On Automorphism Of Non-singular Hypersurface

by

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A thesis submitted in partial fulfilment of the requirements for the degree of

Master of Science

in

Mathematics

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Abstract

We study the automorphism group of a hypersurface in $\mathbb{P}^n$. Firstly we will review the work of Hideyuki Matsumura and Paul Monsky. Then we will construct an isomorphism from Zariski tangent space of $Aut(X)$ and global sections of tangent sheaf T_X. Then by computing the koszul complex of X, we show that $H^0(T_X) = 0$.

Acknowledgments

First I would like to express my sincere gratitude and special appreciation towards my supervisors Prof. Xi Chen for his patience motivation and immense knowledge during the past two years. It's an unforgettable memories for me to do mathematics with my supervisor. He was my first guide in the art of mathematical exposition. My thesis is based on his previous works so I'm sincerely grateful and honored to get involved in this thesis.

Besides my supervisor, I would like to thank the rest of my thesis committee: Prof. James Lewis, Prof. Chuck Doran, Prof. Nicolas Guay, Prof. Feng Dai for serving as my committee members.

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Chapter 1

Introduction

Let X be a subvariety of $\mathbb{P}^{n+1}$. We denote by $\mathrm{Aut}(X)$ the group of Automorphisms of X, and by $\mathrm{Lin}(V)$ the subgroup of $\mathrm{Aut}(X)$ consisting of the elements of automorphisms of $\mathbb{P}^{n+1}$ which leave X invariant.

Let $H_{n,d}$ be a hypersurface of degree d in the $\mathbb{P}^{n+1}$, which is defined by the homogenous polynomial $f(X_0, X_1, ..., X_{n+1})$. Then we have the following results:

- (1) If $H_{n,d}$ is non-singualr and $n \geq 2$, $d \geq 3$, then $\mathrm{Aut}(H_{n,d})$ is finite except the case $n = 2$, $d = 4$.
- (2) If $H_{n,d}$ is generic hypersurface of degree d in $\mathbb{P}^{n+1}$ and if $n \geq 2$, $d \geq 3$, then $\mathrm{Aut}(H_{n,d})$ is finite except the case $n = 2$, $d = 4$.

Chapter 2

Non-singular hypersurfaces

Let k be an algebraically closed field. Assume that the hypersurface $H_{n.d}$ of $\mathbb{P}^{n+1}$ given by

$$f(X_0, X_1, ..., X_{n+1}) = 0$$

is non-singular. We denote by $f_i(X) = \frac{\partial f(X)}{\partial X_i}$. Then we get $Z(f_0(X), f_1(X), ..., f_{n+1}(X), f(X)) = \emptyset$

Theorem 2.0.1. *If $H_{n,d}$ is non-singualr and if $n \geq 2, d \geq 3$, then $Lin(H_{n,d})$ is finite.*

Proof. Since $\mathrm{Lin}(H) \subseteq PGL(n+2)$, we want to show that $\mathrm{Lin}(H)$ is finite. We only need to show that $\widetilde{\mathrm{Lin}(H)} \in GL(n+2)$ is of dimension 1. Here $\widetilde{\mathrm{Lin}(H)}$ is the pre-image of $\mathrm{Lin}(V)$ under the natural map : $GL(n+2) \longrightarrow PGL(n+2)$. It suffices to show that $\widetilde{\mathrm{Lin}(H)} = \{\alpha E | \alpha \in k^*\}$

Case 1. *The characteristic of k is either zero or prime p not dividing the degree d*

Consider the tangent space $T_{G,1}$ of $G = \widetilde{\mathrm{Lin}(H)}$ at 1. We want to show that $dim(T_{G,1}) = 1$. Let $g \in T_{G,1}$. We assume that $g(t) = I + t_1A_1 + t_2A_2 + ... + t_rA_r + O(t^2)$. Then $f(X)$ is semi-invariant under the action of g, and we have

$$f(g(t)(X_0, X_1, ..., X_{n+1})) = \chi(g(t)) f(X_0, X_1, ..., X_{n+1}). \tag{2.0.1}$$

Then we take the partial derivative of t_k and set $t = 0$, we get the following equation:

$$\sum_{0 \leq i \leq n+1} f_i(X) L_i(X) = c' f(X). \tag{2.0.2}$$

Here $L_i(X) = \sum_{0 \leq j \leq n+1} \xi_{ij}^k X_j$. Then by using Euler equation $f(X) = 1/d \sum f_i(X) X_i$ and letting $c = c'/d$, we get

$$\sum_{0 \leq i \leq n+1} f_i(X)(L_i(X) - cX_i) = 0 \tag{2.0.3}$$

Now let $\alpha = (f, f_0, f_1, ..., f_{n+1})$. By euler equation we get that $\alpha = (f_0, f_1, ..., f_{n+1})$ and the depth of α is zero. Let $\alpha_i = (f_0, ..., \widehat{f_i}, ..., f_{n+1})$ for $0 \leq i \leq n+1$. Then depth $\alpha_i \geq 1$. But depth$(\alpha_i,\ f_i) = 0$, therefore depth $\alpha_i = 1$. Since $k[X_0, X_1, ..., X_{n+1}]$ is Cohen-Macaulay ring, the unmixed theorem holds on this ring. α_i is generated by n+1 elements which equals to height α_i. By unmixed theorem, height of all minimal primes in $Ass_A(A/\alpha_i)$ have equal height.

Lemma 2.0.2. *If α is generated by r elements, then the height of all minimal primes in $Ass_A(A/\alpha)$ is less than or equal to r.*

By the lemma we get that the height of minimal primes in $Ass_A(A/\alpha_i)$ is exactly n+1. Furthermore $(\alpha_i : f_i) = Ann(\bar{f}_i)$ where f_i is the image of homomorphism from $A \to A/\alpha_i$. There exists a $P_i \in Ass_A(A/\alpha_i)$, such that $Ann(\bar{f}_i) \subseteq P_i$. But height$(P_i) = n+1$, we get height $Ann(\bar{f}_i) = n+1$. So $(\alpha_i : f_i) = \alpha_i$. Hence we get

$$L_i(X) - cX_i \in \alpha_i \tag{2.0.4}$$

α_i is generated by polynomials of degree $d-1 > 1$. Consequently $L_i(X) - cX_i = 0$. Then we get all matrix $A_i = cE$, which proves that $dim(T_{G,1}) = 1$.

Case 2. *k is of characteristic $p > 0$ and $d \equiv 0 (mod\ p)$*

For any matrix $A \in G$, we can decompose $A = TU$, where T is a torus and U is a unipotent matrix. We are going to prove $T = \alpha E$ and $U = E$.

$i)$ The case of a torus.

Assume that a torus T in $GL(n+2)$ leaves $f(X)$ semi-invariant. After a suitable linear transformation, we may assume that

$$T = \begin{bmatrix} \chi_0(t) & 0 & \cdots & 0 \\ 0 & \chi_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \chi_{n+1}(t) \end{bmatrix}$$

Then we have $f(\chi_0(t)X_0, \cdots, \chi_{n+1}(t)X_{n+1}) = \chi(t)f(X_0, \cdots, X_{n+1})$. If $f(1,0,\cdots,0) \neq 0$, then f contains X_0^d and we have $d\chi_0 = \chi$(we write the product of characters additively). If $f(1,0,\cdots,0) = 0$ we have some $0 \leq i \leq n+1$ such that $f_i(1,0,\cdots,0) \neq 0$. Then f contains $X_0^{d-1}X_i$. We have $(d-1)\chi_0 + \chi_i = \chi$. In any case we have $(d-1)\chi_0 + \chi_i = \chi$ for some i. Similarly, for any $0 \leq i \leq n+1$ there exists index $j = j(i)$ with $\chi_j + (d-1)\chi_i = \chi$. Then there exists a sequence $i_1, \cdots, i_r$ such that

$$\begin{aligned} c\chi_0 + \chi_{i_1} &= \chi \\ c\chi_{i_1} + \chi_{i_2} &= \chi \\ \cdots\cdots\cdots& \\ c\chi_{i_r} + \chi_0 &= \chi \end{aligned} \tag{2.0.5}$$

Eliminating $\chi_{i_1}, \cdots, \chi_{i_r}$, we get

$$(1-(-c)^{r+1})\chi_0 = (1-c+c^2-\cdots+(-c)^r)\chi$$

Hence $(1-(-c)^{r+1})(d\chi_0-\chi)=0$,Then we get $d\chi_0=\chi$. Similarly, $\chi_0=\chi_1=\cdots=\chi_{n+1}$. So $T=\{\alpha E\}$.

ii) The unipotents case

Let U be the unipotent matrix which leave $f(X)$ semi-invariant. Since U has no non-trivial rational character, U actually leaves $f(X)$ invariant. The same with case i we get the polynomial equation:

$$\sum_{i=0}^{n} f_i(X)L_i(X) = 0 \tag{2.0.6}$$

where $L_i(X)$ is a linear term of X_i. On the other hand, the Euler equation shows that $\sum_{i=0}^{n+1} f_i(X)X_i = 0$. Now we have the following lemma:

Lemma 2.0.3. *Let k be a field and let $f_0(X), \cdots, f_{n+1}(X)$ be the forms of the same degree d' in $k[X_0, \cdots, X_{n+1}]$. Put $\alpha = \sum_{i=0}^{n+1} f_i(X)k[X]$ and assume*

i) depth $\alpha \leq 1$

ii) $\sum_{i=0}^{n+1} f_i(X)X_i = 0$

iii) $n \geq 2$, $d' \geq 2$

Then ii) is the only linear relation between $\{f_i(X)\}$ with linear forms as coefficient. That is, if $l_0(X), \cdots, l_{n+1}(X)$ are linear forms satisfying $\sum f_i(X)l_i(X) = 0$, then there exists a constant $c \in k$ such that $l_0(X) = cX_0, \cdots, l_{n+1}(X) = cX_{n+1}$.

Proof. Firstly we note that depth $\alpha = 1$. Otherwise depth $\alpha = 0$. Then depth $\alpha_i = 0$ for some $\alpha_i = (f_0, \cdots, \hat{f}_i, \cdots, f_{n+1})$. We get $X_i \in (\alpha_i : f_i) = \alpha_i$, which is absurd. Then depth $\alpha = 1$. Without loss of generality, we can choose a sufficiently general matrix $(s_{ij}) \in GL(n+2,k)$, and let $f_i' = \sum_{j=0}^{n+1} s_{ij} f_j$, such that depth $(f_1', \cdots, f_{n+1}') = 1$. Then let $(a_{ij}) = (s_{ij})^{-1}$. We have $f_i = \sum a_{ij} f_j'$. Let $\sum a_{ij} x_i = Y_j$, $\sum a_{ij} l_i(X) = h_j(X)$, $f_j'(X) = G_j(Y)$. We have

$$\begin{aligned}\sum G_j(Y)Y_j &= \sum f_j'(X)(\sum a_{ij}X_i)\\ &= \sum X_i(\sum f_j'(X)a_{ij})\\ &= \sum f_i(X)X_i = 0\end{aligned} \tag{2.0.7}$$

$$\begin{aligned}\sum G_j(Y)h_j(Y) &= \sum f_i'(X)(\sum a_{ij}l_i(X)\\ &= \sum l_i(X)(\sum a_{ij}f_j'(X))\\ &= \sum f_i(X)l_i(X) = 0\end{aligned} \tag{2.0.8}$$

Suppose that $(l_0(X), \cdots, l_{n+1}(X)) \neq \lambda(X_0, \cdots, X_{n+1})$.

Then $(h_0(Y),\cdots,h_{n+1}(Y)) \neq \lambda(Y_0,\cdots,Y_{n+1})$. By renumbering $Y_1,\cdots,Y_{n+1}$, we assume that $h_0(Y)$ contains Y_1. Then

$$\begin{aligned}&\sum_{j=1}^{n+1} G_j(Y)(Y_jh_0 - Y_0h_j) = \\ &h_0(-G_0(Y)Y_0) + Y_0(G_0(Y)h_0(Y)) = 0\end{aligned} \tag{2.0.9}$$

Since depth$(G_1,\cdots,G_{n+1}) = 1$, $(Y_jh_0 - Y_0h_j) \in (G_1,\cdots,G_{n+1})$. It's a contradiction when $d' \geq 2$. When $d' = 2$, let $\varphi_j = Y_JH_0 - Y_0h_j$. $\{\varphi_j\}$ is linear combination of $G_1,\cdots,G_{n+1}$. But φ_j contains Y_jY_1, does not contain Y_iY_1. So φ_j are linearly independent. Then $(f_1'(X),\cdots,f_{n+1}'(X)) = (G_1(X),\cdots,G_{n+1}(X))$ $= (\varphi_1(Y),\cdots,\varphi_{n+1}(Y)) \subseteq (Y_0,h_0(Y))$ and depth $(f_1',\cdots,f_{n+1}') = 1 \geq$ depth $(Y_0,h_0(Y)) \geq n+2-2 = n$, which contradicts with the assumption $n \geq 2$ □

Once we have the lemma we can claim that $U = \alpha E$. But U has no non-trival rational character, so $U = E$. □

Theorem 2.0.4. *Let $H_{n,d}$ $(n \geq 2,\ d \geq 3)$ be non-singular. Then $Aut(H_{n,d}) = Lin(H_{n,d})$ except the the case $n = 2,\ d = 4$.*

Proof. Let $X = H_{n,d}$ and k_X be the canonical sheaf of X.

Case i) When $n \geq 3$.

According to a theorem of Severi-Lefschetz-Andreotti, the Picard group of X is isomorphic to Z. Then for any $f \in \mathrm{Aut}(X)$, f induces an automorphism of $Pic(X)$. We have an isomorphism f^*: $\mathscr{O}_X(1) \longrightarrow \mathscr{O}_X(1)$. Consider the short exact sequence of sheaves of modules:

$$0 \longrightarrow I_X(1) \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(1) \longrightarrow \mathscr{O}_X(1) \longrightarrow 0 \tag{2.0.10}$$

Since $I_X \cong \mathscr{O}_X(-d)$, we get

$$0 \longrightarrow \mathscr{O}_X(1-d) \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(1) \longrightarrow \mathscr{O}_X(1) \longrightarrow 0 \tag{2.0.11}$$

This induces a long exact sequence of cohomology groups:

$$H^0(\mathscr{O}_{\mathbb{P}^{n+1}}) \longrightarrow H^0(\mathscr{O}_X(1)) \longrightarrow H^1(\mathscr{O}_X(1-d)) \longrightarrow \cdots \tag{2.0.12}$$

We know that $H^1(\mathscr{O}_X(1-d)) = 0$. Then we have $H^0(\mathscr{O}_{\mathbb{P}^{n+1}}(1)) \cong H^0(\mathscr{O}_X(1))$. The map f^* induced by f gives an isomorphism of $H^0(\mathscr{O}_{\mathbb{P}^{n+1}}(1))$ and $H^0(\mathscr{O}_X(1))$. From this we know that the automorphism of X catually induces an automorphism of $\mathbb{P}^{n+1}$, which leaves X invariant.

Case ii) When $n = 3,\ d \neq 4$

X is a hypersurface in $\mathbb{P}^3$. Since f is automorphism of X, $f^*k_X \cong k_X$. From the Euler sequence:

$$0 \longrightarrow \Omega_{\mathbb{P}^{n+1}} \longrightarrow \mathscr{O}_X(-1)^{n+2} \longrightarrow \mathscr{O}_X \longrightarrow 0 \tag{2.0.13}$$

and take highest wedge product of this exact sequence we get $k_{\mathbb{P}^{n+1}} \cong \mathscr{O}_{\mathbb{P}^{n+1}}(-n-2)$.

Furthermore, from:

$$0 \longrightarrow I/I_2 \longrightarrow f^*\Omega_{\mathbb{P}^{n+1}} \longrightarrow \Omega_X \longrightarrow 0 \tag{2.0.14}$$

and taking the highest wedge product of this exact sequence we get $k_X \otimes \mathscr{O}_X(-d) \cong f^*k_{\mathbb{P}^3}$. So $k_X \cong \mathscr{O}_X(d-4)$. We know that $Pic(X)$ is torsion free. Thus $f^*k_X \cong f^*\mathscr{O}_X(d-4) \cong \mathscr{O}_X(d-4)$. Dividing $d-4$ on the both side we get $f^*\mathscr{O}_X(1) \cong \mathscr{O}_X(1)$. With the same proof of case i, we get that f is an Automorphism of $\mathbb{P}^{n+1}$ which leaves X invariant. $\square$

Chapter 3

Finiteness of Automorphism of hypersurfaces

The Zariski tangent space of $\mathrm{Aut}(X)$ at a point can be identified by global sections of holomorphic tangent sheaf T_X of X, i.e., $H^0(T_X)$. In other words, every morphism $\mathrm{Spec}\, k[t]/(t^2) \to \mathrm{Aut}(X)$ that sends the closed point to a fixed $f \in \mathrm{Aut}(X)$ (we may just take $f = \mathrm{id}$) is uniquely determined by a global section of T_X. This follows from the construction below.

For every morphism $\varphi : S \to \mathrm{Aut}(X)$, we have a morphism $X \times S \to X$ given by the diagram

$$\begin{array}{ccc} X \times S & \longrightarrow & X \times \mathrm{Aut}(X) \\ \downarrow & \overset{\phi}{\searrow} & \downarrow \\ S & \xrightarrow{\varphi} & X \end{array} \tag{3.0.1}$$

where $X \times \mathrm{Aut}(X) \to X$ is the map sending $(x, g) \to g(x)$. The morphism ϕ induces

$$\begin{array}{ccc} T_{X \times S} & \xrightarrow{\phi_*} & T_X \\ \| & & \\ \pi_X^* T_X \oplus \pi_S^* T_S & & \end{array} \tag{3.0.2}$$

where π_X and π_S are the projections of $X \times S$ to X and S, respectively. Fixing a closed point $0 \in S$, we obtain

$$\pi_S^* T_{S,0} \longrightarrow T_{X \times S}\big|_{X_0} \xrightarrow{\phi_*} T_X \tag{3.0.3}$$

Its induced map on global sections is

$$T_{S,0} \xrightarrow{\kappa} H^0(T_X) \tag{3.0.4}$$

Taking $S = \mathrm{Spec}\, k[t]/(t^2)$, we obtain a k-linear map $\kappa : k \to H^0(T_X)$, which is obviously determined by a global tangent vector $v \in H^0(T_X)$. Every morphism $\varphi : \mathrm{Spec}\, k[t]/(t^2) \to \mathrm{Aut}(X)$ gives rise to some

$v \in H^0(T_X)$ in this way. We claim

Theorem 3.0.1. *Let X be a quasi-projective variety over an algebraically closed field k. Then the Zariski tangent space of* $\mathrm{Aut}(X)$ *at id is isomorphic to* $H^0(T_X) = \mathrm{Hom}(\Omega_X, \mathscr{O}_X)$. *More precisely, every morphism* $\varphi : \mathrm{Spec}\, k[t]/(t^2) \to \mathrm{Aut}(X)$ *satisfying* $\varphi(0) = id$ *is uniquely determined by* $v \in H^0(T_X)$ *through* (3.0.1)-(3.0.4).

Proof. Every morphism $\varphi : \mathrm{Spec}\, k[t]/(t^2) \to \mathrm{Aut}(X)$ induces a morphism $\phi : X \times \mathrm{Spec}\, k[t]/(t^2) \to X$ by the diagram (3.0.1). Indeed, every morphism $\varphi : \mathrm{Spec}\, k[t]/(t^2) \to \mathrm{Aut}(X)$ satisfying $\varphi(0) = \mathrm{id}$ corresponds uniquely to a morphism $\phi : X \times \mathrm{Spec}\, k[t]/(t^2) \to X$ satisfying $\phi(x,0) = x$.

From (3.0.1)-(3.0.4), we have constructed a map

$$\left\{\phi : X \times \mathrm{Spec}\, k[t]/(t^2) \to X,\ \phi(x,0) = x\right\} \longrightarrow H^0(T_X) \tag{3.0.5}$$

We need to construct the inverse of (3.0.5). It goes as follows.

Let us fix $v \in H^0(T_X)$. It is equivalent to fixing a map $v : \Omega_X \to \mathscr{O}_X$. For every affine open set $\mathrm{Spec}\, A \subset X$, we define a ring homomorphism $\phi_{A,\#} : A \to A \otimes_k k[t]/(t^2)$ by

$$\phi_{A,\#}(a) = a + v(da)t \tag{3.0.6}$$

for all $a \in A$, where $d : A \to \Omega_A$ is the derivative. So $\phi_{A,\#}$ induces a morphism $\phi_A : \mathrm{Spec}\, A \times k[t]/(t^2) \to \mathrm{Spec}\, A$ and it is easy to check that ϕ_A is the identity map when restricted to $t = 0$. It is also easy to check that ϕ_A glues to a morphism $\phi : X \times \mathrm{Spec}\, k[t]/(t^2) \to X$ satisfying $\phi(x,0) = x$. This gives a map

$$H^0(T_X) \longrightarrow \left\{\phi : X \times \mathrm{Spec}\, k[t]/(t^2) \to X,\ \phi(x,0) = x\right\} \tag{3.0.7}$$

It is easy to check that the maps (3.0.5) and (3.0.7) are inverse to each other. □

Since $\dim Y \le \max \dim_k T_{Y,p}$ for a scheme Y over an algebraically closed field k, we can show that $\dim \mathrm{Aut}(X) = 0$ by proving that $H^0(T_X) = 0$.

Theorem 3.0.2. *Let X be a smooth hypersurface in $\mathbb{P}_k^{n+1}$. If $n \ge 2$ and* $\deg X \ge 3$, *then* $H^0(T_X) = 0$.

We need the following two lemmas.

Lemma 3.0.3. *Let $F_1, F_2, ..., F_r$ be $r \le n+2$ homogeneous polynomials in $k[z_0, z_1, ..., z_{n+1}]$ of degree $d_1, d_2, ..., d_r$, respectively. Suppose that the intersection $Z = \{F_1 = F_2 = ... = F_r = 0\}$ of the hypersurfaces $\{F_i = 0\}$ has codimension r in $P = \mathbb{P}_k^{n+1}$. Let*

$$\begin{aligned} N_d &= \{(G_1, G_2, ..., G_r) : F_1G_1 + F_2G_2 + ... + F_rG_r = 0\} \\ &\subset \bigoplus_{i=1}^{r} H^0(\mathscr{O}_P(d - d_i)) \end{aligned} \tag{3.0.8}$$

for $d \in \mathbb{Z}$. Then N_d is spanned by

$$(G_1, G_2, ..., G_r) = (0, .., 0, -\lambda F_j, 0, ..., 0, \lambda F_i, 0, ..., 0) \tag{3.0.9}$$

with $G_i = -\lambda F_j$ and $G_j = \lambda F_i$ for some $\lambda \in H^0(\mathscr{O}_P(d - d_i - d_j))$ and all $1 \le i < j \le r$. In particular, $N_d = 0$ if $d < d_i + d_j$ for all $1 \le i < j \le r$.

Proof. Clearly, N_d is the kernel of the map

$$\bigoplus_{i=1}^{r} H^0(\mathscr{O}_P(d - d_i)) \longrightarrow H^0(\mathscr{O}_P(d)) \tag{3.0.10}$$

which sends $(G_1, G_2, ..., G_r)$ to $F_1G_1 + F_2G_2 + ... + F_rG_r$. The map (3.0.10) is induced by the map on sheaves:

$$\bigoplus_{i=1}^{r} \mathscr{O}_P(-d_i) \xrightarrow{\ \xi\ } \mathscr{O}_P \tag{3.0.11}$$

which is similarly defined by $\xi(s_1, s_2, ..., s_2) = F_1s_1 + F_2s_2 + ... + F_rs_r$ for local sections s_i of $\mathscr{O}_P(-d_i)$. The image of ξ is the ideal sheaf I_Z of Z since Z is cut out by $F_i = 0$. That is, we have the right exact sequence

$$\begin{array}{ccccccc} \bigoplus_{i=1}^{r} \mathscr{O}_P(-d_i) & \xrightarrow{\ \xi\ } & \mathscr{O}_P & \longrightarrow & \mathscr{O}_Z & \longrightarrow & 0 \\ & & & & \| & & \\ & & & & \mathscr{O}_P/I_Z & & \end{array} \tag{3.0.12}$$

Let

$$V = \bigoplus_{i=1}^{r} \mathscr{O}_P(-d_i). \tag{3.0.13}$$

The sequence (3.0.12) can be extended to a Koszul complex:

$$0 \longrightarrow \wedge^r V \longrightarrow ... \longrightarrow \wedge^2 V \longrightarrow V \longrightarrow I_Z \longrightarrow 0 \tag{3.0.14}$$

Since Z has the expected dimension $n+1-r$, it is a local complete intersection. Therefore, $F_1/z_j^{d_1}, F_2/z_j^{d_2}, ..., F_r/z_j^{d_r}$ is a regular sequence in $\mathscr{O}_{P,p}$ at every point $p \in \{z_j \neq 0\}$ for all j. It follows that the Koszul complex (3.0.14) is exact.

Let us twist (3.0.14) by $\mathscr{O}(d)$ and break it up into short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \wedge^r V\otimes \mathscr{O}_P(d) & \longrightarrow & \wedge^{r-1}V\otimes \mathscr{O}_P(d) & \longrightarrow & M_{r-3} & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & M_1 & \longrightarrow & \wedge^2 V\otimes \mathscr{O}_P(d) & \longrightarrow & M_0 & \longrightarrow & 0 \\
 & & & & & & & & \\
0 & \longrightarrow & M_0 & \longrightarrow & V\otimes \mathscr{O}_P(d) & \longrightarrow & I_Z(d) & \longrightarrow & 0 \\
 & & & & \| & & & & \\
 & & & & \bigoplus_{i=1}^{r} \mathscr{O}_P(d-d_i) & & & &
\end{array} \tag{3.0.15}$$

Obviously, the kernel of the map (3.0.10) is exactly $H^0(M_0)$. That is,

$$N_d = H^0(M_0). \tag{3.0.16}$$

And the space spanned by(3.0.9) is exactly the image of the map

$$\begin{array}{ccc}
H^0(\wedge^2 V\otimes \mathscr{O}_P(d)) & \longrightarrow & H^0(V\otimes \mathscr{O}_P(d)) \\
\| & & \| \\
\bigoplus_{1\le i<j\le r} H^0(\mathscr{O}_P(d-d_i-d_j)) & & \bigoplus_{i=1}^{r} H^0(\mathscr{O}_P(d-d_i))
\end{array} \tag{3.0.17}$$

So it suffices to prove that

$$H^0(M_0) = \mathrm{Im}(H^0(\wedge^2 V\otimes \mathscr{O}_P(d)) \to H^0(V\otimes \mathscr{O}_P(d))) \tag{3.0.18}$$

which is in turn equivalent to the surjection

$$H^0(\wedge^2 V\otimes \mathscr{O}_P(d)) \twoheadrightarrow H^0(M_0) \tag{3.0.19}$$

Furthermore, (3.0.19) is a surjection if $H^1(M_1) = 0$, which holds in turn if

$$H^1(\wedge^3 V\otimes \mathscr{O}_P(d)) = H^2(\wedge^4 V\otimes \mathscr{O}_P(d)) = \ldots = H^{r-2}(\wedge^r V\otimes \mathscr{O}_P(d)) = 0. \tag{3.0.20}$$

To see that (3.0.20) implies $H^1(M_1) = 0$, we argue inductively

$$\begin{aligned} H^{r-2}(\wedge^r V \otimes \mathscr{O}_P(d)) = H^{r-3}(\wedge^{r-1} V \otimes \mathscr{O}_P(d)) = 0 &\Rightarrow H^{r-3}(M_{r-3}) = 0 \\ H^{r-3}(M_{r-3}) = H^{r-4}(\wedge^{r-2} V \otimes \mathscr{O}_P(d)) = 0 &\Rightarrow H^{r-4}(M_{r-4}) = 0 \\ \vdots \qquad\qquad &\Rightarrow \qquad \vdots \\ H^2(M_2) = H^1(\wedge^3 V \otimes \mathscr{O}_P(d)) = 0 &\Rightarrow H^1(M_1) = 0 \end{aligned} \tag{3.0.21}$$

using the short exact sequence (3.0.15).

So it remains to verify (3.0.20). Note that

$$\wedge^m V \otimes \mathscr{O}_P(d) = \bigoplus_{|J|=m} \mathscr{O}_P(d - \sum_{j \in J} d_j) \tag{3.0.22}$$

where J runs over all subsets of $\{1, 2, ..., r\}$ with cardinality $|J| = m$.

Since $H^a(\mathscr{O}_P(b)) = 0$ for all $1 \le a \le n$ and all $b \in \mathbb{Z}$,

$$H^{m-2}(\wedge^m V \otimes \mathscr{O}_P(d)) = 0 \tag{3.0.23}$$

for $m = 3, ..., r$ and all d, which is exactly (3.0.20). □

Lemma 3.0.4. *Let $F_0, F_1, ..., F_r$ be $r + 1 \le n + 2$ homogeneous polynomials in $k[z_0, z_1, ..., z_{n+1}]$ of degree d. Suppose that $Z = \{F_0 = F_1 = ... = F_r = 0\}$ has codimension $\ge r$ in $P = \mathbb{P}_k^{n+1}$ and*

$$F_0 L_0 + F_1 L_1 + ... + F_r L_r = 0 \tag{3.0.24}$$

for some linearly independent linear forms $L_0, L_1, ..., L_r \in H^0(\mathscr{O}_P(1))$. If $d \ge 2$ and $r \ge 3$, then the space

$$\begin{aligned} N = \{&(G_0, G_1, ..., G_r) : F_0 G_0 + F_1 G_1 + ... + F_r G_r = 0\} \\ &\subset H^0(\mathscr{O}_P(1))^{\oplus r+1} \end{aligned} \tag{3.0.25}$$

is spanned by $(L_0, L_1, ..., L_r)$.

Proof. By Lemma 3.0.3 and (3.0.24), $F_0, F_1, ..., F_r$ are linearly independent.

We claim that there exist $F_1', F_2', ..., F_r'$ in $V = \mathrm{Span}\{F_0, F_1, ..., F_r\}$ such that

$$\dim\{F_1' = F_2' = ... = F_r' = 0\} = n + 1 - r. \tag{3.0.26}$$

We construct such sequence inductively such that

$$\dim\{F_1' = F_2' = ... = F_l' = 0\} = n + 1 - l. \tag{3.0.27}$$

for $l = 1, 2, ..., r$. This is obvious when $l = 1$.

Suppose that (3.0.27) holds for some $l < r$. Let W be an irreducible component of $X_l = \{F_1' = F_2' = ... = F_l' = 0\}$. Let us consider

$$V_W = \{F = c_0F_0 + c_1F_2 + ... + c_rF_r : c_i \in k,\ W \subset \{F = 0\}\} \subset V \tag{3.0.28}$$

Clearly, V_W is a Zariski closed subset of V. If $V_W = V$, then for every $F \in V$, $W \subset \{F = 0\}$ and hence

$$W \subset Z = \{F_0 = F_1 = ... = F_r = 0\} \tag{3.0.29}$$

which implies that $\dim Z \geq \dim W = n + 1 - l > n + 1 - r$ and contradicts our hypothesis on Z. Therefore, V_W is a proper Zariski closed subset of V. That is, $V \backslash V_W$ is a nonempty Zariski open set of V. For all irreducible components W of X_l,

$$V \backslash \bigcup_{W \subset X_l} V_W \neq \emptyset \tag{3.0.30}$$

and hence there exists $F_{l+1}' \in V$ such that $\{F_{l+1}' = 0\}$ does not contain any irreducible components of X_l and consequently

$$\dim X_{l+1} = \dim(X_l \cap \{F_{l+1}' = 0\}) = \dim X_l - 1 = n - l \tag{3.0.31}$$

for $X_{l+1} = \{F_1' = F_2' = ... = F_l' = F_{l+1}' = 0\}$. This proves our claim. That is, the set

$$\begin{aligned} U = \{(F_1', F_2', ..., F_r') : \dim\{F_1' = F_2' = ... = F_r' = 0\} = n + 1 - r\} \\ \subset V^r \end{aligned} \tag{3.0.32}$$

is nonempty. In addition, $V^r \backslash U$ is a Zariski closed subset of V^r. We see this by constructing the correspondence

$$\begin{aligned} Y = \{(F_1', F_2', ..., F_r', p) : p \in \{F_1' = F_2' = ... = F_r' = 0\}\} \\ \subset V^r \times P \end{aligned} \tag{3.0.33}$$

Every fiber of $Y \to V^r$ has dimension at least $n + 1 - r$. Then $V^r \backslash U$ is exactly the locus of points $(F_1', F_2', ..., F_r')$ over which the fibers of Y have dimension $> n + 1 - r$, which is a Zariski closed subset of V^r [**?**, Exercise 3.22, p. 95]. So U is a nonempty Zariski open set of V^r.

Let us consider $V^{r+1} = \{(F_0', F_1', ..., F_r')\}$ with the following loci removed

- the locus $(F_0', F_1', ..., F_r')$ where $F_0', F_1', ..., F_r'$ are linearly dependent,
- the locus $(F_0', F_1', ..., F_r')$ where $(F_0', F_1', ..., \widehat{F}_i', ..., F_r')$ lies in $V^r \backslash U$ for some $i = 0, 1, ..., r$.

All the loci removed are proper Zariski closed subset of V^{r+1}. Therefore, the complement is nonempty.

That is, there exists a basis $F_0', F_1', ..., F_r'$ of V such that

$$\dim\{F_0' = F_1' = ... = \widehat{F_i'} = ... = F_r' = 0\} = n+1-r \tag{3.0.34}$$

for $i = 0, 1, ..., r$.

Let us simply replace $F_0, F_1, ..., F_r$ by $F_0', F_1', ..., F_r'$ and assume

$$\dim\{F_0 = F_1 = ... = \widehat{F_i} = ... = F_r = 0\} = n+1-r \tag{3.0.35}$$

for $i = 0, 1, ..., r$.

Suppose that $\dim N \geq 2$. Then there exist $M_0, M_1, ..., M_r \in H^0(\mathscr{O}_P(1))$ such that

$$F_0M_0 + F_1M_1 + ... + F_rM_r = 0 \tag{3.0.36}$$

with M_i and L_i linearly independent for some i. WLOG, let us assume that M_0 and L_0 are linearly independent. Then

$$F_1(L_0M_1 - L_1M_0) + F_2(L_0M_2 - L_2M_0) + ... + F_r(L_0M_r - L_rM_0) = 0. \tag{3.0.37}$$

We necessarily have that $L_0M_1 - L_1M_0, L_0M_2 - L_2M_0, ..., L_0M_r - L_rM_0$ are linearly independent. Otherwise, we have

$$L_0(c_1M_1 + c_2M_2 + ... + c_rM_r) = (c_1L_1 + c_2L_2 + ... + c_rL_r)M_0 \tag{3.0.38}$$

for some $c_1, c_2, ..., c_r \in k$, not all zero. But the two pairs $\{L_0, \sum c_iL_i\}$ and $\{L_0, M_0\}$ are both linearly independent so (3.0.38) cannot hold.

If $d \geq 3$, then $L_0M_i - L_iM_0 = 0$ for $i = 1, 2, ..., r$ by Lemma 3.0.3. Contradiction.

If $d = 2$, again by Lemma 3.0.3, $L_0M_i - L_iM_0 \in \text{Span}\{F_1, F_2, ..., F_r\}$ for $i = 1, 2, ..., r$. And since $L_0M_i - L_iM_0$ are linearly independent, we have

$$\begin{aligned} &\text{Span}\{L_0M_1 - L_1M_0, L_0M_2 - L_2M_0, ..., L_0M_r - L_rM_0\} \\ =\ &\text{Span}\{F_1, F_2, ..., F_r\} \end{aligned} \tag{3.0.39}$$

and hence

$$\begin{aligned} &\{L_0M_1 - L_1M_0 = L_0M_2 - L_2M_0 = ... = L_0M_r - L_rM_0 = 0\} \\ =\ &\{F_1 = F_2 = ... = F_r = 0\}. \end{aligned} \tag{3.0.40}$$

But the left hand side of (3.0.40) contains the subset $\{L_0 = M_0 = 0\}$ which has dimension $n-1$. This contradicts (3.0.35) for $r \geq 3$. □

Proof of Theorem 3.0.2. Since X is smooth, we have the exact sequence [**?**, Theorem 8.17, p. 178]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_X & \longrightarrow & T_P \otimes \mathscr{O}_X & \longrightarrow & N_{X/P} & \longrightarrow & 0 \\ & & & & & & \Big\| & & \\ & & & & & & \mathscr{O}_X(d) & & \end{array} \tag{3.0.41}$$

where $P = \mathbb{P}^{n+1}$, $d = \deg X$ and $N_{X/P}$ is the normal sheaf of X in P. To show that $H^0(T_X) = 0$, it suffices to show that the induced map

$$H^0(X, T_P) \xrightarrow{\xi} H^0(N_{X/P}) \tag{3.0.42}$$

is injective.

By Euler's sequence

$$0 \longrightarrow \mathscr{O}_P \longrightarrow \mathscr{O}_P(1)^{\oplus n+2} \longrightarrow T_P \longrightarrow 0 \tag{3.0.43}$$

we see that $H^0(X, T_P)$ is given by the induced long exact sequence

$$0 \to H^0(\mathscr{O}_X) \to H^0(\mathscr{O}_X(1))^{\oplus n+2} \to H^0(X, T_P) \to H^1(\mathscr{O}_X) \tag{3.0.44}$$

When $n \geq 2$, we have $H^1(\mathscr{O}_X) = 0$ by the short sequence

$$0 \longrightarrow \mathscr{O}_P(-d) \longrightarrow \mathscr{O}_P \longrightarrow \mathscr{O}_X \longrightarrow 0 \tag{3.0.45}$$

and its induced long exact sequence

$$\begin{array}{ccccc} H^1(\mathscr{O}_P) & \longrightarrow & H^1(\mathscr{O}_X) & \longrightarrow & H^2(\mathscr{O}_P(-d)) \\ \Big\| & & & & \Big\| \\ 0 & & & & 0 \end{array} \tag{3.0.46}$$

So when $n \geq 2$, (3.0.44) becomes

$$0 \longrightarrow H^0(\mathscr{O}_X) \xrightarrow{\eta} H^0(\mathscr{O}_X(1))^{\oplus n+2} \longrightarrow H^0(X, T_P) \longrightarrow 0 \tag{3.0.47}$$

where the map η is given by

$$\eta(1) = (z_0, z_1, \ldots, z_{n+1}) \tag{3.0.48}$$

with $(z_0, z_1, \ldots, z_{n+1})$ being the homogeneous coordinates of $\mathbb{P}_k^{n+1}$. By convention, we use $\partial/\partial z_i$ as a basis for $H^0(\mathscr{O}_X(1))^{\oplus n+2}$ and (3.0.48) becomes

$$\eta(1) = \sum_{i=0}^{n+1} z_i \frac{\partial}{\partial z_i} \tag{3.0.49}$$

and hence

$$\begin{aligned} H^0(X, T_P) &= \frac{H^0(\mathscr{O}_X(1))^{\oplus n+2}}{\eta(H^0(\mathscr{O}_X))} \\ &= \left\{ \sum_{i=0}^{n+1} L_i \frac{\partial}{\partial z_i} : L_i \in H^0(\mathscr{O}_X(1)) \right\} / \left(\sum_{i=0}^{n+1} z_i \frac{\partial}{\partial z_i} \right) \end{aligned} \tag{3.0.50}$$

With $H^0(X, T_P)$ identified as above, we see that the map ξ in (3.0.42) is

$$\xi \left(\sum_{i=0}^{n+1} L_i \frac{\partial}{\partial z_i} \right) = \sum_{i=0}^{n+1} L_i \frac{\partial F}{\partial z_i} \tag{3.0.51}$$

where $F(z_0, z_1, ..., z_{n+1})$ is the homogeneous polynomial defining X. Thus,

$$\begin{aligned} & H^0(T_X) = \ker(\xi) \\ &= \left\{ \sum_{i=0}^{n+1} L_i \frac{\partial}{\partial z_i} : L_i \in H^0(\mathscr{O}_X(1)),\ \sum_{i=0}^{n+1} L_i F_i = 0 \right\} / \left(\sum_{i=0}^{n+1} z_i \frac{\partial}{\partial z_i} \right) \\ &= \left\{ \sum_{i=0}^{n+1} L_i \frac{\partial}{\partial z_i} : L_i \in H^0(\mathscr{O}_P(1)),\ \sum_{i=0}^{n+1} L_i F_i \in \mathrm{Span}\{F\} \right\} / \left(\sum_{i=0}^{n+1} z_i \frac{\partial}{\partial z_i} \right) \end{aligned} \tag{3.0.52}$$

for $F_i = \partial F / \partial z_i$. To show that $H^0(T_X) = 0$, it suffices to show that

$$\begin{aligned} & \left\{ \sum_{i=0}^{n+1} L_i \frac{\partial}{\partial z_i} : L_i \in H^0(\mathscr{O}_P(1)),\ \sum_{i=0}^{n+1} L_i F_i \in \mathrm{Span}\{F\} \right\} \\ &= \mathrm{Span} \left\{ \sum_{i=0}^{n+1} z_i \frac{\partial}{\partial z_i} \right\}. \end{aligned} \tag{3.0.53}$$

Note that since X is smooth,

$$\{F_0 = F_1 = ... = F_{n+1} = F = 0\} = \emptyset. \tag{3.0.54}$$

Suppose that $L_0, L_1, ..., L_{n+1} \in H^0(\mathscr{O}_P(1))$ satisfy

$$\sum_{i=0}^{n+1} L_i F_i = \lambda F \tag{3.0.55}$$

for some $\lambda \in k$. Note that

$$\sum_{i=0}^{n+1} z_i F_i = dF. \tag{3.0.56}$$

When char $k \nmid d$, $F = (1/d)\sum z_i F_i$ and (3.0.54) becomes

$$\{F_0 = F_1 = ... = F_{n+1} = 0\} = \emptyset. \tag{3.0.57}$$

And by (3.0.55),

$$\sum_{i=0}^{n+1} L_i F_i = \lambda F = \frac{\lambda}{d} \sum_{i=0}^{n+1} z_i F_i. \tag{3.0.58}$$

Then by Lemma 3.0.3,

$$L_i - \frac{\lambda}{d} z_i = 0 \tag{3.0.59}$$

for $i = 0, 1, ..., n+1$ and (3.0.53) follows.

When char $k \mid d$,

$$\sum_{i=0}^{n+1} z_i F_i = 0. \tag{3.0.60}$$

By (3.0.54),

$$\dim\{F_0 = F_1 = ... = F_{n+1} = 0\} \leq 0. \tag{3.0.61}$$

If $\lambda = 0$, then (3.0.53) follows directly from Lemma 3.0.4.

If $\lambda \neq 0$, then

$$\begin{aligned} &\{F_0 = F_1 = ... = F_{n+1} = 0\} \\ &= \left\{F_0 = F_1 = ... = F_{n+1} = \sum_{i=0}^{n+1} L_i F_i = 0\right\} \\ &= \{F_0 = F_1 = ... = F_{n+1} = \lambda F = 0\} \\ &= \{F_0 = F_1 = ... = F_{n+1} = F = 0\} = \emptyset \end{aligned} \tag{3.0.62}$$

by (3.0.54). Then by Lemma 3.0.3, there do not exist $G_i \in H^0(\mathscr{O}_P(1))$, not all zero, such that

$$G_0 F_0 + G_1 F_1 + ... + G_{n+1} F_{n+1} = 0. \tag{3.0.63}$$

This contradicts (3.0.60). □

More general, let X be a smooth hypersurface in a smooth projective variety P. We have the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_X & \longrightarrow & T_P \otimes \mathscr{O}_X & \longrightarrow & \mathscr{N}_X & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & \mathscr{O}_X(X) & & \end{array}$$

where $\mathscr{N}_X$ is the normal bundle of $X \subset P$. Then the Koszul complex associated to

$$T_P \otimes \mathscr{N}_X^{-1} \longrightarrow \mathscr{O}_X \longrightarrow 0$$

gives rise to

$$0 \longrightarrow \wedge^n T_P \otimes \mathscr{N}_X^{1-n} \longrightarrow \dots \longrightarrow \wedge^2 T_P \otimes \mathscr{N}_X^{-1} \longrightarrow T_P \otimes \mathscr{O}_X \longrightarrow \mathscr{N}_X \longrightarrow 0 \tag{3.0.64}$$

where $n = \dim P$ and T_X is the image of $\wedge^2 T_P \otimes \mathscr{N}_X^{-1} \to T_P \otimes \mathscr{O}_X$. Thus, by breaking down (3.0.64) into short exact sequence, we can show that

$$H^0(T_X) = 0 \text{ if } H^{r-2}(\wedge^r T_P \otimes \mathscr{N}_X^{1-r}) = 0 \text{ for } r \geq 2. \tag{3.0.65}$$

Using the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_P^r(-rX) & \longrightarrow & T_P^r((1-r)X) & \longrightarrow & T_P^r \otimes \mathscr{O}_X((1-r)X) & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & \wedge^r T_P \otimes \mathscr{N}_X^{1-r} & & \end{array} \tag{3.0.66}$$

we further reduce (3.0.65) to

$$\begin{gathered} H^0(T_X) = 0 \\ \text{if } H^{r-1}(T_P^r(-rX)) = H^{r-2}(T_P^r((1-r)X)) = 0 \text{ for } r \geq 2 \end{gathered} \tag{3.0.67}$$

where $T_P^\bullet = \wedge^\bullet T_P$.

Now let us consider $P = P_1 \times P_2 \times \dots \times P_s$ and $X \subset P$ a smooth hypersurface given by a global section of $\pi_1^* L_1 \otimes \pi_2^* L_2 \otimes \dots \otimes \pi_s^* L_s$, where π_i is the projection $P \to P_i$ and L_i is a line bundle on P_i for $i = 1, 2, \dots, s$.

Then by Künneth,

$$\begin{aligned} & H^{r-1}(T_P^r(-rX)) \\ &= \bigoplus_{m_1+m_2+\ldots+m_s=r} H^{r-1}(\bigotimes_{i=1}^{s} \pi_i^* T_{P_i}^{m_i} \otimes \mathscr{O}_P(-rX)) \\ &= \bigoplus_{\substack{m_1+m_2+\ldots+m_s=r \\ l_1+l_2+\ldots+l_s=r-1}} \left(\bigotimes_{i=1}^{s} H^{l_i}(T_{P_i}^{m_i} \otimes L_i^{-r}) \right) \end{aligned} \tag{3.0.68}$$

Among l_i and m_i in (3.0.68), $l_i < m_i$ for at least one i. Therefore,

$$\begin{gathered} H^{r-1}(T_P^r(-rX)) = 0 \\ \text{if } H^l(T_{P_i}^m \otimes L_i^{-r}) \text{ for all } l < m \le r \text{ and } i = 1, 2, \ldots, s. \end{gathered} \tag{3.0.69}$$

Similarly,

$$\begin{gathered} H^{r-2}(T_P^r((1-r)X)) = 0 \\ \text{if } H^l(T_{P_i}^m \otimes L_i^{1-r}) = 0 \text{ for all } l < m \le r,\ l \le r-2 \text{ and } i = 1, 2, \ldots, s. \end{gathered} \tag{3.0.70}$$

Combining (3.0.67), (3.0.69) and (3.0.70), we conclude

$$\begin{gathered} H^0(T_X) = 0 \\ \text{if } H^a(T_{P_i}^b \otimes L_i^{-c}) = 0 \text{ for all } a < b \le c+1,\ a < c \text{ and } i = 1, 2, \ldots, s. \end{gathered} \tag{3.0.71}$$

Now let $P_i = \mathbb{P}^{n_i}$ and $L_i = \mathscr{O}_{P_i}(d_i)$. From the Euler sequence,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_{P_i} & \longrightarrow & \mathscr{O}_{P_i}(1)^{\oplus(n_i+1)} & \longrightarrow & T_{P_i} & \longrightarrow & 0 \\ & & & & \| & & & & \\ & & & & V_i & & & & \end{array}$$

we have the exact sequence

$$0 \longrightarrow \wedge^{\bullet-1} T_{P_i} \longrightarrow \wedge^{\bullet} V_i \longrightarrow \wedge^{\bullet} T_{P_i} \longrightarrow 0 \tag{3.0.72}$$

Then inductively, we have

$$\begin{gathered} H^a(T_{P_i}^b \otimes L_i^{-c}) = H^{a-1}(T_{P_i}^{b+1} \otimes L_i^{-c}) = \ldots = H^0(T_{P_i}^{a+b} \otimes L_i^{-c}) = 0 \\ \text{if } H^a(\wedge^{b+1} V_i \otimes L_i^{-c}) = H^{a-1}(\wedge^{b+2} V_i \otimes L_i^{-c}) = \ldots = H^0(\wedge^{a+b+1} V_i \otimes L_i^{-c}) = \\ H^a(\wedge^{b} V_i \otimes L_i^{-c}) = H^{a-1}(\wedge^{b+1} V_i \otimes L_i^{-c}) = \ldots = H^0(\wedge^{a+b} V_i \otimes L_i^{-c}) = 0. \end{gathered}$$

So the condition $cd_i > a+b+1$ guarantees $H^a(T_{P_i}^b \otimes L_i^{-c}) = 0$. Clearly, the numerical condition on

(a,b,c) implies that $a+b \leq 2c$. So $cd_i > a+b+1$ if $d_i \geq 4$. When $d_i = 3$, $cd_i \leq a+b+1$ if and only if $(a,b,c) = (0,2,1)$. In this case, we again apply (3.0.72) to conclude

$$\begin{aligned} H^0(T^2_{P_i}(-3)) &= H^1(T_{P_i}(-3)) = 0 \\ \text{if } H^0(\wedge^2 V_i \otimes \mathscr{O}_{P_i}(-3)) &= H^1(\wedge^2 V_i \otimes \mathscr{O}_{P_i}(-3)) \\ &= H^1(V_i \otimes \mathscr{O}_{P_i}(-3)) = H^2(\mathscr{O}_{P_i}(-3)) = 0 \end{aligned}$$

This holds as long as $n_i = \dim P_i \geq 3$.

So we arrive at a statement about the automorphism group of a smooth hypersurface in the product of projective spaces.

Theorem 3.0.5. *For a smooth hypersurface X in $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_s}$ of multi-degree $(d_1, d_2, \ldots, d_s)$, $H^0(T_X) = 0$ if either $d_i \geq 4$ or $d_i = 3$ and $n_i \geq 3$ for each $i = 1, 2, \ldots, s$.*

Chapter 4

Generic hypersurface

Let k be a field of characteristic $p \geq 0$. And let k_0 be the subfield of k generated by 1. A hypersurface $H_{n,d}$ is called generic if it is generic over k_0. That is, if it is defined by a homogenous equation $f(X_0, X_1, \cdots, X_{n+1}) = 0$ of which the $\binom{n+d+1}{d}$ coefficients are algebraically independent over k_0. A generic hypersurface is non-singular.

Theorem 4.0.1. *If $H_{n,d}$ is generic and if $n \geq 2, d \geq 3$, then $Lin(H_{n,d}) = \{e\}$.*

Proof. For convenience, we denote $m = n+2$. We assume that $f(X) \in k[X_1, X_2, \cdots, X_m]$ define the generic hypersurface of degree $d \geq 3$ for $m \geq 4$. We need to prove that if $A = (a_{ij}) \in GL(m,k)$ that leaves $f(X)$ semi-invariant:

$$f(A(X)) = \alpha f(X), \ \alpha \in k^* \tag{4.0.1}$$

Then $A = cE_m$ for some $c \in k^*$. Firstly we can decompose $A = A_s A_u$, where A_s and A_u are respectively semi-simple and unipotent. A_s and A_u also leave $f(X)$ semi-invariant. Then we only need to consider about the following two cases:

I. Semi-simple case. Let $A \in GL(m,k)$ be semi-simple and assume $f(A(X)) = cf(X)$. Then we can find a matrix T such that

$$TAT^{-1} = B = \begin{bmatrix} \alpha_1 E_{r_1} & 0 & \cdots & 0 \\ 0 & \alpha_2 E_{r_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_s E_{r_s} \end{bmatrix}$$

where E_{r_i} is the identity matrix of size r_i. So we have

$$\sum_{i=1}^{s} r_i = m$$

and we assume that $\alpha_i \neq \alpha_j (i \neq j)$. Now let $g(X) = f(T^{-1}X)$. If we apply linear transformation B to polynomial $g(X)$, we get $g(BX) = f(T^{-1}BX) = f(AT^{-1}X) = cf(T^{-1}X) = cg(X)$. Now we change the notation of variables:

$$X_{1,1}, \cdots, X_{1,r_1}, X_{2,1}, \cdots, X_{2,r_2}, \cdots, X_{s,1}, \cdots, X_{s,r_s}. \tag{4.0.2}$$

If we use X_i to denote $(X_{i,1}, \cdots, X_{i,r_i})$, then we get

$$g(\alpha_1(X_1), \cdots, \alpha_s(X_s)) = cg(X_1, \cdots, X_s) \tag{4.0.3}$$

by $g(BX) = cg(X)$. Now we try to prove that $g(X)$ will miss more than $2\sum r_i r_j$ monomials of degree d if $s > 1$. Since we have $f(T^{-1}X) = g(X)$, then $f(X)$ is impossibly generic, contrary to our assumption.

Now let us consider the monomials of degree d that are divisible by $X_{1,1}^{d-3}$. WLOG, we may assume $d = 3$. Then we classify the cubic monomials into four classes.

- $C_i = \{\text{cubic in } (X_i)\}$. $\#C_i = r_i(r_i+1)(r_i+2)/6$
- $C_{ij} = \{\text{quadratic in } (X_i) \text{ and linear in } (X_j)\}(i < j)$
$$\#C_{ij} = r_i r_j (r_i + 1)/2$$
- $D_{ij} = \{\text{linear in } (X_i) \text{ and quadratic in } (X_j)\}(i < j)$
$$\#D_{ij} = r_i r_j (r_j + 1)/2$$
- $C_{ijl} = \{\text{linear in } (X_i), (X_j) \text{ and } (X_l)\}(i < j < l)$
$$\#C_{ijl} = r_i r_j r_l$$

C_{ij} and D_{ij} cannot co-exist in $g(X)$. If we have $X_i^2 X_j$ and $X_i X_j^2$ in $g(X)$ at the same time, we must have $\alpha_i^2 \alpha_j = \alpha_i \alpha_j^2$ which is impossible. Similarly, for each $1 \neq i \leq s$, at most one out of classes

$$D_{1,i}, \cdots, D_{i-1,i}, C_i, C_{i,i+1}, \cdots, C_{i,s} \tag{4.0.4}$$

can appear in $g(X)$. Now we define E_{ij} as follows:

- a) If both C_{ij} and D_{ij} are absent in $g(X)$, then $E_{ij} = C_{ij} \cup D_{ij}$
- b) If C_{ij} is absent but D_{ij} is present in $g(X)$, then $E_{ij} = C_{ij} \cup C_j$
- c) If C_{ij} is present in $g(X)$, then $E_{ij} = D_{ij} \cup C_i$

If $C_i \subset E_{ij} \cup E_{il} (i < j < l)$, then C_{ij} and C_{il} co-exist in $g(X)$, which is impossible. Similarly $C_i \subset E_{ji} \cup E_{li} (j < l < i)$ is impossible. If $C_i \subset E_{ij} \cup E_{li} (l < i < j)$, then both C_{ij} and D_{li} appear in $g(X)$, which is again impossible. Therefore the sets $E_{ij} (i < j)$ are disjoint. On the other hand, we can check that

$\#E_{ij} \geq 2r_ir_j$, where the equality can holds only when $r_i = r_j = 1$. Since all E_{ij} are absent in $g(X)$, at least $\sum \#E_{ij}$ monomials are missing and we have $\sum \#E_{ij} \leq 2\sum r_ir_j$, the equality holds only when $s = m, r_1 = r_2 = \cdots = r_m = 1$. In that case, since $m \geq 4$, at least one of $X_1X_2X_3$ and $X_1X_2X_4$ is absent in $g(X)$ also.

II Unipotent case. Let $A \in GL(m,k)$ be a unipotent matrix, $A \neq E$ and let $f(X)$ be a polynomial of degree $d \geq 3$ which is semi-invariant under A. Since A is unipotent, we have $f(X)$ is actually invariant under A. Let J be the jordan normal form of A. We assume that the blocks in J are in the order of increasing size. For $1 \leq i \leq m-1$ we have $J(X_i) = X_i + \varepsilon_i X_{i+1}$, and $J(X_m) = X_m$, where $\varepsilon_i = 1$ *or* 0. We say J is of type $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{m-1})$. We call that an index is regular if $\varepsilon_i = 1$. We define a number $\alpha(J)$ by

$$\alpha_(J) = \sum(\binom{i+1}{2} + 1) \tag{4.0.5}$$

where the sum runs over the regular indices of J. Then we have the following estimates.

Lemma 4.0.2. *Let $g(X)$ be a form of degree $d \geq 3$ which is transformed into itself by J. Then the coefficients of g satisfy at least $\alpha(J)$ linearly independent linear relations with coefficients in k_0.*

Lemma 4.0.3. *Let A be a unipotent matrix with Jordan form J. Then $\alpha(J) > t.d.(A/k_0)$.*

With these two lemmas, now we can prove our theorem. Let T be matrix with algebraic coefficients over $k_0(A_{ij})$ such that $TA = JT$. Let $f(X) = g(TX)$. Then $g(JTX) = g(TAX) = f(AX) = f(X) = g(TX)$. So g is transformed into itself by J. By Lemma 5.2, the coefficients of $g(X)$ satisfy at least $\alpha(J)$ linear independent linear relation with coefficients in k_0. Now we define $dim(g(X)) = tr.d(k_0(a_i)/k_0)$, where a_i are coefficients of $g(X)$. So we have $dim(g(X)) \leq \binom{n+1+d}{d} - \alpha(J)$. Now consider the map: $\{g(X) = g(JX)\} \times \{T|TA = JT\} \longrightarrow g(TX) = f(X)$, we have $dim(g(X)) = \binom{n+1+d}{d} - tr.d(A_{ij}/k_0)$. However, we have $tr.d(A_{ij}/k_0) \leq \alpha(J)$. It's a contradiction. $f(X)$ can't be gneric.

Now we only need to prove Lemma 2 and Lemma 3. To prove Lemma 2, we order the monomials of degree d lexicographically; $\prod X_i^{a_i} < \prod X_i^{b_i}$ if $a_i = b_i, a_s < b_s$ $(i < s)$. Now if $\mu = \prod X_I^{a_i}$, then we have:

$$\mu(J(X)) = \prod (X_i + \varepsilon_i X_{i+1})^{a_i} = \mu(X) + \sum_{\nu < \mu} c_{\mu\nu} \cdot \nu(X) \tag{4.0.6}$$

If we regard J as a transformation on the space $H^0(\mathscr{O}_P(d))$, then J has the form $E + \delta$, where $\delta = (c_{\mu\nu})$

is strictly triangular. Now suppose that $g(X) = \sum_{\mu} a_\mu \cdot \mu(X)$ such that $g(JX) = g(X)$. Then we have

$$\begin{aligned} g(J(X)) &= \sum_{\mu} a_\mu \cdot \mu(JX) \\ &= \sum_{\mu} a_\mu (\mu(X) + \sum_{\nu<\mu} c_{\mu\nu} \cdot \nu(X))) \\ &= \sum_{\mu} a_\mu \cdot \mu + \sum_{\nu} (\sum_{\mu} c_{\mu\nu} \cdot a_\mu) \cdot \nu \\ &= \sum_{\mu} a_\mu \cdot \mu \end{aligned} \tag{4.0.7}$$

Comparing coefficients we get that $\sum_{\nu<\mu} c_{\mu\nu} a_\mu = 0$. Thus the coefficients of g satisfy rank $(c_{\mu\nu})$ linearly independent equations with coefficients in k_0. If μ is any monomial, let μ' be its predecessor in the lexicographic order. If μ is regular for J if $c_{\mu\mu'} \neq 0$. Since $(c_{\mu\nu})$ has strict triangular form, rank $(c_{\mu\nu})$ is at least equal to the number of regular μ. Thus we must show:

Lemma 4.0.4. *There at least $\alpha(J)$ regular monomials.*

Suppose s is a regular index for J, and let $\mu = (\prod_{i=1}^{s} X_i^{a_i}) \cdot X_m^{a_m}$, then $\mu' = (X_{s+1}/X_s) \cdot \mu$ and we see easily that $c_{\mu\mu'} = a_s$. In particular, if the characteristic p does not divide a_s, then μ is regular. Now fix a regular index s. If $a_s = 1$, the number of monomials of the form $(\prod_{i=1}^{s} X_i^{a_i}) \cdot X_m^{a_m}$ is the number of monomials of degree $d-1$ in $X_1, \cdots, X_{s-1}$ and X_m, i.e. $\binom{s+d-2}{d-1}$. Furthermore, since $d \geq 3$, there is a regular monomial of the form $X_s^2 X_m^{d-2}$, or $X_s^3 X_m^{d-3}$, depending on the characteristic. Thus there are at least $\sum_s (\binom{s+d-2}{d-1} + 1)$ regualr monomials in all where s runs over the regular indices for J. Since this function is monotonic increasing in d, and $d \geq 3$, the lemma is proved. Now we need to prove Lemma 3. The idea is that each regular index in J gives a contribution of about $m^2/2$ to $\alpha(J)$, and $t.d.(A_{ij}/k_0) \leq m^2$. Then if we have 3 regular indices, we are done. If we have fewer than 3 regular indices, then we need finer estimates on $k_0(A_{ij})/k_0$.

We begin with a lemma about upper bound for $t.d.(k_0(N_{ij})/k_0)$ where N is a nilpotent matix.

Lemma 4.0.5. *Suppose that N is an m by m nilpotent matrix. Let V_i = image of N^i, and $\beta_i = dim V$. Then $\beta_0 > \beta_1 > \beta_2 > \cdots$. We say that N is of type $(\beta_0, \beta_1, \cdots)$. Let $\beta(N) = 2\sum_{0}^{\infty} (\beta_i - \beta_{i+1}) \beta_{i+1}$. Then we have*

$$t.d(k_0(N_{ij}/k_0)) \leq \beta(N) \tag{4.0.8}$$

N is determined by the subspace V_1 of V_0 and by the images in V_1 of a set of generators of V_0/V_1 under N. V_1 depends on at most $(\beta_0 - \beta_1)\beta_1$ parameters, the same holds true for the images of the $\beta_0 - \beta_1$

generators of V_0/V_1. Then use induction we can get the result.

Now we are ready to prove Lemma 5.3. Let $A = E + N'$, $J = E + N$. Then $t.d.(k_0(A_{ij}/k_0)) \leq \beta(N') = \beta(N)$. There are four cases to consider, according to the maximum of the sizes of the blocks of J. (The blocks are arranged in the increasing order of size)

- (1) If J is of type $(\cdots, 1, 1, 1)$ then $t.d.(k_0(A_{ij})/k_0) \leq m^2 - m$. For $\beta(N) \leq m^2 - m$ always.
- (2) If J is of type $(\cdots, 0, 1, 1)$ then $t.d.(k_0(A_{ij})/k_0) \leq \frac{2}{3}m^2$. Fpr $N^3 = 0$ in this case, and N is of type $(m, \gamma, \delta, 0)$
- (3) If J is of type $(\cdots, 1, 0, 1)$ then $t.d.(k_0(A_{ij})/k_0) \leq \frac{1}{2}m^2$. For $N^2 = 0$ in this case, and N is of type $(m, \delta, 0)$
- (4) If J is of type $(\cdots, 0, 0, 1)$ then $t.d.(k_0(A_{ij})/k_0) \leq 2m - 2$. In this case, and N is of type $(m, 1, 0)$

Now let us estimate $\alpha(J)$ in the 4 cases. In case $(1), m-1$, $m-2$ and $m-3$ are all regular for J. Thus $\alpha(J) \geq \binom{m}{2} + \binom{m-1}{2} + \binom{m-2}{2} + 3$. The other cases are similar. Thus we are reduced to proving the following four inequalities. If $m \geq 4$, then:

- (1) $\binom{m}{2} + \binom{m-1}{2} + \binom{m-2}{2} + 3 > m^2 - m$
- (2) $\binom{m}{2} + \binom{m-1}{2} + 2 > \frac{2}{3}m^2$
- (3) $\binom{m}{2} + \binom{m-2}{2} + 2 > \frac{1}{2}m^2$
- (1) $\binom{m}{2} + 1 > 2m - 2$

This completes the proof of Lemma 5.3. $\square$

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