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The development of decidability proofs based on sequent calculi

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ABSTRACT Some logics are decidable, that is, there is an algorithm to determine whether a formula is provable in a calculus formalizing the logic. This paper aims to depict the evolution of decidability proofs based on sequent calculi from the first such proof for intuitionist logic to some of the latest results that use—in an essential way—the *Curry–Kripke technique*.

KEYWORDS Curry's lemma, decidability, intuitionist logic, Kripke's lemma, modal logic, relevance logic, sequent calculus

Introduction

A desire to know if a sentence is true or not may be the original motivation for human inquiry. Famously, Hilbert, in his Paris lecture, in 1900, claimed that there are no unknowable mathematical theorems. Of course, the tools to gain knowledge, in general, or to prove mathematical claims, in particular, has to be narrowed down to make sense of claims concerning the possibility or impossibility of attaining knowledge.

Decidability is a much more specific problem than the informal idea of somehow establishing or refuting mathematical claims. First of all, a fixed language and a formal theory is assumed. Furthermore, the question is simplified to whether a given formula is provable from the formal theory. Lastly, a solution cannot use arbitrary means; it should involve an effective procedure. Notably, the first formal notions of computability were not introduced until the 1920s.

Our focus in this paper is on formal theories that are propositional logics, some of which are sufficiently complex to be undecidable (e.g., the major relevance logics, T (ticket entailment), E (entailment) and R (relevant implication)). Quantification, as a rule, leads to undecidability; indeed, it is well known that even 2-valued predicate logic with predicates of arbitrary finite arity is undecidable. This is so despite the fact that 2-valued propositional logic is decidable, and 2-valued monadic predicate logic is also decidable. For 2-valued (or, in general, a finitely valued) logic, a quick decidability argument can appeal to truth tables, which can be effectively constructed for any n ($n \in \mathbb{N}$). Then it is trivial to scan the final column for all T's (or distinguished values). Of course, this argument relies on the adequacy of the truth assignment interpretation of 2-valued logic or on the claim that a logic is finitely valued. Sometimes, assuming a suitable semantics for a logic, the decidability problem is explicitly phrased as the question if a formula is valid. This may be labeled as the *semantic decidability problem*. There are techniques beyond truth tables (e.g., algebraic methods and filtration) that can be used to answer the semantic version of the question (e.g., for logics that have the finite model property). However, we limit our considerations here to the *syntactic* question. More specifically, we look at decision procedures that use a *sequent calculus*.

The next section presents an outline of Gentzen's original proof of the decidability of propositional intuitionist logic. Section 2 is an application of a similar method by Lambek to his two calculi. Section 3 describes "Curry's turn," which literally, changes the direction of the proof search (in relation to a sequent calculus proof). In Section 4, we briefly recall Kleene's $G3$ approach together with his influential concept of cognate sequents. In Section 5, we outline how Kripke solved, in three short summer months, the decidability problem for $E_{\rightarrow}$, implicational entailment. Kripke introduced new ideas into Curry's framework, which led to a new group of logics shown to be decidable. Then, we devote Section 6 to ways in which the Curry–Kripke method has been used to obtain further decidability results. Finally, we draw some conclusions in the last section.

1 The decidability of propositional intuitionist logic

Intuitionist logic was formulated by Heyting [32], in 1930. Gödel [31] showed that intuitionist logic is not finitely valued. In other words, there is no hope to obtain a truth-table like semantics for it, and then to check validity through a semantical interpretation.

Gentzen [27, 28] provided new formalizations for intuitionist logic in the form of NJ (a natural deduction calculus) and LJ (a logistic calculus).[1] We are interested in the *propositional* part of LJ here, hence, we will use the label to refer to that logic without the quantifiers; furthermore, we will not consider NJ here at all.

[1] [29, 30] are translations of [27, 28], and they may also be found in [43].

Gentzen's LJ (and LK, his logistic calculus for 2-valued logic) has served as the original blueprint for many later sequent calculi, which motivates us to give a full definition of (propositional) LJ.[2]

Definition 1.1. The language of LJ contains *four connectives*, namely, $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\supset$ (intuitionistic implication), as well as a denumerable set of *propositional variables*, that we denote by $p_0, p_1, \ldots, p_n, \ldots$.

The set of formulas is given as the set of strings that can be generated by the following context-free grammar, where $\mathbf{P}$ rewrites to a propositional variable.

$$A \coloneqq \mathbf{P} \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \supset A)$$

Remark 1.2. There are other connectives that we could introduce, for instance, $\supset\subset$ (co-implication) or $\mathbf{F}$ (falsity constant), which are, indeed, mentioned in [29, p. 289]. However, they are definable using the connectives already introduced and they are not used in LJ.

Definition 1.3. A *sequent* is a pair of finite sequences of formulas $\langle \Gamma, \Theta \rangle$, where Θ contains at most one element. We use the symbol $\Vdash$ to separate the two sequences writing $\Gamma \Vdash \Theta$.

All the sequents in a proof in LJ have to satisfy the size restriction on Θ, hence, we will not keep repeating it. As a notational convention, we use A, Γ to indicate the concatenation of the formula A and the sequence Γ (in that order). Similarly, Γ, A and Γ, Θ indicate concatenations too.

The intended informal interpretation of a provable sequent $\Gamma \Vdash \Theta$ is that the conjunction of the elements of Γ implies θ, where θ is either the only element in Θ or it is $\mathbf{F}$, if Θ is empty.

Definition 1.4. The sequent calculus LJ consists of an *axiom* and *rules*, which are divided into three groups: connective rules, structural rules and the cut rule.

$$A \Vdash A \ \text{(Id)} \qquad \frac{\Gamma \Vdash A}{\neg A, \Gamma \Vdash} \ (\neg\Vdash) \qquad \frac{A, \Gamma \Vdash}{\Gamma \Vdash \neg A} \ (\Vdash\neg)$$

$$\frac{A, \Gamma \Vdash \Theta}{A \wedge B, \Gamma \Vdash \Theta} \ (\wedge_1\Vdash) \qquad \frac{B, \Gamma \Vdash \Theta}{A \wedge B, \Gamma \Vdash \Theta} \ (\wedge_2\Vdash) \qquad \frac{\Gamma \Vdash A \quad \Gamma \Vdash B}{\Gamma \Vdash A \wedge B} \ (\Vdash\wedge)$$

$$\frac{A, \Gamma \Vdash \Theta \quad B, \Gamma \Vdash \Theta}{A \vee B, \Gamma \Vdash \Theta} \ (\vee\Vdash) \qquad \frac{\Gamma \Vdash A}{\Gamma \Vdash A \vee B} \ (\Vdash\vee_1) \qquad \frac{\Gamma \Vdash B}{\Gamma \Vdash A \vee B} \ (\Vdash\vee_2)$$

[2] We will not follow Gentzen's notation as to the use of particular symbols such as &.

$$\frac{\Gamma \Vdash A \quad B, \Lambda \Vdash \Theta}{A \supset B, \Gamma, \Lambda \Vdash \Theta} \;(\supset\Vdash) \qquad \frac{A, \Gamma \Vdash B}{\Gamma \Vdash A \supset B} \;(\Vdash\supset)$$

$$\frac{A, A, \Gamma \Vdash \Theta}{A, \Gamma \Vdash \Theta} \;(W\Vdash) \qquad \frac{\Gamma, A, B, \Lambda \Vdash \Theta}{\Gamma, B, A, \Lambda \Vdash \Theta} \;(C\Vdash)$$

$$\frac{\Gamma \Vdash \Theta}{A, \Gamma \Vdash \Theta} \;(K\Vdash) \qquad \frac{\Gamma \Vdash}{\Gamma \Vdash A} \;(\Vdash K) \qquad \frac{\Gamma \Vdash A \quad A, \Lambda \Vdash \Theta}{\Gamma, \Lambda \Vdash \Theta} \;(\text{cut})$$

A *proof* in LJ is a tree, in which the nodes are occurrences of sequents, specifically, the leaves are occurrences of instances of the axiom, and all other nodes result by applications of rules. The root of the tree is the sequent that is *proved*. (Every sequent in a proof tree is the root of a subtree in the proof tree, and every sequent in a proof tree is a provable sequent.)

Remark 1.5. There are four two-premise rules in LJ, and the $(\Vdash\wedge)$ and $(\vee\Vdash)$ rules differ from the $(\supset\Vdash)$ and the (cut) rules in that the former two require the two premises to be the same save the formulas A and B. The following two rules would do just as well.

$$\frac{\Gamma \Vdash A \quad B, \Gamma \Vdash \Theta}{A \supset B, \Gamma \Vdash \Theta} \;(\supset\Vdash') \qquad \frac{\Gamma \Vdash A \quad A, \Gamma \Vdash \Theta}{\Gamma \Vdash \Theta} \;(\text{cut}')$$

Here is a quick argument for the equivalence of the versions of the rules. If Γ happens to be Λ in the $(\supset\Vdash)$ or the (cut) rule, then the lower sequent will contain Γ, Γ, which may be reduced to Γ by finitely many applications of the $(C\Vdash)$ and $(W\Vdash)$ rules. Conversely, Γ and Λ have a (possibly, empty) common part, thus, Γ and Λ may be depicted—with appeal to $(C\Vdash)$—as Γ', Ξ and Ξ, Λ'. Then both can be beefed up (using $(K\Vdash)$ and $(C\Vdash)$) to Γ', Ξ, Λ'. Then, the primed rules yield lower sequents that are missing a Ξ, which may be inserted by finitely many applications of $(K\Vdash)$ and $(C\Vdash)$.

Remark 1.6. The cut rule is an *admissible rule*, hence, LJ may be formulated without it. In [29], an inductive proof was used to show that the *mix* rule is admissible, which is in turn equivalent to the cut rule. The mix rule has the following form:

$$\frac{\Gamma \Vdash A \quad A, \Lambda \Vdash \Theta}{\Gamma, \Lambda^{-A} \Vdash \Theta} \;(\text{mix}),$$

where Λ^{-A} is the sequence of formulas Λ with *all* the occurrences of A deleted. The proof uses an induction on the number of applications of the mix rule in a proof, as well as a double induction on two parameters that characterize the size of the mix formula and the place of the mix within a proof tree. The lower sequent of the cut ($\Gamma, \Lambda \Vdash \Theta$) can be easily restored from $\Gamma, \Lambda^{-A} \Vdash \Theta$ by applications of $(K \Vdash)$ and $(C \Vdash)$. On the other hand, all the occurrences of A is Λ can be reduced to a single occurrence by (finitely many) uses of $(C \Vdash)$ and $(W \Vdash)$, and then one cut yields Λ^{-A}.

We are interested in the decidability proof for LJ, hence, we will not dwell on the proof of the admissibility of the cut rule. The decidability of LJ was an open problem in 1935, moreover, as we already mentioned, it was known from Gödel's [31] that there was no hope to find "intuitionistic truth tables." We turn to presenting the decidability proof for LJ along the lines of [30]. The flexibility of LJ illustrated by the use of the structural rules in Remarks 1.5 and 1.6 plays a crucial role in the proof.

Definition 1.7. A *reduced* sequent is a sequent in which no formula has more than *three* occurrences in the antecedent.

Obviously, for any sequent with a non-empty antecedent there are at least two reduced sequents, which do not contain occurrences of a formula with no occurrences in the starting sequent, that are equivalent to it. (Equivalence means provability from one another.) We call these sequents *reduced versions.*

Lemma 1.8. *If* $\Gamma \Vdash \Theta$ *is provable in* LJ *and* $\Gamma^{\rho} \Vdash \Theta$ *is a reduced version of* $\Gamma \Vdash \Theta$*, then there is a proof of* $\Gamma^{\rho} \Vdash \Theta$*, in which* all *sequents are* reduced.

Proof. (Sketch) The proof utilizes the ideas mentioned in Remarks 1.5 and 1.6. If $\Gamma \Vdash \Theta$ is provable, then it is provable without applications of the cut rule. Hence, continuing any such proof, if necessary, with applications of $(C \Vdash)$ and $(W \Vdash)$, a cut-free proof of $\Gamma^{\rho} \Vdash \Theta$ can be obtained.

Then it remains to show that if there is a sequent in the latter proof that is not reduced, then the proof can be transformed into one that has only reduced sequents. Instances of the axiom are reduced sequents, hence, any sequent that does not meet the condition in Definition 1.7, must have resulted by an application of a rule. As a critical case, we consider the $(\supset \Vdash)$ rule, which is the most capable rule to produce multiple occurrences of a formula in the lower sequent. Here is a chunk of a proof that we will consider.

$$\dfrac{\begin{matrix}\vdots \\ \Gamma^{-(A \supset B)}, (A \supset B)^m \Vdash A\end{matrix} \qquad \begin{matrix}\vdots \\ B, (A \supset B)^n, \Lambda^{-(A \supset B)} \Vdash \Theta\end{matrix}}{\Gamma^{-(A \supset B)}, (A \supset B)^{m+n+1}, \Lambda^{-(A \supset B)} \Vdash \Theta} \ (\supset \Vdash)$$

With the superscripts, we made explicit the number of occurrences of $A \supset B$. (Since $(C \Vdash)$ is a rule, we may assume that all the $A \supset B$'s have been shepherded together.)

The lowest value for $m + n + 1$ is 1, when $m = n = 0$. If $m \geq 1$ or $n \geq 1$, then applications of $(W \Vdash)$ can reduce those numbers to 1, and of course, $1 + 1 + 1 = 3$, the number in Definition 1.7. It is easy to see that the insertion of sufficiently many $(W \Vdash)$ and $(C \Vdash)$ steps will create a proof from a proof. Applications of other rules may be dealt with similarly.

The number of occurrences of a formula can be always increased by applications of the rule $(K \Vdash)$. However, we should note that no connective rule requires more that one occurrence of any formula. It is not so for the structural rules. But if the occurrences of A have already been reduced to 1, then a subsequent application of the $(W \Vdash)$ rule may be simply omitted. Similarly, if $(C \Vdash)$ was applied to $\Gamma, A, A, \Lambda \Vdash \Theta$, then that application could have been omitted in the first place, and if there is only one A left, then $(C \Vdash)$ definitely should be omitted. In other words, applications of rules remain applications of the same rule when the upper sequents are replaced by their reduced versions, or if the application of a rule became superfluous, then it is omitted altogether. □

Remark 1.9. A way to think about proofs that comprise reduced sequents throughout is that there is no need to make detours in a proof, not only via applications of the cut rule, but via accumulating many copies of one and the same formula either. Of course, it is not true that all provable sequents have cut-free proofs comprising reduced sequents only—simply, because non-reduced sequents are provable too. However, it if trivial to prove $\Gamma \Vdash \Theta$ from $\Gamma^\rho \Vdash \Theta$.

A glance at the rules of LJ helps to establish that a proof without cut has the subformula property.

Definition 1.10. A proof has the *subformula property*, when every formula in any sequent is either a formula or a subformula of a formula in the sequent proved.

Formulas can be viewed as *types*, so to speak, rather that *tokens* or occurrences. For example, an application of $(W \Vdash)$ discards a token (an occurrence), but not a type (a formula).

Definition 1.11. Let $\Gamma \Vdash \Theta$ be a given sequent, and let $\Gamma^\rho \Vdash \Theta$ be a reduced version of it. There are finitely many subformulas in $\Gamma^\rho \Vdash \Theta$, the set of which we denote by Σ. There are finitely many instances of (Id) that can be constructed from the elements of Σ, we denote the set of these sequents by Π. The decision procedure is given as the following three steps.

1. The elements of Π can be used as upper sequents in rules. We generate all the possible lower sequents that satisfy the conditions (1) all the formulas in the sequents are elements of Σ, and (2) the sequents are reduced.

2. We update Π with the newly obtained sequents, and return to step 1. If no new sequents were generated, then we proceed to step 3.

3. We check whether $\Gamma^{\rho} \Vdash \Theta$ is or is not an element of Π, accordingly, $\Gamma \Vdash \Theta$ is or is not provable.

Remark 1.12. We stress that this decision procedure is essentially Gentzen's. It is a *top-down* procedure in the sense that it starts with instances of (Id). An inessential difference from the original in [30] is that we limited the generation of the lower sequents by (1) and (2) instead of outright generating all the reduced sequents from Σ, and then moving sequents into Π. Perhaps, in the spirit of intuitionist logic, it is "more constructive" to generate a sequent when it is (known to be) provable.

The decidability proof for LJ could be adapted to obtain a decidability proof for the propositional part of LK. A key modification would be to define reduced sequents to limit the number of occurrences of a formula in the succedent to *two*. Indeed, Gentzen defined reduced sequents, but he permitted three occurrences for any formula in the succedent. If we glance at the rules for LK, then we see that no rule behaves with respect to the succedent as $(\supset \Vdash)$ does with respect to the antecedent. Namely, $(\supset \Vdash)$ combines Γ, Λ and $A \supset B$. We may conclude that Gentzen's three is not optimal, and at the same time we may note that any natural number larger than three in place of three would do just as well. Although the decidability of propositional LK is not a new result, it hints at other potential applications.

2 The decidability of Lambek's calculi

Joachim Lambek published two influential papers [37] and [38] in which he introduced what afterward became to be known as the *associative Lambek calculus* and the *non-associative Lambek calculus*. He can be seen to continue a long tradition, which goes back (at least) to Frege, where sentences are decomposed into a function and its arguments. Semantic and syntactic types had been investigated by Ajdukiewicz [2], Church [19] and Curry [21] before Lambek's work, however, it seems that previously nobody considered using a sequent calculus to specify derivations of compound types.

We will denote the non-associative Lambek calculus by LN and the associative one by LG. Our definitions will closely resemble Lambek's original definitions.[3]

[3]In [14], two calculi, which are labeled LQ and LA, are essentially Lambek's calculi (LN and LG here)—except that the left-hand side of the turnstile was not permitted to be empty, which harmonized well with some sequent calculi there. Lambek's own label for one of his sequent calculi

Definition 2.1. The language of *LN* and *LG* comprises *three binary connectives* $\cdot$, $/$ and $\backslash$ together with denumerably many *propositional variables.*

The set of formulas is given as the set of strings generated by the following CFG, where **P** may be rewritten as any of the propositional variables.

$$A \coloneqq \mathbf{P} \mid (A \cdot A) \mid (A \mathbin{/} A) \mid (A \mathbin{\backslash} A)$$

Definition 2.2. The set of *ropes* is inductively defined by (1)–(3).

(1) The empty rope is a rope;

(2) if A is a formula, then A is a rope;

(3) if Γ and Θ are ropes, then (Γ, Θ) is a rope.

Remark 2.3. We will assume that the empty rope, which may be denoted by a space, behaves as follows: $(\Gamma, \ ) = (\ , \Gamma)$ and $(\ , \Gamma) = \Gamma$. Also, if Θ is a rope that occurs in Γ, then by $\Gamma[\Theta]$ we denote the rope Γ, in which one particular occurrence of Θ has been chosen. If Θ is replaced by another rope, let us say, Λ, then $\Gamma[\Lambda]$ will be the shorthand for the usual $\Gamma[\Theta/\Lambda]$ within the context of a rule.

The way we think of ropes is that they are like *strings* (or finite sequences) but bulkier, due to the presence of parentheses. Thus, if we omit all the parentheses that indicate groupings in a rope we get a string.

Definition 2.4. A *sequent* is a pair $\langle \Gamma, A \rangle$ where Γ is a rope and A is a formula. As before, we use the notation $\Gamma \Vdash A$ for this pair.

Definition 2.5. The sequent calculus *LN* consists of an *axiom* and seven *rules.*

$$A \Vdash A \quad (\mathrm{Id})$$

$$\frac{\Gamma \Vdash A \quad \Theta[B] \Vdash C}{\Theta[B \mathbin{/} A, \Gamma] \Vdash C} \ ({/}\Vdash) \qquad \frac{\Gamma, A \Vdash B}{\Gamma \Vdash B \mathbin{/} A} \ (\Vdash{/})$$

$$\frac{\Gamma \Vdash A \quad \Theta[B] \Vdash C}{\Theta[\Gamma, A \mathbin{\backslash} B] \Vdash C} \ ({\backslash}\Vdash) \qquad \frac{A, \Gamma \Vdash B}{\Gamma \Vdash A \mathbin{\backslash} B} \ (\Vdash{\backslash})$$

$$\frac{\Gamma[A, B] \Vdash C}{\Gamma[A \cdot B] \Vdash C} \ ({\cdot}\Vdash) \qquad \frac{\Gamma \Vdash A \quad \Theta \Vdash B}{\Gamma, \Theta \Vdash A \cdot B} \ (\Vdash{\cdot})$$

$$\frac{\Gamma \Vdash A \quad \Theta[A] \Vdash B}{\Theta[\Gamma] \Vdash B} \ (\mathrm{cut})$$

is Σ_G, while the other has no label. We use here the labels *LN* and *LG* in order to retain some "*L*-labels" introduced by others, e.g., Curry's *LA*.

Definition 2.6. The sequent calculus *LG* is the result of replacing ropes with *sequences* (or strings) of formulas in the antecedent of a sequent.

Remark 2.7. It seems that the introduction of grouping into the antecedent of a sequent is original with Lambek. This idea proved extremely useful later on. Lambek's motivations seem to have come from algebra (where non-associative binary operations are a common place), and from linguistics (where certain groupings of words are preferred over others in the grammatical analysis of sentences).

We illustrate the latter by an example from [38, p. 158]. The English sentence "John likes fresh milk" in phrase structure grammar is viewed as "John (likes (fresh milk))" rather than the four other possible ways to parse the sentence (without changing the order of the words). (E.g., "(John (likes fresh)) milk" is pretty weird.)

The main influential works dealing with sequent calculi in the 1950s, especially, [22] and [33], focused on 2-valued and intuitionist logics. In that context, it is not necessary, indeed, it would be a nuisance to distinguish sequences of formulas with different groupings. Even the order and the (positive) number of occurrences is more than what is needed in a sequent calculus capturing those logics. Lambek took the bold steps of omitting all the structural rules, of using Ketonen's rule for $(\cdot \Vdash)$ and of discarding the assumption that the antecedent of a sequent is a string of formulas. As a result of these changes, there are two residuals to $\cdot$, which is a well-known fact from algebra. Accordingly, Lambek introduced two versions of Gentzen's $(\supset \Vdash)$ rule—with situating the formulas (potentially) affected inside of a sequent. We also have to point out that the $(\Vdash \cdot)$ rule is the appropriate generalization of the $(\Vdash \wedge)$ rule when there is no $(K \Vdash)$ rule in the calculus.

Theorem 2.8. *The cut rule is* admissible *in LN and in LG without the cut rule.*

Proof. (Idea.) Lambek stated this theorem and outlined the proof in his [37] and [38]. His proof is *not* a direct adaptation of Gentzen's proof; mix could not be shown to be equivalent to the cut rule. Nor his proof goes along the lines of Curry's in [22]. The total absence of structural rules implies that every application of every rule (save cut) introduces a new occurrence of a connective into the sequent. Then, the whole (cut-free) proof above a cut can be characterized in terms of the number of occurrences of connectives in a sequent. Lambek calls the sum of the number of occurrences of connectives in the premises of the cut rule the *degree of the cut*.[4]

The induction is on the degree of the cut, which is reduced through modifications of the proof. The latter are straightforward, because there are no structural rules.

[4]This usage differs from usage by others in the literature, where the degree of the cut (or more precisely, of the cut formula) is a number that characterizes the *complexity of the cut formula* per se in terms of its logical components.

Then another induction on the number of cuts in a proof completes the demonstration of the cut theorem. □

Theorem 2.9. *LG and LN are* decidable*.*

Proof. (Idea.) Lambek sketched the proof for his calculi, and in both cases he outlined a proof search starting from the bottom. By the cut theorem, it is sufficient to look for cut-free proofs. Then, each step in a proof is an application of a connective rule, hence, the search for a proof can be seen as a decomposition of the given sequent in every possible way. There are finitely many ways to decompose a sequent. Thus, either a proof is found or there is no proof.[5] □

Remark 2.10. To complete our quick overview of Lambek's LG and LN, we wish to emphasize that the careful selection of the rules is the key to the cut theorem being provable. In the case of these calculi, it is true (what is often a misunderstanding for other sequent calculi) that the admissibility of the cut rule delivers decidability.

Gentzen's LK and LJ, on one hand, and Lambek's LG and LN, on the other, represent two extremes: *all* structural rules and *no* structural rules are available. However, there are interesting logics that are situated in the middle, so to speak. And to prove their decidability—if they are decidable—requires further new ideas. Next we look at how Curry turned proof search around.

3 Curry's lemma

The concept of a formal system was of primary interest in the 1940s. Curry gave a series of lectures in 1949, which became the basis for his [22]. The systems that are considered by him range from sequent calculi for the positive fragments of 2-valued logic and of intuitionist logic to modal systems through type-assignment systems.

Curry introduced two sequent calculi LA and LC, which formalized the negation-free fragments of intuitionist and 2-valued logic, respectively.[6] We focus on these calculi and their variants. The language and the formulas are as in Definition 1.1 except that $\neg$ is omitted.

[5] Lambek [38, p. 155, p. 165] suggests that his proof "follows" Gentzen's, which is, of course, true in the sense that finding a cut-free proof in a sequent calculus is the target. However, the direction of the proof search is similar to Curry's (whose [22] Lambek referenced). Describing a top-down proof search, which would more closely resemble Gentzen's procedure in its direction would be unproblematic.

[6] We present LA and LC as sequent calculi without stressing their epitheoretic part, and we do not follow Curry's terminology everywhere.

Definition 3.1. A *sequent* in LA is $\langle \Gamma, A \rangle$, where Γ is a (finite) sequence of formulas, and A is a formula. A *sequent* in LC is $\langle \Gamma, \Theta \rangle$, where Γ and Θ are (finite) sequences of formulas.

The difference in the definition of a sequent for LJ and LA is explained by the lack of a rule (namely, of the $(\neg \Vdash)$ rule) that could empty the succedent of a sequent. The single succedent restriction on sequents applies to all sequents in LA—even if we do not repeat it every time.

Definition 3.2. The *axioms* and *rules* for LC (for LA) are the following. (We adopt Lambek's notation to indicate an occurrence of a formula. Θ' is *contained* in Θ (in rule $(\supset \Vdash)$) in the sense that every occurrence of every formula in Θ' is in Θ with no requirement that their order be preserved.)

$$\Gamma[A] \Vdash \Theta[A] \quad (\text{Id})$$

$$\frac{\Gamma, A \Vdash \Theta}{\Gamma, A \wedge B \Vdash \Theta}\ (\wedge_1 \Vdash) \qquad \frac{\Gamma, B \Vdash \Theta}{\Gamma, A \wedge B \Vdash \Theta}\ (\wedge_2 \Vdash) \qquad \frac{\Gamma \Vdash A, \Theta \quad \Gamma \Vdash B, \Theta}{\Gamma \Vdash A \wedge B, \Theta}\ (\Vdash \wedge)$$

$$\frac{\Gamma, A \Vdash \Theta \quad \Gamma, B \Vdash \Theta}{\Gamma, A \vee B \Vdash \Theta}\ (\vee \Vdash) \qquad \frac{\Gamma \Vdash A, \Theta}{\Gamma \Vdash A \vee B, \Theta}\ (\Vdash \vee_1) \qquad \frac{\Gamma \Vdash B, \Theta}{\Gamma \Vdash A \vee B, \Theta}\ (\Vdash \vee_2)$$

$$\frac{\Gamma \Vdash A, \Theta' \quad \Gamma, B \Vdash \Theta}{\Gamma, A \supset B \Vdash \Theta}\ (\supset \Vdash) \qquad \frac{\Gamma, A \Vdash B, \Theta}{\Gamma \Vdash A \supset B, \Theta}\ (\Vdash \supset)$$

$$\frac{\Gamma \Vdash \Theta}{\Xi \Vdash \Theta}\ (C \Vdash) \qquad \frac{\Theta \Vdash \Gamma}{\Theta \Vdash \Xi}\ (\Vdash C)^{7}$$

$$\frac{\Gamma[A], A \Vdash \Theta}{\Gamma[A] \Vdash \Theta}\ (W \Vdash) \qquad \frac{\Gamma \Vdash A, \Theta[A]}{\Gamma \Vdash \Theta[A]}\ (\Vdash W)$$

Remark 3.3. Curry considered axiomatic extensions too; we omitted an axiom and a rule that pertain to those components. Looking at the systems above, we should note several interesting details.

First, there is no thinning rule, either on the left- or on the right-hand side of the $\Vdash$ (i.e., $(K \Vdash)$ and $(\Vdash K)$ are dropped). Instead, a more general version of (Id) is postulated, which allows for the inclusion of formulas on either side in LC (on the left in LA), in addition to A appearing on both sides. This is a step toward *dispensing* with the structural rules, which goes against Gentzen's original aim to separate the operational rules from the structural ones. However, Curry is clearly

[7] Ξ is any permutation of Γ in these two rules.

intrigued by possibilities for changing the structural rules. And some variations on the structural rules—introduced by Curry and by others—proved exceptionally fruitful. (Lambek's calculi from the previous section are just the first examples.) The permutation rules ($C\Vdash$) and ($\Vdash C$) are more general than the similar rules in LK in the sense that several LK (LJ) steps can be combined into one step in LC (LA). Indeed, Curry quickly introduced the convention [22, p. 33] that permutation steps are left tacit, which we could view as him using multisets rather than sequences of formulas. The contraction rules here generalize Gentzen's contraction rules in a different way. Several permutation steps together with one contraction step in a proof in LK (LJ) can be performed as one step in LC (LA).

Second, Curry did not introduce any notation for an occurrence of a formula in a sequence of formulas, though this would be useful for the formulation of (Id) and the contraction rules as shown above. Lastly, we may point out that the ($\supset\Vdash$) rule is a blend of Gentzen's rule and Ketonen's rule. Like Gentzen's rule, Curry's rule permits the formulation of a 2-valued and an intuitionistic calculus in one fell swoop (unlike Ketonen's rule does), but Curry's rule is more restrictive than Gentzen's by requiring Γ in both premises.

Curry proved that the *thinning* rules are *admissible* [22, Theorem 2, p. 35] in the following more general forms. (The primed sequences of formulas are contained in their unprimed companions.)

$$\frac{\Gamma' \Vdash \Theta}{\Gamma \Vdash \Theta}\ (K\Vdash) \qquad\qquad \frac{\Gamma \Vdash \Theta'}{\Gamma \Vdash \Theta}\ (\Vdash K)$$

For us, the most interesting development in [22] is Theorem 3, which is the origin of what is called "Curry's lemma" in some of the literature, and what is called "height-preserving admissibility of contraction" in some other publications. To stress the importance of this theorem, we quote it.

> Theorem 3. If the rules Ol and, in LC, Or and Er, are so modified as to require the principal constituents to appear in all the premises on the same side as in the conclusion, then the rules W are redundant. — [22, p. 36]

The modifications affect eight rules in LC (four of which are impossible in LA, just as ($\Vdash W$) is impossible); we restate them all.

$$\frac{\Gamma[A \wedge B], B \Vdash \Theta}{\Gamma[A \wedge B] \Vdash \Theta}\ (\wedge_2\Vdash)^+ \qquad \frac{\Gamma \Vdash A, \Theta[A \wedge B] \quad \Gamma \Vdash B, \Theta[A \wedge B]}{\Gamma \Vdash \Theta[A \wedge B]}\ (\Vdash\wedge)^+$$

$$\frac{\Gamma[A \wedge B], A \Vdash \Theta}{\Gamma[A \wedge B] \Vdash \Theta}\ (\wedge_1\Vdash)^+ \qquad \frac{\Gamma \Vdash B, \Theta[A \vee B]}{\Gamma \Vdash \Theta[A \vee B]}\ (\Vdash\vee_2)^+$$

$$\frac{\Gamma[A \vee B], A \Vdash \Theta \quad \Gamma[A \vee B], B \Vdash \Theta}{\Gamma[A \vee B] \Vdash \Theta}\ (\vee \Vdash)^+ \qquad \frac{\Gamma \Vdash A, \Theta[A \vee B]}{\Gamma \Vdash \Theta[A \vee B]}\ (\Vdash \vee_1)^+$$

$$\frac{\Gamma[A \supset B] \Vdash A, \Theta' \quad \Gamma[A \supset B], B \Vdash \Theta}{\Gamma[A \supset B] \Vdash \Theta}\ (\supset \Vdash)^+ \qquad \frac{\Gamma, A \Vdash B, \Theta[A \supset B]}{\Gamma \Vdash \Theta[A \supset B]}\ (\Vdash \supset)^+$$

An initial reaction to these rules could be a surprise along the lines that if we already have the formula that is being introduced by the rule (i.e., the principal formula), then why to apply the rule at all? Of course, this riposte overlooks that some formulas *disappear* due to the application of the rule. Indeed, Curry notes that now "the essential function of the rules is ... to eliminate components." [22, p. 37]

Lemma 3.4. *If* ($\natural$) *is a connective rule of LC, then it is* derivable *from* $(\natural)^+$ *and thinning.*

Proof. Assuming the premise(s) of ($\natural$), one application of the appropriate thinning rule, ($K\Vdash$) or ($\Vdash K$), to each premise yields the premise(s) of $(\natural)^+$. An application of $(\natural)^+$ gives the same lower sequent that ($\natural$) has. □

Remark 3.5. If we denote by LC^+ the calculus obtained by replacing the ($\natural$) rules with the $(\natural)^+$ ones, then Lemma 3.4 shows that everything provable in LC^+ is provable in LC. Unquestionably, this is the "easy direction" in showing the equivalence of the two calculi. (By omission, we have also shown half of the equivalence between LA and LA^+, where the latter is LA with ($\wedge_1\Vdash$), ($\wedge_2\Vdash$), ($\vee\Vdash$) and ($\supset\Vdash$) replaced by their plussed versions.)

Lemma 3.6 (Admissibility of contraction)**.** *If* $\Gamma[A], A \Vdash \Theta$ (*or* $\Gamma \Vdash A, \Theta[A]$) *is provable in LC, then* $\Gamma[A] \Vdash \Theta$ (*or* $\Gamma \Vdash \Theta[A]$) *is provable in* LC^+*.*

Proof. The proof is by induction on the structure of a proof (or equivalently, on the height of the proof tree).
1. If $\Gamma[A], A \Vdash \Theta$ or $\Gamma \Vdash A, \Theta[A]$ is an instance of (Id), then so is $\Gamma[A] \Vdash \Theta$ and $\Gamma \Vdash \Theta[A]$, because A must occur in Θ or Γ, respectively.
2. If the claim holds for the premise of ($C\Vdash$) or ($\Vdash C$), then it holds for the lower sequent. (In some cases, the application of the permutation rule may be omitted altogether.)

As Curry noted earlier, the permutation steps may be left tacit everywhere. Indeed, counting them is cumbersome. From now one, we use a generalized version of the contraction rules, which are similar to Curry's generalized thinning and generalized permutation rules. These rules are:

$$\frac{\Gamma[A][A] \Vdash \Theta}{\Gamma[A] \Vdash \Theta}\ (W\Vdash) \qquad \frac{\Gamma \Vdash \Theta[A][A]}{\Gamma \Vdash \Theta[A]}\ (\Vdash W)$$

These rules better capture what we want to show about LC^+ (than the earlier rules do). (**1.** and **2.** above remain true.)

3. Let us assume that the $(\wedge\Vdash)^+$ rule has been applied in a proof. We consider whether $\Gamma[A]\,[A]\,[B\wedge C]\Vdash\Theta$ or $\Gamma[B\wedge C]\,[B\wedge C]\Vdash\Theta$ is the provable sequent of LC. In the first case, by the hypothesis of the induction, we have that $\Gamma[A]\,[B\wedge C],B\Vdash\Theta$ or $\Gamma[A]\,[B\wedge C],C\Vdash\Theta$ is provable, whichever was the premise. An application of $(\wedge\Vdash)^+$ gives $\Gamma[A]\,[B\wedge C]\Vdash\Theta$. In the second case, the upper sequent must have had *at least two* occurrences of $B\wedge C$ (e.g., as in $\Gamma[B\wedge C]\,[B\wedge C],B\Vdash\Theta$), and that number can be reduced by one. That is, the inductive hypothesis supplies the provability of $\Gamma[B\wedge C],B\Vdash\Theta$ or $\Gamma[B\wedge C],C\Vdash\Theta$. Hence, by an application of the rule $(\wedge\Vdash)^+$, we get $\Gamma[B\wedge C]\Vdash\Theta$.

If A occurs on the right-hand side of $\Vdash$, then the lower sequent after the application of the rule $(\wedge\Vdash)$ is $\Gamma[B\wedge C]\Vdash\Theta[A]\,[A]$, but the upper sequents must have been $\Gamma[B\wedge C],B\Vdash\Theta[A]\,[A]$ or $\Gamma[B\wedge C],C\Vdash\Theta[A]\,[A]$. By the hypothesis of the induction, $\Gamma[B\wedge C],B\Vdash\Theta[A]$ and $\Gamma[B\wedge C],C\Vdash\Theta[A]$ are provable. The application of the $(\wedge\Vdash)^+$ rule does not depend on the shape of the succedent, nor does the rule affect any change in it.

The other conjunction rules, all the disjunction rules, and $(\Vdash\supset)^+$ follow the same pattern of argument.

4. We consider the $(\supset\Vdash)^+$ rule. First, if A occurs in the antecedent and it is distinct from $B\supset C$ and C, then we only need to appeal to the inductive hypothesis. Second, if A is $B\supset C$, then we may appeal to the inductive hypothesis and an application of the $(\supset\Vdash)^+$ rule. Third, we scrutinize what happens when A occurs in the succedent, since Θ' and Θ may not coincide. If $\Gamma[B\supset C]\Vdash\Theta[A]\,[A]$ is provable in LC, then one of the premises of the rule is $B,\Gamma[B\supset C]\Vdash\Theta[A]\,[A]$, and we may appeal to the hypothesis of induction to have $\Theta[A]$ as the succedent of the sequent. We have to consider the other premise of $(\supset\Vdash)^+$. The largest Θ' is Θ, hence, we consider this case. (If Θ' is lacking some occurrences of formulas, then that surely does not create a possibility for an application of $(\Vdash W)$; at most some steps become superfluous.) If A is B, then by hypothesis, we have $\Gamma[A\supset B]\Vdash B,\Theta[\;\;]$ (where $[\;\;]$ indicates the place where one of the occurrences of B was). If A is not B, then similarly, we have $\Gamma[A\supset B]\Vdash B,\Theta[A]$. Either way, an application of $(\supset\Vdash)^+$ yields the sequent $\Gamma[B\supset C]\Vdash\Theta[A]$ as we needed. □

Remark 3.7. Curry did not list all the rules in their new form (that we indicated by a $^+$). However, presenting further modifications, he listed the principal formulas in upper sequents next to the immediate subformulas. An advantage of using Lambek's notation and generalized versions of the contraction rules is that we can avoid the insertion of a permutation step in the modified proofs, which occasionally would be necessary, and we may obtain the following corollary.

Corollary 3.8 (No height increase)**.** *If the height of the proof of* $\Gamma \Vdash \Theta$ *in* LC *was* n*, then there is a proof of the same sequent in* LC^+ *with height* m *st* $m \leq n$*.*

Proof. (Idea) Notice that due to the context sharing in the two-premise rules in LC, contractions that are not on formulas introduced by connective rules can be eliminated (by adjusting the numbers of formulas in (Id)). If an application of contraction is on formulas that are introduced by rules, then the matching rules with $^+$'s eliminate the contraction without an extra step in the proof. ☐

As an illustration of the above sketch, we give an example, namely, the proof of $(A \supset B) \wedge A \Vdash B$ in LC and LC^+. (φ stands in for the formula on which it is a subscript—to shorten some sequents.)

$$
(W\Vdash)\ \dfrac{(\wedge_2\Vdash)\ \dfrac{(\wedge_1\Vdash)\ \dfrac{(\supset\Vdash)\ \dfrac{A \Vdash A, B \quad A, B \Vdash B}{A \supset B, A \Vdash B}}{A \supset B, (A \supset B) \wedge A \Vdash B}}{(A \supset B) \wedge A, (A \supset B) \wedge A \Vdash B}}{(A \supset B) \wedge A \Vdash B}
\qquad
\dfrac{\dfrac{\dfrac{\varphi, A \supset B, A \Vdash A \quad \varphi, A \supset B, A, B \Vdash B}{\varphi, A \supset B, A \Vdash B}\ (\supset\Vdash)^+}{((A \supset B) \wedge A)_\varphi, A \Vdash B}\ (\wedge_2\Vdash)^+}{(A \supset B) \wedge A \Vdash B}\ (\wedge_1\Vdash)^+
$$

Corollary 3.9. *If* $\Gamma[A]\,[A] \Vdash \Theta$ $(\Gamma \Vdash \Theta[A]\,[A])$ *is provable in* LC^+ *with the height of the proof tree* n*, then* $\Gamma[A] \Vdash \Theta$ $(\Gamma \Vdash \Theta[A])$ *is provable in* LC^+ *with the height of the proof tree* m*, where* $m \leq n$*.*

Proof. It is easy to check that the steps in the inductive proof of Lemma 3.6 do not increase the height of the proof, because we avoided permutations. Thus, a sequent that results by contraction from a provable formula does not require a longer proof. ☐

Remark 3.10. *Curry's Lemma*, in the contemporary literature, often refers to Lemma 3.6 and Corollary 3.9 together. A, perhaps, useful way to think about Curry's lemma is that if a formula could be contracted in a proof at a step, then it could have been contracted when it was introduced. Hence, there is no need to accumulate multiple occurrences of a formula for no reason, and then, in later steps contract them (which is, in general, not possible).

Lemma 3.6, Corollary 3.8 and Corollary 3.9 have their analogs for LA and LA^+. (We will not state them separately.)

Curry went on to consider further modifications of the connective rules. In particular, he considered rules where not only the principal formula occurs in the premises, but the immediate subformulas may occur too. He gives as an example the four versions of the ($\Vdash\wedge$) rule. The conclusion that he draws from the latter modifications [22, Rem. 6, p. 38] is that the premises may be considered to be *pairs of sets of formulas* rather than pairs of sequences of formulas. We will denote these systems by LA^{++} and LC^{++}.

Remark 3.11. We note that LC^{++} is just one short step from *analytic tableaux*, especially, from a version where each node carries the set of all formulas from the previous step. Signs prefixed to formulas—such as $\mathfrak{t}$ and $\mathfrak{f}$—could indicate whether the formula occurs on the left of on the right from $\Vdash$.

In [22, §6], Curry states the decidability of LA and LC. Of course, these are not new results, but the *proof method* Curry uses to establish them is *new*. We quote his reasoning first.

> ... certain decidability properties should be expected ... each of the rules O derives a more complex statement from simpler ones, and the complexity once introduced can never be got rid of at a later stage. ... It ought to be possible by examining an elementary statement to determine what rules it could be a consequence of and from what premises; ... — [22, p. 39]

The sequent calculi LA and LC, as well as their versions are formulated without a cut rule (so far). Thus, once the rules are inspected, it is *immediate* that each calculus has the subformula property.

Lemma 3.12 (Subformula property)**.** *In a proof of* $\Gamma \Vdash \Theta$ *in* LA (LC, LA^+, LC^+, LA^{++} *and* LC^{++})*, if* A *occurs in a sequent in the proof, then* A *occurs as a formula or as a subformula of a formula in* $\Gamma \Vdash \Theta$*.*

Theorem 3.13 (Decidability)**.** *LA and LC are* decidable.

Proof. This is, essentially, Curry's Theorem 7 in [22, §6]. Given the equivalent formulations of the systems, the subformula property and the finiteness of the set of subformulas of all formulas in any sequent, we can start to search for a proof in a *bottom-up* fashion. Curry gives some examples where he relies on the modified versions of the systems, and he also suggests amending the proof search to shorten it. Using the $^{++}$ formulations, it is obvious that after finitely many steps either all the premises are instances of (Id), or no new sequent can be obtained, hence, the search has failed. □

We give a detailed example with a simple formula. (This is not Curry's example.)

Example 3.14. $\Vdash ((A \vee B) \wedge (C \vee B)) \supset ((A \wedge C) \vee B)$ is a theorem of LC; it expresses the distributivity of $\vee$ over $\wedge$ (in lattices). Here is a proof in LC.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Vdash A, B \quad B \Vdash A, B}{A \vee B \Vdash A, B}\,(\vee\Vdash)}{(A \vee B) \wedge (C \vee B) \Vdash A, B}\,(\wedge_1\Vdash) \qquad \dfrac{\dfrac{C \Vdash C, B \quad B \Vdash C, B}{C \vee B \Vdash C, B}\,(\vee\Vdash)}{(A \vee B) \wedge (C \vee B) \Vdash C, B}\,(\wedge_2\Vdash)}{(A \vee B) \wedge (C \vee B) \Vdash A \wedge C, B}\,(\Vdash\wedge)}{(A \vee B) \wedge (C \vee B) \Vdash (A \wedge C) \vee B, B}\,(\Vdash\vee_1)}{(A \vee B) \wedge (C \vee B) \Vdash B, (A \wedge C) \vee B}\,(\Vdash C)}{(A \vee B) \wedge (C \vee B) \Vdash (A \wedge C) \vee B, (A \wedge C) \vee B}\,(\Vdash\vee_2)}{(A \vee B) \wedge (C \vee B) \Vdash (A \wedge C) \vee B}\,(\Vdash W)}{\Vdash ((A \vee B) \wedge (C \vee B)) \supset ((A \wedge C) \vee B)}\,(\Vdash\supset)$$

For the purposes of the proof search, we may use those rules from LC^{++} that permit the largest amount of contraction. Since we "use" the rules in the reverse direction, this means that we will retain as many formulas as possible. This is unproblematic, because (Id) may contain spurious formulas.

For instance, we want to use the left-hand side rule, rather than the right-hand side rule.

$$\dfrac{\Gamma, A, A \wedge B, B \Vdash \Theta}{\Gamma, A \wedge B \Vdash \Theta} \qquad\qquad \dfrac{\Gamma, A \wedge B, B \Vdash \Theta}{\Gamma, A \wedge B \Vdash \Theta}$$

Now, we give some of the branches of the proof-search tree, which is constructed from its *bottom* toward the top. In order to keep the sequences on the page, we will subscript formulas with φ_n, and repeat only these φ's in lieu of the whole formula in the sequents above. (Π is the proof of $(A \vee B) \wedge (C \vee B) \Vdash C, \varphi_2, B, \varphi_1, \varphi_0$ which is similar to the left chunk above the line.)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi_3, \varphi_4, \varphi_5, A \Vdash A, \varphi_2, B, \varphi_1, \varphi_0 \quad \varphi_3, \varphi_4, \varphi_5, B \Vdash A, \varphi_2, B, \varphi_1, \varphi_0}{\varphi_3, (A \vee B)_{\varphi_4}, (C \vee B)_{\varphi_5} \Vdash A, \varphi_2, B, \varphi_1, \varphi_0}\,(\text{Id})}{((A \vee B) \wedge (C \vee B))_{\varphi_3} \Vdash A, \varphi_2, B, \varphi_1, \varphi_0}\,(\vee\Vdash) \qquad \Pi}{(A \vee B) \wedge (C \vee B) \Vdash (A \wedge C)_{\varphi_2}, B, \varphi_1, \varphi_0}\,(\Vdash\wedge)}{(A \vee B) \wedge (C \vee B) \Vdash ((A \wedge C) \vee B)_{\varphi_1}, \varphi_0}\,(\Vdash\vee)}{\Vdash (((A \vee B) \wedge (C \vee B)) \supset ((A \wedge C) \vee B))_{\varphi_0}}\,(\Vdash\supset)$$

The example illustrates "safe choices" for 2-valued logic, because retaining the φ formulas does not preclude getting to an axiom (if that is possible at all). With some squinting, one might find the above proof—again, starting at the root and moving upward—a slightly uneconomical version of an analytic tableaux.

Remark 3.15. *LA* and *LC* were formulated without a cut rule. However, the cut rule is important for any reasonable sequent calculus. Arguably, Curry aimed at providing a general theory of formal systems, and as part of it, he invented a *new method* to prove the admissibility of the cut rule in [22, §7]. The elimination of the mix for LJ and LK typically uses very localized modifications on a given proof such as swapping the application of a pair of rules, or completely eliminating an application of the cut rule, when one of its premises is an axiom. Curry's idea is to formally characterize categories of formulas with respect to their role in a proof, as well as certain kinds of rules, and then, perform modifications in sequents throughout the subproof. The proof still proceeds by induction, but it is, perhaps, fair to say that Curry's proof is more abstract than Gentzen's. We will not go into the details of Curry's proof of ET (the Elimination Theorem [22, p. 45]), because it is somewhat tangential to our goals in this paper (as the details of previous cut theorems were).

4 Kleene's G calculi

Kleene wrote a textbook [33], which appeared in 1952 and turned out to be a very influential introductory text for 2-valued and intuitionist logics. Kleene did not concentrate on sequent calculi, yet in Chapter XV he introduced sequent calculi for 2-valued and intuitionist logics. As it was fashionable—following Gentzen—he defined sequent calculi for the two logics so that intuitionist logic turn out to be the logic obtained by some restriction on 2-valued logic.[8]

Curry already "officially" generalized the exchange rule into a permutation rule that can permute several formulas at once. The idea of permutations is quite old; in group theory it has been used in the 19th century. However, the notion of a *multiset*, which can be thought of as an equivalence class of permutations, seem not to have been considered as a legitimate data type until the 1970s or so.[9] Kleene almost (but not quite) invented the notion of a multiset, and Curry de facto used it when he suppressed permutation steps.

Definition 4.1. Let us assume the structural rules of LK, in particular, the rules $(W \Vdash)$, $(\Vdash W)$, $(C \Vdash)$ and $(\Vdash C)$.[10] The sequents $\Gamma_1 \Vdash \Theta_1$ and $\Gamma_2 \Vdash \Theta_2$ are *cognate sequents* if there is a sequent $\Phi \Vdash \Psi$ s.t. it is derivable from both $\Gamma_1 \Vdash \Theta_1$ and $\Gamma_2 \Vdash \Theta_2$ by applications of the contraction and exchange rules.

[8] An unfortunate side effect of this strategy is that Kleene's labels (e.g., $G3$) refer ambiguously to a particular version of either the 2-valued or intuitionist logic; cf. [33, p. 480]. We add subscripts to $G3$ to distinguish these two sequent calculi "in $G3$-style," which are related but formalize different logics.

[9] See Blizard's [18] for historical remarks.

[10] We did not list $(\Vdash W)$ and $(\Vdash C)$ in LJ, but these rules in LK are just like the $(W \Vdash)$ and $(C \Vdash)$ rules in LJ but operating on the right-hand side of the $\Vdash$, which may contain any finite number of formulas in LK.

Remark 4.2. A way to look at cognate sequents is that they are strings or multisets that setify into the same set. It is easy to see that given $\Gamma \Vdash \Theta$, which is a pair of finite sequences or multisets of formulas, there are infinitely many $\Gamma' \Vdash \Theta'$ that are cognate with $\Gamma \Vdash \Theta$. Also, the shortest sequents that are cognate with $\Gamma \Vdash \Theta$ contain exactly one occurrence of every formula in Γ (on the left-hand side of the $\Vdash$) and exactly one occurrence of every formula in Θ (on the right-hand side of the $\Vdash$).

Although Curry has already noted [22, Remark 6, p. 38] that sets of formulas can be used instead of sequences of formulas in a sequent, Kleene did not change sequences to sets, rather he allowed a certain ambiguity using the concept of cognate sequents.

Definition 4.3. The sequent calculi $G3_k$ and $G3_j$ are defined by the following *axiom* and *rules*, with certain provisos for $G3_j$. First, there is a general restriction that there is at most one formula on the right-hand side of $\Vdash$. Second, the principal formulas of the rules are not required to occur in the premise(s) of the right-introduction rules. Specifically, in $(\Vdash\neg)$, the premise is $A, \Gamma \Vdash\ $. Lastly, in the $(\neg\Vdash)$ rule, the lower sequent is $\neg A, \Gamma \Vdash C$ (instead of $\neg A, \Gamma \Vdash \Theta$), where C is an arbitrary formula or the empty multiset.

$$A, \Gamma \Vdash \Theta, A \quad \text{(Id)}$$

$$\frac{\neg A, \Gamma \Vdash \Theta, A}{\neg A, \Gamma \Vdash \Theta}\ (\neg\Vdash) \qquad \frac{A, \Gamma \Vdash \Theta, \neg A}{\Gamma \Vdash \Theta, \neg A}\ (\Vdash\neg)$$

$$\frac{A, A\wedge B, \Gamma \Vdash \Theta}{A\wedge B, \Gamma \Vdash \Theta} \quad \frac{B, A\wedge B, \Gamma \Vdash \Theta}{A\wedge B, \Vdash \Theta}\ (\wedge\Vdash) \qquad \frac{\Gamma \Vdash \Theta, A\wedge B, A \quad \Gamma \Vdash \Theta, A\wedge B, B}{\Gamma \Vdash \Theta, A\wedge B}\ (\Vdash\wedge)$$

$$\frac{A, A\vee B, \Gamma \Vdash \Theta \quad B, A\vee B, \Gamma \Vdash \Theta}{A\vee B, \Gamma \Vdash \Theta}\ (\vee\Vdash) \qquad \frac{\Gamma \Vdash \Theta, A\vee B, A}{\Gamma \Vdash \Theta, A\vee B} \quad \frac{\Gamma \Vdash \Theta, A\vee B, B}{\Gamma \Vdash \Theta, A\vee B}\ (\Vdash\vee)$$

$$\frac{A\supset B, \Gamma \Vdash \Theta, A \quad B, A\supset B, \Gamma \Vdash \Theta}{A\supset B, \Gamma \Vdash \Theta}\ (\supset\Vdash) \qquad \frac{A, \Gamma \Vdash \Theta, A\supset B, B}{\Gamma \Vdash \Theta, A\supset B}\ (\Vdash\supset)$$

Remark 4.4. The list above, however, is not the full specification of the $G3$ calculi, because Kleene permits the *replacement* of any sequent with another sequent that is *cognate* with the former. This move means that the $G3$ calculi contain *generalized thinning* as part of the axiom, and generalized versions of three structural rules, namely, of *permutation*, *contraction* and *expansion*. (Expansion is a special version of thinning, where new occurrences of old formulas may be introduced.)[11]

[11]Kleene mentions a further variation of the $G3$ calculi; arbitrary formulas may be omitted from

Definition 4.5. A proof is *irredundant* if and only if there are no cognate sequents on any branch.

The notion of an irredundant proof is intended to counterbalance the extremely lenient way of keeping track of the number of copies of formulas. This notion of irredundancy is compatible with logics that contain both thinning and contraction rules.

Theorem 4.6. *The logic $G3_j$ is* decidable*.*

Proof. Kleene's proof combines several steps. First, another sequent calculus ($G2_j$) already has been shown to correspond to intuitionist logic in the sense that every intuitionistic theorem has a cut-free proof in it. Second, there is a match between the rules in the two calculi, since the explicit structural rules of $G2_j$ are built into the connective rules in $G3_j$ (and do not need to be paired). Third, every proof in $G3_j$ that is not irredundant simplifies to one that is, hence, every theorem has an irredundant proof in $G3_j$. The argument for the elimination of redundancies is based on the observation that if $\Gamma \Vdash \Theta$ occurs above $\Gamma' \Vdash \Theta'$, where the two sequents are cognate, then any sequents between these two sequents as well as $\Gamma' \Vdash \Theta'$ itself may be omitted together with any branches rooted in the omitted sequents. The proof remains a proof, because $\Gamma' \Vdash \Theta'$ can be replaced by $\Gamma \Vdash \Theta$ in an application of a rule. Lastly, a decision procedure results from a bottom-up proof search in $G3_j$, when the search is limited to irredundant proofs. It is sufficient to note that there are finitely many formulas in a sequent of the form $\Vdash A$, where A is the purported theorem. Then, there are finitely many sequents that can be obtained from the set of subformulas of A s̃t the sequents are pairwise incognate, or in other words, they belong to different cognation classes.[12] Then the proof-search tree is, obviously, finite. □

Remark 4.7. Kleene's decision procedure does not use Curry's lemma, which focuses on the admissibility of the contraction rules. Rather, Kleene allows using any cognate sequents in place of each other in applications of rules, hence, he simply excludes proofs in which cognate sequents are above each other.

A similar claim and a similar decision procedure can be fabricated for $G3_k$ too. However, the notion of cognate sequences relies on both contraction and thinning (in addition to permutations). It is, perhaps, fair to say that cognate sequents of formulas already have a name—they are called pairs of finite sets of formulas. It is

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the premise(s) in application of the rules. The omissions, presumably, should exclude the subalterns, which license an application of the rule itself, otherwise, soundness is lost.

[12]"Incognate" and "cognation class" are Kleene's terms, possibly, his neologisms.

possible that using sets in the formulation of the rules would appear less plausible informally. E.g., from the premise $\{A, A\wedge B\}\cup\Gamma \Vdash \Theta$ and application of $(\wedge_1\Vdash)$ could yield $\{A, A \wedge B\} \cup \Gamma \Vdash \Theta$ and $\{A \wedge B\} \cup \Gamma \Vdash \Theta$, which would need some motivation. (Of course, the free use of cognate sequents produces the same effect, but perhaps in a less obvious manner.) The thinning rules are, typically, not part of *relevance logics*. Hence, new ideas were required to prove the decidability of some relevance logics, which were the *new logics* introduced in the 1950s.

5 Kripke's lemma

The implicational fragment of the logic of relevant implication, $R_\to$ was introduced by Church [20], in 1951. However, Anderson and Belnap focused on the logic of entailment, E in the later part of the decade. The implicational fragment of E (i.e., $E_\to$) was denoted at the time by I, and its decision problem was of great interest for several reasons.

Remark 5.1. We use $\to$ for the binary connective that is *entailment* in E and *relevant implication* in R. The $\to$ connective has some similarities to $\supset$ (material implication or intuitionistic implication), but also to Lambek's $/$.

There was no finite characteristic matrix for $E_\to$ at hand to bypass the problem via truth tables; indeed, $E_\to$ is not finitely valued. A difference between $R_\to$ and $E_\to$ is that $(A \to (B \to C)) \to (B \to (A \to C))$ is a theorem of $R_\to$, but not of $E_\to$. The latter has only a restricted version of this formula as its theorem, namely, $(A \to ((B \to D) \to C)) \to ((B \to D) \to (A \to C))$. Although both logics have $(A \to B) \to ((B \to C) \to (A \to C))$ as their theorem, $(A \circ (B \circ C)) \to ((A \circ B) \circ C)$ is not a theorem of $E_\to$. The connective $\circ$ is to $\to$ as $\cdot$ is to $/$ in Lambek's calculi, and it is known that $\circ$ may be added to $E_\to$ conservatively. So far all calculi—save LN—contained a sequence of formulas (or something even less structured) in the antecedent of a sequent. Given that $E_\to$—unlike LN—has $(A \to (A \to B)) \to (A \to B)$ as its theorem, it is not immediately clear how to define a sequent calculus for $E_\to$. (See [10] and [11, Ch. 5] for sequent calculi for fragments of E.) Assuming that the problems stemming from the lack of associativity for $\circ$ can be overcome in a sequent calculus formulation, another problem emerges, namely, $A \to (B \to A)$ and $((A \to B) \to A) \to A$ are not theorems, that is, the thinning rules cannot be postulated. Hence, the more relaxed form of the axiom, that is, $A, \Gamma \Vdash \Theta, A$ cannot be included into the system.

Kripke defined sequent calculi for $E_\to$ and $S4_\to$ too. Moreover, he designed a decision procedure not only for $S4_\to$, but also for $E_\to$ and $R_\to$, as he reported in [35].

5.1 Kripke's correspondence around 1959

According to mail preserved in the Kripke Archive at CUNY, Kripke was in contact (through mail) with several logicians around this time. In particular, Alan R. Anderson, Nuel D. Belnap and Timothy J. Smiley each had written to Kripke (responding to communication from him). In a letter, dated January 8, 1958, Anderson discusses his ideas about adding deontic modalities to some of Lewis's modal logics ($S3$, $S4$ and $S5$). However, Anderson and Belnap's interests seem to had been focused, at the time, on their logic of entailment E (and Ackermann's Π', which is one rule away)—as [3] and [5] show. In Π' and in E, necessity can be defined as $\Box A \coloneqq (A \to A) \to A$, and this gives an $S4$-like modality. The idea that entailment—as distinguished from a mere conditional—has a modal component goes back to at least Lewis and his work on strict implication.

The axiomatic system Π' was defined by Wilhelm Ackermann in [1], and it has four rules. The implicational axioms with detachment give implicational ticket entailment, $T_\to$, and this logic T (including other usual connectives) was defined by Anderson in [4]. Ackermann's third rule (γ) is omitted both from T and E, but the latter retains the fourth rule (δ). Anderson and Belnap re-axiomatized E to replace (δ) with an axiom, which strongly suggests that the permutation of the antecedents of an implication (i.e., $(A \to (B \to C)) \to (B \to (A \to C))$) is not a theorem of E, and—indeed—it is not. In a natural deduction system, dependencies between assumptions and further formulas can be expressed using some sort of indexing on the formulas (see e.g., [3]). Let us reiterate that in the absence of permutation, indeed, even of the associativity for fusion (which was not a connective in Π'), a sequent calculus formulation for E or $E_\to$ does not seem to be forthcoming.

In a letter to Kripke, written on May 31, 1959, Belnap describes LI, which is the merge formulation of the implicational fragment of E.[13] It seems that Kripke must have been experimenting with combining the idea of subscripting formulas (as is done in some natural deduction systems) with a sequent calculus. Belnap writes in his letter:

> I find your use of subscripts very interesting, and I think fruitful. It is quite different from anything I have tried, inasmuch as it never occurred to me to assign sets of subscripts to the consequent. My original essays along the lines of subscripting were in the nature of permutation-controls in the antecedent (under a nesting-of-entailments interpretation). I subsequently abandoned these in favor of the notion of the merge, chiefly with an eye on simplicity of proof

[13] A closely related calculus $L_\mu E_\to$ is in [6, §7.3], together with merge calculi for two other relevance logics ($T_\to$ and $R_\to$, as well as for $S4_\to$ and $J_\to$).

of ET.

In the same letter, Belnap called the decidability problem for LI (that is, $E_{\rightarrow}$) to Kripke's attention. We quote him again:

> I have been able to find no decision procedure for LI. Gentzen's techniques seem to require that we have A→B→.(A→.A→B), a theorem which fails for I and LI: contraction is the stumbling block. ... Of peripheral interest is the fact that one can get the L-formulation of the pure implicational fragment of S4 by permitting primes of the form α, A ⊫ A. For this system we would have a decision procedure, as also for the system like LI but without W.

By mid June, 1959, Kripke had a sequent calculus (with no subscripts) for the pure strict implication part of $S4$. In a letter to Belnap (June 13th, 1959), Kripke specifies the rule Pr with a side condition that X comprises implicational formulas (both for $S4$ and I). That is, the right introduction rule for $\rightarrow$ is

$$\frac{X, A \Vdash B}{X \Vdash A \rightarrow B}\ \text{Pr}, \text{ where all elements of } X \text{ have the form } C \rightarrow D.$$

Imposing side conditions on a connective rule in a sequent calculus is not unheard of (though it is not very common). For example, Curry's formulation of the absolute calculus (i.e., J_{+}) with multiple right-hand sides (LA_m in [23, 5C4]) has one side condition, and it pertains to Pr.

Kripke's letter also shows that he was thinking about a sequent calculus for E_{+}. His attention was on the problem of the distributivity of $\wedge$ and $\vee$, and how different variants of the $\wedge$ and $\vee$ rules interact.

Some three months later, Kripke had the *decidability proof for I*. In a letter to Belnap, dated September 21, 1959, he outlined the proof. The letter, which is merely 2 pages long, contains the crucial component in the proof that has become known as "Kripke's lemma." Before we leave this brief historical excursion into the contents of the Kripke Archive, we should mention that Kripke was aware that the lemma was a more general statement (not specific to sequent calculi) and could be stated in terms of positive integers. He also mentioned that the lemma does not permit (in general) the calculation of an upper bound on the length of irredundant sequences of sequents, though it *guarantees finiteness.* As a result, Kripke conjectured, that the whole decision procedure is general recursive, rather than primitive recursive. The lemma is stated and its proof is presented in [6, pp. 138–139], where it is attributed to Kripke with a mention of a letter, which must be the letter we just mentioned.[14]

[14] As far as I know, Kripke did not publish anything about his decidability results after the

5.2 Church's weak implication is decidable

Church who preferred his λI-calculus over the λK-calculus, introduced, what he called "weak implication" by replacing $A \to (B \to A)$ by $A \to A$ in an axiomatization of $J_\to$ in [20]. Nowadays, this logic is better known as the *implicational fragment of the logic of relevant implication* or $R_\to$. We will illustrate the use of Kripke's lemma (together with the new notion of irredundancy) on this logic. First, $R_\to$ extends $E_\to$ with the permutation of the antecedents of an implication (which does not spoil the connection in information between premises and conclusions). Second, $R_\to$ can be formulated as a sequent calculus without side conditions on $(\Vdash \to)$, which allows us to focus on the essence in Kripke's decidability proofs. Third, the decidability of $R_\to$ led to further decidability results, some of which we will mention in Section 6. Fourth, although Belnap pointed Kripke's attention to $E_\to$, which was his and Anderson's favorite logic, Kripke's decidability proof for $E_\to$ is easily seen to simplify to a decidability proof for $R_\to$—cf. [35].

Definition 5.2. The logic $R_\to$ is defined by (A1)–(A4) and the rule of detachment (from A and $A \to B$, infer B). The axioms are (A1) $A \to A$, (A2) $(A \to B) \to ((C \to A) \to (C \to B))$, (A3) $(A \to (B \to C)) \to (B \to (A \to C))$ and (A4) $(A \to (A \to B)) \to (A \to B)$.

Remark 5.3. The notion of a proof is the usual one for axiomatic calculi, that is, there is no requirement that all the formulas in a proof—save the last formula—be used in the proof. However, the axioms do not include the principal type schema of the combinator K. The above axioms can be conceptualized by saying that $R_\to$ is implicational IBCW.[15]

Definition 5.4. The sequent calculus $LR_\to$ is defined by the following axiom and four rules. (A sequent is a pair of a multiset and a formula, that is, it is of the form $\Gamma \Vdash A$.)

$$A \Vdash A \ \text{(Id)} \qquad \frac{\Gamma \Vdash A \quad B, \Theta \Vdash C}{A \to B, \Gamma, \Theta \Vdash C}\ (\to \Vdash) \qquad \frac{A, \Gamma \Vdash B}{\Gamma \Vdash A \to B}\ (\Vdash \to)$$

$$\frac{A, A, \Gamma \Vdash B}{A, \Gamma \Vdash B}\ (W \Vdash) \qquad \frac{\Gamma \Vdash C \quad C, \Theta \Vdash B}{\Gamma, \Theta \Vdash B}\ \text{(cut)}$$

abstract [35], which appeared in the *Journal of Symbolic Logic* in 1959. Belnap and Wallace in [7] (published as [8]) applied Kripke's method to prove the decidability of the negation–implication fragment of E.

[15] The axioms for these combinators are $\mathsf{K}xy \rhd x$, $\mathsf{I}x \rhd x$, $\mathsf{B}xyz \rhd x(yz)$, $\mathsf{C}xyz \rhd xzy$ and $\mathsf{W}xy \rhd xyy$.

Remark 5.5. The notion of a proof in this sequent calculus is usual. The axioms of $R_{\to}$ are provable in the sense that if A is an axiom, then $\Vdash A$ has a proof.

The cut rule helps to emulate applications of the detachment rule in a proof in the axiomatic system. To prove that if $LR_{\to}$ proves a formula, then so does $R_{\to}$, it is useful to start with noting that the Elimination Theorem (ET) holds. Then it remains to show that the moves sanctioned by the three remaining rules can be mimicked in $R_{\to}$. A sequent $\Gamma \Vdash A$ can be taken to be $\overrightarrow{\Gamma} \to A$, where the elements of Γ are the antecedents of the implication in an arbitrary order of the formulas.

For the proof of ET, it is convenient (though not necessary) to use a version of the cut that is intermediate between the single cut rule and the mix rule; let us call this multi-cut. Namely, in the lower sequent of the cut rule, an arbitrary number of occurrences of C (the cut formula) may be omitted from Θ. The equivalence of the single cut rule and of the multi-cut rule is immediate.

As Belnap stressed in his letter (quoted above), the issue for a decision procedure is the rule $(W \Vdash)$. However, the only rule that allows an increase in the number of formulas in the antecedent (once cut is counted out) is $(\to \Vdash)$. Thus, if contraction can be built into that rule cautiously, then one could hope to be able to control the size of a bottom-up proof-search tree.

Definition 5.6. The sequent calculus $[LR_{\to}]$ is defined with the *axiom* (Id), and the *rule* $(\Vdash \to)$ as in 5.4, with the following rules added.

$$\frac{\Gamma \Vdash A \quad B, \Theta \Vdash C}{[A \to B, \Gamma, \Theta] \Vdash C}\ ([\to \Vdash]) \qquad\qquad \frac{\Gamma \Vdash C \quad C, \Theta \Vdash B}{[\Gamma, \Theta^{-}] \Vdash B}\ ([\text{cut}]),$$

where Θ^{-} is Θ from which an arbitrary number of occurrences of C have been omitted.

The bracketing in the rules indicates that some contractions are *permitted* (but not prescribed). For $([\to \Vdash])$ we have the following: if D is not $A \to B$, and D occurs both in Γ and Θ, then $[A \to B, \Gamma, \Theta]$ may omit an occurrence of D. If $A \to B$ occurs in Γ or Δ, then $[A \to B, \Gamma, \Theta]$ may omit an occurrence of $A \to B$. Lastly, if $A \to B$ occurs both in Γ and Θ, then two occurrences of $A \to B$ may be omitted from $[A \to B, \Gamma, \Theta]$. For ([cut]), if D occurs in Γ and Θ, then the number of occurrences of D in $[\Gamma, \Theta]$ may be the sup of those in Γ and Θ separately.

Remark 5.7. The amount of permitted contractions in $([\to \Vdash])$ may be specified slightly differently than above, where we followed Dunn [25, §3.6]. Belnap preferred a more relaxed version of the rules, which he suggested to Kripke in his letter of September 28, 1959 (Kripke Archive, CUNY). Let D occur m times in Γ and n times in Θ. Then the number of occurrences of D in $[\Gamma, \Theta]$ is p, where $\max(m, n) \leq p \leq m + n$. (See also [8, p. 279].)

The latter form of the side condition (sometimes) allows one to "postpone" contractions. On the other hand, the former condition rhymes with the slogan that formulas may be contracted if they could not have been contracted before.

The calculus $[LR_{\to}]$ is intended as a formalization of $R_{\to}$. The cut theorem holds, and so one could consider establishing the equivalence of the logics directly. However, it is perhaps easier to show the equivalence of the sequent calculi. It is obvious that if $[LR_{\to}]$ proves A, then so does $LR_{\to}$ (without cut). To prove the other direction, we can use Curry's lemma.

Lemma 5.8 (Curry's lemma for $[LR_{\to}]$)**.** *If* $\Gamma \Vdash A$ *is provable in* $[LR_{\to}]$ *with the height of the proof being* n*, and* $\Gamma' \Vdash A$ *could be obtained from* $\Gamma \Vdash A$ *by zero or more applications of the rule* $(W \Vdash)$*, then* $\Gamma' \Vdash A$ *is provable in* $[LR_{\to}]$ *and it has a proof with height less than or equal to* n*.*

Proof. We proceed by induction on the height of the proof, considering at each step how the sequent has been obtained.

The case of the axiom is vacuous, and the case of $(\Vdash \to)$ is by an application of the hypothesis of the induction. Thus, we turn to the next case, which is the crucial one.

3. If the last rule applied in a proof is $([\to \Vdash])$, then if $\Gamma' \Vdash A$ is by zero contraction, then the claim obviously holds. There are two possibilities that we consider, namely, a contraction is on the principal formula of the rule or on another formula. (The combination of these can be handled by the combination of the two subcases.)

Let the principal formula be $B \to C$. If it has more than one occurrence in the antecedent of $[B \to C, \Gamma, \Theta] \Vdash A$, we may consider the origin of the non-principal occurrences. The inductive hypothesis ensures that multiple occurrences of $B \to C$ in Γ as well as in Θ can be reduced to a single occurrence. Then by an application of the rule $([\to \Vdash])$, the number of occurrences of $B \to C$ may be reduced to *one*.

If let us say D is not $B \to C$, and D has multiple occurrences in $[B \to C, \Gamma, \Theta]$ in the sequent, then by inductive hypothesis, we can have the number of occurrences of D be reduced to (at most) one in Γ and in Θ. If D did not occur in Γ or Θ, then this suffices for the truth of the claim. If D occurred in the antecedent of both premises, then an application of $([\to \Vdash])$ can further reduce the number from two to *one*. □

Remark 5.9. The proof is very similar to the proof of Lemma 3.6. However, a difference is that the principal formula is *not required* to occur in the premises and the contractions are *not mandatory*. That is, if we have five $B \to C$'s in Γ, and four in Θ, then we still can prove—using the rule ($[\to \Vdash]$)—the sequent $(B \to C)^n, \Gamma^{-(B\to C)}, \Theta^{-(B\to C)} \Vdash A$, where $n = 10$ (or any other n st $1 \leq n < 10$).

For the decision procedure we need Kripke's notion of irredundancy, which is different than the earlier notion.

Definition 5.10. A sequence of cognate sequents is *irredundant* when earlier elements of the sequence are neither identical to later ones nor can be obtained by contractions from later elements in the sequence.

Remark 5.11. It may be useful to consider what irredundancy would mean in a *proof tree*. Let us assume that the root is $\Gamma \Vdash A$, and we have a branch that comprises sequents some of which may be cognate. If we select a particular cognation class of sequents, then the subsequence in the branch will be *redundant* if there is a pair of sequents $\Theta_1 \Vdash B$ and $\Theta_2 \Vdash B$ st $\Theta_1 \Vdash B$ is *above* $\Theta_2 \Vdash B$ in the proof tree and $\Theta_2 \Vdash B$ is the result of contractions on Θ_1 (or Θ_1 and Θ_2 are identical). In other words, contractions may not be postponed in a proof if the branches are to be irredundant with respect to every cognation class exemplified in the proof.

The *proof-search tree*, however, is built from the root sequent $\Gamma \Vdash A$ upward. And the order of sequents in a sequence (in Definition 5.10) should be the order in the proof-search tree, that is, $\Theta_2 \Vdash B$ precedes $\Theta_1 \Vdash B$. So if the sequence is irredundant, then $\Theta_2 \Vdash B$ cannot be a contracted version of $\Theta_1 \Vdash B$.

Lemma 5.12 (Kripke's lemma)**.** *An irredundant sequence of cognate sequents is* finite*.*

We do not repeat the proof of this lemma; a readily available detailed proof by induction is in [6, p. 139] (which is credited to Kripke).

Theorem 5.13. $R_\to$ *is* decidable*.*

Proof. (Sketch) We outline the components of the proof, so that it can be generalized to other logics.

1. $R_\to$ has a sequent calculus formulation, $LR_\to$ in which the cut rule is admissible.

2. $LR_\to$ has a modified version $[LR_\to]$, in which the cut rule is admissible in $[LR_\to]$.

3. Curry's lemma holds for $[LR_\to]$, that is, contraction is admissible with no increase in the height of the proof tree.

4. Given a sequent $\Gamma \Vdash A$, there is a finite proof-search tree, which either contains a proof of the sequent (if the sequent is provable), or it does not (if it is not provable).

The proof-search tree is built from the given sequent upward by iteratively adding sequents that could be premises of rules that could result in the sequent. Branches are discontinued if they result in an instance of the axiom, if they would become irredundant or if there is no way to obtain the top sequent from some premises by the rules.

(1) The cut theorem guarantees that it is sufficient to search for cut-free proofs. (2) Sequents are finite, hence, contain finitely many formulas. (3) Finitely many formulas have finitely many subformulas (because there are no quantifiers). (4) Each rule has finitely many premises (because there are no rules similar to an ω-rule). (5) There are finitely many cognation classes that appear in the proof-search tree.

Given (1)–(5), the only way the proof-search tree could be infinite—according to Kőnig's lemma about trees—is by having an infinite branch. This possibility is what is excluded by Kripke's lemma. □

Remark 5.14. Some components of the proof are easily transferable to other logics, including the use of Kripke's lemma. To put it succinctly, given a sequent calculus formalization of a (propositional) logic, one needs to define an equivalent sequent calculus in which contractions that were part of the first calculus are admissible. The lemmas that have to be proved for each such calculus is the *admissibility of cut* and *Curry's lemma*.

The fact that Kripke's lemma is equivalent to some lemmas, and in this sense, it is not specific to each sequent calculus in itself, does not diminish the ground breaking role it played in the expansion of the range of logics that could be proved decidable using sequent calculi.

6 Extension of the Curry–Kripke method

The Curry–Kripke method has been extended into several directions. We briefly indicate three of them.

Additional connectives. We mentioned above that Belnap and Wallace proved the implication–negation fragment of E decidable in the early 1960s. The addition of $\wedge$ and $\vee$ to $LR_{\rightarrow}$ is not difficult, however, the usual rules without thinning do not permit the proof of the R theorem $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$. The latter problem was known to Belnap and to Kripke. It was solved by Dunn in [24], where two structural connectives were used together with $\boldsymbol{t}$ (intensional truth). A careful use of $\boldsymbol{t}$ prevents proofs of sequents that express irrelevancies to be constructed by a detour through cut. It was a hope of some relevance logicians in the mid 1960s that the solution to the decidability problem of R (and perhaps, E too) was to be found through the decidability of their non-distributive versions. Robert K. Meyer added

$\wedge$, $\vee$, $\sim$ (negation) and N (necessity) to R and defined a sequent calculus LR^N in his dissertation [39]. Meyer proved his logic, what he called lattice-R, decidable.

The addition of $\boldsymbol{t}$ to $LR_{\to}$ is relatively easy, and it was carried out in [15], where $LR^{\boldsymbol{t}}_{\to}$ was proved decidable (as an auxiliary step toward a proof of the decidability of $T_{\to}$). The addition of $\boldsymbol{t}$ as well as of $\boldsymbol{f}$ to LR was carried out in [17], together with proofs of decidability $LR^{\boldsymbol{tf}}$ (as an auxiliary step toward a proof of the decidability of CLL, classical propositional linear logic).

Emulating proofs. We characterized $R_{\to}$ as implicational IBCW. $T_{\to}$, *implicational ticket entailment* can be obtained from $R_{\to}$ by replacing the simple type of C by the simple type of B', that is, $(A \to B) \to ((B \to C) \to (A \to C))$. The three implicational logics $R_{\to}$, $E_{\to}$ and $T_{\to}$ may be compared as follows. $R_{\to}$ allows all permutations, $E_{\to}$ permits permuting implicational formulas, whereas $T_{\to}$ only allows formulas in certain positions to be permuted with re-associating the formula. This may sound a bit abstract, but we should note that it is not possible to characterize permutations by the shape of the formula in $T_{\to}$, which means that a sequent calculus cannot use multisets (possibly, with a side condition, like Kripke's LI does). These major relevance logics can be enriched with $\boldsymbol{t}$, without creating new theorems in the old language. At the same time the differences between R, E and T correspond to properties of $\boldsymbol{t}$.

Lambek invented the first sequent calculus in which the structural connective was not associative. Along similar lines, and essentially using $\boldsymbol{t}$, [9] introduced a sequent calculus for $LT^{\boldsymbol{t}}_{\to}$ (and also for the positive fragment of T with $\boldsymbol{t}$, $\circ$, $\leftarrow$, $\wedge$ and $\vee$). Relying on the insight about $\boldsymbol{t}$, [15] introduced a sequent calculus for $R^{\boldsymbol{t}}_{\to}$, which extends $LT^{\boldsymbol{t}}_{\to}$ with two rules that involve $\boldsymbol{t}$. In $T^{\circ\boldsymbol{t}}_{\to}$, $\boldsymbol{t}$ is left identity for $\circ$, but in $R^{\circ\boldsymbol{t}}_{\to}$, $\boldsymbol{t}$ is left-right identity. The two new rules in $LT^{\textcircled{t}}_{\to}$ add *exactly the difference* between a left identity and a full identity to the sequent calculus. (The rules do not depend on the presence of $\circ$, but it is easier to describe $\boldsymbol{t}$'s properties with reference to the connective $\circ$ of which $\to$ is a residual.) [16] defined a procedure to imitate proofs in the sequent calculus $LR^{\boldsymbol{t}}_{\to}$ by proofs in $LT^{\textcircled{t}}_{\to}$. With a minor extension of Kripke's result, $LR^{\boldsymbol{t}}_{\to}$ is decidable. Every theorem of $T^{\boldsymbol{t}}_{\to}$ has finitely many irredundant proofs in $LR^{\boldsymbol{t}}_{\to}$, each of which can be transformed into a finite set of proofs in $LT^{\textcircled{t}}_{\to}$. To find the theorems of $T_{\to}$ amongst the theorems of $R_{\to}$, it remains to scrutinize a finite set of proofs to see if there is a proof which does not use the rules that add right identity to $\boldsymbol{t}$.

Subsuming special contractions. We already mentioned that modalities had been added to lattice-R by Meyer. In [36], Kripke introduced rules for $\Diamond$ (possibility), which in the context of $S4$, allowed for a proof of the formulas that are customarily used in the definition of one modal connective in terms of the other and negation (e.g., $\Diamond A \coloneqq \neg\Box\neg A$). The same rules may be added to lattice-R and its fragments. Furthermore, the structural rules, especially, the thinning and contraction rules may be varied, including restricting the applicability of these rules to modalized formulas. A whole range of logics was investigated in [17], and all the logics in that paper were proved decidable. The logics that do not contain modalized structural rules, really fall into the first kind of extension considered in this section. That is, the main issue is to appropriately formulate the connective rules, as well as their version that permit some contraction, and then, to prove the cut theorem and Curry's lemma.

The logics that contain modalized structural rules, however, require yet another idea. Such logics can be paired with logics in which the same structural rules are not restricted. This matching may lead to new theorems, but cannot eliminate theorems. Thus, the task is again to sift out from the set of theorems those that are not theorems of the subsumed logic. Let us assume that logic LX is subsumed by logic LY. If A is found to be a theorem of LY, then (in all interesting cases) A has more than one subformula. The Curry–Kripke method yields a finite set of irredundant proofs in LY for A. From these proofs, we can define a *heap number* for the formula by totaling up contractions on ancestors of A and selecting the largest number. Then, the number of modalized contractions is bounded by the heap number in a proof-search in LX. (For example, a modalized formula such as $\Box B$ or $\Diamond B$ has at least two subformulas, and B is not modalized if the number is two.) These ideas were used in [12] to prove the decidability of MELL (the multiplicative–exponential fragment of LL). In [13], I proved a series of semi-lattice based logics decidable, including NLL, the normal fragment of linear logic, which was introduced by Kopylov in [34].

Of course, there may be further ways to extend the Curry–Kripke technique to prove even further (propositional) logics decidable. However, there are also known obstacles. There are sequent calculi for the positive fragments of R, E and T, but these logics (i.e., R_+, E_+ and T_+) are known to be undecidable. Informally speaking, the introduction of two kinds of structural connectives with the resulting interaction of two kinds of contractions may be a reason behind the complex nature of these logics.

7 Conclusions

We traced the main changes in sequent calculus based proofs of decidability for various (propositional) logics. Curry introduced most of the new ideas in proof theory in the second half of the 20th century. These range from a conceptually different proof of the cut theorem to replacing sequences with sets or multisets.

For proofs of decidability, Curry's idea of modifying a sequent calculus so that the contraction rules become admissible rules is paramount. Turning the proof-search tree upside down (or alternatively, building the proof-search tree in a bottom-up fashion) is due to Curry. The next crucial step forward is Kripke's lemma together with a new notion of irredundancy, which avoid the need to calculate the number of possible sequents concretely, yet guarantee the finiteness of the proof-search tree. Another innovation introduced by Kripke that is crucial for decidability results in relevance and some other substructural logics is the careful handling of contraction in connective rules—no contractions are obligatory, but some are permitted.

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