Exploring Gaussian elimination based solution methods of electric systems using circuit principles

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In power system simulation, the mathematical solution methods are typically regarded as pure numerical operations, which are independent of the circuit principles. In this letter, it is demonstrated that the computational procedures in the commonly used numerical solution methods for circuit equations are essentially equivalent to the basic principles of electricity. The computational procedures of the representative Gaussian elimination based solution methods (including GE and pivoting method for sparsity of GE/LU) are comprehensively analyzed and their deep relationships with the circuit principles are revealed: the computation process of these mathematics based numerical solution methods can be completely explained using circuit principles. The interesting findings could help understand the computational process of solution methods from the circuit perspective, which may also guide the design of new circuit solution methods.

Introduction: In electric circuits, the laws that determine the currents and voltage behaviors are called circuit principles, mainly including the Ohm's law, Kirchhoff's current and voltage laws (KCL and KVL), and other circuit theorems such as the superposition theorem, Thévenin's theorem, and Norton's theorem. These circuit principles play crucial role in circuit simulation, which is the commonly used approach for checking and verifying the design of electrical and electronic circuits and systems prior to manufacturing and deployment [1, 2].

Typically, given an electric system, the process of circuit simulation is divided into two stages: stage one, circuit principles are leveraged to formulate a set of linear or nonlinear algebraic equations (also called the matrix equations) based on power equipment modelling [3–6] and nodal analysis [7, 8]; stage two, the techniques for solution of these equations are utilized to compute the node voltage values and other electrical parameters. The specific techniques of two stages have been the subject of decades of research in power equipment models and numerical analysis. Although both stages of circuit simulation have drawn focused attention in research, they are usually regarded as independent areas, since the power equipment models and nodal analysis strongly correlate to the circuit principles, while the solution methods are purely numerical operations on the specific values based on mathematical analysis. Therefore, the relationships between electric circuit principles and their numerical solution methods are neglected. After a thorough analysis over the commonly used circuit equation solution methods, we found that the computation process of matrix equation solvers can also be derived purely using circuit principles, i.e. when solving the matrix equations specifically formulated from electric systems, the solution process can be completely explained from the perspective of circuit principles.

In this work, the above discoveries are demonstrated upon the representative Gaussian elimination based solution methods (including GE and pivoting method for sparsity of GE/LU), which show the equivalence of circuit principles and numerical solution methods of electric systems. Although this work does not propose any new solution methods, it provides a new perspective on the interpretation of existing numerical methods of solving linear equations, which also leads to more imaginations that may guide the design of new numerical methods.

Formulations: In this work, the analysis is performed over the electromagnetic transient (EMT) simulation case, which is more complicated and can be easily extend to the steady-state or dynamic power flow simulation. In EMT simulation, each power equipment can be represented by an equivalent circuit by discretizing the formulated differential

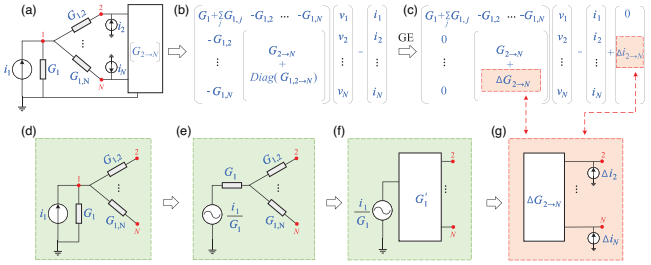

Fig. 1 *Explaining the Gaussian elimination method using circuit principles. (a) Example circuit with N nodes. (b) Formulated matrix equation of the example circuit using nodal analysis. (c) Resulting matrix equation after the first FE step of GE. (d) Left-side circuit connected with nodes 2-N. (e) Using Thevenin's theorem, the current source i_1 in parallel with the conductor G_1 is equivalent to a voltage source $v_1 = i_1/G_1$ in series with a conductance G_1. (f) Using the laws for resistors/conductors in series, the conductor G_1 in series with $G_{1,2\to N}$ is equivalent to the conductor $G_1^{'} = ([G_1^{-1}] + Diag(G_{1,2\to N})^{-1})^{-1}$. (g) Using the Norton's theorem, the voltage source $v_1^{'} = i_1/G_1$ with a series conductance is equivalent to a current source $i_1^{'} = G_1^{'}v_1^{'}$ in parallel with the conductance $G_1^{'}$; where $G_1^{'}$ and $i_1^{'}$ are exactly the same as the results obtained by GE shown in (c).*

equations, such as the transformer [3], machine [4], and transmission line [5] model. For a specific electric system containing different electric apparatus, combining the equivalent circuits together using nodal analysis results in a set of simultaneous linear or nonlinear algebraic equations, which can be expressed in a compact pattern:

$$Gv = i, \tag{1}$$

where G is the equivalent conductance matrix with size of $N \times N$ for a N-node network, $i = \{i_1, i_2, \ldots, i_N\}$ is the N-length current injection vector, and v is an N-vector of the unknown node voltages to be solved. Note that the values of G may be dependent of v (in the nonlinear case) or change over time, and in this article we only focus on the case of constant G since the fundamentals for the G changing over time or over iterations are the same.

Exploring the Gaussian elimination method using circuit principles: Gaussian elimination (GE) is a basic method to solve linear matrix equations, which performs the forward elimination (FE) and backward substitution (BS) operations. We will show that it is possible to find some clues of the underlying connections between GE and circuit principles, although it seems that the GE method is solely a computational procedure.

Here, we start with the first step of FE, as shown in Figure 1a, node 1 is connected with nodes 2-N via branch conductances $G_{1,2}, \ldots, G_{1,N}$, which is reflected in the conductance matrix shown in Figure 1b. It is assumed here that node 1 is connected with all the other nodes, and the case where node 1 is only connected with part of the other nodes is described in the following sections. Here, $G_{2\to N}$ is the conductance matrix only including nodes 2-N, and $\mathrm{Diag}(G_{1,2\to N})$ is the diagonal matrix with $G_{1,2}, \ldots, G_{1,N}$ on the diagonal. After performing the first step of FE, the elements on the first column of G are eliminated into zero (except for the first element), which result in an extra conductance matrix ($\Delta G_{2\to N}$) and an extra current injection ($\Delta i_{2\to N}$) for nodes 2-N, as shown in Figure 1c. From the computation principle of FE, it is easy to check that:

$$\Delta G_{2\to N} = \mathrm{Diag}(G_{1,2\to N}) - \frac{G_{1,2\to N} G_{1,2\to N}^{\mathrm{T}}}{G_1 + \sum_{j=2}^{N} G_{1,j}}, \tag{2}$$

$$\Delta i_{2\to N} = \frac{i_1}{G_1 + \sum_{j=2}^{N} G_{1,j}} \{G_{1,2}, \ldots, G_{1,N}\}^{\mathrm{T}}, \tag{3}$$

where $G_{1,2\to N} = \{G_{1,2}, \ldots, G_{1,N}\}^{\mathrm{T}}$.

Surprisingly, if we analyze the circuit purely based on circuit principles, we can also get the same format. As shown in Figure 1d,e, applying the Thevenin's theorem, the current source i_1 in parallel with the conductance G_1 can be equivalent to a voltage source $v_1^{'} = i_1/G_1$ in

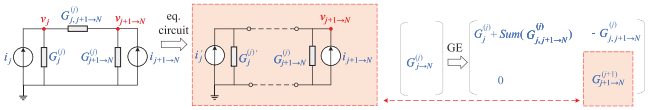

Fig. 2 *In step j of FE for multiple-node networks, only considering the network containing node j to N. The circuit at the left-side of the network containing node $j+1$ to N is equivalent to a current source $i_j^{'}$ in parallel with a conductance $G_j^{(j)'}$, and then the obtained network for node $j+1$ to N is equivalent to the matrix equation after the jth FE step.*

series with a conductance G_1. Next, by applying the laws for resistors in series and in parallel, the conductance G_1 in series with $G_{1,2}, \ldots, G_{1,N}$ is equivalent to the conductance $G_1^{'} = ([G_1^{-1}] + \mathrm{Diag}(G_{1,2\to N})^{-1})^{-1}$, where $[G_1^{-1}]$ is the full matrix with all elements having the same value:

$$[G_1^{-1}] = \begin{bmatrix} G_1^{-1} & \ldots & G_1^{-1} \\ G_1^{-1} & \ldots & G_1^{-1} \\ \vdots & \ddots & \vdots \\ G_1^{-1} & \ldots & G_1^{-1} \end{bmatrix}_{(N-1)\times(N-1)}. \tag{4}$$

Let $A = \mathrm{Diag}(G_{1,2\to N})^{-1}$, $B = \{G_1^{-0.5}, G_1^{-0.5}, .., G_1^{-0.5}\}^{\mathrm{T}}$, and $C = B^{\mathrm{T}}$, it can be observed that $G_1^{'} = (A + BC)^{-1}$. Using the Woodbury identity [10], it can be checked that:

$$\begin{aligned} G_1^{'} &= (A + BC)^{-1} \\ &= A^{-1} - A^{-1}B(I + CA^{-1}B)^{-1}CA^{-1} \\ &= \mathrm{Diag}(G_{1,2\to N}) - \frac{G_{1,2\to N}G_{1,2\to N}^{\mathrm{T}}}{G_1 + \sum_{j=2}^{N} G_{1,j}}. \end{aligned} \tag{5}$$

It can be seen that the equivalent conductance $G_1^{'}$ is exactly equal to $\Delta G_{2\to N}$ that is computed via FE. After applying the Norton's theorem, the voltage source in series with the conductance $G_1^{'}$ is equivalent to a current source $i_1^{'} = G_1^{'}v_1^{'}$ in parallel with the conductance $G_1^{'}$, as shown in Figure 1f,g. It is easy to check that $i_1^{'}$ is also exactly the same as $\Delta i_{2\to N}$. That means, the forward elimination of the conductance matrix can be completely regarded as the process of finding the corresponding equivalent circuit based on circuit principles. If using the proposed equivalent circuit in each iteration, it is equal to conducting the GE solution method.

The equivalence of GE and circuit principles shown above can be easily extended to the case where node 1 is only connected to part of the other nodes. The basic idea is that, in step $j \in \{1, 2, \ldots, N-1\}$, only considering the network containing node $j \to N$ after the $(j-1)$th FE operation, and the corresponding matrix to be operated is $G_{j\to N}^{(j)}$, where the superscript (j) denotes the jth FE operation. Using the same equivalence techniques as proposed above, the circuit at the left-side of the network containing node $j+1$ to N is equivalent to a current source $i_j^{'}$ in parallel with a conductance $G_j^{(j)'}$, and then the obtained network for node $j+1$ to N is equivalent to the matrix equation with $G_{j+1\to N}^{(j+1)}$ after the jth GE step, as illustrated in Figure 2. Here, the coupling conductance $G_{j,j+1\to N}^{(j)}$ is a vector containing the branch conductance values between node j and node $j+1$ to N:

$$G_{j,j+1\to N}^{(j)} = [G_{j,j+1}^{(j)}, G_{j,j+2}^{(j)}, \ldots, G_{j,N}^{(j)}], \tag{6}$$

and $\mathrm{Sum}(G_{j,j+1\to N}^{(j)})$ is the sum of values in the vector. The resulting equivalent circuit after eliminating node j is shown in Figure 2, where the equivalent conductance $G_j^{(j)'}$ is a matrix with the size of $(N-j)\times(N-j)$. The equivalent conductance should be:

$$G_j^{(j)'} = ([G_j^{(j)-1}] + [G_{j,j+1\to N}^{(j)}]^{-1})^{-1}, \tag{7}$$

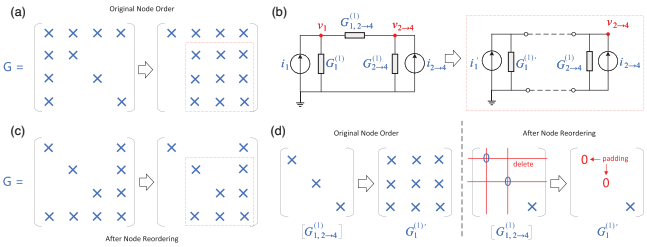

Fig. 3 *Explaining the pivoting methods for sparsity using circuit principles. (a) Conductance matrix of the example circuit with four nodes, where × denotes a non-zero element in the structural representation; after the first FE step, the remaining matrix to be eliminated becomes a full matrix with several fill-ins. (b) In the view of equivalent circuit, the example circuit changes to a three-node circuit with node 1 being eliminated. (c) After node reordering, the first FE step does not introduce extra fill-ins to the matrix. (d) In the view of equivalent circuit, the equivalent conductance $G_1^{(1)'}$ under the original node order is a full matrix, while after the node reordering $G_1^{(1)'}$ becomes a sparse matrix.*

where all the elements in $[G_j^{(j)-1}]$ have the same value:

$$[G_j^{(j)-1}] = \begin{bmatrix} G_j^{(j)-1} & \ldots & G_j^{(j)-1} \\ \vdots & \ddots & \vdots \\ G_j^{(j)-1} & \ldots & G_j^{(j)-1} \end{bmatrix}, \tag{8}$$

and $[G_{j,j+1\to N}^{(j)}]$ is diagonal matrix with branch conductance values in diagonal:

$$[G_{j,j+1\to N}^{(j)}] = \begin{bmatrix} G_{j,j+1}^{(j)} & 0 & \ldots & 0 \\ 0 & G_{j,j+2}^{(j)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & G_{j,N}^{(j)} \end{bmatrix}. \tag{9}$$

However, there may not exist branch between node j and some node $j+p \in \{j+1, \ldots, N\}$, thus $G_{j,j+p}^{(j)} = 0$ and the matrix $[G_{j,j+1\to N}^{(j)}]$ is irreversible. In practical computation, the corresponding $[G_j^{(j)-1}]$ and $[G_{j,j+1\to N}^{(j)}]$ should be modified according to the branches between node j and other nodes. To achieve this purpose, two operations are involved: deleting and padding. (1) Deleting: Assuming there are m branches, then the size of the two matrices should be $m \times m$, by deleting the rows and columns where the diagonal elements with values of zero are located. For example, if $G_{j,j+p}^{(j)} = 0$, then the p^{th} row and column should be deleted from the matrix $[G_j^{(j)-1}]$ and $[G_{j,j+1\to N}^{(j)}]$. (2) Padding: After modifying the matrices, the computed $G_j^{(j)'}$ from Equation (7) also has the size of $m \times m$, then it should be extended to $(N-j)\times(N-j)$ by padding rows and columns with zero-elements according to the deleting locations in $[G_j^{(j)-1}]$ and $[G_{j,j+1\to N}^{(j)}]$. Combining the two operations together, Equation (7) is rewritten as:

$$G_j^{(j)'} = \mathrm{padding}\{([G_j^{(j)-1}]_{\mathrm{del}} + [G_{j,j+1\to N}^{(j)}]_{\mathrm{del}}^{-1})^{-1}\}. \tag{10}$$

Exploring pivoting method for sparsity of GE/LU: In electric power systems, the conductance matrices are generally sparse, with majority elements equal to zero. Therefore, sparse techniques are usually utilized to solve large scale systems to speed up circuit simulation. In Gauss's algorithm for LU factorization (equivalent to GE, denoted as GE/LU [2]), pivoting methods achieve sparsity by reordering the matrix up-front via row and column exchanges. We use a four-node circuit topology to simply demonstrate the principles. The conductance matrix is expressed using the structural representation to avoid showing specific numerical calculations, as shown in Figure 3a, where × denotes a non-zero element. Using the original node order, the matrix changes to a dense matrix after the first FE step, which is not preferred to keep sparsity. And if reorder the nodes by exchanging the node number of 1 and 4, the

resulting matrix could keep sparsity after the first FE step, as shown in Figure 3c.

In this section, we show that the pivoting method of GE/LU could also be derived based on the equivalent circuit described in the section "Exploring the Gaussian elimination method using circuit principles" of the paper. Using the equivalent circuit shown in Figure 3b, the objective of the first FE step (eliminating node 1) to keeping sparsity is equivalent to find a proper reordering to get a sparse $G_1^{(1)'}$. As described in the above section:

$$G_1^{(1)'} = \text{padding}\{([G_1^{(1)-1}]_{\text{del}} + [G_{1,2\to4}^{(1)}]_{\text{del}}^{-1})^{-1}\}. \tag{11}$$

Since $[G_1^{(1)-1}]$ is a full matrix with all elements equal to $G_1^{(1)-1}$, the size of $[G_1^{(1)-1}]_{\text{del}}$ actually determines the number of non-zeros in $G_1^{(1)'}$; i.e. if $[G_1^{(1)-1}]_{\text{del}}$ is a $m \times m$ matrix, then $G_1^{(1)'}$ has m^2 non-zeros. According to the deleting operation, the less branches between node 1 and node 2, 3, 4, the smaller size of $[G_1^{(1)-1}]_{\text{del}}$ and $[G_{1,2\to4}^{(1)}]_{\text{del}}$. As shown in Figure 3d, if using the original node order, node 1 has coupling conductance with node 2, 3, 4, thus $[G_{1,2\to4}^{(1)}]_{\text{del}} = [G_{1,2\to4}^{(1)}] = \text{diag}(G_{1,2}^{(1)}, G_{1,3}^{(1)}, G_{1,4}^{(4)})$ with size of 3×3, and $G_1^{(1)'}$ is a full matrix with size of 3×3. If exchange the indices of node 1 with node 4, then there is no coupling conductance between node 1 and node 2, 3, which results in the deleting operations over $[G_{1,2\to4}^{(1)}]$, and the size of $[G_{1,2\to4}^{(1)}]_{\text{del}}$ changes to 1×1, thus leading to a sparse $G_1^{(1)'}$.

Therefore, to obtain a sparse $G_1^{(1)'}$, the principle is that the number of branches between node 1 and other nodes should be minimized. This principle is just the same as the minimum degree (MD) algorithm proposed by Tinney and Walker [11], which aims to find the minimum degree vertex in the matrix graph. The MD strategy has been very successful in practice and becomes the base of modern sparse techniques such as the multiple minimum degree (MMD) and average minimum degree (AMD).

Conclusion: In this brief express, we used conventional circuit principles to derive the computational process of representative numerical solution methods, mainly including GE and pivoting algorithm for sparsity, which show the surprising equivalence of numerical solution methods and circuit principle based solution methods. The essential reason may be that, the conductance matrix in this case is specially generated from electric systems, which has special features that result in the equivalence. Since the traditional solution methods can be widely used in various forms of matrix equations, the equivalent-circuit based computation procedure in the field of electric system is only a special case of the generic numerical methods. In future work, the circuit principle-based derivation of other numerical solution methods such as Gauss–Jacobi, Gauss–Seidel, and matrix decomposition methods will be investigated; and hopefully, the equivalence could be extended to all numerical methods.

Author contributions: T.D. proposed the idea and performed the theoretical analysis. V.D. organized the paper and did the literature review. The two authors jointly wrote this paper.

Acknowledgments: National Natural Science Foundation of China (No. 62101598). Natural Sciences and Engineering Research Council of Canada.

Conflict of interest: The authors declare no conflict of interest.

Data availability statement: Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Received: *1 April 2023* Accepted: *26 May 2023*
doi: 10.1049/ell2.12836

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