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LOCALIZATION IN THE CATEGORY OF TOPOLOGICAL SPACES

by

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ABSTRACT

In Chapter I the concept of localization in the category of R-modules is discussed. Where R is a commutative ring containing the multiplicative identity. In Section 1 the concept of localization is defined and its interplay with various other functors is discussed. In Section 2 we specialze to the case $R = Z$ and define the concept of localization at a set P of primes.

In Chapter II, localization functor is defined on the category of simple topological spaces, and its interplay with the homotopy and homology functors is discussed in Section 1. In Section 2 the existance problem is settled for the localization of simple spaces. Section 3 contains various results on localization, in particular it is shown how, upto homotopy type, a space can be reconstructed from its localizations at the primes 2, 3, 5, ...

Chapter III contains some results on H-spaces.

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TABLE OF CONTENTS

CHAPTER I

LOCALIZATION IN ALGEBRA

§1.1 LOCALIZATION OF MODULUES

1.1.1 Definition. Let R be a commutative ring with identity, $S \subset R\backslash\{0\}$ is called a *multiplicative* set if $1 \in S$, and $a,b \in S \Longrightarrow ab \in S$.

Define a relation $\sim$ on the product set $A \times S$ by $(a,s) \sim (a',s') \Longleftrightarrow \exists\, u \in S$ with $u(s'a - sa') = o$.

1.1.2 Proposition. $\sim$ is an equivalence relation.

Proof. $\sim$ is reflexive since $1 \in S$, symmetry holds as $\{A,+\}$ is a group. To prove transitivity suppose $(a,s) \sim (a',s')$ and $(a',s') \sim (a'',s'')$ to show $(a,s) \sim (a'',s'')$.

Now $(a,s) \sim (a',s') \Longleftrightarrow \exists\, u \in S$ with $u(s'a - sa') = o$ and $(a's') \sim (a'',s'') \Longleftrightarrow \exists\, v \in S$ with $v(s''a' - s'a'') = o$ hence $vs''u(s'a - sa') = o$ and $usv(s''a' - s'a'') = o$ and as $uvs' \in S$ the result follows by adding $vs''u(s'a - sa')$ and $usv(s''a' - s'a'')$.

1.1.3 Definition. Let a/s denote the equivalence class contianing (a,s) and let $S^{-1}A$ denote the quotient set $\frac{A \times S}{\sim}$.

1.1.4 Remark. We have a map $u_A : A \to S^{-1}A$ defined by $u_A(a) = a/1$. Note that u_A may not be one-to-one.

1.1.5 Note: for all $t \in S$ $a/s = ta/ts$.

Proof is immediate as R is commutative.

1.1.6 $\{S^{-1}A,+,.\}$ is an R-module where $+$ and $.$ are defined as follows:

$$(a/s) + (b|t) = (ta+sb)|st, \quad r.(a/s) = (ra)|s$$

$$a,b \in A, \quad s,t \in S, \quad r \in R .$$

Proof. $+$ is well defined for let $a/s = a'/s'$ and $b/t = b'/t'$, then $\exists\ u_1,u_2 \in S$ with $u_1s'a = u_1sa'$ and $u_2t'b = u_2tb'$ now

$$a'/s' + b'|t' = (t'a'+s'b')|s't' =$$

$$= stu_1u_2(t'a'+s'b')|stu_1u_2s't' \qquad \text{[using 1.1.5]}$$

$$= u_1u_2s't'(ta+sb)|stu_1u_2s't'$$

$$= (ta+sb)|st = a/s + b|t \ .$$

Similarly

$$r(a'/s') = ra'/s' = ru_1sa'|u_1ss'$$

$$= ru_1s'a|u_1ss' = ra/s = r(a|s)$$

so $.$ is well defined. The rest is trivial $0|1 = 0|s$ for all $s \in S$ is the zero element and $-(a/s) = (-a)|s$.

1.1.7 Corollary. $S^{-1}R$ is an algebra over R with multiplication in $S^{-1}R$ defined by $(r_1|s_1)(r_2|s_2) = (r_1r_2)|(s_1s_2)$.

1.1.8 Corollary. $S^{-1}A$ is an $S^{-1}R$ module with the operation defined as $(r/s)(a/t) = ra|st$.

1.1.9 Definition. If $f : A \to B$ is an R-module homomorphism define $S^{-1}f : S^{-1}A \to S^{-1}B$ by

$$S^{-1}f(a/s) = f(a)|s \ .$$

Note that $S^{-1}f$ is well defined for if $a/s = a'/s'$ $\exists u \in S$ with $us'a = usa'$ so

$$S^{-1}f(a'/s') = f(a')/s' = usf(a')/uss' = f(usa')/uss'$$
$$= f(us'a)/uss' = us'f(a)/uss'$$
$$= f(a)/s = S^{-1}f(a/s) .$$

1.1.10 Proposition. $S^{-1}f$ is an $S^{-1}R$ module homomorphism as well as an R-module homomorphism i.e., the following diagrams are commutative

$$\begin{array}{ccc} R \times S^{-1}A & \xrightarrow{1 \times S^{-1}f} & R \times S^{-1}B \\ \downarrow & & \downarrow \\ S^{-1}A & \xrightarrow{S^{-1}f} & S^{-1}B \end{array}$$

$$\begin{array}{ccc} S^{-1}R \times S^{-1}A & \xrightarrow{1 \times S^{-1}f} & S^{-1}R \times S^{-1}B \\ \downarrow & & \downarrow \\ S^{-1}A & \xrightarrow[S^{-1}f]{} & S^{-1}B \end{array}$$

Proof. $rS^{-1}f(a/s) = r(f(a)|s) = rf(a)|s = f(ra)|s$

$$= S^{-1}f[ra\ s] = S^{-1}f[r(a/s)] ,$$

and

$$(r/t)S^{-1}f(a/s) = (r|t)(f(a)|s) = rf(a)|st = f(ra)|st$$
$$= S^{-1}f(ra/st) = S^{-1}f[(r|t)(a|s)] .$$

1.1.11 Remarks. (1) S^{-1} can be regarded as an endofunctor in the category of R-modules, or as a functor from the category of R-modules to the category of $S^{-1}R$-modules.

(2) We have a commutative diagram

$$\begin{array}{ccc} R \times A & \longrightarrow & A \\ {\scriptstyle u_R \times u_A}\downarrow & & \downarrow{\scriptstyle u_A} \\ S^{-1}R \times S^{-1}A & \longrightarrow & S^{-1}A \end{array}$$

1.1.12 Notation. 1) for all $r \in R$ define $r^* : A \to A$ by $r^*(a) = ra$. r^* is an R-endomorphism of A as R is commutative.

2) for all $r \in R$ $S^{-1}(r^*) = r_* : S^{-1}A \to S^{-1}A$ is defined by $r_*(a/s) = ra/s$. r_* is an $S^{-1}R$-endomorphism of $S^{-1}A$.

1.1.13 Theorem. If $r \in S$ then $r_* = S^{-1}(r^*) : S^{-1} \cong S^{-1}A$.

Proof. $r_*(a/s) = ra/s \qquad r \in S$

$r_*^{-1}(a/s) = a/rs \qquad r \in S$ so $rs \in S$.

1.1.14 Definition. An R-module A is called *local away* from S, written local $|S$, iff

$s^* : A \to A$ is an isomorphism for all $s \in S$.

1.1.15 A local $|S \iff u_A : A \to S^{-1}A$ is an isomorphism.

Proof. [$\Longrightarrow$]. Suppose $s^* : A \to A$ is an isomorphism for all $s \in S$. To show u_A is an isomorphism, where $u_A(a) = a/1$.

Now $u_A(a) = 0 \Rightarrow a/1 = 0/1 \Rightarrow \exists s \in S$ with $sa = 0$, i.e., $s^*(a) = 0$ so $a = o$ as s^* is an isomorphism hence u_A is a monomorphism.

Further let $a/s \in S^{-1}A$ now $s^* : A \to A$ isomorphism $\Rightarrow \exists !$ $b \in A$ with $s^*(b) = a$, i.e., $sb = a$ now $u_A(b) = b/1 = sb/s = a/s$ hence u_A is epimorphism.

[<==]. Now suppose u_A is an isomorphism, to show A is local $|S$. Let $s \in S$, consider $s^* : A \to A$. Now $s^*(a) = 0 \Rightarrow sa = o \Rightarrow a/1 = 0/1 \Rightarrow u_A(a) = 0 \Rightarrow a = o$, as u_A is isomorphism it follows that s^* is monomorphism.

On the other hand let $b \in A$. Consider $b/s \in S^{-1}A$. Now as $u_A : A \to S^{-1}A$ is an epimorphism $\exists$ $a \in A$ with $u_A(a) = b/s$ i.e., with $a/1 = b/s$. Further we have

$$u_A(sa) = sa/1 = s(a/1) = s(b/s) = sb/s$$
$$= b/1 = u_A(b) .$$

But as u_A is isomorphism, this means that $sa = b$, i.e., $s^*a = b$. Hence s^* is an epimorphism. So s^* is an isomorphism and A is local $|S$.

1.1.16 <u>Note 1</u>. A local $|S \iff s^* : A \to A$ is an isomorphism for all $s \in S$. Hence given $s \in S$ and $b \in A$ $\exists$ $a \in A$ such that $s^*a = b$, i.e., $sa = b$. In other words we can divide in A by elements of S.

<u>Note 2</u>. by 1.1.13 $S^{-1}A$ is local $|S$.

<u>Note 3</u>. $u_{S^{-1}A} : S^{-1}(A) \to S^{-1}(S^{-1}A)$ is an isomorphism.

<u>Note 4.</u> $S^{-1}A$ is called *localization of* A *away from* S.

1.1.17 <u>Theorem.</u> The functors $S^{-1}(-)$ and $(-) \otimes_R S^{-1}R$ from the category of R-modules and R-module homomorphisms to the category of $S^{-1}R$-modules and $S^{-1}R$-module homomorphisms, are naturally equivalent.

<u>Proof.</u> Define $\lambda_A : S^{-1}A \longleftrightarrow A \otimes_R S^{-1}R : \mu_A$ by

$$\lambda_A(a/s) = a \otimes 1/s$$

$$\mu_A(a \otimes r/s) = ra/s$$

now $\lambda_A \mu_A$ is identity on generators $a \otimes r/s$ of $A \otimes_R S^{-1}R$. Hence $\lambda_A \mu_A$ = identity of $A \otimes_R S^{-1}R$. Also $\mu_A \lambda_A$ = identity of $S^{-1}A$.

Further, it is easy to see that for $f : A \to B$ the following two diagrams are commutative.

$$\begin{array}{ccc} S^{-1}A & \xrightarrow{\lambda_A} & A \otimes S^{-1}R \\ {\scriptstyle S^{-1}f}\downarrow & & \downarrow{\scriptstyle f \otimes 1} \\ S^{-1}B & \xrightarrow[\lambda_B]{} & B \otimes S^{-1}R \end{array}$$

$$\begin{array}{ccc} S^{-1}A & \xleftarrow{\mu_A} & A \otimes S^{-1}R \\ {\scriptstyle S^{-1}f}\downarrow & & \downarrow{\scriptstyle f \otimes 1} \\ S^{-1}B & \xleftarrow[\mu_B]{} & S^{-1}B \otimes S^{-1}R \end{array}$$

Showing that both λ and μ are natural transformations of functors.

1.1.18 Remark. Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{u_A} & S^{-1}A \\ \cong \big\downarrow & & \big\downarrow \cong \; \lambda_A \\ A \otimes R & \xrightarrow[1 \otimes u_R]{} & A \otimes S^{-1}R \end{array}$$

1) u_A is an isomorphism $\iff$ $1 \otimes u_R$ is an isomorphism.

2) R local $|S$ $\iff$ all R-modules are local $|S$.

3) A local $|S$ $\iff$ A and $A \otimes_R S^{-1}R$ are isomorphic as $S^{-1}R$-modules.

4) If A is local $|S$ then

$$S^{-1}A \cong A \cong A \otimes_R S^{-1}R$$

hence

$$S^{-1}(S^{-1}A) \cong S^{-1}A \otimes S^{-1}R .$$

In particular

$$S^{-1}R \cong S^{-1}(S^{-1}R) \cong S^{-1}R \otimes_R S^{-1}R .$$

5) $S^{-1}(A \otimes_R B) \cong (A \otimes_R B) \otimes_R S^{-1}R \cong A \otimes (B \otimes_R S^{-1}R)$

$\cong A \otimes S^{-1}B$ and so on.

Hence

$$S^{-1}(A \otimes_R B) \cong A \otimes_R S^{-1}B \cong S^{-1}A \otimes_R B \cong S^{-1} \otimes_R S^{-1}B .$$

1.1.19 Proposition. 1) If $\{A_\alpha\}$ is a directed system of R-modules then $S^{-1}(\varinjlim_\alpha A_\alpha) = \varinjlim_\alpha (S^{-1}A_\alpha)$, i.e., localization $|S$ commutes with direct limits.

2) $S^{-1}(\bigoplus_\beta B_\beta) \cong \bigoplus_\beta S^{-1}B_\beta$

i.e., localization $|S$ commutes with arbitrary direct summation.

3) If $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is an exact sequence of R-modules then

$$0 \to S^{-1}A \xrightarrow{S^{-1}i} S^{-1}B \xrightarrow{S^{-1}j} S^{-1}C \to 0$$

is an exact sequence of $S^{-1}R$-modules.

Proof. 1) and 2) follow from 1.1.17 and the corresponding property of tensor product. Also in 3) we only have to show that $S^{-1}(i)$ is a monomorphism. Now $S^{-1}i(a/s) = o \Rightarrow i(a)/s = 0/1 \Rightarrow \exists\ u \in S$ with $ui(a) = 0$ but i is a homomorphism, hence $0 = ui(a) = i(ua)$ now this gives $ua = 0$ as i is a monomorphism, but this means $a/s = 0/1$. Hence $S^{-1}i$ is momomorphism.

1.1.20 Notation. If P is a prime ideal of R then $S = R\backslash P$ is multiplicative. For any R-module A, we write A_P for $S^{-1}A$ and say 'local at P' for "local $|S$". We also note that A_P is an R_P-module.

1.1.21 Remark. In A_P can "divide" by all elements not in P.

1.1.22 Examples. 1) If R is an integral domain $\{0\}$ is a prime ideal and $S^{-1}R = R_0$ = field of quotients of R.

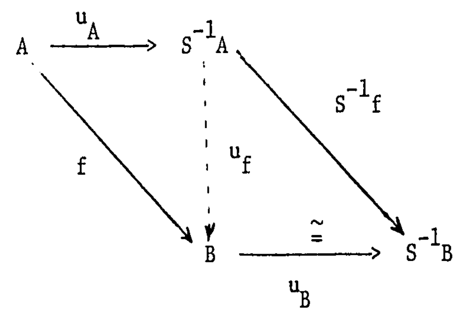
2) If $P = R$ then $S = \{1\}$ and

$$S^{-1}R = R_R = R \ .$$

1.1.23 Remark. $u_R : R \to R_P$ takes P into the ideal of non-units in R_P.

1.1.24 Theorem. Given an R-module homomorphism $f : A \to B$, B local $|S$, then $\exists\,!$ $S^{-1}R$-modules homomorphism $u_f : S^{-1}A \to B$ such that $u_f u_A = f$, as R-module homomorphisms.

Proof.



Define $u_f = (u_B^{-1}) \circ (S^{-1}f)$. Then

$$u_f u_A = u_B^{-1} \cdot S^{-1}f \cdot u_A = u_B^{-1}\, u_B f = f \quad .$$

Also if $g : S^{-1}A \to B$ is any other $S^{-1}R$-module homomorphism with $gu_A = f$ then as $g(a/1) = u_f(a/1)$ we have

$$g(a/s) = g(1/s \cdot a/1) = (1/s)g(a/1)$$

as $1/s \in S^{-1}R$ and g is an $S^{-1}R$-homomorphism then

$$\begin{aligned} g(a/s) &= (1/s)\; g(a/1) = (1/s)\; u_f(a/1) \\ &= u_f(1/s \cdot a/1) = gu_f \quad . \end{aligned}$$

This shows uniqueness of u_f.

1.1.25 Definition. A map $f : A \to B$ with B local $|S$ is called *universal for localization* $|S$ if and only if for any map $g : A \to C$ with C local $|S$, $\exists\,!$ $\bar{u}_g : B \to C$ with $\bar{u}_g f = g$.

Note. 1) $u_A : A \to S^{-1}A$ is universal for localization $|S$.

2) $f : A \to B$ is universal for localization $|S$
$\iff u_f : S^{-1}A \to B$ is an isomorphism.

1.1.26 <u>Theorem</u>. Given a long exact sequence of R-modules

$$\cdots \to A_{n-1} \to A_n \to A_{n+1} \to \cdots$$

such that except possibly for every third module, each module is local $|S$, then every module is local $|S$.

<u>Proof</u>. Since localization $|S$ is an exact functor, we have the following commutative ladder with exact rows

$$\begin{array}{ccccccccc} \cdots \to & A_{n-1} & \to & A_n & \to & A_{n+1} & \to \cdots \\ & \downarrow u_{n-1} & & \downarrow u_n & & \downarrow u_{n+1} & \\ \cdots \to & S^{-1}(A_{n-1}) & \to & S^{-1}(A_n) & \to & S^{-1}(A_{n+1}) & \to \cdots \end{array}$$

where we write u_n for u_{A_n}.

Now, except possibly for every third arrow, each vertical map is an isomorphism, hence by the 5-lemma each vertical map is an isomorphism.

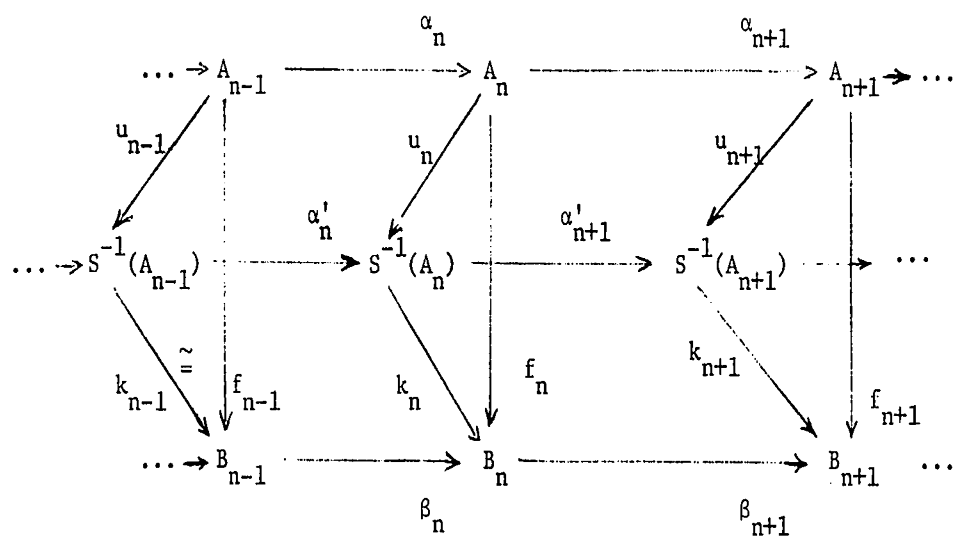
1.1.27 <u>Theorem</u>. Given a commutative ladder of R-modules with exact rows

$$\begin{array}{ccccccccc} \cdots \to & A_{n-1} & \xrightarrow{\alpha_n} & A_n & \xrightarrow{\alpha_{n+1}} & A_{n+1} & \to \cdots \\ & f_{n-1}\downarrow & & f_n\downarrow & & f_{n+1}\downarrow & \\ \cdots \to & B_{n-1} & \xrightarrow{\beta_n} & B_n & \xrightarrow{\beta_{n+1}} & B_{n+1} & \to \cdots \end{array}$$

such that, except possibly for every third vertical map, each vertical map is the universal homomorphism for localization $|S$, then all vertical arrows are universal homomorphisms for localization $|S$.

<u>Proof</u>. Assume $f_{n-2}, f_{n-1}, f_{n+1}$ and f_{n+2} are universal, then $B_{n-2}, B_{n-1}, B_{n+1}, B_{n+2}$ are all local $|S$, hence by 1.1.26 all the B_n's are local $|S$.

Now consider the following diagram, where we write α'_n for $S^{-1}(\alpha_n)$ and k_n for u_{f_n}



Now for all n, $k_n u_n = f_n$ and $u_n \alpha_n = \alpha'_n u_{n-1}$. Also as both f_{n-1} and u_{n-1} are universal, k_{n-1} is an isomorphism. Similarly k_{n-2}, k_{n+1} and k_{n+2} are isomorphism.

We shall first show that the lower ladder is commutative, i.e., $\beta_n k_{n-1} = k_n \alpha'_n$ for all n. Now

$$\beta_n k_{n-1} u_{n-1} = \beta_n f_{n-1} = f_n \alpha_n = k_n u_n \alpha_n$$

$$= k_n \alpha'_n u_{n-1}$$

but as u_{n-1} is universal, this gives

$$\beta_n k_{n-1} = k_n \alpha'_n \quad .$$

Similarly by universality of u_n we get

$$\beta_{n+1} k_n = k_{n+1} \alpha'_{n+1}$$

i.e., the lower ladder is commutative, but $k_{n-2}, k_{n-1}, k_{n+1}, k_{n+2}$ are all isomorphisms, hence k_n is also an isomorphism. Hence by universality of u_n we conclude that $f_n = k_n u_n$ is also universal.

1.1.28 Theorem. $S^{-1}\text{Tor}(A,B) \cong \text{Tor}(S^{-1}A,B) \cong \text{Tor}(A,S^{-1}B)$

$$\cong \text{Tor}(S^{-1}A,S^{-1}B) \quad .$$

Proof. Let $o \to K \to F \to B \to o$ be a free presentation of B. Then by definition we have the exact sequence

$$o \to \text{Tor}(A,B) \to A \otimes K \to A \otimes F \to A \otimes B \to o \quad .$$

This on localization $|S$ gives exact sequence

$$o \to S^{-1}\text{Tor}(A,B) \to S^{-1}(A \otimes K) \to S^{-1}(A \otimes F) \to S^{-1}(A \otimes B) \to o$$

Also by 1.1.18 (5) we get a commutative ladder

$$\begin{array}{ccccccccc} o \to & S^{-1}\text{Tor}(A,B) & \to & S^{-1}(A \otimes K) & \to & S^{-1}(A \otimes F) & \to & S^{-1}(A \otimes B) & \to o \\ & & & \updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong & \\ o \to & \text{Tor}(S^{-1}A,B) & \to & S^{-1}A \otimes K & \to & S^{-1}A \otimes F & \to & S^{-1}A \otimes B & \to o \end{array}$$

as all vertical maps are isomorphisms this gives isomorphism of kernels $S^{-1}\text{Ror}(A,B) \cong \text{Tor}(S^{-1}A,B)$. The rest follows from the fact that $\text{Tor}(A,B) \cong \text{Tor}(B,A)$.

§1.2 In this section we take $R = Z$ and let $\mathbb{P}$ denote the set of all primes in Z, together with zero, i.e., $o \in \mathbb{P}$. Any subset P of $\mathbb{P}$ that we consider will contain zero.

1.2.1 Notation. Let $P \subset \mathbb{P}$ and let $n \in Z$

1) We write $(n,P) = 1$ if $(n,p) = 1$ for all $p \in P$.

2) Let $\langle\mathbb{P}\backslash P\rangle = \{n \in Z,\ n = \text{product of primes not in } P\}$.

3) Note that as $o \in P$ $\langle\mathbb{P}\backslash P\rangle$ is a multiplicative set containing 1 as an empty product.

4) Write $A_P = \langle\mathbb{P}\backslash P\rangle^{-1} A$.

$= \text{localization of } A \text{ at } P.$

1.2.2 Note. 1) In A_P we can divide by every integer n with $(n,P) = 1$.

2) If $P = \{p\} \cup \{o\}$, then (p) is a prime ideal in Z and

$$A_P = A_{(p)} \qquad \text{by 1.1.20.}$$

3) $A_o = \langle P\backslash o\rangle^{-1} A \cong A \otimes Q$

1.2.3 Examples.

1) $Z_P = Z$, $Z_o = Q$

2) $$(Z/p^n Z)_P \cong \begin{cases} 0 & \text{if } p \notin P \\ Z/p^n Z & \text{if } p \in P \end{cases}$$

The proof is by induction on n.

First we show that the result holds for $n = 1$. Consider the exact sequence

$$o \to Z \xrightarrow{m} Z \to Z/pZ \to o$$

where $m(a) = pa$.

This on localization gives exact sequence

$$o \to Z_p \xrightarrow{m_*} Z_p \to (Z/pZ)_p \to o$$

where $m_*(a/s) = (pa)/s$. Now if $p \notin P$ then m_* is an isomorphism (1.1.13) hence $(Z/pZ)_p = 0$. On the other hand if $p \in P$ then $m_*(Z_p) = p(Z_p)$ is a proper subgroup of Z_p. Claim cosets of pZ_p in Z_p are $[0/1], [1|1], \ldots, [(p-1)/1]$ where we write $[i/s]$ for $(i/s) + pZ_p$. For let $a/s \in Z_p$ then $(s,p) = 1$ and $\exists$ integers x,y such that $px + sy = 1$. Now

$$a/s = a(px+sy)/s = apx/s + ay/1 \quad .$$

Now suppose $ay = py' + r$ where $o \leq r < p$, then we get

$$a/s = apx/s + (py' + r)/1$$
$$= p(ax+y's)/s + r/1 \quad .$$

$$\therefore \quad [a/s] = [r/1] \qquad o \leq r < p \; .$$

Also these cosets are distinct for if $[r_1/1] = [r_2/1]$ $o \leq r_1, r_2 < p$ then $r_1/1 + pa_1/s_1 = r_2/1 + pa_2/s_2$ this means $\exists$ $u \in S$ with $u(r_2-r_1)s_1s_2 = up(s_2a_1 - s_1a_2)$ but $(p,us_1s_2) = 1$ hence $p/(r_2-r_1)$ which means $r_2 = r_1$.

<u>Induction Step</u>. Assume the result holds for $k \leq n$. Consider the exact sequence

$$o \to Z/pZ \to Z/p^{n+1}Z \to Z/p^nZ \to o \quad .$$

This gives the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
o & \to & Z/pZ & \to & Z/p^{n+1}Z & \to & Z/p^{n}Z & \to & o \\
 & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_{n+1}} & & \downarrow{\scriptstyle u_n} & & \\
o & \to & (Z/pZ)_P & \to & (Z/p^{n+1}Z)_P & \to & (Z/p^{n}Z)_P & \to & o \quad .
\end{array}$$

Now if $p \notin P$ then by induction $(Z/pZ)_P$ and $(Z/p^nZ)_P$ are zero, hence by exactness of bottom row so also is $(Z/p^{n+1}Z)_P$. If on the other hand $p \in P$, then u_1 and u_n are isomorphisms, hence by 5-lemma so is u_{n+1}.

1.2.4 <u>Lemma</u>. Let $P_1, P_2 \subseteq \mathbb{P}$. If $s \in Z$ is such that $(s, P_1 \cap P_2) = 1$ then we can factorize s uniquely as $s = s_1, s_2\bar{s}$ where $s_1 \in \langle P_1 \backslash P_2 \rangle$, $s_2 \in \langle P_2\ P_1 \rangle$ and $(\bar{s}, P_1 \cup P_2) = 1$.

Note that this implies $(s_1, P_2) = 1 = (s_2, P_1)$ and $(s_1, s_2) = 1 = (s_1 s_2, \bar{s})$.

<u>Proof</u>. Trivial.

1.2.5 <u>Lemma</u>. Let G be any abelian group, and let $P_1, P_2 \subseteq \mathbb{P}$, then

a) $P_1 \subseteq P_2 \Longrightarrow G_{P_2} \subseteq G_{P_1}$

b) $0 = G_{P_2} \subseteq G_{P_1} \Longrightarrow P_1 \subseteq P_2$.

<u>Proof</u>. a) Let $P_1 \subseteq P_2$ then $a/s \in G_{P_2} \Longleftrightarrow (s, P_2) = 1 \Longrightarrow (s, P_1) = 1 \Longleftrightarrow a/s \in G_{P_1}$.

b) If $G_{P_2} \neq \{0\}$ $\exists\, a \in G$ with $0 = a/1 \in G_{P_2}$. Now $P_1 \nsubseteq P_2 \Longrightarrow \exists\, p \in P_1$ such that $p \notin P_2$ so $a/p \in G_{P_2}$ but $a/p \notin G_{P_1}$ so $G_{P_2} \notin G_{P_1}$ this contradiction establishes the result.

1.2.6 <u>Corollary</u>. $P_1 \subseteq P_2 \subseteq \mathbb{P} \Longleftrightarrow Z_{P_2} \subseteq Z_{P_1}$.

1.2.7 Corollary. G_P is a $Z_{P'}$ module for all $P' \supseteq P$.

1.2.8 Theorem. $Z_{P_1} \otimes_Z Z_{P_2} \cong Z_{P_1 \cap P_2}$ as rings.

Proof. Define $\phi : Z_{P_1} \otimes_Z Z_{P_2} \leftrightarrow Z_{P_1 \cap P_2} : \psi$ as follows:

ϕ is defined on generators by

$$\phi((a/s) \otimes (b/t)) = ab/st$$

and extended linearly to $Z_{P_1} \otimes_Z Z_{P_2}$, this makes ϕ an additive homomorphism.

Define ψ by

$$\psi(a/s) = (a/s_2 \otimes 1/s\bar{s})$$

where $s = s_1 s_2 \bar{s}$ is the unique factorization given in 1.2.4. Also note that by 1.1.5 and as tensor product is taken over Z

$$a/s_2 \otimes 1/s\bar{s} = a/s_2\bar{s} \otimes 1/s_1 = 1/s_2\bar{s} \otimes a/s_1 \quad \text{etc.}$$

ϕ is well defined for $(s,P_1) = 1$ $(t,P_2) = 1 \implies (st, P_1 \cap P_2) = 1$ hence $ab/st \in Z_{P_1 \cap P_2}$. Also if $a/s = a'/s'$ and $b/t = b'/t'$ then as $s'a = sa'$ and $t'b = tb'$ we have

$$\begin{aligned}\phi(a'/s' \otimes b'/t') &= ab'/s't' = sta'b'/s't'st \\ &= s'at'b/s't'st = ab/st \\ &= \phi(a/s \otimes b/t) \quad .\end{aligned}$$

Finally ϕ is a ring homomorphism for

$$\begin{aligned}&\phi[(a/s \otimes b/t)(a'/s' \otimes b'/t')] = \\ &\quad \phi[aa'/ss' \otimes bb'/tt'] = aa'bb'/ss'tt' \\ &\quad = (ab/st)(a'b'/s't') = \phi(a/s \otimes b/t)\phi(a'/s' \otimes b'/t')\end{aligned}$$

ψ is well defined for if $a/s = b/t \in Z_{P_1 \cap P_2}$ then

$$ta = sb \quad \text{and} \quad \psi(b/t) = (b/t_2) \otimes (1/t_1\bar{t})$$

$$= (s_2 b/t_2 s_2) \otimes (s_1\bar{s}/s_1\bar{s}t_1\bar{t})$$

$$= (s_1\bar{s}s_2 b/s_2 t_2) \otimes (1/s_1\bar{s}t_1\bar{t})$$

$$= (sb/s_2 t_2) \otimes (1/s_1\bar{s}t_1\bar{t})$$

$$= (ta/s_2 t_2) \otimes (1/s_1\bar{s}t_1\bar{t})$$

$$= (t_1 t_2 \bar{t}a/s_2 t_2) \otimes (1/s_1\bar{s}t_1\bar{t})$$

$$= (a/s_2) \otimes (1/s_1\bar{s}) = \phi(a/s) \quad .$$

ψ is an additive homomorphism for

$$\psi(a/s + b/t) = \psi(ta+sb/st)$$

$$= [(ta+sb)/s_2 t_2] \otimes [1/s_1 t_1 \bar{s}\bar{t}]$$

$$= (ta/s_2 t_2 + sb/s_2 t_2) \otimes (1/s_1 t_1 \bar{s}\bar{t})$$

$$= [(ta/s_2 t_2) \otimes (1/s_1 t_1 \bar{s}\bar{t})] + [(sb/s_2 t_2) \otimes (1/s_1 t_1 \bar{s}\bar{t})]$$

$$= [(a/s_2) \otimes (1/s_1\bar{s})] + [(b/t_2) \otimes (1/t_1\bar{t})]$$

$$= \psi(a/s) + \psi(b/t) \quad .$$

ψ is also a multiplicative homomorphism for

$$\psi(a/s \cdot b/t) = \psi(ab/st) = (ab/s_2 t_2) \otimes (1/s_1 t_1 \bar{s}\bar{t})$$

$$= (a/s_2 \otimes 1/s_1\bar{s})(b/t_2 \otimes 1/t_1\bar{t})$$

$$= \psi(a/s) \cdot \psi(b/t) \quad .$$

<u>Claim</u>. $\psi\phi$ = identity and $\phi\psi$ = identity. Let $(a/s) \otimes (b/t) \in Z_{P_1} \otimes Z_{P_2}$

so

$$(s,P_1) = 1 = (s,P_1 \cap P_2)$$

$$(t,P_2) = 1 = (t,P_1 \cap P_2)$$

hence in factorization $s = s_1 s_2 \bar{s}$ given in 1.2.4. had $s_1 \in \langle P_1 \backslash P_2 \rangle$ $s_2 \in \langle P_2 \backslash P_1 \rangle$, but as $(s,P_1) = 1$ we have $s_1 = 1$, similarly $t_2 = 1$, i.e., $s = s_2 \bar{s}$ $t = t_1 \bar{t}$. Now

$$\begin{aligned}
\psi\phi(a/s \times b/t) &= \psi(ab/st) \\
&= (ab/s_2 t_2) \otimes (1/s_1 t_1 \bar{s}\bar{t}) \\
&= (ab/s_2) \otimes (1/t_1 \bar{s}\bar{t}) \\
&= (a\bar{s}/s_2\bar{s}) \otimes (b/t_1 \bar{s}\bar{t}) \\
&= (a/s_2\bar{s}) \otimes (b\bar{s}/t_1 \bar{s}\bar{t}) \\
&= (a/s) \otimes (b/t)
\end{aligned}$$

so $\psi\phi$ = identity.

Also $\phi\psi(a/s) = \phi(a/s_2 \otimes 1/s_1\bar{s}) = a/s_2 s_1 \bar{s} = a/s$, hence $\phi\psi$ = identity.

1.2.9 <u>Construction</u>. Let $P \subseteq \mathbb{P}$, $S = \{s \in Z, (s,P) = 1\}$ define a partial order $\leq$ in S by $s_1 \leq s_2 \iff s_1$ is a divisor of s_2. Now for $s_1, s_2 \in S$, $s_1, s_2 \leq s_1 s_2 \in S$ hence $\{S, \leq\}$ is a directed set.

Let G be any abelian group. Form a directed system of groups $\{G_s\}$ indexed over S, where $G_s = G$ for all $s \in S$.

If $s_1 \leq s_2$ define a homomorphism

$$\phi_{s_1}^{s_2} : G_{s_1} = G \to G_{s_2} = G$$

by $\phi_{s_1}^{s_2}(a) = (\frac{s_2}{s_1})\, a$, for a in G, note that $\frac{s_2}{s_1}$ is an integer. It is easy to see that $\phi_{s_2}^{s_3}\phi_{s_1}^{s_2} = \phi_{s_1}^{s_3}$ and ϕ_s^s = identity.

For the next lemma take $G = Z$.

1.2.10 Lemma. $\varinjlim_{s} Z_s \cong Z_P$.

Proof. Let $H = \varinjlim_{s} Z_s$, so there are maps $\phi_s : Z_s \to H$ such that for

$$s_1 \le s_2 \quad \phi_{s_2}\phi_{s_1}^{s_2} = \phi_{s_1} .$$

Define $\psi_s : Z_s = Z \to Z_P$ by $\psi_s(a) = a/s \quad a \in Z$. Then

$$\psi_{s_2}\phi_{s_1}^{s_2}(a) = \psi_{s_2}(\frac{s_2}{s_1}a) = \frac{s_2}{s_1}a/s_2 = a/s_1$$

$$= \psi_{s_1}(a) .$$

i.e., $\psi_{s_2}\phi_{s_1}^{s_2} = \psi_{s_1}$.

Now as

$$H = \varinjlim_{s} Z_s \quad \exists ! \quad \psi : H \to Z_P$$

such that $\psi\phi_s = \psi_s \quad s \in S$. As each ψ_s is injective so is ψ. Also if $a/s \in Z_P$ then $a/s = \psi_s(a) = \psi[\phi_s(a)]$ so ψ is onto. Hence ψ is an isomorphism.

1.2.11 Theorem. $\varinjlim_{s} G_s \cong G \otimes Z_P \cong G_P$.

Proof. From the lemma we have

$$\varinjlim Z_s \cong Z_P .$$

Tensoring both sides with G we get

$$G \otimes \varinjlim Z_s \cong G \otimes Z_p \quad ,$$

but

$$G \otimes \varinjlim Z_s \cong \varinjlim (G \otimes Z_s) = \varinjlim G_s \quad .$$

Hence the result follows.

1.2.12 Definition. A square of abelian groups

$$\begin{array}{ccc} A & \xrightarrow{L_1} & B \\ {\scriptstyle L_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ C & \xrightarrow[j_2]{} & D \end{array}$$

is called a *fibre square* iff the following sequence is exact

$$o \longrightarrow A \xrightarrow{<L_1,L_2>} B \oplus C \xrightarrow{\{j_1,-j_2\}} D \longrightarrow o$$

where

$$<L_1,L_2> (a) = (L_1a,L_2a) \in B \oplus C$$

$$\{j_1,-j_2\}(b,c) = j_1(b) - j_2(c) \in D \quad .$$

.2.13 Lemma. If

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ C' & \longrightarrow & D' \end{array}$$

are fibre squares, then so also is

$$\begin{array}{ccc} A \oplus A' & \longrightarrow & B \oplus B' \\ \downarrow & & \downarrow \\ C \oplus C' & \longrightarrow & D \oplus D' \end{array}$$

Proof. Exactness of

$$o \to A \to B \oplus C \to D \to o$$

and

$$o \to A' \to B' \oplus C' \to D' \to o$$

implies exactness of

$$o \to A \oplus A' \to (B + B') \oplus (C + C') \to D \oplus D' \to o \quad .$$

1.2.14 Lemma. If $\{A_s\}$ $\{B_s\}$ $\{C_s\}$ $\{D_s\}$ are directed families of abelian groups, such that for each s

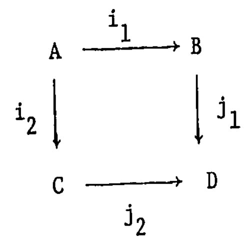
$$\begin{array}{ccc} A_s & \longrightarrow & B_s \\ \downarrow & & \downarrow \\ C_s & \longrightarrow & D_s \end{array}$$

is a fibre square then so also is

$$\begin{array}{ccc} \varinjlim A_s & \longrightarrow & \varinjlim B_s \\ \downarrow & & \downarrow \\ \varinjlim C_s & \longrightarrow & \varinjlim D_s \end{array}$$

Proof. $\underline{\lim}$ takes exact sequences into exact sequences.

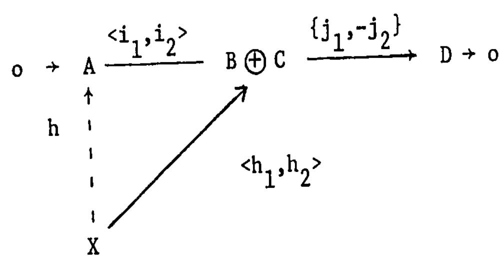
1.2.15 If



is a fibre square then it is both a pull back and a push out.

<u>Proof</u>. First we show that the above diagram is a pull back, we note that as the composite $\{j_1,-j_2\} \langle i_1,i_2\rangle = o$ $j_1i_1 = j_2i_2$.

Now let $h_1 : X \to B$ $h_2 : X \to C$ be such that $j_1h_1 = j_2h_2$. Consider the following diagram



Now $\{j_1,-j_2\} \langle h_1,h_2\rangle = o$ hence as A is the kernel of $\{j_1,-j_2\}$ $\exists!$ $h : X \to A$ such that $\langle i_1,i_2\rangle h = \langle h_1,h_2\rangle$, i.e., $i_1h = h_1$ and $i_2h = h_2$.

This h is unique for if there is also an $h' : X \to A$ with $i_1h' = h_1$ and $i_2h' = {}_2$ then $\langle i_1,i_2\rangle h' = \langle h_1,h_2\rangle = \langle i_1,i_2\rangle h$. But then as $\langle i_1,i_2\rangle$ is monic, we get $h' = h$.

Now we show that the diagram is a push out. Let $k_1 : B \to Y$ and $k_2 : C \to Y$ be such that $k_1i_1 = k_2i_2$. Consider the following diagram

$$\begin{array}{ccccccc} 0 \to A & \xrightarrow{\langle i_1,i_2\rangle} & B \oplus C & \xrightarrow{\{j_1,-j_2\}} & D & \to 0 \\ & & & \searrow^{\{k_1,-k_2\}} & \downarrow{\scriptstyle k} & \\ & & & & Y & \end{array}$$

Now $\{k_1,-k_2\}\ \langle i_1,i_2\rangle(a) = k_1i_1(a) - k_2i_2(a) = o$ but $D = \operatorname{coker} \langle i_1,i_2\rangle$ hence $\exists !\ k : D \to Y$ such that $k\{j_1,-j_2\} = \{k_1,-k_2\}$. This gives $kj_1 = k_1$ and $kj_2 = k_2$. Hence the diagram is a push out.

1.2.16 <u>Theorem.</u> Let $P_1, P_2 \subseteq \mathbb{P}$ then

$$\begin{array}{ccc} G_{P_1 \cup P_2} & \xrightarrow{i_1} & G_{P_1} \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ G_{P_2} & \xrightarrow[j_2]{} & G_{P_1 \cap P_2} \end{array}$$

is a fibre square.

<u>Proof.</u>

<u>Case I.</u> $G = Z$. Suffices to show that

$$o \to Z_{P_1 \cup P_2} \xrightarrow{\langle i_1,i_2\rangle} Z_{P_1} \oplus Z_{P_2} \xrightarrow{\{j_1,-j_2\}} Z_{P_1 \cap P_2} \to o$$

is exact where $\langle i_1,i_2\rangle(a/s) = (a/s,a/s)$ and $\{j_1,-j_2\}(a/s,b/t) = (a/s) - (b/t)$

(i) Exactness at $Z_{P_1 \cup P_2}$: - As each of i_1, i_2 is inclusion so is $\langle i_1, i_2 \rangle$.

(ii) Exactness at $Z_{P_1 \cap P_2}$. Let $c/s \in Z_{P_1 \cap P_2}$ then $(s, P_1 \cap P_2) = 1$ let $s = s_1 s_2 \bar{s}$ be the unique factorization given in 1.2.4. Consider the diaphantine equation (1) $s_1 x - x_2 \bar{s} y = c$.

This always has integral solutions as $(s_1, s_2 \bar{s}) = 1$ which is a divisor of c. In fact if x_o, y_o are solutions of $s_1 x - x_2 \bar{s} y = 1$ then all solutions of (1) are given by

$$x = cx_0 + s_2 \bar{s} n \qquad y = cy_0 + s_1 n$$

where n is an integer.

Now if x_1, y_1 is a solution of (1) then consider

$$(x_1/s_2\bar{s},\ y_1/s_1) \in Z_{P_1} \oplus Z_{P_2} .$$

Then

$$\{j_1, -j_2\}(x/s_2\bar{s},\ y_1/s_1) =$$

$$= (x_1/s_2\bar{s}) - (y_1/s_1) = (s_1 x_1 - s_2 \bar{s} y_1)/s_1 s_2 \bar{s}$$

$$= c/s$$

so $\{j_1, -j_2\}$ is onto.

(iii) Exactness at $Z_{P_1} \oplus Z_{P_2}$.

Firstly $\{j_1, -j_2\} \langle i_1, i_2 \rangle (a/s) = a/s - a/s = 0$.

Conversely if $(a/s, b/t) \in \ker \{j_1, -j_2\}$ then $(a/s) - (b/t) = o$ i.e., $(ta - sb)/st = 0$ hence $ta - sb = 0$ i.e., $a/s = b/t$ Z_{P_1}, Z_{P_2} hence $(s, P_1) = 1 = (s, P_2)$ which implies $(s, P_1 \cup P_2) = 1$. Hence $a/s = b/t \in Z_{P_1 \cup P_2}$ and

$$\langle i_1, i_2\rangle(a/s) = (a/s,\ a/s) = (a/s,\ b/t)$$

so $\ker\{j_1, -j_2\} \subseteq \text{image } \langle i_1, i_2\rangle$.

Case II. $G = Z/p^{\alpha}Z$.

Now if $p \in P_1 \cap P_2 \subseteq P_1, P_2 \subseteq P_1 \cup P_2$ then

$$(Z/p^{\alpha}Z)_{P_1 \cup P_2} = (Z/p^{\alpha}Z)_{P_1} = (Z/p^{\alpha}Z)_{P_2}$$

$$= (Z/p^{\alpha}Z)_{P_1 \cap P_2} = Z/p^{\alpha}A$$

and we have just to show the exactness of

$$o \to Z/p^{\alpha}Z \xrightarrow{\Delta} Z/p^{\alpha}Z \oplus Z/p^{\alpha}Z \xrightarrow{d} Z/p^{\alpha}Z \to o$$

where $\Delta(a) = (a,b)$ and $d(a,b) = a-b$. This is trivial

Next if

$$p \in P_1 \backslash P_2 \subseteq P_1 \cup P_2$$

$$(Z/p^{\alpha}Z)_{P_1} = Z/p^{\alpha}Z = (Z/p^{\alpha}Z)_{P_1 \cup P_2}$$

and

$$(Z/p^{\alpha}Z)_{P_2} = 0 = (Z/p^{\alpha}Z)_{P_1 \cap P_2}$$

and we have to show exactness of

$$0 \to Z/p^{\alpha}Z \xrightarrow{id} Z/p^{\alpha}Z \to o \to o$$

this is again trivial.

The case $p \in P_2 \backslash P_1$ is similar. Finally $p \notin P_1 \cup P_2$ is also trivial.

Case III. If G is a finitely generated abelian group then

$$G = \bigoplus_{p} Z \oplus \left(\bigoplus_{\alpha} Z/p^{\alpha}Z \right)$$

and the result follows from 1.2.13.

Case IV. If G is any abelian group, then G is the direct limit of its finitely generated subgroups and the result follows from 1.2.14.

1.2.17 Corollary.

$$G_{P_1 \cup P_2} \cong G_{P_1} \underset{G_{P_1 \cap P_2}}{\times} G_{P_2}$$

i.e., $G_{P_1 \cup P_2}$ is fiber product over $G_{P_1 \cap P_2}$ of the groups G_{P_1} and G_{P_2}.

Proof. The proof follows from 1.2.16 and 1.2.15.

1.2.18 Corollary. If $P_1 \cap P_2 = 0$ then

$$Z_{P_1 \cup P_2} \cong Z_{P_1} \underset{Q}{\times} Z_{P_2} .$$

1.2.19 Corollary. If G is any abelian group then

$$G \cong G_{(2)} \underset{G_o}{\times} G_{(3)} \underset{G_o}{\times} G_{(5)} \times \cdots$$

i.e., G is isomorphic to fibre product over $G_o = G \otimes Q$ of its localizations at the primes.

CHAPTER II

LOCALIZATION IN THE CATEGORY OF TOPOLOGICAL SPACES

§2.1 In this section we discuss localization of a class of topological spaces, and its interplay with homology and homotopy functors.

2.1.1 Throughout this section $P \subseteq \mathbb{P}$ will be a fixed set of primes and zero, and 'localization' would mean "localization at P", 'local' would mean "local at P".

Also our discussion would be confined to the class of 'simple' spaces defined below.

2.1.2 Definition. A *simple space* is a connected space having the homotopy type of a CW complex, and an abelian fundamental group which acts trivially on the homotopy and homology groups of the universal covering space.

2.1.3 Definition. A space X is *local* if and only if $\Pi_*(X)$ is local, i.e., $\Pi_i(X)$ is local for all $i > o$.

2.1.4 Definition. Let X be any space, a *localization* of X is a map u from X into a local space L such that u is universal with respect to maps into local spaces, i.e., if L' is any local space and if $f : X \to L'$ is any map then $\exists !\ u_f : L \to L'$ such that

$$u_f\, u = f \quad .$$

We shall later establish that for any simple space X, a localization (at P) $u : X \to X_P$ does exist. Assuming existence for a moment we establish

2.1.5 The Functorial Character of Localization. Let $f : X \to X'$ be map of simple spaces. Let $u : X \to X_P$ and $u' : X \to X'_P$ be localizations. Since u is a localization and since X'_P is local, $u'f : X \to X'_P$ induces uniquely a map $f_P : X_P \to X'_P$ such that

$$u'f = f_P\, u.$$

It is easy to see that $1_P = 1$ and that

$$(g\, f)_P = g_P\, f_P \quad .$$

Thus $(\ )_P$ is a functor from the category of simple spaces to a category whose objects are local (at P) spaces and whose maps are maps between local spaces.

2.1.6 Convention

1) By the statement "$f : X \xrightarrow{\to} Y$ localizes homology" we mean that $H_*(Y)$ is local and that $f_* : H_*(X) \to H_*(Y)$ is universal for localization at P. [see 1.1.25]

2) "$f : X \to Y$ localizes homotopy" means that $\Pi_*(Y)$ is local and $f_\# : \Pi_*(X) \to \Pi_*(Y)$ is universal for localization.

2.1.7 Lemma. Let $f : X \to X'$ be a map of simple spaces and suppose that $H_*(X')$ is local, then f localizes homology if and only if

$$f_{*(P)} : (H_*(X))_P \to (H_*(X'))_P$$

is an isomorphism.

Proof. The proof follows from Note 2 in 1.1.25.

2.1.8 Lemma. Consider the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & F' \\ \downarrow & & \downarrow \\ E & \xrightarrow{g} & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{h} & B' \end{array}$$

where each vertical sequence is a fibration.

(i) If the spaces are connected, Π_1 abelian and any two of f, g, h localize homotopy then so does the third.

(ii) If $\Pi_1(B)$ acts trivially on the homology of fibre and any two of f, g, h localize homology then so does the third.

Proof. (i) Apply 1.1.27 to homotopy exact sequence of fibrations.

(ii) is similar.

2.1.9 Let Π be an abelian group.

1) An *Eilenberg-MacLane complex* $K(\Pi,n)$ is a topological space having homotopy type of a CW complex, with exactly one non-zero homotopy group $\Pi_n(K(\Pi,n)) = \Pi$.

2) We recall that X is called n-connected if and only if $\Pi_i(X) = 0$ for all $i \leq n$.

3) Proposition. If X is (n-1)-connected then

$$H^n(X,\Pi) \cong \text{Hom}\,(H_n(X),\Pi)$$

for proof of this and subsequent results see [5].

Taking $\Pi = \Pi_n(X)$ this gives an isomorphism

$$H^n(X,\Pi_n(X)) \cong \text{Hom}(H_n(X),\Pi_n(X)) \quad .$$

Now let $h : \Pi_n(X) \to H_n(X)$ be the Hurewicz isomorphism. We define the *fundamental class* of X $i = i_n X$ as that element of $H^n(X,\Pi_n(X))$ which corresponds to h^{-1} in the isomorphism

$$H^n(X,\Pi_n(X)) \cong \text{Hom}(H_n(X), \Pi_n(X)) \quad .$$

4) Now take $X = K(\Pi,n)$, so $\Pi = \Pi_n(X)$. Writting $H_i(\Pi,n)$ for $H_i(K(\Pi,n))$ etc. we have an isomorphism

$$H^n((\Pi,n),\Pi) \cong \text{Hom}(H_n(\Pi,n),\Pi)$$

5) i, the fundamental class of $K(\Pi,n)$ provides us with a bijection

$$H^n(X,\Pi) \longleftrightarrow [X, K(\Pi,n)] \quad .$$

For proof again see [5].

6) We also have a bijection $[K(\Pi,n), K(\Pi',n)] \longleftrightarrow \text{Hom}(\Pi,\Pi')$. Thus a homomorphism $f : \Pi \to \Pi'$ induces a map

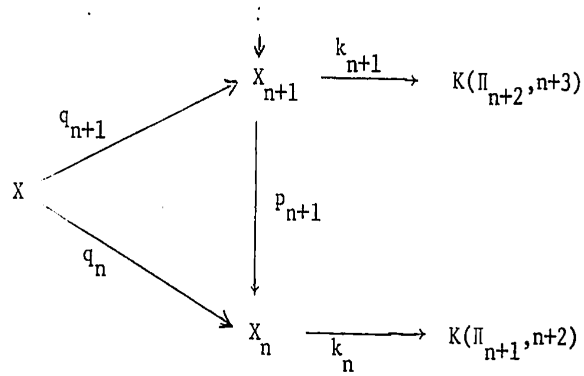
$$\hat{f} : K(\Pi,n) \to K(\Pi',n)$$

such that $(\hat{f}_\#)_n = f$.

We now give a brief description of the right side up and upside down Postnikov systems for a space X. For details see [5] and [6].

2.1.10 The right side up Postnikov system for X is a system

(X_n, p_n, q_n, k_n) constituting the diagram



where we write Π_n for $\Pi_n(X)$. The following conditions are to be satisfied

(i) If X is n-connected then $X_0 = X_1 = \ldots = X_n = *$ and $X_{n+1} = K(\Pi_{n+1}, n+1)$.

(ii) $q_{n\#} : \Pi_i(X) \to \Pi_i(X_n)$ is an isomorphism for $i \leq n$, in other words X_n is n-equivalent to X. Also $\Pi_i(X_n) = 0$ for $i > n$.

(iii) k_n $[X_n, K(\Pi_{n+1}, n+2)] \longleftrightarrow H^{n+2}(X_n, \Pi_{n+1})$ is called the n^{th} k-invariant of X.

(iv) $p_n : X_n \to X_{n-1}$ is the principal fibration induced, by the map k_{n-1}, from the path space fibration

$$K(\Pi_n, n) = \Omega K \to PK \to K = K(\Pi_n, n+1) \quad .$$

i.e., X_n is the pullback of $PK \to K$ and $X_{n-1} \to K$.

(v) Naturality Condition. $F : X \to X'$ induces maps $\{f_i\}$ of the Postnikov system for X into that of X' such that

$$f_n\, q_n = q'_n\, f \qquad f_{n-1}\, p_n = p'_n\, f_n \quad .$$

We call f_n the n^{th} stage Postinkov decomposition of f.

2.1.11 The upside down Postnikov decomposition (X^n, p^n, q^n, k^n) of X. Let (X_n, p_n, q_n, k_n) be the right side up Postnikov decomposition for X described in 2.1.10.

Take $X' = X$ and inductively define X^{n+1} and q^{n+1} by taking $q^{n+1} : X^{n+1} \to X$ to be the fibre of $q_n : X \to X_n$. Note that as $\Pi_i(X) \cong \Pi_i(X_n)$ for $i \leq n$ and as $\Pi_i(X_n) = o$ for $i > n$, we have $\Pi_i(X^{n+1}) = o$ for $i \leq n$ and $\Pi_i(X^{n+1}) = \Pi_i(X)$ for $i \geq n+1$.

To define $p^{n+1} : X^{n+1} \to X^n$ we take the right side up Postnikov decomposition of X^n and note that $(X^n)_j = *$ for $j < n$, and we take p^{n+1} to be the fibre of

$$k_n : X^n \longrightarrow K(\Pi_n, n) \quad .$$

For maps $f : X \to X'$, we have a naturality condition for the upside down Postnikov system as for the right side up system.

2.1.12 Lemma. Let Π be any group and let Π_P be its localization at $P \subseteq \mathbb{P}$. Let $f : K(\Pi, n) \to K(\Pi_P, n)$ be any map, then f localizes homology if and only if f localizes homotopy.

Proof. [$\Longrightarrow$] This is trivial for by Hurewicz theorem

$$H_n(K(\Pi, n)) \cong \Pi \qquad \text{and} \qquad H_n(K(\Pi_P, n)) \cong \Pi_P \quad ,$$

and

$$\Pi_i(K(\Pi, n)) = o \qquad \text{for} \qquad i \neq n \; .$$

Thus

$$f_{\#} : \Pi_i(K(\Pi,n)) \longrightarrow \Pi_i(K(\Pi_P,n))$$

is zero for $i \neq n$ and equal to f_* for $i = n$. Thus if f localizes homology then it localizes homotopy.

[<==] Assume f localizes homotopy, we have to show that f localizes homology. We use induction on n.

Case I. $[n = 1]$ $\Pi = Z$ so $\Pi_P = Z_P$. In this case $K(Z,1) = S^1$ and $H_1(Z,1) = \Pi_1(K(Z,1)) = Z$, $H_i(Z,1) = 0$ for $i > 1$. Also

$$H_1(Z_P,1) = \Pi_1(K(Z_P,1)) = Z_P .$$

Thus the map $f_* : H_i(\Pi,1) \to H_i(\Pi_P,1)$ is in fact a localization for $i = 1$ in which case it is just the map $Z \to Z_P$. Also for $i > 1$ $f_* = 0$ is trivially a localiztion.

Case II. $\Pi = Z/p^kZ$ so

$$\Pi_P = \begin{cases} 0 & \text{if } p \notin P \\ \Pi & \text{if } p \notin P \end{cases} .$$

Thus if $p \notin P$ $\Pi_P = 0$ $K(\Pi_P,1) = *$ and

$$f_* : H_*(\Pi,1) \to H_*(\Pi_P,1) = 0 ,$$

is the trivial map, hence is trivially a localization. If, on the other hand, $p \in P$ then $\Pi_P = \Pi$ thus

$$f : K(\Pi,1) \to K(\Pi_P,1) = K(\Pi,1)$$

is a homotopy equivalence and hence $f_* : H_*(\Pi,1) \to H_*(\Pi_P,1)$ is an isomorphism and a localization.

Case III. If Π is any finitely generated abelian group then it is a direct sum of copies of Z and $Z/p^k Z$. Now as the direct sum of local groups is local, as direct summation commutes with homotopy and as $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$. The result follows.

Case IV. If Π is any abelian group then it is the direct limit of its finitely generated subgroups under inclusion, and the result again follows.

This completes the inductive step for $n = 1$. Now assume that the result holds for $n = m-1$ and consider the fibrations

$$\begin{array}{ccc}
\Omega(K) = K(\Pi,m-1) & \xrightarrow{f_1} & K(\Pi_p,m-1) = \Omega K(\Pi_p,m) \\
\downarrow & & \downarrow \\
P(K) & \xrightarrow{f_2} & PK(\Pi_p,m) \\
\downarrow & & \downarrow \\
K = K(\Pi,m) & \xrightarrow{f_3} & K(\Pi_p,m)
\end{array} .$$

By inductive argument f_1 localizes homology. Also as both $PK(\Pi,m)$ and $PK(\Pi_p,m)$ are contractible f_2 is homotopy trivial and hence localizes homology trivially, therefore by 2.1.8 f_3 localizes homology.

The main result of this section is

2.1.13 Theorem. Let $f: X \to Y$ be a map of simple spaces. Then the following are equivalent

(i) f is a localization (see 2.1.4).

(ii) f localizes integral homology (see 2.1.6).

(iii) f localizes homotopy (see 2.1.6).

Proof. [(iii) $\Longrightarrow$ (ii)] Assume f localizes homotopy, we have to show that f localizes homology. We use induction on the n^{th} stage Postnikov decomposition of f, and shall show that $f_n : X_n \to Y_n$ localizes homology for all n. Then as $f \simeq f_\infty$, the result would follow.

Induction starts by 2.1.12 as $X_1 = K(\Pi_1(X),1)$ and $Y_1 = K(\Pi_1(Y),1)$.

Now consider

$$\begin{array}{ccc} K(\Pi_n(X),n) & \xrightarrow{\hat{f}} & K(\Pi_n(Y),n) \\ \downarrow & & \downarrow \\ X_n & \xrightarrow{f_n} & Y_n \\ \downarrow & & \downarrow \\ X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \end{array}$$

As f localizes homotopy, so does $\hat{f}$ for

$$(\hat{f}_\#)_n = (f_\#)_n \qquad \text{and} \qquad (\hat{f}_\#)_i = 0 \quad \text{if } i \neq n .$$

Hence by 2.1.12 f localizes homology. Also f_{n-1} localizes homology by inductive argument. Hence by 2.1.8 f_n localizes homology.

[(ii) $\Longrightarrow$ (iii)] Assume f localizes homology, we have to show that f localizes homotopy. Now as X and Y are simple spaces Hurewicz theorem gives $\Pi_1(X) = H_1(X)$, $\Pi_1(Y) = H_1(Y)$ and $f_* = f_\#$. Hence $f_\#$ localizes Π_1.

We now use the upside down Postnikov decomposition of f. Consider first

$$\begin{array}{ccccc} X^2 & \longrightarrow & X = X^1 & \longrightarrow & K(\Pi_1(X),1) \\ \Big\downarrow f^2 & & \Big\downarrow f^1 = f & & \Big\downarrow k^1 \\ Y^2 & \longrightarrow & Y = Y^1 & \longrightarrow & K(\;_1(Y),1) \end{array}$$

Now $(k^1_{\#})_1 = (f_{\#})_1$ localizes Π_1, and $(k^1_{\#})_i = o$ for $i \neq 1$ therefore k^1 localizes homotopy hence by 2.1.12 k^1 localizes homology. Also f localizes homology, hence by 2.1.8 f^2 localizes homology. But now X^2 and Y^2 are simply connected, so Hurewicz theorem gives us that f^2 localizes Π_2. Further

$$(f^2_{\#})_2 = (f_{\#})_2 : \Pi_2(X^2) = \Pi_2(X) \to \Pi_2(Y^2) = \Pi_2(Y)$$

therefore f localizes Π_2.

Now assume by induction that $f' = f, \ldots, f^n$ localize homology and that f^n (hence f) localizes Π_n and that k^{n-1} localizes homology.

To complete inductive step we have to show that f^{n+1} and k^n localize homology, and that f^{n+1} (hence f) localizes Π_{n+1}.

Consider the n^{th} and $(n+1)^{st}$ stage in the upside down Postnikov decomposition of f, in the following diagram the rows are fibrations.

$$\begin{array}{ccccc} X^{n+1} & \longrightarrow & X^n & \longrightarrow & K(\Pi_n(X),n) \\ \Big\downarrow f^{n+1} & & \Big\downarrow f^n & & \Big\downarrow k^n \\ Y^{n+1} & \longrightarrow & Y^n & \longrightarrow & K(\Pi_n(Y),n) \end{array}$$

Now $(k^n{}_{\#})_i = o$ for $i \neq n$ and $(k^n{}_{\#})_n = (f_{\#})_n : \Pi_n(X) \to \Pi_n(Y)$, f localizes Π_n hence k^n localizes homotopy, therefore by 2.1.12 k^n localizes homology. Also f^n localizes homology hence by 2.1.8 f^{n+1} localizes homology. But X^{n+1} is n-connected hence by Hurewicz theorem f^{n+1} localizes Π_{n+1}, but

$$(f^{n+1}{}_{\#})_{n+1} = (f_{\#})_{n+1} : \Pi_{n+1}(X^{n+1})$$
$$= \Pi_{n+1}(X) \to \Pi_{n+1}(Y^{n+1}) = \Pi_{n+1}(Y) \quad .$$

Hence f localizes Π_{n+1}. This completes the inductive step. Hence proof of (ii) $\Longrightarrow$ (iii).

[(i) $\Longrightarrow$ (ii)] Assume $f : X \to Y$ is a localization, i.e., Y is local and for any local space Z and any map $g : X \to Z$ $\exists!$ $u_g : Y \to Z$ such that $u_g f = g$.

Now taking $Z = K(\Pi,n)$ where Π is a local abelian group, and $n = 1,2,\ldots$ we get bijections

$$H^n(X,\Pi) \longleftrightarrow [X,K(\Pi,n)] \longleftrightarrow [Y,K(\Pi,n)] \longleftrightarrow H^n(Y,\Pi) \quad .$$

Now if we take $\Pi = Q = Z_o$ and $\Pi = Z/pZ = (Z/pZ)_P$ for $p \in P$, the above gives isomorphisms

$$H^*(X,Q) \cong H^*(Y,Q)$$
$$H^*(X,Z/pZ) \cong H^*(Y,Z/pZ) \qquad p \in P \quad .$$

Universal coefficient theorem now gives us that the homomorphisms

$$f_* : H_*(X,Q) \to H_*(Y,Q)$$
$$f_* : H_*(X,Z/pZ) \to H_*(Y,Z/pZ)$$

are isomorphisms.

Now using the Bockstein sequence

$$0 \to Z/pZ \to Z/p^nZ \to Z/p^{n-1}Z \to 0$$

the induced homology sequence ladder and the five lemma, we see that

$$f_* : H_*(X,A/p^nZ) \to H_*(Y,Z/^nZ)$$

is an isomorphism for $p \in P$.

Now we note that

$$Z_{p^\infty} \cong \varinjlim (Z/p^nZ)$$

and

$$Q/Z_P \cong \bigoplus_{p \in P} Z_{p^\infty} \quad ,$$

and that $\varinjlim$ and $\oplus$ are exact functors, hence

$$f_* : H_*(X,Q/Z_P) \to H_*(Y,Q/Z_P)$$

is an isomorphism. Now from the sequence $o \to Z_P \to Q \to Q/Z_P \to o$ we conclude that $f_* : H_*(X,Z_P) \to H_*(Y,Z_P)$ is an isomorphism.

But as Z_P is torsion free, universal coefficient theorem gives us the following commutative diagram

$$\begin{array}{ccc} H_*(X,Z_P) & \cong & H_*(X) \otimes Z_P \\ \downarrow f_* & & \downarrow (f_*)_P \\ H_*(Y,Z_P) & \cong & H_*(Y) \otimes Z_P \end{array}$$

and the result follows from 2.1.7.

[(ii) $\Longrightarrow$ (i)] We assume now that $f_* : H_*(X) \to H_*(Y)$ is a localization, and show that $f : X \to Y$ is universal for maps into local spaces L.

First note that we can regard $f : X \to Y$ as an inclusion, [taking mapping cylinder]. Hence given $f : X \to Y$ and $g : X \to L$ the obstruction to uniquely extending g to y lies in

$$H^*(Y,X;\Pi_*(L))$$

where $\Pi_*(L)$ being local at P is a Z_p module.

Now look at the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Ext}(H_i(Y,Z_p),Z_p) & \to & H^{i+1}(Y,Z_p) & \to & \mathrm{Hom}(H_{i+1}(Y,Z_p),Z_p) & \to & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Ext}(f_*,id)} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \mathrm{Hom}(f_*,id)} & & \\
0 & \to & \mathrm{Ext}(H_i(X,Z_p),Z_p) & \to & H^{i+1}(X,Z_p) & \to & \mathrm{Hom}(H_{i+1}(X,Z_p),Z_p) & \to & 0
\end{array}$$

Now as f_* is an isomorphism, so also are $Hom(f_*,id)$ and $Ext(f_*,id)$ therefore it follows by the 5-lemma that

$$f^* : H^*(Y,Z_p) \longrightarrow H^*(X,Z_p)$$

is also an isomorphism. Hence from the exact chomology sequence of the pair (Y,X) with coeffients in Z_p we get

$$H^{i+1}(Y,X,Z_p) = 0$$

for all i. Thus all obstruction groups are zero.

Hence g extends uniquely to Y, showing that f is universal.

2.1.14 Corollary. For a simple space X the following are equivalent.

(i) X is its own localization

(ii) X has local homology

(iii) X has local homotopy.

Proof. The proof follows from 2.1.13 by taking $f = id : X \to X$.

2.1.15 Corollary. If $f : X \to X'$ is a map of local simple spaces then the following are equivalent

(i) f is a homotopy equivalence

(ii) f induces isomorphism of local homotopy

(iii) f induces isomorphism of local homology.

Proof. (i) $\Longrightarrow$ (ii) and (iii) is trivial.

[(ii) $\Longrightarrow$ (i)] Simple spaces have homotopy type of CW complexes, for which homotopy equivalence is the same as weak homotopy equivalence (see [6], §7.6.24, p. 405).

[(ii) <==> (iv)] This again follows from ([6], §7.6.25).

2.1.16 Proposition. $H_*(X_P) \cong H_*(X) \otimes Z_P \cong H_*(X;Z_P)$.

Proof. The last isomorphism follows from the universal coefficient theorem for homology, as Z_P is torsion free.

We shall show

$$H_*(X_P) \cong H_*(X) \otimes Z_P \quad .$$

Let $u : X \to X_P$ be localization, then by 2.1.13 $u_* : H_*(X) \to H_*(X_P)$ is also localization and hence universal, But $H_*(X) \to H_*(X) \otimes Z_P$ is also universal. Hence the result follows from uniqueness of universal objects.

2.1.17 Proposition. $\Pi_*(X_P) \cong \Pi_*(X) \otimes Z_P$.

Proof. The proof is similar to the proof of 2.1.16.

§2.2 In this section we shall show existence of the localization functor for simple spaces.

2.2.1 Construction. Let S be an i-sphere for $i \geq 1$, and let $P \subseteq \mathbb{P}$. We shall construct a P-localization of S.

Let $P' = \mathbb{P} \backslash P$ and $S = \{n \in \langle P' \rangle\} = \{n \in Z; (n,P) = 1\}$. Let $\{s_1, s_2, \ldots,\}$ be any cofinal sequence in S. i.e., s_i is a divisor of s_{i+1} and for any $s \in S$ $\exists$ i such that s is a divisor of s_i.

Let $f_n : S \to S$ be a map of degree s_n. Define T_n

inductively as follows $T_0 = \emptyset$ $T_1 = S \times I$, $T_{n+1} = T_n \underset{f_n}{\cup} S \times I$ for $n \geq 1$. Thus T_2 is obtained by identifying bottom of T_1 with top of $S \times I$ using the map f_1. T_{n+1} is obtained by identifying bottom of the last cylinder in T_n with top of $S \times I$ using the map f_n. Milnor has called this construction "the telescope construction". Note that for finite values of n, T_n has the same homotopy type as S. Define

$$T = \lim_{n \to \infty} T_n$$

and define

$$u : S \longrightarrow T$$

by inclusion into the top of the first cylinder.

2.2.2 Claim. $u : S \to T$ is a localization.

Proof. By 2.1.13 it is sufficient to show that u localizes homology.

Now $H_j(S) = o$ if $j \neq i$ and $H_i(S) = Z$.

$$\therefore \quad \tilde{H}_j(T) = \tilde{H}_j\,(\underrightarrow{\lim}\, T_n) \cong \underrightarrow{\lim}\, \tilde{H}_j(T_n) \cong \underrightarrow{\lim}\, \tilde{H}_j(S) \quad .$$

$$\therefore \quad \tilde{H}_j(T) = 0 \quad \text{if} \quad j \neq i \quad \text{and}$$

$$\tilde{H}_i(T) = \underrightarrow{\lim}\, \tilde{H}_i(S) = \underrightarrow{\lim}\, Z \cong Z_P \quad ,$$

where this last isomorphism follows from 1.2.10. Thus $H_*(T)$ is local at P.

Now $u_* : H_*(S) \to H_*(T)$ is either the zero map or is

$$u_* : \tilde{H}_i(S) = Z \to Z_p = \tilde{H}_i(T) \quad .$$

Thus u_* is a localization.

2.2.3 Notation. We write $T = S_p = S_p^i$ and call it the localization of the i-sphere $S = S^i$. The cone on S_p^i is called a local (i+1)-cell.

Note that we do not have any local 0-sphere and hence no local 1-cell.

2.2.4 Definition. A *local CW complex* is built inductively from a point or a local 1-sphere by attatching local cells.

2.2.5 Theorem. If X is a CW complex with one zero cell and one 1-cell there is a local CW complex X_p and a cellular map $u : X \to X_p$ such that

(i) u induces bijection between cells of X and the local cells of X_p.

(ii) u is a localization.

Proof. We shall build X_p inductively, with induction on the dimension n of X. Let $X^{(n)}$ denote the n-skeleton of X. If $n = 2$ i.e., if X is a 2-complex with $X^{(1)} = *$ then X is a wedge of 2-spheres, $X = \vee S^2$, and we take $X_p = \vee S_p^2$, i.e., we take X_p to be wedge of as many copies of S_p^2, we define

$$\bar{u} = \vee u : \vee S^2 \to \vee S_p^2$$

where $u : S^2 \to S_p^2$ is defined in 2.2.1. Then this $\bar{u}$ satisfies (i) and (ii).

Next we note that if $f : A \to A_p$ is a localization satisfying (i) and (ii), then so also is $\Sigma f : \Sigma A \to \Sigma A_p$ for in this case (i) is clear, and to show (ii), i.e., to show that Σf is a localization, we consider

$$(\Sigma f)_* : H_*(\Sigma A) = H_{*-1}(A) \to H_{*-1}(A_p) = H_*(\Sigma A_p) \quad .$$

Thus $(\Sigma f)_*$ localizes homology and hence by 2.1.13 Σf is a localization.

Now we assume inductively that the theorem is true for all complexes of dimension $\leq n-1$, and note that the n-dimensional complex X is formed by attaching the n-cells, i.e., the cone on $V\, S^{n-1}$, to the n-1 skeleton $X^{(n-1)}$ by the map

$$f : V\, S^{n-1} \to X^{(n-1)} \quad .$$

Thus X is the mapping cone of f, and we consider the following diagram, the top row of which is the Puppe sequence of the map f, and the bottom row that of f_p.

2.2.6

$$\begin{array}{ccccccccccc}
VS^{n-1} & \xrightarrow{f} & X^{(n-1)} & \xrightarrow{c} & X & \xrightarrow{d} & VS^n = \Sigma VS^{n-1} & \to & \Sigma X^{(n-1)} & \to & \cdots \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle u^{(n-1)}} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \Sigma i} & & \downarrow{\scriptstyle \Sigma u^{(n-1)}} & & \\
VS_p^{n-1} & \xrightarrow[f_p]{} & X_p^{(n-1)} & \longrightarrow & X_p & \to & VS_p^n = \Sigma VS_p^{n-1} & \to & \Sigma X_p^{n-1} & \to & \cdots
\end{array}$$

i and $u^{(n-1)}$ are defined by inductive hypothesis and f_p exists because of the functorial character of localization. X_p is the coefibre of f_p, ie.,

$$X_P = X_P^{(n-1)} \underset{f_P}{\cup} C(V\, S_P^{n-1})$$

and u is defined by piecing together the cone on i and $u^{(n-1)}$ that is

$$u = u^{n-1} \cup C(i) : X = X^{n-1} \underset{f}{\cup} c(V\, S^{n-1}) \to X_P^{(n-1)} \underset{f_P}{\cup} C(V\, S_P^{n-1}) = X_P .$$

It is clear from the way u is defined as map of cofibres that it respects identifications, hence it is well defined. Also as $u^{(n-1)}$ induces bijection of cells so does u.

Now in the bottom row of the commutative diagram 2.2.6 all spaces except possibly X_P have local homology, hence by exactness X_P does also. Further all vertical maps localize homology except possibly u, hence it does too, (1.1.26 and 1.1.27).

This settles the case of finite dimensional complexes. If X is infinite dimensional then we have $X = \underset{n}{\cup} X^{(n)}$ and we take $X_P = \underset{n}{\cup} X_P^{(n)}$ and $u = \underset{n}{\cup} u^{(n)}$. (i) and (ii) are then easily seen to be satisfied.

2.2.7 Corollary. Any simply connected space X has a localization.

Proof. Chose a CW decomposition $\bar{X}$ with one zero cell and no one cells, and consider $u : \bar{X} \to \bar{X}_P$ defined in 2.2.5. Then $X \simeq \bar{X} \to \bar{X}_P$ gives localization.

2.2.8 Definition. A *local Postnikov tower* is a Postnikov tower with the X_n's constructed inductively from a point using fibrations with local $K(\Pi,n)$'s, i.e., $K(\Pi,n)$ with Π local.

2.2.9 Theorem. If $X \equiv (X_n, p_n, q_n, k_n)$ is any Postnikov tower then there is a local Postnikov tower $X' \equiv (X'_n, p'_n, q'_n, k'_n)$ and a Postnikov map $u : X \to X'$ which localizes homotopy groups.

Proof. We use induction on the number of stages in the Postnikov tower X. Induction starts easily since the first stage is a point.

Assume now that we have localization upto the $(n-1)^{st}$ stage. i.e., for $m < n$ we have localization $u_m : X_m \to X'_m$.

Let Π_n denote $\Pi_n(X)$ and let Π'_n denote $\Pi_n(X)_P$. Then the universal map $u : \Pi_n \to \Pi'_n = (\Pi_n)_P$ induces

$$K(u) : K(\Pi_n, n+1) \to K(\Pi'_n, n+1)$$

where $K(\Pi'_n, n+1)$ is local as $\Pi_*(K(\Pi'_n, n+1))$ is local. Now consider the diagram

$$\begin{array}{ccc} X_{n-1} & \xrightarrow{u_{n-1}} & X'_{n-1} \\ {\scriptstyle k_n}\downarrow & & \downarrow{\scriptstyle \hat{k}_n} \\ K(\Pi_n, n+1) & \xrightarrow[k(u)]{} & K(\Pi'_n, n+1) \end{array}$$

u_{n-1} is universal, and

$$K(u)\, k_n : X_{n-1} \to K(\Pi'_n, n+1)$$

maps X_{n-1} into the local space $K(\Pi'_n, n+1)$ hence

$$\exists\, ! \hat{k}_n : X'_{n-1} \to K(\Pi'_n, n+1)$$

that makes the diagram commutative. Now consider the following diagram in which the vertical sequences are fibrations.

$$\begin{array}{ccc} X_n & \overset{u_n}{\dashrightarrow} & X'_n \\ \downarrow & & \downarrow \\ X_{n-1} & \xrightarrow{u_{n-1}} & X'_{n-1} \\ \downarrow & & \downarrow \\ K(\Pi_n,n) & \longrightarrow & K(\Pi'_n,n) \end{array} .$$

From the homotopy sequence of

$$X'_n \to X'_{n-1} \to K(\Pi'_n,n)$$

we conclude that X'_n is local. Also in the commutative ladder

$$\begin{array}{ccccccccc} \to & \Pi_*(X_n) & \longrightarrow & \Pi_*(X_{n-1}) & \longrightarrow & \Pi_*(K(\Pi_n,n)) & \to & \cdots \\ & {\scriptstyle u_{n\#}}\downarrow & & {\scriptstyle u_{n-1\#}}\downarrow & & \downarrow & & \\ \to & \Pi_*(X'_n) & \longrightarrow & \Pi_*(X'_{n-1}) & \longrightarrow & \Pi_*(K(\Pi'_n,n)) & \to & \cdots \end{array}$$

all vertical maps are localizations except possibly $(u_n)_\#$ hence it is too. Thus $u_n : X_n \to X'_n$ is a localization.

2.2.10 Corollary. Any simple space has a localization.

Proof. Choose a Postnikov decomposition for the simple space X, localize the tower by 2.2.9 to obtain a local Postnikov tower (X'_n) then $X' = X'_\infty$ is a simple space localizing X.

§2.3 In this section we list some results for local spaces.

2.3.1 Proposition. There is a bijection

$$[S^i_p, X] \longleftrightarrow [S^i, X]$$

for local spaces X.

Proof. Let $u : S^i \to S^i_P$ be the localization map. Define

$$\phi : [S^i, X] \longleftrightarrow [S^i_P, X] : \psi$$

by

$$\phi(f) = u_f \qquad \text{for} \qquad f : S^i \to X$$

and

$$\psi(g) = gu \qquad \text{for} \qquad g : S^i_P \to X .$$

The result follows.

2.3.2 Corollary. If X is local

$$\Pi_i(X) \cong [S^i_P, X) \qquad i > 1 .$$

2.3.3 Theorem. If $P_1, P_2 \subseteq \mathbb{P}$ then

$$\begin{array}{ccc} X_{P_1 \cup P_2} & \longrightarrow & X_{P_1} \\ \downarrow & & \downarrow \\ X_{P_2} & \longrightarrow & X_{P_1 \cap P_2} \end{array}$$

is a fibre square.

Proof. It is sufficient to show that the following sequence is exact.

$$0 \to \Pi_i(X_{P_1 \cup P_2}) \to \Pi_i(X_{P_1}) \oplus \Pi_i(X_{P_2}) \to \Pi_i(X_{P_1 \cap P_2}) \to 0$$

but as $\Pi_i(X_P) \cong \Pi_i(X) \otimes Z_P$, it sufficient to show exactness of

$$0 \to \Pi_i(X) \otimes Z_{P_1 \cup P_2} \to (\Pi_i(X) \otimes Z_{P_1}) \oplus (\Pi_i(X) \otimes Z_{P_2}) \to \Pi_i(X) \otimes Z_{P_1 \cap P_2} \to 0$$

or the exactness of

2.3.4
$$o \to \Pi_i(X) \otimes Z_{P_1 \cup P_2} \to \Pi_i(X) \otimes [Z_{P_1} \oplus Z_{P_2}] \to \Pi_i(X) \otimes Z_{P_1 \cap P_2} \to o \ .$$

But by 1.2.16 the sequence

$$o \to Z_{P_1 \cap P_2} \longrightarrow Z_{P_1} \oplus Z_{P_2} \longrightarrow Z_{P_1 \cap P_2} \to o$$

is exact, and as all these groups are torsion free, on tensoring with $_i(X)$ we get exactness of 2.3.4.

2.3.5 Corollary. $X_{P_1 \cup P_2} \simeq X_{P_1} \underset{X_{P_1 \cap P_2}}{\times} X_{P_2}$

2.3.6 Corollary. $X \simeq X_2 \underset{X_o}{\times} X_3 \underset{X_o}{\times} X_5 \underset{X_o}{\times} X_7 \cdots$

where we note that

$$\Pi_*(X_o) \cong \Pi_*(X) \otimes Q \ .$$

2.3.7 Theorem. $(X \times Y)_P \simeq X_P \times Y_P$.

Proof. $\Pi_*((X \times Y)_P) \cong \Pi_*(X \times Y) \otimes Z_P$

$$\cong [\Pi_*(X) \oplus \Pi_*(Y)] \otimes Z_P$$

$$\cong [\Pi_*(X) \otimes Z_P] \oplus [\Pi_*(Y) \otimes Z_P]$$

$$\cong \Pi_*(X_P) \oplus \Pi_*(Y_P) \cong \Pi_*(X_P \times Y_P) \ .$$

Thus

$$\Pi_*(X \times Y) \longrightarrow \Pi_*((X \times Y)_P)$$

and

$$\Pi_*(X \times Y) \longrightarrow \Pi_*(X_P \times Y_P)$$

are both localizations $\therefore$ by 2.1.13 so also are

$$X \times Y \rightarrow (X \times Y)_P \quad ,$$

and

$$X \times Y \longrightarrow X_P \times Y_P \quad ,$$

and by uniqueness of universal object it follows that

$$(X \times Y)_P \simeq X_P \times Y_P \quad .$$

2.3.7 Theorem. $\quad (X \vee Y)_P \simeq X_P \vee Y_P \quad .$

Proof. The proof is the same as above using H_* instead of Π_*.

CHAPTER III

LOCALIZATION AND H-SPACES

3.1.1 Definition. An H-space is a pointed space X with a map $m : X \times X \to X$ such that $mj \simeq \nabla$ where $j : X \vee X \to X \times X$ is the natural inclusion and $\nabla : X \vee X \to X$ is the folding map.

3.1.2 If X is an H-space then so also is X_P.

Proof. Let $m : X \times X \to X$ be multiplication. By 2.3.7 we have a map $\lambda : X_P \times X_P \to (X \times X)_P$ which is a homotopy equivalence, and by 2.3.8 we have a map $\mu : X_P \vee X_P \to (X \vee X)_P$ which is a homotopy equivalence.

Define

$$n = m_P \circ \lambda : X_P \times X_P \to X_P$$

it remains to show that the composite

$$X_P \vee X_P \xrightarrow{j} X_P \times X_P \xrightarrow{n} X_P$$

is homotopic to the map

$$\nabla : X_P \vee X_P \to X_P \quad .$$

Now

$$nj = m_P \lambda j = m_P j_P \mu = (mj)_P \mu$$
$$\simeq \nabla_P \mu = \nabla.$$

3.1.3 Theorem. X_0 is an H-space $\Longleftrightarrow$ it is equivalent to a product of Eilenberg-MacLane complexes.

Proof. [<==] This is trivial as Eilenberg-MacLane complexes are H-spaces.

[==>] By a theorem of Hopf we know that

$$H^*(X_0;Q) \cong H^*(S^{n_1} \times \ldots \times S^{n_r};Q)$$

for $n_1,\ldots,n_r$ odd, which in turn is an exterior algebra on r-generators which can be chosen to be primitive. Let $\alpha_1,\ldots,\alpha_r$ be the primitive generators of $H^*(X_0;Q)$ with

$$\alpha_i \in H^{n_i}(X_0,Q) \longleftrightarrow [X_0,K(Q,n_i)]$$

therefore we get H-maps

$$\hat{\alpha}_i : X_0 \to K(Q;n_i) \quad .$$

This gives the H-map

$$(\hat{\alpha}_1 \times \ldots \times \hat{\alpha}_r)\Delta : X_0 \to \prod_{i=1}^{r} K(Q,n_i) \quad .$$

This induces isomorphisms of homology and cohomology and as the fundamental groups on both sides are abelian, it is a homotopy equivalence. Therefore

$$X_0 \simeq \Pi \; K(Q,n_i)$$

as H-spaces.

3.1.4 Corollary. S_0^n is an H-space <==> n is odd.

Proof. [==>] It is sufficient to show that S_0^{2n} is not an H-space, but this is trivial as S_0^{2n} does not have the cohomology of an H-space.

[$\Longrightarrow$] For this we first note that if n is odd then $\Pi_m(S^n)$ is finite for $m \neq n$, (see [6], 9.7.7, p. 515). Hence when we localize at $\{0\}$ the Postnikov tower for S_o^{2n-1} we get

$$(S_o^{2n-1})_k = \begin{cases} * & \text{if } k < 2n-1 \\ \Omega K(Q,2n) = K(Q,2n-1) & \text{if } k \geq 2n-1 \end{cases}$$

Thus $S_o^{2n-1} \simeq K(Q,2n-1)$ and the result follows from 3.1.3.

3.1.5 Theorem. X is an H-space $\Longleftrightarrow$ X_p is an H-space for each prime p and $H_*(X_p,Q)$ is isomorphic to $H_*(X_q,Q)$ as a ring, for all primes p and q.

Proof. [$\Longrightarrow$] First we note that $(X_p)_o \simeq X_o$ for all primes p. Thus we can give two H-structures to X_o, one directly from the H-structure of X, and the other via the H-structure of X_p, and the equivalence $(X_p)_o \simeq X_o$ is an H-space equivalence. Thus we have ring isomorphism

$$H_*((S_p)_o;Q) \cong H_*(X_o;Q)$$

on the other hand by 2.1.16

$$H_*((X_p)_o,Q) \cong H_*(X_p;Q \quad Q)$$
$$\cong H_*(X_p,Q) \quad .$$

This shows

$$H_*(X_p;Q) \cong H_*(X_o,Q) \cong H_*(X_q;Q) \quad .$$

[<==] Assume that X_p is an H-space for each prime p and

$$H_*(X_p, Q) \cong H_*(X_q, Q)$$

for all primes p,q. The isomorphism being a ring isomorphism then $(X_p)_o \simeq (X_q)_o$ as H-spaces, since the H-space structure on a rational space is determined by its Pontrjagin ring.

Thus we have compatible multiplications for the spaces X_2, X_3, X_5, ... This induces H-space structure on the fibre product X.

3.1.6 Remarks.

(1) Let

$$i_1, i_2 : \Sigma X \to (\Sigma X) \vee (\Sigma X)$$

be embeddings. Make the Whitehead produce

$$[i_1, i_2] : \Sigma(X \wedge X) \to (\Sigma X) \vee (\Sigma X)$$

into a confibration to get

$$\begin{array}{ccccc} \Sigma(X\wedge X) & \xrightarrow{[i_1,i_2]} & (\Sigma X)\vee(\Sigma X) & \xrightarrow{q} & [(\Sigma X \vee \Sigma X) \cup_{[i_1,i_2]} \Sigma(X\wedge X)] = (\Sigma X)\times(\Sigma X) \\ & & \Big\downarrow{\scriptstyle \nabla(i \vee i)} & \swarrow & \\ & & \Sigma X & & \end{array}$$

Then $\nabla(i \vee i)$ extends to $(\Sigma X) \times (\Sigma X)$ if and only if

$$\nabla(i \vee i) \simeq hq \iff \nabla(i \vee i)[i_1, i_2] = *$$

$$\iff [1,1] = 0 \text{ in } [\Sigma(X \wedge X), \Sigma X] .$$

(2) Y is an H-space

$$\iff \nabla : \Sigma Y \vee \Sigma Y \to \Sigma Y$$

extends to $(\Sigma Y) \times (\Sigma Y)$, and the obstruction to this extension is the Whitehead product

$$[i,i] \in [\Sigma(Y \wedge Y), \Sigma Y] .$$

3.1.7 Theorem. S_2^{2n-1} is an H-space $\iff n = 1,2,4$.

Proof. [$\Leftarrow$] If $n = 1,2,4$ then S^{2n-1} is an H-space hence so also is S_2^{2n-1}.

[$\Rightarrow$] First we recall the result proved by Adam that S^{2n-1} is an H-space $\iff n = 1,2$ or 4. Now consider

$$i : S_2^{2n-1} \longrightarrow S_2^{2n-1} .$$

First

$$S_2^{2n-1} = \Sigma(S_2^{2n-1})$$

hence by remark (2) in 3.1.6 S_2^{2n-1} is an H-space

$$\iff o = [i,i] \in [\Sigma(S_2^{2n-2} \wedge S_2^{2n-2}), S_2^{2n-1}]$$

$$= [S_2^{4n-4}, S_2^{2n-2}] = [S_2^{4n-3}, S_2^{2n-1}]$$

$$\simeq \Pi_{4n-3} (S_2^{2n-1}) = \Pi_{4n-3} (S^{2n-1}) \otimes Z_{\{2\}}$$

now if $i' : S^{2n-1} \to S^{2n-1}$ is the identity map then under the isomorphism

$$\Pi_{4n-3}(S_2^{2n-1}) \cong \Pi_{4n-3}(S^{2n-1}) \otimes Z_2$$

$[i,i]$ corresponds to $[i',i'] \otimes 1$. Also Adams' result gives $[i',i'] = 0$ if $n = 1,2$ or 4 and of order 2 otherwise. Therefore $[i,i] = 0 \iff n = 1,2$ or 4.

3.1.8 Theorem. S_2^{2n-1} is a loop space $\iff n = 1$ or 2.

Proof. $[\Leftarrow]$ If $n = 1$ or 2 $S^{2n-1} = S^1$ or S^3 and these are topological groups, therefore they have classifying spaces and are themselves of the same homotopy type as the loops of the classifying spaces.

$[\Rightarrow]$ Suffices to show that for $n = 4$ $S_2^{2n-1} = S_2^7$ is not a loop space.

Assume that S_2^7 is a loop space, so it is an A_∞ space [7], i.e., it has A_p structure for every prime p, and hence has a projective p-space $S_2^7\,P(p)$ and

$$H^*(X,Z/pZ) = (Z/pZ)[x]/x^{p+1}$$

where x $H^8(X,Z/pZ)$ and $x^p \neq 0$. Consider the mod p Steenrod operation P^4 [8] and [5] then $P^4\,x = x^p \neq 0$. Now by Adem relations $P^1\,P^3 = -(3p-4)P^4$ choosing $p = 3$ this says $P^1\,P^3 = -5P^4 = P^4$ in mod 3 arithmetic.

Now

$$P^3(x) \in H^{20}(X,Z/3Z)$$

and as

$$x \in H^8(X,Z/3Z) = (Z/3Z)[x]/x^4 = H^*(X,Z/3Z)$$

is non-zero only in dimensions that are multiples of 8, therefore $H^2(X,Z/3Z) = 0$ and hence $0 \neq x^3 = P^4(x) = PP^3(x) = 0$.

This contradiction proves the result.

Note. The above theorem implies that S_2^7 does not even have an A_3-structure, i.e., it is not even a homotopy associative H-space.

3.1.9 Remark. The number of non-homotopic multiplications on S_2^{2n-1} for $n = 1,2,4$, is the order of

$$[S_2^{2n-1} \wedge S_2^{2n-1}, S_2^{2n-1}] = \Pi_{4n-2}(S_2^{2n-1})$$
$$\cong \Pi_{4n-2}(S^{2n-1}) \otimes Z_{\{2\}} .$$

Therefore, if $n = 1$ the number is $|\Pi_2(S') \times Z_{\{2\}}| = 1$. If $n = 2$, then

$$\Pi_6(S^3) \otimes Z_2 = Z/12Z \otimes Z_{\{2\}}$$
$$\cong (Z/4Z \oplus Z/3Z) \otimes Z_{\{2\}} .$$

Now $Z_{\{2\}}$ is collection of all rationals with odd denominators, i.e., in $Z_{\{2\}}$ we can divide by any odd number. Hence

$$Z/3Z \otimes Z_{\{2\}} = 0$$

and

$$(Z/4Z) \otimes Z_{\{2\}} \cong Z/4Z .$$

Hence the numer of non-homotopic multiplications in S_2^3 is order of $Z/4Z$, i.e., 4.

Similarly as $\Pi_{14}(S^7) = Z/120Z$ S_2^7 has 8 non-homotpic multiplications.

3.1.10 Theorem. S_p^{2n-1} is an H-space for all odd primes p and for all $n \geq 1$.

Proof. If $n = 1,2$ or 4 we see that S^1, S^3 and S^7 are H-spaces hence so also are S_p^1 S_p^3 and S_p^7.

Now for $n > 4$ we use a map

$$\phi : S^{2n-1} \times S^{2n-1} \to S^{2n-1}$$

which is of degree 2 on each factor. This gives rise to a map

$$\hat{\phi} : S_p^{2n-1} \times S_p^{2n-1} \to S_p^{2n-1}$$

such that on each factor it is twice the identity map. This makes sense as for $n > 1$, the suspension structure $S_p^{2n-1} = \Sigma S_p^{2n-2}$ induces a group structure on $[S_p^{2n-1}, S_p^{2n-1}]$. Thus the obstruction to extending $\nabla(2 \vee 2)$ is zero, i.e.,

$$\begin{aligned} 0 = [2,2] &\in [\Sigma(S_p^{2n-2} \wedge S_p^{2n-2}), S_p^{2n-1}] \\ &= \Pi_{4n-3}(S_p^{2n-1}) = \Pi_{4n-3}(S^{2n-1}) \otimes Z_{\{p\}} . \end{aligned}$$

But $[2,2] = 4[1,1]$ and $\Pi_{4n-3}(S^{2n-1})$ is finite and $\Pi_{4n-3}(S^{2n-1}) \otimes Z_{\{p\}}$ ontains only p-torsion for odd p.

$$\therefore \quad [1,1] = 0 \quad \text{i.e.,} \quad \nabla(1 \vee 1) : S_p^{2n-1} \vee S_p^{2n-1} \to S_p^{2n-1}$$

extends to $S_p^{2n-1} \times S_p^{2n-1}$, i.e., S_p^{2n-1} is an H-space.

3.1.11 Proposition. If S_p^{2n-1} is a loop space then $p \equiv 1 (\text{mod } n)$, where p is an odd prime.

Proof. If S_p^{2n-1} is a loop space it admits A_∞ structure and hence A_p structure. Then $\exists$ projective p-space X for S_p^{2n-1} and

$$H^*(X;Z/pZ) \cong (Z/pZ)[x]/x^{p+1}$$

where $x \in H^{2n}(X,Z/pZ)$ and $x^p \neq 0$. If $\mathcal{P}^n$ is Steenrod p-operation, then

$$\mathcal{P}^n x = x^p \neq 0 .$$

Thus

$$\mathcal{P}^{p^i} \neq 0 \text{ in } H^*(X;Z/pZ)$$

for some i. Let r be the smallest i with this property, i.e.,

$$\mathcal{P}^{p^r} \neq 0 \text{ in } H^*(X;Z/pZ)$$

but

$$\mathcal{P}^{p^i} = 0 \quad \text{for} \quad i < r .$$

Further observe that $H^*(X;Z/pZ)$ is non-zero only in dimensions which are multiplies of 2n. Now $\mathcal{P}^{p^r}$ can be factored in terms of secondary operations for $r > 0$. This means that the action of $\mathcal{P}^{p^r}$ will go through intermediate dimensions which will be zero. Therefore since $\mathcal{P}^{p^r} \neq 0$ we conclude that $r = 0$, that is $P' \neq 0$ in $H^*(X;Z/pZ)$. Now since $H^*(X;Z/pZ)$ is truncated polynomial algebra in x, it follows that $\exists$ r such that $P' x^r \neq 0$ that means dimension of $P' x^r$ is some multiple of 2n, say 2nk. But dimension of

$$P' x^r = 2nr + 2(p-1) .$$

Therefore

$$2nr + 2(p-1) = 2nk$$

or

$$n(r-k) + (p-1) = 0$$

i.e., $p = 1 + n(r-k)$ that is

$$p \equiv 1 \pmod{n} \quad .$$

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