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THEORETICAL AND NUMERICAL ANALYSIS OF CHEMOTACTIC MODELS

by

Guangjun Cao

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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ABSTRACT

Two chemotactic models are studied in this thesis, one with mixed Neumann and Dirichlet boundary conditions, refered to as the Anderson-Chaplain model, and one with Dirichlet boundary conditions, refered to as the Chaplain-Stuart model. For the Anderson-Chaplain model, the time-dependent problem is studied first, with or without limitation on its cross-diffusion. In the case of restricted cross-diffusion, proved are not only the global existence and uniqueness of solutions, but also a sufficient condition on the parameters, under which the system transits into a steady state; these are followed by a semi-discrete finite element analysis of the system, in which convergence is established and the error estimate is obtained. In the case of unrestricted cross-diffusion, the global existence of solution is proved in one spatial dimension. It is based on a gradient a priori estimate, a generalized Nirenberg-Gagliardo type inequality, and Fourier's method of solving a parabolic equation. The steady-state Anderson-Chaplain model is also considered. The system is shown to have a unique solution under appropriate conditions. The technique used in the proof is a combination of the concept of upper/lower solutions with a fixed point argument, in which we 'freeze' a non-local term so that the obtained system has quasi-monotone right-hand sides. This technique can also be applied to other steady-state chemotactic models, such as the Keller-Segel model. For the

Chaplain-Stuart model, its steady state is approximated by a finite difference scheme: first, existence and uniqueness of the numerical solutions are proved on the basis of discrete maximum principles; then, the numerical solutions are classified into two categories: *type I* and *type II*, and a sufficient condition on the system parameters is obtained to ensure a solution to be of *type I*; Finally, error estimates and numerical simulations are given.

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BASIC NOTATION

R^m	m-dimensional Euclidean space
$x = (x_1, ..., x_m)$	a variable point in R^m
D_j	$\frac{\partial}{\partial x_j}$
D^α	$D_1^{\alpha_1} ... D_m^{\alpha_m}$ where $\alpha = (\alpha_1, ..., \alpha_m)$, $\|\alpha\| = \alpha_1 + ... + \alpha_m$
Ω	a bounded domain in R^m
$\partial\Omega$	topological boundary of Ω
$\overline{\Omega}$	closure of Ω
$\|\Omega\|$	Lebesgue measure of $\Omega \subset R^m$
$\|\vec{f}\|$	$(\sum_{i=1}^m f_i^2)^{\frac{1}{2}}$ for $\vec{f} = (f_1, ..., f_m)$
$u'(t)$	$\frac{du}{dt}$
u_t	$\frac{\partial u}{\partial t}$
$u_{h,t}$	$\frac{\partial u_h}{\partial t}$
∇u	$(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_m})^T$, gradient of u
$\triangle u$	$\sum_{i=1}^m \frac{\partial^2 u}{\partial x_i^2}$, Laplacian of u
$\int_\Omega u(x)\,dx$	integration with respect to Lebesgue measure
$\overline{u}$	$\|\Omega\|^{-1} \int_\Omega u(x)\,dx$
$L^p(\Omega)$	p^{th}-power Lebesgue integrable functions defined on Ω
$\\|u\\|_p$	L^p norm of u on Ω
$\\|u\\|$	$\\|u\\|_p$ when $p = 2$
$\|u\|_{l,p}$	$\left(\int_\Omega \sum_{\|\alpha\|=l} \|D^\alpha u\|^p\,dx\right)^{\frac{1}{p}}$
$\\|u\\|_{l,p}$	$\left(\int_\Omega \sum_{\|\alpha\|\le l} \|D^\alpha u\|^p\,dx\right)^{\frac{1}{p}}$
$W_p^l(\Omega)$	the Sobolev space of functions having up to l order weak derivatives in $L^p(\Omega)$, endowed with $\\|.\\|_{l;p}$ norm
$H^l(\Omega)$	$W_p^l(\Omega)$ when $p = 2$
V^+	the set of all nonnegative elements in a topological vector space V
V^*	the space of linear bounded functionals on V, its dual space
$< f, u >$	the duality between $u \in V$ and $f \in V^*$

(u,v)	$\int_\Omega u(x)v(x)\,dx$
Q_T	$\Omega \times [0,T]$
$L^q((0,T);X)$	the space of L^q functions on $(0,T)$ with values in the Banach space X
$L^2(Q_T)$	$L^2((0,T);L^2(\Omega))$
$W_2^{1,0}(Q_T)$	the Hilbert space with scalar product $(u,v)_{W_2^{1,0}(Q_T)} = \int_{Q_T}(uv + \nabla u \cdot \nabla v)\,dxds$
$W_2^{1,1}(Q_T)$	the Hilbert space with scalar product $(u,v)_{W_2^{1,1}(Q_T)} = \int_{Q_T}(uv + \nabla u \cdot \nabla v + u_t v_t)\,dxds$
$V_2^{1,0}(Q_T)$	the space of functions in $W_2^{1,0}(Q_T)$ that are continuous in t in the norm of $L^2(\Omega)$, and have finite norm $\|u\|_{V_2^{1,0}(Q_T)} = \max_{0\le t\le T}\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)}$
$C^k[0,T]$	the space of functions having up to k-th order continuous derivatives on the interval $[0,T]$
$C^k[0,T]^l$	$\overbrace{C^k[0,T] \times \cdots \times C^k[0,T]}^{l \text{ times}}$
$V_h^{(0)}$	the piecewise linear finite element space in $H^1(\Omega)$
P_h	projection from $H^1(\Omega)$ to $V_h^{(0)}$ such that $< P_h u - u, v_h >= 0\ \forall v_h \in V_h^{(0)}$
$(a_{i,j})_{l\times l}$	l by l matrix with entry $a_{i,j}$ in the i-th row and j-th column
A^{-1}	inverse matrix of A
a^T (or A^T)	transpose of a vector a (or matrix A)
u^+	$\max\{u,0\}$
u^-	$\max\{-u,0\}$

Chapter 1

Introduction

1.1 A Survey of the Basic Chemotactic Models

All living organisms sense and respond to their surrounding environment. If the external stimulus is due to chemicals, the mechanism for response is called chemotaxis. Typical consequences of chemotaxis are cell aggregation and pattern formation. Several mathematical models were proposed to describe such phenomena with the one (KS) introduced by Keller and Segel (1970, [41]) having been attracting the most interest so far. Let $u(x,t)$ and $v(x,t)$ be the cell density and the concentration of the chemical substance at the position $x \in \Omega$ and the time $t \in (0,T)$, respectively, one version of the KS system is given by

$$(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u\nabla\chi(v)) & \text{in } \Omega \times (0,T), \\ \tau\dfrac{\partial v}{\partial t} = \Delta v - \gamma v + \alpha u & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \vec{\nu}} = \dfrac{\partial v}{\partial \vec{\nu}} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega, \end{cases}$$

where τ, γ, and α are positive constants; $\chi(v)$, called the **sensitivity function**, is a smooth function of v; Ω is a bounded domain in R^m $(m = 1,2,3)$ with smooth boundary $\partial\Omega$; $\vec{\nu}$ denotes the unit outer-normal vector of $\partial\Omega$;

and u_0 and v_0 are smooth, non-negative, and non-trivial initial values in Ω. The first equation describes the conservation of mass. Flux of u is given by $\nabla u - u\nabla\chi(v)$, so that the effect of diffusion $\nabla \cdot \nabla u$ and that of chemotaxis $-\nabla \cdot (u\nabla\chi(v))$ are competing for u to vary. The second equation is linear, and v is produced proportionally to u (the term αu), diffuses (term Δv), and is destroyed by a certain rate (term $-\gamma v$). Note that homogeneous Neumann boundary conditions are imposed on the KS.

A slightly different model (CS) was formulated by Chaplain and Stuart (1993, [13]) to describe the angiogenesis process, a process through which new blood vessels are produced. Here the solid tumor secretes a diffusible substance known as tumor angiogenesis factor (TAF) which causes the nearby endothelial cells (ECs)—cells that form the lining of normal body tissue—to migrate and proliferate under the angiogenesis stimulus. The 1-D CS is given by

$$
(1.2) \quad \begin{cases}
\dfrac{\partial u}{\partial t} - u_{xx} + \lambda u = -\dfrac{avu}{\gamma + u} & \text{in } (0,1) \times (0,T), \\
\dfrac{\partial v}{\partial t} - Dv_{xx} + \beta v + \kappa(vu_x)_x = b(1-v)vG(u) & \text{in } (0,1) \times (0,T), \\
u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0 \quad \text{in } (0,1), \\
u(0,t) = 1, u(1,t) = 0; v(0,t) = 0, v(1,t) = 1 \quad \text{in } (0,T),
\end{cases}
$$

where $u = u(x,t)$ and $v = v(x,t)$ are the TAF concentration and EC density, respectively; λ, a, γ, D, β, κ, and b are positive constants; $G(u) \equiv \max(u - c^*, 0)$, with $c^* \in (0,1)$ being the threshold concentration level of TAF below which proliferation does not occur. As with the KS model, we assume here smooth and compatible initial and boundary conditions. The first equation in (1.2) essentially says that the change of TAF concentration is due to diffusion $(-u_{xx})$, uptake of TAF by the ECs $\left(-\frac{avu}{\gamma+u}\right)$, and decay of chemical (λu). The second equation in (1.2) tells us that the following factors contribute to the change in EC density: diffusion $(-Dv_{xx})$, natural decay (βv), chemotactic motion $(\kappa(vu_x)_x)$, and reproduction $(b(1-v)vG(u))$. We see here that Dirichlet boundary conditions are used in the CS model. The steady-state CS model will be the research subject of Chapter 6.

Another model (AC) was introduced by Anderson and Chaplain (2000, [6]) to explain how secondary tumors can remain undetected in the presence of the primary tumor yet suddenly appear upon surgical removal of the primary tumor. It turns out that the cells of the primary tumor secrete an anti-angiogenic factor called angiostatin that suppresses EC migration and proliferation in a dose-dependent manner. Let $n(x,t)$, $c(x,t)$, and $a(x,t)$ be the EC tip density, TAF concentration, and the concentration of angiostatin, respectively. Then, the system is defined as

$$(1.3)\qquad \begin{cases} n_t - D_1\nabla\cdot\left(\nabla n - n\vec{f}\right) = 0 \qquad \text{in } \Omega\times(0,T), \\ c_t - \Delta c + \gamma_1 c + \beta_1 nc = 0 \qquad \text{in } \Omega\times(0,T), \\ a_t - D_2\Delta a + \gamma_2 a + \beta_2 na = 0 \qquad \text{in } \Omega\times(0,T), \\ c(x,0) = c_0(x),\ c(x,0) = c_0(x),\ a(x,0) = a_0(x), (\geq 0) \ \text{ in } \Omega, \\ \left(\nabla n - n\vec{f}\right)\cdot\vec{\nu} = 0 \text{ on } \partial\Omega\times(0,T), \\ c = 1 \text{ on } \Gamma_1\times(0,T),\ \dfrac{\partial c}{\partial\vec{\nu}} = 0 \text{ on } \Gamma_2\times(0,T), \\ \dfrac{\partial a}{\partial\vec{\nu}} = 0 \text{ on } \Gamma_1\times(0,T), a = 1 \text{ on } \Gamma_2\times(0,T), \end{cases}$$

where $\vec{f} \equiv \nabla\left(\frac{\chi}{\kappa D_1}\ln(1+\kappa c) + \frac{\alpha}{2D_1}a^2\right)$, or more generally,

$$(1.4)\qquad \vec{f} \equiv \chi_1(\nabla c, c, x)\nabla c + \chi_2(\nabla a, a, x)\nabla a$$

with χ_1 and χ_2 are smooth functions of their arguments; Ω is a bounded smooth domain in R^m $(m=1,2,3)$ with boundary $\partial\Omega = \Gamma_1\cup\Gamma_2$; χ, κ, α, D_1, D_2, γ_1, γ_2, β_1, and β_2 are positive constants; $\vec{\nu}$ is the unit outer-normal vector of $\partial\Omega$. Once again, we assume smooth and compatible initial and boundary conditions for the AC model, by which we mean

$$(1.5)\qquad \begin{cases} \left(\nabla n_0 - n_0\vec{f_0}\right)\cdot\vec{\nu} = 0 \text{ on } \partial\Omega, \\ c_0 = 1 \text{ on } \Gamma_1,\ \dfrac{\partial c_0}{\partial\vec{\nu}} = 0 \text{ on } \Gamma_2, \\ \dfrac{\partial a_0}{\partial\vec{\nu}} = 0 \text{ on } \Gamma_1, a_0 = 1 \text{ on } \Gamma_2, \end{cases}$$

where $\vec{f}_0$ is the $\vec{f}$ in (1.3) evaluated at c_0 and a_0. The first equation in (1.3) says the ECs, besides diffusion ($\nabla \cdot \nabla n$), react to both TAF and angiostatin in a chemotactic way $\left(\nabla \cdot (n\nabla(\frac{\chi}{\kappa D_1}\ln(1+\kappa c) + \frac{\alpha}{2D_1}a^2)\right)$. The second and third equations in (1.3) have the same structures. They tell us that the change in both chemicals (TAF and angiostatin) is due to diffusion, natural decay, and uptake of them by the ECs. It needs mentioning that mixed Dirichlet and Neumann boundary conditions are imposed on the AC model. If $\vec{f}$ in (1.4) is bounded by a constant, we call the system **desensitized at high chemical gradients,** or simply **desensitized**; If $\vec{f}$ is the gradient of a scalar function, we refer to that function as sensitivity function, as we did with the KS model. The AC model, with or without desensitization, will be studied in Chapter 3, and Chapters 4 and 5, respectively.

A summary: Our choices of the chemotactic models have been typical and complete, with boundary conditions ranging from Dirichlet's, to Neumann's, to mixed type. We point out that, for each model, the boundary conditions were chosen to simulate the situations encountered in practice.

1.2 Previous Work and Techniques Utilized

On the KS model. The first non-linear analysis of the KS was carried out by Nanjundiah ([65]) who suggested that aggregaton of cells may eventually lead to the formation of δ-functions in cell density, a phenomenon refered to as **chemotactic collapse**. His arguments, however, did not include the possible dependence of such collapse on the dimension of the space in which aggregation occurs. In fact, Childress and Percus ([15], [14]) showed that singular behavior is not possible in one dimension; while in higher dimensions, they presented results supporting Nanjundiah's contention that collapse can occur; As for the two dimensional case, they argued that chemotactic collapse requires a threshold number of cells in the system: precisely, there are numbers c_* and c^* such that the solution exists globally in time when $\|u_0\|_{L^1(\Omega)} < c_*$, and forms a δ-function singularity in finite time when $\|u_0\|_{L^1(\Omega)} > c^*$. While the arguments given by these authors were heuristic, making use of numerical computations for the steady-state problem, later studies supported their validity rigorously. Jäger and Luckhaus ([39]) proved the existence of such numbers for radially symmetric solutions for a simplified system ($\tau = 0$). For global existence, they obtained an a priori estimate for u by considering the test function $\phi = (u-k)_+^{m-1}$ with $k \geq 0$ and $m > 1$. They proved chemotactic collapse by constructing a radially symmetric lower solution for the first equation of KS. Then, Nagai ([64]) refined the above work by pointing out that $8\pi/(\alpha\chi)$ is the exact threshold number for radially symmetric solutions. He first established an L^∞ estimate for ∇v from the second equation of the KS and then applied Moser's technique to the first equation to get an L^∞ bound for u. By this way he showed global existence of solutions in the case $\|u_0\|_{L^1(\Omega)} < 8\pi/(\alpha\chi)$. For blow-up of solutions, he first established a differential inequality for the momentum $M_2(t) = \frac{1}{2\pi}\int_\Omega u(x,t)|x|^2\,dx$. Then he showed there exists $T_0 \in (0,\infty)$ such that $M_2(t) \to 0$ as $t \to T_0$. This leads to finite time blow-up. Global existence or blow-up results can also be found in the references [61], [60], [11], and [21], all of which deal with parabolic-elliptic versions of the KS with different sensitivity functions. In the case of no diffusion in the second equation, Rascle and Ziti ([73]) showed some examples of collapse by constructing self-similar solutions. Several articles were devoted to the steady-state KS. In these studies, the steady-state KS system was first reduced to a parameter-dependent single equation. Schaaf ([76]) analyzed the solutions via bifurcation of stable non-homogeneous aggregation patterns. Different sensitivity functions were considered there. Lin et al. ([51]) gave conditions for the system with *log* sensitivity function to have non-constant and constant solutions, respectively. In the case of linear sensitivity function, Wang et al. ([86]) established the existence of solution by combining Struwe's technique with the blow-up analysis for a problem with Neumann boundary condition. Coming to the full system KS ($\tau > 0$), Herrero and Velázquez ([35]) constructed a radially symmetric solution with u collapsing at the origin in finite time and having a concentrated mass equal to $8\pi/(\alpha\chi)$. On the other hand, Nagai et al. ([62]) proved radially symmetric solutions exist globally in time provided $\|u_0\|_{L^1(\Omega)} < 8\pi/(\alpha\chi)$. As for the general case, they gave $\|u_0\|_{L^1(\Omega)} < 4\pi/(\alpha\chi)$ as a criterion for the existence of global solutions. The basic techniques they used were Lyapunov functional, Trudinger-Moser inequality, and Moser's technique to obtain an L^∞ bound. Also by using Lyapunov functionals, Biler ([12]) and Gajewski et al. ([29]) obtained independently the same criterion above for existence of global solutions for the general case. Note that there is a discrepancy between radial $(8\pi/(\alpha\chi))$ and non-radial thresholds $(4\pi/(\alpha\chi))$. This was clarified by Senba et al. ([58], [78], [77], [32]). They showed that if $4\pi/(\alpha\chi) < \|u_0\|_{L^1(\Omega)} < 8\pi/(\alpha\chi)$ and the solution blows up in finite time, then the concentration toward $\partial\Omega$ occurs to u. Finally, much insight can be gained into the blow-up mechanism of the KS model through the work of Nagai et al. ([59]). They proved the following: If the solution (u, v) blows up in finite time, then u forms a δ-function singularity at each isolated blow-up point, and any blow-up point is isolated provided a certain Lyapunov functional is bounded; only the origin can be a blow-up point of radially symmetric solutions.

Comment: The purpose of the above survey is to examine the mathematical techniques used for the KS model as a means of gaining some insight into what might be done for other chemotactic models. In particular a global existence result is established for the AC model in 1-D, as was the case for the 1-D KS model. But we used a priori estimates instead of a Lyapunov functional, which does not apply to our situation.

On the CS model. Besides formulating the CS model, the authors of [13] also did some numerical simulations of both the time-dependent and the steady-state systems. But no rigorous analysis, theoretical or numerical, was presented there. Allegretto et al. ([4]) studied this system from a purely mathematical point of view. They first tackled the question of existence and uniqueness of solution for the steady-state system. A compact argument was used to show existence, while the proof of uniqueness was based on some integral inequalities. Since the u component of a solution is always decreasing while the v component, as they showed, has at most one pair of extrema, they classified the steady-state solutions into two types, according to whether v is monotone or not. Sufficient conditions were found to ensure a solution to be of a given type. As for the time-dependent problem, they considered both the smooth and non-smooth data. For the former, Schauder's fixed point theorem was used to show existence of solution; while for the latter, a limiting process and Lions' theorem were employed to achieve the same goal. It needs mentioning that the maximum principles are the basis for the existence proof in both the time-dependent case and the steady-state one.

On the AC model. Originally, the AC was formulated as a 1-D model in [6], where an analysis of the time-dependent solution was made by using the steady-state solution as an approximation. Since an explicit expression of n in terms of c and a can be obtained by integrating the first equation of the AC, solution profiles become available with simple numerical simulations.

A Summary: As in the case of the KS model, we expect a chemotactic system to be simplified by restricting the dimension of its domain Ω, by considering only radially symmetric solutions, or by studying extreme cases where diffusion is very large or very small. Another efficient way to reduce the difficulty inherent with chemotactic non-linearity is to allow the **cross-diffusion** $u\nabla\chi(v)$ to be desensitized at high cell density or high gradient of chemicals. The former is what Hillen et al. did in [36] where the system admits a positively invariant region; while the latter is exactly what we are going to do for the AC model in the first chapter of this thesis. Two questions need to be answered at this junction. (**a**) Is the model with desensitized cross-diffusion a realistic one? Even a casual thinking of what happens in real biological systems will lead us to the conclusion that the mechanisms or assumptions which were used to set up the chemotactic model are no longer valid long before its solution blows up. Hence there is a good reason for desensitization. (**b**) Does the soluton of such a desensitized system still exhibit profiles that we are interested in? This was answered in the affirmative for the KS model in [36] where numerical simulations show interesting phenomena of pattern formation and formation of stable aggregates. We believe the same is true of the AC model.

1.3 Topics of This Thesis

With work on the KS model having been done to a quite satisfactory degree, we will mainly study the AC model, with or wothout desensitization, and the steady-state CS model in this thesis.

Chapter 2: We collect all the prerequisites in Chapter 2. We first list all the inequalities we need: the Basic, Young's, Hölder's, Minkowki's, Nirenberg-Gagliardo's, and Gronwall-Bellman type inequalities; we then state different versions of the maximum principle; these are followed by the Divergence Theorem and Green's three identities. Next is Moser's technique to obtain an L^∞ bound for a function satisfying a certain differential inequality. Finally, the positivity proof of the solution to the AC model is presented.

Chapter 3: Chapter 3 deals with the desensitized AC model. Note that no rigorous analysis whatsoever has been made of this system. We allow the chemotactic response from the cells to be desensitized at high chemical gradient. Since it is not clear at the moment whether the original system admits a blow-up mechanism or not, it is essential to retain this restriction to achieve global existence of solution. This can be done by successively using the contraction mapping argument for local existence, which is made possible by the uniform a priori L^∞ estimate obtained through Moser's technique. Note that this result is comparable to the global existence given in [36], but we have used a quite different approach, and we have also considered the long time behavior of the system solution. While we believe the system will transit into a steady state without any condition on the system parameters, we do put such restrictions to attain this. The proof essentially depends on an energy-like inequality $\frac{d}{dt}E(n_t, c_t, a_t) \leq -\delta_0 E(n_t, c_t, a_t)$. Once the theoretical properties of the system are clear, we turn our attention to its numerical aspect. Since only a semi-discrete scheme is considered, it turns out that we can resort to the standard theory of ordinary differential equations for the existence of numerical solutions. But this theory only ensures us a time-dependent a priori L^∞ bound for the solution, and we point out that Moser's technique is no longer applicable to the numerical case because of the restriction on test functions. We manage to obtain a uniform a priori L^2 bound basing on an inequality with an arbitrary parameter $\epsilon > 0$ which relates the L^2 norm of a non-negative function u, its L^1 norm, and the L^2 norm of ∇u together:

$$||n||^2_{L^2(\Omega)} \leq C_\epsilon ||n||^2_{L^1(\Omega)} + \epsilon ||\nabla n||^2_{L^2(\Omega)}.$$

While this inequality is obtainable from the Nirenberg-Gagliardo's inequality, as is shown in Chapter 4, we gave a direct proof using the compact imbedding of $H^1(\Omega)$ into $L^2(\Omega)$. As usual, the proof of uniqueness of solution needs a priori bounds for the numerical solutions. It is an interesting fact that the convergence of the numercal solution to the theoretical one is proved under essentially the same conditions as that which ensure the system to evolve into a steady state. In obtaining the error analysis, we use the idea of elliptic projection ([83]). We emphasize that an L^2 error bound of only $O(h)$ is proved when compared to that of $O(h^2)$ for a linear system, even the same linear finite elements are used in both cases. We believe this is due to the chemotactic nature of the system and the projections we used. It is not clear at the moment how this can be improved, if it can be at all.

Chapter 4: The principal goal of Chapter 4 is to tackle the problem of global existence for the AC model with free cross-diffusion, and success is achieved in the 1-D case. While the KS system admits Lyapunov functionals, which greatly facilitates the proof of global existence, this is unlikely to happen in the AC model due to the inhomogeneous nature of its boundary conditions ([37]). Our main tools have been the Fourier's method to express the solution of a time-dependent problem in terms of the eigenfunctions of the corresponding elliptic problem, the Nirenberg-Gagliardo's inequality, and the eigenvalue estimates for some elliptic problems. For completeness, we also include local in time existence of solution in this chapter, in which case Schauder's fixed point theorem is utilized. The ideas used in the proof of this part were suggested by [29].

Chapter 5: Chapter 5 treats the existence and the uniqueness of solution to the steady-state AC model. The divergence form of the first equation allows us to transform the system through substitution and integration into a system with two unknowns, which to our great advantage has quasi-monotone right-hand sides after a non-local term in it is "frozen". The method of monotone sequence (Pao, [69]) can then be applied to give the existence of solution for this variation of the simplified system. By defining an operator T and showing it has a fixed point, we prove that the simplified system, and thus equivalently the original system, has a solution. The only thing that needs special care during this process is the mixed boundary conditions. We give a direct proof of the maximum principle for such mixed boundary conditions. As for the regularity of solution, we cite reference (Miranda, [56]) for a detailed proof. For the uniqueness of solution, we point out that several conditions were given in [69] for similar systems, but none of these is satisfied by the steady-state AC model. We are able to show a uniqueness result through a different approach, though. The proof is elementary, and the condition is mild when compared to those in the literature for similar systems. What is more, we find that the proof equally applies to the other cases considered in [69].

Chapter 6: Chapter 6 begins with a brief review of the mathematical properties of the steady-state CS model (Allegratto et al. [4]). A sufficient condition in [4] is then improved which ensures a solution to be of *type I*. The proof is based on a non-linear version of the maximum principle. While both theoretical and numerical aspects of the CS model are treated in this chapter, its emphasis is on the latter. The discrete maximum principle is the right place where to start our finite difference numerical analysis. We then use it to set up the positivity of solution for a particular type of second order difference equation, to which both equations of our discretized system belong. The key part of the numerical analysis is the a priori estimates established with the aid of the discrete maximum principle. We first consider difference equations with homogeneous boundary conditions, with or without a first order term. In the latter case, a sharp estimate on the magnitude of solutions, among the uniform estimate class, is obtained. These estimates are then generalized to the cases of inhomogeneous boundary conditions. We point out that these estimates only fit our situations. For example, we require the coefficients involved keep sign, which is not true for general difference equations. With all these preparations at hand, we formulate the difference scheme and prove the existence of solution using Schauder's fixed point theorem. The main part of the proof is the verification of the continuity of an operator defined there, which is done through the a priori estimates set up so far. The uniqueness proof we give in this part is essentially a parallel of that in the continuous case. We then establish convergence and error estimates. While restrictions are put on the parameters to obtain error bounds, convergence does not assume such conditions. In fact, convergence is a direct consequence of the unform boundedness and equi-continuity of the system solutions. The numerical simulations that follow verify the theory developed in the previous sections.

Summary: We have considered chemotactic models with three typical boundary conditions: Neumann's, Dirichlet's, and mixed type, with the first case being thoroughly studied by other authors. We have seen that quite different techniques have been used in different cases. The questions we addressed range from the existence and uniqueness of both the theoretical and numerical solutions, either in the time-dependent case or in the steady-state case, to the long time behavior of a time-dependent model, and to the convergence and convergence rate of a finite difference scheme for a steady-state model. These are the basic questions asked of a system of partial differential equations. In each case a relatively satisfactory answer is obtained.

Chapter 2

Preliminaries

In this Chapter we gather all the preparatory results that will be used in the thesis. Unless otherwise specified, Ω will be a bounded domain in R^m $(m = 1, 2, 3)$.

2.1 Inequalities and Imbedding Theorems

Theorem 2.1. *(The Basic) For any real numbers a, b, and any $\epsilon > 0$, we have*

$$ab \leq \frac{a^2}{4\epsilon} + \epsilon b^2. \tag{2.1}$$

Proof. Expanding $\left(\frac{a}{2\epsilon^{1/2}} - \epsilon^{1/2} b\right)^2 \geq 0$, we obtain the desired result. □

Theorem 2.2. *([54]) (Young) For $a, b \geq 0$ and $1 < p < \infty$, $q = p/(p-1)$, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.2}$$

Theorem 2.3. *([54]) (Hölder) Let $p, q \geq 1$ with $1/p + 1/q = 1$. If $u \in L^p(\Omega)$, and $v \in L^q(\Omega)$, then*

$$\int_\Omega |uv|\, dx \leq \|u\|_{L^p} \|v\|_{L^q}. \tag{2.3}$$

Remark 2.1. Hölder's inequality can be extended to the case of k functions, u_1, ...,u_k lying respectively in spaces $L^{p_1}(\Omega)$, ..., $L^{p_k}(\Omega)$ where $\sum_{i=1}^k 1/p_i = 1$.

Theorem 2.4. *([54]) (Minkowski) Let* $p \geq 1$. *If* $u, v \in L^p(\Omega)$, *then* $u + v \in L^p(\Omega)$, *and*

$$\|u+v\|_{L^p} \leq \|u\|_{L^p} + \|v\|_{L^p}. \tag{2.4}$$

Theorem 2.5. *([24], Page 66) (Poincaré) There is a constant* $\lambda > 0$ *such that for all* $h \in H^1(\Omega)$ *with* $\overline{h} = 0$, *it holds that* $\|h\| \leq \lambda \|\nabla h\|$.

Theorem 2.6. *([27], Page 30) (Imbedding) Let* Ω *satisfy the cone condition. If a function* u *belongs to* W_p^j *with* $j > l + m/p$ *for some nonnegative number* l, *then* $u \in C^l(\overline{\Omega})$.

Remark 2.2. If $\partial\Omega$ is of class C^1, then Ω satisfies the cone condition. A convex domain also has the cone property.

Theorem 2.7. *([27], Page 27) (Nirenberg-Gagliardo) Let* $\partial\Omega$ *be in* C^l, *and* u *be any function in* $W_q^l(\Omega) \cap L^r(\Omega)$, $1 \leq q, r \leq \infty$. *For any integer* k, $0 \leq k < l$, *and for any number* θ *in the interval* $k/l \leq \theta \leq 1$, *set*

$$\frac{1}{p} = \frac{k}{m} + \theta\left(\frac{1}{q} - \frac{l}{m}\right) + (1-\theta)\frac{1}{r}$$

If $l - k - m/q$ *is not a nonnegative integer, then*

$$|u|_{k,p} \leq C_0 \|u\|_{l,q}^{\theta} \|u\|_{0,r}^{1-\theta}. \tag{2.5}$$

If $l - k - m/q$ *is a nonnegative integer, then the above inequality holds for* $\theta = k/l$. *The constant* C_0 *depends only on* Ω, q, r, l, k, *and* θ.

Corollary 2.1. *Let* $\partial\Omega$ *be in* C^1. *For any* $\epsilon > 0$ *there exists a constant* $C_\epsilon > 0$ *such that for all* $u \in H^1(\Omega)$ *we have* $\|u\|_{0,2}^2 \leq \epsilon |u|_{1,2}^2 + C_\epsilon \|u\|_{0,1}^2$.

Proof. See the proof of Lemma 4.4; or see Lemma 3.2 for an alternative (direct) proof. ☐

Theorem 2.8. *([68]) (Gronwall) Let* u *be a continuous function defined on the interval* $I = [\alpha, \alpha + h]$ *and*

$$0 \leq u(t) \leq \int_\alpha^t [bu(s) + a]\, ds,\ t \in I, \tag{2.6}$$

where a and b are constants. Then

$$0 \le u(t) \le ahe^{bh}, \ t \in I. \tag{2.7}$$

Proof. Put $u = z(t)\exp(b(t-\alpha))$, and let the maximum of z on I occur at $t = t_1$. For this value of t, (2.6) gives

$$0 \le z_{max}\exp(b(t_1-\alpha)) \le \int_\alpha^{t_1} [bz(s)\exp(b(s-\alpha)) + a]\, ds,$$

whence by the Mean Value Theorem

$$\begin{aligned} 0 \le z_{max}\exp(b(t_1-\alpha)) &\le z_{max}\int_\alpha^{t_1} b\exp(b(s-\alpha))\, ds + \int_\alpha^{t_1} a\, ds \\ &= z_{max}(\exp(b(t_1-\alpha)-1) + a(t_1-\alpha), \end{aligned}$$

or finally

$$0 \le z_{max} \le a(t_1-\alpha) \le ah,$$

from which (2.7) follows at once. □

Theorem 2.9. *([68]) (Bellman) Let u and f be continuous and nonnegative functions defined on $I = [\alpha, \beta]$, and let c be a nonnegative constant. Then the inequality*

$$u(t) \le c + \int_\alpha^t f(s)u(s)\, ds, \ t \in I \tag{2.8}$$

implies that

$$u(t) \le c\exp\left(\int_\alpha^t f(s)\, ds\right), \ t \in I. \tag{2.9}$$

Proof. Define a function z(t) by the right-hand side of (2.9); then we observe that $z(\alpha) = c$, $u(t) \le z(t)$ and

$$z'(t) = f(t)u(t) \le f(t)z(t), \ t \in I. \tag{2.10}$$

Multiply (2.10) by $\exp(-\int_\alpha^t f(s)\, ds)$ and apply the identity

$$z'(t)\exp(-\textstyle\int_\alpha^t f(s)\, ds) - z(t)f(t)\exp(-\int_\alpha^t f(s)\, ds)$$

$$= \tfrac{d}{dt}\left[z(t)\exp(-\textstyle\int_\alpha^t f(s)\,ds)\right],$$

to obtain

$$\frac{d}{dt}\left[z(t)\exp(-\int_\alpha^t f(s)\,ds)\right] \le 0. \tag{2.11}$$

Integrating (2.11) from α to t gives

$$z(t)\exp(-\int_\alpha^t f(s)\,ds) - z(\alpha) \le 0. \tag{2.12}$$

Using $z(\alpha) = c$ and $u(t) \le z(t)$ in (2.12) we get the desired inequality in (2.9). □

But more frequently used are the various generalizations of the above inequalities. Here we state two of them without proof.

Theorem 2.10. *([68]) Let u, g and h be nonnegative continuous functions defined on $I = [\alpha, \beta]$, $n(t)$ be a continuous, positive and nondecreasing function defined on I and*

$$u(t) \le n(t) + g(t)\int_\alpha^t h(s)u(s)\,ds,\ t \in I; \tag{2.13}$$

then

$$u(t) \le n(t)\left[1 + g(t)\int_\alpha^t h(s)\exp\left(\int_s^t h(r)g(r)\,dr\right)ds\right],\ t \in I. \tag{2.14}$$

Theorem 2.11. *([68]) Let u, p, q, f and g be nonnegative continuous functions defined on $I = [\alpha, \beta]$, and*

$$u(t) \le p(t) + q(t)\int_\alpha^t [f(s)u(s) + g(s)]\,ds,\ t \in I. \tag{2.15}$$

Then

$$u(t) \le p(t) + q(t)\int_\alpha^t [f(s)p(s) + g(s)]\exp\left(\int_s^t f(r)q(r)\,dr\right)ds,\ t \in I. \tag{2.16}$$

Remark 2.3. As pointed out by Beesack ([10]), if the integrals in Theorem 2.8-2.11 are Lebesgue integrals, the hypotheses can be relaxed to: the functions involved are measurable and certain products of them are integrable. The equality and inequality conditions are then understood to hold almost everywhere.

2.2 The Maximum Principles

We first state various versions of the **maximum principle.**

Theorem 2.12. *([72]) Let*

$$\Delta u \geq 0 \text{ in } \Omega.$$

If u attains its maximum M at any interior point of Ω, then $u \equiv M$ in Ω.

Theorem 2.13. *([72]) Let $u \in C^2(\Omega)$ satisfy the differential inequality*

$$(L+h)[u] \equiv \sum_{i,j=1}^{m} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i}^{m} b_i \frac{\partial u}{\partial x_i} + hu \geq 0$$

with $h \leq 0$, L uniformly elliptic in Ω, and the coefficients of L and h bounded. If u attains a nonnegative maximum M at an interior point of Ω, then $u \equiv M$.

Theorem 2.14. *([72]) Let $u \in C^2(\Omega)$ satisfy the differential inequality*

$$(L+h)[u] \equiv \sum_{i,j=1}^{m} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i}^{m} b_i \frac{\partial u}{\partial x_i} + hu \geq 0$$

with $h \leq 0$, and L uniformly elliptic in Ω. Suppose that $u \leq M$ in Ω, that $u = M$ at a boundary point P, and that $M \geq 0$. Assume that P lies on the boundary of a ball in Ω. if u is continuous in $\Omega \cup P$, any outward directional derivative of u at P is positive unless $u \equiv M$ in Ω.

2.3 Green's Identities

Now we give the divergence theorem and Green's three identities ([72]). Recall that Γ is the boundary of Ω. Let $\vec{w}$ be a smooth vector field defined in $\overline{\Omega}$. The **divergence theorem** states that

$$\int_\Omega \nabla \cdot \vec{w} \, d\Omega = \int_\Gamma \vec{w} \cdot \vec{\nu} \, d\Gamma. \tag{2.17}$$

We choose $\vec{w} = v \nabla u$ in (2.17) to obtain **Green's first identity**

$$\int_\Omega v \Delta u \, d\Omega + \int_\Omega \nabla v \cdot \nabla u \, d\Omega = \int_\Gamma v \frac{\partial u}{\partial \vec{\nu}} \, d\Gamma. \tag{2.18}$$

Interchanging u and v in (2.18), we get

$$\int_\Omega u \Delta v \, d\Omega + \int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Gamma u \frac{\partial v}{\partial \vec{\nu}} \, d\Gamma.$$

Subtracting this equation from (2.18), we obtain **Green's second identity**

$$\int_\Omega (v\Delta u - u\Delta v)\,d\Omega = \int_\Gamma (v\frac{\partial u}{\partial\vec{\nu}} - u\frac{\partial v}{\partial\vec{\nu}})\,d\Gamma. \tag{2.19}$$

Let P and Q be two points in Ω with coordinates x_i and y_i $(i = 1, 2, ..., m)$, $r_{PQ} = (\sum_{i=1}^m (x_i - y_i)^2)^{1/2}$ be their distance. Let ψ be a harmonic function throughout Ω. By applying Green's second identity to the function $W = \frac{1}{4\pi r_{PQ}} + \psi$, we obtain **Green's third identity**

$$u(Q) = \int_\Gamma (W\frac{\partial u}{\partial\vec{\nu}} - u\frac{\partial W}{\partial\vec{\nu}})\,d\Gamma - \int_\Omega W\Delta u\,d\Omega. \tag{2.20}$$

Now let us use Green's third identity to solve the following mixed boundary value problem:

$$\begin{cases} \Delta u = h \text{ in } \Omega, \\ u|_{\Gamma_1} = h_1,\ \frac{\partial u}{\partial\vec{\nu}}|_{\Gamma_2} = h_2. \end{cases} \tag{2.21}$$

We first seek a function ψ such that

$$\Delta\psi = 0 \text{ in } \Omega,$$
$$\psi|_{\Gamma_1} = -\frac{1}{4\pi r_{PQ}},$$
$$\frac{\partial\psi}{\partial\vec{\nu}}|_{\Gamma_2} = -\frac{\partial(\frac{1}{4\pi r_{PQ}})}{\partial\vec{\nu}}.$$

Then we define

$$G(P;Q) = \frac{1}{4\pi r_{PQ}} + \psi$$

as the **Green's function** for the problem (2.21). Choosing this function as W in Green's third identity, we get the solution formula for (2.21):

$$u(Q) = -\int_\Omega G(P;Q)h(P)\,d\Omega_P - \int_{\Gamma_1} h_1(P)\frac{\partial G(P;Q)}{\partial\vec{\nu}_P})\,d\Gamma_P + \int_{\Gamma_2} G(P;Q)h_2(P)\,d\Gamma_P \tag{2.22}$$

It can be shown ([72]), by means of the maximum principle, that the Green's function for the problem (2.21) satisfies

$$\begin{cases} \frac{\partial G}{\partial\vec{\nu}} < 0 \text{ in } \Gamma_1, \\ G > 0 \text{ in } \Omega \cup \Gamma_2. \end{cases} \tag{2.23}$$

Conversely, the above properties of the Green's function allow us to read off properties of the solution u of (2.21) from formula (2.22). For example, if $h \leq 0$ in Ω, $h_1 \geq 0$ on Γ_1, and $h_2 \geq 0$ on Γ_2, then (2.22) and (2.23) show that $u \geq 0$ in Ω. Moreover, if h, h_1, and h_2 are not all identically zero, then $u > 0$ in Ω.

2.4 Moser's Technique

Theorem 2.15. *([3]) Assume the following inequality is true of $n = n(x,t) \geq 0$ on $Q_T \equiv \Omega \times (0,T)$*

$$\frac{1}{\lambda+1}\frac{d}{dt}\int_\Omega n^{\lambda+1}\,dx \leq -\frac{2\lambda}{(\lambda+1)^2}\int_\Omega |\nabla n^{\frac{\lambda+1}{2}}|^2\,dx + \frac{K^2\lambda}{2}\int_\Omega n^{\lambda+1}\,dx \tag{2.24}$$

where K is a constant, and $\lambda \in [0,\infty)$. Then there exists a constant N, independent of T, such that $\sup\limits_{t>0}\|n\|_{L^\infty(\Omega)} \leq N$ for.

Proof. Let $E_\lambda(t) \equiv \int_\Omega n^{\lambda+1}\,dx$. Since (2.24) implies $\frac{d}{dt}E_\lambda(t) \leq \frac{K^2\lambda(\lambda+1)}{2}E_\lambda(t)$, for any $t > 0$, we have: (a) $E_0(t) \leq E_0(0)$, that is, $\sup\limits_{t>0}\|n\|_{L^1(\Omega)}$ is finite; and (b) $E_\lambda(t) \leq E_\lambda(0)\exp\left(\frac{K^2\lambda(\lambda+1)}{2}t\right) < \infty$ for any $\lambda > 0$, that is, $n^{\lambda+1}$ is integrable.

Now Let $\lambda_i = 2^i - 1$ in (2.24) ($i = 1,2,...$), we will estimate $\int_\Omega n^{2^i}\,dx$ in terms of $\int_\Omega n^{2^{i-1}}\,dx$ and thus recursively obtain an estimate depending on the uniform $L^1(\Omega)$ bound of n above. By controlling carefully the constants then we will pass to the limit and obtain the desired L^∞ bound.

Let us fix i and define $v \equiv n^{2^{i-1}}$. Then (2.24) for $\lambda = \lambda_i$ takes the form

$$\frac{d}{dt}\int_\Omega v^2\,dx \leq -\nu_i\int_\Omega |\nabla v|^2\,dx + a_i\int_\Omega v^2\,dx, \tag{2.25}$$

where $\nu_i \equiv 2 - \frac{1}{2^{i-1}}$ and $a_i \equiv K^2 2^{i-1}(2^i - 1)$. By Corollary 2.1, for any $\epsilon > 0$ we have

$$\|v\|_{L^2}^2 \leq \epsilon\|\nabla v\|_{L^2}^2 + C_\epsilon\|v\|_{L^1}^2. \tag{2.26}$$

Now choose $\epsilon = \epsilon_i \equiv \frac{2}{\max\{1,K^2\}} \frac{1}{4^i}$ so that

$$(a_i + \epsilon_i)\epsilon_i \leq \nu_i. \tag{2.27}$$

By defining $c_i \equiv C_{\epsilon_i}$, we rewrite (2.26) as

$$\|v\|_{L^2}^2 \leq \epsilon_i \|\nabla v\|_{L^2}^2 + c_i \|v\|_{L^1}^2. \tag{2.28}$$

Multiplying (2.28) by $-(a_i + \epsilon_i)$ and rearranging its terms, we obtain

$$-(a_i + \epsilon_i)\|v\|_{L^2}^2 + (a_i + \epsilon_i)c_i\|v\|_{L^1}^2 \geq -(a_i + \epsilon_i)\epsilon_i\|\nabla v\|_{L^2}^2 \tag{2.29}$$

Combining (2.27) and (2.29) with (2.25) we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega v^2\,dx \quad & \leq -(a_i + \epsilon_i)\epsilon_i \int_\Omega |\nabla v|^2\,dx + a_i \int_\Omega v^2\,dx \\ & \leq -\epsilon_i \int_\Omega v^2\,dx + (a_i + \epsilon_i)c_i \left(\int_\Omega v\,dx\right)^2 \\ & \leq -\epsilon_i \int_\Omega v^2\,dx + (a_i + \epsilon_i)c_i \left(\sup_{t\geq 0} \int_\Omega v\,dx\right)^2. \end{aligned} \tag{2.30}$$

Substituting $v = n^{2^{i-1}}$ back into (2.30) we get

$$\frac{d}{dt}\int_\Omega n^{2^i}\,dx \leq -\epsilon_i \int_\Omega n^{2^i}\,dx + (a_i + \epsilon_i)c_i \left(\sup_{t\geq 0}\int_\Omega n^{2^{i-1}}\,dx\right)^2, \tag{2.31}$$

which gives

$$\int_\Omega n^{2^i}\,dx \leq \max\left\{\delta_i \left(\sup_{t\geq 0}\int_\Omega n^{2^{i-1}}\,dx\right)^2, \int_\Omega n^{2^i}(x,0)\,dx\right\}, \tag{2.32}$$

where $\delta_i = \dfrac{(a_i + \epsilon_i)c_i}{\epsilon_i}$. Now by defining

$$\alpha = \max\{\sup_{t\geq 0} \|n\|_{L^1(\Omega)}, \max\{1, |\Omega|\} \times \|n(x,0)\|_{L^\infty(\Omega)}\}$$

and assuming, without loss of generality, that $\delta_i \geq 1$, from (2.32) we obtain

$$
\begin{aligned}
(2.33)\quad \sup_{t\geq 0}\int_\Omega n^{2^i}\,dx \quad &\leq \max\left\{\delta_i\left(\sup_{t\geq 0}\int_\Omega n^{2^{i-1}}\,dx\right)^2, \alpha^{2^i}\right\} \\
&\leq \delta_i \max\left\{\left(\sup_{t\geq 0}\int_\Omega n^{2^{i-1}}\,dx\right)^2, \alpha^{2^i}\right\} \\
&= \delta_i \max\left\{\sup_{t\geq 0}\int_\Omega n^{2^{i-1}}\,dx, \alpha^{2^{i-1}}\right\}^2 \\
&\leq \delta_i \max\left\{\delta_{i-1}\left(\sup_{t\geq 0}\int_\Omega n^{2^{i-2}}\,dx\right)^2, \alpha^{2^{i-1}}\right\}^2 \\
&\leq \delta_i\left(\delta_{i-1}\max\left\{\left(\sup_{t\geq 0}\int_\Omega n^{2^{i-2}}\,dx\right)^2, \alpha^{2^{i-1}}\right\}\right)^2 \\
&= \delta_i\delta_{i-1}^2 \max\left\{\sup_{t\geq 0}\int_\Omega n^{2^{i-2}}\,dx, \alpha^{2^{i-2}}\right\}^{2^2} \\
&\leq \ldots\ldots \\
&\leq \delta_i\delta_{i-1}^2\delta_{i-2}^{2^2}\ldots\delta_1^{2^{i-1}} \max\left\{\sup_{t\geq 0}\int_\Omega n^{2^0}\,dx, \alpha^{2^0}\right\}^{2^i} \\
&\leq \delta_i\delta_{i-1}^2\delta_{i-2}^{2^2}\ldots\delta_1^{2^{i-1}}\alpha^{2^i}.
\end{aligned}
$$

But

$$
(2.34)\qquad \delta_i = \frac{(a_i+\epsilon_i)c_i}{\epsilon_i} = (a_i+\epsilon_i)\epsilon_i\cdot\frac{c_i}{\epsilon_i^2} \leq \nu_i\cdot\frac{c_i}{\epsilon_i^2} \leq \frac{2c_i}{\epsilon_i^2} \leq r\cdot 2^{(m+4)i},
$$

where

$$
r = \frac{4}{m}C_0^{m+2}\left(\frac{m}{m+2}\right)^{\frac{m+2}{2}}\cdot\ \max\{1, K^{m+4}\}.
$$

Combining (2.34) with (2.33) and using a simple computation we get

$$
(2.35)\qquad \int_\Omega n^{2^i}\,dx \leq \sup_{t\geq 0}\int_\Omega n^{2^i}\,dx \leq r^{2^i-1}2^{(m+4)(2^{i+1}-i-1)}\alpha^{2^i}.
$$

Finally, by taking the $2^i th$ root on both sides of (2.35) and passing to the limit as $i\to\infty$ we obtain

$$
(2.36)\qquad n \leq \|n\|_{L^\infty} \leq N \equiv r\cdot 2^{2(m+4)}\cdot\alpha.
$$

□

Chapter 3

Analysis of the AC Model with Desensitization

Throughout this chapter, we study the AC model of a general dimension m ($m = 1, 2, 3$), but make the following **desensitization assumption**:

$$(3.1) \qquad |\vec{f}| \equiv |\chi_1(\nabla c, c, x)\nabla c + \chi_2(\nabla a, a, x)\nabla a| \leq K,$$

where K is a constant, and we require $\vec{f}$ to have bounded derivatives with respect to its arguments so that it satisfies a Lipschitz condition:

$$(3.2) \quad |\vec{f_2} - \vec{f_1}| \leq L(|\nabla(c_2 - c_1)| + |c_2 - c_1| + |\nabla(a_2 - a_1)| + |a_2 - a_1| + |x_2 - x_1|),$$

where $\vec{f_i} = \vec{f}(\nabla c_i, \nabla a_i, c_i, a_i, x_i)$ ($i = 1, 2$). A simple example of $\vec{f}$ that satisfies (3.1) and (3.2) is given by

$$\vec{f} = \frac{C_1}{1 + |\nabla c|}\nabla c + \frac{C_2}{1 + |\nabla a|}\nabla a.$$

Also, the $\vec{f}$ in model (1.3):

$$\vec{f} \equiv \nabla\left(\frac{\chi}{\kappa D_1}\ln(1 + \kappa c) + \frac{\alpha}{2D_1}a^2\right)$$

will satisfy (3.1) and (3.2) automatically if we can prove that $|\nabla c|$ and $|\nabla a|$ are bounded, as is the case in Chapter 4. We set the constants D_1 and D_2

in (1.3) to be 1 for convenience. We assume compatible initial and boundary conditions (1.5).

Though a vast amount of work can be found on the KS model, no rigorous analysis has been made of the AC model. This chapter represents our first attempt to study the AC model. We start with the formulation of the weak solutions. By obtaining a uniform a priori $L^\infty(\Omega)$ bound on the weak solutions, we prove both the local and the global existence of solution. We have noticed that a global existence result was obtained in [36] under a similar assumption but through a quite different approach (semi-group theory of the heat equation). The long time behavior of a system is one of the interesting topics to work on. We obtain a sufficient condition on the parameters of the system for it to transit into a steady state, which indicates that the system is not likely to admit non-constant periodic solutions. Our semi-discrete finite element analysis of this system ends up with error estimate of first order in space, which is not optimal, but the question of how to improve it still remains unanswered.

The outline for this chapter is as follows. In §3.1, we define the weak solutions, and then use contraction mapping theorem to show existence and uniqueness of solutions. We consider the long time behavior of the system in §3.2, and prove that under appropriate conditions, it transits into a steady state. In the remaining two sections (§3.3 and §3.4), we study a semi-discrete finite element approximation of the desensitized AC model and give the error estimates.

For convenience, we repeat the model equations here:

$$
(3.3) \quad \begin{cases}
n_t - D_1\nabla\cdot\left(\nabla n - n\vec{f}\right) = 0 \qquad \text{in } \Omega\times(0,T),\\
c_t - \Delta c + \gamma_1 c + \beta_1 nc = 0 \qquad \text{in } \Omega\times(0,T),\\
a_t - D_2\Delta a + \gamma_2 a + \beta_2 na = 0 \qquad \text{in } \Omega\times(0,T),\\
c(x,0) = c_0(x),\ c(x,0) = c_0(x),\ a(x,0) = a_0(x), (\geq 0) \ \text{ in } \Omega,\\
\left(\nabla n - n\vec{f}\right)\cdot\vec{\nu} = 0 \text{ on } \partial\Omega\times(0,T),\\
c = 1 \text{ on } \Gamma_1\times(0,T),\ \dfrac{\partial c}{\partial\vec{\nu}} = 0 \text{ on } \Gamma_2\times(0,T),\\
\dfrac{\partial a}{\partial\vec{\nu}} = 0 \text{ on } \Gamma_1\times(0,T), a = 1 \text{ on } \Gamma_2\times(0,T),
\end{cases}
$$

3.1 Existence and Uniqueness of the Exact Solutions

3.1.1 The Weak Formulation of the System Solutions

We first give a formulation of the weak solutions to the system (Page 168, [46]). Let

$$
\begin{aligned}
&W \equiv C((0,T);L^2(\Omega))\cap V_2^{1,0}(Q_T),\\
&V \equiv W_2^{1,1}(Q_T),\\
&V_1 \equiv \{v\in V | v = 0 \text{ on } \Gamma_1\times(0,T)\},\\
&V_2 \equiv \{v\in V | v = 0 \text{ on } \Gamma_2\times(0,T)\}.
\end{aligned}
$$

A weak solution of the AC model (3.3) is a triple (n,c,a) with n, c, a in W, and $c|_{\Gamma_1\times(0,T)} = 1$, and $a|_{\Gamma_2\times(0,T)} = 1$, such that $\forall\ (v,v_1,v_2)\in V\times V_1\times V_2$ which vanish at $t=T$,

$$
(3.4) \qquad -\int_0^T < n, v_t > ds + \int_0^T (\nabla n - \vec{f}n, \nabla v)\, ds = \int_\Omega n_0(x)v(x,0)\, dx,
$$

$$
\begin{aligned}
(3.5) \quad -\int_0^T < c, (v_1)_t > ds + \int_0^T [(\nabla c, \nabla v_1) + ((\gamma_1 + \beta_1 n)c, v_1)]\, ds \\
= \int_\Omega c_0(x)v_1(x,0)\, dx,
\end{aligned}
$$

and

$$(3.6)\quad -\int_0^T < a, (v_2)_t > ds + \int_0^T [(\nabla a, \nabla v_2) + ((\gamma_2 + \beta_2 n)a, v_2)]\, ds = \int_\Omega a_0(x) v_2(x, 0)\, dx,$$

where $\vec{f} = \chi_1(\nabla c, c, x)\nabla c + \chi_2(\nabla a, a, x)\nabla a$.

Remark 3.1. It can be shown that the weak solution defined above satisfies:

$$n_t \in L^2((0,T); (H^1(\Omega))^*),\ c_t \in L^2(Q_T),\ a_t \in L^2(Q_T).$$

Now we establish the positiveness of solutions to the AC model, a fact which will be frequently used later.

Theorem 3.1. *Let (n, c, a) be a solution to the AC model given by (1.3). Then we have $0 \leq n$, $0 \leq c \leq 1$, $0 \leq a \leq 1$.*

Proof. Multiplying the first equation in (1.3) by n^- and integrating by parts, we obtain

$$(3.7)\qquad < n_t, n^- > + D_1(\nabla n - \vec{f} n, \nabla n^-) = 0.$$

From $n = n^+ - n^-$ we know $n_t = (n^+)_t - (n^-)_t$, and since $< (n^+)_t, n^- >= 0$ we obtain

$$(3.8)\qquad < n_t, n^- >=< (n^+)_t, n^- > - < (n^-)_t, n^- >= - < (n^-)_t, n^- > .$$

Similarly, we have

$$(3.9)\qquad < \nabla n - \vec{f} n, \nabla n^- >=< \nabla n^- - \vec{f} n^-, \nabla n^- > .$$

In view of (3.8) and (3.9), equation (3.7) becomes

$$(3.10)\qquad < (n^-)_t, n^- > + D_1(\nabla n^- - \vec{f} n^-, \nabla n^-) = 0,$$

from which we get

$$(3.11)\quad \frac{1}{2}\frac{d}{dt}\int_\Omega (n^-)^2\, dx + D_1 \int_\Omega |\nabla n^-|^2\, dx \leq D_1 \int_\Omega |\vec{f}||n^-||\nabla n^-|\, dx \leq \frac{D_1}{4}\int_\Omega |\vec{f}|^2 |n^-|^2\, dx + D_1 \int_\Omega |\nabla n^-|^2\, dx.$$

Let $E(t) = \int_\Omega (n^-)^2\,dx$ and $F(t) = \max_{Q_t} |\vec{f}|^2$. Then (3.11) implies

$$\frac{dE(t)}{dt} \le \frac{D_1 F(t)}{2} E(t), \tag{3.12}$$

from which we know

$$E(t) \le E(0) \exp(\frac{D_1}{2} \int_0^t F(s)\,ds).$$

But

$$E(0) = \int_\Omega (n_0^-)^2\,dx = 0$$

because $n_0^- = 0$ for a function $n_0 \ge 0$, so

$$E(t) = \int_\Omega (n^-)^2\,dx = 0,$$

from which we obtain $n^- = 0$ a. e., and $n = n^+ \ge 0$.

Now we multiply the second equation in (1.3) by c^- and integrate over Ω to obtain

$$< c_t, c^- > -(\Delta c, c^-) + ((\gamma_1 + \beta_1 n)c, c^-) = 0. \tag{3.13}$$

Note that

$$< c_t, c^- >= - < (c^-)_t, c^- >,$$

$$((\gamma_1 + \beta_1 n)c, c^-) = -((\gamma_1 + \beta_1 n)c^-, c^-).$$

Also, since $c^-|_{\Gamma_1} = 0$ and $\frac{\partial c}{\partial \nu}|_{\Gamma_2} = 0$, an integration by parts of the second term in the left side of (3.13) shows

$$(\Delta c, c^-) = -(\nabla c, \nabla c^-) = (\nabla c^-, \nabla c^-).$$

Hence (3.13) gives

$$\frac{1}{2}\frac{d}{dt} \int_\Omega (c^-)^2\,dx + \int_\Omega |\nabla c^-|^2\,dx + \int_\Omega (\gamma_1 + \beta_1 n)(c^-)^2\,dx = 0. \tag{3.14}$$

Since $n \ge 0$, we know from (3.14) that

$$\frac{1}{2}\frac{d}{dt} \int_\Omega (c^-)^2\,dx \le 0,$$

and so

$$\int_\Omega (c^-)^2\,dx \le \int_\Omega (c_0^-)^2\,dx = 0.$$

Therefore, $c^- = 0$ *a.e.*, and $c = c^+ \ge 0$.

To prove $c \le 1$, we set $c = 1 - c*$ in the second equation of (1.3) and then show $c* \ge 0$ following a similar procedure as above, which we omit here.

The statement about a is proved in the same way. □

Remark 3.2. Theorem 3.1 is clearly still true, if the n in the second (third) equation of the AC model is replaced by some $\tilde{n} \ge 0$ which is not necessarily a component of the system solution.

3.1.2 A Priori L^∞ Bound for n

Both the uniqueness and global existence of solution need the fact that n has an L^∞ bound. The technique used in the proof of the following lemma is from [3].

Lemma 3.1. *Assume* (3.1) *and* (3.2) *are true. Then for any weak solution of* (3.4) – (3.6) $n, c, a \in W$, *we have* $\|n\|_{L^\infty(\Omega)} \le N$, *where* N *is a constant independent of* T.

Proof. We first note that $n \ge 0$ by Theorem 3.1. By setting $v = 1$ in (3.4) we have

$$\frac{d}{dt}\int_\Omega n\,dx = < n_t, 1 > = -(\nabla n - n\vec{f}, \nabla 1) = 0, \tag{3.15}$$

which means $\int_\Omega n\,dx$ is a constant and therefore uniformly bounded with respect to t. By choosing $v = n^\lambda$ in (3.4) with $\lambda \ge 0$ we have

$$\frac{1}{\lambda+1}\frac{d}{dt}\int_\Omega n^{\lambda+1}\,dx = -\lambda\int_\Omega n^{\lambda-1}|\nabla n|^2\,dx + \lambda\int_\Omega \vec{f} n^\lambda \nabla n\,dx \equiv T_1 + T_2. \tag{3.16}$$

Noting that

$$T_1 = -\lambda\int_\Omega |n^{\frac{\lambda-1}{2}}\nabla n|^2\,dx = -\frac{4\lambda}{(\lambda+1)^2}\int_\Omega |\nabla n^{\frac{\lambda+1}{2}}|^2\,dx \tag{3.17}$$

and

$$(3.18)\ T_2 \le K\lambda\left|\int_\Omega n^\lambda \nabla n\, dx\right| = 2\int_\Omega \left(\frac{\sqrt{2\lambda}}{\lambda+1}|\nabla n^{\frac{\lambda+1}{2}}|\right)\left(K\sqrt{\frac{\lambda}{2}}n^{\frac{\lambda+1}{2}}\right) dx$$
$$\le \frac{2\lambda}{(\lambda+1)^2}\int_\Omega |\nabla n^{\frac{\lambda+1}{2}}|^2\, dx + \frac{K^2\lambda}{2}\int_\Omega n^{\lambda+1}\, dx,$$

we obtain from (3.16) the following inequality

$$(3.19)\quad \frac{1}{\lambda+1}\frac{d}{dt}\int_\Omega n^{\lambda+1}\, dx \le -\frac{2\lambda}{(\lambda+1)^2}\int_\Omega |\nabla n^{\frac{\lambda+1}{2}}|^2\, dx + \frac{K^2\lambda}{2}\int_\Omega n^{\lambda+1}\, dx.$$

Now we apply Theorem 2.15 to (3.19) to obtain

$$(3.20)\qquad \| n \|_\infty \le r\cdot 2^{2(m+4)}\cdot \alpha \equiv N$$

where r and α are constants independent of T. □

3.1.3 Existence and Uniqueness

Theorem 3.2. *For any $T > 0$, system $(3.4) - (3.6)$ has a unique solution $(n, c, a) \in W \times W \times W$, where $W = C((0,T); L^2(\Omega)) \cap V_2^{1,0}(Q_T)$.*

Proof. We use the contraction mapping principle to prove the theorem. We first choose $\tilde{n} \in W$. Set $n = \tilde{n}$ in (3.5) and (3.6) and let c and a be their (unique) solutions, respectively. We then denote by n the (unique) solution of (3.4), where c and a are determined in the way above. The existence of such n,c, and a in $V_2^{1,0}(Q_T)$ follows from Theorem 5.1 of [46] (Page 170). Also see the proof of Theorem 4.1. By Lemma 3.1 we have $\|n\|_\infty \le N$ for any $t > 0$, and hence $n \in W$.

Next, we define a mapping $P : W \to W$ with $P\tilde{n} = n$. We shall show that there exists a $T > 0$ such that P is a contraction and thus has a unique fixed point, which means that system $(3.4) - (3.6)$ has a unique solution. We denote by $d(.,.)$ the distance on W induced by its norm, that is, $d(u_1, u_2) = \|u_1 - u_2\|_{L^\infty((0,T);L^2)}$ for any $u_1, u_2 \in W$. Letting $\tilde{n}_1, \tilde{n}_2 \in W$ and $n_1 = P\tilde{n}_1, n_2 = P\tilde{n}_2$, we need to show that there exists a constant $0 < \rho < 1$ such $d(n_1, n_2) \le \rho\, d(\tilde{n}_1, \tilde{n}_2)$ for proper choice of T. Let c_1 and c_2 be the solutions of (3.5) corresponding to $\tilde{n}_1$ and $\tilde{n}_2$, respectively. By taking the difference of these two equations we have

$$((c_1-c_2)_t, v_1)+(\nabla(c_1-c_2), \nabla v_1)+((\gamma_1+\beta_1\tilde{n}_1)c_1-(\gamma_1+\beta_1\tilde{n}_2)c_2, v_1) = 0. \tag{3.21}$$

Setting $v_1 = c_1 - c_2 (\in V_1)$ in (3.21) and rearranging its terms we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (c_1-c_2)^2\,dx + \int_\Omega |\nabla(c_1-c_2)|^2\,dx \\ &+ \int_\Omega (\gamma_1+\beta_1\tilde{n}_1)(c_1-c_2)^2\,dx = \beta_1\int_\Omega c_2(\tilde{n}_1-\tilde{n}_2)(c_1-c_2)\,dx. \end{aligned} \tag{3.22}$$

Note that $0 \le c_2 \le 1$ and $\tilde{n}_1 \ge 0$ by Theorem 3.1. With the aid of the Basic inequality (Theorem 2.1) we obtain from (3.22)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (c_1-c_2)^2\,dx + \int_\Omega |\nabla(c_1-c_2)|^2\,dx + \gamma_1\int_\Omega (c_1-c_2)^2\,dx \\ &\le \frac{\beta_1^2}{4\gamma_1}\int_\Omega (\tilde{n}_1-\tilde{n}_2)^2\,dx + \gamma_1\int_\Omega (c_1-c_2)^2\,dx. \end{aligned} \tag{3.23}$$

Hence,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (c_1-c_2)^2\,dx + \int_\Omega |\nabla(c_1-c_2)|^2\,dx \\ &\le \frac{\beta_1^2}{4\gamma_1}\int_\Omega (\tilde{n}_1-\tilde{n}_2)^2\,dx \le \frac{\beta_1^2}{4\gamma_1} d(\tilde{n}_1,\tilde{n}_2)^2. \end{aligned} \tag{3.24}$$

Using the initial condition $(c_1-c_2)|_{t=0} = 0$, we integrate (3.24) from 0 to t, obtaining

$$d(c_1,c_2)^2 \le \frac{\beta_1^2 T}{2\gamma_1} d(\tilde{n}_1,\tilde{n}_2)^2, \int_0^T\int_\Omega |\nabla(c_1-c_2)|^2\,dx\,dt \le \frac{\beta_1^2 T}{4\gamma_1} d(\tilde{n}_1,\tilde{n}_2)^2. \tag{3.25}$$

Similarly, we have

$$d(a_1,a_2)^2 \le \frac{\beta_2^2 T}{2\gamma_2} d(\tilde{n}_1,\tilde{n}_2)^2, \int_0^T\int_\Omega |\nabla(a_1-a_2)|^2\,dx\,dt \le \frac{\beta_2^2 T}{4\gamma_2} d(\tilde{n}_1,\tilde{n}_2)^2. \tag{3.26}$$

Now by taking the difference of the two equations governing n_1 and n_2 we get

$$((n_1-n_2)_t, v) + (\nabla(n_1-n_2) - (n_1\vec{f_1} - n_2\vec{f_2}), \nabla v) = 0. \tag{3.27}$$

Putting $v = n_1 - n_2$ in (3.27) and rearranging its terms we have

$$\begin{aligned}(3.28)\qquad &\frac{1}{2}\frac{d}{dt}\int_\Omega (n_1-n_2)^2\,dx + \int_\Omega |\nabla(n_1-n_2)|^2\,dx \\ &= \int_\Omega \vec{f_1}\cdot\nabla(n_1-n_2)(n_1-n_2)\,dx \\ &+\int_\Omega n_2\nabla(n_1-n_2)\cdot(\vec{f_1}-\vec{f_2})\,dx.\end{aligned}$$

Using the facts that $|\vec{f_1}| \le K$ (desensitization assumption (3.1)) and $|n_2| \le N$ (Lemma 3.1), we obtain

$$\begin{aligned}(3.29)\qquad &\frac{1}{2}\frac{d}{dt}\int_\Omega (n_1-n_2)^2\,dx + \int_\Omega |\nabla(n_1-n_2)|^2\,dx \\ &\le (K+N)\epsilon\int_\Omega |\nabla(n_1-n_2)|^2\,dx \\ &+\frac{K}{4\epsilon}\int_\Omega (n_1-n_2)^2\,dx + \frac{N}{4\epsilon}\int_\Omega |\vec{f_1}-\vec{f_2}|^2\,dx.\end{aligned}$$

Choose ϵ so that $(K+N)\epsilon \le 1$. Then we have

$$(3.30)\qquad \frac{1}{2}\frac{d}{dt}\int_\Omega (n_1-n_2)^2\,dx \le \frac{K}{4\epsilon}\int_\Omega (n_1-n_2)^2\,dx + \frac{N}{4\epsilon}\int_\Omega |\vec{f_1}-\vec{f_2}|^2\,dx.$$

It follows that

$$(3.31)\qquad \int_\Omega (n_1-n_2)^2\,dx \le \frac{N}{2\epsilon}\exp\left(\frac{K}{2\epsilon}t\right)\int_0^t\int_\Omega |\vec{f_1}-\vec{f_2}|^2\,dx\,dt.$$

Using the Lipschitz condition (3.2) on $\vec{f}$ we estimate the integral on the right as

$$\begin{aligned}(3.32)\quad &\int_0^t\int_\Omega |\vec{f_1}-\vec{f_2}|^2\,dx\,dt \le \int_0^T\int_\Omega |\vec{f_1}-\vec{f_2}|^2\,dx\,dt \\ &\le L^2\int_0^T\int_\Omega [\,|(c_1-c_2)| + |(a_1-a_2)| + |\nabla(c_1-c_2)| + |\nabla(a_1-a_2)|\,]^2\,dx\,dt \\ &\le 4L^2\int_0^T\int_\Omega [(c_1-c_2)^2 + (a_1-a_2)^2 + |\nabla(c_1-c_2)|^2 + |\nabla(a_1-a_2)|^2]\,dx\,dt \\ &\le 4L^2\left(\frac{\beta_1^2T^2}{2\gamma_1} + \frac{\beta_2^2T^2}{2\gamma_2} + \frac{\beta_1^2T}{4\gamma_1} + \frac{\beta_2^2T}{4\gamma_{2L^2}}\right) d(\tilde{n}_1,\tilde{n}_2)^2 \\ &\le 2L^2\left(\frac{\beta_1^2}{\gamma_1} + \frac{\beta_2^2}{\gamma_2}\right) T d(\tilde{n}_1,\tilde{n}_2)^2\end{aligned}$$

where estimates (3.25) and (3.26) have been used and the fact $T < \frac{1}{2}$ has been assumed. Combining (3.31) with (3.32) we have

$$\int_\Omega (n_1 - n_2)^2\, dx \le \frac{NL^2T}{\epsilon} \exp\left(\frac{K}{2\epsilon}T\right)\left(\frac{\beta_1^2}{\gamma_1} + \frac{\beta_2^2}{\gamma_2}\right) d(\tilde{n}_1, \tilde{n}_2)^2 \tag{3.33}$$

from which we derive the relation

$$d(n_1, n_2) \le \rho\, d(\tilde{n}_1, \tilde{n}_2) \tag{3.34}$$

where $\rho = \left[\frac{NL^2T}{\epsilon} \exp\left(\frac{K}{2\epsilon}T\right)\left(\frac{\beta_1^2}{\gamma_1} + \frac{\beta_2^2}{\gamma_2}\right)\right]^{\frac{1}{2}}$. Finally, by choosing T so small as to make $\rho < 1$ we have a contraction $P : W \to W$, and the conclusion of the theorem follows on $(0, T)$. The theorem is true of any time interval $(0, \infty)$, because the T above depends only on the uniform bound of n and other constants independent of time, and we can apply the above procedure arbitrarily many times. □

3.2 Transition of the System into a Steady State

Now we consider the long time behavior of an AC solution, and investigate if it is possible for the system to admit a periodic solution, or, under which conditions the system will evolve into a steady state. The discussion in this section partially answers these questions.

Theorem 3.3. *Under conditions*

$$\delta \le \alpha, \tag{3.35}$$

and

$$0 \le \delta - \frac{L^2N^2}{2\alpha} - \frac{(LN + \alpha\beta_1\lambda)^2}{4(\alpha - \delta)\gamma_1} - \frac{(LN + \alpha\beta_2\lambda)^2}{4(\alpha - \delta)\gamma_2}, \tag{3.36}$$

the desensitized AC system (3.3) will transit into a steady state.

Proof. We assume the solutions concerned are smooth enough so that we can differentiate the first equation (with respect to t) of the AC model to obtain

$$n_{tt} - \nabla \cdot (\nabla n - \vec{f}n)_t = 0, \tag{3.37}$$

where we have used the fact that spatial and time derivatives commute. Bearing the same fact in mind and in particular

$$(\nabla n - fn)_t \cdot \vec{\nu} = [(\nabla n - \vec{f}n) \cdot \vec{\nu}]_t = 0, \tag{3.38}$$

we multiply (3.38) by n_t and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}||n_t||^2 + ||\nabla n_t||^2 - ((\vec{f}n)_t, \nabla n_t) = 0. \tag{3.39}$$

In view of the following bounds

$$|\tfrac{\partial \vec{f}}{\partial c}, \tfrac{\partial \vec{f}}{\partial c_{x_i}}| \leq L, |\tfrac{\partial \vec{f}}{\partial a}, \tfrac{\partial \vec{f}}{\partial a_{x_i}}| \leq L,\ 0 \leq n \leq N,\ |\vec{f}| \leq K,$$

and the fact

$$\vec{f}_t = \textstyle\sum_i \frac{\partial \vec{f}}{\partial c_{x_i}}(c_t)_{x_i} + \sum_i \frac{\partial \vec{f}}{\partial a_{x_i}}(a_t)_{x_i} + \frac{\partial \vec{f}}{\partial c}c_t + \frac{\partial \vec{f}}{\partial a}a_t,$$

the last term in (3.39) is estimated:

$$\begin{aligned} |((\vec{f}n)_t, \nabla n_t)| &\leq |(\vec{f}n_t, \nabla n_t)| + |(\vec{f}_t n, \nabla n_t)| \leq K||n_t||\,||\nabla n_t|| \\ &+LN(||\nabla c_t||\,||\nabla n_t|| + ||\nabla a_t||\,||\nabla n_t|| + ||c_t||\,||\nabla n_t|| + ||a_t||\,||\nabla n_t||). \end{aligned} \tag{3.40}$$

Since $\overline{n}_t = \frac{1}{|\Omega|} < n_t, 1 >= 0$ by (3.15), by Poincaré inequality (Theorem 2.5) we know

$$||n_t|| \leq \lambda ||\nabla n_t||. \tag{3.41}$$

Hence we obtain from (3.39), (3.40), and (3.41) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||n_t||^2 + ||\nabla n_t||^2 &\leq K\lambda||\nabla n_t||^2 + LN(||\nabla c_t||\,||\nabla n_t|| \\ &+||\nabla a_t||\,||\nabla n_t|| + ||c_t||\,||\nabla n_t|| + ||a_t||\,||\nabla n_t||). \end{aligned} \tag{3.42}$$

Now we repeat the process for c_t and a_t. Notice that

$$c_t|_{\Gamma_1} = (c|_{\Gamma_1})_t = 0,$$

and

$$\tfrac{\partial c_t}{\partial \vec{\nu}}|_{\Gamma_2} = (\tfrac{\partial c}{\partial \vec{\nu}}|_{\Gamma_2})_t = 0.$$

Integrating by parts yields

$$\frac{1}{2}\frac{d}{dt}||c_t||^2 + ||\nabla c_t||^2 + (\gamma_1 + \beta_1 n, c_t^2) = -(\beta_1 c n_t, c_t). \tag{3.43}$$

It follows from (3.41), (3.43), $0 \le c \le 1$, and $n \ge 0$ that

$$\frac{1}{2}\frac{d}{dt}||c_t||^2 + ||\nabla c_t||^2 + \gamma_1||c_t||^2 \le \beta_1\lambda||\nabla n_t||\,||c_t||. \tag{3.44}$$

Similarly, we have

$$\frac{1}{2}\frac{d}{dt}||a_t||^2 + ||\nabla a_t||^2 + \gamma_2||a_t||^2 \le \beta_2\lambda||\nabla n_t||\,||a_t||. \tag{3.45}$$

Now we multiply (3.44) and (3.45) by

$$\alpha \equiv \min\{\frac{(\gamma_1)^{\frac{1}{2}}}{\beta_1}, \frac{(\gamma_2)^{\frac{1}{2}}}{\beta_2}\} \tag{3.46}$$

and add them to (3.42). After rearranging terms we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(||n_t||^2 + \alpha||c_t||^2 + \alpha||a_t||^2) + \delta||\nabla n_t||^2 + \delta\gamma_1||c_t||^2 + \delta\gamma_2||a_t||^2 \\ \le -\{\delta||\nabla n_t||^2 + \alpha||\nabla c_t||^2 + \alpha||\nabla a_t||^2 + (\alpha-\delta)\gamma_1||c_t||^2 \\ +(\alpha-\delta)\gamma_2||a_t||^2 - LN||\nabla c_t||\,||\nabla n_t|| - LN||\nabla a_t||\,||\nabla n_t|| \\ -(LN+\alpha\beta_1\lambda)||c_t||\,||\nabla n_t|| - (LN+\alpha\beta_2\lambda)||a_t||\,||\nabla n_t||\}, \end{aligned} \tag{3.47}$$

where we have set $\delta = \frac{1-K\lambda}{2}$. By writing the right-hand side of (3.47) as a complete square and assuming (3.35) and (3.36):

$$\delta \le \alpha,$$

and

$$0 \le \delta - \frac{L^2N^2}{2\alpha} - \frac{(LN+\alpha\beta_1\lambda)^2}{4(\alpha-\delta)\gamma_1} - \frac{(LN+\alpha\beta_2\lambda)^2}{4(\alpha-\delta)\gamma_2},$$

we obtain

$$\frac{1}{2}\frac{d}{dt}(||n_t||^2 + \alpha||c_t||^2 + \alpha||a_t||^2) + \delta\lambda||\nabla n_t||^2 + \delta\gamma_1||c_t||^2 + \delta\gamma_2||a_t||^2 \le 0, \tag{3.48}$$

where again we have used (3.41). Let

$$\gamma = \min\{\gamma_1, \gamma_2\},$$

and

$$\delta_0 = \begin{cases} \delta\lambda^2 & \text{if } \frac{\gamma}{\lambda^2} \geq \alpha, \\ \frac{\delta\gamma}{\alpha} & \text{otherwise.} \end{cases}$$

From (3.48) we have

$$\frac{1}{2}\frac{d}{dt}(||n_t||^2 + \alpha||c_t||^2 + \alpha||a_t||^2) + \delta_0(||n_t||^2 + \alpha||c_t||^2 + \alpha||a_t||^2) \leq 0, \tag{3.49}$$

which implies

$$||n_t||^2 + \alpha||c_t||^2 + \alpha||a_t||^2 \leq Ae^{-\delta_0 t}, \tag{3.50}$$

where A is a constant. It follows from (3.50) that $||n_t||^2$, $||c_t||^2$ and $||a_t||^2 \to 0$ exponentially, and the system approaches a steady state. □

Remark 3.3. Note that the first and second order time-derivatives of the solution component $n(x,t)$ in the proof can always be safely understood in a distributional sense (Proposition 1.1 on Page 7 of [52]). But the existence of all the integrals involved presumes a sufficiently smooth system solution. It is not clear what minimum regularity conditions on the domain and on the initial and boundary data can guarantee this. However, we believe smooth and compatible initial and boundary conditions (1.5) and a C^2-smooth domain Ω will do.

Remark 3.4. To see how the conditions in the theorem are satisfied, we assume all the parameters except K, γ_1, and γ_2 are given. From the definition (3.46) of α we know (3.35) is satisfied for large γ_1 and γ_2. To have (3.36) satisfied, we first need $\delta = \frac{1-K\lambda}{2} > 0$. Noticing that λ is actually an eigenvalue which depends only on the domain geometry, we have to make K small. Assume now we have $\delta = \frac{1-K\lambda}{2} > 0$. Then it is not hard to see that (3.36) can always be satisfied by making both γ_1 and γ_2 large. Physically, this means the decaying rates of both chemicals have to be sufficiently large. It is 'plausible' to expect a steady state in this case, because the 'driving force' for any chemotactic movement, which comes from the chemicals, are disappearing rapidly.

Remark 3.5. The fact that $\|n_t\|^2$, $\|c_t\|^2$ and $\|a_t\|^2 \to 0$ exponentially shows that under the conditions given the system does not admit non-constant periodic solutions. Otherwise, the above quantities would also be periodic, which is obviously not true. This should discourage us enough to try to obtain periodic solutions for the desensitized system, even after limitations on the parameters are removed. Actually, we believe the system goes to a steady state without any condition on the parameters.

3.3 Existence and Uniqueness of the Numerical Solutions

3.3.1 Semi-Discrete Finite Element Formulation

Let $V_h \subset H^1(\Omega)$ be a piecewise linear finite element space corresponding to a particular triangular partition of the domain Ω. We define

$$V_{1,h} \equiv V_h \cap V_1,$$
$$V_{2,h} \equiv V_h \cap V_2,$$

where V_1, V_2 are defined in §3.1.1. A finite element solution of the system is a triple (n_h, c_h, a_h) in $(V_h)^3$ with $c_h - 1 \in V_{1,h}$ and $a_h - 1 \in V_{2,h}$ such that $\forall\ (v_h, v_{1,h}, v_{2,h}) \in V_h \times V_{1,h} \times V_{2,h}$ we have

$$< n_{h,t}, v_h > + (\nabla n_h - \vec{f}_h n_h, \nabla v_h) = 0, \tag{3.51}$$

$$< c_{h,t}, v_{1,h} > + (\nabla c_h, \nabla v_{1,h}) + ((\gamma_1 + \beta_1 n_h) c_h, v_{1,h}) = 0, \tag{3.52}$$

$$< a_{h,t}, v_{2,h} > + (\nabla a_h, \nabla v_{2,h}) + ((\gamma_2 + \beta_2 n_h) a_h, v_{2,h}) = 0, \tag{3.53}$$

and

$$n_h(x,0) = P_h n_0(x),\ c_h(x,0) = P_h c_0(x),\ a_h(x,0) = P_h a_0(x) \text{ in } \mathrm{L}^2(\Omega), \tag{3.54}$$

where $\vec{f}_h = \chi_1(\nabla c_h, c_h, x)\nabla c_h + \chi_2(\nabla a_h, a_h, x)\nabla a_h$.

Remark 3.6. The above definition implies a stationary finite element space, and hence $\forall t \in (0,T)$ we have $n_h, c_h, a_h \in H^1(\Omega)$. Furthermore, we will see that the coefficients of the solution actually have bounded time-derivatives.

3.3.2 A Priori Estimate for n_h

Moser's technique does not apply readily to the numerical case to ensure an L^∞ estimate for n_h, but a uniform $L^2(\Omega)$ estimate is still obtainable with the help of the following:

Lemma 3.2. *Let $\epsilon > 0$ be chosen. Then there exists C_ϵ such that $\forall n \in H^1(\Omega)$ and $n \geq 0$, the following is true:*

$$\int_\Omega n^2\,dx \leq C_\epsilon \left(\int_\Omega n\,dx\right)^2 + \epsilon \int_\Omega |\nabla n|^2\,dx. \tag{3.55}$$

Proof. Suppose (3.55) is not true. Then we can find ϵ, a sequence $\{C_i\}$ with $C_i \to \infty$, and a sequence $\{n_i\}$ with $n_i \geq 0$ such that

$$\int_\Omega n_i^2\,dx \geq C_i \left(\int_\Omega n_i\,dx\right)^2 + \epsilon \int_\Omega |\nabla n_i|^2\,dx. \tag{3.56}$$

By dividing both sides of (3.56) by $\int_\Omega n_i^2\,dx$ and setting $\tilde{n}_i = n_i/\left(\int_\Omega n_i^2\,dx\right)^{1/2}$ we obtain

$$1 \geq C_i \left(\int_\Omega \tilde{n}_i\,dx\right)^2 + \epsilon \int_\Omega |\nabla \tilde{n}_i|^2\,dx, \tag{3.57}$$

where $\tilde{n}_i$ satisfies

$$\int_\Omega \tilde{n}_i^2\,dx = 1. \tag{3.58}$$

Note that (3.57) and (3.58) imply that $\{\tilde{n}_i\}$ is bounded in H^1 and thus relatively compact in L^2, indicating that there is $n \in L^2$ such that a subsequence of $\tilde{n}_i$, still denoted by $\tilde{n}_i$, converges to n in L^2. As a consequence we have

$$\int_\Omega n^2\,dx = 1. \tag{3.59}$$

On the other hand, we know from (3.57) that $\int_\Omega \tilde{n}_i\,dx \to 0$, that is, $\tilde{n}_i \to 0$ in L^1. Therefore, $n = 0$ *a.e.* and $\int_\Omega n^2\,dx = 0$, which contradicts (3.59). □

Remark 3.7. Lemma 3.2 is obtainable from Nirenberg-Gagliardo's inequality (Theorem 2.7). See Lemma 4.4 for a proof. But the direct proof we gave above is much more simple when compared to the sophisticated techniques used in the proof of Nirenberg-Gagliardo's inequality ([28]).

Remark 3.8. Lemma 3.2 is false if n changes sign. Let n be the eigenfunction corresponding to the second eigenvalue of the problem:

$$-\Delta n = \lambda_2 n \quad \text{in } \Omega, \ \frac{\partial n}{\partial \vec{\nu}} = 0 \quad \text{on } \partial\Omega. \tag{3.60}$$

In view of the boundary condition, we obtain $0 = \lambda_2 \int_\Omega n\, dx$ by integrating the governing equation for n, from which we know $\int_\Omega n\, dx = 0$. Next we multiply the equation by n and do an integration by parts, to obtain $\int_\Omega |\nabla n|^2\, dx = \lambda_2 \int_\Omega n^2\, dx$. Note that this last equation implies $\lambda_2 > 0$. Now Choose ϵ so that $0 < \epsilon < 1/\lambda_2$. Then we have

$$\int_\Omega n^2\, dx = 1/\lambda_2 \int_\Omega |\nabla n|^2\, dx > \epsilon \int_\Omega |\nabla n|^2\, dx = C_\epsilon \left(\int_\Omega n\, dx\right)^2 + \epsilon \int_\Omega |\nabla n|^2\, dx$$

for any C_ϵ, which violates (3.55). It is thus important that $n \geq 0$ (or $n \leq 0$).

Now we prove

Lemma 3.3. *There is a constant N', independent of h and t, such that $\|n_h\|_2 \leq N'$.*

Proof. By Setting $v_h = 1$ in (3.51) we obtain

$$\frac{d}{dt}\int_\Omega n_h\, dx =< n_{h,t}, 1 >= -(\nabla n_h - \vec{f}_h n_h, \nabla 1) = 0,$$

from which we deduce that $\int_\Omega n_h\, dx = \int_\Omega P_h n_0\, dx \equiv \delta$, a constant. Choose ϵ so that $1 - K^2\epsilon > 0$. Since $n_h \in H^1(\Omega)$, from (3.55) we know

$$\int_\Omega n_h^2\, dx \leq C_\epsilon \delta^2 + \epsilon \int_\Omega |\nabla n_h|^2\, dx. \tag{3.61}$$

Next, set $v_h = n_h$ in (3.51). In view of saturation assumption (3.1), we obtain

$$\left(\int_\Omega n_h^2\, dx\right)_t + \int_\Omega |\nabla n_h|^2\, dx \leq K\left(\int_\Omega n_h^2\, dx\right)^{1/2}\left(\int_\Omega |\nabla n_h|^2\, dx\right)^{1/2} \tag{3.62}$$

which, together with (3.61), gives

$$\left(\int_\Omega n_h^2\,dx\right)_t + \left[\left(\int_\Omega |\nabla n_h|^2\,dx\right)^{1/2} - K\left(C_\epsilon\delta^2 + \epsilon\int_\Omega |\nabla n_h|^2\,dx\right)^{1/2}\right]$$

$$\times\left(\int_\Omega |\nabla n_h|^2\,dx\right)^{1/2} \le 0. \tag{3.63}$$

Now it follows that if

$$\left(\int_\Omega |\nabla n_h|^2\,dx\right)^{1/2} - K\left(C_\epsilon\delta^2 + \epsilon\int_\Omega |\nabla n_h|^2\,dx\right)^{1/2} \ge 0,$$

that is, if

$$\int_\Omega |\nabla n_h|^2\,dx > \left(\frac{C_\epsilon K^2\delta^2}{1-K^2\epsilon}\right)^2 \equiv M, \tag{3.64}$$

then $\left(\int_\Omega n_h^2\,dx\right)_t < 0$, indicating that $\int_\Omega n_h^2\,dx$ is decreasing. If, on the other hand, $\int_\Omega |\nabla n_h|^2\,dx \le M$, we obtain immmediately from (3.55) that

$$\int_\Omega n_h^2\,dx \le C_\epsilon\delta^2 + \epsilon M \equiv N'. \tag{3.65}$$

With the above two cases being combined, it turns out that (3.65) is always true. □

3.3.3 Existence of Solutions

Theorem 3.4. *System* (3.51) – (3.54) *has a solution* (n_h, c_h, a_h) *in* V_h *with* $n_h, c_h, a_h \ge 0$.

Proof. Let l be the number of partition of the domain Ω; $\{\phi_k\}_{k=1}^l$ be the basis for the space of piece-wise linear polynomial functions corresponding to the above partition; M_l be the space of $l \times l$ real matrices.

To obtain existence of solution, we apply Schauder's fixed point theorem. So we choose $\tilde{n}^T \in (C[0,T])^l$ and set $\tilde{n}_h = \sum_{k=1}^l \tilde{n}_k(t)\phi_k$. We shall solve

$$\begin{cases} < c_{h,t}, v_{1,h} > +(\nabla c_h, \nabla v_{1,h}) + ((\gamma_1 + \beta_1\tilde{n}_h)c_h, v_{1,h}) = 0 \\ < a_{h,t}, v_{2,h} > +(\nabla a_h, \nabla v_{2,h}) + ((\gamma_2 + \beta_2\tilde{n}_h)a_h, v_{2,h}) = 0 \end{cases}$$

(with initial approximations). Note that c_h and a_h have expressions:

$$c_h = \sum_{k=1}^{l} c_k(t)\phi_k, \quad a_h = \sum_{k=1}^{l} a_k(t)\phi_k.$$

By letting

$$A = (\phi_i, \phi_j)_{l\times l},$$
$$B = (\nabla\phi_i, \nabla\phi_j)_{l\times l} + ((\gamma_1 + \beta_1\tilde{n}_h)\phi_i, \phi_j)_{l\times l},$$

and with the notation

$$c(t) = (c_1(t), ..., c_l(t)),$$

we rewrite (the first part of) the system to be solved as

$$Ac'(t) + B(t)c(t) = 0,$$

from which we have

$$c'(t) = -A^{-1}B(t)c(t).$$

Since

$$-A^{-1}B(t) : [0,T] \to M_l$$

is continuous, the above system has a solution $c(t)^T \in (C^1[0,T])^l$ (see, for example, [44]). Similarly, we have $a(t)^T \in (C^1[0,T])^l$. Then, with $c_h = \sum_{k=1}^{l} c_k(t)\phi_k$ and $a_h = \sum_{k=1}^{l} a_k(t)\phi_k$, we solve n_h from

$$< n_{h,t}, v_h > +(\nabla n_h - \vec{f}_h n_h, \nabla v_h) = 0$$

(with initial approximation), where

$$\vec{f}_h = \chi_1(\nabla c_h, c_h, x)\nabla c_h + \chi_2(\nabla a_h, a_h, x)\nabla a_h.$$

Again, the above system can be reduced to the form $n'(t) = D(t)n(t)$ with $D(t) \equiv -A^{-1}C(t) : [0,T] \to M_l$ continuous. Also, $\|D\|_{M_l}$ is bounded by a costant because of the restriction $|\vec{f}_h| \leq K$ in (3.1). Hence we obtain a solution $n(t)^T \in (C^1[0,T])^l$. Thus we have defined a map

$$P : (C[0,T])^l \to (C^1[0,T])^l \subset (C[0,T])^l$$

with $n = P\tilde{n}$.

From the proof of the theorem on Page 79 of [44] we know

$$\|n\|_{(C[0,T])^l} \leq \|n(0)\|_{(C[0,T])^l} \exp\left(\|D\|_{M_l} T\right).$$

Since

$$\|n'\|_{(C[0,T])^l} = \|Dn\|_{(C[0,T])^l} \leq \|D\|_{M_l} \|n\|_{(C[0,T])^l},$$

we have

$$\begin{aligned} \|n\|_{(C^1[0,T])^l} &= \|n\|_{(C[0,T])^l} + \|n'\|_{(C[0,T])^l} \\ &\leq (\|D\|_{M_l} + 1)\|n(0)\|_{(C[0,T])^l} \exp\left(\|D\|_{M_l} T\right). \end{aligned}$$

Thus we have obtained for $n(t)$ an estimate in $(C^1[0,T])^l$, which is uniformly valid for all choice of $\tilde{n}$. So the map P is compact.

While not going into detail, we point out that the continuity of P is the result of the following facts: $\nabla c(t)$ and $\nabla a(t)$, together with $c(t)$ and $a(t)$, depend continuously on their coefficient $\tilde{n}$, and $n(t)$ depends continuously on its coefficient $\vec{f_h}(\nabla c(t), \nabla a(t), c(t), a(t))$. Therefore, being compact and continuous, P has a fixed point. That is, system (3.51) – (3.54) has a solution. For the proof of the positivity of solutions, we refer to Theorem 3.1 □

Remark 3.9. From the facts that all the $n(t) = (n_1(t), ..., n_l(t))^T$ are bounded in $(C^1[0,T])^l$ by a constant and that $n_h = \sum_{k=1}^{l} n_k(t)\phi_k$ we deduce immediately that $|n_h| \leq N$, where N is a constant (possibly T dependent).

3.3.4 Uniqueness of Solutions

Theorem 3.5. *The positive solution of system* (3.51) – (3.54) *is unique.*

Proof. We assume that $(n_h^{(1)}, c_h^{(1)}, a_h^{(1)})$ and $(n_h^{(2)}, c_h^{(2)}, a_h^{(2)})$ are two positive solutions of (3.51) – (3.54). Let

$$\vec{f}_h^{(i)} \equiv \chi_1(\nabla c_h^{(i)}, c_h^{(i)}, x)\nabla c_h^{(i)} + \chi_2(\nabla a_h^{(i)}, a_h^{(i)}, x)\nabla a_h^{(i)} (i = 1, 2),$$

$$n_h \equiv n_h^{(1)} - n_h^{(2)}, \; c_h \equiv c_h^{(1)} - c_h^{(2)}, \; a_h \equiv a_h^{(1)} - a_h^{(2)}.$$

Then we have

$$< n_{h,t}, v_h > -(\nabla n_h - n_h \vec{f}_h^{(1)} + n_h^{(2)}(\vec{f}_h^{(2)} - \vec{f}_h^{(1)}), \nabla v_h) = 0, \tag{3.66}$$

$$< c_{h,t}, v_{1,h} > +(\nabla c_h, \nabla v_{1,h}) + (\gamma_1 c_h + \beta_1 n_h^{(1)} c_h + \beta_1 c_h^{(2)} n_h, v_{1,h}) = 0, \tag{3.67}$$

$$< a_{h,t}, v_{2,h} > +(\nabla a_h, \nabla v_{2,h}) + (\gamma_2 a_h + \beta_2 n_h^{(1)} a_h + \beta_2 a_h^{(2)} n_h, v_{2,h}) = 0. \tag{3.68}$$

We set $v_h = n_h$, $v_{1,h} = c_h$, and $v_{2,h} = a_h$ in (3.66) – (3.68) respectively, to obtain

$$\begin{aligned} \frac{1}{2}\left(\int_\Omega n_h^2\,dx\right)_t + \int_\Omega |\nabla n_h|^2\,dx \le \int_\Omega \left| n_h\ \vec{f}_h^{(1)} \cdot \nabla n_h \right| dx \\ + \int_\Omega \left| n_h^{(2)}\ (\vec{f}_h^{(2)} - \vec{f}_h^{(1)}) \cdot \nabla n_h \right| dx, \end{aligned} \tag{3.69}$$

$$\begin{aligned} \frac{1}{2}\left(\int_\Omega c_h^2\,dx\right)_t + \int_\Omega |\nabla c_h|^2\,dx + \gamma_1 \int_\Omega c_h^2\,dx \\ +\beta_1 \int_\Omega n_h^{(1)} c_h^2\,dx \le \beta_1 \int_\Omega \left| c_h^{(2)} n_h c_h \right| dx, \end{aligned} \tag{3.70}$$

$$\begin{aligned} \frac{1}{2}\left(\int_\Omega a_h^2\,dx\right)_t + \int_\Omega |\nabla a_h|^2\,dx + \gamma_2 \int_\Omega a_h^2\,dx \\ +\beta_2 \int_\Omega n_h^{(1)} a_h^2\,dx \le \beta_2 \int_\Omega \left| a_h^{(2)} n_h a_h \right| dx. \end{aligned} \tag{3.71}$$

Note that

$$\left| \vec{f}_h^{(1)} \right| \le K$$

by (3.1),

$$\left| \vec{f}_h^{(2)} - \vec{f}_h^{(1)} \right| \le L(|\nabla c_h| + |c_h| + |\nabla a_h| + |a_h|)$$

by (3.2), and

$$\left| n_h^{(2)} \right| \le N$$

by Remark 3.9. So we have

$$\int_\Omega \left| n_h\ \vec{f}_h^{(1)} \cdot \nabla n_h \right| dx \le \epsilon \int_\Omega |\nabla n_h|^2\,dx + \frac{K^2}{4\epsilon} \int_\Omega n_h^2\,dx, \tag{3.72}$$

$$\int_\Omega \left| n_h^{(2)} \, (\vec{f}_h^{(2)} - \vec{f}_h^{(1)}) \cdot \nabla n_h \right| \, dx \le \epsilon \int_\Omega |\nabla n_h|^2 \, dx \tag{3.73}$$
$$+\frac{L^2N^2}{\epsilon} \int_\Omega (|\nabla c_h|^2 + c_h^2 + |\nabla a_h|^2 + a_h^2) \, dx.$$

Also we have

$$\beta_1 \int_\Omega \left| c_h^{(2)} n_h c_h \right| \, dx \le \beta_1 \int_\Omega |n_h c_h| \, dx \le \frac{\beta_1}{2} \int_\Omega n_h^2 \, dx + \frac{\beta_1}{2} \int_\Omega c_h^2 \, dx, \tag{3.74}$$

$$\beta_2 \int_\Omega \left| a_h^{(2)} n_h a_h \right| \, dx \le \beta_2 \int_\Omega |n_h a_h| \, dx \le \frac{\beta_2}{2} \int_\Omega n_h^2 \, dx + \frac{\beta_2}{2} \int_\Omega a_h^2 \, dx. \tag{3.75}$$

Taking an appropriate linear combination of (3.69) – (3.71) and using (3.72) – (3.75), we obtain

$$\frac{d}{dt} \int_\Omega \left(n_h^2 + \frac{L^2N^2\gamma_1}{\epsilon} c_h^2 + \frac{L^2N^2\gamma_2}{\epsilon} a_h^2 \right) \, dx + \int_\Omega |\nabla n_h|^2 \, dx \tag{3.76}$$
$$\le 2\epsilon \int_\Omega |\nabla n_h|^2 \, dx + \frac{K^2 + 2L^2N^2(\beta_1 + \beta_2)}{4\epsilon} \int_\Omega n_h^2 \, dx$$
$$+\frac{L^2N^2}{2\epsilon} \int_\Omega (\beta_1 c_h^2 + \beta_2 a_h^2) \, dx$$

Choose ϵ so that

$$0 < \epsilon < [K^2 + 2L^2N^2(\beta_1 + \beta_2)] \min\left\{ \frac{\gamma_1}{2\beta_1}, \frac{\gamma_2}{2\beta_2} \right\}.$$

Then, with

$$E(t) = \int_\Omega \left(n_h^2 + \frac{L^2N^2\gamma_1}{\epsilon} c_h^2 + \frac{L^2N^2\gamma_2}{\epsilon} a_h^2 \right) \, dx,$$

it follows from (3.76) that

$$E'(t) \le \frac{K^2 + 2L^2N^2(\beta_1 + \beta_2)}{4\epsilon} E(t), \tag{3.77}$$

The last inequality and the initial condition $E(0) = 0$ imply that $n_h = 0$, $c_h = 0$, and $a_h = 0$ for $t \in [0, T]$. Therefore $n_h^{(1)} = n_h^{(2)}$, $c_h^{(1)} = c_h^{(2)}$, and $a_h^{(1)} = a_h^{(2)}$, and the solution is unique. □

Remark 3.10. The above proof of uniqueness applies equally to the continuous case in §3.1, where we used a contraction mapping argument to obtain both existence and uniqueness.

3.4 Error Estimate for the Numerical Solutions

Throughout this section, we assume that $\partial\Omega$ is sufficiently smooth, and initial and boundary values are compatible and smooth enough such that n, c, a, n_t, c_t, and a_t are in $L^\infty((0,T);H^2)$. Our work is motivated by the semi-discrete finite element analysis for a linear evolution problem in [83]. But unlike the situation there, we are able to show only a first order accuracy result for the linear finite element numerical solution to this nonlinear problem, which we believe is due to the gradient-dependent (chemotactic) nature of the 'n' equation. Before we can prove the error estimates below, we need to define some elliptic projection operators $P_h^{(1)}$, $P_h^{(2)}$, and $P_h^{(3)}$, and set up some of their properties. $P_h^{(1)}$ is deifned as $P_h^{(1)} : n \in V \to P_h^{(1)} n \in V_h$ with

$$(\nabla P_h^{(1)} n, \nabla v_h) = (\nabla n, \nabla v_h) \ \forall v_h \in V_h \tag{3.78}$$

and

$$\int_\Omega P_h^{(1)} n \, dx = \int_\Omega P_h n_0 \, dx. \tag{3.79}$$

The definition for $P_h^{(2)}$ is: $c \to P_h^{(2)} c$ with $c - 1 \in V_1$ and $P_h^{(2)} c - 1 \in V_{1,h}$, and $\forall v_{1,h} \in V_{1,h}$

$$(\nabla P_h^{(2)} c, \nabla v_{1,h}) + (\gamma_1 P_h^{(2)} c, v_{1,h}) = (\nabla c, \nabla v_{1,h}) + (\gamma_1 c, v_{1,h}) \ \forall v_{1,h} \in V_{1,h}. \tag{3.80}$$

The operator $P_h^{(3)}$ is defined similarly for the 'a' equation.

Lemma 3.4. *The above defined operators $P_h^{(i)}$ $(i = 1, 2, 3)$ are uniquely determined.*

Proof. Note that $P_h^{(2)} c$ is actually the finite element solution of the elliptic problems (3.80) with c its exact solution. A similar statement holds true for $P_h^{(3)} a$. The conclusion then follows from the uniqueness of these numerical solutions (Lax-Milgram Lemma, [55], Page 118). The solution of (3.78) is unique only up to a constant, hence an additional equation (3.79) is there to ensure uniqueness. □

Remark 3.11. The value $\int_\Omega P_h n_0 \, dx$ is chosen for $\int_\Omega P_h^{(1)} n \, dx$ in (3.79) so that the error $\theta_1 \equiv n_h - P_h^{(1)} n$ safisfies

$$\bar{\theta}_1 = \frac{1}{|\Omega|} \int_\Omega \theta_1 \, dx = 0.$$

Then we know $\|\theta_1\| \leq \lambda \|\nabla \theta_1\|$ by Poincaré inequality (Theorem 2.5), which will be used in the proof of the error estimates in Theorem 3.6.

Lemma 3.5. *The time-derivative and the projections $P_h^{(i)}$ commute, that is, we have $(P_h^{(1)} n)_t = P_h^{(1)} n_t$, $(P_h^{(2)} c)_t = P_h^{(2)} c_t$, $(P_h^{(3)} a)_t = P_h^{(3)} a_t$.*

Proof. We only prove the first equality, the other two can be done similarly. Let $\{\phi_k\}_{k=1}^l$ be the base functions corresponding to a partition of the domain Ω. Note that (3.78) is equivalent to

$$(\nabla P_h^{(1)} n, \phi_k) = (\nabla n, \nabla \phi_k), \ \ k = 1, ..., l. \tag{3.81}$$

Now we take the time derivative on both sides of (3.81). Since the ϕ_ks are time independent, we obtain

$$(\nabla (P_h^{(1)} n)_t, \phi_k) = (\nabla n_t, \nabla \phi_k), \ \ k = 1, ..., l. \tag{3.82}$$

On the other hand, by definition we have

$$(\nabla P_h^{(1)} n_t, \phi_k) = (\nabla n_t, \nabla \phi_k), \ \ k = 1, ..., l. \tag{3.83}$$

The first equation in the lemma then follows from the uniqueness of the elliptic projection. □

Remark 3.12. The fact that the time-derivative and the projections $P_h^{(i)}$ commute will be used in the proof of the error estimates in Theorem 3.6. We could have included those nonlinear terms from the AC system in the definitions of $P_h^{(i)}$ $(i = 1, 2, 3)$, but in that case Lemma 3.5 no longer holds true.

Theorem 3.6. *Let (n, c, a) and (n_h, c_h, a_h) be the solutions of* (3.4) – (3.6) *and* (3.51) – (3.54), *respectively. Then for any $t \in [0, T]$ we have*

$$\|n - n_h\| \leq Ch, \ \ \|c - c_h\| \leq Ch, \ \ \|a - a_h\| \leq Ch, \tag{3.84}$$

where C is a constant depending on T.

Proof. We decompose the error $e_1 \equiv n_h - n$ into two parts:

$$e_1 = \theta_1 + \rho_1 \equiv (n_h - P_h^{(1)} n) + (P_h^{(1)} n - n). \tag{3.85}$$

Similarly we have

$$e_2 = \theta_2 + \rho_2 \equiv (c_h - P_h^{(2)} c) + (P_h^{(2)} c - c), \tag{3.86}$$

and

$$e_3 = \theta_2 + \rho_2 \equiv (a_h - P_h^{(3)} a) + (P_h^{(3)} a - a). \tag{3.87}$$

We estimate ρ_i $(i = 1, 2, 3)$ first. From elliptic finite element analysis ([55]) we know

$$\|\rho_i\| + h\|\nabla \rho_i\| \leq Dh^2, (i = 1, 2, 3) \tag{3.88}$$

and

$$\|P_h^{(i)} u_t^{(i)} - u_t^{(i)}\| \leq Dh^2, (i = 1, 2, 3) \tag{3.89}$$

where $u^{(1)} \equiv n$, $u^{(2)} \equiv c$, $u^{(3)} \equiv a$, and D is a constant depending on $\|n\|_{H^2}$, $\|c\|_{H^2}$, $\|a\|_{H^2}$, $\|n_t\|_{H^2}$, $\|c_t\|_{H^2}$, and $\|a_t\|_{H^2}$.

Next, we estimate θ_i $(i = 1, 2, 3)$. The equation for θ_1 is derived this way: $\forall v_h \in V_h \subset V$, by (3.85) we have

$$\begin{aligned}(\theta_{1,t}, v_h) + (\nabla \theta_1 - \vec{f}_h \theta_1, \nabla v_h) &= (n_{h,t}, v_h) - ((P_h^{(1)} n)_t, v_h) \\ &+ (\nabla n_h - \vec{f}_h n_h, \nabla v_h) - (\nabla P_h^{(1)} n - \vec{f}_h P_h^{(1)} n, \nabla v_h)\end{aligned}$$

By (3.51) and Lemma 3.5, the right-hand side of the above equation can be simplified into

$$-(P_h^{(1)} n_t, v_h) - (\nabla P_h^{(1)} n - \vec{f}_h P_h^{(1)} n, \nabla v_h),$$

which, in view of (3.78), gives

$$-(P_h^{(1)} n_t, v_h) - (\nabla n - \vec{f} n, \nabla v_h) + (\vec{f}_h P_h^{(1)} n - \vec{f} n, \nabla v_h).$$

By using $v = v_h$ in (3.4), we can rewrite the last expression as

$$-(P_h^{(1)}n_t - n_t, v_h) + (\vec{f}_h(P_h^{(1)}n - n), \nabla v_h) + ((\vec{f}_h - \vec{f})n, \nabla v_h).$$

The four expressions above being combined, we have

$$(\theta_{1,t}, v_h) + (\nabla\theta_1 - \vec{f}_h\theta_1, \nabla v_h) = -(P_h^{(1)}n_t - n_t, v_h) + (\vec{f}_h(P_h^{(1)}n - n), \nabla v_h) + ((\vec{f}_h - \vec{f})n, \nabla v_h). \tag{3.90}$$

Now by setting $v_h = \theta_1$ ($\in V_h$) in (3.90) and using the following estimates

$$\begin{aligned} |(\vec{f}_h\theta_1, \nabla\theta_1)| &\le K\|\theta_1\|\,\|\nabla\theta_1\| \le K\lambda\|\nabla\theta_1\|^2, \\ |(P_h^{(1)}n_t - n_t, \theta_1)| &\le Dh^2\|\theta_1\| \le D\lambda h^2\|\nabla\theta_1\|, \\ |(\vec{f}_h(P_h^{(1)}n - n), \nabla\theta_1)| &\le DKh^2\|\nabla\theta_1\|. \end{aligned}$$

$$\begin{aligned} |((\vec{f}_h - \vec{f})n, \nabla\theta_1)| &\le LN\|\nabla(c_h - c) + (c_h - c) + \nabla(a_h - a) + (a_h - a)\|\,\|\nabla\theta_1\| \\ &\le LN(\|\nabla\theta_2\| + \|\theta_2\| + \|\nabla\theta_3\| + \|\theta_3\| + 2D(h + h^2))\,\|\nabla\theta_1\|. \end{aligned}$$

we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta_1\|^2 + \|\nabla\theta_1\|^2 \le{}& R^{(1)}(h)\|\nabla\theta_1\| + K\lambda\|\nabla\theta_1\|^2 + \\ & LN(\|\nabla\theta_2\| + \|\theta_2\| + \|\nabla\theta_3\| + \|\theta_3\|)\,\|\nabla\theta_1\|, \end{aligned} \tag{3.91}$$

where $R^{(1)}(h)$ has the dominant term h. Similarly, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta_2\|^2 + \|\nabla\theta_2\|^2 + \gamma_1\|\theta_2\|^2 \le R^{(2)}(h)\|\theta_2\| + \beta_1\lambda\|\nabla\theta_1\|\,\|\theta_2\|, \tag{3.92}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\theta_3\|^2 + \|\nabla\theta_3\|^2 + \gamma_2\|\theta_3\|^2 \le R^{(3)}(h)\|\theta_3\| + \beta_2\lambda\|\nabla\theta_1\|\,\|\theta_3\|. \tag{3.93}$$

Notice the similarity between (3.91) – (3.93) and (3.42) – (3.45). Using the same technique and under similar conditions as (3.35) and (3.36) we obtain

$$\|\theta\|^2 \le e^{-\delta_0 t}\|\theta(0)\|^2 + R(h)^2(1 - e^{-\delta_0 t}), \tag{3.94}$$

where $\|\theta\|^2 = \|\theta_1\|^2 + \alpha\|\theta_2\|^2 + \alpha\|\theta_3\|^2$, $R(h) = O(h)$, and α and δ_0 are constants. Now the conclusion of the theorem follows from (3.88), (3.94), and the fact that $\|\theta(0)\| = O(h^2)$. □

Remark 3.13. Note that the error estimates established for the semi-discrete finite element method are only sub-optimal. The author believes these estimates can be improved, though it is not clear how this can be done at the moment.

Chapter 4

Theoretical Analysis of the One-Dimensional AC Model

We continue our study of the AC model in this chapter. To obtain more general results, we remove the desensitization assumption (3.1). As before, we still work with solutions in the weak sense because of the readiness to obtain various estimates. We assume initial and boundary data are smooth and compatible, and domain $\Omega = [0, 1]$, i. e., we only consider the 1-D system.

We notice that the Lyapunov functionals, which have been the key to obtain global existence of solution to the KS model, are not available to us because of the inhomogeneous mixed boundary conditions. Instead, we mainly rely on a priori estimates obtained through Sobolev-type inequalities. Two preparatory results are specifically proved for this, one as a new consequence of the general Nirenberg-Gagliardo inequality, and the other as a generalization of its particular 1-D case. We include the proof of local existence of solution for completeness. We have found that, in the 1-D case, substitutions like those in [29] are not necessary, and the whole proof can be greatly simplified once we know the 'alien' gradient term in the first equation of (1.3) is bounded.

The outline for this chapter is as follows. §4.1 deals with the local in time existence of solution. The basic tool is Schauder's fixed point theorem. Results on a compact set in the space $L^p((0,T);X)$ and the existence of solution for parabolic equations are needed for this. §4.2 sketches the proof of uniqueness of solution. §4.3 is devoted to the global existence of solution.

4.1 Existence of Local in Time Solution

From now on, we study the 1-D time-dependent AC model, but we keep using general notations like ∇, Ω and etc. in this section in order for an easy adaptation and generalization of available results to higher dimensions at a later stage.

The proof of Theorem 4.1, suggested by [29], needs some preliminary results which we state below as lemmas.

Lemma 4.1. *([79], Page 85, Corollary 4) Assume $W \subset X \subset Y$ with compact imbedding $W \to X$, where W, X, and Y are Banach spaces. Let F be bounded in $L^p((0,T);W)$ where $1 \leq p \leq \infty$, and $\frac{\partial F}{\partial t} = \left\{ \frac{\partial f}{\partial t} \middle| f \in F \right\}$ be bounded in $L^1((0,T);Y)$. Then F is relatively compact in $L^p((0,T);X)$.*

Lemma 4.2. *([46], Page 170, Theorem 5.1 and the comments that follow it) Let $\Omega \in R^n$ with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and unit outer-normal direction $\vec{\iota}$. Let $Q_T = \Omega \times (0,T)$, and $S_T = \partial\Omega \times [0,T]$. Consider in Q_T the following problem:*

$$(4.1) \qquad \begin{cases} L[u] = -f, \\ u|_{\Gamma_1 \times [0,T]} = 0, \quad \left(\dfrac{\partial u}{\partial \vec{\iota}} + \sigma u \right) \Big|_{\Gamma_2 \times [0,T]} = \psi, \\ u|_{t=0} = \psi_0(x), \end{cases}$$

where

$$L[u] \equiv u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} a_{ij}(x,t) u_{x_j} + a_i(x,t) u \right) + \sum_{i=1}^{n} b_i(x,t) u_{x_i} + a(x,t) u.$$

Assume

$$(4.2) \qquad \nu \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x,t) \xi_i \xi_j \leq \mu \sum_{i=1}^{n} \xi_i^2, \quad \nu, \ \mu = constant > 0;$$

$$(4.3)\qquad \left\|\sum_{i=1}^{n} a_i^2\right\|_{q,r,Q_T} \le \mu_1,\ \left\|\sum_{i=1}^{n} b_i^2\right\|_{q,r,Q_T} \le \mu_1,\ \|b\|_{q,r,Q_T} \le \mu_1,$$

in which

$$(4.4)\qquad \begin{cases} \dfrac{1}{r} + \dfrac{n}{2q} = 1, \\ q \in (n/2, \infty],\ r \in [1, \infty)\ for\ n \ge 2, \\ q \in [1, \infty],\ r \in [1, 2]\ for\ n = 1; \end{cases}$$

$$(4.5)\qquad \|f\|_{q_1,r_1,Q_T} \le \mu_2,$$

in which

$$(4.6)\qquad \begin{cases} \dfrac{1}{r_1} + \dfrac{n}{2q_1} = 1 + n/4, \\ q_1 \in [2n/(n+2), 2],\ r_1 \in [1, 2]\ for\ n \ge 3, \\ q_1 \in (1, 2],\ r_1 \in [1, 2)\ for\ n = 2, \\ q_1 \in [1, 2],\ r_1 \in [1, 4/3]\ for\ n = 1. \end{cases}$$

Also, we assume (when $n = 1$)

$$(4.7)\qquad \|\sigma\|_{r_2,S_T} \le \mu_3,\ r_2 = 2;$$

and

$$(4.8)\qquad \|\psi\|_{r_3,S_T} \le \mu_3,\ r_3 = 4/3.$$

Then there exists a unique solution of problem (4.1) in the class $V_2^{1,0}(Q_T)$.

Lemma 4.3. *([52], Page 37, Theorem 6.2) Let $\Omega \in R^n$ with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Let $Q_T = \Omega \times (0, T)$, and $S_T = \partial\Omega \times [0, T]$. Consider the following problem in Q_T:*

$$(4.9)\qquad \begin{cases} Au + u' = f, \\ B_j u = g_j,\ 0 \le j \le m-1, \\ u(0) = u_0, \end{cases}$$

where

$$A(t) = A(x, t, D_x) = \sum_{|p|,|q| \le m} (-1)^{|p|} D_x^p (a_{pq}(x, t) D_x^q),$$

and

$$B_j(t) = B_j(x,t,D_x) = \sum_{|h|\le m_j} b_{jh}(x,t)D_x^q.$$

Let

$$r \ge 0, \;\; 2rm = integer. \tag{4.10}$$

Let g_j, u_0 and f be given with the compatibility condition, and

$$g_j \in H^{2(r+1)m-m_j-1/2,(r+1)-(m_j+1/2)/2m}(S_T), \;\; 0 \le j \le m-1, \tag{4.11}$$

$$u_0 \in H^{2(r+1/2)m}(\Omega), \tag{4.12}$$

$$f \in H^{2rm,r}(Q_T). \tag{4.13}$$

If, in addition, some ellipticity and regularity conditions are satisfied, then problem (4.9) *admits a unique solution in the space $H^{2(r+1)m,r+1}(Q_T)$.*

Remark 4.1. In the above lemma, we omitted the compatibility, ellipticity and regularity conditions for space reason. It can be checked easily that these conditions are satisfied by the equations to be considered. Also, the boundary conditions in the lemma are general enough to include that of mixed Dirichlet-Neumann type (cf. [8] and [2]).

Theorem 4.1. *System* (1.3) *has a local in time solution in $V_2^{1,0}(Q_T)$ for sufficiently small $T > 0$.*

Proof. We are going to use Schauder's fixed point theorem ([88], Page 61, Theorem 1.C.) to prove the existence. We divide the proof into four parts:

(i) Definition of the map A. We first define a map A for appropriate $T > 0$:

$$A : w \in \tilde{W} \to v = Aw \in \tilde{V} \subset \tilde{W}, \;\; \tilde{V} = \left(V_2^{1,0}(Q_T)\right)^+, \;\; \tilde{W} = \left(L^2(Q_T)\right)^+,$$

where v is the unique $V_2^{1,0}(Q_T)$ solution of

$$(4.14)\qquad \begin{cases} v_t - D_1\nabla\cdot\left(\nabla v - v\vec{f}\right) = 0 & \text{in } Q_T \equiv \Omega\times(0,T), \\ v(x,0) = v_0(x) \geq 0 & \text{in } \Omega, \\ \left(\nabla v - v\vec{f}\right)\cdot\vec{\nu} = 0 & \text{on } S \equiv \partial\Omega\times(0,T), \end{cases}$$

where

$$(4.15)\quad \vec{f} = \nabla g \equiv \nabla\left(\frac{\chi}{\kappa D_1}\ln(1+\kappa c) + \frac{\alpha}{2D_1}a^2\right) = \frac{\chi}{D_1(1+\kappa c)}\nabla c + \frac{\alpha a}{D_1}\nabla a,$$

and c and a are the unique $V_2^{1,0}(Q_T)$ solutions of

$$(4.16)\qquad \begin{cases} c_t - \Delta c + \gamma_1 c = -\beta_1 w\, c & \text{in } Q_T, \\ c(x,0) = c_0(x) \geq 0 & \text{in } \Omega, \\ c = 1 \text{ on } S_1 \equiv \Gamma_1\times(0,T), \ \dfrac{\partial c}{\partial\vec{\nu}} = 0 & \text{on } S_2 \equiv \Gamma_2\times(0,T), \end{cases}$$

and

$$(4.17)\qquad \begin{cases} a_t - D_2\Delta a + \gamma_2 a = -\beta_2 w\, a & \text{in } Q_T, \\ a(x,0) = a_0(x) \geq 0 & \text{in } \Omega, \\ \dfrac{\partial a}{\partial\vec{\nu}} = 0 \text{ on } S_1, \quad a = 1 & \text{on } S_2, \end{cases}$$

respectively. Note that the solutions v, c, and a, if exist, must satisfy

$$v \geq 0,\ 0 \leq c \leq 1,\ 0 \leq a \leq 1,$$

as a consequence of the maximum principle (Theorem 3.1 and Remark 3.2). We use Lemma 4.2 to verify that (4.16) (and (4.17)) has a unique solution. In fact, a transform such as $c = 1-u$ will change (4.16) into (4.1), and conditions (4.2) – (4.8) in Lemma 4.2 are satisfied, with

$$\nu = \mu = 1$$

in (4.2),

$$q = 2,\ r \geq \frac{4}{3}$$

in (4.3) – (4.4) (see Remark 1.1 on Page 135 of [46]),

$$q_1 = 2,\ 1 \leq r_1 \leq \infty$$

in (4.5) – (4.6), and

$$\sigma = \psi = 0$$

in (4.7) – (4.8). Furthermore, $c \in H^{2,1}(Q_T)$ by Lemma 4.3, where conditions (4.10) – (4.13) are satisfied with

$$m = 1,\ r = 0,\ m_0 = 1.$$

By the imbedding theorem (Theorem 2.6), when the space dimension $m = 1$, $\nabla c = c_x$ (and $\nabla a = a_x$) is actually in $C(\overline{Q}_T)$. Hence, from (4.15) we know $\vec{f} = f \in C(\overline{Q}_T)$. Then, by applying Lemma 4.2 to (4.14), we know the solution v exists and is unique. This time, the conditions are satisfied with

$$\nu = \mu = D_1$$

in (4.2),

$$q \geq 1,\ r \geq 1 \text{ (arbitrary)}$$

in (4.3) – (4.4),

$$q_1 \geq 1,\ r_1 \geq 1 \text{ (arbitrary)}$$

in (4.5) – (4.6), and

$$\sigma \in C(\overline{Q}_T),\ \psi = 0$$

in (4.7) – (4.8). Therefore, the map A is well-defined.

(ii) The map A sends a bounded, closed, and convex set into itself. Next, we show the map A sends the set $B = \left\{ w \in \tilde{W} \middle| \, \|w\|_{L^2(Q_T)} \leq R \right\}$ into itself for sufficiently small $T > 0$. Assume $w \in L^2(Q_T)$ with $w \geq 0$ and $\|w\|_{L^2(Q_T)} \leq R$. Since $\vec{f} = f \in C(\overline{Q}_T)$, as we have shown in Part **(i)**, there exists a constant F such that $|\vec{f}| \leq F$. Now, multiplying the first equation of (4.14) by v and integrating by parts over Q_T we have

$$\frac{1}{2}\int_\Omega v^2\,dx - \frac{1}{2}\int_\Omega v_0^2\,dx + D_1\left[\iint_{Q_T} |\nabla v|^2\,dxds - \int_{Q_T} v\vec{f}\nabla v\,dxds\right] = 0, \tag{4.18}$$

from which we obtain for any $\epsilon > 0$,

$$\begin{aligned} \frac{1}{2}\|v\|^2 - \frac{1}{2}\|v_0\|^2 + D_1\|\nabla v\|^2_{Q_T} &\leq D_1 F \int_{Q_T} |v||\nabla v|\,dxds \\ &\leq D_1 F\left(\epsilon\|\nabla v\|^2_{Q_T} + \frac{1}{4\epsilon}\|v\|^2_{Q_T}\right) = D_1 F\epsilon\|\nabla v\|^2_{Q_T} + \frac{D_1 F}{4\epsilon}\|v\|^2_{Q_T}. \end{aligned} \tag{4.19}$$

If we choose $\epsilon = \dfrac{1}{F}$ in (4.19), then we have

$$\|v\|^2 \le \|v_0\|^2 + \frac{D_1 F^2}{2}\|v\|^2_{Q_T}. \tag{4.20}$$

We integrate (4.20) over $[0,T]$ to get

$$\|v\|^2_{Q_T} \le \|v_0\|^2 T + \frac{D_1 F^2}{2} T \|v\|^2_{Q_T}. \tag{4.21}$$

If $T > 0$ is such that $\frac{D_1 F^2}{2} T < 1$, then we have

$$\|v\|^2_{Q_T} \le \frac{T}{1 - \frac{D_1 F^2}{2} T}\|v_0\|^2 \to 0 \ (T \to 0).$$

Now it is clear that sufficiently small T can be chosen so that $\|v\|_{Q_T} \le R$, and therefore, A maps the ball B into itself.

(iii) The map A is pre-compact. We show the map A is pre-compact by setting up a bound for $\tilde{B} \equiv \{v = Aw | w \in B\}$ in $L^2((0,T);H^1)$, and a bound for $\{v_t = (Aw)_t | w \in B\}$ in $Y = L^2((0,T);(H^1)^*)$. Note that we have shown $\|v\|_{Q_T} \le R$ in Part **(ii)**. If we choose $\epsilon = \dfrac{1}{2F}$ in (4.19), then we have

$$\|\nabla v\|^2_{Q_T} \le F^2 \|v\|^2_{Q_T} + \frac{1}{D_1}\|v_0\|^2 \le F^2 R^2 + \frac{1}{D_1}\|v_0\|^2 \equiv \tilde{R}^2. \tag{4.22}$$

So we have

$$\|v\|_{L^2((0,T);H^1)} = \left(\|\nabla v\|^2_{Q_T} + \|v\|^2_{Q_T}\right)^{\frac{1}{2}} \le \left(\tilde{R}^2 + R^2\right)^{\frac{1}{2}}.$$

To see $\{v_t = (Aw)_t | w \in B\}$ is bounded in $Y = L^2((0,T);(H^1)^*)$, we take any $u \in L^2((0,T);H^1)$ and proceed from (4.14) to do the following calculations:

$$< v_t, u >= D_1 \int u \nabla \cdot (\nabla v - v\vec{f})\, dx = -D_1 \int \left(\nabla v - v\vec{f}\right) \nabla u \, dx.$$

$$< v_t, u >= D_1 \int_\Omega u \nabla \cdot (\nabla v - v\vec{f})\, dx = -D_1 \int_\Omega \left(\nabla v - v\vec{f}\right) \nabla u \, dx.$$

Using the fact that $|\vec{f}| \le F$ (a constant, see Part **(i)** of this proof), we have

$$\begin{aligned} \int_0^T | < v_t, u > | \, ds &\le D_1 \left(\|\nabla v\|_{Q_T} + F\|v\|_{Q_T}\right) \|\nabla u\|_{Q_T} \\ &\le D_1 \left(\tilde{R} + FR\right) \|\nabla u\|_{Q_T} \le D_1 \left(\tilde{R} + FR\right) \|u\|_{L^2((0,T);H^1)}. \end{aligned} \tag{4.23}$$

Note that (4.23) implies

$$\|v_t\|_{L^2((0,T);(H^1)^*)} \leq D_1\left(\tilde{R}+FR\right),$$

that is,

$$v_t \in L^2((0,T);(H^1)^*) \subset L^1((0,T);(H^1)^*).$$

Now we have $H^1 \subset L^2 \subset (H^1)^*$ with compact imbedding $H^1 \subset L^2$. By Lemma 4.1 (with $p=2$, $W=H^1$, $X=L^2$, and $Y=(H^1)^*$), the image $\tilde{B}$ of B under A is pre-compact, that is, A is a pre-compact mapping.

(iv) The map A is continuous. Assume $w^{(k)} \to w$ in $L^2(Q_T)$ with $w^{(k)} \geq 0$, $w \geq 0$. Then we have $\|w^{(k)}-w\|_{Q_T} \to 0$. Let $c^{(k)}$ and c ($a^{(k)}$ and a) be the corresponding solutions of (4.16) ((4.17)), and $v^{(k)}$ and v be the corresponding solutions of (4.14). By taking the difference of

$$c_t^{(k)} - \Delta c^{(k)} + \gamma_1 c^{(k)} = -\beta_1 w^{(k)}\, c^{(k)},$$

and

$$c_t - \Delta c + \gamma_1 c = -\beta_1 w\, c,$$

we have

$$\left(c^{(k)}-c\right)_t - \Delta\left(c^{(k)}-c\right) + \left(\gamma_1+\beta_1 w^{(k)}\right)\left(c^{(k)}-c\right) = -\beta_1\left(w^{(k)}-w\right)c, \tag{4.24}$$

together with the following homogeneous initial and boundary conditions:

$$\begin{cases} \left(c^{(k)}-c\right)|_{t=0} = 0 \text{ in } \Omega, \\ \left(c^{(k)}-c\right)|_{S_1} = 0, \quad \frac{\partial\left(c^{(k)}-c\right)}{\partial \vec{\nu}}|_{S_2} = 0. \end{cases} \tag{4.25}$$

We multiply (4.24) by $\left(c^{(k)}-c\right)$ and do an integration by parts over Q_t ($t \in [0,T]$), obtaining

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left(c^{(k)}-c\right)^2 dx + \int_{Q_t} \left|\nabla\left(c^{(k)}-c\right)\right|^2 dxds \\ &+ \int_{Q_t} \left(\gamma_1+\beta_1 w^{(k)}\right)\left(c^{(k)}-c\right)^2 dxds \\ &= -\beta_1 \int_{Q_t} \left(w^{(k)}-w\right) c\left(c^{(k)}-c\right) dxds \\ &\leq \beta_1\left[\epsilon\int_{Q_t}\left(c^{(k)}-c\right)^2 dxds + \frac{1}{4\epsilon}\int_{Q_t}\left(w^{(k)}-w\right)^2 dxds\right] \end{aligned} \tag{4.26}$$

Let $\epsilon = \dfrac{\gamma_1}{\beta_1}$ so that $\gamma_1 + \beta_1 w^{(k)} \geq \beta_1 \epsilon$. Then we obtain from (4.26)

$$\frac{1}{2}\|c^{(k)} - c\|^2 + \|\nabla\left(c^{(k)} - c\right)\|^2_{Q_t} \leq \frac{\beta_1^2}{4\gamma_1}\|\left(w^{(k)} - w\right)\|^2_{Q_t}. \tag{4.27}$$

Note that (4.27) implies

$$\|\nabla\left(c^{(k)} - c\right)\|^2_{Q_T} \leq \frac{\beta_1^2}{4\gamma_1}\|\left(w^{(k)} - w\right)\|^2_{Q_T} \to 0 \ (k \to \infty),$$

and

$$\|\left(c^{(k)} - c\right)\|^2_{Q_T} \leq \frac{\beta_1^2 T}{2\gamma_1}\|\left(w^{(k)} - w\right)\|^2_{Q_T} \to 0 \ (k \to \infty),$$

which in turn give

$$\|\vec{f}^{(k)} - \vec{f}\|_{Q_T} \to 0 \ (k \to \infty). \tag{4.28}$$

The proof of (4.28) uses the fact that $\vec{f}$ satisfies a Lipschitz condition with respect to all of its arguments when $|\nabla c|$ and $|\nabla a|$ are bounded, see Theorem 4.2 for a proof.

Next, we use (4.28) to show $\|v^{(k)} - v\|_{Q_T} \to 0$. In fact, the equation for $v^{(k)} - v$ is given by

$$\left(v^{(k)} - v\right)_t - D_1 \nabla \cdot \left[\nabla\left(v^{(k)} - v\right) - \vec{f}^{(k)}\left(v^{(k)} - v\right) - v\left(\vec{f}^{(k)} - \vec{f}\right)\right] = 0. \tag{4.29}$$

We multiply (4.29) by $\left(v^{(k)} - v\right)$ and do an integration by parts over Q_T to get

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left(v^{(k)} - v\right)^2 dx + D_1 \int_{Q_T} \left|\nabla\left(v^{(k)} - v\right)\right|^2 dxds \\ &= D_1 \int_{Q_T} \vec{f}^{(k)}\left(v^{(k)} - v\right)\nabla\left(v^{(k)} - v\right) dxds \\ &+ D_1 \int_{Q_T} v\left(\vec{f}^{(k)} - \vec{f}\right)\nabla\left(v^{(k)} - v\right) dxds \end{aligned} \tag{4.30}$$

Note that $|\vec{f}^{(k)}| \leq F$, $|\vec{f}| \leq F$, and they satisfy a Lipschitz condition. By Lemma 3.1, we know $\|v\|_{L^\infty(Q_T)} \leq N$, a constant. Hence we obtain from

(4.30) that

$$(4.31)\qquad \begin{aligned}&\frac{1}{2}\|v^{(k)}-v\|^2+D_1\|\nabla\left(v^{(k)}-v\right)\|_{Q_T}^2\\ &\leq D_1F\|\nabla\left(v^{(k)}-v\right)\|_{Q_T}\|\left(v^{(k)}-v\right)\|_{Q_T}\\ &+D_1N\|\nabla\left(v^{(k)}-v\right)\|_{Q_T}\|\vec{f}^{(k)}-\vec{f}\|_{Q_T}\\ &\leq D_1F\left(\epsilon\|\nabla\left(v^{(k)}-v\right)\|_{Q_T}^2+\frac{1}{4\epsilon}\|\left(v^{(k)}-v\right)\|_{Q_T}^2\right)\\ &+D_1N\left(\epsilon\|\nabla\left(v^{(k)}-v\right)\|_{Q_T}^2+\frac{1}{4\epsilon}\|\vec{f}^{(k)}-\vec{f}\|_{Q_T}^2\right).\end{aligned}$$

Let $\epsilon=\frac{1}{F+N}$ so that

$$D_1\geq D_1F\epsilon+D_1N\epsilon.$$

Then we obtain from (4.31)

$$(4.32)\qquad \begin{aligned}\|v^{(k)}-v\|^2&\leq D_1\frac{F(F+N)}{2}\|v^{(k)}-v\|_{Q_T}^2\\ &+D_1\frac{N(F+N)}{2}\|\vec{f}^{(k)}-\vec{f}\|_{Q_T}^2\\ &\equiv C_1\|v^{(k)}-v\|_{Q_T}^2+C_2\|\vec{f}^{(k)}-\vec{f}\|_{Q_T}^2.\end{aligned}$$

We integrate (4.32) over $[0,T]$ to get

$$(4.33)\qquad \|v^{(k)}-v\|_{Q_T}^2\leq C_1T\|v^{(k)}-v\|_{Q_T}^2+C_2T\|\vec{f}^{(k)}-\vec{f}\|_{Q_T}^2.$$

Let T be such that $1-C_1T>0$. Then we have

$$(4.34)\qquad \|v^{(k)}-v\|_{Q_T}^2\leq\frac{C_2T}{1-C_1T}\|\vec{f}^{(k)}-\vec{f}\|_{Q_T}^2\to 0\ (k\to\infty).$$

That is,

$$\|v^{(k)}-v\|_{Q_T}\to 0\ (k\to\infty),$$

and the map A is continuous.

Conclusion: The map A, defined on a bounded, closed, and convex set in the Banach space $L^2(Q_T)$, is pre-compact and continuous (and thus compact). Hence it has a fixed point $n\in L^2(Q_T)$ by Schauder's theorem. From the definition of the operator A, we know that n is actually in $V_2^{1,0}(Q_T)$. That is, system (1.3), without the desensitization assumption, has a local in time solution with n, c, $a\in V_2^{1,0}(Q_T)$. □

Remark 4.2. The proof of Theorem 4.1 uses the fact that $|\vec{f}| \leq F$. This is a consequence of the imbedding theorem (Theorem 2.6) when the space dimension $m = 1$. There is no guarantee that this is true when the spatial dimension $m > 1$.

Remark 4.3. Note that the $V_2^{1,0}(Q_T)$-solution in Theorem 4.1 only exists locally in time, because to have the map A well-defined (its image to be within itself), T has to be sufficiently small. Hence the bound F for $|\vec{f}|$ is also known to exist only locally in time. Furthermore, if the solution n blows up when $t \to T$, then by Lemma 3.1 (through an argument of contradiction), we must have

$$|\vec{f}|_{L^\infty(Q_t)} \to \infty \; (t \to T).$$

4.2 Uniqueness of Solution

The proof of uniqueness essentially depends on the boundedness of the chemical gradients, that is, ∇c and ∇a.

Theorem 4.2. *The local in time $V_2^{1,0}(Q_T)$ solution in Theorem 4.1 is unique.*

Proof. We first show that (3.1) and (3.2) are actually satisfied. In Part **(i)** of the proof of Theorem 4.1 we have shown $|\vec{f}| \leq F \; (\equiv K)$, as a result of the imbedding theorem. Then from (4.15) we have

$$\begin{aligned}
(4.35) \; & |\vec{f}_2 - \vec{f}_1| \\
& \leq \left| \frac{\chi}{D_1(1+\kappa c_2)} \nabla c_2 - \frac{\chi}{D_1(1+\kappa c_1)} \nabla c_1 \right| + \left| \frac{\alpha a_2}{D_1} \nabla a_2 - \frac{\alpha a_1}{D_1} \nabla a_1 \right| \\
& \leq \frac{\chi}{D_1(1+\kappa c_2)} |\nabla c_2 - \nabla c_1| + \left| \frac{\chi}{D_1(1+\kappa c_2)} - \frac{\chi}{D_1(1+\kappa c_1)} \right| |\nabla c_1| \\
& + \frac{\alpha a_2}{D_1} |\nabla a_2 - \nabla a_1| + \left| \frac{\alpha a_2}{D_1} - \frac{\alpha a_1}{D_1} \right| |\nabla a_1| \\
& \leq \frac{\chi}{D_1} |\nabla(c_2 - c_1)| + \frac{\alpha}{D_1} |\nabla(a_2 - a_1)| + \frac{\chi \kappa F}{D_1} |c_2 - c_1| + \frac{\alpha F}{D_1} |a_2 - a_1| \\
& \leq L \left(|\nabla(c_2 - c_1)| + |\nabla(a_2 - a_1)| + |c_2 - c_1| + |a_2 - a_1| \right),
\end{aligned}$$

where

$$L \equiv \max \left\{ \frac{\chi}{D_1}, \frac{\alpha}{D_1}, \frac{\chi \kappa F}{D_1}, \frac{\alpha F}{D_1} \right\}.$$

Then, following exactly the proof of Theorem 3.5, we know the solution in Theorem 4.1 is unique. □

4.3 Global in Time Existence of Solution

We are going to extend the local in time solution in Theorem 4.1 to any time interval in this section. By obtaining a uniform $L^\infty((0,T);L^2(\Omega))$ bound for the n component, we show the solution exists globally, thus excluding the possibility of any δ-function blow-up. Some lemmas are needed for this purpose.

The first lemma says that, under appropriate conditions, the L^2 norm of a function can be bounded by its L^1 norm together with the L^2 norm of its gradient. This is important because the cell mass ($\|n(x,t)\|_{L^1(\Omega)}$) in the chemotactic system is conserved (a constant).

Lemma 4.4. *Let $\partial\Omega$ be in C^1. Then for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that for all $u \in H^1(\Omega)$ we have*

$$\|u\|^2_{L^2(\Omega)} \le \epsilon\|\nabla u\|^2_{L^2(\Omega)} + C_\epsilon\|u\|^2_{L^1(\Omega)}. \tag{4.36}$$

Proof. We have given a direct proof of this result in Chapter 3 (Lemma 3.2). But an alternative proof is also available by using Nirenberg-Gagliardo's inequality ((2.5) in Theorem 2.7). To do this, we let $k = 0$, $l = 1$, $p = q = 2$, and $r = 1$ in the theorem. Then we have $\theta = m/(m+2)$, and the inequality (2.5) becomes

$$\|u\|_{0,2} \le C_0\|u\|_{1,2}^{\frac{m}{m+2}}\|u\|_{0,1}^{\frac{2}{m+2}} \tag{4.37}$$

from which, with the help of Young's inequality (Theorem 2.2), we obtain

$$\begin{aligned}\|u\|^2_{0,2} &\le C_0^2\|u\|_{1,2}^{\frac{2m}{m+2}}\|u\|_{0,1}^{\frac{4}{m+2}} = \left(\chi\|u\|_{1,2}^{\frac{2m}{m+2}}\right)\left(\frac{C_0^2}{\chi}\|u\|_{0,1}^{\frac{4}{m+2}}\right) \\ &\le \frac{\left(\chi\|u\|_{1,2}^{\frac{2m}{m+2}}\right)^{\frac{m+2}{m}}}{\frac{m+2}{m}} + \frac{\left(\frac{C_0^2}{\chi}\|u\|_{0,1}^{\frac{4}{m+2}}\right)^{\frac{m+2}{2}}}{\frac{m+2}{2}} \\ &= \frac{\epsilon}{1+\epsilon}\|u\|^2_{1,2} + \frac{C_\epsilon}{1+\epsilon}\|u\|^2_{0,1},\end{aligned} \tag{4.38}$$

where

$$\chi \equiv \left(\frac{\epsilon}{1+\epsilon}\right)^{\frac{m}{m+2}} \left(\frac{m+2}{m}\right)^{\frac{m}{m+2}},$$

and

$$C_\epsilon \equiv (1+\epsilon)\frac{\frac{2}{m}C_0^{m+2}\left(\frac{m}{m+2}\right)^{\frac{m+2}{2}}}{\left(\frac{\epsilon}{1+\epsilon}\right)^{\frac{m}{2}}}.$$

Using the relation

$$\|u\|_{1,2}^2 = |u|_{1,2}^2 + \|u\|_{0,2}^2,$$

we obtain

$$\|u\|_{0,2}^2 \le \epsilon |u|_{1,2}^2 + C_\epsilon \|u\|_{0,1}^2.$$

That is, equation (4.36) is true. □

Remark 4.4. Note that the above lemma is true of any $H^1(0,1)$ function, because the condition on the domain is automatically satisfied.

The second lemma gives a Sobolev type imbedding inequality for any nonnegative H^1 function defined on a 1-D interval. It generalizes a result from [15], which is true only for functions that satisfy the homogeneous Dirichlet boundary condition.

Lemma 4.5. *Let* $\Omega = (0,1)$. *Then for any nonnegative* $u \in H^1(\Omega)$, *we have*

$$\|u\|_{L^\infty(\Omega)} \le \|u\|_{L^1(\Omega)}^{1/3} \left(\|u\|_{L^1(\Omega)} + \frac{3}{2}\|u_x\|_{L^2(\Omega)}\right)^{2/3}. \tag{4.39}$$

Proof. We first observe that $u \in C[0,1]$ by the imbedding theorem. Since $u \ge 0$, we set $u = \rho(x)^2$. Then using Hölder's inequality we obtain

$$\begin{aligned} \|u_x\|_{L^2(\Omega)}^2 \|u\|_{L^1(\Omega)} &= \int_0^1 u_x^2\,dx \int_0^1 u(x)\,dx \\ &= 4\int_0^1 \rho^2 \rho_x^2\,dx \int_0^1 \rho^2\,dx \\ &\ge 4\left(\int_0^1 \rho^2 |\rho_x|\,dx\right)^2. \end{aligned} \tag{4.40}$$

Note that for an unordered pair (α, β) of adjacent maximum and minimum we have

$$\int_\alpha^\beta \rho^2 |\rho_x|\,dx = \frac{1}{3}\left|\rho(\alpha)^3 - \rho(\beta)^3\right| = \frac{1}{3}\left|u(\alpha)^{3/2} - u(\beta)^{3/2}\right|.$$

Hence

$$\int_0^1 \rho^2 |\rho_x|\, dx \geq \frac{1}{3} Var\, u(x)^{3/2}, \tag{4.41}$$

where $Var\, f(x)$ denotes the total variation of $f(x)$. But it is easily seen that

$$Var\, u(x)^{3/2} \geq ||u||^{3/2}_{L^\infty(\Omega)} - \overline{u}^{3/2} = ||u||^{3/2}_{L^\infty(\Omega)} - ||u||^{3/2}_{L^1(\Omega)}. \tag{4.42}$$

Now, (4.40), (4.41), and (4.42) give

$$||u_x||^2_{L^2(\Omega)} ||u||_{L^1(\Omega)} \geq \frac{4}{9} \left(||u||^{3/2}_{L^\infty(\Omega)} - ||u||^{3/2}_{L^1(\Omega)} \right)^2,$$

from which we solve for $||u||_{L^\infty(\Omega)}$ to obtain (4.39). □

Remark 4.5. Lemma 4.5 is not implied by Nirenberg-Gagliardo's inequality, that is, $|u|_{k,p} \leq C_0 ||u||^\theta_{l,q} ||u||^{1-\theta}_{0,r}$, because $||u||^\theta_{l,q}$ involves the L^2 norm of u (when $l = 1$ and q=2). Another closely related result: $|u|_{k,p} \leq C_0 |u|^\theta_{l,q} |u|^{1-\theta}_{0,r}$ in [81] (Page 90, Theorem 3.5), where only semi-norms of u are involved, requires u to be a C^1 function, which is not necessarily true for a function u in $H^1(\Omega)$.

The third lemma, which uses Fourier's method to solve a 1-D parabolic problem with constant coefficients ([46], Page 252-255), allows us to give a gradient estimate of the unknown function.

Lemma 4.6. *Consider the following problem:*

$$\begin{cases} w_t - w_{xx} + \gamma_1 w = h(x,t) & in\ (0,1) \times (0,T), \\ w(0,t) = 0,\ w_x(1,t) = 0 & in\ (0,T), \\ w(x,0) = 0 & in\ (0,1) \end{cases} \tag{4.43}$$

where $h(x,t) \in L^2((0,T); L^2(0,1))$. *Assume there exists a constant* C*, which is independent of* T*, such that in* $[0,T)$

$$\int_0^1 |h(x,t)|\, dx \leq C.$$

Let λ_k *and* $\psi_k(x)$ $(k = 1, 2, ...)$*, be eigenvalues and (orthonormal) eigenfunctions of*

$$-\psi_{xx} + \gamma_1 \psi = 0,\ \ \psi(0) = 0,\ \ \psi_x(1) = 0,$$

and

$$a_k(t) \equiv \int_0^t \exp(-\lambda_k(t-\tau)) \left[\int_0^1 h(x,\tau)\psi_k(x)\,dx\right] d\tau,$$

then the unique $V_2^{1,0}(Q_T)$ *solution of* (4.43) *is given by*

$$w(x,t) = \sum_{k=1}^{\infty} a_k(t)\psi_k(x). \tag{4.44}$$

Also, there is a constant D, *independent of* T, *such that the inequality*

$$\int_0^1 w_x^2(x,t)\,dx \le D^2 \tag{4.45}$$

holds in $[0,T)$.

Proof. The existence of a unique $V_2^{1,0}(Q_T)$ solution of (4.43) is assured by Lemma 4.2. Note that the conditions on $h(x,t)$ suffice to guarantee the convergence of $w(x,t)$ in (4.44) and its spatial derivative as well as the convergence of the Fourier expansion

$$h(x,t) = \sum_{k=1}^{\infty} \left[\int_0^1 h(x,t)\psi_k(x)\,dx\right] \psi_k(x)$$

([23] §8.2.1, Page 131). Then it is trivial to check that $w(x,t)$ satisfies (4.43). Obviously, $w(x,t) \in V_2^{1,0}(Q_T)$.

Next, we estimate $\int_0^1 w_x^2\,dx$. A direct calculation shows

$$\lambda_k = (k-1/2)^2\pi^2 + \gamma_1,$$

and

$$\psi_k(x) = \sqrt{2}\sin[(k-1/2)\pi x] = \sqrt{2}\sin[\sqrt{\lambda_k - \gamma_1}\,x],$$

with the $\psi_k(x)$s being made to be orthonormal. Noticing that the $\psi_k'(x)$s are also orthogonal and

$$\int_0^1 \psi_k'(x)^2\,dx = (k-1/2)^2\pi^2 = \lambda_k - \gamma_1,$$

we have

$$\begin{aligned} \int_0^1 w_x^2\,dx &= \int_0^1 \left(\sum_{k=1}^{\infty} a_k(t)\psi_k'(x)\right)^2 dx = \sum_{k=1}^{\infty} \int_0^1 a_k^2 \psi_k'^2\,dx \\ &= \sum_{k=1}^{\infty} a_k^2 \int_0^1 \psi_k'^2\,dx = \sum_{k=1}^{\infty} (\lambda_k - \gamma_1) a_k^2. \end{aligned} \tag{4.46}$$

On the other hand, using the expression for $a_k(t)$, we have

$$
\begin{aligned}
(4.47) \qquad |a_k(t)| \quad &\le \left| \int_0^t \exp(-\lambda_k(t-\tau)) \left[\int_0^1 h(x,\tau)\psi_k(x)\, dx \right] d\tau \right| \\
&\le \sqrt{2} \int_0^t \exp(-\lambda_k(t-\tau)) \left(\int_0^1 |h(x,\tau)|\, dx \right) d\tau \\
&\le \sqrt{2} C \int_0^t \exp(-\lambda_k(t-\tau))\, d\tau.
\end{aligned}
$$

Hence we obtain from (4.46) and (4.47)

$$
\begin{aligned}
(4.48) \quad \int_0^1 w_x^2\, dx \quad &\le 2C^2 \sum_{k=1}^{\infty} (\lambda_k - \gamma_1) \left(\int_0^t \exp(-\lambda_k(t-\tau))\, d\tau \right)^2 \\
&\le 2C^2 \sum_{k=1}^{\infty} \lambda_k \left(\int_0^t \exp(-\lambda_k(t-\tau))\, d\tau \right)^2 \\
&= 2C^2 \sum_{k=1}^{\infty} \frac{(1-\exp(-\lambda_k t))^2}{\lambda_k} \\
&\le 2C^2 \sum_{k=1}^{\infty} \frac{1}{(k-1/2)^2\pi^2 + \min\{\gamma_1, \gamma_2\}} \equiv D^2 < \infty.
\end{aligned}
$$

□

We are ready to prove the main result of this section. The underlying theory contained in the following theorem is: If uniformly (that is, independent of T) a priori estimates in appropriate spaces can be obtained in $[0,T)$, then the solution can always be extended to $[T, T+\delta_T)$ for some $\delta_T > 0$, so that any $[0,T)$ with $T < \infty$ is not the maximum interval of existence (see Corollary 3.5 on Page 250 of [87]).

Theorem 4.3. *Suppose we have smooth and compatible initial and boundary conditions in the 1-D AC model* $((1.3), m = 1)$*. Then there is a constant* M*, independent of* T*, such that for* $t \in [0,T)$ *we have*

$$
(4.49) \qquad \|n(x,t)\|_{L^2(\Omega)} \le M.
$$

Furthermore, for any $T > 0$ *we have* $n(x,t) \in L^\infty(\overline{Q}_T)$.

Proof. We first notice that a substitute like $w(x,t) = c(x,t) - c(x,0) \equiv c(x,t) - c_0(x)$ will transform the 'c' equation (together with the initial and boundary conditions) of the AC model into (4.43), where

$$h(x,t) \equiv c_0''(x) - \gamma_1 c_0(x) - \beta_1 n(x,t) c(x,t).$$

Note that $n(x,t), c(x,t) \in L^2((0,T); L^2(0,1))$. If, in addition, we assume $c_0(x)$ is so smooth that $c_0^{(2)}(x) \in L^2(0,1)$, then all the conditions in Lemma 4.6 are satisfied, with

$$\int_0^1 |h(x,t)|\, dx \leq \int_0^1 |c_0''(x) - \gamma_1 c_0(x)|\, dx + \beta_1 \int_0^1 n_0(x)\, dx \equiv C, \tag{4.50}$$

where C is constant independent of T, and where we have used the facts that

$$0 \leq c(x,t) \leq 1,$$

and

$$\int_0^1 n(x,t)\, dx = \int_0^1 n(x,0)\, dx \equiv \int_0^1 n_0(x)\, dx$$

for any $t > 0$. By Lemma 4.6 we know $\int_0^1 w_x^2(x,t)\, dx$ is uniformly bounded, and hence the relation $c(x,t) = w(x,t) + c_0(x)$ allows us to conclude that the same is true of $\int_0^1 c_x^2(x,t)\, dx$ (and similarly, of $\int_0^1 a_x^2(x,t)\, dx$). That is, there exists a constant D, independent of T, such that

$$\int_0^1 c_x^2(x,t)\, dx \leq D^2, \quad \int_0^1 a_x^2(x,t)\, dx \leq D^2.$$

Next, from the 'n' equation

$$n_t - D_1 \left(n_x - n\vec{f}\right)_x = 0$$

and its boundary condition we do an integration by parts to get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 n^2\, dx = D_1 \left[\int_0^1 n n_x \vec{f}\, dx - \int_0^1 n_x^2\, dx\right]. \tag{4.51}$$

By applying Lemma 4.5 to $n(x,t)$, we obtain from (4.51) that

$$\begin{aligned}
(4.52)\qquad \frac{1}{2}\frac{d}{dt}\int_0^1 n^2\,dx &\le D_1\left[\|n\|_{L^\infty}\int_0^1 |n_x|\,|f|\,dx - \int_0^1 n_x^2\,dx\right] \\
&\le D_1\left[\|n\|_{L^\infty}\left(\int_0^1 f^2\,dx\right)^{1/2}\left(\int_0^1 n_x^2\,dx\right)^{1/2} - \int_0^1 n_x^2\,dx\right] \\
&\le D_1\left[E\|n\|_{L^\infty}\|n_x\|_{L^2} - \|n_x\|_{L^2}^2\right] \\
&= D_1\|n_x\|_{L^2}\left[E\|n\|_{L^\infty} - \|n_x\|_{L^2}\right], \\
&\le D_1\|n_x\|_{L^2}\left[E\|n\|_{L^1}^{1/3}\left(\|n\|_{L^1} + \frac{3}{2}\|n_x\|_{L^2}\right)^{2/3} - \|n_x\|_{L^2}\right].
\end{aligned}$$

Note that

$$D_1\|n_x\|_{L^2}\left[E\|n\|_{L^1}^{1/3}\left(\|n\|_{L^1} + \frac{3}{2}\|n_x\|_{L^2}\right)^{2/3} - \|n_x\|_{L^2}\right] \le 0$$

is equivalent to the cubic inequality in $\|n_x\|_{L^2}$:

$$\|n_x\|_{L^2}^3 - E^3\|n\|_{L^1}\left(\|n\|_{L^1} + \frac{3}{2}\|n_x\|_{L^2}\right)^2 \ge 0,$$

from which it is clear that there exists a constant $\overline{N} > 0$ such that when $\|n_x\|_{L^2} \ge \overline{N}$ the above inequalities hold true. In this case, we know from (4.52) that $\frac{d}{dt}\int_0^1 n^2\,dx \le 0$, that is, $\int_0^1 n^2\,dx$ is decreasing. And so is $\|n\|_{L^2} = \left(\int_0^1 n^2\,dx\right)^{1/2}$.

If, on the other hand, $\|n_x\|_{L^2} \le \overline{N}$, then by Lemma 4.4 we know

$$(4.53)\qquad \|n\|_{L^2}^2 \le \epsilon\overline{N}^2 + C_\epsilon\|n\|_{L^1}^2.$$

That is, $\|n\|_{L^2}$ is uniformly bounded with respect to T. This fact allows us to conclude that c_x and a_x (and hence f) are uniformly bounded in $L^\infty(0,1)$. Then by applying Moser's technique we know the $L^\infty(Q_T)$ bound for n is unform with respect to T, and therefore, the solution can be extended to any $T > 0$ and never blows up. □

Remark 4.6. We notice that the chemotactic equations in both the AC model and the KS model have the same structure, and it is shown in [87] (Page 52, Proposition 5.1) that the $H^2(\Omega)$ norm of the maximal solution u to the KS model is substantially controlled by $\|u\|_{L^2((0,t);L^2)}$, from which we deduce that, for the AC model also, a uniform $L^2(\Omega)$ bound is sufficient to exclude any δ-function singularity.

Remark 4.7. We point out that the proof of Lemma 4.6 does not apply to higher spatial dimensions ($m = 2$ and $m = 3$). This is because we need the convergence of the series $\sum_{k=1}^{\infty} \frac{1}{\lambda_k}$ to get estimate (4.48). In general, we have the following eigenvalue estimates:

$$\lim_{k\to\infty} \lambda_k/k^{2/m} = C$$

([17] Page 432-434, Theorems 12 and 13), where m ($m = 1, 2, 3$) is the dimension of the domain Ω, and C is a constant. It is easily seen that $\sum_{k=1}^{\infty} \frac{1}{\lambda_k}$ converges when $m = 1$, while diverges when $m = 2$ or $m = 3$. But we emphasize that in higher dimensions the following facts are still true: Fourier's expression (4.44), orthogonality of eigenfunctions and their derivatives ([24], Page 136, Theorem 4), and Nirenberg-Gagliardo's inequality. It seems, therefore, that for the same line of proof to succeed in higher dimensions, we need better approximation of the term $\int_\Omega h(x,\tau)\psi_k(x)\,dx$: instead of being satisfied with a simple constant bound, we have to show something like

$$\sup_{0\le\tau\le T} \int_\Omega h(x,\tau)\psi_k(x)\,dx = O\left((1/\lambda_k)^p\right)$$

for some $0 < p \le 1$. We believe this is possible because, for any fixed $\tau > 0$, $h(x,\tau)$ is a function in $L^1(\Omega)$. Hence, by Theorem 1 on Page 11 of [84], we have

$$\int_\Omega h(x,\tau)\psi_k(x)\,dx = O(1/\lambda_k)$$

for any fixed τ. The problem is how to make the estimate uniform.

Remark 4.8. We have seen that the 1-D AC model in this chapter satisfies all the assumptions in Chapter 3. This implies that all the results from that chapter, for example, uniform $L^\infty((0,T);L^2)$ bound, transition into steady state under proper conditions, and semi-discrete finite analysis are obtainable for the model studied in this chapter.

Chapter 5

Theoretical Analysis of the Steady-State AC Model

We study the steady-state AC system in this chapter. According to model (1.3), its steady state is defined by

$$
(5.1)\qquad \begin{cases} -\nabla\cdot[\nabla n - n\nabla(\frac{\chi}{\kappa D_1}\ln(1+\kappa c)+\frac{\alpha}{2D_1}a^2)] = 0 \quad \text{in } \Omega, \\ -\triangle c + (\gamma_1+\beta_1 n)c = 0 \quad \text{in } \Omega, \\ -\triangle a + (\gamma_2+\beta_2 n)a = 0 \quad \text{in } \Omega, \\ [\nabla n - n\nabla(\frac{\chi}{\kappa D_1}\ln(1+\kappa c)+\frac{\alpha}{2D_1}a^2)]\cdot\vec{\nu} = 0 \text{ on } \partial\Omega = \Gamma_1\cup\Gamma_2, \\ c|_{\Gamma_1} = 1,\ \frac{\partial c}{\partial\vec{\nu}}|_{\Gamma_2} = 0, \\ \frac{\partial a}{\partial\vec{\nu}}|_{\Gamma_1} = 0,\ a|_{\Gamma_2} = 1, \end{cases}
$$

where $\vec{\nu}$ is the unit outer-normal vector of $\partial\Omega$. (We have re-defined γ_2 and β_2 to be γ_2/D_2 and β_2/D_2, respectively, in terms of the original parameters.) We are mainly concerned with the existence and uniqueness of solution to system (5.1). Motivated by the observation that the system has a pair of lower and upper solutions we choose to work in the classical framework in which Pao's technique of upper/lower solutions (also refered to as the method of monotone sequence, [69]) will be utilized. The $C^{2+\alpha}$-regularity of $\partial\Omega$ is assumed for this purpose. But before we start, a few comments could prove helpful.

A Priori Estimates for Mixed Boundary Value Problem:

Miranda ([56]) derived the first Schauder-type a priori estimate for second order elliptic equations with mixed boundary conditions. For elliptic equations of arbitrary order with general boundary conditions which include the mixed type as special case, a priori estimates of both $C^{2+\alpha}(\overline{\Omega})$ and $W_p^2(\Omega)$ types can be found in [2]. All these results assume not only smooth domain ($C^{2+\alpha}$), smooth boundary data ($C^{2+\alpha}(\overline{\Omega})$ or $C^{1+\alpha}(\overline{\Omega})$), and smooth coefficients and source term ($C^{\alpha}(\overline{\Omega})$), but also a 'complementing' algebraic condition between the elliptic operator and the boundary operator. (We point out, without giving the calculating details, that all these conditions are satisfied by system (5.1) and its equivalence (5.3).)

Later, Azzam et al. ([8], Page 257, Theorem 1) showed that these estimates still hold true in 2-D for a plane domain with corners where the two parts of the boundary, Γ_1 and Γ_2, meet at an angle $\theta < \pi/(4+2\alpha)$ or $\theta = \pi/4$. Lieberman ([50], [49]), Savaré ([75]), and Jochmann ([40]) obtained Hölder continuity and H^s-regularity in the cases of non-smooth data and/or non-smooth domain. These results are weaker than Azzam's but are best possible (optimal) with the data given.

Strategy of the Proof of Existence:

Instead of working on the original system (5.1), we work on its equivalence (5.3) below. Although neither system has quasi-monotone right-hand sides, the latter can be made so by freezing its non-local term $B = M/\int_\Omega F(u,v)\,dx$. This changes system (5.3) into system (5.12), to which the method of monotone sequences can be applied. For system (5.12), we first show that for any $B > 0$ it has a solution; we then show that, under appropriate conditions, its solution is unique. This allows us to define, on a closed interval of real numbers, a compact operator

$$T : B \to T(B) = M/\int_\Omega F(u(x,B), v(x,B))\,dx.$$

Existence of solution follows from the fact that the operator T has a fixed point.

We emphasize that though both the fixed-point argument and the concept of upper/lower solutions are classical, the idea to combine them is new. What is more, it allows us to apply the method of monotone sequences to a situation where the system does not have monotone source terms.

The outline for this chapter is as follows. In §5.1 we simplify the system by introducing appropriate substitutions, and use maximum principle to obtain a priori bounds for its solutions. In §5.2 we first define the monotone sequences and establish their relevant properties; then we prove existence of solution. We present the condition for uniqueness of solution in §5.3.

5.1 The Steady-State System and Its Reduction

We first note that no-flux boundary condition is being imposed on the 'n' equation of the steady-state AC system (5.1). This makes its solution not uniquely determined. But by recalling the physical significance of this model, we know that any steady state is the result of the time-evolution of an initial state, and during this process the mass $m(t) \equiv \int_\Omega n(x,t)\,dx$ is conserved. This suggests that a complementary condition is needed:

$$\int_\Omega n(x)\,dx = M \equiv \int_\Omega n_0(x)\,dx, \tag{5.2}$$

where M is the total mass of the endothelial cells. This makes (5.1) a well-posed system.

Next, the divergence form of the first equation in the steady-state AC model (5.1) allows us to remove this equation from the system through integration, resulting in a reaction-diffusion system (5.12), which has quasi-monotone right-hand sides. Hence the technique of upper/lower solutions is applicable to the steady-state AC model.

We say two systems have **equivalent solvability** when their solutions are mutually determined.

Lemma 5.1. *The solvability of the system* (5.1) *is equivalent to that of the* *system*

$$(5.3)\qquad \begin{cases} -\Delta u = (\gamma_1 + \beta_1 FM/\int_\Omega F\,dx)(1-u) & \text{in } \Omega, \\ -\Delta v = (\gamma_2 + \beta_2 FM/\int_\Omega F\,dx)(1-v) & \text{in } \Omega, \\ u|_{\Gamma_1} = 0,\ \frac{\partial u}{\partial \vec{\nu}}|_{\Gamma_2} = 0;\ \frac{\partial v}{\partial \vec{\nu}}|_{\Gamma_1} = 0,\ v|_{\Gamma_2} = 0, \end{cases}$$

where

$$u = 1 - c, \qquad v = 1 - a,$$

and

$$(5.4)\qquad F \equiv e^h = (1 + \kappa(1-u))^{\frac{\chi}{\kappa D_1}} \exp\left(\frac{\alpha}{2D_1}(1-v)^2\right).$$

Proof. We note that the steady-state chemotactic system (5.1) can be simplified first through a substitution and then by an integration. In fact, by letting

$$(5.5)\qquad h = \frac{\chi}{\kappa D_1}\ln(1+\kappa c) + \frac{\alpha}{2D_1}a^2,$$

and introducing a new unknown

$$(5.6)\qquad b \equiv e^{-h} n,$$

we have

$$(5.7)\qquad n = e^h b = (1+\kappa c)^{\frac{\chi}{\kappa D_1}} \exp\left(\frac{\alpha}{2D_1}a^2\right) b,$$

and

$$(5.8)\qquad e^h \nabla b \overset{(5.6)}{=\!=} e^h[e^{-h}(-\nabla h)n + e^{-h}\nabla n] = \nabla n - n\nabla h.$$

In view of (5.5) – (5.8), system (5.1) becomes

$$(5.9)\qquad \begin{cases} -\nabla \cdot (e^h \nabla b) = 0 & \text{in } \Omega, \\ -\Delta c + (\gamma_1 + \beta_1 e^h b)c = 0 & \text{in } \Omega, \\ -\Delta a + (\gamma_2 + \beta_2 e^h b)a = 0 & \text{in } \Omega, \end{cases}$$

with boundary conditions

$$(5.10)\qquad \frac{\partial b}{\partial \vec{\nu}}|_\Gamma = 0;\ c|_{\Gamma_1} = 1,\ \frac{\partial c}{\partial \vec{\nu}}|_{\Gamma_2} = 0;\ \frac{\partial a}{\partial \vec{\nu}}|_{\Gamma_1} = 0,\ a|_{\Gamma_2} = 1.$$

We multiply the first equation in (5.9) by b and integrate by parts over Ω. In view of the boundary condition for b in (5.10) we obtain

$$\int_\Omega e^h |\nabla b|^2 \, dx = 0, \tag{5.11}$$

from which we deduce $|\nabla b|^2 = 0$. Therefore $b = B$, a constant. With $b = B$ and $F \equiv e^h$ having been used in the second and third equations of (5.9), a further transformation $(c, a) = (1 - u, 1 - v)$ will change system (5.9) and (5.10) into

$$\begin{cases} -\Delta u = (\gamma_1 + \beta_1 FB)(1 - u) \equiv H_1(u, v) & \text{in } \Omega, \\ -\Delta v = (\gamma_2 + \beta_2 FB)(1 - v) \equiv H_2(u, v) & \text{in } \Omega, \\ u|_{\Gamma_1} = 0, \ \frac{\partial u}{\partial \vec{\nu}}|_{\Gamma_2} = 0, \\ \frac{\partial v}{\partial \vec{\nu}}|_{\Gamma_1} = 0, \ v|_{\Gamma_2} = 0. \end{cases} \tag{5.12}$$

To determine the constant B, we integrate (5.7) and solve for b to obtain

$$B = \frac{M}{\int_\Omega F \, dx}. \tag{5.13}$$

Finally, by substituting $B = M / \int_\Omega F \, dx$ into (5.12), we obtain system (5.3). Notice that the transformations we have used are invertible. Therefore, the solvability of system (5.1) is equivalent to that of system (5.3). □

Remark 5.1. A simple calculation shows that system (5.3) (and (5.1)) does not have quasi-monotone source terms (right-hand sides). This implies that Pao's technique does not directly apply to system (5.3) (and system (5.1)).

The method of monotone sequences requires (non-negative) bounds to be given for any solution (n, c, a) of (5.1). To do this we need the following lemma, which is a consequence of the maximum principle.

Lemma 5.2. *Let H be bounded and nonnegative. Then any function $w \in C^2(\Omega) \cap C(\overline{\Omega})$ that satisfies*

$$\begin{cases} -\Delta w + H(x, w) w \geq 0 & \text{in } \Omega, \\ w|_{\Gamma_1} \geq 0, \ \frac{\partial w}{\partial \vec{\nu}}|_{\Gamma_2} = 0, \end{cases} \tag{5.14}$$

must be nonnegative throughout $\overline{\Omega}$.

Proof. Suppose the opposite is true, then $W \equiv \max_{x\in\overline{\Omega}}\{-w\} > 0$. Let $P \in \overline{\Omega}$ be such that $-w(P) = W$. Taking at least two values $(-w|_{\Gamma_1} \leq 0, -w(P) = W > 0)$, $-w$ can not be a constant, and at the point P we have $\triangle(-w) \geq H(-w) \geq 0$. So, by Theorem 2.12, P can not be an interior point of Ω. Then as a boundary point, P must lie either on Γ_1 or on Γ_2. P is not on Γ_1 because $-w|_{\Gamma_1} \equiv 0 < -w(P) = W$. But if P were on Γ_2, we would have $\frac{\partial(-w)}{\partial\vec{\nu}}(P) > 0$ by Theorem 2.14, contradicting the boundary condition $\frac{\partial w}{\partial\vec{\nu}}|_{\Gamma_2} = 0$. Therefore, the assumption was wrong and $w \geq 0$. □

Remark 5.2. The lemma is clearly true if we switch Γ_1 and Γ_2 in the boundary conditions.

Lemma 5.3. *Any solution* (n, c, a) *of* (5.1) *must satisfy* $0 \leq c \leq 1$, $0 \leq a \leq 1$, *and* $n = (1+\kappa c)^{\frac{\chi}{\kappa D_1}} \exp\left(\frac{\alpha}{2D_1}a^2\right) B$, *where* $B > 0$ *is a constant given by* (5.13).

Proof. The positivity of c follows immediately from Lemma 5.2 once we notice $c|_{\Gamma_1} = 1 \geq 0$ and $H \equiv \gamma_1 + \beta_1 e^h B \geq 0$. From the first equation of (5.3) we see $u = 1 - c$ satisfies $-\Delta u + Hu = H \geq 0$, with corresponding homogeneous boundary condition. Hence $u \geq 0$ by Lemma 5.2, and $c = 1 - u \leq 1$. The proof for a is similar. The last equation follows from equation (5.7), the fact that $b = B$ is a constant, and equation (5.13). □

5.2 Existence of Solution

We have seen that neither the original AC model ((5.1)) nor its equivalent ((5.3)) has monotone source terms. This prevents us from applying results from [69] to them. One way to solve the difficulty is to treat the non-local term

$$B = M / \int_\Omega F(u, v)\, dx$$

as a constant, unrelated to the solution (u, v). This leads to a system (system (5.12)), which will refered to as the **variational system**, for which monotone sequences can be defined and their properties be proved. The existence result of the original system is then proved using a fixed point argument.

5.2.1 The Monotone Iteration

Now we apply the method of monotone sequence ([69]) to system (5.12). We assume $B > 0$ is any given constant unrelated to the solution (u, v). Let N be a constant such that

$$N > \max_{0 \le u,v \le 1} \left\{ \left|\frac{\partial H_i}{\partial u}\right|, \left|\frac{\partial H_i}{\partial v}\right| \;\; i = 1, 2. \right\}.$$

Starting with $(\overline{u}^{(0)}, \underline{v}^{(0)}) = (1, 0)$ and $(\underline{u}^{(0)}, \overline{v}^{(0)}) = (0, 1)$, we define two sequences $(\overline{u}^{(k)}, \underline{v}^{(k)})$ and $(\underline{u}^{(k)}, \overline{v}^{(k)})$ with components in $C^{2+\alpha}(\overline{\Omega})$ by

$$(5.15) \qquad \begin{cases} -\Delta \overline{u}^{(k)} + N\overline{u}^{(k)} = N\overline{u}^{(k-1)} + H_1(\overline{u}^{(k-1)}, \underline{v}^{(k-1)}) & \text{in } \Omega, \\ -\Delta \underline{v}^{(k)} + N\underline{v}^{(k)} = N\underline{v}^{(k-1)} + H_2(\overline{u}^{(k-1)}, \underline{v}^{(k-1)}) & \text{in } \Omega, \\ -\Delta \underline{u}^{(k)} + N\underline{u}^{(k)} = N\underline{u}^{(k-1)} + H_1(\underline{u}^{(k-1)}, \overline{v}^{(k-1)}) & \text{in } \Omega, \\ -\Delta \overline{v}^{(k)} + N\overline{v}^{(k)} = N\overline{v}^{(k-1)} + H_2(\underline{u}^{(k-1)}, \overline{v}^{(k-1)}) & \text{in } \Omega, \\ \overline{u}^{(k)}|_{\Gamma_1} = 0, \; \frac{\partial \overline{u}^{(k)}}{\partial \vec{\nu}}|_{\Gamma_2} = 0; \; \frac{\partial \underline{v}^{(k)}}{\partial \vec{\nu}}|_{\Gamma_1} = 0, \; \underline{v}^{(k)}|_{\Gamma_2} = 0, \\ \underline{u}^{(k)}|_{\Gamma_1} = 0, \; \frac{\partial \underline{u}^{(k)}}{\partial \vec{\nu}}|_{\Gamma_2} = 0; \; \frac{\partial \overline{v}^{(k)}}{\partial \vec{\nu}}|_{\Gamma_1} = 0, \; \overline{v}^{(k)}|_{\Gamma_2} = 0. \end{cases}$$

To justify the definition of the above sequences, we need:

Lemma 5.4. *The sequences in* (5.15) *are well-defined.*

Proof. We have to show each equation in (5.15) has a unique solution for its right-hand side given. We prove this only for the first equation, and the other proofs are similar.

For simplicity, we first work with weak solutions; then we quote references for $C^{2+\alpha}$-regularity. We write the first equation as

$$(5.16) \qquad \begin{cases} -\Delta w + Nw = \phi & \text{in } \Omega, \\ w|_{\Gamma_1} = 0, \; \frac{\partial w}{\partial \vec{\nu}}|_{\Gamma_2} = 0, \end{cases}$$

and define

$$(5.17) \qquad V_1 \equiv \{v \in H^1(\Omega) | \; v_{|\Gamma_1} = 0\}, V_2 \equiv \{v \in H^1(\Omega) | \; v_{|\Gamma_2} = 0\}.$$

Then, we multiply the first equation in (5.16) by $v \in V_1$ and do an integration by parts to obtain

$$(5.18) \qquad a(w, v) \equiv \int_\Omega (\nabla w \nabla v + Nwv)\, dx = (\phi, v) \equiv \int_\Omega \phi v \, dx \;\; \forall v \in V_1.$$

Since

$$|a(w,v)| \leq (N+1)\|w\|_{H^1}\|v\|_{H^1},$$

and

$$|(\phi,v)| \leq \|\phi\|_{L^2}\|v\|_{L^2} \leq \|\phi\|_{L^2}\|v\|_{H^1},$$

we know $a(w,v)$ and (ϕ,v) are both continuous as bilinear form on $H^1(\Omega)\times H^1(\Omega)$ and linear functional on $H^1(\Omega)$ respectively. Furthermore, $a(w,v)$ defines an inner product on the Hilbert Space $H^1(\Omega)$. The existence of a unique w in $H^1(\Omega)$ satisfying (5.18) is assured by Riesz's representation theorem ([88] p. 167). The fact that w is actually classical follows from a $C^{2+\alpha}(\overline{\Omega})$ estimate of w for mixed boundary value problem of elliptic type ([56]) because all the data here are sufficiently smooth. □

Remark 5.3. It needs mentioning that in at least one of the references ([8]) Γ_1 and Γ_2 are not required to have a non-zero distance (be disjoint) to have the normal regularity result. A smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ is sufficient. In some special cases where $\partial\Omega$ is not smooth, it is still possible to have classical solutions. For example, when both Γ_1 and Γ_2 are smooth, but they meet at an angle $\theta = \pi/4$, or $\theta \leq \pi/(4+2\alpha)$, the solution w can still be shown to be in $C^{2+\alpha}(\overline{\Omega})$. See [8] for details.

Now we show the monotone properties of these sequences. But unlike what was done in [69] where lower and upper solutions are defined, we apply the maximum principle directly.

Lemma 5.5. *([69]) The sequences in* (5.15) *are monotone in the sense that they satisfy*

$$\begin{cases} 0 = \underline{u}^{(0)} \leq \underline{u}^{(1)} \leq \ldots \leq \underline{u}^{(k)} \leq \overline{u}^{(k)} \leq \ldots \leq \overline{u}^{(1)} \leq \overline{u}^{(0)} = 1, \\ 0 = \underline{v}^{(0)} \leq \underline{v}^{(1)} \leq \ldots \leq \underline{v}^{(k)} \leq \overline{v}^{(k)} \leq \ldots \leq \overline{v}^{(1)} \leq \overline{v}^{(0)} = 1. \end{cases} \tag{5.19}$$

Proof. We use induction on k, and hence the proof depends on the following facts: **(I-1)** $0 = \underline{u}^{(0)} \leq \overline{u}^{(0)} = 1$, $0 = \underline{v}^{(0)} \leq \overline{v}^{(0)} = 1$; **(I-2)** For any k, if $\underline{u}^{(k-1)} \leq \overline{u}^{(k-1)}$ and $\underline{v}^{(k-1)} \leq \overline{v}^{(k-1)}$ are true, then $\underline{u}^{(k)} \leq \overline{u}^{(k)}$ and $\underline{v}^{(k)} \leq \overline{v}^{(k)}$ are also true; **(II-1)** $0 = \underline{u}^{(0)} \leq \underline{u}^{(1)}$, $\overline{v}^{(1)} \leq \overline{v}^{(0)} = 1$; **(II-2)** For any k, if

$\underline{u}^{(k-1)} \le \underline{u}^{(k)}$ and $\overline{v}^{(k)} \le \overline{v}^{(k-1)}$ are true, then $\underline{u}^{(k)} \le \underline{u}^{(k+1)}$ and $\overline{v}^{(k+1)} \le \overline{v}^{(k)}$ are also true; **(III-1)** $\overline{u}^{(1)} \le \overline{u}^{(0)} = 1$, $0 = \underline{v}^{(0)} \le \underline{v}^{(1)}$; **(III-2)** For any k, if $\overline{u}^{(k)} \le \overline{u}^{(k-1)}$ and $\underline{v}^{(k-1)} \le \underline{v}^{(k)}$ are true, then $\overline{u}^{(k+1)} \le \overline{u}^{(k)}$ and $\underline{v}^{(k)} \le \underline{v}^{(k+1)}$ are also true.

Proof of **(I-2)**: We subtract the third equation from the first one in (5.15). By using the mean value theorem on the right-hand side, we get

$$(5.20)\quad -\Delta(\overline{u}^{(k)} - \underline{u}^{(k)}) + N(\overline{u}^{(k)} - \underline{u}^{(k)}) = (N + \frac{\partial H_1}{\partial u})(\overline{u}^{(k-1)} - \underline{u}^{(k-1)}) + \frac{\partial H_1}{\partial v}(\underline{v}^{(k-1)} - \overline{v}^{(k-1)}).$$

Note that $\frac{\partial H_1}{\partial u} \le 0$ and $\frac{\partial H_1}{\partial v} \le 0$. In view of the induction assumption in **(I-2)** and the choice of N: $N + \frac{\partial H_1}{\partial u} \ge 0$, the right-hand side of the above equation is nonnegative, and the conclusion $\underline{u}^{(k)} \le \overline{u}^{(k)}$ follows from Lemma 5.2.

Proof of **(II-1)**: The first part of **(II-1)** is true because the equation for $\underline{u}^{(1)}$ becomes $-\Delta \underline{u}^{(1)} + N\underline{u}^{(1)} = H_1(\underline{u}^{(0)}, \overline{v}^{(0)}) \ge 0$. Applying Lemma 5.2 again, we know $\underline{u}^{(0)} = 0 \le \underline{u}^{(1)}$. The second part of **(II-1)** is true because the equation for $\overline{v}^{(1)}$ becomes $-\Delta\overline{v}^{(1)} + N\overline{v}^{(1)} = N$, or as we rewrite it,

$$-\Delta(1 - \overline{v}^{(1)}) + N(1 - \overline{v}^{(1)}) = 0.$$

Once more from Lemma 5.2 we know $1 - \overline{v}^{(1)} \ge 0$, or $\overline{v}^{(1)} \le 1 = \overline{v}^{(0)}$.

To prove **(II-2)**, the following equation suffices:

$$(5.21)\quad -\Delta(\underline{u}^{(k+1)} - \underline{u}^{(k)}) + N(\underline{u}^{(k+1)} - \underline{u}^{(k)}) = (N + \frac{\partial H_1}{\partial u})(\underline{u}^{(k)} - \underline{u}^{(k-1)}) + \frac{\partial H_1}{\partial v}(\overline{v}^{(k)} - \overline{v}^{(k-1)}).$$

Statements **(III-1)** and **(III-2)** are proved in exactly the same way as **(II-1)** and **(II-2)**. □

5.2.2 Existence and Uniqueness Results for the Variational System

Since the variational system, that is, system (5.12), has quasi-monotone free terms after its non-local term is 'frozen', it turns out that the technique of upper/lower solutions is applicable to it. The key part of the proof of the following lemma is the argument that point-wise convergence of the source term satisfying a Lipschitz condition eventually leads to $C^{2+\alpha}(\overline{\Omega})$ convergence of its solution ([69], Page 102, Theorem 2.1).

Lemma 5.6. *The variational system* (5.12) *has a solution for any given* $B \in (0, \infty)$.

Proof. For any given $B > 0$ we know, by Lemma 5.4, the sequences in (5.15) are well-defined; By Lemma 5.5, $\underline{u}^{(k)}$ and $\underline{v}^{(k)}$ are nondecreasing and bounded above, while $\overline{u}^{(k)}$ and $\overline{v}^{(k)}$ are nonincreasing and bounded below. Hence their limits exist, which we denote respectively by $\underline{u}$, $\underline{v}$, $\overline{u}$, $\overline{v}$. Obviously, $0 \leq \underline{u} \leq \overline{u} \leq 1$, $0 \leq \underline{v} \leq \overline{v} \leq 1$.

Now we write any equation in (5.15) as

$$-\Delta u^{(k)} = H(u^{(k)}, v^{(k)}).$$

Note that $u^{(k)}$ and $v^{(k)}$ are uniformly bounded, and H satisfies a Lipschitz condition. This implies that the sequence $H(u^{(k)}, v^{(k)})$ is uniformly bounded in $L^p(\Omega)$ for any $p \geq 1$. By an L^p-estimate from [2] (Page 701, Theorem 14.1) we conclude that $u^{(k)}$ is uniformly bounded in $W_p^2(\Omega)$. Choose $p > m$ (where m is the dimension of the domain Ω) so that $\alpha \equiv 1 - m/p > 0$. Then by the embedding theorem (Theorem 2.6), $u^{(k)}$ and $v^{(k)}$ are uniformly bounded in $C^{1+\alpha}(\overline{\Omega})$. This, together with the fact that $\frac{\partial H}{\partial u}$ and $\frac{\partial H}{\partial v}$ are bounded, implies that $H(u^{(k)}, v^{(k)})$ is uniformly bounded in $C^{\alpha}(\overline{\Omega})$. It follows from a Schauder-type estimate ([2], Page 668, Theorem 7.3) that $u^{(k)}$ is uniformly bounded in $C^{2+\alpha}(\overline{\Omega})$. Then from the Arzela-Ascoli theorem ([16], Page 569) we know there exists a subsequence of $u^{(k)}$ which converges in $C^2(\overline{\Omega})$ to a function $\tilde{u} \in C^{2+\alpha}(\overline{\Omega})$.

On the other hand, $u^{(k)}$ converges to u point-wise. Therefore, we have $u = \tilde{u}$, and moreover the whole sequence $u^{(k)}$ converges to u in $C^2(\overline{\Omega})$. This gives the facts that

$$-\Delta u^{(k)} \to -\Delta u, \quad \text{and} \quad H(u^{(k)}, v^{(k)}) \to H(u, v),$$

and that

$$u^{(k)}|_{\Gamma_1} \to u|_{\Gamma_1}, \quad \frac{\partial u^{(k)}}{\partial \vec{\nu}}|_{\Gamma_2} \to \frac{\partial u}{\partial \vec{\nu}}|_{\Gamma_2}; \quad v^{(k)}|_{\Gamma_1} \to v|_{\Gamma_1}, \quad \frac{\partial v^{(k)}}{\partial \vec{\nu}}|_{\Gamma_2} \to \frac{\partial v}{\partial \vec{\nu}}|_{\Gamma_2}.$$

By letting $k \to \infty$ in (5.15), we know the limit functions, $\underline{u}$, $\underline{v}$, $\overline{u}$, $\overline{v}$, satisfy

$$\begin{cases} -\Delta\overline{u} = H_1(\overline{u},\underline{v}), & -\Delta\underline{v} = H_2(\overline{u},\underline{v}) \quad \text{in } \Omega, \\ -\Delta\underline{u} = H_1(\underline{u},\overline{v}), & -\Delta\overline{v} = H_2(\underline{u},\overline{v}) \quad \text{in } \Omega, \\ \overline{u}|_{\Gamma_1} = 0, \ \frac{\partial \overline{u}}{\partial \vec{\nu}}|_{\Gamma_2} = 0; & \frac{\partial \underline{v}}{\partial \vec{\nu}}|_{\Gamma_1} = 0, \ \underline{v}|_{\Gamma_2} = 0, \\ \underline{u}|_{\Gamma_1} = 0, \ \frac{\partial \underline{u}}{\partial \vec{\nu}}|_{\Gamma_2} = 0; & \frac{\partial \overline{v}}{\partial \vec{\nu}}|_{\Gamma_1} = 0, \ \overline{v}|_{\Gamma_2} = 0. \end{cases} \tag{5.22}$$

That is, both $(\overline{u},\underline{v})$ and $(\underline{u},\overline{v})$ are solutions of system (5.12). □

A uniqueness result for the variational system (5.12) is needed in order for the function T in Theorem 5.1 to be properly defined.

Lemma 5.7. *Let λ be defined as*

$$\lambda_1 \equiv \inf_{\substack{w\in V_1 \\ \|w\|=1}} \|\nabla w\|^2, \quad \lambda_2 \equiv \inf_{\substack{w\in V_2 \\ \|w\|=1}} \|\nabla w\|^2, \quad \lambda = \min\{\lambda_1, \lambda_2\}, \tag{5.23}$$

where V_1 and V_2 are given by

$$V_1 \equiv \{v \in H^1(\Omega) | \ v_{|\Gamma_1} = 0\}, \quad V_2 \equiv \{v \in H^1(\Omega) | \ v_{|\Gamma_2} = 0\}.$$

Also, we define L, P, R, and Q as

$$L \equiv (1+\kappa)^{\frac{\chi}{\kappa D_1}} \exp\left(\frac{\alpha}{2D_1}\right), \tag{5.24}$$

$$P \equiv \min\left\{\gamma_1 + \frac{\beta_1 M}{L|\Omega|}, \ \gamma_2 + \frac{\beta_2 M}{L|\Omega|}\right\}, \tag{5.25}$$

$$R \equiv \max\left\{\frac{\alpha L}{D_1}, \ \frac{\chi}{D_1}\exp\left(\frac{\alpha}{2D_1}\right), \ \frac{\chi L}{(1+\kappa)D_1}\right\}, \tag{5.26}$$

$$Q \equiv \max\{\beta_1, \beta_2\}\frac{RM}{|\Omega|}. \tag{5.27}$$

Then under condition

$$\lambda + P > Q, \tag{5.28}$$

the solution to system (5.12) is unique for any B with $\frac{M}{L|\Omega|} \le B \le \frac{M}{|\Omega|}$.

Proof. Let (u_1, v_1) and (u_2, v_2) be two solutions of (5.3), and $(u, v) \equiv (u_2, v_2) - (u_1, v_1)$. By taking the difference of the equations for u_2 and u_1, we have

$$-\Delta(u_2 - u_1) = H_1(u_2, v_2) - H_1(u_1, v_1) = \frac{\partial H_1}{\partial u}(u_2 - u_1) + \frac{\partial H_1}{\partial v}(v_2 - v_1),$$

which we rewrite as

$$-\Delta u + \left(-\frac{\partial H_1}{\partial u}\right) u = \frac{\partial H_1}{\partial v} v.$$

We multiply the above equation by u, then integrate on both sides over Ω, obtaining

$$\int_\Omega (-\Delta u) u \, dx + \int_\Omega \left(-\frac{\partial H_1}{\partial u}\right) u^2 \, dx = \int_\Omega \frac{\partial H_1}{\partial v} uv \, dx. \tag{5.29}$$

We estimate the first term in (5.29) as

$$\int_\Omega (-\Delta u) u \, dx = \int_\Omega |\nabla u|^2 \, dx \geq \lambda \|u\|^2 \, dx. \tag{5.30}$$

To estimate the second and third terms in (5.29), we first calculate $-\frac{\partial H_1}{\partial u}$ and $\frac{\partial H_1}{\partial v}$ to get

$$-\frac{\partial H_1}{\partial u} = \gamma_1 + \beta_1 F B - \beta_1 \frac{\partial F}{\partial u} B(1 - u),$$

$$\frac{\partial H_1}{\partial v} = \beta_1 \frac{\partial F}{\partial v} B(1 - u).$$

Since

$$1 \leq F \leq L, \quad \frac{M}{L|\Omega|} \leq B \leq \frac{M}{|\Omega|}, \quad \frac{\partial F}{\partial u} \leq 0,$$

and a simple calculation shows

$$\left|\frac{\partial F}{\partial v}\right| \leq R,$$

we have

$$-\frac{\partial H_1}{\partial u} \geq \gamma_1 + \frac{\beta_1 M}{L|\Omega|} \geq P,$$

$$\left|\frac{\partial H_1}{\partial v}\right| \leq \beta_1 R B \leq \beta_1 \frac{RM}{|\Omega|} \leq Q.$$

Hence,

$$\int_\Omega \left(-\frac{\partial H_1}{\partial u}\right) u^2\, dx \geq P\|u\|^2\, dx, \tag{5.31}$$

and

$$\int_\Omega \frac{\partial H_1}{\partial v} uv\, dx \leq Q \int_\Omega |uv|\, dx \leq \frac{Q}{2}(\|u\|^2 + \|v\|^2). \tag{5.32}$$

Combining estimates (5.30), (5.31), and (5.32) with (5.29), we get

$$(\lambda + P)\|u\|^2 \leq \frac{Q}{2}(\|u\|^2 + \|v\|^2). \tag{5.33}$$

Similarly, we obtain from the 'v' equation

$$(\lambda + P)\|v\|^2 \leq \frac{Q}{2}(\|u\|^2 + \|v\|^2). \tag{5.34}$$

By adding (5.33) and (5.34) together, we have

$$(\lambda + P)(\|u\|^2 + \|v\|^2) \leq Q(\|u\|^2 + \|v\|^2). \tag{5.35}$$

Now, if we assume that

$$\lambda + P > Q,$$

then we have $\|u\|^2 = 0$ and $\|v\|^2 = 0$. Therefore, $u = u_2 - u_1 = 0$ and $v = v_2 - v_1 = 0$. That is, the solution of (5.3) is unique for any given constant B of interest. □

Remark 5.4. It is easy to see that a system with the parameter γ_1 and γ_2 sufficiently large will satisfy the condition of this theorem, and therefore such a system has a unique solution. Physically, this means that when the decay rate of both chemicals are very large, the solution of the system is unique.

Remark 5.5. To estimate the magnitude of λ in Lemma 5.7, we first note that λ_1 and λ_2 are actually the first non-zero eigenvalues of

$$-\Delta w = 0 \text{ in } \Omega,\ w|_{\Gamma_1} = 0,\ \tfrac{\partial w}{\partial \nu}|_{\Gamma_2} = 0$$

and

$$-\Delta w = 0 \text{ in } \Omega,\ \frac{\partial w}{\partial \nu}|_{\Gamma_1} = 0,\ w|_{\Gamma_2} = 0$$

respectively (see [24], pp. 133-134, for a proof). Then, in the 1-D case, a direct calculation gives $\lambda = \frac{\pi^2}{4l^2}$, where l is the length of the interval (domain). In 2-D or 3-D, when Γ_2 is empty, the boundary condition for u becomes that of Dirichlet's, and it was shown ([19]) that $\lambda_1 \geq \frac{1}{4\rho^2}$, where ρ is the radius of the largest disk or ball contained in (the simply connected domain) Ω. But in this case we have $\lambda_2 = 0$, because the function $w \equiv 1/|\Omega|^{\frac{1}{2}}$ minimizes $\inf_{\substack{w\in V_2 \\ ||w||=1}} ||\nabla w||^2$. Hence $\lambda = \min\{\lambda_1, \lambda_2\} = 0$. The same is true when Γ_1 is empty. As for the case when both Γ_1 and Γ_2 have non-empty interior, no estimate is available, to our best knowledge, for the first eigenvalue of such a mixed boundary value problem. Nevertheless, we can still prove that $\lambda > 0$. In fact, Friedrich's inequality ([55] p. 82), which applies also to the spaces V_1 and V_2 in (5.17) ([53] pp. 149-150), asserts that there are constants $C_1, C_2 > 0$ such that

$$||v|| \leq C_1 ||\nabla v|| \ \forall v \in V_1,$$

and

$$||v|| \leq C_2 ||\nabla v|| \ \forall v \in V_2.$$

Then from (5.23) we have

$$\lambda_1 \geq 1/C_1^2,\ \lambda_2 \geq 1/C_2^2,$$

and therefore,

$$\lambda \geq \min\{1/C_1^2, 1/C_2^2\} > 0.$$

Remark 5.6. Two sets of sufficient conditions were given in [69] (p. 409 and p. 415) to ensure uniqueness of solution for a class of reaction-diffusion systems. Unfortunately, neither is satisfied by our system. The first set of conditions requires that at least one of the two inequalities

$$\inf_{x\in\Omega} \frac{\partial H_1}{\partial u} > \lambda_1$$

and

$$\inf_{x\in\Omega} \frac{\partial H_2}{\partial v} > \lambda_2$$

holds. None of the above inequalities is true, because a simple calculation shows $\frac{\partial H_1}{\partial u} \leq 0$, $\frac{\partial H_2}{\partial v} \leq 0$. The other set of conditions requires either

$$\frac{\partial(H_1/u)}{\partial u} > 0, \quad \text{or} \quad \frac{\partial(H_2/v)}{\partial v} > 0$$

to be true. But in our case both quantities are negative. In fact, our sufficient conditon for uniqueness is also mild when compared to those for other similar systems found in the literature. See [57], [20], and [74] etc. where, besides monotonicity, the convexity of the right-hand side H_1 and H_2 is also required. In addition, a slight modification of Lemma 5.7 will give similar sufficient conditions for the other two cases considered in [69] (p. 402) where the monotone sequence method applies.

We summarize the above lemmas in the following theorem.

Theorem 5.1. *Under condition* (5.28), *system* (5.1) *has at least one classical solution* (n, c, a) *with*

$$0 \leq c, a \leq 1,$$

and

$$n = (1 + \kappa c)^{\frac{\chi}{\kappa D_1}} \exp\left(\frac{\alpha}{2D_1} a^2\right) M / \int_\Omega F\, dx.$$

Proof. By Lemma 5.1, to solve (5.1), we need only solve (5.3). By freezing its the non-local term $B = M/\int_\Omega F(u, v)$, system (5.3) becomes (5.12). By Lemma 5.6 and Lemma 5.7, for any B with

$$C_1 \equiv \frac{M}{L|\Omega|} \leq B \leq C_2 \equiv \frac{M}{|\Omega|},$$

system (5.12) has a unique solution $(u(x, B), v(x, B))$ under condition (5.28).

Now we consider the quantity $T(B) = M/\int_\Omega F(u(x, B), v(x, B))\, dx$. A simple calculation shows $C_1 \leq T(B) \leq C_2$. We claim that the function $T : B \in [C_1, C_2] \to T(B) \in [C_1, C_2]$ is continuous. Let $B_k, B_0 \in (C_1, C_2)$, $B_k \to B_0$, and

$$(u_k, v_k) \equiv (u(x, B_k), v(x, B_k)), \ (u_0, v_0) \equiv (u(x, B_0), v(x, B_0)),$$

we need to show

$$T(B_k) = M/\int_\Omega F(u_k, v_k)\,dx \to T(B_0) = M/\int_\Omega F(u_0, v_0)\,dx.$$

Since $F(u,v) > 0$ is a continuous function of its components, it suffices to have

$$(u_k, v_k) \to (u_0, v_0).$$

In fact, from the boundedness of $\{B_k\}_{k=1}^{\infty}$ we know $\{(u_k, v_k)\}_{k=1}^{\infty}$ is a relatively compact sequence, following from a uniform $C^{2+\alpha}(\Omega)$ a priori estimate mentioned in Lemma 5.6. Note that any limit point of the above set is also a solution of system (5.12), by a similar argument as that in Lemma 5.6, any such limit point has the form $(\tilde{u}(x, B_0), \tilde{v}(x, B_0))$ because of the fact $B_k \to B_0$. Since the solution of (5.12) is unique under condition (5.28), we conclude that $(\tilde{u}(x, B_0), \tilde{v}(x, B_0)) = (u(x, B_0), v(x, B_0))$, and (u_0, v_0) is the only limit point of the relatively compact sequence $\{(u_k, v_k)\}_{k=1}^{\infty}$. It follows that $(u_k, v_k) \to (u_0, v_0)$. Thus as a compact operator, T has a fixed point by Schauder's Theorem ([88], Page 61, Theorem 1.C.). Such a fixed point $B = M/\int_\Omega F(u,v)\,dx$ makes system (5.12) become system (5.3). This proves the existence of solution for system (5.3). The remaining part of the theorem follows from Lemma 5.3. □

5.3 Uniqueness of Solution

To obtain a condition on the uniqueness of solution to the original system (5.1) (or its equivalent (5.3)), the non-local term $B = M/\int_\Omega F(u,v)\,dx$ has to be treated as a variable depending on (u,v) instead of as a constant. This results in a severer uniqueness condition than for the variational system (5.12).

Theorem 5.2. *Let λ, L, P, R, and Q be defined by* (5.23), (5.24), (5.25), (5.26), *and* (5.27), *respectively. Then under condition*

$$\lambda + P > Q(1 + 2L), \tag{5.36}$$

the solution to the steady-state system (5.1) *is unique.*

Proof. For simplicity, we consider the equations for (c, a) instead of (u, v) (a substitution $(u, v) = (1 - c, 1 - a)$ in (5.3) will suffice.) Suppose (c_1, a_1) and (c_2, a_2) are its two solutions. Let

$$c \equiv c_2 - c_1, \quad a \equiv a_2 - a_1,$$

$$F_1 \equiv F(c_1, a_1), \quad F_2 \equiv F(c_2, a_2),$$
$$B_1 \equiv M / \int_\Omega F_1\, dx, \quad B_2 \equiv M / \int_\Omega F_2\, dx.$$

Then, by taking the difference of the two equations for c_2 and c_1, we have

$$\text{(5.37)} \qquad \begin{aligned} \triangle(c_2 - c_1) &= \gamma_1(c_2 - c_1) + \beta_1 B_2 F_2 (c_2 - c_1) \\ &+\beta_1 B_2 c_1 (F_2 - F_1) + \beta_1 F_1 c_1 (B_2 - B_1). \end{aligned}$$

From

$$F_2 - F_1 = \frac{\partial F}{\partial c}(c_2 - c_1) + \frac{\partial F}{\partial a}(a_2 - a_1),$$

and

$$B_2 - B_1 = M / \int_\Omega F_2\, dx - M / \int_\Omega F_1\, dx = -\frac{B_1 B_2}{M} \int_\Omega (F_2 - F_1)\, dx,$$

we have

$$B_2 - B_1 = -\frac{B_1 B_2}{M} \int_\Omega \left[\frac{\partial F}{\partial c}(c_2 - c_1) + \frac{\partial F}{\partial a}(a_2 - a_1)\right] dx.$$

With $c_2 - c_1 = c$, $a_2 - a_1 = a$, we rewrite equation (5.37) as

$$\text{(5.38)} \qquad \begin{aligned} &-\Delta c + \left(\gamma_1 + \beta_1 B_2 F_2 + \beta_1 B_2 c_1 \frac{\partial F}{\partial c}\right) c \\ &= -\beta_1 B_2 c_1 \frac{\partial F}{\partial a} a + \beta_1 \frac{B_1 B_2}{M} F_1 c_1 \int_\Omega \left(\frac{\partial F}{\partial c} c + \frac{\partial F}{\partial a} a\right) dx. \end{aligned}$$

We multiply (5.38) by c and then integrate over Ω to get

$$\text{(5.39)} \qquad \begin{aligned} &\int_\Omega -\Delta c \cdot c\, dx + \int_\Omega \left(\gamma_1 + \beta_1 B_2 F_2 + \beta_1 B_2 c_1 \frac{\partial F}{\partial c}\right) c^2\, dx \\ &= -\beta_1 B_2 \int_\Omega c_1 \frac{\partial F}{\partial a} c a\, dx \\ &+\beta_1 \frac{B_1 B_2}{M} \int_\Omega \left(\frac{\partial F}{\partial c} c + \frac{\partial F}{\partial a} a\right) dx \int_\Omega F_1 c_1 c\, dx. \end{aligned}$$

Recall the definitions of λ, L, P, R, and Q. Noticing again that

$$1 \le F_i \le L, \quad \frac{M}{L|\Omega|} \le B_i \le \frac{M}{|\Omega|},$$

and that

$$\frac{\partial F}{\partial c} \ge 0, \quad \frac{\partial F}{\partial a} \ge 0,$$
$$0 \le c_1 \le 1, \quad 0 \le a_1 \le 1,$$

we estimate each term in (5.39) as follows:

$$\begin{aligned}
&\int_\Omega -\Delta c \cdot c\, dx = \int_\Omega |\nabla c|^2\, dx \ge \lambda \|c\|^2, \\
&\int_\Omega \left(\gamma_1 + \beta_1 B_2 F_2 + \beta_1 B_2 c_1 \frac{\partial F}{\partial c}\right) c^2\, dx \ge P\|c\|^2, \\
&-\beta_1 B_2 \int_\Omega c_1 \frac{\partial F}{\partial a} ac\, dx \le \frac{Q}{2}(\|c\|^2 + \|a\|^2), \\
&\beta_1 \frac{B_1 B_2}{M} \int_\Omega \left(\frac{\partial F}{\partial c} c + \frac{\partial F}{\partial a} a\right)\, dx \int_\Omega F_1 c_1 c\, dx \le QL(\|c\| + \|a\|)\|c\|,
\end{aligned}$$

where Hölder's inequality has been used to obtain the last estimate, which is in turn bounded by

$$\frac{3}{2} QL\|c\|^2 + \frac{1}{2} QL\|a\|^2.$$

Making use of these estimates in (5.39), we obtain

$$(\lambda + P)\|c\|^2 \le \frac{Q}{2}(1 + 3L)\|c\|^2 + \frac{Q}{2}(1 + L)\|a\|^2. \tag{5.40}$$

Similarly, from the 'a' equation we obtain

$$(\lambda + P)\|a\|^2 \le \frac{Q}{2}(1 + 3L)\|a\|^2 + \frac{Q}{2}(1 + L)\|c\|^2. \tag{5.41}$$

We then derive from (5.40) and (5.41) the inequality

$$[\lambda + P - Q(1 + 2L)](\|c\|^2 + \|a\|^2) \le 0, \tag{5.42}$$

from which we know that if

$$\lambda + P - Q(1 + 2L) > 0,$$

then

$$\|c\|^2 = \|a\|^2 = 0,$$

or

$$c \equiv 0, \;\; a \equiv 0.$$

That is, the solution of the steady-state AC system is unique. □

Remark 5.7. Condition (5.36) obviously implies condition (5.28), from which we know that, under condition (5.36), the solution to the steady-state AC system exists and is unique.

Chapter 6

Analysis of the Steady-State CS Model

Throughout this chapter we study the following steady-state of the CS model

$$(6.1) \qquad \begin{cases} -u_{xx} + \lambda u = -\dfrac{avu}{\gamma + u}, \\ -Dv_{xx} + \beta v + \kappa(vu_x)_x = b(1-v)vG(u), \\ u(0) = 1, u(1) = 0; v(0) = 0, v(1) = 1. \end{cases}$$

The main aim of this chapter is to study a finite difference approximation of the above system. Except for a general maximum principle for matrices, we have not found the discrete a priori estimates we need. We derive these estimates systematically, and use them to prove existence, uniqueness, and error estimates for the numerical solutions, which, to our best knowledge, has not been done before. We have found that the numerical solutions can be classified into two basic types, exactly as was shown in the continuous case ([4]). The only piece of work on the theoretical aspect of the model is Theorem 6.5 that improves a sufficient condition from [4] for *type I* solution.

The outline for this chapter is as follows. In §6.1, we give the mathematical properties of the CS model as background information. In §6.2, we give the improved sufficient condition on the system parameters for the solutions to be of *type I*, and give the a priori estimates for the exact solutions. In §6.3, we first develop some discrete a priori estimates based on the maximum principle;

we then set up a finite difference scheme for the CS; and using those estimates as a tool, we show the existence, uniqueness, and convergence of the numerical solution. Error estimates and numerical simulations are also included in this section. Some remarks are made in the last section.

6.1 A Brief Review

The steady-state CS model has been studied by Allegretto et al.([4]). Now we introduce their work, on which our numerical analysis will be based. First, in view of the physical significance of u,v, it is clear that we need only consider solution with $0 \leq u, v \leq 1$. As for the existence and uniqueness of such solution of (6.1), we have

Theorem 6.1. *([4]) System* (6.1) *has at least one classical solution* (u, v) *which stisfies* $0 < u(x), v(x) < 1$ *for* $x \in (0, 1)$.

Theorem 6.2. *([4]) Suppose*

$$\beta > b(1 - c^*) + \frac{\kappa^2 a^2}{16 D\lambda} + \frac{ab}{4\lambda}. \tag{6.2}$$

Then the nonnegative solution of system (6.1) *is unique.*

Before we give the remaining results, we need the following definition.

We call a solution (u, v) of *type I* if the function v is monotonically increasing in $(0, 1)$; or call it of *type II* if v has a pair of extrema.

Theorem 6.3. *([4]) Any nonnegative solution of system* (6.1) *is either of type I or of type II.*

Theorem 6.4. *([4])* (**a**) *Suppose the condition*

$$\beta + \lambda\kappa c^* \geq b(1 - c^*) \tag{6.3}$$

is satisfied. Then v is of type I. (**b**) *Let*

$$c_1 \equiv \frac{1 - c^*}{\left[1 + \left(\frac{a}{\gamma} + \lambda\right)\right]^{1/2}}, \quad c_2 \equiv \frac{c^*}{\left[1 + \left(\frac{a}{\gamma} + \lambda\right)\right]^{1/2}},$$

and

$$D_1 \equiv \frac{\frac{16\pi^2}{c_1^4} + \frac{\kappa}{D}\left(\lambda + \frac{a}{\gamma} + \frac{\beta}{\kappa}\right)}{b(1-c^*)/2}.$$

If

$$\exp\left(\frac{\kappa}{D}\right) < (1 - D_1)\left(1 + \frac{\beta c_2^4}{2D}\right), \tag{6.4}$$

then v is of type II.

Remark 6.1. Since (6.2):

$$\beta > b(1-c^*) + \frac{\kappa^2 a^2}{16D\lambda} + \frac{ab}{4\lambda}$$

implies (6.3):

$$\beta + \lambda\kappa c^* \geq b(1-c^*),$$

we know that the uniqueness conclusion applies only to *type I* solution.

6.2 The Theoretical Work

In this section we first give a result which improves the sufficent condition in Theorem 6.4 (**a**) for *type I* solution; we then give the gradient estimates for the general solutions. We need the following nonlinear version of the maximum principle first.

Lemma 6.1. *([72]) Suppose $v = v(x) \in C^2[a,b]$ satisfies the differential inequality $L[v] \equiv v_{xx} + H(x, v, v_x) \leq 0$ in $[a,b]$, where $H = H(x,y,z)$, $\frac{\partial H}{\partial y}$, and $\frac{\partial H}{\partial z}$ are continuous in their domains; $H(x,0,0) = 0$, $\frac{\partial H(x,y,z)}{\partial y} \leq 0$. If v assumes a nonnegative maximum value M at an interior point of $[a,b]$, then $v(x) \equiv M$.*

Theorem 6.5. *Suppose*

$$\beta + \lambda\kappa \geq b(1-c^*), \tag{6.5}$$

then v is of type I.

Proof. We apply the above maximum principle to the steady-state equation for v, which we rewrite as

$$v_{xx} - \frac{\kappa}{D}u_x v_x + \left[-\frac{\beta}{D} - \frac{\kappa}{D}\left(\frac{av}{\gamma+u} + \lambda\right)u + \frac{b}{D}(1-v)G(u)\right]v = 0, \tag{6.6}$$

where the equation for u has been used. Let

$$H = -\frac{\kappa}{D}u_x z + \left[-\frac{\beta}{D} - \frac{\kappa}{D}\left(\frac{ay}{\gamma+u} + \lambda\right)u + \frac{b}{D}(1-y)G(u)\right]y.$$

In our case, the conditions in Lemma 6.1 are all satisfied except the one:

$$\frac{\partial H}{\partial y} \leq 0.$$

Since

$$\frac{\partial H}{\partial y} = -\frac{\beta}{D} - \frac{\kappa\lambda}{D}u + \frac{b}{D}G(u) - \left(\frac{2\kappa a u}{D(\gamma+u)} + \frac{2b}{D}G(u)\right)y \tag{6.7}$$

where $y = v(x) \geq 0$, to guarantee $\frac{\partial H(x,y,z)}{\partial y} \leq 0$ it suffices to have $\max_{x\in[0,1]} f(x) \leq 0$, where

$$f(x) \equiv -\frac{\beta}{D} - \frac{\kappa\lambda}{D}u + \frac{b}{D}G(u). \tag{6.8}$$

Note that $u(0) = 1$ and $u(1) = 0$, and u is decreasing on $[0,1]$. Since $0 < c^* < 1$, we know there is a unique $x_0 \in (0,1)$ such that $u(x_0) = c^*$. Hence,

$$f(x) = \begin{cases} -\frac{\beta+bc*}{D} + \frac{b-\kappa\lambda}{D}u \text{ for } x \in [0, x_0], \\ -\frac{\beta}{D} - \frac{\kappa\lambda}{D}u \text{ for } x \in (x_0, 1]. \end{cases}$$

It follows that $f(x)$ is monotone on both invervals $[0, x_0]$ and $(x_0, 1]$. Therefore,

$$\max_{x\in[0,1]} f(x) = \max\{f(0), f(x_0), f(1)\}. \tag{6.9}$$

Since

$$f(1) = -\frac{\beta}{D} < 0, \ f(x_0) = -\frac{1}{D}(\beta + \kappa\lambda c^*) < 0,$$

while

$$f(0) = -\frac{1}{D}[\beta + \kappa\lambda - b(1-c^*)], \tag{6.10}$$

it is clear that $\max_{x\in[0,1]} f(x) \leq 0$ if and only if $\beta + \kappa\lambda \geq b(1-c^*)$, that is, (6.5) is true. Under the above condition, the function $v(x)$ can not assume any nonnegative maximum, and it must be of *type I*. □

Remark 6.2. Note that condition (6.3) implies (6.5), indicating that Theorem 6.5 is an improvement of Theorem 6.4 (**a**).

Lemma 6.2. *Let* (u, v) *be a positive solution of* (6.1), *and*

$$\mu \equiv \left(\lambda + \frac{a}{\gamma}\right)^{\frac{1}{2}}, \quad m \equiv \frac{\mu \exp(\mu)}{\exp(2\mu) - 1}, \quad M \equiv \mu \frac{\exp(2\mu) + 1}{\exp(2\mu) - 1},$$

$$N \equiv \frac{\kappa}{D} M + \frac{3\beta}{2D} + \frac{3b(1 - c^*)}{8D} + 1.$$

Then we have the following estimates:

$$-M \leq u_x \leq -m, \ |v_x| \leq N. \tag{6.11}$$

Proof. Noticing that

$$0 \leq u, \ v \leq 1,$$

we obtain from the first equation in (6.1) that

$$\lambda u \leq u_{xx} = \left(\lambda + \frac{av}{\gamma + u}\right) u \leq \left(\lambda + \frac{a}{\gamma}\right) u = \mu^2 u. \tag{6.12}$$

Now suppose w is the solution of the following problem

$$\begin{cases} w_{xx} = \mu^2 w, \\ w(0) = 1, w(1) = 0. \end{cases} \tag{6.13}$$

By the maximum principle, w has the properties:$(0 \leq w \leq u)$, which results in

$$0 \leq w \leq u, \ \ w_x(0) \leq u_x(0), \ \ u_x(1) \leq w_x(1). \tag{6.14}$$

Direct solution of (6.13) gives us

$$w(x) = -\frac{\exp(\mu x)}{\exp(2\mu) - 1} + \frac{\exp(2\mu - \mu x)}{\exp(2\mu) - 1},$$

and thus

$$w_x(0) = -\mu \frac{\exp(2\mu) + 1}{\exp(2\mu) - 1}, \ \ w_x(1) = -\frac{\mu \exp(\mu)}{\exp(2\mu) - 1}. \tag{6.15}$$

Then, the estimate for u_x follows from the fact that

$$u_x(0) \le u_x \le u_x(1), \tag{6.16}$$

which is a result of $u_{xx} \ge 0$.

Next, we estimate v_x. From the second equation in (6.1) we have

$$v_{xx} = \frac{\kappa}{D}(vu_x)_x + \frac{\beta}{D}v - \frac{b}{D}(1-v)vG(u). \tag{6.17}$$

We integrate the above equation from 0 to x and use the fact that $v(0) = 0$ to get

$$v_x - v_x(0) = \frac{\kappa}{D}vu_x + \int_0^x \left[\frac{\beta}{D}v - \frac{b}{D}(1-v)vG(u)\right] dy. \tag{6.18}$$

Integrating the above equation again from 0 to 1 and using the fact $v(1) = 1$ (plus $v(0) = 0$), we obtain

$$v_x(0) = 1 - \int_0^1 \frac{\kappa}{D}vu_x dx - \int_0^1 \int_0^x \left[\frac{\beta}{D}v - \frac{b}{D}(1-v)vG(u)\right] dydx. \tag{6.19}$$

Application of the Mean-value Theorem to the first integral and direct estimation of the second one on the right-hand side of the above equation give us:

$$|v_x(0)| \le \frac{\kappa}{D} + \frac{\beta}{2D} + \frac{b(1-c^*)}{8D} + 1. \tag{6.20}$$

The estimate for v_x is then obtained easily from (6.20), (6.18), (6.16), (6.15), and (6.14). □

Remark 6.3. We could have used the barrier function method to conduct the gradient estimate for v, but the divergence form of its equation and the boundedness of u, v save us from that way, which involves more complicated work. See Gilbarg and Trudinger ([31]) for more information.

Remark 6.4. We see that u is always decreasing and concaving up, while with v the situation is not that simple. But as we saw in last section, v has at most one pair of extrema.

6.3 The Numerical Work

We will give a finite difference analysis of the CS in this section. So we divide the interval $[0,1]$ into n equal subintervals with the nodes being denoted by $x_i = \frac{i}{n} \equiv ih, \ i = 0, ..., n$. Let V_h be the space of functions that are piece-wise linear on $[x_{i-1}, x_i], \ i = 1, ..., n$. We will identify any sequence $\{w_i\}_{i=0}^n$ with an element $w \in V_h$ with $w(x_i) = w_i, \ i = 0, ..., n$, because they are mutually determined. Now we define some difference operators. The Euler forward-difference operator F_h is given by:

$$F_h w(x) = \frac{w(x+h) - w(x)}{h}.$$

So that

$$F_h w(x_i) = \frac{w(x_{i+1}) - w(x_i)}{h}, \ F_h w_i = \frac{w_{i+1} - w_i}{h}.$$

The second order center-difference operator L_h is defined by:

$$L_h w(x) = \frac{w(x+h) - 2w(x) + w(x-h)}{h^2}$$

So that

$$L_h w(x_i) = \frac{w(x_{i+1}) - 2w(x_i) + w(x_{i-1})}{h^2}, \ L_h w_i = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}.$$

For any $w = \{w_i\}_{i=i_1}^{i_n}$, we define:

$$\min w \equiv \min_{i_1 \le i \le i_n} w_i, \ \ \max w \equiv \max_{i_1 \le i \le i_n} w_i, \ \ \|w\| \equiv \max_{i_1 \le i \le i_n} |w_i|;$$

and for a constant c, $w \equiv c$ means $w_i = c, \ i = i_1, ..., i_n$; and $w \le (\ge) c$ means $w_i \le (\ge) c, \ i = i_1, ..., i_n$.

6.3.1 The Discrete Maximum Principle and A Priori Estimates

The discrete maximum principle and the a priori estimates established as a consequence will play the key role in our numerical analysis. We start with the discrete maximum principle.

Lemma 6.3. *Let $w = \{w_i\}_{i=0}^n$ satisfy*

(6.21) $$-L_h w_i + p_i F_h w_i + q_i w_i \geq 0,\ i = 1, ..., n-1$$

where $p \leq 0$ and $q \geq 0$. If there exists some j $(1 \leq j \leq n-1)$ such that $\min w = w_j \leq 0$, then $w \equiv w_j$.

Proof. From

$$0 \leq -L_h w_j + p_j F_h w_j + q_j w_j \leq -L_h w_j + p_j F_h w_j =$$
$$\tfrac{1}{h^2}(w_j - w_{j-1}) + (\tfrac{1}{h^2} - \tfrac{p_j}{h})(w_j - w_{j+1}) \leq 0,$$

we obtain $w_{j-1} = w_j = w_{j+1}$. Continue the above process with $j-1$ and $j+1$ until we have $w_k = w_j$, $0 \leq k \leq n$, that is, $w \equiv w_j$. □

Remark 6.5. A more general version of this lemma, developed for general matrices associated with elliptic problems, can be found in [38].

Remark 6.6. In Lemma 6.3, if, in addition, we assume $w_0 \geq 0$, $w_n \geq 0$, and $w \not\equiv 0$, then we know $w_k > 0$, $k = 1, ..., n-1$. Otherwise we would have $\min w = w_j \leq 0$ for some $1 \leq j \leq n-1$, and hence $w \equiv w_j \leq 0$ by Lemma 6.3. On the other hand, $w_0 \geq 0$ and $w_n \geq 0$ implies $w \equiv w_j \geq 0$. So the only possibility is $w \equiv 0$. But this contradicts the assumption $w \not\equiv 0$.

We then set up the positivity of solutions to a particular type of second order difference equations.

Lemma 6.4. *Let $w = \{w_i\}_{i=0}^n$ satisfy*

(6.22) $$\begin{cases} -L_h w_i + p_i F_h w_i + q_i w_i = r_i(M - w_i),\ 1 \leq i \leq n-1, \\ 0 \leq w_0 \leq M,\ 0 \leq w_n \leq M, \end{cases}$$

where $p \leq 0$, $q \geq 0$, $r \geq 0$, and $M > 0$ (a constant). Then, either $w \equiv 0$ (in which case $r \equiv 0$), or

(6.23) $$0 < w_i < M,\ i = 1, ..., n-1.$$

Proof. To get the left half of (6.23) we need only to note that

$$-L_h w_i + p_i F_h w_i + (q_i + r_i) w_i = r_i M \geq 0,\ w_0 \geq 0,\ w_n \geq 0;$$

and then apply Lemma 6.3 to w_i. To obtain the right half of (6.23), we apply Lemma 6.3 to $v_i = M - w_i$. □

Next, we proceed to derive some more general a priori estimates. We begin with difference equation without a first order term. The following lemma will be needed.

Lemma 6.5. *Let $w(x)$ and $\{w_i\}_{i=0}^n$ be the solution of*

$$-w_{xx} + qw = 1;\ w(0) = 0,\ w(1) = 0 \tag{6.24}$$

and

$$-L_h w_i + q w_i = 1,\ i = 1, ..., n-1;\ w_0 = 0,\ w_n = 0 \tag{6.25}$$

respectively, where $q \geq 0$ is a constant. Then $w_i \leq w(x_i), i = 0, ..., n$.

Proof. We first show $w_{xx} \leq 0$. By the maximum principle we have $w(x) > 0$ for $0 < x < 1$. Since $w(0) = w(1) = 0$, we know there exists an x_0, $0 < x_0 < 1$, such that $w(x_0) = \max_{[0,1]} w(x)$. From elementary calculus we know $-w_{xx}(x_0) \geq 0$. Then we have $qw(x) \leq qw(x_0) \leq 1$, and therefore, $w_{xx} = qw(x) - 1 \leq 0$. Now we differentiate (6.24) twice to get $w^{(4)} = qw_{xx} \leq 0$.

Next, let $e_i \equiv w(x_i) - w_i$. By using Taylor's expansion

$$w_{xx}(x_i) = L_h w(x_i) - \frac{w^{(4)}(\xi)}{12} h^2,$$

we rewrite (6.24) as

$$\begin{cases} -L_h w(x_i) + qw(x_i) = 1 - \frac{w^{(4)}(\xi)}{12} h^2, & i = 1, ..., n-1; \\ w(x_0) = 0, \quad w(x_n) = 0. \end{cases} \tag{6.26}$$

Now we subtract (6.25) from (6.26) to obtain

$$-L_h e_i + q e_i = -\frac{w^{(4)}(\xi)}{12} h^2 \geq 0,\ i = 1, ..., n-1;\ e_0 = 0,\ e_n = 0. \tag{6.27}$$

Hence by the discrete maximum principle, $e_i \geq 0$, and therefore, $w_i \leq w(x_i)$, $i = 0, ..., n$. □

It turns out that for equation without the first order term and with homogeneous boundary conditions, a sharp uniform estimate on the magnitude of its solution can be obtained.

Lemma 6.6. *There exists a constant* $\Phi(q)$, *where* $q = \{q_i\}_{i=0}^n \geq 0$, *such that any solution of*

$$-L_h w_i + q_i w_i = r_i, \;\; i = 1, ..., n-1; \;\; w_0 = 0, \;\; w_n = 0 \tag{6.28}$$

satisfies

$$\|w\| \leq \Phi \|r\| \tag{6.29}$$

and

$$\|F_h w\| \leq (1 + \Phi \|q\|) \|r\|. \tag{6.30}$$

Further more, Φ *can be made independent of* q.

Proof. Let v be the solution of

$$-L_h v_i + \min q \, v_i = \|r\|, \;\; i = 1, ..., n-1; \;\; v_0 = 0, \;\; v_n = 0.$$

Clearly, $v \geq 0$. Since

$$-L_h(v_i \pm w_i) + q_i(v_i \pm w_i) = \|r\| \pm r_i + (q_i - \min q) v_i \geq 0,$$

and $v_0 - w_0 = 0$, $v_n - w_n = 0$, by the discrete maximum principle we have $-v_i \leq w_i \leq v_i$, $i = 0, ..., n$. That is, $\|w\| \leq \|v\|$.

We then estimate $\|v\|$. Noticing that $\dfrac{v_i}{\|r\|}$ satisfies

$$-L_h \frac{v_i}{\|r\|} + \min q \, \frac{v_i}{\|r\|} = 1, \;\; i = 1, ..., n-1; \;\; \frac{v_0}{\|r\|} = 0, \;\; \frac{v_n}{\|r\|} = 0,$$

we use Lemma 6.5 to get $\dfrac{\|v\|}{\|r\|} \leq \|u(x)\|$, or $\|v\| \leq \|u(x)\| \|r\|$, where $u(x)$ is the solution of (6.24) with q being replaced by $\min q$ (we use $\{u_i\}_{i=0}^n$ to denote the solution of (6.25)). A direct calculation shows

$$u(x) = \frac{1}{q} - \frac{\cosh\left[\frac{\sqrt{q}}{2}(1-2x)\right]}{q \cosh \frac{\sqrt{q}}{2}},$$

and

$$\|u(x)\| = \Phi(q) \equiv \frac{1}{q}\left(1 - \frac{1}{\cosh\frac{\sqrt{q}}{2}}\right)$$

where

$$q = \min_{0\le i\le n} q_i.$$

Therefore, we have $\|w\| \le \|v\| \le \Phi(q)\|r\|$.

Now let $d_i \equiv F_h w_{i-1}$, $i = 1, ..., n$, we rewrite (6.28) as

$$d_{i+1} - d_i = (q_i w_i - r_i)h,$$

so that we have

$$|d_{i+1} - d_i| \le (\|q\|\|w\| + \|r\|)h.$$

Then, for any $1 \le k, l \le n-1$, we have

$$|d_k - d_l| \le |k-l|(\|q\|\|w\| + \|r\|)h \le \|q\|\|w\| + \|r\|.$$

Note that $w_0 = w_n = 0$. It is easily seen that there are m_1, m_2 with $m_1 \ne m_2$ such that $\min d = d_{m_1} \le 0$ and $\max d = d_{m_2} \ge 0$. Hence, we get

$$|d_{m_1}| \le |d_{m_1} - d_{m_2}| \le \|q\|\|w\| + \|r\|,$$

and

$$|d_{m_2}| \le |d_{m_1} - d_{m_2}| \le \|q\|\|w\| + \|r\|.$$

Therefore, we have

$$\|d\| \le \|q\|\|w\| + \|r\| \le (1 + \Phi(q)\|q\|)\,\|r\|,$$

where we have used the fact that $\|w\| \le \Phi(q)\|r\|$. The reason why $\Phi(q)$ can be made independent of q is that it is a decreasing function in $(0, \infty)$ and

$$\Phi(0^+) = \lim_{q\to 0^+} \Phi(q) = \frac{1}{8},$$

so that any contant $\Phi \ge \frac{1}{8}$ works uniformly for all $h > 0$. □

Remark 6.7. Note that the constant Φ in Lemma 6.6 make (6.29) hold uniformly for all q, r, w, and n. Of all such constants, Φ is the best (least). This is because the solution of a difference scheme like

$$-L_h w_i + w_i = 1, \;\; i = 1, ..., n-1; \;\; w_0 = 0, \;\; w_n = 0$$

will converge to the solution of

$$-w_{xx} + w = 1; \;\; w(0) = 0, \;\; w(1) = 0$$

when $n \to \infty$, and Φ provides a sharp estimate for the latter (see the derivation of Φ above). Nonetheless, the gradient estimate (6.30) is still q dependent.

Now we generalize the above lemma to the case with nonhomogeneous boundary conditions.

Lemma 6.7. *Let the conditions in Lemma 6.6 be satisfied except that* $w_0 = \alpha$ *and* $w_n = \beta$. *Then we have*

$$(6.31) \qquad ||w|| \leq \Phi||r|| + (1 + \Phi||q||) \max\{|\alpha|, |\beta|\}$$

and

$$(6.32) \qquad ||F_h w|| \leq (1 + \Phi||q||)||r|| + (1 + \Phi||q||)||q|| \max\{|\alpha|, |\beta|\} + |\beta - \alpha|.$$

Proof. We apply Lemma 6.6 to $v_i \equiv w_i - (\alpha + \frac{\beta-\alpha}{n} i)$ to obtain the estimates above. □

Now we consider difference equations with a first order term.

Lemma 6.8. *Let* $p \leq 0$, $q \geq 0$. *Then the system*

$$(6.33) \qquad -L_h w_i + p_i F_h w_i + q_i w_i = r_i, \;\; i = 1, ..., n-1; \;\; w_0 = \alpha, \;\; w_n = \beta$$

has a unique solution w. *When* $\alpha = \beta = 0$, *it satisfies*

$$(6.34) \qquad ||w|| \leq \Psi||r||$$

and

$$(6.35) \qquad ||F_h w|| \leq (2 + \Psi||q||)||r||,$$

where Ψ *is a constant that can be made independent of* p, q, r, w, *and* n.

Proof. We first show the homogeneous system corresponding to (6.33) has only the trivial solution $w \equiv 0$. In fact, when $r \equiv 0$ and $\alpha = \beta = 0$, we apply Lemma 6.3 and Remark 6.6 to w_i and then to $-w_i$ to get $w_i \geq 0$ and $w_i \leq 0$ at the same time. Thus, $w \equiv 0$. This indicates that the system matrix is invertible. Therefore the system has a unique solution.

Now let $\alpha = \beta = 0$. To find a constant Ψ that works for the lemma, we first observe that the solution of (6.33) is bounded by the solution of

$$-L_h v_i + p_i F_h v_i = \|r\|, \quad i = 1, ..., n-1; \quad v_0 = 0, \quad v_n = 0.$$

So we need only to estimate $\|v\|$. Let $a_i \equiv \frac{1}{1-p_i h}$, $i = 1, ..., n-1$, and $a_0 = 1$. We solve the above equation for v by reducing it into two first order difference equations, in which case we get

$$(6.36) \qquad v_{l+1} = \|r\|h^2 \left[\frac{\sum_{j=0}^{l} \Pi_{i=0}^{j} a_i}{\sum_{j=0}^{n-1} \Pi_{i=0}^{j} a_i} \sum_{k=1}^{n-1} \left(\Pi_{j=1}^{k} a_j \sum_{j=1}^{k} \Pi_{i=0}^{j-1} a_i \right) - \sum_{k=1}^{l} \left(\Pi_{j=1}^{k} a_j \sum_{j=1}^{k} \Pi_{i=0}^{j-1} a_i \right) \right]$$

for $l = 2, ..., n-1$; and $v_1 = \|r\|h^2 \frac{\sum_{k=1}^{n-1} \left(\Pi_{j=1}^{k} a_j \sum_{j=1}^{k} \Pi_{i=0}^{j-1} a_i \right)}{\sum_{j=0}^{n-1} \Pi_{i=0}^{j} a_i}$, $v_0 = 0$.

Also,

$$F_h v_0 = \frac{v_1 - v_0}{h} = \|r\|h \frac{\sum_{k=1}^{n-1} \left(\Pi_{j=1}^{k} a_j \sum_{j=1}^{k} \Pi_{i=0}^{j-1} a_i \right)}{\sum_{j=0}^{n-1} \Pi_{i=0}^{j} a_i}.$$

Noticing that $a_i \leq 1$, we find an easy bound for $\|v\|$:

$$\|v\| \leq \|r\|h^2 \sum_{k=1}^{n-1} \left(\sum_{j=1}^{k} 1 \right) \leq \frac{1}{2}\|r\|.$$

Now, since $\|w\| \leq \|v\| \leq \frac{1}{2}\|r\|$, we know the constant Ψ in the lemma must exist and satisfy $\Psi \leq \frac{1}{2}$.

Next, we set out to estimate $F_h w_i$. Note that although $F_h w_0 \leq F_h v_0$ and $-F_h w_{n-1} \leq -F_h v_{n-1}$ follows immediately from the fact that $\|w\| \leq \|v\|$ (and plus, of course, the fact that $w_0 = v_0, w_n = v_n$), the relation $||F_h w|| \leq ||F_h v||$ does not necessarily hold true. So we proceed in the following way: Let $b_i \equiv (r_i - q_i w_i)h$, and rewrite equation (6.33) as

$$F_h w_i = a_i(F_h w_{i-1} - b_i),$$

from which we find the relation

$$F_h w_i = \Pi_{j=1}^{i} a_j F_h w_0 - \sum_{j=1}^{i} b_j \Pi_{k=j}^{i} a_k, \; i = 1, ..., n-1.$$

We then have

$$\begin{aligned} |F_h w_i| &\leq F_h w_0 + \textstyle\sum_{j=1}^{i} |b_j| \Pi_{k=j}^{i} a_k \leq F_h v_0 + \sum_{j=1}^{i} |b_j| \Pi_{k=j}^{i} a_k \\ &\leq ||r|| + (||r|| + ||q|| ||w||) \leq (2 + \Psi ||q||) ||r||. \end{aligned}$$

Note that the inequality

$$|F_h w_i| \leq (2 + \Psi ||q||) ||r||$$

holds true for all $0 \leq i \leq n-1$. □

Remark 6.8. Though estimate (6.34) is uniform, it does not incorporate the factor q. There is a remedy for this shortcoming: When $0 \leq \min q \leq 2$ we will still use (6.34); but when $\min q > 2$ we adopt the following estimate instead: $||w|| \leq \frac{||r||}{\min q}$. The derivation goes this: We first note that the solution of (6.33) is bounded by the solution of

$$-L_h v_i + p_i F_h v_i + \min q \, v_i = ||r||, \; i = 1, ..., n-1; \; v_0 = 0, \; v_n = 0.$$

Let $1 \leq m \leq n-1$ be such that $v_m = \max v$. Then apparently we have $-L_h v_m \geq 0$, and $p_m F_h v_m \geq 0$. Hence, $\min q \, v_m \leq ||r||$, and $v_m \leq \frac{||r||}{\min q}$. Therefore,

$$||w|| \leq ||v|| \leq v_m \leq \frac{||r||}{\min q}.$$

Remark 6.9. The proof given above involves direct solution of a second order difference equation with only second and first difference terms. By factorizing the difference operator, we can reduce such an equation into two first order ones, which in turn can be solved by the summation factor method. But this is no longer true with the presence of the q terms. See [1] for details.

The above lemma can be generalized to the case of nonhomogeneous boundary conditions.

Lemma 6.9. *Let the conditions in Lemma 6.8 be satisfied except that* $w_0 = \alpha$ *and* $w_n = \beta$ *(not necessarily zero). Then we have*

$$\|w\| \leq \Psi\left(\|r\| + \|p\||\beta - \alpha|\right) + (1 + \Psi\|q\|)\max\{|\alpha|, |\beta|\} \tag{6.37}$$

and

$$\begin{aligned} \|F_h w\| \quad & \leq (2 + \Psi\|q\|)\left(\|r\| + \|q\|\max\{|\alpha|, |\beta|\}\right) \\ & + (1 + 2\|p\| + \Psi\|p\|\|q\|)\,|\beta - \alpha|. \end{aligned} \tag{6.38}$$

Proof. We apply Lemma 6.8 to $v_i \equiv w_i - (\alpha + \frac{\beta-\alpha}{n} i)$ to obtain the estimates above. □

6.3.2 The Difference Scheme

The following finite difference scheme will be used to analyze the chemotaxis system numerically:

$$\begin{cases} -L_h u_i + \lambda u_i = -\frac{a v_i}{\gamma + u_i} u_i, \\ -D L_h v_i + \kappa(F_h u_i F_h v_i + v_i L_h u_i) + \beta v_i = b(1 - v_i) v_i G(u_i), \\ u_0 = 1, \;\; u_n = 0; \;\; v_0 = 0, \;\; v_n = 1, \end{cases} \tag{6.39}$$

where u_i is the approximation of $u(x_i)$, $i = 0, ..., n$. In the following subsections we are going to show that the solution to the above difference system exists and is unique, and it converges to the exact solution of (6.1).

Remark 6.10. We have rewritten the term $\kappa(vu_x)_x$ as $\kappa v_x u_x + \kappa v u_{xx}$ in the original equation of v before we approximate each term. Due to the negative sign of u_x, we adopt a forward diference for v_x so that the maximum principle will apply, and there is no such guarantee otherwise.

Remark 6.11. We see that the truncation error for the first difference equation is due to approximating $u_{xx}(x_i)$ by $L_h u(x_i) = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}$, and therefore is $O(h^2)$; while for the second equation the truncation error is $O(h)$, which is due to the approximation of the first order derivatives by a forward (backward) difference scheme. This indicates that a consistent difference scheme has been used.

6.3.3 The Existence of Numerical Solution

Theorem 6.6. *The finite difference system* (6.39) *has at least one solution pair* (u, v) *with* $0 < u_i < 1, 0 < v_i < 1$ $(i = 1, ..., n-1)$.

Proof. We are going to use Schauder's fixed point theorem to prove the existence of solutions. We break this into two parts:

(a) Definition and Compactness of the Operator T. Note that for any $(U, V) \in V_h^2$, the following system

$$(6.40) \quad \begin{cases} -L_h u_i + \lambda u_i = -\frac{aV_i^+}{\gamma + U_i^+} u_i, \\ -DL_h v_i + \kappa(F_h u_i F_h v_i + v_i L_h u_i) + \beta v_i = b(1 - v_i) V_i^+ G(U_i^+), \\ u_0 = 1, \;\; u_n = 0; \;\; v_0 = 0, \;\; v_n = 1 \end{cases}$$

has a unique solution (u, v) because each equation in (6.40) can be treated as a linear one, and Lemma 6.8 applies. Furthermore, we know $(0, 0) < (u_i, v_i) < (1, 1)$ $(i = 1, ..., n-1)$ by Lemma 6.4. Thus we have defined an operator T: $V_h^2 \to M \subset V_h^2$ with $T(U, V) = (u, v)$, where M is defined by:

$$M \equiv \{(u, v) \in V_h^2 | (0, 0) \le (u, v) \le (1, 1), (u_0, v_0) = (1, 0), (u_n, v_n) = (0, 1)\}.$$

Note that M is a bounded, convex, closed, and nonempty subset of V_h^2; and T clearly maps M into itself. Also, T transforms any bounded set B into a relatively compact set $T(B) \subset M$ becasue M itself is compact as a closed bounded set in a finite dimensional Banah space V_h^2. Below it will be shown that T is also continuous. Thus T has a fixed point (u, v) in M by Schauder's fixed point theorem ([88]). That is, (u, v) is the solution of the following system

$$(6.41) \quad \begin{cases} -L_h u_i + \lambda u_i = -\frac{av_i^+}{\gamma + u_i^+} u_i, \\ -DL_h v_i + \kappa(F_h u_i F_h v_i + v_i L_h u_i) + \beta v_i = b(1 - v_i) v_i^+ G(u_i^+), \\ u_0 = 1, \;\; u_n = 0; \;\; v_0 = 0, \;\; v_n = 1. \end{cases}$$

Since (u, v) is nonnegative, (6.41) is reduced to (6.39). Therefore, system (6.39) has a solution as desired.

(b) Continuity of T. Since U^+ and V^+ are continuous functions of U and V respectively, we need only show T is continuous in $[0, \infty)^2$. Let $(0,0) \leq (U^{(k)}, V^{(k)}), (0,0) \leq (U, V)$ in V_h^2, and $(U^{(k)}, V^{(k)}) \to (U, V)$. We have to show $(u^{(k)}, v^{(k)}) = T(U^{(k)}, V^{(k)}) \to (u, v) = T(U, V)$. That is, the solutions of

$$(6.42) \qquad \begin{cases} -L_h u_i^{(k)} + \lambda u_i^{(k)} = -\frac{aV_i^{(k)}}{\gamma + U_i^{(k)}} u_i^{(k)}, \\ -DL_h v_i^{(k)} + \kappa(F_h u_i^{(k)} F_h v_i^{(k)} + v_i^{(k)} L_h u_i^{(k)}) + \beta v_i^{(k)} \\ = b(1 - v_i^{(k)}) V_i^{(k)} G(U_i^{(k)}) \\ u_0^{(k)} = 1, \ \ u_n^{(k)} = 0; \ \ v_0^{(k)} = 0, \ \ v_n^{(k)} = 1 \end{cases}$$

converge to the solution of

$$(6.43) \qquad \begin{cases} -L_h u_i + \lambda u_i = -\frac{aV_i}{\gamma + U_i} u_i, \\ -DL_h v_i + \kappa(F_h u_i F_h v_i + v_i L_h u_i) + \beta v_i = b(1 - v_i) V_i G(U_i) \\ u_0 = 1, \ \ u_n = 0; \ \ v_0 = 0, \ \ v_n = 1. \end{cases}$$

To this end, we need first set up bounds for $(F_h u^{(k)}, F_h v^{(k)})$. To estimate $\|F_h u^{(k)}\|$, we rewrite the first equation in (6.42) as

$$-L_h u_i^{(k)} + \left(\lambda + \frac{aV_i^{(k)}}{\gamma + U_i^{(k)}} \right) u_i^{(k)} = 0.$$

Note that $(U^{(k)}, V^{(k)}) \to (U, V)$ implies that there are positive numbers $\delta_k \to 0$ and ρ such that

$$\|U^{(k)} - U\| \leq \delta_k, \ \|V^{(k)} - V\| \leq \delta_k, \ \|U^{(k)}\| \leq \rho, \ \|V^{(k)}\| \leq \rho.$$

By Lemma 6.7 we have $\|F_h u^{(k)}\| \leq \zeta(\rho)$, where

$$(6.44) \qquad \zeta(\rho) \equiv 1 + \left(\lambda + \frac{a\rho}{\gamma} \right) + \Phi \left(\lambda + \frac{a\rho}{\gamma} \right)^2.$$

Also, we have

$$0 \leq L_h u^{(k)} \leq \lambda + \frac{a\rho}{\gamma}.$$

To estimate $\|F_h v^{(k)}\|$, we rewrite the second equation in (6.42) as

$$-L_h v_i^{(k)} + \tfrac{\kappa}{D} F_h u_i^{(k)} F_h v_i^{(k)} + \tfrac{1}{D}(\kappa L_h u_i^{(k)} + \beta) v_i^{(k)} = \tfrac{b}{D}(1 - v_i^{(k)}) V_i^{(k)} G(U_i^{(k)}).$$

By Lemma 6.9, we have $\|F_h v^{(k)}\| \le \eta(\rho)$, where

$$\text{(6.45)} \qquad \begin{aligned} \eta(\rho) &\equiv \left[2 + \frac{\kappa\Psi}{D}\left(\lambda + \frac{a\rho}{\gamma}\right) + \frac{\beta\Psi}{D}\right] \\ &\quad \times \left[\frac{\kappa}{D}\left(\zeta(\rho) + \left(\lambda + \frac{a\rho}{\gamma}\right)\right) + \frac{\beta}{D} + \frac{b\rho^2}{D}\right] + 1. \end{aligned}$$

Note that the same bounds apply to $(F_h u, F_h v)$, too. Now let $\epsilon_i^{(k)} \equiv u_i^{(k)} - u_i$. By taking the difference of the first equation in (6.42) with that in (6.43) and rearranging terms, we get

$$\text{(6.46)} \qquad -L_h \epsilon_i^{(k)} + \left(\lambda + \frac{a V_i^{(k)}}{\gamma + U_i^{(k)}}\right) \epsilon_i^{(k)} = r_i$$

with r_i being estimated by

$$\text{(6.47)} \qquad \|r\| \le \frac{a}{\gamma^2}\|U^{(k)} - U\|\|V^{(k)}\| + \frac{a}{\gamma}\|V^{(k)} - V\| \le \frac{a}{\gamma}\left(\frac{\rho}{\gamma} + 1\right)\delta_k \to 0.$$

We apply Lemma 6.6 to $\epsilon^{(k)}$ to obtain

$$\|\epsilon^{(k)}\| \le \Phi\|r\| \to 0,$$

$$\|F_h \epsilon^{(k)}\| \le \left(1 + \Phi\left(\lambda + \tfrac{a\rho}{\gamma}\right)\right)\|r\| \to 0,$$

and then

$$\|L_h \epsilon^{(k)}\| = \|q\epsilon^{(k)} - r\| \le \left(1 + \Phi\left(\lambda + \tfrac{a\rho}{\gamma}\right)\right)\|r\| \to 0.$$

Now let

$$\rho_i^{(k)} \equiv v_i^{(k)} - v_i.$$

By taking the difference of the second equation in (6.42) with that in (6.43) and rearranging terms, we have

$$\begin{aligned} \text{(6.48)} \quad & -L_h \rho_i^{(k)} + \frac{\kappa}{D} F_h u_i F_h \rho_i^{(k)} + \frac{1}{D}\left(\kappa L_h u_i^{(k)} + b V_i^{(k)} G(U_i^{(k)}) + \beta\right) \rho_i^{(k)} = s_i \\ & \equiv \frac{b}{D}(1 - v_i)\left(V_i^{(k)} G(U_i^{(k)}) - V_i G(U_i)\right) - \frac{\kappa}{D} F_h v_i^{(k)} F_h \epsilon_i^{(k)} - \frac{\kappa}{D} v_i L_h \epsilon_i^{(k)}. \end{aligned}$$

Since s_i is bounded by

$$\|s\| \le \frac{2b}{D}\rho\delta_k + \frac{\kappa}{D}\eta\|F_h\epsilon^{(k)}\| + \frac{\kappa}{D}\|L_h\epsilon^{(k)}\| \to 0,$$

we apply Lemma 6.8 to equation (6.48) to obtain $\|\rho^{(k)}\| \le \Psi\|s\| \to 0$. Thus we have proved the continuity of the operator T. □

Remark 6.12. Suggested by the operator T, we use the following iterative procedure to obtain a numerical solution: Starting with an initial point $(u_i^{(0)}, v_i^{(0)})$ with

$$0 \le u_i^{(0)} \le 1, \;\; 0 \le v_i^{(0)} \le 1,$$

we define $(u_i^{(k)}, v_i^{(k)})$ recursively by

$$(6.49) \qquad \begin{cases} -L_h u_i^{(k)} + \lambda u_i^{(k)} = -\frac{a v_i^{(k-1)}}{\gamma + u_i^{(k-1)}} u_i^{(k)}, \\ -DL_h v_i^{(k)} + \kappa F_h u_i^{(k)} F_h v_i^{(k)} + \kappa v_i^{(k)} L_h u_i^{(k)} + \beta v_i^{(k)} \\ = b(1 - v_i^{(k)}) v_i^{(k-1)} G(u_i^{(k-1)}), \\ u_0^{(k)} = 1, \;\; u_n^{(k)} = 0; \;\; v_0^{(k)} = 0, \;\; v_n^{(k)} = 1. \end{cases}$$

Note that the above procedure always gives a solution between 0 and 1 by the discrete maximum principle. Numerical simulations will be conducted later to show the efficiency of this scheme.

To prove convergence, we need set up bounds for the gradient of the numerical solutions. This has actually been done in Theorem 6.6.

Lemma 6.10. *Let (u, v) be the solution of* (6.39)*. Then we have the following:*

$$(6.50) \qquad 0 \le u \le 1, 0 \le v \le 1,$$

and

$$(6.51) \qquad \|F_h u\| \le \zeta(1), \|F_h v\| \le \eta(1),$$

where $\zeta(\rho)$ and $\eta(\rho)$ are defined in Theorem 6.6.

Proof. See the proof of Theorem 6.6 for a more general derivation. □

Remark 6.13. The bounds in Lemma 6.10 are independent of the numerical solutions and the partition size n, and depend only on the parameters.

Remark 6.14. As we saw in the proof of Theorem 6.6: $u_i \geq 0$ and $L_h u_i \geq 0$. Hence, $F_h u_i$ is increasing. But $F_h u_{n-1} = -\frac{u_{n-1}}{h} \leq 0$. So we know $F_h u_i \leq 0$ for all $i = 0, ..., n-1$. Now from $L_h u_i \geq 0$ and $F_h u_i \leq 0$, we know u_i is decreasing and concaving up. The profile of v_i is not so simple, but we still can show that they fall into two classes: type I and type II, according to whether v_i is monotone or not, just like the continuous case. The proof depends on establishing the parallels of Lemma 2.2 and Lemma 2.3 in [4].

Lemma 6.11. *Let*

$$\Delta_i \equiv \beta + \kappa L_h u_i + b(v_i - 1)G(u_i).$$

If $\Delta_{i_0} \leq 0$ and if v_i is increasing from $i = i_0'$ to $i = i_0$ for some $i_0' < i_0$, then $\Delta_i < 0$ for $i_0' \leq i < i_0$.

Proof. Since u_i is decreasing by Remark 6.14, $u_0 = 1$, $u_n = 0$, and $0 < c^* < 1$, we know there exists i^*, $0 < i^* \leq n$, such that $i^* = \min\{i | u_i \leq c^*\}$. Then for $i \geq i^*$ we have $G(u_i) = \max\{u_i - c^*, 0\} = 0$, and hence $\Delta_i = \beta + \kappa L_h u_i > 0$. By assumption, $\Delta_{i_0} \leq 0$, so we deduce that $i_0 < i^*$. We claim that the quantity

$$\frac{\Delta_i}{u_i} = \frac{\beta}{u_i} + \kappa \frac{L_h u_i}{u_i} + b(v_i - 1)\frac{G(u_i)}{u_i}$$

is increasing from $i = i_0'$ to $i = i_0$. In fact, $\frac{1}{u_i}$ is increasing; so is $\frac{L_h u_i}{u_i} = \lambda + \frac{a v_i}{\gamma + u_i}$ by the assumption of increasing v_i. Also, since $i \leq i_0 < i^*$, we have

$$(v_i - 1)\frac{G(u_i)}{u_i} = (v_i - 1)\left(1 - \frac{c^*}{u_i}\right).$$

Notice that $v_i - 1$ is increasing but negative, while $\left(1 - \frac{c^*}{u_i}\right)$ is decreasing but positive. Hence $(v_i - 1)\frac{G(u_i)}{u_i}$ is increasing, and so is $\frac{\Delta_i}{u_i}$. But $\frac{\Delta_{i_0}}{u_{i_0}} \leq 0$, so we know $\frac{\Delta_i}{u_i} < 0$ for $i_0' \leq i < i_0$. Therefore, we have $\Delta_i < 0$ for $i_0' \leq i < i_0$. ☐

Lemma 6.12. *Internal minimal and maximal extreme points, if any, of v_i appear in pairs. Moreover, there is at most one pair of extreme points.*

Proof. The first conclusion follows from the fact that $0 = v_0 \le v_i \le v_n = 1$. Next, we show there is at most one pair of extreme points. Let $i_1 < i_2$ be a pair of extreme points of v_i. We assume that there is no other extreme point on the right side of i_1 except i_2. From the equation for v_i in (6.39) we obtain

$$\Delta_{i_1} = \frac{DL_h v_{i_1} - \kappa F_h u_{i_1} F_h v_{i_1}}{v_{i_1}}.$$

Note that

$$L_h v_{i_1} = \frac{v_{i_1-1} - 2v_{i_1} + v_{i_1+1}}{h^2} \le 0, \quad F_h v_{i_1} = \frac{v_{i_1+1} - v_{i_1}}{h} \le 0$$

because i_1 is a maximal point, and $F_h u_{i_1} < 0$ because u_i is decreasing. Hence, $\Delta_{i_1} \le 0$. Suppose there is another adjacent minimal point $i_3 < i_1$. Then we know $\Delta_{i_3} \ge 0$. Now v_i is increasing from $i = i_3$ to $i = i_1$, and $\Delta_{i_1} \le 0$, so we know $\Delta_{i_3} < 0$ by Lemma 6.11, which contradicts the fact that $\Delta_{i_3} \ge 0$. This proves the lemma. □

We summarize the above two lemmas in the following theorem.

Theorem 6.7. *There are only two types of solution for* (6.39)*: either v_i is increasing (type I), or v_i has exactly one pair of extreme points (type II).*

We then give the following sufficient condition for *type I* solution.

Theorem 6.8. *Under condition* (6.5)*:*

$$\beta + \lambda\kappa \ge b(1 - c^*)$$

the solution of (6.39) *is of type I.*

Proof. The proof depends on the following facts: (1) The difference equation for $w_i \equiv -v_i$ can be written as

$$-L_h w_i + p_i F_h w_i + q_i w_i = 0, \;\; i = 1, ..., n-1,$$

with $p_i \le 0$. When $q_i \ge 0$, by Lemma 6.3, w_i can not have any interior non-positive minimum, or equivalently, v_i can not have any interior non-negative maximum (*type I*). (2) To guarantee $q_i \ge 0$, it suffices for (6.5) to be true, just as the proof in the continuous case. □

6.3.4 The Uniqueness of Numerical Solution

The proof of the uniqueness of numerical solution follows a similar idea as that in the continuous case. But first, we need the following lemmas, which are discrete conterparts of the formula of integration by parts, and Green's first identity.

Lemma 6.13. *For the sequences $\{u_i\}_{i=0}^n$ and $\{v_i\}_{i=0}^n$, we have*

$$\sum_{i=1}^{n} v_{i-1} F_h u_{i-1} = \frac{u_i v_i}{h}\Big|_0^n - \sum_{i=1}^{n} u_i F_h v_{i-1}. \tag{6.52}$$

When $u_0 v_0 = u_n v_n = 0$, we have

$$\sum_{i=1}^{n} v_{i-1} F_h u_{i-1} = -\sum_{i=1}^{n} u_i F_h v_{i-1}. \tag{6.53}$$

Proof. The proof is trivial. □

Lemma 6.14. *For $\{u_i\}_{i=0}^n$ and $\{v_i\}_{i=0}^n$, we have*

$$\sum_{i=1}^{n-1} v_i L_h u_i = \frac{v_i F_h u_{i-1}}{h}\Big|_1^n - \sum_{i=1}^{n-1} F_h u_{i-1} F_h v_{i-1}. \tag{6.54}$$

When $v_n F_h u_{n-1} = v_1 F_h u_0 = 0$ we have

$$\sum_{i=1}^{n-1} v_i L_h u_i = -\sum_{i=1}^{n} F_h u_{i-1} F_h v_{i-1}. \tag{6.55}$$

Proof. The proof is trivial. □

Theorem 6.9. *Under condition* (6.2)*: $\beta > b(1-c^*) + \frac{\kappa^2 a^2}{16 D\lambda} + \frac{ab}{4\lambda}$, the positive solution to the difference scheme* (6.39) *is unique.*

Proof. Let $(u_i^{(1)}, v_i^{(1)})$, $(u_i^{(2)}, v_i^{(2)})$ be two nonnegative solutions of system (6.39), and

$$u_i \equiv u_i^{(2)} - u_i^{(1)}, \ v_i \equiv v_i^{(2)} - v_i^{(1)}.$$

Then (u_i, v_i) satisfies the following equations

$$-L_h u_i + \lambda u_i = -\frac{a\gamma u_i v_i^{(1)}}{(\gamma + u_i^{(1)})(\gamma + u_i^{(2)})} - \frac{a u_i^{(2)} v_i}{\gamma + u_i^{(2)}}, \tag{6.56}$$

and

$$(6.57)\quad -L_h v_i + \frac{\beta}{D} v_i + \frac{\kappa}{D}\left(F_h u_i^{(2)} F_h v_i + F_h u_i F_h v_i^{(1)} + v_i L_h u_i^{(2)} + v_i^{(1)} L_h u_i\right)$$
$$= \frac{b}{D}(1 - v_i^{(1)} - v_i^{(2)}) G(u_i^{(2)}) v_i + \frac{b}{D}(1 - v_i^{(1)}) v_i^{(1)} (G(u_i^{(2)} - G(u_i^{(1)})),$$

with boundary conditions:

$$(6.58)\qquad u_0 = 0,\ \ u_n = 0;\ \ v_0 = 0,\ \ v_n = 0.$$

We multiply (6.56) by u_i, and sum over $i = 1, \ldots, n-1$. (Conditions (6.58) will also be used throughout the proof.) With the help of partial summation, we have

$$(6.59)\qquad \sum_{i=1}^{n} (F_h u_{i-1})^2 + \lambda \sum_{i=1}^{n-1} u_i^2 \le a \sum_{i=1}^{n-1} |u_i v_i|\,.$$

Since(the Basic inequality)

$$(6.60)\qquad a \sum_{i=1}^{n-1} |u_i v_i| \le \lambda \sum_{i=1}^{n-1} u_i^2 + \frac{a^2}{4\lambda} \sum_{i=1}^{n-1} v_i^2,$$

we obtain from (6.59) and (6.60) that

$$(6.61)\qquad \sum_{i=1}^{n} (F_h u_{i-1})^2 \le \frac{a^2}{4\lambda} \sum_{i=1}^{n-1} v_i^2.$$

Also, from (6.59) we have

$$(6.62)\qquad \lambda \sum_{i=1}^{n-1} u_i^2 \le a \left(\sum_{i=1}^{n-1} u_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} v_i^2\right)^{\frac{1}{2}},$$

which gives

$$(6.63)\qquad \left(\sum_{i=1}^{n-1} u_i^2\right)^{\frac{1}{2}} \le \frac{a}{\lambda} \left(\sum_{i=1}^{n-1} v_i^2\right)^{\frac{1}{2}}.$$

So we obtain

$$(6.64)\qquad \sum_{i=1}^{n-1} |u_i v_i| \le \left(\sum_{i=1}^{n-1} u_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} v_i^2\right)^{\frac{1}{2}} \le \frac{a}{\lambda} \sum_{i=1}^{n-1} v_i^2.$$

Similarly, from (6.57) we have

$$\sum_{i=1}^{n}(F_h v_{i-1})^2 + \frac{\beta}{D}\sum_{i=1}^{n-1} v_i^2 + \frac{\kappa}{D}\sum_{i=1}^{n-1} v_i^2 L_h u_i^{(2)} \tag{6.65}$$
$$+\frac{\kappa}{D}\sum_{i=1}^{n-1} v_i F_h u_i^{(2)} F_h v_i + \frac{\kappa}{D}\sum_{i=1}^{n-1}\left(F_h u_i F_h v_i^{(1)} + v_i^{(1)} L_h u_i\right) v_i$$
$$= \frac{b}{D}\sum_{i=1}^{n-1}(1 - v_i^{(1)} - v_i^{(2)}) G(u_i^{(2)}) v_i^2$$
$$+\frac{b}{D}\sum_{i=1}^{n-1}(1 - v_i^{(1)}) v_i^{(1)} (G(u_i^{(2)}) - G(u_i^{(1)})) v_i.$$

Next, using Lemma 6.13 and Lemma 6.14, we make the following estimates. For the fourth term on the left of (6.65), we have

$$\sum_{i=1}^{n-1} v_i F_h u_i^{(2)} F_h v_i = \frac{1}{2}\sum_{i=1}^{n-1} F_h u_i^{(2)} F_h v_i \left[(v_i - v_{i+1}) + (v_i + v_{i+1})\right] \tag{6.66}$$
$$= -\frac{h}{2}\sum_{i=1}^{n-1} F_h u_i^{(2)} (F_h v_i)^2 + \frac{1}{2}\sum_{i=1}^{n-1} F_h u_i^{(2)} F_h(v_i^2) \geq \frac{1}{2}\sum_{i=1}^{n-1} F_h u_i^{(2)} F_h(v_i^2)$$
$$= \frac{1}{2}\sum_{i=1}^{n} F_h u_{i-1}^{(2)} F_h(v_{i-1}^2) - \frac{1}{2} F_h u_0^{(2)} F_h(v_0^2)$$
$$= -\frac{1}{2}\sum_{i=1}^{n-1} v_i^2 L_h u_i^{(2)} - \frac{1}{2h} v_1^2 F_h u_0^{(2)}$$
$$\geq -\frac{1}{2}\sum_{i=1}^{n-1} v_i^2 L_h u_i^{(2)} \geq -\sum_{i=1}^{n-1} v_i^2 L_h u_i^{(2)};$$

For the fifth term, we have

$$\frac{\kappa}{D}\sum_{i=1}^{n-1}\left(F_h u_i F_h v_i^{(1)} + v_i^{(1)} L_h u_i\right) v_i = \frac{\kappa}{D}\sum_{i=1}^{n-1} v_i F_h\left(v_i^{(1)} u_{i-1}\right) \tag{6.67}$$
$$= -\frac{\kappa}{D}\sum_{i=1}^{n} v_i^{(1)} F_h u_{i-1} F_h v_{i-1} \geq -\sum_{i=1}^{n}\left|\frac{\kappa}{D} F_h u_{i-1}\right| |F_h v_{i-1}|$$
$$\geq -\frac{\kappa^2}{4D^2}\sum_{i=1}^{n}(F_h u_{i-1})^2 - \sum_{i=1}^{n}(F_h v_{i-1})^2$$
$$\geq -\frac{\kappa^2 a^2}{16 D^2 \lambda}\sum_{i=1}^{n-1} v_i^2 - \sum_{i=1}^{n}(F_h v_{i-1})^2;$$

As for the two terms on the right-hand side of (6.65), we have

$$\frac{b}{D}\sum_{i=1}^{n-1}\left|(1-v_i^{(1)}-v_i^{(2)})G(u_i^{(2)})v_i^2\right| \le \frac{b}{D}(1-c^*)\sum_{i=1}^{n-1}v_i^2, \tag{6.68}$$

and

$$\begin{aligned} \frac{b}{D}\sum_{i=1}^{n-1}\left|(1-v_i^{(1)})v_i^{(1)}(G(u_i^{(2)}-G(u_i^{(1)}))v_i\right| &\le \frac{b}{4D}\sum_{i=1}^{n-1}|u_iv_i| \\ &\le \frac{ab}{4D\lambda}\sum_{i=1}^{n-1}v_i^2, \end{aligned} \tag{6.69}$$

where a Lipschitz condition on the function $G(\cdot)$ and (6.64) have been used. After substituting (6.66), (6.67), and (6.68), and (6.69) into (6.65), we obtain

$$\frac{1}{D}\left[\beta - b(1-c^*) - \frac{\kappa^2a^2}{16D\lambda} - \frac{ab}{4\lambda}\right]\sum_{i=1}^{n-1}v_i^2 \le 0 \tag{6.70}$$

which implies that $v_i = 0$, and then $u_i = 0$ in view of (6.2). Therefore, the solution is unique. □

Remark 6.15. The uniqueness result applies only to *type I* solution, just as in the continuous case.

6.3.5 Convergence

Convergence is guaranteed by the uniqueness of exact solution and the uniform boundness of the numerical solutions and their first order differences.

Theorem 6.10. *Under condition* (6.2)*:* $\beta > b(1-c^*) + \frac{\kappa^2a^2}{16D\lambda} + \frac{ab}{4\lambda}$, *the numerical soluton of* (6.39) *converges to the exact solution of* (6.1) *with an error of* $O(h)$.

Proof. We deal with convergence and rate of convergence separately in our proof.

Convergence. As before, we let $u(n) = \{u_i\}_{i=0}^n$ and $v(n) = \{v_i\}_{i=0}^n$, where we use n to indicate the dependence of the numerical solution (u, v) on the partition number n. We know from Lemma 6.10 that (u, v) and $(F_h u, F_h v)$ are uniformly bounded. Hence the set $\{(u(n), v(n))\}_{n=1}^{\infty} \subset C[0,1]$ is bounded and equi-continuous and thus relatively compact. Therefore there exists a subsequence $\{(u(n_k), v(n_k))\}_{k=1}^{\infty}$ that converges to an element $(u(x), v(x))$ in $C[0,1]$, which is the unique solution of the original system of equations. We claim that the sequence $\{(u(n), v(n))\}_{n=1}^{\infty}$ itself converges to $(u(x), v(x))$, because otherwise we would conclude that there exists an $\epsilon > 0$ such that for any k there is an $n_k > k$ with the property $\|(u(n_k), v(n_k) - (u(x), v(x))\| \geq \epsilon$. But $\{(u(n_k), v(n_k))\}_{k=1}^{\infty} \subset \{(u(n), v(n))\}_{n=1}^{\infty}$ is itself relatively compact. Following the argument above, we know a limit point $(\tilde{u}(x), \tilde{v}(x))$ exists for this subsequence, and it is also a solution of the original system. Since $\|(\tilde{u}(x), \tilde{v}(x)) - (u(x), v(x))\| \geq \epsilon$, this contradicts the uniqueness of solution.

Rate of Convergence. We need to rewrite the system of differential equations into a discrete form. From the first equation we solve u_{xx} to get

$$u_{xx} = \lambda u + \frac{av}{\gamma + u} u.$$

We substitute it into the second equation. Then, in view of the following approximations:

$$u_{xx}(x_i) = L_h u(x_i) + R_1, \; v_{xx}(x_i) = L_h v(x_i) + R_2,$$

$$u_x(x_i) = F_h u(x_i) + R_3, \; v_x(x_i) = F_h v(x_i) + R_4;$$

and the following notation:

$$R_5 \equiv F_h v(x_i) R_3 + F_h u(x_i) R_4 + R_3 R_4,$$

system (6.1) becomes

$$(6.71) \qquad \begin{cases} -L_h u + \lambda u = -\frac{av}{\gamma+u} u + R_1, \\ -D L_h v + \kappa(F_h u F_h v + v L_h u) + \beta v + \kappa(\lambda u v + \frac{auv^2}{\gamma+u}) \\ = b(1-v) v G(u) + D R_2 - \kappa R_5, \\ u(x_0) = 1, u(x_n) = 0; v(x_0) = 0, v(x_n) = 1, \end{cases}$$

where u and v and all their first and second order differences are evaluated at $x = x_i,\ i = 1, ..., n-1$. Now we take the difference of (6.71) with (6.39). By letting

$$e_i^{(1)} \equiv u(x_i) - u_i,\ e_i^{(2)} \equiv v(x_i) - v_i,$$
$$p_i \equiv \lambda + \frac{a\gamma v(x_i)}{(\gamma + u(x_i))(\gamma + u_i)},\ q_i \equiv \frac{au_i}{\gamma + u_i},$$
$$r_i \equiv \beta + \kappa L_h u(x_i) + \frac{\kappa a v_i u_i}{\gamma + u_i} - b(1 - v(x_i) - v_i)G(u_i),$$
$$T(e_i^{(1)}) \equiv \left(b(1 - v(x_i))v(x_i)\frac{G(u(x_i)) - G(u_i)}{u(x_i) - u_i} - \kappa v_i p_i \right) e_i^{(1)} - \kappa F_h v_i F_h e_i^{(1)}$$

and rearranging terms, we obtain

$$(6.72)\qquad \begin{cases} -L_h e_i^{(1)} + p_i e_i^{(1)} = -q_i e_i^{(2)} + R_1, \\ -DL_h e_i^{(2)} + \kappa F_h u(x_i) F_h e_i^{(2)} + r_i e_i^{(2)} \\ = T(e_i^{(1)}) + \kappa v_i R_1 + DR_2 - \kappa R_5, \\ e_0^{(1)} = 0, e_n^{(1)} = 0; e_0^{(2)} = 0, e_n^{(2)} = 0. \end{cases}$$

Noticing that $\kappa F_h u(x_i) \leq 0$ and that condition (6.2) implies $r_i \geq 0$, we apply Lemma 6.8 to the second equation in (6.72) to obtain

$$(6.73)\qquad \|e^{(2)}\| \leq \Psi(\|T(e^{(1)})\| + \kappa|R_1| + D|R_2| + \kappa|R_5|)$$

(Note that the coefficient D before the term $L_h e_i^{(2)}$ does not affect the estimate). Now we apply Lemma 6.6 to the first equation in (6.72) to get

$$(6.74)\qquad \|e^{(1)}\| \leq \Phi\|q\|\|e^{(2)}\| + \Phi|R_1|,$$

$$(6.75)\qquad \|F_h e^{(1)}\| \leq (1 + \Phi\|p\|)\|q\|\|e^{(2)}\| + (1 + \Phi\|p\|)|R_1|.$$

With the help of (6.74), (6.75), we estimate $\|T(e^{(1)})\|$ as follows:

$$(6.76)\qquad \begin{aligned} \|T(e^{(1)})\| &\leq \left(\frac{b}{4} + \kappa\|p\|\right)\|e^{(1)}\| + \kappa\|F_h v\|\|F_h e^{(1)}\| \\ &\leq \left[\Phi\left(\frac{b}{4} + \kappa\|p\|\right) + \kappa\eta(1 + \Phi\|p\|)\right]\|q\|\|e^{(2)}\| \\ &+ \left[\Phi\left(\frac{b}{4} + \kappa\|p\|\right) + \kappa\eta(1 + \Phi\|p\|)\right]|R_1|. \end{aligned}$$

Note that

$$\|p\| \le \lambda + \frac{a}{\gamma}, \ \|q\| \le \frac{a}{\gamma+1}.$$

We use these in the above estimate to obtain

$$\|T(e^{(1)})\| \le \Delta\|e^{(2)}\| + \Theta|R_1|, \tag{6.77}$$

where

$$\Theta \equiv \left[\Phi\left(\frac{b}{4}+\kappa\left(\lambda+\frac{a}{\gamma}\right)\right)+\kappa\eta\left((1+\Phi\left(\lambda+\frac{a}{\gamma}\right)\right)\right], \tag{6.78}$$

and

$$\Delta \equiv \frac{a\Theta}{\gamma+1}. \tag{6.79}$$

So (6.73) becomes

$$\|e^{(2)}\| \le \Delta\Psi\|e^{(2)}\| + \Psi\left((\Theta+\kappa)|R_1|. + |D|R_2| + \kappa|R_5|\right). \tag{6.80}$$

Now it is clear that under condition

$$1 - \Delta\Psi > 0, \tag{6.81}$$

we have

$$\|e^{(2)}\| \le \frac{\Psi}{1-\Delta\Psi}\left((\Theta+\kappa)|R_1| + |D|R_2| + \kappa|R_5|\right). \tag{6.82}$$

Using this estimate in (6.74), we obtain

$$\|e^{(1)}\| \le \frac{a\Phi\Psi}{(\gamma+1)(1-\Delta\Psi)}\left((\Theta+\kappa)|R_1| + |D|R_2| + \kappa|R_5|\right) + \Phi|R_1|. \tag{6.83}$$

Since the truncation error R_1 is of second order, while R_2 and R_5 are of first, we know our difference scheme converges with a rate of $O(h) = O(\frac{1}{n})$ when $n \to \infty$. □

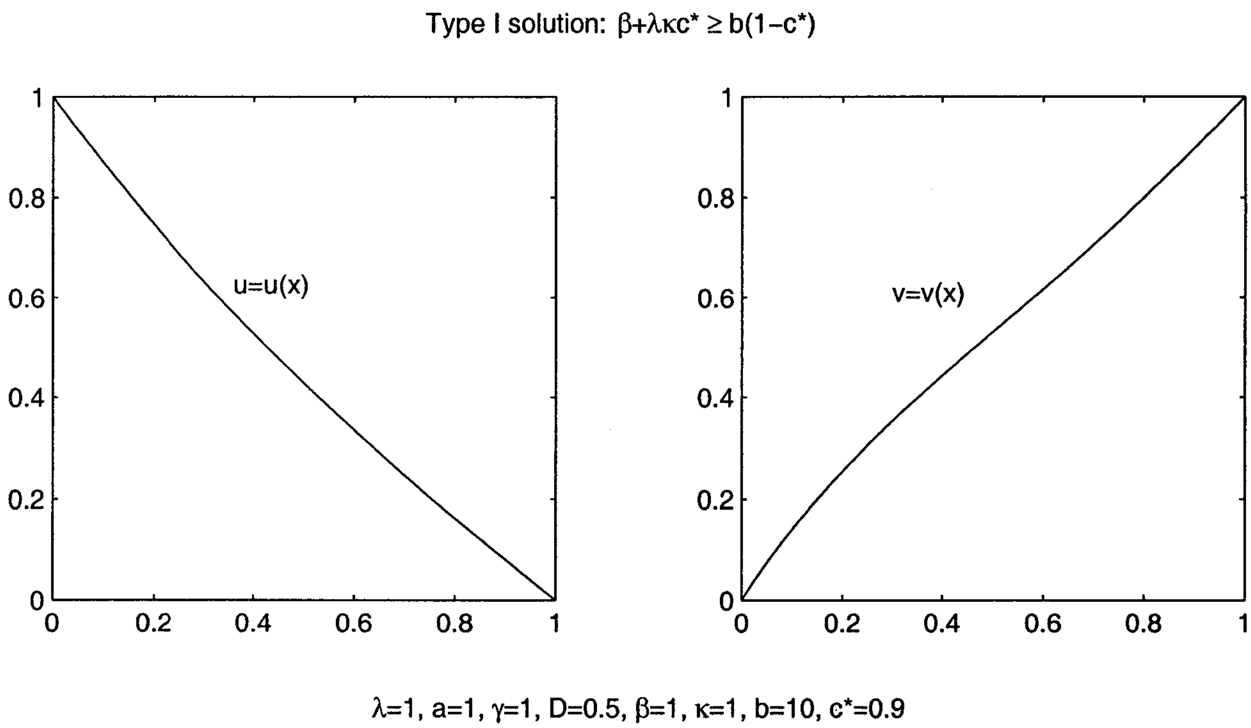
Remark 6.16. It is easily seen that Δ is decreasing as a function of a in $[0, \infty]$, and $\lim_{a\to 0^+} \Delta = 0$. This shows that condition (6.81) can be satisfied by parameters with relatively small a. It should be emphasized that this condition is technical only and not required by the convergence of the difference scheme at all, as we showed above. Note that condition (6.81) can also be satisfied by parameters with large γ as well as by other choices.

Remark 6.17. We point out that we can not improve the order of accuracy from $O(h)$ to $O(h^2)$ by simply adopting center difference scheme for the first order derivatives. This is because $G(u)$, and hence v_{xx}, is a continuous but not differentiable function of $x \in (0,1)$. We expect this difficulty to be solved by using the finite element method.

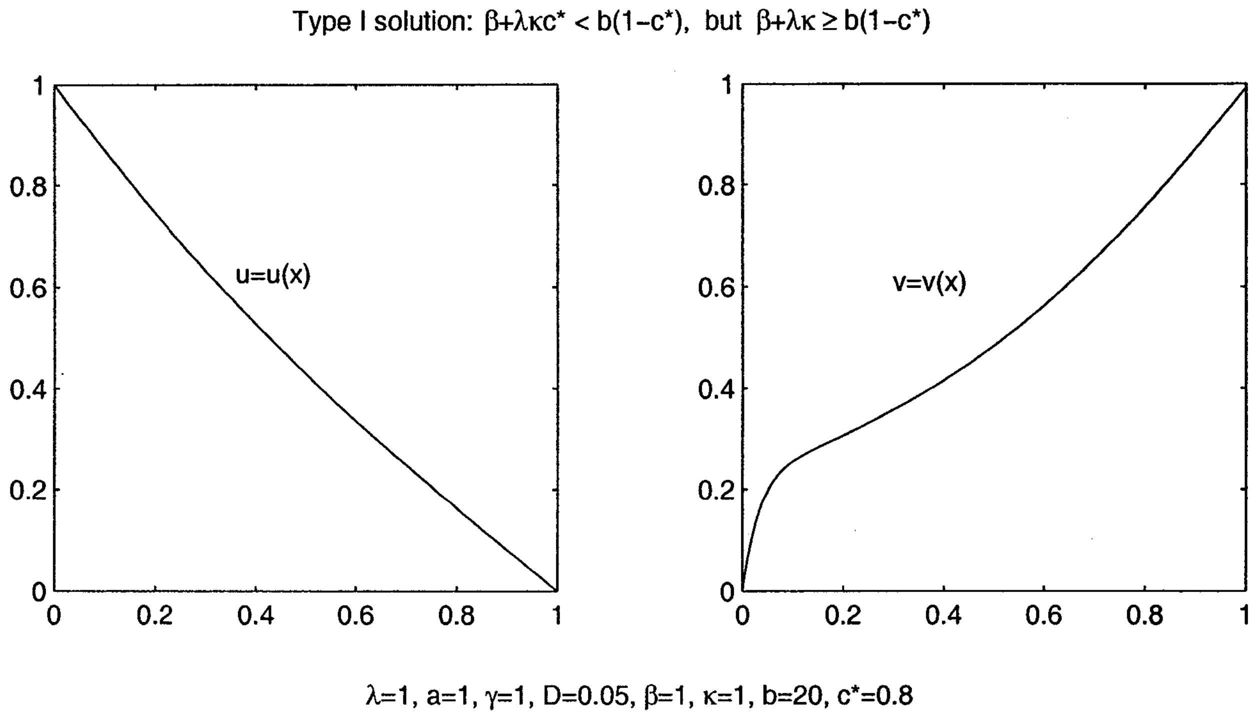
6.3.6 Numerical Simulations

To verify the condition on *type I* or *type II* solution, we use three sets of parameters in our numerical realizations:

(a) Condition (6.3), $\beta + \lambda\kappa c^* \geq b(1 - c^*)$, is satisfied. The solution is of *type I* by Theorem 6.4.

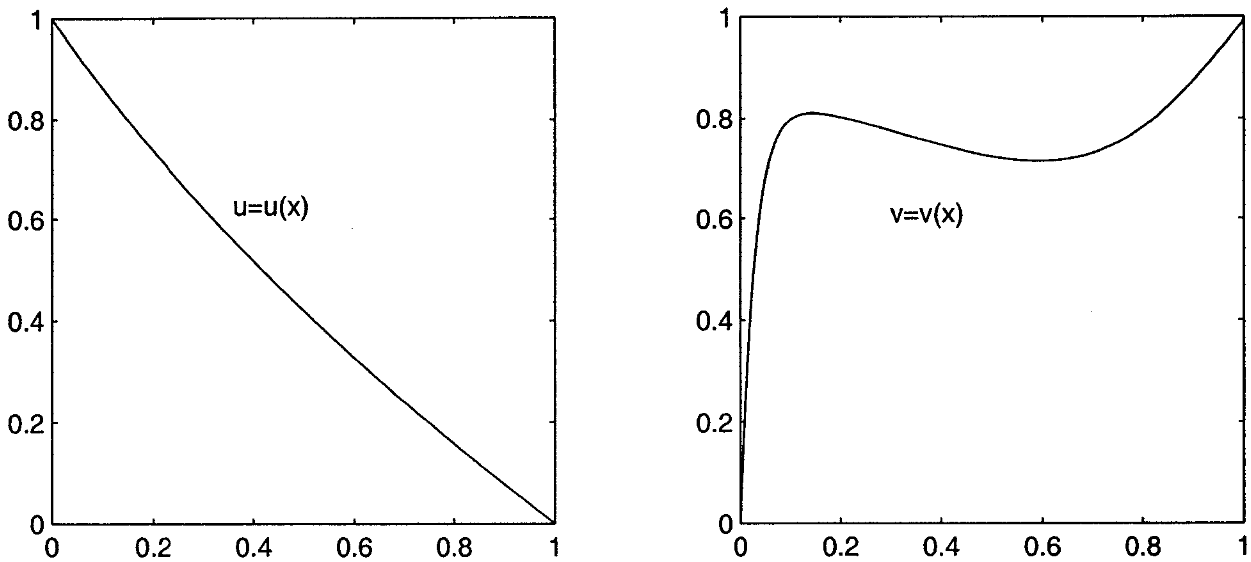


(b) Condition (6.3) is not satisfied, but condition (6.5), $\beta + \lambda\kappa \geq b(1 - c^*)$, is. The solution is still of *type I* by Theorem 6.5.



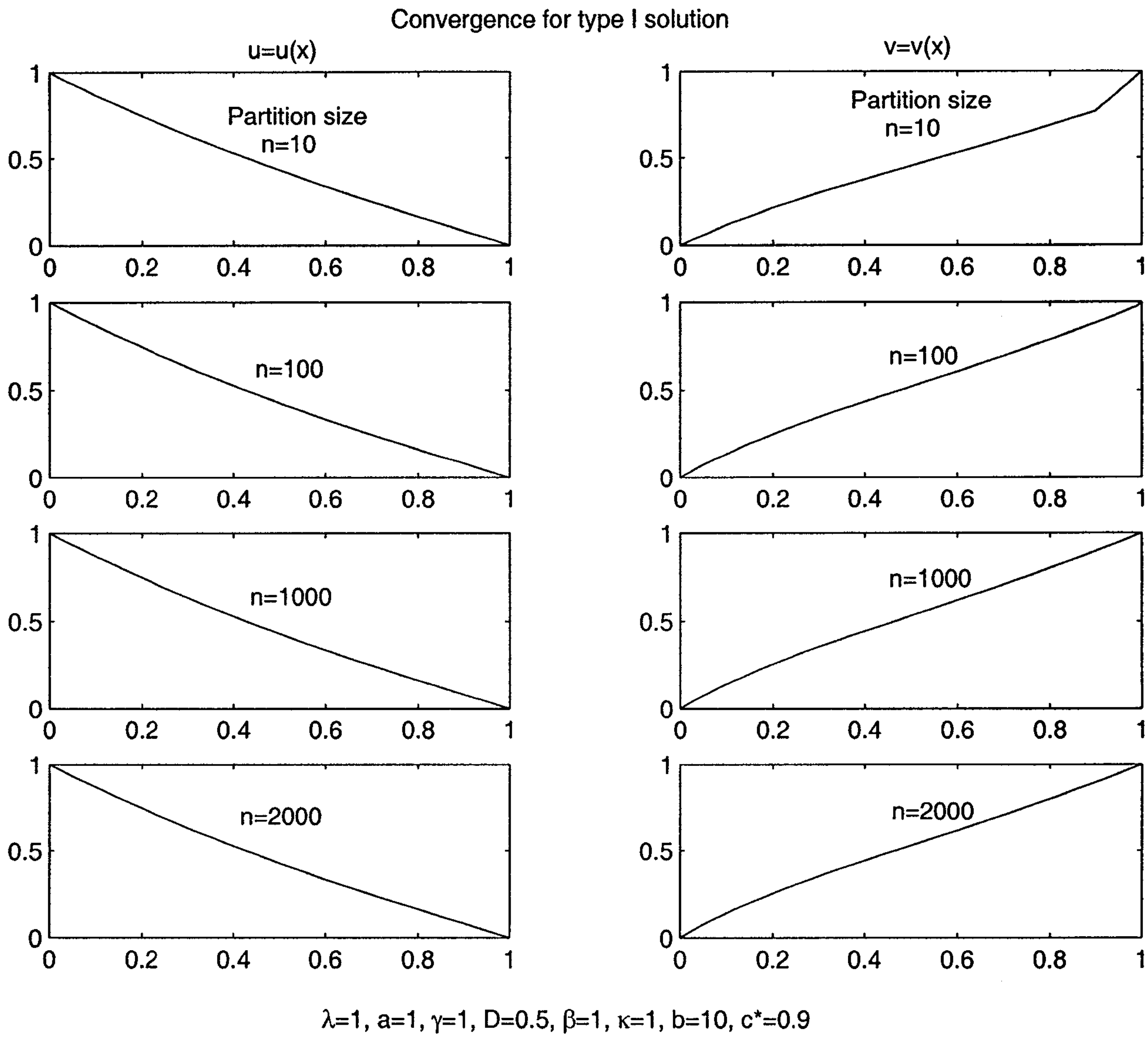
(c) Condition (6.4), $\exp\left(\frac{\kappa}{D}\right) < (1 - D_1)\left(1 + \frac{\beta c_2^4}{2D}\right)$, in Theorem 6.4 is not satisfied, but the graph gives *type II* solution, indicating that this sufficient condition is not a necessary one.

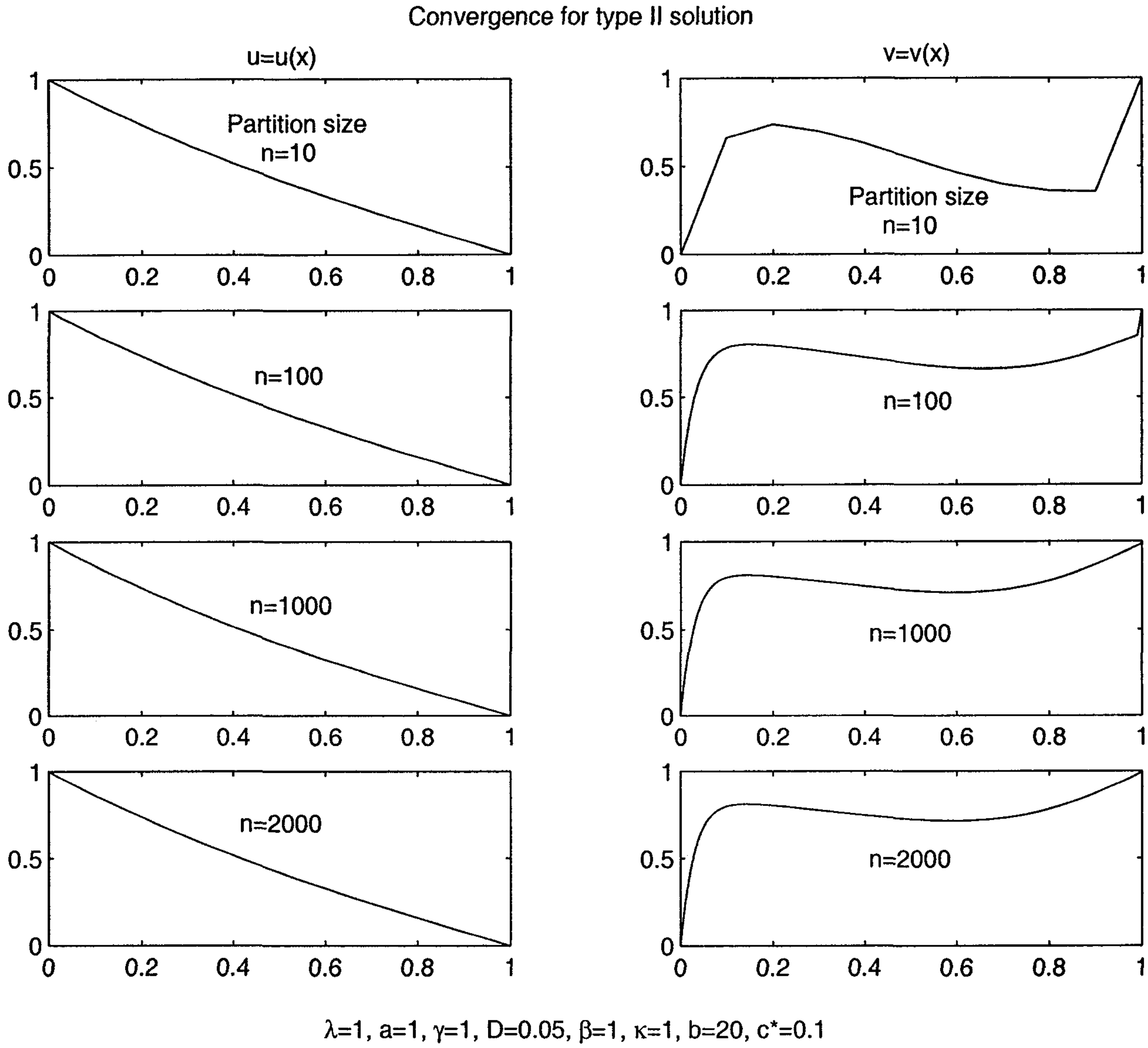
Type II solution: $\exp(\kappa/D) \geq (1-D_1)\,(1+0.5\beta c_2^4/D)$



λ=1, a=1, γ=1, D=0.05, β=1, κ=1, b=20, c*=0.1

To test the convergence of difference scheme (6.39), a typical combination of parameters for each of *type I* and *type II* solutions is considered. In both cases convergence is obvious from the graphs. Also, our calculation shows that the above method actually performs with essentially a rate of $O(h)$, matching our error estimates.


Convergence for type I solution
u=u(x)
v=v(x)
Partition size
n=10
n=100
n=1000
n=2000
0
0.2
0.4
0.6
0.8
1
0.5
λ=1, a=1, γ=1, D=0.5, β=1, κ=1, b=10, c*=0.9



All the numerical solutions are obtained using the iterative procedure (6.49), which typically converges in between 4 to 40 steps when the tolerance is set to be $1.0E - 8$.

6.4 Further Comments

In this chapter a finite difference numerical analysis has been studied for a chemotaxis system. For first order derivatives the Euler forward is chosen so that the maximum principle will be applicable. Problems from existence and uniqueness of numerical solutions to gradient estimate and convergence have been addressed. Particularly, we have seen that convergence is the result of uniqueness of the exact solution and equi-continuity of the numerical ones.

There still leaves some work to be done on this model.

In theoretical aspect:

- Uniqueness result for *type II* solution

Note that the convergence established so far is only for *type I* solution because of the lack of such a result.

- Condition on the parameters to ensure *type II* solution

Though I believe it is too restrictive, I have not succeeded in improving it yet.

In numerical aspect:

- Finite element (or finite volume element) analysis of the CS

The key is to set up the maximum principle. With the basic framework done in this chapter, progress is being made in this direction toward success. We will report this later.

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