n$, then

$$\mathrm{d}(\mathbf{B}, [-1,1]^n) = \|e_1 + ... + e_n\|_{\mathbf{B}}.$$

Proof. We will borrow lemma 2.37 which is from a later subsection in the preliminaries of this thesis for this proof.

Using the notation $C_1^n = [-1,1]^n$:

$$\begin{aligned}\mathrm{d}(\mathbf{B}, C_1^n) &= \min\{\lambda \geq 1 : \mathbf{B} \subset rC_1^n \subset \lambda\mathbf{B} \text{ for some } r > 0\} \\ &= \min\left\{\frac{\lambda_2}{\lambda_1} \geq 1 : \lambda_1\mathbf{B} \subset C_1^n \subset \lambda_2\mathbf{B} \text{ for some } \lambda_1, \lambda_2 > 0\right\}.\end{aligned}$$

Recall that by our normalization, $\|e_i\|_{\mathbf{B}} = 1$ for every $i \in [n]$. We want to minimize $\frac{\lambda_2}{\lambda_1}$; therefore we want to maximize λ_1. By our normalization and the 1-unconditional condition, $\mathbf{B} \subset C_1^n$ thus $\lambda_1 \geq 1$. Fix $i \in [n]$. If $\lambda_1 > 1$, then $\lambda_1 e_i \in \lambda_1\mathbf{B} \subset C_1^n$. This contradicts the definition of C_1^n. Hence, it must be the case that $\lambda_1 = 1$.

On the other hand, for any $s \geq 0$, lemma 2.37 implies that

$$s(e_1 + ... + e_n) \in \mathbf{B} \iff sC_1^n = [-s,s]^n \subset \mathbf{B}.$$

We need to minimize λ_2, so we want to find the largest s_0 so that $s_0 C_1^n \subset \mathbf{B}$.

$$\begin{aligned}\max\{s \geq 0 : s(e_1 + ... + e_n) \in \mathbf{B}\} &= \frac{1}{\min\{t \geq 0 : (e_1 + ... + e_n) \in t\mathbf{B}\}} \\ &= \frac{1}{\|e_1 + ... + e_n\|_{\mathbf{B}}}.\end{aligned}$$

Overall, we have

$$\mathbf{B} \subset C_1^n \subset \frac{1}{s_0}\mathbf{B}.$$

Does there exist a better choice (larger value) for λ_2 than S_0? In the method of how we attained s_0, we can see that $\frac{1}{s_0}$ is the best choice (smallest value) for λ_2. Hence

$$\mathrm{d}(\mathbf{B}, C_1^n) = \frac{1}{s_0} = \|e_1 + ... + e_n\|_{\mathbf{B}}.$$

□

Here are some important facts about lines and convex bodies which we will use throughout the thesis. Since these results are rather elementary, we will state them without proof.

Fact 2.18.

1. If a is an interior point and b a boundary point of a convex body $\mathbf{B}$, then all points of segment ab except b are interior points of $\mathbf{B}$.

2. If the endpoint of a ray (line segment) is on the interior of a convex body $\mathbf{B}$, then the ray (line segment) will only intersect $\partial\mathbf{B}$ exactly (at most) once.

Example 2.19. Suppose $\mathbf{B} \subset \mathbb{R}^n$ is convex and contains $\pm e_i$ for every $i \in [n]$, then $\mathbf{B}$ contains the convex hull of those points (which is the cross-polytope). This means that $\mathbf{0} \in \operatorname{int} \mathbf{B}$. Fix i and let $c \in \mathbb{R}$ so that $0 < c < 1$. We can create a ray which starts from $\mathbf{0}$ and passes ce_i then through e_i, so ce_i must be interior to $\mathbf{B}$ by fact 2.18.

Definition 2.20. Let $\mathbf{B} \subset \mathbb{R}^n$ be a convex body and $x \in \mathbf{B}$, then x is called an *extreme point of* $\mathbf{B}$ if it does not lie between any two distinct points of $\mathbf{B}$. That is, if there do not exist $y, z \in \mathbf{B}$ and $0 < \lambda < 1$ such that $y \neq z$ and $x = \lambda y + (1 - \lambda)z$. We denote the *extreme points of* $\mathbf{B}$ by:

$$\operatorname{ext} \mathbf{B} := \{x \in \mathbf{B} : x \text{ extreme}\}.$$

Example 2.21.

1. The vertices of convex polytopes are the only extreme points.

2. The boundary points of the ball are precisely its extreme points.

We will use the following fact implicitly throughout the thesis.

Fact 2.22. Let $\mathbf{B} \subset \mathbb{R}^n$ be a convex body. A hyperplane supports $\mathbf{B}$ if and only if it supports $\operatorname{ext} \mathbf{B}$, meaning $\mathbf{B}$ lies in the negative half-space of a hyperplane if and only if its extreme points lie in that half-space as well.

Lemma 2.23. Let $\mathbf{B}_1, \mathbf{B}_2 \subset \mathbb{R}^n$ be convex bodies, then

$$\operatorname{ext} \operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2) \subset \operatorname{ext} \mathbf{B}_1 \cup \operatorname{ext} \mathbf{B}_2.$$

Proof. First, we claim that $\operatorname{ext} \operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2) \subset \mathbf{B}_1 \cup \mathbf{B}_2$. Let $x \in \operatorname{ext} \operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2)$, and suppose $x \notin \mathbf{B}_1 \cup \mathbf{B}_2$. However, this means that $x \in \operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2) \backslash (\mathbf{B}_1 \cup \mathbf{B}_2)$ and thus, it is a convex combination of at least two different points from $\mathbf{B}_1$ and $\mathbf{B}_2$. This contradicts x being an extreme point, hence it must be that $x \in \mathbf{B}_1 \cup \mathbf{B}_2$.

In addition, suppose $x \in \mathbf{B}_1$, since $x \in \operatorname{ext} \operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2)$, then there does not exist such $y, z \in \operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2) \supset \mathbf{B}_1$ and $0 < \lambda < 1$ such that $y \neq z$ and $x = \lambda y + (1 - \lambda) z$; this means $x \in \operatorname{ext} \mathbf{B}_1$. Similarly if $x \in \mathbf{B}_2$, then $x \in \operatorname{ext} \mathbf{B}_2$, which proves the claim. $\square$

Definition 2.24. In $\mathbb{R}^n$, an *affine set* is the translation of a vector subspace.

Definition 2.25. In $\mathbb{R}^n$, a *hyperplane* is an affine set of dimension $n - 1$.

Definition 2.26. Let $S \subset \mathbb{R}^n$ be a set. The *affine hull of* S is the smallest affine set containing S, or equivalently, the intersection of all affine sets containing S. It can be shown that

$$\operatorname{aff} S = \left\{ \sum_{i=1}^{k} \lambda_i x_i : \; k > 0, \; x_i \in S, \; \sum_{i=1}^{k} \lambda_i = 1, \; \lambda_i \in \mathbb{R} \right\}.$$

This is the set of all possible affine combination of points from S.

Definition 2.27. The relative interior of a set S is given by

$$\operatorname{relint} S := \{\xi \in S : \exists \varepsilon > 0, \; B_\varepsilon(\xi) \cap \operatorname{aff} S \subset S\}.$$

Example 2.28. The ball of $\mathbb{R}^2$ which lives in $\mathbb{R}^3$ described by

$$B = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 + \xi_2 \leq 1,\ \xi_3 = 0\}$$

is its own closure in $\mathbb{R}^3$, therefore its interior is empty. The affine set which contains this ball will be the copy of $\mathbb{R}^2$ in $\mathbb{R}^3$ i.e.

$$\operatorname{aff} B = \operatorname{span}(e_1, e_2).$$

So, we get

$$\operatorname{relint} B = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 + \xi_2 < 1,\ \xi_3 = 0\}.$$

Remark 2.29. The notion of relative interior can be thought of as giving a notion of interior to a set of $\mathbb{R}^n$ which has empty interior by giving it a relative topology.

Lemma 2.30. Let $\mathbf{B} \subset \mathbb{R}^n$ be a convex body and H be an affine set. If $x \in \operatorname{relint}(H \cap \mathbf{B})$ and $H \cap \operatorname{int} \mathbf{B} \neq \varnothing$ then $x \in \operatorname{int} \mathbf{B}$.

Proof. Let $x \in \operatorname{relint}(H \cap \mathbf{B})$ and suppose $H \cap \operatorname{int} \mathbf{B} \neq \varnothing$. By way of contradiction, assume $x \in \partial \mathbf{B}$, and since $H \cap \operatorname{int} \mathbf{B}$ is non empty and convex, we can travel from x along H to enter the interior of $\mathbf{B}$, i.e. there exists a d in the subspace parallel to H so that $x + d \in H \cap \operatorname{int} \mathbf{B}$. We identify a ray with end point $x + d \in \operatorname{int} \mathbf{B}$, passing through $x \in \partial \mathbf{B}$. So by fact 2.18, $x + d - \varepsilon d \notin \mathbf{B}$ for every $\varepsilon > 1$ or equivalently, $x - \varepsilon d \notin \mathbf{B}$ for every $\varepsilon > 0$. This means that $x \notin \operatorname{relint}(H \cap \operatorname{int} \mathbf{B})$ so we have reached a contradiction. □

This ends the preliminary section part for convex geometry, we will now introduce some basics and lemmas ideas for illumination.

2.3 Illumination

Definition 2.31. Let $\mathbf{B}_1, \mathbf{B}_2 \subset \mathbb{R}^n$ and $\mathcal{I} \subset \mathbb{R}^n \backslash \{0\}$. A point $x \in \partial \mathbf{B}_1$ is $\mathbf{B}_2$*-illuminated by* $\mathcal{I}$ if there exists $d \in \mathcal{I}$ and $\varepsilon > 0$ so that $x + \varepsilon d \in \operatorname{int} \mathbf{B}_2$. Similarly, we say that $\mathbf{B}_1$ is

$\mathbf{B}_2$-*illuminated by* $\mathcal{I}$ if for every $x \in \partial \mathbf{B}_1$, x is $\mathbf{B}_2$ illuminated by $\mathcal{I}$. In the case $\mathcal{I} = \{d\}$, we will say x or $\mathbf{B}_1$ *is* $\mathbf{B}_2$-*illuminated by* d instead.

If the convex body here is not important or if it is obvious that $\mathbf{B}_1 = \mathbf{B}_2$, we will use the terminology, x or $\mathbf{B}_1$ *is illuminated by* $\mathcal{I}$. In fact, in most of our cases, $\mathbf{B}_1 = \mathbf{B}_2$; however, we will see why we sometimes need to distinguish between $\mathbf{B}_1$ and $\mathbf{B}_2$ as in lemma 2.36.

Definition 2.32. We denote the *illumination number* by:

$$\mathcal{I}(\mathbf{B}) := \min_{\mathcal{I} \text{ illuminates } \mathbf{B}} |\mathcal{I}|\,.$$

Let's simplify the problem a little bit. The number required to cover a set using homothetic covering does not change due to shifting, shearing or stretching of a convex body, and thus affine transformations. Therefore, the problem of illumination is also invariant of affine transformations. We should note that the affine transformations we are looking at will be invertible, and in our cases, due to the symmetries imposed we will not need translations, so we will only be looking at invertible linear transformations (linear homeomorphisms). Let T be our linear homeomorphism and $x \in \operatorname{ext}\mathbf{B}$, $d \in \mathbb{R}^n$ so that $x + \varepsilon d \in \operatorname{int}\mathbf{B}$ for some $\varepsilon > 0$, then $T(x) + \varepsilon T(d) = T(x + \varepsilon d) \in \operatorname{int}\mathbf{B}$. We can see that if $\mathcal{I}$ is an illuminating set for $\mathbf{B}$, then $T(\mathcal{I})$ is an illuminating set for $T(\mathbf{B})$. Hence, we can simplify our process: we will only look at 1-symmetric convex bodies $\mathbf{B} \subset \mathbb{R}^n$ which satisfy $\|e_i\|_{\mathbf{B}} = 1$ for every $i \in [n]$.

Notation 2.33. Let $n \in \mathbb{N}$, we use the following notation for Tikhomirov's large illuminating set:

$$\mathcal{I}_T^n := \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_i \in \{0, 1, -1\}\} \backslash \{0\}.$$

Note that $|\mathcal{I}_T^n| = 3^n - 1$.

Lemma 2.34. Let $n \geq 2$ and $\mathbf{B} \subset \mathbb{R}^n$ be a convex body. A direction $d \in \mathbb{R}^n \backslash \{0\}$ illuminates $x \in \partial \mathbf{B}$ if and only if $\langle d, \mathbf{n} \rangle < 0$ for all $\mathbf{n} \in \nu(\mathbf{B}, x)$.

Proof.

$\Rightarrow$ Suppose $d \in \mathbb{R}^n \backslash \{0\}$ illuminates $x \in \partial \mathbf{B}$. Then there exists $\varepsilon > 0$ so that $x + \varepsilon d \in \operatorname{int} \mathbf{B}$. Now let $\mathbf{n} \in \nu(\mathbf{B}, x)$, then $H_x(\mathbf{n})$ is a supporting hyperplane of $\mathbf{B}$ at x. Since $x + \varepsilon d \in \operatorname{int} \mathbf{B}$,

$$\langle x, \mathbf{n} \rangle > \langle x + \varepsilon d, \mathbf{n} \rangle = \langle x, \mathbf{n} \rangle + \varepsilon \langle d, \mathbf{n} \rangle \implies \langle d, \mathbf{n} \rangle < 0.$$

$\Leftarrow$ Assume that $\langle d, \mathbf{n} \rangle < 0$ for all $\mathbf{n} \in \nu(\mathbf{B}, x)$. Suppose for contradiction that for any $\varepsilon > 0$, $x + \varepsilon d \notin \operatorname{int} \mathbf{B}$. One possibility is that there is $\varepsilon_0 > 0$ such that $x + \varepsilon_0 d \in \partial \mathbf{B}$. Then $x + \frac{\varepsilon_0}{2} d$ is also in $\partial \mathbf{B}$ (since by convexity it must be in $\mathbf{B}$, and by our assumption it cannot be in $\operatorname{int} \mathbf{B}$), and thus we can find a unit vector $\mathbf{n}_0$ such that $H_{x+\frac{\varepsilon_0}{2}d}(\mathbf{n}_0)$ is a supporting hyperplane of $\mathbf{B}$ at $x + \frac{\varepsilon_0}{2} d$. Since $x, x + \frac{\varepsilon_0}{2} d$ and $x + \varepsilon_0 d$ must be found in the negative half-space of $H_{x+\frac{\varepsilon_0}{2}d}(\mathbf{n}_0)$, we have that x and $x + \varepsilon_0 d$ are also found on $H_{x+\frac{\varepsilon_0}{2}d}(\mathbf{n}_0)$. Thus $\mathbf{n}_0 \in \nu(\mathbf{B}, x)$, and moreover

$$\langle x, \mathbf{n}_0 \rangle = \langle x + \varepsilon_0 d, \mathbf{n}_0 \rangle \implies \langle d, \mathbf{n}_0 \rangle = 0,$$

which contradicts our assumption about d.

The only other possibility is that $x + \varepsilon d \notin \mathbf{B}$ for every $\varepsilon > 0$. Then for each $\varepsilon_m = \frac{1}{m}$, we can find a hyperplane $H_{y_m}(\mathbf{n}_m)$ separating $\mathbf{B}$ and $x + \varepsilon_m d$, where $\mathbf{n}_m$ is some unit vector and $y_m \in \partial \mathbf{B}$. In other words, for every point $z \in \mathbf{B}$

$$\langle z, \mathbf{n}_m \rangle \leq \langle y_m, \mathbf{n}_m \rangle = \alpha_m < \langle x + \varepsilon_m d, \mathbf{n}_m \rangle .$$

Since the vectors $\mathbf{n}_m$ are all unit, we can find a subsequence of them which converges to a unit vector $\mathbf{n}_0$. Passing to this subsequence if needed, we have that

$$\lim_{m \to \infty} \langle x, \mathbf{n}_m \rangle = \langle x, \mathbf{n}_0 \rangle \leq \liminf_{m \to \infty} \alpha_m = \liminf_{m \to \infty} \max_{z \in \mathbf{B}} \langle z, \mathbf{n}_m \rangle .$$

At the same time, $\langle x + \varepsilon_m d, \mathbf{n}_m \rangle = \langle x, \mathbf{n}_m \rangle + \varepsilon_m \langle d, \mathbf{n}_m \rangle \to \langle x, \mathbf{n}_0 \rangle$ (since $| \langle d, \mathbf{n}_m \rangle | \leq \|d\|_2$ and $\varepsilon_m \to 0$), and thus

$$\langle x, \mathbf{n}_0 \rangle = \lim_{m \to \infty} \langle x + \varepsilon_m d, \mathbf{n}_m \rangle \geq \limsup_{m \to \infty} \max_{z \in \mathbf{B}} \langle z, \mathbf{n}_m \rangle .$$

It follows that $\mathbf{n}_0 \in \nu(\mathbf{B}, x)$, given that, for every $z' \in \mathbf{B}$,

$$\langle z', \mathbf{n}_0 \rangle = \lim_{m\to\infty} \langle z', \mathbf{n}_m \rangle \le \limsup_{m\to\infty} \max_{z\in\mathbf{B}} \langle z, \mathbf{n}_m \rangle \le \langle x, \mathbf{n}_0 \rangle .$$

On the other hand, for every d we have that $\langle d, \mathbf{n}_m \rangle > 0$ (since $\langle x, \mathbf{n}_m \rangle \le \alpha_m < \langle x + \varepsilon_m d, \mathbf{n}_m \rangle$), and thus $\langle d, \mathbf{n}_0 \rangle = \lim_{m\to\infty} \langle d, \mathbf{n}_m \rangle \ge 0$. This contradicts the assumption about d again.

□

Lemma 2.35. Let $\mathbf{B} \subset \mathbb{R}^n$ be a convex body, then $\mathbf{B}$ is illuminated by $\mathcal{I}$ if and only if $\operatorname{ext} \mathbf{B}$ is illuminated by $\mathcal{I}$.

Here, by illuminated, we mean $\mathbf{B}$-illuminated as $\operatorname{ext} \mathbf{B} \subset \mathbf{B}$.

Proof.

$\Rightarrow$ $\operatorname{ext} \mathbf{B} \subset \mathbf{B}$.

$\Leftarrow$ Suppose $\operatorname{ext} \mathbf{B}$ is illuminated by $\mathcal{I}$. Let $x \in \partial\mathbf{B} \setminus \operatorname{ext} \mathbf{B}$. Then there exists $y_1, ..., y_m \in \operatorname{ext} \mathbf{B}$ so that

$$x = \sum_{i=1}^{m} \lambda_i y_i, \quad \text{where } \sum_{i=1}^{m} \lambda_i = 1 \text{ and } 0 \le \lambda_1, ..., \lambda_m \le 1.$$

Let $\mathbf{n} \in \nu(\mathbf{B}, x)$, then $H_x(\mathbf{n})$ supports $\mathbf{B}$. In particular, since $y_i \in \mathbf{B}$ for $i \in [m]$, $\langle y_i, \mathbf{n} \rangle \le \langle x, \mathbf{n} \rangle$. Suppose for some $j \in [m]$, $\lambda_j > 0$ and $\langle y_j, \mathbf{n} \rangle < \langle x, \mathbf{n} \rangle$, then

$$\langle x, \mathbf{n} \rangle = \lambda_j \langle y_j, \mathbf{n} \rangle + \sum_{i\in[m]\setminus\{j\}} \lambda_i \langle y_i, \mathbf{n} \rangle < \lambda_j \langle x, \mathbf{n} \rangle + \sum_{i\in[m]\setminus\{j\}} \lambda_i \langle x, \mathbf{n} \rangle = \langle x, \mathbf{n} \rangle .$$

This means that $\langle y_j, \mathbf{n} \rangle = \langle x, \mathbf{n} \rangle$, i.e. $y_j \in H_x(\mathbf{n})$. Thus every supporting hyperplane which contains x will also contain y. Hence $\nu(\mathbf{B}, x) \subset \nu(\mathbf{B}, y)$. In view of lemma 2.34, if $y \in \mathcal{I}$ illuminates y, then $\langle y, \mathbf{n} \rangle < 0$ for every $\mathbf{n} \in \nu(\mathbf{B}, y) \supset \nu(\mathbf{B}, x)$. Hence, x is illuminated by $\mathcal{I}$.

□

Since for the most part, we will only show that the vertices are illuminated, we will be using lemma 2.35 implicitly.

Lemma 2.36. Let $m \in \mathbb{N}$, $i \in [m]$ and $\mathbf{B}_i \subset \mathbb{R}^n$ be convex bodies which can be $\mathbf{B}_i$-illuminated by $\mathcal{I}$. Then $\operatorname{conv}\left(\bigcup_{i=1}^m \mathbf{B}_i\right)$ is $\operatorname{conv}\left(\bigcup_{i=1}^m \mathbf{B}_i\right)$-illuminated by $\mathcal{I}$.

Proof. Let $m = 2$, and let $\mathbf{B}_1, \mathbf{B}_2 \subset \mathbb{R}^n$ be convex bodies and suppose they are $\mathbf{B}_1, \mathbf{B}_2$-illuminated by $\mathcal{I}$, respectively. By lemma 2.35, $\operatorname{ext} \mathbf{B}_1$ and $\operatorname{ext} \mathbf{B}_2$ are $\mathbf{B}_1$ and $\mathbf{B}_2$-illuminated by $\mathcal{I}$, respectively. Since $\mathbf{B}_1, \mathbf{B}_2 \subset \operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2)$, they are both $\operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2)$-illuminated by $\mathcal{I}$. And thus $\operatorname{ext} \mathbf{B}_1 \cup \operatorname{ext} \mathbf{B}_2$ is $\operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2)$-illuminated by $\mathcal{I}$. From lemma 2.23, we get

$$\operatorname{ext} \operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2) \subset \operatorname{ext} \mathbf{B}_1 \cup \operatorname{ext} \mathbf{B}_2.$$

So $\operatorname{ext} \operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2)$ is also $\operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2)$-illuminated by $\mathcal{I}$. Again, apply lemma 2.35 to see that $\operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2)$ is $\operatorname{conv}(\mathbf{B}_1 \cup \mathbf{B}_2)$-illuminated by $\mathcal{I}$. Finally, for convex bodies $\mathbf{B}_i \subset \mathbb{R}^n$ which are $\mathbf{B}_i$-illuminated by $\mathcal{I}$, use induction to see that $\operatorname{conv}\left(\bigcup_{i=1}^m \mathbf{B}_i\right)$-illuminates $\operatorname{conv}\left(\bigcup_{i=1}^m \mathbf{B}_i\right)$ using $\mathcal{I}$. □

Lemma 2.37 is an observation for any $\mathbf{B} \in \mathcal{U}^n$: if we have points $x \in \mathbb{R}^n$ and $y \in \mathbf{B}$ so that $|x_i| \leq |y_i|$ for all $i \in [n]$, then $x \in \mathbf{B}$. If we want to construct $\mathbf{B}'$ so that $\mathbf{B} \subsetneq \mathbf{B}'$, we have to add points $x \in \mathbb{R}^n$, where for any $y \in \mathbf{B}$, $x_i > y_i$ for some $i \in [n]$.

Lemma 2.37. Let $\mathbf{B} \in \mathcal{U}^n$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $y = (y_1, ..., y_n) \in \mathbf{B}$ with $|x_i| \leq |y_i|$ for every $i \in [n]$. Excluding the case of when $|x_i| = |y_i|$ for every $i \in [n]$, then $x \in \mathbf{B} \backslash \operatorname{ext} \mathbf{B}$.

Proof. Let $x \in \mathbb{R}^n$, $y \in \operatorname{ext} \mathbf{B}$. If $|x_i| = |y_i|$ for every $i \in [n]$, then $x \in \operatorname{ext} \mathbf{B}$ as $\mathbf{B} \in \mathcal{U}^n$. Now suppose $y \in \mathbf{B}$ and WLOG assume $0 < x_j < y_j$ for exactly one $j \in [n]$, then

$$\lambda y + (1-\lambda) \underbrace{\left(-y_j e_j + \sum_{i \in [n] \backslash j} y_i \right)}_{\mathbf{B} \in \mathcal{U}^n} = x, \quad \text{where } \lambda y_j + (1-\lambda)(-y_j) = x_j \text{ and } 0 < \lambda < 1.$$

A similar argument using induction can be used to get the claim for the other cases. □

Lemma 2.38. Let $\mathbf{B} \in \mathcal{U}^n$ and $x = (x_1, ..., x_n) \in \mathbf{B}$, $y = (y_1, ..., y_n) \in \mathbf{B}$ with $|x_i| < |y_i|$ for every $i \in [n]$. Then $x \in \operatorname{int} \mathbf{B}$.

Proof. Take $\varepsilon = \min_{i \in [n]}(|y_i| - |x_i|)$, construct the cross-polytope with vertices

$$\{x \pm \varepsilon e_i : i \in [n]\}$$

centered at x. We have constructed a cross-polytope in $\mathbb{R}^n$ contained in $\mathbf{B}$ due to the convexity of $\mathbf{B}$. Thus $x \in \operatorname{int} \mathbf{B}$. □

Corollary 2.39. Let $\mathbf{B} \in \mathcal{U}^n$ and $x = (x_1, ..., x_n) \in \partial \mathbf{B}$ with $|I_0^x| = 0$. If $d_i > 0$ for every $i \in [n]$, then $d = (-\operatorname{sign}(x_1)d_1, ..., -\operatorname{sign}(x_n)d_n)$ illuminates x.

Proof. Let $d_i > 0$ for $i \in [n]$. Since $n < \infty$, pick a small $\varepsilon > 0$ so that

$$\varepsilon d_i < |x_i| \quad \text{for every } i \in [n].$$

Then

$$|x_i - \varepsilon \operatorname{sign}(x_i) d_i| = ||x_i| - \varepsilon d_i| < |x_i| \quad \text{for every } i \in [n].$$

Now, we can apply lemma 2.38 to see that $x + \varepsilon d \in \operatorname{int} \mathbf{B}$ and thus x is illuminated by d. □

Definition 2.40. Let $d \in \mathbb{R}^n \backslash \{0\}$ be an illuminating direction, we call the following *a perturbation of* d:

$$d + \underbrace{\sum_{i=1}^{n} \delta_i e_i}_{\text{Perturbation}}, \quad \text{where } \delta_i \in \mathbb{R} \text{ for every } i \in [n].$$

In this above setting, we say that d *is perturbed by* $\sum_{i=1}^{n} \delta_i e_i$.

Notice if we only look at directions $d \in \mathbb{R}^n$ with $\|d\|_\infty = 1$, this is enough to account for any direction in $\mathbb{R}^n$ via scaling. If we perturb directions from $\mathcal{I}_T^n$, the larger we allow the perturbations to be, the more directions in $\mathbb{R}^n$ we account for when attempting to illuminate our convex body. However, we will see that we can use small (and it is necessary for them to be small) perturbations to illuminate multiple vertices and attain many of our results. So in a sense, Tikhomirov's original set $\mathcal{I}_T^n$ is very close to containing sufficiently small subsets to illuminate even more 1-symmetric and 1-unconditional convex bodies.

Lemma 2.41. Let $\mathbf{B} \subset \mathbb{R}^n$ be a convex body and d be a illuminating direction of some $x \in \partial\mathbf{B}$. Then there exists $\delta > 0$ so that, whenever $|\delta_i| < \delta$ for every $i \in [n]$, we will have that $d + \sum_{i=1}^n \delta_i e_i$ is also an illuminating direction of $x \in \partial\mathbf{B}$.

Proof. Let $x \in \partial\mathbf{B}$ and suppose it is $\mathbf{B}$-illuminated by d, then there exists $\varepsilon > 0$ so that $x + \varepsilon d \in \operatorname{int}\mathbf{B}$. Then there exists $0 < \eta < \varepsilon$ so that every point of the open ball $B_\eta(x + \varepsilon d)$ lies in $\operatorname{int}\mathbf{B}$. Let $y \in B_\eta(x+\varepsilon d)$, then $y = x+\varepsilon d+\sum_{i=1}^n \eta_i e_i$, where $|\eta_i| < \eta$ for every $i \in [n]$. Setting $\delta_i = \frac{\eta_i}{\varepsilon}$ for every $i \in [n]$, $\delta = \frac{\eta}{\varepsilon}$, and $\sum_{i=1}^n \frac{\eta_i}{\varepsilon} e_i$ to be the perturbation of d satisfies the claim. $\square$

The next lemma (2.42) will be the main lemma which we use to justify a direction illuminates a point. It will be the main tool used in sections 5, 6 and 7. This lemma is an extension of lemma 2.38 and corollary 2.39.

Lemma 2.42. Let $\mathbf{B} \subset \mathbb{R}^n$ be a convex body, H be an m-dimensional affine set defined by $\operatorname{span}(b_1, ..., b_m) + t$, where $b_i \in \mathbb{R}^n \backslash \{0\}$, $t \in \mathbb{R}^n$ such that $t, b_1, ..., b_m$ are mutually orthogonal. Let $\mathcal{B} = \{b_1, ..., b_n\}$ be an orthonormal basis for $\mathbb{R}^n$ and suppose $p \in H \cap \mathbf{B}$, then we can express $p = (p_1, ..., p_m, 0, ..., 0)_{\mathcal{B}} + t$. Now consider the projection

$$P : H \cap \mathbf{B} \to \mathbb{R}^m, \quad t + (x_1, ..., x_m, 0, ..., 0)_{\mathcal{B}} \mapsto (x_1, ..., x_m),$$

and let $P(p) \in P(H \cap \mathbf{B})$.

1. If $P(p) \in \operatorname{int}(P(H \cap \mathbf{B}))$, then $p \in \operatorname{relint}(H \cap \mathbf{B})$.

2. In addition to 1, if $H \cap \operatorname{int}\mathbf{B} \neq \varnothing$, then $p \in \operatorname{int}\mathbf{B}$.

3. In particular, suppose $\mathbf{B} \in \mathcal{U}^n$ and $\|e_i\|_{\mathbf{B}} = 1$ for every $i \in [n]$. Let $x = (x_1, ..., x_n) \in \partial\mathbf{B}$. Pick a rearrangement of $[n]$, $\{i_1, ..., i_n\} = [n]$, and assume

$$\sum_{j=m+1}^{n} |x_{i_j}| < 1.$$

Set $b_j = e_{i_j}$ for $j \in [m]$ and

$$H := \operatorname{span}(b_1, ..., b_m) + \sum_{m+1}^{n} x_{i_j} e_{i_j}.$$

If $d \in \operatorname{span}(b_1, ..., b_m)$ and $P(x + \varepsilon d) \in \operatorname{int}(P(H \cap \mathbf{B}))$ for some $\varepsilon > 0$, then x is illuminated by d.

4. In the context of part 3, assume $\mathbf{B} \in \mathcal{U}^n$, if $[n] \backslash I_0^x = \{i_1, ..., i_m\}$, then the direction

$$d = \sum_{j=1}^{m} -\operatorname{sign}(x_{i_j}) d_j e_{i_j} \quad \text{illuminates } x \text{ where } d_j > 0.$$

The main idea of this lemma is to look at cross sections, $H \cap \mathbf{B}$, of convex bodies, $\mathbf{B}$, where H is some affine set. If these cross sections are "$P(H \cap \mathbf{B})$-illuminated" ($P(x + \varepsilon d) \in \operatorname{int} P(H \cap \mathbf{B})$), then they are also $\mathbf{B}$-illuminated. Simply put, fix H, we use directions to illuminate the cross section, $H \cap \mathbf{B}$ in its own context, and these directions still work when we piece the cross section back into $\mathbf{B}$.

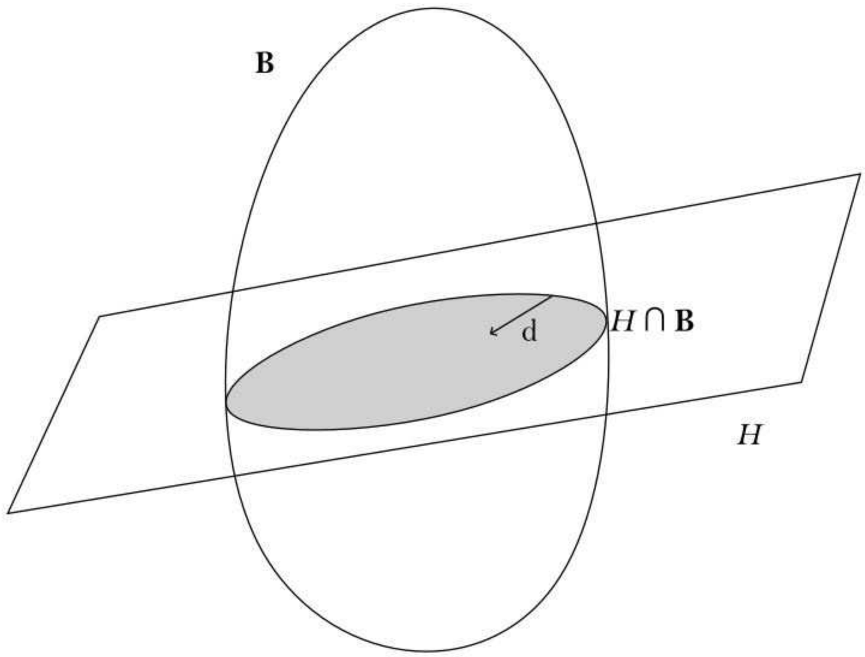


Figure 3: Illustration of lemma 2.42.

Proof of lemma 2.42.

1. The lemma follows from P being continuous and the preimages of open sets in the codomain of continuous mappings being relatively open in their domain.

2. This is lemma 2.30.

3. Since $P(x + \varepsilon d) \in \operatorname{int}(P(H \cap \mathbf{B}))$, this satisfies the condition in the first part of the lemma. Also, $\|e_i\|_{\mathbf{B}} = 1$ for all $i \in [n]$, so we know that $CP_1^n \subset \mathbf{B}$ (see remark 5.1). Then

$$\sum_{j=m+1}^{n} \left|x_{i_j}\right| < 1 \implies \sum_{j=m+1}^{n} x_{i_j} e_j \in \operatorname{int} \mathbf{B}.$$

But this means that $H \cap \operatorname{int} \mathbf{B} \neq \varnothing$, which satisfies the second part of this lemma. Therefore, $x + \varepsilon d \in \operatorname{int} \mathbf{B}$ and d illuminates x.

4. This part follows from part 3 and corollary 2.39.

□

Example 2.43. In the context of the second part of the previous lemma, if we only have $H \cap \mathbf{B} \neq \varnothing$, consider the case of the cube, C_1^n and the hyperplane defined by $x_1 = 1$, then we do not get that $p \in \operatorname{int} \mathbf{B}$.

Lemma 2.44. Let $\mathbf{B} \subset \mathbb{R}^n$ be a convex body, $x \in \operatorname{ext} \mathbf{B}$ and ι be an isomorphism. If $\iota(x)$ is $\iota(\mathbf{B})$-illuminated by d (or I), then x is $\mathbf{B}$-illuminated by $\iota^{-1}(d)$ (or $\iota^{-1}(I)$). And since ι is an isomorphism, $|\iota^{-1}(I)| = |I|$.

Proof. Let $\varepsilon > 0$, since isomorphisms are linear and continuous

$$\iota(x) + \varepsilon d \in \operatorname{int} \iota(\mathbf{B}) \implies x + \varepsilon \iota^{-1}(d) = \iota^{-1}(\iota(x) + \varepsilon d) \in \operatorname{int} \mathbf{B}.$$

□

Lemma 2.45. Let $\mathbf{B} \in \mathcal{U}^n$ which satisfies the following conditions:

1. $\|e_i\|_{\mathbf{B}} = 1$ for every $i \in [n]$.

2. $\|e_i + e_j\|_{\mathbf{B}} > 1$ for every $i, j \in [n]$.

Then any vertex $x = (x_1, ..., x_n) \in \operatorname{ext} \mathbf{B}$ which has at most two non zero entries can be illuminated by $-\operatorname{sign}(x_i)e_i$ for some $i \in [n]$.

Proof. If $|I_0^x| = n - 1$, then x is a vertex of the cross-polytope centered at the origin with diameter 2. This is illuminated by $-x = -x_i e_i$ for some $i \in [n]$. Now consider when $|I_0^x| = n - 2$. Pick k so that $|x_k| = \|x\|_\infty$, then $x - x_k e_k = x_j e_j$, where $j \in [n]$ is the other index for which $x_j \neq 0$. Since $\|e_k + e_j\|_{\mathbf{B}} > 1$, it must be the case that $|x_j| < 1$. But $x_j e_j$ is an interior point of $\mathbf{B}$ as $|x_j| < 1$ and $\|e_i\|_{\mathbf{B}} = 1$. Hence $-\operatorname{sign}(x_k)e_k$ illuminates x. □

Example 2.46. Let $\mathbf{B} \subset \mathbb{R}^4$ be a 1-unconditional convex body. Additionally suppose $\|e_i\|_{\mathbf{B}} = 1$ for every $i \in [n]$.

1. If $(\frac{1}{2}, \frac{3}{4}, 0, 0) \in \operatorname{ext} \mathbf{B}$, then we claim

$$\left(\tfrac{1}{2}, \tfrac{3}{4}, 0, 0\right) - \tfrac{3}{4} e_2 = \left(\tfrac{1}{2}, 0, 0, 0\right) \in \mathbf{B}.$$

   By remark 5.1, $\mathbf{0} \in \operatorname{int} \mathbf{B}$ and by assumption, $(1, 0, 0, 0) \in \mathbf{B}$ since $\|e_i\|_{\mathbf{B}} = 1$ for every $i \in [n]$. With fact 2.18 in mind, consider the ray which has the endpoint at $\mathbf{0}$, and passes through $(\frac{1}{2}, 0, 0, 0)$ and then through $(1, 0, 0, 0)$. From this, we deduce that $(\frac{1}{2}, 0, 0, 0) \in \operatorname{int} \mathbf{B}$.

2. If $(1, 1, 0, 0) \in \operatorname{ext} \mathbf{B}$, then $\|e_i + e_j\|_{\mathbf{B}} > 1$ for every $i, j \in [n]$, will not be satisfied.

3 Tikhomirov's Results

Because we heavily rely on some of Tikhomirov's results, we will present the proofs of some of them in this section. The main theorem of Tikhomirov's paper is:

Theorem 3.1. There is a universal constant $C > 0$ with the following property: let $n \geq C$ and let $\mathbf{B} \in \mathcal{S}^n$. Assume that $\mathbf{B}$ is the affine image of a cube, then $\mathcal{I}(\mathbf{B}) < 2^n$.
In fact, an illuminating set for $\mathbf{B}$ with cardinality $< 2^n$ can be chosen out of the subsets of $\{-1, 0, 1\}^n$.

The proof of this theorem is split into two main cases. The first case is when $1 \neq \mathrm{d}(\mathbf{B}, [-1, 1]^n) < 2$ and Tikhomirov has a deterministic approach for this case. This is the part that we will present proofs for as they are more relevant to our results. As for the second case, $\mathrm{d}(\mathbf{B}, [-1, 1]^n) \geq 2$, we will simply present the results and outline the ideas of the proofs. First, we will give an overview of the main lemmas that he uses.

It should also be noted that in the first part of Tikhomirov's results, there was an error regarding conjecture 3.7. We will discuss this error in section 4 and we will fix this error in section 5.

Lemma 3.2. Let $n \geq 2$, $\mathbf{B} \in \mathcal{U}^n$, $x \in \partial \mathbf{B}$ and $\mathbf{n} \in \nu(\mathbf{B}, x)$. Then $x_i \mathbf{n}_i \geq 0$ for all $i \in [n]$.

Proof. Suppose that $x_i \mathbf{n}_i < 0$ for some $i \in [n]$, now consider $H_x(\mathbf{n})$. Notice

$$\langle x - x_i e_i, \mathbf{n} \rangle = \langle x, \mathbf{n} \rangle - \langle x_i e_i, \mathbf{n} \rangle = \langle x, \mathbf{n} \rangle - x_i \mathbf{n}_i > \langle x, \mathbf{n} \rangle$$

Since $x \in \mathbf{B}$ and $\mathbf{B} \in \mathcal{S}^n$, we know that $x - x_i e_i \in \mathbf{B}$ by lemma 2.37. This contradicts the derived inequality. □

Lemma 3.3. Let $n \geq 2$, $\mathbf{B} \in \mathcal{S}^n$ and $x \in \partial \mathbf{B}$. Then for all $i, j \in [n]$ such that $|x_i| > |x_j|$, we have $|\mathbf{n}_i| \geq |\mathbf{n}_j|$ for any $\mathbf{n} \in \nu(\mathbf{B}, x)$.

Proof. Fix $\mathbf{n} \in \nu(\mathbf{B}, x)$. Suppose for contradiction that for some $i, j \in [n]$, we have $|x_i| > |x_j|$ and $|\mathbf{n}_i| < |\mathbf{n}_j|$. Consider $H_x(\mathbf{n})$, a supporting hyperplane for $\mathbf{B}$. Let $c_1, c_2 \in \{-1, 1\}$ so that

$c_i x_i \mathbf{n}_j$, $c_j x_j \mathbf{n}_i \geq 0$, and denote:

$$y := c_i x_i e_j + c_j x_j e_i + \sum_{k\in[n]\setminus\{i,j\}} x_k e_k$$

Then

$$\begin{aligned}\langle y, \mathbf{n}\rangle &= \langle x, \mathbf{n}\rangle + |x_i \mathbf{n}_j| + |x_j \mathbf{n}_i| - x_i \mathbf{n}_i - x_j \mathbf{n}_j \\ &= |\langle x, \mathbf{n}\rangle| + (|x_i| - |x_j|)(|\mathbf{n}_j| - |\mathbf{n}_i|) \\ &> |\langle x, \mathbf{n}\rangle| .\end{aligned}$$

Hence $y \notin \mathbf{B}$. This contradicts the definition of the class $\mathcal{S}^n$ as

$$x \in \mathbf{B} \implies y = c_i x_i e_j + c_j x_j e_i + \sum_{k\in[n]\setminus\{i,j\}} x_k e_k \in \mathbf{B}.$$

□

Based on these tools, Tikhomirov now proves the first subcase of Theorem 3.1, that is, lemma 3.4. He begins with the preliminary lemma (for the sake of completeness, we include his proofs for these results):

Lemma 3.4. Let $n \geq 2$, $\mathbf{B} \in \mathcal{S}^n$, $x \in \partial\mathbf{B}$ and $d \in \{-1, 0, 1\}^n$ be a vector such that

1. $I_0^d \subset I_0^x$ and
2. for any $i \in [n]$ such that $x_i \neq 0$, we have $d_i = -\operatorname{sign}(x_i)$.

Finally, assume that x is not illuminated in the direction d. Then necessarily

$$\left\| \sum_{i\in[n]\setminus I_0^d} e_i \right\|_{\mathbf{B}} \geq \frac{2}{\|x\|_\infty}.$$

Proof. In view of lemma 2.34, if d does not illuminate x then there is a vector $\mathbf{n} \in \nu(\mathbf{B}, x)$

such that $\langle d, \mathbf{n} \rangle \geq 0$. By Lemma 3.2, we have

$$\sum_{i\in[n]\setminus I_0^x} d_i \mathbf{n}_i = \sum_{i\in[n]\setminus I_0^x} -\operatorname{sign}(x_i)\mathbf{n}_i = -\sum_{i\in[n]\setminus I_0^x} |\mathbf{n}_i|\,.$$

Thus, the condition $\langle d, \mathbf{n} \rangle \geq 0$ gives

$$\begin{aligned}
0 \leq \sum_{i\in[n]} d_i\mathbf{n}_i &= \sum_{i\in I_0^x\setminus I_0^d} d_i\mathbf{n}_i + \sum_{i\in[n]\setminus I_0^x} d_i\mathbf{n}_i = \sum_{i\in I_0^x\setminus I_0^d} d_i\mathbf{n}_i - \sum_{i\in[n]\setminus I_0^x} |\mathbf{n}_i| \\
\implies \sum_{i\in I_0^x\setminus I_0^d} d_i\mathbf{n}_i &\geq \sum_{i\in[n]\setminus I_0^x} |\mathbf{n}_i|\,.
\end{aligned}$$

On the other hand,

$$\left\langle \sum_{i\in[n]\setminus I_0^x} (-d_i)\, e_i + \sum_{i\in I_0^x\setminus I_0^d} d_i e_i, \mathbf{n} \right\rangle \geq 2 \sum_{i\in[n]\setminus I_0^x} |\mathbf{n}_i| \underset{x_i\mathbf{n}_i\geq 0}{\geq} 2 \sum_{i\in[n]\setminus I_0^x} \mathbf{n}_i \frac{x_i}{\|x_i\|_\infty} = \frac{2\langle x, \mathbf{n}\rangle}{\|x\|_\infty}.$$

And since $H_x(\mathbf{n})$ is a supporting hyperplane for $\mathbf{B}$,

$$\left\| \sum_{i\in[n]\setminus I_0^x} (-d_i)\, e_i + \sum_{i\in I_0^x\setminus I_0^y} d_i e_i \right\|_{\mathbf{B}} \geq \frac{2}{\|x\|_\infty}.$$

Using that $d \in \{-1, 0, 1\}^n$ and $\mathbf{B} \in \mathcal{S}^n$ gives the claim. □

Lemma 3.4 only uses lemmas 2.34 and 3.2. Furthermore in the final sentence of proof, we can weaken the condition $\mathbf{B} \in \mathcal{S}^n$ to $\mathbf{B} \in \mathcal{U}^n$ and the proof still works. Hence, this lemma also works for 1-unconditional convex bodies.

Lemma 3.5. Let $n \geq 2$, $\mathbf{B} \in \mathcal{S}^n$ so that $1 \neq \mathrm{d}(\mathbf{B}, [-1,1]^n) < 2$. Then at least one of the following is true:

1. $\mathbf{B}$ can be illuminated in directions

$$T_1 := \{(d_1, ..., d_n) \in \{-1, 1\}^n : \exists i \in [n-1] \text{ with } d_i = -1\} \cup \{e_1 + ... + e_{n-1}\}\,.$$

2. For every $i, j \in [n]$

$$\|e_i + e_j\|_{\mathbf{B}} > \|e_i\|_{\mathbf{B}} = 1.$$

Proof. Recall that $\|e_i\|_{\mathbf{B}} = 1$ (this implies that $\mathbf{B} \subset [-1,1]^n$, i.e., $\|\cdot\|_{\mathbf{B}} \geq \ \|\cdot\|_{\infty}$). Assume that the first condition is not satisfied, then there is a vector $x \in \partial\mathbf{B}$ which is not illuminated in directions from T_1.

Case 1: $I_0^x \neq \varnothing$. There exists a direction $d \in T_1$ such that $I_0^d \subset I_0^x$ and $d_i = -\operatorname{sign}(x_i)$ for all $i \in [n]$ with $x_i \neq 0$. By lemma 3.4,

$$\left\| \sum_{i \in [n]} e_i \right\|_{\mathbf{B}} \geq \frac{2}{\|x\|_{\infty}} \geq 2,$$

contradicting the assumption $d(\mathbf{B}, [-1,1]^n) < 2$.

Case 2: $I_0^x = \varnothing$ and $|x_n| \leq |x_i|$ for all $i \in [n]$. Define d so that for $i \in [n-1]$, $d_i := -\operatorname{sign}(x_i)$ and

$$d_n := \begin{cases} 0, & d_1 = \dots = d_{n-1} = 1 \\ -\operatorname{sign}(x_n), & \text{otherwise.} \end{cases}$$

By definition, $d \in T_1$, so d does not illuminate x. By lemma 2.34, there exists a $\mathbf{n} \in \nu(\mathbf{B}, x)$ such that $\langle d, \mathbf{n} \rangle \geq 0$. Using lemma 3.2,

$$0 \leq \langle d, \mathbf{n} \rangle = \sum_{i \in [n-1]} - |\mathbf{n}_i|.$$

This implies that $\mathbf{n}_i = 0$ for all $i \in [n-1]$ (and $\mathbf{n}_n \neq 0$), so $H_x(\operatorname{sign}(x_n)e_n)$ is a supporting hyperplane of $\mathbf{B}$. On the other hand, $e_n \in \mathbf{B}$ by our assumption, implying that $|x_n| \geq 1$. Thus, $|x_1|, \dots, |x_n| \geq 1$ but since $\mathbf{B} \subset [-1,1]^n$, it must be the case that $\mathbf{B} = [-1,1]^n$ which contradicts our choice of $\mathbf{B}$.

Alternatively, with corollary 2.39 in mind, T_1 illuminates all such $x \in \partial\mathbf{B}$ which satisfies the hypothesis in the case, except those when

$$\operatorname{sign}(x_1) = \cdots = \operatorname{sign}(x_{n-1}) = -1.$$

Since $|x_n| < 1$ (otherwise we reach the same contradiction as in the previous paragraph), with lemma 2.42 in mind, look in the affine set

$$\{(\xi_1, ..., \xi_n) \in \mathbb{R}^n : \xi_n = x_n\}$$

to see that $e_1 + \cdots + e_{n-1}$ illuminates $x \in \partial \mathbf{B}$ when

$$\operatorname{sign}(x_1) = \cdots = \operatorname{sign}(x_{n-1}) = -1.$$

Case 3: $I_0^x = \varnothing$ and there is $j \in [n-1]$ such that $|x_j| \leq |x_i|$ for all $i \in [n]$ (j does not have to be unique). Define d in the following way:

$$d_i := \begin{cases} -\operatorname{sign}(x_i), & i \neq j \\ -1, & i = j \end{cases}.$$

Again, $d \in T_1$, so there exists $\mathbf{n} \in \nu(\mathbf{B}, x)$ so that $\langle d, \mathbf{n} \rangle \geq 0$. Using Lemma 3.2, we get

$$\begin{aligned} 0 \leq \langle d, \mathbf{n} \rangle &= -\mathbf{n}_j + \sum_{i \in [n] \setminus \{j\}} -\operatorname{sign}(x_i)\mathbf{n}_i \\ \implies |\mathbf{n}_j| &\geq -\mathbf{n}_j \geq \sum_{i \in [n] \setminus \{j\}} |\mathbf{n}_i| \\ \implies 2|\mathbf{n}_j| &\geq \sum_{i \in [n]} |\mathbf{n}_i| \neq 0. \end{aligned}$$

By lemma 3.3, we have $|\mathbf{n}_j| \leq |\mathbf{n}_i|$ for all $i \leq n$ such that $|x_i| > |x_j|$. The last two conditions can be simultaneously fulfilled only if the set

$$J := \{i \in [n] : |x_i| > |x_j|\}$$

has cardinality at most 1. The case $J = \varnothing$ (when all coordinates of x are equal by absolute value) was covered in case 2. Thus, we only need to consider the situation $|J| = 1$. Let $k \in [n]$ so that $|x_k| > |x_j|$. Since $2\,|\mathbf{n}_j| \geq \sum_{i\in[n]} |\mathbf{n}_i|$, by lemma 3.3, we have $|\mathbf{n}_k| = |\mathbf{n}_j|$ and $\mathbf{n}_i = 0$ for all $i \neq j, k$. Define

$$\tilde{x} := |x_j|\, e_j + |x_k|\, e_k + \sum_{i\in[n]\setminus\{j,k\}} x_i e_i.$$

Then $\tilde{x} \in \mathbf{B}$ as $\mathbf{B} \in \mathcal{S}^n$. Then

$$H_{\tilde{x}}(\mathbf{n}) = \{(\xi_1, ..., \xi_n) \in \mathbb{R}^n : \xi_j + \xi_k = |x_j| + |x_k|\}$$

is a supporting hyperplane for $\mathbf{B}$. At the same time,

$$1 = \|\tilde{x}\|_{\mathbf{B}} \geq \||x_k|\, e_k\|_{\mathbf{B}} \underset{\|e_i\|_{\mathbf{B}}=1}{=} |x_k| > |x_j|$$

whence $|x_j| + |x_k| < 2$. Thus, $e_k + e_j$ lies in the positive half-space of $H_{\tilde{x}}(\mathbf{n})$ which means that $e_k + e_j \notin \mathbf{B}$, i.e., $\|e_k + e_j\|_{\mathbf{B}} > 1$.

□

Since our research also involves 1-unconditional bodies, one of our aims is to see if there are any parts from Tikhomirov's proofs that are salvageable for those purposes. In view of these, notice that in lemma 3.5, for case 1 and 2, we can extend the argument to $\mathbf{B} \in \mathcal{U}^n$, as lemmas 2.34, 3.4 and corollary 2.39 only require $\mathbf{B} \in \mathcal{U}^n$. However, the method in case 3 requires the use of lemma 3.3, which heavily relies on the condition $\mathbf{B} \in \mathcal{S}^n$. In fact, the statement itself of lemma 3.5 does not hold for all 1-unconditional convex bodies. Consider the following example:

Example 3.6. Suppose $x = (1, -\frac{3}{4}, 1) \in \partial\mathbf{B}$. So $\mathrm{d}(\mathbf{B}, [-1, 1]^3) < 2$. Let $\varepsilon > 0$ and $d = (-1, -1, -1)$. Consider

$$x + \varepsilon d = \left(1 - \varepsilon,\, -\tfrac{3}{4} - \varepsilon,\, 1 - \varepsilon\right).$$

If the coordinate reflections of $(1, \frac{3}{4}, 1)$ and $(0, 1, 0)$ are the only vertices of $\mathbf{B}$ and we want to show that $x + \varepsilon d \in \operatorname{int} \mathbf{B}$, then our only option is to consider the convex combination

$$y := (1 - 4\varepsilon)\left(1, -\tfrac{3}{4}, 1\right) + 4\varepsilon(0, -1, 0) = \left(1 - 4\varepsilon,\ -\tfrac{3}{4} - \varepsilon,\ 1 - 4\varepsilon\right) \in \mathbf{B}.$$

This is because if $x + \varepsilon d \in \operatorname{int} \mathbf{B}$, there must be a point in $\mathbf{B}$, where the second coordinate has absolute value no less than the second coordinate of $x + \varepsilon d$. Based on the definition of y and $\mathbf{B}$ and the fact $\mathbf{B}$ only has the above mentioned vertices, we can see that it is a boundary point of $\mathbf{B}$. However, if we suppose $x + \varepsilon d \in \mathbf{B}$, then, with lemma 2.42 in mind, we can look in the affine set

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_2 = -\tfrac{3}{4} - \varepsilon\}$$

to see that $y \in \operatorname{int} \mathbf{B}$, which contradicts the definition of y. Hence, it must be the case that $x + \varepsilon d \notin \mathbf{B}$.

Now we turn to the second subcase of Theorem 3.1. A slightly stronger version of lemma 3.8 below, which omitted the assumption that there are no vertices of $\mathbf{B}$ with zero coordinates, was originally used to prove conjecture 3.7 (see [22, Lemma 8]); however, there was an oversight in some minor but crucial details. In section 4, we will give a counter example to the stronger version of lemma 3.8, which will show why conjecture 3.7 cannot rely on this version. We will also analyze why the proof given in [22] for lemma 3.8 (and its stronger version) cannot really be extended in that instance. For the time being, we will just give the statements.

Conjecture 3.7. Let $n \geq 2$, $\mathbf{B} \in \mathcal{S}^n$ and suppose $\|e_i + e_j\|_{\mathbf{B}} > 1$ for every $i, j \in [n]$. Then $\mathcal{I}(\mathbf{B}) < 2^n$.

Proof. Proven by corollary 5.20. □

Lemma 3.8. In addition to the assumptions in conjecture 3.7, if for every $x \in \operatorname{ext} \mathbf{B}$, $|I_0^x| = 0$, then $\mathbf{B}$ can be illuminated by

$$T_2 := \left(\{-1, 1\}^{n-1} \times \{0\}\right) \cup \{\pm e_n\}.$$

This concludes the first half of Tikhomirov's paper. The second half of Tikhomirov's paper, which deals with convex bodies $\mathbf{B} \in \mathcal{S}^n$ with $\mathrm{d}(\mathbf{B}, [-1,1]^n) \geq 2$, takes a probabilistic approach to the problem.

Let $n \geq 2$ and $X \in \mathbb{R}^n$, where $X_i \in \{-1, 1\}$, having equal probability to take on each value. Let $\{X^\ell\}_{\ell=1}^\infty$ be copies of X. Define a random projection

$$P^{(m)} : \mathbb{R}^n \to \mathbb{R}^n, \quad \sum_{i=1}^n x_i e_i \to \sum_{k=1}^m x_{j_k} e_{j_k}.$$

We impose that for fixed $e_{i_1}, ..., e_{i_m}$, $\operatorname{Im} P^{(m)} = \operatorname{span}\{e_{i_j}, ..., e_{i_m}\}$ has probability $\binom{n}{m}^{-1}$. In other words, the chance that $\{j_1, ..., j_m\} = \{i_1, ..., i_m\}$ has probability $\binom{n}{m}^{-1}$. Therefore, the projection does not favor any m dimensional subspace spanned by a subset of the standard basis vectors. Let $\{P_\ell^{(m)}\}_{\ell=1}^\infty$ be copies of $P^{(m)}$. For $\ell \in [m]$, we impose that X^ℓ and $P_\ell^{(m)}$ be jointly independent. Now for every $k \leq [n/2]$, define the random (multi)set of vectors

$$\mathcal{S}_k := \{P_\ell^{(2k-1)}(X^\ell)\}_{\ell=1}^{\lfloor 2^n/n^2 \rfloor}.$$

Then

$$\left| \bigcup_{k=1}^{\lceil n/2 \rceil} \mathcal{S}_k \right| \leq \frac{2^n}{n^2} \lceil \frac{n}{2} \rceil \leq \frac{2^{n-1}(n+1)}{n^2} < 2^{n-1}.$$

First, Tikhomirov proves the following technical lemma:

Lemma 3.9. There is a universal constant $C > 0$ so that for any $n \geq C$ and $k \in [\lceil n/2 \rceil]$, the event

"for any $y = \{-1, 0, 1\}^n$ with $|I_0^y| = n - k$, there is $\ell \leq 2^n/n^2$ so that $P_\ell^{2k-1}(y) = y$ and $X_i^\ell = y_i$ for all $i \in [n] \backslash I_0^y$"

has probability at least $1 - \exp(-2n)$.

Assume $\mathrm{d}(\mathbf{B}, [-1,1]^n) \geq 2$. Tikhomirov is able to show that for $x \in \partial \mathbf{B}$ which satisfy $\left| I^x_{\pm \|x\|_\infty} \right| > \lceil n/2 \rceil$, we have that x is illuminated by $\{-1, 1\}^{n-1} \times \{0\}$. On the other hand, assume $x \in \partial \mathbf{B}$ satisfies $\left| I^x_{\pm\|x\|_\infty} \right| \leq \lceil n/2 \rceil$ and let $p_\ell^{(2k-1)}$ and x^ℓ be realizations of $P_\ell^{(2k-1)}$ and X^ℓ, respectively, from the good event described in lemma 3.9. He shows that one of the vectors $p_\ell^{\left(2\left|I^x_{\pm\|x\|_\infty}\right|-1\right)}(x^\ell)$ which works for $d := -\sum_{i \in I^x_{\pm\|x\|_\infty}} \operatorname{sign}(x_i) e_i$, in the way that lemma 3.9 states, will illuminate this point x. This gives the following proposition:

Proposition 3.10. There is a universal constant $C > 0$ so that when $n \geq C$, $\mathbf{B} \in \mathcal{S}^n$ and $\mathrm{d}(\mathbf{B}, [-1,1]^n) \geq 2$, with probability of at least $1 - \exp(n)$, $\mathbf{B}$ is illuminated by

$$(\{-1,1\}^{n-1} \times \{0\}) \cup \bigcup_{k=1}^{\lceil n/2 \rceil} \mathcal{S}_k.$$

4 Motivation

Recall that the beginning of this research stems from the natural question which we formulated from Tikhomirov's results: can we find examples of 1-symmetric convex bodies in lower dimensions where Theorem 3.1 does not hold? Or where at least the "in fact" part of this theorem does not hold? Lemmas 4.1 and 4.2 are positive answers to the latter question. The first example is in dimension 3, and the second example is the direct generalization of the first example in dimension 4.

Lemma 4.1. Let $\mathbf{B} \subset \mathbb{R}^n$ be the 1-symmetric body which is the convex hull of $\operatorname{ext} C^3_{\frac{1}{2}} \cup \operatorname{ext} CP^3_1$ $\mathbf{B}$ requires more than 8 of Tikhomirov's directions from his large illuminating set, $\mathcal{I}^3_T$, in order to be illuminated, no matter which combinations we consider.

The vertices of this polytope are the coordinate permutations of $(1,0,0)$ and $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ and their coordinate reflections. The idea of the proof is to show that the vertices contributed by the cross-polytope can only be illuminated by 6 specific directions. Then we show that those 6 directions cannot be used to illuminate the vertices which are contributed by the cube. **Click here** for a picture.

Proof. Let $x \in \operatorname{ext} \mathbf{B}$. For any indices, $i, j \in [3]$, $i \neq j$, consider the hyperplanes defined by

$$\pm\xi_i \pm \xi_j = 1, \quad (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.$$

Notice that these are a supporting hyperplanes of $\mathbf{B}$ since

$$\pm x_i \pm x_j \leq |\pm x_i \pm x_j| \leq |\pm x_i| + |\pm x_j| \leq |x_i| + |x_j| \leq 1.$$

Consider any vertex from $\operatorname{ext} CP^3_1$; WLOG pick the vertex $(1,0,0)$. Consider any illuminating direction, $d \in \mathcal{I}^3_T$, such that $\left|I^d_0\right| \leq 1$. It must be the case that $d_1 = -1$ (if $d_1 = 0$, in the best case scenario, for any small $\varepsilon > 0$, we get $x + \varepsilon d \in \partial\mathbf{B}$; if $d_1 = 1$ then $x + \varepsilon d \notin \mathbf{B}$). Since $\left|I^d_0\right| \leq 1$, it must be the case that $|d_2| = 1$ or $|d_3| = 1$. If $d_3 \neq 0$, then

$$(1,0,0) + \varepsilon d = (1-\varepsilon,\, \varepsilon d_2,\, \operatorname{sign}(d_3)\varepsilon) \in \{(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 : \xi_1 + \operatorname{sign}(d_3)\xi_3 = 1\}.$$

We get a similar result if $d_2 \neq 0$. So $(1,0,0) + \varepsilon d$ lies in a supporting hyperplane of $\mathbf{B}$. In view of the link before the proof, the red line and blue lines on the hyperplane in the picture represent the attempt to illuminate our vertices with directions $d \in \mathbb{R}^3$ which satisfy $\left|I_0^d\right| \leq 1$. Thus, any illuminating direction $y \in \mathbb{R}^3$ which satisfies $|I_0^d| \leq 1$ is not an illuminating direction for any vertex of CP_1^3. Our illuminating direction from $\mathcal{I}_T^3$ for any $x \in \operatorname{ext} CP_1^3$ must be $-x = \pm e_i$ for some $i \in [3]$.

On the other hand, consider any vertex $x \in \operatorname{ext} C_{\frac{1}{2}}^3$. Denote $[3] = \{i, j, k\}$ and consider $x \pm \varepsilon e_i$, where $\varepsilon > 0$. Then

$$x \pm \varepsilon e_i = \left(\pm\tfrac{1}{2} \pm \varepsilon\right) e_i \pm \tfrac{1}{2} e_j \pm \tfrac{1}{2} e_k \in \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \operatorname{sign}(x_j)\xi_j + \operatorname{sign}(x_k)\xi_k = 1\}.$$

So $x \pm \varepsilon e_i$ lies in a supporting hyperplane of $\mathbf{B}$. Again in view of the link, this case is demonstrated by the green line on the hyperplane. Then e_i is not an illuminating direction for any $x \in \operatorname{ext} CP_1^3$ and its coordinate reflections.

We have illuminated 6 vertices (those of CP_1^3) using the least amount of directions possible, which was also 6 directions ($\{\pm e_1, \pm e_2, \pm e_3\}$); however to illuminate the remaining 8 vertices, if we pick directions from $\mathcal{I}_T^3$, we definitely require more than 2 different directions than the previous 6. Hence to illuminate $\mathbf{B}$, we require more than 8 directions from $\mathcal{I}_T^3$. $\square$

Lemma 4.2. Let $\mathbf{B} = \operatorname{conv}(\operatorname{ext} C_{\frac{1}{2}}^4 \cup \operatorname{ext} CP_1^4)$. $\mathbf{B}$ requires no less than 16 of directions from $\mathcal{I}_T^4$ to be illuminated.

The above lemma is the dimension 4 analogue of lemma 4.1. The proof is essentially the same idea.

Proof. As in lemma 4.1, let $i, j \in [4]$, $i \neq j$. Then the hyperplanes defined by

$$\pm\xi_i \pm \xi_j = 1, \quad (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4$$

are supporting hyperplanes of $\mathbf{B}$. For similar reasoning to lemma 4.1, vertices of $\mathbf{B}$ which come from $\operatorname{ext} CP_1^4$ can only be illuminated by $\pm e_i$, $i \in [n]$ (8 directions).

Again, consider the vertices $x \in \operatorname{ext} C_{\frac{1}{2}}^4$. Let $i, j \in [4] = \{i, j, k, l\}$, $\varepsilon > 0$ and consider $x + \varepsilon \operatorname{sign} x_i e_i$ or $x + \varepsilon(\operatorname{sign} x_i e_i + \operatorname{sign} x_j e_j)$. Both of these points lie on the supporting hyperplane

$$\{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbb{R}^4 : \operatorname{sign}(x_k)\xi_k+\operatorname{sign}(x_l)\xi_l=1\}.$$

This rules out the directions $d\in\mathcal{I}_T^4$ with $|I_0^d|\geq 2$ as illuminating directions for x.

First, we will try to illuminate x using directions $y\in\mathcal{I}_T^4$ with $|I_0^y|=0$. By lemma 2.39, we can illuminate x using d if we define $d_i=-\operatorname{sign}(x_i)$ for every $i\in[n]$. If we fix $\tilde{x}\in\operatorname{ext}C_{\frac{1}{2}}^4$ and define $\tilde{d}\in\mathcal{I}_T^4$ so that $\tilde{d}_i=-\operatorname{sign}(\tilde{x}_i)$ for all $i\in[4]$, then we claim that $\tilde{x}$ is the only vertex in $\operatorname{ext}C_{\frac{1}{2}}^4$ that $\tilde{d}$ illuminates. For the following cases: let $x\in\operatorname{ext}C_{\frac{1}{2}}^4\backslash\{\tilde{x}\}$.

Case 1: Let $i,j\in[4]$, $i\neq j$. Suppose x and $\tilde{d}$ disagree in sign in at least one coordinate, i, but not in every coordinate, j. For $\varepsilon>0$, the point $x+\varepsilon d$ lies in the supporting hyperplane

$$\{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbb{R}^4 : \operatorname{sign}(x_i)\xi_i+\operatorname{sign}(x_j)\xi_j=1\}$$

since

$$\operatorname{sign}(x_i)(x_i+\varepsilon d_i)+\operatorname{sign}(x_j)(x_j+\varepsilon d_j)=|x_i|-\varepsilon+|x_j|+\varepsilon=1.$$

Case 2: Suppose x and $\tilde{d}$ agree in sign in every coordinate. By way of contradiction, suppose that $\tilde{d}$ is an illuminating direction of x, then for some small $\varepsilon>0$, $u+\varepsilon\tilde{d}\in\operatorname{int}\mathbf{B}$. On the other hand, since $\operatorname{sign}(x_i)=\operatorname{sign}(\tilde{d}_i)$ for all $i\in[4]$, $(x+\varepsilon\tilde{d})_i>x_i$ for every $i\in[4]$. By lemma 2.38, x is an interior point which contradicts $x\in\operatorname{ext}C_{\frac{1}{2}}^4$.

Now we try to illuminate $x\in\operatorname{ext}C_{\frac{1}{2}}^4$ using directions $d\in\mathcal{I}_T^4$ which satisfy $\left|I_0^d\right|=1$. Let $d:=\sum_{i\in[4]\backslash\{j\}}-\operatorname{sign}(x_i)e_i$, which means $\{j\}=I_0^d$. We can illuminate x using d as

$$x+\tfrac{1}{2}d=x_je_j=\pm\tfrac{1}{2}e_j \quad\text{and}\quad \left\|\tfrac{1}{2}e_j\right\|_{\mathbf{B}}<\|e_j\|_{\mathbf{B}}=1.$$

We can also illuminate $x-2x_je_j\in\operatorname{ext}C_{\frac{1}{2}}^4$ with the same reasoning. For example, in the case $j=4$, $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ are both illuminated by $(-1,-1,-1,0)$. If we fix $\tilde{j}\in[4]$, $\tilde{x}\in\operatorname{ext}C_{\frac{1}{2}}^4$ and $\tilde{d}=\sum_{i\in[4]\backslash\{\tilde{j}\}}-\operatorname{sign}(x_i)e_i\in\mathcal{I}_T^4$, then we claim that the only vertices that are illuminated by $\tilde{d}$ are $\tilde{x}$ and $\tilde{x}-2\tilde{x}_je_j$. For the following cases: let $x\in\operatorname{ext}C_{\frac{1}{2}}^4\backslash\{\tilde{x}\}$.

Case 1′: Suppose x and $\tilde{d}$ disagree in sign in at least 1 non-zero coordinate of $\tilde{d}$, but not every non-zero coordinate of $\tilde{d}$. We can use the same reasoning from case 1 to see that $x + \varepsilon\tilde{d}$ lies in a supporting hyperplane for any $\varepsilon > 0$.

Case 2′: Suppose x and $\tilde{d}$ agree in sign in every non-zero coordinate of $\tilde{d}$ and that $I_0^{\tilde{d}} = \{i\}$ for some fixed index $i \in [4]$. By way of contradiction, suppose that $\tilde{d}$ is an illuminating direction of x, then $x + \varepsilon\tilde{d} \in \operatorname{int}\mathbf{B}$ for some $\varepsilon > 0$.

$$x + \varepsilon\tilde{d} = \tfrac{1}{2}\operatorname{sign}(x_i)e_i + (\tfrac{1}{2} + \varepsilon)\tilde{d}, \quad x = \tfrac{1}{2}\operatorname{sign}(x_i)e_i + \tfrac{1}{2}\tilde{d}.$$

Consider the ray defined by $\frac{1}{2}\operatorname{sign}(x_i)e_i + c\tilde{d}$ for $c \geq 0$. Notice that the ray starts at $\frac{1}{2}\operatorname{sign}(x_i)e_i \in \operatorname{int}\mathbf{B}$ and as we increase c, the ray passes through x before it passes through $x + \varepsilon\tilde{d}$. In view of fact 2.18, since $x + \varepsilon\tilde{d}$ was an interior point, x must also be an interior point as well. This contradicts our choice of x. For example, if our vertex is $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $\tilde{d} = (1, 1, 1, 0)$, then notice that $(0, 0, 0, \frac{1}{2})$, x, $x + \varepsilon\tilde{d}$ all lie on the ray: $(0, 0, 0, \frac{1}{2}) + c\tilde{d}$, where $c \geq 0$.

From these cases, at best, we can only illuminate two vertices from $\operatorname{ext} C^4_{\frac{1}{2}}$ at a time using directions from $\mathcal{I}^4_T$. Since $\left|\operatorname{ext} C^4_{\frac{1}{2}}\right| = 16$ we can use at best 8 directions from $\mathcal{I}^4_T$ to illuminate the vertices contributed by $\operatorname{ext} C^4_{\frac{1}{2}}$. Additionally, these directions are all different from those which illuminate the vertices contributed by CP^4_1 and CP^4_1 requires 8 directions from $\mathcal{I}^4_T$ to illuminate. Conversely, the directions which illuminate CP^4_1 cannot illuminate those from $C^4_{\frac{1}{2}}$. At best, we can only illuminate $\mathbf{B}$ using 16 directions from $\mathcal{I}^4_T$. □

Remark 4.3. We will later show that when we perturb the directions from $\mathcal{I}^3_T$ and $\mathcal{I}^4_T$, the polytopes in lemmas 4.1 and 4.2 can be illuminated by 6 (Theorem 6.9) and 8 (lemma 5.5) directions, respectively.

The original statement from Tikhomirov's paper [22] ([22, Lemma 8]) claimed that when $n \geq 2$, $\mathbf{B} \in \mathcal{S}^n$ which satisfies $\|e_i + e_j\|_{\mathbf{B}} > \|e_i\|_{\mathbf{B}}$ for every $i, j \in [n]$, then $\mathbf{B}$ could be illuminated by T_2 from lemma 3.8. However, the examples in lemmas 4.1 and 4.2 satisfy the hypothesis of this statement, yet not the conclusion. In fact, we even showed that we cannot find sufficiently small subsets of $\mathcal{I}^4_T$ which are illuminating sets. This means we must resort to perturbations in these low dimensions. Before going in that direction, let's take a look at why Tikhomirov's method does not hold for the convex body in lemma 4.1.

Proof of lemma 3.8 as given in [22]. Recall $\|e_i\|_{\mathbf{B}} = 1$. Let $x \in \operatorname{ext} \mathbf{B}$ and recall

$$T_2 := \left(\{-1,1\}^{n-1} \times \{0\}\right) \cup \{\pm e_n\}\,.$$

Case 1: $|x_n| > |x_i|$ for all $i \in [n-1]$. In view of lemmas 3.2 and 3.3, for any $\mathbf{n} \in \nu(\mathbf{B}, x)$, we have $\mathbf{n}_n \neq 0$ and $\operatorname{sign}(\mathbf{n}_n) = \operatorname{sign}(x_n)$.

$$\langle \mathbf{n}, -\operatorname{sign}(x_n)e_n \rangle = -\operatorname{sign}(x_n)\mathbf{n}_n = -|\mathbf{n}_n| < 0 \quad \text{for every } \mathbf{n} \in \nu(\mathbf{B}, x).$$

Hence by lemma 2.34, x is illuminated by the direction $-\operatorname{sign}(x_n)\, e_n \in T_2$.

Case 2: There is $j \in [n-1]$ such that $|x_j| \geq |x_i|$ for all $i \in [n]$. Define d in the following way:

$$d_i := \begin{cases} -\operatorname{sign}(x_i), & i \in [n-1] \\ 0, & i = n \end{cases}$$

By definition, $d \in T_2$. If d illuminates x, then we are done. Otherwise, by lemmas 2.34 and 3.2, for some $\mathbf{n} \in \nu(\mathbf{B}, x)$,

$$0 \leq \langle d, \mathbf{n} \rangle = -\sum_{i \in [n-1]} |\mathbf{n}_i|$$

Hence, $\mathbf{n} = \pm e_n$ and $H_x(\mathbf{n})$ supports $\mathbf{B}$. This means that $\|x_n e_n\|_{\mathbf{B}} = 1$. On the other hand, in view of the assumptions of the lemma,

$$\|x\|_{\mathbf{B}} \geq \|x_j e_j + x_n e_n\|_{\mathbf{B}} > \|x_n e_n\|_{\mathbf{B}} = 1$$

which contradicts the choice of $x \in \operatorname{ext} \mathbf{B}$.

□

We now compare with how this proof would play out for the convex body from lemma 4.1. Consider the boundary point $x = e_1$. Notice that this has the outer normal vectors $\frac{(1,1,0)}{\sqrt{2}}$ and $\frac{(1,-1,0)}{\sqrt{2}}$ (we can see this if we view $\mathbf{B}$ from a bird's eye view on the $x - y$ plane).

If we choose $d = (-1, \pm 1, 0)$, then $x + \varepsilon d$ lies on the boundary of the body. Notice that $\mathbf{n} = \frac{(1,\pm 1,0)}{\sqrt{2}} \in \nu(\mathbf{B}, x)$ and $\langle \mathbf{n}, d \rangle = 0$. So it is not necessarily true that $\mathbf{n} = \pm e_n$ as required for the contradiction in case 2 of the previous proof.

We have touched upon a very important subtlety in Tikhomirov's method, that is, when $x \in \partial \mathbf{B}$ and has entries of value 0, it is not clear what $\mathbf{n} \in \nu(\mathbf{B}, x)$ will look like. This makes it difficult to test if a direction d illuminates x using lemma 2.34. This is the major motivation for why we need to use a different method later on.

Notice that if $\mathbf{B} = \operatorname{conv}(\operatorname{ext} C^n_{\frac{1}{2}} \cup \operatorname{ext} CP^n_1)$, then $\mathcal{I}(\mathbf{B}) \leq 2^{n-1} + 2n$. This is because the vertices from the cross-polytope can always be illuminated by the set $\{e_i : i \in [n]\}$ which has cardinality $2n$, and the vertices from the cube can always be illuminated by the set $\{-1, 1\}^{n-1} \times \{0\}$ which has cardinality 2^{n-1} (by using the same idea as in lemma 4.2). Notice

$$n \leq 4 \iff 2^{n-1} + 2n \geq 2^n.$$

So it could be possible that the universal constant to which Tikhomirov referred in his paper is 5. This is one avenue to explore; however, we will instead try to illuminate 1-symmetric and 1-unconditional convex bodies using perturbations of the diretions from $\mathcal{I}^n_T$ and will also investigate possible replacements of the set in Tikhomirov's claim. As is probably clear from these, we will be using his directions as a stepping stone for more results.

5 Main Results for 1-Symmetric Convex Bodies

In this section, we are will be focused on the Illumination Conjecture regarding 1-symmetric convex bodies so let $\mathbf{B} \in \mathcal{S}^n$. It should be noted that conjecture 3.7 does not require the dimension to be high and Tikhomirov's approach to it does not use the condition $1 \neq \mathrm{d}(\mathbf{B}, [-1,1]^n) < 2$. It was discovered that his approach to conjecture 3.7 (lemma 3.8) does not work, as lemmas 4.1 and 4.2 serve as counterexamples. The first subsection of results was done under the assumption that conjecture 3.7 holds. In the second subsection, we first prove some preliminary lemmas. Then we use those lemmas then prove both our first two main results:

1. the Illumination Conjecture for $\mathbf{B} \in \mathcal{S}^n$ short of the equality cases (Theorem 5.18) and
2. proof of conjecture 3.7 (corollary 5.22).

The latter gives Tikhomirov's Theorem 3.1. Then immediately when coupled with the results from the first subsection, we are able to get that for $n = 3$ and $n = 4$, $\mathcal{I}(\mathbf{B}) \leq 7$ (corollary 5.21) and $\mathcal{I}(\mathbf{B}) \leq 15$ (corollary 5.22), respectively.

5.1 Large Distance to the Cube for 1-Symmetric in Dimension 4

Let $\mathbf{B} \in \mathcal{S}^4$. Notice that for any $\mathbf{B} \in \mathcal{S}^4$, we cannot satisfy $\mathrm{d}(\mathbf{B}, [-1,1]^n) > 2$ as $\|e_i\|_{\mathbf{B}} = 1$ forces $\mathbf{B}$ to contain the cross-polytope and $\mathrm{d}(CP_1^4, [-1,1]^n) = 2$. If conjecture 3.7 is correct, then we only need to consider the case when $\mathbf{B}$ satisfies $\|e_i + e_j\|_{\mathbf{B}} = 1$ for every $i, j \in [4]$, $i \neq j$, and $\mathrm{d}(\mathbf{B}, [-1,1]^4) = 2$. It turns out imposing both these conditions on 1-symmetric bodies, will force $\mathbf{B} = C_1^4 \cap CP_2^4$ (lemma 5.7). Such $\mathbf{B}$ can be illuminated by 10 directions from $\mathcal{I}_T^4$ or 8 directions which are perturbations of directions from $\mathcal{I}_T^4$ (lemma 5.4).

Remark 5.1. Any convex body $\mathbf{B} \subset \mathbb{R}^n$ will contain CP_1^n when $\mathbf{B}$ satisfies $\|e_i\|_{\mathbf{B}} = 1$ for all $i \in [n]$.

Notation 5.2. Define the set:

$$S := \{s \in \mathbb{R}^4 : |I_0^s| = 2,\ s_i \in \{-1, 1\}\ for\ i \in [4] \backslash I_0^s\}.$$

This is the set of points that have exactly 2 entries with 1 or -1, and 0 in the remaining entries. For example $(1,0,0,1),(-1,0,1,0) \in S$.

Lemma 5.3.

$$\operatorname{conv}(S) = C_1^4 \cap CP_2^4$$

Click here for a picture.

Proof. Consider the supporting hyperplanes of CP_2^4 defined by

$$\pm\xi_1 \pm \xi_2 \pm \xi_3 \pm \xi_4 = 2, \quad (\xi_1,\xi_2,\xi_3,\xi_4) \in \mathbb{R}^4.$$

Let $s \in S$, and $s_i, s_j \in \{-1,1\}$, where $i,j \in [4] = \{i,j,k,l\}$. So $\{k,l\} = I_0^s$.

$$\operatorname{sign}(s_i)s_i + \operatorname{sign}(s_j)s_j \pm s_k \pm s_l = |s_i| + |s_j| = 2.$$

So every element of S lies in the negative half-space of every supporting hyperplane of CP_2^4 which means $S \subset CP_2^4$. Also since $S \subset [-1,1]^4$,

$$S \subset \underbrace{C_1^4 \cap CP_2^4}_{\text{convex}} \implies \operatorname{conv}(S) \subset C_1^4 \cap CP_2^4.$$

Now, we will show that $\operatorname{ext}(CP_1^4 \cap CP_2^4) = S$. Note that $C_1^4 \cap CP_2^4$ can also be described by the hyperplanes which support C_1^4 and CP_2^4. For fixed $i \in [4]$, the following are both supporting hyperplanes of C_1^4:

$$\{(\xi_1,\xi_2,\xi_3,\xi_4) \in \mathbb{R}^4 : \xi_i = 1\}, \quad \{(\xi_1,\xi_2,\xi_3,\xi_4) \in \mathbb{R}^4 : \xi_i = -1\}.$$

Let's look at $(1,1,0,0)$; does there exist $x = (x_1,x_2,x_3,x_4)$, $y = (y_1,y_2,y_3,y_4) \in \mathbb{R}^n \backslash S$ which satisfy

$$\pm\xi_1 \pm \xi_2 \pm \xi_3 \pm \xi_4 \leq 2 \quad \text{and} \quad \|(\xi_1,\xi_2,\xi_3,\xi_4)\|_\infty \leq 1$$

so that $\lambda x + (1-\lambda)y = (1,1,0,0)$ for $0 < \lambda < 1$? Since $x_1, x_2, y_1, y_2 \leq 1$, it must be the case that $x_1 = x_2 = y_1 = y_2 = 1$; however, this also forces $x_3 = x_4 = y_3 = y_4 = 0$, and thus

$(1,1,0,0) = x = y$ which contradicts the choice of x and y. Since $(1,1,0,0) \in \text{ext}(C_1^4 \cap CP_2^4)$, by 1-symmetry, it must be the case that $S \subset \text{ext}(C_1^4 \cap CP_2^4)$.

For the other inclusion, by way of contradiction, suppose there exists $x \in \text{ext}(C_1^4 \cap CP_2^4) \backslash S$. WLOG, assume that $x_i \in [0,1]$ for all $i \in [4]$. Moreover, since S does not favor any entries, WLOG again, assume $x_1 \leq x_2 \leq x_3 \leq x_4$. In the case $x_4 \geq x_1 + x_2 + x_3$, it follows that $x_1 + x_2 + x_3 \leq 1$ so

$$(x_1, x_2, x_3, 1) = x_1(1,0,0,1) + x_2(0,1,0,1) + x_3(0,0,1,1) + (1 - (x_1 + x_2 + x_3))(0,0,0,1),$$

which contradicts the choice of x. Suppose $x_4 < x_1 + x_2 + x_3$. Solve the following system of equations:

$$\begin{aligned}
&(x_1, x_2, x_3, x_4) \\
&= \lambda_1(1,1,0,0) + \lambda_2(0,0,1,1) + \lambda_3(1,0,0,1) + \lambda_4(0,1,1,0) + \lambda_5(1,0,1,0) + \lambda_6(0,1,0,1) \\
&= (\lambda_1 + \lambda_3 + \lambda_5, \lambda_1 + \lambda_4 + \lambda_6, \lambda_2 + \lambda_4 + \lambda_5, \lambda_2 + \lambda_3 + \lambda_6).
\end{aligned}$$

If we set $\lambda_1 = \lambda_3 = 0$, then

$$\begin{aligned}
&(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \\
&= \tfrac{1}{2}(0, -x_1 - x_2 + x_3 + x_4, 0, -x_1 + x_2 + x_3 - x_4, 2x_1, x_1 + x_2 - x_3 + x_4)
\end{aligned}$$

Notice that by the assumption $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4$, we automatically get $\lambda_2, \lambda_5, \lambda_6 \geq 0$. Also note that in the case $x_4 - x_3 \leq x_2 - x_1$ then $\lambda_4 \geq 0$. In addition, $\sum_{i=1}^{6} \lambda_i = \frac{1}{2} \sum_{i=1}^{4} x_i \leq 1$. So if $x_4 - x_3 \leq x_2 - x_1$, then

$$x = \underbrace{\lambda_2(0,0,1,1) + \lambda_4(0,1,1,0) + \lambda_5(1,0,1,0) + \lambda_6(0,1,0,1) + \left(1 - \sum_{i=1}^{6} \lambda_i\right) \vec{0}}_{\text{convex combination}}.$$

This contradicts the choice of x. If $x_4 - x_3 > x_2 - x_1$, set $\lambda_1 = \lambda_4 = 0$, then

$$\begin{aligned}&(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)\\&= \frac{1}{2}(0, -x_1 - x_2 + x_3 + x_4, x_1 - x_2 - x_3 + x_4, 0, x_1 + x_2 + x_3 - x_4, 2x_2).\end{aligned}$$

Notice that $\lambda_2, \lambda_6 \geq 0$. Since $x_1 + x_2 + x_3 > x_4$, we get $\lambda_5 > 0$. Also, since $x_4 - x_3 > x_2 - x_1$, $\lambda_3 > 0$. Finally, we again have $\sum_{i=1}^{6} \lambda_i = \frac{1}{2} \sum_{i=1}^{4} x_i \leq 1$. Then

$$x = \underbrace{\lambda_2(0,0,1,1) + \lambda_3(1,0,0,1) + \lambda_5(1,0,1,0) + \lambda_6(0,1,0,1) + \left(1 - \sum_{i=1}^{6} \lambda_i\right) \vec{0}}_{\text{convex combination}}.$$

This also contradicts the choice of x. Hence, it must be the case that $\text{ext}(C_1^4 \cap CP_2^4) \subset S$. $\square$

Lemma 5.4. $C_1^4 \cap CP_2^4$ can be illuminated by 10 directions from $\mathcal{I}_T^4$. We can show that $\mathcal{I}(\mathbf{B}) \leq 8$ if we allow for perturbations.

Proof.

Method 1: We restrict ourselves to using a subset of $\mathcal{I}_T^4$. Consider vertex $(1,0,0,1)$; we can illuminate this with $(-1,1,0,-1)$ as

$$(1,0,0,1) + \tfrac{1}{2}(-1,1,0,-1) = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0, \tfrac{1}{2}\right).$$

This is because $(\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}) \in \operatorname{int} C_1^4$ and $(\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}) \in \operatorname{int} CP_2^4$. In the table below you can see that we can use this reasoning to illuminate $C_1^4 \cap CP_2^4$ in using only 10 directions.

Vertices			Illuminating direction
$(1,0,0,1)$,	$(1,-1,0,0)$,	$(0,-1,0,1)$	$(-1,1,0,-1)$
$(0,0,1,1)$,	$(0,1,1,0)$,	$(0,1,0,1)$	$(0,-1,-1,-1)$
$(1,0,1,0)$,	$(1,0,0,-1)$,	$(0,0,1,-1)$	$(-1,0,-1,1)$
$(1,1,0,0)$,	$(1,0,-1,0)$,	$(0,1,-1,0)$	$(-1,-1,1,0)$
$(0,1,0,-1)$,	$(-1,0,0,-1)$,	$(-1,1,0,0)$	$(1,-1,0,1)$
$(0,-1,0,-1)$,	$(0,0,-1,-1)$,	$(0,-1,-1,0)$	$(0,1,1,1)$
$(-1,0,-1,0)$,	$(-1,-1,0,0)$		$(1,1,1,0)$
$(-1,0,0,1)$,	$(-1,0,1,0)$		$(1,0,-1,-1)$
$(0,-1,1,0)$			$(0,1,-1,-1)$
$(0,0,-1,1)$			$(0,-1,1,-1)$

Table 1: Illuminating $C_1^4 \cap CP_2^4$ using a subset of $\mathcal{I}_T^4$.

Method 2: We allow for perturbations of directions from $\mathcal{I}_T^4$. Again consider the direction $(1,0,0,1)$. For small $\varepsilon > 0$, the direction $(-1-\delta,\ -1,\ -1,\ -1)$ illuminates this direction as

$$(1,0,0,1) + \tfrac{1}{2}(-1-\delta,\ -1,\ -1,\ -1) = \left(\tfrac{1}{2} - \tfrac{\delta}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{2}\right) \in \operatorname{int} C_1^4 \cap CP_2^4.$$

In the table below, we can use this reasoning to illuminate $C_1^4 \cap CP_2^4$ while using only 8 directions (maybe an interested reader can come up with an even smaller number of directions needed, but 8 is good enough for the purposes of this thesis).

Vertices			Illuminating direction
$(1,0,1,0)$,	$(1,1,0,0)$,	$(1,0,0,1)$	$(-1-\delta,-1,-1,-1)$
$(0,0,1,1)$,	$(0,1,1,0)$,	$(0,1,0,1)$	$(-1+\delta,-1,-1,-1)$
$(0,1,0,-1)$,	$(0,0,1,-1)$,	$(1,0,0,-1)$	$(-1,-1,-1,1+\delta)$
$(0,1,-1,0)$,	$(1,0,-1,0)$,	$(0,0,-1,1)$	$(-1,-1,1+\delta,-1)$
$(0,-1,1,0)$,	$(1,-1,0,0)$,	$(0,-1,0,1)$	$(-1,1+\delta,-1,-1)$
$(-1,0,0,1)$,	$(-1,0,1,0)$,	$(-1,1,0,0)$	$(1+\delta,-1,-1,-1)$
$(-1,0,-1,0)$,	$(-1,-1,0,0)$,	$(-1,0,0,-1)$	$(1+\delta,1,1,1)$
$(0,0,-1,-1)$,	$(0,-1,0,-1)$,	$(0,-1,-1,0)$	$(1-\delta,1,1,1)$

Table 2: Illuminating $C_1^4 \cap CP_2^4$ using perturbed vectors from a subset of $\mathcal{I}_T^4$.

□

Lemma 5.5. The convex body in lemma 4.2 can be illuminated with 8 directions.

Proof. Observe that

$$2\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \frac{1}{2} & -\frac{1}{2}\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}(\operatorname{ext} C_{\frac{1}{2}}^4 \cup \operatorname{ext} CP_1^4) = S.$$

Recall by lemma 5.4 that we can illuminate $\operatorname{conv}(S)$ using only 8 directions, hence we can illuminate $\operatorname{conv}(\operatorname{ext} C_{\frac{1}{2}}^4 \cup \operatorname{ext} CP_1^4)$ by 8 directions.

□

Remark 5.6. Recall also that in lemma 5.4 we showed that $\operatorname{conv}(S)$ can be illuminated by 10 directions from $\mathcal{I}_T^4$. So doesn't that mean we can illuminate $\operatorname{ext} C_{\frac{1}{2}}^4 \cup \operatorname{ext} CP_1^4$ by 10 directions from $\mathcal{I}_T^4$ as well? Wouldn't this contradict our counterexample (lemma 4.2)?

There is a subtlety here. Observe that

$$\left(2\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \frac{1}{2} & -\frac{1}{2}\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\right)^{-1} = \begin{pmatrix}1/2 & 1/2 & 0 & 0\\ -1/2 & 1/2 & 0 & 0\\ 0 & 0 & 1/2 & 1/2\\ 0 & 0 & -1/2 & 1/2\end{pmatrix}.$$

So,

$$\begin{pmatrix}1/2 & 1/2 & 0 & 0\\ -1/2 & 1/2 & 0 & 0\\ 0 & 0 & 1/2 & 1/2\\ 0 & 0 & -1/2 & 1/2\end{pmatrix} S = \operatorname{ext} C^4_{\frac{1}{2}} \cup \operatorname{ext} CP^4_1.$$

Recall that one of the directions we used in lemma 5.4 to illuminate $\operatorname{conv}(S)$ was $(-1, 1, 0, -1)$. Observe that

$$\begin{pmatrix}1/2 & 1/2 & 0 & 0\\ -1/2 & 1/2 & 0 & 0\\ 0 & 0 & 1/2 & 1/2\\ 0 & 0 & -1/2 & 1/2\end{pmatrix}\begin{pmatrix}-1\\ 1\\ 0\\ 1\end{pmatrix} = \begin{pmatrix}0\\ 1\\ -\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}.$$

The problem here is that $(0, 1, -\frac{1}{2}, -\frac{1}{2})$ is not a scalar multiple of any element from $\mathcal{I}^4_T$. So being able to illuminate $\operatorname{conv}(S)$ with 10 directions from $\mathcal{I}^4_T$ does not imply that we can illuminate $\operatorname{conv}(\operatorname{ext} C^4_{\frac{1}{2}} \cup \operatorname{ext} CP^4_1)$ by 10 directions from $\mathcal{I}^4_T$ as well.

Lemma 5.7. If $\mathbf{B} \in \mathcal{S}^4$ satisfies the following conditions:

1. $\|e_i + e_j\|_{\mathbf{B}} = 1$ for every $i, j \in [4]$, $i \neq j$ and
2. $\mathrm{d}(\mathbf{B}, [-1, 1]^4) \geq 2$,

then $\mathbf{B} = C^4_1 \cap CP^4_2$ and $\mathrm{d}(\mathbf{B}, [-1, 1]^4) = 2$.

Recall that we impose $\|e_i\|_{\mathbf{B}} = 1$ for every $i \in [4]$. If we do not impose this condition and we adapt the first condition in the statement of the above lemma to be $\|e_i + e_j\| = \|e_i\|_{\mathbf{B}}$, $\mathbf{B}$ will instead be a scalar multiple of $C_1^4 \cap CP_2^4$.

Proof. Recall

$$\mathrm{d}(C_1^4 \cap CP_2^4, [-1,1]^4) = \|e_1 + e_2 + e_3 + e_4\|_{C_1^4 \cap CP_2^4}.$$

Notice $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ lies in the supporting hyperplane

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 + \xi_2 + \xi_3 + \xi_4 = 2\}.$$

$$\mathrm{d}(C_1^4 \cap CP_2^4, [-1,1]^4) = \|e_1 + e_2 + e_3 + e_4\|_{C_1^4 \cap CP_2^4} = \|(1,1,1,1)\|_{C_1^4 \cap CP_2^4} = \frac{1}{\frac{1}{2}} = 2$$

Recall $\|e_i\|_{\mathbf{B}} = 1$. Consider any $\mathbf{B} \in \mathcal{S}^4$ which contains, but is not equal to $C_1^4 \cap CP_2^4$, (so we satisfy the condition $\|e_i + e_j\|_{\mathbf{B}} = 1$ for every $i, j \in [4]$, $i \neq j$). We will show that $\mathrm{d}(\mathbf{B}, [-1,1]^4) < 2$. There exists some point $x \in \mathbf{B}$ which must lie outside CP_2^4 but inside the unit cube. WLOG, consider the case, where $x_i \geq 0$ for $i \in [4]$. By the 1-symmetry condition, we can create at most 4! distinct points that reside in our current octant and in $\mathbf{B}$ by rearranging the entries of x. They all satisfy

$$x_1 + x_2 + x_3 + x_4 = C > 2.$$

We impose $C > 2$ because otherwise these points lie on CP_2^4. Consider the convex combination

$$\tfrac{1}{4}(x_1, x_2, x_3, x_4) + \tfrac{1}{4}(x_2, x_3, x_4, x_1) + \tfrac{1}{4}(x_3, x_4, x_1, x_2) + \tfrac{1}{4}(x_4, x_1, x_2, x_3) = \tfrac{1}{4}(C, C, C, C),$$

which by convexity, is contained in $\mathbf{B}$.

$$\mathrm{d}(\mathbf{B}, [-1,1]^4) = \|e_1 + e_2 + e_3 + e_4\|_{\mathbf{B}} = \|(1,1,1,1)\|_{\mathbf{B}} \leq \frac{1}{\frac{C}{4}} < 2,$$

which is what we wanted ∎

It turns out we do not even need the 1-symmetry condition in lemma 5.7, but 1-symmetry simplifies the proof of lemma 5.7 significantly. In lemma 5.9, we can weaken the 1-symmetry condition to 1-unconditional and get the same result.

Remark 5.8. By adjusting the proof in lemma 5.7, we can see that for $\mathbf{B} \in \mathcal{S}^n$ which satisfy $\|e_i + e_j\|_{\mathbf{B}} = 1$ for every $i, j \in [n]$, $i \neq j$ (meaning $(\frac{2}{n}, ..., \frac{2}{n}) \in \mathbf{B}$), we get:

$$\mathrm{d}(\mathbf{B}, [-1,1]^4) = \|e_1 + ... + e_n\|_{\mathbf{B}} \leq \frac{1}{\frac{2}{n}} = \frac{n}{2}.$$

In the case of $n = 3$, we get $\mathrm{d}(\mathbf{B}, [-1,1]^3) \leq \frac{3}{2} < 2$.

If conjecture 3.7 is true, we can couple it with lemma 3.4, giving that any 1-symmetric convex body $\mathbf{B} \in \mathcal{S}^3$ which satisfies $1 \neq \mathrm{d}(\mathbf{B}, [-1,1]^3) < 2$, can be illuminated by 7 directions or less. If $\mathrm{d}(\mathbf{B}, [-1,1]^3) \geq 2$, then by the previous remark, $\|e_i + e_j\|_{\mathbf{B}} > 1$ for every $i, j \in [3]$. Conjecture 3.7 states that in this case, $\mathbf{B}$ can be illuminated by 6 directions. Thus, if we prove conjecture 3.7, we will immediately get that for any $\mathbf{B} \in \mathcal{S}^3$, $\mathcal{I}(\mathbf{B}) < 7$.

Although this next lemma is the generalization of lemma 5.7, the proof for lemma 5.9 gives geometric intuition of why the 1-symmetric condition is much stronger and easier to work with than the 1-unconditional condition.

Lemma 5.9. If $\mathbf{B} \in \mathcal{U}^4$ satisfies the following conditions:

1. $\|e_i + e_j\|_{\mathbf{B}} = 1$ for every $i, j \in [4]$, $i \neq j$ and
2. $\mathrm{d}(\mathbf{B}, [-1,1]^4) \geq 2$,

then $\mathbf{B} = C_1^4 \cap CP_2^4$ and $\mathrm{d}(\mathbf{B}, [-1,1]^4) = 2$.

Proof. By way of contradiction, consider any 1-unconditional convex body $\mathbf{B} \subset \mathcal{U}^4$ which contains, but is not equal to $C_1^4 \cap CP_2^4$. So, there must exist some point x which must lie outside CP_2^4 but on C_1^4. With the 1-unconditional condition in mind, WLOG, we let $x = (x_1, x_2, x_3, x_4) \in \mathbf{B}$ such that $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1$ and $x_1 + x_2 + x_3 + x_4 = C > 2$. Geometrically this point is located nearest to the plane defined by $x_4 = 1$;

therefore, we should use the points $(1,1,0,0)$, $(1,0,1,0)$, $(0,1,1,0) \in \mathbf{B}$ to create some convex combinations $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) \in \mathbf{B}$ such that $\tilde{x}_i = \tilde{x}_j > \frac{1}{2}$ for every $i, j \in [4]$, $i \neq j$.

$$\begin{aligned}
&(x_2 - x_1)(1,0,1,0) + (x_1, x_2, x_3, x_4) \\
&\qquad = (x_2,\, x_2,\, x_3 + x_2 - x_1,\, x_4) \\
&(x_3 - x_1)(1,1,0,0) + (x_2,\, x_2,\, x_3 + x_2 - x_1,\, x_4) \\
&\qquad = (x_3 + x_2 - x_1,\, x_3 + x_2 - x_1,\, x_3 + x_2 - x_1,\, x_4)
\end{aligned}$$

Positive scalar combinations can always be scaled into convex combinations. Hence, we can denote $y = (y_1, y_1, y_1, y_4) \in \mathbf{B}$ to be the scaling of the latter vector above, so that

$$y = \lambda_1(1,1,0,0) + \lambda_2(1,0,1,0) + \lambda_3(x_1, x_2, x_3, x_4) \in \mathbf{B}, \quad \sum_{i \in [3]} \lambda_i = 1, \quad 0 < \lambda_i < 1.$$

Then

$$\sum_{i \in [4]} y_i = 2\lambda_1 + 2\lambda_2 + \lambda_3 \sum_{i \in [4]} x_i > 2.$$

If $y_4 = y_1$, we are done. So, consider $y_1 > y_4$. Since $(1,0,0,1)$, $(0,1,0,1)$, $(0,0,1,1) \in \mathbf{B}$,

$$\tfrac{1}{3}(1,0,0,1) + \tfrac{1}{3}(0,1,0,1) + \tfrac{1}{3}(0,0,1,1) = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 1\right) \in \mathbf{B}.$$

Consider the positive scalar combination

$$\mu_1 \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 1\right) + (y_1, y_1, y_1, y_4),$$

which satisfies

$$y_1 + \tfrac{1}{3}\mu_1 = y_4 + \mu_1 > 0.$$

Since $0 < y_4 < y_1$ and $\mu_1 > 0$, there exists $0 < \mu_2 < 1$ so that

$$\mu_2\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 1\right) + (1-\mu_2)(y_1, y_1, y_1, y_4) \in \mathbf{B},$$

where

$$(1-\mu_2)y_1 + \mu_2\tfrac{1}{3} = (1-\mu_2)y_4 + \mu_2.$$

In fact, solving for μ_2 gives $\mu_2 = \frac{\mu_1}{1+\mu_1}$. Setting $\tilde{x}$ to be the above vector gives the claim as $\sum_{i\in[4]} y_i > 2$. Now consider the case $y_1 < y_4$. This time, we consider the vectors $(1,1,0,0), (1,0,1,0)$ and $(0,1,1,0)$ and proceed similarly to get $\tilde{x} \in B$ which satisfies our desired conditions.

Since $\mathbf{B} \subset [-1,1]^4$, notice $\mathrm{d}(\mathbf{B}, [-1,1]^4) \leq \|(1,1,1,1)\|_{\mathbf{B}} \leq \frac{1}{\tilde{x}_1} < 2$ which contradicts our choice of $\mathbf{B}$.

□

5.2 Main Results for 1-Symmetric Convex Bodies.

The first main result of this subsection is Theorem 5.18 which proves the Illumination Conjecture short of the equality cases for 1-symmetric convex bodies. Furthermore, we have proven conjecture 3.7, which gives not only Theorem 3.1, but also the Illumination Conjecture for 1-symmetric bodies in dimensions 3 and 4 (corollaries 5.21 and 5.22) when coupled with the results from the previous subsections.

Lemma 5.10. Let $n \geq 3$, $\mathbf{B} \in \mathcal{S}^n$ and $x \in \partial\mathbf{B}$. If there exists $i \in [n]$ so that $|x_i| > |x_j|$ for $j \in [n]\backslash\{i\}$. Then we can illuminate x using $\{\pm e_i\}$.

It should be noted that Tikhomirov's proof for the first case of lemma 3.8 (see proof from section 4) can also be generalized to this statement. The proof above is simply an alternative proof for this result. We give this alternative result to emphasize on the fact that we use the definition of illumination, rather than its equivalent formulation (lemma 2.34).

Proof. Let $x \in \partial\mathbf{B}$ and WLOG, in the above setting, let $i = 1$, so $|x_2|, ..., |x_n| < 1$. Using the 1-symmetry of $\mathbf{B}$, consider the following convex combination:

$$\frac{1}{n-1}[(|x_2|, |x_1|, |x_3|, ..., |x_n|) + (|x_3|, |x_2|, |x_1|, |x_4|, ..., |x_n|) + ...$$
$$... + (|x_n|, |x_2|, ..., |x_{n-1}|, |x_1|)] \in \mathbf{B}.$$

In the above convex combination, the j^{th} vector swaps the first coordinate and the $j+1$ coordinate of $(|x_1|, ..., |x_n|)$ for every $j \in [n]\backslash\{1\}$. The first coordinate of the convex combination is $\frac{1}{n-1}\sum_{k=2}^{n}|x_k| \geq 0$. The j^{th} coordinate is

$$\frac{1}{n-1}[(n-2)|x_j| + |x_1|] = \frac{n-2}{n-1}|x_j| + \frac{1}{n-1}|x_1|.$$

Since $n \geq 3$, we know that $0 < \frac{n-2}{n-1}, \frac{1}{n-1} < 1$. Additionally, $\frac{n-2}{n-1} + \frac{1}{n-1} = 1$ and $|x_1| > |x_j|$, so

$$\frac{1}{n-1}[(n-2)|x_j| + |x_1|] > |x_j|.$$

If there exists $j \in [n]\backslash\{1\}$ so that $x_j \neq 0$, then $\frac{1}{n-1}\sum_{k=2}^{n}|x_k| > 0$, so we can use lemma 2.38 to see $-\operatorname{sign}(x_1)e_1$ illuminates x. On the other hand, if $x_j = 0$ for every $j \in [n]\backslash\{1\}$, then we again see that $-\operatorname{sign}(x_1)e_1$ illuminates x. □

The lemma just below is not needed in what follows (it was being used in a previous version of the thesis), however it still seems worth mentioning to motivate the sets used in proposition 5.13 and in the results following from that proposition.

Lemma 5.11. Let $n \geq 3$, $\mathbf{B} \in \mathcal{S}^n$, and $x \in \partial\mathbf{B}$. Consider $k \in [n]$ such that $x_k \neq 0$. Then for some $\delta > 0$ (in fact, δ small enough compared to $|x_k|$ and to the dimension n), x is illuminated by

$$-\operatorname{sign}(x_k)e_k + \sum_{i\in[n]\backslash\{k\}} -\delta\operatorname{sign}(x_i)e_i$$

(here, if $x_i = 0$, we simply make a choice for $\operatorname{sign}(x_i)$, setting it equal to either $+1$ or -1; no matter the choice of signs for these coordinates, the conclusion remains the same).

Proof. Let $x \in \partial \mathbf{B}$ and fix $k \in [n]$ such that $x_k \neq 0$. WLOG, suppose $k = 1$ and $|x_i| \geq |x_{i+1}|$ for $2 \leq i \leq n-1$ (we can always reorder these indices in this argument if this is not the case). Consider the case when $|I_0^x| = 0$. Let $0 < \delta < \frac{|x_{n-1}|}{|x_1|}$, and define

$$d := (-\operatorname{sign}(x_1), -\operatorname{sign}(x_2)\delta, ..., -\operatorname{sign}(x_n)\delta, 0).$$

Consider

$$x + x_1 d = (0, x_2 - x_1 \operatorname{sign}(x_2)\delta, ..., x_{n-1} - x_1 \operatorname{sign}(x_n)\delta, x_n)$$

Compare this vector with $(|x_{n-1}|, |x_1|, ..., |x_{n-2}|, x_n) \in \mathbf{B}$. Since $|x_{n-1}| > 0$ and for $i \in [n-2]$ we have

$$|x_i| \geq |x_{i+1}| > ||x_i| - \delta x_1| = |x_i - x_1 \operatorname{sign}(x_i)\delta|.$$

In view of lemma 2.42, look in the affine set

$$\{(\xi_1, ..., \xi_n) \in \mathbb{R}^n : \xi_n = x_n\}$$

to deduce that we can illuminate x using d. This proof also works when $x_n = 0$. If $x_2 = 0$, then $x_1 \neq 0$ by the assumptions in the lemma and at the beginning of the proof. In view of the proof of lemma 2.45, we can see that $-\operatorname{sign}(x_1)e_1$ illuminates x. Then we can use lemma 2.41 to get the claim. So, let's suppose $x_m = 0$ and $x_{m-1} \neq 0$ for some $m > 2$. Let $0 < \delta < \min(\frac{|x_{m-1}|}{|x_1|}, \frac{1}{n-m})$ and $d := (-\operatorname{sign}(x_1), -\operatorname{sign}(x_2)\delta, ..., -\operatorname{sign}(x_n)\delta, 0)$. We will show that $x + \varepsilon d \in \operatorname{int} \mathbf{B}$ by comparing it with the convex combination

$$\begin{aligned}
&\frac{1}{n-m}[(0, x_2, ..., x_{m-1}, x_1, 0, ..., 0, x_n) + (0, x_2, ..., x_{m-1}, 0, x_1, 0, ..., 0, x_n) + ... \\
&\qquad ... + (0, x_2, ..., x_{m-1}, 0, ..., 0, x_1, x_n)] \\
&= (0, x_2, ..., x_{m-1}, \frac{1}{n-m}x_1, ..., \frac{1}{n-m}x_1, x_n) \in \mathbf{B}.
\end{aligned}$$

With lemma 2.42 in mind, look in the affine set

$$\{(\xi_1, ..., \xi_n) \in \mathbb{R}^n : \xi_1 = 0,\ \xi_n = x_n\}.$$

Notice that for $i \in [m-1]\backslash\{1\}$, we have $|x_i| > ||x_i| - x_1\delta|$ and for $i \in [n-1]\backslash[m-1]$, we have $\left|\frac{1}{n-m}x_1\right| > |\delta x_1|$. Thus, $x + \varepsilon d \in \operatorname{int}\mathbf{B}$. Let $x \in \operatorname{ext}\mathbf{B}$ such that $|x_n| < 1$, and WLOG suppose that $k = 1$, that is, the first coordinate of x is non-zero. For simplicity in notation, we assume that $|x_i| \geq |x_{i+1}|$ for $i \in [n-1]\backslash\{1\}$ (we can always reorder these indices in this argument if this is not the case). We first deal with the case where $x_i \neq 0$ for all other $i \in [n-1]$ as well, that is, $I_0^x \subseteq \{n\}$. Let $0 < \delta < \frac{|x_{n-1}|}{|x_1|}$, and define $d := (-\operatorname{sign}(x_1), -\operatorname{sign}(x_2)\delta, ..., -\operatorname{sign}(x_{n-1})\delta, 0)$. Consider

$$x + |x_1|d = (0,\ x_2 - |x_1|\operatorname{sign}(x_2)\delta,\ ...,\ x_{n-1} - |x_1|\operatorname{sign}(x_{n-1})\delta,\ x_n).$$

Compare this vector with the following one, which is found in $\mathbf{B}$ because $\mathbf{B} \in \mathcal{S}^n \subset \mathcal{U}^n$:

$$(0, |x_2|, ..., |x_{n-1}|, x_n).$$

Both the above point and $x + |x_1|d$ are found in the affine set

$$\{(\xi_1, ..., \xi_n) \in \mathbb{R}^n : \xi_1 = 0,\ \xi_n = x_n\}$$

which also contains $x_n e_n \in \operatorname{int}\mathbf{B}$. Since $\delta < \frac{|x_{n-1}|}{|x_1|} \leq \frac{|x_i|}{|x_1|}$ for all $i \in [n-1]\backslash\{1\}$, we have that

$$|x_i - |x_1|\operatorname{sign}(x_i)\delta| = ||x_i| - |x_1|\delta| = |x_i| - |x_1|\delta < |x_i|,$$

and thus $x + |x_1|d$ is in the relative interior of the above affine set. We can then apply lemma 2.42 to conclude that $x_1 + |x_1|d \in \operatorname{int}\mathbf{B}$.

In the more general case, assume that

$$|x_2| \geq |x_3| \geq ...|x_{m-1}| > |x_m| = 0 = |x_{m+1}| = ... = |x_{n-1}|$$

for some $m > 2$ (the case $x_i = 0$ for all $i \in [n-1]\backslash\{1\}$ is analogous and slightly simpler).

Again, we will consider the point $x + |x_1|d$ with d of the above form for a suitable $\delta > 0$, and we will try to find another point in $\mathbf{B}$ for coordinate comparison.

Since $\mathbf{B} \in \mathcal{S}^n$, the following is a convex combination of points in $\mathbf{B}$:

$$\begin{aligned}
&\frac{1}{n-m}[(0, x_2, ..., x_{m-1}, x_1, 0, ..., 0, x_n) + (0, x_2, ..., x_{m-1}, 0, x_1, 0, ..., 0, x_n) + ... \\
&\qquad ... + (0, x_2, ..., x_{m-1}, 0, ..., 0, x_1, 0, x_n) + (0, x_2, ..., x_{m-1}, 0, ..., 0, x_1, x_n)] \\
&= (0,\ x_2,\ ...,\ x_{m-1},\ \frac{1}{n-m}x_1\ , ...,\ \frac{1}{n-m}x_1\ ,\ \frac{1}{n-m}x_1,\ x_n) \in \mathbf{B}.
\end{aligned}$$

Let $0 < \delta < \min(\frac{|x_{m-1}|}{|x_1|}, \frac{1}{n-m})$. Notice that for $i \in [m-1]\backslash\{1\}$, we have

$$|x_i| > ||x_i| - |x_1|\,\delta| = |x_i - |x_1|\operatorname{sign}(x_i)\delta|$$

and for $i \in [n-1]\backslash[m-1]$, we have

$$\left|\frac{1}{n-m}x_1\right| > |\delta x_1|\,.$$

In view of lemma 2.42, look in the affine set

$$\{(\xi_1, ..., \xi_n) \in \mathbb{R}^n : \xi_1 = 0,\ \ \xi_n = x_n\}$$

to conclude that $x + |x_1|\,d \in \operatorname{int}\mathbf{B}$, and thus d illuminates x. □

The following lemma is a stronger version of the lemma we just stated, in that it allows δ to depend only on the dimension n.

Lemma 5.12. Let $n \geq 3$, $\mathbf{B} \in \mathcal{S}^n$. Consider $k \in [n]$ such that $x_k \neq 0$. Then for any $\delta \in (0, \frac{1}{n})$, x is illuminated by

$$-\operatorname{sign}(x_k)e_k + \sum_{i \in [n]\backslash\{k\}} -\delta\operatorname{sign}(x_i)e_i$$

(here, if $x_i = 0$, we simply make a choice for $\operatorname{sign}(x_i)$, setting it equal to either $+1$ or -1; no matter the choice of signs for these coordinates, the conclusion remains the same).

Proof. Denote the direction in the statement of the lemma by d. WLOG, suppose $k = 1$ and also that $|x_i| \geq |x_{i+1}|$ for $2 \leq i \leq n-1$ (we can always reorder the indices in this argument if this is not the case; note also that the second assumption does not hinge on whether $|x_1| = \|x\|_\infty$ or not, we simply order the remaining coordinates of x).

If $|I_0^x| = 0$, then the conclusion follows immediately by lemma 2.38 (note that $x + \varepsilon d$ will have strictly smaller respective entries in absolute value compared to x, as long as $\varepsilon > 0$ is sufficiently small).

Thus we focus on the case where

$$|x_2| \geq |x_3| \geq \cdots \geq |x_{m-1}| \ > \ |x_m| = 0 = |x_{m+1}| = \cdots = |x_n|$$

for some $2 \leq m \leq n$. Let us examine the entries of $x + |x_1|d$:

- $(x + |x_1|d)_1 = 0$;
- for every $2 \leq i \leq m-1$, $|(x + |x_1|d)_i| = \max(|x_i| - |x_1|\delta,\ |x_1|\delta - |x_i|)$ (and the maximum should probably be equal to the first argument, unless $|x_i|$ is very small);
- for every $m \leq i \leq n$, $|(x + |x_1|d)_i| = |x_1|\delta$.

Let m' be the smallest index ≥ 2 such that $|x_{m'}| < \frac{1}{n}|x_1|$. We consider the following convex combination, which will be in $\mathbf{B}$ because of the 1-symmetry:

$$\frac{1}{n-m'+2}\Big[(|x_1|, |x_2|, |x_3|, \ldots, |x_{n-1}|, |x_n|) \ + \ (|x_{m'}|, |x_2|, |x_3|, \ldots, |x_{m'-1}|, |x_1|, \ldots, |x_{n-1}|, |x_n|) \ + \\ + (|x_{m'+1}|, |x_2|, |x_3|, \ldots, |x_{m'-1}|, |x_{m'}|, |x_1|, \ldots, |x_{n-1}|, |x_n|) \ + \\ + \cdots\cdots + (|x_{n-1}|, |x_2|, |x_1|, \ldots, |x_1|, |x_n|) + (|x_n|, |x_2|, |x_1|, \ldots, |x_{n-1}|, |x_1|)\Big].$$

We can observe the following regarding its entries:

- its 1st entry is $\frac{1}{n-m'+2}\left(|x_1| + \sum_{i=m'}^{n} |x_i|\right) \geq \frac{|x_1|}{n-m'+2} > 0$.
- For every $2 \leq i \leq m'-1$, its i^{th} entry is equal to $|x_i|$.

- For every $m' \le i \le m-1$, its i^{th} entry is equal to

$$\frac{n-m'+1}{n-m'+2}|x_i| + \frac{1}{n-m'+2}|x_1| \ge \frac{1}{n-m'+2}|x_1|.$$

- For every $m \le i \le n$, its i^{th} entry is equal to $\frac{1}{n-m'+2}|x_1|$.

Thus, as long as we have chosen $\delta < \frac{1}{n}$, we will have that this convex combination has strictly larger respective entries in absolute value compared to $x + |x_1|d$. Indeed, this is immediately clear for $i = 1$ or $i \ge m$. For $2 \le i \le m'-1$, we have that $|x_i| \ge \frac{1}{n}|x_1| > \delta|x_1|$, and thus $(x + |x_1|d)_i = |x_i| - |x_1|\delta < |x_i|$. Finally, for $m' \le i \le m-1$,

$$\frac{1}{n-m'+2}|x_1| \ge \frac{1}{n}|x_1| > \max(|x_i|, \delta|x_1|) \ge \max(|x_i| - |x_1|\delta,\, |x_1|\delta - |x_i|),$$

which gives the desired inequality. We can then apply lemma 2.38 to see that $x + |x_1|d \in \operatorname{int}\mathbf{B}$. $\square$

Proposition 5.13. Let $n \ge 2$, $\delta > 0$, and

$$S^n = \left\{ \pm e_j + \sum_{i \in [n]\setminus\{j\}} \pm\delta e_i \in \mathbb{R}^n : \; j \in [n] \right\}.$$

There exists a subset, $\mathcal{I}^n \subset S^n$, with $|\mathcal{I}^n| = 2^n$, so that for any vector $x \in \mathbb{R}^n\setminus\{0\}$, there exists a $d \in \mathcal{I}^n$ so that $\operatorname{sign}(x_i) = -\operatorname{sign}(d_i)$ for every $i \in [n]$ such that $x_i \neq 0$ and $|d_j| = \|d\|_\infty = 1$ for an index $j \in [n]$ such that $x_j \neq 0$.

Definition 5.14. In the context of proposition 5.13, for vector $x \in \mathbb{R}^n\setminus\{0\}$ and a set $\mathcal{I} \subset S^n$ (or singleton $\{d\} \subset S^n$), we say $\mathcal{I}$ (or d) *deep illuminates* x if there exists a $d \in \mathcal{I}$ so that $\operatorname{sign}(x_i) = -\operatorname{sign}(d_i)$ for every $i \in [n]$ such that $x_i \neq 0$ and $|d_j| = \|d\|_\infty = 1$ for an index $j \in [n]$ such that $x_j \neq 0$.

We provide some examples here to give some geometric intuition of the proof. See example 5.16.

Proof. We first show that we can find a closed and simple curve/path P^n on the boundary of the n-dimensional cube $[-1,1]^n$ which starts at a vertex w_1 of the cube and has the following two important properties:

- the path P^n passes by every vertex of the cube (exactly once since it will be simple) and returns to w_1;
- consecutive vertices on the path share an edge, or in other words differ in exactly one coordinate. This will also be true for the pair of the last vertex w_{2^n} on the path and w_1; in other words, P^n returns to w_1 through one of the vertices which differ from w_1 in exactly one coordinate.

Given that we can travel from one vertex on the path to the next one along the edge they share, finding such a path is completely equivalent to finding a sequence/ordering $w_1, w_2, ..., w_{2^n}$ of the vertices of the cube

- such that all vertices appear in the sequence
- and such that consecutive vertices in the sequence differ in exactly one coordinate; this should be true also for the pair of vertices w_{2^n} and w_1.

Moreover, since the path is going to be closed, we could start at any vertex of the cube that we want. Also observe that applying a symmetry of the cube to this path (combinations of coordinate permutations and sign changes) will just give us another path with the same properties. Thus it is fine to require that $w_1 = (1,1,1,...,1,1)$, while $w_{2^n} = (1,1,1,...,1,-1)$.

In the dimension 2 case, we can see that the sequence

$$w_1 = (1,1),\ w_2 = (-1,1),\ w_3 = (-1,-1),\ w_4 = (1,-1) \tag{1}$$

satisfies the desired properties. We can use this path to construct desirable paths in higher dimensions via recursion.

In dimension $n > 2$, we use the following method to find a path P^n with the desired properties. Suppose we have already found an analogous path P^{n-1} for the $(n-1)$-dimensional cube, and that the corresponding sequence of vertices is $w_1^{n-1}, w_2^{n-1}, ..., w_{2^{n-1}}^{n-1}$.

As explained before, WLOG we can also assume that

$$w_1^{n-1} = \sum_{i=1}^{n-1} e_i \qquad \text{while} \quad w_{2^{n-1}}^{n-1} = -e_{n-1} + \sum_{i=1}^{n-2} e_i$$

(the e_i's here are the standard basis vectors in $\mathbb{R}^{n-1}$). Then the sequence for the path P^n can be chosen to be

$$\begin{aligned} w_1^n = \big(w_1^{n-1}, 1\big), \quad w_2^n = \big(w_2^{n-1}, 1\big), \ldots \\ \ldots, w_{2^{n-1}}^n = \big(w_{2^{n-1}}^{n-1}, 1\big), \quad w_{2^{n-1}+1}^n = \big(w_{2^{n-1}}^{n-1}, -1\big), \\ w_{2^{n-1}+2}^n = \big(w_{2^{n-1}-1}^{n-1}, -1\big), \quad w_{2^{n-1}+3}^n = \big(w_{2^{n-1}-2}^{n-1}, -1\big), \ldots \\ \ldots, w_{2^n-1}^n = \big(w_2^{n-1}, -1\big), \quad w_{2^n}^n = \big(w_1^{n-1}, -1\big). \end{aligned}$$

It is not hard to check that the constructed sequence has the desired properties, as long as the sequence arising from P^{n-1} did so as well.

Fixing 1 gives us an explicit desirable sequence using the above method. We have a different example of such a sequence as well (see example 5.17).

Next we explain how we construct a subset $\mathcal{I}^n$ of S^n once we have a path P^n on the boundary of the cube with the above properties. Let $w_1^n, w_2^n, ..., w_{2^n}^n$ be the sequence of vertices arising from this path, and let $r \in \{1, 2, ..., 2^n\}$ (from now on we will be suppressing the superscript n which indicates the dimension). Consider the pair of vertices (w_r, w_{r+1}) (if $r = 2^n$, then the pair to consider is (w_{2^n}, w_1)). This leads to the direction

$$\mathbf{d}_r := -\operatorname{sign}(w_{r,i_r}) e_{i_r} + \sum_{i \in [n] \setminus \{i_r\}} (-\delta \cdot \operatorname{sign}(w_{r,i})) e_i,$$

where i_r is the index of the unique entry in which w_r and w_{r+1} differ (in fact, $-\operatorname{sign}(w_{r,i_r}) = w_{r+1,i_r}$). For instance, $\mathbf{d}_{2^n} = (-\delta, -\delta, ..., -\delta, -\delta, +1)$.

Finally we show that the set

$$\mathcal{I}^n \equiv \mathcal{I}^n(\delta) := \{\mathbf{d}_1, \mathbf{d}_2, ..., \mathbf{d}_{2^n-1}, \mathbf{d}_{2^n}\}$$

deep illuminates any vector $x \in \mathbb{R}^n \setminus \{0\}$. Indeed, fix such a vector x, and consider the vector z of its coordinate signs:

$$z \equiv z_x := \big(\widetilde{\text{sign}}(x_1), \widetilde{\text{sign}}(x_2), ..., \widetilde{\text{sign}}(x_{2^n-1}), \widetilde{\text{sign}}(x_{2^n})\big),$$

where $\widetilde{\text{sign}}(x_i) = \text{sign}(x_i)$ if $x_i \neq 0$, and $\widetilde{\text{sign}}(x_i) = 0$ if $x_i = 0$.

Clearly z_x is found on the boundary of the cube $[-1,1]^n$, and if x has no zero coordinates, then z_x is in fact a vertex of the cube. Otherwise there is a unique face F_z of the cube with dimension $k \in \{1, 2, ..., n-1\}$ such that z is in the relative interior of F_z.

Consider now the vertices of the cube which belong to F_z and call them 'good' for z. More simply, these are precisely the vertices v_j of the cube which agree with z in all entries where z has a non-zero coordinate: $v_j \in F_z$ if and only if $v_{j,i} = z_i$ for all $i \in [n]$ such that $z_i \neq 0$. Observe that there will be 2^k vertices which will be 'good' for z, where k is the number of zero coordinates of z (same as the number of zero coordinates of x). Since $x \in \mathbb{R}^n \setminus \{0\}$, there will be at most 2^{n-1} 'good' vertices for z.

Claim. There exists an index $r_0 \in \{1, 2, ..., 2^n\}$ such that the r_0-th vertex w_{r_0} on the path P^n is 'good' for z, while the next vertex w_{r_0+1} is 'bad' for z (here we take $r_0+1 \bmod 2^n$, that is, $r_0 + 1 = 1$ if $r_0 = 2^n$).

Proof of the claim. Since there exists at least one and at most 2^{n-1} 'good' vertices for z, we can set $s_0 \in \{1, 2, ..., 2^n\}$ to be the largest index of a 'good' vertex for z, and we can also set t_0 to be the smallest index of a 'bad' vertex for z. We distinguish the following cases.

1. $s_0 < 2^n$. Then $s_0 + 1 \leq 2^n$ and is larger than s_0, so it cannot be the index of a 'good' vertex for z. Thus we can set $r_0 = s_0$, since w_{s_0} is a 'good' vertex for z and w_{s_0+1} is a 'bad' vertex for z.

2a. $s_0 = 2^n$ and $t_0 = 1$. By our choices for s_0 and t_0, we have that w_{s_0} is a 'good' vertex for z and w_{t_0} is a 'bad' vertex for z. Moreover, in this case $s_0 + 1 \bmod 2^n = t_0 \bmod 2^n$, so we can set $r_0 = s_0$ again.

2b. $t_0 > 1$. Then $t_0 - 1 \geq 1$ and is smaller than t_0, so, by our choice for t_0, $t_0 - 1$ cannot be the index of a 'bad' vertex for z (while w_{t_0} is indeed a 'bad' vertex for z). Thus we can set $r_0 = t_0 - 1$.

Finally, we verify that, if $r_0 \in \{1, 2, ..., 2^n\}$ is an index which satisfies the property in the claim, then the direction $\mathbf{d}_{r_0}$ deep illuminates the points z and x. Indeed, recall that respective entries of $\mathbf{d}_{r_0}$ and of the vertex w_{r_0} have opposite signs. Since w_{r_0} is 'good' for z, for every $i \in [n]$ such that $z_i \neq 0$ we have that $z_i = w_{r_0,i} = -\operatorname{sign}(\mathbf{d}_{r_0,i})$. On the other hand, w_{r_0+1} is a 'bad' vertex for z, so we can find some index i_0 such that $z_{i_0} \neq 0$ and $z_{i_0} \neq w_{r_0+1,i_0}$. But w_{r_0} and w_{r_0+1} are consecutive vertices on the path P^n, so they can only differ in exactly one entry; then this must be the i_0-th entry. By construction of the directions $\mathbf{d}_r$, we see that $\|\mathbf{d}_{r_0}\|_\infty$ is attained in the i_0-th entry for which we have $z_{i_0} \neq 0$. These combined show that $\mathbf{d}_{r_0}$ deep illuminates z (which is equivalent to $\mathbf{d}_{r_0}$ deep illuminating x).

The proof is complete. $\square$

Notation 5.15. Let $n \in \mathbb{N}$, $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $i \in [n-1]$, denote

$$P_i := \mathbb{R}^n \to \mathbb{R}^i, \quad \sum_{k=1}^{n} x_k e_k \mapsto \sum_{k=1}^{i} x_k e_k.$$

Notice in the above definition: $P_i : \mathbb{R}^n \to \mathbb{R}^i$, not $\mathbb{R}^n \to \mathbb{R}^n$, so there is a slight abuse of notation when using e_i in the definition of P_i.

Example 5.16. Lets observe proposition 5.13 in dimensions 3 and 4. We will also use the notation and notions of 'good' and 'bad' from proposition 5.13 .

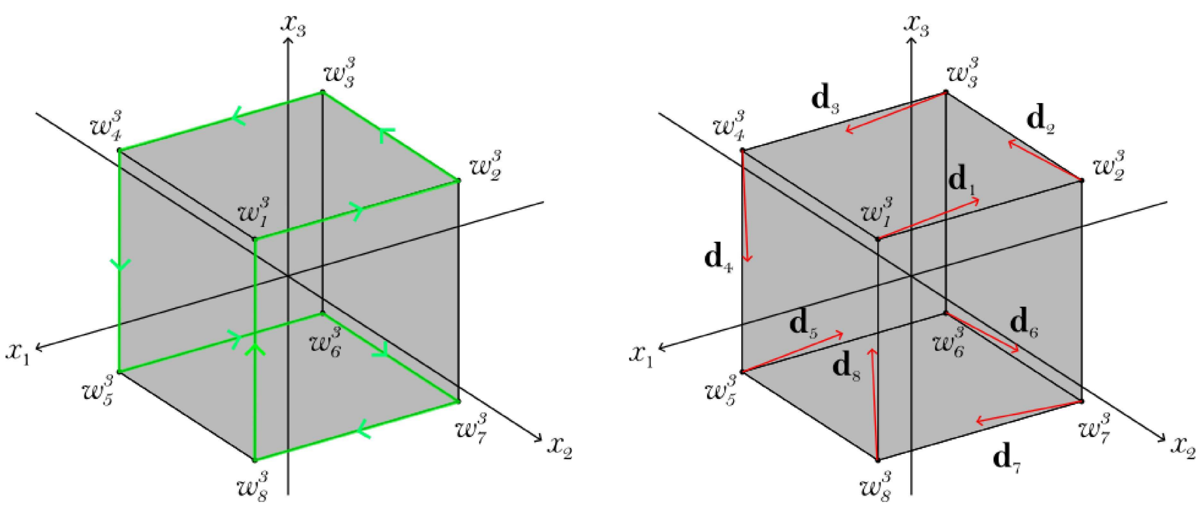
$n = 3$**:** Recall:

$$w_1 = (1,1),\ w_2 = (-1,1),\ w_3 = (-1,-1),\ w_4 = (1,-1).$$

Assign variables to the vertices of the cube using the following table:

Variable	Entries	Variable	Entries
w_1^3	$(1,1,1)$	w_5^3	$(1,-1,-1)$
w_2^3	$(-1,1,1)$	w_6^3	$(-1,-1,-1)$
w_3^3	$(-1,-1,1)$	w_7^3	$(-1,1,-1)$
w_4^3	$(1,-1,1)$	w_8^3	$(1,1,-1)$

Table 3: Dimension 3 case. Sequence representing vertices of the cube.



(a) Path on the cube induced by sequence of vertices, P^3.

(b) Directions created from the sequence of vertices.

Figure 4: Directions based on P^3.

So

$$\mathcal{I}^3 := \{\mathbf{d}_i \in \mathbb{R}^3 : i \in [8]\}.$$

We can see from figure 4b that $\mathcal{I}^3$ deep illuminates any $x \in \mathbb{R}^n$ which satisfies $|I_0^x| = 0$.

Suppose $|I_0^x| \geq 1$. Let's look at the example $x = (x_1, x_2, 0)$, where $x_1, x_2 > 0$, then $z := z_x = (1,1,0)$.

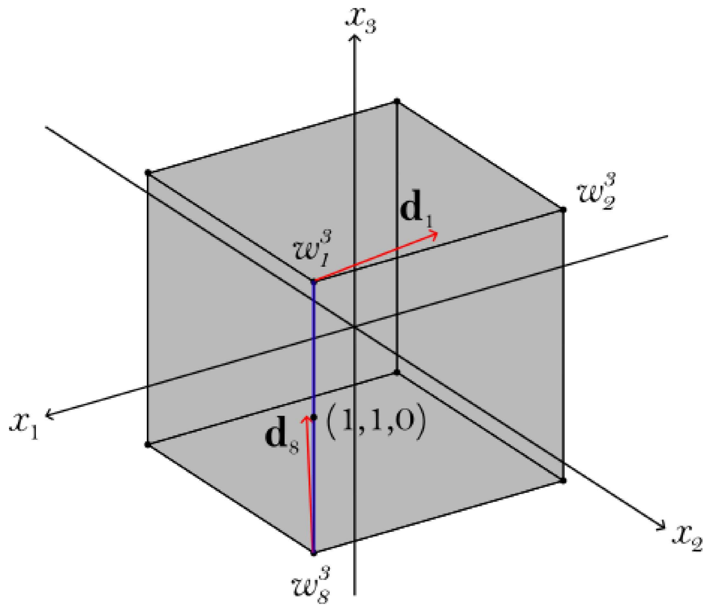


Figure 5: 1-D face on the cube which contains (1,1,0).

Observe from figure 5 that z lies in the relative interior of the 1-D face $F_z := w_8^3 w_1^3$ (the blue line segment). We will use the P^3 from 4a to determine which $i \in [8]$ has the property $\mathbf{d}_i$ deep illuminates z_x (thereby also illuminating x). In figure 4b, observe that the sign of $\mathbf{d}_i$ depends on w_i^3, while the location of the entry with absolute 1 is the same entry that is non zero for the vector $w_{i+1}^3 - w_i$.

So to make sure that

$$\operatorname{sign}((\mathbf{d}_1)_1) = -\operatorname{sign}(x_1) = -1 \quad \text{and} \quad \operatorname{sign}((\mathbf{d}_1)_2) = -\operatorname{sign}(x_2) = -1,$$

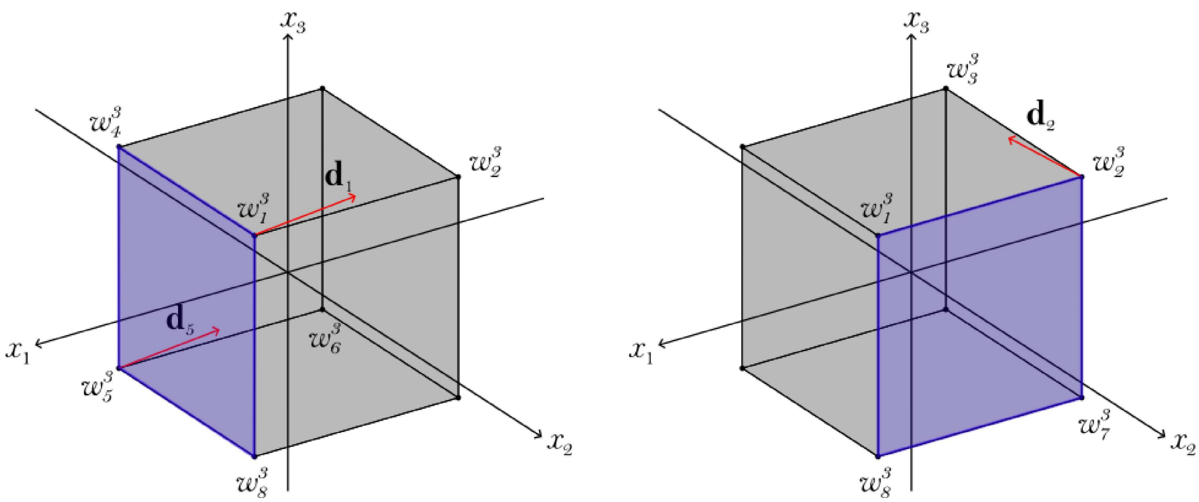
we need to pick a direction which is leaving from either w_8^3 or w_1^3, this means we can only pick either $\mathbf{d}_1$ or $\mathbf{d}_8$.

Now we need to determine whether $\mathbf{d}_1$ or $\mathbf{d}_8$ has absolute value 1 in entries two or three. w_8^3 and w_1^3 are consecutive entries; however, they are both good vertices for z. This means that except for the zero entries of z_x, w_8^3 and w_1^3 agree on those values. So $w_1^3 - w_8^3$ must have zero entries on on a non zero entries of z_x. So, we can rule out $\mathbf{d}_8$. On the other hand, the location of the entry with absolute value 1 of $\mathbf{d}_1$ is determined by w_1^3 and w_2^3. Consider the unique non zero entry of $w_2^3 - w_1^3$. In this entry, w_2^3 must also differ in sign with z_x since w_1^3 and z agree in sign in their non zero entries, and w_2^3 is a bad vertex as $w_2^3 \notin F_z$. So $\mathbf{d}_1$ must be the direction which deep illuminates x. This is the case as $\mathbf{d}_1 = (-1, -\delta, -\delta)$.

The general principle is that for any m-D face ($m < n$), if there exist a part of the path which is leaving the face (meaning for that segment of the path, the beginning node lies in the face and the end node lies outside of the face), we can use that part of the path to define the direction which deep illuminates any z_x which lies inside the face, hence deep illuminating any x which has the same zero entries as z_x and signs in their corresponding non zero entries. A simply connected path has nodes which are precisely every vertex on the cube, and each consecutive node only differs in sign in one entry does the job here. We can see that P^3 from figure 4a satisfies both of these conditions.

Note that the vertices which lie in the face are called the good vertices. The vertex which is the end node to a path leaving the face are actually what we call bad vertices.

Recall that we do not need to worry about vectors that look like $(1, 0, 0)$ or $(0, 1, 0)$. We still present the figures 6a and 6b below to show that these are deep illuminated for geometric intuition.



(a) 2-D face on the cube which contains $(1, 0, 0)$.

(b) 2-D face on the cube which contains $(0, 1, 0)$.

Figure 6: 2-D faces on the cube.

Finally, we can identify any $x \in \mathbb{R}^3 \backslash \{0\}$ with some z_x which lies on the boundary of the cube via

$$x \sim z_x \iff \widetilde{\text{sign}}(x_i) = \widetilde{\text{sign}}(z_i).$$

So we can see that $\mathcal{I}^3$ deep illuminates $\mathbb{R}^3$.

In a similar manor, it can be verified that if we construct

$$\mathcal{J}^3 := \{\mathbf{d}_i \in \mathbb{R}^3 : i \in [16] \backslash [8]\}$$

using the sequence $(w_i^3)_{i=9}^{16}$ (see table 4) and the same method we used to construct $\mathcal{I}^3$, then $\mathcal{J}^3$ also deep illuminates $\mathbb{R}^3 \backslash \{0\}$. The sequence $(w_i^3)_{i=9}^{16}$ is simply $(w_i^3)_{i=8}^{1}$.

Variable	Entries	Variable	Entries
w^3_{16}	$(1, 1, 1)$	w^3_{12}	$(1, -1, -1)$
w^3_{15}	$(-1, 1, 1)$	w^3_{11}	$(-1, -1, -1)$
w^3_{14}	$(-1, -1, 1)$	w^3_{10}	$(-1, 1, -1)$
w^3_{13}	$(1, -1, 1)$	w^3_9	$(1, 1, -1)$

Table 4: Dimension 3 case. Alternative sequence representing vertices of the cube.

$n = 4$**:** We use the sequence from tables 3 and 4 to construct the following sequence.

Variable	Entries	Variable	Entries
w_1	$(1, 1, 1, 1)$	w_9	$(1, 1, -1, -1)$
w_2	$(-1, 1, 1, 1)$	w_{10}	$(-1, 1, -1, -1)$
w_3	$(-1, -1, 1, 1)$	w_{11}	$(-1, -1, -1, -1)$
w_4	$(1, -1, 1, 1)$	w_{12}	$(1, -1, -1, -1)$
w_5	$(1, -1, -1, 1)$	w_{13}	$(1, -1, 1, -1)$
w_6	$(-1, -1, -1, 1)$	w_{14}	$(-1, -1, 1, -1)$
w_7	$(-1, 1, -1, 1)$	w_{15}	$(-1, 1, 1, -1)$
w_8	$(1, 1, -1, 1)$	w_{16}	$(1, 1, 1, -1)$

Table 5: Dimension 4 case. Sequence representing vertices of the cube.

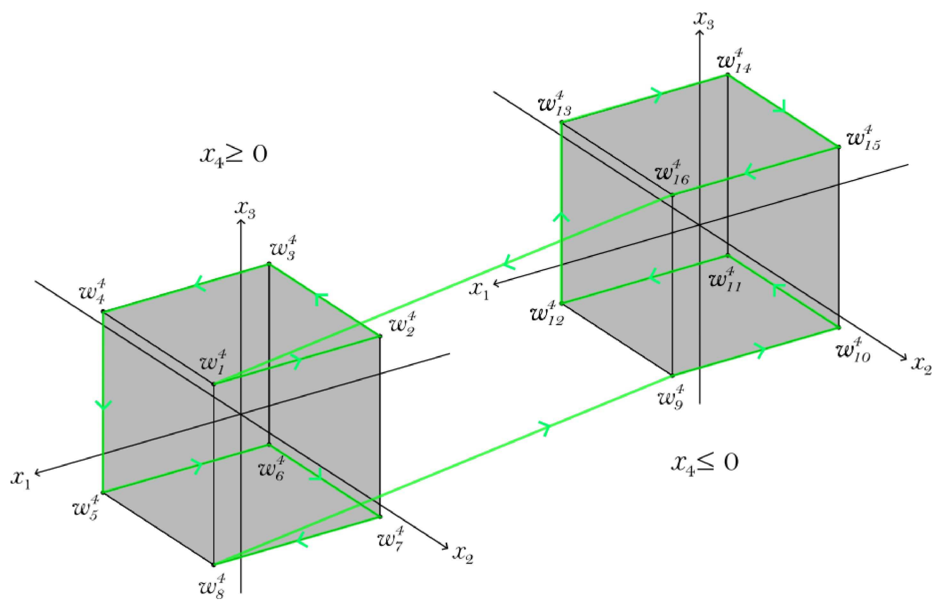


Figure 7: Path on the cube induced by sequence of vertices, P^4.

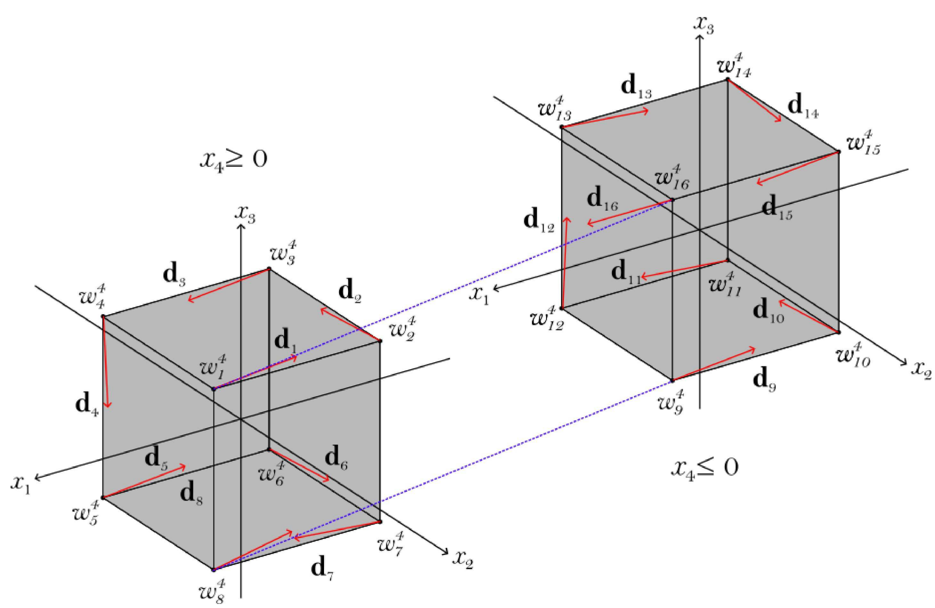


Figure 8: Directions based on P^4.

So

$$\mathcal{I}^4 := \{\mathbf{d}_i^4 \in \mathbb{R}^4 : i \in [16]\}.$$

Here, the superscript notation on the directions indicate the dimension it lives in. So from the previous example,

$$\mathcal{I}^3 = \{\mathbf{d}_i^3 \in \mathbb{R}^4 : i \in [8]\} \quad \text{and} \quad \mathcal{J}^3 = \{\mathbf{d}_i^3 \in \mathbb{R}^4 : i \in [16]\backslash[8]\}.$$

Notice that $P_{n-1}(\mathbf{d}_i^4) = \mathbf{d}_i^3$ for every $i \in [16]\backslash\{8, 16\}$ and $\text{sign}(P_{n-1}(\mathbf{d}_i^4)_j) = \text{sign}((\mathbf{d}_i^3)_j)$ for every $j \in [3]$ and $i \in \{8, 16\}$. This implies that if $x \in \mathbb{R}^4$ and $x_4 > 0$, then we can deep illuminate x using

$$\{\mathbf{d}_i^4 \in \mathbb{R}^4 : i \in [8]\}.$$

If $x_4 < 0$, then we can deep illuminate x using

$$\{\mathbf{d}_i^4 \in \mathbb{R}^4 : i \in [16]\backslash[8]\}.$$

Suppose $x_4 = 0$. Since $\mathcal{I}^3$ and $\mathcal{J}^3$ deep illuminate $\mathbb{R}^3$, we can see that

$$\{\mathbf{d}_i^4 \in \mathbb{R}^4 : i \in [7]\} \text{ deep illuminates } \mathbb{R}^3 \times \{0\}\backslash\{(\xi_1, \xi_2, -\xi_3, 0) : \xi_1, \xi_2, \xi_3 \geq 0\} \tag{2}$$

and

$$\{\mathbf{d}_i^4 \in \mathbb{R}^4 : i \in [16]\backslash[9]\} \text{ deep illuminates } \mathbb{R}^3 \times \{0\}\backslash\{(\xi_1, \xi_2, \xi_3, 0) : \xi_1, \xi_2, \xi_3 \geq 0\}. \tag{3}$$

The remaining vectors in $\mathbb{R}^n\backslash\{0\}$ are in the form of $(x_1, x_2, 0, 0)$ where $x_1, x_2 > 0$. We do not have to worry about the cases $x_1 = 0$ or $x_2 = 0$ since the directions from $\mathcal{I}^4$ are perturbations of $\{\pm e_i : i \in [n]\}$. With similar reasoning from the $n = 3$ example, we can see that deduce that $\mathbf{d}_1^4 = (-1, -\delta, -\delta, -\delta)$ and $\mathbf{d}_9^4 = (-1, -\delta, \delta, \delta)$ deep illuminates $(x_1, x_2, 0, 0)$. Thus, we can deep illuminate $\mathbb{R}^4\backslash\{0\}$ with $\mathcal{I}^4$. If we construct $\mathcal{J}^4$ in similar way to how we construct $\mathcal{J}^3$, one can verify that $\mathcal{J}^4$ also deep illuminates $\mathbb{R}^4\backslash\{0\}$.

So what is going on geometrically? In figure 7, take a look at the cube on the left ($x_4 \geq 0$) and the directions we take, lets call this C_1. We will call other cube ($x_4 \leq 0$) C_2. Notice that the path on C_1,

$$w_1^4 \to w_2^4 \to ... \to w_9^4,$$

is identical to the path in figure 4a, except for the line segments $w_8^3 w_1^3$ and $w_8^4 w_9^4$. The path induced by $(w_i)_{i=9}^{16}$ (call it $-P^3$) has the same trace as the path from 4a; however it has the opposite orientation. This second path is identical to the path on C_2,

$$w_9^4 \to w_{10}^4 \to ... \to w_{16}^4 \to w_1^4,$$

except for the line segments $w_{16}^3 w_9^3$ and $w_{16}^4 w_1^4$. So to construct P^4, we can use P^3 and $-P^3$. We break the connection at the end of P^3 ($w_8^3 w_1^3$) and connect the last node of P^3 (w_8^3) to the first node of $-P^3$ (w_9^3). Then we break the connection at the end of $-P^3$ ($w_8^3 w_1^3$) and connect the last node of $-P^3$ (w_{16}^3) to the first node of P^3 (w_1^3).

To see that the illuminating set induced by this construction $\mathcal{I}^4$ deep illuminates $\mathbb{R}^4\backslash\{0\}$, take a look at the locations of C_1 and C_2. C_1 is located on the side $x \geq 0$, therefore, the directions we derive from it will deep illuminate some $x \in \mathbb{R}^4$ where $x \geq 0$ as these directions will have a negative entry for the last coordinate. However, we know that P^3 induces $\mathcal{I}^3$ with deep illuminates $\mathbb{R}^3$, by the way we construct $\mathcal{I}^4$, $\mathbf{d}_8^4$ and $\mathbf{d}_8^3$ have the same signs in the first three corresponding entries since $\mathbf{d}_8^4$ is derived from the part of the path that is leaving the node w_8^4 and $\mathbf{d}_8^3$ is derived from the part of the path that is leaving the node w_8^3. So we can use (2) to deep illuminate such vertices. Similarly this is why (3) is used to deep illuminate $x \in \mathbb{R}^4$ which satisfy $x_4 < 0$.

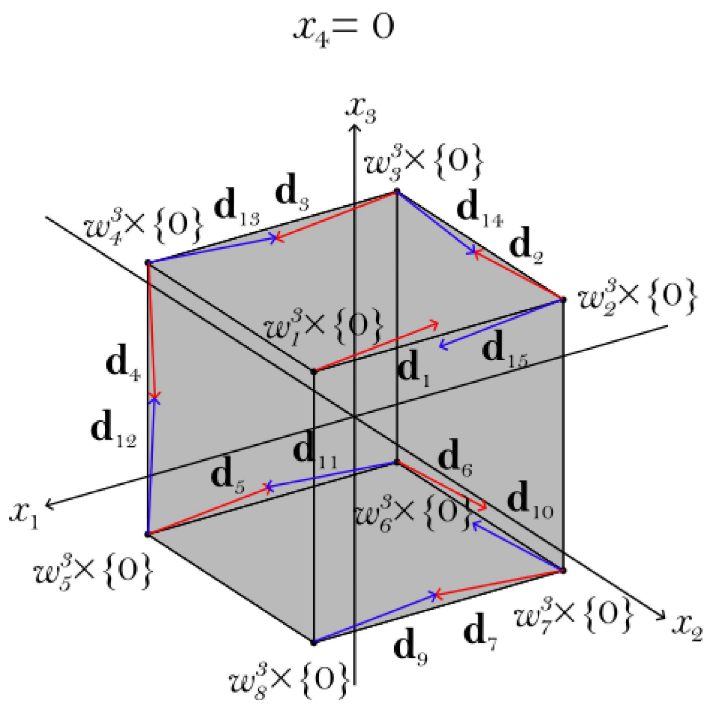


Figure 9: 3-cube living in $\mathbb{R}^4$ on the plane $x_4 = 0$ with $\mathcal{I}^4 \backslash \{\mathbf{d}_8, \mathbf{d}_{16}\}$.

What if $x = 0$? Geometrically, this should be the easiest case (see figure 9). This is because we are now allowed to look at directions from both C_1 and C_2, except for $\mathbf{d}_8^4$ and $\mathbf{d}_{16}^4$ as those are the only directions from $\mathcal{I}^4$ which have last entry of absolute value 1. We only have to consider families of vectors $(x_1, x_2, x_3, 0)$, which are not deep illuminated by both

$$\mathbf{d}_8^4 = (-\delta, -\delta, \delta, -1) \quad \text{and} \quad \mathbf{d}_{16}^4 = (-\delta, -\delta, -\delta, 1).$$

This leaves us with only the family of vectors which look like $(x_1, x_2, 0, 0)$ where $x_1, x_2 \geq 0$. Being able to deep illuminate this family of vectors is equivalent to deep illuminating $(1, 1, 0, 0)$, $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$. Referring back to figures 6a 6b and 5, we can see that we can deep illuminate $(1, 1, 0)$, $(1, 0, 0)$ and $(0, 1, 0)$ without using $\mathbf{d}_8^3$ to deep illuminate any of these vectors; that is, we can just use $\mathcal{I}^3 \backslash \{\mathbf{d}_8^3\}$. Thus, we can see that we don't need $\mathbf{d}_8^4$ to deep illuminate $(1, 1, 0, 0)$, $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ either.

Analogously, we do not need $\mathbf{d}_{16}^3$ as well. Hence, we can deep illuminate $x \in \mathbb{R}^4$ which satisfy $x_4 = 0$ using $\mathcal{I}^4 \backslash \{\mathbf{d}_8^4, \mathbf{d}_{16}^4\}$

Example 5.17. Let $n > 4$. We will construct a desirable sequence for proposition 5.13, that is a sequence which satisifies the following conditions:

- such that all vertices appear in the sequence
- and such that consecutive vertices in the sequence differ in exactly one coordinate; this should be true also for the pair of vertices w_{2^n} and w_1.

First, enumerate the vertices of the n-cube. The following table shows how the first 3 entries of a vertex are determined.

Letter	Entries	Letter	Entries
A	$(1, 1, 1, ...)$	E	$(-1, -1, 1, ...)$
B	$(1, 1, -1, ...)$	F	$(-1, -1, -1, ...)$
C	$(1, -1, -1, ...)$	G	$(-1, 1, -1, ...)$
D	$(1, -1, 1, ...)$	H	$(-1, 1, 1, ...)$

Table 6: Assignment of letters to vertices of the n-cube based on their first 3 entries.

Now we will find a way to determine the other entries. First note that every natural number $k \in \mathbb{N}$ can be expressed as $2^i + 2^{i+1}j$ for some $i, j \in \mathbb{N}_0$. If k is odd, then set $i = 0$ to see this. If k is even, this question is equivalent to: does there exists $i \in \mathbb{N}_0$ so that $k \equiv 2^i \mod 2^{i+1}$? Since k is even, take out the greatest power of 2 which divides k, i.e. $k = 2^{\tilde{i}}\tilde{k}$ for some $\tilde{k} \in \mathbb{N}$ and $\tilde{i} \in \mathbb{N}$. Since $2 \nmid \tilde{k}$, $\tilde{k} \equiv 1 \mod 2$. Thus there exists $\tilde{j} \in \mathbb{N}_0$ so that

$$\tilde{k} = 2\tilde{j} + 1 \implies 2^{\tilde{i}}\tilde{k} = 2^{\tilde{i}+1}\tilde{j} + 2^{\tilde{i}} \implies k \equiv 2^{\tilde{i}} \mod 2^{\tilde{i}+1}.$$

Setting $\tilde{j} = j$ and $\tilde{i} = i$ gives the claim.

Now we will show that this representation is unique. In other words, suppose $k = 2^i + 2^{i+1}j$ for some $i, j \in [n]$, does there exists $i' \in [n] \backslash \{i\}$ and $j' \in [n] \backslash \{j\}$ so that $k = 2^{i'} + 2^{i'+1}j'$?

By way of contradiction, suppose that this is the case, WLOG suppose $i = i' + m$ for some $m \in \mathbb{N}$, then

$$2^i + 2^{i+1}j = 2^{i'} + 2^{i'+1}j' \implies 2^{m-1} + 2^m j = \frac{1}{2} + j'.$$

Since $m \in \mathbb{N}$ and $j, j' \in \mathbb{N}_0$, this is impossible. Thus this representation of k is unique. To emphasize the dependency on k, we will use the notation $i(k)$ and $j(k)$. Define the following sequence:

$$(S_k)_{k=1}^{2^{n-3}} = \begin{cases} e_{i(k)+4}, & j(k) \text{ odd} \\ -e_{i(k)+4}, & j(k) \text{ even or } 0. \end{cases}$$

For $l \geq 2$, $V \in \{A, ..., H\}$ define

$$V_1 := (*, *, *, 1, ..., 1), \quad V_l := V_1 + 2\sum_{k=1}^{l-1} S_k,$$

where the first three entries of V_1 are determined by the letter V is. Now we can label the vertices of the n-cube. We pick the directions in the following way:

$$A_1 \underset{d^n_{A_1}}{\rightarrow} \ldots \underset{d^n_{G_1}}{\rightarrow} H_1 \underset{d^n_{H_1}}{\rightarrow} H_2 \underset{d^n_{H_2}}{\rightarrow} \ldots \underset{d^n_{B_2}}{\rightarrow} A_2 \underset{d^n_{A_2}}{\rightarrow} A_3 \underset{d^n_{A_3}}{\rightarrow} \ldots \ldots \underset{d^n_{B_{2^{n-3}}}}{\rightarrow} A_{2^{n-3}} \underset{d^n_{A_{2^{n-3}}}}{\rightarrow} A_1.$$

One can check that this sequence is "desirable".

Theorem 5.18. Let $n \geq 2$ and let $\mathbf{B} \in \mathcal{S}^n$. Then there is $\delta = \delta_{\mathbf{B}} > 0$ such that $\mathcal{I}^n(\delta)$ illuminates $\mathbf{B}$. It follows that $\mathcal{I}(\mathbf{B}) \leq 2^n$.

Proof. Set $\alpha = (\|e_1 + e_2 + ... + e_n\|_{\mathbf{B}})^{-1}$; note that α is the largest positive constant such that $(\alpha, \alpha, ..., \alpha) \in \mathbf{B}$. We can also observe that, since $e_i \in \partial\mathbf{B}$ for each i, by convexity we have that $(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}) \in \mathbf{B}$, and thus $\alpha \geq \frac{1}{n}$. We will see that the conclusion in the statement can be satisfied as long as we choose $\delta < \frac{\alpha}{2n^2}$.

Because $\mathbf{B}$ is 1-symmetric, by lemma 2.38 we know that, if $y = (y_1, y_2, ..., y_n)$ is such that $|y_i| < \alpha$ for each $i \in [n]$, then $y \in \operatorname{int}\mathbf{B}$. Thus for each $x \in \partial\mathbf{B}$, we will have that

$\|x\|_\infty \geq \alpha$.

As mentioned before, we fix some $\delta < \frac{\alpha}{2n^2}$. Consider $x \in \partial\mathbf{B}$; we will show that there is $\mathbf{d} \in \mathcal{I}^n(\delta)$ which illuminates x (that is, $\mathbf{B}$-illuminates x).

Consider the non-negative numbers $|x_1|, |x_2|, |x_3|, ..., |x_{n-1}|, |x_n|$, and observe that they are found in the interval $[0, \|x\|_\infty]$. Divide this interval into, say, M disjoint subintervals of the form $(\gamma_{i+1}, \gamma_i]$ each of which has length $\frac{\alpha}{2n}$, except for the last one which may have shorter length (and will also be of the form $[\gamma_{M+1}, \gamma_M] = [0, \gamma_M]$). Given that $\|x\|_\infty = \text{length}([0, \|x\|_\infty]) \geq \alpha$, we need at least $2n$ such subintervals. Moreover, the first subinterval $(\gamma_2, \gamma_1] = (\|x\|_\infty - \frac{\alpha}{2n}, \ \|x\|_\infty]$ definitely contains some of the numbers $|x_1|, |x_2|, |x_3|, ..., |x_{n-1}|, |x_n|$. Given that there are at most n different such numbers, we can find an index

$$r_0 \in \{1, 2, ..., n, n+1\} \subseteq \left\{1, 2, ..., \lfloor \tfrac{M}{2} \rfloor, \lfloor \tfrac{M}{2} \rfloor + 1\right\}$$

such that the subinterval $(\gamma_{r_0+1}, \gamma_{r_0}]$ does not contain any of the $|x_i|$, $i \in [n]$.

It follows that, if $x_{i_1}, x_{i_2}, ..., x_{i_s}$ are the coordinates of x which have absolute value $> \gamma_{r_0+1}$, and $x_{j_1}, x_{j_2}, ..., x_{j_t}$ are the coordinates of x which have absolute value $\leq \gamma_{r_0+1}$, then the former have actually absolute value $> \gamma_{r_0} = \gamma_{r_0+1} + \frac{\alpha}{2n}$ and

$$|x_{i_u}| - |x_{j_w}| > \gamma_{r_0} - \gamma_{r_0+1} = \frac{\alpha}{2n} \tag{4}$$

for every $1 \leq u \leq s$ and $1 \leq w \leq t$. Note also that we always have $s \geq 1$ (while in some cases we might not have any 'small' coordinates, but this won't matter).

Consider now the vector

$$\widetilde{x} = \sum_{u=1}^{s} x_{i_u} e_{i_u}$$

(that is, the vector we get if we keep the 'big' coordinates of x, but set the remaining ones equal to 0). By proposition 5.13, we know that there is a direction $\mathbf{d} \in \mathcal{I}^n(\delta)$ which deep illuminates $\widetilde{x}$. We will show that $\mathbf{d}$ $\mathbf{B}$-illuminates x. By the definition of deep illumination, $\|\mathbf{d}\|_\infty$ is attained in one of the coordinates which is non-zero for $\widetilde{x}$, and thus for one of the coordinates which is 'big' for x; let's denote this by i_{u_0}.

We examine the vector

$$x + |x_{i_{u_0}}|\mathbf{d} = \sum_{u \neq u_0} \big(x_{i_u} - \delta \operatorname{sign}(x_{i_u})|x_{i_{u_0}}|\big) e_{i_u} \; + \; \sum_{w=1}^{t} \big(x_{j_w} + \delta \operatorname{sign}(\mathbf{d}_{j_w})|x_{i_{u_0}}|\big) e_{j_w}.$$

First of all, recall that $r_0 \leq n+1$, and thus $\gamma_{r_0} = \|x\|_\infty - (r_0 - 1)\frac{\alpha}{2n} \geq \frac{\alpha}{2}$. It follows that, for each $1 \leq u \leq s$, $|x_{i_u}| > \gamma_{r_0} \geq \frac{\alpha}{2} > \delta$, and thus, if $u \neq u_0$,

$$|x_{i_{u_0}}|\delta \leq \delta < |x_{i_u}| \quad \Rightarrow \quad \big|x_{i_u} - \delta \operatorname{sign}(x_{i_u})|x_{i_{u_0}}|\big| = |x_{i_u}| - \delta|x_{i_{u_0}}| < |x_{i_u}|.$$

In the case that x has only 'big' coordinates, we can conclude that $x + |x_{i_{u_0}}|\mathbf{d} \in \operatorname{int} \mathbf{B}$.

On the other hand, if $t \geq 1$, then we first consider the following convex combination, which will be in $\mathbf{B}$ because $\mathbf{B}$ is 1-symmetric:

$$\begin{aligned}
&\frac{1}{t+1}\Bigg[\Bigg(\sum_{u \neq u_0} |x_{i_u}|e_{i_u} + \sum_{w \neq 1} |x_{j_w}|e_{j_w} \; + \; |x_{i_{u_0}}|e_{j_1}\Bigg) \\
&\qquad + \Bigg(\sum_{u \neq u_0} |x_{i_u}|e_{i_u} + \sum_{w \neq 2} |x_{j_w}|e_{j_w} \; + \; |x_{i_{u_0}}|e_{j_2}\Bigg) + \ldots \\
&\qquad \ldots + \Bigg(\sum_{u \neq u_0} |x_{i_u}|e_{i_u} + \sum_{w \neq t} |x_{j_w}|e_{j_w} \; + \; |x_{i_{u_0}}|e_{j_t}\Bigg) + \Bigg(\sum_{i=1}^{n} |x_i|e_i\Bigg)\Bigg] \\
&= \frac{1}{t+1}|x_{i_{u_0}}|e_{i_{u_0}} \; + \; \sum_{u \neq u_0} |x_{i_u}|e_{i_u} \; + \; \sum_{w=1}^{t} \left(\frac{t}{t+1}|x_{j_w}| + \frac{1}{t+1}|x_{i_{u_0}}|\right) e_{j_w}.
\end{aligned}$$

For each $1 \leq w \leq t$, we have that

$$\begin{aligned}
\frac{t}{t+1}|x_{j_w}| + \frac{1}{t+1}|x_{i_{u_0}}| \; - \; |x_{j_w}| = \frac{|x_{i_{u_0}}| - |x_{j_w}|}{t+1} &> \frac{\alpha/(2n)}{t+1} \qquad \text{because of (4)} \\
&\geq \frac{\alpha}{2n^2} > \delta.
\end{aligned}$$

At the same time, the absolute value of the j_w-th coordinate of the vector $x + |x_{i_{u_0}}|\mathbf{d}$ is

$$\big|x_{j_w} + \delta \operatorname{sign}(\mathbf{d}_{j_w})|x_{i_{u_0}}|\big| \leq |x_{j_w}| + \delta|x_{i_{u_0}}| \leq |x_{j_w}| + \delta,$$

which, based on the above, is strictly smaller than the absolute value of the j_w-th coordinate of the above convex combination. Thus, each of the coordinates of this convex combination is strictly larger in absolute value than the corresponding coordinate of $x + |x_{i_{u_0}}|\mathbf{d}$. We conclude that $x + |x_{i_{u_0}}|\mathbf{d} \in \operatorname{int} \mathbf{B}$.

Since x was an arbitrary point of $\partial\mathbf{B}$, we conclude that $\mathcal{I}^n(\delta)$ illuminates $\mathbf{B}$ for the fixed δ we started with. $\square$

Remark 5.19. Suppose that $\widetilde{\mathbf{B}}$ is a 1-symmetric convex body in $\mathbb{R}^n$, which is not necessarily in $\mathcal{S}^n$. That is, we don't necessarily have $e_i \in \partial\widetilde{\mathbf{B}} \Leftrightarrow \|e_i\|_{\widetilde{\mathbf{B}}} = 1$.

We can still illuminate $\widetilde{\mathbf{B}}$ using a set of the form $\mathcal{I}^n(\delta)$ (simply because $\widetilde{\mathbf{B}}$ is a dilation of a convex body in $\mathcal{S}^n$), but a careful inspection of the above proof can also give us an explicit estimate of what δ work here: as long as

$$\delta < \frac{1}{2n^2} \cdot \frac{\|e_1\|_{\widetilde{\mathbf{B}}}}{\|e_1 + e_2 + ... + e_n\|_{\widetilde{\mathbf{B}}}},$$

$\mathcal{I}^n(\delta)$ illuminates $\widetilde{\mathbf{B}}$. In particular, by convexity it always suffices to take $\delta < \dfrac{1}{2n^3}$.

Corollary 5.20. Let $n \geq 3$, $\mathbf{B} \in \mathcal{S}^n$ and suppose $\|e_i + e_j\|_{\mathbf{B}} > 1$ for every $i, j \in [n]$. There exists $\delta > 0$ such that, if $\mathcal{I}^{n-1}(\delta)$ is the set from proposition 5.13, then the set $(\mathcal{I}^{n-1}(\delta) \times \{0\}) \cup \{\pm e_n\}$ illuminates $\mathbf{B}$. Thus, if $\mathbf{B}$ satisfies the above norm assumption, then $\mathcal{I}(\mathbf{B}) \leq |(\mathcal{I}^{n-1}(\delta) \times \{0\}) \cup \{\pm e_n\}| = 2^{n-1} + 2$.

Proof. Consider $x \in \partial\mathbf{B}$, and assume first that $\|x\|_\infty$ is uniquely attained at the last coordinate of x. In particular, because of the norm assumption of the lemma, this is guaranteed to happen if

$$|x_n| > \beta, \quad \text{where } \beta \text{ is the maximum positive constant such that } \beta e_i + \beta e_j \in \mathbf{B},\ i \neq j.$$

More precisely here, $\beta = (\|e_1 + e_2\|_{\mathbf{B}})^{-1} < 1$. Then by lemma 5.10 one of $\pm e_n$ illuminates x.

Now consider x for which $|x_n| \leq |x_i|$ for some $i < n$. As we said above, this implies that $|x_n| \leq \beta < 1$. Consider then the subset

$$\mathbf{B}_0 = \mathbf{B} \cap \{\xi \in \mathbb{R}^n : \xi_n = x_n\}.$$

We have that $\mathrm{Proj}_{e_n^\perp}(\mathbf{B}_0)$ is a 1-symmetric convex body in $\mathbb{R}^{n-1}$, and thus by Theorem 5.18 and the subsequent remarks we know that $\mathrm{Proj}_{e_n^\perp}(x)$ can be $\mathrm{Proj}_{e_n^\perp}(\mathbf{B}_0)$-illuminated by a direction in $I^{n-1}(\delta)$ as long as $\delta < \dfrac{1}{2(n-1)^3}$; denote this direction by $\mathbf{d}_x^{n-1}$. We obtain that, for some $\varepsilon > 0$, $x + \varepsilon \cdot (\mathbf{d}_x^{n-1}, 0)$ will be contained in the relative interior of $\mathbf{B}_0$, which combined with lemma 2.42 shows that x is illuminated by $(\mathbf{d}_x^{n-1}, 0)$.

This completes the proof. □

Theorem 5.18 gives us $\mathcal{I}(\mathbf{B}) \leq 2^n$ for any $\mathbf{B} \in \mathcal{S}^n$ which proves the Illumination Conjecture up short of the equality cases for 1-symmetric convex bodies! If we want $\mathcal{I}(\mathbf{B}) < 2^n$, corollary 5.20 proves conjecture 3.7 and when combined with Tikhomirov's results, we get Theorem 3.1. Recall that Theorem 3.1 comes at the cost of assuming n to be larger than some universal constant C. We also have the following:

Corollary 5.21. Let $\mathbf{B} \in \mathcal{S}^3$ which is not an affine image of the cube, then $\mathcal{I}(\mathbf{B}) \leq 7$.

Proof.

Case 1: $\|e_i + e_j\|_{\mathbf{B}} = 1$ for every $i, j \in [4]$, $i \neq j$. Recall from remark 5.8 that it must be the case that $\mathrm{d}(\mathbf{B}, [-1,1]^4) < 2$. We can use lemma 3.5 which tells us that we only need 7 directions.

Case 2: $\|e_i + e_j\|_{\mathbf{B}} > 1$ for every $i, j \in [4]$. Using lemma 5.20, we get $\mathcal{I}(\mathbf{B}) \leq 6$.

□

Corollary 5.22. Let $\mathbf{B} \in \mathcal{S}^4$ which is not an affine image of the cube, then $\mathcal{I}(\mathbf{B}) \leq 15$.

Proof.

Case 1: $\|e_i + e_j\|_{\mathbf{B}} = 1$ for every $i, j \in [4]$, $i \neq j$. We separate this case into two subcases. The first subcase is when $\mathrm{d}(\mathbf{B}, [-1,1]^4) \geq 2$. Recall that lemma 5.7 tells us that we only need to consider when $\mathrm{d}(\mathbf{B}, [-1,1]^4) = 2$, and in this subcase, we only need 8 directions (lemma 5.4). In the subcase $1 \neq \mathrm{d}(\mathbf{B}, [-1,1]^4) < 2$, lemma 3.5 tells us that we only need 15 directions.

Case 2: $\|e_i + e_j\|_{\mathbf{B}} > 1$ for every $i, j \in [4]$. Using lemma 5.20, we can see that in this case $\mathcal{I}(\mathbf{B}) \leq 10$.

$\square$

6 Dimension 3, 1-Unconditional Polytopes

Definition 6.1. We say $\mathbf{B} \in \mathcal{U}_p^n$ if $\mathbf{B} \subset \mathbb{R}^n$ is a 1-unconditional polytope.

Now it is time to move onto 1-unconditional convex bodies. We will start with the case: $\|e_i + e_j\|_{\mathbf{B}} = 1$ for some $i, j \in [4]$, $i \neq j$. The geometric implication of imposing this condition for any pair of i, j is adding a 2-d square which will lie on a subspace perpendicular to e_k, where $k \neq i, j$. Then we will continue adding this condition for more pairs until we look at the case $\|e_i + e_j\|_{\mathbf{B}} = 1$ for every $i, j \in [4]$, $i \neq j$. Completing this process gives us the main result of this section (Theorem 6.9). There is even an additional feature of this theorem which Lassak conjectured at the end of his paper [14] (which was more generally about origin-symmetric convex bodies): the directions we find come in pairs of opposite vectors (in our case 3 pairs, compared to the illuminating sets found by Lassak, which consisted of 4 pairs of opposite vectors).

Proposition 6.2. Let $\mathbf{B} \in \mathcal{U}_p^3$ and suppose $\|e_i + e_j\|_{\mathbf{B}} > 1$ for every $i, j \in [3]$. Then there exists $\delta > 0$ so that $\mathbf{B}$ can be illuminated by some coordinate permutation of the set

$$\begin{aligned} \mathcal{I}_\delta := \{&(1,0,0),\ (0,1,-\delta),\ (0,\delta,1),\\ &(-1,0,0),\ (0,-1,\delta),\ (0,-\delta,-1)\}. \end{aligned}$$

Proof. Let $\mathbf{B} \in \mathcal{U}_p^3$ and suppose $\|e_i + e_j\|_{\mathbf{B}} > 1$ for every $i, j \in [3]$. Recall from example 2.46 that for any vertex of $\mathbf{B}$, there can only be one entry with absolute value of 1.

Step 1: Let $x \in \operatorname{ext} \mathbf{B}$ be a vertex so that $|I_0^x| > 0$. So $|I_0^x| \in \{1, 2\}$. Then we can illuminate x by $\mathcal{I}_\delta$ by first applying lemma 2.45 then lemma 2.41.

Step 2: Let $x \in \operatorname{ext} \mathbf{B}$ be a vertex so that $|I_0^x| = 0$ with $x_1, x_2, x_3 > 0$. Suppose $\|x\|_\infty < 1$ or $x_i = 1$ for exactly one $i \in \{2, 3\}$. Consider the case when $x = (x_1, x_2, x_3)$. Let $\delta > 0$ and $d = (0, -\delta, -1)$. With lemma 2.42 in mind, look in the affine set

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 = x_1\}$$

to see that $x + d \in \operatorname{int} \mathbf{B}$. Using this method, we can construct the following table:

Type of vertex		Illuminating direction
(x_1, x_2, x_3),	$(-x_1, x_2, x_3)$	$(0, -\delta, -1)$
$(x_1, -x_2, -x_3)$,	$(-x_1, -x_2, -x_3)$	$(0, \delta, 1)$
$(x_1, x_2, -x_3)$,	$(-x_1, x_2, -x_3)$	$(0, -1, \delta)$
$(x_1, -x_2, x_3)$,	$(-x_1, -x_2, x_3)$	$(0, 1, -\delta)$

Table 7: Summary of method.

Now, suppose that $x_{\max} = (x_{\max,1}, x_{\max,2}, x_{\max,3}) \in \operatorname{ext} \mathbf{B}$ so that $|x_{\max,2}| = 1$ and $1 > |x_{\max,3}| \geq |x_i|$ for all $x \in \operatorname{ext} \mathbf{B}$ which satisfy $|I_0^x| = 0$ and for all $i \in [3]$ except when $|x_i| = 1$. In other words, $x_{\max}$ has an entry which has the largest absolute value out of those entries which are not equal to 1 among vertices with non-zero entries, and we assign this entry to be the third entry. By imposing that $|x_{\max,2}| = 1$, we know that this vertex can be illuminated by some direction in table 7. Since $\mathbf{B} \in \mathcal{U}^3$, WLOG, we can assume that $x_{\max,i} > 0$ for all $i \in [3]$. Let $x \in \operatorname{ext} \mathbf{B}$ so that $|x_1| = 1$. Consider the direction $d := (-\operatorname{sign}(x_1), 0, 0)$ and compare the coordinates of $x + d$ coordinates to those of $x_{\max}$.

$$|x_1 - \operatorname{sign}(x_1)| = 1 - 1 = 0 < |x_{\max,1}|$$
$$|x_2| < 1 = |x_{\max,2}|$$
$$|x_3| \leq |x_{\max,3}| < 1.$$

The second inequality holds as vertices of $\mathbf{B}$ can only have one entry with absolute value of 1. If $|x_3| < |x_{\max,3}|$, then we can apply lemma 2.38 to see that $x + d \in \operatorname{int} \mathbf{B}$. If $|x_3| = |x_{\max,3}|$, in the context of lemma 2.42, look at the affine set

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_3 = x_3\}$$

to see that $x + d \in \operatorname{int} \mathbf{B}$. And thus, x is illuminated by d. Thus we have shown that for arbitrary $x \in \operatorname{ext} \mathbf{B}$, we can illuminate these vertices using $\mathcal{I}_\delta$, given that such a

$x_{\max}$ exists. It should be noted that $\delta > 0$ in table 7 can take on any value in $\mathbb{R}_{>0}$ as this was the case for $\delta > 0$ in corollary 2.39.

Step 3: In the previous step, we made a choice which favored some coordinates when defining $x_{\max}$, we will now show this choice does not affect the outcome if we "adjust **B** accordingly". Consider $\iota(x), \iota(x_{\max}) \in \operatorname{ext} \iota(\mathbf{B})$, where ι is the isomorphism

$$\iota : \mathbb{R}^3 \to \mathbb{R}^3, \quad \sum_{k \in [3]} z_{i_k} e_{i_k} \mapsto (z_{i_1}, z_{i_2}, z_{i_3})$$

and $x_{\max} = (x_{\max,1}, x_{\max,2}, x_{\max,3}) \in \operatorname{ext} \mathbf{B}$ so that $|x_{\max,i_2}| = 1$ and $1 > |x_{\max,i_3}| \geq |x_{i_k}|$ for all $x \in \operatorname{ext} \mathbf{B}$ with $|I_0^x| = 0$ and for all $k \in [3]$ except when $|x_{i_k}| = 1$. Now we are reduced to the previous step, so $\iota^{-1}(\mathcal{I}_\delta)$ illuminates **B**. Note such $x_{\max}$ from this step exists for any $\mathbf{B} \in \mathcal{U}_p^3$.

Recall from step 2 that $\delta > 0$ has no restrictions; however they are bounded by above in step 1. Let $x \in \operatorname{ext} \mathbf{B}$. Since each δ, which allows $\mathcal{I}_\delta$ to illuminate x, depends on x, we will denote them by δ_x. Note that we can also illuminate x, using $\mathcal{I}_{\tilde{\delta}}$ for any $0 < \tilde{\delta} < \delta$. Since each $|\operatorname{ext} \mathbf{B}| < \infty$, we can take a sufficiently small $\tilde{\delta} > 0$ so that $\mathcal{I}_{\tilde{\delta}}$ illuminates any $x \in \operatorname{ext} \mathbf{B}$. □

Example 6.3. The steps in this example refer to the steps from proposition 6.2.

1. Let $\mathbf{B} \in \mathcal{U}_p^3$, suppose

$$x = \left(\tfrac{3}{4}, 1, \tfrac{1}{2}\right), \; y = \left(1, \tfrac{1}{2}, \tfrac{3}{4}\right), \; z = \left(\tfrac{1}{2}, \tfrac{1}{2}, 1\right) \in \operatorname{ext} \mathbf{B}.$$

In the context of step 3, $i_1 = 3$, $i_2 = 2$, $i_3 = 1$ and $x_{\max} = x$. After applying ι, as defined in step 3,

$$\iota(x) = \left(\tfrac{1}{2}, 1, \tfrac{3}{4}\right), \; \iota(y) = \left(\tfrac{3}{4}, \tfrac{1}{2}, 1\right), \; \iota(z) = \left(1, \tfrac{1}{2}, \tfrac{1}{2}\right) \in \operatorname{ext} \iota(\mathbf{B}).$$

Notice $\iota(x)$ and $\iota(y)$ are in the form of vertices from step 2, so we know these are illuminated by the set

$$\{(0, -\delta, -1), \; (0, \delta, 1), \; (0, -1, \delta), \; (0, 1, -\delta)\}$$

for some $\delta > 0$. $\iota(z) - 1e_1 = (0, \frac{1}{2}, \frac{1}{2})$. The entries of $\iota(x)$ are strictly greater than the entries of $\iota(z) - e_1$, so, by lemma 2.38, we know that $\iota(z) - e_1$ is an interior point of $\iota(\mathbf{B})$. Let $\delta > 0$, we can illuminate $\iota(x), \iota(y), \iota(z)$ and their coordinate reflections using

$$\mathcal{I}_\delta = \{(1,0,0),\ (-1,0,0),\ (0,-\delta,-1),\ (0,\delta,1),\ (0,-1,\delta),\ (0,1,-\delta)\}.$$

This means we can illuminate x, y, z and their coordinate reflections using

$$\iota^{-1}(\mathcal{I}_\delta) = \{(0,0,1),\ (0,0,-1),\ (-1,-\delta,0),\ (1,\delta,0),\ (\delta,-1,0),\ (-\delta,1,0)\}.$$

2. In the setting of the previous example, now suppose $z = (\frac{3}{4}, \frac{1}{2}, 1)$. So $\iota(z) = (1, \frac{1}{2}, \frac{3}{4})$. In the context of lemma 2.42, consider the affine set

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_3 = \tfrac{3}{4}\}.$$

Notice that $\iota(x)$ has strictly greater absolute values than $\iota(z) - e_1 = (0, \frac{1}{2}, \frac{3}{4})$ in the first two entries. But this means that $\iota(z) - e_1$ is an interior point of $\iota(\mathbf{B})$.

Proposition 6.4. Let $\mathbf{B} \in \mathcal{U}_p^3$ and suppose $\|e_i + e_j\|_{\mathbf{B}} = 1$ for exactly one pair of distinct $i, j \in [3]$. Then there exists $\delta > 0$ so that $\mathbf{B}$ can be illuminated by some coordinate permutation of the set

$$\begin{aligned}\mathcal{I}_\delta := \{&(1,0,0),\ (0,1,-\delta),\ (0,\delta,1),\\ &(-1,0,0),\ (0,-1,\delta),\ (0,-\delta,-1)\}.\end{aligned}$$

Proof. Suppose that $i = 2$ and $j = 3$, meaning we have vertices of the form $x = (x_1, \pm 1, \pm 1) \in \operatorname{ext} \mathbf{B}$, where $|x_1| < 1$. Let $y \in \operatorname{ext} \mathbf{B}$.

Case 1: $|I_0^y| = 2$. Then we can illuminate y by $\mathcal{I}_\delta$ by first applying lemma 2.45 then lemma 2.41.

Case 2: $|I_0^y| \in \{0, 1\}$. Lemma 2.37 and our initial assumption implies that for vertices $y \in \operatorname{ext} \mathbf{B} \backslash \{x\}$, it must be the case that $I_{\pm 1}^y \notin \{\{1,3\}, \{1,2\}, \{1,2,3\}\}$. Therefore, if

$1 \in I^y_{\pm 1}$, then $I^y_{\pm 1} = \{1\}$, so suppose this is the case. Then, we can apply lemma 5.10 to see that $\pm e_1$ illuminates y. Now suppose $1 \notin I^y_{\pm 1}$, which means that $|y_1| < 1$. If $y_2, y_3 \neq 0$, then in view of lemma 2.42, look in the affine set

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 = y_1\}$$

to see that we can use the following set to illuminate y.

$$\mathcal{J}_\delta := \{(0, 1, -\delta),\ (0, -1, \delta),\ (0, \delta, 1),\ (0, -\delta, -1)\}.$$

Since $|I^y_0| \in \{0, 1\}$, the remaining cases are either $y_2 = 0$ or $y_3 = 0$ but not $y_2 = y_3 = 0$. But in both cases, we can again apply lemma 2.45 then lemma 2.41 to see that $\mathcal{J}_\delta$ illuminates y.

Using the fact that $|\operatorname{ext} \mathbf{B}| < \infty$, minimize δ in cases 1 and 2 to attain an $\mathcal{I}_\delta$ which illuminates every $y \in \operatorname{ext} \mathbf{B}$.

In the general setting, if $i = i_2$ and $j = i_3$, then we can use the method above and the following isomorphism to get the result:

$$\iota : \mathbb{R}^3 \to \mathbb{R}^3, \quad \sum_{k \in [3]} x_{i_k} e_{i_k} \mapsto (x_{i_1}, x_{i_2}, x_{i_3}).$$

□

Notation 6.5. We will need to take perturbations of perturbed directions to illuminate some sets. For $\delta \in \mathbb{R}$, when we write η_δ, we mean that η_δ is a constant which depends on δ, and when we write ζ_{η_δ}, we mean that ζ_{η_δ} is a constant which depends on η_δ.

From this point on, we need to be careful about $\mathbf{B}$ being an affine image of the cube. This means that we impose $(\pm 1, \pm 1, \pm 1) \notin \mathbf{B}$ since $\|e_i\|_{\mathbf{B}} = 1$ for every $i \in [3]$.

Proposition 6.6. Let $\mathbf{B} \in \mathcal{U}^3_p$ which is not an affine image of the cube and suppose $\|e_i + e_j\|_{\mathbf{B}} = 1$ for exactly two pairs of distinct $i, j \in [3]$. Then one of the following holds:

1. There exists $\delta > 0$ and $\eta_\delta > 0$ so that $\mathbf{B}$ can be illuminated by some coordinate permutation of the set

$$\begin{aligned}\mathcal{I}_{\delta,\eta} := \{&(1,-\eta_\delta,-\delta),\ (\eta_\delta,1,\delta),\ (\eta_\delta,-\delta,1),\\ &(-1,\eta_\delta,\delta),\ (-\eta_\delta,-1,-\delta),\ (-\eta_\delta,\delta,-1)\}.\end{aligned}$$

2. There exists $\delta > 0$ so that $\mathbf{B}$ can be illuminated by some coordinate permutation of the set

$$\begin{aligned}\mathcal{I}_{\delta} := \{&(-1,\delta,1),\ (\delta,-1,1),\ (1,1,1),\\ &(1,-\delta,-1),\ (-\delta,1,-1),\ (-1,-1,-1)\}.\end{aligned}$$

Proof. Let $|x_1|, |x_2|, |x_3| < 1$. Suppose that $\|e_1 + e_3\|_{\mathbf{B}} = 1$ and $\|e_2 + e_3\|_{\mathbf{B}} = 1$. This means we must have vertices in the form of $(x_1, \pm 1, \pm 1)$ or $(\pm 1, x_2, \pm 1)$, but no vertices in the form of $(\pm 1, \pm 1, x_3)$.

Case 1: Supposing we have vertices of the form $(x_1, \pm 1, \pm 1)$ and $(\pm 1, x_2, \pm 1)$ (where x_1 must be the largest in absolute value first coordinate of points of the form $(\tilde{x}_1, \pm 1, \pm 1)$ in $\mathbf{B}$, and analogously for x_2), we first consider the possibility that at most one of x_1 and x_2 equals 0; that is, either both $x_1, x_2 \neq 0$ or exactly one of them is $= 0$. WLOG we also assume that $|x_2| \geq |x_1|$.

We first deal with any vertices y of $\mathbf{B}$ which satisfy $\left|I^y_{\pm 1}\right| < 2$. If $\left|I^y_{\pm 1}\right| = 0$, or if $I^y_{\pm 1}$ is equal to $\{1\}$ or to $\{2\}$, then one of $\pm e_i$ will illuminate y. Indeed, if $y = (\pm 1, y_2, y_3)$, then $y - \operatorname{sign}(y_1)e_1 = (0, y_2, y_3) \in \operatorname{int}\mathbf{B}$ given that $\lambda(|x_1|, 1, 1) + (1 - \lambda)(1, |x_2|, 1)$ will have larger respective coordinates for some λ sufficiently close to 1. If $y = (y_1, \pm 1, y_3)$, then it's even simpler to see that $y - \operatorname{sign}(y_2)e_2$ is in $\operatorname{int}\mathbf{B}$ using just the vertex $(1, |x_2|, 1)$ to compare to. If $y = (y_1, y_2, y_3)$ with either y_1 or $y_2 \neq 0$, then similarly we can use one of the directions $\pm e_1, \pm e_2$ to illuminate; the case $y_1 = y_2 = 0$ cannot arise here, because $y_3 e_3$ is not a boundary point of $\mathbf{B}$ given that $\|e_3\|_{\mathbf{B}} = 1$. Finally, if $y = (y_1, 0, \pm 1)$, then $y - \operatorname{sign}(y_3)e_3 \in \operatorname{int}\mathbf{B}$, whereas if $y = (y_1, y_2, \pm 1)$ with

$y_2 \neq 0$, then we can use lemma 2.42 applied with the affine set

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 = y_1\}$$

to see that, for any $\delta > 0$, one of the directions in the set

$$\{(0, 1, \delta), (0, -1, -\delta), (0, -\delta, 1), (0, \delta, -1)\}$$

will illuminate y.

Now we observe that if δ, and subsequently η_δ, are chosen sufficiently small, then the corresponding sets $\mathcal{I}_{\delta,\eta}$ will consist of suitable perturbations of the directions used so far, which should work too to illuminate the finitely many vertices of $\mathbf{B}$ which satisfy $\left|I^y_{\pm 1}\right| < 2$.

It remains to consider the vertices of the form $(x_1, \pm 1, \pm 1)$ and $(\pm 1, x_2, \pm 1)$. Recall that WLOG we have assumed that $|x_2| \geq |x_1|$ and that at least $x_2 \neq 0$.

For vertices of the form $(x_1, \pm 1, \pm 1)$ we can use again lemma 2.42 applied with the affine set

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 = x_1\}$$

to see that perturbations of directions in the set

$$\{(0, 1, \delta),\ (0, -1, -\delta),\ (0, -\delta, 1),\ (0, \delta, -1)\}$$

illuminate them (no matter what $\delta > 0$ is); thus, for sufficiently small η_δ, sets of the form $\mathcal{I}_{\delta,\eta}$ work for them as well.

Finally, vertices of the form $(1, \pm|x_2|, 1)$ or $(-1, \pm|x_2|, -1)$ can be illuminated by the directions $(-\eta_\delta, -1, -\delta), (-\eta_\delta, \delta, -1)$ or $(\eta_\delta, 1, \delta), (\eta_\delta, -\delta, 1)$, respectively, which appear in all sets $\mathcal{I}_{\delta,\eta}$ which we can use. On the other hand, for vertices of the form

$(1, \pm|x_2|, -1)$ or $(-1, \pm|x_2|, 1)$, we can first use lemma 2.42 applied with the affine set

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_2 = x_2\}$$

to see that directions of the form $(-1, 0, \delta)$ or $(1, 0, -\delta)$ will illuminate these vertices (no matter what $\delta > 0$ is). Thus, for sufficiently small η_δ, directions from $\mathcal{I}_{\delta,\eta}$ will also work.

Note also that, if we had $|x_1| > |x_2|$ instead, then suitable choices of the sets $\mathcal{I}_{\delta,\eta}$ would illuminate the set $\iota(\mathbf{B})$, where

$$\iota : \mathbb{R}^3 \to \mathbb{R}^3, \quad \sum_{k \in [3]} x_k e_k \mapsto (x_2, x_1, x_3).$$

Thus $\mathbf{B}$ can be illuminated by $\iota^{-1}(\mathcal{I}_{\delta,\eta})$.

Case 2: First, note that if $\operatorname{ext} \mathbf{B} = \{(0, \pm 1, \pm 1),\ (\pm 1, 0, \pm 1)\}$, then $\mathbf{B}$ is an affine image of the cube.

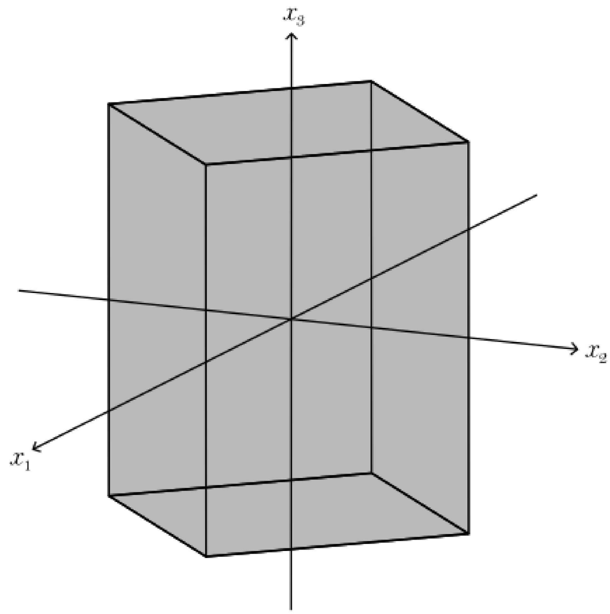


Figure 10: $\operatorname{ext}\mathbf{B} = \{(0, \pm 1, \pm 1),\ (\pm 1, 0, \pm 1)\}$.

If $x_1 = x_2 = 0$ then we must have a vertex in the form of $y = (y_1, y_2, y_3)$, where $|y_1|, |y_2|, |y_3| \leq 1$ and $|y_1| + |y_2| > 1$, otherwise $\operatorname{ext}\mathbf{B} = \{(0, \pm 1, \pm 1),\ (\pm 1, 0, \pm 1)\}$. For example, if $|y_1| + |y_2| = 1$, then

$$|y_1|(1, 0, 1) + |y_2|(0, 1, 1) = (|y_1|, |y_2|, 1).$$

And thus $(y_1, y_2, y_3) \notin \operatorname{ext}\mathbf{B}$ lemma 2.37. The geometric intuition here is that the cross section $\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_3 = c\} \cap \mathbf{B}$ must contain the 2-D cross-polytope C_1^2 for any $-1 \leq c \leq 1$. If we want to add vertices to $\mathbf{B}$, they must lie outside of these cross-polytopes.

Step 1: Suppose we have vertices in the form $(0, \pm 1, \pm 1)$ and $(\pm 1, 0, \pm 1)$. Notice by lemma 2.37, we cannot have any vertices which satisfy $|I_0^y| = 2$. So let $y \in \operatorname{ext}\mathbf{B} \backslash \{(0, \pm 1, \pm 1),\ (\pm 1, 0, \pm 1)\}$ (which we will deal with later) so that $|I_0^y| = 1$.

Since $|y_1| + |y_2| > 1$ and $\|y\|_\infty \leq 1$, it must be the case that $1, 2 \notin I_0^y$; in other words $I_0^y = \{3\}$. It should also be noted that we cannot have vertices of the form $(\pm 1, \pm 1, x_3)$ by assumption.

If we have a vertex of the form $(y_1, y_2, 0)$, where $0 < |y_1|, |y_2| < 1$, notice

$$(y_1, y_2, 0) + y_1(-1, 0, 1) = (0, y_2, y_1).$$

If we compare this point to the vertex $(0, 1, 1)$, in the context of lemma 2.42, consider the affine set

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 = 0\}$$

to see that $(y_1, y_2, 0)$ is illuminated by $\text{sign}(y_1)(-1, 0, 1)$.

Now, we will look at the vertices that still fall into this case which also satisfy $\|y\|_\infty = 1$. Suppose we have a vertex of the form $(y_1, y_2, 0)$, where $|y_2| < 1$ and $|y_1| = 1$. Consider the convex combination

$$\lambda_1(1, |y_2|, 0) + \lambda_2(0, 1, 1) + \lambda_3(1, 0, 1) \in \mathbf{B}.$$

Set $\lambda_1 = \lambda_2$ which means $\lambda_1 < \frac{1}{2}$. So

$$\lambda_1(1, |y_2|, 0) + \lambda_2(0, 1, 1) + \lambda_3(1, 0, 1) = (\lambda_1 + \lambda_3, \lambda_1(|y_2| + 1), \lambda_1 + \lambda_3) \in \mathbf{B}.$$

Set λ_1 so that

$$\lambda_1 > \frac{|y_2|}{(|y_2| + 1)},$$

which is possible since $|y_2| < 1$ and λ_1 is allowed to be arbitrarily close to $\frac{1}{2}$. Now consider

$$(y_1, y_2, 0) + \frac{y_1}{2}(-1, 0, 1) = \left(\pm\tfrac{1}{2}, y_2, \pm\tfrac{1}{2}\right).$$

Since $\sum_{k \in [3]} \lambda_k = 1$, we get $\lambda_1 + \lambda_3 > \frac{1}{2}$, and since $\lambda_1(|y_2| + 1) > |y_2|$ we can apply lemma 2.38 to see that $(y_1, y_2, 0)$ is illuminated by $-\operatorname{sign}(y_1)(-1,0,1)$. We can illuminate vertices of the form $(y_1, y_2, 0)$, where $|y_1| < 1$ and $|y_2| = 1$ by $\operatorname{sign}(y_2)(0,-1,1)$ in a similar fashion.

Step 2: Now suppose that $|I_0^y| = 0$, so our vertex is in the form of $y = (y_1, y_2, y_3)$, where $|y_1|, |y_2|, |y_3| < 1$. Because of corollary 2.39, we can ignore any direction which has opposite signs of the directions in $\mathcal{I}_\delta$. This means we only need to consider the vertices of the form $(y_1, y_2, -y_3)$ and $(-y_1, -y_2, y_3)$, where $0 < y_1, y_2, y_3 \leq 1$. Notice only one of y_1, y_2, y_3 can be 1, or else we contradict the assumption that the only vertices which satisfy $\|e_i + e_j\|_{\mathbf{B}} = 1$ are $(0, \pm 1, \pm 1)$ and $(\pm 1, 0, \pm 1)$. Let's look at vertices of the form $y = (y_1, y_2, -y_3)$. If $y_1 = 1$ or $y_3 = 1$, with lemma 2.42 we can use $(-1, 0, 1)$ and look in the affine set

$$\{(\xi_1, \xi_2, \xi_3) :\in \mathbb{R}^n : \xi_2 = y_2\}$$

to see that $(-1,0,1)$ illuminates y. Similarly, if $y_2 = 1$, then we can use $(0,-1,1)$ to illuminate y. Now use lemma 2.41 to see that $\mathcal{I}_\delta$ illuminates y for sufficiently small $\delta > 0$. Since $\mathcal{I}_\delta$ is a set constructed by pairs of opposite directions, it also illuminates vertices of the form $(-y_1, -y_2, y_3)$.

Step 3: The remaining vertices we need to consider are $(\pm 1, 0, \pm 1)$ and $(0, \pm 1, \pm 1)$. We can see for sufficiently small $\delta > 0$ that the vertex $(1, 0, -1)$ is illuminated by $(-1, \delta, 1)$. Similarly $(-1, 0, 1)$, $(0, 1, -1)$, and $(0, -1, 1)$ are illuminated by $\mathcal{I}_\delta$. However, the vertices $(1, 0, 1)$, $(-1, 0, -1)$, $(0, 1, 1)$, $(0, -1, -1)$ require slightly more attention. Consider

$$(-1, 0, -1) + \tfrac{1}{2}(1, 1, 1) = \left(-\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right).$$

Recall we must have a vertex of the form $y = (y_1, y_2, y_3)$, where $|y_1| + |y_2| > 1$ and $|y_1|, |y_2|, |y_3| \leq 1$. Since $\mathbf{B} \in \mathcal{U}_p^n$, we know that $(|y_1|, |y_2|, |y_3|) \in \operatorname{ext} \mathbf{B}$. Set

$$\lambda_1 = \frac{1}{3}, \quad \lambda_2 = \frac{-|y_1| + |y_2| + 2}{6}, \quad \lambda_3 = \frac{|y_1| - |y_2| + 2}{6},$$

then consider the convex combination

$$\lambda_1(|y_1|, |y_2|, |y_3|) + \lambda_2(1,0,1) + \lambda_3(0,1,1) \\ = \left(\frac{|y_1| + |y_2| + 2}{6}, \frac{|y_1| + |y_2| + 2}{6}, \frac{|y_3| + 2}{3}\right).$$

Since $\frac{|y_1|+|y_2|+2}{6}, \frac{|y_3|+2}{3} > \frac{1}{2}$ we can apply lemma 2.38 to see that $(1,1,1)$ illuminates $(-1,0,-1)$. Similarly, $(1,0,1)$, $(0,1,1)$, $(0,-1,-1)$ are also illuminated by $\mathcal{I}_\delta$.

Suppose $e_{i_2} + e_{i_3}, e_{i_1} + e_{i_3} \in \operatorname{ext} \mathbf{B}$, then the isomorphism to take here is

$$\iota : \mathbb{R}^3 \to \mathbb{R}^3, \quad \sum_{n \in [3]} x_{i_n} e_{i_n} \mapsto (x_{i_1}, x_{i_2}, x_{i_3}).$$

Finally since $\mathbf{B} \in \mathcal{U}_p^n$, we can use a minimizing argument to attain δ and η_δ to get the result.

□

Example 6.7. An example of the second part of proposition 6.6 can be seen pictorially in the figure below.

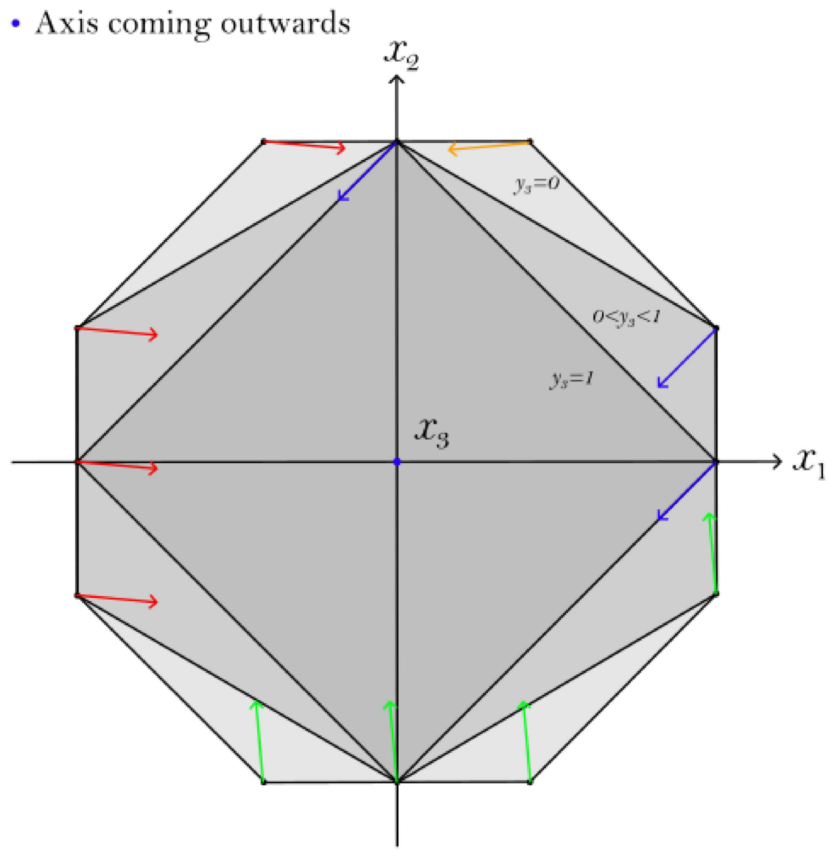


Figure 11: Proposition 6.6 example.

The top most layer ($y_3 = 1$), middle ($0 < y_3 < 1$) and bottom layer ($y_3 = 1$) are a cross-polytope, hexagon and octagon, respectively. The red, green and blue vectors represent the directions $(1, -\delta, -1)$, $(-\delta, 1, -1)$ and $(-1, -1, -1)$, respectively. The orange direction is $-(1, -\delta, -1)$, the negative vector of the red vector. Recall that when we are not looking at what is the affine image of the cube, we required a vertex $y \in \text{ext}\,\mathbf{B}$ so that $|y_1| + |y_2| > 1$. This is precisely why the blue vector $(-1, -1, -1)$ is able to illuminate $(1, 0, 1)$ and $(0, 1, 1)$.

If we flip 11 so that x_3 is entering the page, we can use the negative directions of the directions we have already listed. Since the orange and red directions are opposite directions already, we will only need 3 pairs of opposite directions in total to illuminate this convex body.

Proposition 6.8. Let $\mathbf{B} \in \mathcal{U}_p^3$ which is not an affine image of the cube and suppose $\|e_i + e_j\|_{\mathbf{B}} = 1$ for all $i, j \in [3]$, $i \neq j$. Then there exists $\delta > 0$ and $\eta_\delta > 0$ so that $\mathbf{B}$ can be illuminated by

$$\mathcal{I}_{\delta,\eta} := \{(1, -\eta_\delta, -\delta),\ (\eta_\delta, 1, \delta),\ (\eta_\delta, -\delta, 1),\\ (-1, \eta_\delta, \delta),\ (-\eta_\delta, -1, -\delta),\ (-\eta_\delta, \delta, -1)\}.$$

Proof. First, we should note that it is only possible to have vertices

$$x \in \operatorname{ext} \mathbf{B} \backslash \{(\pm 1, 0, \pm 1),\ (0, \pm 1, \pm 1),\ (\pm 1, \pm 1, 0)\}$$

which satisfy $|I_0^x| = 0$. With lemma 2.37 in mind, we can see this by comparing the points $e_i + e_j \in \mathbf{B}$, where $i, j \in [3]$, $i \neq j$, with any such vectors which do not satisfy $|I_0^x| = 0$. Additionally if a vertex does not satisfy $I_{\pm 1}^x = 2$, then we can use $\{\pm e_i : i \in [3]\}$ to illuminate it. For example $(1, \frac{1}{2}, \frac{1}{2}) - e_1 = (0, \frac{1}{2}, \frac{1}{2}) \in \operatorname{int} \mathbf{B}$ as $(0, 1, 1) \in \mathbf{B}$. Note that the set $\{\pm e_i : i \in [3]\}$ can be perturbed into $\mathcal{I}_{\delta,\eta}$.

Consider a vertex of the form $x = (x_1, x_2, x_3)$, where $|x_1| < 1$ and $|x_2| = |x_3| = 1$. With lemma 2.42 in mind, consider the affine set

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^n, \xi_1 = x_1\}$$

to see that such vertices are illuminated by the set

$$\{(0, 1, \delta),\ (0, -\delta, 1),\ (0, -1, -\delta),\ (0, \delta, -1)\}.$$

Analogously if $|x_2| < 1$, we can also illuminate $(1, x_2, -1)$ and $(-1, x_2, 1)$ using $(-1, 0, \delta)$ and $(1, 0, -\delta)$, respectively. It remains to be seen how to pick δ and η_δ so that the set $\mathcal{I}_{\delta,\eta}$ can illuminate vertices of the form $(1, x_2, 1)$, $(-1, x_2, -1)$ or $(\pm 1, \pm 1, x_3)$.

For a vertex of the form $(1, 1, x_3)$, where $|x_3| < 1$, pick δ so that $|x_3| + \delta < 1$ and

$0 < \eta_\delta < 2$ and consider

$$(1, 1, x_3) + (-\eta_\delta, -1, -\delta) = (1 - \eta_\delta,\ 0,\ x_3 - \delta).$$

With lemma 2.42 in mind, look in the affine set

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_2 = 0\}$$

to see that we can illuminate x using $(-\eta_\delta, -1, -\delta)$. With this line of reasoning, we construct the following table:

Type of vertex	Illuminating direction	Type of vertex	Illuminating direction
$(1, 1, x_3)$	$(-\eta_\delta, -1, -\delta)$	$(-1, -1, x_3)$	$(\eta_\delta, 1, \delta)$
$(-1, 1, x_3)$	$(1, -\eta_\delta, -\delta)$	$(1, -1, x_3)$	$(-1, \eta_\delta, \delta)$
$(1, x_2, 1)$	$(-\eta_\delta, \delta, -1)$	$(-1, x_2, -1)$	$(\eta_\delta, -\delta, 1)$

Table 8: Summary of method.

□

Theorem 6.9. Let $\mathbf{B} \in \mathcal{U}_p^3$ which is not an affine image of the cube, then $\mathcal{I}(\mathbf{B}) \leq 6$. In fact, the illuminating sets are all 3 pairs of opposite directions.

As already mentioned, at the end of Lassak's paper, one of the questions is: for centrally symmetric convex bodies, which are not parallelotopes, does there exist an illuminating set constructed by 3 pairs of opposite directions [14]? Theorem 6.9 gives a positive answer to this question in a less general setting. In the next section, we will see how far we can push these methods into dimension 4.

7 Dimension 4, 1-Unconditional Polytopes

In this section, we will focus on 1-unconditional convex polytopes which are not affine images of the cube in dimension 4. We apply similar methods to that of the previous section and are able to show that to prove the Illumination Conjecture for 1-unconditional convex polytopes up to the equality case: for $\mathbf{B} \in \mathcal{U}_p^4$, we have shown that $\mathcal{I}(\mathbf{B}) \leq 16$ (Theorem 7.15).

Since $\|e_i\|_{\mathbf{B}} = 1$ for any $i \in [4]$, when we say that $\mathbf{B}$ is not an affine image of the cube, this means that $(\pm 1, \pm 1, \pm 1, \pm 1) \notin \mathbf{B}$.

Again, the strategy here is to first solve the problem when $\|e_i + e_j\|_{\mathbf{B}} > 1$ and impose the condition $\|e_i + e_j\|_{\mathbf{B}} = 1$ for pairs of distinct $i, j \in [4]$. In each case, we will gradually increase the number of pairs of distinct i, j. As we impose $\|e_i + e_j\|_{\mathbf{B}} = 1$ for three or more pairs of distinct $i, j \in [4]$, it is possible for the convex body $\mathbf{B}$ to satisfy $\|e_r + e_s + e_t\|_{\mathbf{B}} = 1$ for a triple of distinct $r, s, t \in [4]$. We will first assume that $\|e_r + e_s + e_t\|_{\mathbf{B}} > 1$ for every triple of distinct $r, s, t \in [4]$ and see what happens when we impose the condition $\|e_i + e_j\|_{\mathbf{B}} = 1$ for more pairs of distinct $i, j \in [4]$. Then, we will deal with the case $\|e_r + e_s + e_t\|_{\mathbf{B}} = 1$ for some triple of distinct $r, s, t \in [4]$ after.

Although for $\mathbf{B} \in \mathcal{U}_p^4$ we have shown that $\mathcal{I}(\mathbf{B}) \leq 16$, there also many cases in this section where $\mathcal{I}(\mathbf{B}) \leq 14$. This can be seen in propositions 7.4, 7.5, 7.6 and 7.9. For the sake of transparency, we suspect that with this method of proof, it is possible to construct illuminating sets with cardinality of no greater than 14 for any 1-unconditional polytope. A step in this direction is to analyze propositions 7.7, 7.12 and 7.14 more carefully in the sense that we avoid situations where $\mathbf{B}$ becomes an affine image of the cube. We already assume that $(\pm 1, \pm 1, \pm 1, \pm 1) \notin \mathbf{B}$; however, there are more situations where $\mathbf{B}$ can be an affine image of the cube which we did not account for in the mentioned propositions. This is a good place to continue this research.

A summary of results is given in the following table.

Dimension 4, 1-unconditional (Theorem 7.15)				
$\forall r,s,t \in [4],\ \|e_r+e_s+e_t\|_{\mathbf{B}} > 1$			$\exists r,s,t \in [4],\ \|e_r+e_s+e_t\|_{\mathbf{B}} = 1$	
Pairs of distinct i,j so that $\|e_i+e_j\|_{\mathbf{B}} = 1$	$\mathcal{I}(\mathbf{B})$ upper bound	Proposition	$\mathcal{I}(\mathbf{B})$ upper bound	Proposition
0	16	7.1	16	7.14
1	14	7.4		
2	14	7.5		
3	14	7.6		
4	16	7.7		
5	14	7.9		
6	16	7.12		

Table 9: Summary of results in dimension 4 for 1-unconditional convex polytopes.

Proposition 7.1. Let $\mathbf{B} \in \mathcal{U}_p^4$ and suppose $\|e_i+e_j\|_{\mathbf{B}} > 1$ for every $i,j \in [4]$. Then there exists $\delta > 0$, $\eta_\delta > 0$ so that $\mathbf{B}$ can be illuminated by some coordinate permutations of the set $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_{\delta,\eta}$, where

$$\begin{aligned}
\mathcal{I}_1 &= \{(1,0,1,0),\ (1,0,-1,0),\ (-1,0,1,0),\ (-1,0,-1,0)\},\\
\mathcal{I}_2 &= \{(0,1,0,1),\ (0,1,0,-1),\ (0,-1,0,1),\ (0,-1,0,-1)\},\\
\mathcal{I}_{\delta,\eta} &= \{(0,-1,-\delta,-\eta_\delta),\ (-1,\eta_\delta,0,-\delta),\ (0,-\delta,1,-\eta_\delta),\ (\delta,\eta_\delta,0,-1),\\
&\qquad -(0,-1,-\delta,-\eta_\delta),\ -(-1,\eta_\delta,0,-\delta),\ -(0,-\delta,1,-\eta_\delta),\ -(\delta,\eta_\delta,0,-1)\}.
\end{aligned}$$

Note that $|\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_{\delta,\eta}| = 16$.

Remark 7.2. We will first show that if $x \in \operatorname{ext}\mathbf{B}$ and $|I_0^x| \in \{2,3\}$, then we can illuminate these vertices using perturbations on $\pm e_i$ for every $i \in [4]$. When $|I_0^x| = 1$, we will show that $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_{\delta,\eta}$ is sufficient to illuminate such vertices. In the final case, when $|I_0^x| = 0$, we first will make a key assumption favouring two coordinates over the other two. We will address this assumption by showing that it does not matter in the sense that a bijective transformation of $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_{\delta,\eta}$ will compensate for the assumption when it's more appropriate to favor another pair of coordinates. Additionally, this transformation will simply be a swapping of coordinates, and since the cases $|I_0^x| \in \{1, 2, 3\}$ do not favor any entry, after we apply the transformation onto $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_{\delta,\eta}$, it will not affect the other cases.

Proof. Let $\mathbf{B} \in \mathcal{U}_p^4$ and suppose $\|e_i + e_j\|_{\mathbf{B}} > 1$ for every $i, j \in [4]$.

Case 1: Let $x \in \operatorname{ext} \mathbf{B}$ be a vertex so that $|I_0^x| \in \{2, 3\}$. We can illuminate x by $\mathcal{I}_{\delta,\eta}$ by first applying lemma 2.45 then lemma 2.41.

Case 2: Now, we will look at the vertices $x \in \operatorname{ext} \mathbf{B}$ which satisfy $|I_0^x| = 1$. Consider a vertex in the form of $(x_1, x_2, x_3, 0) \in \operatorname{ext} \mathbf{B}$, where $|x_1| = 1$ and $0 < |x_2|, |x_3| < 1$. By the assumption in the lemma, it cannot be the case that either $|x_2| = 1$ or $|x_3| = 1$. In the setting of lemma 2.42, consider the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_4 = 0,\ \xi_2 = x_2\}$$

to see that $\mathcal{I}_1$ illuminates such vertices. Using this method, we can construct the following table:

Type of vertex	Illuminating set
$(\pm 1, x_2, x_3, 0)$, $(x_1, 0, \pm 1, x_4)$, $(x_1, x_2, \pm 1, 0)$, $(\pm 1, 0, x_3, x_4)$	$\mathcal{I}_1$
$(x_1, \pm 1, 0, x_4)$, $(0, x_2, x_3, \pm 1)$, $(0, \pm 1, x_3, x_4)$, $(x_1, x_2, 0, \pm 1)$	$\mathcal{I}_2$

Table 10: Summary of method.

If $\|x\|_\infty < 1$, the argument for the vertices in table 10 still holds for any such vertex $x \in \operatorname{ext} \mathbf{B}$ that satisfies $|I_0^x| = 1$.

Now let $x \in \operatorname{ext} \mathbf{B}$ so that $|x_2| = 1$, $|x_4| = 0$ and $|x_1|, |x_3| > 0$. So x is in the form of $(x_1, \pm 1, x_3, 0)$. We look in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = x_1,\ \xi_4 = 0\}.$$

We can apply lemma 2.42 to see that for any $\delta > 0$, either $(0, -\operatorname{sign}(x_2), -\delta \operatorname{sign}(x_3), 0)$ or $(0, -\delta \operatorname{sign}(x_2), -\operatorname{sign}(x_3), 0)$ illuminate x, hence x can be illuminated by the set

$$\{(0,-1,-\delta,0),\ (0,-\delta,1,0),\ -(0,-1,-\delta,0),\ -(0,-\delta,1,0)\}.$$

Therefore by lemma 2.41, there exists $\eta_\delta > 0$ so that $\mathcal{I}_{\delta,\eta}$ illuminates x. In a similar fashion, we can illuminate the vertices in the table below using $\mathcal{I}_{\delta,\eta}$.

Type of vertex	Illuminating set
$(x_1, \pm 1, x_3, 0)$, $(0, x_2, \pm 1, x_4)$	$\{(0,-1,-\delta,-\eta_\delta)$, $(0,-\delta,1,-\eta_\delta)$, $-(0,-1,-\delta,-\eta_\delta)$, $-(0,-\delta,1,-\eta_\delta)\}$
$(\pm 1, x_2, 0, x_4)$, $(x_1, 0, x_3, \pm 1)$	$\{(-1,\eta_\delta,0,-\delta)$, $(\delta,\eta_\delta,0,-1)$, $-(-1,\eta_\delta,0,-\delta)$, $-(\delta,\eta_\delta,0,-1)\}$

Table 11: Summary of method.

So we have illuminated all the vertices, $x \in \operatorname{ext} \mathbf{B}$, which satisfy $|x_i| = 1$ for exactly one $i \in [4]$ and $|I_0^x| = 1$. This is because we have illuminated 12 different types of vertices which satisfy this condition, but since we can only choose four positions to put 0 as an entry, with three remaining spots for ± 1, there are only $4 \times 3 = 12$ types of vertices.

Due to the usage of lemma 2.41 in case 1 and the later part of case 2, each δ and η_δ depend on x_i, $i \in [4]$ for each type of vertex. We will address this after case 3.

Case 3: Consider the case when $x = (x_1, x_2, x_3, x_4) \in \operatorname{ext} \mathbf{B}$ and $|I_0^x| = 0$. In addition, suppose $0 < x_1, x_2, x_3, x_4 < 1$ or $x_i = 1$ for exactly one $i \in \{2, 4\}$. Using lemma 2.42 and either the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^n : \xi_1 = x_1\} \quad \text{or} \quad \{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^n : \xi_3 = x_3\}$$

we can construct the following table:

Type of vertex		Illuminating direction
(x_1, x_2, x_3, x_4),	$(-x_1, x_2, x_3, x_4)$	$(0, -1, -\delta, -\eta_\delta)$
$-(x_1, x_2, x_3, x_4)$,	$-(-x_1, x_2, x_3, x_4)$	$-(0, -1, -\delta, -\eta_\delta)$
$(x_1, -x_2, x_3, x_4)$,	$(x_1, -x_2, -x_3, x_4)$	$(-1, \eta_\delta, 0, -\delta)$
$-(x_1, -x_2, x_3, x_4)$,	$-(x_1, -x_2, -x_3, x_4)$	$-(-1, \eta_\delta, 0, -\delta)$
$(x_1, x_2, -x_3, x_4)$,	$(-x_1, x_2, -x_3, x_4)$	$(0, -\delta, 1, -\eta_\delta)$
$-(x_1, x_2, -x_3, x_4)$,	$-(-x_1, x_2, -x_3, x_4)$	$-(0, -\delta, 1, -\eta_\delta)$
$(-x_1, -x_2, x_3, x_4)$,	$(-x_1, -x_2, -x_3, x_4)$	$(\delta, \eta_\delta, 0, -1)$
$-(-x_1, -x_2, x_3, x_4)$,	$-(-x_1, -x_2, -x_3, x_4)$	$-(\delta, \eta_\delta, 0, -1)$

Table 12: Summary of method.

Suppose that $x_{\max} = (x_{\max,1}, x_{\max,2}, x_{\max,3}, x_{\max,4}) \in \operatorname{ext} \mathbf{B}$ so that $|x_{\max,4}| = 1$ and $1 > |x_{\max,2}| \geq |x_i|$ for all $x \in \operatorname{ext} \mathbf{B}$ with $|I_0^x| = 0$ and for all $i \in [4]$ except when $|x_i| = 1$. We imposed that $|x_{\max,4}| = 1$ so we know that this vertex can be illuminated by some direction in table 12. Since $\mathbf{B} \in \mathcal{U}_p^4$, let's assume that $x_{\max,i} > 0$ for all $i \in [4]$. Let $0 < \lambda < 1$, $x \in \operatorname{ext} \mathbf{B}$ so that $|x_1| = 1$ and consider the convex combination

$$\lambda x_{\max} + (1 - \lambda)(|x_1|, |x_2|, |x_3|, |x_4|) \in \mathbf{B}.$$

Comparing the vector above with x,

$$\begin{aligned} x_{\max,1} &< \lambda x_{\max,1} + (1 - \lambda)|x_1| < 1 = |x_1| \\ |x_2| &\leq \lambda x_{\max,2} + (1 - \lambda)|x_2| \leq x_{\max,2} \\ 0 &< \lambda x_{\max,3} + (1 - \lambda)|x_3| \\ |x_4| &< \lambda x_{\max,4} + (1 - \lambda)|x_4| < 1 = x_{\max,4}. \end{aligned}$$

This implies that

$$(\lambda x_{\max,1} + (1 - \lambda)|x_1|, x_2, 0, \lambda x_{\max,4} + (1 - \lambda)|x_4|) \in \mathbf{B}.$$

Choose λ sufficiently close to 0 so that $\lambda x_{\max,1} + (1-\lambda)|x_1| > |x_1| - |x_3|$ and compare the above vector to

$$x + |x_3|(-\operatorname{sign}(x_1), 0, -\operatorname{sign}(x_3), 0) = (x_1 - \operatorname{sign}(x_1)x_3, x_2, 0, x_4).$$

With lemma 2.42 in mind, consider the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^n : \xi_2 = x_2,\ \xi_3 = 0\}$$

to see that $(-\operatorname{sign}(x_1), 0, -\operatorname{sign}(x_3), 0)$ illuminates x. Thus $\mathcal{I}_1$ illuminates x. When $x \in \operatorname{ext}\mathbf{B}$ with $|I_0^x| = 0$ and $|x_3| = 1$, it is analogously illuminated by $\mathcal{I}_1$.

In case 3, we made the key assumptions:

1. $1 > |x_{\max,2}| \geq |x_i|$ for all $i \in [4] \backslash I_{\pm 1}^x$ for all $x \in \operatorname{ext}\mathbf{B}$ with $|I_0^x| = 0$,
2. and $x_{\max,4} = 1$.

Now assume this condition for $x_{\max,i_2}$ in place of $x_{\max,2}$ and that $x_{\max,i_4} = 1$ for some $i_2, i_4 \in [4]$. Consider the linear isomorphism

$$\iota : \mathbb{R}^4 \to \mathbb{R}^4, \quad \sum_{n\in[4]} z_{i_n} e_{i_n} \mapsto (z_{i_1}, z_{i_2}, z_{i_3}, z_{i_4}).$$

We can apply the method from case 3 to see that $\mathcal{I}_1 \cup \mathcal{I}_{\delta,\eta}$ illuminates $\iota(\mathbf{B})$. Hence $\iota^{-1}(\mathcal{I}_1 \cup \mathcal{I}_{\delta,\eta})$ illuminates $\mathbf{B}$.

The last item to address is that δ and η_δ from steps 1 and 2 are not completely arbitrary; however, since $\mathbf{B} \in \mathcal{U}_p^4$, we know that $|\operatorname{ext}\mathbf{B}| < \infty$. Recall lemma 2.41. This means we can choose a sufficiently small $\delta > 0$ so that all the vertices from step 1 and the first part of step 2 are illuminated by $\mathcal{I}_{\delta,0}$. Then choose a sufficiently small $\eta_\delta > 0$ so that all those mentioned vertices are still illuminated by $\mathcal{I}_{\delta,\eta}$. □

The following lemma is an attempt at a partial improvement of proposition 7.1.

Lemma 7.3. Let $\mathbf{B} \in \mathcal{U}_p^4$, suppose $\|e_i + e_j\|_{\mathbf{B}} > 1$ for every $i, j \in [4]$. Suppose that for every $x \in \operatorname{ext} \mathbf{B}$, we have $|I_0^x| \neq 1$. Then there exists $\delta > 0$, $\eta_\delta > 0$, $\zeta_{\eta_\delta} > 0$ so that $\mathbf{B}$ can be illuminated by some coordinate permutation of the set $\mathcal{I}_3 \cup \mathcal{I}_{\delta,\eta,\zeta}$, where

$$\begin{aligned}
\mathcal{I}_3 &= \{(1,0,-1,0),\ (-1,0,1,0)\}\\
\mathcal{I}_{\delta,\eta,\zeta} &= \{(\zeta_{\eta_\delta}, -\delta, 1, -\eta_\delta),\ (-\zeta_{\eta_\delta}, -1, -\delta, -\eta_\delta),\ (-1, \eta_\delta, -\zeta_{\eta_\delta}, -\delta),\ (\delta, \eta_\delta, \zeta_{\eta_\delta}, -1),\\
&\quad -(\zeta_{\eta_\delta}, -\delta, 1, -\eta_\delta),\ -(-\zeta_{\eta_\delta}, -1, -\delta, -\eta_\delta),\ -(-1, \eta_\delta, -\zeta_{\eta_\delta}, -\delta),\ -(\delta, \eta_\delta, \zeta_{\eta_\delta}, -1)\}.
\end{aligned}$$

Note that $|\mathcal{I}_3 \cup \mathcal{I}_{\delta,\eta,\zeta}| = 10$.

Proof. In proposition 7.1, case 1 and case 3 only require $\mathcal{I}_{\delta,\eta,\zeta}$ and $\mathcal{I}_1 \cup \mathcal{I}_{\delta,\eta,\zeta}$, respectively. For $x \in \operatorname{ext} \mathbf{B}$, when $|I_0^x| = 0$, recall that $\mathcal{I}_{\delta,\eta,\zeta}$ can illuminate x as long as $I_{\pm 1}^x \in \{\varnothing, \{2\}, \{4\}\}$. If $I_{\pm 1}^x \in \{\{1\}, \{3\}\}$, with corollary 2.39 in mind, it can be verified that the only vertices which are not illuminated by $\mathcal{I}_{\delta,\eta,\zeta}$ are ones which satisfy $\operatorname{sign}(x_1) = -\operatorname{sign}(x_3)$. These vertices are illuminated by $\mathcal{I}_3$ using a similar method to case 3 of proposition 7.1

□

Proposition 7.4. Let $\mathbf{B} \in \mathcal{U}_p^4$, and suppose $\|e_i + e_j\|_{\mathbf{B}} = 1$ for exactly one pair of distinct $i, j \in [4]$. Then there exists $\delta > 0, \eta_\delta > 0$, and $\zeta_{\eta_\delta} > 0$ so that $\mathbf{B}$ can be illuminated by some coordinate permutation of the set

$$\begin{aligned}
\mathcal{I}_{\delta,\eta} = \{&(1, \delta, \eta_\delta, 0),\ -(1, \delta, \eta_\delta, 0),\ (\delta, -1, \eta_\delta, 0),\ -(\delta, -1, \eta_\delta, 0),\\
&(-\eta_\delta, \delta, 1, 0),\ -(-\eta_\delta, \delta, 1, 0),\ (\eta_\delta, \delta, -1, 0),\ -(\eta_\delta, \delta, -1, 0),\\
&(-\eta_\delta, 0, \delta, 1),\ -(-\eta_\delta, 0, \delta, 1),\ (-\eta_\delta, 0, \delta, -1),\ -(-\eta_\delta, 0, \delta, -1),\\
&(1, 0, 1, 0),\ (-1, 0, -1, 0)\}
\end{aligned}$$

Note that $|\mathcal{I}_{\delta,\eta,\zeta}| = 14$.

Proof. Suppose $i = 1$ and $j = 2$, so we must have a vertex of the form $y = (y_1, y_2, y_3, y_4) \in \operatorname{ext} \mathbf{B}$ so that $|y_1| = |y_2| = 1$ and $|y_3|, |y_4| < 1$. Let $x \in \operatorname{ext} \mathbf{B}$.

Case 1: Suppose $|I_0^x| \in \{2, 3\}$. With the exception of $(\pm 1, \pm 1, 0, 0)$, such vertices are illuminated by $\{\pm e_i : i \in [4]\}$. So by lemma 2.45, these vertices can also be illuminated by $\mathcal{I}_{\delta,\eta}$ for sufficiently small $\delta, \eta_\delta > 0$. For any $\delta > 0$, $(\pm 1, \pm 1, 0, 0)$ can be illuminated by the set

$$\{(1, \delta, 0, 0),\ (\delta, -1, 0, 0),\ -(1, \delta, 0, 0),\ -(\delta, -1, 0, 0)\}.$$

So there exists sufficiently small $\eta_\delta > 0$, so that $(\pm 1, \pm 1, 0, 0)$ can be illuminated by the set

$$\{(1, \delta, \eta_\delta, 0),\ -(1, \delta, \eta_\delta, 0),\ (\delta, -1, \eta_\delta, 0),\ -(\delta, -1, \eta_\delta, 0)\}.$$

Case 2: Suppose $|I_0^x| = 1$. Using the reasoning from step 2 of proposition 7.1, we can construct the following table:

Type of vertex	Illuminating set
$(x_1, \pm 1, x_3, 0)$, $(x_1, \pm 1, 0, x_4)$, $(\pm 1, x_2, x_3, 0)$, $(\pm 1, x_2, 0, x_4)$, $(\pm 1, \pm 1, x_3, 0)$, $(\pm 1, \pm 1, 0, x_4)$	$\{(1, \delta, \eta_\delta, 0)$, $(\delta, -1, \eta_\delta, 0)$, $-(1, \delta, \eta_\delta, 0)$, $-(\delta, -1, \eta_\delta, 0)\}$
$(x_1, x_2, \pm 1, 0)$, $(0, \pm 1, x_3, x_4)$, $(0, x_2, \pm 1, x_4)$	$\{(-\eta_\delta, \delta, 1, 0)$, $(\eta_\delta, \delta, -1, 0)$, $-(-\eta_\delta, \delta, 1, 0)$, $-(\eta_\delta, \delta, -1, 0)\}$
$(x_1, 0, x_3, \pm 1)$, $(0, x_2, x_3, \pm 1)$, $(x_1, 0, \pm 1, x_4)$	$\{(-\eta_\delta, 0, \delta, 1)$, $(-\eta_\delta, 0, \delta, -1)$, $-(-\eta_\delta, 0, \delta, 1)$, $-(-\eta_\delta, 0, \delta, -1)\}$

Set $\eta_\delta = 0$ and use lemma 2.42 with the appropriate affine set. Then use lemma 2.41 to get sufficiently small non zero values of η_δ.

Table 13: Summary of method.

The remaining vertices are of the form

$$(x_1, x_2, 0, \pm 1) \quad \text{and} \quad (\pm 1, 0, x_3, x_4),$$

where $|x_1|, |x_2|, |x_3|, |x_4| < 1$.

First consider a vertex of the form $(x_1, x_2, 0, \pm 1)$. Since $(1, 1, 0, 0) \in \mathbf{B}$ and

$|x_1|, |x_2| < 1$, the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = x_1,\ \xi_2 = x_2\}$$

has non trivial intersection with $\operatorname{int} \mathbf{B}$. Therefore we can use lemma 2.42 to see that vertices of the form $(x_1, x_2, 0, \pm 1)$ can be illuminated by $\pm e_4$. Then we can use lemma 2.41 to see that for sufficiently small $\delta > 0$ and $\eta_\delta > 0$,

$$\{(-\eta_\delta, 0, \delta, 1),\ -(-\eta_\delta, 0, \delta, 1),\ (-\eta_\delta, 0, \delta, -1),\ -(-\eta_\delta, 0, \delta, -1)\}$$

is an illuminating set for such vertices.

Now we will consider a vertex of the form $(\pm 1, 0, x_3, x_4)$. With lemma 2.42 in mind, consider the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_2 = 0\}$$

to see that for any $\delta > 0$ and $\eta_\delta > 0$, the set

$$\{(-\eta_\delta, 0, \delta, 1),\ -(-\eta_\delta, 0, \delta, 1),\ (-\eta_\delta, 0, \delta, -1),\ -(-\eta_\delta, 0, \delta, -1)\}$$

illuminates every vertex of the form $(\pm 1, 0, x_3, x_4)$ which satisfies $\operatorname{sign}(x_1) = -\operatorname{sign}(x_3)$. Using lemma 2.42 again, look in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_2 = 0,\ \xi_4 = x_4\}$$

to see that $\pm(1, 0, 1, 0)$ illuminates every vertex of the form $(\pm 1, 0, x_3, x_4)$ which satisfies $\operatorname{sign}(x_1) = \operatorname{sign}(x_3)$.

Case 3: Suppose $|I_0^x| = 0$. If $|x_4| < 1$, observe that the set

$$\begin{aligned}\{&(1, \delta, \eta_\delta, 0),\ -(1, \delta, \eta_\delta, 0),\ (\delta, -1, \eta_\delta, 0),\ -(\delta, -1, \eta_\delta, 0),\\ &(-\eta_\delta, \delta, 1, 0),\ -(-\eta_\delta, \delta, 1, 0),\ (\eta_\delta, \delta, -1, 0),\ -(\eta_\delta, \delta, -1, 0)\}\end{aligned}$$

has the following property: any two elements from the set differ in sign in at least one of their first three corresponding entries. This means that we can rely on lemma 2.42 and look in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_4 = x_4\}$$

to see that the set above illuminates x.

If $|x_4| = 1$, then $|x_1|, |x_2| < 1$. And since $(1, 1, 0, 0) \in \mathbf{B}$, we can use lemma 2.42 and look in the set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = x_1,\ \xi_2 = x_2\}$$

to see that for any $\delta > 0$,

$$\{(0, 0, \delta, 1),\ -(0, 0, \delta, 1),\ (0, 0, \delta, -1),\ -(0, 0, \delta, -1)\}$$

illuminates x. Now apply lemma 2.41 to see that

$$\{(-\eta_\delta, 0, \delta, 1),\ -(-\eta_\delta, 0, \delta, 1),\ (-\eta_\delta, 0, \delta, -1),\ -(-\eta_\delta, 0, \delta, -1)\}$$

also illuminates x.

To get the result for the general case, if $i = i_1$ and $j = i_2$, then consider the isomorphism:

$$\iota : \mathbb{R}^4 \to \mathbb{R}^4, \quad \sum_{k \in [4]} x_{i_k} e_{i_k} \mapsto (x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}).$$

□

Proposition 7.5. Let $\mathbf{B} \in \mathcal{U}_p^4$, and suppose $\|e_i + e_j\|_{\mathbf{B}} = 1$ for exactly two pairs of distinct $i, j \in [4]$. Then $\mathcal{I}(\mathbf{B}) \leq 14$.

Proof. Let $x \in \operatorname{ext} \mathbf{B}$.

Step 1: Suppose $(1, 1, y_3, y_4), (z_1, z_2, 1, 1) \in \operatorname{ext} \mathbf{B}$, where $0 \leq |z_1|, |z_2|, |y_3|, |y_4| < 1$. Notice we cannot have vertices which satisfy $|I_0^x| = 3$. Suppose $|I_0^x| = 2$. For vertices which are not $(1,1,0,0)$ or $(0,0,1,1)$, by lemma 2.45, we can illuminate them by using $\{\pm e_i : i \in [4]\}$. Let $\delta > 0$. By lemma 2.42, $(1,1,0,0)$ and $(0,0,1,1)$ can be illuminated by

$$\begin{aligned}\mathcal{I}_\delta := \{&(1,\delta,0,0),\ (-\delta,1,0,0),\ (0,0,1,-\delta),\ (0,0,\delta,1),\\ &-(1,\delta,0,0),\ -(-\delta,1,0,0),\ -(0,0,1,-\delta),\ -(0,0,\delta,1)\}.\end{aligned}$$

For the case $|I_0^x| \in \{0,1\}$, let's take a look at the following example: consider $x = (x_1, x_2, x_3, 0)$, where $0 < |x_1|, |x_2| \le 1$ and $0 < |x_3| < 1$. Then x is illuminated by the set

$$\{(1,\delta,0,0),\ (-\delta,1,0,0),\ -(1,\delta,0,0),\ -(-\delta,1,0,0)\}$$

for any $\delta > 0$. If $|x_3| = 1$, then $|x_1|, |x_2| < 1$.

$$x - \operatorname{sign}(x_3)e_3 = (x_1, x_2, 0, 0).$$

With lemma 2.42, compare the above vector with $(1,1,0,0)$ to see that it is an interior point, hence x is illuminated by $-\operatorname{sign}(x_3)e_3$. It turns out for any $x \in \operatorname{ext}\mathbf{B}$ so that $|I_0^x| \in \{0,1\}$, x can be illuminated in a similar method. Finally, minimize δ so that $\mathcal{I}_\delta$ is an illuminating set for $\mathbf{B}$.

Step 2: Suppose $(1,1,y_3,y_4), (z_1,1,1,z_4) \in \operatorname{ext}\mathbf{B}$, where $0 < |y_3|, |y_4|, |z_1|, |z_4| < 1$. For the same reasons from the previous step, there exists sufficiently small $\delta > 0$ and η_δ so that we can use the following set to illuminate vertices, x which satisfy $|I_0^x| \in \{1,2,3\}$:

$$\begin{aligned}I_{\delta,\eta,\zeta} := \{&(1,-\delta,-\eta_\delta,\zeta_{\eta_\delta}),\ (-1,\delta,-\eta_\delta,\zeta_{\eta_\delta}),\ (\delta,1,\eta_\delta,\zeta_{\eta_\delta}),\ (-\delta,-1,\eta_\delta,\zeta_{\eta_\delta}),\\ &(-\eta_\delta,\delta,1,-\zeta_{\eta_\delta}),\ (\eta_\delta,-\delta,1,-\zeta_{\eta_\delta}),\ (-\eta_\delta,-\delta,-1,-\zeta_{\eta_\delta}),\ (\eta_\delta,\delta,-1,-\zeta_{\eta_\delta}),\\ &(\eta_\delta,-\zeta_{\eta_\delta},\delta,1),\ (\eta_\delta,\zeta_{\eta_\delta},-\delta,1),\ (-\eta_\delta,-\zeta_{\eta_\delta},\delta,-1),\ (-\eta_\delta,\zeta_{\eta_\delta},-\delta,-1),\\ &(-\delta,0,0,1),\ (\delta,0,0,-1)\}.\end{aligned}$$

The only exception is if $x = (x_1, 0, x_3, x_4)$, where $|x_1| = 1$, and $|x_3|, |x_4| < 1$. In this case, note that we can use

$$\{(\eta_\delta, -\zeta_{\eta_\delta}, \delta, 1),\ (\eta_\delta, \zeta_{\eta_\delta}, -\delta, 1),\ (-\eta_\delta, -\zeta_{\eta_\delta}, \delta, -1),\ (-\eta_\delta, \zeta_{\eta_\delta}, -\delta, -1)\}$$

to illuminate every vertex which does not satisfy $\operatorname{sign}(x_1) = -\operatorname{sign}(x_4)$. For such vertices, we can use $(-\delta, 0, 0, 1)$ or $(\delta, 0, 0, -1)$.

Suppose $|I_0^x| = 0$ and $|x_4| < 1$. With lemma 2.42 in mind, look at the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^n : \xi_4 = x_4\}.$$

Then we can illuminate x using the set

$$\begin{aligned}\{&(1, -\delta, -\eta_\delta, \zeta_{\eta_\delta}),\ (-1, \delta, -\eta_\delta, \zeta_{\eta_\delta}),\ (\delta, 1, \eta_\delta, \zeta_{\eta_\delta}),\ (-\delta, -1, \eta_\delta, \zeta_{\eta_\delta}),\\ &(-\eta_\delta, \delta, 1, -\zeta_{\eta_\delta}),\ (\eta_\delta, -\delta, 1, -\zeta_{\eta_\delta}),\ (-\eta_\delta, -\delta, -1, -\zeta_{\eta_\delta}),\ (\eta_\delta, \delta, -1, -\zeta_{\eta_\delta})\}.\end{aligned}$$

This is because every different pair of directions above differ in sign in at least one of the first three corresponding entries. If $|x_4| = 1$, then by assumption, it cannot be the case that $|x_i| = 1$ for $i \in [3]$. Use corollary 2.39 to illuminate 12 types of vertices using

$$\begin{aligned}\{&(1, -\delta, -\eta_\delta, \zeta_{\eta_\delta}),\ (-1, \delta, -\eta_\delta, \zeta_{\eta_\delta}),\ (\delta, 1, \eta_\delta, \zeta_{\eta_\delta}),\ (-\delta, -1, \eta_\delta, \zeta_{\eta_\delta}),\\ &(-\eta_\delta, \delta, 1, -\zeta_{\eta_\delta}),\ (\eta_\delta, -\delta, 1, -\zeta_{\eta_\delta}),\ (-\eta_\delta, -\delta, -1, -\zeta_{\eta_\delta}),\ (\eta_\delta, \delta, -1, -\zeta_{\eta_\delta}),\\ &(\eta_\delta, -\zeta_{\eta_\delta}, \delta, 1),\ (\eta_\delta, \zeta_{\eta_\delta}, -\delta, 1),\ (-\eta_\delta, -\zeta_{\eta_\delta}, \delta, -1),\ (-\eta_\delta, \zeta_{\eta_\delta}, -\delta, -1)\}.\end{aligned}$$

It can be checked that the remaining four types of vertices are of the form

$$(-x_1, x_2, x_3, x_4),\ -(-x_1, x_2, x_3, x_4),\ (-x_1, -x_2, -x_3, x_4),\ -(-x_1, -x_2, -x_3, x_4),$$

where $0 < x_1, x_2, x_3, x_4 \leq 1$. Notice the above vertices satisfy $\operatorname{sign}(x_1) = -\operatorname{sign}(x_4)$. Since $0 < |x_2|, |x_3| < 1$, with lemma 2.42 in mind, look in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^n : \xi_2 = x_2,\ \xi_3 = x_3\}$$

to see that such directions can be illuminated by $(-\delta, 0, 0, 1)$, or $(\delta, 0, 0, -1)$.

For the general case, suppose that $\|e_{i_1} + e_{i_2}\|_{\mathbf{B}} = 1$ and $\|e_{i_3} + e_{i_4}\|_{\mathbf{B}} = 1$. In the case $\{i_1, i_2\} \cap \{i_3, i_4\} = \varnothing$, we can use the isomorphism

$$\iota : \mathbb{R}^4 \to \mathbb{R}^4, \quad \sum_{k \in [4]} x_{i_k} e_{i_k} \mapsto (x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}),$$

then apply the method from step 1. In the case $\{i_1, i_2\} \cap \{i_3, i_4\} \neq \varnothing$, WLOG suppose that $i_2 = i_4$ and denote the remaining index by i_5. Then we can use the isomorphism,

$$\iota : \mathbb{R}^4 \to \mathbb{R}^4, \quad x_{i_1} e_{i_1} + x_{i_2} e_{i_2} + x_{i_3} e_{i_3} + x_{i_5} e_{i_5} \mapsto (x_{i_1}, x_{i_2}, x_{i_3}, x_{i_5}),$$

and apply the method from step 2. □

The isomorphisms we have used so far have all been coordinate permutations. From here on, we will only consider cases up to coordinate permutations and the explicit isomorphisms will not be stated as identifying them should be clear based on the previous parts.

Proposition 7.6. Let $\mathbf{B} \in \mathcal{U}_p^4$, and suppose $\|e_i + e_j\|_{\mathbf{B}} = 1$ for exactly three pairs of distinct $i, j \in [4]$ and $\|e_r + e_s + e_t\|_{\mathbf{B}} > 1$ for every triple of distinct $r, s, t \in [4]$. Then $\mathcal{I}(\mathbf{B}) \leq 14$.

Notice that $\|e_r + e_s + e_t\|_{\mathbf{B}} > 1$ for every $r, s, t \in [4]$ means that for every $x \in \partial \mathbf{B} \supset \operatorname{ext} \mathbf{B}$, we have $\left|I_{\pm 1}^x\right| < 3$.

Proof. For $x \in \mathbf{B}$ with $\left|I_{\pm 1}^x\right| = 2$, there are $\binom{4}{2}$ ways to place entries with absolute value of 1 up to sign.

Case 1: Suppose $\|e_1 + e_2\|_{\mathbf{B}} = \|e_3 + e_4\|_{\mathbf{B}} = \|e_2 + e_3\|_{\mathbf{B}} = 1$. So $\mathbf{B}$ has vertices which look like

$$(\pm 1, \pm 1, x_3, x_4), \ (y_1, y_2, \pm 1, \pm 1), \ (z_1, \pm 1, \pm 1, z_4),$$

where $|x_3|, |x_4|, |y_1|, |y_2|, |z_1|, |z_4| < 1$. The other cases only differ by permutation of coordinates (see following table).

Vertex combinations	Coordinate permutation
$(\pm1,\pm1,x_3,x_4),\ (y_1,y_2,\pm1,\pm1),\ (\pm1,z_2,\pm1,z_4)$	$(1\,2)$
$(\pm1,\pm1,x_3,x_4),\ (y_1,y_2,\pm1,\pm1),\ (z_1,\pm1,z_3,\pm1)$	$(3\,4)$
$(\pm1,\pm1,x_3,x_4),\ (y_1,y_2,\pm1,\pm1),\ (\pm1,z_2,z_3,\pm1)$	$(1\,2)(3\,4)$

Table 14: Cases up to coordinate permutation.

Notice that if $e_{i_1}+e_{i_2}, e_{i_3}+e_{i_4} \in \mathbf{B}$ and $\{i_1,i_2,i_3,i_4\}=[4]$, then we can use a coordinate permutation to get a case that is listed in table 14 or the case we are currently working on.

Let $u \in \operatorname{ext}\mathbf{B}$, recall that it cannot be the case that $|I_0^u| = 3$. Suppose $|I_0^u| = 2$; using lemmas 2.42 and 2.41 we can see that $(\pm1,\pm1,0,0)$, $(0,\pm1,\pm1,0)$ and $(0,0,\pm1,\pm1)$ are illuminated by $\mathcal{I}_{\delta,\eta}\cup\mathcal{J}_{\delta,\eta}\cup\mathcal{K}_{\delta}$, where

$$\begin{aligned}
\mathcal{I}_{\delta,\eta} &:= \{(1,\delta,\eta_\delta,0),\ -(1,\delta,\eta_\delta,0),\ (\delta,-1,\eta_\delta,0),\ -(\delta,-1,\eta_\delta,0)\},\\
\mathcal{J}_{\delta,\eta} &:= \{(-\eta_\delta,\delta,1,0),\ -(-\eta_\delta,\delta,1,0),\ (\eta_\delta,\delta,-1,0),\ -(\eta_\delta,\delta,-1,0)\},\\
\mathcal{K}_{\delta} &:= \{(0,0,\delta,1),\ -(0,0,\delta,1),\ (0,0,-\delta,1),\ -(0,0,-\delta,1)\}
\end{aligned}$$

As for other vertices which satisfy $|I_0^u| = 2$, we can use lemma 2.45 then lemma 2.41 to see that $\mathcal{I}_{\delta,\eta}\cup\mathcal{J}_{\delta,\eta}\cup\mathcal{K}_{\delta}$ illuminates such vertices.

Suppose $|I_0^u| = 1$. Let's take a look at the vertex of the form $u=(u_1,u_2,u_3,0)$. If $|u_1|<1$, with lemmas 2.41 and 2.42 in mind, look in the affine set

$$\{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbb{R}^n : \xi_1 = u_1,\ \xi_4 = 0\}$$

to see that we can illuminate u with $\mathcal{J}_{\delta,\eta}$.

If $|u_1| = 1$, by assumption it cannot be the case that $|u_3| = 1$, hence we can illuminate u with $\mathcal{I}_{\delta,\eta}$. With similar reasoning, we can construct the following table:

Type of vertex	Condition	Illuminating set	Condition	Illuminating set
$(u_1, u_2, u_3, 0)$	$\|u_1\| < 1$	$\mathcal{J}_{\delta,\eta}$	$\|u_1\| = 1$	$\mathcal{I}_{\delta,\eta}$
$(u_1, u_2, 0, u_4)$	$\|u_4\| < 1$	$\mathcal{I}_{\delta,\eta}$	$\|u_4\| = 1$	$\mathcal{K}_{\delta}$
$(u_1, 0, u_3, u_4)$	$\|u_1\| < 1$	$\mathcal{K}_{\delta}$	$\|u_1\| = 1$	$\mathcal{I}_{\delta,\eta}$
$(0, u_2, u_3, u_4)$	$\|u_4\| < 1$	$\mathcal{J}_{\delta,\eta}$	$\|u_4\| = 1$	$\mathcal{K}_{\delta}$

Table 15: Summary of method.

If $|I_0^u| = 0$, and $|u_4| \neq 1$, then we can illuminate u using $\mathcal{I}_{\delta,\eta} \cup \mathcal{J}_{\delta,\eta}$ as any two directions from this set differ in sign in at least one of their first three corresponding entries. If $|u_4| = 1$, then by assumption, it must be the case that $0 < |u_1|, |u_2| < 1$. Since $(1, 1, 0, 0) \in \mathbf{B}$, look in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^n : \xi_1 = u_1,\ \xi_2 = u_2\}$$

which contains the interior point $(u_1, u_2, 0, 0)$ to see that we can illuminate u by $\mathcal{K}_\delta$. For this case $\mathcal{I}(\mathbf{B}) \leq 12$.

Case 2: Suppose $\|e_1 + e_2\|_{\mathbf{B}} = \|e_2 + e_3\|_{\mathbf{B}} = 1$. To avoid overlapping with the first case, it must be the case that $\|e_1 + e_3\|_{\mathbf{B}} = 1$ or $\|e_2 + e_4\|_{\mathbf{B}} = 1$. So $\mathbf{B}$ has vertices which look like

$$(1, 1, x_3, x_4),\ (y_1, 1, 1, y_4),\ \text{and}\ (1, z_2, 1, z_4),$$

or

$$(1, 1, x_3, x_4),\ (y_1, 1, 1, y_4),\ \text{and}\ (z_1, 1, z_3, 1),$$

where $|x_3|, |x_4|, |y_1|, |y_2|, |z_1|, |z_2|, |z_3|, |z_4| < 1$. If $e_{i_1} + e_{i_2}, e_{i_3} + e_{i_4}, e_{i_5} + e_{i_6} \in \mathbf{B}$, where $\{i_1, i_2\}$, $\{i_3, i_4\}$, $\{i_5, i_6\}$ are distinct and mutually intersect, then this case only differs by the two cases above by up to coordinate permutations.

In this case we will suppose $(1,1,x_3,x_4),(y_1,1,1,y_4),(1,z_2,1,z_4)\in\mathbf{B}$ and consider the remaining case in case 3. Let $u\in\operatorname{ext}\mathbf{B}$. Suppose $|I_0^u|\in\{2,3\}$. With the exception of $(\pm1,\pm1,0,0)$, $(0,\pm1,\pm1,0)$ and $(\pm1,0,\pm1,0)$, using sufficiently small $\delta>0$, $\eta_\delta>0$ and $\zeta_{\eta_\delta}>0$, we can illuminate u (lemma 2.41) using $\mathcal{I}_{\delta,\eta,\zeta}\cup\mathcal{J}_{\delta,\eta,\zeta}\cup\mathcal{K}_{\delta,\eta}\cup\{\pm e_4\}$, where

$$\begin{aligned}\mathcal{I}_{\delta,\eta,\zeta}&:=\{(1,\delta,\eta_\delta,\zeta_{\eta_\delta}),\ (1,-\delta,\eta_\delta,\zeta_{\eta_\delta}),\ (-1,\delta,-\eta_\delta,-\zeta_{\eta_\delta}),\ (-1,-\delta,-\eta_\delta,-\zeta_{\eta_\delta})\},\\ \mathcal{J}_{\delta,\eta,\zeta}&:=\{(-\eta_\delta,1,\delta,-\zeta_{\eta_\delta}),\ (\eta_\delta,1,-\delta,-\zeta_{\eta_\delta}),\ (-\eta_\delta,-1,\delta,-\zeta_{\eta_\delta}),\ (\eta_\delta,-1,-\delta,-\zeta_{\eta_\delta})\},\\ \mathcal{K}_{\delta,\eta,\zeta}&:=\{(\delta,0,1,-\eta_\delta),\ (\delta,0,-1,\eta_\delta),\ (-\delta,0,1,\eta_\delta),\ (-\delta,0,-1,\eta_\delta)\}.\end{aligned}$$

Using any $\delta>0$ and adjusting η_δ and ζ_{η_δ} to be sufficiently small, we can see that $\mathcal{I}_{\delta,\eta,\zeta}\cup\mathcal{J}_{\delta,\eta,\zeta}\cup\mathcal{K}_{\delta,\eta,\zeta}$ illuminates $(\pm1,\pm1,0,0)$, $(0,\pm1,\pm1,0)$ and $(\pm1,0,\pm1,0)$. If $|I_0^u|=1$, let's look at the vertex of the form $(u_1,u_2,u_3,0)$, where $0<|u_1|,|u_2|,|u_3|\le1$. Assume $|u_1|=1$. Then with sufficiently small η_δ and ζ_{η_δ}, we can illuminate u with $\mathcal{I}_{\delta,\eta,\zeta}$ or $\mathcal{K}_{\delta,\eta,\zeta}$. If $|u_1|<1$, then we can use $\mathcal{J}_{\delta,\eta,\zeta}$ instead. We use similar methods to construct the following table:

Type of vertex	Condition	Illuminating set	Condition	Illuminating set
$(u_1,u_2,u_3,0)$	$\|u_1\|<1$	$\mathcal{J}_{\delta,\eta,\zeta}$	$\|u_1\|=1$	$\mathcal{I}_{\delta,\eta,\zeta}\cup\mathcal{K}_{\delta,\eta,\zeta}$
$(u_1,u_2,0,u_4)$	$\|u_4\|<1$	$\mathcal{I}_{\delta,\eta,\zeta}$	$\|u_4\|=1$	$\{\pm e_4\}$
$(u_1,0,u_3,u_4)$	$\|u_1\|<1$	$\mathcal{K}_{\delta,\eta,\zeta}\cup\{\pm e_4\}$	$\|u_1\|=1$	$\mathcal{K}_{\delta,\eta,\zeta}$
$(0,u_2,u_3,u_4)$	$\|u_4\|<1$	$\mathcal{J}_{\delta,\eta,\zeta}$	$\|u_4\|=1$	$\{\pm e_4\}$

Table 16: Summary of method.

Suppose $I_0^u=0$ and $u_4<1$. Notice that any two directions from $\mathcal{I}_{\delta,\eta,\zeta}\cup\mathcal{J}_{\delta,\eta,\zeta}$ have different sign in at least one of the first three corresponding entries. So for any $\delta>0$, and $\eta_\delta>0$, we can find sufficiently small $\zeta_{\eta_\delta}>0$ so that u is illuminated by $\mathcal{I}_{\delta,\eta,\zeta}\cup\mathcal{J}_{\delta,\eta,\zeta}$. Now suppose $|u_4|=1$ and let $0<u_1,u_2,u_3,<1$. Using lemma 2.42 and corollary 2.39, we construct the following table:

Type of vertex		Illuminating set or direction
$(u_1,u_2,u_3,1)$, $(-u_1,-u_2,-u_3,-1)$,	$(u_1,-u_2,u_3,1)$, $(-u_1,u_2,-u_3,-1)$	$\mathcal{I}_{\delta,\eta,\zeta}$
$(u_1,-u_2,-u_3,1)$, $(-u_1,-u_2,u_3,1)$,	$(u_1,u_2,-u_3,1)$, $(-u_1,u_2,u_3,1)$	$\mathcal{J}_{\delta,\eta,\zeta}$
$(-u_1,u_2,-u_3,1)$,	$(-u_1,-u_2,-u_3,1)$	$(\delta,0,1,-\eta_\delta)$
$(-u_1,u_2,u_3,-1)$,	$(-u_1,-u_2,u_3,-1)$	$(\delta,0,-1,\eta_\delta)$
$(u_1,u_2,-u_3,-1)$,	$(u_1,-u_2,-u_3,-1)$	$(-\delta,0,1,\eta_\delta)$
$(u_1,u_2,u_3,-1)$,	$(u_1,-u_2,u_3,-1)$	$(-\delta,0,-1,\eta_\delta)$

Table 17: Application of corollary 2.39 and lemma 2.42.

Hence $\mathcal{I}_{\delta,\eta,\zeta} \cup \mathcal{J}_{\delta,\eta,\zeta} \cup \mathcal{K}_{\delta,\eta} \cup \{\pm e_4\}$ is an illuminating set in this case, thus $\mathcal{I}(\mathbf{B}) \leq 14$.

Case 3: Now suppose $(1,1,0,0),(0,1,1,0),(0,1,0,1) \in \mathbf{B}$. Let $u \in \operatorname{ext}\mathbf{B}$. Note that it cannot be the case that $|I_0^u| = 3$. Suppose $|I_0^u| = 2$ and $\left|I_{\pm1}^u\right| \leq 1$. If $u \in \operatorname{ext}\mathbf{B}$ and $u_2 = 1$, then exactly one of u_1, u_3, u_4 has non zero absolute value. For example, if u_1 was non zero, then $|u_1| < 1$ as $\left|I_{\pm1}^u\right| \leq 1$. But then $(u_1,1,0,0) \notin \operatorname{ext}\mathbf{B}$ as $(1,1,0,0) \in \operatorname{ext}\mathbf{B}$. Thus, we can see that if $|I_0^u| = 2$ and $\left|I_{\pm1}^u\right| \leq 1$, then that it cannot be the case that $I_{\pm1}^u = \{2\}$. Hence $\{\pm e_1 : i \in [4]\backslash\{2\}\}$ is an illuminating set for such vertices. With sufficiently small $\delta > 0$, $\eta_\delta > 0$ and $\zeta_{\eta_\delta} > 0$, we can see that $\mathcal{I}_{\delta,\eta} \cup \mathcal{J}_{\delta,\eta,\zeta} \cup \mathcal{K}_{\delta,\eta,\zeta}$ is an illuminating set for such vertices, where

$$\begin{aligned}
\mathcal{I}_{\delta,\eta} &:= \{(1,\eta_\delta,-\delta,0),\ (1,-\eta_\delta,\delta,0),\ (-1,\eta_\delta,\delta,0),\ (-1,-\eta_\delta,-\delta,0)\},\\
\mathcal{J}_{\delta,\eta,\zeta} &:= \{(\zeta_{\eta_\delta},\eta_\delta,1,\delta),\ (-\zeta_{\eta_\delta},-\eta_\delta,1,\delta),\ (\zeta_{\eta_\delta},\eta_\delta,-1,-\delta),\ (-\zeta_{\eta_\delta},-\eta_\delta,-1,-\delta)\},\\
\mathcal{K}_{\delta,\eta,\zeta} &:= \{(\zeta_{\eta_\delta},\eta_\delta,-\delta,1),\ (-\zeta_{\eta_\delta},-\eta_\delta,-\delta,1),\ (\zeta_{\eta_\delta},\eta_\delta,\delta,-1),\ (-\zeta_{\eta_\delta},-\eta_\delta,\delta,-1)\}.
\end{aligned}$$

In this case we will also be using the directions

$$(1,-\eta_\delta,-\delta,0) \quad \text{and} \quad (-1,\eta_\delta,-\delta,0)$$

which gives us a total of 14 directions. Let's take a look at the vertex $(1, 1, 0, 0)$. Let $0 < \delta < 1$ and $0 < \eta_\delta < 2$, and take $d = (-1, -\eta_\delta, -\delta, 0)$, then

$$(1, 1, 0, 0) + d = (0,\ 1 - \eta_\delta,\ -\delta,\ 0).$$

Note that $|1 - \eta_\delta| < 1$, $|\delta| < 1$ and $(0, 1, 1, 0) \in \mathbf{B}$. With lemma 2.42 in mind, we can see that $x + d \in \operatorname{int} \mathbf{B}$ when looking in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = 0,\ \xi_4 = 0\}.$$

This calculation shows us that the sign of $d_3 = -\delta$ does not matter for this sub-case, we are only looking at the signs of $d_1 = -1$ and $d_2 = -\eta_\delta$. We can see that $(\pm 1, \pm 1, 0, 0)$ is illuminated by $\mathcal{I}_{\delta,\eta}$. If we treat $\zeta_{\eta_\delta} = 0$, we can illuminate vertices of the form $(0, \pm 1, \pm 1, 0)$ and $(0, 0, \pm 1, \pm 1)$ using $\mathcal{J}_{\delta,\eta,\zeta}$ and $\mathcal{K}_{\delta,\eta,\zeta}$, respectively. Then use lemma 2.41 to get a sufficiently small non zero $\zeta_{\eta_\delta} > 0$ so that $(0, \pm 1, \pm 1, 0)$ and $(0, 0, \pm 1, \pm 1)$ are still illuminated by $\mathcal{J}_{\delta,\eta,\zeta}$ and $\mathcal{K}_{\delta,\eta,\zeta}$ after applying perturbations.

Now suppose $|I_0^u| = 1$. Let's first take a look at a vertex of the form $(0, u_2, u_3, u_4)$, where $0 \le |u_2|, |u_3|, |u_4| \le 1$. Note that any two directions from $\mathcal{J}_{\delta,\eta,\zeta} \cup \mathcal{K}_{\delta,\eta,\zeta}$ differ in sign in at least one of their last three corresponding entries. With lemmas 2.42 and 2.41 in mind, look in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = 0\}.$$

to see that $\mathcal{J}_{\delta,\eta,\zeta} \cup \mathcal{K}_{\delta,\eta,\zeta}$ illuminates u for any $\delta > 0$, $\eta_\delta > 0$ and sufficiently small $\zeta_{\delta,\eta} > 0$.

Suppose $u = (u_1, 0, u_3, u_4)$, where $0 < |u_1|, |u_3|, |u_4| \le 1$. Either $|u_4| = 1$ or $|u_4| < 1$. Suppose $|u_4| = 1$, then by our initial assumptions, $|u_1| < 1$. With lemma 2.42 in mind, look in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_2 = 0,\ \xi_4 = u_4\}$$

to see that $(\pm 1, 0, \pm\delta, 0)$ illuminates u_4 for any $\delta > 0$. Thus for sufficiently small $\eta_\delta > 0$,

$\mathcal{I}_{\delta,\eta}$ also illuminates u. If $|u_1| < 1$, then we can use a similar argument to get that $\mathcal{J}_{\delta,\eta,\zeta} \cup \mathcal{K}_{\delta,\eta,\zeta}$ illuminates u for any $\delta > 0$ and sufficiently small $\eta_\delta > 0$ and $\zeta_{\eta_\delta} > 0$.

Now suppose $u = (u_1, u_2, u_3, 0)$, where $0 < |u_1|, |u_2|, |u_3| \leq 1$. If $|u_3| = 1$, it must be the case that $|u_1| < 1$. Let's look at the example $u = (u_1, u_2, 1, 0)$, where $u_1, u_2, > 0$ (it is possible that $u_2 = 1$). Let $0 < \delta < 1 - u_1$, $0 < \eta_\delta < 1$ and $d = (0, -\eta_\delta, -1, -\delta)$, then

$$u + d = (u_1,\ u_2 - \eta_\delta,\ 0,\ -\delta).$$

Let's compare $u + d$ with

$$\begin{aligned}(1-\delta)(1,1,0,0) + \delta(0,1,0,1) =& (1-\delta,\ 1,\ 0,\ \delta) \in \mathbf{B} \\ &\Longrightarrow (1-\delta,\ 1,\ 0,\ -\delta) \in \mathbf{B}.\end{aligned}$$

By the choice of δ and η_δ, we have $u_1 < 1 - \delta$ and $|u_2 - \eta_\delta| < 1$. Now apply lemma 2.42 with the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_3 = 0,\ \ \xi_4 = -\delta\}$$

to see that $u + d \in \operatorname{int} \mathbf{B}$. Observe that the sign on $d_4 = -\delta$ does not matter here. Now we can use lemma 2.41 to see that for sufficiently small $\zeta_{\eta_\delta} > 0$, $\mathcal{J}_{\delta,\eta,\zeta}$ illuminates vertices of the form $(u_1, u_2, \pm 1, 0)$, where $0 < |u_1| < 1$ and $1 < |u_2| \leq 1$. Now assume $|u_3| < 1$ and let's look at the example $u = (u_1, u_2, -u_3, 0)$, where $0 < u_1, u_2, u_3 \leq 1$. Let $0 < \delta < \frac{1-u_3}{u_1}$, $0 < \eta < 1$ and $d = (-1, -\eta_\delta, -\delta, 0)$, then

$$u + u_1 d = (0,\ u_2 - u_1\eta_\delta,\ -(u_3 + u_1\delta),\ 0).$$

By our choice of δ and η_δ, $|u_2 - u_1\eta_\delta| < 1$ and $|u_3 + u_1\delta| < 1$. Recall that $(0,1,1,0) \in \mathbf{B}$. Now we can use lemma 2.42 and the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = 0,\ \ \xi_4 = 0\}$$

to see that $u + u_1 d \in \operatorname{int} \mathbf{B}$. We choose this specific example for this calculation to emphasize the fact that the sign on $d_3 = -\delta$ does not matter here. Hence we can see that $\mathcal{I}_{\delta,\eta}$ is an illuminating set for vertices of the form $(u_1, u_2, u_3, 0)$, where $1 \leq |u_1|, |u_2| \leq 1$ and $0 < |u_3| < 1$

Now we move onto the case $u = (u_1, u_2, 0, u_4)$, where $0 < |u_1|, |u_2| |u_4| \leq 1$. If $|u_4| = 1$, we can use a similar argument to the previous case ($u = (u_1, u_2, u_3, 0)$ and $|u_3| = 1$) to get that $\mathcal{K}_{\delta,\eta,\zeta}$ is an illuminating set for u for sufficiently small δ, η_δ and $\zeta_{\eta_\delta} > 0$ (the signs of the entries which we care about here are $d_2 = \pm\eta$ and $d_4 = \pm 1$). So suppose $|u_4| < 1$ and let's look at the example $u = (u_1, u_2, 0, u_4)$, where $0 < u_1, u_2, u_4 \leq 1$ (here we allow $u_1 = 1$ or $u_2 = 1$). Let $0 < \delta < \frac{1-u_4}{\delta}$, $0 < \eta_\delta < 1$ and $d = (-1, -\eta_\delta, -\delta, 0)$

$$u + u_1 d = (0,\ u_2 - u_1\eta_\delta,\ -u_1\delta,\ u_4).$$

Compare $u + u_1 d$ with

$$\begin{aligned} u_1\delta(0,1,1,0) + (1 - u_1\delta)(0,1,0,1) =& (0,\ 1,\ u_1\delta,\ 1 - u_1\delta) \in \mathbf{B} \\ &\implies (0,\ 1,\ -u_1\delta,\ 1 - u_1\delta) \in \mathbf{B}. \end{aligned}$$

By the choice of δ and η, $|u_2 - u_1\eta| < 1$ and $u_4 < 1 - u_1\delta$. Now we can apply lemma 2.42 and look in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = 0,\ \xi_3 = -u_1\delta\}$$

to see that $u + u_1 d \in \operatorname{int} \mathbf{B}$. Again, the sign on $d_3 = -\delta$ is not important while the signs on $d_1 = -1$ and $d_2 - \eta_\delta$ are important. So we can see that $\mathcal{I}_{\delta,\eta}$ is an illuminating set in this case.

Finally, consider the when $|I_0^u| = 1$. If $|u_1| < 1$, recall that any two directions from $\mathcal{J}_{\delta,\eta,\zeta} \cup \mathcal{K}_{\delta,\eta,\zeta}$ differ in sign in at least one of their last three corresponding entries.

With lemmas 2.42 and 2.41 in mind, look in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = u_1\}$$

to see that $\mathcal{J}_{\delta,\eta,\zeta} \cup \mathcal{K}_{\delta,\eta,\zeta}$ illuminates u for any $\delta > 0$, $\eta_\delta > 0$ and sufficiently small $\zeta_{\delta_\eta} > 0$. If $|u_1| = 1$, we know that $|u_4| < 1$. Apply lemma 2.39 to see that for any $\delta > 0$, $\eta_\delta > 0$ and $\zeta_{\delta_\eta} > 0$, $\mathcal{J}_{\delta,\eta,\zeta} \cup \mathcal{K}_{\delta,\eta,\zeta}$ illuminates precisely the vertices which satisfy $\operatorname{sign}(u_1) = \operatorname{sign}(u_2)$. Use lemma 2.42 and the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = u_1\}$$

to see that the following directions illuminate u which satisfy $-\operatorname{sign}(u_1) = \operatorname{sign}(u_2)$:

$$\underbrace{(1, -\eta_\delta, \delta, 0)}_{\in \mathcal{I}_{\delta,\eta}}, \ \underbrace{(-1, \eta_\delta, \delta, 0)}_{\in \mathcal{I}_{\delta,\eta}}, \ (1, -\eta_\delta, -\delta, 0), \ \ (-1, \eta_\delta, -\delta, 0),$$

where $\delta > 0$ and $\eta_\delta > 0$.

□

Proposition 7.7. Let $\mathbf{B} \in \mathcal{U}_p^4$ and suppose $\|e_i + e_j\|_{\mathbf{B}} = 1$ for exactly four pairs of distinct $i, j \in [4]$ and $\|e_r + e_s + e_t\|_{\mathbf{B}} > 1$ for every triple of distinct $r, s, t \in [4]$. Then $\mathcal{I}(\mathbf{B}) \leq 16$.

Proof. First we will construct $\mathbf{B}$ with specific vertices which is a representative family of examples of this case. Notice it must be the case that there exists vertices $x, y \in \operatorname{ext} \mathbf{B}$ so that $\left|I_{\pm 1}^x\right| = \left|I_{\pm 1}^y\right| = 2$ and $I_{\pm 1}^x \cap I_{\pm 1}^y = \varnothing$. Let's take a look at the example,

$$S := \{(1, 1, 0, 0), \ (1, 0, 1, 0), \ (1, 0, 0, 1)\} \subset \mathbf{B}.$$

Let $z \in S$, then $1 \in I_{\pm 1}^z$. Notice if we want to add another vertex y so that $I_{\pm 1}^y \neq I_{\pm 1}^z$ for every $z \in S$, then it must be case that $I_{\pm 1}^y \cap I_{\pm 1}^x = \varnothing$ for at least one $x \in S$. There is nothing special about the choice we made, that is: all three elements of S satisfy $1 \in I_{\pm 1}^x$. Hence we can see that it must be the case that if $\|e_i + e_j\|_{\mathbf{B}} = 1$ for exactly four pairs of distinct

$i, j \in [4]$, then there must exist $x, y \in \mathbf{B}$ so that $I^x_{\pm 1} \cap I^y_{\pm 1} = \varnothing$. WLOG assume

$$\tilde{x} \in \{(\pm 1, \pm 1, 0, 0)\} \subset \mathbf{B}, \quad \tilde{y} \in \{(0, 0, \pm 1, \pm 1)\} \subset \mathbf{B}.$$

Since $I^{\tilde{x}}_{\pm 1} \cup I^{\tilde{y}}_{\pm 1} = [4]$, if we want to add another vertex $\tilde{z}$ to $\mathbf{B}$ so that

$$I^{\tilde{x}}_{\pm 1} \neq I^{\tilde{z}}_{\pm 1} \neq I^{\tilde{y}}_{\pm 1},$$

it must be the case that $I^{\tilde{z}}_{\pm 1} \in \{\{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}\}$. Notice that $I^{\tilde{x}}_{\pm 1}$, $I^{\tilde{y}}_{\pm 1}$ and $\{\{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}\}$ are all invariant under combinations of the coordinate transpositions $(1\,2)$ and $(3\,4)$. Hence WLOG, we can assume that in addition to $\tilde{x}, \tilde{y} \in \mathbf{B}$,

$$\tilde{z} \in \{(0, \pm 1, \pm 1, 0)\} \subset \mathbf{B}.$$

Case 1: $\{\pm 1, 0, 0, \pm 1\} \subset \mathbf{B}$. Let $u \in \mathbf{B}$. First notice that $|I^u_0| \neq 3$. Suppose $|I^u_0| = 2$. Notice that if $\left|I^u_{\pm 1}\right| = 2$, then by lemma 2.42 we can use the following set to illuminate u:

$$\{(\pm 1, \pm\delta, 0, 0)\} \cup \{(\pm\delta, 0, 0, \pm 1)\} \cup \{(0, 0, \pm 1, \pm\delta)\} \cup \{(0, \pm\delta, \pm 1, 0)\},$$

where $\delta > 0$. If $\left|I^u_{\pm 1}\right| < 2$, then for sufficiently small $\delta > 0$, we can still use the above directions to illuminate u (lemma 2.45). Suppose $|I^u_0| = 1$, let's take a look at the case $u = (u_1, u_2, u_3, 0)$, where $0 < |u_1|, |u_2|, |u_3| \leq 1$.

If $|u_1| < 1$, with lemma 2.42 in mind, we can use $\{(0, \pm\delta, \pm 1, 0)\}$ to illuminate u. If $|u_1| = 1$, then by assumption, it must be the case that $|u_3| < 1$. In this case, we can use $\{(\pm 1, \pm\delta, 0, 0)\}$ to illuminate u. Using this method we can construct the following table:

Type of vertex	Condition	Illuminating directions	Condition	Illuminating directions
$(x_1, x_2, x_3, 0)$	$\lvert x_1\rvert < 1$	$(0, \pm\delta, \pm 1, 0)$	$\lvert x_1\rvert = 1$	$(\pm 1, \pm\delta, 0, 0)$
$(x_1, x_2, 0, x_4)$	$\lvert x_4\rvert < 1$	$(\pm 1, \pm\delta, 0, 0)$	$\lvert x_4\rvert = 1$	$(\pm\delta, 0, 0, \pm 1)$
$(x_1, 0, x_3, x_4)$	$\lvert x_1\rvert < 1$	$(0, 0, \pm 1, \pm\delta)$	$\lvert x_1\rvert = 1$	$(\pm\delta, 0, 0, \pm 1)$
$(0, x_2, x_3, x_4)$	$\lvert x_4\rvert < 1$	$(0, \pm\delta, \pm 1, 0)$	$\lvert x_4\rvert = 1$	$(0, 0, \pm 1, \pm\delta)$

Table 18: Summary of method.

Suppose $|I_0^u| = 0$. Suppose $|u_1| = \max_{i\in[4]} |u_i|$. By assumption, this implies either $|u_3|, |u_4| < 1$ or $|u_2|, |u_3| < 1$. In the first case, we can use lemma 2.42 and look in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = u_3,\ \xi_4 = u_4\}$$

to see that $(\pm 1, \pm\delta, 0, 0)$ illuminates u. Similarly in the latter case, we can use $(\pm\delta, 0, 0, \pm 1)$. This is one method to illuminate vertices which satisfy $|I_0^u| = 0$. A second method would be to perturb the current directions we have so that any two of them have at least one entry with opposite signs. Then this new set of directions will illuminate all such u. So $\mathcal{I}(\mathbf{B}) \leq 16$ in this case.

Case 2: Suppose $\{(\pm 1, 0, \pm 1, 0)\} \subset \mathbf{B}$. If instead $\{(0, \pm 1, 0, \pm 1)\} \subset \mathbf{B}$, apply the coordinate permutation $(1\,4)(2\,3)$ to see that this is the same as assuming $\{(\pm 1, 0, \pm 1, 0)\} \subset \mathbf{B}$. Let $u \in \mathbf{B}$. Again it cannot be the case that $|I_0^u| = 3$. By lemmas 2.45 and 2.41, we can see that if $|I_0^u| = 2$, and $\delta > 0$ and $\eta_\delta > 0$ are sufficiently small, then

$\mathcal{I}_{\delta,\eta} \cup \mathcal{J}_{\delta,\eta} \cup \mathcal{K}_{\delta,\eta} \cup \mathcal{H}_{\delta}$ illuminates u, where

$$\begin{aligned}
\mathcal{I}_{\delta,\eta} &:= \{(1, \delta, -\eta_\delta, 0),\ (-1, \delta, \eta_\delta, 0), \\
&\qquad -(1, \delta, -\eta_\delta, 0),\ -(-1, \delta, \eta_\delta, 0)\}, \\
\mathcal{J}_{\delta,\eta} &:= \{(\eta_\delta, 1, \delta, 0),\ (-\eta_\delta, -1, \delta, 0), \\
&\qquad -(\eta_\delta, 1, \delta, 0)\ -(-\eta_\delta, -1, \delta, 0)\}, \\
\mathcal{K}_{\delta,\eta} &:= \{(\delta, -\eta_\delta, -1, 0),\ (\delta, -\eta_\delta, 1, 0), \\
&\qquad -(\delta, -\eta_\delta, -1, 0),\ -(\delta, -\eta_\delta, -1, 0)\}, \\
\mathcal{H}_{\delta} &:= \{(0, 0, \pm\delta, \pm 1)\}.
\end{aligned}$$

We can also see that for sufficiently small η_δ, $\mathcal{I}_{\delta,\eta} \cup \mathcal{J}_{\delta,\eta} \cup \mathcal{K}_{\delta,\eta} \cup \mathcal{H}_{\delta}$ also illuminates

$$(\pm 1, \pm 1, 0, 0),\ (\pm 1, 0, \pm 1, 0),\ (0, \pm 1, \pm 1, 0) \text{ and } (0, 0, \pm 1, \pm 1).$$

Now suppose $|I_0^u| = 1$. Let's look at the case when $u = (u_1, u_2, u_3, 0)$, where $0 < |u_1|, |u_2|, |u_3| \leq 1$. If $|u_1| < 1$, then by lemma 2.42, look in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = u_1,\ \xi_4 = 0\}$$

to see that $\{(0, \pm 1, \pm\delta, 0)\}$ illuminates u. By lemma 2.41, with sufficiently small η, $\mathcal{J}_{\delta,\eta}$ illuminates u. Similarly, if $|u_2| < 1$ or $|u_3| < 1$, then we can illuminate u using $\mathcal{K}_{\delta,\eta}$ or $\mathcal{I}_{\delta,\eta}$, respectively. We can use similar reasoning to construct the following table:

Type of vertex	Condition	Illuminating set	Condition	Illuminating set
$(x_1, x_2, 0, x_4)$	$\lvert x_4\rvert < 1$	$\mathcal{I}_{\delta,\eta}$	$\lvert x_4\rvert = 1$	$\mathcal{H}_{\delta}$
$(x_1, 0, x_3, x_4)$	$\lvert x_4\rvert < 1$	$\mathcal{K}_{\delta,\eta}$	$\lvert x_4\rvert = 1$	$\mathcal{H}_{\delta}$
$(0, x_2, x_3, x_4)$	$\lvert x_4\rvert < 1$	$\mathcal{J}_{\delta,\eta}$	$\lvert x_2\rvert = 1$	$\mathcal{H}_{\delta}$

Table 19: Summary of method.

If $|I_0^u| = 0$ and $|u_4| < 1$, then $\mathcal{I}_{\delta,\eta} \cup \mathcal{J}_{\delta,\eta}$ is an illuminating set as any two directions from this set have opposite signs in one of their first three corresponding entries (lemma 2.42). If $|u_4| = 1$, then by assumption, $0 < |u_1|, |u_2| < 1$. With lemma 2.42 in mind, look in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = u_1,\ \xi_2 = u_2\}$$

to see that $\mathcal{H}_\delta$ illuminates u. In this case $\mathcal{I}(\mathbf{B}) \le 16$.

$\square$

Remark 7.8. Consider the product of the following invertible matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_3 - x_4 \\ x_3 + x_4 \end{pmatrix}$$

Notice that if $x_1, x_2, x_3, x_4 \in \{-1, 1\}$ then exactly two of

$$\tfrac{1}{2}(x_1 - x_2),\ \tfrac{1}{2}(x_1 + x_2),\ \tfrac{1}{2}(x_3 - x_4),\ \tfrac{1}{2}(x_3 + x_4)$$

are 0, and the other two are either 1 or -1. Thus,

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \operatorname{ext} C_1^4 \\ &\qquad = \{(\pm 1, 0, \pm 1, 0),\ (0, \pm 1, 0, \pm 1),\ (\pm 1, 0, 0, \pm 1),\ (0, \pm 1, \pm 1, 0)\} \end{aligned}$$

Notice that if we apply the coordinate transposition $(2\,3)$ to the above set of vectors, we can see that if

$$\operatorname{ext} \mathbf{B} = \{(\pm 1, 0, \pm 1, 0),\ (0, \pm 1, 0, \pm 1),\ (\pm 1, 0, 0, \pm 1),\ (0, \pm 1, \pm 1, 0)\},$$

then this case is covered in case 1 of lemma 7.7. Therefore, in order to show that $\mathcal{I}(\mathbf{B}) < 16$ in the context of lemma 7.7, the analysis of case 1 requires some extra special assumptions.

Alternatively, notice that

$$\begin{aligned}
&\{(\pm1,0,\pm1,0),\ (0,\pm1,0,\pm1),\ (\pm1,0,0,\pm1),\ (0,\pm1,\pm1,0)\}\\
&\qquad = \{(\pm1,0),\ (0,\pm1)\} \times \{(\pm1,0),\ (0,\pm1)\}\\
&\qquad = CP_1^2 \times CP_1^2.
\end{aligned}$$

Since the cross-polytope is a rotated cube in dimension 2, the cartesian product of two 2-D cross-polytopes is a 4-D cube.

Proposition 7.9. Let $\mathbf{B} \in \mathcal{U}_p^4$ which is not an affine image of the cube and suppose $\|e_i + e_j\|_{\mathbf{B}} = 1$ for exactly five pairs of distinct $i, j \in [4]$ and $\|e_r + e_s + e_t\|_{\mathbf{B}} > 1$ for every triple $r, s, t \in [4]$. Then $\mathcal{I}(\mathbf{B}) \leq 14$.

Proof. WLOG suppose that $(1,1,0,0) \notin \mathbf{B}$. Let $x \in \operatorname{ext} \mathbf{B}$. First note that it cannot be the case that $|I_0^x| = 3$. In this setting, the following set illuminates $\mathbf{B}$:

$$\begin{aligned}
&\{(\delta,0,-\delta,1),\ (-1,0,-\delta,\delta),\ (-\delta,\eta_\delta,1,\delta),\\
&\ -(\delta,0,-\delta,1),\ -(-1,0,-\delta,\delta)\ -(-\delta,\eta_\delta,1,\delta),\\
&(\eta_\delta,-\delta,\delta,1),\ (0,-1,\delta,-\delta),\ (0,\delta,1,-\delta),\\
&\ -(\eta_\delta,-\delta,\delta,1),\ -(0,-1,\delta,-\delta),\ -(0,\delta,1,-\delta),\\
&(1,1,1,1),\ -(1,1,1,1)\}
\end{aligned}$$

Case 1: $|I_0^x| = 2$. Let $0 < \delta < 1$, observe that

$$(0,1,1,0) \in \mathbf{B} \implies (0,1,0,1) + (0,-\delta,-\delta,-1) = (0,\ 1-\delta,\ -\delta,\ 0) \in \operatorname{int} \mathbf{B}.$$

Using this method, we create the following table:

Type of vertex	Illuminating set
$(0, \pm 1, 0, \pm 1)$	$\{(0, -\delta, \delta, 1),\ (0, -1, \delta, -\delta),\ -(0, -\delta, \delta, 1),\ -(0, -1, \delta, -\delta)\}$
$(0, \pm 1, \pm 1, 0)$	$\{(0, \delta, 1, -\delta),\ (0, -1, \delta, -\delta),\ -(0, \delta, 1, -\delta),\ -(0, -1, \delta, -\delta)\}$
$(\pm 1, 0, \pm 1, 0)$	$\{(-\delta, 0, 1, \delta),\ -(-\delta, 0, 1, \delta),\ (-1, 0, -\delta, \delta),\ -(-1, 0, -\delta, \delta)\}$
$(\pm 1, 0, 0, \pm 1)$	$\{(\delta, 0, -\delta, 1),\ (-1, 0, -\delta, \delta),\ -(\delta, 0, -\delta, 1),\ -(-1, 0, -\delta, \delta)\}$
$(0, 0, \pm 1, \pm 1)$	$\{(0, -\delta, \delta, 1),\ (0, \delta, 1, -\delta),\ -(0, -\delta, \delta, 1),\ -(0, \delta, 1, -\delta)\}$

Table 20: Summary of method.

For vertices of the form $x = (x_1, x_2, 0, 0)$, where $0 \leq |x_1|, |x_2| \leq 1$, it must be the case that $\left|I^x_{\pm 1}\right| \leq 1$, hence $\{\pm e_1, \pm e_2\}$ is an illuminating set for such vertices. By lemma 2.41, we can use the following set instead:

$$\{(-1, 0, -\delta, \delta),\ -(-1, 0, -\delta, \delta),\ (0, -1, \delta, -\delta),\ -(0, -1, \delta, -\delta)\}.$$

Case 2: $|I^x_0| = 1$. Suppose $0 < x_2, < 1$, $0 < x_1, x_3 \leq 1$ and $x = (x_1, x_2, -x_3, 0)$. Let $0 < \delta < \frac{x_1}{|x_3|}$, and $d = (-\delta, 0, 1, \delta)$, then

$$x + |x_3|\, d = (x_1 - |x_3|\, \delta,\ x_2,\ 0,\ |x_3|\, \delta).$$

Consider the convex combination

$$\begin{aligned}&(1 - |x_3|\, \delta)x + |x_3|\, \delta(0, 1, 0, 1)\\ &\qquad = (x_1 - x_1\, |x_3|\, \delta,\ x_2(1 - |x_3|\, \delta) + |x_3|\, \delta,\ (1 - |x_3|\, \delta)x_3,\ |x_3|\, \delta) \in \mathbf{B}\end{aligned}$$

Comparing the entries of the above vector with $x + |x_3|\, d$,

$$\begin{aligned} x_1 - x_1 |x_3|\, \delta &\geq x_1 - |x_3|\, \delta & x_1 &\leq 1 \\ x_2 - x_2 |x_3|\, \delta + |x_3|\, \delta &> x_2 & x_2 &< 1. \end{aligned}$$

And thus,

$$(x_1 - |x_3|\, \delta,\; x_2(1 - |x_3|\, \delta) + |x_3|\, \delta,\; 0,\; |x_3|\, \delta) \in \mathbf{B}.$$

Since $(1, 0, 0, 1) \in \mathbf{B}$, we can see that $(x_1 - |x_3|\, \delta,\; 0,\; 0,\; |x_3|\, \delta) \in \operatorname{int} \mathbf{B}$. Hence, look in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = x_1 - |x_3|\, \delta,\; \xi_3 = 0,\; \xi_4 = |x_3|\, \delta\}$$

to see that d illuminates x. With similar reasoning, we can construct the following table:

Type of vertex	Condition	Illuminating set	Condition	Illuminating set
$(x_1, x_2, x_3, 0)$	$\lvert x_2\rvert < 1$	$\{\pm(-\delta, 0, 1, \delta), \pm(-1, 0, -\delta, \delta)\}$	$\lvert x_2\rvert = 1$	$\{\pm(0, \delta, 1, -\delta), \pm(0, -1, \delta, -\delta)\}$
$(x_1, x_2, 0, x_4)$	$\lvert x_2\rvert < 1$	$\{\pm(\delta, 0, -\delta, 1), \pm(-1, 0, -\delta, \delta)\}$	$\lvert x_2\rvert = 1$	$\{\pm(0, -\delta, \delta, 1), \pm(0, -1, \delta, -\delta)\}$

Table 21: Summary of method.

If $x = (x_1, 0, x_3, x_4)$, where $0 < |x_1|, |x_3|, |x_4| \leq 1$. With the exception of the case $\operatorname{sign}(x_1) = \operatorname{sign}(x_3) = \operatorname{sign}(x_4)$, we can use lemma 2.42 to illuminate x using the set

$$\begin{aligned} &\{(\delta, 0, -\delta, 1),\; (-1, 0, -\delta, \delta),\; (-\delta, 0, 1, \delta), \\ &\; -(\delta, 0, -\delta, 1),\; -(-1, 0, -\delta, \delta),\; -(-\delta, 0, 1, \delta)\}. \end{aligned}$$

Similarly, if $x = (0, x_2, x_3, x_4)$, where $0 < |x_2|, |x_3|, |x_4| \leq 1$, with the exception of the case $\text{sign}(x_2) = \text{sign}(x_3) = \text{sign}(x_4)$, we can illuminate x using

$$(0, -\delta, \delta, 1),\ (0, -1, \delta, -\delta),\ (0, \delta, 1, -\delta),$$
$$-(0, -\delta, \delta, 1),\ -(0, -1, \delta, -\delta),\ -(0, \delta, 1, -\delta).$$

Suppose $x = (x_1, 0, x_3, x_4)$ and $x_1, x_3, x_4 > 0$. If $x_4 < 1$, let $0 < \delta < 1 - x_4$ and $d = (-1, 0, -\delta, \delta)$, then

$$(0, 0, 1, 1) \in \mathbf{B} \implies x + |x_1|\,(-1, 0, -\delta, \delta) = (0, 0, x_3 - \delta, x_4 + \delta) \in \operatorname{int}\mathbf{B}.$$

With this method, we create the following table:

Condition	Illuminating directions	Condition	Illuminating directions	Condition	Illuminating directions
$\lvert x_4\rvert < 1$	$\pm(-1, 0, -\delta, \delta)$	$\lvert x_3\rvert < 1$	$\pm(\delta, 0, -\delta, 1)$	$\lvert x_1\rvert < 1$	$\pm(-\delta, 0, 1, \delta)$
$\lvert x_4\rvert < 1$	$\pm(0, \delta, 1, -\delta)$	$\lvert x_3\rvert < 1$	$\pm(0, -1, \delta, -\delta)$	$\lvert x_2\rvert < 1$	$\pm(0, -\delta, \delta, 1)$

Row 1 and 2 are for vertices of the form $(x_1, 0, x_3, x_4)$ and $(0, x_2, x_3, x_4)$, respectively.

Table 22: Summary of method.

Case 3: $|I_0^x| = 0$. Recall we assumed that $(1, 1, 0, 0) \notin \mathbf{B}$. Since $\mathbf{B} \in \mathcal{U}_p^4$, if $x \in \operatorname{ext}\mathbf{B}$ and $|x_1| = 1$, then $|x_2| < 1$. Similarly, if $|x_2| = 1$, then $|x_1| < 1$. Let's assume $|x_2| < 1$. With lemma 2.42 in mind, we can see that the following directions illuminate x unless $\text{sign}(x_1) = \text{sign}(x_3) = \text{sign}(x_4)$:

$$(\delta, 0, -\delta, 1),\ (-1, 0, -\delta, \delta),\ (-\delta, 0, 1, \delta),$$
$$-(\delta, 0, -\delta, 1),\ -(-1, 0, -\delta, \delta),\ -(-\delta, 0, 1, \delta).$$

Let $x_1, x_2, x_3, x_4 > 0$, let's take a look at the vertex $x = (x_1, -x_2, x_3, x_4)$. In this case, consider $d = (-\eta_\delta, \delta, -\delta, -1)$, where $\delta > 0$ and $\eta_\delta > 0$ are sufficiently small so that all the vertices illuminated by $(0, \delta, -\delta, -1)$ are also illuminated by d. Similarly, we can illuminate $(-x_1, x_2, -x_3, -x_4)$ using $-d$. All that remains are the vertices of the form

(x_1, x_2, x_3, x_4) and $-(x_1, x_2, x_3, x_4)$; for these we will use $\pm(1,1,1,1)$. If $|x_1| < 1$, using the same reasoning from above,

$$(0, -\delta, \delta, 1),\ (0, -1, \delta, -\delta),\ (0, \delta, 1, -\delta),$$
$$-(0, -\delta, \delta, 1),\ -(0, -1, \delta, -\delta),\ -(0, \delta, 1, -\delta)$$

illuminates x unless $\operatorname{sign}(x_2) = \operatorname{sign}(x_3) = \operatorname{sign}(x_4)$. We will use $\pm(-\delta, \eta_\delta, 1, \delta)$ to illuminate x, where $\operatorname{sign}(x_1) = -\operatorname{sign}(x_2)$ and $\pm(1,1,1,1)$ for when $\operatorname{sign}(x_1) = \operatorname{sign}(x_2)$.

Overall we have used the 14 directions at the beginning of the proof to illuminate $\mathbf{B}$. $\square$

Lemma 7.10. Let $\mathbf{B} \in \mathcal{U}^n$, suppose $(\frac{1}{2}, ..., \frac{1}{2}) \in \operatorname{int} \mathbf{B}$. Then we can illuminate $\mathbf{B}$ by $\{-1, 1\}^n$.

Proof. Let $x \in \mathbf{B}$. Consider

$$x + \frac{1}{2}\sum_{i=1}^{n}(-\operatorname{sign}(x_i)e_i),$$

where we choose to set $\operatorname{sign}(x_i) = 1$ if $x_i = 0$ (and as we will now see, any other choice for these coordinates of x would work here). For $i \in [n]$, $\left|x_i - \operatorname{sign}(x_i)\frac{1}{2}\right| = \left||x_i| - \frac{1}{2}\right| \leq \frac{1}{2}$. So the point above lies in $C^n_{\frac{1}{2}} = \operatorname{conv}(\{-\frac{1}{2}, \frac{1}{2}\}^n) \subset \operatorname{int} \mathbf{B}$ (convex combination of interior points is an interior point). $\square$

Corollary 7.11. Let $\mathbf{B} \in \mathcal{U}^4$, and suppose $\{e_{i_1} + e_{i_2} : i, j \in [4],\ i \neq j\} \subset \mathbf{B}$ and $\operatorname{ext} \mathbf{B} \backslash \{e_i + e_j : i, j \in [4],\ i \neq j\} \neq \varnothing$. Then we can illuminate $\mathbf{B}$ by $\{-1, 1\}^4$.

Proof. By the hypothesis of the corollary, $C^4_1 \cap CP^4_2 \subsetneq \mathbf{B}$, so by the proof of lemma 5.9, we can see that $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \operatorname{int} \mathbf{B}$. Then apply lemma 7.10 to get the result. $\square$

Proposition 7.12. Let $\mathbf{B} \in \mathcal{U}^4_p$ and suppose $\|e_i + e_j\|_{\mathbf{B}} = 1$ for every pair of distinct $i, j \in [4]$. Then $\mathcal{I}(\mathbf{B}) \leq 16$.

Proof. By the hypothesis of the proposition,

$$\{e_i + e_j : i, j \in [4],\ i \neq j\} \subset \partial \mathbf{B}.$$

We distinguish two cases. If the only extreme points of $\mathbf{B}$ are in $\{e_i+e_j : i,j \in [4],\ i \neq j\}$, and thus each of the points in this set is an extreme point of $\mathbf{B}$, then we know from lemmas 5.4 and 5.3 that $\mathbf{B} = C_1^4 \cap CP_2^4$ and that $\mathcal{I}(\mathbf{B}) \leq 8$. If there is $x \in \operatorname{ext} \mathbf{B} \backslash \{e_i + e_j : i,j \in [4],\ i \neq j\}$, then corollary 7.11 gives us that $\mathbf{B}$ can be illuminated by $\{-1,1\}^4$, and thus $\mathcal{I}(\mathbf{B}) \leq 16$. $\square$

Lemma 7.13. Let $\mathbf{B} \in \mathcal{U}^n$. If $(1,...,1,x_n) \in \mathbf{B}$ and $|x_n| > 0$, then $(\frac{1}{2},...,\frac{1}{2}) \in \operatorname{int} \mathbf{B}$.

Proof. Suppose $x = (1,...,1,x_n) \in \operatorname{ext} \mathbf{B}$, where $0 < |x_n| < 1$. We know that

$$\tfrac{1}{2}(1,...1,x_n) + \tfrac{1}{2}(0,...,0,1) = \left(\frac{1}{2},...,\frac{1}{2},\frac{x_n+1}{2}\right) \in \mathbf{B} \implies \left(\tfrac{1}{2},...,\tfrac{1}{2}\right) \in \mathbf{B}.$$

Notice that $(\frac{1}{2},...,\frac{1}{2},0) \in \operatorname{int} \mathbf{B}$ as $(1,...,1,0) \in \mathbf{B}$. Then $(\frac{1}{2},...,\frac{1}{2})$ lies in the line segment with end points $(\frac{1}{2},...,\frac{1}{2},0)$ and $(\frac{1}{2},...,\frac{1}{2},\frac{x_n+1}{2})$. Thus $(\frac{1}{2},...,\frac{1}{2}) \in \operatorname{int} \mathbf{B}$. $\square$

Proposition 7.14. Let $\mathbf{B} \in \mathcal{U}_p^n$, and suppose $\|e_r + e_s + e_t\|_{\mathbf{B}} = 1$ for some distinct $r,s,t \in [4]$. Then $\mathcal{I}(\mathbf{B}) \leq 16$.

Proof. Let $x \in \operatorname{ext} \mathbf{B}$. If $u \notin \{r,s,t\}$ and $u \in I_0^x$, then $x = e_r + e_s + e_t$.

Case 1: Suppose $(1,1,1,x_n) \in \mathbf{B}$ and $|x_n| > 0$. This is an application of lemmas 7.13 and 7.10 in dimension 4.

Case 2: Suppose $(1,1,1,0) \in \operatorname{ext} \mathbf{B}$ and $\|e_r + e_s + e_t\|_{\mathbf{B}} > 1$ for any indices $\{r,s,t\} \neq \{1,2,3\}$. Let $x \in \operatorname{ext} \mathbf{B}$ and suppose $|x_4| = \max_{i \in [4]} |x_i|$. Additionally, suppose $|x_1| = \max_{i \in [3]} |x_i|$. Let $0 < \delta < \min(\frac{1-|x_2|}{|x_1|}, \frac{1-|x_3|}{|x_1|}, 1)$, and

$$d = (-\operatorname{sign}(x_1)\delta, -\operatorname{sign}(x_1)\delta, -\operatorname{sign}(x_1)\delta, -\operatorname{sign}(x_4)).$$

Consider

$$x + d = (-\operatorname{sign}(x_1)\left||x_1| - \delta\right|,\ x_2 - \operatorname{sign}(x_1)\delta,\ x_3 - \operatorname{sign}(x_1)\delta,\ 0).$$

Since $\|e_{i_1} + e_{i_2} + e_{i_3}\|_{\mathbf{B}} > 1$ for any indices $\{i_1,i_2,i_3\} \neq \{1,2,3\}$, we know that $|x_2|, |x_3| < 1$. By the choice of δ, we know that

$$\left||x_1| - \operatorname{sign}(x_1)\delta\right|,\ \left|x_2 - \operatorname{sign}(x_1)\delta\right|,\ \left|x_3 - \operatorname{sign}(x_1)\delta\right| < 1.$$

With lemma 2.42 in mind, compare $x+d$ to $(1,1,1,0)$ while looking at the affine set

$$\{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbb{R}^4 : \xi_4 = 0\}.$$

We can deduce that $x+d\in \operatorname{int}\mathbf{B}$. Thus d illuminates x. An anologous proof shows that d illuminates x if $|x_1| = \max_{i\in[3]}|x_i|$ or $|x_2| = \max_{i\in[3]}|x_i|$. Thus far, we only require the following four directions:

$$(\delta,\delta,\delta,1),\ (-\delta,-\delta,-\delta,1),\ (\delta,\delta,\delta,-1),\ (-\delta,-\delta,-\delta,-1).$$

Suppose $|x_4| < \max_{i\in[4]}|x_i|$. If $|I_0^x|\in\{2,3\}$, then we can apply lemma 2.45 to see that $\{\pm e_i : i\in[4]\}$ is an illuminating set for such vertices. The only vertices left to illuminate are in the form of

$$(x_1,x_2,0,x_4),\ (x_1,0,x_3,x_4),\ (0,x_2,x_3,x_4),\ (x_1,x_2,x_3,x_4),$$

where $0<|x_1|,|x_2|,|x_3|\le 1$ and $|x_4|<1$. Let's take a look at vertices of the form $x=(x_1,x_2,0,x_4)$. With lemma 2.42 in mind, look in the affine set

$$\{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbb{R}^4 : \xi_3=0,\ \xi_4=x_4\},$$

we can see that x is illuminated by

$$(-\operatorname{sign}(x_1), -\operatorname{sign}(x_2)\delta,0,0)\quad\text{or}\quad(-\operatorname{sign}(x_1)\delta,-\operatorname{sign}(x_2),0,0)$$

for any $\delta>0$. By lemma 2.41, with sufficiently small $\eta_\delta>0$, we can illuminate x using the following set:

$$\begin{aligned}\mathcal{I}_{\delta,\eta} := \{&(1,\delta,\eta_\delta,0),\ -(1,\delta,\eta_\delta,0),\ (\delta,-1,\eta_\delta,0),\ -(\delta,-1,\eta_\delta,0),\\ &(-\eta_\delta,1,\delta,0),\ -(-\eta_\delta,1,\delta,0),\ (\eta_\delta,\delta,-1,0),\ -(\eta_\delta,\delta,-1,0)\}.\end{aligned}$$

For analogous reasons, vertices of the form $(0,x_2,x_3,x_4)$ can also be illuminated by

$\mathcal{I}_{\delta,\eta}$. It should be noted that there are no two directions in $\mathcal{I}_{\delta,\eta}$ which share signs in every non zero entry. Thus we can illuminate any vertex of the form (x_1, x_2, x_3, x_4) by looking in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_4 = x_4\}$$

and applying lemma 2.42. The last type of vertex left is in the form of $(x_1, 0, x_3, x_4)$. If $\operatorname{sign}(x_1) = \operatorname{sign}(x_3)$, let $d = (-\operatorname{sign}(x_1)\delta, -\operatorname{sign}(x_1)\delta, -\operatorname{sign}(x_1)\delta, -\operatorname{sign}(x_4))$, then

$$\begin{aligned} &x - |x_4|\, d \\ &= (-\operatorname{sign}(x_1)\, |x_1 - |x_4|\, \delta|\,,\ -\operatorname{sign}(x_1)\, |x_4|\, \delta,\ -\operatorname{sign}(x_1)\, |x_3 - |x_4|\, \delta|\,,\ 0) \in \operatorname{int} \mathbf{B} \end{aligned}$$

for sufficiently small $\delta > 0$ as $(1, 1, 1, 0) \in \mathbf{B}$. To illuminate the remaining vertices of the form $(x_1, 0, x_3, x_4)$, we only need

$$\{(1, 0, -1, 0),\ (-1, 0, 1, 0)\},$$

as $|x_4| < 1$. Recall we need small $\delta > 0$ and $\eta_\delta > 0$ so that $\mathcal{I}_{\delta,\eta}$ can still illuminate the vertices which satisfy $|I_0^x| \in \{2, 3\}$. For sufficiently small $\delta > 0$ and $\eta_\delta > 0$, we can see that

$$I_{\delta,\eta} \cup \{(1, 0, -1, 0),\ (-1, 0, 1, 0)\} \cup \{(c_1\delta, c_1\delta, c_1\delta, c_2) \in \mathbb{R}^4\ ,\ c_1, c_2 \in \{-1, 1\}\}$$

is an illuminating set for $\mathbf{B}$, and it has only 14 directions.

Case 3: Suppose $(1, 1, 1, 0) \in \operatorname{ext} \mathbf{B}$ and $\|e_r + e_s + e_t\| = 1$ for some indices $\{r, s, t\} \neq \{1, 2, 3\}$. WLOG, suppose $(0, 1, 1, 1) \in \mathbf{B}$ as it will always be the case that $4 \in \{i_1, i_2, i_3\}$. Let $x \in \operatorname{ext} \mathbf{B}$. First, consider the case $x = (x_1, x_2, x_3, x_4)$, where $|x_1| = |x_4| = 1$ and $|x_2|, |x_3| \leq 1$. Observe

$\frac{1}{2}(1, |x_2|\,, |x_3|\,, 1) + \frac{1}{2}(1, 1, 1, 0) = \frac{1}{2}(2,\ |x_2| + 1,\ |x_3| + 1,\ 1) \in \mathbf{B} \implies (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \mathbf{B}$.
$(0, 1, 1, 1) \in \mathbf{B} \implies (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \operatorname{int} \mathbf{B}$.

Notice that $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ lies in the line segment between $(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \mathbf{B}$. Thus $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is an interior point and lemma 7.10 gives the result.

Suppose that we do not have a vertex in $\mathbf{B}$ in the form of $y = (y_1, y_2, y_3, y_4)$, where $|y_2|, |y_3| \le 1$ and $|y_1| = |y_4| = 1$. Let $x \in \operatorname{ext} \mathbf{B}$. The only type of vertex which can possibly satisfy $|I_0^x| \in \{2, 3\}$ is $(x_1, 0, 0, x_4)$, where $0 < \min(|x_1|, |x_4|) < 1$. We know that $\{\pm e_1, \pm e_4\}$ is an illuminating set for such vertices (lemma 2.45). By lemma 2.41, we can see that for sufficiently small $\delta > 0$ and $\eta_\delta > 0$, $\mathcal{I}_{\delta,\eta} \cup \mathcal{J}_{\delta,\eta}$ illuminates x, where

$$\begin{aligned}
\mathcal{I}_{\delta,\eta} := \{&(1, \delta, \eta_\delta, 0),\ -(1, \delta, \eta_\delta, 0),\ (\delta, -1, \eta_\delta, 0),\ -(\delta, -1, \eta_\delta, 0),\\
&(-\eta_\delta, 1, \delta, 0),\ -(-\eta_\delta, 1, \delta, 0),\ (\eta_\delta, \delta, -1, 0),\ -(\eta_\delta, \delta, -1, 0)\},\\
\mathcal{J}_{\delta,\eta} := \{&(0, 1, \eta_\delta, \delta),\ -(0, 1, \eta_\delta, \delta),\ (0, -1, -\eta_\delta, \delta),\ -(0, -1, -\eta_\delta, \delta),\\
&(0, -\eta_\delta, \delta, 1),\ -(0, -\eta_\delta, \delta, 1),\ (0, -\eta_\delta, \delta, -1),\ -(0, -\eta_\delta, \delta, -1)\}.
\end{aligned}$$

The only vertices which can possibly satisfy $|I_0^x| = 1$ are $(x_1, x_2, 0, x_4)$ and $(x_1, 0, x_3, x_4)$, where $0 < \min(|x_1|, |x_4|) < 1$. First consider $x = (x_1, x_2, 0, x_4)$. Suppose $|x_4| \ge |x_1|$; this means $|x_1| < 1$. With lemma 2.42 in mind, look at the affine set defined by

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = x_1\}$$

to see that x is illuminated by

$$(0, -\operatorname{sign}(x_2), 0, -\operatorname{sign}(x_4)\delta) \quad \text{or} \quad (0, -\operatorname{sign}(x_2)\delta, 0, -\operatorname{sign}(x_4))$$

for any $\delta > 0$. Similarly, If $|x_4| \le |x_1|$, then x is illuminated by

$$(-\operatorname{sign}(x_1), -\operatorname{sign}(x_2)\delta, 0, 0) \quad \text{or} \quad (-\operatorname{sign}(x_1)\delta, -\operatorname{sign}(x_2), 0, 0).$$

We can apply lemma 2.41 to see that there exist sufficiently small $\eta_\delta > 0$ so that $\mathcal{I}_{\delta,\eta} \cup \mathcal{J}_{\delta,\eta}$ illuminates x.

All that's left are $(\pm 1, \pm 1, \pm 1, 0)$, $(0, \pm 1, \pm 1, \pm 1)$ and (x_1, x_2, x_3, x_4), where $0 < |x_1|, |x_2|, |x_3|, |x_4| \leq 1$. $(\pm 1, \pm 1, \pm 1, 0)$ can be illuminated by $\mathcal{I}_{\delta,\eta}$ as any two directions from this set have different sign in at least one of their first three corresponding entries. Similarly $(0, \pm 1, \pm 1, \pm 1)$ can be illuminated by $\mathcal{J}_{\delta,\eta}$.

Assume finally $|I_0^x| = 0$. Recall, it cannot be the case that $|x_1| = |x_4| = 1$. If $|x_1| \leq |x_4|$, we can use $\mathcal{J}_{\delta,\eta}$ to illuminate x. We can see this by looking in the affine set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 = x_1\}.$$

Similarly, if $|x_1| \geq |x_4|$, we can illuminate x by $\mathcal{I}_{\delta,\eta}$.

□

Theorem 7.15. Let $\mathbf{B} \in \mathcal{U}_p^4$, then $\mathcal{I}(\mathbf{B}) \leq 16$.

Proof. If $\|e_r + e_s + e_t\|_{\mathbf{B}} > 1$ for every $r, s, t \in [4]$, then by propositions 7.1, 7.4, 7.5, 7.6, 7.7, 7.9 and 7.12, we can see that $\mathcal{I}(\mathbf{B}) \leq 16$. If $\|e_r + e_s + e_t\|_{\mathbf{B}} = 1$ for some $r, s, t \in [4]$, we can use proposition 7.14. □

8 Remarks About the Method

Let $\mathbf{B} \in \mathcal{U}^n$, $x \in \partial\mathbf{B}$. The first step to the method we are using to construct the illuminating sets to look at the entries of x which have maximum value, and those entries which take on the value 0 and determine if any vertices of $\mathbf{B}$ can take on this form. We try to illuminate as many *potential* vertices as possible using only $\{\pm e_i : i \in [n]\}$. Usually, it is the vertices which satisfy $|I_0^x| = 1$ which are the most tedious to illuminate. In these cases, we use lemma 2.42 and we have to identify a coordinate projection of x that is in the interior of $\mathbf{B}$ and then choose signs for a direction d such that $x + \varepsilon d$ will satisfy the hypothesis of lemma 2.42 (for some $\varepsilon > 0$). Usually to check the latter, we evaluate $x + \varepsilon d$ and compare it with some desirable convex combination in $\mathbf{B}$. The desirable convex combination is determined by the entries of $x + \varepsilon d$ and it can be constructed by using x or whatever vertices are assumed to already be in $\mathbf{B}$ based on whether $\mathbf{B}$ satisfies $\|e_i + e_j\|_{\mathbf{B}} = 1$ for specific pairs of $i, j \in [n]$. Moreover, for any d here which can also be viewed as *small* perturbations of $\pm e_i$, $i \in [n]$, we choose specific perturbations (by also employing lemma 2.41) so that d will illuminate a maximal number of vertices.

The upside to this method is that we are able to reduce this geometric problem into a combinatorial problem using only basic topology. Also, using this method, we are able to avoid the issue of determining the signs of $\mathbf{n} \in \nu(\mathbf{B}, x)$ which may cause problems as seen in section 4.

There are also a few downsides, that is, when using lemma 2.41, we implicitly force $\mathbf{B}$ to be a polytope. This is because we need to "minimize"/optimize over all the permissible perturbations so that all the extremal points of our convex body can be illuminated, but at the same time, we cannot allow the perturbations to essentially have more zero entries than before, since it's usually the combinations of signs on the entries of a perturbation that allow us to use lemma 2.42. Therefore, to generalize our results to non-polytopal/piecewise smooth convex bodies would most probably not be a trivial task.

However, in the case of 1-symmetric convex bodies, a previous version of this thesis focused primarily on results for polytopes from this class using the set $\mathcal{I}^n(\delta)$ from proposition 5.13. These results were essentially the polytope analogue of Theorem 5.18 and corollary 5.20. The δ in those results was found through the use of lemma 2.41, and this was restricting us from generalizing from 1-Symmetric polytopes to general 1-symmetric bodies. Prof. Vritsiou was able to use the same type of sets to get Theorem 5.18 and corollary 5.20, but avoided the use of lemma 2.41, which would impose a dependence of δ on each vertex. Thus in the end it was possible to choose δ, which depended only on the 1-symmetric convex body instead. A good place to continue is to try to see if we can avoid using 2.41 and still illuminate convex bodies using the same sets in our 1-unconditional polytope results.

Another major limitation of our method is the amount of cases that we have to consider. As one can see, in dimension 4, we are already dealing with many cases. We personally did not feel it was reasonable to attempt this method in dimension 5 without some sort of adaptations. If one would want to extend this method into higher dimensions, we recommend using matrices with rotation blocks to reduce some cases to other cases. In fact, we suspect that in dimension 4, we are able to simplify many of our cases using the following isomorphism:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, notice that almost every direction we picked is a small perturbation of $\pm e_i$ for $i \in [n]$. Even the directions which are not small perturbation of $\pm e_i$ for $i \in [n]$ can be replaced by these directions. Because of this, we hypothesize that we can apply a linear transformation onto the directions we used to illuminate any convex polytope which is inscribed in the cube. Since this problem is invariant under affine transformations, showing the previous claim would give the result for any convex polytope in dimensions 3 and 4.

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