Developments of Method of Market Completions in Mathematical Finance

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Mathematical Finance

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Abstract

In present thesis I focus on development of method of market completions and its applications to various problems in pricing and hedging of contingent claims.

Since theory of mathematical finance is well developed on complete markets, and corresponding solutions are well understood, method of market completions emerges as a naturall idea for applying knowledge available for complete markets to the case of incomplete one.

Key approach of proposed method is to introduce family of possible "completed" versions of initially incomplete market, parametrized by the set of special auxiliary assets. This manipulation leads to multiple subproblems, that could be solved by the means of known complete market techniques. Further, having corresponding family of solutions, I demonstrate how one could come up with criteria to choose optimal solution for initial market which would not depend on auxiliary assets.

I start with discussion of reasons for market incompleteness and introduce market completions for parametrization of completed versions of the market.

Then, I demonstrate how proposed method can be applied for fundamental problems of utility maximization, including not-necessarily concave utility function and pricing of contingent claims.

Then, I move to another important group of problems in the field of mathematical finance – hedging of contingent claims. In the modern risk management industry, however it is more common to choose partial hedging, since it allows for more flexibility and money savings. I will start with discussing application of method of market completions for fundamental problems of quantile and effective hedging and then for

modern risk-measures approach. I will also provide numerical examples for solutions on incomplete market.

It is known that the key element of this problem is a risk measure chosen for assessment of risks. Two of the most widely used risk measures in the industry nowadays are Value-at-Risk (VaR) and Expected Shortfall (CVaR). However, it has been demonstrated recently that both of these measures could be incorporated into one two-parametric risk measure called Range Value-at-Risk (RVaR). I will focus on demonstration that partial hedging problem with respect to both CVaR and RVaR in incomplete market could be approached with the help of method of market completions through the Utility Maximization task embedded into RVaR optimization problem.

Conclusions and further research directions in exploring the ideas of Method of market completions are in the last chapter.

Preface

Current thesis is structured in the way that each chapter has its own separate introduction.

Some chapters of this thesis are based on a published and submitted for publication research articles.

Chapter 2 is based on a paper which has been published as Ilia Vasilev and Alexander Melnikov, "On Market Completions Approach to Option Pricing" Review of Business and Economics Studies 9(3).

RVaR part from Chapter 7 has been also published as Ilia Vasilev and Alexander Melnikov, "RVaR Hedging and Market Completions" Springer International Publishing: Mathematical and Statistical Methods for Actuarial Sciences and Finance.

In all joint papers, I was responsible for the proofs of results and their applications. Dr. Melnikov has advised on general approaches for the proofs.

To My Parents

Acknowledgements

I would like to use this opportunity to express my gratitude to people that care and supported me through my PhD program. Without them, I would have been impossible for me to arrive at the current stage of my PhD study.

First of all, I want to express my special appreciation to my supervisor Professor Alexander Melnikov. This endeavor would not have been possible without your continuous support, patience and motivation. Your immense knowledge and guidance were an inspiration and I could not have imagined better supervisor for my PhD studies. I am deeply grateful for your mentorship both inside and outside academia. It has been an honor and pleasure to work with you.

I greatly appreciate the financial support that I have received from the Department of Mathematical and Statistical Sciences, University of Alberta. And would also like to extend my sincere thanks for this opportunity to work on my research. During my program I had the pleasure to be a part of Model Development team at Canadian Western Bank. I would like to thank my team and especially Amir Nosrati for being a role model and motivating me in my studies. I would be remiss in not mentioning support from all my friends and colleagues. I appreciate open and kind atmosphere that allowed us to share ideas and stay inspired. Thank you all very much!

Last, but not least, many thanks to my loved one for always being by my side. I would also like to give special thanks to my parents. Your hard work allowed me to pursue my life goals with the feeling of strong backup. I could not have undertaken this journey without your unconditional love and support. No words can describe how grateful I am for your wise advises and gentle guidance through any life storms.

Thank you for always being there for me.

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Chapter 1

Introduction

1.1 Outline

The main goal of present thesis is to introduce Method of Market Compeltions and investigate its applications to various famous problems of mathematical finance including pricing of contingent claims, utility maximization, perfect and partial hedging.

Chapter 1 is intended to set up a context of discussion. Standard multidimensional market model is introduced. We also elaborate on incompleteness of the model under consideration and provide necessary details regarding concepts of perfect hedging, risk measures and hypothesis testing tasks.

In Chapter 2 we introduce method of market completions which is a core methodology of the thesis. As it follows from the name, the nature of the method is to find equivalent completed versions of the market. In Chapter 2 we demonstrate the steps required to find such completed versions of the initial market with the help of *orthogonal market completions*. Each market completion is, essentially, a set of specific auxiliary market assets of similar structure. We elaborate how each completed market version an be parametrized by the sets of auxiliary assets. In line with it, we obtain the particular market completion that corresponds to the important *minimal martingale measure*.

Having portfolios and assets price processes introduced, we focus on the problem of optimal investment. In Chapter 3 we present the utility maximization problem for both concave and not necessarily concave utility function, that satisfy mild growth condition. We elaborate on criteria that can be used in order to pick particular solution on incomplete market based on completions.

Moving on, we now assume existence of some financial claim in future and present classical pricing approaches for such claims in Chapter 4. We start with demonstration that standard results from martingale approach could be obtained by means of method of market completions. We also present the Utility Based Indifference Pricing method and show how market completions fit into this framework.

Important question of partial hedging is discussed in following chapters. We start with Quantile hedging task in Chapter 5 where we obtain results with the help of famous Neyman-Pearson lemma. And also show how the problem can be solved on incomplete market with the help of method of market completions. Efficient hedging, including both linear and convex loss function is discussed in Chapter 6.

In Chapter 7 we discuss modern approach to partial hedging with focus on distribution-based risk measures. We consider CVaR minimization problem for both complete and incomplete cases with two different approaches. We then move on to RVaR optimization task and demonstrate how solution can be obtained.

Chapter 8 broadens the applications of method of market completions for the defaultable market model. In this chapter we assume that there is a possibility of default. So the market information is generated by additional source of uncertainty which is, by nature, impossible to hedge. In addition, we assume that underlying financial market is incomplete. In this scenario we demonstrate how method of market completions might be applied to find optimal partial hedging strategies.

1.2 Standard Multidimensional Market Model

In this sections we introduce the market model that will be in the focus through the whole thesis. First of all, we introduce a *stochastic basis* $(\Omega, \mathcal{F}_T, \mathbb{F}, P)$ where Ω is a set of elementary outcomes ω, σ-algebra of this outcomes $\mathcal{F}_T$, probability measure P and filtration $\mathbb{F}$ which is an increasing family $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ of sub-σ-algebras $\mathcal{F}_t \in \mathcal{F}_T$ that is right continuous and contains all subsets of $\mathcal{F}_T$ with P-measure zero. Stochastic process X defined on stochastic basis $(\Omega, \mathcal{F}_T, \mathbb{F}, P)$ is called progressively measurable with respect to filtration $\mathbb{F}$ if for every $t \in [0,T]$ function $X(s,\omega) : [0,t] \times \Omega \to \mathbb{R}$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$-measurable function. Process X_t is called adapted to filtration $\mathbb{F}$, or simply $\mathcal{F}_t$-adapted if $X_t \in \mathcal{F}_t$ for all $t \geq 0$.

On introduced stochastic basis we define *Standard Multidimensional Market Model* that consists of one risk-free asset and n risky assets (stocks). This model can be written as $(B,S) = (B_t, S_t^1, ..., S_t^n)_{t\leq T}$ where $(B_t)_{t\leq T}$ represents the value process of a bank account and $S_t = (S_t^1, ..., S_t^n)_{t\leq T}$ describes the prices of n risky assets:

$$dB_t = B_t r_t dt, \quad B_0 = 1 \tag{1.1}$$

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^{k} \sigma_t^{ij} dW_t^j \right) \tag{1.2}$$

Elements of a k-dimensional vector $W = (W^1, ..., W^k)$ are independent standard Brownian motions. All coefficients of the model: r_t, μ_t^i and σ_t^i are assumed to be progressively measurable with respect to filtration $\mathbb{F}$ for all i. In addition, matrix $\Sigma_t = \{\sigma_t^{ij}\}$ has a full rank for all $t \in [0,T]$. We also assume that $\int_0^T \|\mu_t\| dt < \infty$ and $\int_0^T |r_t| dt \leq L$ where L is some real positive constant.

In general, one can define multidimensional market model in a way that each risky asset price is governed by separate Brownian motion and they are correlated among different assets. It was shown in Dhaene *et al.* 2013 that market can be equivalently described by both of mentioned models. We will focus on Standard model with non-correlated "underlying" Browninan motions.

$(\mathcal{F})_{t\leq T}$-adapted process $\pi = (\beta_t, \pi_t^1, ..., \pi_t^n)_{t\leq T}$ is called a *portfolio* (*strategy*) with value process V_t^π defined by

$$V_t^\pi = \beta_t B_t + \sum_{i=1}^{n} \pi_t^i S_t^i.$$

Each π_i element of the vector π represents quantity of asset type i acquired for portfolio. Let us also introduce class of *admissible* portfolios with initial capital x as

$$\mathcal{A}(x) = \{\pi : V_0^\pi = x, \exists K(\pi) \geq 0 \text{ s.t. } V_t^\pi \geq -K \text{ for all } t \leq T\}.$$

Admissible strategy π is called *self-financing* if the following condition holds

$$\int_0^T \sum_{i=1}^n \left(|\pi_t^i \mu_t^i| + (\pi_t^i)^2 \sum_{j=1}^k (\sigma_t^{ij})^2 \right) dt < \infty \tag{1.3}$$

and the associated value process can be written as

$$V_t^\pi = V_0^\pi + \beta_t dB_t + \sum_{i=1}^n \int_0^t \pi_s^i dS_s^i.$$

Further, class of self-financing strategies will be denoted SF_a

For a market model to be consistent, it is required that there is no possibility of *arbitrage.*

Definition 1.1 *Market admits arbitrage if*

$$\exists \pi \in \mathcal{A}(x), \; s.t. \; V_0^\pi = 0 \; and \; P\left(V_T^\pi(x) > 0\right) = 1$$

Absence of arbitrage is closely connected to existence of risk-neutral, or martingale, measure. In Karatzas and Shreve 2016 authors demonstrated criteria for the standard multidimensional market model to be arbitrage-free:

Proposition 1.2 *1. If there exists a $(\mathcal{F}_t)_{t \leq T}-$progressively measurable process $\theta = (\theta_t^1, ..., \theta_t^k)_{t \leq T}$ that satisfies*

$$\sum_{j=1}^k \sigma_t^{ij} \theta_t^j = \mu_t^i - r, \quad i = 1, ..., n, \quad P - a.s. \tag{1.4}$$

and

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \sum_{j=1}^k (\theta_t^j)^2 dt\right)\right] < \infty, \tag{1.5}$$

then the (B, S) market is arbitrage free.

2. *Conversely, if the (B,S) market is arbitrage-free, then there exists a $(\mathcal{F}_t)_{t\le T}-$progressively measurable process $\theta = (\theta_t^1, ..., \theta_t^k)_{t\le T}$ that satisfies conditions (1.4) and (1.5).*

In other words, market is arbitrage-free if and only if system (1.4) has solution. From the first fundamental theorem of mathematical finance, no-arbitrage condition is equivalent to existence of a special equivalent measure, called martingale, or risk-neutral. Such measure is the one, which is equivalent to initial, physical measure in a sense of mutual absolute continuity and under which all risky assets discounted price processes are martingales. As a consequence, discounted value process of any portfolio under the martingale measure is also a martingale. It is well known, that probability density of such equivalent martingale measure can be constructed by the means of stochastic exponential in the form introduced by Girsanov:

$$Z_T^A = \frac{dP^*}{dP} = \exp\left\{-\sum_{i=1}^{n}\int_0^T \theta_t^i dW_t^i - \frac{1}{2}\sum_{i=1}^{n}\int_0^T (\theta_t^i)^2 dt\right\} \tag{1.6}$$

where θ is a solution of (1.4). Such solution can also be written in an explicit form:

$$\theta_t = \Sigma_t^T \cdot \left(\Sigma_t \Sigma_t^T\right)(\mu_t - r\bar{1}_k). \tag{1.7}$$

According to famous Girsanov theorem, process

$$\widehat{W_t} := W_t + \int\limits_0^t \theta_s ds \tag{1.8}$$

is a Brownian motion under measure P^*.

1.3 Market Completeness

In a field of mathematical finance it is common to think about market completeness in terms of attainability of contingent claims. Namely, if any contingent claim payoff can be replicated as a capital of some admissible portfolio, consisting of assets, present on the market, then such market will be called complete. Let us consider probability space $(\Omega, \mathcal{F}_T^W, P)$. Then mentioned attainability can be summarized in the following definition.

Definition 1.3 (Market Completeness) *The market is called complete if for each $\mathcal{F}_T^W-$measurable payment function $g = g_T(\omega) \geq 0$, such that $\mathbb{E}[g] < \infty$ there exists a strategy $\pi \in SF_a$ such that $\mathbb{P}-a.s.$*

$$V_T^\pi = g$$

Generally speaking, all possible states of the market through the investment time interval can be described by filtration $\mathbb{F} = \{\mathcal{F}_t^W\}_{t \leq T}$. Each sigma-algebra in such filtration is generated by some underlying stochastic processes $\{W_t^i\}_{1 \leq i \leq k}$ which could be thought of as sources of uncertainty on the market. In case of model (8.1), set of these underlying processes is represented as a set of independent Brownian motions. In line with it, there are risky assets, available for trading on the market, with stochastic price processes. Trading in such assets leads to existence of another sigma algebras, generated by assets $\mathcal{F}_T^A$. Then, market model is incomplete when there exists $B \in \mathcal{F}_T^W$ such that $B \notin \mathcal{F}_T^A$.

Such difference in sigma algebras might stem from various reasons. For example, market might be incomplete directly due to asymmetric information. In other words, when market agents have different access to information driving the prices of assets on the market. This might happen due to different relation to particular company of some investors (e.g. being employee, contractor and e.t.c.). In Eyraud-Loisel 2019 it was demonstrated how presence of informed, influential agent might lead to market incompleteness for non-informed small investors. From this perspective market can be "completed" if necessary information is provided to non-informed participant. Or, rigorously speaking, by enlargement of filtration.

Another reason for differences in filtration might be presence of non-financial risks which, by their nature, cannot be covered by financial assets. One of such examples is risk of default, which we will cover in Chapter 8. From this perspective, uncertainty in asset prices is driven by underlying Brownian motions. However, there is also a risk of default, represented by Poisson process. In Bielecki and Rutkowski 2002 authors developed methodology of "extending" the filtration, generated by financial assets with filtration from default-related process. Details on such methodology are provided in Chapter 8.

Nevertheless, the most well-known reason for market incompleteness is the "structural" one. In other words, when there is a lack of tradeable assets necessary for coverage of all possible outcomes. According to the first fundamental theorem of Mathematical Finance, we know that markets complete if and only if equivalent martingale measure is unique for such market. Summarizing results from Dhaene *et al.* 2013 and Karatzas and Shreve 2016 we can formulate conditions for *Market completeness* of Standard Multidimensional Market model (8.1) as follows:

Theorem 1.4 (Completeness criteria) *Standard financial market $\mathcal{M}$ is complete $\iff$ number of available stocks $n = k$, where k is a dimension of underlying vector of Brownian motions.*

To switch to mentioned unique martingale measure, one could use special Girsanov exponential, which, in multidimensional case, will be written as in (1.6). It is straightforward to notice that market completeness is connected with the "volatility" matrix $\Sigma_t = \{\sigma_t^{ij}\}_{i=1..n,j=1..k}$. In case of complete market one observes $n = k$, matrix has full rank and we expect to see non-degenerative matrix $n \times n$.

Despite the amount and variety of claims traded on the market, the set of possible outcomes is usually greater than the set of claims, which makes its vital to be able to operate on such markets. In current manuscript we focus on structural incompleteness and propose a Method of Market Completeness (Chapter 2) as a tool for working with structurally incomplete market models. The main idea behind proposed approach is to add necessary information to the market by introducing required amount of financial assets that would complete the set of tradeable assets in order to cover all possible outcomes.

The idea of adding auxiliary assets to the market in order to make it complete is not limited to the diffusion market model. For example, it is well known that jump-diffusion model is naturally incomplete and usually considered with two assets instead of one. There were also some developments towards more general geometric Levy model in which asset price is governed by jumps

$$dB_t = rB_t dt \tag{1.9}$$

$$dS_t = S_{t-}\left(\mu dt + dZ_t\right) \quad S_0 > 0, \tag{1.10}$$

$$Z_t = \sigma W_t + X_t \tag{1.11}$$

where X_t is a pure jump process and W and X are independent. That is well known, that such Levy model is not complete even in one dimensional case as it includes jumps and Brownian motions as two independent sources of risk and only one asset to use. So instead of introducing same structure auxiliary assets, authors in Corcuera *et al.* 2005 offer to enlarge the Levy market with the so-called *ith-power-jump assets* defined as

$$X_t^{(i)} = \sum_{0<s\leq t} \left(\Delta X_s\right)^i, \quad i \geq 2, \tag{1.12}$$

where $\Delta X_s = X_s - X_{s-}$ and $X_t^{(1)} = X_t$. Processes $X^{(i)}$ are again Levy processes. These power-jump processes jumps at the same time as the original Z_t, however, jump sizes are the i-th power of jumps of the original process. Note, that $X_t^{(i)} = Z_t^{(i)}, i \geq 2$. It is convenient to rewrite these assets in compensated form

$$Y_t^{(i)} = Z_t^{(i)} - E\left[Z_t^{(i)}\right] = Z_t^{(i)} - m_i t, \quad i \geq 1. \tag{1.13}$$

Enlargement of the model is then consists of allowing to trade in assets:

$$H_t^{(i)} = e^{rt} Y_t^{(i)}, \quad i \geq 2. \tag{1.14}$$

With these assets available it was demonstrated in Corcuera *et al.* 2005 that any square integrable martingale M_t can be represented as follows:

$$M_t = M_0 + \int_0^t h_s d\tilde{Z}_s + \sum_{i=2}^{\infty} \int_0^t h_s^{(i)} dY_s^{(i)} \tag{1.15}$$

where h_s and $h_s^{(i)}, i \geq 2$ are predictable processes such that

$$\tilde{Z} = Z_t - (\mu - r)t, t \geq 0 \tag{1.16}$$

and

$$E\left[\int_0^t |h_s|^2 ds\right] < \infty \tag{1.17}$$

$$E\left[\int_0^t |h_s^{(i)}|^2 ds\right] < \infty. \tag{1.18}$$

In other words, for any square-integrable contingent claim f (non-negative, $\mathcal{F}_T$ measurable random variable) we can set up a sequence of self-financing portfolios whose final values converge in $L^2(P^*)$. This portfolio will consist of a finite number of bonds, stocks and ith-power-jump assets. Which means that f can be replicated and market is approximately complete.

This interesting result is important to consider within the general idea of market completion because it offers to search for more specific auxiliary assets beyond just structure-preserving additional assets discussed before. In case of Levy market model or other model with jumps it might be more convenient to pick specific types of completing assets for each kind of risks presented. It is also useful in terms of interpretation of the auxiliary assets as power-jump-assets are by nature an instruments that give an exposure to moments like variance (2nd-power-jump asset) or skeweness and kurtosis of distribution (3rd and 4th correspondingly). Assets of such type might be more convenient to introduce to real markets in order to fix its incompleteness.

1.4 Coherent Risk Measures

One of the central questions in Risk management industry will always be the problem of quantification of the risk. It is essential to be able to estimate risk in order to construct more complex problems associated with its management. Therefore, it comes at no surprise that problem of risk quantification lead to considerable theoretical work described in literature. It is common to find some specific function that would use financial position as input and produce risk of this position as a number.

In current manuscript, we focus on the notion of *coherent risk measures* proposed by Artzner *et al.* 1999. According to this approach, risk measure is a mapping from random payoffs to numbers line. In this section we focus on the properties of such mappings and their requirements in order to preserve coherence property. We also introduce risk measures actively used by market participants in their risk management strategies.

Consider probability space of market outcomes $(\Omega, \mathcal{F}_T, P)$ and space of all possible losses $\mathcal{L}$, which are represented as a random variable. If random variable $L \in \mathcal{L}$ is negative – it would imply that position generated gain. We assume the set of all such random variables is a convex cone containing all constants.

Definition 1.5 *A risk measure is a mapping $\rho : \mathcal{L} \to R$. Number $\rho(L)$ for each $L \in \mathcal{L}$ represents how risky the portfolio, which generates L is.*

Following Artzner *et al.* 1999, we focus on *coherent* measures of risk.

Definition 1.6 (from Artzner *et al.* 1999) *A risk measure ρ is called* coherent *if it is:*

1. *monotone(decreasing): $\rho(X) \leq \rho(Y)$ for any $X \leq Y$,*
2. *cash-additive (additive with respect to cash reserves): $\rho(X + c) = \rho(X) + c$ for any $c \in R$,*
3. *positive homogeneous: $\rho(\lambda X) = \lambda \rho(X)$ for any $\lambda > 0$,*
4. *sub-additive (diversification): $\rho(X + Y) \leq \rho(X) + \rho(Y)$.*

In financial industry it is common to use risk measures that are distribution-based, i.e. depend only on distribution of losses from position. Therefore, if two losses are equivalent in distribution, their risk measures will also be equal. Two important examples of such risk measures are Value-at-Risk and Conditional Value-at-Risk, which are popular among market agents and accepted in regulatory documentations. We now provide rigorous definition of these measures which we will use in following chapters.

Definition 1.7 (VaR) *Value-at-Risk (VaR) measure of a loss X can be defined as*

$$VaR_\alpha(X) = \inf\{a : P(X > a) \leq \alpha\}$$

Value-at-Risk (VaR) is a very popular and widely-used risk measure and was accepted as preferred one in industry regulations. However, it becomes unstable and difficult to work with numerically in case when distribution of losses is different from normal. In addition to that, VaR measure of risk is not a coherent one (see Artzner *et al.* 1999). Another shortcoming of Value-at-Risk is that it fails to provide any information regarding severity of losses beyond the chosen threshold. In other words, it has a bias towards optimism as this measure provides lowest bound for losses in the tail, whereas conservatism is ought to be more appreciated among industry professionals. As a response, alternative measure of risk was developed which is coherent and offers a way to quantify tail risk.

Definition 1.8 (CVaR) *Conditional Value-at-Risk (CVaR) measure of a loss X can be defined as*

$$CVaR_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\alpha(x)dx,$$

In coming chapters (precisely Chapter 7) we will use these measures as a risk "estimators" for constructing optimal hedging strategies. In the next section we elaborate on purpose of such strategies and the natural ways of their construction.

1.5 Perfect and Partial Hedging

In the area of mathematical finance it is usually the case that market agent is involved in some deal with derivatives, or, in general, with contingent claims. In other words, she participate in a deal, according to which she obtain some financial obligation desribed as a contingent claim payoff function. We treat this payoff as a $\mathcal{F}_T$-measurable random variable $g_T(\omega)$. In case payoff happens at maturity of the instrument and depends only on the value at the moment T, such contract is called to be a European type.

Without loss of generality, let us assume that investor has entered a financial contract, according to which he will repay amount of $g_T(\omega)$ in T years from current moment. Main goal of hedging is to fund a portfolio π, such that its value V_T^π will be no less than obligation at maturity time T.

$$V_T^\pi \geq f_T, \quad (a.s.)$$

Initial price of such strategy usually defined as mathematical expectation over special risk neutral measure

$$V_0 = E^* \left[\frac{f_T}{B_T} \right],$$

we postpone details of this to the contingent claim pricing chapter of the current manuscript.

If we are able to fund such strategy, it means that investor is *completely hedged*. And strategy that replicates future payoff with probability 1 (almost surely) is called a *perfect hedge*. With the help of pricing techniques he might determine what would be the cost of such strategy and use it as a fair price of the deal he is planning to participate in. However, there is a chance that perfect- or super-hedging strategies are not feasible as their initial price might be above investors budget constraint or the fair price of the contract is above what market participants are ready to propose.

The concept of partial (imperfect) hedging arises precisely when for some reasons, investor cannot, or does not want to invest into perfect hedging strategy. Most of the time it happens when initial price of necessary investments is too large. From the opposite side, having natural desire to spend the least amount of money possible, investor may tolerate some risk in exchange for discount on a hedging strategy. So market agent can focus on partial-hedging strategies, where the goal is to invest capital in the most optimal way considering anticipated loss from hedging. Such type of strategies is extremely important as there are always a huge variety of risks on the market and, clearly, most of the time it is too expensive to eliminate all of them.

Since it is not goal to build a perfect hedge anymore, there emerges a possibility of a *shortfall*, or, chance that terminal value of the a portfolio will not be enough to cover all possible losses from the claim. Consequently, one need to find optimal strategy. Natural question is – what is optimal?

There are several ways to introduce optimality criteria that should be used to build a strategy. One natural idea is to maximize probability of a successful hedging which is called Quantile Hedging (Föllmer and Leukert 1999,Spivak and Cvitanić 1999). Another possible approach is to focus on the size of shortfall in terms of its impact. It is natural to assume that investor possesses different attitude and tolerance to small and large shortfall sizes. Basically, her preferences can be well described by *utility functions*, which are either convex or concave functions, depending on investors attitude towards risk. Introducing Utility functions, one can define optimal strategy as the one which minimizes the expected utility of a shortfall. This idea lies at the core of Efficient hedging approach (Föllmer and Leukert 2000). On top of that, it was first demonstrated in Melnikov 2004a and Melnikov 2004b that mentioned methods of partial hedging could also be useful in applications to problems in the field of equity-linked life insurance.

More broadly, we can try to assess our attitude towards shortfall with the help of *risk measures* described above. Since recently, risk exposure is measured with the help of special measures, widely used by market participants: Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). The latter one is better known as Expected Shortfall (ES) and was introduced in 2016 as a risk-measure recommended in The Market Risk Framework of Basel III – international regulatory accord. Not a surprise that these measures spark a special interest in their application in the optimal partial-hedging problem.

Although solutions to these partial hedging problems are being discussed in the literature, most of them are for complete market case. Thus, market incompleteness is still a challenge and makes existing approaches difficult to apply. In this manuscript we propose a method of market completions as a tool to connect known solutions on complete market with incomplete version of it. We will focus on all fundamental types of partial hedging further and demonstrate application of MMC for them.

1.6 Generalized Neyman-Pearson Lemma

Assume there is a measurable space $(\Omega, \mathcal{F})$. Suppose on this space we are given with two measures Q ("null"-hypothesis H_0) and P ("alternative" hypothesis H_a). It will also be convenient for further explanation to think about measure Q as risk-neutral, or martingale measure, whereas treat P as "real-world" measure which is obtained from observed market data.

The main goal of hypothesis testing approach is to discriminate Q against P. Namely, to assess if one can or cannot reject hypothesis H_0 with some level of certainty. Classical instrument to solve such problem which we are going to focus on is called *randomized* test, which is a random variable $X : \Omega \to [0, 1]$ with an interpretation that hypothesis Q is rejected with probability $X(\omega)$ and not rejected with probability $(1 - X(\omega))$ for each outcome $\omega \in \Omega$. Such instrument, of course allows some mistakes in decision regarding rejection of H_0. Firstly, "null" hypothesis might be rejected when it is actually true. Such case is called *Type I error* and probability of it can be calculated as

$$E^Q[X] = \int X(\omega) Q(d\omega).$$

Secondly, randomized test X can suggest against rejection H_0 when it is not true and, consequently, should be rejected. This is called *Type II error* and its probability is

$$E^P[1-X] = \int (1 - X(\omega))P(d\omega).$$

Naturally, one aims at minimization of both Type I and Type II error. However, considering their definitions, it is impossible to minimize both at the same time. Therefore, the approach is usually to fix some acceptable probability α for Type I error when trying to maximize *test power* $E^P[X]$. Thus, initial discrimination problem can be stated as: Find randomized test of maximum power with probability of Type I error less than or equal to α, or

$$\begin{cases} E^P[X] = \int X(\omega)P(d\omega) \to \max \\ E^Q[X] \leq \alpha. \end{cases} \tag{1.19}$$

1.6.1 Simple Hypothesis

Let us start with the case of Simple Hypothesis – when measures $\mathcal{Q} = \{Q\}$ and $\mathcal{P} = \{P\}$ are singletons.

Then problem (1.19) can be solved with the help of Neyman-Pearson lemma. According to this classical approach, one should consider auxiliary measure μ, such that

$$P \ll \mu, \quad Q \ll \mu.$$

Denote

$$G \equiv \frac{dP}{d\mu}, \quad H \equiv \frac{dQ}{d\mu}.$$

Then, solution to (1.19), or

$$\sup_{X \in \mathcal{X}_\alpha} E^P[X], \text{ where } \mathcal{X}_\alpha = \{X : \Omega \to [0,1]; E^Q[X] \leq \alpha\} \tag{1.20}$$

is attained for every level α by

$$\hat{X} = 1_{\hat{z}H<G} + b \cdot 1_{\hat{z}H=G} \tag{1.21}$$

where

$$\hat{z} = \inf\{u \geq 0; Q(uH < G) \leq \alpha\}$$
$$b = \frac{\alpha - Q(\hat{z}H < G)}{Q(\hat{z}H = G)}.$$

1.6.2 Composite Hypothesis

Moving on, assume that now measures $\mathbb{Q}$ and $\mathbb{P}$ represent *families of measures* rather than single elements. By analogy with simple hypothesis case, we make the following assumptions

$$\mathbb{Q} \cap \mathbb{P} = \emptyset$$

and

$$G_P := \frac{dP}{d\mu}(P \in \mathbb{P}), \quad H_Q := \frac{dQ}{d\mu}(Q \in \mathbb{Q})$$
$$P \ll \mu, \quad Q \ll \mu, \quad \forall P \in \mathbb{P},\ Q \in \mathbb{Q}.$$

With the setting presented, the problem (1.20) can be rewritten in the following form

$$\underline{V}(\alpha) := \sup_{X \in \mathcal{X}_\alpha^{\mathbb{Q}}} \left(\inf_{P \in \mathbb{P}} E^P[X] \right), \tag{1.22}$$

where $\mathcal{X}_\alpha^{\mathbb{Q}} := \{X : \Omega \to [0,1]; E^Q[X] \leq \alpha,\ \forall Q \in \mathbb{Q}\}$. In other words, optimization problem can be interpreted as maximization of the "worst-case" test power

$$\gamma(X) := \inf_{P \in \mathbb{P}} E^P[X]$$

over all randomized tests X of size

$$s(X) := \sup_{Q \in \mathbb{Q}} E^Q[X] \leq \alpha.$$

Definition 1.9 *If such a randomized test $\hat{X} \in \mathcal{X}_\alpha^{\mathbb{Q}}$ exists, it will be called max-min-optimal for testing the (composite) hypothesis $\mathbb{Q}$ against the (composite) alternative $\mathbb{P}$, at the given level of significance $a \in (0,1)$.*

It was demonstrated in Cvitanic and Karatzas 2001 that under certain conditions on the family $\mathbb{P}$ of alternatives, there exist max-min-optimal randomized test that preserves the form (1.21):

$$\hat{X} = 1_{\hat{z}\hat{H}<\hat{G}} + B \cdot 1_{\hat{z}\hat{H}=\hat{G}} \tag{1.23}$$

where B is a random variable with values in $[0,1]$, $\hat{G}$ is of the form $G_{\hat{P}} = \frac{d\hat{P}}{d\mu}$ for some $\hat{P} \in \mathbb{P}$, the random variable $\hat{H}$ is chosen from a suitable family that contains convex hull of $\{H_Q\}_{Q\in\mathbb{Q}}$

$$Co(H;\mathbb{Q}) \coloneqq \{\lambda H_{Q_1} + (1-\lambda)H_{Q_2};\ Q_1 \in \mathbb{Q},\ Q_2 \in \mathbb{Q},\ 0 \le \lambda \le 1\};$$

and $\hat{z}$ is a suitable positive number.

To calculate mentioned quantities methods of non-smooth convex analysis and duality theory will be used. The key observation is that for arbitrary $G \in \mathcal{G}$ and $H \in \mathcal{H}$, one has $G = \frac{dP}{d\mu}$ for some $P \in \mathbb{P}$, consequently

$$E^P[X] = E^\mu[GX] = E^\mu[X(G - zH)] + z \cdot E^\mu(HX) \tag{1.24}$$

$$\le E^\mu[G - zH]^+ + \alpha z; \quad \forall z > 0,\ \forall X \in \mathcal{X}_\alpha^{\mathbb{Q}}. \tag{1.25}$$

Furthermore, equality in (1.25) is achieved for some $\hat{G} \in \mathcal{G},\ \hat{H} \in \mathcal{H},\ \hat{z} \in (0,\infty)$ if and only if both conditions

$$E^\mu[\hat{H}\hat{X}] = \alpha, \tag{1.26}$$

$$\hat{X} = 1_{\hat{z}\hat{H}<\hat{G}} + B \cdot 1_{\hat{z}\hat{H}=\hat{G}}, \quad \mu - a.e. \tag{1.27}$$

hold for some random variable $B : \Omega \to [0,1]$. Then, as $\hat{P} \coloneqq \int \hat{G} d\mu \in \mathbb{P}$, one has

$$E^{\hat{P}}(\hat{X}) = E^\mu(\hat{G}\hat{X}) = E^\mu[\hat{G} - \hat{z}\hat{H}]^+ + \alpha \cdot \hat{z}. \tag{1.28}$$

Proposition 1.10 *Suppose there exists quadruple* $(\hat{G}, \hat{H}, \hat{z}, \hat{X}) \in (\mathcal{G} \times \mathcal{H} \times (0,\infty) \times \mathcal{X}_\alpha^{\mathbb{Q}})$ *that satisfies* (1.26) – (1.27) *and*

$$E^\mu[\hat{X}(\hat{G} - G)] \le 0, \quad \forall G \in \mathcal{G}. \tag{1.29}$$

Then we have

$$E^{\hat{P}}[X] \leq E^{\hat{P}}[\hat{X}] \leq E^{P}[\hat{X}], \quad \forall X \in \mathcal{X}_{\alpha}^{\mathbb{Q}},\ \forall P \in \mathbb{P}. \tag{1.30}$$

So the pair $(\hat{X}, \hat{P})$ *is a saddle-point for the stochastic game with lower value* $\underline{V}(\alpha)$ *as in* (1.22) *and upper value*

$$\overline{V}(\alpha) = \inf_{P \in \mathbb{P}} \left(\sup_{X \in \mathcal{X}_{\alpha}^{\mathbb{Q}}} E^{P}[X] \right) \tag{1.31}$$

namely

$$\underline{V}(\alpha) = \overline{V}(\alpha) = E^{\hat{P}}[\hat{X}] = \int \hat{G}\hat{X} d\mu \tag{1.32}$$

Further, introducing value function

$$\tilde{V}(z) = \tilde{V}(z;\alpha) := \inf_{(G,H) \in (\mathcal{G},\mathcal{H})} E^{\mu}[G - zH]^{+}, \quad z \in (0, \infty). \tag{1.33}$$

From (1.25) one can observe that

$$\overline{V}(\alpha) \leq \inf_{z>0} \left(\tilde{V}(z) + z \cdot \alpha \right) = V_{*}(\alpha) \tag{1.34}$$

Proposition 1.11 *1. The pair* $(\hat{G}, \hat{H})$ *attains infimum in* (1.33) *with* $z = \hat{z}$

2. The triple $(\hat{G}, \hat{H}, \hat{z})$ *attains the first infimum in* (1.34)

3. The number $\hat{z} \in (0, \infty)$ *attains the second infimum in* (1.34)

4. There is no duality gap *in* (1.34)*, namely*

$$V_{*}(\alpha) = \overline{V}(\alpha) = \underline{V}(\alpha) = E^{\hat{P}}[\hat{X}]$$

So in order to find a statistical test, that would satisfy min-max criteria and, consequently, be optimal for composite hypothesis testing problem, one should solve a dual problem (1.34) and use obtained values $(\hat{z}, \hat{G}, \hat{H})$ to construct test in a form of (1.23).

1.7 Minimal Maringale Measure (Locally Risk Minimizing approach)

As there are infinitely many martingale measures on incomplete market, it is reasonable to think of criteria to choose some specific measure to work with.

One of such possible criteria is based on the concept of *local risk minimization*. This concept was introduced in Schweizer 1999 and consists in minimization of the variance of the increases in the hedging costs process. According to local risk minimization approach, optimal hedging strategy can be obtained as conditional expectation of the payoff under special *minimal martingale measure*, if such measure exists.

Market model introduced in (8.1) admits at least one equivalent martingale measure and S is continuous semi-martingale which satisfy

$$S_t = S_0 + M_t + A_t, \tag{1.35}$$

where $M_t^i = \sum_{j=1}^{k} \int_0^t S_t^i \sigma_t^{ij} dW_t^j$ is a square integrable local $P-$martingale and A is a process of finite variation satisfying *structural condition*

$$A_t = \int_0^t \lambda_s d\langle M \rangle_s,$$

for R^k-valued predictable process λ.

Definition 1.12 *In the* (B, S) *market, an ELMM* $\hat{P}$ *is called* minimal *if* $\hat{P} = P$ *on* $\mathcal{F}_0$*, and if any square-integrable* P*-martingale which is orthogonal to* M *under* P *remains a martingale under* $\widehat{P}$

Theorem 1.13 (Adapted Theorem 1 from Schweizer 1999) *In the market* (8.1)*, if measure* $\widehat{P}$*, equivalent to* P *is defined as*

$$\frac{d\widehat{P}}{dP} = \mathcal{E}\left(\int_0^t \lambda_s dM_s\right), \tag{1.36}$$

then $\widehat{P}$ solves

$$\arg\min_{\tilde{P}\in\mathcal{M}} H(P|\tilde{P}), \tag{1.37}$$

where

$$H(Q|P) = \begin{cases} E^Q\left[\ln \frac{dQ}{dP}\right], & if Q << P \\ \infty, & otherwise. \end{cases}$$

Chapter 2

Method of Market Completions

2.1 Introduction

It is known that in practice markets are rarely complete. There might be a lot of different reasons for market incompleteness including but not limited to incomplete information (see Eyraud-Loisel 2019), models with stochastic parameters (Otaka and Yoshida 2003) or structural incompleteness caused by greater amount of sources of risk than amount of risky assets available for investor or even by adding some new type of risk source as f.e. jumps – as reader knows, simple jump diffusion model with one risky asset is already incomplete. In a situation when market incompleteness can be caused by impressive range of reasons and indeed is closer to a standard rather than exception – it is important to develop a toolkit to operate within such models.

As theory of mathematical finance is relatively well developed for complete markets – it is also reasonable to find a way of "transferring" accumulated knowledge to the case of incomplete models. According to Karatzas (Karatzas and Shreve 2016) standard multidimensional diffusion market is proven to be complete if and only if amount of independent risky assets is equivalent to amount of risk sources driving assets prices. It creates a base to use an alternative method of working with incomplete markets called *Method of Market Completions* which was shown, for example, in Karatzas *et al.* 1991 and was already proven to give same pricing results to classical methods in Melnikov and MacKay 2017, Guilan 1999 which makes this method promising for further applications.

The core approach to pricing and hedging problems is to find equivalent no-arbitrage martingale measure, so the issue that appears when market is no longer complete is the fact that equivalent no-arbitrage, or risk-neutral, market measure is not unique anymore. Fair price of contingent claim, obtained with the help of martingale method becomes an interval rather than a single value. As a result, on incomplete market each possible solution can be parametrized by the set of equivalent local martingale measures.

We demonstrate that instead of using abstract set of equivalent local martingale measures as a parameter – agent can also work with easier-to-interpret set of completing assets. For obvious reasons this approach opens a way to a nice flexibility of auxiliary assets and greater practical application as one can potentially find necessary assets to complete the market.

Method of market completions can mainly be used in two different ways. First approach consists in *estimation of the intervals.* As there is almost always set of possible orthogonal completions available, one may aim at estimation of the intervals of optimal strategy price. This approach might be useful for scenario testing or other estimation problems. Second approach is to *pick particular completion.* This is similar to choosing some particular risk measure as Esscher measure or Minimal Relative Entropy measure (Miyahara 2004 and other). Using the second approach allows us to be more specific regarding assets required for the market to be complete and in some cases it might be even possible to reverse-engineer such auxiliary assets to the market, for instance, with the help of BSDE technique Kobylanski 2000.

Apart from pricing a contingent claims, another important practical problem in financial markets and actuary industry is construction of an optimal hedging portfolio. On incomplete market – perfect-hedging strategy which is self-financing and replicates payoff, may not exist. At the same time, a super-hedging strategy, which uses minimal initial budget to completely cover any possible anticipated payoff may be too expensive for investor. For both complete and incomplete markets, there is a chance that perfect- or super-hedging strategies are not feasible as their initial price is out of investors budget. In this case, market agent can focus on partial-hedging strategies that minimize risk exposure measured by some chosen risk-measure. Such type of strategies is extremely important as there are always a huge variety of risks on the market and, clearly, it might be too expensive to eliminate all of them.

There is a well-developed study in the area of partial-hedging. Föllmer&Leukert (Föllmer and Leukert 1999) and Spivac&Cvitanic (Spivak and Cvitanić 1999) considered quantile hedging, or maximization of probability of successful perfect-hedging, in Föllmer and Leukert 2000 authors also investigated shortfall minimization in line with its utility-weighted value minimization. These articles lay a foundation of partial hedging with the help of Neyman-Pearson lemma and Convex optimization methods. Since recently, risk exposure is measured with the help of special measures, widely used by market participants: Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). The latter one is better known as Expected Shortfall (ES) and was recommended in 2016 in The Market Risk Framework of Basel III – international regulatory accord. These measures spark a special interest in their application in the optimal partial-hedging problem. As shown in Melnikov&Smirnov (Melnikov and Smirnov 2012) it is still possible to apply Neyman-Pearson lemma to CVaR optimization. Recent papers by Cong et al. (Cong *et al.* 2014), Li&Xu (Li and Xu 2013), Capinski(Capinski 2014) and Godin (Godin 2015) demonstrate growing interest towards CVaR optimization.

There are some results obtained for incomplete markets as in Deaconu (Deaconu *et al.* 2017), little is done in application of optimal-partial hedging methods to multi-dimensional incomplete market model – model with several risky assets, which price dynamics is driven by several sources of uncertainty, represented, for example, by multi-dimensional Brownian motion. Of course, some steps towards incomplete market were done in mentioned papers on partial-hedging. However, they lead to mathematical results that are oftentimes difficult to interpret and calculate.

In the rest of the chapter we provide introduction of the method of Market Completions in line with its comparison with classical methodologies for risk-neutral price interval estimation on incomplete market. We demonstrate that introduced method indeed gives the same results and therefore is a promising tool for further applications. Method of market completions was developed in Melnikov and Feoktistov 2001 (see also the book by Melnikov 1999, Appendix 3) for discrete markets as well as for jump-diffusion model in Melnikov and MacKay 2017. In addition, introducing necessary technical complications, demonstrated approach and results obtained for diffusion market model can be extended to other models.

2.2 Market Parametrization by Orthogonal Completions

So how can one obtain "completed" version of initially incomplete market?

It was demonstrated in Vasilev and Melnikov 2021 that all equivalent risk-neutral or martingale market measures can be parametrized with the help of sets of special objects called *orthogonal market completions.* For convenience of the reader we remind here the main statements regarding market completions approach necessary for further developments of the paper.

As it was mentioned above, market completeness is closely connected to the shape of matrix $\Sigma = \{\sigma^{ij}\}_{i=1..n,j=1..k}$. In case of incomplete market, one deals with the matrix which rank is not full. Or, roughly speaking, when volatility matrix for tradeable assets has a rectangular shape with more columns (sources of risks represented by independent Brownian motions) than rows (risky assets).

Consequently, to obtain complete market which would correspond to existing incomplete one, it is reasonable to add more "rows" to volatility matrix under consideration. This idea forms a foundation of the *method of market completions.*

Obviously, these "completing" assets should be independent from existing ones and among each other to be suitable for solving the issue of non-full rank volatility matrix. Adding them, we obtain "proper" volatility matrix which corresponds to some complete market where known and well-developed methods can be applied. Now let us formalize this idea.

Denote S^c a $(k-n)-$dimensional $(\mathcal{F}_t)_{t\le T}-$adapted process $S^c = (S_t^{n+1}, ..., S_t^k)_{t\le T}$ with the same structure as primary assets:

$$dS_t^i = S_t^i \left(a_t^i dt + \sum_{j=1}^{k} \rho_t^{ij} dW_t^j \right), \quad i = n+1, ..., k.$$

Note that in definition above we keep indexing for ρ coefficients from $n+1$ to k instead of from 1 to $(k-n)$ just for the ease of merging these auxiliary assets with existing volatility matrix. With the help of new introduced assets, we can "complete" initially rectangular volatility matrix Σ for a set of existing risky assets:

$$\Sigma = \text{n assets} \overbrace{\begin{pmatrix} \sigma_t^{11} & \dots & \sigma_t^{1k} \\ \vdots & \ddots & \vdots \\ \sigma_t^{n1} & \dots & \sigma_t^{nk} \end{pmatrix}}^{\text{k risks}} = (n \times k) \text{ matrix.} \tag{2.1}$$

By adding $k-n$ auxiliary assets introduced, one arrives to properly shaped volatility matrix $\tilde{\Sigma}$.:

$$\tilde{\Sigma} = \begin{bmatrix} \Sigma \\ \rho \end{bmatrix} = \begin{matrix} \text{n assets} \\ \\ \text{k-n assets} \end{matrix} \overbrace{\begin{pmatrix} \sigma_t^{11} & \dots & \sigma_t^{1d} \\ \vdots & \ddots & \vdots \\ \sigma_t^{n1} & \dots & \sigma_t^{nk} \\ & & \\ \rho_t^{n+1,1} & \dots & \rho_t^{n+1,d} \\ \vdots & \ddots & \vdots \\ \rho_t^{k1} & \dots & \rho_t^{kk} \end{pmatrix}}^{\text{k risks}} = (k \times k) \text{ matrix.} \tag{2.2}$$

Definition 2.1 *The* $(k-n)-$*dimensional* $(\mathcal{F}_t)_{t\leq T}-$*adapted process* $S^c = (S_t^{n+1}, ..., S_t^k)_{t\leq T}$ *is called a completion for the (B,S) market if the resulting volatility matrix* $\tilde{\Sigma}_t$ *has full rank for all* $t \leq T$.

Definition 2.2 *A completion* $\bar{S}^c = (\bar{S}^{n+1}, ..., \bar{S}^k)$ *is called orthogonal if it satisfies:*

$$\langle S_t^i, \bar{S}_t^j \rangle = 0, \text{ for all } i = 1, ..., n; j = n+1, ..., k; t \in [0, T]$$

and

$$\langle \bar{S}_t^i, \bar{S}_t^j \rangle = 0, \text{ for all } i, j = n+1, ..., k; t \in [0, T]$$

Further in the paper the set of orthogonal completions will be denoted as $\mathcal{C}^{ort}$. Proof of the following lemma is important as it includes key manipulation with market model that will be extensively used in what follows.

Lemma 2.3 *For any completion* $S^c \in \mathcal{C}$ *of the* (B, S) *market, there is an orthogonal completion* $\bar{S}^c \in \mathcal{C}^{ort}$

Proof. It is enough to show that one can always construct orthogonal Orthogonal completion from non-orthogonal assets. It can be accomplished, for example, with the help of a famous Gram-Schmidt method. Our goal is to construct a process $\bar{S}^c = (\bar{S}^{n+1}, ..., \bar{S}^k)$ that satisfies the definition above.

To do it we first define the stochastic logarithm $H^i = (H_t^i)_{t\leq T}$:

$$dH_t^i = \frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{j=1}^{k} \sigma_t^{ij} dW_t^j \tag{2.3}$$

Considering that $i \neq j$, if $\langle H_t^i, H_t^j \rangle = 0$ for all $t \in [0, T]$ then $\langle S_t^i, S_t^j \rangle$. On the other hand, if row-vectors σ_t^i and σ_t^j of volatility matrix are orthogonal for $i \neq j$ for all $t \in [0, T]$, then

$$\langle H_t^i, H_t^j \rangle = \left\langle H_0^i + \int_0^t \mu_s^i ds + \sum_{l=1}^{k} \int_0^t \sigma^{il} dW_s^l, H_0^j + \int_0^t \mu_s^j ds + \sum_{l=1}^{k} \int_0^t \sigma^{jl} dW_s^l \right\rangle$$
$$= \int_0^t \sum_{l=1}^{k} \sigma_s^{il} \sigma_s^{jl} ds \tag{2.4}$$

Consequently, to complete the proof it is enough to show how to construct orthogonal row-vectors $\bar{\sigma}_t^j$ which would imply orthogonality of assets.

To construct such vectors we will use Gram-Schmidt method of orthogonalization for $\sigma_t^i, i = 1, ..., k$:

$$\bar{\sigma}_t^1 = \sigma_t^1, \tag{2.5}$$
$$\bar{\sigma}_t^i = \sigma_t^i - \sum_{j=1}^{i-1} \alpha_t^{ij} \bar{\sigma}_t^j, \tag{2.6}$$

for $i = 2, ..., k$ with $\alpha_t^{ij} = \frac{\langle \sigma_t^i, \bar{\sigma}_t^j \rangle}{\langle \bar{\sigma}_t^j, \bar{\sigma}_t^j \rangle}$ for $i, j = 2, ..., k; j < i$. It is easy to see that obtained vectors are indeed orthogonal.

Let's also obtain the assets for completion. Defining $\bar{H}^i = (\bar{H}_t^i)_{t \leq T}$ for $i = k+1, ..., n$ as

$$d\bar{H}^i = \bar{\mu}_t^i + \sum_{l=1}^{n} \bar{\sigma}_t^{il} dW_t^j, \tag{2.7}$$

with

$$\bar{\mu}_t^1 = \mu_t^1, \tag{2.8}$$
$$\bar{\mu}_t^i = \mu_t^i - \sum_{j=1}^{i-1} \alpha_t^{ij} \bar{\mu}_t^j, \tag{2.9}$$

for $i = 2, ..., n$. Final completion assets can be obtained from:

$$d\bar{S}_t^j = \bar{S}_t^j d\bar{H}_t^j, \quad j \in \overline{k+1, n} \tag{2.10}$$

■

Remark 2.4 *Orthogonalization of drift terms for assets in the proof of lemma above plays rather technical role. In such form, one would get much simpler solution for the (1.4).*

Now one has everything required in order to show that completed market can be re-written in terms of corresponding orthogonal completion. Consider an orthogonal completion $\bar{S}^c \in \mathcal{C}^{ort}$. With the help of this set of auxiliary assets, one forms a complete market $(B, \tilde{S})$ where $\tilde{S} = \begin{bmatrix} S \\ S^c \end{bmatrix}$. This new market is described by the dynamics of an k-dimensional process

$$d\tilde{S}_t = \tilde{S}_t \left(\tilde{\mu}_t dt + \tilde{\Sigma}_t \cdot dW_t \right) \tag{2.11}$$

where row vectors σ_t^i satisfy $\langle \sigma_t^i, \rho_t^j \rangle = 0$.

Next, one can rewrite dynamics of the orthogonal completion assets in the following way

$$\begin{aligned} dS_t^i =& S_t^i \left(a_t^i dt + \sum_{j=1}^{k} \rho_t^{ij} dW_t^j \right) = & (2.12) \\ =& S_t^i \left(a_t^i dt + \|\rho_t^i\| \sum_{j=1}^{k} \frac{\rho_t^{ij}}{\|\rho_t^i\|} dW_t^j \right) = & (2.13) \\ =& S_t^i \left(a_t^i dt + \|\bar{\sigma}_t^i\| \sum_{j=1}^{k} \frac{\bar{\sigma}_t^{ij}}{\|\bar{\sigma}_t^i\|} dW_t^j \right) = & (2.14) \\ =& S_t^i \left(a_t^i dt + \|\bar{\sigma}_t^i\| d\widehat{W}_t^j \right), & (2.15) \end{aligned}$$

where $\bar{\sigma}$ is from (2.5). Similarly, dynamics of existing market assets can be rewritten in terms of innovative Wiener process $\widehat{W}_t$:

$$\begin{aligned} dS_t^i =& S_t^i \left(\mu_t^i dt + \sum_{j=1}^{k} \sigma_t^{ij} dW_t^j \right) = & (2.16) \\ =& S_t^i \left(\mu_t^i dt + \sum_{j=1}^{k} \left(\bar{\sigma}_t^{ij} + \sum_{l=1}^{i-1} \alpha^{il} \bar{\sigma}_t^{lj} \right) dW_t^j \right) = & (2.17) \\ =& S_t^i \left(\mu_t^i dt + \sum_{l=1}^{i-1} \alpha_t^{ij} \|\bar{\sigma}_t^j\| d\widehat{W}_t^j \right). & (2.18) \end{aligned}$$

Which means dynamics of completed market can be expressed as

$$d\tilde{S}_t = \tilde{S}_t \left(\tilde{\mu} dt + \widehat{\Sigma}_t \cdot d\widehat{W}_t \right) \tag{2.19}$$

with

$$\widehat{\Sigma}_t = \begin{bmatrix} L_t & 0_{n,k-n} \\ 0_{k-n,n} & D_t \end{bmatrix}$$

where $D_t = diag(\|\rho_t\|)$ and L_t is a $n \times n$ lower triangle matrix with values of the form $\alpha_t^{ij}\|\bar{\sigma}_t^j\|$.

In Vasilev and Melnikov 2021 we demonstrated that working with the set of orthogonal completions would be equivalent to working with the set of equivalent local martingale measures (ELMM). Important result stemming from this fact can be summarized in the following lemma.

Lemma 2.5 *1. Each completion S^c uniquely defines a single ELMM in the incomplete market. Moreover, for the equivalent orthogonal complete market (obtained using method of Lemma 2.3), such local martingale measure will be the same.*

2. Each ELMM $\tilde{P}$ in the incomplete market ($\tilde{P} \in \mathcal{M}$) will be a unique ELMM in the associated completed market model.

Therefore, the set $\mathcal{M}$ of ELMMs in the incomplete market is equivalent to the set $\mathcal{M}^c$ of unique ELMMs corresponding to each completion of the market.

This beautiful fact allows us to switch analysis tool from abstract class of Equivalent Martingale measures to class of "completing" assets. The latter is much easier to interpret and also impose different restrictions such as maximal asset volatility or no short selling on the market.

As it was demonstrated in previous sections, each martingale measure can be associated with particular orthogonal market completion. Which implies that, with the help of appropriate orthogonal completion, it is possible to construct completed market with risk-neutral measure corresponding to minimal martingale measure. In other words, there is a way of choosing particular set of orthogonal completions for solving problems of pricing and hedging. In the following theorem we summarize what particular market completion will correspond to such measure.

Theorem 2.6 *In the market* (8.1)*, completion* $\bar{S}^c \in \mathcal{C}^{ort}$ *solving* (1.37) *is such that for all completion assets the drift process* $a^i \equiv r$*,* P*-a.s. The unique ELMM admitted by the resulting completed market is the minimal martingale measure* $\widehat{P}$ *for the incomplete* (B, S) *market.*

Proof. Consider an arbitrary completion $\nu K(\Sigma)$. Using (3.12), one can switch to equivalent measure which is an ELMM for some completed market. Then

$$H(Q^\nu|P) = E^\nu\left[\ln Z_T^A Z_T^\nu\right] = \tag{2.20}$$

$$= E^\nu\left[\int_0^t (\theta_s^T + \nu_s^T)\cdot dW_s + \frac{1}{2}\int_0^t (\|\theta_s\|^2 + \|\nu_s\|^2)ds\right] = \tag{2.21}$$

$$= E^\nu\left[\frac{1}{2}\int_0^t (\|\theta_s\|^2 + \|\nu_s\|^2)ds\right] \tag{2.22}$$

since θ_s depends on the parameters of existing assets on incomplete market, in order to achieve minimality in (2.22), we should choose completing assets such that each $\nu_s^i = 0$, or, accordingly, $a_i = r$ ■

In the following example we demonstrate how presented method could be applied in order to parametrize set of possible completed markets.

Example 2.7 *Assume the following incomplete market parameters*

Description	*Parameter*	*Value*
Interest rate	r	*0*
Stock 1	σ_{11}	*0.2*
	σ_{12}	*0.11*
	σ_{13}	*0.4*
	μ_1	*0.02*
Stock 2	σ_{21}	*0.3*
	σ_{22}	*0.15*
	σ_{23}	*0.2*
	μ_2	*0.08*

In order to make this model complete, it is enough to add only one auxiliary asset. Let us demonstrate how this market model would be described on each completed version of the market, depending on the completion asset we introduce. We first re-write market model in a form introduced in (2.19).

$$\sigma^1 = \begin{bmatrix} 0.2 & 0.11 & 0.4 \end{bmatrix}, \quad \|\sigma^1\| = 0.4605$$
$$\sigma^2 = \begin{bmatrix} 0.3 & 0.15 & 0.2 \end{bmatrix}, \quad \|\sigma^2\| = 0.3905$$
$$\bar{\sigma}^1 = \sigma^1 = \begin{bmatrix} 0.2 & 0.11 & 0.4 \end{bmatrix}$$
$$\alpha^{21} = \frac{\langle \sigma^2, \bar{\sigma}^1 \rangle}{\langle \bar{\sigma}^1, \bar{\sigma}^1 \rangle} = 0.7379$$
$$\bar{\sigma}^2 = \sigma^2 - \alpha^{21}\bar{\sigma}^1 = \begin{bmatrix} 0.1524 & 0.0688 & -0.0951 \end{bmatrix}$$

and consequently

$$\|\bar{\sigma}^1\| = 0.4605$$
$$\|\bar{\sigma}^2\| = 0.1924$$

and

$$\widehat{\Sigma} = \begin{bmatrix} \|\bar{\sigma}^1\| & 0 & 0 \\ \alpha^{21}\|\bar{\sigma}^1\| & \|\bar{\sigma}^2\| & 0 \\ 0 & 0 & \|\rho^3\| \end{bmatrix} = \begin{bmatrix} 0.4605 & 0 & 0 \\ 0.3398 & 0.1924 & 0 \\ 0 & 0 & \|\rho^3\| \end{bmatrix}$$

where ρ^3 comes from some completion asset.

Further, dynamics of the stocks under innovative Brownian Motion will be:

$$dS_t^1 = S_t^1\left(\mu^1 dt + \|\bar{\sigma}^1\| d\widehat{W}_t^1\right)$$
$$dS_t^2 = S_t^2\left(\mu^2 dt + \alpha^{21}\|\bar{\sigma}^1\| d\widehat{W}_t^1 + \|\bar{\sigma}^2\| d\widehat{W}_t^2\right)$$

Corresponding vector θ then becomes:

$$\widehat{\theta} = \begin{bmatrix} 0.0434 \\ 0.2624 \\ \frac{\mu^3 - r}{\|\rho^3\|} \end{bmatrix} = \begin{bmatrix} 0.0434 \\ 0.2624 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{\mu^3 - r}{\|\rho^3\|} \end{bmatrix} = \theta + \nu$$

The purpose of the last decomposition will be described in following chapters, specifically in Utility maximization section.

Remark 2.8 *In Example 2.7, one could notice that completed versions of market are parametrized by vector ν. Each element of such vector depends on particular set of completing assets. Namely, on their Sharpe-type ratio $\frac{\mu^3 - r}{\|\rho^3\|}$.*

Alternative way of solving the issue of non-squared (non-full rank) volatility matrix would be to reduce amount of underlying Brownian motions. This approach was introduced in Zhang 2007 for the case when coefficients of stock prices are deterministic or functions of stock prices. Reducing dimension of Brownian Motion in the following way:

$$dB_t^i = \sum_{j=1}^{k} \frac{\sigma_t^{ij}}{\|\sigma_t^i\|} dW_t^j \tag{2.23}$$

$$dS_t^i = S_t^i\left(b_t^i dt + \|\sigma_t^i\| dB_t^i\right), \quad i = 1..n. \tag{2.24}$$

$$dB_t^i dB_t^j = \rho_t^{ij} \tag{2.25}$$

We arrive to vector with correlated components $B_t = (B_t^1, ..., B_t^n)$. Denote $\Psi_t = \{\rho_t^{il}\}_{i,l=1..n}$, non-singular, symmetric and positive semi-definite matrix, which then has matrix square-root A_t, $\Psi_t = A_t \cdot A_t^T, \quad A_t = \{a_t^{ij}\}_{i,j=1..n}$.

Moreover, $\exists \tilde{W}_t^1, ..., \tilde{W}_t^n$ independent, such that:

$$B_t^i = \sum_{j=1}^{n} \int_0^t a_s^{ij} d\tilde{W}_s^j. \tag{2.26}$$

As a result, risky assets can be presented in a form

$$dS_t^i = S_t^i \left(b_t^i dt + \|\sigma_t^i\| \sum_{j=1}^{n} a_t^{ij} d\tilde{W}_t^j \right), \quad i = 1..n \tag{2.27}$$

$$\tilde{\sigma}_t = \Sigma_t \cdot A_t \tag{2.28}$$

Where Σ_t is a diagonal matrix of $\{||\sigma^i||\}$ and matrix A_t depends on particular decomposition of Ψ_t. According to Zhang 2007, one obtains the following "completed" model:

$$dS_t^i = S_t^i \left(b_t^i dt + \sum_{j=1}^{n} \tilde{\sigma}_t^{ij} d\tilde{W}_t^j \right), \quad i = 1..n \tag{2.29}$$

$$\theta_t = A_t^{-1} \cdot \Sigma_t^{-1} \cdot (b_t - r_t \mathbf{1}_n)$$

$$\|\theta_t\|^2 = (b_t - r_t \mathbf{1}_n)^T \cdot (\sigma_t \sigma_t^T)^{-1} \cdot (b_t - r_t \mathbf{1}_n)$$

Though this methodology presents promising approach for practical use, in what follows we will focus on applications of method of market completions only.

Chapter 3

Method of Market Completions in Application to Utility Maximization Problem

3.1 Introduction

We start demonstration of the applications of method of market completions with the classical problem of *utility maximization*. Assume that investor, having some initial budget v_0 wants to invest in the market (8.1) in such a way that his portfolio capital at termination time T will give him maximal expected utility. In rigorous terms, the goal is to find a portfolio π^* with initial price v_0 such that

$$\mathbb{E}[U(V_T^{\pi^*})] = \max_{\pi \in \mathcal{A}(v_0)} \mathbb{E}[U(V_T^{\pi})].$$

This task of optimal allocation of initial capital among a portfolio of assets is a fundamental problem in mathematical finance and therefore widely covered in literature. Starting with seminal work Merton 1969, maximal expected utility become one of the central criteria in making individual investment decision in continuous time. Classical assumptions in the field of optimizing expected utility are: utility function U is concave (risk-averse investor) and decision maker is rational. Therefore we first focus on providing solution in case of concave utility function, where method of market completions has already proven to be useful (see Karatzas *et al.* 1991) and we build a basis for further discussions. We recall in this chapter how method of market completions becomes a powerful tool for solving classical problem of utility maximization.

Main contribution in this chapter is generalization of a utility function to the case of not necessarily concave one. Inspired by the Reichlin 2012, Bahchedjioglou and Shevchenko 2022 we demonstrate how method of market completions can be used to solve the problem of utility maximization in this case by means of "concavification" procedure.

3.2 Concave Utility Function

Let us start with presentation of the fundamental approach to utility maximization with respect to concave utility function that was demonstrated in Karatzas *et al.* 1991. This approach will form basis for further development of application of method of market completions towards more general, not necessarily concave utility function.

Definition 3.1 *A function $U : (0, \infty) \to \mathbb{R}$, such that it is strictly increasing, strictly concave, of a class C^1 and satisfies*

$$U'(0+) = \infty, \quad U'(\infty) = 0$$

is called a concave utility function.

Having such function, investor might use it in order to assess how attractive some arbitrary portfolio π is based on his subjective perception of its expected terminal capital utility, expressed as $\mathbb{E}[U(V_T^\pi(v_0))]$. In rigorous terms, the goal of agent can be expressed as finding the value function

$$\zeta(x) \coloneqq \max_{\pi \in \mathcal{A}(v_0)} \mathbb{E}[U(V_T^\pi)], \tag{3.1}$$

where $\mathcal{A}(v_0)$ is a set of admissible strategies with initial capital v_0.

3.2.1 Complete Market Case

One of the classical approaches to solving such problem involves application of methods from convex analysis and famous Legendre-Fenchel transformation for utility function U:

$$\tilde{U}(y) := \max_{x>0}\{U(x) - xy\}. \tag{3.2}$$

Conjugate function $\tilde{U}(y)$ is strictly decreasing, strictly convex and it is straightforward to prove that

$$\tilde{U}'(y) = -(U'(x))^{-1} \quad 0 < y < \infty. \tag{3.3}$$

Consequently, denoting inverse function $I(y) = (U'(x))^{-1}$, the following equality holds:

$$\tilde{U}(y) = U(I(y)) - yI(y), \quad 0 < y < \infty \tag{3.4}$$

from which follows

$$U(I(y)) \geq U(x) + y\left(I(y) - x\right), \quad \forall x > 0, y > 0. \tag{3.5}$$

Assumption 3.2 *For some $\alpha \in (0,1), \gamma \in (1,\infty)$ holds*

$$\alpha U'(x) \geq U'(\gamma x), \quad \forall x \in (0,\infty) \tag{3.6}$$

Remark 3.3 *Condition of Assumption 3.2 is essentially the same that was introduced by Kramkov and Schachermayer in Kramkov and Schachermayer 1999 on the concept of Assymptotic Elastisity that*

$$AE_0(\tilde{U}(y)) := \limsup_{y\to 0} \sup_{q\in\partial\tilde{U}} \frac{|q|y}{\tilde{U}(y)} < \infty, \tag{3.7}$$

where $\partial\tilde{U}$ is a subdifferential of convex conjugate $\tilde{U}(y)$.

We would also assume that solution for (3.1) exists and finite. Then such solution in complete market case can be directly calculated. In order to find it, one needs special auxiliary function defined as

$$\mathcal{X}_0(y) := \mathbb{E}\left[\beta_T Z_T^A I(y\beta_T Z_T^A)\right], \quad 0 < y < \infty, \tag{3.8}$$

which is a continuous, strictly decreasing function which has an inverse

$$\mathcal{Y}_0(x) = (\mathcal{X}_0(y))^{-1}.$$

Following Karatzas *et al.* 1991, the optimal terminal capital that maximizes the expected utility, or the solution to (3.1) is given by

$$\xi_0(x) := I\left(\mathcal{Y}_0(x)\beta_T Z_T^A\right). \tag{3.9}$$

Example 3.4 *Assume that investors attitude is described by logarithmic utility function $U(x) = \ln(x)$. Then*

$$\begin{aligned}
I(y\beta_T Z_T^A) &= \frac{1}{y\beta_T Z_T^A} \\
\mathcal{X}_0(y) &= \frac{1}{y}, \quad \mathcal{Y}_0(x) = \frac{1}{x} \\
\xi_0(x) &= \frac{x}{\beta_T Z_T^A} \\
\zeta(x) &= \mathbb{E}\left[\ln\left(\frac{x}{\beta_T Z_T^A}\right)\right]
\end{aligned}$$

3.2.2 Incomplete Market Case

We now demonstrate how method of market completions becomes useful for solving problem (3.1) on incomplete market. Assume now in the market model (8.1) we have more sources of uncertainty than independent assets. In other words $k > n$. In this case market agent does not have one specific vector θ from (1.4) in order to construct Z_T^A and, consequently, a solution $\xi_0(x)$. In this situation method of market completions allows us to parameterize all possible "completed" markets with the help of orthogonal completions. The problem of expected utility maximization can be solved on each of these markets using classical approach described in the previous section. Natural question is then how to use such solutions, parametrized by the set of orthogonal completions in order to get the best expected utility on incomplete market which we started with? Using results from Karatzas *et al.* 1991, we can extract valuable criteria that connects solutions on incomplete market with the set of solutions on completed versions.

We start with adding necessary amount of auxiliary assets in the form of orthogonal completion. Namely, for initially incomplete market we form orthogonal completions consisting of $(k-n)$ assets

$$dS_t^i = S_t^i \left(a_t^i dt + \sum_{j=1}^{k} \rho_t^{ij} dW_t^j \right), \quad i = n+1, ..., k.$$

As a result, completed market, corresponding to particular orthogonal completion will be rewritten as in (2.19)

$$d\tilde{S}_t = \tilde{S}_t \left(\tilde{\mu} dt + \widehat{\Sigma}_t \cdot d\widehat{W}_t \right) \tag{3.10}$$

Consequently, for each completed market, we could rewrite dynamics of the Girsanov exponential in terms of innovative Brownian motion for (2.19). Solving (1.4) on completed market, we arrive to:

$$\begin{aligned} \widehat{\Sigma} \cdot \widehat{\theta} =& \bar{\mu} - \bar{1}_k r \\ \widehat{\theta}_t =& \widehat{\Sigma}_t^T \cdot \left(\widehat{\Sigma}_t \widehat{\Sigma}_t^T \right)^{-1} (\tilde{\mu}_t - r \bar{1}_k) = \\ =& \begin{bmatrix} L_t^T \cdot \left(L_t L_t^T \right)^{-1} (\bar{\mu} - \bar{1}_n r) \\ D_t^T \cdot \left(D_t D_t^T \right)^{-1} (\bar{a} - \bar{1}_{k-n} r) \end{bmatrix} = \begin{bmatrix} \theta \\ \widehat{\nu} \end{bmatrix}. \end{aligned}$$

For orthogonalized market model, the first n elements of vector $\widehat{\theta}$ depends only on existing market assets and stays the same for any completed market. At the same time, last $k-n$ elements will correspond to particular completion. Consequently

$$\widehat{\theta}_t = \begin{bmatrix} \theta \\ 0_{k-n} \end{bmatrix} + \begin{bmatrix} 0_n \\ \widehat{\nu} \end{bmatrix} = \theta_t + \nu_t = \theta_t^\nu. \tag{3.11}$$

And, since each completed market is in correspondence with vector ν, the set of all completed versions of the initial market can be parametrized by this vector. Using θ_t^ν, we construct a Girsanov type exponential for corresponding martingale probability density on completed market

$$Z_t^C = \exp\left\{-\int_0^t (\theta_s^T + \nu_s^T)\cdot dW_s - \frac{1}{2}\int_0^t (\|\theta_s\|^2 + \|\nu_s\|^2)ds\right\} = \tag{3.12}$$

$$= Z_t^A \times Z_t^\nu. \tag{3.13}$$

Then, on each completed version of initially incomplete market, we are able to perform steps (3.8)-(3.9) in order to obtain corresponding optimal solution $\xi_\nu(x)$:

$$\mathcal{X}_\nu(y) := \mathbb{E}\left[\beta_T Z_T^A Z_T^\nu I(y\beta_T Z_T^A Z_T^\nu)\right], \quad 0 < y < \infty, \tag{3.14}$$

$$\mathcal{Y}_\nu(x) = (\mathcal{X}_\nu(y))^{-1}, \tag{3.15}$$

$$\xi_\nu(x) := I\left(\mathcal{Y}_\nu(x)\beta_T Z_T^A Z_T^\nu\right). \tag{3.16}$$

For further progress, it is reasonable to assume, that we will be working only with completions that lead to market where expected utility maximization problem can be solved. Therefore, we introduce the following assumption

Assumption 3.5 *We restrict ourselves to set of completions*

$$K_1(\Sigma_t) = \{\nu \in \ker(\Sigma_t) \mid \mathcal{X}_\nu(y) < \infty,\ \forall y > 0\}$$

where $\ker(\Sigma_t)$ *is a kernel of a volatility matrix* Σ_t *from initial incomplete market.*

In order to preserve consistency, we also provide here theorem from Karatzas *et al.* 1991 (Theorem 8.5) which is an important tool for our further applications. Details of the proof could be found in reference.

Theorem 3.6 *Consider a positive,* $\mathcal{F}_T$*-measurable random variable* B*, for which there exists a process* $\lambda \in \ker\Sigma$ *with*

$$\mathbb{E}[\beta_T Z_T^A Z_T^\nu B] \le x = \mathbb{E}[\beta_T Z_T^A Z_T^\lambda B],\ \forall \nu \in \ker\Sigma. \tag{3.17}$$

Then there exists a portfolio $\pi \in \mathcal{A}(x)$*, such that* $V_T^\pi(x) = B$*, almost surely.*

As this theorem will be used in what follows we provide a brief recoup of the proof below.

Proof. Define $\mathcal{F}_t$-adapted process X_t as

$$\beta_t Z_t^A Z_t^\lambda X_t = M_t := \mathbb{E}\left[\beta_T Z_T^A Z_T^\lambda B | \mathcal{F}_t\right] \tag{3.18}$$

, with $X_0 = x$ and $X_T = B$ a.s. and M_t is a positive martingale with $M_0 = x$. From the martingale representation theorem then, it follows that there exists $\mathcal{F}_t$ adapted process φ_s with $\int_0^T \|\varphi_t\|^2 dt < \infty$ a.s. such that

$$M_t = x + \int_0^t \varphi_s dW_s, \quad 0 \le t \le T \tag{3.19}$$

from continuity and positivity of M_t, process $\psi_t = -\frac{\varphi_t}{M_t}$ is well-defined and

$$M_t = x - \int_0^t M_s \psi_s^* dW_s = x - \int_0^t \beta_s Z_s^A Z_s^\lambda X_s \psi_s^* dW_s, \quad 0 \le t \le T. \tag{3.20}$$

Further, decompose $\psi_t = \psi_t^A + \psi_t^\lambda$ with $\psi_t^A \in \ker^\perp \Sigma$ and $\psi_t^\lambda \in \ker \Sigma$. For some arbitrary portfolio π, its value process $V_t^\pi(x)$ will satisfy

$$\beta_t Z_t^A Z_t^\lambda V_t^\pi = x + \int_0^t \beta_s Z_s^A Z_s^\lambda V_s^\pi \left(\Sigma_s^* \pi_s - (\theta_s + \lambda_s)\right)^* dW_s. \tag{3.21}$$

Therefore, in order to show that there is a portfolio financing $\xi_\lambda(x,\omega)$, one should show

$$-X_t(\psi_t^A + \psi_t^\lambda) = V_t^\pi \left(\Sigma_t^* \pi_t - (\theta_t + \lambda_t)\right). \tag{3.22}$$

As we could choose π_t in a way that $\Sigma_t^* \pi_t = \theta_t - \psi_t^A$, it is enough to prove that $\psi_t^\lambda = \lambda_t$ in order to verify financiability of $\xi_\lambda(x,\omega)$.

For arbitrary ν introduce the sequence of stopping times $\{\tau_n\}_{n=1}^\infty$ as

$$\tau_n = T \wedge \inf\left\{t \in [0,T]; M_t \ge n, \text{ or } \int_0^t (\|\psi_s\|^2 + \|\lambda_s\|^2) ds \ge n, \text{ or } \int_0^t \|\nu_s\|^2 ds \ge, n \text{ or } \left|\int_0^t \nu_s^* dW_s\right| \ge n\right\}$$

for all $n \geq 1$. Notice that $\lim_{n\to\infty} \tau_n = T$ a.s. and denote $\nu_t^n = \nu_t 1_{t\leq\tau_n}$. Then, for $\lambda_t + \varepsilon\nu_t^n \in \ker\Sigma$ it holds

$$Z_t^{\lambda_t+\varepsilon\nu_t^n} = Z_t^A \cdot \exp\left\{-\varepsilon \int_0^t (\nu_s^n)^*(dW_s + \lambda_s ds) - \frac{\varepsilon^2}{2}\int_0^t \|\nu_s^n\|^2 ds\right\}. \tag{3.23}$$

for $-1 \leq \varepsilon \leq 1,\ n \geq 1$. And also

$$e^{-3n|\varepsilon|} \leq \frac{Z_t^{\lambda_t+\varepsilon\nu_t^n}}{Z_t^\lambda} \leq e^{3n|\varepsilon|} \quad -1 \leq \varepsilon \leq 1. \tag{3.24}$$

Then, from dominated convergence theorem and (3.17) it follows

$$0 = \frac{\partial}{\partial\varepsilon}\mathbb{E}\left[\beta_T Z_T^{\lambda_t+\varepsilon\nu_t^n} B\right]\Big|_{\varepsilon=0} = \mathbb{E}\left[\beta_T \cdot \frac{\partial}{\partial\varepsilon} Z_T^{\lambda_t+\varepsilon\nu_t^n}\Big|_{\varepsilon=0} \cdot B\right] = \tag{3.25}$$

$$= -\,\mathbb{E}\left[\beta_T Z_T^\lambda B \int_0^{\tau_n} (\nu_s^n)^*(dW_s + \lambda_s ds)\right] = \tag{3.26}$$

$$=\mathbb{E}\left[M_{\tau_n} \int_0^{\tau_n} (\nu_s^n)^*(dW_s + \lambda_s ds)\right] \tag{3.27}$$

Applying Ito rule of integration by parts, we get

$$M_{\tau_n}\int_0^{\tau_n} (\nu_s^n)^*(dW_s + \lambda_s ds) = \int_0^{\tau_n} M_s(\nu_s^n)^*(\lambda_s - \psi_s^\lambda)ds + \int_0^{\tau_n} M_s(\nu_s^n)^* dW_s - \tag{3.28}$$

$$- \int_0^{\tau_n} M_s \left(\int_0^s (\nu_t^n)^*(dW_t + \lambda_t dt)\right)(\psi_s^A + \psi_s^\lambda)^* dW_s.$$

As expected value of the last two integrals equals 0, $\lambda_t = \psi_t^\lambda$ a.s. Which proves the statement. ■

With all these assumptions the following criteria was developed in Karatzas *et al.* 1991 in order to find connect parametrized problem on a set of orthogonal completions to the initial expected utility maximization on incomplete market

Theorem 3.7 *Assume $U(0) > -\infty$. Then, for conditions*

1. *Optimality of $\hat{\pi}$: $EU(V_T^\pi) \leq EU(V_T^{\hat{\pi}}) \quad \forall \pi \in \mathcal{A}(x)$*

2. Financiability of $\xi_\lambda(x)$: $\exists \hat{\pi} \in \mathcal{A}(x)$ such that $V_T^{\hat{\pi}} = \xi_\lambda(x)$

3. Least Favorability of λ: $EU(\xi_\lambda(x)) \leq EU(\xi_\nu(x)) \quad \forall \nu \in K_1(\Sigma)$

4. Dual optimality of λ: $\forall \nu \in K_1(\Sigma)$,

$$\mathbb{E}\left[\tilde{U}(\mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\lambda)\right] \leq \mathbb{E}\left[\tilde{U}(\mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\nu)\right]$$

5. Parsimony of λ: $E[\beta_T Z_T^A Z_T^\nu \xi_\lambda(x)] \leq x, \quad \forall \nu \in K_1(\Sigma)$

the following holds: (2)-(5) are equivalent. Furthermore, if (2) holds, then portfolio $\hat{\pi}$ in (2) satisfies (1).

In other words, in order to obtain solution for (3.1) on incomplete market, one may find a particular orthogonal market completion, such that it would satisfy one of the conditions (2)-(5) and as a consequence will produce a result that will stay optimal on initial market.

Example 3.8 *Consider utility function $U(x) = \ln(x)$ for which:*

$$\mathcal{X}_\nu(y) = \frac{1}{y}, \quad \mathcal{Y}_\nu(x) = \frac{1}{x} \tag{3.29}$$

and optimal terminal capital can be calculated as

$$\xi_\nu^x = \frac{x}{\beta_T Z_T^\nu}. \tag{3.30}$$

One could check that completion with parameter $\lambda = 0$ satisfies 5.

$$E[\beta_T Z_T^\nu \xi_0^x] = x \cdot E\left[\exp\left\{-\int_0^T \nu_s^T dW_s - \frac{1}{2}\int_0^T \|\nu_s\|^2 ds\right\}\right] \leq x \quad \forall \nu \in K(\Sigma) \tag{3.31}$$

as the process under expectation is a supermartingale. It means that investor would not use auxiliary stocks ($\lambda = 0$) to form an optimal portfolio even for hedging purposes.

3.3 Non-Concave Utility Function

The idea that utility function might not necessarily be concave, or, in other words, investors on the market are not necessarily risk-averse, emerges from significant empirical evidences, provided, for instance, in Tversky and Kahneman 1992. It suggests that, depending on the context, agents on the market might switch from being risk-averse to risk-taking perspective. Such behavior could be described by "mixed" utility function, which is not necessarily concave on all intervals of its domain. Generalization of approaches to solving the utility maximization problem in such case was generalized and discussed in details in Reichlin 2012.

In this section we demonstrate how classical approach described in previous sections can be modified in order to accommodate not necessarily concave utility function. The approach was inspired by Bahchedjioglou and Shevchenko 2022.

$$\bar{\zeta}(x) := \max_{\pi \in \mathcal{A}(v_0)} \mathbb{E}[U(V_T^{\pi})], \tag{3.32}$$

where now utility function $U(x)$ of investor is not necessarily concave. Instead, we are working with some non-decreasing, upper-semicontinuous function that satisfies mild growth condition

$$\lim_{x \to 0} \frac{U(x)}{x} = 0.$$

For such function we define the concave envelope in the following way

Definition 3.9 *$U_c(x)$ is called a concave envelope of function $U(x)$ if it is a smallest non-decreasing, continuous and concave function such that*

$$U_c(x) \geq U(x), \quad \forall x \in \mathbb{R}.$$

We provide important fact from Reichlin 2012 regarding the structure of such "concavification" of utilility function.

Lemma 3.10 (Reichlin 2012 Lemma 2.8) *Concave envelope $U_c(x)$ is finite, continuous on $(0, \infty)$ and satisfy mild growth condition. The set $\{U_c > U\} = \{x \in \mathbb{R}^+ :$*

$U_c(x) > U(x)\}$ *is open and its (countable) connected components are bounded (open) intervals. Moreover,* $U_c(x)$ *is locally affine on the set* $\{U_c > U\}$*, in the sense that it is affine on each of the above intervals.*

From this lemma it follows that we can treat set on which concave envelope is strictly greater than source utility function as a countable union of open intervals $\bigcup_i (a_i, b_i)$.

Lemma 3.11 *Concavified objective function* $U_c(x)$ *has both one-sided derivatives at any point* $x > 0$*, and the right derivative is less or equal than left derivative. Derivatives coincide everywhere except at most countable set. Moreover, for every* $x, y \in \mathbb{R}$ *one has that*

$$U_c(x) - U_c(y) \leq U_c'(y+)(x - y)$$
$$U_c(x) - U_c(y) \leq U_c'(y-)(x - y)$$

We denote the set of points where function $U_c(x)$ is non-differentiable as $\{c_i,\ i \geq 1\}$. It is also fair to notice that derivative of a concave envelope $U_c(x)$ is decreasing and $\lim_{x\to\infty} U_c'(x) = 0$. At the same time, $U_c'(x)$ is constant on each interval (a_i, b_i).

From this lemma it follows that $U_c(a_i-) \geq U_c(a_i+) = U_c(b_i-) \geq U_c(b_i+)$. Then, following the idea from Bahchedjioglou and Shevchenko 2022, one could define the inverse function to concave envelope as

$$i(y) = \begin{cases} (U_c'(x))^{-1} & , \ y \in \mathbb{R}^+ \setminus [\{\bigcup_i [U_c'(b_i+), U_c'(a_i-)]\} \cup \{\bigcup_i [U_c'(c_i+), U_c'(c_i-)]\}] \\ a_i & , \ y \in [U_c'(a_i+), U_c'(a_i-)] \\ b_i & , \ y \in [U_c'(b_i+), U_c'(b_i-)] \\ c_i & , \ y \in [U_c'(c_i+), U_c'(c_i-)] \\ 0 & , \ y \geq U_c'(0+) \end{cases} \tag{3.33}$$

3.3.1 Complete Market Case

Similarly to classical approach introduced in the previous section, in order to solve expected utility maximization problem we construct, with the help of inverse function introduced in (3.33)

$$\mathcal{C}_0(y) := \mathbb{E}\left[\beta_T Z_T^A \cdot i(y\beta_T Z_T^A)\right], \quad 0 < y < \infty, \tag{3.34}$$

Proposition 3.12 *Function $\mathcal{C}_0(y)$ is continuous and decreasing.*

Proof. The statement that the function of interest is decreasing follows directly from the fact that $i(y)$ is decreasing.

Let us prove the continuity of $\mathcal{C}_0(y)$. First, notice that the set A of discontinuity points of the function $i(y)$ is at most countable. Let y_n be a decreasing sequence such that

$$\lim_{n\to\infty} y_n = y.$$

If, for some $\omega \in \Omega$

$$\lim_{n\to\infty} i\left(y_n \beta_T Z_T^A(\omega)\right) \neq i\left(y\beta_T Z_T^A(\omega)\right),$$

then $y\beta_T Z_T^A(\omega) \in A$.

Since $\beta_T Z_T^A$ has a continuous distribution, it follows that $P\left(y\beta_T Z_T^A(\omega) \in A\right) = 0$ and therefore

$$P\left(\lim_{n\to\infty} i\left(y_n \beta_T Z_T^A(\omega)\right) \neq i\left(y\beta_T Z_T^A(\omega)\right)\right) = 0.$$

As $i(y)$ is decreasing and right-continuous, it follows from Monotone Convergence theorem that

$$\lim_{n\to\infty} \mathbb{E}^0\left[\beta_T i\left(y_n \beta_T Z_T^A(\omega)\right)\right] = \mathbb{E}^0\left[\beta_T i\left(y\beta_T Z_T^A(\omega)\right)\right] \tag{3.35}$$

which proves the statement. ■

Further, due to its continuity, function $\mathcal{C}_0(y)$ has continuous and decreasing inverse $\mathcal{J}_0(x) = (\mathcal{C}_0(y))^{-1}$. Then, we define

$$\psi_0(x) = i\left(\mathcal{J}_0(x)\beta_T Z_T^A\right) \tag{3.36}$$

Lemma 3.13 *Random variable $\psi_0(x)$ satisfies*

$$\mathbb{E}\left[\beta_T Z_T^A \psi_0(x)\right] = x. \tag{3.37}$$

For every portfolio $\pi \in \mathcal{A}(x)$

$$\mathbb{E}\left[U(V_T^\pi(x))\right] \leq \mathbb{E}\left[U_c(V_T^\pi(x))\right] \leq \mathbb{E}\left[U_c(\psi_0(x))\right] \tag{3.38}$$

Proof. Equation (3.37) follows directly from constructions of $\psi_0(x)$ and $\mathcal{J}_0(x)$.

Let us focus the optimality of $\psi_0(x)$ in terms of expected utility. Consider some portgolio $\pi \in \mathcal{A}(x)$. Then, applying (3.5) to $U_c(\psi_0(x))$ we get

$$\mathbb{E}\left[U_c(\psi_0(x))\right] = \mathbb{E}\left[U_c\left(i\left(\mathcal{J}_0(x)\beta_T Z_T^A\right)\right)\right] \geq \tag{3.39}$$

$$\geq \mathbb{E}\left[U_c(V_T^\pi(x)) + \mathcal{J}_0(x)\beta_T Z_T^A(i\left(\mathcal{J}_0(x)\beta_T Z_T^A\right) - V_T^\pi(x))\right] = \tag{3.40}$$

$$= \mathbb{E}\left[U_c(V_T^\pi(x)) + \mathcal{J}_0(x)\beta_T Z_T^A(\psi_0(x) - V_T^\pi(x))\right] \geq \tag{3.41}$$

(Applying (3.37) and the fact that $\beta_t Z_t^A V_t^\pi(x)$ is a supermartingale)

$$\geq \mathbb{E}\left[U_c(V_T^\pi(x))\right] \geq \mathbb{E}\left[U(V_T^\pi(x))\right] \tag{3.42}$$

■

From Lemma 3.13 it follows that if there exists portfolio such that its terminal capital replicates the $\psi_0(x)$, it would be optimal for expected utility maximization problem.

Theorem 3.14 *On complete market* (8.1)*, for fixed initial capital v_0, the optimal portfolio that maximizes the expected utility would be the one, which terminal capital satisfies $V_T^\pi(v_0) = \psi_0(v_0) = i\left(\mathcal{J}_0(v_0)\beta_T Z_T^A\right)$*

Proof. From (3.13) follows that $\psi_0(x)$ is a maximizer for the function $\mathbb{E}[U_c(V_T^\pi)]$. For consistency, let us denote this fact as $\psi_0(x) = \arg\bar{\zeta}_c(x)$. In Reichlin 2012 if was demonstrated that $\bar{\zeta}_c(x)$ is itself a concave envelope for value function $\zeta(x)$ of an original maximization problem (3.32). Since distribution of Z_T^A is continuous, from Lemma 5.7 in Reichlin 2012 follows the statement of theorem. ■

3.3.2 Incomplete Market Case

Similarly to Section 3.2.2, in this one we demonstrate how method of market completions can be applied to solving the problem of maximization of expected not necessarily concave utility on incomplete market.

Following the procedure of the method, we introduce $(k-n)$ auxiliary assets. With the same steps, we obtain parameterization of completed versions of the initial market by possible orthogonal completions, each of which corresponds to particular vector ν as in (3.11). Consequently, with the help of (3.12), we can construct necessary functions on each of completed markets

$$\mathcal{C}_\nu(y) := \mathbb{E}\left[\beta_T Z_T^A Z_T^\nu i(y\beta_T Z_T^A Z_T^\nu)\right], \quad 0 < y < \infty, \tag{3.43}$$

$$\mathcal{J}_\nu(x) = (\mathcal{C}_\nu(y))^{-1}, \tag{3.44}$$

$$\psi_\nu(x) := i\left(\mathcal{J}_\nu(x)\beta_T Z_T^A Z_T^\nu\right). \tag{3.45}$$

In case of availability of completion assets, the maximal expected utility that could be achieved on market would be, following Theorem 3.14, the one, that replicates $\psi_\nu(x)$. However, these assets are not available for trading, which means the following inequality holds:

$$\sup_{\pi\in\mathcal{A}(x)} \mathbb{E}\left[U_c(V_T^\pi(x))\right] \leq \mathbb{E}\left[U_c(\psi_\nu(x))\right]. \tag{3.46}$$

And the equality would be achieved in case there exists a portfolio of existing stocks that replicates $\psi_\nu(x)$ at maturity time T. Following the approach from Karatzas *et al.* 1991, let us show that the same criteria as in Theorem 3.7 will take place in case of not necessarily concave utility function. We start with formulating conditions for the case of concavified function $U_c(x)$:

A. Optimality of $\hat\pi$: $EU_c(V_T^\pi) \leq EU_c(V_T^{\hat\pi}) \quad \forall\pi \in \mathcal{A}(x)$

B. Financiability of $\psi_\lambda(x)$: $\exists\hat\pi \in \mathcal{A}(x)$ such that $V_T^{\hat\pi} = \psi_\lambda(x)$

C. Least Favorability of λ: $EU_c(\psi_\lambda(x)) \leq EU_c(\psi_\nu(x)) \quad \forall\nu \in K_1(\Sigma)$

D. Parsimony of λ: $E[\beta_T Z_T^A Z_T^\nu \psi_\lambda(x)] \leq x, \quad \forall \nu \in K_1(\Sigma)$

Following Karatzas *et al.* 1991, $\beta_T Z_T^A Z_T^\nu V_T^\pi(x)$ is a local martingale under physical measure P for every completion $\nu \in K(\Sigma)$. Moreover, $\mathbb{E}\left[\beta_T Z_T^A Z_T^\nu V_T^\pi(x)\right] \leq x,\ \forall \nu \in K(\Sigma)$. From this fact it directly follows that if capital $\psi_\nu(v_0)$ is financeable, then its initial price is no more than required initial capital v_0 on all possible completed markets under consideration, which is nothing but Parsimony condition [E]. Similar to Karatzas *et al.* 1991 we could formulate criteria for choosing market completion for which optimal solution would be financiable on initial incomplete market.

Theorem 3.15 *Conditions [B] and [D] are equivalent and imply [C]. Moreover, portfolio from [B] satisfy [A].*

Proof. Idea and logic of the proof is similar to the one for concave function.

[B.→D.]

For any portfolio $\pi \in \mathcal{A}(x)$, $\beta_t Z_t^A Z_t^\nu V_t^\pi$ is a local martingale under P for every vector $\nu \in \ker(\Sigma)$. Which follows directly from the following representation

$$\beta_t Z_t^A Z_t^\nu V_t^\pi = x + \int_0^t \beta_s Z_s^A Z_s^\nu V_s^\pi \left(\Sigma_s^* \cdot \pi_s - (\theta_s + \nu_s)\right)^* dW_s. \tag{3.47}$$

Since the process $\beta_t Z_t^A Z_t^\nu V_t^\pi$ is a local martingale, one could write

$$\mathbb{E}[\beta_T Z_T^A Z_T^\nu V_T^\pi] \leq x,\ \forall \nu \in \ker\Sigma. \tag{3.48}$$

Consequently, if on some completed market we were able to find a portfolio π^* of existing assets that finances $\xi_\nu(x)$, or, in other words, that $V_T^{\pi^*} = \xi_\nu(x)$ and condition [D] holds.

[D.→B.]

Is a direct consequence of application of Theorem 3.6 with $B = \xi_\lambda(x)$

■

Having solution for concavified utility function, we could use the result from Reichlin 2012 that in case of atomless probability space and continuous distribution of densities $\frac{dP^\nu}{dP}$, solution to utility maximization problem with respect to not-necessarily concave utility coincides with the solution obtained for its concave envelope. Therefore, completion, that satisfies [D] will simultaneously be a solution for utility maximization for not necessarily concave function on incomplete market.

Chapter 4

Pricing of contingent claims by means of Market Completions

4.1 Introduction

Assume now that instead of trying to maximize his utility at some moment T in future, market agent has some financial obligation at that time. In other words, there is some *contingent claim* which agent agrees to sell and provide payoff according to agreed $\mathcal{F}_T$-measurable function $g(\omega)$. Let us here focus on European type claims only. Namely ones for which payoff happens at maturity time only. One of the core questions in mathematical finance is how should one price such contingent claim?

Classical approach to pricing contingent claims is based on no-arbitrage principle. In other words, the price of a contingent claim will be a value that allows no-arbitrage on the market. Such price would be equivalent to a minimal investment required, in order to build a minimal hedge for claim under interest (see Karatzas and Shreve 2016, Merton 1973), which can be calculated as an expectation of the claim discounted value under special equivalent risk-neutral or martingale measure. Which, in case of complete market will be unique and in line with the fact that any claim is perfectly replicable on complete markets, presents no difficulties in pricing the claim.

However, on incomplete markets such equivalent measure is not unique anymore. Which implies that no-arbitrage price is not unique on incomplete market as well. Instead investors can estimate an *interval of no-arbitrage prices* $[C_*(g_T, P), C^*(g_T, P)]$.

For any price within this interval, there are no arbitrage opportunities, whereas for any price outside of it there is one (see Karatzas and Kou 1996). Assessment of the boundaries of such interval is an important task and is covered in literature as in Karatzas and Kou 1996, or, alternatively, by embedding this question into suitable stochastic control problem as in El Karoui and Quenez 1995 or Cvitanic and Karatzas 1993.

Such equivalent martingale measure is also closely connected to risk-free bond as a numeraire (discounting factor used to express relative prices) and, in fact, depends on it. Therefore, it is natural to believe, that instead of working with infinite set of equivalent martingale measures, one could focus on "objective", or observed measure P and come up with some "discounting portfolio" which will play a role of market numeraire (Bajeux-Besnainou and Portait 1997).

There also an interesting question of how to choose a unique price within interval of no-arbitrage prices. The answer to this question depends on many factors and there are several approaches proposed for answering Davis 1997, Foellmer *et al.* 1985.

In current section we will show how method of market completions can be used to price contingent claims. We start with application for classical approach for description of no-arbitrage prices by the means of proposed orthogonal completions. To be consistent, we also provide change of numeraire approach, which was performed in Guilan 1999 for American claims on incomplete market. We then move on to useful way of finding a "fair" price with the helo of utility maximization problem. The latter was proposed Davis in Davis 1997 and will be used in further applications described below.

4.2 Equivalent Martingale Measure Approach

In classical martingale approach to pricing contingent claims is to find minimal hedge for a contingent claim and use its initial value as a price of a claim. Such price would be calculated as an expectation under special *martingale measure*, under which discounted stock price process and, consequently, discounted portfolio value process are martingales. Then, the initial price can be calculated as an expected value with respect to martingale measure.

In case of complete market, no-arbitrage condition (1.4) is enough to find a unique equivalent martingale measure and obtain single initial price of minimal hedge. However, when market is incomplete, there are infinitely many equivalent martingale measures. As a result, one expects to face an interval of "fair" prices as it was desribed in Karatzas and Kou 1996. Consequently, pricing contingent claim on incomplete market based on martingale approach is equivalent to describing interval of fair prices.

In Vasilev and Melnikov 2021 we demonstrated how such unterval can be found with the help of market completions. With the help of Lemma 2.5 it is possible to state the following theorem.

Theorem 4.1 *In the incomplete (B, S) market, asssume that $r = 0$ and let $\bar{\mu}_t^i$ and $\bar{\sigma}_t^i = (\bar{\sigma}_t^{i1}, ..., \bar{\sigma}_t^{ik})$ be as defined in the proof of Lemma 2.3 for $i = 1, ..., n$. Let also $\bar{W}$ be a standard k-dimensional Brownian motion, with the first n elements given by*

$$\bar{W}_t^i = \frac{1}{\|\bar{\sigma}_t^i\|} \sum_{j=1}^{k} \bar{\sigma}_t^{ij} W_t^j \tag{4.1}$$

for $i = 1, ..., n$, $t \in [0, T]$, where $\|\bar{\sigma}_t^i\| = \sqrt{\sum_{j=1}^{k} (\bar{\sigma}_t^{ij})^2}$. Then the upper hedging price can be expressed as

$$C^*(f_T, P) = \sup_{\frac{\bar{\mu}^i}{\|\bar{\sigma}^i\|}, i=\overline{n+1,k}} \mathbb{E}^P \left[\exp\left\{ -\sum_{i=1}^{k} \int_0^T \frac{\bar{\mu}_t^i}{\|\bar{\sigma}_t^i\|} d\bar{W}_t - \frac{1}{2} \sum_{i=1}^{k} \int_0^T \left(\frac{\bar{\mu}_t^i}{\|\bar{\sigma}_t^i\|} \right)^2 dt \right\} f_T(\bar{W}) \right]. \tag{4.2}$$

Moreover, for each orthogonal completion, there is a corresponding "completed" market for which volatility matrix $\tilde{\Sigma}$ has a proper square shape and system (1.4) produces a unique solution. Which means there should exist unique equivalent local martingale measure, parametrized with the help of solution $\{\theta_t\}_{t \geq 0}$. As each "completed" volatility matrix corresponds to particular market completion, there is a one-to-one correspondence between the set of ELMM for initial incomplete model and set of orthogonal completions.

In other words, for completed matrix (2.2) there is a unique solution for

$$\tilde{\Sigma} \cdot \tilde{\theta} = \bar{\mu} - \bar{1}_k r \tag{4.3}$$

where $\bar{\mu} = \begin{bmatrix} \mu \\ a \end{bmatrix}$ - column vector of shift coefficients for original and completing assets. This solution can be obtained similarly to (1.7) as

$$\tilde{\theta}_t = \tilde{\Sigma}_t^T \cdot \left(\tilde{\Sigma}_t \tilde{\Sigma}_t^T \right)^{-1} (\tilde{\mu}_t - r\bar{1}_k) = \tag{4.4}$$

$$= \theta_t + \nu_t \tag{4.5}$$

where

$$\nu_t = \rho_t^T \cdot (a_t - \bar{1}_k r)$$

and $\theta_t^T \cdot \nu = 0$ by definition of orthogonal completion. Assuming that

$$\int_0^T \|\nu_t\|^2 dt < \infty, \quad \text{(a.s.)}$$

one may define corresponding martingale measure density as

$$Z_t^C = \exp\left\{ - \int_0^t (\theta_s^T + \nu_s^T) \cdot dW_s - \frac{1}{2} \int_0^t (\|\theta_s\|^2 + \|\nu_s\|^2) ds \right\} = \tag{4.6}$$

$$= Z_t^A \times Z_t^\nu \tag{4.7}$$

Using decomposition (4.4) implies that

$$Z_t^C = \widehat{Z}_t^{\widehat{\theta}} \times \widehat{Z}_t^{\widehat{\nu}} \tag{4.8}$$

Corollary 4.2 *If contingent claim depends only on the first n components of innovative Brownian motion, then its price could be uniquely defined on incomplete market.*

In other words, if payoff from contingent claim only depends on the existing assets, then it will be replicable on incomplete market and will have a unique risk-neutral price.

Remark 4.3 *It is possible to further decompose expression (4.8) with respect to particular asset. Without loss of generality, assume that investor focuses on the asset* $\tilde{S}^1_t$*. Dynamics of the price of this asset is described by the following equation:*

$$d\tilde{S}^1_t = \tilde{S}^1_t \left(\tilde{\mu}^1_t dt + \|\bar{\sigma}^1_t\| d\widehat{W}^1_t \right), \tag{4.9}$$

where $\widehat{W}^i_t = \sum_{j=1}^{k} \frac{\bar{\sigma}^{ij}_t}{\|\bar{\sigma}^i_t\|} dW^j_t$.

Consequently, in case of constant coefficients $\tilde{\mu}^1$ *and* $\bar{\sigma}^1$ *one can obtain the following solution*

$$\tilde{S}^1_t = \tilde{S}^1_0 \cdot \exp\left(\left(\tilde{\mu}^1 - \frac{\|\bar{\sigma}^1\|^2}{2} \right) t + \|\bar{\sigma}^1\| \widehat{W}^1_t \right) \tag{4.10}$$

Solving (1.4) in this case, it is clear that $\widehat{\theta}^1 = \frac{\tilde{\mu}^1 - r}{\|\bar{\sigma}^1\|}$*. Then*

$$\tilde{S}^1_t = \tilde{S}^1_0 \cdot \exp\left(\left(r - \frac{\|\bar{\sigma}^1\|^2}{2} \right) t + \|\bar{\sigma}^1\| \widehat{W}^{1\widehat{\theta}}_t \right) \tag{4.11}$$

where $\widehat{W}^{1\widehat{\theta}}_t = \widehat{W}^1_t + \widehat{\theta}^1 t$ *is a Brownian motion under the risk-neutral measure with density* Z^C_t*. Consequently,*

$$(Z^C_t)^{-1} = \frac{dP}{dP^{\widehat{\theta}}} = (Z^{\widehat{\theta}}_t \times Z^{\widehat{\nu}}_t)^{-1} = (Z^{\widehat{\nu}}_t)^{-1} \times \exp\left(\sum_{j=1}^{n} \widehat{\theta}^j \widehat{W}^{j\widehat{\theta}}_t - \frac{1}{2} \sum_{j=1}^{n} (\widehat{\theta}^j)^2 t \right) = \tag{4.12}$$

$$= (Z^{\widehat{\nu}}_t \times Z^{\widehat{\theta}2+}_t)^{-1} \times \exp\left(\widehat{\theta}^1 \widehat{W}^{1\widehat{\theta}}_t - \frac{1}{2} (\widehat{\theta}^1)^2 t \right) = \tag{4.13}$$

$$= (Z^{\widehat{\nu}}_t \times Z^{\widehat{\theta}2+}_t)^{-1} \times \exp\left(\frac{\widehat{\theta}^1}{\|\bar{\sigma}^1\|} \left(\|\bar{\sigma}^1\| \widehat{W}^{1\widehat{\theta}}_t - \frac{1}{2} \tilde{\mu}^1 t + \frac{1}{2} r t \right) \right) = \tag{4.14}$$

$$= (Z^{\widehat{\nu}}_t \times Z^{\widehat{\theta}2+}_t)^{-1} \times \left(\tilde{S}^1_t \right)^{\frac{\widehat{\theta}^1}{\|\bar{\sigma}^1\|}} \exp\left(\frac{\widehat{\theta}^1}{\|\bar{\sigma}^1\|} \left(-\ln \tilde{S}^1_0 - \frac{1}{2} \left(\tilde{\mu}^1 + r + \frac{\|\bar{\sigma}^1\|}{2} \right) t \right) \right) = \tag{4.15}$$

$$= (Z^{\widehat{\nu}}_t \times Z^{\widehat{\theta}2+}_t)^{-1} \times \Psi(t) \times \left(\tilde{S}^1_t \right)^{\frac{\widehat{\theta}^1}{\|\bar{\sigma}^1\|}} \tag{4.16}$$

4.3 Change of numeraire approach

In line with Equivalent Martingale Measure approach, it is worth mentioning also so-called change of numeraire pricing approach. Here we provide review of this approach for the completeness. Connection between change of numeraire approach and method of market completions was described in Guilan 1999, we briefly provide main steps below for informational purposes and to complete overview of the method of Market Completions.

According to this approach, instead of trying to "re-weight" probability of events by choosing some risk-neutral measure, one is searching for special portfolio that could be used as discounting factor instead of classical bank account. However, the choice criteria for such discounting portfolio stays the same – discounted strategy prices should be martingales.

More formally, the main goal is to find portfolio, which value process X_t is a strictly positive, continuous Ito process such that:

$$dX_t = X_t(r_t dt + \pi_t^* \sigma_t (dW_t + u_t dt)). \tag{4.17}$$

Remark: *Here we will intentionally use notation u instead of θ just to distinguish approaches. However, they both represents same idea of price of the risk.*

We want to use this portfolio as numeraire, such that risk-premiums with respect to this numeraire are constrained to be equal 0. In other words, price process, discounted by mentioned portfolio will be local martingale w.r.t. "objective" probability P.

Theorem 4.4 *Let $\alpha_t = \left(\sigma_t \sigma_t^T\right)^{-1}(\mu_t - r_t 1)$, i.e. $u_t = \sigma_t^T \cdot \alpha_t$. Consider the self-financing strategy $\pi_t = (\alpha_t^i)_{i=1}^n$ in the risky-assets. Denote by M_t the present value of this admissible strategy. Then M_t satisfies SDE:*

$$dM_t = M_t(r_t dt + (u_t)^T (dW_t + u_t dt)) = M_t(r_t dt + \|u_t\|^2 dt + (u_t)^T dW_t) \tag{4.18}$$

In the market with M_t as numeraire, investors are risk-neutral. M-price process $S_t^M = \frac{S_t}{M_t}$ of any asset S_t is a local martingale. We refer to it as a market numeraire.

Proposition 4.5 *If m is a strategy that corresponds to M_t, then:*

- *m maximizes the expected logarithm of terminal wealth*
- *m is unique even in incomplete market*
- *m maximizes the expected growth rate*

Details about mentioned properties can be found in Bajeux-Besnainou and Portait 1997.

Price of European contingent claim f_T on the complete market, according to market numeraire approach could be found as:

$$V_0 = \mathbb{E}^P \left(\frac{f_T}{M_T} \right). \tag{4.19}$$

Working with incomplete market case, it is obvious that *there are several risk-neutral prices* as we can find several α_t that fit conditions of Theorem 4.4. So let's apply market completions approach and show that it can be used for estimation of option price boundaries on incomplete market.

Let's consider some market completion S^c. Then coefficients of these fictitious assets satisfy

$$\det(\sigma(\rho)) = \det \begin{pmatrix} \sigma_t \\ \rho_t \end{pmatrix} \neq 0 \quad \text{and} \quad u(\rho, a, t) = \begin{pmatrix} \sigma_t \\ \rho_t \end{pmatrix}^{-1} \begin{pmatrix} b_t - r_t I_n \\ a_t - r_t I_{k-n} \end{pmatrix} \tag{4.20}$$

with

$$\int_0^T \|u(\rho, a, t)\|^2 dt < \infty, \quad P - a.s. \tag{4.21}$$

On completed market one can define market numeraire as in (4.18):

$$dM(\rho, a, t) = M(\rho, a, t)(r_t dt + \|u(\rho, a, t)\|^2 dt + (u(\rho, a, t))^T dW_t) \tag{4.22}$$

In the completed market we have the fair price of CC f_T calculated similar to (4.19):

$$V_0(\rho, a) = \mathbb{E}^P \left(\frac{f_T}{M(\rho, a, T)} \right). \tag{4.23}$$

Let

$$V_1(\rho) = \inf_{a\in\mathcal{D}_\rho} V_0(\rho, a), \quad V_2(\rho) = \sup_{a\in\mathcal{D}_\rho} V_0(\rho, a) \tag{4.24}$$

where $\mathcal{D}_\rho = \{a : \mathbb{R}^{k-n}$-valued progressively measurable processes such that $\int_0^T \|u(\rho, a, t)\|^2 dt < \infty$ a.s.$\}$. According to Guilan 1999, the following holds

Proposition 4.6 *$V_1(\rho)$ and $V_2(\rho)$ are independent of ρ*

Proposition 4.6 serves as another proof that it is enough to work with orthogonal completions only. Let's pick the orthogonal completion $\sigma\rho^T = 0$, $\rho\rho^T = I$. For such ρ and $a \in \mathcal{D}_\rho$:

$$u(\rho, a) = \begin{pmatrix} \sigma \\ \rho \end{pmatrix}^{-1} \begin{pmatrix} \mu - rI_n \\ a - rI_{k-n} \end{pmatrix} = \sigma^T(\sigma\sigma^T)^{-1}(\mu - rI_n) + \rho^T(a - rI_{k-n}) = u + \psi = u_\psi. \tag{4.25}$$

And this u_ψ would be used for construction of market numeraire. Also, it follows that:

$$\sigma\psi = 0, \tag{4.26}$$

and

$$a = \rho\psi + rI_{k-n} \tag{4.27}$$

Which means that "non-arbitrage" vector on completed market can be decomposed into u from incomplete source market and ψ which is completion-dependent. If we define class $K(\sigma) = \{\psi : \psi$ is $\mathbb{R}^k$-valued progressively measurable, $\sigma_t\psi_t = 0, \forall t \in [0, T], a.s.$ and $\int_0^T \|\psi_t\|^2 dt < \infty, a.s.\}$, then this class will be a parameter space for fictitious completions of the incomplete market. For each $\psi \in K(\sigma)$ one can find a fair price in a completed market. It implies that option price boundaries will be

$$J(t) = \sup_{\psi\in K(\sigma)} \mathbb{E}\left[M_\psi(t)\frac{f_T}{M_\psi(T)}|\mathcal{F}_t\right] \quad \text{or} \quad \inf_{\psi\in K(\sigma)} \mathbb{E}\left[M_\psi(t)\frac{f_T}{M_\psi(T)}|\mathcal{F}_t\right] \tag{4.28}$$

Noting results of Guilan 1999, it is possible to show that these price boundaries coincide with boundaries from classical approach:

$$V(t) = \sup_{\tilde{P}\in\mathcal{M}} \mathbb{E}^{\tilde{P}}\left[B_t \frac{f_T}{B_T}|\mathcal{F}_t\right] \quad \text{or} \quad \inf_{\tilde{P}\in\mathcal{M}} \mathbb{E}^{\tilde{P}}\left[B_t \frac{f_T}{B_T}|\mathcal{F}_t\right] \tag{4.29}$$

4.4 Utility Based Indifference Pricing

As it was mentioned in previous sections, classical approach for pricing contingent claims in a field of mathematical finance is to find a portfolio, which terminal capital replicates (a.s.) such contingent claim and use its initial value as the price for contingent claim. However, in incomplete markets, such replication often troublesome or might not be possible at all. Holding option in incomplete market is unavoidably a risky business, which means it makes sense to consider investors attitude towards risk and consider some pricing methodology which does not require replication of contingent claim.

One of such pricing methodologies was introduced in Davis 1997 and further developed in, for example, Karatzas *et al.* 1991 and Karatzas and Kou 1996. This approach consists in embedding option pricing problem into Utility Maximization task where utility function reflects investor attitude towards risk.

Let us assume that investors appetite is described with concave, non-decreasing utility function U with $U \in C^2$ on $\mathbb{R}^+$ with $U' > 0$, $\lim_{x\to 0} U'(x) = \infty$ and $\lim_{x\to\infty} U'(x) = 0$. Having an initial capital x he forms a portfolio π which terminal value is $V_T^\pi(x)$. His objective is to maximize expected utility of terminal wealth:

$$\zeta(x) = \sup_{\pi\in\mathcal{A}(x)} \mathbb{E}\left[U(V_T^\pi(x))\right] \tag{4.30}$$

In order to get initial price of any financial claim, we construct special function in the following way: assume capital δ is diverted towards purchase of claim with payoff

g_T and current price p, then maximal expected utility can be defined as:

$$w(\delta, x, p) = \sup_{\pi \in \mathcal{A}(x)} \mathbb{E}\left[U\left(V_T^{\pi}(x-\delta) + \frac{\delta}{p} g_T\right)\right] \tag{4.31}$$

Definition 4.7 *Suppose that for each (x, p) the function $\delta \mapsto w(\delta, x, p)$ is differentiable at $\delta = 0$ and there is a unique solution $\hat{p}(x)$ of the equation*

$$\frac{\partial w}{\partial \delta}(0, x, p) = 0 \tag{4.32}$$

then $\hat{p}(x)$ is a fair option price at time 0.

It was demonstrated in both Davis 1997 and Karatzas and Kou 1996, that fair price then can be obtained as:

Theorem 4.8 *Suppose that $\zeta(x)$ is differentiable at each $x \in \mathbb{R}^+$ and that $\zeta'(x) > 0$. Then the fair price $\hat{p}(x)$ from Definition 4.7 is given by*

$$\hat{p}(x) = \frac{\mathbb{E}\left[U'\left(V_T^{\pi^*}(x)\right) g_T\right]}{\zeta'(x)} \tag{4.33}$$

From Section 3.2, we know that $V_T^{\pi^*}(x) = \xi_0(x) = I\left(\mathcal{Y}_0(x)\beta_T Z_T^A\right)$. Therefore, $\zeta(x) = \mathbb{E}\left[U(I\left(\mathcal{Y}_0(x)\beta_T Z_T^A\right))\right]$ and $\zeta'(x) = \mathcal{Y}_0(x)$. Similarly, on incomplete market we obtain

$$\zeta_\lambda(x) = \mathbb{E}\left[U\left(I\left(\mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\lambda\right)\right)\right], \tag{4.34}$$

$$\zeta_\lambda'(x) = \mathcal{Y}_\lambda(x). \tag{4.35}$$

Then, following Karatzas and Kou 1996, initial UBIP price of the claim with payoff g_T can be found with the help of the following theorem.

Theorem 4.9 *For all $x > 0$, the UBIP price $\hat{p}(x)$ can be found as*

$$\hat{p}(x) = \mathbb{E}[\beta_T Z_T^A Z_T^\lambda g_T] \tag{4.36}$$

Proof. We provide sketch of the proof. One should use (4.33).

$$\begin{aligned}\mathbb{E}\left[U'\left(V_T^{\pi^*}(x)\right)g_T\right] &= \mathbb{E}\left[U'\left(I\left(\mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\lambda\right)\right)g_T\right] = \\ &= \mathbb{E}\left[\mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\lambda g_T\right] = \\ &= \zeta'_\lambda(x)\mathbb{E}\left[\beta_T Z_T^A Z_T^\lambda g_T\right].\end{aligned}$$

The rest follows from (4.33). ■

It was shown in Karatzas and Kou 1996, that with mild restrictions on concave utility function, if market is structurally incomplete, then, for deterministic coefficients, optimal completion λ will be found from

$$\lambda = \arg\min_{y\in\ker\Sigma}\|\theta(t)+y\|^2 \tag{4.37}$$

and will be independent from both initial budget of investor and particular utility function.

Remark 4.10 *It is straightforward to notice that completion λ found from* (4.37) *will correspond to $\lambda = 0$ and to Equivalent minimal martingale measure.*

Proof. Follows directly from definition 2.22. ■

Moreover, noticing connection with Minimal martingale measure approach, we can summarize obtained results in the following proposition.

Proposition 4.11 *If market model* (8.1) *is structurally incomplete, then UBIP price will be determined as an expectation under equivalent minimal martingale measure.*

Proof. Follows from Theorem 4.9, (4.37) and Remark 4.10. ■

To better demonstrate how method of market completions can be used for UBIP methodology, we provide the following example of implementation in case of logarithmic utility function.

Example 4.12 *Consider utility function $U(x) = \ln(x)$ for which:*

$$\mathcal{X}_\nu(y) = \frac{1}{y}, \quad \mathcal{Y}_\nu(x) = \frac{1}{x} \tag{4.38}$$

and optimal terminal capital can be calculated as

$$\xi^x_\nu = \frac{x}{\beta_T Z^\nu_T}. \tag{4.39}$$

One could check that completion with parameter $\lambda = 0$ satisfies [5] in Theorem 3.7.

$$E[\beta_T Z^\nu_T \xi^x_0] = x \cdot E\left[\exp\left\{-\int_0^T \nu_s^T dW_s - \frac{1}{2}\int_0^T \|\nu_s\|^2 ds\right\}\right] \leq x \quad \forall \nu \in K(\Sigma) \tag{4.40}$$

as the process under expectation is a supermartingale. It means that investor would not use auxiliary stocks ($\lambda = 0$) to form an optimal portfolio even for hedging purposes.

Note also, that $\lambda = 0$ corresponds to the version of the market for which equivalent local martingale measure will be minimal martingale measure.

Chapter 5

Application of Method of Market Completions (MMC) to quantile hedging

5.1 Problem of Quantile Hedging

In previous chapters we demonstrated how method of market completions can be applied to classical problems of mathematical finance that

Knowing the initial price g_0 of a perfect hedging strategy for a contingent claim with payoff $g_T = g_T(\omega)$ it is often the case that investor is unwilling, or unable to invest such amount of money. Then, the question is: What will be the best hedging portfolio the investor can achieve given initial budget constraint $v_0 < g_0$?

As long as initial budget is not enough for the perfect hedging strategy, there emerges a probability of *shortfall*. We assume that the goal of investor is to maximize the probability of successful hedging – or, in other words, minimize the probability of a shortfall. Or in other words, having capital v_0, what would be the best portfolio, such that the hedge is successful with maximal possible probability? In this section we will demonstrate how to construct strategy which maximizes the probability of a successful hedge under the observed measure P given some budget restriction in place. We follow approach from Föllmer and Leukert 1999 for complete market case and further show how method of market completions is useful for solving this problem in case of incomplete market.

We will call the following set

$$\{V_T^\pi(v_0) \geq g_T\}$$

a "successful hedging set" that corresponds to the strategy (v_0, π). So the problem of quantile hedging becomes

$$\begin{cases} \max\limits_{\pi \in \mathcal{A}(x)} P\left(V_T^\pi(x) \geq g_T\right) \\ s.t. \quad \Pi\left(V_T^\pi(x)\right) \leq v_0 \end{cases} \tag{5.1}$$

In Föllmer and Leukert 1999 it was demonstrated that this problem can be solved

Proposition 5.1 *Let $\tilde{A} \in \mathcal{F}_T$ be a solution of the problem*

$$\begin{cases} \max\limits_{\pi \in \mathcal{A}(x)} P\left(A\right) \\ s.t. \quad \Pi\left(g_T 1_A\right) \leq v_0. \end{cases} \tag{5.2}$$

Then perfect hedge $\tilde{\pi}$ for the knockout option $\tilde{g} = g 1_{\tilde{A}}$ solves the optimization problem (5.1) *and the corresponding successful hedging set coincides almost surely with $\tilde{A}$.*

5.2 Complete Market Case

The problem of construction of the maximum probability successful hedging set in (5.2) can be solved by the means of famous Neyman-Pearson lemma for hypothesis testing task.

Assuming that $\varphi = 1_A$ for successful hedging set A in a problem (5.2). Then this problem can be re-written in the following form

$$\begin{cases} \max\limits_{\pi \in \mathcal{A}(x)} E\left[\varphi\right] \\ s.t. \quad E^*\left[\varphi g_T\right] \leq v_0. \end{cases} \tag{5.3}$$

Introducing auxiliary measure Q^* as $\frac{dQ^*}{dP^*} = \frac{g_T}{E^*[g_T]}$ and the problem becomes

$$\begin{cases} \max\limits_{\pi \in \mathcal{A}(x)} E\left[\varphi\right] \\ s.t. \quad E^{Q^*}\left[\varphi\right] \leq \frac{v_0}{E^*[g_T]} = \alpha. \end{cases}$$

which, according to Neyman-Pearson lemma, has a solution with $G = 1$ and $H = \frac{dQ^*}{dP} = \frac{g_T}{E^*[g_T]} \times \frac{dP^*}{dP}$ as

$$\hat{\varphi} = 1_{\hat{z}g_T < \frac{dP}{dP^*}} + b \cdot 1_{\hat{z}g_T = \frac{dP}{dP^*}} \tag{5.4}$$

where

$$\hat{z} = \inf\left\{u \geq 0; Q^*\left(ug_T < \frac{dP}{dP^*}\right) \leq \alpha\right\}$$
$$b = \frac{\alpha - Q^*\left(\hat{z}g_T < \frac{dP}{dP^*}\right)}{Q^*\left(\hat{z}g_T = \frac{dP}{dP^*}\right)}.$$

Theorem 5.2 (from Föllmer and Leukert 1999) *Let π^* denote the perfect hedge for the modified contingent claim $\tilde{g} = \tilde{\varphi}g$ where*

$$\tilde{\varphi} = 1_{\hat{z}g_T < \frac{dP}{dP^*}} + b \cdot 1_{\hat{z}g_T = \frac{dP}{dP^*}}, \tag{5.5}$$

as in (5.4). *Then*

1. *π^* maximizes the expected success ratio $\mathbb{E}[\varphi]$ under all admissible strategies $\pi \in \mathcal{A}(v_0)$ with $v_0 < g_0$*

2. *The corresponding success ratio is given by $\tilde{\varphi}$*

5.3 Incomplete Market Case

Assume now we are in a situation of incomplete market. Since in this case equivalent martingale measure is not unique anymore, the problem of quantile hedging (5.3) will be written in the following form

$$\begin{cases} \max\limits_{\pi \in \mathcal{A}(x)} E\left[\varphi\right] \\ s.t. \quad \sup\limits_{P^* \in \mathcal{P}} E^*\left[\varphi g_T\right] \leq v_0. \end{cases} \tag{5.6}$$

In order to find solution to this problem in incomplete market, we would also use the Method of Market Completions. First of all, we introduce required amount of auxiliary assets

$$dS_t^i = S_t^i \left(a_t^i dt + \sum_{j=1}^{k} \rho_t^{ij} dW_t^j \right), \quad i = n+1, ..., k.$$

As was demonstrated in Section 2.2, each completed version of the market can be parametrized by the corresponding vector ν. Therefore, using such parametrization, one gets the following sub-problem on each of completed markets:

$$\begin{cases} \max\limits_{\pi \in \mathcal{A}(x)} E\left[\varphi\right] \\ s.t. \quad E^{\nu}\left[\varphi g_T\right] \leq v_0. \end{cases} \tag{5.7}$$

Further, on each completed market, we define an equivalent measure Q^{ν}

$$\frac{dQ^{\nu}}{dP^{\nu}} = \frac{g}{\mathbb{E}^{\nu}[g]}. \tag{5.8}$$

With the help of this auxiliary measure the subproblem becomes

$$\begin{cases} \max\limits_{\pi \in \mathcal{A}(x)} E\left[\varphi\right] \\ s.t. \quad E^{\nu}\left[\varphi\right] \leq \frac{v_0}{\mathbb{E}^{\nu}[g]} = \alpha^{\nu}. \end{cases} \tag{5.9}$$

It is straight forward to see that finding solution to this problem will be the most powerful randomized test which produces error of first type no more than α^{ν}. Such problem can be solved with the help of Neyman-Pearson lemma.

Proposition 5.3 *Solution to the problem* (5.9) *exists and takes the form*

$$\hat{\varphi}^{\nu} = 1_{\hat{z}\frac{1}{\mathbb{E}^{\nu}[g_T]}g_T < \frac{dP}{dP^{\nu}}} + b \cdot 1_{\hat{z}\frac{1}{\mathbb{E}^{\nu}[g_T]}g_T = \frac{dP}{dP^{n}u}} = \tag{5.10}$$

$$= 1_{\hat{z}^{\nu} g_T < \frac{dP}{dP^{\nu}}} + b \cdot 1_{\hat{z}^{\nu} g_T = \frac{dP}{dP^{\nu}}} \tag{5.11}$$

where

$$\hat{z}^{\nu} = \inf \left\{ u \geq 0; Q\left(u g_T < \frac{dP}{dP^{\nu}} \right) \leq \alpha \right\} \tag{5.12}$$

$$b = \frac{\alpha - Q^{\nu}\left(\hat{z}^{\nu} g_T < \frac{dP}{dP^{\nu}}\right)}{Q^{\nu}\left(\hat{z}^{\nu} g_T = \frac{dP}{dP^{\nu}}\right)}. \tag{5.13}$$

As a result, optimal strategy, that would maximize the probability of successful hedging on the completed market will be to replicate $\tilde{g}^{\nu} = \hat{\varphi}^{\nu} g$. Obviously, for any strategy with terminal capital V_T^{π}, constructed from assets, available on incomplete market.

Proposition 5.4 *Successful hedging ratio will be not greater than for any completed version of the market:*

$$\sup_{\pi \in \mathcal{A}(x)} \mathbb{E}[\varphi^{\pi}] \leq \mathbb{E}[\hat{\varphi}^{\nu}], \quad \forall \nu \in \ker(\Sigma), \tag{5.14}$$

where φ^{π} is a successful hedge ratio of strategy π, defined as

$$\varphi^{\pi} = 1_{V_T^{\pi} \geq g_T} + \frac{V_T^{\pi}}{g_T} 1_{V_T^{\pi} < g_T}, \ \pi \in \mathcal{A}(x). \tag{5.15}$$

Proof. Notice, that each φ represents indicator function for successful hedging set. Assume that there exists a parameter $\tilde{\nu} \in \ker \Sigma$ for which $\sup_{\pi \in \mathcal{A}(x)} E[\varphi^{\pi}] > \mathbb{E}[\hat{\varphi}^{\nu}]$. In other words, on completed market maximal corresponding probability of successful hedge will be $\mathbb{E}[\hat{\varphi}^{\nu}]$, however, it contradicts the fact that existing assets are available on any completed market and therefore investor will always be able to achieve probability of successful hedging at least $\sup_{\pi \in \mathcal{A}(x)} E[\varphi^{\pi}]$ if he decides not to use any completing assets at all. ∎

Consequently, if optimal claim from some completed market version $\hat{\varphi}^{\lambda} g$ will be financeable on initial incomplete market, then it would also be a solution to quantile hedging problem (5.6). With the help of method of market completions we can develop a criteria to choose particular completion λ as in Theorem 3.7

Theorem 5.5 *If for a $\mathcal{F}_T$-measurable contingent claim g_T on incomplete market there exists a completion $\lambda \in \ker(\Sigma)$ such that for modified contingent claim $\hat{\varphi}^{\lambda} g_T$ holds*

$$\mathbb{E}\left[\beta_T Z_T^A Z_T^{\nu} \hat{\varphi}^{\lambda} g_T\right] \leq v_0 = \mathbb{E}\left[\beta_T Z_T^A Z_T^{\lambda} \hat{\varphi}^{\lambda} g_T\right], \tag{5.16}$$

where $\hat{\varphi}^{\lambda}$ is defined in (5.10).

Then there exists portfolio $\pi \in \mathcal{A}(v_0)$ which is a perfect hedge for $\hat{\varphi}^{\lambda} g_T$ and solves quantile hedging problem (5.6) on incomplete market.

Proof. To prove it is enough to use Theorem 3.6 taking $B = \hat{\varphi}^{\lambda} g_T$ together with Proposition 5.4. ■

Remark 5.6 *It is also possible to pick particular market completion based on some criteria. For instance, one could choose a completion that corresponds to market version, for which equivalent local martingale measure will coincide with minimal martingale measure, that was introduced in Section 1.7.*

Let us demonstrate how it can be done. Following results from Theorem 2.6, we remind that vector $\nu^{min} \in \ker(\Sigma)$ that corresponds to the completed market on which equivalent martingale measure will coincide with minimal martingale measure could be written as

$$\nu^{min} = \begin{bmatrix} \widehat{\theta} \\ 0_{k-n} \end{bmatrix}. \tag{5.17}$$

Consequently, picking this version of completed market based on locally risk minimization criteria, one solves the problem (5.7) for $\nu = \underline{\nu} := \nu^{min}$. It is clear, that

$$Z_T^{\underline{\nu}} = 1 \tag{5.18}$$

and, consequently,

$$Z_T^A Z_T^{\underline{\nu}} = Z_T^A. \tag{5.19}$$

Therefore, corresponding solution will be a perfect hedge for modified claim $\varphi^{min} g_T$, where

$$\varphi^{min} = 1_{\hat{z}^{\underline{\nu}} g_T < \frac{dP}{dP^{\underline{\nu}}}} + b \cdot 1_{\hat{z}^{\underline{\nu}} g_T = \frac{dP}{dP^{\underline{\nu}}}} = \tag{5.20}$$

$$1_{\hat{z}^{\underline{\nu}} g_T < (Z_T^A)^{-1}} + b \cdot 1_{\hat{z}^{\underline{\nu}} g_T = (Z_T^A)^{-1}} \tag{5.21}$$

and

$$\hat{z}^{\nu} = \inf\left\{u \geq 0; Q\left(ug_T < (Z_T^A)^{-1}\right) \leq \alpha\right\} \tag{5.22}$$

$$b = \frac{\alpha - Q\left(\hat{z}^{\nu} g_T < (Z_T^A)^{-1}\right)}{Q\left(\hat{z}^{\nu} g_T = (Z_T^A)^{-1}\right)}. \tag{5.23}$$

Summarizing obtained results we conclude that method of market completions offers convenient tool for solving the problem of Quantile Hedging on incomplete market. As it was demonstrated in Theorem 5.5, in order to find solution, one should find a completion parameter $\lambda \in \ker \Sigma$ satisfying parsimony condition. Assembling together all provided facts, we can conclude this section with the following result.

Theorem 5.7 *Solution to quantile hedging problem* (5.6) *on structurally incomplete market with deterministic coefficients will be a perfect hedge for modified contingent claim* $\varphi^{min} g_T$*, where*

$$\varphi^{min} = 1_{\hat{z}^{\nu} g_T < (Z_T^A)^{-1}} + b \cdot 1_{\hat{z}^{\nu} g_T = (Z_T^A)^{-1}} \tag{5.24}$$

$$\hat{z}^{\nu} = \inf\left\{u \geq 0; Q\left(ug_T < (Z_T^A)^{-1}\right) \leq \alpha\right\} \tag{5.25}$$

$$b = \frac{\alpha - Q\left(\hat{z}^{\nu} g_T < (Z_T^A)^{-1}\right)}{Q\left(\hat{z}^{\nu} g_T = (Z_T^A)^{-1}\right)}. \tag{5.26}$$

Proof. It is clear from Remark 5.6 and (5.7) that modified claim $\varphi^{min} g_T$ maximizes the probability of successful hedge for completed version of the market corresponding to parameter $\lambda = 0$. Therefore, based on Theorem 5.5, to prove the statement of the theorem it is enough to show that modified claim $\varphi^{min} g_T$ satisfy condition (5.16) which would imply existence of the hedging portfolio for modified claim. Optimality of such portfolio is a consequence from Proposition 5.4.

From continuity of Z_T^A and (5.25) it follows that $\mathbb{E}\left[\beta_T Z_T^A \varphi^{min} g_T\right] = v_0$. Then, for auxiliary market completion with corresponding parameter $\nu \in \ker \Sigma$, it is fair to write that

$$\mathbb{E}\left[\beta_T Z_T^A Z_T^{\nu} \varphi^{min} g_T\right] = v_0 \cdot \mathbb{E}\left[\exp\left\{-\int_0^T \nu_s^* dW_s - \frac{1}{2}\int_0^T \|\nu_s\|^2 ds\right\}\right] \leq v_0 \tag{5.27}$$

since $\exp\left\{-\int\limits_0^t \nu_s^* dW_s - \frac{1}{2}\int\limits_0^t \|\nu_s\|^2 ds\right\}$ is non-negative local martingale and hence supermartingale.

From (5.27) follows (5.16). ■

5.4 Numerical Example

Example 5.8 *Consider the market introduced in Example 2.7. Following steps described above, for incomplete market, let us assume several completions on the market, corresponding to parameters ν. For demonstration purposes we will consider small amount of possible market completions, parametrized by vector ν following Example 2.7. For each of completions we calculate initial price on completed market and optimal $\hat{z}^\nu$ parameter from* (5.12)*:*

Market Completion Parameter ID	ν_3 *Value*	$\hat{z}$	*Risk-Neutral Price, Completed Market*
Z_0	*0.0*	*0.004773*	*14.5429*
Z_1	*0.1*	*0.004733*	*14.5353*
Z_2	*0.2*	*0.004463*	*14.5251*
Z_3	*0.3*	*0.004096*	*14.5122*
Z_4	*0.4*	*0.003620*	*14.4963*
Z_5	*0.5*	*0.003138*	*14.4771*
Z_6	*0.6*	*0.002657*	*14.4543*
Z_7	*0.7*	*0.002189*	*14.4276*
Z_8	*0.8*	*0.001823*	*14.3965*
Z_9	*0.9*	*0.001460*	*14.3607*

Let us use initial budget 13.7512 which is less than capital required for perfect hedging on each of completed market versions. Using obtained parameters, we are able to construct modified claims according to (5.10)*. According to Theorem 5.2, strategy, solving quantile hedging problem will be to perfectly hedge corresponding modified* *claims. Probabilities of successful hedging for all mentioned completed markets are presented on the Figure 5.1 below.*

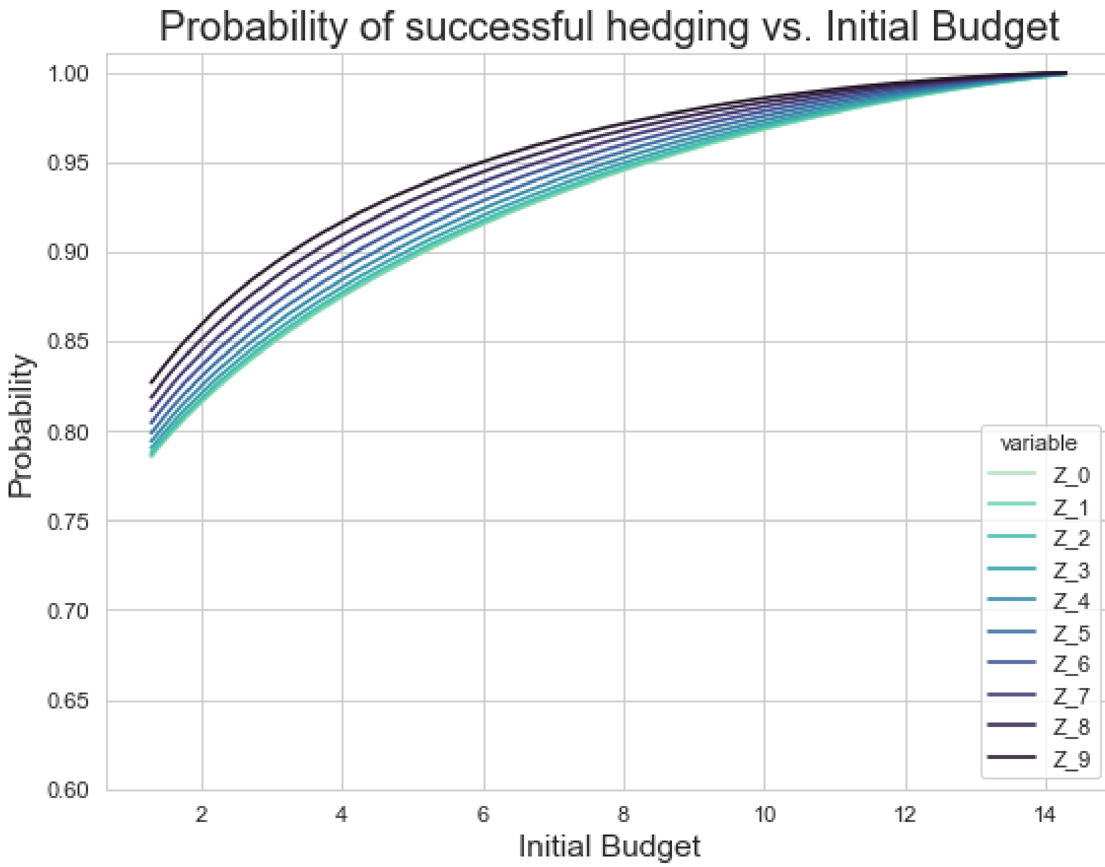

Figure 5.1: Quantile Hedging: Probability of successful hedging vs. Initial budget constraint by Market Completions

It is also worth highlighting, that completion 'Z_0' corresponds to the minimal martingale measure. In the presented example, using this measure would also mean considering the "worst case scenario" since solution on completed market that corresponds to 'Z_0' gives lowest probability of successful hedging across completions. This is an expected behaviour since 'Z_0' corresponds to minimal martingale measure with least favorable solution.

Therefore, picking 'Z_0' as a completed version of market and assuming $P(\hat{z}^{\nu} g_T = Z_T^A) = 0$, *we arrive at conclusion, that solution to quantile hedging problem on in-*

complete market would be to perfectly hedge modified claim, defines as:

$$\tilde{g} = g_T \cdot 1_{\left\{0.004773 \cdot g_T < (Z_T^A)^{-1}\right\}}$$

Chapter 6

Application of MMC to efficient hedging

6.1 Problem of Efficient Hedging

Another important step in developing optimal partial hedging strategy is to move on from analyzing the probability of shortfall to working with actual size of possible expected loss. Having positive probability of future losses from hedging due to insufficient capital to fund a perfect hedging, or super-hedging strategy investor may now assess impact of coming shortfall with the help of some loss function $l(x)$. Such loss function should reflect that investor may be tolerant to some small losses and would want to avoid large ones. That is why it is reasonable to assume that loss function is increasing, convex function defined for positive losses from $[0, \infty)$ and $l(0) = 0$.

Then, having some financial obligation $g(\omega)$ in future, investor forms hedging portfolio with capital available. Assuming, naturally, that

$$\mathbb{E}[l(g))] < \infty, \tag{6.1}$$

the *shortfall risk* of investor can be defined as follows

Definition 6.1 *The expectation of shortfall, weighted by loss function l*

$$\mathbb{E}\left[l\left((g - V_T^\pi(x))^+\right)\right] \tag{6.2}$$

is called a shortfall risk.

Then, the goal of efficient hedging is to construct hedging strategy $\hat{\pi}(x)$ such that will minimize shortfall risk and its initial cost will be within investors budget. In other words, the problem can be stated as

$$\begin{cases} \min\limits_{\pi \in \mathcal{A}(x)} \mathbb{E}\left[l\left((g - V_T^{\pi}(x))^{+}\right)\right] \\ s.t. \quad \Pi\left(V_T^{\pi}(x)\right) \leq v_0, \end{cases} \tag{6.3}$$

where $\Pi\left(V_T^{\pi}(x)\right)$ is an initial price of the claim assessed with the help of some pricing functional Π. It was demonstrated in Föllmer and Leukert 2000 that, similarly to quantile hedging, this problem can be reduced to the search of the randomized statistical test with the highest power. Namely, consider the class of *randomized tests*

$$\mathcal{R} = \left\{\varphi : \Omega \to [0,1] | \varphi \text{ is } \mathcal{F}_T \text{ measurable}\right\}.$$

Then, following Föllmer and Leukert 2000, the following result holds

Proposition 6.2 *There exist a solution $\tilde{\varphi} \in \mathcal{R}$ for the problem*

$$\begin{cases} \min\limits_{\varphi \in \mathcal{R}} \mathbb{E}\left[l\left((1-\varphi)g)\right)\right] \\ s.t. \quad \Pi\left(\varphi g\right) \leq v_0, \end{cases} \tag{6.4}$$

and a super-hedging strategy for modified claim $\tilde{g} = \tilde{\varphi} g$ solves the optimization problem (6.3).

In what follows we discuss separately two cases of the efficient hedging problem. Firstly, when loss function is linear, which implies that the problem becomes minimization of expected shortfall. Secondly, for general case of convex loss function.

6.2 Minimizing expected shortfall

One important case of loss function is $l(x) = x$. In other words, when investor estimate losses "as is" no matter how large they are. In such case (6.4) is written as

$$\begin{cases} \min\limits_{\varphi \in \mathcal{R}} \mathbb{E}\left[((1-\varphi)g)\right] \\ s.t. \quad \Pi\left(\varphi g\right) \leq v_0, \end{cases} \equiv \begin{cases} \max\limits_{\varphi \in \mathcal{R}} \mathbb{E}\left[\varphi g\right] \\ s.t. \quad \Pi\left(\varphi g\right) \leq v_0. \end{cases} \tag{6.5}$$

As it was noted in Föllmer and Leukert 2000, the latter can be solved by the means of famous Neyman-Pearson lemma.

6.2.1 Complete Market Case

When market is complete then there exists precisely one, unique equivalent martingale, or risk-neutral market measure. It implies that, under martingale approach the pricing functional in problem (6.3) can be thought of as an expectation under this martingale measure. So, on complete market the problem becomes

$$\begin{cases} \max\limits_{\varphi \in \mathcal{R}} \mathbb{E}\left[\varphi g\right] \\ s.t. \quad \mathbb{E}^*\left(\varphi g\right) \leq v_0, \end{cases} \tag{6.6}$$

where

$$\frac{dP^*}{dP} = \exp\left(\int_0^t \theta_s' dW(s) - \frac{1}{2}\int_0^t \|\theta_s\|^2 ds\right).$$

Introducing

$$\frac{dQ}{dP} = \frac{g}{\mathbb{E}[g]}, \quad \frac{dQ^*}{dP^*} = \frac{g}{\mathbb{E}^*[g]}$$

and from Föllmer and Leukert 2000, the solution can be obtained explicitly as

Proposition 6.3 *The optimal randomized test for the problem* (6.6) $\tilde{\varphi} \in \mathcal{R}$ *is given by*

$$\tilde{\varphi} = 1_{\left\{\frac{dP}{dP^*} > \tilde{a}\right\}} + \gamma 1_{\left\{\frac{dP}{dP^*} = \tilde{a}\right\}} \tag{6.7}$$

where

$$\tilde{a} = \inf\left\{a : \mathbb{E}^*[1_{\left\{\frac{dP}{dP^*} > a\right\}} g] \leq v_0\right\} \tag{6.8}$$

and

$$\gamma = \frac{v_0 - \mathbb{E}^*[1_{\left\{\frac{dP}{dP^*} > \tilde{a}\right\}} g]}{\mathbb{E}^*[1_{\left\{\frac{dP}{dP^*} = \tilde{a}\right\}} g]} \tag{6.9}$$

In other words, when market is complete, optimal hedging strategy can be uniquely determined as a perfect hedge for modified claim $\tilde{\varphi} g$

6.2.2 Incomplete Market Case

Let us now move on to the case when market is no longer complete. Consequently, there are infinitely many equivalent martingale measures and no-arbitrage principle is no longer enough to determine a unique price of contingent claim. So the optimization problem on incomplete market becomes

$$\begin{cases} \max\limits_{\varphi \in \mathcal{R}} \mathbb{E}\left[\varphi g\right] \\ s.t. \quad \sup\limits_{P^* \in \mathcal{P}} \mathbb{E}^*\left[\varphi g\right] \leq v_0, \end{cases} \tag{6.10}$$

where $\mathcal{P}$ is a set of all martingale measures equivalent to "observed" measure P.

The natural way of dealing with such incompleteness is then to use generalized version of Neyman-Pearson lemma. Namely, on incomplete market one would have a simple alternative hypothesis, represented by the "physical" Wiener measure μ and a family of null hypothesis as a family of special equivalent risk-neutral measures.

In order to re-write the problem (6.10) in a form of hypothesis testing, we introduce auxiliary equivalent measures

$$\mathbb{G} = \frac{dP}{d\mu} = \frac{g}{\mathbb{E}[g]}, \quad \mathbb{Q} = \left\{\frac{dQ^\nu}{d\mu}, \nu \in \ker(\Sigma)\right\} = \left\{\frac{g}{\mathbb{E}^\nu[g]} Z_T^A Z_T^\nu, \nu \in \ker(\Sigma)\right\} \tag{6.11}$$

Rigorously speaking, one is working with P as a simple alternative hypothesis and the family of measures $\{\mu_\nu, \nu \in \ker(\Sigma)\}$ given by

$$\frac{d\mu_\nu}{d\mu} = Z_T^A Z_T^\nu = \exp\left(\int_0^t (\theta_s^\nu)' dW(s) - \frac{1}{2}\int_0^t (\|\theta_s\|^2 + \|\nu_s\|^2) ds\right). \tag{6.12}$$

Consequently, associated value function (1.33) will be written as

$$\tilde{V}(z) = \inf_{\nu \in \ker(\Sigma)} E\left[\frac{g}{\mathbb{E}[g]} - z \cdot \frac{g}{\mathbb{E}^\nu[g]} Z_T^A Z_T^\nu\right] = \inf_{\nu \in \ker(\Sigma)} E\left[\frac{g}{\mathbb{E}[g]}\left(1 - z \cdot \frac{\mathbb{E}[g]}{\mathbb{E}^\nu[g]} Z_T^A Z_T^\nu\right)\right] \tag{6.13}$$

Though proposed optimization problem admits a solution, it is not easy to solve it in practice. Therefore, we can apply method of market completions for finding shortfall-minimizing strategy on incomplete market. Similarly to Section 5.3, we parametrize set of completed versions of the market by vectors $\nu \in \ker(\Sigma)$. Then, on each completed market we define sub-problem

$$\begin{cases} \max\limits_{\varphi \in \mathcal{R}} \mathbb{E}\left[\varphi g\right] \\ s.t. \quad \mathbb{E}^{\nu}\left[\varphi g\right] \leq v_0. \end{cases} \tag{6.14}$$

Defining

$$\frac{dQ}{dP} = \frac{g}{\mathbb{E}[g]}, \quad \frac{dQ^{\nu}}{dP^{\nu}} = \frac{g}{\mathbb{E}^{\nu}[g]}, \tag{6.15}$$

we get a standard simple hypothesis testing problem

$$\begin{cases} \max\limits_{\varphi \in \mathcal{R}} \mathbb{E}^{Q}\left[\varphi\right] \\ s.t. \quad \mathbb{E}^{Q^{\nu}}\left[\varphi\right] \leq \frac{v_0}{\mathbb{E}^{\nu}[g]} = \alpha^{\nu}. \end{cases} \tag{6.16}$$

Using results from Proposition 6.3, one obtains for each completed market

$$\tilde{\varphi}^{\nu} = 1_{\left\{\frac{dP}{dP^{\nu}} > \tilde{a}^{\nu}\right\}} + \gamma^{\nu} 1_{\left\{\frac{dP}{dP^{\nu}} = \tilde{a}^{\nu}\right\}} \tag{6.17}$$

where

$$\tilde{a}^{\nu} = \inf\left\{a : \mathbb{E}^{\nu}[1_{\left\{\frac{dP}{dP^{\nu}} > a\right\}} g] \leq v_0\right\} \tag{6.18}$$

and

$$\gamma^{\nu} = \frac{v_0 - \mathbb{E}^{\nu}[1_{\left\{\frac{dP}{dP^{\nu}} > \tilde{a}^{\nu}\right\}} g]}{\mathbb{E}^{\nu}[1_{\left\{\frac{dP}{dP^{\nu}} = \tilde{a}^{\nu}\right\}} g]} \tag{6.19}$$

Similarly to Quantile hedging section, we can summarize obtained results in the following theorem.

Theorem 6.4 *Minimal expected shortfall on incomplete market* (8.1) *will be achieved by investing in perfect hedging strategy for modified claim* $\varphi^{\underline{\nu}} g_T$, *where*

$$\tilde{\varphi}^{\underline{\nu}} = 1_{\left\{Z_T^A > \tilde{a}^{\underline{\nu}}\right\}} + \gamma^{\underline{\nu}} 1_{\left\{Z_T^A = \tilde{a}^{\underline{\nu}}\right\}} \tag{6.20}$$

$$\tilde{a}^{\underline{\nu}} = \inf\left\{a : \mathbb{E}^{\underline{\nu}}[1_{\left\{Z_T^A > a\right\}} g] \leq v_0\right\} \tag{6.21}$$

$$\gamma^{\underline{\nu}} = \frac{v_0 - \mathbb{E}^{\underline{\nu}}[1_{\left\{Z_T^A > \tilde{a}^{\underline{\nu}}\right\}} g]}{\mathbb{E}^{\underline{\nu}}[1_{\left\{Z_T^A = \tilde{a}^{\underline{\nu}}\right\}} g]} \tag{6.22}$$

Proof. On each completed version of the market, optimal strategy could be uniquely defined by the means of complete market approach. Therefore, for each completed market corresponding to $\nu \in \ker\Sigma$, one obtains optimal randomized test $\tilde{\varphi}^{\nu}$ from (6.17). Noting Proposition 5.4, we can deduce from similar logic that

$$\sup_{\pi \in \mathcal{A}(v_0)} \mathbb{E}[\varphi^{\pi} g_T] \leq \mathbb{E}[\varphi^{\nu} g_T], \ \forall \nu \in \ker\Sigma,$$

where φ^{π} is a successful hedge ratio of strategy π, defined as

$$\varphi^{\pi} = 1_{V_T^{\pi} \geq g_T} + \frac{V_T^{\pi}}{g_T} 1_{V_T^{\pi} < g_T}, \ \pi \in \mathcal{A}(x).$$

Moreover, equality is achieved if there exists strategy $\pi^* \in \mathcal{A}(v_0)$ such that its successful hedge ratio coincide with some φ^{ν}.

Consider market completion corresponding to parameter $\underline{\nu} = 0$. Then from continuity of Z_T^A and (6.21) it follows that

$$\mathbb{E}\left[\beta_T Z_T^A Z_T^{\nu} \varphi^{\underline{\nu}} g_T\right] = v_0 \cdot \mathbb{E}\left[\exp\left\{-\int_0^T \nu_s^* dW_s - \frac{1}{2}\int_0^T \|\nu_s\|^2 ds\right\}\right] \leq v_0, \ \forall \nu \in \ker\Sigma, \tag{6.23}$$

since $-\int_0^t \nu_s^* dW_s - \frac{1}{2}\int_0^t \|\nu_s\|^2 ds$ is non-negative local martingale and, consequently, supermartingale.

Therefore, from Theorem 5.5 it follows that there exists a strategy $\pi^* \in \mathcal{A}(v_0)$ that replicates $\varphi^{\underline{\nu}} g_T$ which proves the statement of the theorem. ■

6.3 Working with convex loss function

Now assume that investor does apply some convex loss function to assess shortfall risk. In other words, in this section we will discuss the problem (6.4) in general case of "proper" loss function.

Following steps proposed by Föllmer & Leukert in Föllmer and Leukert 2000, we will apply convex duality methods in order to derive solution. First of all, define *state-dependent utility function* as

$$u_l(x,\omega) = l(g(\omega)) - l((g(\omega) - x)^+). \tag{6.24}$$

It is straightforward to see, that initial problem of minimization of shortfall risk (6.3) can be defined as the following utility maximization task

$$\begin{cases} \max\limits_{\pi \in \mathcal{A}(x)} \mathbb{E}[u_l(V_T^\pi, \omega)] \\ \Pi(V_T^\pi) \leq v_0 \end{cases} \tag{6.25}$$

Before we move on and demonstrate how this problem can be solved in both complete and incomplete markets, we should agree on some necessary technical assumptions.

Assumption 6.5 *Function $u_l(x,\omega)$ is a non-decreasing, concave in x, strictly concave on $[0, g(\omega)]$ and continuously differentiable on $[0, g(\omega)]$. In addition*

$$-\infty < \mathbb{E}[u_l(0,\omega)] \quad \textit{and} \quad \mathbb{E}[u_l(g,\omega)] < \infty.$$

Following classical approach of convex duality we introduce inverse of marginal utility function, which also is a state-dependent function:

$$I_l(y,\omega) = \inf\{z \in [0, g(\omega)] | u_l'(z,\omega) < y\}, \tag{6.26}$$

in other words, $I_l(y,\omega) = (u_l'(x,\omega))^{-1}$ and it is continuous, strictly decreasing function in y. Then, applying famous Legendre transformation to $-u_l(-x)$, one gets a conjugate

$$\tilde{u}_l(y,\omega) = \max_{x>0}[u_l(x,\omega) - xy] = u_l(I_l(y,\omega),\omega) - yI_l(y,\omega). \tag{6.27}$$

From the latter equation, it follows that the following inequalities hold

$$u_l(I_l(y,\omega),\omega) \geq u_l(x,\omega) + y\left(I_l(y,\omega) - x\right), \quad \forall x > 0, y > 0 \tag{6.28}$$

$$\tilde{u}_l(u_l'(x,\omega),\omega) \leq \tilde{u}_l(y,\omega) - x\left(u_l'(x,\omega) - y\right), \quad \forall x > 0, y > 0. \tag{6.29}$$

Assumption 6.6 *For some* $\alpha \in (0,1),\ \gamma \in (1,\infty)$, *we have*

$$\alpha u_l'(x,\omega) \geq u_l'(\gamma x,\omega)$$

6.3.1 Complete Market Case

In fact, when market is complete, then one can claim that there exists a unique risk-neutral market measure. In such case, pricing functional Π in (6.25) becomes nothing but an expectation under this martingale market measure.

$$\begin{cases} \max\limits_{\pi\in\mathcal{A}(x)} \mathbb{E}[u_l(V_T^\pi,\omega)] \\ \mathbb{E}^*\left[V_T^\pi\right] \leq v_0 \end{cases} \tag{6.30}$$

This implies that in order to find optimal efficient hedge, one might first solve the utility maximization problem (6.30). In order to accomplish this, let us use approach, proposed in Karatzas *et al.* 1991.

Assume that $\beta_t = \frac{1}{B_t}$ is a discounting factor and

$$\mathbb{E}\left[\beta_T Z_T^A I_l(y\beta_T Z_T^A,\omega)\right] < \infty, \quad \forall y \in (0,\infty).$$

Then, define

$$\mathcal{X}_0(y) \coloneqq \mathbb{E}\left[\beta_T Z_T^A I_l(y\beta_T Z_T^A,\omega)\right], \quad 0 < y < \infty, \tag{6.31}$$

which is a continuous, strictly decreasing function which has an inverse

$$\mathcal{Y}_0(x) = (\mathcal{X}_0(y))^{-1}.$$

In Karatzas *et al.* 1991 it was shown that the optimal terminal capital (in this case state-dependent random variable) that maximizes the expected utility is defined as

$$\xi_0(x) := I_l\left(\mathcal{Y}_0(x)\beta_T Z_T^A\right) \tag{6.32}$$

Now, knowing what terminal capital would be desirable for investor in order to achieve maximal expected utility, one should be able to construct optimal strategy in a classical way. Namely, as from definition it follows that $\xi_0(x)$ is a martingale, we introduce:

$$M_t := \mathbb{E}\left[\beta_t Z_T^A \xi_0(x)|\mathcal{F}_t\right] \tag{6.33}$$

then, M_t admits stochastic integral representation

$$M_t = x + \int_0^t \psi_s^* dW_s \tag{6.34}$$

for some $\mathcal{F}_t$-adapted process ψ_t, satisfying $\int_0^T \|\psi_s\|^2 ds < \infty$. Further, define a wealth process

$$\hat{V}_t := \frac{M_t}{\beta_t Z_t^A} = \frac{1}{\beta_t}\left(x + \int_0^t \frac{1}{Z_s^A}(\psi_s + M_s\theta_s)^* d\hat{W}_s\right). \tag{6.35}$$

Consequently, the optimal replicating strategy can be found as

$$\hat{\pi}_t = \frac{1}{\beta_t Z_t^A \hat{V}_t}(\Sigma_t^*)^{-1}\left(\psi_t + M_t\theta_t\right). \tag{6.36}$$

Proposition 6.7 *Strategy* $\hat{\pi}_t$ *from* (6.36) *will be optimal for state-dependent utility maximization problem* (6.30) *and, consequently, for efficient hedging problem* (6.3) *defined on complete market.*

6.3.2 Incomplete Market Case

Moving on to the case of incomplete market, we are again facing an issue of infinitely many equivalent martingale market measures, which presents the problem of applying martingale approach for finding optimal strategy. However, this problem can be solved by the means of proposed method of market completions.

In particular, let us now assume that we are working with incomplete market. It would imply that volatility matrix contains more columns than rows in it. Applying MMC, we are able to parametrize each completed market by a special vector ν as it was described in (4.8).

Obviously, on each completed market, which corresponds to particular parameter ν, one would be able to apply approach described in the previous section. Namely, for each completed market, we assume

$$\mathbb{E}\left[\beta_T Z_T^A Z_T^\nu I_l(y\beta_T Z_T^A Z_T^\nu, \omega)\right] < \infty, \quad \forall y \in (0, \infty).$$

Moving on, we define corresponding

$$\mathcal{X}_\nu(y) := \mathbb{E}\left[\beta_T Z_T^A Z_T^\nu I_l(y\beta_T Z_T^A Z_T^\nu, \omega)\right], \quad 0 < y < \infty, \tag{6.37}$$

which is a continuous, strictly decreasing function which has an inverse

$$\mathcal{Y}_\nu(x) = (\mathcal{X}_\nu(y))^{-1}. \tag{6.38}$$

Finally, on each completed market, we define optimal terminal capital (state-dependent) that maximizes the utility function as

$$\xi_\nu(x) := I_l\left(\mathcal{Y}_\nu(x)\beta_T Z_T^A Z_T^\nu\right) \tag{6.39}$$

In terms of initial loss function, we could formulate the following criteria in order to choose a particular optimal strategy which won't require usage of auxiliary assets and will be replicable by the means of existing assets only. At the same time, this strategy should give us the lowest expected shortfall risk. In the spirit of Theorem (3.7) and approach from Karatzas *et al.* 1991 we formulate the following conditions for the problem of efficient hedging.

A. Optimality of $\hat{\pi}$: $El(g(\omega) - V_T^\pi)^+ \leq El(g(\omega) - V_T^{\hat{\pi}})^+ \quad \forall \pi \in \mathcal{A}(x)$

B. Financiability of $\xi_\lambda(x)$: $\exists \hat{\pi} \in \mathcal{A}(x)$ such that $V_T^{\hat{\pi}} = \xi_\lambda(x)$

C. Worst case loss associated with λ: $El(g(\omega)-\xi_\lambda(x))^+ \geq El(g(\omega)-\xi_\nu(x))^+ \quad \forall \nu \in K_1(\Sigma)$

D. Dual optimality of λ: $\forall \nu \in K_1(\Sigma)$,

$$\mathbb{E}\left[\tilde{u}_l(\mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\lambda)\right] \leq \mathbb{E}\left[\tilde{u}_l(\mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\nu)\right]$$

E. Parsimony of λ: $E[\beta_T Z_T^A Z_T^\nu \xi_\lambda(x)] \leq x, \quad \forall \nu \in K_1(\Sigma)$

Theorem 6.8 *Criterias B.-E. are equivalent and portfolio from B. satisfy A.*

For the sake of completeness, we provide proof that was originally described in Karatzas *et al.* 1991. Here we apply the same idea for state-dependent utility function.

Proof.

[B.→E.]

First of all, we notice that for any portfolio $\pi \in \mathcal{A}(x)$, $\beta_t Z_t^A Z_t^\nu V_t^\pi$ is a local martingale under P for every vector $\nu \in \ker(\Sigma)$. Which follows directly from the following representation

$$\beta_t Z_t^A Z_t^\nu V_t^\pi = x + \int_0^t \beta_s Z_s^A Z_s^\nu V_s^\pi \left(\Sigma_s^* \cdot \pi_s - (\theta_s + \nu_s)\right)^* dW_s. \tag{6.40}$$

Since the process $\beta_t Z_t^A Z_t^\nu V_t^\pi$ is a local martingale, which implies, among other things, that

$$\mathbb{E}[\beta_T Z_T^A Z_T^\nu V_T^\pi] \leq x, \ \forall \nu \in \ker \Sigma. \tag{6.41}$$

Consequently, if on some completed market we were able to find a portfolio π^* of existing assets that finances $\xi_\nu(x)$, it would mean that $V_T^{\pi^*} = \xi_\nu(x)$ and (6.41) means exactly condition E.

[E.→B.]

Is a direct consequence of application of Theorem 3.6 with $B = \xi_\lambda(x)$

[B.→C.]

In case all completing assets, corresponding to particular vector ν, were available for trading, optimal expected loss from hedging would have been equal $\mathbb{E}l\left((g(\omega)-\xi_\lambda(x))^+\right)$, however, since they are not, we have the following

$$\mathbb{E}\left[l\left((g(\omega)-\xi_\lambda(x))^+\right)\right] \le \inf_{\pi\in\mathcal{A}(x)} \mathbb{E}\left[l\left((g(\omega)-V_T^\pi(x))^+\right)\right] = v(x), \tag{6.42}$$

and equality holds if, for some portfolio $V_T^\pi = \xi_\lambda(x)$ (a.s). So, if B. holds, then

$$v(x) = \mathbb{E}\left[l\left((g(\omega)-\xi_\lambda(x))^+\right)\right] \ge \mathbb{E}\left[l\left((g(\omega)-\xi_\nu(x))^+\right)\right],\ \forall\nu\in\ker\Sigma. \tag{6.43}$$

[C.→D.]

Since $u_l(x,\omega)$ is concave for each ω, its conjugate function $\tilde{u}_l(x,\omega)$ is convex for each outcome, from this fact, in line with the fact that $(\tilde{u}_l(y,\cdot))'_y = -(u_l'(x,\cdot))^{-1}$, it is fair to write for $y\in\mathbb{R}^+$, where $\tilde{u}_l(y,\omega) = u_l(I_l(y,\omega)) - yI_l(y,\omega),\quad 0<y<\infty$ exists and, since $\tilde{u}_l'(y,\cdot) = -I_l(y,\cdot)$, one has

$$\frac{1}{|\varepsilon|}\left|\tilde{u}_l\left((y+\varepsilon)\beta_T Z_T^A Z_T^\nu\right) - \tilde{u}_l\left(y\beta_T Z_T^A Z_T^\nu\right)\right| \le \beta_T Z_T^A Z_T^\nu\cdot I_l\left(\frac{y}{2}\beta_T Z_T^A Z_T^\nu\right) \tag{6.44}$$

for $\varepsilon>0$. Then, by definition, $\mathbb{E}\left[\beta_T Z_T^A Z_T^\nu\cdot I_l\left(\frac{y}{2}\beta_T Z_T^A Z_T^\nu\right)\right] = \mathcal{X}_\nu(\frac{y}{2})<\infty$.

Consequently, applying dominated convergence theorem, we can claim that

$$\frac{d}{dy}\mathbb{E}\left[\tilde{u}_l(y\beta_T Z_T^A Z_T^\nu)\right] = -\mathcal{X}_\nu(y) \tag{6.45}$$

Noting that condition [C] of least favourability means $\mathbb{E}[u_l(\xi_\lambda(x,\omega))]\le\mathbb{E}[u_l(\xi_\nu(x,\omega))]$, $\forall\nu\in\ker\Sigma$, one obtains

$$\mathbb{E}\left[\tilde{u}_l(\mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\lambda)\right] = \mathbb{E}\left[\tilde{u}_l(\mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\lambda) + \mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\lambda\xi_\lambda(x,\omega)\right] - x\mathcal{Y}_\lambda(x) = \tag{6.46}$$

$$=\mathbb{E}\left[u_l(\xi_\lambda(x,\omega))\right] - x\mathcal{Y}_\lambda(x)\le \tag{6.47}$$

$$\le\mathbb{E}\left[u_l(\xi_\nu(x,\omega))\right] - x\mathcal{Y}_\lambda(x)\le \tag{6.48}$$

$$\le\mathbb{E}\left[\tilde{u}_l(\mathcal{Y}_\nu(x)\beta_T Z_T^A Z_T^\nu)\right] + x\mathcal{Y}_\nu(x) - x\mathcal{Y}_\lambda(x) \tag{6.49}$$

Since (6.45), minumum of $\mathbb{E}\left[\tilde{u}_l(y\beta_T Z_T^A Z_T^\nu)\right] + xy$ attained at $\mathcal{Y}_\nu(x)$

$$\le\mathbb{E}\left[\tilde{u}_l(\mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\nu)\right] + x\mathcal{Y}_\lambda(x) - x\mathcal{Y}_\lambda(x) = \tag{6.50}$$

$$=\mathbb{E}\left[\tilde{u}_l(\mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\nu)\right]. \tag{6.51}$$

[D.→B.]

Define $\mathcal{F}_t$-adapted process X_t as

$$\beta_t Z_t^A Z_t^\lambda X_t = M_t := \mathbb{E}\left[\beta_T Z_T^A Z_T^\lambda \xi_\lambda(x,\omega)|\mathcal{F}_t\right] \tag{6.52}$$

, with $X_0 = x$ and $X_T = \xi_\lambda(x,\omega)$ a.s. and M_t is a positive martingale with $M_0 = x$. From the martingale representation theorem then, it follows that there exists $\mathcal{F}_t$ adapted process φ_s with $\int_0^T \|\varphi_t\|^2 dt < \infty$ a.s. such that

$$M_t = x + \int_0^t \varphi_s dW_s, \quad 0 \le t \le T \tag{6.53}$$

from continuity and positivity of M_t, process $\psi_t = -\frac{\varphi_t}{M_t}$ is well-defined and

$$M_t = x - \int_0^t M_s \psi_s^* dW_s = x - \int_0^t \beta_s Z_s^A Z_s^\lambda X_s \psi_s^* dW_s, \quad 0 \le t \le T. \tag{6.54}$$

Further, decompose $\psi_t = \psi_t^A + \psi_t^\lambda$ with $\psi_t^A \in \ker^\perp \Sigma$ and $\psi_t^\lambda \in \ker \Sigma$. For some arbitrary portfolio π, its value process $V_t^\pi(x)$ will satisfy

$$\beta_t Z_t^A Z_t^\lambda V_t^\pi = x + \int_0^t \beta_s Z_s^A Z_s^\lambda V_s^\pi \left(\Sigma_s^* \pi_s - (\theta_s + \lambda_s)\right)^* dW_s. \tag{6.55}$$

Therefore, in order to show that there is a portfolio financing $\xi_\lambda(x,\omega)$, one should show

$$-X_t(\psi_t^A + \psi_t^\lambda) = V_t^\pi \left(\Sigma_t^* \pi_t - (\theta_t + \lambda_t)\right). \tag{6.56}$$

As we could choose π_t in a way that $\Sigma_t^* \pi_t = \theta_t - \psi_t^A$, it is enough to prove that $\psi_t^\lambda = \lambda_t$ in order to verify financiability of $\xi_\lambda(x,\omega)$.

For arbitrary ν introduce the sequence of stopping times $\{\tau_n\}_{n=1}^\infty$ as

$\tau_n =$

$T \wedge \inf\left\{t \in [0,T]; M_t \ge n, \text{ or } \int_0^t (\|\psi_s\|^2 + \|\lambda_s\|^2) ds \ge n, \text{ or } \int_0^t \|\nu_s\|^2 ds \ge, n \text{ or } \left|\int_0^t \nu_s^* dW_s\right| \ge n\right\}$

for all $n \ge 1$. Notice that $\lim_{n\to\infty} \tau_n = T$ a.s. and denote $\nu_t^n = \nu_t 1_{t \le \tau_n}$. Then, for $\lambda_t + \varepsilon \nu_t^n \in \ker \Sigma$ it holds

$$Z_t^{\lambda_t + \varepsilon \nu_t^n} = Z_t^A Z_T^\lambda \cdot \exp\left\{-\varepsilon \int_0^t (\nu_s^n)^*(dW_s + \lambda_s ds) - \frac{\varepsilon^2}{2} \int_0^t \|\nu_s^n\|^2 ds\right\}. \tag{6.57}$$

for $-1 \le \varepsilon \le 1,\ n \ge 1$. And also

$$e^{-3n|\varepsilon|} \le \frac{Z_t^{\lambda_t+\varepsilon\nu_t^n}}{Z_t^A Z_t^\lambda} \le e^{3n|\varepsilon|} \quad -1 \le \varepsilon \le 1. \tag{6.58}$$

From dual optimality it follows that

$$\frac{\partial}{\partial\varepsilon}\mathbb{E}\left[\tilde{u}(\beta_T Z_T^{\lambda_t+\varepsilon\nu_t^n}\xi_\lambda(x,\omega))\right]\Big|_{\varepsilon=0} = 0 \tag{6.59}$$

From mean value theorem there exists random variable $\gamma_\varepsilon \in [0,1]$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon}\left(\tilde{u}(y\beta_T Z_T^{\lambda+\varepsilon\nu_t^n}) - \tilde{u}\left(y\beta_T Z_T^A Z_T^\lambda\right)\right) = \\
&= \frac{1}{\varepsilon}y\beta_T(Z_T^{\lambda+\varepsilon\nu_t^n} - Z_T^A Z_T^\lambda)\left(\tilde{u}'\left(y\beta_T(Z_T^A Z_T^\lambda + \gamma_\varepsilon(Z_T^{\lambda+\varepsilon\nu_t^n} - Z_T^A Z_T^\lambda))\right)\right) \\
&= -\frac{1}{\varepsilon}y\beta_T Z_T^A Z_T^\lambda\left(\exp\left\{-\varepsilon\int_0^t(\nu_s^n)^*(dW_s+\lambda_s ds) - \frac{\varepsilon^2}{2}\int_0^t\|\nu_s^n\|^2 ds\right\} - 1\right)\cdot \\
&\cdot I_l\left(y\beta_T(Z_T^A Z_T^\lambda + \gamma_\varepsilon(Z_T^{\lambda+\varepsilon\nu_t^n} - Z_T^A Z_T^\lambda))\right)
\end{aligned} \tag{6.60}$$

Also notice that due to convexity of $\tilde{u}(y)$:

$$|\tilde{u}(y\beta_T Z_T^{\lambda+\varepsilon\nu_t^n}) - \tilde{u}\left(y\beta_T Z_T^A Z_T^\lambda\right)| \le \tag{6.61}$$

$$\le y\beta_T I_l\left(y\beta_T\left(Z_T^{\lambda+\varepsilon\nu^{tn}} \wedge Z_T^A Z_T^\lambda\right)\right)|Z_T^{\lambda+\varepsilon\nu^{tn}} - Z_T^A Z_T^\lambda| \tag{6.62}$$

$$\le K_n|\varepsilon|\cdot y\beta_T Z_T^A Z_T^\lambda I_l\left(y\beta_T e^{-3n} Z_T^A Z_T^\lambda\right) \tag{6.63}$$

where $K_n = \sup\limits_{0<\varepsilon<1}\frac{e^{3n\varepsilon}-1}{\varepsilon}$. Therefore, finding expected value of both sides,

$$\mathbb{E}\left[|\tilde{u}(y\beta_T Z_T^{\lambda+\varepsilon\nu_t^n}) - \tilde{u}\left(y\beta_T Z_T^A Z_T^\lambda\right)|\right] \le yK_n|\varepsilon|\mathcal{X}_\lambda\left(ye^{-3n}\right) < \infty. \tag{6.64}$$

and from dominated convergence theorem and (6.60) it follows that

$$\begin{aligned}
0 = \frac{\partial}{\partial\varepsilon}\mathbb{E}\left[\tilde{u}(\beta_T Z_T^{\lambda_t+\varepsilon\nu_t^n}\xi_\lambda(x,\omega))\right]\Big|_{\varepsilon=0} &= \mathbb{E}\left[\frac{\partial}{\partial\varepsilon}\tilde{u}(\beta_T Z_T^{\lambda_t+\varepsilon\nu_t^n}\xi_\lambda(x,\omega))\right]\Big|_{\varepsilon=0} = \\
=&\mathbb{E}\left[\beta_T Z_T^\lambda \xi_\lambda(x,\omega)\int_0^{\tau_n}(\nu_s^n)^*(dW_s+\lambda_s ds)\right] = \\
=&\mathbb{E}\left[M_{\tau_n}\int_0^{\tau_n}(\nu_s^n)^*(dW_s+\lambda_s ds)\right]
\end{aligned} \tag{6.65}$$

Applying Ito rule of integration by parts, we get

$$M_{\tau_n}\int\limits_0^{\tau_n}(\nu_s^n)^*(dW_s+\lambda_s ds)=\int\limits_0^{\tau_n}M_s(\nu_s^n)^*(\lambda_s-\psi_s^\lambda)ds+\int\limits_0^{\tau_n}M_s(\nu_s^n)^*dW_s- \tag{6.66}$$

$$-\int\limits_0^{\tau_n}M_s\left(\int\limits_0^{s}(\nu_t^n)^*(dW_t+\lambda_t dt)\right)(\psi_s^A+\psi_s^\lambda)^*dW_s.$$

As expected value of the last two integrals equals 0 and parameter ν was taken arbitrary, $\lambda_t=\psi_t^\lambda$ a.s. Which proves the statement.

■

As we mentioned at the beginning of this chapter, the problem of efficient hedging is equivalent to solving optimization problem (6.4). In what follows, we summarize how to use method of market completions to obtain final structure of optimal randomized test that solves (6.4) and, consequently, problem of efficient hedging.

We start with reminding important result demonstrated in Föllmer and Leukert 2000.

Theorem 6.9 (Theorem 5.1 from Föllmer and Leukert 2000) *The solution* $\tilde\varphi$ *to the optimization problem* (6.4) *on complete market is given by*

$$\tilde\varphi=1-\left(\frac{J(cZ_T^A)}{g}\wedge 1\right)\ \text{on}\ \{g>0\}, \tag{6.67}$$

where c *is determined by the condition*

$$\mathbb{E}^*[\tilde\varphi g]=v_0. \tag{6.68}$$

In other words, using proposed structure of modified contingent claim $\tilde g=\tilde\varphi g$ one could write

$$\min_{\varphi\in\mathcal{R}}\mathbb{E}\left[l\left((1-\varphi)g\right)\right]=\mathbb{E}\left[l\left((1-\tilde\varphi)g)\right)\right]=\mathbb{E}\left[l\left(J(cZ_T^A)\wedge g\right)\right]. \tag{6.69}$$

On the other hand, in Section 6.3.1 it was demonstrated that, by representing efficient hedging problem in a state-dependent utility maximization form, we obtain that it would be optimal to construct portfolio in such a way that its terminal capital would be equal to $\xi_\lambda(v_0)$. Consequently, it is fair to notice that

$$\xi_0(v_0) = \tilde{\varphi} g = \left(g(\omega) - J(cZ_T^A Z_T^\nu)\right)^+ . \tag{6.70}$$

We summarize result provided above in the following theorem

Theorem 6.10 *The strategy that solves efficient hedging problem* (6.3) *on complete market would be to replicate a modified claim $\tilde{g}_\nu$ defined as*

$$\tilde{g}_0 = \xi_0(v_0) = I_l\left(\mathcal{Y}_0(x)\beta_T Z_T^A\right). \tag{6.71}$$

Similarly to complete market case, we can apply results obtained in Section 6.3.2 in order to derive optimal claim for efficient hedging problem on incomplete market. Introducing market completions to the market we recall that solution on each completed version on such market will be written as

$$\xi_\lambda(x) := I_l\left(\mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\lambda\right),$$

where particular completion corresponding to parameter λ is chosen in accordance to criteria in Theorem 6.8. Consequently, optimal randomized test for problem on incomplete market would be

$$\varphi_\lambda = \frac{I_l\left(\mathcal{Y}_\lambda(x)\beta_T Z_T^A Z_T^\lambda\right)}{g} \tag{6.72}$$

6.4 Numerical Examples

Example 6.11 (Linear Loss Function (Shortfall Minimization)) *Consider the following incomplete market model*

Description	*Parameter*	*Value*
Interest rate	r	*0*
Stock 1	σ_{11}	*0.2*
	σ_{12}	*0.11*
	σ_{13}	*0.4*
	μ_1	*0.02*
Stock 2	σ_{21}	*0.3*
	σ_{22}	*0.15*
	σ_{23}	*0.2*
	μ_2	*0.08*
Initial Prices	S_0^1, S_0^2	*100, 80*
Option Strike Price	K	*110*
Maturity Time	T	*1*

Similar to Example 5.8, we introduce several market completions for demonstration purposes. And then calculate corresponding parameters $\tilde{a}^{\nu}$ *required for construction of maximal successful hedging sets in* (6.18)*:*

Market Completion Parameter ID	ν_3 *Value*	$\tilde{a}$ *from* (6.18)	*Risk-Neutral Price, Completed Market*
Z_0	*0.0*	*0.6778*	*14.5429*
Z_1	*0.1*	*0.6562*	*14.5353*
Z_2	*0.2*	*0.5974*	*14.5251*
Z_3	*0.3*	*0.5237*	*14.5122*
Z_4	*0.4*	*0.4443*	*14.4963*
Z_5	*0.5*	*0.3687*	*14.4771*
Z_6	*0.6*	*0.2998*	*14.4543*
Z_7	*0.7*	*0.2407*	*14.4276*
Z_8	*0.8*	*0.1870*	*14.3965*
Z_9	*0.9*	*0.1434*	*14.3607*

Constructing modified claims with (6.17)*, expected shortfall is presented in Figure 6.1 below.*

Figure 6.1: Linear Loss: Expected Shortfall vs. Initial budget constraint by Market Completions

As in Example 5.8, completed market version for which unique risk-neutral market measure is a Minimal Martingale Measure on initially incomplete market, presents "worst-case scenario". Choosing minimal martingale measure, we arrive at optimal claim:

$$\tilde{g} = g_T \cdot 1_{\left\{0.6778<(Z_T^A)^{-1}\right\}}$$

Example 6.12 (Convex Loss Function) *Let us consider loss function* $l(x) = x^2$ *and problem of efficient hedging for the market from Example 2.7. For this loss* *function, we can write*

$$u_l(x,\omega) = (g_T(\omega))^2 - \left((g_T(\omega) - x)^+\right)^2,$$

then, derivative for this state-dependent utility will be

$$u_l'(x,\omega) = 2(g - x)^+$$

and $I_l()$ can be found numerically from (6.26).

Consider set of completed versions of market, parametrized by vector $\nu = [00\nu^3]$. For different initial budgets, we calculate corresponding $\mathcal{Y}_\nu(x)$:

Market Completion Parameter ID	ν_3 *Value*	$\mathcal{Y}_\nu(x)$ *from* (6.38)
Z_0	*0.0*	*4.4454*
Z_1	*0.2*	*4.1095*
Z_2	*0.4*	*3.4874*
Z_3	*0.6*	*2.7113*
Z_4	*0.8*	*1.9258*

Then, for each completed market version, we obtain expected loss, measured with $l(x) = x^2$. Results are demonstrated in Figure 6.2

In this case, if we assume that this small set of market completions is the only one available, we should choose "worst-case" scenario, or least favorable one, which, in this particular case, will correspond to parameter $\lambda = 0$.

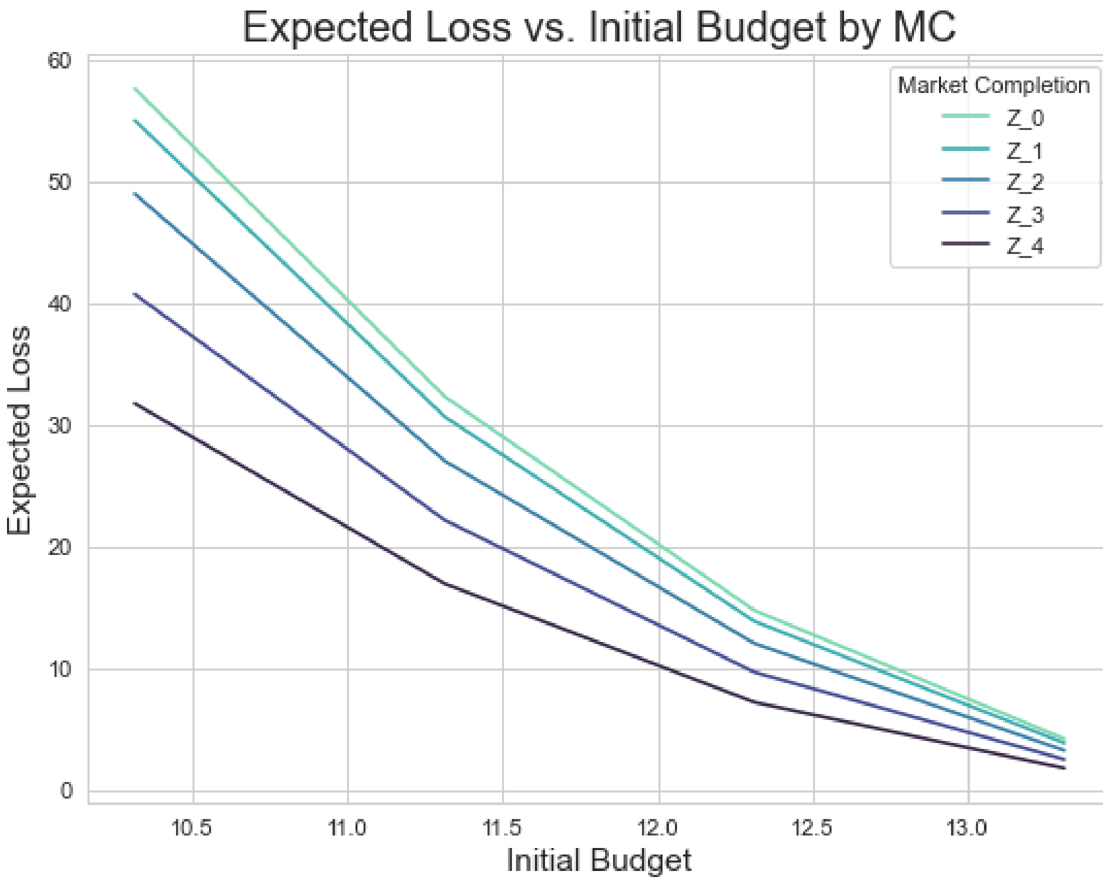

Figure 6.2: Efficient Hedging: Expected Loss vs. Initial budget constraint by Market Completions

Chapter 7

MMC for Hedging with respect to Modern Risk Measures

7.1 Introduction

In this section we describe how one can construct optimal partial hedging strategy with respect to chosen risk measure. As it was described in introduction, risk measures are functions used to represent exposure to risk for market agent. In modern risk management industry, two of such functions are especially important in practice, since they were presented in The Market Risk Framework of Basel III – international regulatory accord. In other words, estimation of risk exposure with these measures is a best practice on current markets and therefore presents an important practical problem.

In the current section we focus on Conditional value at risk. As shown in Melnikov&Smirnov (Melnikov and Smirnov 2012) it is possible to represent the problem of CVaR minimization as a two-step optimization task. Each step in this case will be an independent one-parametric problem that could be solved separately with the help of Neyman-Pearson lemma.

At the same time, recent papers by Cong et al. (Cong *et al.* 2014), Li&Xu (Li and Xu 2013), Capinski(Capinski 2014) and Godin (Godin 2015) demonstrate growing interest towards CVaR optimization. Namely, in Cong *et al.* 2014 authors prove that the best (CVaR optimal) hedging strategy would be to construct static strategy that replicates special bull-call-spread option.

Moving on, we focus on more general risk measure called *Range Value-at-Risk*, or RVaR which was demonstrated in Embrechts *et al.* 2018 and Cont *et al.* 2008. It was stated in Cont *et al.* 2008 that risk measure can not be both robust and coherent. However, RVaR offers a trade-off between the sensitivity of CVaR and the robustness of VaR.

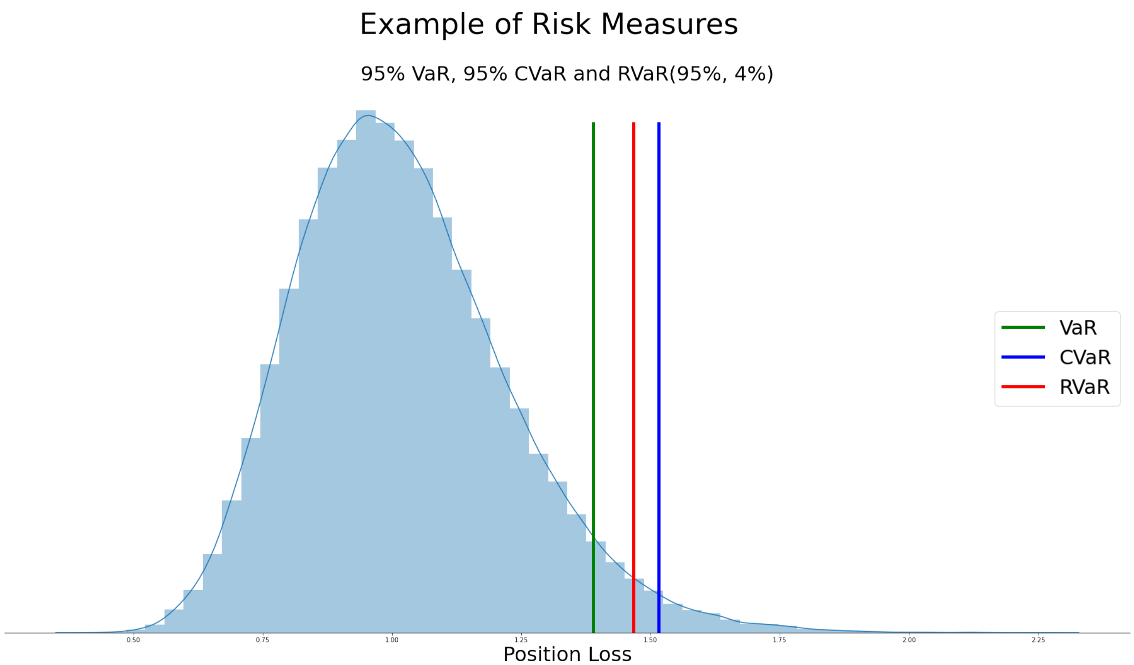

Figure 7.1: Illustration of Risk Measures

Figure 7.1 illustrates the idea behind the industry accepted risk measures. Assume that after hedging with portfolio π investor anticipates that its terminal capital will not be enough to construct perfect hedge for claim payoff $X(\omega)$. In other words, he is expecting loss. Let distribution of such loss be described with some positively skewed distribution curve (blue curve on Figure 7.1). Value-at-Risk (VaR) measure is a green line and represents a loss, that is not exceeded with predetermined probability (95% in current example). Essentially, RVaR measure is a two-parametric risk measure that includes both VaR and CVaR measures as a limit cases. This fact makes it efficient to solve the partial hedging problem with respect to this measure in order to obtain insights regarding both widely used industry measures at the same time.

In what follows we first demonstrate how method of market completions can be conveniently used in tandem with approach of Cong *et al.* 2014 to solve the problem on incomplete market for both RVaR and CVaR risk measures. We show how local risk minimization criteria can be used to obtain an exact result on incomplete market.

Then, we focus on method of two-step optimization from Melnikov and Smirnov 2012 and apply method of market completions in order to find a solution on incomplete market.

7.2 CVaR Hedging Approach

It is known that technique of mathematical finance is well developed for complete markets. With unique risk-neutral measure it is possible to unambiguously solve both perfect and partial hedging problems via martingale approach. Partial hedging strategy means that there is a probability of not having enough capital generated by strategy to cover all possible losses from future obligation. Then, investor can choose how to express his attitude towards expected loss. In modern finance industry this is done with the help of risk measures which we already briefly touched in Introduction.

Keeping in mind Definition 1.8, the problem of CVaR optimal hedging therefore can be stated in terms of hedging loss function $L(x,\pi)$ as

$$\begin{cases} CVaR_\alpha(L(x,\pi)) \to \min_{(x,\pi)} \\ x \le v_0 \end{cases} \tag{7.1}$$

One can approach this problem from the perspective of optimal split into hedged/unhedged proportions of the claim $H = f(H) + R_f(H)$ with payoff H, where $f(H)$ describes the optimal hedged proportion of the claim. This method was offered in Cong *et al.* 2014.

Considering European type contingent claim, we expect to have a payoff $X(\omega)$ at maturity time T, so the total *risk exposure* of the investor is going to be

$$T_f(X) = R_f(X) + e^{rT}\Pi(f(X)), \tag{7.2}$$

where $\Pi(f(X))$ is some pricing functional for hedged part.

Given the initial budget constraint, investor is pursuing the goal of minimizing risk measure of total exposure (7.31), given the restriction on initial cost of hedging:

$$\begin{cases} \min_{f\in\Omega} CVaR(T_f(X)) \\ s.t. \quad \Pi(f(X)) \leq v_0 \end{cases} \tag{7.3}$$

According to Cong *et al.* 2014, under particular assumptions, explicit way of identifying the optimal hedged loss function is stated in the following theorem

Theorem 7.1 *Assume that pricing functional is linear for any time-t contingent payout X. Then, the optimal hedged loss function g_f^* is given by*

$$g_f^*(x) = (x - d^*)^+ - (x - u^*)^+ \tag{7.4}$$

where (d^*, u^*) satisfies the following equations

$$\begin{cases} e^{-rT} \int_{d^*}^{u^*} \mathbb{Q}(X > x)dx = \pi_0 \\ \mathbb{P}(X > u^*) = \alpha \cdot \frac{\mathbb{Q}(X>u^*)}{\mathbb{Q}(X>d^*)} \end{cases} \tag{7.5}$$

where $\mathbb{Q}$ is a risk-neutral measure.

In Cong *et al.* 2014 it was also shown that one possible candidate of the optimal hedged loss function can be found in more simple form by assuming $u^* = \infty$:

Corollary 7.2 *One of the possible optimal hedging functions is given by*

$$g_f^*(x) = (x - d^*)^+ \tag{7.6}$$

where d^* is the solution to the following equation

$$E^Q \left[e^{-rT}(X - d^*)^+\right] = v_0 \tag{7.7}$$

In other words, for solving Optimal CVaR Hedging problem, one should focus on finding corresponding fixed probability measure Q in order to obtain optimal value d^*.

Example 7.3 *Consider the following market*

Description	*Parameter*	*Value*
Interest rate	r	*0*
Stock 1	σ_{11}	*0.2*
	σ_{12}	*0.11*
	μ_1	*0.02*
Stock 2	σ_{21}	*0.3*
	σ_{22}	*0.15*
	μ_2	*0.08*
Initial Prices	S_0^1, S_0^2	*100, 80*
Option Strike Price	K	*110*
Maturity Time	T	*1*

Investor is working with Call European Option on the Stock 1 with strike price $K = 110$. *As the market is complete, one can obtain unique risk-neutral price for such claim, which is equal to 5.4.*

Assume that the goal of the investor is to find CVaR-optimal hedging function for initial budgets 5, 4, 3, 2 and 1. According to Corollary 7.2,

Initial Budget	*% of Risk-Neutral Price*	*Optimal Hedging Claim*
\$1	*18.5*	$(S_1 - 137.68)^+$
\$2	*37*	$(S_1 - 126.78)^+$
\$3	*56*	$(S_1 - 120.12)^+$
\$4	*74*	$(S_1 - 115.23)^+$
\$5	*93*	$(S_1 - 111.29)^+$

In case of incomplete market, however, investor cannot assess initial price of claim with certainty anymore. As not all assets might be available for trading, constructions of replicating portfolio becomes difficult. On the other hand, this lack of tools on the market becomes the reason for having infinitely many equivalent risk-neutral measures. Furthermore, by focusing on incomplete market, one should not expect unique fair price similar to complete market. Instead, same no-arbitrage considerations lead to an interval of "fair" prices (see f.e. Karatzas and Kou 1996).

In rigorous terms, exponential of the form (1.6) can be uniquely defined on complete market as system (1.4) has only one solution θ. However, when switching to incomplete market, one can not enjoy this uniqueness of solution to (1.4) anymore. Usually, mentioned system has infinitely many solutions and, as a consequence, infinitely many possible equivalent martingale measures.

Having multiple "completed" versions of initial market, agent can solve CVaR optimization problem on each of them separately. For this optimization task we will use UBIP approach introduced in Section 4.4 as a pricing functional of choice.

In order to find a specific market completion and corresponding completed market on which one can solve optimal hedging problem and optimal strategy will also be a solution for originally incomplete market. To move on with this idea, lets introduce $k-n$ fictitious assets in addition to n existing assets on incomplete market, driven by the same k-dimensional Brownian motion as n real tradeable assets.

As we presented in Section 4.4, unique fair price on incomplete market will be written as the following expected value, constructed with the help of particular completions λ:

$$\Pi(x) = \mathbb{E}[\beta_T Z_T^A Z_T^\lambda x] \tag{7.8}$$

We summarize these findings in the form of the following theorem.

Theorem 7.4 *CVaR minimizing hedging strategy for contingent claim $X(\omega)$ on incomplete market for the problem* (7.3) *will be to finance a static strategy that replicates knock-out option with payoff function*

$$\tilde{X} = (X - d^*)^+ - (X - u^*)^+ \tag{7.9}$$

where (d^*, u^*) satisfies the following equations

$$\begin{cases} e^{-rT}\mathbb{E}\left[Z_T^A\left((X-d^*)^+ - (X-u^*)^+\right)\right] = v_0 \\ \mathbb{P}(X > u^*) = \alpha \cdot \frac{\mathbb{Q}(X>u^*)}{\mathbb{Q}(X>d^*)} \end{cases} \tag{7.10}$$

and $\mathbb{Q}$ is a minimal martingale measure.

Proof. From (7.8) and Theorem 7.1 it is clear that (7.9)–(7.7) is a solution on completed version of the market that corresponds to $\lambda = 0$, or, as it follows from Theorem 2.6, on the market for which ELMM coincides with minimal martingale measure. Moreover, from Proposition 4.11 it follows that expectation under minimal martingale measure will coincide with UBIP pricing functional for structurally incomplete model (8.1).

Then, since Z_T^ν is a supermartingale under actual probability measure, it follows that

$$
\begin{aligned}
&E\left[\beta_T Z_T^A \left((X-d^*)^+ - (X-u^*)^+\right)\right] = v_0 \\
&\qquad\qquad \geq E\left[\beta_T Z_T^A Z_T^\nu \left((X-d^*)^+ - (X-u^*)^+\right)\right], \ \forall \nu \in \ker \Sigma.
\end{aligned}
$$

Which implies, according to Theorem 3.6 with $B = (X-d^*)^+ - (X-u^*)^+$ it follows that claim $\tilde{X}$ will be financeable on initially incomplete market. ■

7.3 CVaR: Alternative Approach

Inspired by Rockafellar&Urasev, the following approach was developed by Melnikov and Smirnov in Melnikov and Smirnov 2012. Denote $L(x,\pi) = g_T - V_T^\pi(x)$ the loss function, defined by initial capital and chosen strategy. General idea consists of usage of the following representation:

$$
F_\alpha\left((x,\pi),z\right) = z + \frac{1}{1-\alpha}\mathbb{E}\left[(L(x,\pi)-z)^+\right] \tag{7.11}
$$

In case of European contingent claim g_T under consideration, loss function $L(x,\pi)$ can be written as $L(x,\pi) = g_T - V_T^\pi(x)$. Then

$$
\begin{aligned}
F_\alpha\left((x,\pi),z\right) &= z + \frac{1}{1-\alpha}\mathbb{E}\left[(g_T - V_T^\pi(x) - z)^+\right] &(7.12)\\
&= z + \frac{1}{1-\alpha}\mathbb{E}\left[(g_T(z) - V_T^\pi(x))^+\right] &(7.13)
\end{aligned}
$$

where $g_T(z) = (g_T - z)^+$ z-dependent contingent claim.

So, it is easy to see that

$$CVaR_\alpha(V) = \min_{z \in \mathbb{R}} F_\alpha(V, z) \tag{7.14}$$

And the optimization problem (7.1) can be rewritten as:

$$F_\alpha((x, \pi), z) \to \min_{\pi \in (A)_{\tilde{V}}} \min_{z \in \mathbb{R}} \tag{7.15}$$

Introducing a special function

$$c(z) = z + \frac{1}{1-\alpha} \min_{(x,\pi)} \mathbb{E}\left[(g(z) - V_T^\pi(x))^+\right] \tag{7.16}$$

$$\min_{z \in \mathbb{R}} c(z) = \min_{(x,\pi)} CVaR_\alpha(x, \pi). \tag{7.17}$$

Solution of (7.1) now can be decomposed into consequent optimization by z after solving:

$$\begin{cases} \mathbb{E}\left[(g_T(z) - V_T^\pi(x))^+\right] \to \min_{\pi \in \mathcal{A}} \\ x < \tilde{V} \end{cases} \tag{7.18}$$

In Section 6.2 we demonstrated how (7.18) could be solved both on complete and incomplete markets. In former case, according to Proposition 6.3 the optimal strategy will be to perfectly hedge modified claim $\tilde{\varphi}(z) g_T(z, \omega)$ where

$$\tilde{\varphi}(z) = 1_{\left\{\frac{dP}{dP^*} > \tilde{a}(z)\right\}} + \gamma 1_{\left\{\frac{dP}{dP^*} = \tilde{a}(z)\right\}}, \tag{7.19}$$

$$\tilde{a}(z) = \inf\left\{a : \mathbb{E}^*[1_{\left\{\frac{dP}{dP^*} > a\right\}} g(z)] \le v_0\right\}, \tag{7.20}$$

$$\gamma = \frac{v_0 - \mathbb{E}^*[1_{\left\{\frac{dP}{dP^*} > \tilde{a}*(z)\right\}} g(z)]}{\mathbb{E}^*[1_{\left\{\frac{dP}{dP^*} = \tilde{a}(z)\right\}} g(z)]}. \tag{7.21}$$

This result was summarized in Melnikov and Smirnov 2012 in a form of theorem

Theorem 7.5 (Theorem 2.4 in Melnikov and Smirnov 2012) *The optimal strategy π^* for the CVaR minimization problem* (7.1) *is a perfect hedge for the contingent* *claim* $\tilde{g}(\hat{z}) = (g-\hat{z})^+\tilde{\varphi}(\hat{z})$ *where* $\tilde{\varphi}$ *is defined by* (7.19), $\hat{z}$ *is a point of global minimum of function*

$$c(z) = \begin{cases} z + \frac{1}{1-\alpha}\mathbb{E}\left[(g-z)^+(1-\tilde{\varphi}(z))\right], & z < z^*, \\ z, & z \geq z^*, \end{cases} \tag{7.22}$$

on interval $z < z^*$, *and* z^* *is a real root of equation*

$$v_0 = \mathbb{E}^*\left[(g-z^*)^+\right]. \tag{7.23}$$

Besides, one always has

$$CVaR_\alpha(v_0, \pi^*) = c(\hat{z}), \tag{7.24}$$

$$VaR_\alpha(v_0, \pi^*) = \hat{z}. \tag{7.25}$$

For the case of incomplete market, we will apply method of market completions again. It was demonstrated in Section 6.2.2 that solution to the problem (7.18) can be obtained on incomplete market. Denote this solution as $\hat{\varphi}^\lambda(z)$. Therefore, we can summarize results in a form of the following theorem.

Theorem 7.6 *Optimal strategy for CVaR minimization problem* (7.1) *on incomplete market will be a perfect hedge for contingent claim* $\tilde{g}^\lambda(\hat{z}) = (g-\hat{z})^+\hat{\varphi}^\lambda(\hat{z})$, *where* $\hat{z}$ *is a point of global minimum of a function*

$$c(z) = \begin{cases} z + \frac{1}{1-\alpha}\mathbb{E}\left[(g-z)^+(1-\tilde{\varphi}^\lambda(z))\right], & z < z^*, \\ z, & z \geq z^*, \end{cases} \tag{7.26}$$

where z^* *is a root of equation* $v_0 = \mathbb{E}^\lambda\left[(g-z^*)^+\right]$.

Corollary 7.7 *Combining results of Theorem 6.4 and Theorem 7.6, the solution to problem* (7.1) *will be a perfect hedge for* $\tilde{g}^{\underline{\nu}}(\underline{z}) = (g-\underline{z})^+\hat{\varphi}^{\underline{\nu}}(\underline{z})$ *where* $\underline{z}$ *is a point of global minimum of a function*

$$c(z) = \begin{cases} z + \frac{1}{1-\alpha}\mathbb{E}\left[(g-z)^+(1-\tilde{\varphi}^{\underline{\nu}}(z))\right], & z < z^*, \\ z, & z \geq z^*, \end{cases} \tag{7.27}$$

where z^* *is a root of equation* $v_0 = \mathbb{E}^{\underline{\nu}}\left[(g-z^*)^+\right]$.

7.4 RVaR hedging Approach

We now formulate the RVaR Optimization Problem (Vasilev and Melnikov 2022). Define PL for hedging position as a random variable $B = g_T - V_T^{\pi},\ \pi \in \mathcal{A}(v_0)$, where $B > 0$ if loss occurred and negative if hedging position generated gain. Recall definition of two industry-accepted risk measures:

$$VaR_{\alpha}(B) = \inf\{v \in \mathbb{R} : P(B > v) \leq 1 - \alpha\} \tag{7.28}$$

$$CVaR_{\alpha}(B) = \frac{1}{1-\alpha}\int_{\alpha}^{1} VaR_s(B)ds \tag{7.29}$$

It was demonstrated in, f.e. Melnikov and Wan 2022, that both of these measures are incorporated as a limiting cases in more general, two-parametric risk measure RVaR, defined as

$$RVaR_{\alpha,\beta}(B) = \begin{cases} \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} VaR_s(B)ds & ,\beta > 0 \\ VaR_{\alpha}(B) & ,\beta = 0 \end{cases} \tag{7.30}$$

RVaR hedging problem consists in finding a strategy, that satisfies budget constraint and at the same time minimizes RVaR of future obligations.

Following the steps by Cong *et al.* 2014, first we split the future obligation g_T into two natural parts from the perspective of optimal split into hedged/unhedged proportions of the claim $g_T = f(g_T) + R_f(g_t)$, where $f(g_T)$ describes the optimal hedged proportion of the claim.

Considering European type contingent claim, we expect to have a payoff at maturity time T, so the total *risk exposure* of the investor is going to be

$$T_f(g_T) = R_f(g_T) + e^{rT}\Pi(f(g_T)), \tag{7.31}$$

where $\Pi(f(g_T))$ is some pricing functional for hedged part.

According to Cong *et al.* 2014, we should search for an optimal hedged-loss function within the set of functions, satisfying the following assumptions:

1. Not globally over-hedged: $f(x) \le x, \forall x \ge 0$;

2. Not locally over-hedged: $f(x_2) - f(x_1) \le x_2 - x_1 \quad \forall 0 \le x_1 \le x_2$;

3. Non-negativity of a hedged loss: $f(x) \ge 0, \forall x \ge 0$;

Which, in other words means, that we are working with the class

$$\mathcal{D} = \{0 \le f(x) \le x : R_f(x) = xf(x) \text{ is a non-decreasing and left continuous function}\}. \tag{7.32}$$

Given the initial budget constraint, investor is pursuing the goal of minimizing risk measure of total exposure (7.31), given the restriction on initial cost of hedging:

$$\begin{cases} \min_{f\in\mathcal{D}} RVaR(T_f(g_T)) \\ s.t. \quad \Pi(f(g_T)) \le v_0 < \Pi(g_T) \end{cases} \tag{7.33}$$

As in the case of CVaR, problem is solved in two steps:

1. Find an optimal function for determining hedged-loss proportion of risk

2. Use function that replicates this proportion of risk as optimal for RVaR hedging problem.

It was demonstrated in Melnikov and Wan 2022, that solution to this problem depends on the size of initial capital that investor possess. Namely, if investor has enough money to pay for hedging of $f^*(x) = x \cdot I_{x \le VaR_{\alpha+\beta}(x)}$ then one should use it as an optimal hedged loss proportion, otherwise, optimal hedged loss function will be $f^*(x) = f(x, d^*, u^*) = ((x - d^*)^+ + (x - u^*)^+) \cdot I_{x \le VaR_{\alpha+\beta}(B)}$ which we, following Melnikov and Wan 2022 will repeat in a form of theorems here.

Theorem 7.8 *If* $\Pi(g_T \cdot I_{B \le VaR_{\alpha+\beta}(g_T)}) \le v_0$*, then the optimal hedged loss function is:*

$$f^*(x) = x \cdot I_{x \le VaR_{\alpha+\beta}(x)} \tag{7.34}$$

And

Theorem 7.9 *If* $\Pi(g_T \cdot I_{g_T \leq VaR_{\alpha+\beta}(g_T)}) > v_0$*, then the optimal hedged loss function is:*

$$f^*(x) = f(x, d^*, u^*) = \big((x-d^*)^+ + (x-u^*)^+\big) \cdot I_{x \leq VaR_{\alpha+\beta}(g_T)}, \tag{7.35}$$

where (d^*, u^*) *is the solution to*

$$\begin{cases} \min\limits_{\substack{0 \leq d \leq VaR_\alpha(g_T), \\ d \leq u \leq VaR_{\alpha+\beta}(g_T)}} \left\{ d + \frac{1}{\beta} \int_\alpha^{\alpha+\beta} (VaR_s(g_T) - u)^+ ds \right\} \\ s.t. \quad \Pi(f(g_T, d, u)) \leq v_0 \end{cases} \tag{7.36}$$

As one can see, to choose proper optimal hedged loss function, it is crucial to assess initial capital with the help of pricing functional. Such pricing functional should preserve stop-loss ordering and allow no-arbitrage on the market. Applying the same logic as in case of CVaR optimization part, we use UBIP pricing functional in order to assess condition in Theorems 7.8-7.9.

Since in case of complete market solution to RVaR optimization task can be obtained directly, we further elaborate on application of method of market completions for solving the problem on incomplete market. Summarizing results above we can write solution to RVaR hedging problem with the help of UBIP pricing functional, which, according to Proposition 4.11 will not depend on either initial capital or particular concave utility function and will correspond to $\lambda = 0$.

Therefore, summarizing results demonstrated above, we formulate the following theorem.

Theorem 7.10 *For contingent claim with payoff* g_T*, if* $\mathbb{E}[g_T \cdot I_{g_T \leq VaR_{\alpha+\beta}(g_T)}] \leq v_0$*, then RVaR-minimizing strategy on incomplete market will be to replicate*

$$f^*(g_T) = g_T \cdot I_{g_T \leq VaR_{\alpha+\beta}(g_T)}.$$

Otherwise, optimal strategy will be to perfectly hedge bull-call spread

$$f(x, d^*, u^*) = \big((x-d^*)^+ + (x-u^*)^+\big) \cdot I_{x \leq VaR_{\alpha+\beta}(g_T)},$$

where (d^*, u^*) is a solution to optimization problem

$$\begin{cases} \min\limits_{\{0 \le d \le VaR_\alpha(g_T), d \le u \le VaR_{\alpha+\beta}(g_T)\}} & \left\{ d + \frac{1}{\beta} \int_\alpha^{\alpha+\beta} (VaR_s(g_T) - u)^+ ds \right\} \\ & s.t. \quad E[\beta_T Z_T^A f(g_T, d, u)] \le v_0. \end{cases} \tag{7.37}$$

7.5 Numerical Examples

We further demonstrate how method of market completions can be applied to finding optimal hedging strategies. In all examples that follow, we will use the following market set-up:

Description	Parameter	Value
Interest rate	r	0
Stock 1	σ_{11}	0.2
	σ_{12}	0.11
	σ_{13}	0.4
	μ_1	0.02
Stock 2	σ_{21}	0.3
	σ_{22}	0.15
	σ_{23}	0.2
	μ_2	0.08
Initial Prices	S_0^1, S_0^2	100, 80
Call Option Strike Price	K	110
Maturity Time	T	1

Example 7.11 (CVaR minimization on incomplete market) *Assume that in Example 7.3, there is an additional source of risk W_t^3, which makes market incomplete.*

Initial Budget	% of Risk-Neutral Price	Optimal Hedging Claim
\$2.69	18.5	$(S_1 - 183.48)^+$
\$5.38	37	$(S_1 - 153.13)^+$
\$8.14	56	$(S_1 - 135.19)^+$
\$10.75	74	$(S_1 - 123.13)^+$
\$13.51	93	$(S_1 - 113.18)^+$

Table 7.1: CVaR minimizing hedging claims for initial budgets. Incomplete Market example

Following the standard procedure described in Example 2.7, dynamics of the stocks under innovative Brownian Motion will be:

$$dS_t^1 = S_t^1 \left(\mu^1 dt + \|\bar{\sigma}^1\| d\widehat{W}_t^1 \right)$$
$$dS_t^2 = S_t^2 \left(\mu^2 dt + \alpha^{21} \|\bar{\sigma}^1\| d\widehat{W}_t^1 + \|\bar{\sigma}^2\| d\widehat{W}_t^2 \right)$$

Corresponding vector θ then becomes:

$$\theta = \begin{bmatrix} 0.0434 \\ 0.2624 \\ \frac{\mu^3 - r}{\|\rho^3\|} \end{bmatrix}$$

According to Theorem 2.6, for minimal martingale measure, one should use completion for which $\frac{\mu^3 - r}{\|\rho^3\|} \equiv 0$. The risk neutral price in this case will be 14.5298 and the resulting optimal hedging claims are:

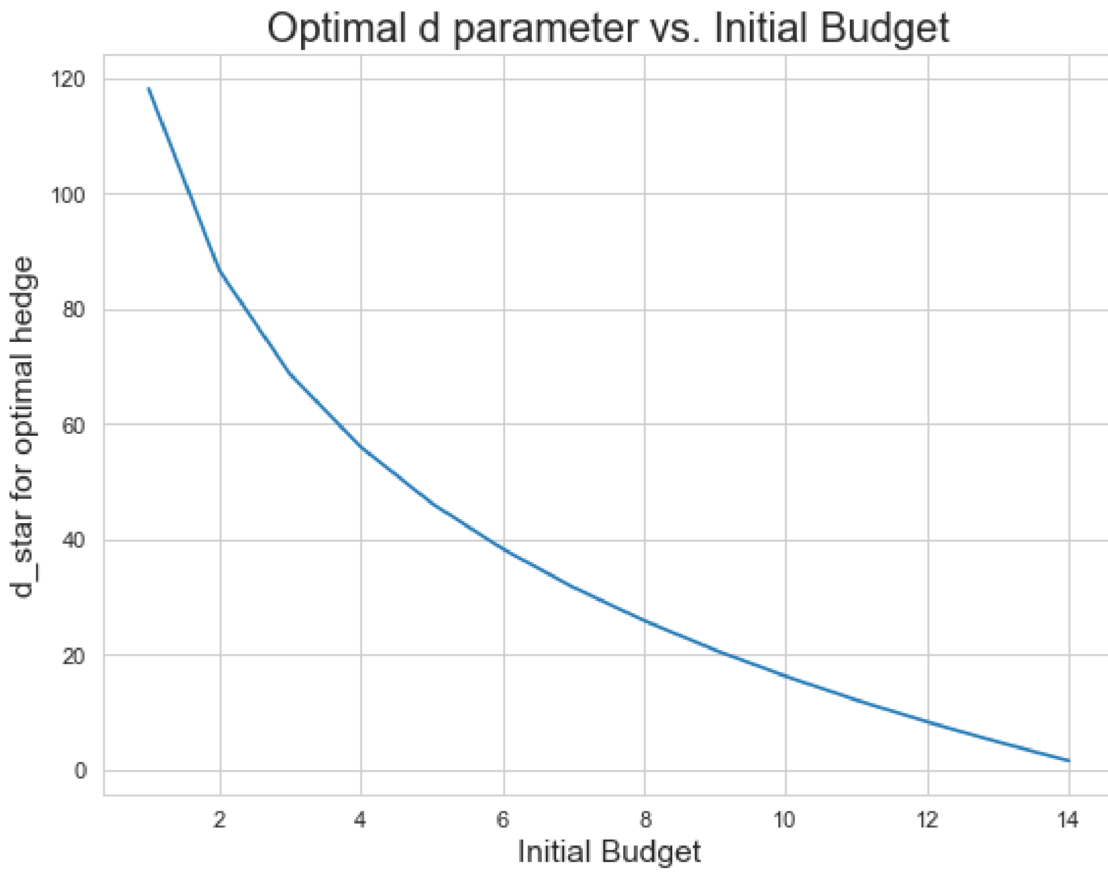

Figure 7.2: Optimal parameter for CVaR hedge vs. Initial budget constraint

Example 7.12 (RVaR minimization on incomplete market) *From Section 7.4, first step in determining the optimal RVaR hedging strategy would be to verify if one has enough initial capital in order to replicate*

$$g_T \cdot I_{B \leq VaR_{\alpha+\beta}(g_T)}. \tag{7.38}$$

Assume the market introduced in Example 2.7. We would also use Utility-based indifference pricing functional for assessment of initial price of the claim. Since all coefficients are deterministic, as we mentioned in Section 4.4, optimal completion parameter λ will not depend neither on initial budget, nor on particular utility function. Therefore, we perform all calculations with respect to minimal martingale measure which would be equivalent to using UBIP functional.

Corresponding completion parameter is then $\nu = [0\ 0\ 0]$*. Initial price of modified claim* (7.38) *is then approximately 12.70. Since with initial capital greater than 12.7 investor can achieve RVaR=0, we assume that agent has only 95% of required capital, namely 12.07. Solving optimization problem* (7.37) *we obtain optimal parameters* $d^* = 2.03$*,* $u^* = 159.20$*. Therefore, RVaR-optimal hedging strategy will be to replicate bull-call spread* $(S_T - 112.0313)^+ + (S_T - 269.1985)^+$*.*

Chapter 8

Application of MMC to the markets with defaults

8.1 Markets with default

In case of defaultable markets we add another source of uncertainty generated by the possibility of default. Following the logic of Bielecki and Rutkowski 2002, assume that positive random variable τ such that $P(\tau = 0) = 0$ and $P(\tau > t) > 0, \forall t \geq 0$ describes time of default.

As in previous chapters, we will be working with standard multidimensional diffusion market model:

$$dB_t = B_t r_t dt, \quad B_0 = 1 \tag{8.1}$$

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^{k} \sigma_t^{ij} dW_t^j \right) \tag{8.2}$$

where W_t is a standard k-dimensional Brownian Motion on a complete probability space $(\Omega, \mathcal{G}, P)$. Filtration $\mathbb{F}$ is generated by the $\{W_t\}_{t\geq 0}$ and augmented by P-null sets of $\mathcal{G}$. Processes $\mu_t \in \mathbb{R}^n$, r_t and $\Sigma_t = \{\sigma_t^{ij}\}_{t\geq 0} \in \mathbb{R}^n \times \mathbb{R}^k$ are all assumed to be bounded $\mathbb{F}$-predictable processes. In line with it we assume that matrix $\Sigma_t^* \Sigma_t$ is non-degenerate and positive semi definite, so that this matrix is invertible (here and further * denotes transposed matrix).

We will also assume that the model is arbitrage free which means that there exists at least one process $\theta_t \in \mathbb{R}^k$ such that:

$$\theta_t = \Sigma_t^*(\Sigma_t^*\Sigma_t)^-1(\mu_t - 1_n \cdot r_t), \tag{8.3}$$

where $1_k = (1, ..., 1) \in \mathbb{R}^\ltimes, \mu_t = (\mu_t^1, ..., \mu_t^n) \in \mathbb{R}^n$. Alternatively, this can be written

$$\sum_{j=1}^{n} \sigma_t^{ij}\theta_t^j = \mu_t^i - r_t, \quad \forall i \in \overline{1, n}. \tag{8.4}$$

Now, to add default possibility to the model, introduce positive random variable τ, satisfying $P(\tau > t) > 0$ for all $t \geq 0$ and construct special counting process

$$N_t = 1_{\{\tau \leq t\}}, t \geq 0$$

This process generates filtration that covers all default information $\mathbb{H} = \{\mathcal{H}_t\}$. Consequently, full market information will be described by filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. With the help of time of default τ random variable, one can introduce survival process as

$$G_t = P(\tau > t|\mathcal{F}_t), \quad 0 \leq t \leq T.$$

Assume that $G_t > 0$ for all $t > 0$ and consider hazard process $\Gamma_t = -\ln G_t, t \geq 0$. It is a common assumption that $\Gamma_t = \int_0^t \gamma_s ds, t \geq 0$ for some non-negative and $\mathbb{F}$-predictable random variable $\{\gamma_t\}_{t \geq 0}$, which is called $\mathbb{F}$-intensity of the random time τ. It was demonstrated in Bielecki and Rutkowski 2002, that the process

$$M_t = N_t - \int_0^t \gamma_s(1 - N_{s-})ds = N_t - \int_0^{t\wedge\tau} \gamma_s ds, \quad t \geq 0, \tag{8.5}$$

follows a $\mathbb{G}$-martingale. Let us assume that default is independent of the "financial" uncertainty generated by Brownian motion W_t. This implies that W_t is not only $(\mathbb{F}, P)$-standard Brownian motion, but also remains standard Brownian motion under the "extended" filtration $\mathbb{G}$.

For markets with defaults, introduce the process

$$Z_t^\kappa = (1 + \kappa_\tau 1_{\{\tau \leq t\}}) \exp\left(-\int_0^{t\wedge\tau} \kappa_s\gamma_s ds\right), \quad 0 \leq t \leq T, \tag{8.6}$$

$$Z_t^\kappa = 1 + \int_0^t \kappa_s Z_{s-}^\kappa dM_s \tag{8.7}$$

where $\{\kappa_t\}_{0\le t\le T} > -1, dt\times dP - a.e.$ is taken from the class of Bounded, $\mathbb{G}$-predictable processes and M_t was introduced in (8.5).

Let us also remind a risk-neutral measure density for model (8.1) which was given in (1.6):

$$Z_t^A = \frac{dP^*}{dP} = \exp\left\{-\sum_{i=1}^{n}\int_0^T \theta_t^i dW_t^i - \frac{1}{2}\sum_{i=1}^{n}\int_0^T (\theta_t^i)^2 dt\right\} =$$
$$= \exp\left\{-\int_0^T \theta_t^* dW_t - \frac{1}{2}\int_0^T \|\theta_t\|^2 dt\right\}.$$

Note that

$$d(Z_t^A Z_t^\kappa) = Z_t^A Z_{t-}^\kappa(-\theta_t dW_t + \kappa_t dM_t) \tag{8.8}$$

since $[Z_t^A, Z_t^\kappa] = 0$. Consequently, process $Z_t^A Z_t^\kappa$ is a positive $\mathbb{G}$-martingale for κ_t taken from the corresponding class. Let us analyze dynamics of portfolio wealth process under new measure P^k, defined as $\frac{dP^\kappa}{dP} = Z_t^A Z_t^\kappa$.

As before we describe portfolio as a $\mathbb{G}$-predictable process $\{\pi_t\}_{0\le t\le T} \in \mathbb{R}^n$ that satisfy $\int_0^T \|\pi_t\|^2 dt < \infty$, a.s. Where each π_t^i denotes capital invested in i^{th} stock. Then the wealth process $\{V_t^\pi(x)\}_{0\le t\le T}$ of self-financing portfolio π_t with initial capital $x \ge 0$ can be described as

$$dV_t^\pi = \sum_{i=1}^{n} \pi_t^i\left((\mu_t^i - r_t)dt + \sigma_t^i dW_t\right) + V_t^\pi r_t dt, \tag{8.9}$$
$$V_0^\pi = x. \tag{8.10}$$

Further, define special process

$$L_t^\kappa = \frac{Z_t^A Z_t^\kappa}{B_t}. \tag{8.11}$$

Applying Ito lemma for process $L_t^\kappa V_T^\pi$, we obtain

$$dL_t^\kappa V_t^\pi = L_{t-}^\kappa dV_t^\pi + V_{t-}^\pi dL_t^\kappa + d[L_t^\kappa, V_t^\pi] = \tag{8.12}$$

$$= L_{t-}^\kappa \left(r_t V_t^\pi dt + \sum_{i=1}^n \pi_t^i (\mu_t^i - r_t) dt + \sum_{i=1}^n \pi_t^i (\bar{\sigma}_t^i \cdot d\bar{W}_t) \right) - \tag{8.13}$$

$$- r_t L_t^\kappa V_t^\pi dt + L_{t-}^\kappa V_t^\pi \left(\kappa_t dM_t - \bar{\theta}_t \cdot \bar{W}_t \right) - \tag{8.14}$$

$$- L_{t-}^\kappa \sum_{i=1}^n \pi_t^i \sum_{j=1}^n \bar{\sigma}_t^{ij} \theta_t^j dt = \tag{8.15}$$

$$= L_{t-}^\kappa \left(\sum_{i=1}^n \pi_t^i (\mu_t^i - r_t) dt + \sum_{i=1}^n \pi_t^i (\bar{\sigma}_t^i \cdot d\bar{W}_t) \right) + \tag{8.16}$$

$$+ L_{t-}^\kappa V_t^\pi \left(\kappa_t dM_t - \bar{\theta}_t \cdot \bar{W}_t \right) - \tag{8.17}$$

$$- L_{t-}^\kappa \sum_{i=1}^n \pi_t^i (\mu_t^i - r_t) dt. \tag{8.18}$$

Therefore discounted portfolio dynamics can be written as

$$dL_t^\kappa V_t^\pi = L_{t-}^\kappa \left(\sum_{i=1}^n \pi_t^i \sigma_t^i dW_t - V_t^\pi (\theta_t^* \cdot dW_t) + V_t^\pi \kappa_t dM_t \right), \quad 0 \le t \le T. \tag{8.19}$$

From non-negativity of wealth process V_t^π it follows that process $\{L_t^\kappa V_t^\pi\}$ is a supermartingale for each portfolio $\pi_t \in \mathcal{A}(x)$, where $\mathcal{A}(x)$ is the set of all portfolio processes with initial capital x and non-negative wealth process for $\forall t \in [0, T]$. Denote also $\mathcal{L}$ as a set of all random variables $L_t^\kappa, \kappa \in \mathcal{D}$.

We will consider contingent claims of the form

$$H = Y 1_{\{\tau > T\}}. \tag{8.20}$$

It was shown in Nakano 2011 that the conservative price for such claim would be a super-hedging price for default-free claim. However, it is reasonable to assume that price should reflect the risk of default, and, therefore, be lower due to risk premium. However, in such a case there is a possibility of a shortfall and we can use partial hedging technique in order to derive optimal hedging strategy for defaultable case.

8.2 Quantile Hedging

We first consider the situation in which investor is interested in maximizing the probability of a perfect hedge for the claim. In other words,

$$\max_{\pi\in\mathcal{A}(x)} P(V_T^\pi \geq g_T) \tag{8.21}$$

This problem, as in the default-free case, could be solved by the means of Neyman-Pearson lemma. For current scenario, when default event is possible, the Neyman-Pearson type task becomes

$$\begin{cases} \max_{\varphi\in\mathcal{R}} \mathbb{E}[\varphi], \\ \mathcal{R} = \{\varphi : 0 \leq \varphi \leq 1 \ a.s., \ \sup_{L\in\mathcal{L}} \mathbb{E}[Lg_T\varphi] \leq v_0\}. \end{cases} \tag{8.22}$$

To solve aforementioned problem, we can again use famous Neyman-Pearson Lemma in order to obtain solution as it was described in Chapter 5. However, in sake of demonstrating applications of method of market completions, in this section we demonstrate alternative path which was described in Nakano 2011. The nature of proposed approach is to solve specific "dual" problem which can be stated based on the following consideration

$$\mathbb{E}[\varphi] = \mathbb{E}[\varphi(1 - yLg_T)] + y\mathbb{E}[\varphi Lg_T] \leq \mathbb{E}\left[(1 - yLg_T)^+\right] + yx. \tag{8.23}$$

The idea is to focus on minimization problem

$$\inf_{y\geq 0,\ L\in\mathcal{L}} \left\{\mathbb{E}\left[(1 - yLg_T)^+\right] + yx\right\}. \tag{8.24}$$

To guarantee the existence of the solution for (8.24), instead of using the class $\mathcal{L}$ one should the closed under almost sure convergence and convex set $\bar{\mathcal{L}}$ defined as

$$\bar{\mathcal{L}} = \left\{L \in L^1 :\ \mathbb{E}[\beta_T^{-1}L] \leq 1,\ \mathbb{E}[Lg_T] \leq \sup_{L'\in\mathcal{L}}[L'g_T],\ \mathbb{E}[Lg_T\varphi] \leq x\right\} \tag{8.25}$$

for some random variable C.

Then, solution $(\hat{L}, \hat{y})$ to dual problem (8.24) will always exist in $\bar{\mathcal{L}} \times R^+$ and there won't be any duality gap between dual and initial problems. In other words, their solutions will coincide. Therefore, optimal randomized test for (8.22) will be written in the following form

$$\hat{\varphi} := 1_{\hat{y}\hat{L}g_T<1} + C \cdot 1_{\hat{y}\hat{L}g_T=1}. \tag{8.26}$$

However, the class $\bar{\mathcal{L}}$ is abstract enough to make it challenging to find an explicit solution. That is why, author in Nakano 2011 proposes to consider closed convex hull of $\mathcal{L}$ with respect to L^1 norm. Denoting the hull as $\bar{\mathcal{L}}_1$, we define

$$\mathcal{R}_1 = \{\varphi : 0 \leq \varphi \leq 1 \; a.s., \; \sup_{L\in\bar{\mathcal{L}}_1} \mathbb{E}[Lg_T\varphi] \leq v_0\} \tag{8.27}$$

and rewrite quantile hedging problem as

$$\max_{\varphi\in\mathcal{R}_1}[\varphi]. \tag{8.28}$$

Proposition 8.1 *Suppose $\hat{\varphi}$ solves the modified problem (8.28) and there exists portfolio $\hat{\pi}$ such that $V_T^{\pi}(x) \geq g_T$, (a.s.). Then $\hat{\pi}$ is optimal for the quantile hedging problem (8.22).*

Assume first that market is complete. Then, defining

$$\hat{L} = \beta_T Z_T^A 1_{\{\tau>T\}} e^{\int_0^T \gamma_t dt}, \tag{8.29}$$

one could notice the following relation

$$\mathbb{E}[1 \wedge yL_T^{\kappa}g_T] = \mathbb{E}[1 \wedge y\beta_T Z_T^A Z_T^{\kappa} g_T 1_{\{\tau>T\}}] = \tag{8.30}$$

$$= \mathbb{E}[1 \wedge y\beta_T Z_T^A (1+\kappa_\tau 1_{\{\tau\leq T\}}) e^{-\int_0^T \kappa_t\gamma_t dt} g_T 1_{\{\tau>T\}}] \leq \tag{8.31}$$

$$\leq \mathbb{E}[1 \wedge y\beta_T Z_T^A 1_{\{\tau>T\}} e^{\int_0^T \gamma_t dt} g_T 1_{\{\tau>T\}}] = \mathbb{E}[1 \wedge y\hat{L}g_T]. \tag{8.32}$$

Consequently, as $\mathbb{E}\left[(1 - yL_T^{\kappa}g_T)^+\right] = 1 - \mathbb{E}[1 \wedge yL_T^{\kappa}g_T]$, $\hat{L}$ is a candidate to be the optimal for minimization problem (8.24). The following theorem was formulated in Nakano 2011 and we provide it for reference

Theorem 8.2 *Suppose $H = g_T 1_{\{\tau > T\}}$ and $\mathbb{E}^*[\beta_T g_T] < \infty$ then $\hat{L}$ defined by* (8.29) *solves*

$$\inf_{L \in \hat{\mathcal{L}}_1} \left\{ \mathbb{E}\left[(1 - yLH)^+ \right] \right\}. \tag{8.33}$$

Moreover, there exists $\hat{y}_1 > 0$ that minimizes

$$h(y) = \mathbb{E}\left[(1 - y\hat{L}H)^+ \right] + yx, \ y \geq 0. \tag{8.34}$$

The pair $(\hat{L}, \hat{y})$ is optimal for the problem

$$\inf_{y \geq 0, \ L \in \bar{\mathcal{L}}_1} \left\{ \mathbb{E}\left[(1 - yLH)^+ \right] + yx \right\}. \tag{8.35}$$

By the means of this theorem it is possible to find optimal solution to quantile hedging problem (8.21).

$$\xi_1 = \hat{y}_1 \beta_T Z_T^A e^{\int_0^T \gamma_t dt} g_T \tag{8.36}$$

Following Nakano 2011, the optimal strategy for quantile hedging problem (8.21) will be a perfect hedging strategy for modified claim $g_T 1_{\{\xi_1 < 1\}}$.

8.2.1 Incomplete Market Case

We now move on to demonstration of application of method of market completions for quantile hedging on defaultable markets. Of course, defaultable market is incomplete due to appearance of the risk of default, which we assume independent from financial market. In current section we consider scenario when underlying financial market is also incomplete. In this case, we introduce orthogonal market completions that parametrize completed versions of the initial market. Therefore, we arrive at the set of processes (8.11) different for each completed version of the market:

$$L_t^{\kappa,\nu} = \frac{Z_t^A Z_t^\nu Z_t^\kappa}{B_t}. \tag{8.37}$$

It is straightforward to check that for each $\nu \in \ker(\Sigma)$, process $L_t^{\kappa,\nu} V_t^\pi$ is a supermartingale. Therefore, following the same steps from previous section on each completed market, solution to quantile hedging problem will be constructed with the help of the following completion-dependent quantity

$$\xi_1^{\nu} = \hat{y}_1 \beta_T Z_T^A Z_T^{\nu} e^{\int_0^T \gamma_t dt} g_T \tag{8.38}$$

By choosing market completion, corresponding to minimal martingale measure, we obtain solution to quantile hedging problem on incomplete market.

Assembling together results provided above, we can state the following theorem.

Theorem 8.3 *For incomplete market with defaults, solution to quantile hedging problem* (8.21) *will be a perfect hedge for modified claim* $\tilde{H} = g_T 1_{\{\xi_1^{\lambda} < 1\}}$.

We illustrate application of the proposed methodology in the following example.

Example 8.4 *Consider financial market from Example 2.7. In addition to structural market incompleteness we add possibility of default which is independent from any financial information. Let us assume that* $\gamma = 0.05$, *which implies that for* $T = 1$, *probability* $P(\tau > T) = 0.9512$.

With fair price of the claim at approximately 14, in Figure 8.1 we compare the probabilities of successful hedging for claim on defaultable market with default-free case.

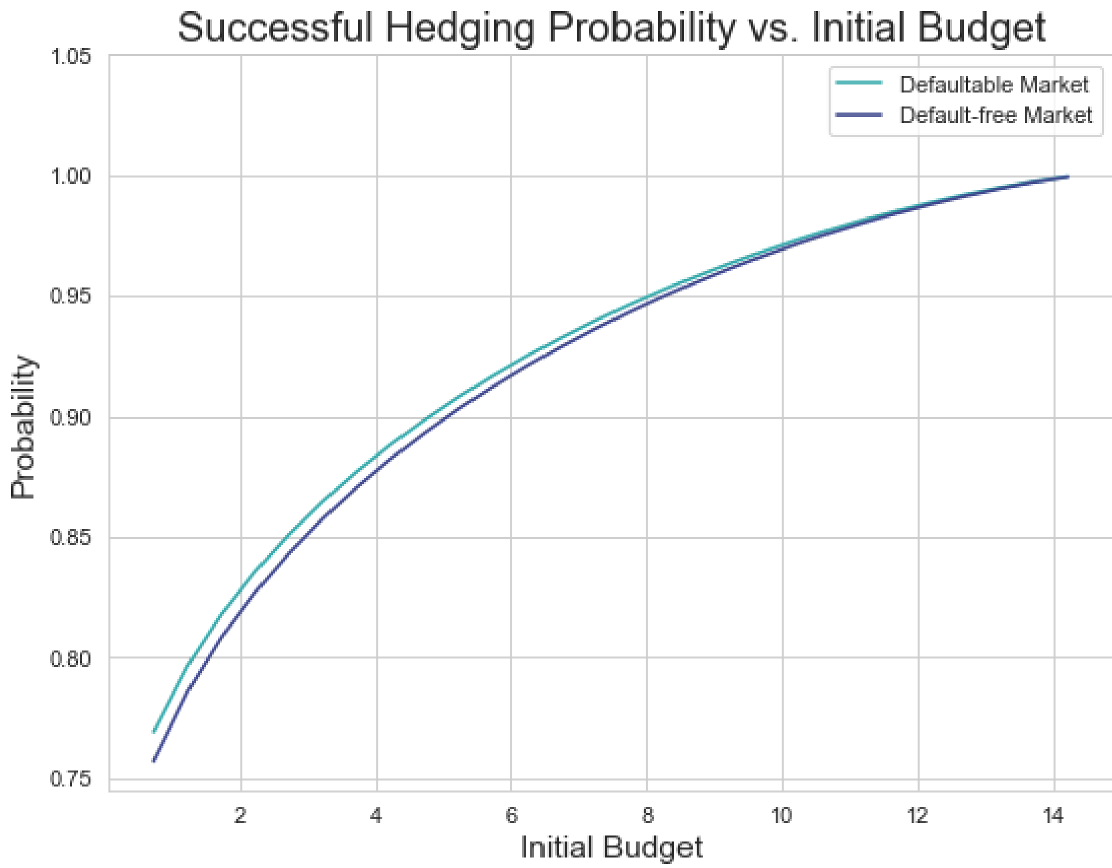

Figure 8.1: Probability of Successful hedging vs. Initial budget constraint

8.3 CVaR Hedging

Assume now, that investor with available initial budget v_0 is willing to construct optimal hedging strategy with respect to CVaR risk measure. Since this measure is constructed on the distribution of losses from hedging, first step would be to define such loss function for defaultable market scenario. As P&L will be conditional on default event occurrence, it is straightforward to define loss from hedging as:

$$L^{\tau}(x,\pi) = g_T 1_{\{\tau > T\}} - V_T^{\pi}(x).$$

Consequently, we arrive at the following representation of CVaR for hedging position on defaultable market:

$$c(z,\tau) = z + \frac{1}{1-\alpha} \min_{(x,\pi)} \mathbb{E}\left[(g^{\tau}(z) - V_T^{\pi}(x))^{+}\right] \tag{8.39}$$

where $g^\tau(z) = (g_T - z)^+ 1_{\{\tau > T\}} = g(z) 1_{\{\tau > T\}}$.

$$\min_{z \in \mathbb{R}} c(z, \tau) = \min_{(x,\pi)} CVaR_\alpha(x, \pi). \tag{8.40}$$

Therefore, in order to construct CVaR-minimizing hedging portfolio on the defaultable market, we could use approach presented in Section 7.3 of two-parametric optimization. Therefore, we first should solve the following minimization problem:

$$\begin{cases} \mathbb{E}\left[(g^\tau(z) - V_T^\pi(x))^+\right] \to \min_{\pi \in \mathcal{A}} \\ x < v_0. \end{cases} \tag{8.41}$$

Considering $\hat{L}$ defined in (8.29), from Nakano 2011 it is known that there exists $\hat{y}_2 > 0$ that maximizes

$$\mathbb{E}\left[g_T 1_{\{\tau > T\}} \left(\beta_T \wedge y\hat{L}\right)\right] - yv_0 \tag{8.42}$$

over all $y > 0$.

On complete market, the following theorem holds.

Theorem 8.5 (Theorem 4.6 from Nakano 2011) *Suppose that* $H = g_T 1_{\{\tau > T\}}$ *with* $\mathbb{E}^*[\beta g_T] < \infty$ *and that for*

$$\xi_2 = \hat{y}_2 Z_T^A e^{\int_0^T \gamma_t dt} g_T, \tag{8.43}$$

where $\hat{y}_2$ *is from* (8.42), *holds* $P(\xi_2 = 1) = 0$.

Then the prefect hedging portfolio for $g_T 1_{\{\xi_2 < 1\}}$ *is optimal for*

$$\min_{\pi \in \mathcal{A}(v_0)} \mathbb{E}\left[(H - V_T^\pi)^+\right].$$

Then, for each value $z > 0$, it is fair to derive from Theorem 8.5 with $H = H(z) = g_T(z) 1_{\{\tau > T\}}$, that solution to (8.41) will be constructed as a perfect hedging portfolio for modified claim $g_T(z) 1_{\{\xi_2(z) < 1\}}$, where

$$\xi_2(z) = \hat{y}_2(z) Z_T^A e^{\int_0^T \gamma_t dt} g_T(z), \tag{8.44}$$

$$\hat{y}_2(z) = \arg\max_{y > 0} \left(\mathbb{E}\left[g_T(z) 1_{\{\tau > T\}} \left(\beta_T \wedge y\hat{L}\right)\right] - yv_0\right). \tag{8.45}$$

Consequently, in order to find solution to CVaR hedging problem, it is left to find $\hat{z}$, that minimizes

$$c(z) = \begin{cases} z + \frac{1}{1-\alpha}\mathbb{E}\left[(g_T - z)^+(1 - 1_{\{\xi_2(z)<1\}})\right], & z < z^*, \\ z, & z \geq z^*, \end{cases} \tag{8.46}$$

on interval $z < z^*$, and z^* is a real root of equation

$$v_0 = \mathbb{E}^*\left[(g_T - z^*)^+\right]. \tag{8.47}$$

8.3.1 Incomplete Market Case

In this section we will demonstrate how method of market completions can be applied in order to solve CVaR hedging problem on defaultable market, which is at the same time structurally incomplete. As it was mentioned in the previous section, problem of finding optimal CVaR hedge can be decomposed into two sequential optimization problems.

In Section 6.2.2 we demonstrated how this problem can be solved with the help of market completions for the case of incomplete market. As before, we parametrize completed versions of the market with vector ν. Therefore, for each z, solution to sub-problem (8.41) on corresponding completed markets will be to perfectly hedge portfolio for modified claim $g_T(z)1_{\{\xi_2^\nu(z)<1\}}$, where

$$\xi_2^\nu(z) = \hat{y}_2^\nu(z) Z_T^A Z_T^\nu e^{\int_0^T \gamma_t dt} g_T(z), \tag{8.48}$$

$$\hat{y}_2^\nu(z) = \arg\max_{y>0}\left(\mathbb{E}\left[g_T(z)1_{\{\tau>T\}}\left(\beta_T \wedge y\hat{L}^\nu\right)\right] - yv_0\right). \tag{8.49}$$

where

$$\hat{L}^\nu = \beta_T Z_T^A Z_T^\nu 1_{\{\tau>T\}} e^{\int_0^T \gamma_t dt}, \tag{8.50}$$

And corresponding solution to CVaR hedging problem on each completed market would be found with $\hat{z}$, that minimizes

$$c(z,\nu) = \begin{cases} z + \frac{1}{1-\alpha}\mathbb{E}\left[(g-z)^+(1-1_{\{\xi_2^\nu(z)<1\}})\right], & z < z^*, \\ z, & z \geq z^*, \end{cases} \tag{8.51}$$

on interval $z < z^*$, and z^* is a real root of equation

$$v_0 = \mathbb{E}^\nu\left[(g-z^*)^+\right]. \tag{8.52}$$

As it was demonstrated earlier, if we decide to use UBIP as a pricing functional for assessing initial price of the claim, we should pick market completion with $\nu = 0$ that corresponds to completed version on which unique equivalent martingale measure will coincide with minimal martingale one. We summarize proposed results in a form of the following theorem.

Theorem 8.6 *Strategy that minimizes CVaR from hedging defaultable claim $g_T 1_{\{\tau>T\}}$, will be a perfect hedge for modified claim $\bar{g_T} = g_T \cdot 1_{\{\xi_2^0(\hat{z})<1\}}$, where $\hat{z}$ is a point of global minimum of the function*

$$c(z,0) = \begin{cases} z + \frac{1}{1-\alpha}\mathbb{E}\left[(g_T-z)^+(1-1_{\{\xi_2^0(z)<1\}})\right], & z < z^*, \\ z, & z \geq z^*, \end{cases} \tag{8.53}$$

on interval $z < z^*$, and z^* is a real root of equation

$$v_0 = \mathbb{E}\left[Z_T^A(g_T-z^*)^+\right]. \tag{8.54}$$

Chapter 9

Conclusions, Recommendations, & Future Work

In this thesis I have introduced Method of Market Completions as a useful framework for solving various problems in a field of mathematical finance for structurally incomplete market models with the main focus on standard multidimensional market model.

Starting with utility maximization task, where I have demonstrated how existing approach, developed for concave utility functions, can be extended towards the case of not necessarily concave ones. Then, we moved on to pricing of contingent claims, where it was shown that method of market completions provides dual approach for obtaining the same results as a classical martingale and numerarie approaches. Moreover, proposed method becomes extremely useful element for utility-based indifference pricing approach, which proves itself to be a powerful pricing technique for tasks on incomplete market. In addition to demonstration of application of the method, we also show the relationship between UBIP pricing functional and famous Minimal Martingale measure for special cases of market models. It would be an interesting direction of further research to analyze more complex constraints existing on the market such as convex sets of admissible strategies, or different interest rates.

Further, I discuss how method of market completions can be applied for problems of quantile and effective hedging on incomplete market. Since these problems are well described in literature, I propose a way of transferring this knowledge to incomplete market situation. This is achieved by parametrization of all possible completed markets with the family of orthogonal completions (special sets of auxiliary assets introduced to the market). Then, having corresponding set of solutions on each completed version of the initially incomplete market, I demonstrate how one could choose particular solution from this set, that will be attainable and optimal for incomplete market. I present two perspectives on choosing such solution. Firstly, one could work directly with the set of obtained solutions "bottom-up", by analyzing which solution will be "the best fit" according to both investors budget constraint and availability of the assets for trade. Alternatively, it is possible to choose with "top-down" approach, by imposing external requirements, such as finding solution from assumption of locally risk minimizing criteria, or, in other words, choosing completion that leads to equivalent martingale measure that coincides with minimal martingale measure. In both cases method of market completions is demonstrated to be a convenient way of parametrizing markets and working with them. Since method has proven to be effective in this tasks it is an interesting to look closely at the question of choice of completing assets, possibly by the means of machine learning algorithms.

Problems of quantile and effective hedging are fundamental for further development of partial hedging. One of such directions, actively used in modern risk-management industry is partial hedging with respect to distribution-based risk measures. Therefore, I discuss the question of CVaR optimal hedging with budget constraint in place. With two different approaches I demonstrate how working from market completions perspective helps to solve the problem on structurally incomplete market. Then, I also presented more general, two-parametric risk-measure, that is gaining its popularity in recent research – RVaR.

Finally, I briefly described how proposed methodology can be used for market models where possibility of default is present.

In current manuscript I have set up a framework of parametrization of incomplete market model by means of market completions. It would be reasonable to further develop method of market completions to more general case of models that are based on Levy processes.

Another interesting question would be to find parameters of completions which would help investors achieve their specific goals. For example, if some agent aims at maintaining particular level of risk – what parameters of additional market assets he should be searching for in order to get it? With the growing interest in machine learning algorithms, looking at incomplete markets from the perspective of method of market completions opens opportunity for researching angles of application of some techniques for the problems of mathematical finance. One of such directions might be to use Principal Component Analysis in order to arrive at specific volatility matrix for model.

As in the present work I have only focused on structural incompleteness, it would be useful to work with other "types" of incomplete markets and see if method of market completions could be extended for them.

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