Proportional Reinsurance for Models with Stochastic Cash Reserve Rate

by

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Abstract

This thesis investigates a problem of risk control for a financial corporation. Precisely, the thesis considers the case of proportional reinsurance for an insurance company. The objective is to find the optimal policy, that consists of risk control, which maximizes the total expected discounted value of cash reserve up to the bankruptcy time.

The models for the cash reserve process, considered in this thesis, have stochastic drifts per unit time (that we call stochastic cash reserve rate hereafter) and constant volatility. These models extend the literature on proportional reinsurance, to the case of stochastic cash reserve rate that is either fully or partially observed. Precisely, I address three principal models. The first model deals with the case when the cash reserve rate is time dependent but deterministic. The second model assumes that the cash reserve rate process has an observable noise, while the third model assumes that the cash reserve rate is stochastic and is not observable.

Thanks to the Bellman's principle, for each of these three models, I derive the Hamilton-Jacobi-Bellman equation that corresponds to the stochastic control problem. Then I solve these equations as explicitly as possible. Afterwards, I describe the optimal policy for each model in terms of the obtained optimal value function, and I state the verification theorem. Finally, I consider the case where the insurance company pays liability at a constant rate per unit time.

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Chapter 1

Introduction

Actuarial science originally became a formal mathematical discipline for long-term insurance coverage such as burial, life insurance and annuities in the end of 17th century. These long term coverage required that money be reserved to pay future benefits, such as annuity and death benefits many years into the future. Moreover, actuarial science is also applied to property, casualty, liability and general insurance which are short-term forms of insurance. Rigorous models for insurance in actuarial science were born in 1903, when Filip Lundberg defended his Ph.D thesis and proposed the collective risk model for insurance claim data. In his thesis, Lundberg introduced the compounded Poisson process and developed some results on the central limit theorem. At this time, mathematical finance was already established in the Ph.D thesis of Louis Bachelier in 1900. One of the first attempts to describe the stock price fluctuation via a Brownian motion can be traced back to Bachelier's Ph.D. thesis. The limitations of the arithemetic Brownian motion, used by this PhD thesis hindered further development of this model. Both Poisson processes and

Brownian motion are the primary examples of the wider class of stochastic processes with stationary and independent increments, called Levy processes in the probability literature. However, the two areas began to fall apart during the first half of the last century, while Cramer, Essher and many other mathematicians pushed actuarial science into a new level. Fortunately, in the second half of the last century, there were significant advances in mathematical finance and modern finance due to the works of Paul Samuelson, Robert Merton, Black, Scholes and Markowitz. During the recent decades, actuarial science has started to embrace more sophisticated mathematical modelling of finance. In fact, there has been an upsurge of interest in applying diffusion models to financial mathematics and in particular in reinsurance modeling setting. For more details, we refer the reader to Asmussen and Taksar (1997), Boyle, Elliott and Yang (1998), Højgaard and Taksar (1998) and Taksar and Zhou (1998). In these models the liquid assets processes of the corporation are driven by a Brownian motion with constant drift and diffusion coefficients. The drift term corresponds to the expected profit per unit time, while the diffusion term is interpreted as risk. The larger the diffusion coefficient the greater the business risk the company takes on.

If a company wants to increase the potential profit from its business activities, it should also face an increase in risk. This shows that the controls, in an optimal stochastic control model, affect not only the drift but also the diffusion part of the dynamic of the system.

Insurance is one of the natural areas where those models become widely applied. Risk control in insurance takes on a natural form of reinsurance. Thus, the natural questions that arise here are

(1) what the reinsurance means?

(2) when do we need reinsurance?

1.1 Reinsurance

Reinsurance is the process of controlling revenues by diverting part of the premium to another insurance company, thus reducing its own risk as well as profit.

In other words, reinsurance is purchased by an insurance company (called "cedent") from one or more other insurance companies (called the "reinsurer") directly or through a broker. The ceding company and the reinsurer enter into a reinsurance agreement which details the conditions upon which the reinsurer would pay a share of the claims incurred by the ceding company. The reinsurer is paid a "reinsurance premium" by the ceding company, which issues insurance policies to its own policyholders.

Recently, this becomes one of the most popular and effective way such that insurance companies could successfully increase their firms value at the cost of bearing reasonable risk. Some of the reasons to employ reinsurance are summarized in the following.

- Hedging adverse fluctuations that may incur in the course of business.
- The appearance of excessively large claims such as catastrophic risks, or an unusually large number of claims. The most dangerous risk comes from the large claims and also a large number of claims can lead to a disastrous situation.
- Reinsurance can be considered to increase the capacity of the company by offering more services to its clients.
- Financial distress due to unexpected changes in premium collection or profit.

In conclusion, one of the most common reasons why an insurance companies would employ reinsurance is to diminish the impact of large claims.

Reinsurance process can take various forms. Thus, reinsurance can be classified into proportional reinsurance and non-proportional reinsurance. The most common example of non-proportional reinsurance is the excess-of-loss reinsurance.

Proportional Reinsurance: the reinsurer is required to pay a certain fraction of each claim, while in return the cedent (insurer) diverts the same or a larger fraction of all the premium to the reinsurer. If the safety loading of the reinsurer and the cedent are the same, that is, if the fraction of the premium diverted to the reinsurer is the same as the fraction of each claim covered by the reinsurer, then the contract is called a cheap reinsurance. If the safety loading of the reinsurer is higher than that of the cedent, then such a contract is called a noncheap reinsurance.

Excess-of-loss Reinsurance: this reinsurance responds only if the loss suffered by the insurer exceeds a certain amount, which is called the “retention” or “priority” level.

This thesis focuses only on the case of cheap proportional reinsurance. In this case, the firm would need to pay $a\%$ of the claim and as well receive $a\%$ of the premium, and reinsurance company would have the obligation for the rest of $(1-a)\%$ claim size and correspondingly receive $(1-a)\%$ premium.

In general, companies's objective is to maximize profit. We should know that there is always a trade-off between increasing firm's value and bearing risk, i.e in order to increase firm's value, we have to bear more risk. Therefore, how could we make as much money by bearing certain level of risk becomes the ultimate optimization problem that we are interested in.

1.2 Cramer-Lunderberg models

The first model for the cash reserve process of an insurance company was proposed by Cramer-Lundberg and is denoted by

$$R(t) = R(0) + pt - \sum_{i=1}^{N_t} U_i$$

Here $R(0)$ is the initial capital, p is the expect premium per unit time and U_i is a random variable that describe the size of the i^{th} claim $(i \geq 0)$. We are assuming claims arrive or occur at a Poissonian rate. N_t is a random variable that follows a Poisson process and represents the number of claims occurred up to time t.

This model shows the case when the insurance company takes the full risk. When the insurance company considers reinsurance, it means that the company assume the risk with sizes $U_i^{(a)}$ where a is the retention level, while the company divert $U_i - U_i^{(a)}$ to another company. Therefore, the reserve process for the cedent becomes:

$$R^{a,\eta}(t) = \mu + p^{a,\eta}t - \sum_{i=1}^{N_t} U_i^{(a)},$$

where η represents the safety loading and the premium rate $p^{a,\eta}$ is given by $p^{a,\eta} = (1 + \eta)E(U_i^{(a)})$. We can see from the above that the risk $U_i^{(a)}$ is crucial in reinsurance model. As a result, the two popular types of reinsurance (proportional and excess-of-loss reinsurance) can be obtained as follows.

(a) The proportional reinsurance can be obtained by putting

$$U_i^{(a)} = aU_i, \qquad 0 \le a \le 1.$$

(b) The excess-of-loss reinsurance can be obtained by putting

$$U_i^{(a)} = min(a, U_i), \qquad a > 0.$$

1.3 Diffusion models for reinsurance

Some insurance companies may have big portfolios and hence the size of an individual claim is too small compared to the size of total cash reserve. Thus, in this case, the Cramer-lunderberg model is not suitable and becomes inadequate. This is one of the reason that we need diffusion models to model the business activities of an insurance company. The diffusion models for reinsurance are based on the Brownian motion which we introduce in the next chapter. Specifically, as η goes to zero, $\left(\eta R_{t/\eta^2}^{(a,\eta)}\right)_{t\ge 0}$ converges to BM$(\mu(a), \sigma^2(a))$ in the space of $D[0,\infty)$. For more details about this, we refer the reader to Asmussen and Taksar (1997).

1.4 Summary of the thesis

This thesis contains six chapters including the current chapter. In the next chapter (Chapter 2), we give some mathematical tools that will be used throughout the thesis.

Chapter 3 addresses the case when the cash reserve rate is time independent but deterministic. For this model, I derive the Hamilton-Jacobie-Bellman equation (HJB equation for short hereafter), and I propose a solution to it as explicitly as possible. Then, I describe the optimal policy and the optimal cash reserve process associated to this policy. The work of this chapter extends the paper of Højgaard and Taksar (1998).

Chapter 4 deals with the case when the cash reserve rate is stochastic and its dynamics follow an Ornstein-Uhlenbeck process with positive volatility. Herein, I assume that this cash reserve rate is fully observed, and hence is adapted to the filtration (information) used to find the optimal policy (risk control). For this model, I derive the corresponding HJB equation. Then, I propose a solution to this equation as explicitly as possible, and describe the optimal policy. In this chapter, I distinguish two cases depending on whether there is a correlation or not between the noise of the cash reserve rate and the noise of the cash reserve itself.

Chapter 5 considers the case when the cash reserve rate follows the Ornstein-Uhlenbeck process but cannot be observed, and the selection of the policies is based on the observation of the cash reserve process only. This leads to the problem of proportional reinsurance under partial information. To solve this problem, I make appeal to the filtering techniques to transfer the optimization problem under partial information to an optimization problem with full information. Then, I derive the HJB equation for the latter control problem with full information. Similarly as in the previous chapters, I suggest a solution to the HJB equation and describe the optimal policy.

Finally, the last chapter (Chapter 6) studies the case when the insurance company pays liability at a constant rate δ per unit time. Here, I derive the HJB equation and I connect it to an HJB equation of a model without liability.

Chapter 2

Mathematical Preliminaries

In this chapter, we introduce some mathematical and statistical tools that will be used throughout the rest of the thesis. This chapter is divided into two sections. The first section recalls stochastic calculus (including Ito's formula and topics on Brownian motion and martingales). The second section recalls the filtering techniques and namely the Kalman-Bucy filter.

2.1 Stochastic basis

The financial modeling of our system starts with a given filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P),$$

also called stochastic basis in the probabilistic literature. Here, P is a probability measure and $\mathcal{F}$ is a σ-algebra that contains all negligible sets. The family: $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is called filtration, where $\mathcal{F}_t$ is a σ-fields and

$$\mathcal{F}_0 \subseteq \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}, \qquad 0 \leq s \leq t.$$

The set σ-field $\mathcal{F}_t$ represents the cumulative information about the market under consideration up to time t. This explains the fact that the family of $\sigma - fields$, is a non-decreasing family. In economics, one of the most important random time is the bankruptcy time, which is a stopping time. Below, we recall the definition of this mathematical concept of stopping time.

Definition 2.1.1. *For a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, a stopping time is any nonnegative random variable τ satisfying*

$$\{\tau \leq t\} \in \mathcal{F}_t$$

for any $t \geq 0$. In some literature, stopping time is also called Markovian time.

Since an insurer can make a policy decision based on the previous information up to present time, then the policy factor should be measurable in some way with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Below, we recall the definition of adapted process.

Definition 2.1.2. *Consider a filtered probability space such as $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, and a stochastic process $X = (X_t)_{t\geq 0}$. Then X is said to be adapted to $(\mathcal{F}_t)_{t\geq 0}$ if X_t is $\mathcal{F}_t$-measurable for any $t \geq 0$.*

2.1.1 Martingales and Brownian motion

Definition 2.1.3. *A real-valued stochastic process* $W = \{W_t\}_{t\in[0,\infty)}$ *is called one-dimensional Brownian motion if it has continuous sample paths (for all* $w \in \Omega$*,* $t \to W_t(w)$ *is continuous) and satisfies the following properties:*

1) $P\{W_0 = 0\} = 1$

2) For every $0 \le s < t$*:*$(W_t - W_s)/\sqrt{t-s}$ *has a standard normal distribution.*

3) For every $0 \le r \le u \le s \le t$*:* $W_t - W_s$ *is independent of* $W_u - W_r$

An n-dimentional Brownian motion is an R^n-valued stochastic process

$$W = (W_1, W_2, \cdots, W_n)$$

with components W_i being independent one-dimensional Brownian motions.

Definition 2.1.4. *A filtration* $(\mathcal{F}_t)_{t\ge0}$ *satisfies the usual conditions if it is right-continuous and* $\mathcal{F}_0$ *contains all P-null sets of* $\mathcal{F}$*.*

In other words, if we have information up to time t, then nothing more can be learned by peeking infinitesimally far into the future. Throughout this thesis, I shall assume that the filtrations always satisfy the usual conditions.

In the following part, we introduce the martingale concept.

Definition 2.1.5. *Consider a real-valued stochastic process* $X = (X_t)_{t\ge0}$ *and* $(\mathcal{F}_t)_{t\ge0}$ *adapted process satisfying* $E|X_t| < \infty$ *for all* $t \ge 0$*.*

1) X is a super-martingale if

$$E[X_t|\mathcal{F}_s] \leq X_s, \quad 0 \leq s \leq t.$$

2) X is a sub-martingale if

$$E[X_t|\mathcal{F}_s] \geq X_s, \quad 0 \leq s \leq t.$$

3) X is a martingale if

$$E[X_t|\mathcal{F}_s] = X_s, \quad 0 \leq s \leq t.$$

Corollary 2.1.1. *A process $X = (X_t)_{t\geq 0}$ is a martingale if and only if is a super-martingale and a sub-martingale.*

Theorem 2.1.1. *A one-dimensional Brownian motion $W = \{W_t\}_{t\geq 0}$ is a martingale.*

Definition 2.1.6. *The Brownian motion with drift $\mu \in (-\infty, \infty)$ and volatility $\sigma \in (0, \infty)$ is the process $X = (X_t)_{t\geq 0}$ given by*

$$X(t) := \mu t + \sigma W(t), \qquad t \geq 0.$$

Corollary 2.1.2. *The Brownian motion with drift μ is a martingale if $\mu = 0$, a super-martingale if $\mu \leq 0$, and a sub-martingale if $\mu \geq 0$.*

2.1.2 Stochastic integral and Itô's formula

Before introducing the Itô's formula, we need to define the stochastic integral. We shall start with constructing it for so-called simple process X_t.

Definition 2.1.7. *A stochastic process $X = (X_t)_{t\geq 0}$ is called a simple process if there exist real number $0 = t_0 < t_1 < .... < t_p < +\infty, p \in \mathbb{N}$, and bounded random variables $\Phi_i \colon \Omega \to \mathbb{R}$*

$$\Phi_0 \qquad \mathcal{F}_0 - measurable, \qquad \Phi_i \qquad \mathcal{F}_{t_{i-1}} - measurable,$$

and

$$X_t(w) = X(t,w) = \Phi_0(w)I_0(t) + \sum_{i=1}^{p} \Phi_i(w)I_{(t_{i-1},t_i]}(t), \qquad w \in \Omega, t \geq 0$$

for each $w \in \Omega$.

Definition 2.1.8. *For a simple process $X = (X_t)_{t\geq 0}$ the stochastic integral $I_t(X)$ for $t \in (t_k, t_{k+1}]$ is defined according to*

$$I_t(X) := \int_0^t X_s dW_s := \sum_{1\leq i\leq k} \Phi_i(W_{t_i} - W_{t_{i-1}}) + \Phi_{k+1}(W_t - W_{t_k})$$

or more generally for $t \geq 0$:

$$I_t(X) := \int_0^t X_s dW_s := \sum_{1\leq i\leq p} \Phi_i(W_{t_i\wedge t} - W_{t_{i-1}\wedge t}).$$

In most of cases, simple process is a strict condition. So we need to define the stochastic integral for a more general process. We have to take a closer look at measurability assumptions for the stochastic process X to be able to define the stochastic integral for more general integrands in a reasonable way.

Definition 2.1.9. *Let $(X_t, \mathcal{F}_t)_{t\geq 0}$ be a stochastic process. This stochastic process will be called measurable if the mapping*

$$[0,\infty) \times \Omega \to \mathbb{R}^n : \qquad (s,w) \mapsto X_s(w)$$

is $\mathcal{B}([0,\infty]) \otimes \mathcal{F} - \mathcal{B}(\mathbb{R}^n)$ measurable.

Remark 2.1.1. *Measurability of the process* X *in particular implies that for a fixed* $w \in \Omega, X(\cdot, w)$ *is* $\mathcal{B}([0,\infty]) - \mathcal{B}(\mathbb{R}^n)$*-measurable. Thus, for all* $t \geq 0$*, the integral* $\int_0^t X_t^2(s)ds$ *is defined.*

Definition 2.1.10. *Let* $X = (X_t)_{t\geq 0}$ *be a stochastic process. This stochastic process will be called progressively measurable if for all* $t \geq 0$ *the mapping*

$$[0,t) \times \Omega \to \mathbb{R} : (s,w) \mapsto X_s(w)$$

is $\mathcal{B}([0,\infty]) \otimes \mathcal{F} - \mathcal{B}(\mathbb{R}^n)$ *measurable.*

Remark 2.1.2. *Every progressively measurable process is measurable.*

According to the above discussion, we require integrands to be progressively measurable when we want to extend the stochastic integral for a larger class of integrands than simple processes. Further to be able to define a norm for stochastic integrals, we consider the following vector space:
$L^2[0,T] := L^2\left([0,T], \Omega, \mathbb{F}, P\right)$
:=$(X_t)_{t\geq 0}$ real-valued progressively measurable process and $E\left(\int_0^T X_t^2 dt\right) < \infty$

Theorem 2.1.2. *(Construction of the Ito integral for process in* $L^2[0,T]$*) There exist a unique linear mapping* J *from* $L^2[0,T]$ *into the space of continuous martingales on [0,T] with respect to* $(\mathcal{F})_{t\in[0,T]}$ *satisfying*
(1) For any simple process, $X = \{X_t\}_{t\geq 0}$,

$$J_t(X) = I_t(X), \qquad t \geq 0.$$

(2) For any $X \in L^2[0,T]$*, we have*

$$E\left(J_t(X)^2\right) = E\left(\int_0^t X_s^2 ds\right).$$

Definition 2.1.11. *For $X \in L^2[0,T]$ and J as in Theorem 2.1.2, we denote*

$$\int_0^t X_s dW_s := J_t(X), \qquad t \geq 0.$$

J(X) is called the stochastic integral or the Ito integral of X with respect to W.

Then, we introduce Itô's formula for an n-dimensional Itô's process having the form of

$$X_i(t) = X_i(0) + \int_0^t K_i(s)ds + \sum_{j=1}^{m} \int_0^t H_{ij}(s)dW_j(s), \qquad i = 1, 2, ...n.$$

where K_i, H_{ij} are progressively measurable process such that

$$\int_0^t \left[|K_i(s)| + H_{ij}(s)^2 \right] ds < +\infty \qquad t \geq 0.$$

Theorem 2.1.3. *Let f be a continuous function that is continuously differentiable with respect to the first variable, and twice continuously differentiable with respect to the last n variables. Then, for every $t \geq 0$, the following holds.*

$f(t, X_1(t), ...X_n(t))$

$= f(0, X_1(0), ...X_n(0))$

$+ \int_0^t f_t(s, X_1(s), ...X_n(s)ds + \sum_{i=1}^n \int_0^t f_{x_i}(s, X_1(s), ..., X_n(s)))dX_i(s)$

$\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_0^t f_{x_i x_j}(s, X_1(s), ..., X_n(s))d < X_i, X_j >_s$

Theorem 2.1.4. *Let $W = (W_t)_{t \geq 0}$ be an m-dimensional Brownian motion $x \in R$, and A, a, S_j, σ_j be progressively measurable,real-valued stochastic process such that*

$$P\left\{ \forall t \geq 0 : \int_0^t (|A(s)| + |a(s)|)ds < \infty \right\} = 1,$$

$$P\left\{ \forall t \geq 0 : \int_0^t (\left|S_j^2(s)\right| + \left|\sigma_j^2(s)\right|)ds < \infty \right\} = 1.$$

Then the stochastic differential equation

$$dX_t = [A(t)X(t) + a(t)]\, dt + \sum_{j=1}^{m} [S_j(t)X(t) + \Sigma_j(t)]\, dW_j(t), \tag{2.1.2.1}$$
$$X(0) = x,$$

has a unique solution given by

$$X(t) = Z(t) \left(x + \int_0^t \frac{1}{Z_u} \left[a(u) - \sum_{j=1}^{m} S_j(u)\Sigma_j(u) \right] du + \sum_{j=1}^{m} \int_0^t \frac{\Sigma_j(u)}{Z(u)} dW_j(u) \right), \qquad t \geq 0. \tag{2.1.2.2}$$

Here

$$Z(t) := exp \left\{ \int_0^t (A(u) - \frac{1}{2} \|S(u)\|^2) du + \int_0^t S(u) dW(u) \right\}, \qquad t \geq 0.$$

For the proof of this theorem, we refer the reader to stanley (1997).

2.2 Filtering techniques

In chapter 3, we establish a model under full information. Actually, it is more realistic to assume that companies have only partial information. The drifts and paths of Brownian motions are just mathematical tools for model description and not observable.
Filtering problems consider estimating something about an unobserved stochastic process Y, given observations of a related process Λ. It is an important tool to transform a partial information problem into a complete information problem.

Given a probability space $(\Omega, \mathbb{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. All processes are assumed to be $\mathbb{F}$-adapted. Note that $\mathbb{F}$ is not the observation filtration. We call $\mathbb{F}$ the background filtration. Let us consider two one-dimensional processes:

(1) a signal process $Y := (Y_t)_{t\in[0,T]}$ which is not directly observable;

(2) an observation process $\Lambda = (\Lambda_t)_{t\in[0,T]}$ which is observable and somehow correlated with Y, so that by observing Λ we can know something about the distribution of Y.

Let $\mathbb{F}^\Lambda := (\mathcal{F}_t^\Lambda)_{t\geq 0}$ denote the observation filtration generated by Λ and is given by

$$\mathcal{F}_t^\Lambda := \sigma(\Lambda_s; 0 \leq s \leq t), \qquad t \geq 0.$$

The filtering problem consists of calculating the conditional distribution of the signal Y_t, given observations up to that time represented by $\mathcal{F}_t^\Lambda$.

To proceed further, we need to specify some particular model for the observation process.

2.2.1 Observation process

Let $W = (W_t)_{t\in[0,T]}$ be an $\mathbb{F}$-Brownian motion, let $G = (G_t)_{t\in[0,T]}$ be an $\mathbb{F}$-adapted process satisfying

$$E\int_0^T G_t^2 dt < \infty,$$

We assume that the observation process Λ has following the linear form

$$d\Lambda_t = G(t)Y_t dt + dW_t, \qquad t \in [0,T].$$

2.2.2 Innovation process

We introduce the filter estimate process $\widehat{Y}$, for any $\mathbb{F}$-adapted process Y, as the optional projection of Y onto the $\mathbb{F}^{\Lambda}$ filtration, i.e.

$$\widehat{Y}_t = E[Y_t|\mathcal{F}_t^{\Lambda}], \qquad t \in [0, T]. \tag{2.2.1}$$

Define the $\mathbb{F}^{\Lambda}$-adapted innovation process

$$N_t := \Lambda_t - \int_0^t (\widehat{G(t)Y_t})ds, \qquad t \in [0, T]. \tag{2.2.2}$$

Theorem 2.2.1. *The innovation process N is an $\mathbb{F}^{\Lambda}$-Brownian motion.*

The proof of this theorem can be found in [14] (Monoyios. (2009)).

2.2.3 The One-dimensional Kalman-Bucy filter

On a filtered probability space $(\Omega, \mathbb{F}, P)$, with background filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, let $Y = (Y_t)_{t\in[o,T]}$ be an $\mathbb{F}$-adapted signal process satisfying

$$dY_t = A(t)Y_t dt + C(t)dB_t.$$

and let $\Lambda = (\Lambda_t)_{t\in[0,T]}$ be an $\mathbb{F}$-adapted observation process satisfying

$$d\Lambda_t = G(t)Y_t dt + dW_t.$$

Here W and B are $\mathbb{F}$-Brownian motions with correlation coefficient ρ, and the coefficients $A(\cdot), C(\cdot)$ and $G(\cdot)$ are deterministic functions satisfying

$$\int_0^T (|A(t)| + C^2(t) + G^2(t))dt < \infty.$$

Define the observation filtration $\mathbb{F}^{\Lambda} = (\mathcal{F}_t^{\Lambda})_{t\in[0,T]}$ by

$$\mathcal{F}_t^{\Lambda} = \sigma(\Lambda_s : 0 \leq s \leq t).$$

Theorem 2.2.2. *Suppose Y_0 is an $\mathcal{F}_0$-measurable random variable, independent of W and B and its distribution is Gaussian with mean μ_0 and variance η_0.*
Then, the conditional expectation $\widehat{Y}_t := E[Y_t|\mathcal{F}_t^{\Lambda}]$, for $t \in [0,T]$ satisfies

$$d\widehat{Y}_t = A(t)\widehat{Y}_t dt + [G(t)V_t + \rho C(t)]dN_t, \qquad \widehat{Y}_t = \eta_0.$$

Here $N = (N_t)_{t\in[0,T]}$ is the innovations process, which is an $\mathbb{F}^{\Lambda}$-Brownian motion, given by (2.2.2).

$$dN_t = d\Lambda_t - G(t)\hat{Y}_t dt.$$

Furthermore, the conditional variance given by

$$V_t = Var[Y_t|\mathcal{F}_t^{\Lambda}] = E[(Y_t - \hat{Y}_t)^2|\mathcal{F}_t^{\Lambda}], \qquad t \in [0,T],$$

is independent of $\mathcal{F}_t^{\Lambda}$ and satisfies the deterministic Riccati equation

$$\frac{dV_t}{dt} = (1-\rho^2)C^2(t) + 2[A(t) - \rho C(t)G(t)]V_t - G^2(t)V_t^2, \qquad V_0 = \theta_0.$$

The proof of this theorem can be found in Monoyios.(2009).

Chapter 3

Deterministic Cash Reserve Rate

In this chapter, we consider the optimization problem of finding the optimal proportional reinsurance policy. For optimization of proportional reinsurance, we refer the reader to Gerber (1970), Sundt (1993).

In this thesis, we consider diffusion models for proportional reinsurance. These models shaped their ways into the proportional reinsurance problems via the works of Whittle (1983) and Dayananda (1993) and Højgaard and Taksar (1997). For more applications of control theory in insurance mathematics, we refer the reader to Højgaard and Taksar (1997) and the references therein.

In this chapter, we extend the work of Højgaard and Taksar (1998) to the case when the cash reserve rate of the insurance company is time independent but deterministic.

This chapter contains four sections. The first section discusses the model for the cash reserve process and formulates mathematically the objective. This goal takes the form of a stochastic control problem for which we need to find the optimal value function. The second section gives useful properties for this optimal value function, and describes it as a solution of an HJB equation. The third section constructs, as explicitly as possible, a solution to this resulting HJB equation. The last section (Section 4) describes the optimal policy and solves finally the stochastic control problem via a verification theorem.

3.1 Mathematical and Economic model

Herein, we start our mathematical model by a given filtered probability space that we denote by $(\Omega, \mathcal{F}, \mathbb{F} := \mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The σ-field $\mathcal{F}_t$ represents the information available up to time t and any decision is made based on this information. On this filtered probability space, we consider a one-dimensional standard Brownian motion $W = (W_t)_{t\geq 0}$. In this chapter, we consider an insurance company whose cash reserve at time t is denoted by R_t, for all $t \geq 0$. This gives us a stochastic process $R = (R_t)_{t\geq 0}$, which is also referred as the risk process in the insurance literature. The dynamics for the cash reserve process are given by

$$dR_t = \mu dt + \sigma dW_t, \quad t \geq 0, \quad R_0 = \text{initial cash reserve},$$

where $\mu > 0, \sigma > 0$.

The policy π is a proportional reinsurance policy that chooses a fraction $a_\pi(t)$ of the incoming claims that the company insures itself and the rest $1 - a_\pi(t)$ of the incoming claims to reinsurance. Let $R^\pi = (R^\pi_t)_{t\geq 0}$ be the risk process associated to the policy π. Under the assumption that the reinsuring companies have the same safety loadings as the insurance company, the dynamics for the cash reserve process $R^\pi = (R_t^\pi)_{t\geq 0}$ are given by

$$\begin{cases} dR_t^\pi = \mu_t a_t^\pi dt + a_t^\pi \sigma dW_t, \qquad t \geq 0 \\ \qquad\qquad R_0^\pi = R. \end{cases} \tag{3.1.1}$$

The policy π is admissible if the process $a_\pi = (a_\pi(t))_{t\geq 0}$ is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and we refer the stochastic process $a_\pi = (a_\pi(t))_{t\geq 0}$ as the control process.
The cash reserve rate process $(\mu_t)_{t\geq 0}$ satisfies the linear ordinary differential equation

$$d\mu_t = -\lambda\left(\mu_t - \bar{\mu}\right)dt, \qquad \mu_0 > 0, \tag{3.1.2}$$

where $\bar{\mu}$ is a positive constant.

Lemma 3.1.1. *The solution to (3.1.2) is given by*

$$\mu_t = \bar{\mu} + (\mu_0 - \bar{\mu})e^{-\lambda t} = e^{-\lambda t}\mu_0 + (1 - e^{-\lambda t})\bar{\mu}, \qquad t \geq 0. \tag{3.1.3}$$

Furthermore, the following properties holds:
(1) $\mu_t > 0$ *for all* $t \geq 0$.
(2) μ_t *belongs to the segment* $\left[\mu_0 \wedge \bar{\mu}, \mu_0 \vee \bar{\mu}\right]$.
(3) μ_t *is increasing if* $\mu_0 < \bar{\mu}$*;* μ_t *is decreasing if* $\mu_0 > \bar{\mu}$*, and is a constant if* $\mu_0 = \bar{\mu}$.

Proof. Notice that the equation (3.1.2) is a particular case of (2.1.2.1). Indeed, by putting

$$m = 1, \qquad \Sigma_1 = 0, \qquad S = 0, \qquad A = -\lambda, \qquad \textit{and } a = \lambda\bar{\mu},$$

on (2.1.2.1), we obtain equation (3.1.2). Hence, we deduce

$$Z(t) = e^{-\lambda t}, \qquad t \geq 0,$$

and (3.1.3) follows immediately from (2.1.2.2). Now, we prove the remaining property of μ. By differentiating (3.1.3), we get

$$(\mu_t)^{'} = -\lambda(\mu_0 - \bar{\mu})e^{-\lambda t}.$$

Therefore, $(\mu_t)' > 0$ if $\mu_0 < \bar{\mu}$ and hence μ_t is increasing in this case; $(\mu_t)' < 0$ if $\mu_0 > \bar{\mu}$ and hence μ_t is decreasing. If $\mu_0 = \bar{\mu}$, we have $\mu_t = 0$, and hence $\mu_t = \mu_0 = \bar{\mu}$.
Since $1 - e^{-\lambda t} \geq 0$, and both μ_0 and $\bar{\mu}$ are positive, we conclude that μ_t is always positive. This ends the proof of the lemma. □

We end this subsection by formulating mathematically our objective. To this end, we consider a discount factor, $c > 0$. Let π be an admissible policy and $R^\pi = (R_t^\pi)_{t\geq 0}$ be the corresponding cash reserve process given by the SDE (3.1.1). Then, bankruptcy time for this cash reserve is defined by

$$\tau_\pi = inf\left\{t \geq 0 : R_t^\pi = 0\right\}.$$

Here, by convention we put $inf(\phi) = +\infty$. Thus, the return function associated to the policy π is denoted by $J_\pi(R, \eta)$, and is given by

$$J_\pi(R, \eta) := E\left(\int_0^{\tau_\pi} e^{-ct} R_t^\pi dt | R_0^\pi = R, \mu_0 = \eta\right).$$

Therefore, our objective lies in describing the optimal value function

$$V(R, \eta) := \sup_\pi J_\pi(R, \eta), \qquad \forall R \geq 0, \quad \eta \geq 0, \tag{3.1.5}$$

and finding the optimal policy π^* that satisfies

$$V(R, \eta) = J_{\pi^*}(R, \eta), \qquad \forall R \geq 0, \quad \eta \geq 0. \tag{3.1.6}$$

3.2 Properties of the optimal value function

Proposition 3.2.1. *The optimal value function V, defined by (3.1.5), is concave and satisfies*

$$0 \leq V(R,\eta) \leq \frac{R}{c} + \frac{\bar{\mu}}{c^2} + \frac{\eta - \bar{\mu}}{c(c+\lambda)}, \qquad \forall R \geq 0, \quad \eta \geq 0. \tag{3.2.1}$$

Proof. Let R_1, R_2 and λ be three positive numbers such that $\lambda \in (0,1)$. Consider three admissible policies π_1, π_2 and π that corresponds to the initial reserve R_1, R_2 and R respectively, where

$$R := \lambda R_1 + (1-\lambda)R_2, \qquad \textit{and} \qquad \pi = \lambda\pi_1 + (1-\lambda)\pi_2.$$

Then it is clear that

$$R_t^{\pi} = \lambda R_t^{\pi_1} + (1-\lambda)R_t^{\pi_2}, \qquad \textit{and} \qquad \tau_\pi = \tau_{\pi_1} \vee \tau_{\pi_2}.$$

Since $R_1^{\pi_1} = 0$ on $[\tau_{\pi_1}, \tau_\pi]$ and $R_2^{\pi_2} = 0$ on $[\tau_{\pi_2}, \tau_\pi]$, we obtain

$$\begin{aligned} J_\pi(R,\eta) &= E\left(\int_0^{\tau_\pi} e^{-ct} R_t^{\pi} dt | R_0^{\pi} = R, \mu_0 = \eta\right) \\ &= E\left(\int_0^{\tau_{\pi_1}\vee\tau_{\pi_2}} e^{-ct}\left[\lambda R_t^{\pi_1} + (1-\lambda)R_t^{\pi_2}\right] dt | R_0^{\pi} = R, \mu_0 = \eta\right) \\ &= \lambda J_{\pi_1}(R_1,\eta) + \lambda J_{\pi_2}(R_2,\eta). \end{aligned}$$

For any $\varepsilon > 0$ we can choose π_i such that $V(R_i,\eta) - \varepsilon < J_\pi(R_i,\eta) \leq V(R_i,\eta)$. Since π is suboptimal, then we have

$$\begin{aligned} \lambda V(R_1,\eta) + (1-\lambda)V(R_2,\eta) - \varepsilon &< \lambda J_{\pi_1}(R_1,\eta) + (1-\lambda)J_{\pi_2}(R_2,\eta) \\ &= J_\pi(R,\eta) \leq V(R,\eta). \end{aligned}$$

Then, by letting ε go to zero, we prove the concavity of V. Hence the rest of proof focuses on proving (3.2.1). To this end, due to

$$R^{\pi}_{t\wedge\tau_\pi} = R^{\pi}_0 + \int_0^{\tau_\pi\wedge t} \mu_s a^{\pi}_s ds + \int_0^{\tau_\pi\wedge t} \sigma a^{\pi}_s dW_s,$$

we derive

$$\begin{aligned} E\left(R^{\pi}_{t\wedge\tau_\pi}|R^{\pi}_0 = R, \mu_0 = \eta\right) &= R + E\left(\int_0^{t\wedge\tau_\pi} \mu_s a^{\pi}_s ds|R^{\pi}_0 = R, \mu_0 = \eta\right) \\ &\leq R + \int_0^t \left[\bar{\mu} + (\eta - \bar{\mu})e^{-\lambda s}\right] ds. \end{aligned}$$

As a result, for any admissible policy π, we get

$$\begin{aligned} J_\pi(R,\eta) &\leq \int_0^\infty e^{-ct}\left[R + \bar{\mu}t + \frac{\eta - \bar{\mu}}{\lambda}(1 - e^{-\lambda t})\right] dt \\ &= \frac{R}{c} + \frac{\bar{\mu}}{c^2} + \frac{\eta - \bar{\mu}}{c(c+\lambda)}. \end{aligned}$$

Then, (3.2.1) follows immediately from this inequality. This ends the proof of the proposition. □

In the following, we connect the optimal value function to an HJB equation under conditions of smoothness.

Theorem 3.2.1. *If the optimal value function $V(R,\eta)$, defined by (3.1.5), is twice continuously differentiable on $(0,\infty)$, then V satisfies the Hamilton-Jacobi-Bellman equation*

$$\begin{cases} R - cV - \lambda\left(\eta - \bar{\mu}\right)V_\eta + \max_{a\in[0,1]}\left[a\eta V_R + \frac{1}{2}a^2\sigma^2 V_{RR}\right] = 0, \\ V(0) = 0. \end{cases} \tag{3.2.2}$$

Proof. By applying Ito's formula, we derive

$$dV\left(R^{\pi}_t, \mu_t\right) = V_R dR^{\pi}_t + V_\eta d\mu_t + \frac{1}{2}V_{RR} d\langle R^{\pi}, R^{\pi}\rangle_t + \frac{1}{2}V_{\eta\eta} d\langle \mu, \mu\rangle_t + V_{R\eta} d\langle R^{\pi}, \mu\rangle_t$$

$$= V_R\left(\mu_t a_t^\pi dt + \sigma a_t^\pi dW_t\right) + V_\eta\left[-\lambda\left(\mu_t - \bar{\mu}\right)dt\right] + \frac{1}{2}V_{RR}\sigma^2 a_t^2 dt. \tag{3.2.3}$$

Put

$$Y_t^\pi := \int_0^t e^{-cs}R_s^\pi ds + e^{-ct}V(R_t^\pi, \mu_t), \qquad t \geq 0.$$

Thanks to Bellman's principle (dynamic programming principle), the optimal value function $V(R,\eta)$, is such that $(Y_t^\pi)_{t\geq 0}$ is a supermartingale for any admissible policy π and is a local martingale for the optimal and admissible policy π^*. Hence, in virtue of (3.2.3), Y^π is a supermartingale if

$$R - cV - \lambda\left(\eta - \bar{\mu}\right)V_\eta + \underset{0\leq a\leq 1}{max}\left[a^\pi \eta V_R + \frac{1}{2}(a^\pi)^2\sigma^2 V_{RR}\right] \leq 0. \tag{3.2.4}$$

and Y^{π^*} is a local martingale if

$$R^{\pi^*} - cV - \lambda\left(\eta - \bar{\mu}\right)V_\eta + \left[a^{\pi^*}\eta V_R + \frac{1}{2}(a^{\pi^*})^2\sigma^2 V_{RR}\right] = 0.$$

Thus, by combining this equation with (3.2.4), we deduce that V is a solution to the HJB equation (3.2.2). This ends the proof of the theorem. □

Now, our goal is to construct a solution to the HJB equation (3.2.2). This is the aim of the following subsection.

3.3 Construction of the solution to the HJB

Assume there exists an open set $O \subseteq [0,\infty)$ such that $a(R,\eta)$ satisfies $0 < a(R,\eta) < 1$ for all $R \in O$. Then for any $R \in O$ and $\eta > 0$, the maximizer

$$a^*(R,\eta) := \underset{a}{argmax}\Big[a\eta V_R(R,\eta) + \frac{1}{2}a^2\sigma^2 V_{RR}(R,\eta)\Big]$$

is given by

$$a^*(R,\eta) = -\frac{\eta V_R(R,\eta)}{\sigma^2 V_{RR}(R,\eta)}. \tag{3.3.1}$$

By inserting (3.3.1) into (3.2.2), we get

$$R - cV - \lambda(\eta - \bar{\mu})V_\eta - \frac{\eta^2}{2\sigma^2}\frac{V_R^2}{V_{RR}} = 0. \tag{3.3.2}$$

In order to solve this partial differential equation, we assume

$$V_R\Big(R(z,\eta),\eta\Big) = exp\Big(-z + m(\eta)\Big). \tag{3.3.3}$$

This transformation is a slight modification of a transformation used frequently in the finance literature, e.g. Karatzas et al.(1998) and Højggard and Taskar (1998).

Then, we calculate the following derivatives

$$V_{RR} = -\frac{V_R}{R_z}, \qquad V_{R\eta} = (m' + \frac{R_\eta}{R_z})V_R.$$

By substituting these equations in (3.3.2) and using (3.3.3), we obtain

$$R(z,\eta) - cV(R(Z,\eta),\eta) - \lambda(\eta - \bar{\mu})V_\eta(R(z,\eta),\eta) + \frac{\eta^2}{2\sigma^2}R_z(z,\eta)V_R(R(z,\eta),\eta) = 0. \tag{3.3.4}$$

By differentiating this equation with respect to z and using (3.3.3) again, we derive

$$R_z\left\{1 - cV_R - \frac{\eta^2}{2\sigma^2}V_R\right\} - \lambda(\eta - \bar{\mu})(R_z m' + R_\eta)V_R = -\frac{\eta^2}{2\sigma^2}R_{zz}V_R.$$

This can be written as

$$\frac{R_{zz}}{R_z} = (-\frac{2\sigma^2}{\eta^2}e^{z-m} + \frac{2\sigma^2 c}{\eta^2} + 1) + \frac{2\sigma^2}{\eta^2}\lambda(\eta - \bar{\mu})(m' + \frac{R_\eta}{R_z}). \tag{3.3.5}$$

In order to simplify the problem, we put

$$R_z(z,\eta) = exp\Big[z - m(\eta) + L(e^{z-m(\eta)},\eta)\Big], \tag{3.3.6}$$

where L is a function to be described later on.

The solution to this equation is given by

$$R(z,\eta) = \int_{-\infty}^{z} exp\Big[x - m(\eta) + L(e^{x-m(\eta)}, \eta)\Big] dx + k(\eta).$$

By changing the variable, we get

$$R(z,\eta) = \int_{0}^{e^{z-m}} exp\Big[L(x,\eta)\Big] dx + k(\eta) = G(e^{z-m(\eta)}, \eta) + k(\eta). \tag{3.3.7}$$

where $G(x,\eta)$ is given by

$$G(x,\eta) = \int_{0}^{x} exp\left\{L(y,\eta)\right\} dy. \tag{3.3.8}$$

By mimicking the proof of Højggard and Taskar (1998), we deduce that $k(\eta) = 0$. By differentiating (3.3.8) with respect to the variable x, we get

$$G_x(x,\eta) = exp\Big[L(x,\eta)\Big], \qquad \forall \eta > 0, \quad x > 0. \tag{3.3.9}$$

From (3.3.9), we deduce that $G_x > 0$ for all $x > 0$ and hence $G(x,\eta)$ is strictly increasing on $(0,\infty)$ and continuous. Therefore $G(x,\eta)$ is invertible, and (3.3.7) leads to

$$exp\Big[z - m(\eta)\Big] = G^{-1}(R,\eta).$$

Then by plugging this resulting equation in (3.3.3), we obtain

$$V_R(R,\eta) = \frac{1}{G^{-1}(R,\eta)}, \qquad \forall R > 0, \quad \eta > 0. \tag{3.3.10}$$

To determine completely the function V on a neighborhood of zero, we need to describe the function L introduced in (3.3.6). This is the aim of the following.

Lemma 3.3.1. *The function L, given by (3.3.6), satisfies the following PDE*

$$x^2\left[L_x^2 + L_{xx}\right] + 2xL_x = \frac{2\sigma^2 c}{\eta^2} - \frac{4\sigma^2}{\eta^2}x + \frac{2\sigma^2}{\eta^2}(c-x)xL_x + \frac{2\sigma^2\lambda}{\eta^2}(\eta - \bar{\mu})L_\eta. \tag{3.3.11}$$

Proof. From (3.3.6), we calculate the following:

$$R_{zz}(z,\eta) = R_z + e^{z-m}L_xR_z = R_z(1+e^{z-m}L_x), \tag{3.3.12}$$

$$R_{z\eta}(z,\eta) = (-m' - m'e^{z-m}L_x + L_\eta)R_z.$$

By plugging (3.3.6) and (3.3.12) into (3.3.5), we get

$$R_z(1+e^{z-m}L_x) = \left\{1 - \frac{2\sigma^2}{\eta^2}e^{z-m} + \frac{2\sigma^2 c}{\eta^2} + \frac{2\sigma^2\lambda}{\eta^2}(\eta-\bar{\mu})m'\right\}R_z + \frac{2\sigma^2\lambda}{\eta^2}(\eta-\bar{\mu})R_\eta. \tag{3.3.13}$$

By differentiating this equation with respect to z again, we obtain

$$(1+e^{z-m}L_x)^2 + e^{z-m}L_x + e^{2(z-m)}L_{xx} = 1 - \frac{2\sigma^2}{\eta^2}e^{z-m} + \frac{2\sigma^2 c}{\eta^2} + \frac{2\sigma^2\lambda}{\eta^2}(\eta-\bar{\mu})m'$$
$$+e^{z-m}L_x\left(1 - \frac{2\sigma^2}{\eta^2}e^{z-m} + \frac{2\sigma^2 c}{\eta^2}\right) + \frac{2\sigma^2\lambda}{\eta^2}(\eta-\bar{\mu})m'e^{z-m}L_x - \frac{2\sigma^2}{\eta^2}e^{z-m} - \frac{2\sigma^2\lambda}{\eta^2}(\eta-\bar{\mu})m'$$
$$-\frac{2\sigma^2\lambda}{\eta^2}(\eta-\bar{\mu})m'e^{z-m}L_x + \frac{2\sigma^2\lambda}{\eta^2}(\eta-\bar{\mu})L_\eta.$$

After simplifying this equation, we get

$$(1+e^{z-m}L_x)^2 + e^{z-m}L_x + e^{2(z-m)}L_{xx} = 1 - \frac{4\sigma^2}{\eta^2}e^{z-m} + \frac{2\sigma^2 c}{\eta^2} + e^{z-m}\left(1 - \frac{2\sigma^2}{\eta^2}e^{z-m} + \frac{2\sigma^2 c}{\eta^2}\right)L_x$$
$$+\frac{2\sigma^2\lambda}{\eta^2}(\eta-\bar{\mu})L_\eta.$$

This can be written as

$$e^{2(z-m)}\left[L_x^2 + L_{xx}\right] + 2e^{z-m}L_x = \frac{2\sigma^2 c}{\eta^2} - \frac{4\sigma^2}{\eta^2}e^{z-m} + \frac{2\sigma^2}{\eta^2}(c - e^{z-m})e^{z-m}L_x + \frac{2\sigma^2\lambda}{\eta^2}(\eta-\bar{\mu})L_\eta.$$

By changing the variable (i.e. $x = e^{z-m}$), we obtain

$$x^2\left[L_x^2 + L_{xx}\right] + 2xL_x = \frac{2\sigma^2 c}{\eta^2} - \frac{4\sigma^2}{\eta^2}x + \frac{2\sigma^2}{\eta^2}(c-x)xL_x + \frac{2\sigma^2\lambda}{\eta^2}(\eta-\bar{\mu})L_\eta.$$

which is exactly the PDE (3.3.12). This ends the proof of the lemma. □

By differentiating (3.3.10) with respect to R, we deduce that (3.3.1) becomes

$$a^*(R,\eta) = \frac{\eta}{\sigma_2} G^{-1}(R,\eta) G_x(G^{-1}(R,\eta),\eta), \qquad R > 0, \tag{3.3.14}$$

and $a^*(0,\eta) = 0$. Then, Put

$$y = G^{-1}(R,\eta), \qquad and \qquad a_1(y,\eta) = \frac{\eta}{\sigma^2} y G_x(y,\eta).$$

By differentiating $a_1(y,\eta)$ with respect to y, we get

$$\frac{\partial}{\partial y} a_1(y,\eta) = \frac{\eta}{\sigma^2}\Big[1 + yL_y\Big] exp\Big[L(y,\eta)\Big].$$

Assumption 3.3.1. *Assume (3.3.11) has a solution $L(y,\eta)$ such that the equation*

$$y exp\Big[L(y,\eta)\Big] = \frac{\sigma^2}{\eta} \tag{3.3.15}$$

has a root $y_1(\eta) \in (0,c)$. Then, under this assumption, we have

$$R_1(\eta) = G(y_1(\eta),\eta). \tag{3.3.16}$$

According to our assumptions $a^*(R,\eta) = 1$ for $R > R_1(\eta)$. By substituting $a = 1$ into (3.3.1), we obtain the following equation:

$$R - cV + \eta V_R - \lambda(\eta - \bar{\mu})V_\eta + \frac{1}{2}\sigma^2 V_{RR} = 0. \tag{3.3.17}$$

In order to solve this partial differential equation explicitly, we propose the following

$$V(R,\eta) = C_1(\eta)e^{d(\eta)R} + C_2(\eta) + \frac{R}{c}.$$

Then, we calculate the derivatives as follows

$$V_R = C_1 d e^{dR} + \frac{1}{c}, \qquad V_{RR} = C_1 d^2 e^{dR}, \qquad V_\eta = C_1' e^{dR} + C_1 d' e^{dR} + C_2'.$$

By inserting these in (3.3.17), we get

$$e^{dR}[-cC_1 + C_1\eta d - \lambda(\eta-\bar{\mu})(C_1' + d') + \frac{1}{2}\sigma^2 C_1 d^2] + [-cC_2 + \frac{\eta}{c} - \lambda(\eta-\bar{\mu})C_2'] = 0.$$

Since the above equation holds for all values of $R > R_1(\eta)$, we must have:

$$-cC_1 + C_1\eta d - \lambda\left(\eta-\bar{\mu}\right)\left(C_1' + d'\right) + \frac{1}{2}\sigma^2 C_1 d^2 = 0, \tag{3.3.18}$$

and

$$-cC_2 + \frac{\eta}{c} - \lambda(\eta-\bar{\mu})C_2' = 0. \tag{3.3.19}$$

It is clear that the general solution to this ODE is given by

$$C_2(\eta) = \frac{1}{c(c+\lambda)}\left|\eta-\bar{\mu}\right| + \frac{\bar{\mu}}{c^2} + C_3\left|\eta-\bar{\mu}\right|^{-\frac{c}{\lambda}}.$$

where C_3 is an arbitrary constant.

Therefore, for $R > R_1(\eta)$, the optimal return function takes the form of

$$V(R,\eta) = C_1(\eta)e^{d(\eta)R} + \frac{1}{c(c+\lambda)}\left|\eta-\bar{\mu}\right| + \frac{\bar{\mu}}{c^2} + C_3\left|\eta-\bar{\mu}\right|^{-\frac{c}{\lambda}} + \frac{R}{c}. \tag{3.3.20}$$

Thanks to Proposition 3.2.1, for $R > R_1(\eta)$, we get $0 \leq \underset{\eta\to\bar{\mu}}{limsup} V(R,\eta) < \infty$. By combining this with (3.3.20), we deduce that $C_3 = 0$. As a result, we get

$$V(R,\eta) = \begin{cases} \displaystyle\int_0^R \frac{1}{G^{-1}(x,\eta)}dx, & \text{if } \ 0 \leq R \leq R_1(\eta) \\ \\ C_1(\eta)\exp\left[d(\eta)(R-R_1)\right] + \dfrac{c\left|\eta-\bar{\mu}\right|}{c(c+\lambda)} + \dfrac{\bar{\mu}+cR}{c^2} & \text{if } \ R \geq R_1(\eta). \end{cases} \tag{3.3.21}$$

To ensure that V is twice continuously differentiable, it is necessary and sufficient that its value, first and second derivative are continuous at the point R_1. To this end, we put

$$V_1(R,\eta) := \int_0^R \frac{1}{G^{-1}(x,\eta)}dx, \qquad V_2(R,\eta) := C_1(\eta)\exp\left[d(\eta)(R-R_1)\right] + \frac{c\left|\eta-\bar{\mu}\right|}{c(c+\lambda)} + \frac{\bar{\mu}+cR}{c^2}.$$

We consider the first and second derivative of the return function V at the point R_1.

$$V_1^{'}(R_1) = \frac{1}{y_1}, \qquad V_2^{'}(R_1) = \frac{1}{c} + C_1 d.$$

$$V_1^{''}(R_1) = -\frac{\eta}{\sigma_1^2} V_1^{'}(R_1) = -\frac{\eta}{\sigma_1^2}\frac{1}{y_1}, \qquad V_2^{''}(R_1) = C_1 d^2.$$

Then V_R and V_{RR} are continuous at $R = R_1(\eta)$ if and only if

$$\frac{1}{y_1} = \frac{1}{c} + C_1 d, \qquad -\frac{\eta}{\sigma^2}\frac{1}{y_1} = C_1 d^2.$$

This implies that

$$d^2 + \frac{\eta}{\sigma^2} d + \frac{\eta}{c\sigma^2 C_1} = 0. \tag{3.3.22}$$

Since $d < 0$, the solution to this equation is

$$d(\eta) = \frac{1}{2}\left\{ -\frac{\eta}{\sigma^2} - \sqrt{\frac{\eta^2}{\sigma^4} - \frac{4\eta}{c\sigma^2 C_1(\eta)}} \right\}. \tag{3.3.23}$$

Then, we calculate the derivative

$$d'(\eta) = \frac{1}{2}\left\{ -\frac{1}{\sigma^2} - \frac{\frac{\eta}{\sigma^4} - \frac{2}{c\sigma^2 C_1(\eta)} + \frac{2\eta C_1'(\eta)}{c\sigma^2 C_1(\eta)^2}}{\sqrt{\frac{\eta^2}{\sigma^4} - \frac{4\eta}{c\sigma^2 C_1(\eta)}}} \right\}. \tag{3.3.24}$$

By plugging (3.3.23) and (3.3.24) into (3.3.17), we get

$$C_1'(\eta) + d'(\eta) = \frac{C_1(\eta)\Big(\eta d(\eta) - c + \frac{1}{2}\sigma^2 d^2(\eta)\Big)}{\lambda(\eta - \bar{\mu})},$$

and

$$C_1'(\eta) = \left\{ \frac{1}{2\sigma^2} + \frac{\frac{\eta}{\sigma^4} - \frac{2}{c\sigma^2 C_1}}{2\sqrt{\frac{\eta^2}{\sigma^4} - \frac{4\eta}{c\sigma^2 C_1}}} - \left(\frac{\eta^2}{4\sigma^2} + \frac{\eta}{4}\sqrt{\frac{\eta^2}{\sigma^4} - \frac{4\eta}{c\sigma^2 C_1}} + c \right) \frac{C_1}{\lambda(\eta - \bar{\mu})} \right\} \frac{1}{1 - \frac{\eta}{c\sigma^2 C_1}\sqrt{\frac{\eta^2}{\sigma^4} - \frac{4\eta}{c\sigma^2 C_1}}}. \tag{3.3.25}$$

This proves the following theorem.

Theorem 3.3.1. *Suppose that Assumption 3.3.1 holds, then the solution to the HJB equation (3.2.1) is given by*

$$V(R,\eta) = \begin{cases} \displaystyle\int_0^R \frac{1}{G^{-1}(R,\eta)}dx, & \text{if } \ 0 \le R \le R_1(\eta) \\ \\ -\dfrac{\sigma^2\eta \exp\left[d(\eta)(R - R_1(\eta))\right]}{c\sigma^2(\sigma^2 d^2(\eta) + \eta d(\eta))} + \dfrac{|\eta - \bar{\mu}|}{c(c+\lambda)} + \dfrac{\bar{\mu} + cR}{c^2}, & \text{if } \ R \ge R_1(\eta). \end{cases} \tag{3.3.26}$$

Here

$$R_1(\eta) := G(y_1(\eta),\eta), \ \ G(x,\eta) = \int_0^x exp\left\{L(y,\eta)\right\}dy, \ \ d(\eta) = \frac{1}{2}\left\{-\frac{\eta}{\sigma^2} - \sqrt{\frac{\eta^2}{\sigma^4} - \frac{4\eta}{c\sigma^2 C_1(\eta)}}\right\},$$

where $y_1(\eta)$ is the root of (3.3.15), and $C_1(\eta)$ is the solution to ODE (3.3.25).

3.4 Optimal policy and the verification theorem

In this section we construct the optimal policy based on the solution to the HJB equation obtained in the previous section. Recall that $R_1(\eta) = G^{-1}(y_1(\eta),\eta)$, where $y_1(\eta)$ is root of (3.3.13). For $R \le R_1(\eta)$, we obtain

$$a^*(R,\eta) := \underset{a}{argmax}\Big[a\eta V_R(R,\eta) + \frac{1}{2}a^2\sigma^2 V_{RR}(R,\eta)\Big].$$

As evident from below the function $a^*(R,\eta)$ represents the optimal feedback control function for the control component $a_\pi = \Big(a_\pi(t)\Big)_{t\ge 0}$. More precisely, the value $a^*(R,\eta)$ is the optimal risk that one should take when the value of the current reserve is R and the reserve rate is η. From the analysis of the previous section, it follows that $a^*(R,\eta)$ can be represented as

$$a^*(R,\eta) = \begin{cases} \dfrac{\eta}{\sigma^2}G^{-1}(R,\eta)G_x(G^{-1}(R,\eta),\eta) & \text{if } \ \ 0 \le R \le R_1(\eta) \\ \quad 1, & \text{if } \ \ R \ge R_1(\eta). \end{cases} \tag{3.4.1}$$

For any $0 \le a \le 1$, we define the differential operator $\mathcal{L}^a$ by

$$\mathcal{L}^a f(R,\eta) = \frac{\sigma^2 a^2}{2} f_{RR}(R,\eta) + a\eta f_R - cf(R,\eta) - \lambda(\eta-\mu) f_\eta(R,\eta) \tag{3.4.2}$$

For any $f \in C^{2\times 1}((0,\infty)\times(0,\infty))$. From the previous section, it is clear that

$$\mathcal{L}^{a^*(R,\eta)} V(R,\eta) = -R. \tag{3.4.3}$$

Let $R^*_t = (R^*_t)_{t\ge 0}$ be a solution to the following Skorohod problem :

$$R^*_t = R + \int_0^t a(R^*_s,\mu_s)\mu_s ds + \int_0^t \sigma a(R^*_s,\mu_s) dW_s, \tag{3.4.4}$$

$$R^*_t \le R_1(\mu_t).$$

Theorem 3.4.1. *Let V be a concave, twice continuously differentiable solution of the HJB equation (3.2.1) and $(R^*_t)_{t\ge 0}$ be a solution to the Skorohod problem (3.4.4).*
Then for $\pi^* := \Big(a^*(R^*_t,\mu_t)\Big)_{t\ge 0}$, we have

$$J_{\pi^*}(R,\eta) = V(R,\eta), \qquad \forall R \ge 0, \quad \eta \ge 0.$$

Proof. Notice that $R^* = R^{\pi^*}$. Let $R_0 = R$ and $\mu_0 = \eta$. Choose $0 < \varepsilon < R$ and let $\tau^\varepsilon_* = inf\,\{t : R^*_t = \varepsilon\}$, then Ito's formula yields,

$$\begin{aligned} e^{-c(t\wedge\tau^\varepsilon_*)} V(R^*_{t\wedge\tau^\varepsilon_*},\mu_t) &= V(R,\eta) + \int_0^{t\wedge\tau^\varepsilon_*} e^{-cs}\mathcal{L}^{a_{\pi^*}(s)} V(R^*_s,\mu_s) ds + \int_0^{t\wedge\tau^\varepsilon_*} e^{-cs}\sigma a_{\pi^*}(s) V_R(R^*_s,\mu_s) dW_s \\ &= V(R,\eta) - \int_0^{t\wedge\tau^\varepsilon_*} e^{-cs} R^*_s ds + \int_0^{t\wedge\tau^\varepsilon_*} e^{-cs}\sigma a_{\pi^*}(s) V_R(R^*_s,\mu_s) dW_s \end{aligned} \tag{3.4.5}$$

Since $V_R(R^*_s,\mu_s) \le V_R(\varepsilon,\mu_s) < \infty$ on $[0, t\wedge\tau^\varepsilon_*]$, the last term on the r.h.s. is a zero-mean martingale. Taking expectations in (3.4.5), we obtain

$$\mathbb{E}\left[e^{-c(t\wedge\tau^\varepsilon_*)} V(R^*_{t\wedge\tau^\varepsilon_*},\mu_t)\right] + \mathbb{E}\int_0^{t\wedge\tau^\varepsilon_*} e^{-cs} R^*_s ds = V(R,\eta). \tag{3.4.6}$$

Since $\tau_*^\varepsilon \to \tau_{\pi^*}$ when $\varepsilon \to 0$

$$\int_0^{t\wedge\tau_*^\varepsilon} e^{-cs} R_s^* ds \to \int_0^{t\wedge\tau_{\pi^*}} e^{-cs} R_s^* ds.$$

Thus letting $\varepsilon \to 0$ in (3.4.6), we use dominated and monotone convergence theorems for the first and the second terms in (3.4.6) respectively, we get

$$\mathbb{E}\left[e^{-c(t\wedge\tau_{\pi^*})} V(R_{t\wedge\tau_{\pi^*}^{\pi^*}, \mu_t)})\right] + \mathbb{E}\int_0^{t\wedge\tau_{\pi^*}} e^{-cs} R_s^{\pi^*} ds = V(R,\eta). \tag{3.4.7}$$

Since $V(0,\eta) = 0$, we have

$$\mathbb{E}\left[e^{-c(t\wedge\tau_{\pi^*})} V(R_{t\wedge\tau_{\pi^*}}^{\pi^*}, \mu_t)\right] = e^{-ct}\mathbb{E}\left[V(R_t^{\pi^*}, \mu_t); t < \tau_{\pi^*}\right] \to 0.$$

as $t \to \infty$. Letting $t \to \infty$ on (3.4.7), we have

$$\mathbb{E}\int_0^{\tau_{\pi^*}} e^{-cs} R_s^* ds = V(R,\eta).$$

Therefore, $V(R,\eta) = J_{\pi^*}(R,\eta)$.

This ends the proof of the theorem. □

Chapter 4

Stochastic Cash Reserve Rate: The Case of Full Information

This chapter extends the analysis of Chapter 3 to the case when the dynamics of the cash reserve rate $\mu = (\mu_t)_{t\geq 0}$ has a noise that is observable. For this model of cash reserve process, we address the same objective as in Chapter 3.

This chapter contains three sections. In the first section, we define our model for the cash reserve process, and derive the corrsponding HJB equation. The second section deals with solving this obtained HJB equation. The last section describes the optimal policy and gives a verification theorem.

4.1 The model and its HJB equation

This section is divided into three subsections. The first subsection describes the model and the objective. The second subsection discusses some properties of the resulting optimal value function, while the third subsection derives the HJB equation for this optimal value function.

4.1.1 The model and the objective

For a given insurance company, we let $R^\pi = (R_t^\pi)_{t\geq 0}$ be the risk process associated to the policy π. Under the assumption that the reinsuring companies have the same safety loadings as the insurance company, the dynamic for the cash reserve process $R^\pi = (R_t^\pi)_{t\geq 0}$ is given by

$$dR_t^\pi = \mu_t a_\pi(t)dt + \sigma_1 a_\pi(t)dW_t, \qquad t \geq 0. \tag{4.1.1.1}$$

$$R_0^\pi = R.$$

The policy π is admissible if the process $a_\pi = (a_\pi(t))_{t\geq 0}$ is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. The drift process $(\mu_t)_{t\geq 0}$ satisfies the Ornstein Uhlenbeck stochastic differential equation

$$d\mu_t = -\lambda(\mu_t - \bar{\mu})dt + \sigma_2 dB_t, \qquad \mu_0 > 0. \tag{4.1.1.2}$$

Here, $(W_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are two $(\mathcal{F}_t)_{t\geq 0}$-adapted Brownian motions with correlation coefficient $\rho \in [-1, 1]$. The initial value of the drift process μ_0 is assumed to be a $\mathcal{F}_0$-measurable Gaussian random variable, which is independent of Brownian motions W and B. We also assume that all coefficients $\sigma_1 > 0$, $\lambda > 0$, $\bar{\mu} > 0$, $\sigma_2 > 0$ are constants.

Lemma 4.1.1. *The solution to (4.1.1.2) is given by*

$$\mu_t = \mu_0 e^{-\lambda t} + \bar{\mu}(1 - e^{-\lambda t}) + \sigma_2 e^{-\lambda t} \int_0^t e^{\lambda s} dB_s. \tag{4.1.1.3}$$

Proof. Notice that the equation (4.1.1.3) is a particular case of (2.1.2.1). Indeed, by putting

$$m = 1, \qquad W = B \qquad \Sigma_1 = \sigma_2, \qquad S = 0, \qquad A = -\lambda, \qquad and \quad a = \lambda\bar{\mu},$$

in (2.1.2.1), we obtain equation (4.1.1.3). Hence, we deduce

$$Z(t) = exp\Big[\int_0^t -\lambda ds\Big] = e^{-\lambda t}, \qquad t \geq 0,$$

and (4.1.1.3) follows immediately from (2.1.2.2). This ends the proof of the Lemma. □

We end this subsection by formulating mathematically our objective. To this end, we consider a discount factor $c > 0$. Let π be an admissible policy and $R^\pi = (R_t^\pi)_{t\geq 0}$ be the corresponding cash reserve process given by the SDE (4.1.1.1). Then, bankruptcy time for this cash reserve is defined by

$$\tau_\pi = inf\left\{t \geq 0 : R_t^\pi = 0\right\}.$$

Here, by convention we put $inf(\phi) = +\infty$. Thus, the return function associated to π is denoted by $J_\pi(R, \eta)$, and is given by

$$J_\pi(R, \eta) := E\left(\int_0^{\tau_\pi} e^{-ct} R_t^\pi dt | R_0^\pi = R, \mu_0 = \eta\right).$$

Therefore, our objective lies in describing the optimal value function

$$V(R, \eta) := \sup_\pi J_\pi(R, \eta), \qquad \forall R \geq 0, \quad \eta \geq 0, \tag{4.1.1.4}$$

and finding the optimal policy π^* that satisfies

$$V(R, \eta) = J_{\pi^*}(R, \eta), \qquad \forall R \geq 0, \quad \eta \geq 0. \tag{4.1.1.5}$$

4.1.2 Properties of the Optimal Value Function

Proposition 4.1.1. *The optimal value function V, defined by (4.1.1.4), is concave and satisfies*

$$0 \le V(R,\eta) \le \frac{R}{c} + \frac{\bar{\mu}\sqrt{2\lambda}+\sigma_2}{c^2\sqrt{2\lambda}} + \frac{|\eta-\bar{\mu}|}{c(c+\lambda)}, \qquad \forall R \ge 0, \quad \eta \ge 0. \tag{4.1.2.1}$$

Proof. The proof of the concavity follows from the same arguments as in the proof of Proposition 3.2.1. Thus, the rest of the proof focuses on proving (4.1.1.6). To this end, due to

$$R^{\pi}_{t\wedge\tau_\pi} = R^{\pi}_0 + \int_0^{\tau_\pi\wedge t} \mu_s a^{\pi}_s ds + \int_0^{\tau_\pi\wedge t} \sigma a^{\pi}_s dW_s,$$

we obtain

$$E\left(R^{\pi}_{t\wedge\tau_\pi}|R^{\pi}_0 = R, \mu_0=\eta\right) = R + E\left(\int_0^{t\wedge\tau_\pi} \mu_s a^{\pi}_s ds | R^{\pi}_0 = R, \mu_0 = \eta\right)$$
$$\le R + \int_0^t \left[\bar{\mu} + |\eta-\bar{\mu}|e^{-\lambda s}\right] ds + \sigma_2 \int_0^t e^{-\lambda s} E\Big(|\int_0^s e^{\lambda\mu} dB_s||R^{\pi}_0 = R, \mu_0 = \eta\Big) ds.$$

As a result, we get

$$\int_0^t \left[\bar{\mu} + |\eta-\bar{\mu}|e^{-\lambda s}\right] ds = \bar{\mu}t + \frac{|\eta-\bar{\mu}|}{\lambda}(1-e^{-\lambda t}),$$

and

$$\begin{aligned}
\sigma_2 E\Big[\int_0^t e^{-\lambda s}|\int_0^s e^{\lambda\mu} dB_s\big||R^{\pi}_0 = R, \mu_0=\eta\Big] ds &= \sigma_2 \int_0^t e^{-\lambda s} E\left(\left|\int_0^s e^{\lambda\mu} dB_\mu\right|^2\right)^{\frac{1}{2}} ds \\
&= \frac{\sigma_2}{\sqrt{2\lambda}} \int_0^t e^{-\lambda s} \left(\int_0^s e^{2\lambda\mu} d\mu\right)^{\frac{1}{2}} ds \\
&\le \frac{\sigma_2}{\sqrt{2\lambda}} t.
\end{aligned}$$

Therefore, for any admissible policy π, we get

$$\begin{aligned} J_\pi(R,\eta) &\le \int_0^\infty e^{-ct}\left[R+\bar{\mu}t+\frac{|\eta-\bar{\mu}|}{\lambda}(1-e^{-\lambda t})+\frac{\sigma_2}{\sqrt{2\lambda}}t\right]dt \\ &= \frac{R}{c}+\frac{\bar{\mu}\sqrt{2\lambda}+\sigma_2}{c^2\sqrt{2\lambda}}+\frac{|\eta-\bar{\mu}|}{c(c+\lambda)}. \end{aligned}$$

Then, (4.2.1.1) follows immediately from this inequality. This proves the proposition. □

4.1.3 The HJB equation

The goal of this subsection is to derive an HJB equation that the optimal value function shall satisfy under smoothness conditions.

Theorem 4.1.1. *If the optimal value function $V(R,\eta)$, defined by (4.1.1.4), is twice continuously differentiable on $(0,\infty)$, then V satisfies the Hamilton-Jacobi-Bellman equation*

$$R-cV-\lambda(\eta-\bar{\mu})V_\eta+\frac{1}{2}\sigma_2^2V_{\eta\eta}+\max_{0\le a\le 1}\left(a\eta V_R+\frac{1}{2}a^2\sigma_1^2V_{RR}+a\sigma_1\sigma_2\rho V_{R\eta}\right)=0, \quad (4.1.3.1)$$

$$V(0)=0.$$

Proof. By applying Ito's formula (see chapter 2) to $V(R_t^\pi,\mu_t)$, we derive

$$dV(R_t^\pi,\mu_t)=V_RdR_t^\pi+V_\eta d\mu_t+\frac{1}{2}V_{RR}d\langle R^\pi,R^\pi\rangle_t+\frac{1}{2}V_{\eta\eta}d\langle\mu_t,\mu_t\rangle+V_{R\eta}d\langle R_t^\pi,\mu_t\rangle_t$$

$$=V_R(a_t^\pi\eta dt+\sigma_1a_t^\pi dW_t)+V_\eta\left[-\lambda(\eta-\bar{\mu})dt+\sigma_2dB_t\right]+\frac{1}{2}V_{RR}(a_t^\pi)^2\sigma_1^2dt+\frac{1}{2}\sigma_2^2V_{\eta\eta}dt+a_t^\pi\sigma_1\sigma_2\rho V_{R\eta}dt \quad (4.1.3.2)$$

Put

$$Y_t^\pi:=\int_0^t e^{-cs}R_sds+e^{-ct}V(R_t,\mu_t).$$

Thanks to Bellman's principal (dynamics programming principal), the optimal value function $V(R, \eta)$, is such that $(Y_t^{\pi})_{t\geq 0}$ is a supermartingale for any admissible policy π and is a local martingale for the optimal and admissible policy π^*. Hence, in virtue of (3.2.3), Y^{π} is a supermartingale if

$$R - cV - \lambda(\eta - \bar{\mu})V_\eta + \frac{1}{2}\sigma_2^2 V_{\eta\eta} + \underset{0\leq a\leq 1}{max}\left(a^{\pi^*}\eta V_R + \frac{1}{2}(a^{\pi^*})^2\sigma_1^2 V_{RR} + a^{\pi^*}\sigma_1\sigma_2\rho V_{R\eta}\right) \leq 0. \tag{4.1.3.3}$$

and $Y_t^{\pi^*}$ is a local martingale if

$$R_t^{\pi^*} - cV - \lambda(\eta - \bar{\mu})V_\eta + \frac{1}{2}\sigma_2^2 V_{\eta\eta} + \left(a^{\pi^*}\eta V_R + \frac{1}{2}(a^{\pi^*})^2\sigma_1^2 V_{RR} + a^{\pi^*}\sigma_1\sigma_2\rho V_{R\eta}\right) = 0$$

Thus, by combining this equation with (4.1.3.3), we deduce that V is a solution to the HJB equation (4.1.3.1). This ends the proof of the theorem. □

Now, our goal is to construct a solution to the HJB equation (4.1.3.1). This is the aim of the following subsection.

4.2 Construction of the solution to HJB

The goal of this section is to construct as explicitly as possible, a solution to the HJB equation. We divide this section into two subsections, where we consider the cases whether the two noises B and W are correlated or not.

4.2.1 The Case of Orthogonal Noise (i.e. $\rho = 0$)

Assume there exists an opent set $O \subseteq [0, \infty)$ such that $a(R, \eta)$ satisfies $0 < a(R, \eta) < 1$ for all $R \in O$. Then for any $R \in O$, and $\eta > 0$, the maximizer

$$a^*(R, \eta) := \underset{a}{argmax}\Big[a\eta V_R(R, \eta) + \frac{1}{2}a^2\sigma_1^2 V_{RR}(R, \eta)\Big]$$

is given by

$$a^*(R, \eta) = -\frac{\eta V_R(R, \eta)}{\sigma_1^2 V_{RR}(R, \eta)}. \tag{4.2.1.1}$$

By inserting (4.2.1.1) into (4.1.1.7), we get

$$R - cV - \lambda(\eta - \bar{\mu})V_\eta + \frac{1}{2}\sigma_2^2 V_{\eta\eta} - \frac{\eta^2}{2\sigma_1^2}\frac{V_R^2}{V_{RR}} = 0 \tag{4.2.1.2}$$

In order to solve this PDE, we assume

$$V_R(R(z, \eta), \eta) = e^{-z}. \tag{4.2.1.3}$$

Then, we calculate the following derivatives

$$V_{RR} = -\frac{V_R}{R_z}, \qquad V_{R\eta} = \frac{R_\eta}{R_z}V_R.$$

By substituting these in (4.2.1.2) and using (4.2.1.3), we obtain

$$R(z, \eta) - cV(R(z, \eta), \eta) - \lambda(\eta - \bar{\mu})V_\eta(R(z, \eta), \eta) + \frac{1}{2}\sigma_2^2 V_{\eta\eta} + \frac{\eta^2}{2\sigma_1^2}R_z(z, \eta)V_R(R(z, \eta), \eta) = 0. \tag{4.1.2.4}$$

By differentiating this equation with respect to z and using (4.2.1.3) again, we derive

$$R_z - cR_zV_R - \lambda(\eta - \bar{\mu})R_zV_{R\eta} + \frac{1}{2}\sigma_2^2 R_z V_{R\eta\eta} + \frac{\eta^2}{2\sigma_1^2}R_{zz}V_R - \frac{\eta^2}{2\sigma_1^2}R_zV_R = 0. \tag{4.2.1.5}$$

Since

$$V_{R\eta} = \frac{R_\eta}{R_z} V_R.$$

By differentiating this with respect to η, we get

$$V_{R\eta\eta} = \left(\frac{R_{\eta\eta}}{R_z} - 2\frac{R_\eta R_{z\eta}}{R_z^2} + \frac{R_\eta^2 R_{zz}}{R_z^3} + \frac{R_\eta^2}{R_z^2} \right) V_R. \tag{4.2.1.6}$$

By plugging (4.2.1.6) into (4.2.1.5), we get

$$R_z - cR_zV_R - \lambda(\eta - \bar{\mu})R_\eta V_R + \frac{1}{2}\sigma_2^2 R_z \left[\frac{R_{\eta\eta}}{R_z} - 2\frac{R_\eta R_{z\eta}}{R_z^2} + \frac{R_\eta^2 R_{zz}}{R_z^3} + \frac{R_\eta^2}{R_z^2} \right] V_R$$
$$+ \frac{\eta^2}{2\sigma_1^2} R_{zz} V_R - \frac{\eta^2}{2\sigma_1^2} R_z V_R = 0.$$

This can be written as

$$\frac{R_{zz}}{R_z} = \left(1 + \frac{2\sigma_1^2 c}{\eta^2} - \frac{2\sigma_1^2}{\eta^2} e^z \right) + \frac{2\sigma_1^2}{\eta^2}\lambda(\eta - \bar{\mu})\frac{R_\eta}{R_z} - \frac{\sigma_1^2\sigma_2^2}{\eta^2} \left[\frac{R_{\eta\eta}}{R_z} - 2\frac{R_\eta R_{z\eta}}{R_z^2} + \frac{R_\eta^2 R_{zz}}{R_z^3} + \frac{R_\eta^2}{R_z^2} \right] \tag{4.2.1.7}$$

In order to simplify the problem, we put

$$R_z(z, \eta) = exp\Big[z + L(e^z, \eta) \Big], \tag{4.2.1.8}$$

where L is a function to be described later on.

The solution to this equation is given by

$$R(z, \eta) = \int_{-\infty}^{z} exp\Big[x + L(e^x, \eta) \Big] dx + k(\eta).$$

By changing the variable, we get

$$R(z, \eta) = \int_{0}^{e^z} exp\Big[x + L(x, \eta) \Big] dx + k(\eta) = G(e^z, \eta) + k(\eta). \tag{4.2.1.9}$$

where $G(x, \eta)$ is given by

$$G(x, \eta) = \int_{0}^{x} exp\Big[L(y, \eta) \Big] dy. \tag{4.2.1.10}$$

By mimicking the proof of Højggard and Taskar (1998), we deduce that $k(\eta) = 0$. By differentiating (4.2.1.10) with respect to x, we get

$$G_x(x, \eta) = exp\Big[L(x, \eta)\Big], \qquad \forall \eta > 0, \quad x > 0. \tag{4.2.1.11}$$

From (4.2.1.11), we deduce that $G_x > 0$ for all $x > 0$ and hence $G(x, \eta)$ is strictly increasing on $[0, \infty)$ and continuous. Therefore $G(x, \eta)$ is invertible, and (4.2.1.9) leads to

$$e^z = G^{-1}(R, \eta).$$

Then by plugging this resulting equation in (4.2.1.3), we obtain

$$V_R(R, \eta) = \frac{1}{G^{-1}(R, \eta)}, \qquad \forall R > 0, \quad \eta > 0. \tag{4.2.1.12}$$

To determine completely the function V on a neighborhood of zero, we need to describe the function L introduced in (4.2.18).

Lemma 4.2.1. *The function L given by (4.2.1.8) satisfies the following equation.*

$$\left(\frac{\eta^2}{2\sigma_1^2} x L_x + x - c\right) x^2 e^{2L} + \frac{1}{2}\sigma_2^2 \left(2 + x L_x\right) \left(\int_0^x L_\eta e^L dy\right)^2 + \left[-\lambda(\eta - \bar{\mu}) - \sigma_2^2 L_\eta\right] x e^L \int_0^x L_\eta e^L dy$$
$$+ \frac{1}{2}\sigma_2 x e^L \int_0^x (L_{\eta\eta} e^L + L_\eta^2) dy = 0. \tag{4.2.1.13}$$

Proof. By simplifying (4.2.1.7), we obtain

$$\left(\frac{\sigma_2^2}{2}\frac{R_\eta^2}{R_z^2} + \frac{\eta^2}{2\sigma_1^2}\right)\frac{R_{zz}}{R_z} + e^z - c - \lambda(\eta - \bar{\mu})\frac{R_\eta}{R_z} + \frac{\sigma_2^2}{2}\frac{R_{\eta\eta}}{R_z} - \sigma_2^2 \frac{R_\eta R_{\eta z}}{R_z^2} + \frac{\sigma_2^2}{2}\frac{R_\eta^2}{R_z^2} - \frac{\eta^2}{2\sigma_1^2} = 0. \tag{4.2.1.14}$$

Then, we calculate the derivatives

$$R_{zz}(z, \eta) = (1 + e^z L_x) R_z, \qquad R_{z\eta} = L_\eta R_z. \tag{4.2.1.15}$$

$$R_\eta = \int_0^x L_\eta e^L dy, \qquad R_{\eta\eta} = \int_0^x (L_{\eta\eta} e^L + L_\eta^2 e^L) dy. \tag{4.2.1.16}$$

By multiplying R_z^2 and plugging (4.2.1.15) into (4.2.1.14), we get

$$\left(\frac{\sigma_2^2}{2}\frac{R_\eta^2}{R_z^2} + \frac{\eta^2}{2\sigma_1^2}\right)(1 + e^z L_x) R_z^2 + e^z R_z^2 - \left(c + \frac{\eta^2}{2\sigma_1^2}\right) R_z^2 - \lambda(\eta - \bar{\mu}) R_\eta R_z$$

$$+\frac{\sigma_2^2}{2} R_{\eta\eta} R_z - \sigma_2^2 L_\eta R_\eta R_z + \frac{\sigma_2^2}{2} R_\eta^2 = 0.$$

This can be written as

$$\left(\frac{\eta^2}{2\sigma_1^2} e^z L_x + e^z - c\right) R_z^2 + 2\sigma_2^2 (2 + e^z L_x) R_\eta^2 + \left[-\lambda(\eta - \bar{\mu}) - \sigma_2^2 L_\eta\right] RzR_\eta + \frac{1}{2}\sigma_2^2 R_z R_{\eta\eta} = 0. \tag{4.2.1.17}$$

By plugging (4.2.1.16) in (4.2.1.17), we obtain

$$\left(\frac{\eta^2}{2\sigma_1^2} e^z L_x + e^z - c\right) e^{2(z+L)} + \frac{1}{2}\sigma_2^2 (2 + e^z L_x) \left(\int_0^x L_\eta e^L dy\right)^2$$

$$+ \left[-\lambda(\eta - \bar{\mu}) - \sigma_2^2 L_\eta\right] e^{z+L} \int_0^x L_\eta e^L dy + \frac{1}{2}\sigma_2 e^{z+L} \int_0^x (L_{\eta\eta} e^L + L_\eta^2) dy = 0.$$

By changing the variable (i.e.$x = e^z$), we get

$$\left(\frac{\eta^2}{2\sigma_1^2} x L_x + x - c\right) x^2 e^{2L} + \frac{1}{2}\sigma_2^2 (2 + x L_x) \left(\int_0^x L_\eta e^L dy\right)^2 + \left[-\lambda(\eta - \bar{\mu}) - \sigma_2^2 L_\eta\right] x e^L \int_0^x L_\eta e^L dy$$

$$+\frac{1}{2}\sigma_2 x e^L \int_0^x (L_{\eta\eta} e^L + L_\eta^2) dy = 0. \tag{4.2.1.18}$$

which is the exactly the PDE (4.2.1.13). This ends the proof of the lemma. □

By differentiating (4.2.1.12) with respect to R, we deduce that (4.2.1.1) becomes

$$a^*(R, \eta) = \frac{\eta}{\sigma_2} G^{-1}(R, \eta) G_x(G^{-1}(R, \eta), \eta), \qquad R > 0 \tag{4.2.1.19}$$

and $a^*(0,\eta) = 0$. Put

$$y = G^{-1}(R,\eta), \qquad and \qquad a_1(y,\eta) = \frac{\eta}{\sigma^2} y G_x(y,\eta).$$

By differentiating $a_1(y,\eta)$ with respect to y, we get

$$\frac{\partial}{\partial y} a_1(y,\eta) = \frac{\eta}{\sigma^2}\Big[1 + yL_y\Big] exp\Big[L(y,\eta)\Big].$$

Assumption 4.2.1. *Assume (4.2.1.13) has a solution $L(y,\eta)$ such that the equation*

$$y exp\Big[L(y,\eta)\Big] = \frac{\sigma_1^2}{\eta} \tag{4.2.1.20}$$

has a root $y_1(\eta) \in (0,c)$.

Then, under this assumption, we have

$$R_1(\eta) = G(y_1(\eta),\eta).$$

Then $a_1(R,\eta)$ is strictly increasing on $[0, R_1(\eta)]$. As a result, we have the following solution

$$V(R,\eta) = \int_0^R \frac{1}{G^{-1}(y,\eta)} dy, \qquad 0 \le R \le R_1(\eta). \tag{4.2.1.21}$$

According to our assumptions $a^*(R,\eta) = 1$ for $R > R_1(\eta)$. By substituting $a = 1$ into (4.2.1.1), we obtain the following equation

$$R - cV - \lambda(\eta - \bar{\mu})V_\eta + \frac{1}{2}\sigma_2^2 V_{\eta\eta} + \eta V_R + \frac{1}{2}\sigma_1^2 V_{RR} = 0 \tag{4.2.1.22}$$

In order to solve this PDE explicitly, we propose the following function for $V(R,\eta)$

$$V(R,\eta) = C_1(\eta)e^{d(\eta)R} + C_2(\eta) + \frac{R}{c},$$

Then, we calculate the derivatives

$$V_R = C_1 d e^{dR} + \frac{1}{c}, \qquad V_{RR} = C_1 d^2 e^{dR}, \qquad V_\eta = C_1^{'} e^{dR} + C_1 d' e^{dR} + C_2^{'},$$

$$V_{\eta\eta} = C_1'' d e^{dR} + 2C_1' d' e^{dR} + C_1 d'' e^{dR} + C_1 d'^2 e^{dR} + C_2''.$$

By inserting these in (4.2.1.22), we get

$$e^{dR}[-cC_1 + C_1\eta d - \lambda(\eta - \bar{\mu})(C_1' + C_1 d') + \frac{1}{2}\sigma_2^2 C_1'' + \sigma_2^2 C_1' d' + \frac{1}{2}\sigma_2^2 C_1 d'' + \frac{1}{2}\sigma_2^2 C_1 d'^2$$

$$+\frac{1}{2}\sigma_1^2 C_1 d^2] + [-cC_2 + \frac{\eta}{c} + \frac{1}{2}\sigma_2^2 C_2'' - \lambda(\eta - \bar{\mu})C_2^{'}] = 0.$$

Since the above equation holds for all values of $R > R_1$, we must have:

$$-cC_1 + C_1\eta d - \lambda(\eta - \bar{\mu})(C_1' + C_1 d') + \frac{1}{2}\sigma_2^2 C_1'' + \sigma_2^2 C_1' d' + \frac{1}{2}\sigma_2^2 C_1 d'' + \frac{1}{2}\sigma_2^2 C_1 d'^2 + \frac{1}{2}\sigma_1^2 C_1 d^2 = 0, \tag{4.2.1.23}$$

and

$$-cC_2 + \frac{\eta}{c} + \frac{1}{2}\sigma_2^2 C_2'' - \lambda(\eta - \bar{\mu})C_2' = 0. \tag{4.2.1.24}$$

Lemma 4.2.2. *The solution to the ODE (4.2.1.24), $C_2(\eta)$, is given by*

$$C_2(\eta) = exp\left(\int_0^\eta g(x)dx + \frac{\lambda(\eta - \bar{\mu})^2}{2\sigma_2^2}\right) + \frac{\eta}{c(c+\lambda)} + \frac{\lambda\bar{\mu}}{c^2(c+\lambda)}, \tag{4.2.1.25}$$

where $g(\eta)$ is the solution to the following Riccati equation

$$g'(\eta) + g^2(\eta) = \frac{2c - \lambda}{\sigma_2^2} + \frac{\lambda^2(\eta - \mu)^2}{\sigma_2^4}. \tag{4.2.1.26}$$

Proof. To find a particular solution to (4.2.1.24), we assume that

$$p_1(\eta) = a\eta + b. \tag{4.2.1.27}$$

Here, a and b are constants. By inserting (4.2.1.27) into (4.2.1.24), we get

$$a = \frac{1}{c(c+\lambda)}, \qquad b = \frac{\lambda\bar{\mu}}{c^2(c+\lambda)}.$$

Therefore, the particular solution to (4.1.2.24) is

$$p_1(\eta) = \frac{\eta}{c(c+\lambda)} + \frac{\lambda\bar{\mu}}{c^2(c+\lambda)}.$$

Then, we need to find the general solution to

$$-cC_2 + \frac{1}{2}\sigma_2^2 C_2'' - \lambda(\eta - \bar{\mu})C_2' = 0. \tag{4.2.1.28}$$

To solve this equation, we put

$$ln\,(C_2(\eta)) = K(\eta) \tag{4.2.1.29}$$

Then, we calculate the derivatives

$$K'(\eta) = \frac{C_1'(\eta)}{C_1(\eta)}, \qquad K''(\eta) = \frac{C_1''}{C_1} - \left(\frac{C_1'}{C_1}\right)^2.$$

By plugging these into (4.2.1.28), we obtain

$$K'' + \left(K' - \frac{\lambda(\eta-\mu)}{\sigma_2^2}\right)^2 = \frac{2c}{\sigma_2^2} + \frac{\lambda^2(\eta-\mu)^2}{\sigma_2^4}. \tag{4.2.1.30}$$

Then, we put

$$g(\eta) = K'(\eta) - \frac{\lambda(\eta-\mu)}{\sigma_2^2}. \tag{4.2.1.31}$$

By differentiating (4.2.1.31) with respect to η, we get

$$g'(\eta) = k''(\eta) - \frac{\lambda}{\sigma_2^2},$$

and hence (4.2.1.30) becomes

$$g'(\eta) + g^2(\eta) = \frac{2c-\lambda}{\sigma_2^2} + \frac{\lambda^2(\eta-\mu)^2}{\sigma_2^4}.$$

This ends the proof of the lemma. □

Therefore, for $R > R_1(\eta)$, the optimal return function takes the form of

$$V(R,\eta) = C_1(\eta)e^{d(\eta)R} + exp\left(\int_0^\eta g(x)dx + \frac{\lambda(\eta-\bar{\mu})^2}{2\sigma_2^2}\right) + \frac{\eta}{c(c+\lambda)} + \frac{\lambda\bar{\mu}}{c^2(c+\lambda)} + \frac{R}{c}. \tag{4.2.1.31}$$

As a result, we obtain

$$V(R,\eta) = \begin{cases} \displaystyle\int_0^R \frac{1}{G^{-1}(x,\eta)}dx & \text{if} \quad 0 \le R \le R_1(\eta) \\ \\ C_1(\eta)e^{d(\eta)R} + exp\left(\displaystyle\int_0^\eta g(x)dx + \frac{\lambda(\eta-\bar{\mu})^2}{2\sigma_2^2}\right) + \dfrac{\eta c + \lambda\bar{\mu} + c(c+\lambda)R}{c(c+\lambda)} & \text{if} \quad R \ge R_1(\eta). \end{cases} \tag{4.2.1.32}$$

To ensure that V is twice continuously differentiable, it is necessary and sufficient that its value, first and second derivative are continuous at the point R_1. To this end, we put

$$V_1 := \int_0^R \frac{1}{G^{-1}(x,\eta)}dx, \quad V_2 := C_1(\eta)e^{d(\eta)R} + exp\left(\int_0^\eta g(x)dx + \frac{\lambda(\eta-\bar{\mu})^2}{2\sigma_2^2}\right) + \frac{\eta c + \lambda\bar{\mu} + c(c+\lambda)R}{c(c+\lambda)}$$

Then, we calculate the first and second derivative of V_1 and V_2 at the point $R_1(\eta)$.

$$V_1^{'}(R_1) = \frac{1}{y_1}, \qquad V_2^{'}(R_1) = \frac{1}{c} + C_1 d.$$

$$V_1^{''}(R_1) = -\frac{\eta}{\sigma_2^2}V_1^{'}(R_1) = -\frac{\eta}{\sigma_2^2}\frac{1}{y_1}, \qquad V_2^{''}(R_1) = C_1 d^2.$$

Then, V_R and V_{RR} are continuous at $R = R_1(\eta)$ if and only if

$$\frac{1}{y_1} = \frac{1}{c} + C_1 d \qquad \text{and} \quad -\frac{\eta}{\sigma_1^2}\frac{1}{y_1} = C_1 d^2.$$

This implies

$$C_1\left(d^2 + \frac{\eta d}{\sigma_1^2}\right) = -\frac{\eta}{c\sigma_1^2},$$

which can be written as

$$d^2 + \frac{\eta}{\sigma_1^2}d + \frac{\eta}{c\sigma_1^2 C_1} = 0.$$

Since $d < 0$, the solution to this equation is

$$d(\eta) = \frac{1}{2}\left\{-\frac{\eta}{\sigma_1^2} - \sqrt{\frac{\eta^2}{\sigma_1^4} - \frac{4\eta}{c\sigma_1^2 C_1(\eta)}}\right\}. \tag{4.2.1.34}$$

Then, we calculate the derivative

$$d'(\eta) = \frac{1}{2}\left\{-\frac{1}{\sigma_2^2} - \frac{\frac{\eta}{\sigma_1^4} - \frac{2}{c\sigma_1^2 C_1(\eta)} + \frac{2\eta C_1'(\eta)}{c\sigma_1^2 C_1(\eta)^2}}{\sqrt{\frac{\eta^2}{\sigma_1^4} - \frac{4\eta}{c\sigma_1^2 C_1(\eta)}}}\right\}. \tag{4.2.1.35}$$

$$\begin{aligned} d''(\eta) = & -\frac{1}{2}\left(\frac{\eta^2}{\sigma_1^4} - \frac{4\eta}{c\sigma_1^2 C_1(\eta)}\right)^{-\frac{1}{2}}\left(\frac{1}{\sigma_1^4} + \frac{4C_1'(\eta)}{c\sigma_1^2 C_1(\eta)^2} + \frac{2\eta C_1''(\eta)}{c\sigma_1^2 C_1(\eta)^2} - \frac{4\eta C_1'(\eta)^2}{c\sigma_1^2 C_1(\eta)^3}\right) \\ & + \frac{1}{2}\left(\frac{\eta}{\sigma_1^4} - \frac{2}{c\sigma_1^2 C_1(\eta)} + \frac{2\eta C_1'(\eta)}{c\sigma_1 C_1(\eta)^2}\right)^2\left(\frac{\eta^2}{\sigma_1^4} - \frac{4\eta}{c\sigma_1^2 C_1(\eta)}\right)^{-\frac{3}{2}} \end{aligned} \tag{4.2.1.36}$$

By plugging (4.2.1.34), (4.2.1.35) and (4.2.1.36) into (4.2.1.23), we get

$$\begin{aligned} 0 \;=\; & \frac{1}{2}\sigma_2^2 C_1''(\eta) + \left[-\lambda(\eta-\bar{\mu}) + \frac{1}{2}\sigma_1^2\left(-\frac{1}{\sigma_1^2} - \frac{\frac{\eta}{\sigma_1^4} - \frac{2}{c\sigma_1^2 C_1(\eta)} + \frac{2\eta C_1'(\eta)}{c\sigma_1^2 C_1(\eta)^2}}{\sqrt{\frac{\eta^2}{\sigma_1^4} - \frac{4\eta}{c\sigma_1^2 C_1}}}\right)\right] C_1'(\eta) \\ + \; & \left\{-c + \frac{1}{2}\eta\left(-\frac{\eta}{\sigma_1^2} - \sqrt{\frac{\eta^2}{\sigma_1^4} - \frac{4\eta}{c\sigma_1^2 C_1(\eta)}}\right) - \frac{1}{2}\lambda(\eta-\bar{\mu})\left(-\frac{1}{\sigma_1^2} - \frac{\frac{\eta}{\sigma_1^4} - \frac{2}{c\sigma_2^2 C_1(\eta)} + \frac{2\eta C_1'(\eta)}{c\sigma_1^2 C_1(\eta)^2}}{\sqrt{\frac{\eta^2}{\sigma_1^4} - \frac{4\eta}{c\sigma_1^2 C_1(\eta)}}}\right)\right\} C_1(\eta) \\ + \; & \frac{1}{4}\sigma_2^2\left(-\frac{1}{\sigma_1^2} - \frac{\frac{\eta}{\sigma_1^4} - \frac{2}{c\sigma_1^2 C_1(\eta)} + \frac{2\eta C_1'(\eta)}{c\sigma_1^2 C_1(\eta)^2}}{\sqrt{\frac{\eta^2}{\sigma_1^4} - \frac{4\eta}{c\sigma_1^2 C_1(\eta)}}}\right)^2 C_1(\eta) + \frac{1}{4}\sigma_1^2\left(-\frac{\eta}{\sigma_1^2} - \sqrt{\frac{\eta^2}{\sigma_1^4} - \frac{4\eta}{c\sigma_1^2 C_1(\eta)}}\right)^2 C_1(\eta) \\ + \; & \frac{1}{4}\sigma_2^2\left\{-\left(\frac{\eta^2}{\sigma_1^4} - \frac{4\eta}{c\sigma_1^2 C_1(\eta)}\right)^{-\frac{1}{2}}\left(\frac{1}{\sigma_1^4} + \frac{4C_1'(\eta)}{c\sigma_1^2 C_1(\eta)^2} + \frac{2\eta C_1''(\eta)}{c\sigma_1^2 C_1(\eta)^2} - \frac{4\eta C_1'(\eta)^2}{c\sigma_1^2 C_1(\eta)^3}\right)\right\} C_1(\eta) \\ + \; & \frac{1}{4}\sigma_2^2\left\{\left(\frac{\eta}{\sigma_1^4} - \frac{2}{c\sigma_1^2 C_1(\eta)} + \frac{2\eta C_1'(\eta)}{c\sigma_1 C_1(\eta)^2}\right)^2\left(\frac{\eta^2}{\sigma_1^4} - \frac{4\eta}{c\sigma_1^2 C_1(\eta)}\right)^{-\frac{3}{2}}\right\} C_1(\eta). \end{aligned} \tag{4.2.1.37}$$

This proves the following theorem.

Theorem 4.2.1. *Suppose that Assumption 4.2.1 holds, then the solution to the HJB (4.1.3.1) is given by*

$$V(R,\eta)=\begin{cases}\displaystyle\int_0^R \frac{1}{G^{-1}(x,\eta)}dx & if \quad 0\le R\le R_1(\eta)\\[2ex] \displaystyle -\frac{\eta}{c\sigma_2^2(d^2+\frac{\eta d}{\sigma_2^2})}e^{d(\eta)(R-R_1(\eta))}+exp\left(\int_0^\eta g(x)dx+\frac{\lambda(\eta-\bar{\mu})^2}{2\sigma_2^2}\right) & \\[2ex] \displaystyle +\frac{\eta c+\lambda\bar{\mu}+c(c+\lambda)R}{c(c+\lambda)}, & if \quad R\ge R_1(\eta).\end{cases} \tag{4.2.1.38}$$

Here

$$R_1(\eta):=G(y_1(\eta),\eta),\ \ G(x,\eta)=\int_0^x exp\Big[L(y,\eta)\Big]dy,\ \ d(\eta)=\frac{1}{2}\left\{-\frac{\eta}{\sigma^2}-\sqrt{\frac{\eta^2}{\sigma^4}-\frac{4\eta}{c\sigma^2C_1(\eta)}}\right\},$$

where $y_1(\eta)$ is the root of (4.2.1.20) and $C_1(\eta)$ is the solution to PDE (4.2.1.37).

4.2.2 The Case of Correlated Noise (i.e. $\rho\neq 0$)

Assume there exists an opent set $O\subseteq[0,\infty)$ such that $a(R,\eta)$ satisfies $0<a(R,\eta)<1$ for all $R\in O$. Then for any $R\in O$, and $\eta>0$, the maximizer

$$a^*(R,\eta):=\underset{a}{argmax}\Big[a\eta V_R(R,\eta)+\frac{1}{2}a^2\sigma_1^2V_{RR}(R,\eta)+a\sigma_1\sigma_2\rho V_{R\eta}(R,\eta)\Big]$$

is given by

$$a^*(R,\eta)=-\frac{\eta V_R(R,\eta)+\sigma_1\sigma_2\rho V_{R\eta}(R,\eta)}{\sigma_1^2V_{RR}(R,\eta)}. \tag{4.2.2.1}$$

By inserting (4.2.2.1) into (4.2.1.6), we obtain

$$R-cV-\lambda(\eta-\bar{\mu})V_\eta+\frac{1}{2}\sigma_2V_{\eta\eta}-\frac{1}{2}\frac{(\eta V_R+\rho\sigma_1\sigma_2V_{R\eta})^2}{\sigma_1^2V_{RR}}=0 \tag{4.2.2.2}$$

In order to solve this PDE, we assume

$$V_R(R(z,\eta),\eta) = e^{-z} \tag{4.2.2.3}$$

Then, we calculate the following derivatives

$$V_{RR} = -\frac{V_R}{R_z}, \qquad V_{R\eta} = \frac{R_\eta}{R_z} V_R, \qquad V_{R\eta\eta} = \left(\frac{R_{\eta\eta}}{R_z} - 2\frac{R_\eta R_{z\eta}}{R_z^2} + \frac{R_\eta^2 R_{zz}}{R_z^3} + \frac{R_\eta^2}{R_z^2} \right) V_R.$$

By substituting these in (4.2.2.2) and using (4.2.2.3), we obtain

$$R(z,\eta) - cV(R(z,\eta),\eta) - \lambda(\eta - \bar{\mu})V_\eta(R(z,\eta),\eta) + \frac{1}{2}\sigma_2 V_{\eta\eta}(R(z,\eta),\eta) +$$

$$\frac{R_z(z,\eta)V_R(R(z,\eta),\eta)\left[\eta + \sigma_1\sigma_2\rho\frac{R_\eta(z,\eta)}{R_z(z,\eta)})\right]^2}{2\sigma_1^2} = 0 \tag{4.3.2.4}$$

By differentiating with respect to z and using (4.2.2.3) again lead to

$$R_z - cV_R R_z - \lambda(\eta - \bar{\mu})R_z V_{R\eta} + \frac{1}{2}\sigma_2 R_z V_{R\eta\eta} + \frac{1}{2}\frac{\left[\eta + \frac{R_\eta}{R_z}\sigma_1\sigma_2\rho\right]^2}{\sigma_1^2}(R_{zz}V_R - R_z V_R)$$

$$+\frac{\left[\eta + \frac{R_\eta}{R_z}\sigma_1\sigma_2\rho\right](\sigma_1\sigma_2\rho\frac{R_{z\eta}R_z - R_\eta R_{zz}}{R_z^2})}{\sigma_1^2} R_z V_R = 0 \tag{4.2.2.5}$$

That is equivalent to

$$e^{z-m} - c - \lambda(\eta - \bar{\mu})\frac{R_\eta}{R_z} + \frac{1}{2}\sigma_2^2\left[\frac{R_{\eta\eta}}{R_z} - \frac{2R_\eta R_{\eta z}}{R_z^2} + \frac{R_\eta^2 R_{zz}}{R_z^3} + + \frac{R_\eta^2}{R_z^2}\right]$$

$$+\frac{1}{2\sigma_1^2}\left[\eta^2 + \sigma_1^2\sigma_2^2\rho^2\frac{R_\eta^2}{R_z^2} + 2\eta\sigma_1\sigma_2\rho\frac{R_\eta}{R_z}\right]\frac{R_{zz}}{R_z} - \frac{1}{2\sigma_1^2}\left[\eta^2 + \sigma_1^2\sigma_2^2\rho^2\frac{R_\eta^2}{R_z^2} + 2\eta\sigma_1\sigma_2\rho\frac{R_\eta}{R_z}\right]$$

$$+\frac{1}{\sigma_1^2}\left[\eta\sigma_1\sigma_2\rho\frac{R_{\eta z}}{R_z} - \eta\sigma_1\sigma_2\rho\frac{R_\eta R_{zz}}{R_z^2} + \sigma_1^2\sigma_2^2\rho^2\left(\frac{R_\eta R_{\eta z}}{R_z^2} - \frac{R_\eta^2 R_{zz}}{R_z^3}\right)\right] = 0 \tag{4.2.2.6}$$

In order to simplify the problem, put

$$R_z(z,\eta) = exp\Big[z + L(e^z,\eta)\Big], \tag{4.2.2.7}$$

where L is a function to be described later on.

The solution to this equation is given by

$$R(z,\eta) = \int_{-\infty}^{z} exp\Big[x + L(e^x,\eta)\Big]dx + k(\eta).$$

By changing the variable, we get

$$R(z,\eta) = \int_0^{e^z} exp\Big[x + L(x,\eta)\Big]dx + k(\eta) = G(e^z,\eta) + k(\eta). \tag{4.2.2.8}$$

where $G(x,\eta)$ is given by

$$G(x,\eta) = \int_0^x exp\Big[L(y,\eta)\Big]dy. \tag{4.2.2.9}$$

By mimicking the proof of Højggard and Taskar (1998), we deduce that $k(\eta) = 0$. By differentiating (4.2.2.9) with respect to x, we get

$$G_x(x,\eta) = exp\Big[L(x,\eta)\Big], \qquad \forall \eta > 0, \quad x > 0. \tag{4.2.2.10}$$

From (4.2.2.10), we deduce that $G_x > 0$ for all $x > 0$ and hence $G(x,\eta)$ is strictly increasing on $(0,\infty)$ and continuous. Therefore $G(x,\eta)$ is invertible, and (4.2.2.8) leads to

$$e^z = G^{-1}(R,\eta).$$

Then by plugging this resulting equation in (4.2.2.3), we obtain

$$V_R(R,\eta) = \frac{1}{G^{-1}(R,\eta)}. \qquad \forall R > 0, \quad \eta > 0. \tag{4.2.2.11}$$

To describe completely the function V, we need to describe the function L introduced in (4.2.2.7). This is the aim of the following.

Lemma 4.2.3. *L is given by (4.2.2.7) satisfies the following equation.*

$$\left(\frac{\eta^2}{2\sigma_1^2}xL_x + x - c + \frac{\eta\sigma_2\rho}{\sigma_1}L_\eta\right)x^2e^{2L} + \frac{1}{2}\sigma_2^2\left(2+\rho^2+xL_x\right)\left(\int_0^x L_\eta e^L dy\right)^2$$

$$+\left[\sigma_2^2(\rho^2-1)L_\eta-\lambda(\eta-\bar{\mu})-\frac{\eta\sigma_2\rho}{\sigma_1}\right]xe^L\int_0^x L_\eta e^L dy+\frac{1}{2}\sigma_2 xe^L\int_0^x (L_{\eta\eta}e^L+L_\eta^2)dy=0. \tag{4.2.2.12}$$

Proof. By simplifying (4.1.2.6), we obtain

$$\left(\frac{\sigma_2^2}{2}\frac{R_\eta^2}{R_z^2}+\frac{\eta^2}{2\sigma_1^2}\right)\frac{R_{zz}}{R_z}+e^z-c-\lambda(\eta-\bar{\mu})\frac{R_\eta}{R_z}+\frac{\sigma_2^2}{2}\frac{R_{\eta\eta}}{R_z}-\sigma_2^2\frac{R_\eta R_{\eta z}}{R_z^2}+\frac{\sigma_2^2}{2}\frac{R_\eta^2}{R_z^2}-\frac{\eta^2}{2\sigma_1^2}$$

$$-\frac{\sigma_2^2\rho^2}{2}\frac{R_\eta^2}{R_z^2}-\frac{\eta\sigma_2\rho}{\sigma_1}\frac{R_\eta}{R_z}+\frac{\eta\sigma_2\rho}{\sigma_1}\frac{R_{z\eta}}{R_z}+\sigma_2^2\rho^2\frac{R_\eta R_{\eta z}}{R_z^2}=0. \tag{4.2.2.13}$$

Then, we calculate the derivatives

$$R_{zz}(z,\eta)=(1+e^zL_x)R_z, \qquad R_{z\eta}=L_\eta R_z. \tag{4.2.2.14}$$

$$R_\eta=\int_0^x L_\eta e^L dy, \qquad R_{\eta\eta}=\int_0^x (L_{\eta\eta}e^L+L_\eta^2e^L)dy. \tag{4.2.2.15}$$

By multiplying R_z^2 and plugging (4.2.2.14) into (4.2.2.13), we get

$$\left(\frac{\sigma_2^2}{2}\frac{R_\eta^2}{R_z^2}+\frac{\eta^2}{2\sigma_1^2}\right)(1+e^zL_x)R_z^2+e^zR_z^2-\left(c+\frac{\eta^2}{2\sigma_1^2}\right)R_z^2-\lambda(\eta-\bar{\mu})R_\eta R_z+\frac{\sigma_2^2}{2}R_{\eta\eta}R_z$$

$$-\sigma_2^2L_\eta R_\eta R_z+\frac{\sigma_2^2}{2}R_\eta^2-\frac{\sigma_2^2\rho^2}{2}R_\eta^2-\frac{\eta\sigma_2\rho}{\sigma_1}R_zR_\eta+\frac{\eta\sigma_2\rho}{\sigma_1}L_\eta R_\eta R_z+\sigma_2^2\rho^2R_\eta R_{\eta z}=0.$$

This can be written as

$$\left(\frac{\eta^2}{2\sigma_1^2}e^zL_x+e^z-c\right)R_z^2+2\sigma_2^2\left(2+\rho^2+e^zL_x\right)R_\eta^2+\left[-\lambda(\eta-\bar{\mu})-\sigma_2^2L_\eta-\frac{\eta\sigma_2\rho}{\sigma_1}+\sigma_2^2\rho^2L_\eta\right]RzR_\eta$$

$$+\frac{1}{2}\sigma_2^2R_zR_{\eta\eta}+\frac{\eta\sigma_2\rho}{\sigma_1}L_\eta R_z^2=0. \tag{4.2.2.16}$$

By plugging (4.2.2.15) in (4.2.2.16), we obtain

$$\left(\frac{\eta^2}{2\sigma_1^2}e^zL_x+e^z-c+\frac{\eta\sigma_2\rho}{\sigma_1}L_\eta\right)e^{2(z+L)}+\frac{1}{2}\sigma_2^2\left(2+\rho^2+e^zL_x\right)\left(\int_0^x L_\eta e^L dy\right)^2$$

$$+\left[\sigma_2^2\rho^2 L_\eta - \lambda(\eta-\bar{\mu}) - \sigma_2^2 L_\eta - \frac{\eta\sigma_2\rho}{\sigma_1}\right] e^{z+L}\int_0^x L_\eta e^L dy + \frac{1}{2}\sigma_2 e^{z+L}\int_0^x (L_{\eta\eta}e^L + L_\eta^2)dy = 0.$$

By changing the variable (i.e.$x = e^z$), we get

$$\left(\frac{\eta^2}{2\sigma_1^2}xL_x + x - c + \frac{\eta\sigma_2\rho}{\sigma_1}L_\eta\right)x^2e^{2L} + \frac{1}{2}\sigma_2^2\left(2+\rho^2+xL_x\right)\left(\int_0^x L_\eta e^L dy\right)^2$$

$$+\left[\sigma_2^2\rho^2 L_\eta - \lambda(\eta-\bar{\mu}) - \sigma_2^2 L_\eta - \frac{\eta\sigma_2\rho}{\sigma_1}\right] xe^{L}\int_0^x L_\eta e^L dy + \frac{1}{2}\sigma_2 xe^{L}\int_0^x (L_{\eta\eta}e^L + L_\eta^2)dy = 0. \tag{4.2.2.17}$$

which is exactly the PDE (4.2.2.12). This ends the proof of the lemma. □

By differentiating (4.2.2.11) with respect to R, we deduce that (4.2.2.1) becomes

$$a^*(R,\eta) = \frac{\eta}{\sigma_1^2}G^{-1}(R,\eta)G_x(G^{-1}(R,\eta),\eta) - \frac{\sigma_2\rho}{\sigma_1}G_x(G^{-1}(R,\eta),\eta)G_\eta(G^{-1}(R,\eta),\eta), \qquad R > 0.$$

and $a^*(0,\eta) = 0$. Then put

$$y = G^{-1}(R,\eta), \qquad \textit{and} \qquad a_1(R,\eta) = \frac{\eta}{\sigma_1^2}yG_x(y,\eta) - \frac{\sigma_2\rho}{\sigma_1}G_x(y,\eta)G_\eta(y,\eta).$$

Assumption 4.2.2. *Assume (4.2.2.12) has a solution $L(y,\eta)$ such that the equation*

$$exp\{L(y,\eta)\}\left(\frac{\eta}{\sigma_1^2}y - \frac{\sigma_2\rho\int_0^y L_\eta e^{L(x,\eta)}dx}{\sigma_1}\right) = 1 \tag{4.2.2.18}$$

has a root $y_1(\eta) \in (0,c)$.

Then, we have

$$R_1(\eta) = G(y_1(\eta),\eta) \qquad (4.2.2.19).$$

As a result, we have the following solution

$$V(R,\eta) = \int_0^R \frac{1}{G^{-1}(R,\eta)}dy, \qquad 0 \le R \le R_1(\eta).$$

According to our assumptions $a^*(R,\eta) = 1$ for $R > R_1(\eta)$. By substituting $a = 1$ into (4.2.1.6), we obtain the following equation:

$$R - cV - \lambda(\eta - \bar{\mu})V_\eta + \frac{1}{2}\sigma_2^2 V_{\eta\eta} + \eta V_R + \frac{1}{2}\sigma_1^2 V_{RR} + \sigma_1\sigma_2\rho V_{R\eta} = 0. \qquad (4.2.2.15)$$

In order to solve this PDE explicitly, we propose the following function for $V(R,\eta)$

$$V(R,\eta) = C_1(\eta)e^{d(\eta)R} + C_2(\eta) + \frac{R}{c},$$

and calculate the following derivatives:

$$V_R = C_1 d e^{dR} + \frac{1}{c}, \qquad V_{RR} = C_1 d^2 e^{dR}, \qquad V_\eta = C_1' e^{dR} + C_1 d' e^{dR} + C_2',$$

$$V_{\eta\eta} = C_1'' d e^{dR} + 2C_1' d' e^{dR} + C_1' d d' e^{dR} + C_1 d'' e^{dR} + C_1 d'^2 e^{dR} + C_2''.$$

By inserting them in (4.2.2.15), we get

$$e^{dR}[-cC_1 - \lambda(\eta - \bar{\mu})(C_1' + C_1 d') + \frac{1}{2}\sigma_2^2(C_1'' + 2C_1'd' + C_1 d'' + C_1 d'^2) + \eta C_1 d + \frac{1}{2}\sigma_1^2 C_1 d^2$$
$$+\sigma_1\sigma_2\rho(C_1'd + C_1 d' + C_1 d d')] + [-cC_2 + \frac{\eta}{c} + \frac{1}{2}\sigma_2^2 C_2'' - \lambda(\eta - \bar{\mu})C_2'] = 0.$$

Since the above equation holds for all values of $R > R_1$, we must have:

$$-cC_1 - \lambda(\eta - \bar{\mu})(C_1' + C_1 d') + \frac{1}{2}\sigma_2^2(C_1'' + 2C_1'd' + C_1 d'' + C_1 d'^2) + \eta C_1 d + \frac{1}{2}\sigma_1^2 C_1 d^2$$
$$+\sigma_1\sigma_2\rho(C_1'd + C_1 d' + C_1 d d')] = 0, \qquad (4.2.2.16)$$

and

$$-cC_2 + \frac{\eta}{c} + \frac{1}{2}\sigma_2^2 C_2'' - \lambda(\eta - \bar{\mu})C_2' = 0. \qquad (4.2.2.17)$$

It is clear that (4.2.2.17) is the same as (4.2.1.28), we have already solved it.

Therefore, for $R > R_1(\eta)$, the optimal return function takes the form of

$$V(R,\eta) = C_1(\eta)e^{d(\eta)R} + exp\left(\int_0^\eta g(x)dx + \frac{\lambda(\eta - \bar{\mu})^2}{2\sigma_2^2}\right) + \frac{\eta}{c(c+\lambda)} + \frac{\lambda\bar{\mu}}{c^2(c+\lambda)} + \frac{R}{c}. \qquad (4.2.2.18)$$

As a result, we obtain

$$V(R,\eta) = \begin{cases} \displaystyle\int_0^R \frac{1}{G^{-1}(x,\eta)}dx & \text{if} \quad 0 \le R \le R_1(\eta) \\ \\ C_1(\eta)e^{d(\eta)R} + exp\left(\displaystyle\int_0^\eta g(x)dx + \frac{\lambda(\eta-\bar{\mu})^2}{2\sigma_2^2}\right) + \frac{\eta c + \lambda\bar{\mu} + c(c+\lambda)R}{c(c+\lambda)} & \text{if} \quad R \ge R_1(\eta). \end{cases} \tag{4.2.2.19}$$

To ensure that V is twice continuously differentiable, it is necessary and sufficient that its value, first and second derivative are continuous at the point R_1. To this end, we put

$$V_1 := \int_0^R \frac{1}{G^{-1}(x,\eta)}dx, \quad V_2 := C_1(\eta)e^{d(\eta)R} + exp\left(\int_0^\eta g(x)dx + \frac{\lambda(\eta-\bar{\mu})^2}{2\sigma_2^2}\right) + \frac{\eta c + \lambda\bar{\mu} + c(c+\lambda)R}{c(c+\lambda)}.$$

Then, we calculate the first and second derivative of V_1 and V_2 at the point $R_1(\eta)$.

$$V_1^{'}(R_1) = \frac{1}{y_1}, \qquad V_2^{'}(R_1) = \frac{1}{c} + C_1 d.$$

$$V_1^{''}(R_1) = -\frac{\eta}{\sigma_2^2}V_1^{'}(R_1) = -\frac{\eta}{\sigma_2^2}\frac{1}{y_1} - \frac{\sigma_2\rho}{\sigma_1}\frac{G_\eta(y_1,\eta)}{G_x(y_1,\eta)}, \qquad V_2^{''}(R_1) = C_1 d^2.$$

Then, V_R and V_{RR} are continuous at $R = R_1(\eta)$ if and only if

$$\frac{1}{y_1} = \frac{1}{c} + C_1 d.$$

$$-\frac{\eta}{\sigma_2^2}\frac{1}{y_1} - \frac{\sigma_2\rho}{\sigma_1}\frac{G_\eta(y_1,\eta)}{G_x(y_1,\eta)} = C_1 d^2.$$

This implies

$$d(\eta) = \frac{c\eta}{\sigma_1(y_1-c)} + \frac{cy_1}{c-y_1}\frac{\rho\sigma_2}{\sigma_1}\frac{G_\eta(y_1,\eta)}{G_x(y_1,\eta)} \tag{4.2.2.20}$$

This proves the following theorem.

Theorem 4.2.2. *The solution to the HJB (4.2.2.12) is given by*

$$V(R,\eta) = \begin{cases} \displaystyle\int_0^R \frac{1}{G^{-1}(x,\eta)}dx & \text{if} \quad 0 \le R \le R_1(\eta) \\ \\ \displaystyle C_1(\eta)e^{d(\eta)(R-R_1(\eta))} + exp\left(\int_0^\eta g(x)dx + \frac{\lambda(\eta-\bar{\mu})^2}{2\sigma_2^2}\right) \\ \displaystyle \quad + \frac{\eta c + \lambda\bar{\mu} + c(c+\lambda)R}{c(c+\lambda)} & \text{if} \quad R \ge R_1(\eta) \end{cases} \tag{4.2.2.21}$$

Here

$$R_1(\eta) := G(y_1(\eta),\eta), \;\; G(x,\eta) = \int_0^x exp\Big[L(y,\eta)\Big]dy, d(\eta) = \frac{c\eta}{\sigma_1(y_1-c)} + \frac{cy_1}{c-y_1}\frac{\rho\sigma_2}{\sigma_1}\frac{G_\eta(y_1,\eta)}{G_x(y_1,\eta)}.$$

where $y_1(\eta)$ is the root of (4.2.2.18) and $C_1(\eta)$ is the solution to PDE (4.2.2.16).

4.3 Optimal policy and the verification theorem

In this section we construct the optimal policy based on the solution to the HJB equation obtained in the previous section. Recall that $R_1(\eta) = G^{-1}(y_1(\eta),\eta)$, where $y_1(\eta)$ is root of (4.2.2.18). For $R \le R_1(\eta)$, we obtain

$$a^*(R,\eta) := \underset{a}{argmax}\Big[a\eta V_R(R,\eta) + \frac{1}{2}a^2\sigma^2 V_{RR}(R,\eta) + a\sigma_1\sigma_2\rho V_{R\eta}(R,\eta)\Big].$$

As evident from below the function $a^*(R,\eta)$ represents the optimal feedback control function for the control component $a_\pi = \Big(a_\pi(t)\Big)_{t\ge 0}$. More precisely, the value $a^*(R,\eta)$ is the optimal risk that one should take when the value of the current reserve is R and the reserve rate is η. From the analysis of the previous section, it follows that $a^*(R,\eta)$ takes the following form

$$a^*(R,\eta) = \begin{cases} \dfrac{\eta}{\sigma_1^2} G^{-1}(R,\eta) G_x(G^{-1}(R,\eta),\eta) - \dfrac{\sigma_2\rho}{\sigma_1} G_x(G^{-1}(R,\eta),\eta) G_\eta(G^{-1}(R,\eta),\eta) & \text{if } 0 \le R \le R_1(\eta) \\ 1, & \text{if } \quad R \ge R_1(\eta). \end{cases} \tag{4.3.1}$$

For any $0 \le a \le 1$, we define a differential operator $\mathcal{L}^a$ by

$$\mathcal{L}^a f(R,\eta) = \frac{\sigma^2 a^2}{2} f_{RR}(R,\eta) + a\eta f_R + a\sigma_1\sigma_2\rho f_{R\eta} - cf(R,\eta) - \lambda(\eta - \mu) f_\eta(R,\eta) \tag{4.4.2}$$

For any $f \in C^{2\times 1}((0,\infty)\times(0,\infty))$. Thus, thanks to the previous section, we have

$$\mathcal{L}^{a^*(R,\eta)} V(R,\eta) = -R. \tag{4.4.3}$$

Let $R_t^* = (R_t^*)_{t\ge 0}$ be a solution to the following Skorohod problem :

$$R_t^* = R_0^* + \int_0^t a(R_s^*,\mu_s)\mu_s ds + \int_0^t \sigma_1 a(R_s^*,\mu_s) dW_s, \tag{4.4.4}$$

$$R_t^* \le R_1(\mu_t).$$

Theorem 4.3.1. *Let V be a concave, twice continuously differentiable solution of the HJB equation (4.1.3.1) and $(R_t^*)_{t\ge 0}$ be a solution to the Skorohod problem (4.4.4). Then for $\pi^* := \Big(a(R_t^*,\mu_t)\Big)_{t\ge 0}$,we have*

$$J_{\pi^*}(R,\eta) = V(R,\eta), \qquad \forall R \ge 0, \quad \eta \ge 0.$$

Proof. Notice that $R_t^{\pi^*} = R_t^*$. Let $R_0 = R$ and $\mu_0 = \eta$. Choose $0 < \varepsilon < R$ and let $\tau_\pi^\varepsilon = inf\,\{t : R_t^* = \varepsilon\}$, then Ito's formula yields,

$$e^{-c(t\wedge\tau_\pi^\varepsilon)} V(R_{t\wedge\tau_*^\varepsilon}^\pi,\mu_t) = V(R,\eta) + \int_0^{t\wedge\tau_*^\varepsilon} e^{-cs}\mathcal{L}^{a_{\pi^*}(s)} V(R_s^*,\mu_s) ds + \int_0^{t\wedge\tau_*^\varepsilon} e^{-cs}\sigma_1 a_{\pi^*}(s) V_R(R_s^*,\mu_s) dW_s$$

$$= V(R,\eta) - \int_0^{t\wedge\tau_*^\varepsilon} e^{-cs} R_s^* ds + \int_0^{t\wedge\tau_*^\varepsilon} e^{-cs}\sigma_1 a_{\pi^*}(s) V_R(R_s^*,\mu_s) dW_s \tag{4.4.5}$$

Since $V_R(R_s^*,\mu_s) \le V_R(\varepsilon,\mu_s) < \infty$ on $[0, t\wedge\tau_*^\varepsilon]$, the last term on the r.h.s. is a zero-mean martingale. Taking expectations in (4.4.5), we obtain

$$\mathbb{E}\left[e^{-c(t\wedge\tau_*^\varepsilon)}\right] V(R^\pi_{t\wedge\tau_*^\varepsilon},\mu_t) + \mathbb{E}\int_0^{t\wedge\tau_\pi^\varepsilon} e^{-cs} R_s^\pi ds = V(R,\eta). \tag{4.4.6}$$

Since $\tau_*^\varepsilon \to \tau_{\pi^*}$ when $\varepsilon \to 0$

$$\int_0^{t\wedge\tau_*^\varepsilon} e^{-cs} R_s^* ds \to \int_0^{t\wedge\tau_{\pi^*}} e^{-cs} R_s^* ds.$$

Thus letting $\varepsilon \to 0$ in (4.4.6), we use dominated and monotone convergence theorems for the first and the second terms in (4.4.5) respectively, we get

$$\mathbb{E}\left[e^{-c(t\wedge\tau_{\pi^*})} V(R^*_{t\wedge\tau_{\pi^*}},\mu_t)\right] + \mathbb{E}\int_0^{t\wedge\tau_*} e^{-cs} R_s^* ds = V(R,\eta). \tag{4.4.7}$$

Since $V(0,\eta) = 0$, we have

$$\mathbb{E}\left[e^{-c(t\wedge\tau_*)} V(R^*_{t\wedge\tau_*},\mu_t)\right] = e^{-ct}\mathbb{E}\left[V(R_t^*,\mu_t); t < \tau_{\pi^*}\right] \to 0.$$

as $t\to\infty$. Letting $t\to\infty$ on (4.4.7), we have

$$\mathbb{E}\int_0^{\tau_{\pi^*}} e^{-cs} R_s^{\pi^*} ds = V(R,\eta).$$

Therefore, $V(R,\eta) = J_{\pi^*}(R,\eta)$. This ends the proof of the theorem. □

Chapter 5

Stochastic Cash Reserve Rate: The Case of Partial Information

This chapter investigates the proportional reinsurance problem under partial information. Precisely, we suppose that the cash reserve rate is not observable and follows an Ornstein-Uhlenbeck process with positive volatility. The policy selection for the reinsured proportion $1 - a_\pi$ is based on the observations of the cash reserve process itself, while its noise and its cash reserve rate are not observable.

This chapter has five sections. The first section outlines the model for the cash reserve process. The second section applies the filtering techniques to the model and defines the objective. The third section derives the HJB equation associated to this objective, while Section 4 gives a solution to this HJB equation. The last section describes the optimal policy and gives a verification theorem.

5.1 Mathematical and economic model

We start our mathematical model with a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. On this filtered probability space, we assume given two one-dimensional $\mathbb{F}$-adapted Brownian motions $W = (W_t)_{t\geq 0}$ and $B = (B_t)_{t\geq 0}$ that are correlated with the correlation coefficient $\rho \in [-1, 1]$.

In this chapter, we assume that the dynamics of the cash reserve process $R = (R_t)_{t\geq 0}$ (cash reserve process with no-reinsurance) are given by

$$dR_t = \mu_t a_\pi(t)dt + \sigma_1 a_\pi(t)dW_t, \qquad R_0 = \text{initial cash reserve}.$$

where the cash reserve rate process, $\mu = (\mu_t)_{t\geq 0}$, is given by

$$d\mu_t = -\lambda(\mu_t - \bar{\mu})dt + \sigma_2 dB_t, \qquad \mu_0 \quad \text{is given}, \tag{5.1.00}$$

and $\sigma_1 > 0$, $\lambda > 0$, $\bar{\mu} \geq 0$, $\sigma_2 > 0$ are constants.

In this chapter, we assume that the policy maker (the insurance company) observes the cash reserve process $R = (R_t)_{t\geq 0}$ only, and does not observe the processes μ, W, and B.

A policy π is a rule which for each $t \geq 0$ associates a random variable $0 \leq a_\pi(t) \leq 1$ ($1 - a_\pi(t)$ is the reinsured proportion of the claims). The policy π is admissible if the process $a_\pi = \Big(a_\pi(t)\Big)_{t\geq 0}$ is adapted to the information generated by the process R. In other words, a_π is adapted to the filtration

$$(\mathcal{F}^R_t)_{t\geq 0}, \qquad \mathcal{F}^R_t := \sigma\left(R_s,\ 0 \leq s \leq t\right).$$

Hereafter, we denote by $\mathcal{A}$ the set of all admissible policies. Then, the objective of the insurance company is the same as in the previous chapters. However the policy selection is made under restricted information. Thus, our main first task lies in transforming the maximization problem under partial information into a maximization problem with full information using the filtering techniques. This is the aim of the following section.

5.2 Filtering

We apply the filtering techniques in chapter 2. We define the innovation process $\widehat{W}$ associate to W as follows

$$d\widehat{W}_t \triangleq \frac{1}{\sigma_1}\left[(\mu_t - \widehat{\mu}_t)dt + \sigma_1 dW_t\right] = \frac{1}{\sigma_1}\left(\frac{dR_t}{a(t)} - \widehat{\mu}_t dt\right), \qquad t \geq 0.$$

Here $\widehat{\mu}_t := E\left[\mu_t|\mathcal{F}_t^R\right]$.

Thanks to Proposition 2.2.1, $\widehat{W} := (\widehat{W}_t)_{t\in[0,T]}$ is a Brownian motion under $\mathbb{F}^R$, and due to Kalman-Bucy filter, the process $\widehat{\mu}_t$ satisfies

$$d\widehat{\mu}_t = -\lambda\left(\widehat{\mu}_t - \bar{\mu}\right)dt + \left(\frac{\widehat{\Omega}_t + \sigma_1\sigma_2\rho}{\sigma_1}\right)d\widehat{W}_t, \qquad t \geq 0. \tag{5.2.1}$$

with $\widehat{\mu}_0 = E[\mu_0|F_0^R] = \eta_0$.

Moreover, the conditional variance $\widehat{\Omega}_t = E[(\mu_t - \widehat{\mu}_t)^2|\mathcal{F}_t^R]$ satisfied the deterministic Riccati ordinary differential equation (ODE):

$$d\widehat{\Omega}_t = [-\frac{1}{\sigma_1^2}\widehat{\Omega}_t^2 + \left(-\frac{2\sigma_2\rho}{\sigma_1} - 2\lambda\right)\widehat{\Omega}_t + \left(1-\rho^2\right)\sigma_2^2]dt, \qquad t \geq 0.$$

with $\widehat{\Omega}(0) = E[(\mu_0 - \eta)^2|\mathcal{F}_0^R] = \theta_0$, which has an explicit solution

$$\widehat{\Omega}_t = \widehat{\Omega}(t;\theta_0) = \sqrt{k}\sigma_1\frac{k_1 exp(2(\frac{\sqrt{k}}{\sigma_1})t) + k_2}{k_1 exp(2(\frac{\sqrt{k}}{\sigma_1})t) - k_2} - (\lambda + \frac{\sigma_2\rho}{\sigma_1})\sigma_1^2.$$

and

$$k = \lambda^2\sigma_1^2 + 2\sigma_1\sigma_2\lambda\rho + \sigma_2^2$$

$$k_1 = \sqrt{k}\sigma_1 + \left(\lambda\sigma_1^2 + \sigma_1\sigma_2\rho\right) + \theta_0$$

$$k_2 = -\sqrt{k}\sigma_1 + \left(\lambda\sigma_1^2 + \sigma_1\sigma_2\rho\right) + \theta_0$$

Under the observation filtration $\left(\mathcal{F}_t^R\right)_{t\geq 0}$, the cash reserve dynamics

$$dR_t^\pi = \widehat{\mu}_t a_\pi(t)dt + \sigma_1 a_\pi(t)d\widehat{W}_t, \qquad t \geq 0. \tag{5.2.2}$$

We end this subsection by formulating mathematically our objectives. To this end, we consider a discount factor. Let π be an admissible policy and $R = (R_t^\pi)_{t\geq 0}$ corresponding to cash reserve process given by the SDE (5.2.2). For a given control policy π, the time of bankruptcy for the cash reserve, is defined by

$$\tau_\pi = inf\left\{t \geq 0 : R_t^\pi = 0\right\}.$$

Thus, the return function associated to the policy π deduces $J_\pi(R,\eta,\theta)$, and is given by

$$J_\pi(R,\eta,\theta) := \mathbb{E}\left(\int_0^{\tau_\pi} e^{-ct}R_t^\pi dt | R_0 = R, \widehat{\mu}_0 = \eta, \widehat{\Omega}(0) = \theta\right).$$

Our objective lies in the describing the optimal value function under the partial observations filtration

$$V(R,\eta,\theta) := \sup_{\pi\in\mathcal{A}} J_\pi(R,\eta,\theta), \qquad \forall R \geq 0, \quad \eta \geq 0, \quad \theta \geq 0. \tag{5.2.3}$$

and finding the optimal policy $\pi^* \in \mathcal{A}$ that satisfies

$$V(R,\eta,\theta) = J_{\pi^*}(R,\eta,\theta), \qquad \forall R \geq 0, \quad \eta \geq 0, \quad \theta \geq 0. \tag{5.2.4}$$

5.3 The Hamilton-Jacobi-Bellman equation

In this section, we derive the HJB equation for the optimal return function.

Theorem 5.3.1. *The optimal value function $V(R,\eta,\theta)$, defined by (5.2.3), is twice continuously differentiable on $(0,\infty)$, then V satisfies the Hamilton-Jacobi-Bellman equation*

$$R - cV - \lambda(\eta-\bar{\mu})V_\eta + \left[-\frac{1}{\sigma_1^2}\theta^2 + (-\frac{2\sigma_2\rho}{\sigma_1} - 2\lambda)\theta + (1-\rho^2)\sigma_2^2\right]V_\theta + \frac{1}{2}\left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 V_{\eta\eta}$$
$$+\max_{a\in[0,1]}\left[a\eta V_R + \frac{1}{2}a^2\sigma_1^2 V_{RR} + a(\theta+\sigma_1\sigma_2\rho)V_{R\eta}\right] = 0, \tag{5.3.1}$$
$$V(0) = 0.$$

Proof. By applying Itô's formula ,we derive $V(R,\eta,\theta)$,

$$dV\left(R_t^\pi, \widehat{\mu}_t, \widehat{\Omega}_t,\right) = V_R dR_t^\pi + V_\eta d\widehat{\mu}_t + V_\theta d\widehat{\Omega}_t + \frac{1}{2}V_{RR}d\langle R^\pi, R^\pi\rangle_t + \frac{1}{2}V_{\eta\eta}d\langle\widehat{\mu},\widehat{\mu}\rangle_t + V_{R\eta}d\langle R_t^\pi, \widehat{\mu}_t\rangle$$
$$= V_R\Big[(a_t^\pi\eta-\delta)dt + a_t^\pi\sigma_1 d\widehat{W}_t\Big] + V_\eta\Big[-\lambda(\eta-\bar{\mu})dt + (\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1})d\widehat{W}_t\Big]$$
$$= V_\theta\left[-\frac{1}{\sigma_1^2}\theta^2 + (-\frac{2\sigma_2\rho}{\sigma_1} - 2\lambda)\theta + (1-\rho^2)\sigma_2^2\right]dt + \frac{1}{2}V_{RR}(a_t^\pi)^2\sigma_1^2 dt$$
$$+\frac{1}{2}V_{\eta\eta}(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1})^2 dt + V_{R\eta}a_t^\pi(\theta+\sigma_1\sigma_2\rho)dt \tag{5.3.2}$$

Put

$$Y_t^\pi := \int_0^t e^{-cs}R_s ds + e^{-ct}V(R_t, \widehat{\mu}_t, \widehat{\Omega}_t),$$

Thanks to Bellman's principal (dynamics programming principal), the optimal value function, $V(R,\eta,\theta)$ is such that $(Y_t^\pi)_{t\geq 0}$ is a supermartingale for any admissible policy π and is a local martingale for the optimal admissible policy π^*. Hence, in virtue of (5.3.2), Y_t^π is a supermartingale if

$$R - cV - \lambda(\eta-\bar{\mu})V_\eta + \left[-\frac{1}{\sigma_1^2}\theta^2 + (-\frac{2\sigma_2\rho}{\sigma_1} - 2\lambda)\theta + (1-\rho^2)\sigma_2^2\right]V_\theta$$

$$+\frac{1}{2}\left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 V_{\eta\eta}+\max_{a\in[0,1]}\left[a\eta V_R+\frac{1}{2}a^2\sigma_1^2V_{RR}+a(\theta+\sigma_1\sigma_2\rho)V_{R\eta}dt\right]\leq 0, \qquad (5.3.3)$$

and Y^{π^*} is a local martingale if

$$\int_0^t e^{-cs}R_s^{\pi^*}ds+e^{-ct}V(R_t^{\pi^*},\widehat{\mu}_t,\widehat{\Omega}_t),$$

is a local martingale if and only if V satisfies

$$R_t^{\pi^*}-cV-\lambda\left(\eta-\bar{\mu}\right)V_\eta+\left[-\frac{1}{\sigma_1^2}\theta^2+(-\frac{2\sigma_2\rho}{\sigma_1}-2\lambda)\theta+(1-\rho^2)\sigma_2^2\right]V_\theta$$

$$+\frac{1}{2}\left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 V_{\eta\eta}+\left[a_t^{\pi^*}\eta V_R+\frac{1}{2}(a_t^{\pi^*})^2\sigma_1^2V_{RR}+a_t^{\pi^*}(\theta+\sigma_1\sigma_2\rho)V_{R\eta}\right]=0.$$

Thus, by combining this equation and (5.3.3), we deduce that V is a solution to the HJB equation (5.3.1). This ends the proof of the theorem. □

Now, our goal is to construct a solution to HJB equation (5.3.1). This is the aim of the following subsection.

5.4 Construction of the solution to HJB

Assume there exists an open set $O\subseteq[0,\infty)$ such that $a(R,\eta,\theta)$ satisfies $0<a(R,\eta,\theta)<1$ for all $R\in O$. Then for any $R\in O$, $\eta>0$ and $\theta>0$, the maximizer

$$a^*(R,\eta,\theta):=\underset{a}{argmax}\Big[a\eta V_R(R,\eta,\theta)+\frac{1}{2}a^2\sigma_1^2V_{RR}(R,\eta,\theta)+a\sigma_1\sigma_2\rho V_{R\eta}(R,\eta,\theta)\Big]$$

is given by

$$a^*(R,\eta,\theta)=-\frac{\eta V_R(R,\eta,\theta)+\sigma_1\sigma_2\rho V_{R\eta}(R,\eta,\theta)}{\sigma_1^2V_{RR}(R,\eta,\theta)}. \qquad (5.4.1)$$

By inserting (5.4.1) into (5.3.1), we get

$$R - cV - \lambda(\eta - \bar{\mu})V_\eta + \left[-\frac{1}{\sigma_1^2}\theta^2 + \left(-\frac{2\sigma_2\rho}{\sigma_1} - 2\lambda \right)\theta + \left(1-\rho^2\right)\sigma_2^2 \right] V_\theta$$

$$+\frac{1}{2}\left(\frac{\theta + \sigma_1\sigma_2\rho}{\sigma_1}\right)^2 V_{\eta\eta} - \frac{1}{2}\frac{\left[\eta V_R + (\sigma_1\sigma_2\rho + \theta)V_{R\eta}\right]^2}{\sigma_1^2 V_{RR}} = 0 \tag{5.4.2}$$

In order to solve this PDE, we assume

$$V_R(R(z,\eta,\theta),\eta,\theta) = e^{-z} \tag{5.4.3}$$

Then, we calculate the following derivatives

$$V_{RR} = -\frac{V_R}{R_z}, \qquad V_{R\eta} = \frac{R_\eta}{R_z}V_R, \qquad V_{R\theta} = \frac{R_\theta}{R_z}V_R.$$

$$V_{R\eta\eta} = [\frac{R_{\eta\eta}}{R_z} - 2\frac{R_\eta R_{\eta z}}{R_z^2} + \frac{R_\eta^2 R_{zz}}{R_z^3} + \frac{R_\eta^2}{R_z^2}]V_R$$

By substituting $R = R(z,\eta,\theta)$ in (5.4.2) and using (5.4.3), we obtain

$$R - cV - \lambda(\eta - \bar{\mu})V_\eta + \left[-\frac{1}{\sigma_1^2}\theta^2 + \left(-\frac{2\sigma_2\rho}{\sigma_1} - 2\lambda \right)\theta + \left(1-\rho^2\right)\sigma_2^2 \right] V_\theta$$

$$+\frac{1}{2}\left(\frac{\theta + \sigma_1\sigma_2\rho}{\sigma_1}\right)^2 V_{\eta\eta} + \frac{\left[\eta + (\sigma_1\sigma_2\rho + \theta)(m_\eta + \frac{R_\eta}{R_z})\right]^2}{2\sigma_1^2} R_z V_R = 0 \tag{5.4.4}$$

By differentiating with respect to z leads to,

$$R_z - cV_R R_z - \lambda(\eta - \bar{\mu})R_z V_{R\eta} + R_z V_{\theta R}\left[-\frac{1}{\sigma_1^2}\theta^2 + \left(-\frac{2\sigma_2\rho}{\sigma_1} - 2\lambda \right)\theta + \left(1-\rho^2\right)\sigma_2^2 \right]$$

$$+\frac{1}{2}\left(\frac{\theta + \sigma_1\sigma_2\rho}{\sigma_1}\right)^2 R_z V_{R\eta\eta} + \frac{1}{2}\frac{\left[\eta + (\sigma_1\sigma_2\rho + \theta)\frac{R_\eta}{R_z}\ \right]^2}{\sigma_1^2}(R_{zz}V_R - R_z V_R)$$

$$+\frac{\left[\eta + \frac{R_\eta}{R_z}(\sigma_1\sigma_2\rho + \theta)\right]((\sigma_1\sigma_2\rho + \theta)\frac{R_{z\eta}R_z - R_\eta R_{zz}}{R_z^2})}{\sigma_1^2} R_z V_R = 0 \tag{5.4.5}$$

That is equivalent to

$$
\begin{aligned}
&e^{z-m} - c - \lambda(\eta - \bar{\mu})\frac{R_\eta}{R_z} + \frac{R_\theta}{R_z}\left[-\frac{1}{\sigma_1^2}\theta^2 + \left(-\frac{2\sigma_2\rho}{\sigma_1} - 2\lambda\right)\theta + \left(1-\rho^2\right)\sigma_2^2\right] \\
&+\frac{1}{2}\left(\frac{\theta + \sigma_1\sigma_2\rho}{\sigma_1}\right)^2\left[\frac{R_{\eta\eta}}{R_z} - \frac{2R_\eta R_{\eta z}}{R_z^2} + \frac{R_\eta^2 R_{zz}}{R_z^3} + + \frac{R_\eta^2}{R_z^2}\right] \\
&-\frac{1}{2\sigma_1^2}\left[\eta^2 + (\sigma_1\sigma_2\rho + \theta)^2\frac{R_\eta^2}{R_z^2} + 2\eta(\sigma_1\sigma_2\rho + \theta)\frac{R_\eta}{R_z}\right] \\
&+\frac{1}{2\sigma_1^2}\left[\eta^2 + (\sigma_1\sigma_2\rho + \theta)^2\frac{R_\eta^2}{R_z^2} + 2\eta(\sigma_1\sigma_2\rho + \theta)\frac{R_\eta}{R_z}\right]\frac{R_{zz}}{R_z} \\
&+\frac{1}{\sigma_1^2}\left[\eta(\sigma_1\sigma_2\rho + \theta)\frac{R_{\eta z}}{R_z} - \eta(\sigma_1\sigma_2\rho + \theta)\frac{R_\eta R_{zz}}{R_z^2} + (\sigma_1\sigma_2\rho + \theta)^2\left(\frac{R_\eta R_{\eta z}}{R_z^2} - \frac{R_\eta^2 R_{zz}}{R_z^3}\right)\right] = 0
\end{aligned}
\tag{5.4.6}
$$

In order to simplify the problem, we put

$$
R_z(z, \eta, \theta) = exp\Big[z + L(e^z, \eta, \theta)\Big], \tag{5.4.7}
$$

where L is a function to be described later on.

The solution to this equation is given by

$$
R(z, \eta, \theta) = \int_{-\infty}^{z} e^x exp\Big[L(e^x, \eta, \theta)\Big]dx + k(\eta, \theta).
$$

By changing the variable (i.e. $x = e^z$), we get

$$
R(z, \eta, \theta) = \int_0^{e^z} exp\left\{L(x, \eta, \theta)\right\}dx + k = G(e^z, \eta, \theta) + k(\eta, \theta). \tag{5.4.8}
$$

where $G(x, \eta, \theta)$ is given by

$$
G(x, \eta, \theta) = \int_0^x exp\Big[L(y, \eta, \theta)\Big]dy, \qquad x > 0, \quad \eta > 0, \quad \theta > 0. \tag{5.4.9}
$$

By mimicking the proof of Højggard and Taskar (1998), we deduce that $k(\eta,\theta)$=0. By differentiating (5.4.9) with respect to x, we get

$$G_x(x,\eta,\theta) = exp\Big[L(x,\eta,\theta)\Big]. \tag{5.4.10}$$

From (5.4.10), we get $G_x > 0$ for all $x > 0$, $\eta > 0$ and $\theta > 0$. $G(x,\eta,\theta)$ is increasing on $(0,\infty)$ and continuous, therefore $G(x,\eta,\theta)$ is invertible. As a result, we get

$$e^z = G^{-1}(R,\eta,\theta).$$

Then by plugging this resulting equation in (5.4.3), we obtain

$$V_R(R,\eta,\theta) = \frac{1}{G^{-1}(R,\eta,\theta)}. \qquad R > 0, \quad \eta > 0, \quad \theta > 0. \tag{5.4.11}$$

To describe completely the function V, we need to describe the function L introduced in (5.4.7). This is the aim of the following.

Lemma 5.4.1. *The function L given by (5.4.7) satisfies the following PDE.*

$$\left(\frac{\eta^2}{\sigma_1^2} + x - c\right) xL_x + \frac{\eta^2}{2\sigma_1^2}x^2 L_{xx} + \frac{\eta^2}{2\sigma_1^2}x^2 L_x^2 + 2x - c - \lambda(\eta - \bar{\mu})L_\eta + fL_\theta$$
$$+\frac{1}{2}\beta^2\left(L_{\eta\eta} + L_\eta^2\right) + \frac{\eta\beta}{\sigma_1}x\left(L_{x\eta} + L_x L_\eta\right) = 0 \tag{5.4.12}$$

Here,

$$f(\theta) = -\frac{1}{\sigma_1^2}\theta^2 + (-\frac{2\sigma_2\rho}{\sigma_1} - 2\lambda)\theta + (1-\rho^2)\sigma_2^2,$$
$$\beta(\theta) = \frac{\sigma_1\sigma_2\rho + \theta}{\sigma_1},$$

Proof. By simplifying (5.4.7), we obtain

$$\frac{\eta^2}{2\sigma_1^2}R_{zz} + e^z R_z - (c + \frac{\eta^2}{2\sigma_1^2})R_z - \left[\lambda(\eta - \bar{\mu}) + \frac{\eta\beta}{\sigma_1}\right]R_\eta + fR_\theta + \frac{1}{2}\beta^2 R_{\eta\eta} + \frac{\eta\beta}{\sigma_1}R_{z\eta} = 0 \tag{5.4.13}$$

Then, we calculate the derivatives

$$R_{zz} = (1 + e^z L_z)R_z, \qquad R_{z\eta} = L_\eta R_z, \qquad R_{z\eta\eta} = \left(L_{\eta\eta} + L_\eta^2\right) R_z.$$

By plugging these into (5.4.13), we obtain

$$\frac{\eta^2}{2\sigma_1^2}(1+e^z)R_z + e^z R_z - (c + \frac{\eta^2}{2\sigma_1^2})R_z - \left[\lambda(\eta - \bar{\mu}) + \frac{\eta\beta}{\sigma_1}\right] R_\eta + fR_\theta + \frac{1}{2}\beta^2 R_{\eta\eta} + \frac{\eta\beta}{\sigma_1} L_\eta R_z = 0 \tag{5.4.14}$$

By differentiating (5.4.14) with respect to z, we get

$$\frac{\eta^2}{2\sigma_1^2}R_{zz} + \frac{\eta^2}{2\sigma_1^2}e^z L_x R_z + \frac{\eta^2}{2\sigma_1^2}e^{2z} L_{xx} R_z + \frac{\eta^2}{2\sigma_1^2}e^z L_x R_{zz} + e^z R_z + e^z R_{zz} - (c + \frac{\eta^2}{2\sigma_1^2})R_{zz}z$$
$$- \left[\lambda(\eta - \bar{\mu}) + \frac{\eta\beta}{\sigma_1}\right] R_{z\eta} + fR_{z\theta} + \frac{1}{2}\beta^2 R_{z\eta\eta} + \frac{\eta\beta}{\sigma_1}\left(e^z L_{x\eta} R_z + L_\eta R_{zz}\right) = 0$$

This can be written as

$$\frac{\eta^2}{\sigma_1^2}e^z L_x + \frac{\eta^2}{2\sigma_1^2}e^{2z} L_{xx} + \frac{\eta^2}{2\sigma_1^2}e^{2z} L_x^2 + 2e^z + e^{2z}L_x - c - ce^z L_x - \lambda(\eta - \bar{\mu})L_\eta + fL_\theta$$
$$+ \frac{1}{2}\beta^2 (L_{\eta\eta} + L_\eta^2) + \frac{\eta\beta}{\sigma_1}e^z L_{x\eta} + \frac{\eta\beta}{\sigma_1}e^z L_x L\eta = 0$$

By changing the varible, we derive

$$\left(\frac{\eta^2}{\sigma_1^2} + x - c\right) xL_x + \frac{\eta^2}{2\sigma_1^2}x^2 L_{xx} + \frac{\eta^2}{2\sigma_1^2}x^2 L_x^2 + 2x - c - \lambda(\eta - \bar{\mu})L_\eta$$
$$+ \frac{1}{2}\beta^2 \left(L_{\eta\eta} + L_\eta^2\right) + \frac{\eta\beta}{\sigma_1} x \left(L_{x\eta} + L_x L_\eta\right) = 0.$$

which is the exactly the PDE (5.4.12). This ends the proof of the Lemma.

□

By differentiating with (5.4.11) respect to R and we deduce that (5.4.1) becomes

$$a^*(R,\eta,\theta) = \left\{ \frac{\eta}{\sigma_1^2} G^{-1}(R,\eta,\theta) + \frac{(\theta+\sigma_1\sigma_2\rho)}{\sigma_1^2}\left(\frac{1}{G^{-1}(R,\eta,\theta)}\right)_\eta \right\} G_x\left(G^{-1}(R,\eta,\theta),\eta,\theta\right). \tag{5.4.15}$$

for $R \geq 0$ and $a(0,\eta,\theta) = 0$.

Let $y = G^{-1}(R,\eta,\theta)$, then we have

$$a_1(R,\eta) = \frac{\eta}{\sigma_1^2} y G_x(y,\eta) - \frac{\sigma_2\rho}{\sigma_1} G_x(y,\eta) G_\eta(y,\eta).$$

Assumption 5.4.1. *Assume (5.4.12) has a solution $L(y,\eta,\theta)$ such that the equation*

$$exp\left\{L(y,\eta,\theta)\right\}\left(\frac{\eta}{\sigma_1^2} y - \frac{\sigma_2\rho\int_0^y L_\eta e^{L(x,\eta,\theta)}dx}{\sigma_1}\right) = 1 \tag{5.4.16}$$

has a root $y_1(\eta,\theta) \in (0,c)$.

Then, we have

$$R_1(\eta,\theta) = G(y_1(\eta,\theta),\eta,\theta) \tag{5.4.17}$$

As a result, we have the following solution

$$V(R,\eta,\theta) = \int_0^R \frac{1}{G^{-1}(y,\eta,\theta)} dy, \qquad 0 < R < R_1(\eta,\theta) \tag{5.4.18}$$

According to our assumptions $a^*(R,\eta,\theta) = 1$ for $R > R_1(\eta,\theta)$. By substituting $a = 1$ into (5.4.1), we obtain the following equation:

$$R - cV - \lambda(\eta-\bar{\mu})V_\eta + V_\theta f(\theta) + \frac{1}{2}\left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 V_{\eta\eta} + \left[\eta V_R + \frac{1}{2}\sigma_1^2 V_{RR} + (\theta+\sigma_1\sigma_2\rho)V_{R\eta}\right] = 0. \tag{5.4.19}$$

where, $f(\theta) = -\frac{1}{\sigma_1^2}\theta^2 + (-\frac{2\sigma_2\rho}{\sigma_1} - 2\lambda)\theta + (1-\rho^2)\sigma_2^2$.

In order to solve this PDE explicitly, we propose the following function for $V(R,\eta,\theta)$

$$V(R,\eta,\theta) = M(\eta,\theta)e^{d(\eta,\theta)R} + N(\eta,\theta) + \frac{R}{c}.$$

We calculate the following derivatives:

$$V_R = Mde^{dR} + \frac{1}{c}, \qquad V_{RR} = Md^2e^{dR},$$

$$V_\eta = M_\eta e^{dR} + Md_\eta e^{dR} + N_\eta, \qquad V_{\eta\eta} = M_{\eta\eta}e^{dR} + 2m_\eta d_\eta e^{dR} + Md_{\eta\eta}e^{dR} + Md_\eta^2 + N_{\eta\eta},$$

$$V_\theta = M_\theta e^{dR} + Md_\theta e^{dR} + N_\theta, \qquad V_{R\eta} = M_\eta de^{dR} + Md_\eta e^{dR} + Mdd_\eta e^{dR}.$$

By inserting them in (5.4.18), we get

$$\Big[-cM - \lambda(\eta-\bar{\mu})M_\eta - \lambda(\eta-\bar{\mu})Md_\eta + fM_\theta + Md_\theta f\Big]e^{dR}$$

$$+\left[\frac{1}{2}\left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 M_{\eta\eta}\left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 M_\eta d_\eta\right]e^{dR}$$

$$+\left[\left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 Md_{\eta\eta} + \eta Md + \left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 Md_\eta^2 + \frac{1}{2}\sigma_1^2Md^2\right]e^{dR}$$

$$+\left[(\theta+\sigma_1\sigma_2\rho)(M_\eta d + Md_\eta + Mdd_\eta)\right]e^{dR}$$

$$+\left[-N - \lambda(\eta-\bar{\mu})N_\eta + fN_\theta + \frac{1}{2}\left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 N_{\eta\eta} + \frac{\eta}{c}\right] = 0$$

Since the above equation holds for all values of $R > R_1(\eta,\theta)$, we must have:

$$-cM - \lambda(\eta-\bar{\mu})M_\eta - \lambda(\eta-\bar{\mu})Md_\eta + fM_\theta + Md_\theta f + \frac{1}{2}\left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 M_{\eta\eta}$$

$$+\left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 M_\eta d_\eta + \left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 Md_{\eta\eta} + \eta Md + \left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 Md_\eta^2$$

$$+\frac{1}{2}\sigma_1^2Md^2 + (\theta+\sigma_1\sigma_2\rho)(M_\eta d + Md_\eta + Mdd_\eta) = 0 \tag{5.4.20}$$

and

$$\frac{1}{2}\left(\frac{\theta+\sigma_1\sigma_2\rho}{\sigma_1}\right)^2 N_{\eta\eta}-\lambda(\eta-\bar{\mu})N_\eta+\beta N_\theta-cN+\frac{\eta}{c}=0 \tag{5.4.21}$$

As a result, we get the optimal return function

$$V(R,\eta,\theta)=\begin{cases}\displaystyle\int_0^R \frac{1}{G^{-1}(x,\eta,\theta)} & \text{if} \quad 0\le R\le R_1(\eta,\theta)\\ \\ M(\eta,\theta)e^{d(\eta,\theta)(R-R_1)}+N(\eta,\theta)+\frac{R}{c} & \text{if} \quad R\ge R_1(\eta,\theta)\end{cases} \tag{5.4.22}$$

To ensure that V is twice continuously differentiable, it is necessary and sufficient that its value, first and second derivative are continuous at the point $R_1(\eta,\theta)$. To this end, we put

$$V_1(R,\eta,\theta):=\int_0^R \frac{1}{G^{-1}(x,\eta,\theta)}, \qquad V_2(R,\eta,\theta):=M(\eta,\theta)e^{d(\eta,\theta)(R-R_1)}+N(\eta,\theta)+\frac{R}{c}.$$

Then, we consider the first and second derivative of V_1 and V_2 at the point $R_1(\eta,\theta)$.

$$V_1^{'}(R_1)=\frac{1}{y_1}, \qquad V_2^{'}(R_1)=\frac{1}{c}+Md.$$

$$V_1^{''}(R_1)=-\frac{\eta}{\sigma_2^2}V_1^{'}(R_1)=-\frac{\eta}{\sigma_2^2}\frac{1}{y_1}-\frac{\sigma_2\rho}{\sigma_1}, \qquad V_2^{''}(R_1)=Md^2.$$

Then, V_R and V_{RR} are continuous at $R(\eta,\theta)=R_1(\eta,\theta)$ if only

$$\frac{1}{y_1}=\frac{1}{c}+Md.$$

$$-\frac{\eta}{\sigma_2^2}\frac{1}{y_1}-\frac{\sigma_2\rho}{\sigma_1}G_x(y,\eta,\theta)G_\eta(y,\eta,\theta)=Md^2.$$

This implies

$$d(\eta)=\frac{c\eta}{\sigma_1(y_1-c)}+\frac{cy_1}{c-y_1}\frac{\rho\sigma_2}{\sigma_1}\frac{G_\eta(y_1,\eta,\theta)}{G_x(y_1,\eta,\theta)} \tag{5.4.24}$$

This proves the following theorem.

Theorem 5.4.1. *Suppose that Assumption 5.4.1, then the solution to the HJB (5.3.1) is given by*

$$V(R,\eta,\theta)=\begin{cases}\displaystyle\int_0^R \frac{1}{G^{-1}(R,\eta,\theta)}dx & \text{if} \quad 0\le R\le R_1(\eta,\theta)\\ \\ M(\eta,\theta)e^{d(\eta,\theta)(R-R_1(\eta,\theta))}+N(\eta)e^{d(\eta,\theta)R}+\frac{R}{c} & \text{if} \quad R\ge R_1(\eta,\theta)\end{cases} \tag{5.4.25}$$

Here

$$R_1(\eta,\theta)=G(y_1(\eta),\eta,\theta), \quad G(x,\eta,\theta)=\int_0^x exp\Big[L(y,\eta,\theta)\Big]dy,$$
$$d(\eta)=\frac{c\eta}{\sigma_1(y_1-c)}+\frac{cy_1}{c-y_1}\frac{\rho\sigma_2}{\sigma_1}\frac{G_\eta(y_1,\eta,\theta)}{G_x(y_1,\eta,\theta)},$$

where $y_1(\eta,\theta)$ is the root of (5.4.16) and $M(\eta,\theta)$ is the solution to PDE (5.4.20).

5.5 Optimal policy and the verification theorem

In this section we construct the optimal policy based on the solution to the HJB equation obtained in the previous sections. Recall that $R_1(\eta,\theta)=G(y_1(\eta,\theta),\eta,\theta)$, where $y_1(\eta,\theta)$ is root of (5.4.16). For $R\le R_1(\eta,\theta)$, we obtain

$$a^*(R,\eta,\theta):=arg\max_{0\le a\le 1}\Big[a\eta V_R(R,\eta,\theta)+\frac{1}{2}a^2\sigma^2 V_{RR}(R,\eta,\theta)+a\sigma_1\sigma_2\rho V_{R\eta}(R,\eta,\theta)\Big].$$

As evident from below the function $a^*(R,\eta,\theta)$ represents the optimal feedback control function for the control component $a_\pi=\Big(a_\pi(t)\Big)_{t\ge 0}$. From the analysis in previous section, it follows that $a^*(R,\eta,\theta)$ can be represented as

$$a^*(R,\eta,\theta)=\Big[\frac{\eta}{\sigma_1^2}G^{-1}(R,\eta)G_x(G^{-1}(R,\eta,\theta),\eta,\theta)$$
$$-\frac{\sigma_2\rho}{\sigma_1}G_x(G^{-1}(R,\eta,\theta),\eta,\theta)G_\eta(G^{-1}(R,\eta,\theta),\eta,\theta)\Big]I_{\{0\le R\le R_1(\eta,\theta)\}}+I_{\{R\ge R_1(\eta,\theta)\}}. \tag{5.5.1}$$

For any $0 \le a \le 1$, we define the differential operator $\mathcal{L}^a$ by

$$\begin{aligned}\mathcal{L}^a g(R,\eta,\theta) &= \frac{\sigma^2 a^2}{2} g_{RR}(R,\eta) + a\eta g_R + a\sigma_1\sigma_2\rho g_{R\eta} - cg(R,\eta) \\ &- \lambda(\eta-\mu)g_\eta(R,\eta) + f(\theta)g_\eta + \frac{1}{2}\beta^2 g_{\eta\eta} \end{aligned} \tag{5.5.2}$$

For any $g \in C^{2\times 1}((0,\infty)\times(0,\infty))$. As a result, due to the previous section, we get

$$\mathcal{L}^{a^*(R,\eta,\theta)} V(R,\eta,\theta) = -R. \tag{5.5.3}$$

Let $R_t^* = (R_t^*)_{t\ge 0}$ be a solution to the following Skorohod problem :

$$R_t^* = R + \int_0^t a(R_s^*, \widehat{\mu}_s, \widehat{\Omega}_s)\widehat{\mu}_s ds + \int_0^t \sigma_1 a(R_s^*, \widehat{\mu}_s, \widehat{\Omega}_s) d\widehat{W}_s, \tag{5.5.4}$$

$$R_t^* \le R_1(\widehat{\mu}_t, \widehat{\Omega}_s).$$

Theorem 5.5.1. *Let V be a concave, twice continuously differentiable solution of the HJB equation (5.3.1) and $(R_t^*)_{t\ge 0}$ be a solution to the Skorohod problem (5.5.4).*
Then for $\pi^* := \Big(a^*(R_t^*, \widehat{\mu}_t, \widehat{\Omega}_t)\Big)_{t\ge 0}$, we have

$$J_{\pi^*}(R,\eta,\theta) = V(R,\eta,\theta), \qquad \forall R \ge 0, \quad \eta \ge 0, \quad \theta \ge 0.$$

Proof. Notice that $R_t^{\pi^*} = R_t^*$. Let $R_0 = R$, $\widehat{\mu}_0 = \eta$ and $\widehat{\Omega}_0 = \theta$. Choose $0 < \varepsilon < R$ and let $\tau_\pi^\varepsilon = inf\,\{t : R_t^* = \varepsilon\}$, then Ito's formula yields,

$$\begin{aligned} e^{-c(t\wedge\tau_*^\varepsilon)} V(R^*_{t\wedge\tau_*^\varepsilon}, \widehat{\mu}_s, \widehat{\Omega}_s) &= V(R,\eta,\theta) + \int_0^{t\wedge\tau_*^\varepsilon} e^{-cs}\mathcal{L}^{a_{\pi^*}(s)} V(R_s^*, \widehat{\mu}_s, \widehat{\Omega}_s) ds \\ &+ \int_0^{t\wedge\tau_*^\varepsilon} e^{-cs}\sigma_1 a_{\pi^*}(s) V_R(R_s^*, \widehat{\mu}_s, \widehat{\Omega}_s) d\widehat{W}_s \\ = V(R,\eta,\theta) - \int_0^{t\wedge\tau_*^\varepsilon} e^{-cs} R_s^* ds &+ \int_0^{t\wedge\tau_*^\varepsilon} e^{-cs}\sigma_1 a_{\pi^*}(s) V_R(R_s^*, \widehat{\mu}_s, \widehat{\Omega}_s) d\widehat{W}_s \end{aligned} \tag{5.5.5}$$

Since $V_R(R_s^*, \widehat{\mu}_s, \widehat{\Omega}_s) \le V_R(\varepsilon, \widehat{\mu}_s, \widehat{\Omega}_s) < \infty$ on $[0, t \wedge \tau_*^\varepsilon]$, the last term on the r.h.s. is a zero-mean martingale. Taking expectations in (5.5.5), we obtain

$$\mathbb{E}\left[e^{-c(t\wedge\tau_*^\varepsilon)} V(R^*_{t\wedge\tau_*^\varepsilon}, \widehat{\mu}_s, \widehat{\Omega}_s)\right] + \mathbb{E}\int_0^{t\wedge\tau_*^\varepsilon} e^{-cs} R_s^* ds = V(R, \eta, \theta). \tag{5.5.6}$$

Since $\tau_*^\varepsilon \to \tau_{\pi^*}$ when $\varepsilon \to 0$

$$\int_0^{t\wedge\tau_*^\varepsilon} e^{-cs} R_s^* ds \to \int_0^{t\wedge\tau_{\pi^*}} e^{-cs} R_s^* ds.$$

Thus letting $\varepsilon \to 0$ in (5.5.6), we use dominated and monotone convergence theorems for the first and the second terms in (5.5.5) respectively, we get

$$\mathbb{E}\left[e^{-c(t\wedge\tau_{\pi^*})}\right] V(R^*_{t\wedge\tau_{\pi^*}}, \widehat{\mu}_s, \widehat{\Omega}_s) + \mathbb{E}\int_0^{t\wedge\tau_{\pi^*}} e^{-cs} R_s^* ds = V(R, \eta, \theta). \tag{5.5.7}$$

Since $V(0, \eta, \theta) = 0$, we have

$$\mathbb{E}\left[e^{-c(t\wedge\tau_{\pi^*})} V(R^\pi_{t\wedge\tau_{\pi^*}}, \widehat{\mu}_s, \widehat{\Omega}_s)\right] = e^{-ct}\mathbb{E}\left[V(R_t^{\pi^*}, \widehat{\mu}_s, \widehat{\Omega}_s); t < \tau_{\pi^*}\right] \to 0.$$

as $t \to \infty$. Letting $t \to \infty$ on (5.5.7), we have

$$\mathbb{E}\int_0^{\tau_{\pi^*}} e^{-cs} R_s^* ds = V(R, \eta, \theta).$$

Therefore, $V(R, \eta, \theta) = J_{\pi^*}(R, \eta, \theta)$. This ends the proof of the theorem. □

Chapter 6

The Case of Nonzero Liability

This chapter considers the case when the insurance company pays liability at a constant rate per unit time. Here, we consider the cash reserve model of Chapter 3.

6.1 Mathematical and economic model

We start with a filtered probability space $(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and one-dimensional standard Brownian motion W_t adapted to $(\mathcal{F}_t)_{t\geq 0}$. We denote by R_t^π the cash reserve of the company at time t under a control policy π. The dynamics of the cash reserve process R_t^π is described by

$$dR_t^\pi = (a_t^\pi \mu_t - \delta)dt + a_t^\pi \sigma dW_t, \qquad R_0 = \text{initial cash reserve},$$

where the process μ_t satisfies the linear ODE

$$d\mu_t = -\lambda \left(\mu_t - \bar{\mu}\right) dt.$$

and the initial value of the drift process μ_0 is given.

We define the value function $J_\pi^\delta(R,\eta)$ under a given admissible policy π by

$$J_\pi^\delta(R,\eta) := \mathbb{E}\left(\int_0^{\tau_\pi} e^{-ct} R_t^\pi dt | R_0 = R, \mu_0 = \eta\right).$$

Our goal is to to find the optimal value function

$$V^\delta(R,\eta) := \underset{\pi}{sup} J_\pi^\delta(R,\eta), \qquad \forall R \geq 0, \qquad \eta \geq 0. \tag{6.1.1}$$

and find the optimal policy π^* that satisfies

$$V^\delta(R,\eta) = J_{\pi^*}^\delta(R,\eta), \qquad \forall R \geq 0, \qquad \eta \geq 0. \tag{6.1.2}$$

6.2 The Hamilton-Jacobi-Bellman equation

In this subsection, we derive the HJB equation for the optimal value function.

Theorem 6.2.1. *The optimal value function $V^\delta(R,\eta)$ defined by (6.1.1) is twice continuously differentiable on $(0,\infty)$. Then V^δ satisfies the Hamilton-Jacobi-Bellman equation*

$$\begin{cases} R - cV^\delta - \lambda\left(\eta - \bar{\mu}\right)(V^\delta)_\eta + \underset{a\in[0,1]}{max}\left[(a\eta - \delta)(V^\delta)_R + \frac{1}{2}a^2\sigma_1^2(V^\delta)_{RR}\right] = 0. \\ V(0) = 0 \end{cases} \tag{6.2.1}$$

Proof. By applying the Ito's formula, we derive

$$dV^\delta\left(R,\mu\right) = (V^\delta)_R dR + (V^\delta)_\eta d\mu + \frac{1}{2}(V^\delta)_{RR} d\langle R^\pi, R^\pi\rangle_t + \frac{1}{2}(V^\delta)_{\eta\eta} d\langle \mu,\mu\rangle_t + (V^\delta)_{R\eta} d\langle R^\pi,\mu\rangle_t$$

$$= (V^\delta)_R\left((\eta a_t^\pi - \delta)dt + \sigma_1 a_t^\pi dW_t\right) + (V^\delta)_\eta\left(-\lambda\left(\eta - \bar{\mu}\right)dt\right) + \frac{1}{2}(V^\delta)_{RR}\sigma_1^2(a_t^\pi)^2 dt \tag{6.2.2}$$

Put

$$Y_t^\pi =: \int_0^t e^{-cs} R_s^\pi ds + e^{-ct} V^\delta(R_t^\pi, \mu_t), \qquad t \geq 0.$$

Thanks to Bellman's principal, the optimal value function, $V^\delta(R,\eta)$, is such that $(Y_t^\pi)_{t\geq 0}$ is a supermartingale for any admissible policy π and is a local martingale for the optimal admissible policy π^*. Hence, in virtue of (6.2.2), Y^π is a supermartingale if

$$R - cV^\delta - \lambda\left(\eta - \bar{\mu}\right)\left(V^\delta\right)_\eta + \underset{a}{max}\left[(a\eta - \delta)(V^\delta)_R + \frac{1}{2}a^2\sigma_1^2(V^\delta)_{RR}\right] \leq 0. \qquad (6.2.3)$$

and Y^π is a local martingale if

$$R_t^{\pi^*} - cV^\delta - \lambda\left(\eta - \bar{\mu}\right)\left(V^\delta\right)_\eta + \left[(a_t^\pi\eta - \delta)(V^\delta)_R + \frac{1}{2}(a_t^\pi)^2\sigma_1^2(V^\delta)_{RR}\right] = 0.$$

Thus, by combining this equation and (6.2.3), we deduce that V is a solution to the HJB equation (5.3.1). This ends the proof of the theorem. □

6.3 Construction of the solution to HJB equation

Proposition 6.3.1. *Let V^δ be the solution to (6.2.1) for $\delta \geq 0$, then the following hold:*

(1) V^0 is a solution to (3.2.1)

(2) For any $R \geq 0$, $\eta \geq 0$,

$$V^\delta(R,\eta) = V^0(R + \frac{\delta}{\lambda} ln\left|\eta - \bar{\mu}\right|, \eta).$$

Proof. (1) V^0 represents the value function with $\delta = 0$.

Then, by inserting $\delta = 0$ into (6.2.1), we get

$$R - cV^0 - \lambda(\eta - \bar{\mu})(V^0)_\eta + \underset{0\leq a\leq 1}{max}\left[a\eta(V_0)_R + \frac{1}{2}a^2\sigma_1^2(V^0)_{RR}\right] = 0.$$

which is exactly the same as (3.2.1). Thus, we can conclude that V_0 is a solution to (3.2.1).

Now we prove (2) in the following.

We know that V_δ is the solution to (6.2.1).

Let

$$V^\delta(R,\eta) = M(R - \frac{\delta}{\lambda} ln \left|\eta - \bar{\mu}\right|, \eta).$$

Then, by differentiating this equation, we get

$$(V^\delta)_R = M_R, \qquad (V^\delta)_{RR} = M_{RR}, \qquad (M^\delta)\eta = \frac{\delta}{\lambda(\eta - \bar{\mu})} V_R + V_\eta.$$

By inserting these derivatives into (6.2.1), we get

$$R - cM - \lambda(\eta - \bar{\mu})M_\eta + \max_{0 \le a \le 1} \left[a\eta M_R + \frac{1}{2}a^2\sigma_1^2 M_{RR} \right] = 0.$$

which is the same as (3.2.1), it means M is the solution to this equation. Since we already proved V_0 is the solution to (3.2.1), we can conclude that $M(R,\eta) = V^0(R,\eta)$, while $M(R - \frac{\delta}{\lambda} ln \left|\eta - \bar{\mu}\right|, \eta) = V^\delta(R,\eta)$. Therefore, $V^0(R - \frac{\delta}{\lambda} ln \left|\eta - \bar{\mu}\right|, \eta) = V_\delta(R,\eta)$.

This ends to the proof of the proposition. □

Proposition 6.3.2. *The following holds:*

(1) Suppose that $|\eta - \bar{\mu}| < 1$*,then* V^δ *will increase when* δ *increases and* V^δ *will decrease when* δ *decreases.*

(2) Suppose that $|\eta - \bar{\mu}| > 1$*, then* V^δ *will decrease when* δ *increases and* V^δ *will increase when* δ *decreases.*

Proof. From (3.1.3), we know that

$$\mu_t = \bar{\mu} + (\mu_0 - \bar{\mu})e^{-\lambda t}, \qquad t \ge 0.$$

if $\mu_0 \leq \bar{\mu}$, it means $\eta \leq \bar{\mu}$, then when δ increases, $R - \frac{\delta}{\lambda} ln\,|\eta - \bar{\mu}|$ is increasing. Therefore, V^δ will increase.

if $\mu_0 > \bar{\mu}$, $\eta > \bar{\mu}$, then when δ increases, $R - \frac{\delta}{\lambda} ln\,|\eta - \bar{\mu}|$ is decreasing. Therefore, V^δ will decrease.

$\square$

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