University of Alberta

Some Inequalities in Convex Geometry

by

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Abstract

We present some inequalities in convex geometry falling under the broad theme of quantifying complexity, or deviation from particularly pleasant geometric conditions: we give an upper bound for the Banach–Mazur distance between an origin-symmetric convex body and the n-dimensional cube which improves known bounds when $n \geq 3$ and is "small"; we give the best known upper and lower bounds (for high dimensions) for the maximum number of points needed to hit every member of an intersecting family of positive homothets (or translates) of a convex body, a number which quantifies the complexity of the family's intersections; we give an exact upper bound on the VC-dimension (a measure of combinatorial complexity) of families of positive homothets (or translates) of a convex body in the plane, and show that no such upper bound exists in any higher dimension; finally, we introduce a novel volumetric functional on convex bodies which quantifies deviation from central symmetry, establish the fundamental properties of this functional, and relate it to classical volumetric measures of symmetry.

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Chapter 1

Introduction

We present some inequalities in convex geometry falling under the broad theme of quantifying complexity, or deviation from particularly pleasant geometric conditions. We generally consider only affine properties, that is, properties which are invariant under invertible affine transformations of $\mathbb{R}^n$; when dealing with the important special class of centrally symmetric convex bodies, we often further take all bodies to be centred at the origin, and then consider only properties which are invariant under linear transformations.

A linear-invariant notion of distance between origin-symmetric convex bodies is Banach–Mazur distance, which is customarily defined between bodies K and L rather than their equivalence classes, as

$$d_{\mathrm{BM}}(K, L) = \inf \{\lambda\colon \lambda > 0 \text{ and } K \subseteq TL \subseteq \lambda K \text{ for some } T \in GL(\mathbb{R}^n)\} .$$

(See figure 1.1.) This is a multiplicative distance: $d_{\mathrm{BM}}(K, K) = 1$, and the triangle inequality is $d_{\mathrm{BM}}(K, M) \leq d_{\mathrm{BM}}(K, L) d_{\mathrm{BM}}(L, M)$. The logarithm of this distance induces a metric on the space of linear equivalence classes of origin-symmetric convex bodies, yielding a compact metric space on which many interesting linear-invariant functionals are continuous (for example, volume ratio, MM*, and the K-convexity constant). Thus estimates on Banach–Mazur distance are of fundamental interest.

A seminal result of this type is John's theorem that, for any origin-symmetric convex body K in $\mathbb{R}^n$,

$$d_{\mathrm{BM}}(K, B_2^n) \leq \sqrt{n} ,$$

where B_2^n denotes the unit Euclidean ball. More specifically, every origin-symmetric convex body K has a linear image TK such that B_2^n is the ellipsoid of maximum volume in TK, and then $B_2^n \subseteq TK \subseteq \sqrt{n} B_2^n$. (See section 2.2, and figure 1.2.)

The analogous question for the n-dimensional cube B_∞^n remains open

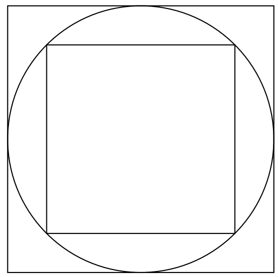


Figure 1.1: Banach–Mazur distance between the square and the circle.

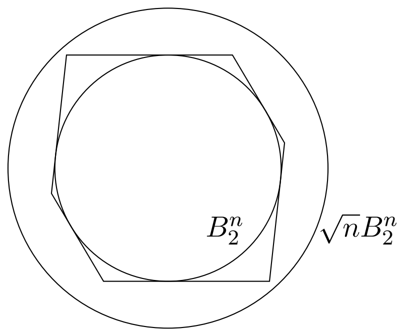


Figure 1.2: John's theorem on Banach–Mazur distance from the ball.

for $n \geq 3$. As in the case of B_2^n, one can consider a linear image TK such that B_1^n (the unit ball of ℓ_1^n norm, that is, a regular cross-polytope) is the cross-polytope of maximum volume in TK; it follows that $B_1^n \subseteq TK \subseteq B_\infty^n$, which yields $d_{\mathrm{BM}}(K, B_\infty^n) \leq n$. (This is essentially the proof of the classical Auerbach lemma.) Perhaps surprisingly, this estimate is not sharp: in $\mathbb{R}^2$, the exact inequality is

$$d_{\mathrm{BM}}(K, B_\infty^2) \leq \frac{3}{2} \ ,$$

as shown by Asplund [3]; for $n \to \infty$ it is known that

$$d_{\mathrm{BM}}(K, B_\infty^n) \leq cn^{5/6} \ .$$

(See chapter 2 for references and more information.) The exact upper bound remains open for all $n \geq 3$.

The main result of chapter 2, Theorem 2.3.4, asserts that, for any origin-symmetric convex body K in $\mathbb{R}^n$,

$$d_{\mathrm{BM}}(K, B_\infty^n) \leq \sqrt{n^2 - 2n + 2 + \frac{2}{\sqrt{n+2} - 1}} \ ,$$

which improves previous estimates for small $n \geq 3$. (For large n, the afore-

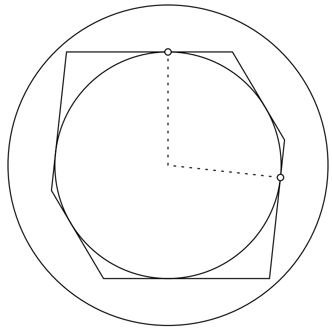


Figure 1.3: Nearly orthogonal contact points in John's theorem.

mentioned estimate $cn^{5/6}$ is superior.) The argument relies heavily on information from John's theorem about origin-symmetric convex bodies for which B_2^n is the maximum volume ellipsoid, specifically about the contact points between the body and B_2^n. We show that there always exist two contact points u and v which are nearly orthogonal, in the sense that

$$|\langle u, v\rangle| \leq \frac{1}{\sqrt{n+2}} \; .$$

(See figure 1.3.) This inequality is sharp in some cases, and is closely related to a well-known estimate of Gerzon on how many lines there can be in an equiangular system of lines through the origin in $\mathbb{R}^n$.

In chapter 3 we consider transversal properties of families of convex bodies, a topic with a long history in convex geometry. The most famous classical result on transversals is Helly's theorem, which asserts that if, in a family of convex bodies, every $(n+1)$-element subfamily has a common point, then the whole family has a common point. A well-known consequence is that, if, in a family $\mathcal{F}$ of translates of a cube (more generally, images of a cube under diagonal affine maps), every pair of translates has a common point, then the whole family has a common point.

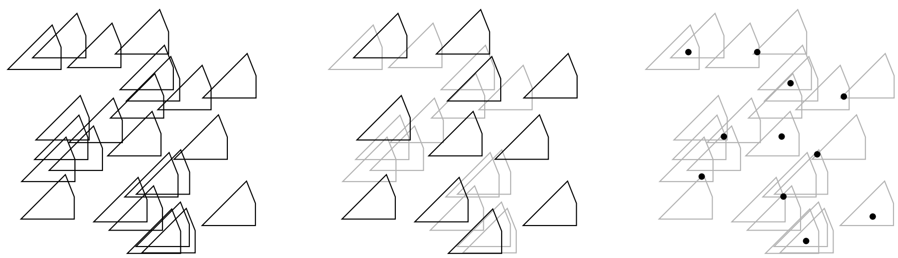
In other words, for such $\mathcal{F}$, if $\nu(\mathcal{F}) = 1$ then $\tau(\mathcal{F}) = 1$, where

$$\nu(\mathcal{F}) = \sup \{\#(\mathcal{E})\colon \mathcal{E} \subseteq \mathcal{F} \text{ and } \mathcal{E} \text{ is pairwise disjoint}\}$$

is the independence number of $\mathcal{F}$ and

$$\tau(\mathcal{F}) = \inf \{\#(T)\colon T \subseteq \mathbb{R}^n \text{ and } (\forall F \in \mathcal{F}\colon F \cap T \neq \varnothing)\}$$

is its transversal number. (See figure 1.4.) Note that $\nu(\mathcal{F}) \leq \tau(\mathcal{F})$ by the pigeonhole principle. In view of the above result for the cube, it is natural to ask the maximum value of $\tau(\mathcal{F})$ among families $\mathcal{F}$ of translates of a



(a) A family $\mathcal{F}$ of translates of a convex body.

(b) A pairwise disjoint subfamily of $\mathcal{F}$, showing $\nu(\mathcal{F}) \geq 11$.

(c) A transversal of $\mathcal{F}$, showing $\tau(\mathcal{F}) \leq 11$.

Figure 1.4: Transversal and independence numbers.

convex body having $\nu(\mathcal{F}) = 1$, or more generally, the maximum value of the ratio $\tau(\mathcal{F})/\nu(\mathcal{F})$. which in a sense quantifies the complexity of intersections among the elements of $\mathcal{F}$.

Passing from K to the cube in this problem, so as to exploit the cube's pleasant packing and covering properties, yields

$$\tau(\mathcal{F}) \leq \lceil 2 d_{\mathrm{BM}}(K, B_\infty^n) \rceil^n \nu(\mathcal{F}) ,$$

as shown by Kim et al. [35]. Combined with estimates of the type considered in chapter 2, this yields an inequality of the type $\tau(\mathcal{F}) \leq c^n n^n \nu(\mathcal{F})$. Chapter 3 presents asymptotically superior estimates using volumetric arguments for covering numbers; the main result of the chapter, Theorem 3.3.2, asserts that for any family $\mathcal{F}$ of translates (or positive homothets) of a convex body K in $\mathbb{R}^n$,

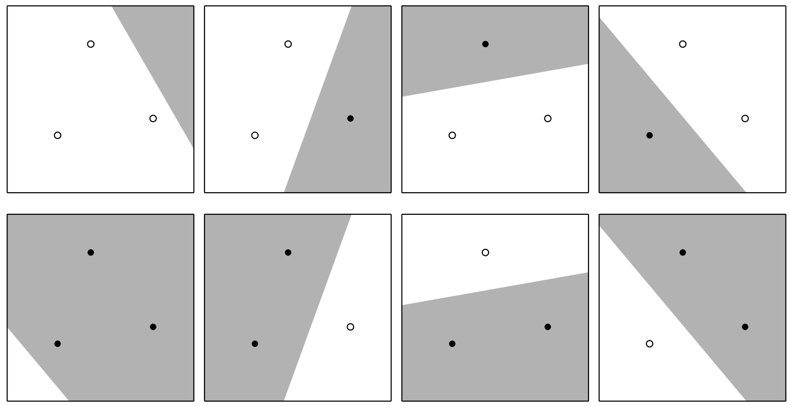
$$\tau(\mathcal{F}) \leq c^n \nu(\mathcal{F}) ,$$

with $c = 3 + 2\sqrt{2} + o(1)$ (as $n \to \infty$). An estimate with $c = 6 + o(1)$ was also given independently (and roughly simultaneously) by Dumitrescu and Jiang [18]; the constant given here is the best known. We also give an example to show that we must have $c \geq 2 - o(1)$, which again is the best known constant.

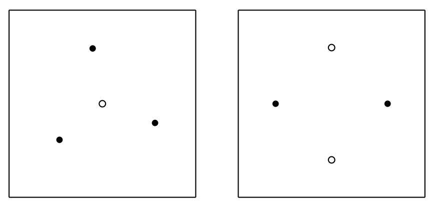
Also in chapter 3, we consider estimates on the VC-dimension of a family of translates (or positive homothets) of a convex body. The VC-dimension of a family $\mathcal{F}$ of sets is

$$\operatorname{vcdim}(\mathcal{F}) = \sup \{\#(X) \colon \mathcal{F} \text{ shatters } X\},$$

where a family $\mathcal{F}$ is said to *shatter* a set of points X if for every subset $X' \subseteq X$, there exists a set $F \in \mathcal{F}$ such that $X \cap F = X'$. (See figure 1.5.)



(a) The family of halfplanes in $\mathbb{R}^2$ shatters this set of three points.



(b) The family of halfplanes in $\mathbb{R}^2$ does not shatter any set of four points.

Figure 1.5: The VC-dimension of the family of halfplanes in $\mathbb{R}^2$ is 3.

Note that if there is no upper bound on the sizes of sets shattered by $\mathcal{F}$, then this definition yields $\operatorname{vcdim}(\mathcal{F}) = \infty$. This combinatorial measure of complexity was introduced by Vapnik and Červonenkis [61, 62] for probabilistic applications: families of events having finite VC-dimension satisfy a uniform law of large numbers.

Our main motivation in studying the VC-dimension is its involvement in upper bounds on transversal numbers (see the Epsilon Net Theorem of Haussler and Welzl [31] and [43, Corollary 10.2.7]) and related phenomena (see [44], for example). We show, however, that $\operatorname{vcdim}(\mathcal{F})$ is bounded from above in dimension two but not in any higher dimension.

Our example for higher dimensions also conclusively settles the conjecture of Grünbaum for dual VC-dimension (see [43, Section 10.3] for this notion), that $\operatorname{vcdim}(\mathcal{F}^*) \leq n + 1$ for any family $\mathcal{F}$ of positive homothets of a convex body in $\mathbb{R}^n$. Naiman and Wynn [46] disproved this conjecture by giving an example with $\operatorname{vcdim}(\mathcal{F}^*) = \lfloor \frac{3n}{2} \rfloor$; our example implies that no upper bound exists if $n \geq 3$.

Chapter 4 deals with measures of (central) symmetry, that is, affine-invariant functionals on the space of convex bodies which quantify how far a convex body deviates from being centrally symmetric. Three classical

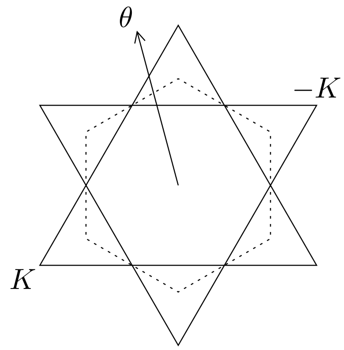


Figure 1.6: The radius of $\frac{1}{2} \diamond K \mathbin{\widehat{+}} \frac{1}{2} \diamond (-K)$ in a direction θ is the harmonic mean of the radii of K and $-K$ in that direction.

measures of symmetry arise by considering the volume of some combination of the body K and its negative $-K$: their Minkowski average $\frac{1}{2}K - \frac{1}{2}K$; their intersection $K \cap -K$; and the convex hull of their union $\mathrm{conv}(K \cup -K)$. (In the latter two cases, for affine invariance we optimize over translates of K.)

We introduce the *harmonic mean measure of symmetry*, which combines K and $-K$ by an operation which is dual to Minkowski average in the same way that intersection is dual to convex hull of union:

$$m_{\mathrm{HM}}(K) = \sup_{x \in K} \frac{\mathrm{vol}(\frac{1}{2} \diamond (K - x) \mathbin{\widehat{+}} \frac{1}{2} \diamond (x - K))}{\mathrm{vol}(K)} ,$$

where $\diamond$ and $\widehat{+}$ denote *harmonic linear combination*, that is, linear combination of gauge functions:

$$\| \cdot \|_{\alpha \diamond K \widehat{+} \beta \diamond L} = \alpha \| \cdot \|_K + \beta \| \cdot \|_L .$$

(See figure 1.6.) Harmonic linear combinations are dual to the usual Minkowski linear combinations, in the sense that

$$\alpha \diamond K \mathbin{\widehat{+}} \beta \diamond L = (\alpha K^o + \beta L^o)^o$$

if K and L are convex bodies with the origin in their interiors. (The term "harmonic" evokes this duality, suggesting an analogy to harmonic and arithmetic means.) Harmonic linear combinations were first considered by Steinhardt [54], who obtained the *dual Brunn–Minkowski inequality*

$$\mathrm{vol}(\alpha \diamond K \mathbin{\widehat{+}} \beta \diamond L)^{-1/n} \geq \alpha \, \mathrm{vol}(K)^{-1/n} + \beta \, \mathrm{vol}(L)^{-1/n} .$$

Firey [21], apparently independently, considered them in a slightly more general form, and Lutwak [41] treated them together with his dual mixed volumes [40] in an integrated dual Brunn–Minkowski theory. See also the thorough survey of Gardner [23, §18.9] and references therein.

We prove that, for any convex body K in $\mathbb{R}^n$,

$$\frac{\sqrt{\pi(n+1)}}{2^{n+1}} \leq m_{\mathrm{HM}}(K) \leq 1 \ ,$$

and that, for the simplex $\triangle$,

$$m_{\mathrm{HM}}(\triangle) = \frac{2^n n!}{(n+1)^n} \sim \sqrt{\frac{\pi n}{2}} \left(\frac{2}{e}\right)^{n+1} \qquad \text{as } n \to \infty.$$

Thus the range of values of the harmonic mean measure of symmetry is exponential in the dimension n, as is the case for the three classical measures of symmetry mentioned above.

We also prove sharp inequalities between the volume of the harmonic mean of K and $-K$ and the volume of their intersection, namely that

$$\mathrm{vol}(K \cap -K) \leq \mathrm{vol}(\tfrac{1}{2} \diamond K \mathbin{\widehat{+}} \tfrac{1}{2} \diamond (-K)) \leq (n+1)\,\mathrm{vol}(K \cap -K)$$

if K is a convex body with the origin in its interior. (A dual version of these inequalities is implicit in the work of Rogers and Shephard.) Since the factor $n+1$ is small in comparison to the exponential range of these quantities, these inequalities show that the harmonic mean of K and $-K$ and their intersection have roughly the same volume.

Finally, we show that, if K is in John's position, then the harmonic mean of K and $-K$ has the smallest volume exactly when K is the Euclidean ball and the largest volume exactly when K is a simplex.

Notation and terminology

Throughout the thesis, we work in $\mathbb{R}^n$, with its standard basis $(e_i)_1^n$, its usual inner product $\langle \cdot, \cdot \rangle$, the associated norm $|\cdot|$, and origin o. The boundary of a set $A \subseteq \mathbb{R}^n$ is denoted ∂A; its interior is $\operatorname{int} A$; its closure is $\operatorname{cl} A$; its convex hull is $\operatorname{conv} A$; and its polar is

$$A^\circ = \{x \in \mathbb{R}^n \colon \langle x, a \rangle \leq 1 \text{ for all } a \in A\} \ .$$

The cardinality of A is denoted $\#(A)$, and its n-dimensional volume is denoted $\mathrm{vol}(A)$, or $\mathrm{vol}_n(A)$ to emphasize the dimension.

A *convex body* is a convex, compact set in $\mathbb{R}^n$ with non-empty interior. A set $A \subseteq \mathbb{R}^n$ is said to be *centrally symmetric* if there exists a point $x \in \mathbb{R}^n$, the centre of A, such that $A = 2x - A$; in particular, it is *origin-symmetric* if $A = -A$. Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, its closed unit ball $\{x \in \mathbb{R}^n \colon \|x\| \leq 1\}$ is an origin-symmetric convex body; conversely, given an origin-symmetric convex body K, the Minkowski functional, or gauge function,

$$\|x\|_K = \inf \{\lambda \colon \lambda > 0 \text{ and } x \in \lambda K\}$$

is a norm. (We also use this notation when K is not origin-symmetric.) We write

$$B_p^n = \{x \in \mathbb{R}^n \colon \|x\|_p \leq 1\}$$

for the unit ball of the ℓ_p^n norm. Thus $B_\infty^n = [-1, 1]^n$ is the n-dimensional cube, and B_2^n is the Euclidean ball.

Chapter 2

The Banach–Mazur distance to the cube in low dimensions*

Abstract

We show that the Banach–Mazur distance from any symmetric convex body in $\mathbb{R}^n$ to the n-dimensional cube is at most

$$\sqrt{n^2 - 2n + 2 + \frac{2}{\sqrt{n+2} - 1}}\ ,$$

which improves previously known estimates for "small" $n \geq 3$. The proof uses an idea of Lassak and the existence of two nearly orthogonal contact points in John's decomposition of the identity. Our estimate on such contact points is closely connected to a well-known estimate of Gerzon on equiangular systems of lines.

2.1 Introduction

The Banach–Mazur distance between two origin-symmetric convex bodies K and L in $\mathbb{R}^n$ is

$$d_{\mathrm{BM}}(K, L) = \inf \{\lambda > 0\colon K \subseteq TL \subseteq \lambda K \text{ for some } T \in \mathrm{GL}(\mathbb{R}^n)\}\ .$$

(See figure 1.1.) This distance is multiplicative, in that $d_{\mathrm{BM}}(K, K) = 1$, and $d_{\mathrm{BM}}(K, M) \leq d_{\mathrm{BM}}(K, L)\, d_{\mathrm{BM}}(L, M)$. Asplund [3] showed that, for any

*A version of this chapter has been published [59].

origin-symmetric convex body K in the plane,

$$d_{\mathrm{BM}}(K, B_\infty^2) \leq \frac{3}{2} ,$$

with equality for the regular hexagon. (See [29, §7] for a survey of this area in classical convex geometry.) The analogous problem in $\mathbb{R}^n$ for $n \geq 3$ is open.

The estimate

$$d_{\mathrm{BM}}(K, B_\infty^n) \leq n$$

follows from the classical Auerbach lemma, which yields that every origin-symmetric convex body K in $\mathbb{R}^n$ has a linear image TK such that $B_1^n \subseteq TK \subseteq B_\infty^n$. (For the generalization to convex bodies which are not necessarily centrally symmetric, see [15, Lemma 1].) This estimate also follows from John's theorem that $d_{\mathrm{BM}}(K, B_2^n) \leq \sqrt{n}$ (see section 2.2) and the (multiplicative) triangle inequality.

Lassak [37] improved the argument via John's theorem to give

$$d_{\mathrm{BM}}(K, B_\infty^n) \leq \sqrt{n^2 - n + 1} \,. \tag{2.1}$$

Lassak's idea was to pass from the ball B_2^n to a ball truncated by parallel hyperplanes, a body which more closely resembles B_∞^n. The main result in this chapter, Theorem 2.3.4, extends Lassak's idea by truncating B_2^n twice, obtaining the estimate

$$d_{\mathrm{BM}}(K, B_\infty^n) \leq \sqrt{n^2 - 2n + 2 + \frac{2}{\sqrt{n+2} - 1}} \,, \tag{2.2}$$

which improves Lassak's bound for all $n \geq 3$. (For example, for $n = 3$, Lassak's bound gives $\sqrt{7}$, while (2.2) gives $\sqrt{6.618\ldots}$.)

As in Lassak's proof, the truncating hyperplanes are chosen based on the contact points between the given body K and its inscribed ellipsoid of maximum volume. For our purposes it is desirable to choose two contact points which are as nearly orthogonal as possible; we show, in Lemma 2.3.1, that there always exist contact points u_i and u_j with

$$|\langle u_i, u_j \rangle| \leq \frac{1}{\sqrt{n+2}} \,. \tag{2.3}$$

The case of equality here is closely related to the question of how many lines there can be in an equiangular system of lines in $\mathbb{R}^n$. A theorem due to Gerzon (stated below as Theorem 2.4.1) asserts that such a system contains at most $\frac{1}{2}n(n+1)$ lines and gives necessary conditions on n for this bound to be achieved (see section 2.4 for details). We show that (2.3) is sharp in $\mathbb{R}^n$ if and only if Gerzon's bound is sharp.

The estimate (2.2) is of interest only for small n because, like Lassak's bound, it is asymptotically linear in n but $\max_K d_{\mathrm{BM}}(K, B_\infty^n)$ is known to be of lower order; in fact, there are absolute positive constants c and C such that

$$c\sqrt{n}\log n \leq \max_K d_{\mathrm{BM}}(K, B_\infty^n) \leq Cn^{5/6} \ .$$

The lower bound is due to Szarek [56]. The upper bound is the culmination of work by several authors: Bourgain and Szarek [12] obtained the first estimate of order $o(n)$; Szarek and Talagrand [57] proved $O(n^{7/8})$; Giannopoulos [24] improved their argument to give $O(n^{5/6})$. See [25, §7.2] for a survey of this area in asymptotic geometric analysis.

The analogous, more general, problem when K is not required to be centrally symmetric (and the notion of Banach–Mazur distance is suitably generalized), was solved by Chakerian and Stein [15, Lemma 1], who showed that $d_{\mathrm{BM}}(K, B_\infty^n) \leq n$, and this is sharp for the simplex. A broad generalization was obtained by Gordon, Litvak, Meyer, and Pajor [28], who showed that $d_{\mathrm{BM}}(K, L) \leq n$ for any convex body K and *any* centrally symmetric convex body L, and that equality is attained if K is the simplex; see [28, Theorem 5.5, Corollary 5.8].

Section 2.2 sketches the necessary background on John's theorem. Section 2.3 proves the main results, and section 2.4 discusses the case of equality in (2.3).

2.2 John's theorem

Definition 2.2.1. A finite collection $(u_i)_1^m$ of unit vectors in $\mathbb{R}^n$ is a *John configuration* if there exist positive real numbers $(c_i)_1^m$ such that

$$\sum_i c_i u_i \otimes u_i = \mathrm{Id} \ , \tag{2.4}$$

where $u \otimes v$ denotes the map $\mathbb{R}^n \to \mathbb{R}^n, x \mapsto \langle u, x\rangle v$. A John configuration is *centred* if, moreover,

$$\sum_i c_i u_i = o \ .$$

Theorem 2.2.2 (John [33])**.** *Let K be an origin-symmetric (resp., arbitrary) convex body in $\mathbb{R}^n$. Then B_2^n is the ellipsoid of maximum volume in K if and only if $B_2^n \subseteq K$ and $\partial B_2^n \cap \partial K$ contains a John configuration (resp., a centred John configuration). Every convex body has one such affine image (up to orthogonal transformations).*

A body satisfying the conditions in Theorem 2.2.2 is said to be *in John's position*.

The original paper of John is difficult to obtain; for proofs of Theorem 2.2.2, see, for example, [4, Lecture 3], [25, §2.3], or [60, §15]. For generalizations to two arbitrary convex bodies, see [26], [9], and for the strongest form, [28, Theorem 3.8].

We will use the following well-known facts about John configurations. It is easy to show that the defining condition (2.4) is equivalent to the condition that

$$\sum_i c_i \langle u_i, x\rangle^2 = |x|^2 \qquad \text{for all } x \in \mathbb{R}^n. \tag{2.5}$$

Taking traces in (2.4) yields

$$\sum_i c_i = n \ , \tag{2.6}$$

and so $(u_i)_1^m$ is a John configuration if and only if

$$\frac{1}{n}\,\mathrm{Id} \in \operatorname{conv}\,\{u_i \otimes u_i \colon i = 1, \ldots, m\} \ . \tag{2.7}$$

Carathéodory's theorem (applied to the space of symmetric trace-one linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$) yields that any John configuration has a subset with at most $\frac{1}{2}n(n+1)$ elements which is also a John configuration.

These facts together yield that $d_{\mathrm{BM}}(K, B_2^n) \le \sqrt{n}$ for any origin-symmetric convex body K, as follows. (See figure 1.2.) If K and $(u_i)_1^m \subseteq \partial B_2^n \cap \partial K$ are as in Theorem 2.2.2, then any halfspace that supports K at u_i also supports B_2^n there, and so must be $\{u_i\}^\circ$; thus $K \subseteq \{\pm u_1, \ldots, \pm u_m\}^\circ$. Finally, $\{\pm u_1, \ldots, \pm u_m\}^\circ \subseteq \sqrt{n} B_2^n$ by a straightforward computation using (2.5) and (2.6). These observations will also be used in the proof of Theorem 2.3.4.

John's theorem is also an essential ingredient in the proof of Ball's volume ratio theorem, which is stated here for convenience, though it will not be used until section 4.6.

Theorem 2.2.3. *Let K be a convex body in John's position in $\mathbb{R}^n$. If K is origin-symmetric, then*

$$\mathrm{vol}(K) \le 2^n \ ,$$

with equality if and only if K is a cube. In general,

$$\mathrm{vol}(K) \le \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!} \ ,$$

with equality if and only if K is a simplex.

The inequalities are due to Ball [5, 6] (see also the more self-contained proof in [7]); both cases of equality are due to Barthe [8, Proposition 10]. In the proof of the general case, the following observation by Ball plays a key role. We state it here as a separate lemma for use in section 4.6.

Lemma 2.2.4 (Ball [6]). *Let $(u_i)_{i=1}^m$ be a sequence of unit vectors in $\mathbb{R}^n$ and let $(c_i)_{i=1}^m$ be associated positive scalars. If the vectors u_i form a centred John configuration with the weights c_i, then the vectors*

$$v_i = \left(\sqrt{\frac{n}{n+1}}\, u_i, -\frac{1}{\sqrt{n+1}} \right)$$

form a John configuration in $\mathbb{R}^n \oplus \mathbb{R} = \mathbb{R}^{n+1}$ with weights $d_i = \frac{n+1}{n} c_i$.

Proof. By direct computation. □

2.3 Nearly orthogonal contact points and distance to the cube

Lemma 2.3.1. *If $(u_i)_1^m$ is a John configuration in $\mathbb{R}^n$, $n \geq 2$, then*

$$\min_{i,j} |\langle u_i, u_j \rangle| \leq \frac{1}{\sqrt{n+2}} \ .$$

Proof. As noted in section 2.2, we may assume $m \leq \frac{1}{2}n(n+1)$. Let $(c_i)_1^m$ be positive real numbers such that $\sum_i c_i u_i \otimes u_i = \mathrm{Id}$. Let X be a random variable in $\{1, \dots, m\}$ with

$$\mathrm{Prob}(X = j) = \frac{c_j}{n} \ .$$

Since

$$\max_i c_i \geq \frac{1}{m} \sum_i c_i = \frac{n}{m} \geq \frac{2}{n+1} \ ,$$

we have

$$\begin{aligned}
\min_i \min_j \ \langle u_i, u_j \rangle^2 &= \min_i \min_{j \neq i} \ \langle u_i, u_j \rangle^2 \\
&\leq \min_i \mathbb{E}(\langle u_i, u_X \rangle^2 \mid X \neq i) = \min_i \frac{1}{1 - c_i/n} \sum_{j \neq i} \frac{c_j}{n} \langle u_i, u_j \rangle^2 \\
&= \min_i \frac{1}{n - c_i} \Big(|u_i|^2 - c_i \langle u_i, u_i \rangle^2 \Big) = \min_i \frac{1 - c_i}{n - c_i} \\
&\leq \frac{1 - 2/(n+1)}{n - 2/(n+1)} = \frac{1}{n+2} \ ,
\end{aligned}$$

as desired. □

The case of equality in Lemma 2.3.1 is discussed in section 2.4.

Remark 2.3.2. The idea of averaging over the u_i according to their weights (as in the proof of Lemma 2.3.1) goes back to John [33], who used a similar argument to prove Jung's inequality [11], which asserts that if r is the minimum radius of a sphere enclosing some set $A \subseteq \mathbb{R}^d$, then

$$r \leq \sqrt{\frac{d}{2(d+1)}}\,\operatorname{diam} A\ ,$$

and equality is attained by regular simplices. In fact, one can deduce Lemma 2.3.1 from this inequality by taking A to be the set of operators $u_i \otimes u_i$ (with geometry given by the Hilbert–Schmidt inner product, as in the proof of Proposition 2.4.2). The proof above, however, gives slightly more, since it shows that we may take u_i to be any point with above-average weight; John's method cannot provide such information because he computes the average over all pairs (i, j) with $i \neq j$ instead of fixing i and averaging over j. (Compare also [55, Proposition 2], which uses a method resembling John's to estimate the *maximum* inner product between two vectors in a slightly different context.)

Remark 2.3.3. The weaker, though asymptotically equivalent, bound

$$\min_{i,j} |\langle u_i, u_j\rangle| \leq \frac{1}{\sqrt{n}}$$

is obtained in the first step of the proof of the classical Dvoretzky–Rogers lemma [19] (also [25, Theorem 2.3.3]). The argument of Theorem 2.3.4 can be given using this bound, but the resulting bound on $d_{\mathrm{BM}}(K, B_\infty^n)$ does not improve Lassak's result (2.1) in the case $n = 3$.

Theorem 2.3.4. *For any origin-symmetric convex body K in $\mathbb{R}^n$, $n \geq 3$,*

$$d_{BM}(K, B_\infty^n) \leq \sqrt{n^2 - 2n + 2 + \frac{2}{\sqrt{n+2} - 1}}\ .$$

Proof. Applying an affine transformation if necessary, we may assume B_2^n is the maximum volume ellipsoid of K. Let $(u_i)_1^m$ be a John configuration of contact points, as in Theorem 2.2.2. By Lemma 2.3.1, we may assume

$$|\langle u_1, u_2\rangle| \leq \frac{1}{\sqrt{n+2}}\ .$$

Let $V = \operatorname{span}\{u_1, u_2\}$. Rotating if necessary, we may assume that $V = \operatorname{span}\{e_1, e_2\}$ and that $u_1 + u_2$ is a multiple of $e_1 + e_2$.

Let

$$U = \{\pm u_1, \pm u_2\}^\circ \ , \quad L = U \cap \sqrt{n} B_2^n \ , \quad \text{and} \quad M = U \cap \sqrt{n} B_\infty^n \ .$$

We will show that

$$d_{\mathrm{BM}}(K, B_\infty^n) \le d_{\mathrm{BM}}(K, L)\, d_{\mathrm{BM}}(L, M)\, d_{\mathrm{BM}}(M, B_\infty^n) \le \sqrt{n} \cdot \frac{r}{\sqrt{n}} \cdot 1 = r \ ,$$

where $r = \max_{x \in M} |x|$.

To show that $d_{\mathrm{BM}}(K, L) \le \sqrt{n}$, note that, by the remarks in section 2.2, $K \subseteq L \subseteq \sqrt{n} B_2^n \subseteq \sqrt{n} K$.

To show that $d_{\mathrm{BM}}(L, M) \le \frac{r}{\sqrt{n}}$, note first that $\sqrt{n} e_3 \in M$, so $r \ge \sqrt{n}$. Thus $M \subseteq \frac{r}{\sqrt{n}} U$. Furthermore, $M \subseteq r B_2^n = \frac{r}{\sqrt{n}} \sqrt{n} B_2^n$ by the definition of r. So $L \subseteq M \subseteq \frac{r}{\sqrt{n}} L$.

To show that $d_{\mathrm{BM}}(M, B_\infty^n) = 1$, we will show that M is a parallelotope. Consider a vertex p of the rhombus $U \cap V$. (See figure 2.1.) Since p is the intersection of the tangents to the circle $B_2^n \cap V$ at u_1 and u_2, the line ℓ joining u_1 and u_2 is $\partial(\{p\}^\circ)$, so

$$|p|^2 = \frac{1}{\mathrm{dist}(o, \ell)^2} = \frac{1}{|\frac{1}{2}(u_1 + u_2)|^2} = \frac{2}{1 + \langle u_1, u_2 \rangle} \le \frac{2}{1 - \frac{1}{\sqrt{n+2}}} = 2 + \frac{2}{\sqrt{n+2} - 1} \ .$$

In particular, $|p|^2 < 4 < 2n$, so the vertices of the rhombus $U \cap V$ are closer to the origin than the vertices of the square $\sqrt{n} B_\infty^n \cap V$ are. Recall that $u_1 + u_2$ is a multiple of $e_1 + e_2$, so the vertices of the rhombus lie on the

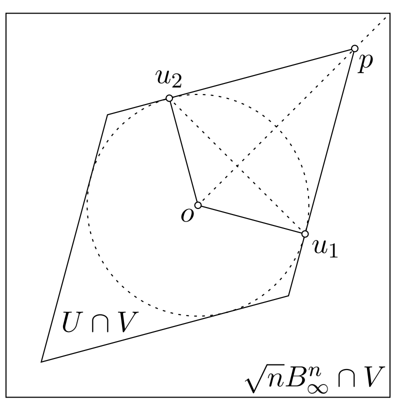


Figure 2.1: The rhombus $U \cap V$, in the proof of Theorem 2.3.4.

same rays through the origin as the vertices of the square; thus

$$U \cap V \subseteq \sqrt{n} B_\infty^n \cap V \ ,$$

as figure 2.1 shows. It then follows easily that

$$M = U \cap \sqrt{n} B_\infty^n = U \cap V + \sqrt{n} B_\infty^n \cap V^\perp \ ,$$

which is the sum of a rhombus and an $(n-2)$-dimensional cube in complementary subspaces, hence a parallelotope.

It remains only to estimate r. Let $x = v + w \in M$, where $v \in U \cap V$ and $w \in \sqrt{n} B_\infty^n \cap V^\perp$. Then

$$|x|^2 = |v|^2 + |w|^2 \leq 2 + \frac{2}{\sqrt{n+2} - 1} + n(n-2) \ .$$

Taking square roots yields the desired result. □

2.4 Equiangular lines

Theorem 2.4.1 (Gerzon [38, Theorem 3.5]). *An equiangular system of lines in $\mathbb{R}^n$ contains at most $\frac{1}{2}n(n+1)$ lines. If equality is achieved, then the angle between any pair of lines is $\arccos(1/\sqrt{n+2})$, and moreover, either $n = 2$, $n = 3$, or $n + 2$ is an odd square.*

In fact, Gerzon's theorem as stated in [38] does not include the value of the angle, but the proof given there does establish it.

Systems of lines achieving Gerzon's bound are known to exist for the first few n satisfying the conditions stated in Theorem 2.4.1, namely $n = 2, 3, 7, 23$; see [38]. The case of equality (or near-equality) in Gerzon's theorem is also of interest in the theory of frames; see [55] for a recent survey from this point of view.

Proposition 2.4.2. *Let $(u_i)_1^m$ be a finite collection of unit vectors in $\mathbb{R}^n$, $n \geq 2$. Then the following are equivalent:*

(i) $(u_i)_1^m$ is a John configuration and $\min_{i,j} |\langle u_i, u_j \rangle| = 1/\sqrt{n+2}$;

(ii) the lines spanned by the u_i are equiangular, and $m = \frac{1}{2}n(n+1)$.

Proof. Examining the inequalities in the proof of Lemma 2.3.1, we see that condition (i) holds if and only if these three conditions are satisfied:

1. (u_i) is a John configuration with all weights equal;
2. $m = \frac{1}{2}n(n+1)$; and

3. all the values $\langle u_i, u_j\rangle^2$ with $i \neq j$ are equal.

The last two statements are exactly condition (ii), which thus clearly follows from condition (i).

Conversely, suppose condition (ii) holds, so that $m = \frac{1}{2}n(n+1)$ and the points $(u_i)_1^m$ satisfy, say,

$$\langle u_i, u_j\rangle^2 = \begin{cases} 1 & \text{if } i = j, \\ a & \text{if } i \neq j. \end{cases}$$

We wish to show that the points $(u_i)_1^m$ are a John configuration with all weights equal, that is,

$$\sum_i \frac{n}{m} u_i \otimes u_i = \mathrm{Id} \ .$$

Consider the space of real symmetric $n \times n$ matrices, under the Hilbert–Schmidt inner product

$$\langle\!\langle A, B\rangle\!\rangle = \mathrm{tr}(AB) \ ,$$

and the associated norm $\|\cdot\|_{\mathrm{HS}}$ (which in this finite-dimensional setting is sometimes called the Frobenius norm). With this inner product structure, the hyperplane $\{A\colon \mathrm{tr}(A) = 1\}$ is at distance $1/\sqrt{n}$ from the origin, and the (unique) closest point to the origin in this hyperplane is $\frac{1}{n}\,\mathrm{Id}$. On the other hand, $\mathrm{tr}(\frac{1}{m}\sum_i u_i \otimes u_i) = 1$, and

$$\begin{aligned} \left\| \frac{1}{m} \sum_i u_i \otimes u_i \right\|_{\mathrm{HS}}^2 &= \frac{1}{m^2} \sum_i \sum_j \langle\!\langle u_i \otimes u_i, u_j \otimes u_j \rangle\!\rangle = \frac{1}{m^2} \sum_i \sum_j \langle u_i, u_j\rangle^2 \\ &= \frac{1}{m^2} \left(\sum_i |u_i|^4 + \sum_i \sum_{j \neq i} \langle u_i, u_j\rangle^2 \right) = \frac{1}{m} + \left(1 - \frac{1}{m}\right) a = \frac{1}{n} \ , \end{aligned}$$

since $m = \frac{1}{2}n(n+1)$ (by hypothesis) and $a = 1/(n+2)$ (by Theorem 2.4.1). Thus $\frac{1}{m}\sum_i u_i \otimes u_i$ is also a closest point to the origin in the trace-one hyperplane, so by uniqueness, $\frac{1}{m}\sum_i u_i \otimes u_i = \frac{1}{n}\,\mathrm{Id}$, as desired. □

Remark 2.4.3. The direction (ii) ⇒ (i) in the above proof essentially shows from condition (ii) that $\frac{1}{n}\,\mathrm{Id}$ is the centroid of the operators $u_i \otimes u_i$. With a little more computation, one may actually show from condition (ii) that

$$\frac{\langle\!\langle u_i \otimes u_i - \frac{1}{n}\,\mathrm{Id}, u_j \otimes u_j - \frac{1}{n}\,\mathrm{Id}\rangle\!\rangle}{\left\|u_i \otimes u_i - \frac{1}{n}\,\mathrm{Id}\right\|_{\mathrm{HS}} \left\|u_j \otimes u_j - \frac{1}{n}\,\mathrm{Id}\right\|_{\mathrm{HS}}} = \begin{cases} 1 & \text{if } i = j, \\ -1/d & \text{if } i \neq j, \end{cases}$$

where $d = \frac{1}{2}n(n+1) - 1$ is the dimension of the ambient space of symmetric trace-one $n \times n$ matrices; thus the operators $u_i \otimes u_i$ are the vertices of a regular d-simplex centred at $\frac{1}{n}\,\mathrm{Id}$, which accounts for the extremality of the situation.

Example 2.4.4. Lemma 2.3.1 is sharp for the contact points of the Platonic dodecahedron in $\mathbb{R}^3$, but Theorem 2.3.4 is not sharp for this body. Indeed, let K be a Platonic dodecahedron in $\mathbb{R}^3$, centred at the origin. Then K^o is a Platonic icosahedron. Without loss of generality, K^o is the convex hull of the points

$$(0,1,a),(a,0,1),(1,a,0),(0,1,-a),(-a,0,1),(1,-a,0)$$

and their negatives, where $a=\frac{1}{2}(\sqrt{5}-1)$. Then $B_1^3\subseteq K^o\subseteq(a+1)B_1^3$, so $d_{\mathrm{BM}}(K,B_\infty^3)=d_{\mathrm{BM}}(K^o,B_1^3)\leq a+1=\frac{1}{2}(1+\sqrt{5})$.

Chapter 3

Complexity of families of positive homothets*

Abstract

We investigate two approaches to measuring the combinatorial complexity of a family of positive homothets (or translates) of a convex body K in $\mathbb{R}^n$. First, we give an upper bound on the transversal number of the family in terms of its independence number and the dimension, extending previous results to the case of infinitely large families and giving the best known dependence on the dimension. Our example for a lower bound also gives the best known dependence on the dimension, making use of an asymptotically sharp estimate on the number of translates of a simplex needed to cover its negative.

Second, we consider VC-dimension, a standard combinatorial measure of complexity. We show that any family of positive homothets of a convex body in the plane has VC-dimension at most 3, but that no such upper bound exists in any higher dimension, conclusively settling a conjecture of Grünbaum.

3.1 Introduction

A *positive homothet* of a set $S \subseteq \mathbb{R}^n$ is a set of the form $\lambda S + x$, where $\lambda > 0$ and $x \in \mathbb{R}^n$. Let $\mathcal{F}$ be a family of positive homothets (or translates) of a given convex body K in $\mathbb{R}^n$. In this chapter we study two approaches to

*A version of this chapter has been published [47]. It is joint work with Márton Naszódi.

measuring the complexity of $\mathcal{F}$.

First, we bound the transversal number $\tau(\mathcal{F})$ in terms of the dimension n and the independence number $\nu(\mathcal{F})$. The *transversal number* $\tau(\mathcal{F})$ of a family of sets $\mathcal{F}$ is defined as

$$\tau(\mathcal{F}) = \min \{\#(S)\colon S \subseteq \mathbb{R}^n \text{ and } S \cap F \neq \varnothing \text{ for all } F \in \mathcal{F}\}.$$

The *independence number* $\nu(\mathcal{F})$ of $\mathcal{F}$ is defined as

$$\nu(\mathcal{F}) = \max \{\#(S)\colon S \subseteq \mathcal{F} \text{ and } S \text{ is pairwise disjoint}\}.$$

Clearly $\nu(\mathcal{F}) \leq \tau(\mathcal{F})$. Karasev [34] showed that, if $\mathcal{F}$ is a family of translates of a convex body in $\mathbb{R}^2$ with $\nu(\mathcal{F}) = 1$, then $\tau(\mathcal{F}) \leq 3$, as had been conjectured by Grünbaum [17]. One of the main results of [35] is that if $\mathcal{F}$ is a family of positive homothets of a convex body in $\mathbb{R}^n$ then $\tau(\mathcal{F}) \leq 2^{n-1}n^n\nu(\mathcal{F})$. We improve the dependence on n to exponential, showing that

$$\tau(\mathcal{F}) \leq \begin{cases} 3^n(n\log n + n\log\log n + 5n)\nu(\mathcal{F}) & \text{if } K = -K, \\ (3+2\sqrt{2})^n(n\log n + n\log\log n + 5n)^2\nu(\mathcal{F}) & \text{otherwise.} \end{cases}$$

In the general case, the published version of this chapter [47] gave an estimate of order roughly 8^n; Dumitrescu and Jiang [18], roughly simultaneously, published an estimate of order roughly 6^n (for finite families) by applying a result of Rogers [17, footnote to item 7.10]. The order $(3+2\sqrt{2})^n$ given here is the best known.

We also show that an exponential bound is the best possible, even when $\mathcal{F}$ contains only translates of K, by constructing an example of such $\mathcal{F}$ with

$$\tau(\mathcal{F}) \geq \frac{2^n}{\sqrt{\pi(n+\frac{1}{2})}}\nu(\mathcal{F}) \ .$$

Other examples with $\tau(\mathcal{F}) \geq c^n\nu(\mathcal{F})$ are known ([47, Proposition 2], [18, Theorem 5]); the value of c in the example given here is the best known.

Our second approach is to investigate the *VC-dimension* of a family $\mathcal{F}$ of positive homothets (or translates) of a convex body K. This combinatorial measure of complexity was introduced by Vapnik and Červonenkis [61, 62], and is defined as

$$\operatorname{vcdim}(\mathcal{F}) = \sup \{\#(X)\colon \mathcal{F} \text{ shatters } X\},$$

where a family $\mathcal{F}$ is said to *shatter* a set of points X if for every subset $X' \subseteq X$, there exists a set $F \in \mathcal{F}$ such that $X \cap F = X'$. Note that if there is no upper bound on the sizes of sets shattered by $\mathcal{F}$, then this definition yields $\operatorname{vcdim}(\mathcal{F}) = \infty$.

Our main motivation in studying the VC-dimension is its involvement in upper bounds on transversal numbers (see the Epsilon Net Theorem of Haussler and Welzl [31] and [43, Corollary 10.2.7]) and related phenomena (see [44], for example). We show, however, that $\operatorname{vcdim}(\mathcal{F})$ is bounded from above in dimension two but not in any higher dimension.

Our example for higher dimensions also settles a conjecture of Grünbaum on dual VC-dimension (see [43, Section 10.3] for this notion). He showed [30] that if $\mathcal{F}$ is a family of positive homothets of a convex body in $\mathbb{R}^2$, then $\operatorname{vcdim}(\mathcal{F}^*) \leq 3$, and conjectured [30, point (7), p. 21] the upper bound $\operatorname{vcdim}(\mathcal{F}^*) \leq n+1$ for such families in $\mathbb{R}^n$. (Grünbaum uses a different terminology: instead of dual VC-dimension, he writes "the maximal number of sets in independent families", where "independence" is *not* as we defined above.) Naiman and Wynn [46] disproved this conjecture by giving an example with $\operatorname{vcdim}(\mathcal{F}^*) = \left\lfloor \frac{3n}{2} \right\rfloor$; our example shows that no upper bound exists, since $\operatorname{vcdim}(\mathcal{F}) < 2^{\operatorname{vcdim}(\mathcal{F}^*)+1}$ [43, Lemma 10.3.4]. (We are grateful to Leonard Schulman, who upon learning of our Example 3.4.2, brought this conjecture of Grünbaum to our attention.)

The construction of our example shares some principles with the constructions given in [32] and [27, Theorem 2.9] to show that certain Helly-type and Hadwiger-type theorems for line transversals of families of translates of a convex set in the plane do not generalize to $\mathbb{R}^3$. These examples and ours show that, in some sense, translates of a convex set in $\mathbb{R}^3$ may form families of high complexity. They also suggest that finding good bounds for the transversal numbers of such families is a difficult task.

Section 3.2 reviews some background on covering and separation numbers in preparation for section 3.3, which presents the results on transversal and independence numbers. Section 3.4 presents the results on VC-dimension.

3.2 Covering and separation numbers

For any sets $A, B \subseteq \mathbb{R}^n$, define the *covering number* of A by B to be

$$N(A, B) = \inf \{\#(T)\colon A \subseteq T + B\} .$$

Equivalently, $N(A, B)$ is the smallest number of translates of B that can cover A. Also say that a set T is *separated by B* if

$$\forall t_1, t_2 \in T\colon t_1 \neq t_2 \implies (t_1 + B) \cap (t_2 + B) = \varnothing ,$$

and define the *separation number* of A by B to be

$$M(A, B) = \sup \{\#(T)\colon T \subseteq A \text{ and } T \text{ is separated by } B\} .$$

Equivalently, $M(A,B)$ is the largest number of pairwise disjoint translates of B by vectors in A. Note that if $\mathcal{F}$ is a family of translates of a set K, say, $\mathcal{F} = \{t + K : t \in T\}$, then

$$\tau(\mathcal{F}) = N(T, -K) \qquad \text{and} \qquad \nu(\mathcal{F}) = M(T,K) \ .$$

The elementary facts about covering and separation numbers are consequences of the following geometric lemma.

Lemma 3.2.1. *For any sets $A, B \subseteq \mathbb{R}^n$ and vectors $x, y \in \mathbb{R}^n$, the following are equivalent:*

1. $x - y \in B - A$;
2. $(x + A) \cap (y + B) \neq \varnothing$;
3. $(x - B) \cap (y - A) \neq \varnothing$;
4. *there exists $z \in \mathbb{R}^n$ such that $x \in z + B$ and $y \in z + A$;*
5. *there exists $z \in \mathbb{R}^n$ such that $x \in z - A$ and $y \in z - B$.*

Proposition 3.2.2. *For any sets $A, B \subseteq \mathbb{R}^n$, we have*

$$N(A, B - B) \leq M(A,B) \leq N(A,B) \ .$$

Proof. Let $T \subseteq A$ be a maximal set as in the definition of $M(A,B)$. By maximality, if $a \in A$ then there exists $t \in T$ such that $(a+B) \cap (t+B) \neq \varnothing$. Condition (1) of the lemma yields that $A \subseteq T + B - B$, and so $N(A, B-B) \leq \#(T)$, which proves the lower inequality. On the other hand, by condition (4) of the lemma, no translate of B can cover any two distinct points of T; by the pigeonhole principle, $N(A,B) \geq \#(T)$, proving the upper inequality. □

We will make use of the inequality

$$N(A + B, C + D) \leq N(A,C)N(B,D) \ ,$$

and of the basic volumetric estimates for covering numbers, that is, the obvious bound

$$\frac{\mathrm{vol}(K)}{\sup_{x \in \mathbb{R}^n} \mathrm{vol}(K \cap (x + L))} \leq N(K,L) \ ,$$

and the Rogers–Zong lemma:

Theorem 3.2.3 (Rogers [48], Rogers–Zong [51])**.** *Let $K, L \subset \mathbb{R}^n$ be convex sets. Then*

$$N(K,L) \leq \frac{\mathrm{vol}(K - L)}{\mathrm{vol}(L)}(n \log n + n \log\log n + 5n) \ .$$

3.3 Transversal and independence numbers

Lemma 3.3.1. *Let K be a convex body in $\mathbb{R}^n$ and let $a > 0$. Then*

$$N(aK - K, -K) \leq (\sqrt{a+1} + 1)^{2n}(n \log n + n \log\log n + 5n)^2 \ .$$

Proof. Let $\lambda > a$ be chosen later. Then

$$\begin{aligned} N(aK - K, -K) &\leq N(aK - K, \lambda K)N(\lambda K, -K) \\ &= N(aK - K, aK + (\lambda - a)K)N(\lambda K, -K) \\ &\leq N(-K, (\lambda - a)K)N(\lambda K, -K) \\ &\leq \left(\frac{(\lambda - a + 1)(\lambda + 1)}{\lambda - a}\right)^n (n \log n + n \log\log n + 5n)^2 \end{aligned}$$

by Theorem 3.2.3. Taking $\lambda = a + \sqrt{a+1}$ yields the desired result. □

Theorem 3.3.2. *Let $\mathcal{F}$ be a family of positive homothets of a convex body K in $\mathbb{R}^n$. Then*

$$\begin{aligned} \tau(\mathcal{F}) &\leq \nu(\mathcal{F}) \inf_{a>1} N(aK - K, -K) \\ &\leq \begin{cases} 3^n(n \log n + n \log\log n + 5n)\nu(\mathcal{F}) & \textit{if } K = -K, \\ (3 + 2\sqrt{2})^n(n \log n + n \log\log n + 5n)^2\nu(\mathcal{F}) & \textit{otherwise.} \end{cases} \end{aligned}$$

Proof. The second inequality is a direct application of Theorem 3.2.3 (when $K = -K$) and of Lemma 3.3.1 (in general).

For the first inequality, fix $a > 1$. For any $\mathcal{A} \subseteq \mathcal{F}$, define $\lambda(\mathcal{A}) = \inf \{\lambda \colon \lambda K + x \in \mathcal{A}\}$. If $\lambda(\mathcal{A}) > 0$, then say that $F \in \mathcal{A}$ is a *small* member of $\mathcal{A}$ if $F = \mu K + x$ and $\mu < a\lambda(\mathcal{A})$.

First consider the case $\lambda(\mathcal{F}) > 0$. Iteratively choose $F_1, \ldots, F_m \in \mathcal{F}$ as follows. First let F_1 be a small member of $\mathcal{F}$; then let F_2 be a small member of $\{F \in \mathcal{F} \colon F \cap F_1 = \varnothing\}$; then let F_3 be a small member of $\{F \in \mathcal{F} \colon F \cap (F_1 \cup F_2) = \varnothing\}$; and so on, until $\{F \in \mathcal{F} \colon F \cap (F_1 \cup \cdots \cup F_m) = \varnothing\} = \varnothing$. Let

$$\mathcal{F}_i = \{F \in \mathcal{F} \colon \min \{j \colon F \cap F_j \neq \varnothing\} = i\} \ .$$

Note that, by construction, F_i is a small member of $\mathcal{F}_i$.

For each $F \in \mathcal{F}_i$ (including F_i itself), choose a point $z \in F \cap F_i$, and shrink F with center z to obtain a translate of $\lambda(\mathcal{F}_i)K$. Let $\mathcal{F}_i'$ be the family of the shrunken copies, say, $\mathcal{F}_i' = \{\lambda(\mathcal{F}_i)K + t \colon t \in T_i\}$. Since every element of $\mathcal{F}_i$ is a superset of a set in $\mathcal{F}_i'$, any transversal of $\mathcal{F}_i'$ is a transversal of $\mathcal{F}_i$. Furthermore, every element of $\mathcal{F}_i'$ intersects F_i, so $T_i \subseteq F_i - \lambda(\mathcal{F}_i)K$ (by

Lemma 3.2.1). Thus

$$\begin{aligned}\tau(\mathcal{F}_i) &\le \tau(\mathcal{F}_i') \\ &= N(T_i, -\lambda(\mathcal{F}_i)K) \\ &\le N(F_i - \lambda(\mathcal{F}_i)K, -\lambda(\mathcal{F}_i)K) \\ &\le N(a\lambda(\mathcal{F}_i)K - \lambda(\mathcal{F}_i)K, -\lambda(\mathcal{F}_i)K) \\ &= N(aK - K, -K)\end{aligned}$$

Since $\mathcal{F} = \bigcup_{i=1}^m \mathcal{F}_i$, it follows that $\tau(\mathcal{F}) \le mN(aK - K, -K)$, which completes the proof of this case since the F_i are pairwise disjoint by construction, so $m \le \nu(\mathcal{F})$.

Finally, consider the case $\lambda(\mathcal{F}) = 0$. Let $(\delta_m)_{m=1}^\infty$ be a sequence of positive real numbers with $\delta_m \downarrow 0$. For $m \in \mathbb{Z}^+$, define $\mathcal{F}^{(m)} = \{\lambda K + x \in \mathcal{F} \colon \lambda > \delta_m\}$. By the previous case, for each m there is a transversal $T^{(m)}$ of $\mathcal{F}^{(m)}$ with $\#T^{(m)} \le \nu(\mathcal{F})N(aK-K, -K)$. Let $k = \nu(\mathcal{F})N(aK-K, -K)$, and write $T^{(m)} = \{t_1^m, \dots, t_k^m\}$ (duplicating entries if necessary). Let

$$\Lambda = \{i \colon \lim_{m\to\infty} t_i^m \text{ exists}\} \qquad \text{and} \qquad T = \{\lim_{m\to\infty} t_i^m \colon i \in \Lambda\} \ .$$

We claim that if T is not a transversal of $\mathcal{F}$, then we can replace $T^{(m)}$ with a subsequence such that the set Λ is (strictly) enlarged. Since $\#\Lambda \le k$, repeating this procedure will, after at most k repetitions, yield a transversal of $\mathcal{F}$ with at most k elements, as desired.

So suppose T is not a transversal of $\mathcal{F}$. Let $F \in \mathcal{F}$ be such that $F \cap T = \varnothing$. On the other hand, for each m, $F \cap T^{(m)} \ne \varnothing$. By the pigeonhole principle, there exists $i \in \{1, \dots, k\}$ such that $t_i^m \in F$ for infinitely many m. If $i \in \Lambda$ then, since F is closed, $\lim_{m\to\infty} t_i^m \in F$, contrary to the assumption that $F \cap T = \varnothing$; so $i \notin \Lambda$. Passing to a subsequence, we obtain that $t_i^m \in F$ for all m; by the compactness of F, passing to a further subsequence, we obtain that $\lim_{m\to\infty} t_i^m$ exists, which proves the claim. □

Remark 3.3.3. In some special cases of Theorem 3.3.2, we have the stronger inequality

$$\tau(\mathcal{F}) \le \nu(\mathcal{F})N(K - K, K) \ .$$

(The inequality $N(K - K, K) \le \inf_{a>1} N(aK - K, -K)$ may indeed be strict, as when $K = B_\infty^n$: then $N(K - K, K) = 2^n$ but $N(aK - K, -K) \ge M((a+1)B_\infty^n, B_\infty^n) \ge 3^n$, as taking $T = \{-(a+1), 0, a+1\}^n$ in the definition of $M(\cdot, \cdot)$ shows.) First, if $\mathcal{F}$ consists only of translates of K, then the observations of section 3.2 yield

$$\begin{aligned}\tau(\mathcal{F}) = N(T,-K) &\le N(T,K-K)N(K-K,-K) \\ &\le M(T,K)N(K-K,-K) = \nu(\mathcal{F})N(K-K,-K)\ .\end{aligned}$$

Second, if $\mathcal{F}$ is finite, then in the proof of Theorem 3.3.2 we have only the case $\lambda(\mathcal{F}) > 0$, and we can moreover take F_i to be the smallest member of $\mathcal{F}_i$, hence a translate of $\lambda(\mathcal{F}_i)K$ rather than of $a\lambda(\mathcal{F}_i)K$. The proof for this case appears, essentially, in [35, Lemma 13], and for the subcase of $\nu(\mathcal{F}) = 1$, in [17, item 7.9]; the generalization to infinite families $\mathcal{F}$ given here first appeared in [47].

Lemma 3.3.4. *Let K be a convex body in $\mathbb{R}^n$, and let A be any set in $\mathbb{R}^n$. The function*

$$\begin{aligned}[0,1] &\to \mathbb{N} \\ \lambda &\mapsto N((1-\lambda)K + \lambda A, K)\end{aligned}$$

is increasing.

Proof. Indeed, let $0 \le \lambda_1 \le \lambda_2 \le 1$, and set

$$t = 1 - \frac{\lambda_1}{\lambda_2} \in [0,1]\ .$$

Then

$$\begin{aligned}&N((1-\lambda_1)K + \lambda_1 A, K) \\ &= N\big((1-t)\big((1-\lambda_2)K + \lambda_2 A\big) + tK, (1-t)K + tK\big) \\ &\le N((1-t)((1-\lambda_2)K + \lambda_2 A), (1-t)K)\, N(tK, tK) \\ &= N((1-\lambda_2)K + \lambda_2 A, K)\ .\end{aligned}$$

□

Proposition 3.3.5. *Let $\triangle$ be an n-dimensional simplex. Then*

$$\frac{2^n}{\sqrt{\pi(n+\frac{1}{2})}} \le N(\triangle, -\triangle) \le 2^n(n\log n + n\log\log n + 5n)\ .$$

Proof. The upper bound is a direct application of Theorem 3.2.3. For the lower bound,

$$\begin{aligned}
N(-\triangle,\triangle) &\geq N(\tfrac{1}{2}(\triangle-\triangle),\triangle) && \text{(Lemma 3.3.4: } K, A := \triangle, -\triangle;\ \lambda = \tfrac{1}{2}, 1) \\
&\geq \frac{\operatorname{vol}(\frac{1}{2}(\triangle-\triangle))}{\operatorname{vol}(\triangle)} \\
&= \frac{1}{2^n}\binom{2n}{n} && \text{(see [49])} \\
&\geq \frac{2^n}{\sqrt{\pi(n+\frac{1}{2})}} && \text{(Lemma 4.2.4)}
\end{aligned}$$

□

Remark 3.3.6. Chakerian and Stein [14] gave the cruder estimate

$$N(\triangle,-\triangle) \geq \frac{\operatorname{vol}(\triangle)}{\max_{x\in\mathbb{R}^n}\operatorname{vol}(\triangle\cap(2x-\triangle))} \sim \frac{1}{\sqrt{3}}\left(\frac{e}{2}\right)^{n+1} .$$

(See Proposition 4.2.3.) In the terminology of chapter 4, Chakerian and Stein estimate $N(\triangle,-\triangle)$ using $m_{\mathrm{KB}}(\triangle)$ while the nearly sharp result of Proposition 3.3.5 uses $m_{\mathrm{DB}}(\triangle)$.

Example 3.3.7. Let $\triangle$ be an n-dimensional simplex. The observations of section 3.2 yield that if $\mathcal{F} = \{t+\triangle : t \in \triangle\}$ then $\nu(\mathcal{F}) = 1$ and $\tau(\mathcal{F}) = N(\triangle,-\triangle)$, which is roughly 2^n by Proposition 3.3.5.

3.4 VC-dimension

Theorem 3.4.1. *If $K \subseteq \mathbb{R}^2$ is a convex body and $\mathcal{F}$ is a family of positive homothets of K, then* $\operatorname{vcdim}(\mathcal{F}) \leq 3$.

Proof. Let $\mathcal{F}$ be a family of positive homothets of a convex body $K \subseteq \mathbb{R}^2$. Suppose, for contradiction, that $\mathcal{F}$ shatters some set of four points, say, $X = \{x_1, x_2, x_3, x_4\}$.

Case 1: One of the points of X is in the convex hull of the other three, say, $x_1 \in \operatorname{conv}\{x_2, x_3, x_4\}$. By hypothesis, there is an $F \in \mathcal{F}$ such that $X \cap F = \{x_2, x_3, x_4\}$. But since F is convex, it follows that $x_1 \in F$, which is a contradiction.

Case 2: The points of X are in convex position, forming the vertices of a convex quadrilateral in, say, the order $x_1x_2x_3x_4$. (See figure 3.1.) Without loss of generality, $X \cap K = \{x_1, x_3\}$ and $X \cap TK = \{x_2, x_4\}$, where $T\colon \mathbb{R}^2 \to \mathbb{R}^2, Tx = \lambda x + t$ is a homothety with ratio $\lambda \geq 1$.

First suppose $\lambda > 1$. Let

$$p = \frac{1}{1-\lambda}t ,$$

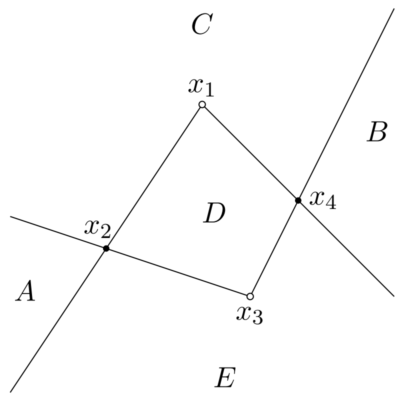


Figure 3.1: Theorem 3.4.1, Case 2.

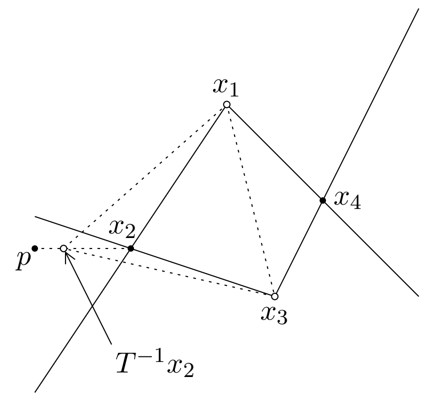


Figure 3.2: Why $p \notin A$.

which is the centre of the homothety T. If p is in the (closed) region A shown in figure 3.1, then $x_2 \in \operatorname{conv}\{x_1, x_3, p\}$. On the other hand, $T^{-1}x_2$ is a convex combination of p and x_2; thus $x_2 \in \operatorname{conv}\{x_1, x_3, T^{-1}x_2\}$. (See figure 3.2.) But $\{x_1, x_3, T^{-1}x_2\} \subseteq K$, so by convexity, $x_2 \in K$, a contradiction.

Similarly, if $p \in B$ then $x_4 \in \operatorname{conv}\{x_1, x_3, T^{-1}x_4\} \subseteq K$; if $p \in C \cup D$ then $x_3 \in \operatorname{conv}\{x_2, x_4, Tx_3\} \subseteq TK$; and if $p \in D \cup E$ then $x_1 \in \operatorname{conv}\{x_2, x_4, Tx_1\} \subseteq TK$. In all cases we obtain a contradiction.

The case $\lambda = 1$, when T is a translation, succumbs to essentially the same argument, with p an ideal point corresponding to the direction of the translation. We omit the details. □

Example 3.4.2. We construct a convex body $K \subseteq \mathbb{R}^3$ and a countable family $\mathcal{F}$ of translates of K such that $\operatorname{vcdim}(\mathcal{F}) = \infty$. (This example can, of course, be embedded in $\mathbb{R}^n$ for $n > 3$ as well.)

The construction is based on the example in figure 3.3, which shows a set whose translates shatter 4 points. The example is non-convex, and Theorem 3.4.1 shows that in the plane the non-convexity is essential. In $\mathbb{R}^3$, however, we can wrap such a set around a strictly convex surface (see figure 3.4), and then take the convex hull.

Now, we present Example 3.4.2. Let $\mathcal{E}$ be the family of all finite subsets of $\mathbb{N}$, and let $E\colon \mathbb{N} \to \mathcal{E}$ be a bijection. Set

$$A = \{(m, n) \in \mathbb{N}^2 \colon m \in E(n)\} \ .$$

For $m, n \in \mathbb{N}$, let $u_m = (\frac{1}{m}, 0, \frac{1}{m^2})$ and $v_n = (0, \frac{1}{n}, \frac{1}{n^2})$, and define

$$p\colon \mathbb{N}^2 \to \mathbb{R}^3 \ , \quad p(m, n) = u_m + v_n \ .$$

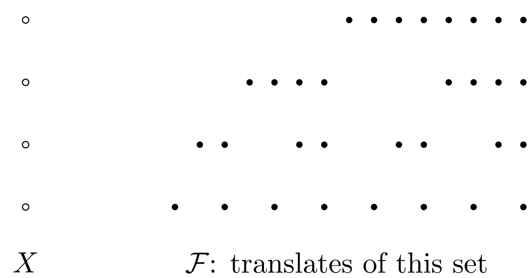


Figure 3.3: A (non-convex) set in the plane whose translates shatter four points.

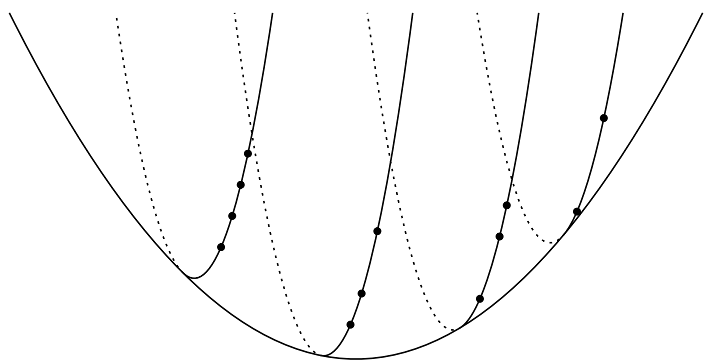


Figure 3.4: The paraboloid $z = x^2 + y^2$ and a few sections of it.

Let $K = \operatorname{conv} \operatorname{cl} p(A)$ and $\mathcal{F} = \{K - v_n \colon n \in \mathbb{N}\}$. We claim that $\operatorname{vcdim}(\mathcal{F}) = \infty$.

Let $P \subseteq \mathbb{R}^3$ be the paraboloid with equation $z = x^2 + y^2$. Since P is the boundary of a strictly convex set, $P \cap \operatorname{conv} S = S$ for any $S \subseteq P$. Since $p(\mathbb{N}^2)$ is a discrete set, $p(\mathbb{N}^2) \cap \operatorname{cl} S = S$ for any $S \subseteq p(\mathbb{N}^2)$. So if $T \subseteq p(\mathbb{N}^2)$, then

$$T \cap K = T \cap p(\mathbb{N}^2) \cap P \cap K = T \cap p(\mathbb{N}^2) \cap \operatorname{cl} p(A) = T \cap p(A) \ .$$

Now, let $M \in \mathbb{N}$, $X = \{u_1, \ldots, u_M\}$, and $X' \subseteq X$. Let $n \in \mathbb{N}$ be such that $X' = \{u_m \colon m \in E(n)\}$. Then

$$(X + v_n) \cap K = (X + v_n) \cap p(A) = X' + v_n \ ,$$

that is, $X \cap (K - v_n) = X'$. Thus $\mathcal{F}$ shatters X, so $\operatorname{vcdim}(\mathcal{F}) \geq M$.

Corollary 3.4.3. *There is a convex body $K \subseteq \mathbb{R}^3$ and a countable family $\mathcal{F}$ of translates of K such that* $\operatorname{vcdim}(\mathcal{F}^*) = \infty$.

Chapter 4

The harmonic mean measure of symmetry*

Abstract

We introduce a measure of (central) symmetry for a convex body K in $\mathbb{R}^n$ based on the volume of the harmonic mean of K and $-K$ and compare it to other classical volumetric measures of symmetry. We prove sharp inequalities between the volume of the harmonic mean of K and $-K$ and the volume of their intersection, and we show that, if K is in John's position, then the volume of the harmonic mean of K and $-K$ is minimized for the Euclidean ball and maximized for the simplex.

4.1 Introduction

A *measure of symmetry* is a functional on the space of convex bodies which is continuous with respect to the Hausdorff metric and invariant under invertible affine transformations, and which attains one of its extreme values exactly for centrally symmetric convex bodies. This definition was first made explicit by Grünbaum in his survey of the classical results on such functionals [29].

Three classical measures of symmetry arise by considering the volume of some combination of the body K and its negative $-K$: the difference body measure of symmetry

$$m_{\mathrm{DB}}(K) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(\frac{1}{2}K - \frac{1}{2}K)} \ ; \tag{4.1}$$

*A version of this chapter has been accepted for publication [58].

the Kovner–Besicovitch measure of symmetry

$$m_{\mathrm{KB}}(K) = \sup_{x\in\mathbb{R}^n} \frac{\mathrm{vol}((K-x)\cap(x-K))}{\mathrm{vol}(K)} \ ; \qquad (4.2)$$

and Estermann's measure of symmetry

$$m_{\mathrm{E}}(K) = \sup_{x\in\mathbb{R}^n} \frac{\mathrm{vol}(K)}{\mathrm{vol}(\mathrm{conv}((K-x)\cup(x-K)))} \ . \qquad (4.3)$$

We follow here the nomenclature proposed by Grünbaum. Some known results on these functionals are summarized in section 4.2.

We introduce the *harmonic mean measure of symmetry*, which is dual to the difference body measure of symmetry in the same way that the Kovner–Besicovitch measure of symmetry is dual to Estermann's:

$$m_{\mathrm{HM}}(K) = \sup_{x\in K} \frac{\mathrm{vol}(\frac{1}{2}\diamond(K-x)\,\widehat{+}\,\frac{1}{2}\diamond(x-K)))}{\mathrm{vol}(K)} \ ,$$

where $\diamond$ and $\widehat{+}$ denote *harmonic linear combination*, that is, linear combination of gauge functions:

$$\|\cdot\|_{\alpha\diamond K\widehat{+}\beta\diamond L} = \alpha\|\cdot\|_K + \beta\|\cdot\|_L \ .$$

Harmonic linear combinations were first considered by Steinhardt [54], then, apparently independently, by Firey [21], in a slightly more general form. Lutwak [41] treated them together with his dual mixed volumes [40] in an integrated dual Brunn–Minkowski theory. We follow the notation and terminology of Lutwak [41]. See also the thorough survey of Gardner [23, §18.9] and references therein.

We prove that, for any convex body K in $\mathbb{R}^n$,

$$\frac{\sqrt{\pi(n+1)}}{2^{n+1}} \le m_{\mathrm{HM}}(K) \le 1 \ , \qquad (4.4)$$

and that, for the simplex $\triangle$,

$$m_{\mathrm{HM}}(\triangle) = \frac{2^n n!}{(n+1)^n} \sim \sqrt{\frac{\pi n}{2}}\left(\frac{2}{e}\right)^{n+1} \qquad \text{as } n\to\infty.$$

Thus the range of values of the harmonic mean measure of symmetry is exponential in the dimension n, as is the case for the three classical measures of symmetry listed above.

We also prove the sharp volume inequalities

$$\mathrm{vol}(K \cap -K) \le \mathrm{vol}(\tfrac{1}{2} \diamond K \mathbin{\widehat{+}} \tfrac{1}{2} \diamond (-K)) \le (n+1)\,\mathrm{vol}(K \cap -K)\ . \tag{4.5}$$

The dual result, that if K contains the origin then

$$\frac{1}{n+1}\,\mathrm{vol}(\mathrm{conv}(K \cup -K)) \le \mathrm{vol}(\tfrac{1}{2}(K-K)) \le \mathrm{vol}(\mathrm{conv}(K \cup -K))\ , \tag{4.6}$$

is implicit in the work of Rogers and Shephard; see Remark 4.4.6 below. The factor $n+1$ in (4.5) and (4.6) is "small" in comparison to the exponential range of these phenomena, so these inequalities indicate that the bodies considered have roughly the same volume (for large n). In contrast, we give examples to show that, although the harmonic mean measure of symmetry and the Kovner–Besicovitch measure of symmetry are close (by (4.5)), and the difference body measure of symmetry and Estermann's measure of symmetry are close (by (4.6)), the former pair may differ from the latter pair by an exponential factor, in either direction.

It immediately follows from (4.5) that

$$m_{\mathrm{KB}}(K) \le m_{\mathrm{HM}}(K) \le (n+1)m_{\mathrm{KB}}(K)\ , \tag{4.7}$$

but it does not follow that the upper inequality is sharp. In fact, the example which shows the upper inequality in (4.5) is sharp relies on a badly centred convex body, while the definitions of m_{KB} and m_{HM} choose good centres. We conjecture that the sharp upper bound in (4.7) is of order $\sqrt{n}$, and give three pieces of evidence in favour of this conjecture: first, it is true for the simplex (see Remark 4.5.6); next, it is true for the analogous statement with expectations in place of suprema (see Remark 4.5.10); and finally, in a corollary of (4.5) for bodies in John's position, the factor $n+1$ can be improved to a factor of order $\sqrt{n}$ (see Remark 4.6.2).

Some of the results stated above for $K \mathbin{\widehat{+}} (-K)$ generalize to $K \mathbin{\widehat{+}_p} (-K)$ (see section 4.2 for definitions); in the interest of generality we define the *p-harmonic mean measure of symmetry*

$$m_{\mathrm{HM}}^{(p)}(K) = \sup_{x \in K} \frac{\mathrm{vol}(\frac{1}{2} \diamond (K-x) \mathbin{\widehat{+}_p} \frac{1}{2} \diamond (x-K))}{\mathrm{vol}(K)}\ ,$$

and prove results below for suitable ranges of p.

Section 4.2 introduces notation and summarizes the known results on the classical measures of symmetry. Section 4.3 proves the technical fact that the harmonic mean measure of symmetry is continuous. Section 4.4 proves the main volume inequality (4.5), and gives examples for which the difference body and harmonic mean measures of symmetry differ greatly. Section 4.5

proves that the volume of the harmonic mean (with a suitable exponent) is a concave function of the choice of centre, which yields (4.4) as well as the value of m_{HM} for the simplex. Section 4.6 gives a sharp volume-ratio-type theorem for $K \mathbin{\widehat{+}} (-K)$ when K is in John's position.

4.2 Notation and background

Let $\mathcal{K}^n$ denote the space of compact convex sets in $\mathbb{R}^n$, endowed with the Hausdorff metric δ^H. The radial function $\rho_K\colon S^{n-1} \to \mathbb{R}$ of a convex body K with $o \in \operatorname{int} K$ is given by

$$\rho_K(\theta) = \sup \left\{\lambda\colon \lambda\theta \in K\right\} ,$$

and its support functional $h_K\colon \mathbb{R}^n \to \mathbb{R}$ is

$$h_K(x) = \sup \left\{\langle x, y\rangle\colon y \in K\right\} .$$

Note that, for $\theta \in S^{n-1}$,

$$h_K(\theta) = \|\theta\|_{K^o} = \frac{1}{\rho_{K^o}(\theta)} . \tag{4.8}$$

For any convex body K in $\mathbb{R}^n$ with $o \in \operatorname{int} K$, and any $p > 0$, we have the standard volume formula

$$\int_K \|x\|_K^p \, dx = \frac{n}{n+p} \operatorname{vol}(K) . \tag{4.9}$$

Indeed,

$$\begin{aligned}
\int_K \|x\|_K^p \, dx &= \int_K \int_0^{\|x\|_K} pt^{p-1} \, dt \, dx \\
&= \int_0^1 \int_{K\setminus tK} pt^{p-1} \, dx \, dt = \int_0^1 pt^{p-1}(1-t^n) \operatorname{vol}(K) \, dt .
\end{aligned}$$

If K and L are convex sets containing the origin, then we define their p-Minkowski sum $K +_p L$, for $p \geq 1$, via

$$h_{K+_pL} = (h_K^p + h_L^p)^{1/p} .$$

We extend this definition as usual to $p = \infty$ by

$$h_{K+_\infty L} = \max(h_K, h_L) ,$$

which yields $K +_\infty L = \operatorname{conv}(K \cup L)$. The case $p = 1$ is the usual Minkowski sum; the generalization to $p \geq 1$ is due to Firey [22], who obtained the analogue of the Brunn–Minkowski inequality, namely

$$\operatorname{vol}(K +_p L)^{p/n} \geq \operatorname{vol}(K)^{p/n} + \operatorname{vol}(L)^{p/n} .$$

Lutwak [42] developed a full Brunn–Minkowski-type theory for such combinations. See also the survey of Gardner [23, §18.3] and references therein.

The p-harmonic linear combination $\alpha \diamond K \mathbin{\widehat{+}_p} \beta \diamond L$ is defined by

$$\|\cdot\|_{\alpha\diamond K \widehat{+}_p \beta\diamond L} = (\alpha\|\cdot\|_K^p + \beta\|\cdot\|_L^p)^{1/p} .$$

Note that $K \mathbin{\widehat{+}_p} L = (K^o +_p L^o)^o$ and that $K \mathbin{\widehat{+}_\infty} L = K \cap L$ (again using the usual convention for this case). Harmonic linear combinations satisfy a dual Brunn–Minkowski inequality

$$\operatorname{vol}(\alpha \diamond K \mathbin{\widehat{+}_p} \beta \diamond L)^{-p/n} \geq \alpha \operatorname{vol}(K)^{-p/n} + \beta \operatorname{vol}(L)^{-p/n} . \tag{4.10}$$

Standard inequalities involving pth powers yield that $K \mathbin{\widehat{+}_p} L$ is increasing in p and $\frac{1}{2} \diamond K \mathbin{\widehat{+}_p} \frac{1}{2} \diamond L$ is decreasing in p. In particular,

$$K \cap L \subseteq \tfrac{1}{2} \diamond K \mathbin{\widehat{+}_p} \tfrac{1}{2} \diamond L \subseteq 2^{1/p}(K \cap L) . \tag{4.11}$$

These inclusions immediately yield the volume estimates

$$\operatorname{vol}(K \cap -K) \leq \operatorname{vol}(\tfrac{1}{2} \diamond K \mathbin{\widehat{+}_p} \tfrac{1}{2} \diamond (-K)) \leq 2^{n/p} \operatorname{vol}(K \cap -K) ,$$

which are refined below for $p < n$ in Theorem 4.4.5.

The difference body measure of symmetry (4.1) satisfies

$$\frac{2^n}{\binom{2n}{n}} \leq m_{\mathrm{DB}}(K) \leq 1 .$$

The upper bound is an instance of the Brunn–Minkowski inequality; we have equality there exactly when K is centrally symmetric. The lower bound is a well-known result of Rogers and Shephard [49], who also showed that we have equality there exactly when K is a simplex.

For Estermann's measure of symmetry (4.3), we have

$$\frac{1}{2^n} \leq m_{\mathrm{E}}(K) \leq 1 .$$

The upper bound is by inclusion, and we have equality there exactly when K is centrally symmetric. The lower bound is a consequence of a result of

Rogers and Shephard [50], who actually showed that

$$\frac{1}{2^n} \leq \inf_{x\in K} \frac{\mathrm{vol}(K)}{\mathrm{vol}(\mathrm{conv}((K-x)\cup(x-K)))} \ . \tag{4.12}$$

(Bianchini and Colesanti [10] recently proved a result analogous to (4.12) for the p-difference body $K +_p (-K)$ in the case $n = 2$.) Note that, since $K - K$ is invariant under translations of K, the inequalities (4.6) imply that, for any K, the infimum in (4.12) and the supremum in the definition of m_{E} differ at most by the small factor $n + 1$.

The exact lower bound for m_{E} is not known, but Fáry and Rédei [20] showed that for the n-dimensional simplex $\triangle$ we have

$$\sup_{x\in\mathbb{R}^n} \frac{\mathrm{vol}(\triangle)}{\mathrm{vol}(\mathrm{conv}((\triangle-x)\cup(x-\triangle)))} = \frac{1}{\binom{n}{\lfloor n/2\rfloor}} \sim \frac{\sqrt{\pi n/2}}{2^n} \qquad \text{as } n\to\infty,$$

so the Rogers and Shephard bound on m_{E} is nearly sharp.

For the Kovner–Besicovitch measure of symmetry (4.2), we have

$$\frac{1}{2^n} \leq m_{\mathrm{KB}}(K) \leq 1 \ . \tag{4.13}$$

The upper bound is by inclusion, and we have equality there exactly when K is centrally symmetric. The lower bound is due to Stein [53]. The exact lower bound is not known, but it must be exponential in n since for the n-dimensional simplex $\triangle$ we have the following asymptotic result, which essentially goes back to Laplace. The argument is very briefly sketched in [13, §11]; we give a more detailed proof for the reader's convenience.

Lemma 4.2.1. *Let X be a uniform random variable in $[-\frac{1}{2},\frac{1}{2}]$, let $(X_i)_1^n$ be i.i.d. copies of X, let $S_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n X_i$, and let p_n be the probability density function of S_n. Then, for any $x\in\mathbb{R}$,*

$$\lim_{n\to\infty} p_n(x) = \sqrt{\frac{6}{\pi}} e^{-6x^2} \ .$$

Proof. The characteristic function of X is $\varphi_X(t) = \mathbb{E}e^{itX} = \mathrm{sinc}(\frac{t}{2})$. By the central limit theorem,

$$\mathrm{sinc}\left(\frac{t}{2\sqrt{n}}\right)^n = \varphi_{S_n}(t) \to e^{-t^2/24}$$

pointwise as $n\to\infty$. We will show that if $n \geq 2$ then

$$|\varphi_{S_n}(t)| \leq 1 \wedge \frac{12}{t^2} \tag{4.14}$$

and so φ_{S_n} (for such n) is integrable. By the Fourier inversion theorem and dominated convergence,

$$p_n(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\varphi_{S_n}(t)\,dt \to \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}e^{-t^2/24}\,dt = \sqrt{\frac{6}{\pi}}\,e^{-6x^2}\ ,$$

pointwise as $n \to \infty$.

It remains to show (4.14). First, if $|t| \le \pi$ then

$$\begin{aligned}\operatorname{sinc}(t) &= \prod_{k=1}^{\infty}\Big(1 - \frac{t^2}{\pi^2 k^2}\Big) && \text{(see, e.g., [2, eq. (4.6)])}\\ &\le \exp\Big(-\frac{t^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\Big)\\ &= e^{-t^2/6}\ .\end{aligned}$$

So, if $|t| \le 2\pi\sqrt{n}$ then

$$\operatorname{sinc}\left(\frac{t}{2\sqrt{n}}\right)^n \le (e^{-(t/2\sqrt{n})^2/6})^n = e^{-t^2/24} \le 1 \wedge \frac{24}{et^2} \le 1 \wedge \frac{12}{t^2}\ .$$

Suppose, on the other hand, that $|t| > 2\pi\sqrt{n}$. Then

$$\left|\operatorname{sinc}\Big(\frac{t}{2\sqrt{n}}\Big)\right|^n \le \left|\frac{2\sqrt{n}}{t}\right|^n \le \left(\frac{1}{\pi}\right)^n \le 1\ .$$

Furthermore, if $n \ge 2$ then

$$\left|\operatorname{sinc}\Big(\frac{t}{2\sqrt{n}}\Big)\right|^n \le \left|\frac{2\sqrt{n}}{t}\right|^n = \left|\frac{2\sqrt{n}}{t}\right|^{n-2}\frac{4n}{t^2} \le \left(\frac{1}{\pi}\right)^{n-2}\frac{4n}{t^2} \le \frac{12}{t^2}\ ,$$

which completes the proof of (4.14). □

Lemma 4.2.2 (Fáry and Rédei [20])**.** *For any convex bodies K and L in $\mathbb{R}^n$, the function $x \mapsto \operatorname{vol}_n(K \cap (x+L))^{1/n}$ is concave on its support.*

Proposition 4.2.3. *Let $\triangle^n$ denote a simplex in $\mathbb{R}^n$. Then*

$$m_{KB}(\triangle^n) \sim \sqrt{3}\left(\frac{2}{e}\right)^{n+1} \qquad \textit{as } n \to \infty.$$

Proof. We may assume $\triangle^n$ is a regular simplex with centroid at the origin; let its size be chosen later. By Lemma 4.2.2, the map $x \mapsto \operatorname{vol}(\triangle^n \cap (2x - \triangle^n))$ attains its supremum $m_{\mathrm{KB}}(\triangle^n)$ at a point fixed under all affine symmetries of $\triangle^n$, that is, at the origin. Now, let $v = (1, 1\ldots, 1) \in \mathbb{R}^{n+1}$, let $H = \{x \in$

$\mathbb{R}^{n+1}\colon \langle x, v\rangle = 0\}$, and write $\mathbb{R}_+^{n+1}$ for the positive orthant in $\mathbb{R}^{n+1}$. The orthant $\mathbb{R}_+^{n+1}$ is a cone whose cross-sections by hyperplanes parallel to H are homothets of $\triangle^n$. Direct calculations show that $(\mathbb{R}_+^{n+1} - \frac{1}{2}v) \cap H$ has circumradius $\frac{1}{2}\sqrt{n(n+1)}$; we may assume $\triangle^n$ has this size. Thus

$$\tfrac{1}{2}B_\infty^{n+1} \cap H = ((\mathbb{R}_+^{n+1} - \tfrac{1}{2}v) \cap H) \cap ((\tfrac{1}{2}v - \mathbb{R}_+^{n+1}) \cap H)$$

is congruent to $\triangle^n \cap -\triangle^n$, and so

$$\begin{aligned} m_{\mathrm{KB}}(\triangle^n) &= \frac{\mathrm{vol}(\triangle^n \cap -\triangle^n)}{\mathrm{vol}(\triangle^n)} = \frac{\mathrm{vol}(\frac{1}{2}B_\infty^{n+1} \cap H)}{\mathrm{vol}((\mathbb{R}_+^{n+1} - \frac{1}{2}v) \cap H)} \\ &= \frac{2^n n!}{(n+1)^{n+1/2}} \mathrm{vol}(\tfrac{1}{2}B_\infty^{n+1} \cap H) \sim \sqrt{\frac{\pi}{2}} \left(\frac{2}{e}\right)^{n+1} \mathrm{vol}(\tfrac{1}{2}B_\infty^{n+1} \cap H) \end{aligned}$$

as $n \to \infty$. Finally, in the notation of Lemma 4.2.1,

$$\mathrm{vol}(\tfrac{1}{2}B_\infty^{n+1} \cap H) = p_n(0) \to \sqrt{\frac{6}{\pi}} \ ,$$

yielding the desired result. □

We will also make use of the following concrete numerical estimates on central binomial coefficients, which can be extracted from a traditional proof of Stirling's formula; for the reader's convenience we sketch the argument.

Lemma 4.2.4. *For any $n \geq 1$, we have*

$$\frac{4^n}{\sqrt{\pi(n+\frac{1}{2})}} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{\pi n}} \ .$$

Proof. Let $I_n = \int_0^\pi \sin^n(x)\,dx$. Integrating by parts yields $I_n = \frac{n-1}{n} I_{n-2}$; by induction, $I_{2n} = \pi\binom{2n}{n}/4^n$ and $I_{2n+1} = 4^n/\binom{2n}{n}(n+\frac{1}{2})$. The desired inequalities then follow from $I_{2n+1} \leq I_{2n} \leq I_{2n-1} = \frac{2n+1}{2n} I_{2n+1}$. □

4.3 Continuity

We consider $\mathcal{K}^n$ with the Hausdorff metric δ^H.

Lemma 4.3.1. *Let $K, L \in \mathcal{K}^n$. If $o \in \mathrm{int}\, K$ and $o \in \mathrm{int}\, L$ then*

$$\delta^H(K^o, L^o) \leq \frac{\delta^H(K, L)}{r(K)r(L)} \ ,$$

where $r(K) = \inf_{\theta \in S^{n-1}} \rho_K(\theta)$ is the inradius of K (with respect to the origin).

Proof. Note that, since $\lambda B_2^n \subseteq K$ if and only if $K^o \subseteq \frac{1}{\lambda} B_2^n$,

$$r(K) = \frac{1}{\sup_{\theta \in S^{n-1}} \rho_{K^o}(\theta)} = \inf_{\theta \in S^{n-1}} h_K(\theta) \ . \tag{4.15}$$

Thus

$$\begin{aligned} \delta^H(K^o, L^o) &\le \sup_{\theta \in S^{n-1}} |\rho_{K^o}(\theta) - \rho_{L^o}(\theta)| \\ &\le \frac{1}{r(K) r(L)} \sup_{\theta \in S^{n-1}} |h_K(\theta) - h_L(\theta)| \quad \text{(by (4.15) and (4.8))} \\ &= \frac{\delta^H(K, L)}{r(K) r(L)} \qquad \text{(see [52, Theorem 1.8.11])} \quad \square \end{aligned}$$

Lemma 4.3.2. *If $K \in \mathcal{K}^n$ and $o \in \partial K$, then* $\dim(K \mathbin{\widehat{+}_p} (-K)) < n$.

Proof. By standard separation theorems, there exists a halfspace H with $o \in \partial H$ and $K \subseteq H$. Then $K \mathbin{\widehat{+}_p} (-K) \subseteq H \mathbin{\widehat{+}_p} (-H) = H \cap -H$. (We abuse notation slightly here, since H is not a convex body and doesn't have the origin in its interior, but formal application of the definitions yields that $\|x\|_H = 0$ if $x \in H$ and $\|x\|_H = \infty$ if $x \notin H$, and $H \mathbin{\widehat{+}_p} (-H)$ can then be interpreted following the usual conventions for arithmetic involving ∞.) $\square$

Lemma 4.3.3. *For any compact set C, the set $\{K \in \mathcal{K}^n : C \subseteq \operatorname{int} K\}$ is open in $\mathcal{K}^n$.*

Proof. Let $K \in \mathcal{K}^n$ satisfy $C \subseteq \operatorname{int} K$ and let $(K_m)_{m=1}^\infty$ be a sequence in $\mathcal{K}^n$ such that $K_m \to K$. Since C is compact, there exists $\varepsilon > 0$ such that $C + \varepsilon B_2^n \subseteq K$. For all sufficiently large m, we have $K \subseteq K_m + \frac{\varepsilon}{2} B_2^n$, and so $C + \varepsilon B_2^n \subseteq K_m + \frac{\varepsilon}{2} B_2^n$; since Minkowski addition is cancellative for convex sets, it follows that $\operatorname{conv} C + \frac{\varepsilon}{2} B_2^n \subseteq K_m$, and in particular, $C \subseteq \operatorname{int} K_m$. $\square$

Proposition 4.3.4. *The function $f \colon \mathcal{K}^n \to \mathbb{R}$ given by*

$$f(K) = \begin{cases} \operatorname{vol}(K \mathbin{\widehat{+}_p} (-K)) & \text{if } o \in K, \\ 0 & \text{if } o \notin K \end{cases}$$

is continuous.

Proof. The set $\{K \in \mathcal{K}^n : o \in \operatorname{int} K\}$ is open in $\mathcal{K}^n$ by Lemma 4.3.3, and on this set the function is

$$f(K) = \operatorname{vol}((K^o +_p (-K)^o)^o) \ ,$$

which is a composition of continuous functions, namely volume [52, §1.8], p-Minkowski combinations [22], and taking polars (Lemma 4.3.1).

The set $\{K \in \mathcal{K}^n \colon o \notin K\}$ is open in $\mathcal{K}^n$ (since it is the preimage of the open interval $(0, \infty)$ under the continuous function $K \mapsto \operatorname{dist}(o, K)$), and f vanishes on this set.

It remains to show that f is continuous at any $K \in \mathcal{K}^n$ with $o \in \partial K$. Let H be a hyperplane supporting K at o. Let $(K_m)_{m=1}^{\infty}$ be a sequence in $\mathcal{K}^n$ with $K_m \to K$. If $o \notin K_m$ then $f(K_m) = 0$ by the definition of f. If $o \in \partial K_m$ then $f(K_m) = 0$ by Lemma 4.3.2. If $o \in \operatorname{int} K_m$ then

$$\begin{aligned} K_m \,\widehat{+}_p\, (-K_m) &\subseteq K_m \cap -K_m \\ &\subseteq (K + \delta^H(K_m, K) B_2^n) \cap (H + \delta^H(K_m, K) B_2^n) \ . \end{aligned}$$

By the continuity of measure for decreasing sequences of sets with finite measure,

$$\lim_{\delta \downarrow 0} \operatorname{vol}((K + \delta B_2^n) \cap (H + \delta B_2^n)) = \operatorname{vol}(K \cap H) = 0 \ .$$

Thus $f(K_m) \to 0$, which is as desired since $f(K) = 0$ by Lemma 4.3.2. □

Proposition 4.3.5. $m_{HM}^{(p)}$ *is continuous.*

Proof. Define $f \colon \mathbb{R}^n \times \mathcal{K}^n \to \mathbb{R}$ by

$$f(x, K) = \begin{cases} \operatorname{vol}((K - x) \,\widehat{+}_p\, (x - K)) / \operatorname{vol}(K) & \text{if } x \in K, \\ 0 & \text{if } x \notin K. \end{cases}$$

Then f is continuous by Proposition 4.3.4, and we wish to show that the function

$$m_{\mathrm{HM}}^{(p)} = \sup_{x \in \mathbb{R}^n} f(x, \cdot)$$

is continuous. Let $K_0 \in \mathcal{K}^n$, and set $G = \{K \in \mathcal{K}^n \colon \delta^H(K, K_0) \le 1\}$. Note that G is a neighbourhood of K_0 and compact by the Blaschke selection theorem. Now, if $K \in G$ and $f(x, K) \neq 0$ then $x \in K \subseteq K_0 + B_2^n$; so $f|_{\mathbb{R}^n \times G}$ is supported on the compact set $(K_0 + B_2^n) \times G$. Therefore $f|_{\mathbb{R}^n \times G}$ is uniformly continuous, and so $m_{\mathrm{HM}}^{(p)}$ is continuous on G, hence continuous at K_0. □

4.4 Intersection of two cones

Definition 4.4.1. For any set $A \subseteq \mathbb{R}^n$ and any nonzero p, define

$$\operatorname{cone}_p(A) = \{(x, t) \in \mathbb{R}^n \oplus \mathbb{R} \colon x \in (1 + t)^{1/p} A \text{ and } t \ge -1\} \ .$$

For a convex body K with the origin in its interior, $\mathrm{cone}_p(K)$ is the epigraph of the function $\|\cdot\|_K^p - 1$.

Lemma 4.4.2. *For any convex bodies K and L in $\mathbb{R}^n$ with $o \in \mathrm{int}(K \cap L)$,*

$$\mathrm{vol}_n(K \mathbin{\widehat{+}_p} (-L)) = \frac{\frac{n}{p}+1}{2^{\frac{n}{p}+1}} \mathrm{vol}_{n+1}(\mathrm{cone}_p(K) \cap -\mathrm{cone}_p(L)) \ .$$

Proof. Since

$$(x,t) \in \mathrm{cone}_p(K) \cap -\mathrm{cone}_p(L) \iff \|x\|_K^p - 1 \le t \le 1 - \|x\|_{-L}^p \ ,$$

the one-dimensional section of $\mathrm{cone}_p(K) \cap -\mathrm{cone}_p(L)$ by the line $\{(x,t) \colon t \in \mathbb{R}\}$ is a line segment of length

$$(2 - \|x\|_K^p - \|x\|_{-L}^p)^+ = 2(1 - \|x\|^p_{\frac{1}{2}\diamond K \widehat{+}_p \frac{1}{2}\diamond(-L))})^+$$

(where a^+ denotes $\max(a,0)$). By Fubini's theorem and (4.9),

$$\begin{aligned}\mathrm{vol}_{n+1}(\mathrm{cone}_p(K) \cap -\mathrm{cone}_p(L)) &= 2\int_{2^{1/p}(K \widehat{+}_p(-L))} (1 - \|x\|^p_{\frac{1}{2}\diamond K \widehat{+}_p \frac{1}{2}\diamond(-L)})\,dx \\ &= \frac{2p}{n+p}\mathrm{vol}_n(\tfrac{1}{2}\diamond K \mathbin{\widehat{+}_p} \tfrac{1}{2}\diamond(-L)) = \frac{2p}{n+p}\mathrm{vol}_n(2^{1/p}(K \mathbin{\widehat{+}_p} (-L))) \ ,\end{aligned}$$

as desired. (See figure 4.1.) □

Remark 4.4.3. The body $\mathrm{cone}_1(K) \cap -\mathrm{cone}_1(L)$ is dual to one considered by Rogers and Shephard [50]. Indeed, one can check that if K is a convex body in $\mathbb{R}^n$ with $o \in K$, then

$$(K \times \{-1\})^o = \mathrm{cone}_1(K^o)$$

(where the polar on the left is taken in $\mathbb{R}^{n+1}$ and that on the right in $\mathbb{R}^n$). So, for bodies as in Lemma 4.4.2,

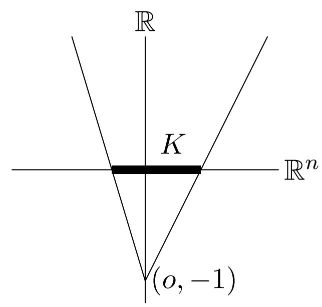
$$(\mathrm{cone}_1(K) \cap -\mathrm{cone}_1(L))^o = \mathrm{conv}\left((K^o \times \{-1\}) \cup -(L^o \times \{-1\})\right) \ ,$$

which is essentially $C(K^o, L^o)$ in Rogers and Shephard's notation.

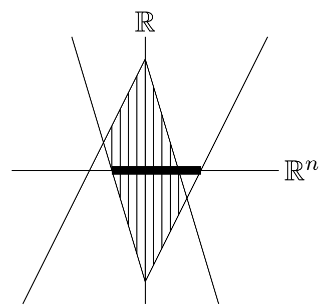
Example 4.4.4. For any simplex $\triangle$ in $\mathbb{R}^n$, we have

$$\mathrm{vol}((\triangle - \triangle)^o)\,\mathrm{vol}(\triangle) = \frac{n+1}{n!} \ .$$

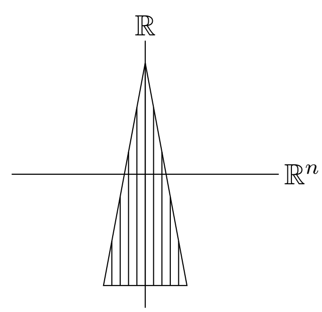
Indeed, the left-hand side is invariant under affine transformations, so we may assume $\triangle$ is a regular simplex, centred at the origin and with cir-



(a) $\text{cone}_1(K)$.



(b) $\text{cone}_1(K) \cap -\text{cone}_1(K)$.



(c) A cone of height 2 with base $\frac{1}{2} \diamond K \mathbin{\widehat{+}} \frac{1}{2} \diamond (-K)$.

Figure 4.1: The idea of Lemma 4.4.2, in the case $p = 1$, $K = L$. Lining up the "vertical" fibers of the set in (b) yields the cone in (c).

cumradius $\sqrt{n}$. Then $\triangle^o = -\triangle$ and $\mathrm{cone}_1(\triangle)$ is congruent to an orthant in $\mathbb{R}^{n+1}$. (Recall that the convex hull of the standard basis vectors in $\mathbb{R}^{n+1}$ is a regular n-dimensional simplex.) Thus $\mathrm{cone}_1(\triangle) \cap -\mathrm{cone}_1(\triangle)$ is an $(n+1)$-dimensional cube with main diagonal of length 2, hence volume $2^{n+1}(n+1)^{-(n+1)/2}$. By Lemma 4.4.2,

$$\mathrm{vol}((\triangle - \triangle)^o) = \mathrm{vol}(\triangle \,\widehat{+}\, (-\triangle)) = \frac{1}{(n+1)^{(n-1)/2}} \ .$$

Finally,

$$\mathrm{vol}(\triangle) = \frac{(n+1)^{(n+1)/2}}{n!} \ .$$

(See [16, §8.8] and note that, by the equality case of Jung's theorem [11], a regular simplex with circumradius 1 has edge length $\sqrt{2(n+1)/n}$.)

Theorem 4.4.5. *For any convex body K in $\mathbb{R}^n$, and any $p \in [1, \infty]$,*

$$\mathrm{vol}(K \cap -K) \leq \mathrm{vol}(\tfrac{1}{2} \diamond K \,\widehat{+}_p\, \tfrac{1}{2} \diamond (-K)) \leq \min(\tfrac{n}{p} + 1, 2^{n/p})\, \mathrm{vol}(K \cap -K) \ .$$

Proof. The lower inequality, and the upper inequality with $2^{n/p}$, follow from the inclusions (4.11). For the upper inequality with $\frac{n}{p} + 1$, take $K = L$ in Lemma 4.4.2 and write $H_t = \{(x, t) \colon x \in \mathbb{R}^n\}$. Then $\mathrm{cone}_p(K) \cap -\mathrm{cone}_p(K)$ is an o-symmetric convex body in $\mathbb{R}^{n+1}$, whose section by the hyperplane H_0 is congruent to $K \cap -K$; by Brunn's theorem and Lemma 4.4.2,

$$\begin{aligned}
\mathrm{vol}_n(K \cap -K) &= \max_{t \in \mathbb{R}} \mathrm{vol}_n(\mathrm{cone}_p(K) \cap -\mathrm{cone}_p(K) \cap H_t) \\
&\geq \frac{1}{2} \int_{-1}^{1} \mathrm{vol}_n(\mathrm{cone}_p(K) \cap -\mathrm{cone}_p(K) \cap H_t)\, dt \\
&= \frac{1}{2} \mathrm{vol}_{n+1}(\mathrm{cone}_p(K) \cap -\mathrm{cone}_p(K)) \\
&= \frac{2^{\frac{n}{p}}}{\frac{n}{p} + 1} \mathrm{vol}_n(K \,\widehat{+}_p\, (-K)) \ .
\end{aligned}$$

□

Remark 4.4.6. The case $p = 1$ of Theorem 4.4.5 has a dual version: if K is a convex body in $\mathbb{R}^n$ which contains the origin, then

$$\frac{1}{n+1} \mathrm{vol}(\mathrm{conv}(K \cup -K)) \leq \mathrm{vol}(\tfrac{1}{2}(K - K)) \leq \mathrm{vol}(\mathrm{conv}(K \cup -K)) \ .$$

The upper inequality is by inclusion. The lower inequality is implicit in [50]; for the reader's convenience we give the proof. Define the $(n+1)$-dimensional body

$$C(K) = \mathrm{conv}((K \times \{1\}) \cup -(K \times \{1\})) \ .$$

Let $V = \operatorname{span}\{e_{n+1}\} \subset \mathbb{R}^{n+1}$. Note that since K contains the origin, $C(K) \cap V$ is a line segment of length 2, that $C(K) \cap V^\perp$ is congruent to $\frac{1}{2}(K - K)$, and that the orthogonal projection of $C(K)$ onto $V^\perp$ is congruent to $\operatorname{conv}(K \cup -K)$. Therefore

$$\begin{aligned}
&\frac{2}{n+1} \operatorname{vol}_n(\operatorname{conv}(K \cup -K)) \\
&\leq \operatorname{vol}_{n+1}(C(K)) && \text{(by [50, Theorem 1])} \\
&= \int_{-1}^{1} \operatorname{vol}_n(C(K) \cap (V^\perp + t e_{n+1}))\, dt \\
&\leq 2 \operatorname{vol}_n(C(K) \cap V^\perp) && \text{(by Brunn's theorem)} \\
&= 2 \operatorname{vol}_n(\tfrac{1}{2}(K - K))
\end{aligned}$$

which yields the desired inequality.

Example 4.4.7. In the case $p = 1$, the value $n + 1$ in the upper inequality of Theorem 4.4.5 is sharp. Indeed, let $x = (1, 1, \ldots, 1) \in \mathbb{R}^n$ and $0 \leq \lambda < 1$, and take $K = B_\infty^n + \lambda x$. Then, by Lemma 4.4.2,

$$\begin{aligned}
\operatorname{vol}_n(\tfrac{1}{2} \diamond K \,\widehat{+}\, \tfrac{1}{2} \diamond (-K)) &= \frac{n+1}{2} \operatorname{vol}_{n+1}(\operatorname{cone}_1(K) \cap -\operatorname{cone}_1(K)) \\
&= (n+1) \int_0^1 \operatorname{vol}_n((1+t)K \cap (1-t)(-K))\, dt \\
&= 2^n (n+1) \int_0^1 (1 - \max(\lambda, t))^n\, dt \\
&= 2^n (1-\lambda)^n (\lambda n + 1) \\
&= (\lambda n + 1) \operatorname{vol}_n(K \cap -K) \ .
\end{aligned}$$

Letting $\lambda \to 1$ demonstrates the desired result.

As noted in the introduction, it follows from Theorem 4.4.5 that, for any convex body K in $\mathbb{R}^n$,

$$m_{\mathrm{KB}}(K) \leq m_{\mathrm{HM}}(K) \leq (n+1) m_{\mathrm{KB}}(K) \ , \tag{4.16}$$

which since m_{KB} and m_{HM} have exponential range (see (4.13) and Corollary 4.5.9) means that m_{KB} and m_{HM} are roughly equal. In contrast, the following examples show that m_{DB} and m_{HM} may differ by an exponential factor, in either direction.

Example 4.4.8. As noted in section 4.2,

$$m_{\mathrm{KB}}(\triangle) \sim \sqrt{3} \left(\frac{2}{e}\right)^{n+1} \qquad \text{and} \qquad m_{\mathrm{DB}}(\triangle) = \frac{2^n}{\binom{2n}{n}} \sim \frac{\sqrt{\pi n}}{2^n}$$

as $n \to \infty$, where $\triangle$ denotes the n-dimensional simplex. So, in this case, the difference body measure of symmetry is exponentially smaller than the Kovner–Besicovitch measure of symmetry. (It follows by (4.16) that the difference body measure of symmetry is also exponentially smaller than the harmonic mean measure of symmetry; we compute $m_{\mathrm{HM}}(\triangle)$ explicitly below, as Example 4.5.5.)

The reverse example requires the following simple lemma.

Lemma 4.4.9. *The functions m_{KB} and m_{DB} are multiplicative, that is, if K is a convex body in $\mathbb{R}^m$ and L is a convex body in $\mathbb{R}^n$ then for the convex body $K \times L$ in $\mathbb{R}^{m+n}$ we have $m_{KB}(K \times L) = m_{KB}(K)m_{KB}(L)$ and $m_{DB}(K \times L) = m_{DB}(K)m_{DB}(L)$.*

Example 4.4.10. Let K be the pentagon

$$K = \mathrm{conv}\ \{(-1,-1),(-1,1),(1,1),(5,0),(1,-1)\}\ .$$

Then $\mathrm{vol}(K) = 8$ and $\mathrm{vol}(\frac{1}{2}K - \frac{1}{2}K) = 10$, so $m_{\mathrm{DB}}(K) = \frac{4}{5}$. To compute $m_{\mathrm{KB}}(K)$, by Lemma 4.2.2 and symmetry we need only consider x of the form te_1 for some $t \in \mathbb{R}$. Let $F(t) = K \cap (2te_1 - K)$ and $f(t) = \mathrm{vol}_2(F(t))$. Then F is increasing (in the sense that $s \le t$ implies $F(s) \subseteq F(t)$) on $(-\infty, 0]$ and decreasing (in the analogous sense) on $[2,\infty)$, so we need only consider $t \in [0,2]$. Direct computations show that

$$f(t) = \begin{cases} -2t^2 + 4t + 4 & \text{if } t \in [0,1] \\ -\frac{3}{2}t^2 + 3t + \frac{9}{2} & \text{if } t \in [1,2] \end{cases}$$

This function attains its maximum at $t = 1$, so $m_{\mathrm{KB}}(K) = \frac{1}{8}f(1) = \frac{3}{4}$.

So, for this K in $\mathbb{R}^2$,

$$m_{\mathrm{KB}}(K) < m_{\mathrm{DB}}(K)\ .$$

For any $n \ge 2$, let L_n be the convex body in $\mathbb{R}^n$ which is the Cartesian product of $\lfloor \frac{n}{2} \rfloor$ copies of K (and a line segment, if n is odd). By Lemma 4.4.9,

$$\frac{m_{\mathrm{DB}}(L_n)}{m_{\mathrm{KB}}(L_n)} = \left(\frac{m_{\mathrm{DB}}(K)}{m_{\mathrm{KB}}(K)}\right)^{\lfloor n/2 \rfloor}\ ,$$

and so in this case the difference body measure of symmetry is exponentially larger than the Kovner–Besicovitch measure of symmetry (and hence, by (4.16), exponentially larger than the harmonic mean measure of symmetry).

This contruction gives an exponential factor c^n with $c = (15/16)^{1/2} \approx 0.9682$. The slightly better constant $c = ((5\sqrt{5}+11)/24)^{1/2} \approx 0.9613$ can be obtained by replacing $(5,0)$ in the definition of K with $(3+2\sqrt{5},0)$, at the cost of a slightly more complicated computation.

4.5 A body of Rogers and Shephard

Definition 4.5.1. Given a set $A \subseteq \mathbb{R}^n$, define

$$G(A) = \{(x,y,t) \in \mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R} \colon \\ x+y \in (1+t)A \text{ and } -x+y \in (1-t)A \text{ and } -1 \le t \le 1\} \ .$$

The body $G(A)$ was, up to an affine transformation, introduced by Rogers and Shephard [50], whose computations yield, for convex sets K,

$$\operatorname{vol}_{2n+1}(G(K)) = \frac{2^{n+1}(n!)^2}{(2n+1)!} \operatorname{vol}_n(K)^2 \ . \tag{4.17}$$

Proposition 4.5.2. *For any convex body K in $\mathbb{R}^n$ and any $y \in K$,*

$$\operatorname{vol}_n((K-y) \mathbin{\widehat{+}} (y-K)) = \frac{n+1}{2^{n+1}} \operatorname{vol}_{n+1}(G(K) \cap V_y) \ ,$$

where $V_y = \{(x,y,t) \in \mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R} \colon x \in \mathbb{R}^n, t \in \mathbb{R}\}$.

Proof. First consider the case $y \notin \operatorname{int} K$. In this case, the left-hand side is zero by Lemma 4.3.2. On the other hand, $G(K) \cap V_y \neq \varnothing$ if and only if $y \in K$, that is, the orthogonal projection of $G(K)$ onto $V_y^\perp$ is $\{o\} \times K \times \{o\}$. It therefore follows from $y \notin \operatorname{int} K$ that $V_y \cap \operatorname{int} G(K) = \varnothing$, and so $\operatorname{vol}_{n+1}(G(K) \cap V_y) = 0$.

Now consider the case $y \in \operatorname{int} K$. Define

$$\Phi \colon \mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R} \to \mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R} \ , \qquad (x,y,t) \mapsto (x+ty, y, t) \ .$$

Note that $\Phi(V_y) = V_y$ and that, on V_y, Φ acts as a shear and so preserves volumes. Note also that $(x,t) \in \operatorname{cone}_1(K-y) \cap -\operatorname{cone}_1(K-y)$ if and only if $\Phi(x,y,t) \in G(K)$, and so

$$\begin{aligned} &\operatorname{vol}_n(\tfrac{1}{2} \diamond (K-y) \mathbin{\widehat{+}} \tfrac{1}{2} \diamond (y-K)) \\ &= \frac{n+1}{2} \operatorname{vol}_{n+1}(\operatorname{cone}_1(K-y) \cap -\operatorname{cone}_1(K-y)) \quad \text{(by Lemma 4.4.2)} \\ &= \frac{n+1}{2} \operatorname{vol}_{n+1}(\Phi^{-1}(G(K)) \cap V_y) \\ &= \frac{n+1}{2} \operatorname{vol}_{n+1}(\Phi^{-1}(G(K) \cap V_y)) \\ &= \frac{n+1}{2} \operatorname{vol}_{n+1}(G(K) \cap V_y) \ . \end{aligned}$$

□

Corollary 4.5.3. *Let K be a convex body in $\mathbb{R}^n$. The function*

$$\begin{aligned} K &\to \mathbb{R} \\ x &\mapsto \mathrm{vol}_n((K-x) \mathbin{\widehat{+}} (x-K)) \end{aligned}$$

is $\frac{1}{n+1}$-concave.

Proof. By Proposition 4.5.2 and Brunn's theorem. □

Remark 4.5.4. When K is the Euclidean ball or the simplex, we can improve the exponent in Corollary 4.5.3 to $\frac{1}{n}$, as follows. First, note that the expression

$$\mathrm{vol}((L-L)^o)\,\mathrm{vol}(L)$$

is invariant under invertible affine transformations of the convex body L. Write $K^x = (K-x)^o$. When K is the Euclidean ball or the simplex, the sets K^x for $x \in \mathrm{int}\,K$ are all affine, and so the value of

$$\mathrm{vol}((K-x) \mathbin{\widehat{+}} (x-K))\,\mathrm{vol}(K^x) = \mathrm{vol}((K^x - K^x)^o)\,\mathrm{vol}(K^x)$$

is constant for $x \in \mathrm{int}\,K$. But as noted by Aleksandrov [1], $\mathrm{vol}(K^x)$ is a $(-\frac{1}{n})$-concave function of x. (This is also an immediate consequence of the dual Brunn–Minkowski inequality: in (4.10), take $p = 1$ and $K, L := K^x, K^y$.) It follows that $\mathrm{vol}((K-x) \mathbin{\widehat{+}} (x-K))$ is $\frac{1}{n}$-concave.

Example 4.5.5. Let $\triangle$ be an n-dimensional simplex in $\mathbb{R}^n$ with centroid at the origin. By concavity, the expression $\mathrm{vol}((\triangle - x) \mathbin{\widehat{+}} (x - \triangle))$ attains its maximum value at a point which is fixed under all affine symmetries of $\triangle$. The only such point is the centroid of $\triangle$, so

$$m_{\mathrm{HM}}(\triangle) = \frac{\mathrm{vol}((\frac{1}{2}\triangle^o - \frac{1}{2}\triangle^o)^o)}{\mathrm{vol}(\triangle)} = \frac{2^n(n+1)}{n!\,\mathrm{vol}(\triangle)\,\mathrm{vol}(\triangle^o)} = \frac{2^n n!}{(n+1)^n}\ .$$

(See Example 4.4.4.)

Remark 4.5.6. As noted in section 4.2, for the n-dimensional simplex $\triangle$ we have

$$m_{\mathrm{KB}}(\triangle) \sim \sqrt{3}\left(\frac{2}{e}\right)^{n+1} \qquad \text{as } n \to \infty,$$

while Example 4.5.5 and Stirling's formula yield

$$m_{\mathrm{HM}}(\triangle) \sim \sqrt{\frac{\pi n}{2}}\left(\frac{2}{e}\right)^{n+1} \qquad \text{as } n \to \infty.$$

Thus, for the simplex, these measures of symmetry are in ratio of order $\sqrt{n}$, as conjectured for all convex bodies in the introduction.

Corollary 4.5.7. *Let* $(x_0, y_0, t_0) \in \mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R}$ *be the centroid of* $G(K)$*; then* $y_0 \in K$ *and*

$$\mathrm{vol}((K - y_0) \,\widehat{+}\, (y_0 - K)) \geq \binom{2n+1}{n}^{-1} \mathrm{vol}(K) \ .$$

Proof. As shown by Milman and Pajor [45, Lemma 1], the volume of a section of a convex body through its centroid is at least as large as the average volume of all parallel sections. Thus, using the notation V_y introduced in Proposition 4.5.2,

$$\begin{aligned} \mathrm{vol}_{n+1}(G(K) \cap V_{y_0}) &\geq \frac{1}{\mathrm{vol}(K)} \int_K \mathrm{vol}_{n+1}(G(K) \cap V_y)\, dy \\ &= \frac{\mathrm{vol}_{2n+1}(G(K))}{\mathrm{vol}_n(K)} = \frac{2^{n+1}(n!)^2}{(2n+1)!} \mathrm{vol}_n(K) \end{aligned}$$

by (4.17). The desired result follows by Proposition 4.5.2. □

Remark 4.5.8. Due to the symmetries of $G(K)$, in Corollary 4.5.7 we in fact have $x_0 = o$ and $t_0 = 0$.

Corollary 4.5.9. *For any convex body* K *in* $\mathbb{R}^n$*,*

$$\frac{\sqrt{\pi(n+1)}}{2^{n+1}} \leq m_{HM}(K) \leq 1 \ .$$

Proof. The upper inequality follows from the dual Brunn–Minkowski inequality: in (4.10), take $p = 1$ and $L = -K$. The lower inequality follows from Corollary 4.5.7 and the observation that

$$\frac{2^n}{\binom{2n+1}{n}} = \frac{2^{n+1}}{\binom{2n+2}{n+1}} \geq \frac{\sqrt{\pi(n+1)}}{2^{n+1}} \ .$$

(For the inequality, see Lemma 4.2.4.) □

Remark 4.5.10. Corollary 4.5.9 improves slightly the estimate $m_{\mathrm{HM}}(K) \geq \frac{1}{2^n}$, which follows from combining (4.16) with Stein's result (4.13). Stein in fact showed that if X is a random vector uniformly distributed in K then

$$\mathbb{E}\, \mathrm{vol}(K \cap (X - K)) = \frac{1}{2^n} \mathrm{vol}(K) \ ;$$

similarly the proof of Corollary 4.5.7 yields

$$\mathbb{E}\, \mathrm{vol}(\tfrac{1}{2} \diamond (K - X) \,\widehat{+}\, \tfrac{1}{2} \diamond (X - K)) \sim \frac{\sqrt{\pi n}}{2} \cdot \frac{1}{2^n} \mathrm{vol}(K) \qquad \text{as } n \to \infty.$$

Thus the average volumes of these two functions of X are asymptotically in ratio of order $\sqrt{n}$, as conjectured for the suprema m_{KB} and m_{HM} in the introduction.

4.6 Bodies in John's position

Let K be a convex body in $\mathbb{R}^n$. If K is in John's position (see section 2.2), then so is $K \cap -K$; by Theorem 2.2.3, $\mathrm{vol}(K \cap -K) \leq 2^n$, and Theorem 4.4.5 then yields

$$\mathrm{vol}(\tfrac{1}{2} \diamond K \,\widehat{+}\, \tfrac{1}{2} \diamond (-K)) \leq 2^n(n+1) \ . \tag{4.18}$$

This inequality is, however, not sharp; we obtain an exact inequality for this situation by exploiting Lemma 2.2.4.

Theorem 4.6.1. *Let K be a convex body in $\mathbb{R}^n$. If K is in John's position, then*

$$\frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})} \leq \mathrm{vol}(\tfrac{1}{2} \diamond K \,\widehat{+}\, \tfrac{1}{2} \diamond (-K)) \leq \frac{2^n n^{n/2}}{(n+1)^{(n-1)/2}} \ ,$$

with equality on the left if and only if $K = B_2^n$ and equality on the right if and only if K is a simplex.

Proof. If K is in John's position then $K \supseteq B_2^n$; setting $L = -K$ in (4.11) and taking volumes yields the lower inequality (and its equality case).

For the upper inequality, let $(u_i)_{i=1}^m$ and $(c_i)_{i=1}^m$ be as in John's theorem, and let $\widetilde{K} = \{u_1, \ldots, u_m\}^o$. Also let $(v_i)_{i=1}^m$ and $(d_i)_{i=1}^m$ be as in Lemma 2.2.4, and let $V = \{v_1, \ldots, v_m\}$. A direct computation shows that

$$V^o = \sqrt{n+1}\,\mathrm{cone}_1(n^{-1/2}\widetilde{K}) \ . \tag{4.19}$$

Therefore

$$\begin{aligned}
&\mathrm{vol}_n(\tfrac{1}{2} \diamond K \,\widehat{+}\, \tfrac{1}{2} \diamond (-K)) \\
&\leq \mathrm{vol}_n(\tfrac{1}{2} \diamond \widetilde{K} \,\widehat{+}\, \tfrac{1}{2} \diamond (-\widetilde{K})) && \text{(since } K \subseteq \widetilde{K}\text{)} \\
&= \frac{n+1}{2} \cdot \frac{n^{n/2}}{(n+1)^{(n+1)/2}} \mathrm{vol}_{n+1}(V^o \cap -V^o) && \text{(by (4.19) and Lemma 4.4.2)} \\
&\leq \frac{n+1}{2} \cdot \frac{n^{n/2}}{(n+1)^{(n+1)/2}} \cdot 2^{n+1} && \text{(by Theorem 2.2.3)}
\end{aligned}$$

(For the last step, note that $V^o \cap -V^o$ is, by construction, an origin-symmetric convex body in John's position in $\mathbb{R}^{n+1}$.) This proves the upper inequality.

Next, if K is a simplex in John's position, then $K = \widetilde{K}$, and the contact points satisfy

$$\langle u_i, u_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ -1/n & \text{if } i \neq j. \end{cases}$$

A direct computation then shows that $(v_i)_{i=1}^m$ is an orthonormal basis, and so $V^o \cap -V^o$ is an $(n+1)$-dimensional cube. Therefore we have equality in the upper inequality.

Finally, suppose we have equality in the upper inequality. Then $K = \widetilde{K}$ and $\mathrm{vol}(V^o \cap -V^o) = 2^{n+1}$. By the equality case of Theorem 2.2.3, the set $V^o \cap -V^o$ is a cube circumscribed about B_2^{n+1}. Since the v_i are unit vectors, it follows that $V \cup -V$ consists of the $2(n+1)$ contact points between the cube and B_2^{n+1}. Since V and $-V$ are disjoint (indeed, they are separated by $\{(x, 0) \in \mathbb{R}^n \oplus \mathbb{R} \colon x \in \mathbb{R}^n\}$), it follows that V contains exactly $n+1$ points, that is, $m = n + 1$. Therefore $\widetilde{K}$ is a simplex, and so is K. $\square$

Remark 4.6.2. The fact that the factor $n + 1$ in (4.18) can be strengthened to the factor in Theorem 4.6.1 (which is asymptotically $\sqrt{n/e}$ as $n \to \infty$) is suggestive evidence in favour of our conjecture that $m_{\mathrm{HM}}(K) \leq c\sqrt{n}\, m_{\mathrm{KB}}(K)$.

Chapter 5

Conclusions

Chapter 2 concerns the problem of determining $\sup_K d_{\mathrm{BM}}(K, B_\infty^3)$, where the supremum is over all origin-symmetric convex bodies K in $\mathbb{R}^3$; Theorem 2.3.4 states a new upper bound for this quantity. Although the method of Theorem 2.3.4 could obviously be extended — say, by trying to choose three nearly orthogonal contact points — it does not seem likely that such an approach would yield sharp estimates for $d_{\mathrm{BM}}(K, B_\infty^3)$. Indeed, the method of Theorem 2.3.4 involves attempting to align the facets of the cube B_∞^3 with the facets of K, but the example of the Platonic dodecahedron shows that for some bodies this approach yields very suboptimal positions: it is far better to align the facets of the cube with the edges of dodecahedron, as in Example 2.4.4.

An intriguing by-product of the proof of Theorem 2.3.4 is the connection between John configurations and systems of equiangular lines established in Proposition 2.4.2. Further exploration of this connection could yield insight into the conditions under which Gerzon's bound, Theorem 2.4.1, is sharp.

Chapter 3 considers two problems on the complexity of families of positive homothets of a convex body. The first is to bound the transversal number of such a family in terms of the independence number and the dimension. The dependence on the dimension is known to be exponential; Theorem 3.3.2 gives the best known base, $3 + 2\sqrt{2}$, for the upper bound, and Example 3.3.7 gives the best known base, 2, for the lower bound. The major open problem here is to close this gap.

The analysis of Example 3.3.7 raises a tangential but very interesting question: for which convex body is $N(K, -K)$ maximal? Proposition 3.3.5 gives strong asymptotic evidence in favour of the obvious conjecture that it is the simplex. (Note that the upper bound in Proposition 3.3.5 holds for all K.)

The second problem considered in chapter 3 was to determine the maximal VC-dimension of a family of homothets of a convex body. It is well-known that the family of all convex sets has infinite VC-dimension; our Example 3.4.2 shows that this is true also for the much simpler families considered in this chapter. It seems, then, that the pleasant consequences of finite VC-dimension (such as Vapnik and Červonenkis' uniform law of large numbers) are not typically available even for quite simple families of convex sets.

Chapter 4 introduces the harmonic mean measure of symmetry and establishes its fundamental properties. The most salient open problem is whether the simplex the least centrally symmetric convex body according to this measure of symmetry. A positive answer to that question would immediately yield, by (4.16), substantial progress on the analogous question for the Kovner–Besicovitch measure of symmetry; on the other hand, a counterexample would be a striking violation of the intuition that the simplex is in all reasonable senses the least centrally symmetric convex body. One possible line of inquiry is based on the observation that, writing

$$H(K) = \tfrac{1}{2} \diamond K \mathbin{\widehat{+}} \tfrac{1}{2} \diamond (-K) \ ,$$

we have

$$H((1-\lambda) \diamond K \mathbin{\widehat{+}} \lambda \diamond L) = (1-\lambda) \diamond H(K) \mathbin{\widehat{+}} \lambda \diamond H(L) \ .$$

Taking volumes yields, by (4.10), that $\mathrm{vol}(H(K))$ is a $\frac{1}{n}$-concave function of K, in the sense of harmonic convex combinations. What are the extreme points of $\mathcal{K}^n$ in the sense of harmonic convex combinations?

A few other questions arise naturally from chapter 4. First, what is the correct dependence on the dimension in (4.16)? In particular, is it the case that

$$m_{\mathrm{HM}}(K) \le c\sqrt{n}\, m_{\mathrm{KB}}(K) \ ,$$

as conjectured in the remarks after (4.7)? Second, can the exponent in Corollary 4.5.3 be improved to $\frac{1}{n}$, as in Remark 4.5.4? Third, do exponential lower bounds such as that in Corollary 4.5.7 hold for any natural central point of K, such as its centroid, its Santaló point, or the centre of its John ellipsoid?

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