University of Alberta

STRUCTURAL PROPERTIES OF FINITE AND INFINITE DIMENSIONAL BANACH SPACES

by

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To My Family

ABSTRACT

In this thesis we discuss two separate topics from the theory of Banach spaces. One comes from the local theory of finite dimensional spaces (Part I), the other from infinite dimensional Banach spaces (Part II).

Part I is concerned with the study of certain structural properties of finite dimensional normed spaces. It is shown that a finite dimensional Banach space has the Euclidean distance of maximal order if and only if it contains a proportional dimensional subspace (and a quotient of a subspace) of a very special form. This is joint work with N. Tomczak-Jaegermann and R. Anisca and was published in Houston Journal of Mathematics.

In Chapter 1 we recall basic concepts in Banach space theory as well as more specific results from Local Theory. Chapter 2 contains the main result of this part of the thesis.

Part II is of a infinite dimensional nature and presents a new result on the asymptotic structure of Banach spaces. We prove that if a Banach space is saturated with infinite dimensional subspaces in which all special n-tuples of vectors are equivalent, uniformly in n, then the space contains asymptotic-l_p subspaces, for some $1 \leq p \leq \infty$. The proof reflects a technique used by Maurey in the context of unconditional basic sequence problem and extends a result by Figiel, Frankiewicz, Komorowski and Ryll-Nardzewski.

In Chapter 3 we introduce typical infinite dimensional concepts and discuss in more detail the notion of asymptotic structure. Chapter 4 is devoted to the main result.

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Part I

Structure of normed spaces with extremal distance to the Euclidean space

Chapter 1

Introduction, Finite Dimensional Banach Spaces

The *Local Theory* is the part of Banach space theory that investigates the structure of finite dimensional spaces and the connections between infinite dimensional Banach spaces and their finite dimensional subspaces. Since the isomorphic classification of finite dimensional normed spaces is trivial, with two normed spaces being isomorphic if and only if they have the same dimension, meaningful results in Local Theory are quantitative in nature. Finite dimensional methods involve the study of certain isometric invariants and their behavior as the dimension grows to infinity.

In order to describe our main result of this part of the thesis let us recall the fundamental notion of the *Banach-Mazur distance.* For two n-dimensional normed spaces X and Y, the Banach-Mazur distance between X and Y is defined as the infimum of expressions $\|w\|\|w^{-1}\|$ over all isomorphisms $w : X \to Y$. It follows from a classical result of F.John that for any n-dimensional normed space X there is an isomorphism $w : X \to l_2^n$ such that $\|w\|\|w^{-1}\| \leq$

$\sqrt{n}$, thus the Euclidean distance of X (which measures how far the isomorphic structure of a space X is from Euclidean space), denoted by d_X, satisfies $d_X \le \sqrt{n}$. Well known examples for which this estimate is sharp are l_1^n and l_∞^n. In fact the class of spaces for which $d_X = \sqrt{n}$ is much larger. In this thesis, we first introduce a family of spaces for which the Euclidean distance is maximal; the construction is described in Example 2.1. We also consider the isomorphic version of this situation and study n-dimensional normed spaces X for which $d_X \ge c\sqrt{n}$, where c is an absolute constant (hence independent on n). We show that a space satisfies this conditions if and only if the space contains a large part which "resembles" the family of spaces previously introduced in Example 2.1. But first we shall put the problem into a wider perspective.

Since for any normed spaces X, Y, Z we have that $d(X,Y) \le d(X,Z)d(Y,Z)$ we can find an upper bound for $d(X,Y)$ by first bounding the distance of an n-dimensional space from l_2^n. Therefore we have that

$$d(X,Y) \le d(X,l_2^n)d(Y,l_2^n) \le n^{1/2}n^{1/2} = n.$$

Although this estimate seems somewhat crude, since the Banach-Mazur distance between X and Y is estimated by going through l_2^n, it is, in fact, close to being the best possible. In 1981 E.D. Gluskin proved that there is a constant $c > 0$ such that for every n there are n-dimensional normed spaces X_n and Y_n with $d(X_n, Y_n) > cn$. At the time the result was extremely surprising; the proof was based on a probabilistic argument which since has become an important tool in Local Theory.

As we mentioned before, the estimate for d_X obtained from F. John's Theorem is sharp; it can be checked that $d_{l_1^n} = d_{l_\infty^n} = \sqrt{n}$ for any n. What can be said about subspaces and quotients? Given an n-dimensional normed space

X, can we get "closer" to the Euclidean space by passing to subspaces or quotients? More precisely, given X and ε, for which k does there exists a subspace $E \subseteq \mathbb{R}^n$ with dimension k and an ellipsoid $D \subseteq E$ such that

$$D \subset B_X \cap E \subset (1+\varepsilon)D \tag{1.1}$$

where B_X is the unit ball of X? A celebrated result of Dvoretzky answers this question.

Theorem 1.1 (Dvoretzky). *Let X be an n-dimensional normed space and $\varepsilon > 0$. There exists an integer $k \geq c(\varepsilon)\log n$, with $c(\varepsilon) > 0$ depending only on ε, and a k-dimensional subspace E of X which satisfies $d_E \leq 1+\varepsilon$.*

This estimate is the best possible in general. It can be shown that for $X = l_\infty^n$ the $\log n$ bound cannot be improved. However, if the unit ball of X is, in a certain analytic sense, far from the cube then Figiel, Lindenstrauss and Milman showed in [F-L-M] that the estimate can be improved to $c(\varepsilon)n^\alpha$ for some $\alpha > 0$. More precisely, if X has cotype q $(2 \leq q < \infty)$ with constant C then this holds with $\alpha = 2/q$ and $c(\varepsilon)$ depending only on ε and C. The most interesting case is the case $q = 2$, hence $\alpha = 1$ for which we find an answer to (1.1) with k proportional to n. In particular, this covers the case $X = l_p^n$ for $1 \leq p \leq 2$. We mention in passing that the notions of *type* and *cotype* are very important in Local Theory and they have been developed in close connection with the geometry of Banach spaces. We shall not make use of them here and for more information and a detailed presentation of type and cotype we direct the reader to [TJ], Section 4.

By duality, (1.1) implies that B_X°, the polar of B_X, admits a projection onto E which is $(1+\varepsilon)$-equivalent to an ellipsoid. Namely, if P_E is the orthogonal projection from X onto E, we have

$$(1+\varepsilon)^{-1}D^\circ \subset P_E(B_X^\circ) \subset D^\circ. \tag{1.2}$$

Since B_X is arbitrary, we can replace B_X by B_X° in (1.2). Thus, Dvoretzky's Theorem says that for any n-dimensional normed space X, the unit ball B_X admits k-dimensional sections and k-dimensional projections which are almost ellipsoids. However, in general k is small compared to n.

One of the striking discoveries of Milman [Mi] (known as *Quotient of a Subspace Theorem*) is that if we consider the class of all projections of sections of B_X (instead of either sections or projections), then we can always find a projection of a section (equivalently, a quotient of a subspace) which is $(1+\varepsilon)$-equivalent to an ellipsoid and has dimension $k = c(\varepsilon)n$, with $c(\varepsilon) > 0$ depending only on $\varepsilon > 0$. Thus we find again k proportional to n, but this time without any assumptions on X.

At the other end of the "spectrum" we have the n-dimensional normed spaces that are as far from the Euclidean space as possible: their Euclidean distance is asymptotically of order $\sqrt{n}$, as $n \to \infty$. For example, if we consider the space l_∞^n it can be easily shown that it contains isometric copies of l_1^k, with $k \to \infty$ as $n \to \infty$. Milman and Wolfson showed in ([M-W]) that this is true in a more general situation.

Theorem 1.2. *Let X be an n-dimensional normed space such that $d_X = \sqrt{n}$. Then X has a k-dimensional subspace E, with $k \geq c \log n$, which is isometric to l_1^k. Here c is a universal constant.*

The estimate is exact, since l_∞^n contains an l_1^k with k not greater than $\log_2 n$. In the same paper, Milman and Wolfson also proved an isomorphic version of this result.

Theorem 1.3. *Let $0 < \alpha < 1$. There exists $c \geq 1$ such that every n-dimensional normed space with $d_X \geq \alpha\sqrt{n}$ contains a k-dimensional subspace E, with $k \to \infty$ as $n \to \infty$, such that $d(E, l_1^k) < c$.*

The original estimate for k was $k \sim \log\log\log n$ and was later improved to the best asymptotic estimate $k \sim \log n$ through work of Kashin, Bourgain, Tomczak-Jaegermann (see [TJ], Section 31 for details). What about the proportional-dimensional structure of such spaces? Using a deep combinatorial results of Elton [E] and Pajor [Pa] it can be shown that if the n-dimensional normed space X is of type 2 and d_X is of maximal order then it contains a copy of l_1^k with k proportional to n.

We will show in this part of the thesis that the proportional-dimensional structure of spaces with Euclidean distance of maximal order (without any additional assumptions on the space) is surprisingly regular as well and it contains subspaces (and quotients of subspaces) of a very special form. This is joint work with N. Tomczak-Jaegermann and R. Anisca and appeared in a paper [A-T-TJ] published in Houston Journal of Mathematics.

We shall briefly describe the organization of this part of the thesis. Section 2.1 brings a few comments about spaces whose Euclidean distance is equal to $\sqrt{n}$. It is easy to see that the spaces $X = l_1^n$ and $X = l_\infty^n$ satisfy this condition (the unit balls of these spaces are the octahedron and the cube, respectively). An interesting example shows that any space X whose unit ball is squeezed between the cube and an octahedron spanned by an orthogonal system of vertices of the cube, also satisfies $d_X = \sqrt{n}$.

Section 2.2 contains the main result of this part of the thesis, Theorem 2.2. It shows that X has the Euclidean distance of maximal order if and only if X contains a subspace (and a quotient of a subspace) of proportional dimension which is an isomorphic analogue of the above example of a space squeezed between the cube and an octahedron. Our main theorem follows from a more general result, Theorem 2.5, on spaces whose Euclidean distance is large, but not necessarily of maximal order. Apart of basic properties of operator ideals related to l_2-factorizations, the main ingredient in the proofs is the Bourgain-Tzafriri restricted invertibility theorem ([B-T]).

For the remaining of this chapter we recall basic concepts in Banach space theory as well as more specific results that we will be using in Chapter 2.

1.1 Basic Concepts

A *normed space* is a pair $(X, \|\cdot\|)$, where X is a vector space over $\mathbb{R}$ or $\mathbb{C}$ and $\|\cdot\|$ is a real valued function such that the following conditions are satisfied by all vectors x and y of X and each scalar α:

(i) $\|x\| \geq 0$, and $\|x\| = 0$ if and only of $x = 0$;

(ii) $\|\alpha x\| = |\alpha|\|x\|$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality);

Every normed space is a metric space with the induced metric given by $d(x, y) := \|x - y\|$. The induced metric in turn, defines a topology on X, called the *norm topology.*

Let $(X, \|\cdot\|)$ be a normed space. A subspace of $(X, \|\cdot\|)$ is a linear subspace Y of the underlying vector space, endowed with the restriction to Y of the norm on X. A subspace is *closed* if it is closed in the norm topology.

A normed space is called a *Banach space* if it is complete as a metric space, i.e. if every Cauchy sequence is convergent: if $(x_n)_{n\geq 1} \subset X$ is such that

$||x_n - x_m|| \to 0$ as $\min\{n, m\} \to \infty$ then $(x_n)_{n\geq 1}$ converges to some point x_0 in X (i.e., $||x_n - x_0|| \to 0$). It is easy to see that a subspace of a Banach space is complete if and only if it is closed.

If X and Y are two normed spaces over the same field we define a *linear operator* from X to Y to be a map $T : X \longrightarrow Y$ such that

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$$

for all $x_1, x_2 \in X$ and scalars λ_1, λ_2. A linear operator $T : X \longrightarrow Y$ is bounded if there exists $M > 0$ such that

$$||Tx|| \leq M||x||$$

for all $x \in X$. The smallest constant M satisfying the above inequality is denoted by $||T||$ and is called *norm* of T.

Two normed spaces X and Y are said to be *isomorphic* if there is a one-to-one operator from X onto Y such that T and T^{-1} are both bounded. In this context T is called a (linear) isomorphism. We call X and Y *isometrically isomorphic* if there is a linear isomorphism from X to Y such that $||T(x)|| = ||x||$ for all x in X. For isomorphic Banach spaces X and Y the Banach-Mazur distance is defined by $d(X, Y) := \inf ||T||\,||T^{-1}||$, where the infimum runs over all isomorphisms T from X onto Y.

Two norms, $||\cdot||_1$ and $||\cdot||_2$, on the same vector space X are said to be *equivalent* if they induce the same topology on X. Alternatively, the two norms are equivalent if there are constants $C, D > 0$ such that

$$C||x||_1 \leq ||x||_2 \leq D||x||_1 \tag{1.3}$$

for all $x \in X$. The equivalence of norms has an intuitive geometrical interpretation. Let B_1 and B_2 be the closed unit balls in $(X, ||\cdot||_1)$ and $(X, ||\cdot||_2)$ respectively. Then (1.3) holds if and only if $B_2 \subset (1/C)B_1$ and $B_1 \subset DB_2$.

Among the first "classical" Banach spaces to be studied were the sequence spaces l_p and c_0. For $1 \leq p < \infty$ the space l_p consists of all scalar sequences $x = (x_1, x_2, \dots)$ for which

$$\left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty.$$

The norm of an element $x \in l_p$ is

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}.$$

The space l_∞ consists of all bounded scalar sequences with

$$\|x\|_\infty = \sup_i |x_i|$$

and c_0 is the space of all scalar sequences tending to 0, with the same norm $\|x\|_\infty$. Among all Banach spaces, the Hilbert space l_2 is the "nicest" and most "regular". It provides a natural generalization of the n-dimensional Euclidean space l_2^n.

1.2 Finite Dimensional Normed Spaces

If we consider just the crude classification of norms, finite dimensional normed spaces are very simple in the sense that any two norms on a finite dimensional vector space are equivalent. Moreover, every finite dimensional normed space is complete. In particular, if Y is a finite dimensional subspace of a normed space X, then Y must be closed in X.

An important characterization of a finite dimensional normed space is the fact that a normed space X is finite dimensional if and only if $B_X = \{x \in X : \|x\| \leq 1\}$, the closed unit ball in X is compact. It can also be easily proved that every linear map on a finite dimensional normed space is continuous.

To quote N. Carothers [C], every finite dimensional normed space over $\mathbb{R}$ is just "$\mathbb{R}^n$ in disguise". To see this, suppose that $(X, \|\cdot\|_X)$ is a finite dimensional normed space with basis $x_1, x_2, \ldots, x_n$ and let $e_1, e_2, \ldots, e_n$ be the standard unit vector basis in $\mathbb{R}^n$. We define a norm on $\mathbb{R}^n$ by setting

$$\|\sum_{i=1}^{n} a_i e_i\| = \|\sum_{i=1}^{n} a_i x_i\|_X$$

where $a_1, a_2, \ldots, a_n$ are any scalars in $\mathbb{R}^n$. It is easy to see that the basis-to-basis map $x_i \longmapsto e_i$ extends to a linear isomorphism from $(X, \|\cdot\|_X)$ onto $(\mathbb{R}^n, \|\cdot\|)$ and these spaces are isometrically isomorphic.

We consider the standard Euclidean norm on $\mathbb{R}^n$, that is,

$$\|x\|_2 := (\sum_{i=1}^{n} |a_i|^2)^{1/2}$$

for $x = (a_i) \in \mathbb{R}^n$. By B_2^n we denote the corresponding unit ball and by $(\cdot, \cdot)$ the corresponding inner product. By an ellipsoid we mean a set of the form $\mathcal{E} = w(B_2^n)$ for any one-to-one operator $w : \mathbb{R}^n \to \mathbb{R}^n$. As noted before, for any n-dimensional normed space X we can always identify X with $\mathbb{R}^n$ by selecting a basis in X. We shall then write $X = (\mathbb{R}^n, \|\cdot\|_X)$ or $(\mathbb{R}^n, B_X)$, depending whether we would like to emphasize the norm or the unit ball in X, and we call it a position of X (or of B_X). Of course for every X there is a multitude of such positions. In particular for any position $X = (\mathbb{R}^n, B_X)$ and any ellipsoid $\mathcal{E}$ on $\mathbb{R}^n$ we can apply a linear invertible operator which takes $\mathcal{E}$ into B_2^n; it then takes B_X into some $\tilde{B}_X$, which is another position of B_X (so the spaces $(\mathbb{R}^n, B_X)$ and $(\mathbb{R}^n, \tilde{B}_X)$ are isometric). Conversely, if B is a compact convex symmetric subset of $\mathbb{R}^n$ with nonempty interior (in the sequel we will call these sets simply *balls*), and if we denote by $\|\cdot\|_B$ the Minkowski functional of B then the space $\mathbb{R}^n$ equipped with $\|\cdot\|_B$ is a normed space having B as its unit ball. For a ball K we denote by $\mathrm{Vol}(K)$ the Lebesgue measure of K in the appropriate dimension.

It will be useful to geometrically identify the balls of subspaces and quotient spaces of an n-dimensional normed space X having unit ball $B_X \subset \mathbb{R}^n$. This is intuitively clear for subspaces: if Y is a subspace of X then the section $B_X \cap Y$ can be viewed as the unit ball of the normed subspace Y. If we consider the quotient space X/Y, geometrically this corresponds not to sections of B_X but to linear projections of B_X. Indeed, let $P : X \longrightarrow X$ be any linear projection such that $\ker P = Y$ and let Z be the range of P. We equip Z with the norm that admits $P(B_X)$ as its unit ball. Then Z is isometric to X/Y. In particular, if we use the orthogonal projection $P_{Y^\perp}$ onto $Y^\perp$ (orthogonal with respect to a inner product on X, fixed in advance) then $P_{Y^\perp}(B_X)$ can be naturally identified with the unit ball of the normed space X/Y.

Let $\|\cdot\|_X$ be a norm on $\mathbb{R}^n$, and X be the corresponding Banach space. Every isomorphism $u : l_2^n \to X$ induces an inner product $[\cdot,\cdot]$ on X defined by

$$[x, y] = (u^{-1}x, u^{-1}y) \quad \text{for} \quad x, y \in X \tag{1.4}$$

and the Euclidean norm $|\cdot|_2$ on X defined by

$$|x|_2 = [x, x]^{1/2} = (u^{-1}x, u^{-1}x)^{1/2} \quad \text{for} \quad x \in X. \tag{1.5}$$

Then the ellipsoid $\mathcal{E} = \{x \in E : |x|_2 \leq 1\}$ is equal to $u(B_2^n)$. Conversely, every inner product on X determines an isomorphism $u : l_2^n \to X$ such that (1.4) holds.

Given an inner product $[\cdot,\cdot]$ on X and $K \subset \mathbb{R}^n$ we define the *polar* of K by

$$K^\circ = \{y : [x, y] \leq 1 \quad \text{for any} \quad x \in K\}$$

K° is convex and if K is symmetric then K° is symmetric as well and

$$K^\circ = \{y : |[x, y]| \leq 1 \quad \text{for any} \quad x \in K\}$$

For any given inner product $[\cdot,\cdot]$ on X, there is a natural identification between the dual space X^* and $\mathbb{R}^n$. More precisely, if $X = (\mathbb{R}^n, B_X)$ then $X^* = (\mathbb{R}^n, B_X^\circ)$. We also have the following natural identifications for all subspaces $Y \subset \mathbb{R}^n$:

$$(Y \cap B_X)^\circ = P_Y(B_X^\circ) \qquad (P_Y(B_X))^\circ = Y \cap (B_X^\circ)$$

1.3 More specific concepts and facts

Since all finite dimensional spaces of the same dimension over the same scalar field are isomorphic, for results on finite dimensional normed spaces to be meaningful they must be of a quantitative nature. The Banach-Mazur distance is of central importance in this context.

Recall that for isomorphic spaces X and Y the Banach-Mazur distance is defined as

$$d(X,Y) = \inf\{\|T\|\|T^{-1}\|\} \tag{1.6}$$

where the infimum runs over all isomorphisms $T : X \longrightarrow Y$. Let $\mathcal{M}_n$ denote the set of all normed spaces of dimension n. If X and Y are in $\mathcal{M}_n$ a simple compactness argument shows that the infimum is attained in (1.6), in particular X and Y are isometric if and only if $d(X,Y) = 1$. The relation *X and Y are isometric* is an equivalence relation on $\mathcal{M}_n$. If we denote by $\tilde{\mathcal{M}}_n$ the set of all classes modulo this equivalence then it is not hard to check that $\tilde{\mathcal{M}}_n$ equipped with the metric $\log d(X,Y)$ is a compact metric space, called the *Banach-Mazur compactum.*

Estimating the distance of an n-dimensional Banach space X to l_2^n is of particular interest. In the sequel we denote $d(X, l_2^n)$ by d_X. From the definition of Banach-Mazur distance it follows that there exists an ellipsoid $\mathcal{E}$ such that

$$\mathcal{E} \subseteq B_X \subseteq d_X \mathcal{E}$$

Such an ellipsoid is referred to as the *distance ellipsoid.* The following important theorem of F. John (1948) shows that one can obtain a good upper bound for d_X by considering the ellipsoid of maximal volume contained in B_X. By compactness the existence of such an ellipsoid is clear, but F.John also proved its uniqueness and, more importantly, characterized it.

Theorem 1.4 (F. John, 1948). *Let $(X, \|\cdot\|)$ be an n-dimensional Banach space. Then there exists a unique ellipsoid $\mathcal{E}_{max}$ of maximal volume contained in B_X. Furthermore, if we denote by $[\ ,\ ]$ the inner product and by $|\cdot|_2$ the Euclidean norm induced by $\mathcal{E}_{max}$, then there exists vectors $u_1, u_2, \ldots, u_N$ and constants $c_1, c_2, \ldots, c_N$ such that*

(i) $\|x\| \leq |x|_2$ *for* $x \in X$

(ii) $\|u_i\| = |u_i|_2 = \|u_i\|_* = 1$ *for* $i = 1, 2, \ldots, N$

(iii) $x = \sum_{i=1}^{N} c_i[x, u_i]u_i$ *for any* $x \in X$

It follows from the John's theorem that $B_X \subseteq \sqrt{n}\mathcal{E}_{max}$ which implies immediately (since $\mathcal{E}_{max} \subseteq B_X$ by definition) that $d_X \leq \sqrt{n}$. The estimate is sharp in the sense that for the n-dimensional cube and n-dimensional octahedron we have

$$d(l_\infty^n, l_2^n) = d(l_1^n, l_2^n) = \sqrt{n}.$$

The theory of absolutely summing operators was developed mainly by Pietsch in the late sixties, although the idea was present in the work of Grothendieck [G] under another name. Let $u : X \longrightarrow Y$ be an operator between Banach spaces and let $0 < p < \infty$. We say that u is *p-summing* if there is a constant C such that, for all finite sequences $\{x_i\}$ in X we have

$$(1.7) \qquad \left(\sum_i \|ux_i\|^p\right)^{1/p} \leq C \sup\left\{\left(\sum_i |f(x_i)|^p\right)^{1/p} \middle| f \in B_{X^*}\right\}$$

The smallest constant C satisfying (1.7) is denoted by $\pi_p(u)$ and we denote by $\Pi_p(X, Y)$ the set of all p-summing operators $u : X \longrightarrow Y$. It is easy to see that, if $1 \leq p < \infty$, π_p is a norm on $\Pi_p(X, Y)$ which turns this space in a Banach space. If $0 < p < 1$ it is only a quasi-Banach space. Moreover, the pair $(\Pi_p(X, Y), \pi_p)$ is an *operator ideal*, that is if $u : X \longrightarrow Y$ is p-summing and if $v_1 : W \longrightarrow X$ and $v_1 : Y \longrightarrow Z$ are bounded operators between Banach spaces, then the composition v_2uv_1 is p-summing and we have $\pi_p(v_2uv_1) \leq \|v_2\|\pi_p(u)\|v_1\|$.

If $X = (\mathbb{R}^n, \|\cdot\|_X)$ and $Y = (\mathbb{R}^n, \|\cdot\|_Y)$ are two normed spaces, we adopt the notation I_{XY} (or $I_{XY} : X \to Y$) for the formal identity operator from X to Y. We shall also write $I_{2X} : l_2^n \to X$ and $I_{X2} : X \to l_2^n$ instead of $I_{l_2^n X}$ and $I_{X l_2^n}$. Let $(X, \|\cdot\|_X)$ be an n-dimensional normed space and consider $\mathcal{E}_{max}$ the ellipsoid of maximal volume contained in B_X. Without loss of generality we can assume that $\mathcal{E}_{max} = B_2^n$ and we have the following important property (cf. [TJ], Proposition 15.5)

$$\pi_2(I_{2X}) = \pi_2(I_{X2}) = \sqrt{n}.$$

In general extremal ellipsoids may be very far from distance ellipsoids. In searching for an ellipsoid that would be "closed" to both we obviously need to relax the extremal conditions and replace them by conditions involving equivalence, up to universal constants. It is known that for any n-dimensional space $X = (\mathbb{R}^n, \|\cdot\|_X)$ there exists an ellipsoid $\mathcal{E}$ which combines properties of the distance ellipsoid and the ellipsoid of maximal volume (cf. e.g., [TJ], Proposition 17.2). Assuming that $\mathcal{E} = B_2^n$, the precise properties are the following:

$$(\sqrt{2}d_X)^{-1}\|x\|_2 \leq \|x\|_X \leq \sqrt{2}\|x\|_2 \quad \text{for } x \in \mathbb{R}^n; \tag{1.8}$$

$$\pi_2(I_{2X}) \leq \sqrt{2n}, \quad \pi_2(I_{X2}) \leq \sqrt{2n}. \tag{1.9}$$

In the sequel we will also make use of results concerning 2-*factorable* operators. We say that an operator $u : X \to Y$ is 2-*factorable* if there exists a Hilbert space H and bounded operators $v_1 : X \to H$ and $v_2 : H \to Y$ such that $u = v_2 v_1$. Let

$$\gamma_2(u) = \inf\{\|v_1\|\|v_2\|\}$$

where the infimum runs over all possible factorizations. We denote by $\Gamma_2(X, Y)$ the space of all 2-factorable operators from X to Y. It is not hard to check that γ_2 is a Banach space norm on $\Gamma_2(X, Y)$ and that $(\Gamma_2(X, Y), \gamma_2)$ is an operator ideal. For more properties of standard operator ideal norms, the 2-summing norm, $\pi_2(\cdot)$, and the 2-factorable norm, $\gamma_2(\cdot)$, we refer the reader to [TJ], Sections 9, 10, 13, 15 and 17. In particular, fundamental connections between these norms as well as to the ellipsoids of maximal and minimal volume can be found there.

The next definition is less standard but it is very convenient in our context. It was first introduced in [P] (see also [TJ], §27 for more information).

Definition 1.5. Let X be a Banach space. For $k \geq 1$ the relative Euclidean factorization constant $e_k(X)$ is defined by

$$e_k(X) = \sup\{e(E, X) : \ E \subset X, \dim E \leq k\}$$

where

$$e(E, X) = \inf\{\gamma_2(P) : \ P : X \to X \ \text{ projection onto } E\}.$$

It is easy to see that for an n-dimensional Banach space X we have $e_n(X) = d_X$.

Chapter 2

Structure of Banach spaces with extremal Euclidean distance

2.1 The isometric case

Let us start with a few comments about n-dimensional spaces with the maximal Euclidean distance. It is easy to check that spaces $X = l_1^n$ and $X = l_\infty^n$ satisfy $d_X = \sqrt{n}$. (The unit balls of these spaces will be denoted by B_1^n and B_∞^n, respectively.) It is also easy to come up with other spaces with the maximal distance which may be small perturbations or combinations of these two basic examples. In fact, the class of spaces with the maximal Euclidean distance is much larger.

The following example which is a version of a result by Bourgain ([B], see also [TJ], Proposition 27.5), introduces an interesting new family of spaces with the maximal Euclidean distance. This example was known to N. Tomczak-Jaegermann in the early 1990's, and it was also observed independently by B. Maurey ([M1]) at about the same time.

Example 2.1. Let n be a natural number such that there exists a system of mutually orthogonal vectors $x_1, \dots, x_n$ in $\mathbb{R}^n$ of the form $x_i = \sum_{k=1}^n \theta_{k,i} e_k$ for $1 \le i \le n$, with $\theta_{k,i} = \pm 1$ for $1 \le i, k \le n$. Set $K_1 := \mathrm{conv}(\{\pm x_i\}_{i=1}^n)$. For any space $X = (\mathbb{R}^n, B_X)$ such that $K_1 \subset B_X \subset B_\infty^n$, the Euclidean distance satisfies $d_X = \sqrt{n}$.

Proof Let $u : l_2^n \to l_2^n$ be an operator defined by $ue_i = x_i$, for $1 \le i \le n$. Then clearly $n^{-1/2}u$ is an isometry of l_2^n. In particular, for all $(a_i) \in \mathbb{R}^n$ we have

$$\Big\|\sum_{i=1}^n a_i x_i\Big\|_2 = \sqrt{n}\Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2}. \tag{2.1}$$

Let $\mathcal{E}$ be any ellipsoid such that $\mathcal{E} \subset B_X \subset d\mathcal{E}$, for some $d \ge d_X$. Since B_2^n is the ellipsoid of maximal volume contained in B_∞^n and $\mathcal{E} \subset B_\infty^n$ we have that $\mathrm{Vol}(\mathcal{E}) \le \mathrm{Vol}(B_2^n)$.

From the fact that B_2^n is the ellipsoid of minimal volume containing B_1^n and $u(B_1^n) = K_1$ it follows that $u(B_2^n)$ is the minimal volume ellipsoid for K_1. Since $K_1 \subset B_X \subset d\mathcal{E}$ we have

$$\mathrm{Vol}(u(B_2^n)) \le \mathrm{Vol}(d\mathcal{E}) = d^n \mathrm{Vol}(\mathcal{E}).$$

Since $n^{-1/2}u$ is an isometry then

$$\mathrm{Vol}(u(B_2^n)) = n^{n/2}\mathrm{Vol}(B_2^n).$$

Combining this with the previous two inequalities we get $d \ge \sqrt{n}$. Passing to the infimum over all $\mathcal{E}$ it follows that $d_X \ge \sqrt{n}$, and hence $d_X = \sqrt{n}$.

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Treating the vectors $\{x_i\}$ as columns of a matrix, we obtain an $n \times n$ matrix $[\theta_{k,i}]$ with ± 1 entries, whose columns are mutually orthogonal. These are so-called Hadamard matrices which exist for many values of n (cf. e.g., [H]). In particular, for $n = 2^k$, such a matrix can be taken as (an appropriate multiple of) the Walsh matrix corresponding to the Walsh system on $\{1, \ldots, 2^k\}$. In this case it is also clear that assuming by relabeling that $x_1, \ldots, x_k$ are the k Rademacher functions and letting $E = \text{span}\,\{x_1, \ldots, x_k\}$, we get the subspace $(E, \|\cdot\|_X)$ of X isometric to l_1^k.

Let us recall that an analogous fact is true in general, namely, as shown in [M-W], any n-dimensional space X with $d_X = \sqrt{n}$ must contain an isometric copy of l_1^k for $k \geq c \log n$, where $c > 0$ is an absolute constant.

Some further properties of spaces with the maximal Euclidean distance were known at the beginning of the 1990's to several people working in the area (Arias, Komorowski, Maurey and Tomczak-Jaegermann). In particular they showed that spaces with the maximal Euclidean distance have a unique distance ellipsoid, and that the only 3-dimensional spaces with the maximal distance are the obvious ones, $X = l_1^3$ and $X = l_\infty^3$. Maurey also proved ([M1]) that if for an n-dimensional space X a distance ellipsoid is not unique then there exists an $(n-1)$-dimensional subspace Y of X such that $d_Y = d_X$. It follows that if an n-dimensional space X satisfies $d_X > \sqrt{n-1}$ then the distance ellipsoid is unique. Furthermore, every finite-dimensional space X has a subspace Y such that $d_Y = d_X$ and Y has a unique distance ellipsoid.

Example 2.1 has a counterpart for spaces with type and cotype properties, which was the original aim of Bourgain's result. Let $1 < p < 2$ and let $q = p/(p-1)$. Replacing B_∞^n by the ball B_q^n of l_q^n, and K_1 by $K_p := n^{-1/q} u(B_p^n)$ (where u is as in the proof of Example 2.1), we get a class of spaces X satisfying $d_X = n^{1/p-1/2}$. Taking X as the interpolation space between l_q^n and $(\mathbb{R}^n, K_p)$ we get a space with type s and cotype t constants independent of n, for appropriate values of s and t, namely $\frac{1}{s} = \frac{1}{2}(\frac{1}{2} + \frac{1}{p})$ and $\frac{1}{t} = \frac{1}{2}(\frac{1}{2} + \frac{1}{q})$.

Note that in this case $d_X = n^{1/s-1/t}$ which was the main point of Bourgain's construction (for details see [B], see also [TJ], Proposition 27.5). The original arguments for the lower estimate for the distance used extra symmetries which an interpolation space inherits from the end spaces.

At the end of this section it is worthwhile to note that Example 2.1 does not use the full strength of the assumption on the orthogonality of vectors $\{x_i\}_{i=1}^n$. The proof works as well if equality (2.1) is replaced by the analogous lower estimate for the norm. Thus the example remains true if the normalized vectors $\{n^{-1/2}x_i\}_{i=1}^n$ merely satisfy the lower l_2 estimate (defined in (2.2) below) with constant 1. This additionally supports an expectation that a characterization of all n-dimensional spaces with maximal Euclidean distance might be in general involved, if possible at all, although it might be perhaps possible for some particular values of n (or for series of n).

2.2 Spaces with Euclidean distance of maximal order

Now we pass to the isomorphic case of n-dimensional spaces whose distance to l_2^n is of the order $\sqrt{n}$, as $n \to \infty$. First note that if X is such a space, with $d_X \geq \delta\sqrt{n}$, then for example, a direct sum $Y = X \oplus_2 l_2^n$ is a $2n$-dimensional space with d_Y of the maximal order as well, $d_Y = d_X \geq (\delta/\sqrt{2})\sqrt{\dim Y}$; at the same time Y contains an n-dimensional Euclidean subspace. This means that in considering the isomorphic case we can only expect a structural result on subspaces (or quotient spaces, or quotients of subspaces). We shall show in this section that isomorphic analogues of spaces considered in Example 2.1 can be "almost" reconstructed as subspaces (or quotients of subspaces) of proportional dimension inside every space with the Euclidean distance of maximal order.

Let us recall that vectors $\{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ are said to satisfy the lower l_2 estimate with constant $c > 0$ whenever for any sequence of scalars $\{a_i\}_{i=1}^m$ we have

$$\Big\|\sum_{i=1}^m a_i x_i\Big\|_2 \geq c\Big(\sum_{i=1}^m |a_i|^2\Big)^{1/2}. \tag{2.2}$$

The following classes of convex bodies will play an important role in the structure of our spaces. Given a subspace $E \subset \mathbb{R}^n$ and $x_1, \ldots, x_m \in E$ we let

$$\begin{aligned} K_\infty(\{x_i\}_{i=1}^m, E) &:= \{z \in E : |(z, x_i)| \leq 1 \quad \text{for all } \ i = 1, \cdots, m\} \\ K_1(\{x_i\}_{i=1}^m) &:= \text{conv } (\{\pm x_i\}_{i=1}^m) \subset \text{span } (\{x_i\}_{i=1}^m). \end{aligned}$$

The main result of this chapter is the following characterization of spaces with the Euclidean distance of maximal order.

Theorem 2.2. *Let X be an n-dimensional normed space. The following conditions are equivalent:*

(i) $d_X \geq \delta\sqrt{n}$, *for some $\delta > 0$ independent of n;*

(ii) *there exist $0 < b \leq a < 1$ and $0 < c \leq 1$ independent of n, a position $X = (\mathbb{R}^n, \|\cdot\|_X)$ and a subspace $E \subset \mathbb{R}^n$ with $\dim E = \lceil an \rceil$, and there exist vectors $y_1, y_2, \ldots, y_{\lceil an \rceil} \in E$ and $z_1, z_2, \ldots, z_{\lceil bn \rceil} \in E$, each set of vectors satisfying lower l_2 estimate with constant c, such that*

$$K_1\Big(\{\sqrt{n}y_i\}_{i=1}^{\lceil an \rceil}\Big) \subset B_X \cap E \subset K_\infty\Big(\{z_i\}_{i=1}^{\lceil bn \rceil}, E\Big);$$

(iii) *there exist $0 < b \leq a < 1$ and $0 < c \leq 1$ independent of n, a position $X = (\mathbb{R}^n, \|\cdot\|_X)$ and subspaces $F \subset E \subset \mathbb{R}^n$ with $\dim E = \lceil an \rceil$ and $\dim F =$*

$\lceil bn \rceil$, *and there exist vectors* $y_1, y_2, \ldots, y_{\lceil an \rceil} \in E$ *and* $z_1, z_2, \ldots, z_{\lceil bn \rceil} \in F$, *each set of vectors satisfying lower* l_2 *estimate with constant* c, *such that denoting by* P_F *the orthogonal projection onto* F *we have*

$$K_1\Big(\{\sqrt{n}\, P_F y_i\}_{i=1}^{\lceil an \rceil}\Big) \subset P_F(B_X \cap E) \subset K_\infty\Big(\{z_i\}_{i=1}^{\lceil bn \rceil}, F\Big).$$

First observe that $B_X \cap E$ is the ball in E treated as a subspace of X, so (ii) is a condition for the existence in X of a (proportional dimensional) subspace with a structure mimicking the construction from Example 2.1. It could be noted however that the body $K_\infty(\{z_i\}_{i=1}^{\lceil bn \rceil}, E)$ may happen to be unbounded (it is definitely so if, for example, $\lceil bn \rceil < \lceil an \rceil$). This "unpleasantness" is removed by condition (iii). Here observe that the space $(F, P_F(B_X \cap E))$ is a quotient space of $(E, B_X \cap E)$, and hence a quotient of a subspace of X. Then $K_\infty(\{z_i\}_{i=1}^{\lceil bn \rceil}, F)$ is a linear image of the cube $B_\infty^{\lceil bn \rceil}$ (since by the lower l_2 estimate the vectors z_i's are linearly independent), and $K_1(\{\sqrt{n}\, P_F y_i\}_{i=1}^{\lceil an \rceil})$ is a projection of an octahedron in E.

Theorem 2.2 will be an easy consequence of a more general result valid for spaces with large Euclidean distance (but not necessarily of the maximal order). To formulate this result we need a few more specialized notions and some preliminary facts.

We shall need the following lemma similar in spirit to [P] and [TJ] (Lemma 27.10).

Lemma 2.3. *Let* X *be an* n*-dimensional Banach space and let* $k \le n$. *For any projection* $P : X \to X$ *with rank* $P \ge k$ *we have*

$$e_n(X) \le \gamma_2(P) + e_{n-k}(X).$$

Proof Let $P : X \to X$ be an arbitrary projection with rank $P \ge k$. From the theory of 2-factorable operators ([TJ], Theorem 27.1) it is enough to prove that for every $v : l_2^n \to X$ such that $\pi_2(v^*) = 1$ we have

$$\pi_2(v) \leq \gamma_2(P) + e_{n-k}(X).$$

Fix v as above and without loss of generality assume that v is one-to-one. Let H be a subspace of $v^{-1}(PX)$ of dimension k. Then dim $v(H^\perp) = n - k$ and denote by $P_1 : X \to X$ a projection onto $v(H^\perp)$ such that $\gamma_2(P_1) \leq e_{n-k}(X)$. Let $Q : l_2^n \to l_2^n$ be the orthogonal projection onto H. Then

$$\pi_2(vQ) = \pi_2(PvQ) \leq \pi_2(Pv) \leq \gamma_2(P)\pi_2(v^*) \leq \gamma_2(P).$$

The next to the last inequality is one of basic connections between the norms γ_2 and π_2 (cf. e.g., [TJ], Theorem 13.11). Similarly, denoting by I the identity on l_2^n we get

$$\pi_2(v(I - Q)) = \pi_2(P_1 v(I - Q)) \leq \pi_2(P_1 v) \leq \gamma_2(P_1)\pi_2(v^*) \leq e_{n-k}(X).$$

Since $l_2^n = H \oplus H^\perp$ the conclusion immediately follows.

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We shall also need the Bourgain-Tzafriri's restricted invertibility result ([B-T], see also [B-S], Lemma B, for a convenient statement and a short proof)

Lemma 2.4. *Let $x_1, x_2, \ldots, x_n \in l_2$ and $\alpha > 0$ be such that*

(i) $\|x_j\|_2 \leq 1$ *for all j.*

(ii) $|(x_j, e_j)| \geq \alpha$ *for all j.*

Then there exists $\sigma \subset \{1, 2, \ldots, n\}$, $|\sigma| > cn$, such that, for all scalars $(t_j)_j$

$$\|\sum t_j x_j\|_2 \geq \frac{\alpha}{4} \left(\sum |t_j|^2\right)^{1/2},$$

where $c = c(\alpha)$ depends only on α.

Our approach to study the structure of spaces with large distance to l_2^n works in a more general context, when the distance d_X is equal to a certain function $\varphi(n)$. Thus $\varphi(n) \leq \sqrt{n}$ and the case of the distance of maximal order can be described as $\varphi(n) \geq \delta\sqrt{n}$, for an absolute constant $\delta > 0$.

Theorem 2.5. *Let X be an n-dimensional normed space in a position $X = (\mathbb{R}^n, \|\cdot\|_X)$ such that the Euclidean norm $\|\cdot\|_2$ satisfies (1.8) and (1.9). Suppose that the Euclidean distance satisfies $d_X = e_n(X) = \varphi(n)$ and that there exist constants $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$ such that $e_{\lceil \delta_1 n \rceil}(X) \leq \delta_2 \varphi(n)$.*

Then the following condition $(*)$ is satisfied for some $0 < b \leq a < 1$, $0 < c \leq 1$ depending on δ_1 and δ_2 only.

$(*)$ *there exist a subspace E of $\mathbb{R}^n$ with $\dim E = \lceil an \rceil$, and two sets of vectors $y_1, y_2, \ldots, y_{\lceil an \rceil} \in E$ and $z_1, z_2, \ldots, z_{\lceil bn \rceil} \in E$, each set satisfying the lower l_2 estimate with constant c, such that*

$$K_1\Big(\{\varphi(n) y_i\}_{i=1}^{\lceil an \rceil}\Big) \subset B_X \cap E \subset K_\infty\Big(\{z_i\}_{i=1}^{\lceil bn \rceil}, E\Big). \tag{2.3}$$

Conversely, let $X = (\mathbb{R}^n, \|\cdot\|_X)$ be an n-dimensional normed space in an arbitrary position and suppose that there exist constants $0 < b \leq a < 1$, $0 < c \leq 1$ such that $(*)$ is satisfied. Then

$$d_X \geq c^2 \sqrt{b/a}\, \varphi(n).$$

Proof Set $m := \lceil (1-\delta_1)n \rceil$. Since $e_k(X)$ is increasing in k, then by Lemma 2.3 and our hypothesis we obtain, for any projection $P : X \to X$ with $\operatorname{rank} P \geq m$

$$\varphi(n) = e_n(X) \leq \gamma_2(P) + e_{\lceil \delta_1 n \rceil}(X) \leq \gamma_2(P) + \delta_2 \varphi(n)$$

and hence

$$\gamma_2(P) \geq (1-\delta_2)\varphi(n).$$

Thus for any orthogonal projection $Q : l_2^n \to l_2^n$ with rank $Q \geq m$ we have the estimate $\gamma_2(I_{2X}QI_{X2}) \geq (1-\delta_2)\varphi(n)$. Since, by (1.8), $\gamma_2(I_{2X}QI_{X2}) \leq \|QI_{X2}\| \, \|I_{2X}\| \leq \sqrt{2}\|QI_{X2}\|$, we get

$$\|QI_{X2}\| \geq \frac{1-\delta_2}{\sqrt{2}}\varphi(n).$$

Next we will present the construction of K_1 and K_∞.

Construction of K_1: The vectors y_i will be constructed in two steps. The first is an induction. Assume that $1 \leq j \leq \lceil \delta_1 n \rceil$ and that $y_1, \dots, y_{j-1}$ have been constructed. let $P : l_2^n \to l_2^n$ be the orthogonal projection onto $(\text{span}\,[y_1, \dots, y_{j-1}])^\perp$. Then rank$P \geq m$ and hence there exist $y_j \in X$ such that $\|y_j\| = 1/\varphi(n)$ and $\|Py_j\|_2 \geq ((1-\delta_2)/\sqrt{2})$. Let $h_j = Py_j/\|Py_j\|_2$. This procedure gives us vectors $y_1, \dots, y_{\lceil \delta_1 n \rceil}$ with $\|y_i\|_X = 1/\varphi(n)$ and orthonormal vectors $h_1, \dots, h_{\lceil \delta_1 n \rceil}$ such that

$$(y_i, h_i) \geq \frac{1-\delta_2}{\sqrt{2}}, \text{ for all } i = 1, \dots, m.$$

We also have

$$\|y_i\|_2 \leq \varphi(n)\sqrt{2}\|y_i\| = \sqrt{2}.$$

Now, Bourgain-Tzafriri's restricted invertibility result yields that there exists a set $\sigma \subset \{1, \dots, \lceil \delta_1 n \rceil\}$ with cardinality $|\sigma| \geq an$ such that, for any choice of scalars $\{a_i\}_i$ we have

$$\Big\|\sum_{i\in\sigma} a_i y_i\Big\|_2 \geq \frac{1-\delta_2}{4\sqrt{2}}\Big(\sum_{i\in\sigma}|a_i|^2\Big)^{1/2}.$$

Here $a > 0$ depends on δ_1 and δ_2 only. Assuming, by relabelling, that $\sigma \supset \{1, \dots, \lceil an \rceil\}$ we clearly have

$$K_1(\{\varphi(n)y_i\}_{i=1}^{an}) \subset B_X \cap E \tag{2.4}$$

where $E = \operatorname{span}[y_i]_{i=1}^{\lceil an \rceil}$.

Construction of K_∞: Since the definition of K_∞ involves vectors z_i acting as functionals on E (with the unit ball $B_X \cap E$) rather than on X itself, it is natural to consider our arguments restricted to the appropriate subspace of X. Thus denote the space $(E, \|\cdot\|_X)$ by Y (so that $B_Y = B_X \cap E$), hence making Y a subspace of X. Let $H := (E, \|\cdot\|_2)$. Then the formal identity operators $I_{HY} : H \to Y$ and $I_{YH} : Y \to H$ are just the restrictions (both in the domain and in the range space) of the operators I_{2X} and I_{X2}, respectively. In particular, it is easy to check that both these operators satisfy estimates (1.9).

Note that $H^* = H$ and let $v := (I_{HY})^* : Y^* \to H$ to be the formal identity operator on E. The vectors $z_i \in E$ will be constructed in two steps analogous to the construction of the y_i's. First assume by induction that $1 \le j \le (a/2)n$ and that $z_1, \dots, z_{j-1}$ have been constructed. Let $P : H \to H$ be the orthogonal projection onto $(\operatorname{span}[z_1, \dots, z_{j-1}])^\perp \subset E$.

Consider the operator $T := (Pv)(I_{YH})^*$. As an operator on H, T is clearly an orthogonal projection with $\operatorname{rank} T = \operatorname{rank} Pv = \operatorname{rank} P$. Thus, by (1.9) and our earlier remarks we have

$$\begin{aligned} \operatorname{rank} T &= \operatorname{tr} T^* \le \pi_2((Pv)^*)\pi_2(I_{YH}) \\ &\le \left(\operatorname{rank}(Pv)\right)^{1/2} \|(Pv)^*\| \sqrt{2n} = (\operatorname{rank} T)^{1/2} \|Pv\| \sqrt{2n}. \end{aligned}$$

We also used the fact that an arbitrary operator $S : H \to Y$ can be factored through the subspace $F = (\ker S)^\perp \subset H$ as $S = S_{|F} Q_F$, where Q_F is the orthogonal projection onto F; so that

$$\pi_2(S) \le \pi_2(Q_F)\|S_{|F}\| \le \sqrt{\dim F}\|S\|.$$

Therefore our main estimate implies

$$\|Pv\| \geq \sqrt{\frac{\operatorname{rank} P}{2n}}.$$

Now recall that rank $P \geq (a/2)n$ and a depends on δ_1, δ_2 only. Thus there exists $z_j \in Y^*$ such that $\|z_j\|_{Y^*} = 1$ and $\|Pz_j\|_2 \geq \sqrt{a}/2$. Let $h_j = Pz_j/\|Pz_j\|_2$. This procedure gives $z_1, \ldots, z_{\lceil an/2 \rceil - 1} \in Y^*$ with $\|z_i\|_{Y^*} = 1$ and orthonormal vectors $h_1, \ldots, h_{\lceil an/2 \rceil - 1}$ such that

$$(z_i, h_i) \geq \frac{\sqrt{a}}{2}, \text{ for all } i = 1, \ldots, \lceil an/2 \rceil - 1.$$

By condition (1.8) we have $\|x\|_Y \leq \sqrt{2}\|x\|_2$ for $x \in Y$ (recall that for $x \in E$, $\|x\|_Y = \|x\|_X$). Therefore, by duality,

$$\|z_i\|_2 \leq \sqrt{2}\|z_i\|_{Y^*} = \sqrt{2}$$

for all $i = 1, \ldots, \lceil an/2 \rceil - 1$. Using again Lemma B in [B-S], there exists $\sigma' \subset \{1, \ldots, \lceil an/2 \rceil - 1\}$, with $|\sigma'| \geq bn$, such that for any choice of scalars $\{a_i\}_i$ we have

$$\Big\|\sum_{i \in \sigma'} a_i z_i\Big\|_2 \geq \frac{\sqrt{a}}{8}\Big(\sum_{i \in \sigma'} |a_i|^2\Big)^{1/2}.$$

Moreover $b = b(\delta_1, \delta_2) > 0$ depends on δ_1, δ_2 only. By relabeling we may assume that $\sigma' = \{1, \ldots, \lceil bn \rceil\}$. Also, for any $z \in B_X \cap E = B_Y$ we get

$$|(z, z_i)| \leq \|z_i\|_{Y^*} \, \|z\|_Y \leq 1,$$

and hence

$$B_X \cap E \subset K_\infty(\{z_i\}_{i=1}^{\lceil bn \rceil}, E). \tag{2.5}$$

Combining (2.4) and (2.5) we conclude the first part of the theorem.

Next we will prove the converse part of the statement.

Let $E_0 := \operatorname{span}[z_i]_{i=1}^{\lceil bn \rceil}$ and let P_{E_0} be the orthogonal projection from E onto E_0. Then

$$P_{E_0} K_\infty\Big(\{z_i\}_{i=1}^{\lceil bn \rceil}, E\Big) = K_\infty\Big(\{z_i\}_{i=1}^{\lceil bn \rceil}, E_0\Big),$$

and

$$P_{E_0} K_1\Big(\{\varphi(n) y_i\}_{i=1}^{\lceil an \rceil}\Big) = K_1\Big(\{\varphi(n)\, P_{E_0} y_i\}_{i=1}^{\lceil an \rceil}\Big).$$

Thus (2.3) implies

$$K_1\Big(\{\varphi(n)\, P_{E_0} y_i\}_{i=1}^{\lceil an \rceil}\Big) \subset P_{E_0}(B_X \cap E) \subset K_\infty\Big(\{z_i\}_{i=1}^{\lceil bn \rceil}, E_0\Big) \tag{2.6}$$

Set $Z := (E_0, K_\infty(\{z_i\}_{i=1}^{\lceil bn \rceil}, E_0))$, and $H_0 := (E_0, \|\cdot\|_2)$. We shall show that the formal identity operator satisfies

$$\pi_2(I_{ZH_0} : Z \to H_0) \le \sqrt{\lceil bn \rceil}/c. \tag{2.7}$$

Fix any orthonormal basis $\{f_i\}_{i=1}^{\lceil bn \rceil}$ in H_0 and define the operator $u : H_0 \to H_0$ by $uf_i = z_i$, for $i = 1, \dots, \lceil bn \rceil$. Then the lower l_2-estimate for $\{z_i\}_{i=1}^{\lceil bn \rceil}$ in (2.2), is equivalent to $\|u^{-1}\| \le 1/c$. On the other hand, if $B_\infty := \{z \in E_0 : |(z, f_i)| \le 1, \text{ for } i = 1, \dots, \lceil bn \rceil\}$ (so that B_∞ is a cube in E_0) then an easy calculation shows that

$$K_\infty(\{z_i\}_{i=1}^{\lceil bn \rceil}, E_0) = (u^*)^{-1} B_\infty.$$

Indeed, for $z \in E_0$ we have equalities

$$|((u^*)^{-1} z, z_i)| = |(z, u^{-1} z_i)| = |(z, f_i)|.$$

This means that for any $z \in E_0$, the condition $(u^*)^{-1} z \in K_\infty(\{z_i\}_{i=1}^{\lceil bn \rceil}, E_0)$ is equivalent to $z \in B_\infty$.

Now, $B_2^n \cap E_0$ is the ellipsoid of maximal volume for B_∞, and therefore, $\mathcal{E} := (u^*)^{-1}(B_2^n \cap E_0)$ is the ellipsoid of maximal volume for $K_\infty(\{z_i\}_{i=1}^{\lceil bn \rceil}, E_0)$.

Thus the formal identity operator $I_{Z\mathcal{E}} : Z \to (E_0, \mathcal{E})$ satisfies

$$\pi_2(I_{Z\mathcal{E}}) = \sqrt{\dim E_0} = \sqrt{\lceil bn \rceil}.$$

On the other hand, since $\mathcal{E} \subset \|(u^*)^{-1}\| B_2^n \cap E_0 \subset (1/c) B_2^n \cap E_0$, then the operator $I_{\mathcal{E}H_0}$ satisfies

$$\|I_{\mathcal{E}H_0} : (E_0, \mathcal{E}) \to H_0\| \leq 1/c.$$

Since $I_{ZH_0} = I_{\mathcal{E}H_0} I_{Z\mathcal{E}}$, putting together the last two estimates we immediately get (2.7)

Now we are ready for the proof of the required distance estimate. Set $W := (E_0, P_{E_0}(B_X \cap E))$. We will first show that

$$d_W = d(W, l_2^{\lceil bn \rceil}) \geq c^2 \sqrt{b/a}\, \varphi(n).$$

Recall that for every m and every operator $w : l_2^m \to W$ we have the estimate $\pi_2(w) \leq d_W \pi_2(w^*)$ (cf. [TJ], Proposition 27.1) (in fact, d_W is equal to the smallest constant satisfying the above inequality for all m and all w).

Clearly we can take as w the operator $I_{H_0W} : H_0 \to W$. Also consider two operators $I_{H_0Z} : H_0 \to Z$ and $I_{WZ} : W \to Z$. Note that, by (2.6), $\|I_{WZ}\| \leq 1$. Since $I_{H_0Z} = I_{WZ} I_{H_0W}$, then, by (2.7) we have

$$\pi_2(w) = \pi_2(I_{H_0W}) \geq \pi_2(I_{H_0W}) \|I_{WZ}\| \geq \pi_2(I_{H_0Z}) \geq \lceil bn \rceil\, \pi_2(I_{ZH_0})^{-1} \geq c\sqrt{\lceil bn \rceil}.$$

Next observe that the dual operator $w^* = (I_{H_0W})^*$ is the formal identity from W^* to $(H_0)^* = H_0$. The unit ball B_{W^*} is the polar $\left(P_{E_0}(B_X \cap E)\right)^\circ$ which in turn is equal to $(B_X \cap E)^\circ \cap E_0$. All this follows from the basic general theory and is easy to check directly. If we set $Y := (E, B_X \cap E)$ (as in the proof of the construction of K_∞ above), then it follows that W^*

is a subspace of Y^*. Thus the operator $(I_{H_0W})^*$ is a restriction of the formal identity operator $v := I_{HY}^* : Y^* \to H$. Thus

$$\pi_2(w^*) = \pi_2((I_{H_0W})^*) \leq \pi_2(v).$$

Also observe the duality between the balls K_1 and K_∞. Precisely, letting $x_i = \varphi(n)y_i$ for $i = 1, \ldots, \lceil an \rceil$, we have

$$K_1(\{x_i\}_{i=1}^{\lceil an \rceil}) = \left(K_\infty(\{x_i\}_{i=1}^{\lceil an \rceil}, E)\right)^\circ.$$

In the language of normed spaces this means that if $V := \left(E, K_\infty(\{x_i\}_{i=1}^{\lceil an \rceil}, E)\right)$, then $V^* = \left(E, K_1(\{x_i\}_{i=1}^{\lceil an \rceil})\right)$.

Now we are ready to finish the estimate for $\pi_2((I_{H_0Y})^*)$. With the notation above we have

$$\pi_2(v) = \pi_2(I_{VH}\, I_{Y^*V}) \leq \pi_2(I_{VH})\|I_{Y^*V}\| \leq \pi_2(I_{VH}).$$

The last inequality is implied by $\|I_{Y^*V}\| = \|I_{V^*Y}\| \leq 1$, by inclusion (2.3). Finally, by an argument similar to (2.7) we get

$$\pi_2(I_{VH}) \leq \frac{\sqrt{an}}{c\varphi(n)}.$$

Putting these estimates together,

$$\pi_2(w^*) \leq \frac{\sqrt{an}}{c\varphi(n)}.$$

Since $\pi_2(w) \leq d_W \pi_2(w^*)$ it follows that $d_W \geq c^2\sqrt{b/a}\varphi(n)$. Since $d_X \geq d_W$, this completes the proof of the second part of the theorem.

∎

Now the proof of Theorem 2.2 is very easy.

Proof [Theorem 2.2] (i) $\Rightarrow$ (ii) Set $\varphi(n) = d_X$. Since $e_k(X) \leq \sqrt{k}$ for all $1 \leq k \leq n$, then setting, for example, $\delta_1 = \delta/2$ we get the assumption of Theorem 2.5 satisfied with $\delta_2 = 1/2$. Then (ii) is the same as condition $(*)$.

(ii) $\Rightarrow$ (iii) is trivial by letting $F := \text{span}\,[z_i]_{i=1}^{[bn]}$ and P_F to be the orthogonal projection onto F.

Finally (iii) $\Rightarrow$ (i) was shown in the proof of the converse part of Theorem 2.5.

■

Remark 2.6. A closer inspection of the proof of Theorem 2.5 shows that the same conclusion follows if in the hypothesis we replace the condition $e_{\delta_1 n}(X) \leq \delta_2\varphi(n)$ by a weaker condition that there exists $0 < \delta_1 \leq 1$ and $0 < \delta_2 \leq 1$ such that for any projection $P : X \to X$ with rank $P \geq \delta_1 n$ we have $\gamma_2(P) \geq \delta_2\varphi(n)$.

Part II

Stabilization and Asymptotic Structure of Banach Spaces

Chapter 3

Introduction, Infinite Dimensional Banach Spaces

This part of the thesis is devoted to the study of certain structural properties of infinite dimensional Banach spaces.

From the early days of Functional Analysis the objective of the classical theory of infinite dimensional spaces was to investigate the linear-topological structure of Banach spaces. Many problems along these lines are concerned with finding subspaces with a "nice" structure. Old questions raised by Banach in the early 1930's remained open for a long time and they turned out to be very important in the development of the infinite dimensional theory of Banach spaces. To name a few: Does every infinite dimensional Banach space contain a subspace isomorphic to one of the classical spaces c_0 or l_p for some $1 \leq p < \infty$? If a Banach space X is isomorphic to every infinite dimensional subspace of itself, does it follow that X is isomorphic to l_2? Is it true that every Banach space is isomorphic to its hyperplanes?

The solution to the first question is particularly important for the development that followed. In the early seventies Tsirelson [T] constructed a counterexample, more precisely he constructed a Banach space that does not contain any isomorphic copy of c_0 or l_p for $1 \leq p < \infty$. Tsirelson's space was the first example of a space where the norm is defined by an inductive procedure that forces a specific property to pass to every infinite dimensional subspace, and this *saturation* prevents the space from containing c_0 or any l_p. Figiel and Johnson [F-J] gave an analytic description for the norm of the dual of Tsirelson's space, and their example is denoted nowadays by T. The idea behind Tsirelson's space, to define the norm implicitly, led to the construction of many variations that answered a multitude of questions in Banach space theory, most of them being counterexamples (cf. [C-S]). About the same time as Tsirelson's example, Krivine [K] proved what can be considered a finite dimensional counterpart, that however goes in the "opposite" direction and roughly says that every Banach space contains l_p^n's of an arbitrarily large finite dimension n.

In the early 1990's Schlumprecht [S] constructed a space which, while a modification of Tsirelson's space, has much more pronounced geometric properties. It is nowadays called by his name and it has initiated a series of spectacular results in Banach space theory. For Gowers and Maurey [GM], Schlumprecht's space was the starting point for their famous construction of a space without an unconditional basic sequence. Their space has in fact a stronger property, it is *hereditarily indecomposable* (H.I.), which means that no closed subspace can be written as a topological direct sum of two infinite dimensional closed subspaces. The connection between H.I. spaces and spaces having unconditional basic sequences was clarified by Gowers in [G1]. His famous dichotomy theorem states that every Banach space has an infinite dimensional subspace with an unconditional basis or has a hereditarily indecomposable subspace. In particular, combined with a result of Komorowski and Tomczak-Jaegermann [K-TJ], it provided the positive solution to the homogeneous space problem: if a Banach space X is isomorphic to all its infinite dimensional closed subspaces, then X must be isomorphic to l_2. Schlumprecht's space was also instrumental in the solution to another old problem, known as the "distortion problem". All these results and examples had a great impact on the understanding of the structure of infinite dimensional Banach spaces and of the classical notion of a "nice" subspace. Quoting from Maurey, Milman and Tomczak-Jaegermann [M-M-TJ], "it has been realized recently that such a nice and elegant structural theory does not exist. Recent examples (or counterexamples to classical problems) due to Gowers and Maurey [GM] and Gowers [G2], [G3] showed much more diversity in the structure of infinite-dimensional Banach spaces than was expected."

On the other hand, in the last three decades there has been deep development in the Local Theory of Banach spaces; structural properties of finite dimensional subspaces of Banach spaces and related local properties have been well understood. This theory has an asymptotic nature: as the dimension increases to infinity, surprising regularities of finite dimensional spaces are revealed (cf. [M-S]). In general, asymptotic methods in the theory of infinite-dimensional Banach spaces look at stabilized information of finite nature by going "far enough" in the space, "at infinity".

The first stabilization of this type was the construction of *spreading models* by Brunel and Sucheston [Br-Su] in 1974, which is based on the combinatorial Ramsey's theorem. They showed that in every Banach space every normalized basic sequence $\{x_i\}$ has a subsequence $\{y_i\}$ on which the norm of any linear combination of n vectors of $\{y_i\}$ stabilizes (they span the same finite dimensional space) provided that they are sufficiently far along $\{y_i\}$. Consequently, the iterated limit

$$\lim_{i_1\to\infty}\dots\lim_{i_k\to\infty}\|\sum_k a_k y_{i_k}\|$$

exists and it defines a norm on the linear space of finite scalars c_{00}. The space c_{00} with this new norm is called a *spreading model* generated by $\{y_i\}$. This new object behaves relatively "better" than the original sequence $\{y_i\}$. For example, the unit vector basis $\{e_i\}$ of a spreading model has the "spreading" property, which means that $\|\sum_{i=1}^k a_i e_{n_i}\| = \|\sum_{i=1}^k a_i e_{m_i}\|$ for all scalars $(a_i)_{i=1}^k$, $n_1 < \dots < n_k$ and $m_1 < \dots < m_k$. Moreover, the basis is often unconditional. Roughly speaking, starting with an arbitrary basic sequence, a spreading model provides subsequences of finite (but arbitrary) length which have "nice" properties.

However, information on subsequences is not enough to reflect properties of all subspaces of a Banach space. An old result of Bessaga and Pelczynski states that every subspace Y of a space X with a basis contains a further subspace Z "very close" to a so-called block subspace (for the precise definition of this and other unexplained notions see Section 3.1). It follows that in many problems it is enough to consider just block subspaces instead of general subspaces. Therefore one has to look at blocks of a basis, rather than its subsequences. Gowers [G1] proved a *block Ramsey* theorem that provides stronger stabilization results than that of spreading models and used it to prove his dichotomy theorem mentioned above. On the other hand, in a striking contrast with the finite dimensional situation, infinite dimensional phenomena may not stabilize, the primary example of this being a distortion (as shown in a breakthrough result of Odell and Schlumprecht [OS]).

In order to bridge the gap between finite and infinite dimensional structures, Maurey, Milman and Tomczak-Jaegermann [M-M-TJ] have introduced a new type of stabilization that gave rise to new notion of asymptotic structures. The theory studies the structure of infinite dimensional Banach spaces by looking at finite dimensional spaces that appear arbitrarily far away and arbitrarily spread out in the space. Such spaces are called *asymptotic spaces.* We will briefly explain this notion; the precise definitions and more detailed explanations will be given in Sections 3.2 and 3.3. For subsets I and J of the natural numbers $\mathbb{N}$, we write $I < J$ if $\max I < \min J$. If X is a Banach space with a basis $\{u_i\}_i$ and $x = \sum_i a_i u_i$ is a vector in X then $\text{supp} x$ is the set of i such that a_i is non-zero. A block vector is a vector with finite support, and blocks are successive, and we write $x_1 < x_2$, if $\text{supp} x_1 < \text{supp} x_2$. If E is an n-dimensional space with a fixed monotone normalized basis $\{e_i\}_{i=1}^n$, we say that E is an asymptotic space for X, and we write $E \in \{X\}_n$, if for any $\varepsilon > 0$, and for any n_1 we can find a block x_1 with $n_1 < \text{supp} x_1$, such that for any n_2 we can find a block x_2 with $n_2 < \text{supp} x_2$ and so on, such that after n steps the blocks $x_1, x_2, \dots, x_n$ are successive and $(1+\varepsilon)$-equivalent to the basis $\{e_i\}_{i=1}^n$. The asymptotic structure of X consists of all asymptotic spaces of X, for all n. From Krivine's theorem it follows that for some $1 \le p \le \infty$, $l_p^n \in \{X\}_n$ for all n, hence $\{X\}_n$ is never empty. If there exists $1 \le p \le \infty$ and a constant C such that for all n and all $E \in \{X\}_n$ the basis in E is C-equivalent with the unit vector basis of l_p^n, then we say that X is an *asymptotic-l_p* space. It was shown by Milman and Tomczak-Jaegermann in [M-TJ2] that the asymptotic structure can be further "stabilized" by passing to a subspace. Stabilized asymptotic-l_p spaces appear naturally in connection with some developments we mentioned before. For example, Tsirelson's space T is a stabilized asymptotic l_1-space.

In Chapter 4 we prove the main result of this part of the thesis. We show that under certain regularity conditions imposed on a Banach space X, one can find a subspace Y which is saturated with stabilized asymptotic-l_p spaces, for some p. More precisely,

Theorem. *Let X be a Banach space with the following property:*

For any infinite dimensional subspace $Y \subseteq X$ there exists a constant M_Y such that for any n there exist infinite dimensional subspaces $U_1, U_2, \cdots, U_n$ of Y such that

$$\frac{1}{M_Y}\|\sum_{i=1}^{n} a_i x_i\| \leq \|\sum_{i=1}^{n} a_i y_i\| \leq M_Y\|\sum_{i=1}^{n} a_i x_i\|$$

for any norm one vectors x_i, y_i in U_i and any scalars a_i.
Then there exists $p \in [1, \infty]$ such that X contains a stabilized asymptotic-l_p subspace (stabilized asymptotic-c_0 when $p = \infty$).

To better describe the result and its motivation it is worthwhile to recall an old theorem of Zippin [Z].

Theorem (Zippin). *Let X be a Banach space with a normalized basis $\{e_n\}_n$. Assume that $\{e_n\}_n$ is equivalent to all its normalized block bases. Then $\{e_n\}_n$ is equivalent to the unit vector basis in c_0 or in some l_p, $1 \leq p < \infty$.*

Let us compare the hypotheses from the two theorems. Note the particularities of our hypothesis: the space X is saturated with a certain geometric property, finite in nature but of an arbitrary length. It is, in a certain sense, an asymptotic version of Zippin's. We impose a similar condition on finite sequences (instead of infinite sequences) with the equivalence constant uniform in n. This theorem is a generalization of a result by Figiel, Frankiewicz, Komorowski and Ryll-Nardzewski [F-F-K-R]. They obtain the same conclusion under much stronger conditions; the finite sequence of vectors considered in the hypothesis were already equivalent to the basis in a space with a norm prescribed in advance (for example l_p^n).

We shall briefly describe the organization of this part of the thesis. In Chapter 3 we present some fundamental facts in Banach space theory and discuss in more detail the notion of asymptotic structure. Chapter 4 is devoted to the main result. We start by presenting in Section 4.1 a few essentially known stabilization techniques which will be used in the subsequent sections. Sections 4.2 and 4.3 contain the proof of our theorem. The proof is rather complicated and it is divided into several parts to emphasize the factors involved, which are of independent interest. We use both analytic and combinatorial techniques. In particular, the argument in Section 4.3 has its roots in Maurey proof of Gowers dichotomy theorem.

3.1 Basic Concepts

In the following chapters, all spaces will be considered to be real, separable Banach spaces and all subspaces will be closed. We shall denote by $X, Y, \ldots$ infinite dimensional Banach spaces and by $E, F, \ldots$ finite dimensional Banach spaces. The sets of natural numbers, rational numbers and real numbers are denoted by $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively.

Let X be a Banach space and let $\{x_n\}_n$ be a non-zero sequence in X. We say that $\{x_n\}_n$ is a *(Schauder) basis* for X if, for each $x \in X$, there is a unique sequence of scalars $\{a_n\}_n$ such that $x = \sum_{n=1}^{\infty} a_n x_n$, where the sum converges in the norm topology. Clearly, a basis for X is linearly independent. Moreover, any basis has a dense linear span. That is, the set

$$\operatorname{span}\{x_i : i \in \mathbb{N}\} = \left\{ \sum_{i=1}^{n} a_i x_i : a_1, \ldots, a_n \in \mathbb{R}, n \in \mathbb{N} \right\}$$

is dense in X. In fact it is easy to check that

$$\left\{ \sum_{i=1}^{n} a_i x_i : a_1, \ldots, a_n \in \mathbb{Q}, n \in \mathbb{N} \right\}$$

is dense in X. We say that $\{x_n\}_n$ is a *basic sequence* if $\{x_n\}_n$ is a basis for the closure of its linear span.

The *basis projections* of a basis $\{x_n\}_n$ defined by $P_n(\sum_{i=1}^{\infty} a_i x_i) = \sum_{i=1}^{n} a_i x_i$ for $n = 1, 2 \ldots$ are uniformly bounded linear operators and the supremum of the norms of these basis projections is called the *basis constant* of $\{x_n\}_n$. If the basis constant is 1, the basis is called *monotone.* A sequence $\{x_n\}_n$ is called normalized if for each n we have that $\|x_n\| = 1$.

A basis $\{x_n\}_n$ is said to be *unconditional* if for every $x \in X$ its expansion $\sum_{n=1}^{\infty} a_n x_n$ converges unconditionally. Being unconditional is equivalent to the fact that there exists a constant $C > 0$ such that for all scalars $\{a_n\}_n$ and signs $\varepsilon_n = \pm 1$, we have

$$\|\sum_n \varepsilon_n a_n x_n\| \leq C \|\sum_n a_n x_n\|.$$

The smallest C is called the unconditional basis constant of $\{x_n\}_n$.

Two sequences $\{x_n\}_n$ and $\{y_n\}_n$, possibly from different Banach spaces, are said to be *equivalent* if we can find constants C_1 and C_2 such that for all scalars $\{a_n\}_n$, we have

$$\frac{1}{C_1}\|\sum_n a_n x_n\| \leq \|\sum_n a_n y_n\| \leq C_2 \|\sum_n a_n x_n\|. \tag{3.1}$$

Let $C = C_1 C_2$. The infimum of C satisfying (3.1) is called the *equivalence constant* and in this case we say that $\{x_n\}_n$ and $\{y_n\}_n$ are C-equivalent.

A basis $\{x_n\}_n$ of a Banach space X is said to be *symmetric* if, for any permutation π of positive integers, $\{x_{\pi(n)}\}_n$ is equivalent to $\{x_n\}_n$. It is standard to see that every symmetric basis is also unconditional. A basis $\{x_n\}_n$ is called *subsymmetric* if for every increasing sequence of positive integers $\{p_n\}_n$, $\{x_{p_n}\}_n$ is equivalent to $\{x_n\}_n$. Note that a subsymmetric basis is not automatically unconditional. As an example, the summing basis of c, the space of converging sequences of scalars, is equivalent to all its subsequences but it is not unconditional. Some authors require a subsymmetric sequence to be unconditional. However, if $\{x_n\}_n$ is bounded and subsymmetric then it follows from [R1] that either it is equivalent to the unit vector basis of l_1 (hence is unconditional) or is weak Cauchy, hence the difference sequence $\{x_n - x_{n+1}\}_n$ is unconditional and subsymmetric.

Let $\{x_n\}_n$ be a basic sequence in a Banach space X. Given an increasing sequence of positive integers $p_1 < p_2 < p_3 < \ldots$, let $y_k = \sum_{i=p_k+1}^{p_{k+1}} a_i x_i$ be any non-zero vector in the span of $x_{p_k+1}, x_{p_k+1}, \ldots, x_{p_{k+1}}$. We say that $\{y_k\}_k$ is a *block basic sequence* of $\{x_n\}_n$. It is easy to see that $\{y_k\}_k$ is indeed a basic sequence whose basis constant does not exceed that of $\{x_n\}_n$. When $\{x_n\}_n$ is fixed, we'll simply call $\{y_k\}_k$ a block basic sequence, or just a *block basis*. The usefulness of this notion rests very much on the following result of Bessaga and Pelczynski (cf, e.g. [L-T]).

Proposition 3.1. *Let X be a Banach space with a basis $\{x_n\}_n$ and let Y be an infinite dimensional subspace of X. Then Y contains a basic sequence equivalent to a block basis of $\{x_n\}_n$.*

3.2 Asymptotic Spaces

We introduce first some more notations which are specific for the study of asymptotic structure of infinite dimensional Banach spaces. Let X be a Banach space with a basis $\{u_n\}_n$ with a basis constant equal to K. For a vector $x = \sum_i a_i u_i$, the support of x, supp$x$, is the set of i for which a_i is non-zero. For two vectors x and y in X we write $x < y$ (and we say that the vector y starts after the vector x) if maxsupp$x <$ minsupp$y$. For subsets I and J of the natural numbers $\mathbb{N}$, we write $I < J$ (and we say that these sets are *successive*) if $\max I < \min J$. In particular for $n \in \mathbb{N}$, $n < J$ if $n < \min J$.

Let $\mathcal{B}$ be the family of all tail subspaces of X with respect to the basis $\{u_n\}_n$, that is the family of all subspaces of the form $X^n = \overline{\text{span}}\{u_i\}_{i>n}$ for some $n \in \mathbb{N}$. It is easy to check that the family $\mathcal{B}$ satisfies the following filtration condition

For every X_1, $X_2 \in \mathcal{B}$, there exists $X_3 \in \mathcal{B}$ such that $X_3 \subset X_1 \cap X_2$.

By $\mathcal{M}_n$ we'll denote the space of all n-dimensional Banach spaces with a fixed normalized basis having the basis constant no greater than K. Even though $\mathcal{M}_n$ depends on K, our notation does not lead to any confusion since the constant K is fixed once the Banach space X is fixed. Given two such spaces, E with basis $\{e_i\}_{i=1}^n$ and F with basis $\{f_i\}_{i=1}^n$, we'll denote by $d_b(E, F)$ the basis distance

$$d_b(E,F) = \inf\{C_1C_2 : \frac{1}{C_1}\|\sum_{i=1}^n a_ie_i\| \leq \|\sum_{i=1}^n a_if_i\| \leq C_2\|\sum_{i=1}^n a_ie_i\|,$$
$$\text{for all } \{a_i\}_i \subseteq \mathbb{R}\}.$$

Note that $d_b(E, F)$ is the equivalence constant between the basis $\{e_i\}_{i=1}^n$ and

$\{f_i\}_{i=1}^n$, as defined in (3.1). It can be shown that $\log d_b$ is a metric on $\mathcal{M}_n$ which makes it in a compact space.

The language of *asymptotic games* was introduced by Maurey, Milman and Tomcazk-Jaegermann in [M-M-TJ] and it represents a convenient way for describing asymptotic structures. Two players **S** and **P** play this game, with respect to a fixed family $\mathcal{B}$, in the following way. In the first step player **S** picks a subspace $X_1 \in \mathcal{B}$ and player **P** chooses a vector $x_1 \in S(X_1)$, where $S(X_1)$ is the unit sphere of X_1. In the second step player **S** picks a subspace $X_2 \in \mathcal{B}$ and **P** a vector $x_2 \in S(X_2)$. They continue in this way choosing alternately subspaces in $\mathcal{B}$ (player **S**) and vectors (player **P**). Player **P** must also ensure that at any step the vectors $x_1, x_2, \ldots, x_k$ form a basic sequence with the basis constant smaller or equal than 2. Additional rules will guarantee that the games will stop after a finite number of steps given in advance and also will dictate the strategy of each player.

Fix a space $E \in \mathcal{M}_n$ with a basis $\{e_i\}_{i=1}^n$ and $\varepsilon > 0$. In a *vector game* associated to E and ε player **P** tries to pick vectors x_i such that after n moves the vectors $\{x_i\}_{i=1}^n$ are $(1+\varepsilon)$-equivalent to $\{e_i\}_{i=1}^n$. If she succeeds we say that **P** wins the game. Of course, **S** tries to prevent **P** from winning by carefully choosing spaces X_i from $\mathcal{B}$. We say that **P** has a *winning strategy* for E and ε if, regardless the strategy of **S**, **P** can win every vector game as above. Also note that the smaller the space player **S** chooses from $\mathcal{B}$, the worse is the chance for player **P** of finding a good vector. Therefore, from the filtration condition, we can assume without loss in generality that $X_{k+1} \subset X_k$ for every $1 \leq k \leq n$.

A space $E \in \mathcal{M}_n$ with a basis $\{e_i\}_i$ is called an *asymptotic space* for X if for any $\varepsilon > 0$, player **P** has a winning strategy for E and ε in any vector game. In other words, for every $\varepsilon > 0$ we have

$$\forall X_1 \in \mathcal{B} \quad \exists x_1 \in S(X_1) \quad \forall X_2 \in \mathcal{B}, X_2 \subset X_1 \quad \exists x_2 \in S(X_2) \ldots$$

$$\ldots \forall X_n \in \mathcal{B}, X_n \subset X_{n-1} \quad \exists x_n \in S(X_n) \text{ such that}$$

$$\{x_1, x_2, \ldots, x_n\} \overset{1+\varepsilon}{\sim} \{e_1, e_2, \ldots, e_n\}.$$

The set of all n-dimensional asymptotic spaces for X, with respect to a fixed $\mathcal{B}$, is denoted by $\{X\}_n$. It is not hard to check that the set $\{X\}_n$ is closed in $\mathcal{M}_n$. As it was showed in [M-M-TJ] this set can be described in terms of a different asymptotic game, called a *subspace game*. Even though it will be not used in the sequel, we present it here in order to build a better intuitive understanding of asymptotic spaces.

Fix a family $\mathcal{F} \subset \mathcal{M}_n$ and $\varepsilon > 0$. In a *subspace game* associated to $\mathcal{F}$ and ε player **S** tries to choose spaces X_i from $\mathcal{B}$ in such a way that after n moves the vectors $\{x_i\}_i$ resulting from the game are $(1+\varepsilon)$-equivalent to the basis in some space from $\mathcal{F}$. If she succeeds we say that **S** wins the game. The filtration condition implies that **S** can always choose spaces from $\mathcal{B}$ satisfying $X_{k+1} \subset X_k$ for $1 \le k \le n$, since by restricting the choices of **P** it increases her own chance to win.

We say that **S** has a *winning strategy* for $\mathcal{F}$ and ε if, regardless of **P** strategy, player **S** can win every subspace game as above. In other words

$$\exists X_1 \in \mathcal{B} \quad \forall x_1 \in S(X_1) \quad \exists X_2 \in \mathcal{B}, X_2 \subset X_1 \quad \forall x_2 \in S(X_2) \ldots$$

$$\ldots \exists X_n \in \mathcal{B}, X_n \subset X_{n-1} \quad \forall x_n \in S(X_n) \text{ such that}$$

$$\exists F \in \mathcal{F} \text{ with basis } \{f_1, f_2, \ldots, f_n\} \text{ and } \{x_1, x_2, \ldots, x_n\} \overset{1+\varepsilon}{\sim} \{f_1, f_2, \ldots, f_n\}.$$

It is proved in [M-M-TJ] that the set $\{X_n\}_n$ is the smallest subset $\mathcal{F} \subset \mathcal{M}_n$ such that, for every $\varepsilon > 0$, **S** has a winning strategy for $\mathcal{F}$ and ε.

Recall the famous Krivine's theorem [K]:

Theorem 3.2. *If $\{x_n\}_{n=1}^{\infty}$ is a basic sequence in a Banach space, then there exists $1 \leq p \leq \infty$ so that for every k and $\varepsilon > 0$ there is a block basis $\{y_n\}_{n=1}^{k}$ of $\{x_n\}_{n=1}^{\infty}$ of length k which is $(1+\varepsilon)$-equivalent to the unit vector basis of l_p^k.*

Moreover the proof of the theorem yields that the sequence $\{y_n\}_{n=1}^{k}$ can be chosen according to the following scheme.

$$\begin{array}{ll} \forall n_1 \in \mathbb{N} & \exists y_1 \in \operatorname{span}\{x_n\}_n \text{ with } y_1 > x_{n_1} \\ \forall n_2 \in \mathbb{N} & \exists y_2 \in \operatorname{span}\{x_n\}_n \text{ with } y_2 > x_{n_2} \\ \vdots & \\ \forall n_k \in \mathbb{N} & \exists y_k \in \operatorname{span}\{x_n\}_n \text{ with } y_k > x_{n_k} \end{array}$$

so that $\{y_n\}_{n=1}^{k}$ is $(1+\varepsilon)$-equivalent to the unit vector basis of l_p^k. It follows immediately that for every Banach space X we can find $1 \leq p \leq \infty$ so that for all n, $l_p^n \in \{X\}_n$, therefore $\{X_n\}_n$ is non-empty.

It is natural to ask whether it is possible to obtain a higher level of stabilization for the asymptotic structure of X, while keeping the same set of all asymptotic spaces? Milman and Tomczak-Jaegermann showed in [M-TJ2] that this can be done by passing to an appropriate subspace. More precisely, there exists an infinite dimensional subspace Z of X with a basis $\{z_i\}_i$ such that for every $\varepsilon > 0$, any n successive blocks of $\{z_i\}_i$ starting after z_n are $(1+\varepsilon)$-equivalent to some asymptotic space, while all asymptotic spaces of X appear also as asymptotic spaces of Z.

3.3 Asymptotic-l_p Spaces

Let X be a Banach space with basis $\{u_i\}_i$. Then X is called an *asymptotic-l_p space*, for $1 \leq p \leq \infty$, if there is a constant K such that for every n and every $E \in \{X\}_n$ we have $d_b(E, l_p^n) \leq K$. In other words, an asymptotic l_p-space has a simple asymptotic structure: there are no other elements in $\{X\}_n$ except the ones whose existence follows from Krivine's theorem. An interesting and nontrivial result proved in [M-M-TJ] shows that for $p > 1$ one only needs to assume that the Banach-Mazur distance $d(E, l_p^n) \leq K$ for some K, meaning that in the asymptotic setting the more general condition of isomorphism already implies the equivalence of the natural basis. It is still open whether this is true for $p = 1$.

Clearly any l_p space is an asymptotic-l_p space (asymptotic-c_0 for $p = \infty$). An example of an asymptotic l_p-space which is not isomorphic to l_p is the infinite l_p-direct sum $(\sum_n \bigoplus E_n)_p$ of any finite dimensional Banach spaces E_n with the dimension growing to ∞ as n goes to ∞.

Let the Banach space X with basis $\{u_i\}_i$ be an asymptotic-l_p space. As explained at the end of the previous section, we can further stabilize the asymptotic structure of X by passing to a subspace. The subspace Z with basis $\{z_i\}_i$ will have the property that there exists a constant K such that for all n, we have that any n successive blocks of $\{z_i\}_i$ starting after z_n are K-equivalent to the unit vector basis of l_p^n. A space with a basis satisfying the above property will be called a *stabilized asymptotic-l_p* space.

A very important example of a stabilized asymptotic-l_p space, which was the cornerstone of many fundamental discoveries of the last decades in Banach space theory, is Tsirelson's space [T]. We recall the definition of this space. Denote by c_{00} the vector space of finitely supported sequences with the norm

$\|(x_i)_i\|_0 = \max_i x_i$. For $x \in c_{00}$ set

$$\|x\|_T = \max\left\{\|x\|_0, \sup_{n \leq E_1 < \ldots E_n} \frac{1}{2}\sum_{i=1}^{n} \|E_i x\|_T\right\}$$

where $\{E_i\}_{i=1}^n$ are successive subsets of $\mathbb{N}$ and $E_i x$ denotes the restriction of x on the set E_i. Note that this is an implicit definition and the existence of such a norm follows by induction (cf. eg. [L-T]). The space T is the completion of $(c_{00}, \|\cdot\|_T)$. This is actually the dual of Tsirelson's original example and is due to Figiel and Johnson [F-J].

The (dual of) space T was the first example of a reflexive Banach space not containing any copy of l_p for $1 \leq p < \infty$ or c_0. It can be shown that any n-blocks of the basis supported after n are equivalent to l_1^n, hence T is a stabilized asymptotic l_1-space. The Tsirelson-type norms have become common place now and, among other things, they provide a more sophisticated class of examples of stabilized asymptotic-l_p spaces not containing isomorphs of l_p.

Chapter 4

Stabilization and Asymptotic Structure

4.1 Stabilization techniques

In this section we introduce some more terminology and present a few essentially known stabilization results that in particular reflect the techniques used in [M2], [F-F-K-R]. Let X be an infinite dimensional Banach space. On the set of infinite dimensional subspaces of X consider the following partial order

(4.1) $\quad Y \preccurlyeq Z \Longleftrightarrow Y \subseteq Z + F$, for some finite dimensional space F.

We shall need the following easy lemma.

Lemma 4.1. *Let Y, Z be infinite dimensional subspaces of a Banach space X. If $Y \preccurlyeq Z$ then $Y \cap Z$ is infinite dimensional.*

Proof From the definition of the partial order $\preccurlyeq$ there exists a finite dimensional space F such that $Y \subseteq Z+F$. Let $W := Z+F$ and let $P : W \longrightarrow W$ be a projection (not necessarily bounded) onto Z. Since Z is finite codimensional in W we have that $\mathrm{Ker}P$ is finite dimensional, hence $Id - P$ is a finite rank projection, where $Id : W \longrightarrow W$ is the identity on W. In particular $(Id - P)(Y)$ is finite dimensional. Since Y is infinite dimensional and $(Id - P)(Y)$ is finite dimensional it follows that $\mathrm{Ker}(Id - P) \cap Y = Z \cap Y$ must be infinite dimensional, as required.

■

Lemma 4.2. *If $\{Y_n\}_n$ is a sequence of infinite dimensional subspaces of X such that $Y_{n+1} \preccurlyeq Y_n$ for each n then there exist an infinite dimensional subspace Y of X such that $Y \preccurlyeq Y_n$ for any n.*

Proof The proof uses a so called *diagonalization technique.* Define, for any n,

$$Z_n = \bigcap_{1 \le i \le n} Y_i. \tag{4.2}$$

Using Lemma 4.1 it can be proved by induction that each Z_n is infinite dimensional. Indeed, (4.2) is trivially true for $n = 1$. Assume that Z_n is infinite dimensional; then $Z_{n+1} = Z_n \cap Y_{n+1}$ and it can be easily checked that $Y_{n+1} \preccurlyeq Z_n$. Applying Lemma 4.1 for Y_{n+1} and Z_n it follows that Z_{n+1} is infinite dimensional. Now, since each Z_n is infinite dimensional, we can build by induction a linearly independent sequence $(y_n)_n$ such that $y_n \in Z_n$ for each n. Denote by Y the closed linear span of $(y_n)_n$. Also note that, since $Z_{n+1} \subseteq Z_n$ for any n, we have that for any n and any $k \ge n$, $y_k \in Z_n$. Then it follows that, for any n, $Y \subseteq \mathrm{span}\{y_1, \dots, y_{n-1}\} + Z_n$ and from $Z_n \subseteq Y_n$ we have that $Y \preccurlyeq Y_n$.

■

Lemma 4.3. *Let φ be a function defined on the set of all infinite dimensional subspaces of X taking values in $[0, \infty]$. If φ is monotone with respect to the* *partial order $\preccurlyeq$ then for any Y infinite dimensional subspace of X there exists Z, an infinite dimensional subspace of Y, such that for any infinite dimensional subspace Z' of Y with $Z' \preccurlyeq Z$ we have that $\varphi(Z') = \varphi(Z)$. In other words, the function φ can be stabilized by passing to a subspace.*

Proof We can assume without loss of generality that the function φ is increasing (otherwise consider $\varphi' = 1/\varphi$).

Fix an infinite dimensional subspace Y of X and assume the conclusion is false for Y. By transfinite induction and diagonalization we shall construct $\{Z_\alpha\}_{\alpha<\omega_1}$ so that

$$\beta < \alpha \Longrightarrow Z_\alpha \preccurlyeq Z_\beta \text{ and } \varphi(Z_\alpha) < \varphi(Z_\beta) \tag{4.3}$$

Recall that the set $\{\alpha < \omega_1\}$ is uncountable and well ordered by " $<$ " and note that relation (4.3) establishes a bijective order preserving correspondence between $\{\alpha < \omega_1\}$ and a subset of $[0, \infty]$ with the natural order on $\mathbb{R}$. But this is a contradiction since $[0, \infty]$ cannot contain an uncountable subset which is well-ordered with respect to the natural order on $\mathbb{R}$.

Suppose that for any subspace of Z of Y we can find another subspace Z' of Y such that $Z' \preccurlyeq Z$ and $\varphi(Z') < \varphi(Z)$. For $\alpha = 0$ put $Z_0 = Y$. Take α to be an ordinal, $\alpha < \omega_1$, and assume we have defined Z_β for all $\beta < \alpha$.

If α is of the form $\beta + 1$, then from the above we can find Z_α subspace of Y such that $Z_\alpha \preccurlyeq Z_\beta$ and $\varphi(Z_\alpha) < \varphi(Z_\beta)$.

Otherwise, α must be a limit ordinal and since $\alpha < \omega_1$, α is the limit of some increasing sequence of ordinal numbers $\{\alpha_n\}_n$. From the induction hypothesis we have that

$$\cdots \preccurlyeq Z_{\alpha_n} \preccurlyeq Z_{\alpha_{n-1}} \preccurlyeq \cdots \preccurlyeq Z_{\alpha_2} \preccurlyeq Z_{\alpha_1}$$

and

$$\cdots < \varphi(Z_{\alpha_n}) < \varphi(Z_{\alpha_{n-1}}) < \cdots < \varphi(Z_{\alpha_2}) < \varphi(Z_{\alpha_1})$$

From Lemma 4.2 it follows that there exists Z_α infinite dimensional subspace of Y such that $Z_\alpha \preccurlyeq Z_{\alpha_n}$ for any n. Since φ is increasing we have, for any n, that $\varphi(Z_\alpha) \leq \varphi(Z_{\alpha_n}) < \varphi(Z_{\alpha_{n-1}})$ which ends the construction.

■

The next Lemma establishes that a countable family of monotone functions can be stabilized by passing to a subspace.

Lemma 4.4. *Let $\{\varphi_n\}_n$ be a family of functions defined on the set of all infinite dimensional subspaces of X taking values in $[0,\infty]$. If each φ_n is monotone with respect to the partial order $\preccurlyeq$ then for any Y infinite dimensional subspace of X there exists Z, an infinite dimensional subspace of Y, such that for any infinite dimensional subspace Z' of Y with $Z' \preccurlyeq Z$ we have that $\varphi_n(Z') = \varphi_n(Z)$ for any n.*

Proof Fix an infinite dimensional subspace Y of X. By applying Lemma 4.3 to Y and φ_1 we obtain Z_1 an infinite dimensional subspace of Y stabilizing for φ_1. We apply now Lemma 4.3 to Z_1 and φ_2 to obtain Z_2 stabilizing for φ_2. Repeating this procedure we obtain an infinite sequence $\{Z_n\}_n$ such that $Z_{n+1} \subset Z_n$ for any n. From Lemma 4.2 it follows that we can find an infinite dimensional subspace Z of Y such that, for any n, $Z \preccurlyeq Z_n$ and since Z_n is stabilizing for φ_n we have that Z is stabilizing for φ_n. This concludes the proof.

■

4.2 Main result

In this section we prove our main structural result.

Theorem 4.5. *Let X be a Banach space with the following property:*

For any infinite dimensional subspace $Y \subseteq X$ there exists a constant M_Y such that for any n there exist infinite dimensional subspaces $U_1, U_2, \cdots, U_n$ of Y such that

$$\frac{1}{M_Y}\|\sum_{i=1}^{n} a_i x_i\| \leq \|\sum_{i=1}^{n} a_i y_i\| \leq M_Y \|\sum_{i=1}^{n} a_i x_i\| \tag{4.4}$$

for any norm one vectors x_i, y_i in U_i and any scalars a_i.
Then there exists $p \in [1, \infty]$ such that X contains a stabilized asymptotic-l_p subspace (stabilized asymptotic-c_0 when $p = \infty$).

We'll prove a slightly different statement from which our result will follow. But first another definition is given.

Definition 4.6. A basic sequence $\{x_n\}_n$ is said to have property (P) if there is a $K < \infty$ such that for every n the following holds: for every sequence $(A_i)_{i=1}^n$ of finite mutually disjoint subsets of $\mathbb{N}$ such that $\min \bigcup_i A_i \geq n$, if $y_i, z_i \in \text{span}\{x_j : j \in A_i\}$ for $i = 1, 2, \ldots, n$ are two finite sequences of norm one vectors then $\{y_i\}_1^n$ is K-equivalent to $\{z_i\}_1^n$.

Theorem 4.7. *Under the hypothesis of Theorem 4.5, the space X contains a basic sequence with property (P).*

We show first how to derive Theorem 4.5 from Theorem 4.7.

Lemma 4.8. *Let $\{x_n\}_n$ be a basic sequence with property (P). Then the closed span of $\{x_n\}_n$ is a stabilized asymptotic-l_p space, for some $1 \leq p \leq \infty$ (stabilized asymptotic-c_0 for $p = \infty$).*

Proof Let $\{x_n\}_n$ be a basic sequence that has property (P) with constant K. We'll show first that for all n, any two normalized block sequences of length n of $\{x_k\}_k$ that start after x_n are K^2-equivalent. Indeed pick two normalized block sequences of length n starting after x_n, call them $(y_1, y_2, \ldots, y_n)$ and $(z_1, z_2, \ldots, z_n)$. Pick now a third block sequence $(t_1, t_2, \ldots, t_n)$ that starts after x_n and has the support disjoint from the previous two block sequences; that is, for any i and j, $\mathrm{supp} t_i \cap \mathrm{supp} y_j = \emptyset$ and $\mathrm{supp} t_i \cap \mathrm{supp} z_j = \emptyset$. For any $i = 1, \ldots, n$ define A_i to be the set

$$A_i := \{j \in \mathbb{N} : x_j \in \mathrm{supp}\ y_i \cup \mathrm{supp}\ t_i\}$$

Then $(A_i)_i$ satisfies the conditions in the definition of property (P), so it follows that $(y_1, y_2, \ldots, y_n)$ and $(t_1, t_2, \ldots, t_n)$ are K-equivalent. Similarly $(z_1, z_2, \ldots, z_n)$ and $(t_1, t_2, \ldots, t_n)$ are K-equivalent. Hence $(y_1, y_2, \ldots, y_n)$ and $(z_1, z_2, \ldots, z_n)$ are K^2-equivalent.

From Krivine's Theorem it follows that there exists $1 \leq p \leq \infty$ such that, for any n, we can find normalized blocks $w_1, w_2, \ldots, w_n$ of $\{x_j\}_j$ that start as far as we want, such that $w_1, w_2, \ldots, w_n$ is 2-equivalent to the standard unit vector basis of l_p^n. Since any two normalized block sequences that start far enough are K^2-equivalent then they must be $2K^2$-equivalent to the standard unit vector basis of l_p^n. Hence the closed span of $\{x_j\}_j$ is a stabilized asymptotic-l_p space. ■

Theorem 4.5 follows easily now. From Theorem 4.7 we have that we can find a basic sequence $\{x_i\}_i$ with property (P) in X and from Lemma 4.8 we conclude that the closed span of $\{x_i\}_i$ is a stabilized asymptotic-l_p subspace of X.

Note that if a space X satisfies the hypothesis of Theorem 4.7, so does every infinite dimensional subspace of X. Therefore it follows that every infinite dimensional subspace contains a further stabilized asymptotic-l_p subspace, possibly for different p's.

Remark 4.9. As this thesis was completed we realized that the previous result can be improved. Under the same hypothesis as in Theorem 4.5 we can obtain that the stabilized asymptotic-l_p subspaces we obtain also have an unconditional basis. More precisely the proof can be modified such that the space X contains an *unconditional* basic sequence with property (P), thus strengthening Theorem 4.7, from which the conclusion follows. Observe that, in particular, the hypothesis implies that the space X cannot contain any H.I. subspaces, therefore, by Gowers dichotomy, there exist unconditional basic sequences in every subspace. Our proof blends Maurey's approach to Gowers dichotomy with the argument from the proof of Theorem 4.7 to build these unconditional basic sequences in such a way that they also have property (P).

We are also investigating whether it is possible to relax the hypothesis of Theorem 4.5 and obtain the same conclusion under the hypothesis that relation (4.4) holds only for vectors with equal coefficients.

4.3 Proof of Theorem 4.7

Now it remains to prove Theorem 4.7. First we introduce some new notations that are convenient for the proof.

Let X be a Banach space. Denote by Δ the set of all pairs of n-tuples of vectors $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_n)$ with the property that $\|x_i\| = \|y_i\|$ for any $i \leq n$ and any $n \geq 1$. If Z is a subspace of X, $\Delta(Z)$ will be the subset of Δ consisting of all pairs $(\vec{x}, \vec{y})$ of n-tuples of vectors from Z, for any $n \geq 1$. Given $\vec{U} = (U_1, \ldots, U_n)$ where $U_1, \ldots, U_n$ are infinite dimensional subspaces of X and $\vec{u} = (u_1, \ldots, u_n)$ an n-tuple of vectors we write $\vec{u} \in \vec{U}$ if $u_i \in U_i$ for $1 \le i \le n$. Then set

$$\Delta(\vec{U}) = \{(\vec{u}, \vec{v}) \in \Delta : \vec{u} \in \vec{U}, \vec{v} \in \vec{U}\}.$$

This notation makes possible a more compact formulation of the hypothesis of Theorem 4.7. Namely, for any infinite dimensional subspace Y of X there exist a constant M_Y such that for any n there exist infinite dimensional subspaces $U_1, U_2, \cdots, U_n$ of Y such that

$$\frac{1}{M_Y}\|\sum_{i=1}^{n} x_i\| \le \|\sum_{i=1}^{n} y_i\| \le M_Y\|\sum_{i=1}^{n} x_i\| \tag{4.5}$$

for any $(\vec{x}, \vec{y}) \in \Delta(\vec{U})$, where $\vec{U} = (U_1, \ldots, U_n)$.

It is standard in this setting to pass to vector spaces over $\mathbb{Q}$ in order to use the countable structure of such a vector space. Without loss of generality we can assume that the Banach space X has a basis $\{e_n\}_n$. Let X_0 denote the set of all vectors of the form $\sum_{i=1}^{n} a_i e_i$ for $n \in \mathbb{N}$, $\{a_i\}_{i=1}^{n} \subseteq \mathbb{Q}$. Then X_0 is a countable vector space over $\mathbb{Q}$. Moreover, since X_0 is dense in X, it is enough to prove the conclusion of the theorem in X_0. Therefore, from this point onward, our argument will take place in X_0.

If Y is an infinite dimensional subspace of X_0, then we denote by $\Sigma(Y)$ the set of all infinite dimensional subspaces of Y and by $\Sigma_f(Y)$ the set of all finite dimensional subspaces of Y. By " $\preccurlyeq$ " we denote the partial order defined in (4.1) restricted to $\Sigma(X_0)$.

For any $n \ge 1$ and $\vec{E} = (E_1, E_2, \cdots, E_n)$, where $E_1, E_2, \cdots E_n$ are finite dimensional subspaces of X_0, and for any $Y \in \Sigma(X_0)$, set $\varepsilon_{\vec{E},Y}$ to be the supremum of all ε for which we can find $U_1, \cdots U_n \in \Sigma(Y)$ such that for any

$(u_1, \cdots u_n) \in \vec{U}$, $(v_1, \cdots v_n) \in \vec{U}$, $(e_1, \cdots, e_n) \in \vec{E}$, $(f_1, \cdots, f_n) \in \vec{E}$, with the property that $(\vec{u} + \vec{e}, \vec{v} + \vec{f}) \in \Delta$ we have that

$$(4.6) \qquad \varepsilon \| \sum_{i=1}^{n} (u_i + e_i) \| \leq \| \sum_{i=1}^{n} (v_i + f_i) \| \leq (1/\varepsilon) \| \sum_{i=1}^{n} (u_i + e_i) \|$$

Note that the condition $(\vec{u} + \vec{e}, \vec{v} + \vec{f}) \in \Delta$ simply means that $\|u_i + e_i\| = \|v_i + f_i\|$ for any $1 \leq i \leq n$. For any n, by $\vec{0}_n$ we understand the n-tuple $(\{0\}, \{0\}, \ldots, \{0\})$, in other words the n-tuple of finite dimensional subspaces of X_0 in which each entry is the trivial $\{0\}$ subspace. For a fixed n, comparing (4.6) with (4.5) observe that $(1/\varepsilon_{\vec{0}_n, Y})$ is simply the "best" constant M_Y appearing in (4.5) for this particular n.

Next, using the stabilization techniques from the previous section, we will stabilize the invariant $\varepsilon_{\vec{E}, Y}$.

Since X_0 is a countable vector space and $\vec{E}$ are finite tuples with entries from $\Sigma_f(X_0)$ we have that the family $\{\varepsilon_{\vec{E}, \cdot}\}$ of functions on $\Sigma(X_0)$, indexed by $\vec{E}$ is also countable. We show next that each $\varepsilon_{\vec{E}, Y}$ is increasing in Y with respect to the partial order $\preccurlyeq$ on $\Sigma(X_0)$. To this end, fix $\vec{E} = (E_1, E_2, \ldots, E_n)$ and let $Y_1 \preccurlyeq Y_2$. Pick any ε that satisfies (4.6) for the definition of $\varepsilon_{\vec{E}, Y_1}$. It follows that we can find $U_1, \cdots U_n \in \Sigma(Y_1)$ such that for any $(u_1, \cdots u_n) \in \vec{U}$, $(v_1, \cdots v_n) \in \vec{U}$, $(e_1, \cdots, e_n) \in \vec{E}$, $(f_1, \cdots, f_n) \in \vec{E}$ with the property that $(\vec{u} + \vec{e}, \vec{v} + \vec{f}) \in \Delta$, relation (4.6) holds for ε. For any $1 \leq i \leq n$ let $U_i' := U_i \cap Y_2$. Since U_i is a (infinite dimensional) subspace of Y_1 and $Y_1 \preccurlyeq Y_2$ we have that $U_1 \preccurlyeq Y_2$. Applying Lemma 4.1 for U_i and Y_2 it follows that $U_i' = U_i \cap Y_2$ is infinite dimensional. Also note that for any $1 \leq i \leq n$, U_i' is an infinite dimensional subspace of Y_2. Let $\vec{U'} = (U_1', \ldots, U_n')$. Therefore we can find $U_1', \cdots U_n' \in \Sigma(Y_2)$ such that for any $(u_1, \cdots u_n) \in \vec{U}'$, $(v_1, \cdots v_n) \in \vec{U}'$, $(e_1, \cdots, e_n) \in \vec{E}$, $(f_1, \cdots, f_n) \in \vec{E}$ with the property that $(\vec{u} + \vec{e}, \vec{v} + \vec{f}) \in \Delta$, relation (4.6) holds for ε. But this means exactly that ε satisfies (4.6) for the definition of $\varepsilon_{\vec{E},Y_2}$. Taking the supremum over all these ε it follows that $\varepsilon_{\vec{E},Y_1} \leq \varepsilon_{\vec{E},Y_2}$, hence $\varepsilon_{\vec{E},Y}$ is increasing in Y.

From Lemma 4.3 we have that there exist a subspace $Z \in \Sigma(X)$ stabilizing for the entire family $\{\varepsilon_{\vec{E},\cdot}\}$. In other words we have that there exists Z such that $\varepsilon_{\vec{E},Z'} = \varepsilon_{\vec{E},Z}$ for any infinite dimensional Z' subspace of Z and any $\vec{E}$. From this moment on we proceed with the argument inside this subspace Z. Since the subspace Z is stabilizing we can drop the subscript Z' in $\varepsilon_{\vec{E},Z'}$; the argument will take place in Z so the notation $\varepsilon_{\vec{E}}$ will be unambiguous.

From the hypothesis together with (4.5) and the definition of $\varepsilon_{\vec{0}_n}$ it follows that

$$\inf_n \varepsilon_{\vec{0}_n} \geq \frac{1}{M_Z} > 0$$

and $\varepsilon_{\vec{0}_n} \leq 1$ for any n.

Pick ε_0 satisfying the following two conditions

(i) $0 < \varepsilon_0 < \inf_n \varepsilon_{\vec{0}_n}$

(ii) For any $\vec{E}$, $\varepsilon_0 \neq \varepsilon_{\vec{E}}$.

The following definition is very important in the logical structure of the argument. Consider the subset $A \subset \Delta(Z)$ defined by

$$A := \{(\vec{x},\vec{y}) \in \Delta(Z) : \ \|\textstyle\sum_i x_i\| < \varepsilon_0 \|\sum_i y_i\|, \text{ or} \quad \|\sum_i x_i\| > (1/\varepsilon_0)\|\sum_i y_i\|\}. \tag{4.7}$$

In other words, A consist of all $(\vec{x},\vec{y}) \in \Delta(Z)$ which are **not** $(1/\varepsilon_0)$- equivalent.

We shall use the following suggestive terminology, similar to the one introduced by Maurey in [M3]. Let $\vec{E} = (E_1, \cdots, E_n)$, where $E_i \in \Sigma_f(Z)$ for $1 \leq i \leq n$. We say that $\vec{E}$ **accepts** a subspace $Y \in \Sigma(Z)$ iff for any $U_1, \cdots, U_n \in \Sigma(Y)$ we can find $(u_1, \cdots u_n) \in \vec{U}$, $(v_1, \cdots v_n) \in \vec{U}$ and

$(e_1, \cdots e_n) \in \vec{E}$, $(f_1, \cdots f_n) \in \vec{E}$ such that $(\vec{u} + \vec{e}, \vec{v} + \vec{f}) \in A$. We say that $\vec{E}$ **rejects** Z if it doesn't accept any subspace Y of Z. The following Lemma clarifies the dichotomy between "accepts" and "rejects".

Lemma 4.10. *For any $Y \in \Sigma(Z)$ we have that $\vec{E}$ accepts Y iff $\varepsilon_{\vec{E}} < \varepsilon_0$.*

Proof Indeed, if $\vec{E}$ accepts Y then for any $U_1, \cdots, U_n \in \Sigma(Y)$ we can find $(u_1, \cdots u_n) \in \vec{U}$, $(v_1, \cdots v_n) \in \vec{U}$ and $(e_1, \cdots e_n) \in \vec{E}$, $(f_1, \cdots f_n) \in \vec{E}$ such that

$$\|\sum_{i=1}^{n}(u_i + e_i)\| < \varepsilon_0 \|\sum_{i=1}^{n}(v_i + f_i)\| \text{ or}$$
$$\|\sum_{i=1}^{n}(u_i + e_i)\| > (1/\varepsilon_0) \|\sum_{i=1}^{n}(v_i + f_i)\|.$$

It follows that ε_0 does not satisfy the condition described in (4.6), hence $\varepsilon_{\vec{E},Y} \le \varepsilon_0$. From stability and from the fact that $\varepsilon_0 \ne \varepsilon_{\vec{E}}$ we have that

$$\varepsilon_{\vec{E}} = \varepsilon_{\vec{E},Y} < \varepsilon_0.$$

Conversely, if $\varepsilon_0 > \varepsilon_{\vec{E}} = \varepsilon_{\vec{E},Y}$, then ε_0 is not in the set of ε's from the definition of $\varepsilon_{\vec{E},Y}$. This means exactly that $\vec{E}$ accepts Y.

■

From Lemma 4.10 we derive the following important remark.

Remark 4.11. If $\vec{E}$ does not accept Z then it does not accept any subspace of Z, hence it rejects Z. Therefore we may simply say accepts or rejects without creating confusion.

In the sequel we shall also use the next simple remarks.

Remark 4.12. For any $n \ge 1$, if $\vec{E} = (E_1, \cdots, E_n)$ accepts (rejects) then so does $\vec{E}_\pi := (E_{\pi(1)}, \cdots, E_{\pi(n)})$ where π is any permutation on $\{1, 2, \cdots, n\}$. Indeed, from the definition of $\varepsilon_{\vec{E},Z}$ we can easily show that $\varepsilon_{\vec{E}} = \varepsilon_{\vec{E}_\pi}$ and the conclusion follows immediately from Lemma 4.11.

Remark 4.13. For any $n \geq 1$, if $\vec{E} = (E_1, \cdots, E_n)$ rejects then for any $\vec{e} = (e_1, \cdots, e_n) \in \vec{E}$, $\vec{f} = (f_1, \cdots, f_n) \in \vec{E}$ with $(\vec{e}, \vec{f}) \in \Delta$ we have that $(\vec{e}, \vec{f}) \notin A$. Indeed, from the definition of the term "rejects" it follows that we can find $U_1, \cdots, U_n \in \Sigma(Z)$ such that for any $\vec{u} = (u_1, \cdots u_n) \in \vec{U}$, $\vec{v} = (v_1, \cdots v_n) \in \vec{U}$ and $\vec{e} = (e_1, \cdots e_n) \in \vec{E}$, $\vec{f} = (f_1, \cdots f_n) \in \vec{E}$ with $(\vec{u} + \vec{e}, \vec{v} + \vec{f}) \in \Delta$ we have that

$$\varepsilon_0 \| \sum_{i=1}^{n} (u_i + e_i) \| \leq \| \sum_{i=1}^{n} (v_i + f_i) \| \leq (1/\varepsilon_0) \| \sum_{i=1}^{n} (u_i + e_i) \|.$$

Our claim follows by choosing $\vec{u}$ and $\vec{v}$ as the n-tuples of null vectors.

The connection between the terminology introduced above and property (P) becomes clear in view of the following simple observation which follows immediately from the previous remark and the definition of property (P).

Remark 4.14. Suppose $(x_j)_j$ is a basic sequence in Z. Fix $n \geq 1$ and let $(A_i)_{i=1}^{n}$ be as in Definition 4.6. Let $E_i := \text{span}\{x_j : j \in A_i\}$ for $i = 1, 2, \cdots, n$. To say that property (P) is satisfied with constant $(1/\varepsilon_0)$ is equivalent to saying that for any $n \geq 1$ any such $\vec{E} = (E_1, \cdots, E_n)$ rejects.

We shall build by induction a basic sequence $\{x_j\}_j$ that satisfies the condition equivalent to property (P), presented in Remark 4.14. But first we prove a key lemma for the inductive step.

Lemma 4.15. *Let $n \geq 2$. If $\vec{E} = (E_1, \cdots, E_n)$ rejects then for every infinite dimensional subspace W of Z there exists an infinite dimensional subspace W' of W such that for every $w' \in W'$ we have that $(E_1 + \text{span}\{w'\}, E_2, \cdots, E_n)$ rejects.*

Proof Assume that the conclusion is false. Then by Remark 4.11 there exists $W \in \Sigma(Z)$ such that for any $U \in \Sigma(W)$, we can find $u_0 \in U$ such that if $F_{u_0} := E_1 + \mathrm{span}\{u_0\}$, then $(F_{u_0}, E_2, \ldots, E_n)$ accepts. Thus, for any $U_2, U_3, \ldots, U_n \in \Sigma(W)$ we can find

$$\vec{u} = (u, u_2, u_3, \ldots, u_n) \in U \times U_2 \times U_3 \times \cdots \times U_n$$

$$\vec{v} = (v, v_2, v_3, \ldots, v_n) \in U \times U_2 \times U_3 \times \cdots \times U_n$$

and

$$\vec{e} = (e_{u_0}, e_2, e_3, \ldots, e_n) \in F_{u_0} \times E_2 \times E_3 \times \cdots \times E_n$$

$$\vec{f} = (f_{u_0}, f_2, f_3, \ldots, f_n) \in F_{u_0} \times E_2 \times E_3 \times \cdots \times E_n$$

such that

$$(\vec{u} + \vec{e}, \vec{v} + \vec{f}) \in A.$$

Since $e_{u_0} \in F_{u_0}$ and $f_{u_0} \in F_{u_0}$ we can write $e_{u_0} = e_1 + \alpha u_0$ and $f_{u_0} = f_1 + \beta u_0$ with $\alpha, \beta \in \mathbb{Q}$ and $e_1, f_1 \in E_1$. Hence we have that for any $(U, U_2, \ldots, U_n) \in (\Sigma(W))^n$ we can find $(u_1, u_2, \ldots, u_n) \in U \times U_2 \times \cdots \times U_n$,$(v_1, v_2, \ldots, v_n) \in U \times U_2 \times \cdots \times U_n$ and $(e_1, e_2, \ldots, e_n) \in E_1 \times E_2 \times \cdots \times E_n$,$(f_1, f_2, \ldots, f_n) \in E_1 \times E_2 \times \cdots \times E_n$ such that

$$((u_1, u_2, \ldots, u_n) + (e_1, e_2, \ldots, e_n), (v_1, v_2, \ldots, v_n) + (f_1, f_2, \ldots, f_n)) \in A \tag{4.8}$$

Indeed, we can take $(u_2, u_3, \ldots, u_n)$,$(v_2, v_3, \ldots, v_n)$, $(e_1, e_2 \ldots e_n)$,$(f_1, f_2 \ldots f_n)$ as above and put $u_1 := u + \alpha u_0$ and $v_1 = u + \beta u_0$. Then the pair in (4.8) is exactly $(\vec{u} + \vec{e}, \vec{v} + \vec{f})$ and it belongs to A. This means that $(E_1, \ldots, E_n)$ accepts. But this is a contradiction since $(E_1, \ldots, E_n)$ rejects W.

■

Proof [Theorem 4.7] We shall build inductively a basic sequence $\{x_j\}_j$ having the following property

(*) For any $n > 1$, and for any disjoint finite subsets $A_1, A_2, \ldots, A_n$ of $\{n-1, n, \ldots\}$, if $E_i := \text{span}\{x_j : j \in A_i\}$ for $i = 1, 2 \ldots, n$ then $(E_1, E_2, \ldots, E_n)$ rejects.

By convention, $\text{span}\{\emptyset\} = \{0\}$. Once we build such a sequence it follows from the Remark 4.14 that the sequence $\{x_j\}_j$ has property (P) and this would conclude the proof.

To have a better intuitive understanding of the proof that will follow some more explanations and clarifications are in order. First note that from Remark 4.12 we have that it is sufficient to check $(*)$ assuming additionally that the sets $\{A_j\}_1^n$ satisfy the following two conditions: (1) if $A_i = \emptyset$ then $A_j = \emptyset$ for all $i < j \leq n$, and (2) if $A_i \neq \emptyset$ and $A_j \neq \emptyset$ for $i < j$ then $\min A_i < \min A_j$. Another important observation is the following: we can always assume that $\min \bigcup_{i \leq n} A_i = n - 1$; indeed, otherwise if $\min \bigcup_{i \leq n} A_i := k > n - 1$ we add the empty sets $A_{n+1}, \ldots, A_k, A_{k+1}$ to the existing sets $A_1, \ldots, A_n$ and the new family $\{A_j\}_{j=1}^{k+1}$ will satisfy the assumption and it is a "valid" family since $\min \bigcup_{i \leq k+1} A_i \geq k$. To exemplify, instead of considering the family $A_1 = \{4, 5\}$ and $A_2 = \{8, 11, 13\}$ for $n = 2$, we consider the family $A_1 = \{4, 5\}$, $A_2 = \{8, 11, 13\}$, $A_3 = A_4 = A_5 = \emptyset$ for $n = 5$.

The fact that $\{x_n\}_n$ will be a basic sequence follows from a standard argument. At each step the choice of x_j will be from an infinite dimensional subspace. Choosing the vectors "far enough" along the basis $\{e_n\}_n$ and using the well known gliding hump argument (cf. eg [L-T]) we can obtain that the sequence $\{x_n\}_n$ is equivalent to a block basis of $\{e_n\}_n$, hence it will be itself a basic sequence.

An important first remark is that, from the choice of ε_0 we have that $\vec{0}_n$ rejects for any n.

Step 1 : Since $\vec{0}_2$ rejects, from Lemma 4.15, we can pick $x_1 \in Z$ such that $(\text{span}\{x_1\}, \{0\})$ rejects.

Step 2 : Next, since $\vec{0}_3$ and the previous pair reject, we can find an infinite dimensional subspace W_0 of Z such that for any $w \in W_0$ we have $(\text{span}\{x_1, w\}, \{0\})$, $(\text{span}\{x_1\}, \text{span}\{w\})$ and $(\text{span}\{w\}, \{0\}, \{0\})$ reject (by applying Lemma 4.15 three times). Take as x_2 any such w, with the provision that x_2 must be also chosen according to the gliding hump procedure, as explained before. We now have that tuples

$$(\text{span}\{x_1\}, \{0\})$$

$$(\text{span}\{x_1, x_2\}, \{0\}) \qquad\qquad (\text{span}\{x_1\}, \text{span}\{x_2\})$$

$$(\text{span}\{x_2\}, \{0\}, \{0\})$$

all reject.

Step 3 : Since all the previous tuples and $\vec{0}_4$ reject, we can find x_3 such that by adding x_3 to any coordinate we obtain tuples $\vec{E}$ that reject. That is, in addition to the ones in *Step 2*, the following tuples will reject.

$$(\text{span}\{x_1, x_3\}, \{0\}) \qquad\qquad (\text{span}\{x_1\}, \{x_3\})$$

$$(\text{span}\{x_1, x_2, x_3\}, \{0\}) \qquad\qquad (\text{span}\{x_1, x_2\}, \{x_3\})$$

$$(\text{span}\{x_1, x_3\}, \text{span}\{x_2\}) \qquad\qquad (\text{span}\{x_1\}, \text{span}\{x_2, x_3\})$$

$$(\text{span}\{x_2, x_3\}, \{0\}, \{0\}) \qquad\qquad (\text{span}\{x_2\}, \{x_3\}, \{0\})$$

$$(\text{span}\{x_3\}, \{0\}, \{0\}, \{0\})$$

The inductive idea is clear now. Suppose we have picked $x_1, x_2, \ldots, x_n$ such that the inductive hypothesis holds. Let $\mathcal{S}_{n-1}$ be the set of "acceptable" tuples $\vec{E}$ built in *Step* $n-1$, from $x_1, x_2, \ldots, x_n$. We have that for any $\vec{E} \in \mathcal{S}_{n-1}$, $\vec{E}$ rejects. We shall find a vector x_{n+1} such that any $\vec{E} \in \mathcal{S}_n$ rejects. For a vector $y \in Z$ denote by $\mathcal{S}_{n-1,y}$ the set obtained by adding y to every entry of every $\vec{E} \in \mathcal{S}_{n-1}$. Since the set $\mathcal{S}_{n-1}$ is finite and $\vec{0}_{n+1}$ rejects, by applying Lemma 4.15 repeatedly, we can find an infinite dimensional subspace W such that for any $w \in W$ we have that any $\vec{E} \in \mathcal{S}_{n-1,w}$ rejects and the $(n+1)$-tuple $\vec{F} = (\text{span}\{w\}, \{0\}, \{0\}, \ldots, \{0\})$ rejects as well. Choose any $x_{n+1} \in W$ which is "good" in the gliding hump procedure. It is easy to see now that any tuple $\vec{E} \in \mathcal{S}_n$ belongs either to $\mathcal{S}_{n-1}$ or to $\mathcal{S}_{n-1,x_{n+1}}$ or is $\vec{F}$, hence rejects. This concludes the inductive step and the proof of Theorem 4.7.

■

Remark 4.16. Note that each x_n is chosen subject to a finite number of conditions, this is why an "acceptable" decomposition must start far enough. What we cannot do in every inductive step is, having a vector $(E_1, E_2, \ldots, E_n)$ that rejects, to find x_{n+1} such that $(E_1, E_2, \ldots, E_n, \text{span}\{x_{n+1}\})$ rejects as well. As it turns out, this is not a mere technical difficulty but rather an genuine obstacle, since there are known example of spaces having a basic sequence with property (P), yet they do not contain l_p, for any $1 \le p < \infty$, or c_0.

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