UNIVERSITY OF ALBERTA

OSCILLATION RESULTS FOR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

by

QINGKAI KONG ©

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ABSTRACT

This thesis is mainly concerned with the oscillation theory of the neutral differential equations of the form

$$\frac{d}{dt}\left[y(t)+g\Big(t,y\big(t-r_1(t)\big),\ldots,y\big(t-r_\ell(t)\big)\Big)\right] + f\Big(t,y(t),y\big(t-\tau_1(t)\big),\ldots,y\big(t-\tau_m(t)\big)\Big)=0.$$

Some new criteria for oscillation and for existence of nonoscillatory solutions are obtained. An asymptotic analysis for a class of nonlinear equations is also obtained.

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1. Introduction

A neutral delay differential equation (NDDE) is a differential equation in which the highest order derivative of the unknown function appears in the equation both with and without delays. This thesis is mainly concerned with the oscillation theory of neutral delay differential equations of the form

$$\frac{d}{dt}\left[y(t)+g\Big(t,y\big(t-r_1(t)\big),\ldots,y\big(t-r_\ell(t)\big)\Big)\right] + f\Big(t,y(t),y\big(t-\tau_1(t)\big),\ldots,y\big(t-\tau_m(t)\big)\Big)=0 \qquad (1.1.1)$$

where for some $\widetilde{t}_0 \in \mathbb{R}$ and some positive integer n

$$\begin{aligned} &g \in C([\widetilde{t}_0,\infty)\times\mathbb{R}^n\times\cdots\times\mathbb{R}^n,\mathbb{R}^n) \\ &f \in C([\widetilde{t}_0,\infty)\times\mathbb{R}^n\times\cdots\times\mathbb{R}^n,\mathbb{R}^n), \end{aligned} \qquad (1.1.2)$$

and for $i=1,\ldots,\ell$ and $j=1,\ldots,m$

$$\begin{aligned} &r_i,\ \tau_j \in C\big([\widetilde{t}_0,\infty),(0,\infty)\big) \quad \text{and} \\ &\lim_{t\to\infty}[t-r_i(t)]=\infty=\lim_{t\to\infty}[t-\tau_j(t)]. \end{aligned} \qquad (1.1.3)$$

When $g \equiv 0$, (1.1.1) is usually called a delay differential equation (DDE). Since Sturm (1836) introduced the concept of oscillation when he studied the problem of heat transmission, oscillation theory has been an important area of research in the qualitative theory of ordinary differential equations (ODEs). Oscillation theory for NDDEs is a natural extension of that for ODEs, while certain known results in the oscillation theory for ODEs carry over to NDDEs. Therefore, some background in the oscillation theory for ODEs is essential for understanding the oscillation theory for NDDEs. Beside its theoretical interest, the study of oscillatory and asymptotic behavior of solutions of NDDEs is of some importance in applications. NDDEs appear in networks containing lossless transmission lines, in the study of vibrating masses attached to an elastic bar and also in the study of population dynamics, see [1-4,6,7] and the references cited therein.

However, the oscillation theory of DDEs and NDDEs is much different from that of ODEs. For instance, while all linear ODEs of first order exhibit the nonoscillatory property, the linear NDDEs of first order may have complicated oscillatory behavior. A simple example of this is that of the equation $y'(t) + y(t - \frac{\pi}{2}) = 0$ which has oscillatory solutions $y = \sin t$ and $y = \cos t$. In general, the delays appearing in the unknown function and its derivatives have the tendency to cause oscillation. Since the oscillation theory of NDDEs presents some new problems that are not relevant to the corresponding ODEs, a study of the oscillation and nonoscillation caused by time delays is most interesting.

For DDEs, i.e., the special case of NDDE (1.1.1) with $g \equiv 0$, the oscillation theory has extensively developed, see [5] and [8] and the references cited therein. For general NDDEs, the research on oscillation is also active, and the oscillation theory has rapidly developed, but it is still not well-established, see [5] and the references cited therein.

In this thesis some recent work on the oscillation of NDDEs will be presented. Chapters 2 and 3 deal with linear scalar NDDEs of first order; some new necessary and sufficient conditions for oscillation are obtained for the autonomous case, and a number of new conditions and comparison results are obtained for the nonautonomous case. Chapter 4 deals with nonlinear scalar NDDEs of second order. In Chapters 5 and 6, we give some explicit conditions for oscillation of linear systems of NDDEs of higher order, either autonomous, or nonautonomous. Chapter 7 is mainly concerned with the nonoscillatory aspect of a class of nonlinear NDDEs in which the asymptotic behavior of solutions is thoroughly investigated.

1.2. Definition and Some Basic Theorems

The content of this section is mainly extracted from [5] and [7].

Consider equation (1.1.1) under the assumptions (1.1.2) and (1.1.3). For a given initial point $t_0 \geq \tilde{t}_0$, we now define t_{-1} to be

$$t_{-1} = \min\left\{\min_{1\leq i\leq \ell}\left\{\inf_{t\geq t_0}\{t - r_i(t)\}\right\}, \min_{1\leq j\leq m}\left\{\inf_{t\geq t_0}\{t - \tau_j(t)\}\right\}\right\}. \tag{1.2.1}$$

With Eq. (1.1.1) we associate the initial conditions

$$x(t) = \varphi(t) \quad \text{for} \quad t_{-1} \leq t \leq t_0 \tag{1.2.2}$$

where $\varphi : [t_{-1}, t_0] \to \mathbb{R}^n$ is a given initial function.

DEFINITION 1.2.1: A function y is said to be a solution of the initial value problem (1.1.1) and (1.2.2) on the interval I, where I is of the form $[t_0, T)$ for $T \leq \infty$, if $y : [t_{-1}, t_0] \cup I \to \mathbb{R}^n$ is continuous, $y(t) + g\Big(t, y(t - r_1(t)), \ldots, y(t - r_\ell(t))\Big)$ is continuously differentiable for $t \in I$ and y satisfies Eq. (1.1.1) for all $t \in I$.

REMARK 1.2.1. In this thesis, since we deal with oscillation theory, a solution of (1.1.1) will be understood as a solution existing on the interval $[t_0, \infty)$ for some $t_0 \geq \widetilde{t}_0$.

The following is the basic global existence and uniqueness theorem for neutral delay differential systems.

THEOREM 1.2.1. *Assume*

i) (1.1.2) and (1.1.3) are satisfied, and $t_0 \geq \widetilde{t}_0$ *and* $\varphi \in C([t_{-1}, t_0], \mathbb{R}^n)$ *are given;*

ii) for every compact set $D \subset [\widetilde{t}_0, \infty) \times \mathbb{R}^{(m+1)n}$ *there exists a constant* $K = K(D) > 0$ *such that*

$$\|f(t, x_0, x_1, \ldots, x_m) - f(t, y_0, y_1, \ldots, y_m)\| \leq K\|x_0 - y_0\|$$

for all $(t, x_0, x_1, \ldots, x_m), (t, y_0, y_1, \ldots, y_m) \in D$.

Then the initial value problem (1.1.1) and (1.2.2) has exactly one solution in the interval $[t_0, T)$ *of existence.*

REMARK 1.2.2.

(i) If the function f in Eq. (1.1.1) has the form

$$f\Big(t, y(t-\tau_1(t)), \ldots, y(t-\tau_m(t))\Big),$$

then condition (i) of Theorem 1.2.1 implies the existence and uniqueness of the solution of (1.1.1) and (1.2.2).

(ii) Any linear NDDE with continuous coefficients satisfies the condition of Theorem 1.2.2.

There are many ways in which one can define the concept of oscillation of solutions. For purposes of this thesis we shall use only the following definitions.

DEFINITION 1.2.2: Let Eq. (1.1.1) be a scalar equation. Then a solution y of (1.1.1) is said to be oscillatory if y has arbitrarily large zeros. Otherewise, y is called nonoscillatory.

DEFINITION 1.2.3: Let Eq. (1.1.1) be a system of NDDEs. Then a solution $y = (y_1, \ldots, y_n)^T$ is said to be oscillatory if every component y_i, $i = 1, \ldots, n$, of the solution y has arbitrarily large zeros. Otherwise, y is called nonoscillatory.

DEFINITION 1.2.4: Let Eq. (1.1.1) be a system of NDDEs. Then a solution $y = (y_1, \ldots, y_n)^T$ is said to be oscillatory if it is eventually trivial or at least one component does not have eventually constant signum (positive, negative, or zero). Otherwise, y is called nonoscillatory.

DEFINITION 1.2.5: Eq. (1.1.1) is said to be oscillatory in the sense of Definition 1.2.2, 1.2.3, or 1.2.4, if all solutions of (1.1.1) are oscillatory in the sense of Definition 1.2.2, 1.2.3, or 1.2.4, respectively.

From the above definitions, the following remarks are obvious.

REMARK 1.2.3.

(i) Eq. (1.1.1) is nonoscillatory, according to Definition 1.2.2, iff there exist a solution $y(t)$ and a $t_0 \geq \tilde{t}_0$ such that $y(t) > 0$ (resp. < 0) for some $i \in \{1, \ldots, n\}$ and $t \geq t_0$; Eq. (1.1.1) is nonoscillatory, according to Definition 1.2.3, iff there exist a solution $y(t)$ and a $t_0 \geq \tilde{t}_0$ such that $y_i(t) > 0$ (resp. < 0) for some $i \in \{1, \ldots, n\}$ and $t \geq t_0$; Eq. (1.1.1) is nonoscillatory, according to Definition 1.2.4, iff there exist an eventually nontrivial solution $y(t)$ and a $t_0 \geq \tilde{t}_0$ such that $y_i(t)$ has eventually constant signum for all $i = 1, \ldots, n$ and $t \geq t_0$.

(ii) Although Definitions 1.2.3 and 1.2.4 are different, they coincide with Definition 1.2.2 when Eq. (1.1.1) is a scalar equation.

In the sequel, oscillation for linear, homogeneous and autonomous systems will be understood in the sense of Definition 1.2.3, and oscillation for other cases will be understood in the sense of Definition 1.2.4. This is because with Definition 1.2.4 it is "easier" to obtain results for oscillation.

For a later application we mention a result on exponential boundedness of solutions of the following linear system of NDDEs

$$\frac{d}{dt}\left[y(t)+\sum_{i=1}^{\ell}P_i y(t-r_i)\right]+Q_0(t)y(t)+\sum_{j=1}^{m}Q_j(t)y(t-\tau_j)=0. \tag{1.2.3}$$

The following theorem is a simple extension of Theorem 1.2.1 in [5] and Theorem 7.3 in [7, Chapter 1] for the autonomous case. The proof is similar so we omit it here.

THEOREM 1.2.2. *Assume*

i) $P_i(i=1,\dots,\ell)\in\mathbb{R}^{n\times n}$, $Q_j(j=0,\dots,m)\in C([\tilde{t}_0,\infty),\mathbb{R}^{n\times n})$ *and* $\|Q_j(t)\|\le k$ *for some constant* k *and* $t\ge\tilde{t}_0$;

ii) $r_i(i=1,\dots,\ell)$ *and* $\tau_j(j=1,\dots,m)$ *are positive constants.*

Let $y(t)$ *be a solution of Eq. (1.2.3) on* $[\tilde{t}_0,\infty)$. *Then there exist positive constants* M *and* α *such that*

$$\|y(t)\|\le Me^{\alpha t}\quad\text{for}\quad t\ge\tilde{t}_0.$$

Although the above discussion is for the first order NDDE (1.1.1), for higher order NDDEs, there are parallel concepts to Definitions 1.2.1 - 1.2.5, and there are similar results to Theorems 1.2.1 and 1.2.2. We do not mention the details here.

References

[1] R.K. Brayton, *Nonlinear oscillations in a distributed network,* Quart. Appl. Math. **XXIV** (1967), 289-301.

[2] R.K. Brayton and R.A. Willoughby, *On the numerical integration of a symmetric system of difference-differential equations of neutral type,* J. Math. Anal. Appl. **18** (1967), 182-189.

[3] R.D. Driver, *A mixed neutral system,* Nonlinear Analysis: TMA, **8** (1984), 155-158.

[4] K. Gopalsamy and B.G. Zhang, *On a neutral delay logistic equation,* Dynamics and Stability of Systems, **2** (1988), 183-195.

[5] I. Györi and G. Ladas, "Oscillation Theory of Delay Differential Equations with Applications", Clarendon Press, Oxford, 1991.

[6] Y. Kuang and A. Feldstein, *Boundedness of solutions of a nonlinear nonautonomous neutral delay equation,* J. Math. Anal. Appl., **156** (1991), 293-304.

[7] J.K. Hale, "Theory of Functional Differential Equations," Springer-Verlag, New York, 1977.

[8] G.S. Ladde, V. Lakshmikantham, and B.G. Zhang, "Oscillation Theory of Differential Equations with Deviating Arguments," Marcel Dekker, New York, 1987.

CHAPTER 2

OSCILLATION OF LINEAR EQUATIONS OF FIRST ORDER

— Autonomous Case

2.1. Introduction

In this chapter we consider the scalar linear first order NDDE with constant coefficients of the form

$$\frac{d}{dt}[y(t) - \sum_{i=1}^{n} p_i y(t - r_i)] + \sum_{j=1}^{m} q_j y(t - \tau_j) = 0 \tag{2.1.1}$$

where p_i $(i = 1, \ldots, n)$, q_j, τ_j $(j = 1, \ldots, m) \in \mathbb{R}_+$, $r_i > 0$ $(i = 1, \ldots, n)$. As mentioned in Section 1.2, a function $y(t)$ is said to be oscillatory on $[t_0, \infty)$ if it has arbitrarily large zeros. Eq. (2.1.1) is said to be oscillatory if all solutions of (2.1.1) are oscillatory.

A number of authors have made contributions to the following result, see, for example, O.Arino and I.Győri [1], [5], and the references cited in [5].

RESULT 2.1.1.. *Eq. (2.1.1) is oscillatory if and only if its characteristic equation*

$$F(\lambda) = \lambda(1 - \sum_{i=1}^{n} p_i e^{-\lambda r_i}) + \sum_{j=1}^{m} q_j e^{-\lambda \tau_j} = 0 \tag{2.1.2}$$

has no real roots, or equivalently, $F(\lambda) > 0$ *for all* $\lambda \in \mathbb{R}$.

But to determine whether Eq. (2.1.2) has a real root is quite a problem itself. To the best of the authors' knowledge, until now there are very few explicit conditions to guarantee the oscillation of Eq. (2.1.1) for the general case. Motivated by W. Huang [5] which gives an alternative necessary and sufficient condition for oscillation of delay differential equations, we derive some new necessary and sufficient conditions for the oscillation of Eq. (2.1.1) for the case that $n = 2$ which can be easily applied to get some explicit sufficient conditions. The reasons that we discuss the case that $n = 2$ are: i) the criteria for $n = 2$ are much different from those for $n = 1$; ii) the discussion for the general case is tedius and is of a similar idea. Here we should mention that even for the special case where p_1 or p_2 vanishes, the sufficient conditions obtained here are still the best so far in the literature, see section 2.4 and [2–4,7–9]. For the case $p_1 = p_2 = 0$, our results coincide with the result given by W. Huang [6].

In the folowing we consider the equation

$$\frac{d}{dt}[y(t) - p_1 y(t - r_1) - p_2 y(t - r_2)] + \sum_{j=1}^{m} q_j y(t - \tau_j) = 0 \qquad (2.1.3)$$

with its characteristic equation

$$F(\lambda) = \lambda(1 - p_1 e^{-\lambda r_1} - p_2 e^{-\lambda r_2}) + \sum_{j=1}^{m} q_j e^{-\lambda \tau_j} = 0. \qquad (2.1.4)$$

2.2. Necessary and sufficient conditions

From (2.1.4) we see that if $p_1+p_2=1$ then every solution of Eq. (2.1.3) oscillates.

For the case that $p_1+p_2<1$, we define a subset D of the space l^1 and a function f on D as follows:

$$D=\{\ell=(\ell_{ijk}):(ijk)\in I_1,\ \ell_{ijk}\geq 0,\ \Sigma_1\ \ell_{ijk}=1\}$$

and

$$f(\ell)=\Sigma_1\ \frac{\ell_{ijk}}{ir_1+(k-i)r_2+\tau_j} \times \ln\left(eC_k^i p_1^i p_2^{k-i} q_j[ir_1+(k-i)r_2+\tau_j]/\ell_{ijk}\right) \tag{2.2.1}$$

where $I_1=\{(ijk):\ j=1,\ldots,m;\ k=0,1,\ldots;\ i=0,\ldots,k\}$, $\Sigma_1=\sum_{(ijk)\in I_1}$, and in (2.2.1) $\ell_{ijk}=0$ implies the corresponding terms vanish. Here $C_k^i=\frac{k!}{i!(k-i)!}$.

It is easy to see that the series in (2.2.1) is convergent in D and then $f(\ell)$ is well-defined on D. In fact, for $\ell\in D$

$$\Sigma_1\ \frac{\ell_{ijk}}{ir_1+(k-i)r_2+\tau_j}\ln\left(eC_k^i p_1^i p_2^{k-i} q_j[ir_1+(k-i)r_2+\tau_j]\right)<\infty$$

by the comparison test and the definition of D. Also

$$|\Sigma_1 \frac{\ell_{ijk}}{ir_1+(k-i)r_2+\tau_j} \ln \ell_{ijk}| \\ \leq \left(\Sigma_1 \frac{1}{(ir_1+(k-i)r_2+\tau_j)^2}\right)^{1/2} \left(\Sigma_1 \ell_{ijk}^2 (\ln \ell_{ijk})^2\right)^{1/2},$$

the right hand side is convergent for $\ell \in D$.

Theorem 2.2.1. *Assume* $p_1+p_2<1$. *Then*

i) $f(\ell)$ *achieves its maximum at a point* $\ell^0=(\ell_{ijk}^0)$ *on* D *which is completely determined by the condition that*

$$\frac{1}{ir_1+(k-i)r_2+\tau_j} \ln \left(C_k^i p_1^i p_2^{k-i} q_j [ir_1+(k-i)r_2+\tau_j]/\ell_{ijk}^0\right) \qquad (2.2.2)$$

has the same value for all $(ijk) \in I_1$, $\ell_{ijk}^0 \neq 0$.

ii) Eq. (2.1.3) is oscillatory if and only if $f(\ell^0)>0$.

For the case that $p_1+p_2>1$, we define the following three subsets D_b, $b=1,2,3$, of the space l^1, and three functions $h_b(S)$ defined on D_b, $b=1,2,3$, respectively: (We have in the expansions below $C_{-(k+1)}^i = \frac{(-k-1)(-k-2)\dots(-k-i)}{i!}$, $i,k=1,2,\dots$.)

$$D_1 = \{S=(S_{ijk}):(ijk)\in I_2,\ S_{ijk}C_{-(k+1)}^i[(k+i+1)r_1-ir_2-\tau_j]\geq 0, \\ \Sigma_2 S_{ijk}=1\},$$

$$D_2 = \{S = (S_{ijk}) : (ijk) \in I_2,\ S_{ijk} C^i_{-(k+1)} [(k+i+1)r_2 - ir_1 - \tau_j] \geq 0,$$
$$\Sigma_2 S_{ijk} = 1\},$$

$$D_3 = \{S = (S_{jk}) : (j,k) \in I_3,\ S_{jk}[(k+1)r_1 - \tau_j] \geq 0,\ \Sigma_3 S_{jk} = 1\},$$

$$h_1(S) = \Sigma_2 \frac{S_{ijk}}{(k+i+1)r_1 - ir_2 - \tau_j} \times \ln\left(eC^i_{-(k+1)} p_1^{-(k+i+1)} p_2^i q_j [(k+i+1)r_1 - ir_2 - \tau_j]/S_{ijk}\right), \tag{2.2.3}$$

$$h_2(S) = \Sigma_2 \frac{S_{ijk}}{(k+i+1)r_2 - ir_1 - \tau_j} \times \ln\left(eC^i_{-(k+1)} p_1^i p_2^{-(k+i+1)} q_j [(k+i+1)r_2 - ir_1 - \tau_j]/S_{ijk}\right), \tag{2.2.4}$$

$$h_3(S) = \Sigma_3 \frac{S_{jk}}{(k+1)r_1 - \tau_j} \ln\left(e(2p_1)^{-(k+1)} q_j [(k+1)r_1 - \tau_j]/S_{jk}\right); \tag{2.2.5}$$

where $I_2 = \{(ijk) : j = 1,\ldots,m;\ k,\ i = 1,2,\ldots\}$, $I_3 = \{(jk) : j = 1,\ldots,m,\ k = 0,1,\ldots\}$, $\Sigma_2 = \sum_{(ijk)\in I_2}$, $\sum_3 = \sum_{(jk)\in I_3}$, where $S_{ijk} = 0$ or $S_{jk} = 0$ implies that the corresponding terms vanish. Different from the series in (2.2.1), we can not guarantee that every series $h_b(S)$ in (2.2.3) - (2.2.5) is convergent in D_b. Instead, as shown in the Remark 2.3.1, we see there exists at least one series $h_b(S)$ which is convergent on D_b.

Theorem 2.2.2. *Assume* $p_1 + p_2 > 1$. *Then*

i) for some $b = 1,2,$ *or* 3, $h_b(S)$ *has a maximum at a point* $S^{(b)} = (S^{(b)}_{ijk})$, $b = 1,2,$ *or* $S^{(3)} = (S^{(3)}_{jk})$ *on* D_b, *which is completely determined by one of the conditions that*

$$\frac{1}{(k+i+1)r_1 - ir_2 - \tau_j} \times \ln\left(C^i_{-(k+1)} p_1^{-(k+i+1)} p_2^i q_j[(k+i+1)r_1 - ir_2 - \tau_j]/S^{(1)}_{ijk}\right), \tag{2.2.6}$$

$$\frac{1}{(k+i+1)r_2 - ir_1 - \tau_j} \times \ln\left(C^i_{-(k+1)} p_1^i p_2^{-(k+i+1)} q_j[(k+i+1)r_2 - ir_1 - \tau_j]/S^{(2)}_{ijk}\right), \tag{2.2.7}$$

or

$$\frac{1}{(k+1)r_1 - \tau_j} \ln\left((2p_1)^{-(k+1)} q_j[(k+1)r_1 - \tau_j]/S^{(3)}_{jk}\right) \tag{2.2.8}$$

has the same value for all $(ijk) \in I_2$ *or* $(jk) \in I_3$, *such that* $S^{(b)}_{ijk} \neq 0$, $b = 1,2$, *or* $S^{(3)}_{jk} \neq 0$.

ii) Eq. (2.1.3) is oscillatory if and only if $0 < h_b(S^{(b)}) < \infty$ *for some* $b = 1,2,$ *or* 3.

Theorems 2.2.1 and 2.2.2 can be applied to get a series of sufficient conditions for oscillation of Eq. (2.1.3). To use Theorem 2.2.1 we need to find a $\ell^* \in D$ such that $f(\ell^*) > 0$, whereas, to use Theorem 2.2.2, we need to first choose a suitable function h_b which is convergent on D_b, and then find a $S^* \in D_b$ such that $h_b(S^*) > 0$. The following corollaries are derived from Theorems 2.2.1 and 2.2.2, where

$$q = \left(\prod_{j=1}^m q_j\right)^{1/m}, \quad \text{and} \quad \tau = \frac{1}{m}\sum_{j=1}^m \tau_j.$$

Corollary 2.2.3. *Assume* $p_1 + p_2 < 1$. *Then each one of the following conditions is sufficient for Eq. (2.1.3) to be oscillatory:*

i) $\Sigma_1 C_k^i p_1^i p_2^{k-i} q_j [ir_1 + (k-i)r_2 + \tau_j] \geq \frac{1}{e}$ *(2.2.9)*

ii) $mq \sum_{k=0}^{\infty} \sum_{i=0}^{k} C_k^i p_1^i p_2^{k-i} [ir_1 + (k-i)r_2 + \tau] \geq \frac{1}{e}$,

iii) let $a = \Sigma_1 C_k^i p_1^i p_2^{k-i} q_j$, $\alpha_{ijk} = \frac{C_k^i p_1^i p_2^{k-i} q_j}{ir_1 + (k-i)r_2 + \tau_j}$, $(ijk) \in I_1$, *and*

$\Sigma_1 \frac{\alpha_{ijk}}{a} \ln(ea[ir_1 + (k-i)r_2 + \tau_j]) \geq 0$.

Corollary 2.2.4. *Assume* $p_1 + p_2 > 1$.

i) Let $r_1 \neq r_2$, *and*

$$\Sigma_3 (2p_1)^{-(k+1)} q_j [(k+1)r_1 - \tau_j] e^{\left(\frac{1}{r_1 - r_2} \ln \frac{p_1}{p_2}\right)[(k+1)r_1 - \tau_j]} = 1. \quad (2.2.10)$$

Then Eq. (2.1.3) is oscillatory if and only if

$$\Sigma_3 (2p_1)^{-(k+1)} q_j e^{\left(\frac{1}{r_1 - r_2} \ln \frac{p_1}{p_2}\right)[(k+1)r_1 - \tau_j]} - \frac{1}{r_1 - r_2} \ln \frac{p_1}{p_2} > 0. \quad (2.2.11)$$

Otherwise, each one of the following conditions is sufficient for Eq. (2.1.3) to be oscillatory:

ii) $\sum_{(ijk) \in I_2^*} C_{-(k+1)}^i p_1^{-(k+i+1)} p_2^i q_j [(k+i+1)r_1 - ir_2 - \tau_j] \geq \frac{1}{e}$, or *(2.2.12)*

$$mq \sum_{(ij)\in J_2^*} C^i_{-(k+1)} p_1^{-(k+i+1)} p_2^i q_j [(k+i+1)r_1 - ir_2 - \tau] \geq \frac{1}{e} \qquad (2.2.13)$$

where

$$I_2^* = \Big\{(ijk) : C^i_{-(k+1)} > 0,\ (k+i+1)r_1 - ir_2 - \tau_j > 0,\ (k+i+1)r_2 - ir_1 - \tau_j > 0,$$

$$\text{and} \qquad \left(\frac{p_1}{p_2}\right)^{k+2i+1} \frac{(k+i+1)r_2 - ir_1 - \tau_j}{(k+i+1)r_1 - ir_2 - \tau_j} > 1\Big\}$$

$$J_2^* = \Big\{(ik) : C^i_{-(k+1)} > 0,\ (k+i+1)r_1 - ir_2 - \tau > 0,\ (k+i+1)r_2 - ir_1 - \tau > 0$$

$$\text{and} \qquad \left(\frac{p_1}{p_2}\right)^{k+2i+1} \frac{(k+i+1)r_2 - ir_1 - \tau}{(k+i+1)r_1 - ir_2 - \tau} > 1\Big\}$$

iii) (2.2.12) and (2.2.13) hold if p_1 *and* p_2, r_1 *and* r_2 *exchange their positions respectively.*

It is easy to see that $I_2^* \neq \phi$ and has infinitely many terms provided $p_1 > p_2$, or $p_1 = p_2$ and $r_1 > r_2$.

2.3. Proofs

To prove Theorem 2.2.1 we need the following lemmas.

Lemma 2.3.1. *Assume* $p_1 + p_2 < 1$. *Define*

$$G_1(\lambda) = \lambda + \sum_{j=1}^{m} q_j e^{-\lambda\tau_j}(1 - p_1 e^{-\lambda r_1} - p_2 e^{-\lambda r_2})^{-1}, \qquad (2.3.1)$$

and let $\lambda^* < 0$ *be such that* $p_1 e^{-\lambda^* r_1} + p_2 e^{-\lambda^* r_2} = 1$. *Then there exists a* $\lambda_0 \in (\lambda^*, \infty)$ *such that*

$$0 < p_1 e^{-\lambda_0 r_1} + p_2 e^{-\lambda_0 r_2} < 1, \qquad (2.3.2)$$

and $G_1(\lambda_0)$ *is the minimal value of* $G_1(\lambda)$ *in* (λ^*, ∞).

Proof: Noting that $G_1(\infty) = \infty$ and $G_1(\lambda^* + 0) = \infty$, the conclusion is obvious by the continuity of $G_1(\lambda)$.

□

Lemma 2.3.2. *Assume* $p_1 + p_2 < 1$, λ_0 *is defined by Lemma 2.3.1, and let*

$$\ell^0_{ijk} = C^i_k p^i_1 p^{k-i}_2 q_j [ir_1 + (k-i)r_2 + \tau_j] e^{-\lambda_0[ir_1 + (k-i)r_2 + \tau_j]}, \ (ijk) \in I_1. \qquad (2.3.3)$$

Then

i) $\ell^0 = (\ell^0_{ijk}) \in D$,

ii) (2.2.2) has the same value for all $(ijk) \in I_1$, $\ell^0_{ijk} \neq 0$,

iii) $f(\ell^0) = G_1(\lambda_0)$,

(iv) $f(\ell^0)$ *is the maximum value of* $f(\ell)$ *on* D.

PROOF: By (2.3.2) there exists a neighborhood Λ on λ_0 such that

$$0 < p_1 e^{-\lambda r_1} + p_2 e^{-\lambda r_2} < 1$$

for all $\lambda \in \Lambda$. Expanding $(1 - p_1 e^{-\lambda r_1} - p_2 e^{-\lambda r_2})^{-1}$ we get for $\lambda \in \Lambda$

$$G_1(\lambda) = \lambda + \sum_{i=1}^{m} q_j e^{-\lambda \tau_j} \sum_{k=0}^{\infty} (p_1 e^{-\lambda r_1} + p_2 e^{-\lambda r_2})^k$$
$$= \lambda + \Sigma_1 C_k^i p_1^i p_2^{k-i} q_j e^{-\lambda[ir_1 + (k-i)r_2 + \tau_j]}.$$

Thus

$$G_1'(\lambda) = 1 - \Sigma_1 C_k^i p_1^i p_2^{k-i} q_j [ir_1 + (k-i)r_2 + \tau_j] e^{-\lambda[ir_1 + (k-i)r_2 + \tau_j]}.$$

Since $G'(\lambda_0) = 0$ we have

$$\Sigma_1 C_k^i p_1^i p_2^{k-i} q_j [ir_1 + (k-i)r_2 + \tau_j] e^{-\lambda_0[ir_1 + (k-i)r_2 + \tau_j]} = 1,$$

i.e., $\Sigma_1 \ell_{ijk}^0 = 1$, or $\ell^0 \in D$. From (2.3.3)

$$\frac{1}{ir_1 + (k-i)r_2 + \tau_j} \ln \left(C_k^i p_1^i p_2^{k-i} q_j [ir_1 + (k-i)r_2 + \tau_j] / \ell_{ijk}^0 \right) = \lambda_0 \qquad (2.3.4)$$

for $\ell^0_{ijk} \neq 0$. Furthermore by (2.3.3) and (2.3.4)

$$\begin{aligned} f(\ell^0) &= \Sigma_1\Big\{\frac{\ell^0_{ijk}}{ir_1+(k-i)r_2+\tau_j} + \frac{\ell^0_{ijk}}{ir_1+(k-i)r_2+\tau_j} \\ &\qquad \times \ln(C^i_k p^i_1 p^{k-i}_2 q_j[ir_1+(k-i)r_2+\tau_j]/\ell^0_{ijk})\Big\} \\ &= \Sigma_1\left\{C^i_k p^i_1 p^{k-i}_2 q_j e^{-\lambda_0[ir_1+(k-i)r_2+\tau_j]} + \lambda_0 \ell^0_{ijk}\right\} \\ &= G_1(\lambda_0). \end{aligned} \tag{2.3.5}$$

For $\ell = (\ell_{ijk}) \in D, \ell \neq \ell^0$, from (2.3.5) and Lemma 1 in [5] we have

$$\begin{aligned} f(\ell^0) &= \Sigma_1\left\{C^i_k p^i_1 p^{k-i}_2 q_j e^{-\lambda_0[ir_1+(k-i)r_2+\tau_j]} + \lambda_0 \ell_{ijk}\right\} \\ &> \Sigma_1 \frac{\ell_{ijk}}{ir_1+(k-i)r_2+\tau_j} \ln\left(eC^i_k p^i_1 p^{k-i}_2 q_j[ir_1+(k-i)r_2+\tau_j]/\ell_{ijk}\right) \\ &= f(\ell). \end{aligned}$$

□

Proof of Theorem 2.2.1: The first part is shown by Lemma 2.3.2. Assume Eq. (2.1.3) is oscillatory. Thus $F(\lambda) > 0$ for F defined by (2.1.4) and all $\lambda \in \mathbb{R}$. In particular,

$$F(\lambda_0) = \lambda_0\left(1 - p_1 e^{-\lambda_0 r_1} - p_2 e^{-\lambda_0 r_2}\right) + \sum_{j=1}^{m} q_j e^{-\lambda_0 \tau_j} > 0$$

where λ_0 satisfies $0 < p_1 e^{-\lambda_0 r_1} + p_2 e^{-\lambda_0 r_2} < 1$ by Lemma 2.3.1. Hence

$$G_1(\lambda_0) = \lambda_0 + \sum_{j=1}^{m} q_j e^{-\lambda_0 \tau_j}\left(1 - p_1 e^{-\lambda_0 r_1} - p_2 e^{-\lambda_0 r_2}\right)^{-1} > 0.$$

From Lemma 2.3.2 we have $f(\ell^0) = G_1(\lambda_0) > 0$.

On the other hand, if $f(\ell^0) > 0$, by Lemma 2.3.2, $G_1(\lambda_0) > 0$. Hence $G_1(\lambda) > 0$ for all $\lambda \in (\lambda^*, \infty)$, where λ^* is given by Lemma 2.3.1. If Eq. (2.1.3) is not oscillatory, then there exists a $\lambda_1 \in R$ such that

$$F(\lambda_1) = \lambda_1\left(1 - p_1 e^{-\lambda_1 r_1} - p_2 e^{-\lambda_1 r_2}\right) + \sum_{j=1}^{m} q_j e^{-\lambda_1 r_j} = 0. \tag{2.3.6}$$

Obviously $\lambda_1 < 0$ and thus $0 < p_1 e^{-\lambda_1 r_1} + p_2 e^{-\lambda_1 r_2} < 1$, i.e., $\lambda_1 \in (\lambda^*, \infty)$. (2.3.6) gives us that

$$0 = \lambda_1 + \sum_{j=1}^{m} q_j e^{-\lambda_1 r_j}\left(1 - p_1 e^{-\lambda_1 r_1} - p_2 e^{-\lambda_1 r_2}\right)^{-1} = G_1(\lambda_1)$$

which contradicts $G_1(\lambda) > 0$ on (λ^*, ∞).

□

To prove Theorem 2.2.2 we need the following lemmas.

Lemma 2.3.3. *Assume* $p_1 + p_2 > 1$. *Define*

$$G_2(\mu) = -\mu + \sum_{j=1}^{m} q_j e^{-\mu r_j}\left(p_1 e^{-\mu r_1} + p_2 e^{-\mu r_2} - 1\right)^{-1},$$

and let $\mu\ > 0$ *be such that* $p_1 e^{-\mu^* r_1} + p_2 e^{-\mu^* r_2} = 1$. *Then there exists a* $\mu_0 \in (-\infty, \mu^*)$ *such that*

$$p_1 e^{-\mu_0 r_1} + p_2 e^{-\mu_0 r_2} > 1, \tag{2.3.7}$$

and $G_2(\mu_0)$ *is the minimum value of* $G_2(\mu)$ *in* $(-\infty, \mu^*)$.

PROOF: Similar to Lemma 2.3.1.

□

LEMMA 2.3.4.1. *Assume* $p_1 + p_2 > 1$, μ_0 *is defined by Lemma 2.3.3. Let*

i) $\frac{p_2}{p_1} e^{-\mu_0(r_2 - r_1)} < 1$, and

ii) $S^{(1)}_{ijk} = C^i_{-(k+1)} p_1^{-(k+i+1)} p_2^i q_j [(k+i+1)r_1 - ir_2 - \tau_j] e^{\mu_0[(k+i+1)r_1 - ir_2 - \tau_j]}$, $(ijk) \in I_2$.

Then

i) $S^{(1)} = (S^{(1)}_{ijk}) \in D_1$,

ii) (2.2.6) has the same value for all $(ijk) \in I_2$, $S^{(1)}_{ijk} \neq 0$,

iii) $h_1(S^{(1)}) = G_2(\mu_0)$,

iv) $h_1(S^{(1)})$ *is the maximal value of* $h_1(s)$ *on* D_1.

PROOF: By (2.3.7) and condition i), there exists a neighbourhood $\mathcal{U}$ of μ_0 such that $0 < \left(p_1 e^{-\mu r_1} + p_2 e^{-\mu r_2}\right)^{-1} < 1$, and $\frac{p_2}{p_1} e^{-\mu(r_2 - r_1)} < 1$ for

$\mu \in \mathcal{U}$.

$$G_2(\mu) = -\mu + \sum_{j=1}^{m} q_j e^{-\mu\tau_j} \left(p_1 e^{-\mu r_1} + p_2 e^{-\mu r_2}\right)^{-1} \left[1 - \left(p_1 e^{-\mu r_1} + p_2 e^{-\mu r_2}\right)^{-1}\right]^{-1}$$

$$= -\mu + \sum_{j=1}^{m} \sum_{k=0}^{\infty} q_j e^{-\mu\tau_j} \left(p_1 e^{-\mu r_1} + p_2 e^{-\mu r_2}\right)^{-(k+1)}$$

$$= -\mu + \sum_{j=1}^{m} \sum_{k=0}^{\infty} p_1^{-(k+1)} q_j e^{\mu[(k+1)r_1 - \tau_j]} \left(1 + \frac{p_2}{p_1} e^{-\mu(r_2 - r_1)}\right)^{-(k+1)} \tag{2.3.8}$$

$$= -\mu + \Sigma_2 C^i_{-(k+1)} p_1^{-(k+1)} p_2^i q_j e^{\mu[(k+1)r_1 - ir_2 - \tau_j]}. \tag{2.3.9}$$

The rest of the proof is similar to that of Lemma 2.3.2 and hence is omitted.

□

LEMMA 2.3.4.2. *Let the conditions of Lemma 2.3.4.1 hold if* p_1 *and* p_2, r_1 *and* r_2 *exchange their positions respectively, and* $S^{(1)}_{ijk}$ *are replaced by* $S^{(2)}_{ijk}$. *Then the conclusion of Lemma 2.3.4.1 holds if* $S^{(1)}$, D_1, h_1, *and (2.2.6) are replaced by* $S^{(2)}$, D_2, h_2, *and (2.2.7) respectively.*

LEMMA 2.3.4.3. *Assume* $p_1 + p_2 > 1$, μ_0 *is defined by Lemma 2.3.3. Let*

i) $\frac{p_2}{p_1} e^{-\mu_0(r_2 - r_1)} = 1$, *and*

ii) $S^{(3)}_{jk} = (2p_1)^{-(k+1)} q_j [(k+1)r_1 - \tau_j] e^{\mu_0[(k+1)r_1 - \tau_j]}$, $(jk) \in I_3$. (2.3.10)

Then

i) $S^{(3)} = (S^{(3)}_{jk}) \in D_3$, *ii) (2.2.8) has the same value for all* $(jk) \in I_3$, $S^{(3)}_{jk} \neq 0$,

iii) $h_3(S^{(3)}) = G_2(\mu_0)$,

iv) $h_3(S^{(3)})$ *is the maximum value of* $h_3(S)$ *on* D_3.

PROOF: From (2.3.8) and condition i)

$$G_2(\mu) = -\mu + \Sigma_3 (2p_1)^{-(k+1)} q_j e^{\mu[(k+1)r_1 - r_j]}. \tag{2.3.11}$$

The rest of the proof is simple and will be omitted.

□

REMARK 2.3.1: There exists at least one of h_b which is well-defined on D_b $(b = 1, 2, 3)$. In fact, there is at least one expansion of (2.3.9), its dual form, and (2.3.11), of $G_2(\mu)$, which converges at $\mu = \mu_0$. Therefore, at least one of $h_b(S^{(b)})$, $(b = 1, 2, 3)$ has a finite value and satisfies $h_b(S^{(b)}) = G_2(\mu_0)$.

PROOF OF THEOREM 2.2.2: The first part is shown by Lemmas 2.3.4.1 - 2.3.4.3.

Assume Eq. (2.1.3) is oscillatory. Then $F(\mu) > 0$ for all $\mu \in \mathbb{R}$. In particular,

$$F(\mu_0) = \mu_0 (1 - p_1 e^{-\mu_0 r_1} - p_2 e^{-\mu_0 r_2}) + \sum_{j=1}^{m} q_j e^{-\mu_0 \tau_j} > 0,$$

where μ_0 satisfies (2.3.7). Hence

$$G_2(\mu_0) = -\mu_0 + \sum_{j=1}^{m} q_j e^{-\mu_0 \tau_j} \left(p_1 e^{-\mu_0 r_1} + p_2 e^{-\mu_0 r_2} - 1\right)^{-1} > 0.$$

By Remark 2.3.1 there exists $b = 1, 2,$ or $3,$ such that $h_b(S^{(b)}) = G_2(\mu_0)$. Therefore

$0 < h_b(S^{(b)}) < \infty$ for some $b = 1, 2,$ or 3.

On the other hand, if $0 < h_b(S^{(b)}) < \infty$ for some $b = 1, 2,$ or $3,$ by Remark 2.3.1, $G_2(\mu_0) > 0$. Hence $G_2(\mu) > 0$ for all $\mu \in (-\infty, \mu^*)$. Assume Eq. (2.1.3) is not oscillatory. Then there exists a $\mu_1 \in \mathbb{R}$ such that

$$F(\mu_1) = \mu_1\left(1 - p_1 e^{-\mu_1 r_1} - p_2 e^{-\mu_1 r_2}\right) + \sum_{j=1}^{m} q_j e^{-\mu_1 \tau_j} = 0.$$

Obviously, $\mu_1 > 0$, and thus $p_1 e^{-\mu_1 r_1} + p_2 e^{-\mu_1 r_2} > 1$, i.e., $\mu_1 \in (-\infty,\ \mu^*)$, and

$$G_2(\mu_1) = -\mu_1 + \sum_{j=1}^{m} q_j e^{-\mu_1 \tau_j} \left(p_1 e^{-\mu_1 r_1} + p_2 e^{-\mu_1 r_2} - 1\right)^{-1} = 0,$$

contradicting that $G_2(\mu) > 0$ on $(-\infty, \mu^*)$.

□

PROOF OF COROLLARY 2.2.3:

i) Choose

$$\ell^*_{ijk} = e C^i_k p^i_1 p^{k-i}_2 q_j [i r_1 + (k-i) r_2 + \tau_j] c, \ (ijk) \in I_1,$$

such that $\Sigma_1 \ell^*_{ijk} = 1$. By (2.2.9) we have $0 < c \leq 1$. Hence

$$f(\ell^*) = \Sigma_1 e C_k^i p_1^i p_2^{k-i} q_j c \ln \tfrac{1}{c} \geq 0.$$

Noting that ℓ^* does not make (2.2.2) have the same value for $(ijk) \in I$, we see $\ell^* \neq \ell^0$. So $f(\ell^0) > 0$. By Theorem 2.2.1, Eq. (2.1.3) is oscillatory. The proofs for ii) and iii) are similar. For ii) we choose

$$\ell^*_{ijk} = e C_k^i p_1^i p_2^{k-i} q[ir_1 + (k-i)r_2 + \tau_j]c,$$

for iii) we choose $\ell^*_{ijk} = C_k^i p_1^i p_2^{k-i} q_j / a$.

□

Proof of Corollary 2.2.4:

i) The conditions of Lemma 2.3.4.3 are satisfied, where $\mu_0 = \frac{1}{r_1 - r_2} \ln \frac{p_1}{p_2}$. Hence from (2.2.5) and (2.3.10)

$$\begin{aligned} h_3(S^{(3)}) &= \Sigma_3 \frac{S^{(3)}_{jk}}{(k+1)r_1 - \tau_j}\left(1 - \ln e^{-\mu_0[(k+1)r_1 - \tau_j]}\right) \\ &= \Sigma_3 (2p_1)^{-(k+1)} q_j e^{\left(\frac{1}{r_1 - r_2} \ln \frac{p_1}{p_2}\right)[(k+1)r_1 - \tau_j]} - \frac{1}{r_1 - r_2} \ln \frac{p_1}{p_2}. \end{aligned}$$

Thus $h_3(S^{(3)}) > 0$ if and only if (2.2.11) holds.

ii) Without loss of generality we may assume the left hand side of (2.2.12) is finite, for otherwise we can replace I_2^* by its suitable subset in (2.2.12) such that the above assumption holds. Choose

$$S^*_{ijk} = \begin{cases} eC^i_{-(k+1)}p_1^{-(k+i+1)}p_2^i q_j[(k+i+1)r_1 - ir_2 - \tau_j]c, & (ijk) \in I_2^* \\ 0, & (ijk) \in I_2 \setminus I_2^* \end{cases}$$

such that $\Sigma_2 S^*_{ijk} = 1$, i.e., $S^* = (S^*_{ijk}) \in D_2$. By (2.2.12) we have $0 < c \leq 1$. From (2.2.3)

$$h_1(S^*) = \sum_{(ijk)\in I_2^*} eC^i_{-(k+1)}p_1^{-(k+i+1)}p_2^i q_j c \ln \tfrac{1}{c} \geq 0.$$

From (2.2.4) and the definition of I_2^*

$$\begin{aligned} h_2(S^*) = & \sum_{(ijk)\in I_2^*} eC^i_{-(k+1)}p_1^{-(k+i+1)}p_2^i q_j \frac{(k+i+1)r_1 - ir_2 - \tau_j}{(k+i+1)r_2 - ir_1 - \tau_j}c \\ & \times \ln \left[\left(\frac{p_1}{p_2}\right)^{k+2i+1} \frac{(k+i+1)r_2 - ir_1 - \tau_j}{(k+i+1)r_1 - ir_2 - \tau_j} \right] \\ & > 0. \end{aligned}$$

By Remark 2.3.1 we get that there exists a $b = 1$ or 2 such that $h_b(S^{(b)}) < \infty$. Noting that S^* does not make (2.2.6) have the same value for $(ijk) \in I_2$, hence $h_b(S^{(b)}) > h_b(S^*) \geq 0$. By Theorem 2.2.2, Eq. (2.1.3) is oscillatory.

iii) Similar.

□

2.4. Remarks on a Special Case

If we let $p_1 = p$, $p_2 = 0$, $r_1 = r$, then Eq. (2.1.3) becomes

$$\frac{d}{dt}[y(t) - py(t-r)] + \sum_{j=1}^{m} q_j y(t-\tau_j) = 0, \tag{2.4.1}$$

and the results in Section 2.2 can be reformulated as follows.

Define two sets

$$D_1 = \Big\{\ell = (\ell_{jk}) : \ell_{jk} \geq 0, \quad j = 1,\ldots,m; \quad k = 0,1,\ldots; \sum_{j=1}^{m}\sum_{k=0}^{\infty} \ell_{jk} = 1\Big\},$$

$$D_2 = \Big\{S = (S_{jk}) : S_{jk}[(k+1)r - \tau_j] \geq 0,$$
$$j = 1,\ldots,m; \quad k = 0,1,\ldots; \sum_{j=1}^{m}\sum_{k=0}^{\infty} S_{jk} = 1\Big\}.$$

On D_1 we define a function

$$f(\ell) = \sum_{j=1}^{m}\sum_{k=0}^{\infty} \frac{\ell_{jk}}{kr + \tau_j} \ln\left(ep^k q_j (kr + \tau_j)/\ell_{jk}\right),$$

on D_2 we define a function

$$h(S) = \sum_{j=1}^{m}\sum_{k=0}^{\infty} \frac{S_{jk}}{(k+1)r_1 - \tau_j} \ln\left(ep^{-(k+1)} q_j [(k+1)r - \tau_j]/S_{jk}\right).$$

where $\ell_{jk} = 0$ or $S_{jk} = 0$ implies that the corresponding terms vanish.

Theorem 2.4.1. *Assume* $0 \leq p < 1$. *Then* *i)* $f(\ell)$ *has a maximum at a point* $\ell^0 = (\ell^0_{jk})$ *on* D_1 *which is determined by the condition that*

$$\frac{1}{kr+\tau_j}\ln\left(p^k q_j[kr+\tau_j]/\ell^0_{jk}\right)$$

has the same value for all $j = 1,\ldots,m,\ k = 0,1,\ldots,\ell^0_{jk} \neq 0.$

ii) Eq. (2.4.1) is oscillatory if and only if $f(\ell^0) > 0.$

THEOREM 2.4.2. *Assume* $p > 1$. *Then*

i) $h(s)$ *has a maximum value at a point* $S^0 = (S^0_{jk})$ *on* D_2, *which is determined by the condition that*

$$\frac{1}{(k+1)r-\tau_j}\ln\left(p^{-(k+1)}q_j[(k+1)r-\tau_j)/S^0_{jk}\right)$$

has the same value for all $j = 1,\ldots,m,\ k = 0,1,\ldots,S^0_{jk} \neq 0.$

ii) Eq. (2.4.1) is oscillatory if and only if $h(S^0) > 0.$

COROLLARY 2.4.3. *Assume* $0 \leq p < 1$. *Then each of the following is sufficient for Eq. (2.4.1) to be oscillatory:*

i) $\sum_{j=1}^{m}\sum_{k=0}^{\infty} p^k q_j(kr+\tau_j) \geq \frac{1}{e},$

ii) $mq\sum_{k=0}^{\infty} p^k(kr+\tau) \geq \frac{1}{e},$

iii) *let* $a = \sum_{j=1}^{m} \sum_{k=0}^{\infty} p^k q_j$, $\alpha_{jk} = \frac{p^k q_j}{kr+\tau_j}$, $j = 1,\ldots,m$, $k = 0,1,\ldots,$ *and*

$$\sum_{j=1}^{m} \sum_{k=0}^{\infty} \frac{\alpha_{jk}}{a} \ln[ea(kr+\tau_j)] \geq 0.$$

COROLLARY 2.4.4. *Assume* $p > 1$. *Then each one of the following is sufficient for Eq. (2.4.1) to be oscillatory:*

i) *there is a* k_j *for each* $j = 1,\ldots,m$, *such that* $(k_j+1)r - \tau_j > 0$, *and*

$$\sum_{j=1}^{m} \sum_{k=k_j}^{\infty} p^{-(k+1)} q_j[(k+1)r - \tau_j] \geq \frac{1}{e},$$

ii) *there is a* k_0 *such that* $(k_0+1)r - \tau > 0$, *and*

$$mq \sum_{k=k_0}^{\infty} p^{-(k+1)}[(k+1)r - \tau] \geq \frac{1}{e}.$$

The above results include the main results in [7]. It is obvious that Corollaries 2.4.3 and 2.4.4 substantially improve the results for oscillation of Eq. (2.4.1) and its special cases given in [2-4,7-9]. We can also see that if $p = 0$ in (2.4.1) then Theorem 2.4.1 is the same as Theorem 1 in [6].

References

[1] O. Arino and I. Győri, ***Necessary and sufficient condition for oscillation of a neutral differential system with several delays.*** J. Differ. Eqns. 81 (1989), 98–105.

[2] Y. Chen, ***Oscillation of a class of neutral functional differential equations.*** Chin. Ann. Math. 10A(5) (1989), 545–553 (Chinese edition).

[3] K. Gopalsamy and B.G. Zhang, *Oscillation and nonoscillation in first order neutral differential equations.* J. Math. Anal. Appl. **151** (1990), 42–57.

[4] M.K. Grammatikopoulos, E.A. Grove and G. Ladas, *Oscillation and asymptotic behavior of neutral differential equations with deviating arguments.* Applicable Anal. **22** (1986), 1–19.

[5] I. Győri and G. Ladas, "Oscillation Theory of Delay Differential Equations with Applications", Clarendon Press, Oxford, 1991.

[6] W. Huang, *Necessary and sufficient conditions for oscillation of first-order functional differential equations*, J. Math. Anal. Appl. **148** (1990), 360–370.

[7] Jaroslav Jaroš, *On characterization of oscillations in first-order linear neutral differential equations*, Funkcialaj Ekvacioj, **34** (1991), 331-342.

[8] G. Ladas and Y.G. Sficas, *Oscillations of neutral delay differential equations*, Can. Math. Bull. **29** (1986), 438-445.

[9] I.P. Stavroulakis, *Oscillations of mixed neutral equations.* Hiroshima Math. J. **19** (1989), 441–456.

CHAPTER 3

OSCILLATION OF LINEAR EQUATIONS OF FIRST ORDER

—Nonautonomous Case

3.1. Introduction

This chapter is concerned with the scalar linear NDDEs with some variable coefficients of the form

$$\frac{d}{dt}[y(t) - py(t-r)] + \sum_{i=1}^{n} q_i(t)y(t-\tau_i) = 0 \tag{3.1.1}$$

where $p \in [0,1]$, q_i $(i = 1, \ldots, n) \in C([t_0, \infty), \mathbb{R}_+)$ and r, τ_i $(i = 1, \ldots, n) \in (0, \infty)$.

Our aim here is to establish some new sufficient conditions for oscillation and existence of nonoscillatory solutions of equation (3.1.1). As corollaries, some results are derived which yield sufficient and necessary conditions for oscillation. We also obtain some comparison criteria and give some explicit conditions for oscillation. Similar results are obtained for delay equations with variable delays. Those results can also be extended to the more general case where the derivative of y appears with several delays.

There is a lot of work done on the oscillation of Eq.(3.1.1). For some recent results in oscillation theory, see [1–17] and the references cited therein. For completeness, we cite the following results:

Result 3.1 [1,16]. Let $0 \le p \le 1$ and $q_i > 0$ $(i = 1, \dots, n)$ be constants. Then (3.1.1) is oscillatory if and only if

$$\lambda(1 - pe^{-\lambda r}) + \sum_{i=1}^{n} q_i e^{-\lambda \tau_i} = 0 \tag{3.1.2}$$

has no real root.

Result 3.2 [12]. Let $0 \le p \le 1$ be a constant, $\tau_i = ir$, $i = 1, \dots, n$, and $q_i(t)$, $i = 1, \dots, n$ be r-periodic functions. Then (3.1.1) is oscillatory if and only if

$$\lambda(1 - pe^{-\lambda r}) + \sum_{i=1}^{n} \frac{1}{r}\left(\int_0^r q_i(s)\, ds\right) e^{-\lambda \tau_i} = 0 \tag{3.1.3}$$

has no real root.

Result 3.3 [6]. Consider the case $n = 1$ in (3.1.1) where $q_1(t) := q(t)$, $\tau_1 := \tau$, and p is replaced by $p(t) \in C([t_0, \infty), \mathbb{R}_+)$.

i) Assume $p(t)$ is bounded, $p(t^* + nr) \le 1$ for a $t^* \ge t_0$ and $n = 0, 1, 2, \dots,$ and $q(t) \ge q > 0$, $t \ge t_0$. If

$$\inf_{\mu > 0, t \ge T} \left[p(t - \tau) \frac{q(t)}{q(t - r)} e^{\mu r} + \frac{1}{\mu} q(t) e^{\mu \tau} \right] > 1,$$

then (3.1.1) is oscillatory.

ii) Assume there exists a $\mu^* > 0$ such that

$$\sup_{t \geq T} \left[p(t-\tau) \frac{q(t)}{q(t-\tau)} e^{\mu^* \tau} + \frac{1}{\mu^*} q(t) e^{\mu^* \tau} \right] \leq 1.$$

Then (3.1.1) has a positive solution.

In Section 3.2 we will obtain sufficient conditions for oscillation of (3.1.1) and for the existence of a nonoscillatory solution for the neutral equation (3.1.1). These conditions cover Result 3.1 for the constant coefficient case and improve Result 3.2 for the periodic coefficient case. They also yield some sufficient and necessary conditions for oscillation even for a class of equations with aperiodic coefficients. In Section 3.3, the above criteria for oscillation are developed for delay equations with variable delays, which substantially improve the established conjecture by Hunt and Yorke [13]. Based on these results, in Section 3.4, we derive some comparison criteria for oscillation and existence of nonoscillatory solutions, and in Section 3.5, we obtain some explicit conditions for oscillation.

Before stating the main results we introduce the following lemmas which will be used in the proofs.

LEMMA 3.1.1. *Let* $a > 0, b > 0,$ *and* $f(t) \geq 0$ *be a locally integrable function on* $\mathbb{R}_+$. *Assume both the limits*

$$I_1 = \lim_{t \to \infty} \frac{1}{a} \int_t^{t+a} f(s)\, ds \quad \text{and} \quad I_2 = \lim_{t \to \infty} \frac{1}{b} \int_t^{t+b} f(s)\, ds$$

exist. Then $I_1 = I_2$.

Proof: We prove it by contradiction. Without loss of generality we only consider the case that $I_1 > I_2$. Choose a positive integer n such that $anI_1 - b(m+1)I_2 > 1$, where $m = [an/b]$ is the integer part of an/b. This is possible since $I_1 > I_2$ and $b[an/b] \le an$. Choose $T \ge 0$ so large that for $t \ge T$

$$\int_t^{t+a} f(s)\,ds > aI_1 - \frac{1}{n+m+1}$$

and

$$\int_t^{t+b} f(s)\,ds < bI_2 + \frac{1}{n+m+1}.$$

This implies that

$$\int_{T+bm}^{T+an} f(s)\,ds > anI_1 - bmI_2 - \frac{n+m}{n+m+1} > bI_2 + \frac{1}{n+m+1}.$$

But this is impossible since

$$(T+an) - (T+bm) = an - [an/b]b \le b,$$

and hence

$$\int_{T+bm}^{T+an} f(s)\,ds \le \int_{T+bm}^{T+(m+1)b} f(s)\,ds \le bI_2 + \frac{1}{n+m+1}.$$

This gives a contradition.

□

With a similar proof we get the following generalization of Lemma 3.1.1.

LEMMA 3.1.2. *Let* $0 < a_1 \le a(t) \le a_2 < \infty$, $0 < b_1 \le b(t) \le b_2 < \infty$, *and let* $f(t) \ge 0$ *be a locally integrable function on* $[0, \infty)$. *Assume both the limits*

$$I_1 = \lim_{t\to\infty} \frac{1}{a(t)} \int_t^{t+a(t)} f(s)\,ds \quad \text{and} \quad I_2 = \lim_{t\to\infty} \frac{1}{b(t)} \int_t^{t+b(t)} f(s)\,ds$$

exist. Then $I_1 = I_2$.

3.2. Criteria for oscillation and nonoscillation

In this section we are concerned with the equation

$$\frac{d}{dt}[y(t) - py(t-r)] + \sum_{i=1}^{n} q_i(t)y(t-\tau_i) = 0 \tag{3.2.1}$$

where

$$\begin{gathered} q_i(t) \in C([t_0,\infty), \mathbb{R}_+),\ i = 1,\dots,n, \\ p \in [0,1],\ r, \tau_i \in (0,\infty),\ i = 1,\dots,n. \end{gathered} \tag{3.2.2}$$

Denote $\sigma = \max\{r, \tau_1, \dots, \tau_n\}$.

The following lemma is needed in the proof.

Lemma 3.2.1. *In addition to (3.2.2) assume*

$$q_1(t) \geq q > 0 \tag{3.2.3}$$

and

$$q_i(t-\tau) \leq q_i(t), \quad t \geq t_0 + r, \; i = 1, \ldots, n. \tag{3.2.4}$$

Let $y(t)$ *be an eventually positive solution of (3.2.1), and let*

$$z(t) = y(t) - py(t-r). \tag{3.2.5}$$

Then eventually $z(t) > 0$, $z'(t) < 0$, *and*

$$z'(t) - pz'(t-r) + \sum_{i=1}^{n} q_i(t) z(t-\tau_i) \leq 0. \tag{3.2.6}$$

Proof: From (3.2.1) it is easy to see that $z(t) > 0$, $z'(t) < 0$ eventually. From (3.2.4) and (3.2.5)

$$\begin{aligned} z'(t) - pz'(t-r) &= -\sum_{i=1}^{n} \left[q_i(t) y(t-\tau_i) - pq_i(t-r) y(t-r-\tau_i) \right] \\ &\leq -\sum_{i=1}^{n} q_i(t) \left[y(t-\tau_i) - py(t-r-\tau_i) \right] \\ &= -\sum_{i=1}^{n} q_i(t) z(t-\tau_i). \end{aligned}$$

Thus (3.2.6) is true eventually.

□

THEOREM 3.2.1. *Assume (3.2.2)–(3.2.4) hold, and for all* $\mu > 0$, *and* $\ell = r, \tau_1, \ldots, \tau_n$

$$\liminf_{t\to\infty} \left[pe^{\mu r} + \frac{1}{\ell\mu} \sum_{i=1}^{n} e^{\mu\tau_i} \int_t^{t+\ell} q_i(s)\,ds \right] > 1. \tag{3.2.7}$$

Then (3.2.1) is oscillatory.

PROOF: Assume (3.2.1) has an eventually positive solution $y(t)$. Define $z(t)$ as given by (3.2.5). Then by Lemma 3.2.1 there exists a $T \geq t_0$ such that $z(t) > 0$, $\dot{z}(t) < 0$, and (3.2.6) holds for $t \geq T$. Let $w(t) = -\frac{z'(t)}{z(t)}$, $t \geq T$. Then $w(t) > 0$, $t \geq T$, and (3.2.6) becomes

$$w(t) \geq pw(t-r)\exp\left(\int_{t-r}^{t} w(s)\,ds\right) + \sum_{i=1}^{n} q_i(t)\exp\left(\int_{t-\tau_i}^{t} w(s)\,ds\right)$$

for $t \geq T + \sigma$, where $\sigma = \max\{r, \tau_1, \ldots, \tau_n\}$.

We now define a sequence of functions $\{w_k(t)\}$ for $k = 1, 2, \ldots,$ and $t \geq T$, and a sequence of numbers $\{\mu_k\}$ for $k = 1, 2, \ldots,$ as follows:

$$w_1(t) \equiv 0, \quad t \geq T$$

and for $k = 1, 2, \ldots,\ t \geq T + k\sigma$

$$w_{k+1}(t) = pw_k(t-r)\exp\left(\int_{t-r}^{t} w_k(s)\,ds\right) + \sum_{i=1}^{n} q_i(t)\exp\left(\int_{t-\tau_i}^{t} w_k(s)\,ds\right); \tag{3.2.8}$$

and $\mu_1 = 0$, and for $k = 1, 2, \ldots$

$$\mu_{k+1} = \inf_{t\geq T}\min_{\ell=r,\tau_1,\ldots,\tau_n}\left\{p\mu_k e^{\mu_k r} + \frac{1}{\ell}\sum_{i=1}^{n} e^{\mu_k \tau_k}\int_t^{t+\ell} q_i(s)\,ds\right\}. \tag{3.2.9}$$

We claim that the following inequalities hold:

i) $0 = \mu_1 < \mu_2 < \cdots;$

ii) $w_k(t) \leq w(t)$ for $t \geq T + (k-1)\sigma$ and $k = 1, 2, \ldots;$

iii) $\frac{1}{\ell}\int_t^{t+\ell} w_k(s)\,ds \geq \mu_k$ for $t \geq T + (k+1)\sigma,\ k = 1, 2, \ldots,$ and $\ell = r, \tau_1, \ldots, \tau_n$.

In fact, since $\mu_2 > \mu_1 = 0$, and $w_1(t) < w(t)$ for $t \geq T$, by induction we see i) and ii) are true. We now show that iii) also holds. Clearly iii) is true for $k = 1$. Assume iii) is true for some k. Then (3.2.8) and (3.2.9) imply that for $t \geq T + k\sigma,\ \ell = r, \tau_1, \ldots, \tau_n$

$$\frac{1}{\ell}\int_t^{t+\ell} w_{k+1}(s)\,ds = \frac{p}{\ell}\int_t^{t+\ell} w_k(s-r)\exp\left(\int_{s-r}^{s} w_k(\theta)\,d\theta\right)ds + \frac{1}{\ell}\sum_{i=1}^{n}\int_t^{t+\ell} q_i(s)\exp\left(\int_{s-\tau_i}^{s} w_k(\theta)\,d\theta\right)ds$$

$$\geq p\mu_k e^{\mu_k r} + \frac{1}{\ell}\sum_{i=1}^{n} e^{\mu_k \tau_i}\int_t^{t+\ell} q_i(s)\,ds$$

$$\geq \inf_{t\geq T}\min_{\ell=r,\tau_1,\ldots,\tau_n}\left\{p\mu_k e^{\mu_k r} + \frac{1}{\ell}\sum_{i=1}^{n} e^{\mu_k \tau_i}\int_t^{t+\ell} q_i(s)\,ds\right\}$$

$$= \mu_{k+1}.$$

Hence iii) holds.

Let $\mu^* = \lim_{k\to\infty}\mu_k$. From (3.2.7) and (3.2.9) there exists an $\alpha > 1$ such that $\mu_{k+1} \geq \alpha\mu_k,\ k = 1, 2, \ldots,$ and this means that $\mu^* = \infty$.

By ii) and iii) we have that $\lim_{t\to\infty}\int_t^{t+\tau_1} w(s)\,ds = \infty$, and so

$$\limsup_{t\to\infty}\int_t^{t+\frac{\tau_1}{2}} w(s)\,ds = \infty.$$

Integrating both sides of the equation $w(t) = -\frac{z'(t)}{z(t)}$ from t to $t+\frac{\tau_1}{2}$ for t sufficiently large we get

$$\frac{z(t)}{z(t+\frac{\tau_1}{2})} = \exp\left(\int_t^{t+\frac{\tau_1}{2}} w(s)\,ds\right).$$

Thus

$$\limsup_{t\to\infty}\frac{z(t)}{z(t+\frac{\tau_1}{2})} = \infty. \qquad (3.2.10)$$

Since

$$z'(t) = -\sum_{i=1}^{n} q_i(t)y(t-\tau_i) \leq -qy(t-\tau_1) \leq -qz(t-\tau_1),$$

integrating both sides from $t+\tau_1/2$ to $t+\tau_1$ and using the decreasing nature of $z(t)$ we find for t sufficiently large

$$0 < z(t+\tau_1) \le z\left(t+\frac{\tau_1}{2}\right) - \frac{q\tau_1}{2} z(t).$$

Thus

$$\frac{z(t)}{z(t+\frac{\tau_1}{2})} \le \frac{2}{q\tau_1}$$

contradicting (3.2.10). This completes the proof.

□

Theorem 3.2.2. *Assume (3.2.2) holds and there exist a* $\mu^* > 0$ *and a* $T \ge t_0$ *such that for* $\ell = r, \tau_1, \ldots, \tau_n$

$$\sup_{t \ge T} \left[p e^{\mu^* r} + \frac{1}{\ell \mu^*} \sum_{i=1}^{n} e^{\mu^* \tau_i} \int_t^{t+\ell} q_i(s)\, ds \right] \le 1, \tag{3.2.11}$$

Then (3.2.1) has a positive solution on $[T+\sigma, \infty)$.

Proof: First we claim that the integral equation

$$v(t) = p v(t-r) \exp\left(\int_{t-r}^{t} v(s)\, ds \right) + \sum_{i=1}^{n} q_i(t) \exp\left(\int_{t-\tau_i}^{t} v(s)\, ds \right) \tag{3.2.12}$$

possesses a positive solution on $[T+\sigma, \infty)$. To this end set

$$v_1(t) \equiv 0, \quad t \ge T,$$

and for $k = 1, 2, \ldots$

$$v_{k+1}(t) = \begin{cases} p v_k(t-r)\exp\left(\int_{t-r}^{t} v_k(s)\,ds\right) \\ \qquad + \sum_{i=1}^{n} q_i(t)\exp\left(\int_{t-\tau_i}^{t} v_k(s)\,ds\right), \quad t \geq T+\sigma \\ \beta_{k+1}(t), \quad T \leq t < T+\sigma \end{cases} \tag{3.2.13}$$

where $\{\beta_k\}$ is given function sequence satisfying

i) $\beta_k \in C^2([T, T+\sigma), \mathbb{R}_+)$ with $\beta_k' \geq 0$ and $\beta_k''(t) \geq 0$, $t \in [T, T+\sigma)$, $k = 1, 2, \ldots,$

ii) $\beta_k(t) = 0$, $t \in [T, T+\sigma-r]$, $\beta_k(T+\sigma) = v_k(T+\sigma)$, and the $\beta_k(t)$ are increasing in k for $t \in [T+\sigma-r, T+\sigma)$ and $k = 1, 2, \ldots,$

iii) for every $\ell = r, \tau_1, \ldots, \tau_n$, for $t \in [T+\sigma-\ell, T+\sigma)$, $k = 1, 2, \ldots,$

$$\int_{t}^{T+\sigma} \beta_k(s)\,ds \leq \int_{t+\ell}^{T+\sigma+\ell} v_k(s)\,ds.$$

Obviously, $v_1(t) \leq v_2(t) \leq \cdots$. By induction we will show that for $k = 1, 2, \ldots$ and $\ell = r, \tau_1, \ldots, \tau_n$

$$\frac{1}{\ell}\int_{t}^{t+\ell} v_k(s)\,ds \leq \mu^*, \quad t \geq T. \tag{3.2.14}$$

In fact, (3.2.14) is true for $k=1$. Assume (3.2.14) is true for some k. Then from (3.2.11) and (3.2.13) we have for $t \geq T+\sigma$, $\ell = r, \tau_1, \dots, \tau_n$

$$\frac{p}{\ell}\int_t^{t+\ell} v_k(s-r)\exp\left(\int_{s-r}^{s} v_k(\theta)\,d\theta\right)ds$$
$$+\frac{1}{\ell}\sum_{i=1}^{n}\int_t^{t+\ell} q_i(s)\exp\left(\int_{s-\tau_i}^{s} v_k(\theta)\,d\theta\right)ds$$
$$\leq p\mu^* e^{\mu^* r} + \frac{1}{\ell}\sum_{i=1}^{n} e^{\mu^*\tau_i}\int_t^{t+\ell} q_i(s)\,ds \leq \mu^*. \tag{3.2.15}$$

For every $\ell = r, \tau_1, \dots, \tau_n$, for $t \in [T+\sigma-\ell, T+\sigma)$, from (3.2.15) and condition iii) for $\{\beta_k\}$,

$$\frac{1}{\ell}\int_t^{t+\ell} v_{k+1}(t) = \frac{1}{\ell}\left[\int_t^{T+\sigma} \beta_{k+1}(s)\,ds + \int_{T+\sigma}^{t+\ell} v_{k+1}(s)\,ds\right]$$
$$\leq \frac{1}{\ell}\int_{T+\sigma}^{T+\sigma+\ell} v_k(s)\,ds \leq \mu^*.$$

From the monotonic property of $\beta_{k+1}(t)$ with respect to t we see that (3.2.14) also holds for $t \in [T, T+\sigma-\ell)$, $\ell = r, \tau_1, \dots, \tau_n$.

Let $v(t) = \lim_{k\to\infty} v_k(t)$. Then $v(t) \equiv 0$, $t \in [T, T+\sigma-r]$, $v(t)$ is increasing on $[T+\sigma-r, T+\sigma)$ and for $t \geq T$ and $\ell = r, \tau_1, \dots, \tau_n$,

$$\frac{1}{\ell}\int_t^{t+\ell} v(s)\,ds \leq \mu^*.$$

Taking limits as $k \to \infty$ on both sides of (3.2.13), and by using the Lebesgue monotone convergence theorem we see that $v(t)$ satisfies (3.2.12) for $t \geq$

$T+\sigma$. It is also easy to see that $v(t)$ is well-defined on $[T,\infty)$. In fact, by condition ii) of $\{\beta_k\}$,

$$v(T+\sigma) = \sum_{i=1}^{n} q_i(T+\sigma)\exp\left(\int_{T+\sigma-\tau_i}^{T+\sigma} v(s)\,ds\right)$$
$$\leq \sum_{i=1}^{n} q_i(T+\sigma)e^{\mu^*\tau_i} := M < \infty,$$

and hence $v(t) \leq M$ for $t \in [T, T+\sigma]$. If $v(t^*) = \infty$ for some $t^* > T+\sigma$, then choose an integer m such that $t^* - mr \in [T+\sigma-r, T+\sigma)$. By (3.2.12) we get an immediate contradiction if $p = 0$ and we have $v(t^*-mr) = \infty$ if $p \neq 0$. This is impossible. Furthermore, from condition i) of $\{\beta_k\}$ we get that $v(t)$ is continuous on $[T, T+\sigma]$, and in view of (3.2.12) we see that $v(t)$ is continuous on the whole interval $[T,\infty)$. Thus $v(t)$ is a positive solution of (3.2.12) on $[T+\sigma,\infty)$. Set

$$y(t) = \exp\left(-\int_{T+\sigma}^{t} v(s)\,ds\right), \quad t \geq T+\sigma.$$

Then $y(t)$ is a positive solution of (3.2.1).

Remark 3.2.1 Theorems 3.2.1 and 3.2.2 partially improve the criteria given by Result 3 since in (3.2.7) and (3.2.11) the "integral averages" of functions are used instead of the functions themselves. Both theorems are sharp since together they give the following result which extends Results 3.1 and 3.2.

Theorem 3.2.3. *Assume (3.2.2)–(3.2.4) hold, and* $\sum_{i=1}^{n} e^{\mu\tau_i} \int_t^{t+\ell} q_i(s)\,ds$ *is a nondecreasing function in* t *for* $\ell = r, \tau_1, \ldots, \tau_n$ *. Then (3.2.1) is oscillatory if and only if for all* $\mu > 0$, *and for* $\ell = r, \tau_1, \ldots, \text{or}, \tau_n$

$$\lim_{t\to\infty}\left[pe^{\mu r} + \frac{1}{\ell\mu}\sum_{i=1}^{n} e^{\mu\tau_i}\int_t^{t+\ell} q_i(s)\,ds\right] > 1.$$

Proof: Denote

$$f(t,\mu,\ell) = pe^{\mu r} + \frac{1}{\ell\mu}\sum_{i=1}^{n} e^{\mu\tau_i}\int_t^{t+\ell} q_i(s)\,ds.$$

From the condition we see that $\lim_{t\to\infty} f(t,\mu,\ell)$ exists for $\ell = r, \tau_1, \ldots, \tau_n$. By Lemma 3.1.1 we have

$$\lim_{t\to\infty} f(t,\mu,r) = \lim_{t\to\infty} f(t,\mu,\tau_1) = \cdots = \lim_{t\to\infty} f(t,\mu,\tau_n).$$

In this case for $\ell = r, \tau_1, \ldots, \tau_n$

$$\lim_{t\to\infty} f(t,\mu,\ell) = \liminf_{t\to\infty} f(t,\mu,\ell) = \sup_{t\geq T} f(t,\mu,\ell).$$

The conclusion is then immediate from Theorems 3.2.1 and 3.2.2.

□

As a special case, we have the following corollary.

Corollary 2.4. *Assume (3.2.2) and (3.2.3) hold, and there exists an* $\ell > 0$ *such that* $r = m_0\ell, \tau_i = m_i\ell$ *for some integers* m_i, $i = 0, \ldots, n$. *Furthermore,*

$$q_i(t) = g_i(t) + h_i(t), \quad i = 1, \ldots, n$$

where $g_i(t)$ *are* ℓ*-periodic functions with* $\frac{1}{\ell}\int_t^{t+\ell} g_i(s)\,ds = g_i^*$, $h_i(t)$ *are nondecreasing functions with* $\lim_{t\to\infty} h_i(t) = h_i^*$, $i = 1, \ldots, n$. *Then (3.2.1) is oscillatory if and only if for all* $\mu > 0$,

$$pe^{\mu r} + \frac{1}{\mu}\sum_{i=1}^{n} e^{\mu\tau_i}(q_i^* + h_i^*) > 1.$$

If $q_i(t) \equiv 0$ and $h_i(t)$ are constants, $i = 1, \ldots, n$, then Corollary 3.2.4 becomes Result 3.1; if $h_i(t) \equiv 0$, $i = 1, \ldots, n$, then Corollary 3.2.4 gives an extension to Result 3.2 since the requirement $\tau_i = ir, i = 1, \ldots, n$ is improved here.

Theorems 3.2.1 and 3.2.2 will also yield necessary and sufficient conditions for oscillation of some equations other than those satisfying the hypotheses of Theorem 3.2.3. To see this, we give the following example.

Example 3.2.1. Consider the equation

$$\frac{d}{dt}[y(t) - py(t-r)] + [1 - \frac{1}{t}(1 - \sin t)]y(t-\tau) = 0, \quad t \geq 2 \qquad (3.2.16)$$

where $0 \le p \le 1$, $r, \tau > 0$. Let $q(t) = 1 - \frac{1}{t}(1 - \sin t)$. It is easy to see that for all $\ell > 0$

$$\frac{1}{\ell}\int_t^{t+\ell} q(s)\,ds = \frac{1}{\ell}\int_t^{t+\ell} \left[1 - \frac{1}{s}(1 - \sin s)\right] ds \to 1, \quad \text{as } t \to \infty,$$

and

$$\sup_{t \ge T} \frac{1}{\ell}\int_t^{t+\ell} q(s)\,ds = 1.$$

According to Theorems 3.2.1 and 3.2.2, (3.2.16) is oscillatory if and only if for all $\mu > 0$

$$pe^{\mu r} + \frac{1}{\mu}e^{\mu\tau} > 1.$$

3.3. Criteria for Delay Equations

Now we are going to extend the criteria in Section 3.2 to the delay equation with variable delays

$$\frac{d}{dt}y(t) + \sum_{i=1}^{n} q_i(t)y(t - \tau_i(t)) = 0 \tag{3.3.1}$$

under the assumptions

(H1) $q_i(t)$, $\tau_i(t)$ $\in C([t_0, \infty), [0, \infty))$, $t - \tau_i(t) \to \infty$, as $t \to \infty$, $i = 1, \ldots, n$;

(H2) $0 \le q_i(t) \le q^*$, $0 \le \tau_i(t) \le \tau^*$, $i = 1, \ldots, n$, where $q^*, \tau^* > 0$.

The following is a conjecture of Hunt and Yorke in [13] which was recently established by Chen and Huang in [2].

RESULT 3.4. Under the assumptions (H1) and (H2), if for all $\mu > 0$

$$\liminf_{t\to\infty}\left\{\frac{1}{\mu}\sum_{i=1}^{n} q_i(t)e^{\mu\tau_i(t)}\right\} > 1,$$

then (3.3.1) is oscillatory.

Employing the method of Section 3.2 we may obtain this as a consequence of the result below. Again, since the "integral average" technique is involved, they greatly improve Result 3.4. Under certain circumstances, they also yield necessary and sufficient conditions. for oscillation.

Denote

$$f(t,\mu,\ell(t)) = \frac{1}{\mu\ell(t)}\sum_{i=1}^{n}\int_t^{t+\ell(t)} q_i(s)e^{\mu\tau_i(s)}\,ds.$$

THEOREM 3.3.1. *Let (H1) and (H2) hold, and for all* $\mu > 0$

$$\liminf_{t\to\infty} f(t,\mu,\tau_i(t)) > 1, \quad i = 1,\dots,n.$$

Then (3.3.1) is oscillatory.

Theorem 3.3.2. *Let (H1) and (H2) hold, and there exist* $\mu > 0$ *and* $T \geq t_0$ *such that*

$$\sup_{t \geq T} f(t, \mu^*, \tau_i(t)) \leq 1, \quad i = 1, \ldots, n.$$

Then (3.3.1) has a positive solution on $[T_1, \infty)$ *for some* $T_1 > T$.

Theorem 3.3.3. *Let (H1) and (H2) hold with* $0 \leq \tau_* \leq \tau_i(t) \leq \tau^*$. *Assume* $f(t, \mu, \tau_i(t))$ *is a nondecreasing function in* t *for* $\mu > 0$ *and* $i = 1, \ldots, n$. *Then (3.3.1) is oscillatory if and only if* $\lim_{t\to\infty} f(t, \mu, \tau_i(t)) > 1$ *for all* $\mu > 0$ *and some* $i = 1, \ldots, n$.

The proofs are basically the same as those of Theorems 3.2.1–3.2.3. Here we only give an outline of the proof of Theorem 3.3.1. Note also that Lemma 3.1.2 is needed in the proof of Theorem 3.3.3.

Proof of Theorem3.3.1: Assume (3.3.1) has an eventually positive solution $y(t)$. From (3.3.1) there exists $t_0 \geq 0$ such that $y(t) > 0, y'(t) \leq 0,\ t \geq t_0$. Denote a sequence $\{t_k\}_{k=0}^{\infty}$ by

$$t_k = \sup\{t : \min_{i=1,\ldots,n} \{t - \tau_i(t)\} \leq t_{k-1}\}.$$

Then $t_k \geq t_{k-1},\ k \geq 1$. Let $w(t) = -\frac{y'(t)}{y(t)}$. From (3.3.1)

$$w(t) = \sum_{i=1}^{n} q_i(t) \exp\left(\int_{t-\tau_i(t)}^{t} w(s)\, ds\right), \quad t \geq t_1. \tag{3.3.2}$$

We now define a sequence of functions $\{w_k(t)\}$ and a sequence of numbers $\{\mu_k\}$ as follows:

$$w_1(t) \equiv 0, \quad t \geq t_0$$

and for $k = 1, 2, \ldots$

$$w_{k+1} = \sum_{i=1}^{n} q_i(t) \exp\left(\int_{t-\tau_i(t)}^{t} w_k(s)\, ds\right), \quad t \geq t_k;$$

and $\mu_1 = 0$, and for $k = 1, 2, \ldots$

$$\mu_{k+1} = \inf_{t \geq t_0} \min_{\ell = \tau_1, \ldots, \tau_n} \left\{ \frac{1}{\ell(t)} \sum_{i=1}^{n} \int_t^{t+\ell(t)} q_i(s) e^{\mu_k \tau_i(s)}\, ds \right\}.$$

Similar to the proof of Theorem 3.2.1 we can show that for $\ell(t) = \tau_1(t), \ldots, \tau_n(t)$

$$\frac{1}{\ell(t)} \int_t^{t+\ell(t)} w_k(s)\, ds \geq \mu_k \to \infty, \quad \text{as } k \to \infty$$

and $w(t) \geq w_k(t)$ for $t \geq t_k$, $k = 1, 2, \ldots$. Hence for $\ell(t) = \tau_1(t), \ldots, \tau_n(t)$

$$\frac{1}{\ell(t)} \int_t^{t+\ell(t)} w(s)\, ds \to \infty, \quad \text{as } t \to \infty. \tag{3.3.3}$$

From Theorem 1 in [2] we get that

$$\liminf_{t \to \infty} \max_{1 \leq i \leq n} \{q_i(t) \tau_i(t)\} > 0,$$

and that if we let $q(t) = q_j(t), \tau(t) = \tau_j(t)$ be such that for each t

$$q_j(t)\tau_j(t) = \max\{q_i(t)\tau_i(t), \ i = 1, \ldots, n\},$$

then $0 < q_* \leq q(t) \leq q^*$ and $0 < \tau_* \leq \tau(t) \leq \tau^*$. From (3.3.3), $\frac{1}{\tau(t)} \int_t^{t+\tau(t)} w(s)\,ds \to \infty$ as $t \to \infty$, hence $\int_t^{t+\tau^*} w(s)\,ds \to \infty$ as $t \to \infty$. Therefore,

$$\limsup_{t\to\infty} \int_t^{t+\tau_*/2} w(s)\,ds = \infty.$$

Since

$$y'(t) = -\sum_{i=1}^{n} q_i(t) y(t - \tau_i(t)) \leq -q(t) y(t - \tau(t)) \leq -q_* y(t - \tau_*),$$

the rest part of the proof is similar to that of Theorem 3.2.1.

□

REMARK 3.3.1 It can be shown that Theorems 3.3.1–3.3.3 still hold if we replace (H2) by:

(H3) there exists a nonempty subset I of the set $\{1, \ldots, n\}$ such that $\tau(t) = \min\{\tau_i(t), i \in I\}$ satisfying $t - \tau(t) \to \infty$ as $t \to \infty$,

$$\liminf_{t\to\infty} \tau(t) = \tau_0 > 0$$

and

$$\liminf_{T\to\infty} \int_{t-\tau_0}^{t} \sum_{i\in I} q_i(s)\,ds > 0.$$

The corresponding results are improvements of Theorem 2 in [2].

3.4. Comparison Results

Using the above theorems we can derive some comparison results for oscillation and for the existence of nonoscillatory solutions for a pair of equations. Here we will only mention the results based on the theorems in Section 3.2. Parallel results based on the theorems in Section 3.3 are obtained in a similar fashion.

Consider two equations

$$\frac{d}{dt}[y(t)-py(t-r)]+\sum_{i=1}^{n}q_i(t)y(t-\tau_i)=0 \tag{3.4.1}$$

and

$$\frac{d}{dt}[y(t)-\tilde{p}y(t-\tilde{r})]+\sum_{i=1}^{n}\tilde{q}_i(t)y(t-\tilde{\tau}_i)=0. \tag{3.4.2}$$

Assume (3.2.2)–(3.2.4) hold for both the sets $\{q_i(t)\}_{i=1}^{n}$ and $\{\tilde{q}_i(t)\}_{i=1}^{n}$. Furthermore, assume the conditions for $q_i(t)$, $i=1,\dots,n$, in Corollary 3.2.4 hold.

THEOREM 3.4.1.

i) Suppose (3.4.1) is oscillatory, and $\tilde{p}\ge p$, $\tilde{r}\ge r$, *and for all* $\mu>0$, $\ell=\tilde{r},\tilde{\tau}_1,\dots,\tilde{\tau}_n$

$$\liminf_{t\to\infty}\frac{1}{\ell}\sum_{i=1}^{n}e^{\mu\tilde{\tau}_i}\int_t^{t+\ell}\tilde{q}_i(s)\,ds\ge\sum_{i=1}^{n}e^{\mu\tau_i}(q_i^*+h_i^*). \tag{3.4.3}$$

Then (3.4.2) is oscillatory.

ii) Suppose (3.4.2) is oscillatory, and $\widetilde{p} \le p$, $\widetilde{r} \le r$, *and there exists a* $T \ge t_0$ *such that for all* $\mu > 0$, $\ell = \widetilde{r}, \widetilde{\tau}_1, \ldots, \widetilde{\tau}_n$

$$\sup_{t \ge T} \frac{1}{\ell} \sum_{i=1}^{n} e^{\mu \tilde{\tau}_i} \int_t^{t+\ell} \widetilde{q}_i(s)\, ds \le \sum_{i=1}^{n} e^{\mu \tau_i}(q_i^* + h_i^*). \tag{3.4.4}$$

Then (3.4.1) is oscillatory.

iii) Suppose (3.4.2) is nonoscillatory, and $\widetilde{p} \ge p$, $\widetilde{r} \ge r$, *and for all* $\mu > 0$, $\ell = \widetilde{r}, \widetilde{\tau}_1, \ldots, \widetilde{\tau}_n$, *(3.4.3) holds. Then (3.4.1) has a nonoscillatory solution.*

iv) Suppose (3.4.1) has a nonoscillatory solution, and $\widetilde{p} \le p$, $\widetilde{r} \le r$, *and there exists a* $T \ge t_0$ *such that for all* $\mu > 0$, $\ell = \widetilde{r}, \widetilde{\tau}_1, \ldots, \widetilde{\tau}_n$, *(3.4.4) holds. Then (3.4.2) has a nonoscillatory solution.*

PROOF: i) Since (3.4.1) is oscillatory, by Corollary 3.2.4 we have for all $\mu > 0$,

$$pe^{\mu r} + \frac{1}{\mu} \sum_{i=1}^{n} e^{\mu \tau_i}(q_i^* + h_i^*) > 1.$$

Then (3.4.3) gives that for all $\mu > 0$, $\ell = \widetilde{r}, \widetilde{\tau}_1, \ldots, \widetilde{\tau}_n$

$$\liminf_{t \to \infty} \left[\widetilde{p} e^{\mu \tilde{r}} + \frac{1}{\ell \mu} \sum_{i=1}^{n} e^{\mu \tilde{\tau}_i} \int_t^{t+\ell} \widetilde{q}_i(s)\, ds \right] > 1.$$

By Theorem 3.2.1, (3.4.2) is oscillatory.

ii) If not, (3.4.1) has a nonoscillatory solution. By Corollary 3.2.4 there exists a $\mu^* > 0$ such that

$$pe^{\mu^* r} + \frac{1}{\mu^*}\sum_{i=1}^{n} e^{\mu^* r_i}(q_i^* + h_i^*) \leq 1.$$

Then (3.4.4) gives that for all $\mu > 0$, $\ell = \widetilde{r}, \widetilde{\tau}_1, \ldots, \tau_n$

$$\sup_{t \geq T}\left[\widetilde{p}e^{\mu^* \widetilde{r}} + \frac{1}{\ell\mu^*}\sum_{i=1}^{n} e^{\mu^* \widetilde{\tau}_i}\int_t^{t+\ell}\widetilde{q}_i(s)\,ds\right] \leq 1.$$

By Theorem 3.2.2, (3.4.2) has a nonoscillatory solution, contradicting the assumption.

iii) and iv) are the converses of i) and ii).

Theorem 3.4.1 provides criteria for oscillation and for existence of nonoscillatory solutions for a certain class of equations by means of a comparison with equations having constant, periodic or even aperiodic coefficients. The criteria are given by investigating the "integral averages" of the coefficients q_i over an interval of length ℓ, and can be easily verified. It is clear that Theorem 3.4.1 substantially improves Theorems 2 and 3 in [11] under the conditions (3.2.2)–(3.2.4) since the latter are only very special cases of parts ii) and iv) in Theorem 3.4.1. Similar discussion for delay differential equations can also improve the results in [14] by using the "integral average" technique.

3.5. Explicit Conditions for Oscillation

We can also obtain some explicit conditions for oscillation from Theorems 3.2.1 and 3.3.1. As an example, we mention the result derived from Theorem 3.2.1.

Theorem 3.5.1. *Assume (3.2.2)–(3.2.4) hold, and for* $\ell = r, \tau_1, \ldots, \tau_n$

$$\liminf_{t\to\infty} \sum_{i=1}^{n} \sum_{k=0}^{\infty} \left(\frac{1}{\ell} \int_t^{t+\ell} q_i(s)\, ds \right) p^k(kr + \tau_i) > \frac{1}{e}. \tag{3.5.1}$$

Then (3.2.1) is oscillatory.

Proof: We show that (3.2.7) is true for all $\mu > 0$, and then (3.2.1) is oscillatory by Theorem 3.2.1.

For any $\mu > 0$, if $pe^{\mu r} \geq 1$, then (3.2.7) is obviously true. So we only consider the values of μ such that $pe^{\mu r} < 1$. From a well-known inequality we see that for $\mu > 0$,

$$e^{\mu(kr+\tau_i)} \geq e\mu(kr + \tau_i). \tag{3.5.2}$$

So we have

$$\frac{1}{\mu}\sum_{i=1}^{n}\left(\frac{1}{\ell}\int_t^{t+\ell} q_i(s)\,ds\right)e^{\mu\tau_i}(1-pe^{\mu r})^{-1}$$
$$=\frac{1}{\mu}\sum_{i=1}^{n}\sum_{k=0}^{\infty}\left(\frac{1}{\ell}\int_t^{t+\ell} q_i(s)\,ds\right)p^k e^{\mu(kr+\tau_i)}$$
$$\geq \sum_{i=1}^{n}\sum_{k=0}^{\infty}\left(\frac{1}{\ell}\int_t^{t+\ell} q_i(s)\,ds\right)p^k e(kr+\tau_i).$$

Then (3.5.1) implies that for $\ell = r, \tau_1, \ldots, \tau_n$

$$\liminf_{t\to\infty}\frac{1}{\mu}\sum_{i=1}^{n}\left(\frac{1}{\ell}\int_t^{t+\ell} q_i(s)\,ds\right)e^{\mu\tau_i}(1-pe^{\mu r})^{-1} > 1$$

or

$$\liminf_{t\to\infty}\frac{1}{\ell\mu}\sum_{i=1}^{n}e^{\mu\tau_i}\int_t^{t+\ell} q_i(s)\,ds > 1-pe^{\mu r}.$$

Hence (3.2.7) holds for all $\mu > 0$, and $\ell = r, \tau_1, \ldots, \tau_n$, and the theorem is proved.

Theorem 3.5.1 gives a sharp condition for oscillation in the sense that for the constant coefficient case it almost coincides with our result in Corollary 2.4.3 and it is better than the corresponding results in [5,7,8,9,15].

References

[1] O. Arino and I. Györi, *Necessary and sufficient condition for oscillation of a neutral differential system with several delays*, J. Differ. Eqns. 81 (1989), 98–105.

[2] S. Chen and Q. Huang, *On a conjecture of Hunt and Yorke,* J. Math. Anal. Appl. **159** (1991), 469–484.

[3] Q. Chuanxi and G. Ladas, *Oscillations of neutral differential equations with variable coefficients,* Applicable Analysis, **32** (1989), 215-228.

[4] ————, *Existence of positive solutions for neutral differential equations,* J. Appl. Math. Simulations, **2** (1989), 267-276.

[5] ————, *Oscillations of first-order neutral equations with variable coefficients,* Mh. Math. **109** (1990), 103–111.

[6] Q. Chuanxi, G. Ladas, B. G. Zhang and T. Zhao, *Sufficient conditions for oscillation and existence of positive solutions,* Applicable Analysis, **35** (1990), 187–194.

[7] M. K. Grammatikopoulos, E. A. Grove and G. Ladas, *Oscillation of first order neutral delay differential equations,* J. Math. Anal. Appl. **120** (1986), 510–520.

[8] M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, *Oscillation and asymptotic behavior of higher order neutral equations with variable coefficients,* Chin. Ann. of Math. **3** (1988), 322–337.

[9] M. K. Grammatikopoulos, G. Ladas and Y. G. Sficas, *Oscillation and asymptotic behavior of neutral equations with variable coefficients,* Radovi Mathematicki, **2** (1986), 279–303.

[10] E. A. Grove, M. R. S. Kulenović and G. Ladas, *Sufficient conditions for oscillation and nonoscillation of neutral equations,* J. Differ. Eqns. **68** (1987), 373–382.

[11] ————, *A Myskis-type comparison result for neutral equations,* Math. Nachr. **146** (1990), 195–206.

[12] Q. Huang and S. Chen, *Oscillation of neutral differential equations with periodic coefficients,* Proc. Amer. Math. Soc. **110** (1990), 997–1001.

[13] B. R. Hunt and J. A. Yorke, *When all solutions of* $x' = -\sum q_i x(t - T_i(t))$ *oscillate,* J. Differential Equations, **53** (1984), 139–145.

[14] G. Ladas, C. Qian and J. Yan, *A comparison result for the oscillation of delay differential equations,* Proc. Amer. Math. Soc. **114** (1992), 939-947.

[15] G. Ladas and Y. G. Sficas, *Oscillations of neutral delay differential equations,* Cand. Math. Bull. 29 (1986), 438–445.

[16] ———, *Oscillation of higher-order neutral equations,* J. Austral. Math. Soc. Ser. **B27** (1986), 502–511.

[17] G. Ladas, Ch. G. Philos and Y. G. Sficas, *Oscillation in neutral equations with periodic coefficients,* Proc. Amer. Math. Soc. **113** (1991), 123–134.

CHAPTER 4

OSCILLATION OF NONLINEAR EQUATIONS OF SECOND ORDER

4.1. Introduction

In this chapter we deal with the oscillatory behaviour of the neutral differential equation

$$\frac{d^2}{dt^2}[y(t) - py(t-r)] + q(t)f\big(y(t-\tau(t))\big) = 0 \tag{4.1.1}$$

under the following assumption

(H) p and r are positive numbers;

q and $\tau \in C(\mathbb{R}_+, \mathbb{R}_+)$, $q(t) \not\equiv 0$, $t - \tau(t)$ is increasing and tends to ∞ as $t \to \infty$, $\tau(t) > r$; $f \in C(\mathbb{R}, \mathbb{R})$ is increasing, $f(-x) = -f(x)$, $f(xy) \geq f(x)f(y)$, for $xy > 0$,

$f(\infty) = \infty$, and $\frac{f(y)}{y} \to \infty$ or 1 as $y \to 0$.

The oscillation problem of equation (4.1.1) has received wide attention [1,2,4-9,11,12]. Much work has been done for the case where $p < 0$. In [7,9,11], the case $p > 0$ was studied for linear equations with constant coefficients and constant delay, some conclusions and conjectures were given, but the oscillation result specialized to the case where $p > 1$ is only a sufficient condition which guarantees that equation (4.1.1) has no bounded nonoscillatory solutions. In

[4] the oscillatory problem of (4.1.1) was considered for the general form of equations, but the results still do not apply to the case $p \geq 1$.

The aim of this paper is to obtain some oscillation criteria for equation (4.1.1) for the case where $p \geq 1$ under the assumptions (H) and, along the way, we establish the conjectures in [11]. The results obtained in this paper can be easily extended to equations of the more general form

$$\frac{d}{dt}[a(t)\frac{d}{dt}(y(t) - py(t-r))] + q(t)f\big(y(t-\tau(t))\big) = 0.$$

4.2. Comparison Results for Oscillation

For comparison purposes we mention the results for the case $0 < p < 1$ obtained in [4]:

Lemma 4.2.0. *Under the assumptions (H), if the equation*

$$z'' + q(t)f\left(\frac{\lambda(t-\tau(t))}{t} z(t)\right) = 0 \tag{4.2.1}$$

is oscillatory for some $0 < \lambda < 1$, *then the nonoscillatory solutions of Eq. (4.1.1) tend to zero as* $t \to \infty$.

Theorem 4.2.0. *In addition to the conditions of Lemma 4.2.0, assume further* *that*

$$\limsup_{t\to\infty}\int_{t-\tau(t)+r}^{t}\big(u-(t-\tau(t)+r)\big)q(u)du > \begin{cases} p & if \ \frac{f(y)}{y}\to 1,\ y\to 0 \\ 0 & if \ \frac{f(y)}{y}\to\infty,\ y\to 0. \end{cases} \qquad (4.2.2)$$

Then Eq. (4.1.1) is oscillatory.

Now we state our results below.

DEFINITION 4.2.1: Let E be a subset of $\mathbb{R}_+$. Define

$$\rho_t(E) = \frac{\mu\{E\cap[0,t]\}}{t}, \quad \text{and} \quad \rho(E) = \limsup_{t\to\infty}\rho_t(E)$$

where μ is the Lebesgue measure.

LEMMA 4.2.1. *Assume (H) holds and $p = 1$. Then the nonoscillatory solutions $y(t)$ of Eq. (4.1.1) are bounded provided the equation*

$$z''(t) + q(t)f\big(Q(t)z(t)\big) = 0 \qquad (4.2.3)$$

is oscillatory, where $Q(t) = \frac{1}{3rt}\big(t-\tau(t)\big)^2$.

LEMMA 4.2.2. *Assume (H) holds and $p > 1$. Then the nonoscillatory solutions $y(t)$ of Eq. (4.1.1) satisfy $y(t) < py(t-r)$ eventually provided the following conditions hold:*

i) $z''(t) + q(t)f\big(R(t,\lambda)z(t)\big) = 0 \qquad (4.2.4)$

is oscillatory for all $0 < \lambda < 1$, *where* $R(t,\lambda) = \frac{\lambda}{t}\, p^{\frac{t-\tau(t)}{r}}$;

ii) $\limsup_{\substack{t\to\infty \\ t \notin E}} p_1^{-t/r} \int_0^t (t-u)q(u)f\big(u-\tau(u)+r\big)du > 0$ (4.2.5)

holds for some $p_1 > p$ *and any set* E *with* $\rho(E) = 0$.

COROLLARY 4.2.1. *In addition to the assumptions of Lemma 4.2.1, assume further that* τ *is a positive constant, and*

$$\sum_{i=0}^{\infty} \int_{T+ir}^{T+ir+\alpha} (u-T)q(u)du = \infty \tag{4.2.6}$$

holds for any $T \in \mathbb{R}_+$, *and* $0 < \alpha \le r$, *then all nonoscillatory solutions of Eq. (4.1.1) tend to zero as* $t \to \infty$.

COROLLARY 4.2.2. *In addition to the assumptions of Lemma 4.2.2, assume further that* τ *is a positive constant,*

$$\int_t^{\infty} (u-t)q(u)du = \infty$$

and

$$\sum_{i=0}^{\infty} f(p^i) \int_{T+ir}^{T+ir+\alpha} (u-T)q(u)du = \infty \tag{4.2.7}$$

holds for any $T \in \mathbb{R}_+$ *and* $0 < \alpha \le r$, *then all nonoscillatory solutions of Eq. (4.1.1) tend to zero as* $t \to \infty$.

Remark 4.2.1. Corollaries 4.2.1 and 4.2.2 establish the conjectures in [11] for $n = 2$ since (4.2.6) and (4.2.7) are true for the case that $q(t)$ is a positive constant. In general (i.e. for even order equations) the conjectures can be established by similar arguments as in this section.

Theorem 4.2.1. *Assume (H) holds and* $p \geq 1$. *In addition to the conditions of Lemmas 4.2.1 and 4.2.2 for the cases* $p = 1$ *and* $p > 1$, *respectively, we assume (4.2.2) holds. Then Eq. (4.1.1) is oscillatory.*

4.3. Proofs

Proof of Lemma 4.2.1: Assume the contrary, and without loss of generality let $y(t)$ be an eventually positive solution of Eq. (4.1.1). Let $z(t) = y(t) - y(t-r)$. Then (4.1.1) becomes

$$z''(t) + q(t)f\big(y(t-\tau(t))\big) = 0 \tag{4.3.1}$$

and $z''(t) \leq 0,\ t \geq t_0 \geq 0$. We claim $z'(t) > 0,\ t \geq t_0$. Otherwise, $z'(t) < 0,\ t \geq t_1 \geq t_0$. Then $z'(t) < -\ell < 0,\ t \geq t_1$. This gives that $z(t) = y(t) - y(t-r) \to -\infty$, contradicting that $y(t)$ is eventually positive.

a) Assume $z(t) > 0,\ t \geq t_2 \geq t_1$. From Erbe's lemma [3] we see that for any $0 < k < 1$ and $i = 0, 1, 2, \ldots,$ there exists $T_i \geq t_0$ such that

$$z\big(t - \tau(t) - ir\big) \geq \frac{k\big(t - \tau(t) - ir\big)}{t}\, z(t), \quad t - \tau(t) \geq T_i. \tag{4.3.2}$$

Without loss of generality we may assume $t_0 = T_0$. Then we can choose $T_i = T_0 + ir$ for a common k. In fact, from the proof of the lemma, it suffices to show that for $i = 0, 1, 2, \ldots,$

$$(1-k)\big(t - \tau(t) - ir\big) \geq \tilde{T}_0 \triangleq (1-k)T_0, \quad t - \tau(t) \geq T_i. \tag{4.3.3}$$

(4.3.3) is obviously true for $i = 0$. And if (4.3.3) is true for some i, then for $t - \tau(t) \geq T_{i+1} = T_i + r$, we have

$$(1-k)\big(t - \tau(t) - (i+1)r\big) \geq (1-k)(T_i - ir) = (1-k)T_0 = \tilde{T}_0.$$

Denote $R_{T_0} = \{t : t + \tau(t) \geq T_0\}$. Then for any $t \in R_{T_0}$, there is a positive integer n satisfying

$$T_0 \leq t - \tau(t) - nr < T_0 + r.$$

Since

$$\begin{aligned} y\big(t - \tau(t)\big) &= \sum_{i=0}^{n-1} z\big(t - \tau(t) - ir\big) + y\big(t - \tau(t) - nr\big) \\ &\geq \sum_{i=0}^{n} z\big(t - \tau(t) - ir\big) \end{aligned}$$

(here $\sum_{i=0}^{-1} = 0$), from Eq. (4.1.1) we have

$$z''(t) + q(t)f\Big(\sum_{i=0}^{n} z(t - \tau(t) - ir)\Big) \le 0.$$

Using (4.3.2) we get

$$z''(t) + q(t)f\Big(\frac{k}{t} \sum_{i=0}^{n} (t - \tau(t) - ir)z(t)\Big) \le 0,$$

i.e.,

$$z''(t) + q(t)f\Big(\frac{k}{t}\,(n+1)(t - \tau(t) - \frac{n}{2}\,r)z(t)\Big) \le 0.$$

Since $n + 1 > \frac{t-\tau(t)-T_0}{r}$, $nr \le t - \tau(t) - T_0$, we get

$$z''(t) + q(t)f\Big(\frac{k}{2rt}\,[(t - \tau(t))^2 - T_0^2]z(t)\Big) \le 0.$$

Choose $T \ge T_0$ large enough, then the above inequality becomes

$$z''(t) + q(t)f\Big(\frac{1}{3rt}\,(t - \tau(t))^2 z(t)\Big) \le 0, \quad t \ge T.$$

Noting that $z(t), z(T)$ are upper and lower solutions of Eq. (4.1.1), respectively, and using Theorem 7.4 in [10], we see there is a solution $y(t)$ satisfying $z(T) \le y(t) \le z(t)$, contradicting the fact that Eq. (4.2.3) is oscillatory.

b) Assume $z(t) < 0,\ t \geq t_2 \geq t_1$. Then $y(t) - y(t-r) < 0,\ t \geq t_2$. It is obvious that $y(t)$ is a bounded solution since $y(t)$ is eventually positive.

□

For the proof of Lemma 4.2.2 we shall need the following lemma.

LEMMA 4.3.1. *Assume set* $E \subset \mathbb{R}_+$ *and* $\rho(E) = \rho > 0$. *Then for any* $t_0 \in \mathbb{R}_+$ *and integer* n, *there exists a* $T \in [t_0, t_0+r)$ *such that* $\{T+ir\}_{i=1}^{\infty}$ *intersects* E *at least* n *times.*

PROOF: Assume that contrary holds, i.e., there exist a $t_0 \in \mathbb{R}_+$ and an integer N, such that $\{T+ir\}_{i=1}^{\infty}$ intersects E at most N times for any $T \in [t_0, t_0+r)$. This implies that $\mu\{E\} < \infty$. But $\rho(E) = \rho > 0$ means there exist $t_n \to \infty$ such that $\rho_{t_n}(E) \geq \frac{\rho}{2} > 0$. Thus

$$\mu\{E \cap [0, t_n]\} \geq \frac{\rho}{2}\, t_n \to \infty, \quad n \to \infty,$$

and this is impossible.

□

PROOF OF LEMMA 4.2.2: Assume the contrary, and without loss of generality let $y(t)$ be an eventually positive solution of Eq. (4.1.1). Let $z(t) = y(t) -$

$py(t-r)$. Then (4.1.1) becomes (4.3.1) and $z''(t) \leq 0$ eventually. There are three possibilities:

a) $z'(t) > 0,\ z(t) > 0,$ b) $z'(t) < 0,\ z(t) < 0,$ c) $z'(t) > 0,\ z(t) < 0$

eventually.

a) Assume $z'(t) > 0,\ z(t) > 0,\ t \geq t_0 \geq 0$. Then (4.3.2) holds and for any $t \in R_{T_0}$ defined as in the proof of Lemma 4.2.1, there is also a positive integer n satisfying

$$T_0 \leq t - \tau(t) - nr < T_0 + r.$$

Since

$$\begin{aligned} y\big(t-\tau(t)\big) &= \sum_{i=0}^{n-1} p^i z\big(t-\tau(t)-ir\big) + p^n y\big(t-\tau(t)-nr\big) \\ &\geq \sum_{i=0}^{n} p^i z\big(t-\tau(t)-ir\big), \end{aligned}$$

from Eq. (4.1.1) we have

$$z''(t) + q(t) f\Big(\sum_{i=0}^{n} p^i z(t-\tau(t)-ir)\Big) \leq 0.$$

(4.3.2) gives that

$$z''(t) + q(t) f\Big(\frac{k}{t} \sum_{i=0}^{n} p^i (t-\tau(t)-ir) z(t)\Big) \leq 0,$$

i.e.,

$$z''(t)+q(t)f[(\frac{k}{t}(t-\tau(t))\frac{p^{n+1}-1}{p-1}-\frac{kr}{t}\sum_{i=1}^{n}ip^{i})z(t)]\leq 0. \qquad (4.3.4)$$

Since

$$\sum_{i=1}^{n}ip^{i}=\frac{np^{n+2}-(n+1)p^{n+1}+p}{(p-1)^{2}},$$

we have

$$\begin{aligned}
&\frac{k}{t}\left(t-\tau(t)\right)\frac{p^{n+1}-1}{p-1}-\frac{kr}{t}\sum_{i=1}^{n}ip^{i}\\
&=\frac{k}{(p-1)^{2}t}\left[\left(t-\tau(t)\right)(p^{n+2}-p^{n+1}-p+1)\right.\\
&\qquad\left.-r\left(np^{n+2}-(n+1)p^{n+1}+p\right)\right]\\
&=\frac{k}{(p-1)^{2}t}\left[\left(t-\tau(t)-nr\right)p^{n+2}\right.\\
&\qquad\left.-\left(t-\tau(t)-(n+1)r\right)p^{n+1}-\left(t-\tau(t)+r\right)p+\left(t-\tau(t)\right)\right]\\
&\geq\frac{k}{(p-1)^{2}t}\left[T_{0}p^{n+2}-T_{0}p^{n+1}-\left(t-\tau(t)+r\right)p+\left(t-\tau(t)\right)\right]\\
&\geq\frac{1}{t}\,p^{n+2}\geq\frac{1}{t}\,p^{\frac{t-\tau(t)-T_{0}+r}{r}}=\frac{\lambda}{t}\,p^{\frac{t-\tau(t)}{r}}
\end{aligned} \qquad (4.3.5)$$

for some $0<\lambda<1$ if T_0 and t are sufficiently large. Substituting (4.3.5) into (4.3.4), we have

$$z''(t)+q(t)f\left(\frac{\lambda}{t}\,p^{\frac{t-\tau(t)}{r}}z(t)\right)\leq 0.$$

Noting that $z(t), z(T_0)$ are upper and lower solutions of (4.2.4), respectively, and using Theorem 7.4 in [10] we see there is a solution $y(t)$ satisfying $z(T_0) \le y(t) \le z(t)$, contradicting the fact that Eq. (4.2.4) is oscillatory for all $0 < \lambda < 1$.

b) Assume $z'(t) < 0$, $z(t) < 0$, $t \ge t_0 \ge 0$. Then $z(t) \le -\ell t$, $t \ge t_0$, for some $\ell > 0$.

We claim $z(t) \ge -p_1^{t/r}$ essentially, where $p_1 > p$ is arbitrary, i.e., if $E = \{t : z(t) < -p_1^{t/r}\}$. Then $\rho(E) = 0$. Otherwise, $\rho(E) = \rho > 0$. By Lemma 4.3.1, for any n, there exists a $T_1 \in [t_0, t_0 + r)$ such that $\{T_1 + ir\}_{i=1}^{\infty}$ intersects E at least n times. Assume $M = \max\limits_{t \in [t_0, t_0 + r]} \{y(t)\}$. Then if n is sufficiently large,

$$\begin{aligned} y(T_1 + nr) &\le p^n y(T_1) + z(T_1 + nr) \le p^n M - p_1^{\frac{T_1 + nr}{r}} \\ &= p^n M - p_1^{n + \frac{T_1}{r}} < 0, \end{aligned}$$

contradicting that $y(t) > 0$ eventually.

It is easy to see that condition ii) implies that

$$\int_0^{\infty} q(t) f\big(u - \tau(u) + r\big)\, du = \infty. \tag{4.3.6}$$

Condition ii) also implies that $z'(t) < -\mu$ for all $\mu > 0$ eventually. For otherwise, there exists a $\mu > 0$ such that $z' \ge -\mu$, $t \ge T_2$. From (4.3.1) and $y(t-r) \geq -\frac{1}{p}\, z(t)$, we get

$$z''(t) + q(t)f\big(-\frac{1}{p}\, z(t-\tau(t)+r)\big) \leq 0 \tag{4.3.7}$$

$$z'(t) + \int_{T_2}^{t} q(u)f\big(-\frac{1}{p}\, z(u-\tau(u)+r)\big)du \leq 0.$$

Noting that $z\big(t-\tau(t)+r\big) \leq -\ell\big(t-\tau(t)+r\big)$ we have

$$z'(t) + \int_{T_2}^{t} q(u)f\big(\frac{\ell}{p}\,(u-\tau(u)+r)\big)du \leq 0$$

$$f(\frac{\ell}{p}) \int_{T_2}^{t} q(u)f\big(u-\tau(u)+r\big)du \leq -z'(t) \leq \mu,$$

which is in contradiction with (4.3.6). Hence from (4.3.7) we see that for any $\mu > 0$, there is a T_μ such that for $t \geq T_\mu$

$$z(t) + \int_{T_\mu}^{t} (t-u)q(u)f\big(\frac{\mu}{p}\,(u-\tau(u)+r)\big)du \leq 0.$$

On $E^C \cap [T_\mu, \infty)$

$$-p_1^{t/r} + \int_{T_\mu}^{t} (t-u)q(u)f\big(\frac{\mu}{p}\,(u-\tau(u)+r)\big)\, du \leq 0,$$

$$f(\frac{\mu}{p})p_1^{-\frac{t}{r}} \int_{T_\mu}^{t} (t-u)q(u)f\big(u-\tau(u)+r\big)du \leq 1.$$

Hence

$$p_1^{-\frac{t}{r}} \int_{T_\mu}^{t} (t-u)q(u)f\big(u-\tau(u)+r\big)du \leq \frac{1}{f(\frac{\mu}{p})}\,, \tag{4.3.8}$$

contradicting (4.2.5) since $\mu > 0$ is arbitrary and $f(\infty) = \infty$.

c) Assume $z'(t) > 0$, $z(t) < 0$, $t \geq t_0 \geq 0$. Then $y(t) < py(t-r)$ is obvious.

□

Proof of Corollary 4.2.1: If not, there exists an eventually positive solution $y(t)$ satisfying $\limsup_{t\to\infty} y(t) > 0$, and this can only occur when $z'' \lesssim 0$, $z'(t) > 0$, and $z(t) < 0$, $t \geq t_0 \geq 0$, hence $z'(t) \to 0$, $z(t) \to 0$ as $t \to \infty$. If $\liminf_{t\to\infty} y(t) > 0$, then $y(t) \geq a > 0$, $t \geq t_1 \geq t_0$. Integrating (4.3.1) twice we get

$$z(t) + \int_t^\infty (u-t)q(u)f(a)du < 0.$$

Taking limit superoirs on both sides as $t \to \infty$ we have

$$\limsup_{t\to\infty} \int_t^\infty (u-t)q(u)du \leq 0,$$

which is in contradiction with (4.2.6). So

$$\limsup_{t\to\infty} y(t) > 0 \quad \text{and} \quad \liminf_{t\to\infty} y(t) = 0. \tag{4.3.9}$$

Then we can choose $t_2 > t_1 \geq t_0$ such that $y(t_2 - \tau) > y(t_1 - \tau)$. We claim

$$\liminf_{n\to\infty} y(t_2 - \tau + nr) > 0. \tag{4.3.10}$$

In fact

$$y(t_2 - \tau + nr) = \sum_{i=1}^{n} z(t_2 - \tau + ir) + y(t_2 - \tau)$$

and

$$y(t_1 - \tau + nr) = \sum_{i=1}^{n} z(t_1 - \tau + ir) + y(t_1 - \tau).$$

Since $z(t_2 - \tau + ir) \geq z(t_1 - \tau + ir)$ for $i = 1, 2, \ldots, n,$ and

$$\liminf_{n \to \infty} y(t_1 - \tau + nr) \geq 0,$$

we have

$$\liminf_{n \to \infty} y(t_2 - \tau + nr) \geq y(t_2 - \tau) - y(t_1 - \tau) > 0.$$

Now, choose $t_0 \leq t_1 < t_2 < t_3$ such that for any $T \in [t_2, t_3]$,

$$y(t_1 - \tau) < y(t_2 - \tau) \leq y(T - \tau).$$

From the above discussion, we see that (4.3.10) holds, i.e., there exists a $\mu > 0$ such that $y(t_2 - \tau + nr) \geq \mu$ for all n. It is easy to see that for $T \in [t_2, t_3]$

$$\begin{aligned} y(T - \tau + nr) &= \sum_{i=1}^{n} z(T - \tau + ir) + y(T - \tau) \\ &\geq \sum_{i=1}^{n} z(t_2 - \tau + ir) + y(t_2 - \tau) \\ &= y(t_2 - \tau + nr) \geq \mu. \end{aligned}$$

From (4.3.1) we have

$$-z'(s) + \int_s^t q(u)f(y(u-\tau))du \le 0, \qquad t_0 \le s \le t,$$

$$z(t_0) + \int_{t_0}^t (u-t_0)q(u)f(y(u-\tau))du \le 0, \qquad t_0 < t.$$

Hence

$$z(t_0) + f(\mu)\sum_{i=0}^{n}\int_{t_2+ir}^{t_3+ir}(u-t_0)q(u)du \le 0,$$

and then

$$z(t_0) + f(\mu)\sum_{i=0}^{n}\int_{t_2+ir}^{t_3+ir}(u-t_2)q(u)du \le 0,$$

contradicting (4.2.6). □

PROOF OF COROLLARY 4.2.2: If not, similar to the proof of Corollary 4.2.1 we see there exists an eventually positive solution $y(t)$ satisfying (4.3.9). From the proof of lemma 4.2.2 we see this can only occur when $z''(t) \le 0$, $z'(t) > 0$ and $z(t) < 0$, $t \ge t_0 \ge 0$. Choose $t_2 > t_1 \ge t_0$ such that $y(t_2 - \tau) > y(t_1 - \tau)$. Since

$$y(t_2 - \tau + nr) = \sum_{i=1}^{n} p^{n-i} z(t_2 - \tau + ir) + p^n y(t_2 - \tau)$$

$$y(t_1 - \tau + nr) = \sum_{i=1}^{n} p^{n-i} z(t_1 - \tau + ir) + p^n y(t_1 - \tau)$$

$$z(t_2 - \tau + ir) \ge z(t_1 - \tau + ir), \quad i = 1, 2, \ldots, n$$

and $y(t_1 - \tau + nr) > 0$, $n = 0, 1, \ldots,$ we see

$$y(t_2 - \tau + nr) \ge p^n[y(t_2 - \tau) - y(t_1 - \tau)] \triangleq Ap^n. \tag{4.3.11}$$

Similar to the proof of Corollary 4.2.1, we can show that there is an interval $[t_2, t_3]$ such that

$$y(T - \sigma + nt) \geq Ap^n$$

for $T \in [t_2, t_3]$ and all n. From (4.3.1) we get

$$z(t_0) + f(A) \sum_{i=0}^{n} \int_{t_2+ir}^{t_3+ir} (u - t_2) q(u) f(p^n) du \leq 0,$$

contradicting (4.2.7).

PROOF OF THEOREM 4.2.1: According to the proofs of Lemmas 4.2.1 and 4.2.2 we have $z'(t) > 0$, $z(t) < 0$ eventually. The remainder of the proof is similar to that of Lemma 2.2 in [4]. We omit it here.

References

[1] D.D. Bainov and A.I. Zahariev, *Oscillating properties of the solutions of a class of neutral type functional differential equations*, Bull. Austral. Math. Soc. 22, (1980), 365-372.

[2] ————, *Oscillating and asymptotic properties of a class of functional differential equations with maxima*, Czechoslavak Math. J. 34, (1984), 247-251.

[3] L. Erbe, *Oscillation criteria for second order nonlinear delay equations*, Canad. Math. Bull. 16, (1973), 49-56.

[4] L.H. Erbe and B.G. Zhang, *Oscillation of second order neutral differential equations*, Bull. Austral. Math. Soc. 39, (1989), 71-80.

[5] S.R. Grace and B.S. Lalli, *Oscillation of nonlinear second order neutral delay differential equations*, Rad. Mat. 3, (1987), 77-84.

[6] M.K. Grammatikopoulas, G. Ladas and A. Meimaridou, *Oscillations of second order neutral delay differential equations,* Rad. Mat. **1**, (1985), 267-274.

[7] ________, *Oscillation and asymptotic behavior of second order neutral differential equations,* Ann. Mat. Pura Appl. **(4)** 148, (1987), 29-40.

[8] ________, *Oscillation and asymptotic behavior of higher order neutral equations with variable coefficients,* Chinese Ann. Math. Ser. **B9**, (1988), No. 3, 322-338.

[9] M.K. Grammatikopoulos, E.A. Grove and G. Ladas, *Oscillation and asymptotic behavior of second order neutral differential equations,* International Conference of Differential Equations at Univ. of Toronto, (July 14-16).

[10] L.K. Jackson, *Subfunctions and second order differential inequalities,* Adv. Math. **2**, (1968), 307-363.

[11] G. Ladas and Y.G. Sficas, *Oscillations of higher-order neutral equations,* J. Austral. Math. Soc. **27** Ser. B, (1986), 502-511.

[12] E.C. Partheniadis, *On bounded oscillations of neutral differential equations,* Appl. Anal. **29**, (1988), No. 1-2, 63-69.

CHAPTER 5

OSCILLATION OF LINEAR SYSTEMS OF HIGHER ORDER

—Autonomous Case

5.1. Introduction and Preliminaries

Consider the neutral delay differential system in the form

$$\frac{d^N}{dt^N}[y(t) - P\,y(t-r)] + \sum_{j=1}^{m} Q_j\,y(t-\tau_j) = 0 \tag{5.1.1}$$

where P, Q_j $(j = 1, \ldots, m)$ are given $n \times n$ constant matrices, r, τ_j $(j = 1, \ldots, m)$ are nonnegative numbers, $\sigma = \max\{r, \tau_1, \ldots, \tau_m\}$ and N is a positive integer.

Here we use Definition 1.2.3 for oscillation of Eq. (5.1.1), i.e., a solution $y(t)$ of (5.1.1) is oscillatory if and only if every component of $y(t)$ has arbitrarily large zeros.

Our results are based on the following lemma which is an extension of Theorem 2.1 in [1].

LEMMA 5.1.1. *Eq. (5.1.1) is oscillatory if and only if its characteristic equation*

$$\det\left[\lambda^N(I - Pe^{-\lambda r}) + \sum_{j=1}^{m} Q_j\,e^{-\lambda\tau_j}\right] = 0 \tag{5.1.2}$$

has no real root.

Since this condition is not easy to verify, some explicit conditions are needed. For systems of DDEs J.M.Ferreire and I.Győri [3] give some criteria for oscillation by using logarithmic norms of matrices. But to the best of the authors' knowledge there are very few results so far for systems of NDDEs; here we mention only the results by I. Győri and G. Ladas [8] for a special system. In this chapter we will give some explicit conditions for oscillation of neutral systems using Lozenskii measures, or logarithmic norms, of matrices from a different point of view. Our results cover the results in [3] for systems of DDEs, and improve the results in [8]. Furthermore, even for the scalar case our results for explicit conditions are still the best up to now.

For the criteria of oscillation we need the following notations and definitions.

For any $n \times n$ real matrix A we denote by $\lambda_i(A)$ $(i = 1, \ldots, n)$ the eigenvalues of A satisfying

$$\operatorname{Re}\lambda_1(A) \geq \operatorname{Re}\lambda_2(A) \geq \cdots \geq \operatorname{Re}\lambda_n(A).$$

We define $\|A\|_i = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|_i}{\|x\|_i}$, $i = 1, 2, \ldots, \infty$, where $x = (x_1, \ldots, x_n)$,

$\|x\|_i = \left(\sum_{j=1}^{n} |x_j|^i\right)^{1/i}$, $i < \infty$ and $\|x\|_\infty = \max_{1\le j\le n}\{|x_j|\}$. For each $i = 1,2,\ldots,\infty$, the Lozenskii measures $\mu_i(A)$ of A is defined as follows:

$$\mu_i(A) = \lim_{h\to 0+} \frac{\|I + hA\|_i - 1}{h},$$

and $\nu_i(A) = -\mu_i(-A)$, $i = 1,2,\ldots,\infty$. In general, without specification, we denote by $\mu(A)$ and $\nu(A)$ any pair of $\mu_i(A)$ and $\nu_i(A)$, $i = 1,2,\ldots,\infty$. It has been shown that $\mu_i(A)$ and $\nu_i(A)$, $i = 1,2,\ldots,\infty$, exist for any $n\times n$ matrix A and can be explicitly calculated for $i = 1,2$, and $i = \infty$:

$$\mu_1(A) = \sup_i \left\{a_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|\right\}, \qquad \nu_1(A) = \inf_i \left\{a_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|\right\};$$

$$\mu_2(A) = \lambda_1\left(\frac{1}{2}(A + A^T)\right), \qquad \nu_2(A) = \lambda_n\left(\frac{1}{2}(A + A^T)\right);$$

$$\mu_\infty(A) = \sup_j \left\{a_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}|\right\}, \qquad \nu_\infty(A) = \inf_j \left\{a_{jj} - \sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}|\right\}.$$

For any $n\times n$ matrices A and B and any Lozenskii measures we have

i) $\mu(A+B) \le \mu(A) + \mu(B)$, $\nu(A+B) \ge \nu(A) + \nu(B)$;

ii) $\nu(-A) = -\mu(A)$, $\mu(-A) = -\nu(A)$;

iii) $\mu(\alpha A) = \alpha\mu(A)$, $\nu(\alpha A) = \alpha\nu(A)$, $\alpha > 0$;

iv) $\mu(A) \ge \mathrm{Re}\,\lambda_1(A)$, $\nu(A) \le \mathrm{Re}\,\lambda_n(A)$. (5.1.3)

For more details concerning Lozenskii measures, see [12]. In the sequel, we will obtain some criteria for oscillation by using Lozenskii measures. Since the criteria are given by the general form of Lozenskii measures μ and ν, we will actually have infinitely many different results corresponding to each criterion in the theorems. Moreover, three of them, which are given by μ_i and ν_i, $i = 1, 2, \infty$, can be expressed explicitly. But for the scalar case, where $\mu(A) = \nu(A) = A$, all of them coincide to give the same results.

5.2. Explicit Conditions for Oscillation

To simplify the discussion and proofs we first consider a simpler equation

$$\frac{d^N}{dt^N}\,[y(t) - P\,y(t-r)] + Q\,y(t-\tau) = 0 \tag{5.2.1}$$

where P, Q are $n \times n$ matrices, $r, \tau \geq 0$. The following conditions will be used to determine the oscillation of Eq. (5.2.1):

$$(\mathrm{A}_1) \qquad \sum_{k=0}^{\infty}[\nu(P)]^k\,\nu(Q)(kr+\tau)^N \geq \left(\frac{N}{e}\right)^N,$$

$$(\mathrm{A}_2) \qquad \sum_{k}{}^{*}[\mu(P)]^{-(k+1)}\,\nu(Q)[(k+1)r-\tau]^N \geq \left(\frac{N}{e}\right)^N,$$

$$(\mathrm{A}_3) \qquad \sum_{k}{}_{*}[\mu(P)]^{-(k+1)}\,\nu(Q)[-(k+1)r+\tau]^N \geq \left(\frac{N}{e}\right)^N$$

where $\sum_k^*$ and $\sum_{k*}$ denote the sums over all the terms for $k \in Z_+$ such that

$(k+1)r - \tau > 0$ and $-(k+1)r + \tau > 0$, respectively.

REMARK 5.2.1. The inequality in (A_3) will become strict if the sum has only one term.

THEOREM 5.2.1. *Assume N is odd and $\nu(Q) > 0$. Then each of the following is sufficient for (5.2.1) to be oscillatory:*

i) $\mu(P) = \nu(P) = 1$,

ii) $0 < \nu(P) \leq \mu(P) \leq 1$, *and* (A_1) *holds,*

iii) $1 \leq \nu(P) \leq \mu(P)$, *and* (A_2) *holds,*

iv) $\nu(P) < 1 < \mu(P)$, *and* (A_1), (A_2) *hold.*

THEOREM 5.2.2. *Assume N is even and $\nu(Q) > 0$. Then each of the following is sufficient for (5.2.1) to be oscillatory:*

i) $0 < \mu(P) \leq 1$, *and* (A_3) *holds,*

ii) $\mu(P) > 1$, *and* (A_2) , (A_3) *hold.*

REMARK 5.2.2 The condition (A_3) in Theorem 5.2.2 is required in the sense that if the set $\{k \in \mathbb{Z}_+ : -(k+1)r + \tau > 0\}$ is empty and $\nu(P) > 0$, then Eq. (5.2.1) must have a nonoscillatory solution. In fact, the above assumption implies that $r \geq \tau$. If (5.2.1) is oscillatory, then (5.3.1) has no real root. Let

$$F(\lambda) = \lambda^N(I - Pe^{-\lambda r}) + Q\,e^{-\lambda\tau}.$$

Then

$$\mu(F(0)) = \mu(Q) > 0$$

implies that $\mu(F(\lambda)) > 0$ for all $\lambda \in \mathbb{R}$. But

$$\mu(F(\lambda)) \leq \lambda^N(1 - \nu(P)e^{-\lambda r}) + \mu(Q)e^{-\lambda\tau} \to -\infty$$

as $\lambda \to -\infty$. This contradiction shows that all solutions cannot be oscillatory.

The above oscillation criteria for Eq. (5.2.1) can be easily extended to the Eq. (5.1.1) by using the following conditions where $q = \left(\prod_{j=1}^{m} \nu(Q_j)\right)^{1/m}$, $\tau = \frac{1}{m}\sum_{j=1}^{m} \tau_j$.

$$(B_1) \quad \sum_{j=1}^{m}\sum_{k=0}^{\infty}[\nu(P)]^k\,\nu(Q_j)(kr + \tau_j)^N \geq \left(\frac{N}{e}\right)^N, \quad \text{or} \tag{5.2.2}$$

$$mq\sum_{k=0}^{\infty}[\nu(P)]^k\,(kr + \tau)^N \geq \left(\frac{N}{e}\right)^N, \tag{5.2.3}$$

$$(B_2) \quad \sum_{j,k}{}^{*}[\mu(P)]^{-(k+1)}\,\nu(Q_j)[(k+1)r - \tau_j]^N \geq \left(\frac{N}{e}\right)^N, \quad \text{or}$$

$$mq\sum_k{}^{*}[\mu(P)]^{-(k+1)}[(k+1)r-\tau]^N \geq \left(\frac{N}{e}\right)^N,$$

(B₃) $$\sum_{j,k}{}_{*}[\mu(P)]^{-(k+1)}\,\nu(Q_j)[-(k+1)r+\tau_j]^N \geq \left(\frac{N}{e}\right)^N, \quad \text{or}$$

$$mq\sum_k{}_{*}[\mu(P)]^{-(k+1)}[-(k+1)r+\tau_j]^N \geq \left(\frac{N}{e}\right)^N$$

where $\sum_k^*$, $\sum_{k*}$ are defined the same as before; $\sum_{j,k}^*$, $\sum_{j,k*}$ denote the sums over all the terms for $1 \leq j \leq m$, $k \geq 0$ such that $(k+1)r - \tau_j > 0$ and $-(k+1)r + \tau_j > 0$, respectively. The inequality of (B_3) will become strict if the sum has only one term.

THEOREM 5.2.3. *Assume* N *is odd and* $\nu(Q_j) \geq 0$ *but not all zero,* $j = 1, \ldots, m$. *Then each one of the following is sufficient for (5.1.1) to be oscillatory:*

i) $\mu(P) = \nu(P) = 1$,

ii) $0 \leq \nu(P) \leq \mu(P) \leq 1$, *and* (B_1) *holds,*

iii) $1 \leq \nu(P) \leq \mu(P)$, *and* (B_2) *holds,*

iv) $\nu(P) < 1 < \mu(P)$, *and* (B_1) , (B_2) *hold.*

THEOREM 5.2.4. *Assume* N *is even and* $\nu(Q_j) \geq 0$ *but not all zero,*

$j = 1, \ldots, m$. *Then each one of the following is sufficient for (5.1.1) to be oscillatory:*

i) $0 < \mu(P) \leq 1$, *and* (B_3) *holds,*

ii) $\mu(P) > 1$, *and* (B_2) , (B_3) *hold.*

The idea in the proofs of the above theorems may also be applied to the case that $\nu(Q_j)$ are not all nonnegative. As an example we give a result for the equations of the form

$$\frac{d^N}{dt^N}[y(t) - P\,y(t-r)] + \sum_{j=1}^{m}[G_j\,y(t-\sigma_j) - H_j\,y(t-\tau_j)] = 0 \qquad (5.2.4)$$

where P, G_j, H_j are $n \times n$ matrices, $\nu(P) > 0$, $\nu(G_j) - \mu(H_j) \geq 0$ and not all zero for $j = 1, \ldots, m$, r, σ_j, τ_j $(j = 1, \ldots, m) \geq 0$.

THEOREM 5.2.5. *Assume* N *is odd. Then each one of the following is sufficient for (5.2.4) to be oscillatory.*

i) $\nu(P) \geq 1$, $\sigma_j < \tau_j < r$, *and* (B_2) *holds for the case where* $\nu(Q_j)$ *are replaced by* $\nu(G_j) - \mu(H_j)$, $j = 1, \ldots, m$. *Furthermore,*

$$-\left(\frac{N}{er}\right)^N + a^N\,\nu(P) + \sum_{j=1}^{m}\left[\nu(G_j)\,e^{-b_j(r-\sigma_j)} - \mu(H_j)\,e^{-b_j(r-\tau_j)}\right] > 0 \quad (5.2.5)$$

where

$$a = \min_{1 \le j \le m} \left\{ \frac{1}{\tau_j - \sigma_j} \ln \frac{\nu(G_j)}{\mu(H_j)} \right\},$$
$$b_j = \frac{1}{\tau_j - \sigma_j} \ln \frac{\nu(G_j)(r - \sigma_j)}{\mu(H_j)(r - \tau_j)}, \; j = 1, \ldots, m.$$

ii) $\nu(P) > 0$, $\mu(P) \le 1$, $\tau_j > \sigma_j > 0$, and (B_1) *holds for the case where* $\nu(Q_j)$ *are replaced by* $\nu(G_j) - \mu(H_j)$, $j = 1, \ldots, m$. *Furthermore,*

$$-\left(\frac{N}{er}\right)^N + a^{*N} [\mu(P)]^{-1} + \sum_{j=1}^{m} [\nu(P^{-1}G_j) e^{-b_j^* \sigma_j} - \mu(P^{-1}H_j) e^{-b_j^* \tau_j}] > 0,$$

where

$$a^* = \min_{1 \le j \le m} \left\{ \frac{1}{\sigma_j - \tau_j} \ln \frac{\nu(P^{-1}G_j)}{\mu(P^{-1}H_j)} \right\},$$
$$b_j^* = \frac{1}{\sigma_j - \tau_j} \ln \frac{\nu(P^{-1}G_j)\sigma_j}{\mu(P^{-1}H_j)\tau_j}, \; j = 1, \ldots, m.$$

5.3. Proofs

The following lemma will be needed in the proofs of the results.

LEMMA 5.3.1. *Let* A *be a* $n \times n$ *real matrix. If either* $\nu(A) > 0$ *or* $\mu(A) < 0$, *then* $\det(A) \neq 0$.

PROOF: From (5.1.3), if $\nu(A) > 0$, then $\operatorname{Re}\lambda_n(A) > 0$. Hence $\operatorname{Re}\lambda_i(A) > 0$ for $i = 1, 2, \ldots, n$. Thus

$$\det(A) = \lambda_1(A) \cdots \lambda_n(A) \neq 0.$$

The case that $\mu(A) < 0$ is similar.

□

PROOF OF THEOREM 5.2.1: The characteristic equation of (5.2.1) is

$$\det(\lambda^N(I - Pe^{-\lambda r}) + Q\, e^{-\lambda r}) = 0. \tag{5.3.1}$$

i) Assume $\mu(P) = \nu(P) = 1$. Let

$$F(\lambda) = \lambda^N(I - Pe^{-\lambda r}) + Q\, e^{-\lambda r}.$$

Then $\nu(F(0)) = \nu(Q) > 0$, and

$$\nu(F(\lambda)) \geq \nu(\lambda^N(I - P\, e^{-\lambda r})) + \nu(Q)\, e^{-\lambda r}.$$

For $\lambda > 0$

$$\nu(F(\lambda)) \geq \lambda^N(1 - \mu(P)\, e^{-\lambda r}) + \nu(Q)\, e^{-\lambda r} \tag{5.3.2}$$

$$= \lambda^N(1 - e^{-\lambda r}) + \nu(Q)\, e^{-\lambda r} > 0.$$

For $\lambda < 0$

$$\nu(F(\lambda)) \geq |\lambda|^N \nu(-I + P e^{-\lambda r}) + \nu(Q) e^{-\lambda r}$$

$$\geq |\lambda|^N(-1 + \nu(P) e^{-\lambda r}) + \nu(Q) e^{-\lambda r} > 0.$$

Thus $\nu(F(\lambda)) > 0$ for all $\lambda \in \mathbb{R}$. By Lemma 5.3.1 $\det F(\lambda) \neq 0$ for $\lambda \in \mathbb{R}$, i.e., (5.3.1) has no real root.

ii) Assume $0 \leq \nu(P) \leq \mu(P) \leq 1$. Clearly $\lambda = 0$ is not a root of (5.3.1). For $\lambda > 0$, by (5.3.2)

$$\nu(F(\lambda)) \geq \lambda^N(1 - \mu(P) e^{-\lambda r}) + \nu(Q) e^{-\lambda r} > 0.$$

Hence $\det(F(\lambda)) \neq 0$ for $\lambda > 0$ by Lemma 5.3.1. Let $\lambda = -s$ and denote

$$G(s) = -s^N(I - P e^{sr}) + Q e^{sr}. \tag{5.3.3}$$

Then $\lambda < 0$ is a root of (5.3.1) if and only if $s > 0$ is a root of $\det(G(s)) = 0$. By Lemma 5.3.1, if $\det(G(s)) = 0$ has a root $s > 0$, then $\nu(G(s)) \leq 0$. Since N is odd,

$$0 \geq \nu(G(s)) \geq s^N(-1 + \nu(P) e^{sr}) + \nu(Q) e^{sr}$$

which implies that $0 \le \nu(P)\, e^{sr} < 1$. As a result

$$\begin{aligned} s^N &\ge \nu(Q)\, e^{s\tau}\, (1 - \nu(P)\, e^{sr})^{-1} \\ &= \sum_{k=0}^{\infty} [\nu(P)]^k\, \nu(Q)\, e^{s(kr+\tau)} \\ &> \sum_{k=0}^{\infty} [\nu(P)]^k\, \nu(Q) \left[\frac{s(kr+\tau)e}{N}\right]^N . \end{aligned} \tag{5.3.4}$$

The equality can not hold since $e^{s(kr+\tau)}$ attains its minimal value at different point s for different k. Thus

$$\sum_{k=0}^{\infty} [\nu(P)]^k\, \nu(Q) (kr+\tau)^N < \left(\frac{N}{e}\right)^N ,$$

contradicting (A_1) . Therefore (5.3.1) has no real root.

iii) Assume $1 \le \nu(P) \le \mu(P)$. Similar to ii) we see that $\lambda \le 0$ is not a root of (5.3.1). Assume $\lambda > 0$ is a root of (3.1). By Lemma 5.3.1 we have

$$0 \ge \nu(F(\lambda)) \ge \lambda^N (1 - \mu(P)\, e^{-\lambda r}) + \nu(Q)\, e^{-\lambda \tau} \tag{5.3.5}$$

which implies that $\mu(P)\, e^{-\lambda r} > 1$. So

$$\begin{aligned} \lambda^N &\ge [\mu(P)]^{-1} \nu(Q)\, e^{\lambda(r-\tau)} (1 - [\mu(P)]^{-1}\, e^{\lambda r})^{-1} \\ &= \sum_{k=0}^{\infty} [\mu(P)]^{-(k+1)}\, \nu(Q)\, e^{\lambda[(k+1)r-\tau]} \\ &> \sum_k{}^{*} [\mu(P)]^{-(k+1)}\, \nu(Q) \left[\frac{\lambda((k+1)r-\tau)e}{N}\right]^N , \end{aligned}$$

that is,

$$\sum_k{}^{*}[\mu(P)]^{-(k+1)}\,\nu(Q)[(k+1)r-\tau]^N < \left(\frac{N}{e}\right)^N,$$

contradicting (A_2) .

iv) Clearly, $\lambda = 0$ is not a root of (5.3.1). From the proofs of ii) and iii) we see that (A_1) and (A_2) imply that any $\lambda > 0$ and $\lambda < 0$ can not be a root of (5.3.1).

□

PROOF OF THEOREM 5.2.2:

i) Assume $\mu(P) \leq 1$. Similar to the proof of Theorem 5.2.1 ii), we see any $\lambda \geq 0$ is not a root of (5.3.1). Assume $\lambda = -s < 0$ is a root of (5.3.1). Then by Lemma 5.3.1 and from (5.3.1)

$$\nu(F(-s)) = \nu(s^N(1 - Pe^{sr}) + Q\,e^{s\tau}) \leq 0.$$

Since N is even,

$$0 \geq \nu(F(-s)) \geq s^N(1 - \mu(P)\,e^{sr}) + \nu(Q)\,e^{s\tau}$$

which implies that $\mu(P)\,e^{sr} > 1$. As a result

$$
\begin{aligned}
s^N &\geq [\mu(P)]^{-1}\nu(Q)\,e^{s(-r+\tau)}(1-[\mu(P)]^{-1}\,e^{-sr})^{-1} \\
&= \sum_{k=0}^{\infty}[\mu(P)]^{-(k+1)}\,\nu(Q)\,e^{s[-(k+1)r+\tau]} \\
&> \sum_{k}{}_{*}[\mu(P)]^{-(k+1)}\,\nu(Q)\left[\frac{s[-(k+1)r+\tau]e}{N}\right]^N ,
\end{aligned}
$$

that is,

$$
\sum_{k}{}_{*}[\mu(P)]^{-(k+1)}\,\nu(Q)[-(k+1)r+\tau]^N < \left(\frac{N}{e}\right)^N .
$$

Note that the inequality may become an equality if the sum has only one term, and this contradicts (A_3) .

ii) Assume $\mu(P) > 1$. If λ is a real root of (5.3.1), then $\lambda \neq 0$ and $\lambda^N > 0$. By the proof of i), (A_3) implies that $\lambda < 0$ cannot be a root of (5.3.1). By the proof of Theorem 5.2.1 iii), (A_2) implies that $\lambda > 0$ cannot be a root of (5.3.1).

□

PROOF OF THEOREM 5.2.3 AND 5.2.4: Similar to those of Theorem 5.2.1 and 5.2.2. To show the difference we only give an outline of the proof of Theorem 5.2.3 ii) as an example.

Corresponding to (5.3.4) we now have

$$s^N \geq \sum_{j=1}^{m}\sum_{k=0}^{\infty}[\nu(P)]^k\,\nu(Q_j)\,e^{s(kr+\tau_j)} \tag{5.3.6}$$

$$> \sum_{j=1}^{m}\sum_{k=0}^{\infty}[\nu(P)]^k\,\nu(Q_j)\,\left[\frac{s(kr+\tau_j)e}{N}\right]^N,$$

that is,

$$\sum_{j=1}^{m}\sum_{k=0}^{\infty}[\nu(P)]^k\,\nu(Q_j)[kr+\tau_j]^N < \left(\frac{N}{e}\right)^N,$$

contradicting (5.2.2). From (5.3.6) we also have

$$s^N \geq \left(\sum_{j=1}^{m}\nu(Q_j)\,e^{s\tau_j}\right)\left(\sum_{k=0}^{\infty}[\nu(P)]^k\,e^{skr}\right)$$

$$\geq m\left(\prod_{j=1}^{m}\nu(Q_j)\,e^{s\tau_j}\right)^{1/m}\left(\sum_{k=0}^{\infty}[\nu(P)]^k\,e^{skr}\right)$$

$$= mq\sum_{k=0}^{\infty}[\nu(P)]^k\,e^{s(kr+\tau)}$$

$$> mq\sum_{k=0}^{\infty}[\nu(P)]^k\left[\frac{s(kr+\tau)e}{N}\right]^N,$$

that is,

$$mq\sum_{k=0}^{\infty}[\nu(P)]^k(kr+\tau)^N < \left(\frac{N}{e}\right)^N,$$

contradicting (5.2.3).

□

PROOF OF THEOREM 5.2.5: The characteristic equation of Eq. (5.2.4) is

$$\det\left(\lambda^N(I - Pe^{-\lambda r}) + \sum_{j=1}^{m}(G_j e^{-\lambda\sigma_j} - H_j e^{-\lambda\tau_j})\right) = 0. \tag{5.3.7}$$

Let

$$F(\lambda) = \lambda^N(I - Pe^{-\lambda r}) + \sum_{j=1}^{m}(G_j e^{-\lambda\sigma_j} - H_j e^{-\lambda\tau_j}).$$

Then $\lambda = 0$ is not a root of (5.3.7) since the assumption before Theorem 5.2.5 implies that

$$\nu(F(0)) = \nu\left(\sum_{j=1}^{m}(G_j - H_j)\right) \geq \sum_{j=1}^{m}(\nu(G_j) - \mu(H_j)) > 0.$$

Hence $\det F(0) \neq 0$.

i) Assume $\nu(P) \geq 1$ and $\sigma_j < \tau_j < r$. If $\lambda > 0$ is a root of (5.3.7), then by Lemma 5.3.1

$$0 \geq \nu(F(\lambda)) \geq \lambda^N(1 - \mu(P)e^{-\lambda r}) + \sum_{j=1}^{m}\left(\nu(G_j)e^{-\lambda\sigma_j} - \mu(H_j)e^{-\lambda\tau_j}\right)$$

$$\geq \lambda^N(1 - \mu(P)e^{-\lambda r}) + \sum_{j=1}^{m}\left(\nu(G_j) - \mu(H_j)\right)e^{-\lambda\tau_j}.$$

This is a similar inequality to (5.3.5) for Eq. (5.2.1). By a similar discussion we can get a contradiction to (B_2) where $\nu(Q_j)$ are replaced by $\nu(G_j) - \mu(H_j)$, $j = 1, \ldots, m$.

If $\lambda < 0$ is a root of (5.3.7), let $\lambda = -s$, and denote

$$\begin{aligned}\phi(s) &= -s^N(Ie^{-sr} - P) + \sum_{j=1}^{m}(G_j e^{-s(r-\sigma_j)} - H_j e^{-s(r-\tau_j)}), \\ \alpha(s) &= -s^N(Ie^{-sr} - P) \\ \beta_j(s) &= G_j e^{-s(r-\sigma_j)} - H_j e^{-s(r-\tau_j)}, \quad j = 1, \ldots, m.\end{aligned} \tag{5.3.8}$$

Then $\det(\phi(s)) = 0$, and since N is odd,

$$\nu(\alpha(s)) \geq s^N(-e^{-sr} + \nu(P)) > 0,$$

$$\nu(\beta_j(s)) \geq \nu(G_j)e^{-s(r-\sigma_j)} - \mu(H_j)e^{-s(r-\tau_j)} \triangleq \ell_j(s), \quad j = 1, \ldots, m.$$

$\nu(\beta_j(s)) \geq 0$ if and only if

$$s \leq \frac{1}{\tau_j - \sigma_j} \ln \frac{\nu(G_j)}{\mu(H_j)} \triangleq a_j, \qquad j = 1, \ldots, m.$$

Set $a = \min_{1 \leq j \leq m} \{a_j\}$. We have $\nu(\phi(s)) > 0$ for $0 < s \leq a$. Consider the case that $s > a$. Then

$$\nu(\alpha(s)) \geq -\left(\frac{N}{er}\right)^N + a^N \nu(P),$$

and since

$$\ell_j'(s) = -\nu(G_j)(r - \sigma_j)e^{-s(r-\sigma_j)} + \mu(H_j)(r - \tau_j)e^{-s(r-\tau_j)},$$

we see the minimum of ℓ_j is attained at

$$s_j = \frac{1}{\tau_j - \sigma_j} \ln \frac{\nu(G_j)(r-\sigma_j)}{\mu(H_j)(r-\tau_j)} = b_j,$$

and thus

$$\nu(\beta_j(s)) \geq \ell_j(s) \geq \ell_j(s_j) = \nu(G_j)e^{-b_j(r-\sigma_j)} - \mu(H_j)e^{-b_j(r-\tau_j)}, \quad j = 1, \ldots, m.$$

Therefore by (5.3.8) and (5.2.5)

$$\nu(\phi(s)) \geq \nu(\alpha(s)) + \sum_{j=1}^{m} \nu(\beta_j(s))$$

$$\geq -\left(\frac{N}{er}\right)^N + a^N \nu(P) + \sum_{j=1}^{m} [\nu(G_j)e^{-b_j(r-\sigma_j)} - \mu(H_j)e^{-b_j(r-\tau_j)}] > 0$$

contradicting that $\det(\phi(s)) = 0$.

ii) By (5.1.3) $\nu(P) > 0$ implies that P^{-1} exists. (5.3.7) is equivalent to

$$\det\left(-\lambda^N (I - P^{-1} e^{\lambda r}) + \sum_{j=1}^{m} [P^{-1} G_j e^{\lambda(r-\sigma_j)} - P^{-1} H_j e^{\lambda(r-\tau_j)}] \right) = 0.$$

With $\lambda = -s$, we have

$$\det\left(s^N (I - P^{-1} e^{-sr}) + \sum_{j=1}^{m} [P^{-1} G_j e^{-s(r-\sigma_j)} - P^{-1} H_j e^{-s(r-\tau_j)}] \right) = 0. \tag{5.3.9}$$

Then we have a duality between (5.3.7) and (5.3.9) as follows:

$$(P, G_j, H_j, \sigma_j, \tau_j) \longleftrightarrow (P^{-1}, P^{-1}G_j, P^{-1}H_j, r - \sigma_j, r - \tau_j).$$

Using this duality and part i) we obtain the desired result.

□

5.4. Remarks on Special Cases

Here we mention the work by Győri and Ladas in [8] where Eq. (5.1.1) is considered for the case that $N = 1$, P is a diagonal matrix with diagonal entries $p_1, \ldots, p_n$ such that $0 \leq p_i \leq 1$, $i = 1, \ldots, n$. $\nu_1(Q_j) \geq 0$ where $\nu_1(Q_j)$ denotes the particular Lozenskii measure defined in Section 5.1. Some comparison results with delay equations are obtained. An explicit condition for oscillation is given there:

If $P = pI$ for $p \in [0,1]$ (hence $\nu_1(P) = p$), and

$$\sum_{j=1}^{m} \sum_{k=0}^{\infty} [\nu_1(P)]^k \, \nu_1(Q_j) \, (kr + \tau_j) > \frac{1}{e},$$

then Eq. (5.1.1) is oscillatory according to Definition 1.2.3.

Obviously this result is included in Theorem 5.2.1 ii), and the equality in (B_1) is not valid there.

A special case for Eq. (5.1.1) is that P and Q_j, $j = 1, \ldots, m$, are symmetric matrices. Since for any symmetric matrix A, $\mu_2(A) = \lambda_1(A)$,

$\nu_2(A) = \lambda_n(A)$ and $\mu(A) \geq \lambda_1(A)$, $\nu(A) \leq \lambda_n(A)$ for any Lozenskii measures, then if we use μ_2 and ν_2 in the previous theorems, they will give the best results among those using all the Lozenskii measures.

Another special case for Eq. (5.1.1) is the scalar case, i.e., P and Q_j, $j = 1, \ldots, m$ are constants, hence

$$\mu(P) = \nu(P) = P, \qquad \mu(Q_j) = \nu(Q_j) = Q_j, \quad j = 1, \ldots, m.$$

Therefore, if we substitute P and Q_j into $\mu(P)$ or $\nu(P)$, $\mu(Q_j)$ or $\nu(Q_j)$, $j = 1, \ldots, m$, respectively, the previous theorems will give criteria for oscillations of scalar equations. It is easy to see that they coincide with some results obtained in Section 2.4 for the case that $n = 1$, and they include and improve the following known sufficient conditions for oscillation, see [2,4–7, 10, 11]:

A. $N = 1$, $0 < p < 1$, $\sum_{j=1}^{m} q_j \tau_j > \frac{1-p}{e}$ or $\left(\prod_{j=1}^{m} q_j\right)^{1/m} \left(\sum_{j=1}^{m} \tau_j\right) > \frac{1-p}{e}$, or $p \geq 1$, $r > \tau_j$ $(j = 1, \ldots, m)$ and $\sum_{j=1}^{m} q_j(r - \tau_j) > \frac{p-1}{e}$.

B. $N = 2$, $m = 1$, $0 < p < 1$, $p^{-1} q(\tau - r)^2 > \left(\frac{2}{e}\right)^2$.

C. N is odd, $m = 1$, $0 < p < 1$, and either one of the following:

i) $q\tau^N > \left(\frac{N}{e}\right)^N$,

ii) $\frac{q}{1-p}(\tau - r)^N > \left(\frac{N}{e}\right)^N$,

iii) $q[\tau^N + p(\tau + r)^N] > \left(\frac{N}{e}\right)^N$,

iv) $q\frac{p^{n+1}-1}{p-1}\tau^N > \left(\frac{N}{e}\right)^N$, or

v) $\max_i (ir + \tau)^N qp^i > \left(\frac{N}{e}\right)^N$.

D. N is even, $m = 1$, $0 < p \le 1$ and $p^{-1}q(\tau - r)^N > \left(\frac{N}{e}\right)^N$.

5.5. Criteria for Systems of Delay Equations

In this section, we obtain a criterion for oscillation of the following delay defferential system of first order

$$\dot{y}(t) + Q_0 y(t) + \sum_{j=1}^{m} Q_j y(t - \tau_j) = 0 \tag{5.5.1}$$

where $Q_j, j = 0, \ldots, m$, are $n \times n$ matrices, and $\tau_j \ge 0, j = 1, \ldots, m$. The main result of this section is given in the following.

THEOREM 5.5.1. *Assume* $\nu(Q_j) \ge 0$, $j = 0, 1, \ldots, m$, *and* $\sum_{j=1}^{m} \nu(Q_j) > 0$. *Then either one of the following guarantees that (5.5.1) is oscillatory:*

i) $\sum_{j=1}^{m} \nu(Q_j) e^{\nu(Q_0)\tau_j} \tau_j e \geq 1,$ *(5.5.2)*

ii) $\big(\prod_{j=1}^{m} \nu(Q_j)\big)^{1/m} \exp\big(\frac{\nu(Q_0)}{m}\tau^*\big)\tau^* e \geq 1, \quad \text{where} \quad \tau^* = \sum_{j=1}^{m} \tau_j.$ *(5.5.3)*

The above inequalities become strict if $m = 1$.

Proof: The characteristic equation of (5.5.1) is

$$\det\big(\lambda I + Q_0 + \sum_{j=1}^{m} Q_j e^{-\lambda \tau_j}\big) = 0. \tag{5.5.4}$$

Let $F(\lambda) = \lambda I + Q_0 + \sum_{j=1}^{m} Q_j e^{-\lambda \tau_j}$. Then $\det\big(F(\lambda)\big) \neq 0$ for $\lambda > 0$. If (5.5.4) has a real root $\lambda_0 < 0$, then let $\lambda_0 = -s_0$ and let

$$G(s) = -sI + Q_0 + \sum_{j=1}^{m} Q_j e^{s\tau_j}.$$

Then $\det\big(G(s_0)\big) = 0$ implies that $\nu\big(G(s_0)\big) \leq 0$. Hence

$$0 \geq \nu\big(G(s_0)\big) \geq -s_0 + \nu(Q_0) + \sum_{j=1}^{m} \nu(Q_j) e^{s_0 \tau_j} \tag{5.5.5}$$

$$s_0 - \nu(Q_0) \geq \sum_{j=1}^{m} \nu(Q_j) e^{\nu(Q_0)\tau_j} e^{(s_0 - \nu(Q_0))\tau_j}.$$

Obviously, $s_0 - \nu(Q_0) > 0$ and then

$$1 \geq \sum_{j=1}^{m} \nu(Q_j) e^{\nu(Q_0)\tau_j} e^{(s_0 - \nu(Q_0))\tau_j} / \big(s_0 - \nu(Q_0)\big)$$

$$> \sum_{j=1}^{m} \nu(Q_j) e^{\nu(Q_0)\tau_j} \tau_j e,$$

contradicting (5.5.2). Note that this last inequality may become an equality if $m = 1$. Therefore, (5.5.4) has no real root, and (5.5.1) is oscillatory.

The case corresponding to (5.5.3) can be similarly proved.

□

REMARK 5.5.1. Theorem 5.5.1 improves the results in[3], and for the scalar case, it coincides with the result obtained by W. Huang [9].

References

[1] O. Arino and I. Győri, *Necessary and sufficient condition for oscillation of a neutral differential system with several delays,* J. Differential Equations 81 (1989), 95–105.

[2] Y. Chen, *Oscillation of a class of neutral functional differential equations,* Chin. Ann. Math. 10A (5) (1989), 545–553.

[3] J. M. Fereira and I. Győri, *Oscillation behavior in linear retarded functional differential equations,* J. Math. Anal. Appl. 128 (1987), 332–346.

[4] K. Gopalsamy, *Oscillation and nonoscillation in neutral differential equations with variable parameters,* Jour. Math. Phy. Sci. 21 (1987), 593–611.

[5] K. Gopalsamy and B. G. Zhang, *Oscillation and nonoscillation in first order neutral differential equations,* J. Math. Anal. Appl. 151 (1990), 42–57.

[6] M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, *Oscillation and asymptotic behavior of second order neutral differential equations,* Ann. Mat. Pure Appl. (1987), [illegible]–40.

[7] E. A. Grove, G. Ladas and J. Schinas, *Sufficient conditions for the oscillation of delay and neutral delay equations,* Canad. Math. Bull. 31(4) (1988), 459–466.

[8] I. Györi and G. Ladas, *Oscillations of systems of neutral differential equations*, Diff. Integral Eqns. 3 (1988), 281–286.

[9] W. Huang, *Necessary and sufficient conditions for oscillation of first order functional differential equations*, J. Math. Anal. Appl. 148 (1990), 360–370.

[10] G. Ladas and Y. G. Sficas, *Oscillations of higher-order neutral equations*, J. Austral. Math. Soc. Ser. B27 (1986), 502–511.

[11] I. P. Stavroulakis, *Oscillations of mixed neutral equations*, Hiroshima Math. J. 19 (1989), 441–456.

[12] M. Vidyasagar, "Nonlinear Systems Analysis," Prentice-Hall, Inc., Englewood Cliffs, 1978.

CHAPTER 6

OSCILLATION OF LINEAR SYSTEMS OF HIGHER ORDER

—Nonautonomous Case

6.1. Criteria for Oscillation

Now we turn to the oscillation of the linear nonautonomous system

$$\frac{d^N}{dt^N}[y(t)-Py(t-r)]+\sum_{j=1}^{m}Q_j(t)y(t-\tau_j)=0 \tag{6.1.1}$$

where $P \in \mathbb{R}^{n\times n}, Q_j \in C(\mathbb{R}_+,\mathbb{R}^{n\times n}), j=1,\ldots,m;\ r,\tau_j, j=1,\ldots,m,$ are nonnegative constants. Furthermore assume $Q_j(t)=\big((q_j)_{\ell k}(t)\big)$, where $(q_j)_{\ell k}(t)$, $\ell,k=1,\ldots,n,\ j=1,\ldots,m,$ are continuous, bounded and do not change signs on $\mathbb{R}_+$.

The definition for oscillation used in this chapter is the Definition 1.2.4.

As indicated by Theorem 1.2.2, all solutions of Eq. (6.1.1) have the exponential order and hence the Laplace transforms of the solutions exist. Denote

$$(\bar{q}_j)_{\ell k}=\inf_{t\in\mathbb{R}_+}(q_j)_{\ell k}(t),\ (\hat{q}_j)_{\ell k}=\sup_{t\in\mathbb{R}_+}(q_j)_{\ell k}(t),\ \ell,k=1,\ldots,n,\ j=1,\ldots,m.$$

For any $(q_j^*)_{\ell k}\in\big((\bar{q}_j)_{\ell k},(\hat{q}_j)_{\ell k}\big)$, $\ell,k=1,\ldots,n,\ j=1,\ldots,m,$ define

$$Q_j^*=\big((q_j^*)_{\ell k}\big).$$

We will determine oscillation properties of (6.1.1) by comparing (6.1.1) with the following equation

$$\frac{d^N}{dt^N}[y(t) - Py(t-r)] + \sum_{j=1}^{m} Q_j^* y(t-\tau_j) = 0. \qquad (6.1.2)$$

This is done in the following theorem.

THEOREM 6.1.1. *If for every* $(q_j^*)_{\ell k} \in \left((\bar{q}_j)_{\ell k}, (\hat{q}_j)_{\ell k}\right)$, $\ell, k = 1, \ldots, n$, $j = 1, \ldots, m$, *Eq. (6.1.2) is oscillatory according to Definition 1.2.3, then Eq. (6.1.1) is oscillatory according to Definition 1.2.4.*

The proof is mainly by Laplace transform which has been used in [1] for the systems with constant coefficients.

PROOF: We only prove it for the case that $N = 1$, since the general proof is similar. Assume the contrary. Then there exist an eventually nontrivial solution $y(t)$ of (6.1.1) and $t_0 \geq 0$ such that $y_i(t)$ has eventually constant signum for $t \geq t_0 - \sigma$, $i = 1, \ldots, n$, where $\sigma = \max\{r, \tau_1, \ldots, \tau_m\}$. Without loss of generality we may assume $y_1(t) > 0$ for $t \geq t_0 - \sigma$.

The Laplace transforms of $y(t)$ exists for $s \geq s_0$, where $s_0 \in (-\infty, +\infty)$. Let

$$u(s) = \mathcal{L}[y(t)] \triangleq \int_{t_0}^{\infty} e^{-st} y(t)dt, \quad s \geq s_0,$$

and $u(s) = \left(u_1(s), \ldots, u_n(s)\right)^T$. Then multiplying (6.1.1) with e^{-st}, integrating it from t_0 to ∞, and using the Mean Value Theorem we find that there exist $(q_j^*)_{\ell k} \in \left((\bar{q}_j)_{\ell k}, (\hat{q}_j)_{\ell k}\right)$ $\ell, k = 1, \ldots, n$, $j = 1, \ldots, n$ such that $Q_j^* = \left((q_j^*)_{\ell k}\right)$ and

$$F(s)u(s) = \varphi(s), \quad s \geq s_0.$$

where

$$\begin{aligned} F(s) =& sI - sPe^{-sr} + \sum_{j=1}^{m} Q_j^* e^{-s\tau_j}, \\ \varphi =& y_0 - Py(-r) + sPe^{-sr} \int_{t_0 - r}^{t_0} e^{-st} y(t)dt \\ & - \sum_{j=1}^{m} Q_j^* e^{-s\tau_j} \int_{t_0 - \tau_j}^{t_0} e^{-st} y(t)dt. \end{aligned}$$

Obviously, $F(s)$ and $\varphi(s)$ are continuous on $\mathbb{R}$. Since (6.1.2) is oscillatory according to Definition 1.2.3 for the above Q_j^*, $j = 1, \ldots, m$, we have

$$\det\left(F(s)\right) \neq 0 \quad \text{for} \quad s \in \mathbb{R}.$$

Hence $F^{-1}(s)$ exists and

$$u(s) = F^{-1}(s)\varphi(s), \quad s \geq s_0 \tag{6.1.3}$$

where $F^{-1}(s)\varphi(s)$ is continuous on $\mathbb{R}$. We claim that $s_0 = -\infty$. Otherwise, from the definition of $u(s)$ we have $\lim_{s \to s_0^+} |u(s)| = +\infty$, contradicting (6.1.3). $s_0 = -\infty$ implies that (6.1.3) holds for all $s \in \mathbb{R}$.

It is easy to see that there exist $a_1 > 0$ and $a_2 > 0$ such that

$$|\varphi(s)| \le a_1 e^{a_2|s|}, \quad s \in \mathbb{R}, \tag{6.1.4}$$

and since $\det\big(F(s)\big) > 0$,

$$|F^{-1}(s)| \le \frac{b}{\det F(s)}\, |F(s)|^{n-1} \tag{6.1.5}$$

where b is a constant depending only on n. It is also seen that there exist $C_1 > 0$ and $C_2 > 0$ such that

$$|F(s)| \le C_1 e^{C_2|s|}, \quad s \in \mathbb{R}.$$

Moreover, since $\det\big(F(s)\big)$ is a polynomial of s and $e^{-s\tau_j}$ $(j = 1, \ldots, m)$ and $\det\big(F(s)\big) > 0$ for $s \in \mathbb{R}$, we know that $\lim_{s\to-\infty} \det\big(F(s)\big) = +\infty$, and hence there exists an $s_1 < 0$ such that $\det\big(F(s)\big) \ge 1$ for $s \le s_1$. Thus from (6.1.5) we have

$$|F^{-1}(s)| \le bC_1^{n-1} e^{(n-1)C_2|s|}, \quad s \le s_1. \tag{6.1.6}$$

Noting that $u_1(s) = L[y_1](s) > 0$, from (6.1.3), (6.1.4) and (6.1.6) we get that there exist $d_1 > 0$ and $d_2 > 0$ such that

$$0 \le u_1(s) \le |u(s)| \le d_1 e^{d_2|s|}, \quad s \le s_1.$$

Therefore

$$0 \le \limsup_{s \to -\infty} e^{-d_2|s|} u_1(s) \le d_1. \tag{6.1.7}$$

But for $t_1 \ge \max\{t_0, d_2\}$ we obtain that

$$\begin{aligned} e^{-d_2|s|} u_1(s) &= e^{-d_2|s|} \int_{t_0}^{\infty} e^{-st} y_1(t) dt \\ &\ge e^{-d_2|s|} \int_{t_1}^{\infty} e^{-st} y_1(t) dt \\ &\ge e^{-(d_2 - t_1)|s|} \int_{t_1}^{\infty} y_1(t) dt \\ &\longrightarrow +\infty \end{aligned}$$

as $s \to -\infty$ since $y_1(t) > 0$, contradicting (6.1.7). The proof is complete.

□

REMARK 6.1.1 Theorem 6.1.1 is a correction and an extension of Theorem 3.1 in [3].

By combining Theorems 6.1.1 and 5.5.1, we can consider the following equation

$$y'(t) + Q_0(t)y(t) + \sum_{j=1}^{m} Q_j(t) y(t - \tau_j) = 0. \tag{6.1.8}$$

Assume $Q_j(t) = ((q_j)_{\ell k}(t))$, $j = 0, \ldots, m$, where $(q_j)_{\ell k}(t)$, $\ell, k = 1, \ldots, n$, are continuous, bounded, and do not change signs on $\mathbb{R}_+$, $\tau_j > 0$,

$j = 1, \ldots, m$. Denote

$$(\bar{q}_j)_{\ell k} = \inf_{t \in \mathbb{R}_+} \big((q_j)_{\ell k}(t)\big), \quad (\hat{q}_j)_{\ell k} = \sup_{t \in \mathbb{R}_+} \big((q_j)_{\ell k}(t)\big).$$

For any $(q_j^*)_{\ell k} \in \big((\bar{q}_j)_{\ell k}, (\hat{q}_j)_{\ell k}\big)$, define $Q_j^* = \big((q_j^*)_{\ell k}\big)$.

Corollary 6.1.1. *Assume for any* $(q_j^*)_{\ell k} \in \big((\bar{q}_j)_{\ell k}, (\hat{q}_j)_{\ell k}\big)$, $\ell, k = 1, \ldots, n$, $j = 0, \ldots, m$, *we have*

i) $\nu(Q_j^*) \geq 0 \, (j = 0, \ldots, m)$ *and* $\sum_{j=1}^{m} \nu(Q_j^*) > 0$,

and one of the following conditions holds:

ii) $\sum_{j=1}^{m} \nu(Q_j^*) e^{\nu(Q_0^*)\tau_j} \tau_j e \geq 1$, *or*

ii)' $\big(\prod_{j=1}^{m} \nu(Q_j^*)\big)^{1/m} \exp\big(\frac{\nu(Q_0^*)}{m}\tau^*\big)\tau^* e \geq 1$, *where* $\tau^* = \sum_{j=1}^{m} \tau_j$,

the above inequalities become strict if $m = 1$.

Then Eq. (6.1.8) is oscillatory.

For the scalar case we have the following result.

Consider the scalar equation

$$y'(t) + q_0(t)y(t) + \sum_{j=1}^{m} q_j(t) y(t - \tau_j) = 0 \tag{6.1.9}$$

where $q_j(t)$, $j = 0, \ldots, m$ are continuous, bounded and do not change signs, $0 \le \bar{q}_j \le q_j(t) \le \hat{q}_j$, $j = 0, \ldots, m$, and $\tau_j > 0$, $j = 1, \ldots, m$. Then our results reduce to the following

COROLLARY 6.1.2. *Assume that the equation*

$$y'(t) + \bar{q}_0 y(t) + \sum_{j=1}^{m} \bar{q}_j y(t - \tau_j) = 0 \tag{6.1.10}$$

is oscillatory. Thus Eq. (6.1.9) is also oscillatory. In particular, assume

$$\sum_{j=1}^{m} \bar{q}_j e^{\bar{q}_0 \tau_j} \tau_j e \ge 1 \quad \text{or}$$

$$\Big(\prod_{j=1}^{m} \bar{q}_j\Big)^{1/m} e^{\frac{\bar{q}_0}{m}\tau^*} \tau^* e \ge 1, \quad \text{where} \quad \tau^* = \sum_{j=1}^{m} \tau_j,$$

the inequalities become strict if $m = 1$. *Then Eq. (6.1.9) is oscillatory.*

PROOF: The characteristic equation of (6.1.10) is

$$F(\lambda) \equiv \lambda + \bar{q}_0 + \sum_{j=1}^{m} \bar{q}_j e^{-\lambda \tau_j} = 0.$$

Eq. (6.1.10) is oscillatory implies that $F(\lambda) > 0$ for all $\lambda \in \mathbb{R}$. Then for any

$q_j^* \in [\bar{q}_j, \hat{q}_j]$, $j = 0, \ldots, m$, we see that

$$\lambda + q_0^* + \sum_{j=1}^{m} q_j^* e^{-\lambda \tau_j} \ge \lambda + \bar{q}_0 + \sum_{j=1}^{m} \bar{q}_j e^{-\lambda \tau_j} > 0.$$

This means that equation

$$y'(t) + q_0^* y(t) + \sum_{j=1}^{m} q_j^* y(t - \tau_j) = 0$$

is oscillatory. By Theorem 6.1.1, Eq. (6.1.9) is oscillatory. The rest of the proof is immediate from the first part and Corollary 6.1.1.

□

Finally we show how our results may be extended to certain nonlinear systems of the form

$$y'(t) + \sum_{j=1}^{m} Q_j(t, y(t))y(t - \tau_j) = 0 \tag{6.1.11}$$

where $\tau_j > 0$, $j = 1, \ldots, m$. Let $Q_j(t, y) = \left((q_j)_{\ell k}(t, y)\right)$, $j = 0, \ldots, m$, be $n \times n$ matrices, where the $(q_j)_{\ell k}(t, y(t))$ are continuous, uniformly bounded and do not change signs on $\mathbb{R}_+$ for all solutions $y(t)$.

Denote

$$(\bar{q}_j)_{\ell k} = \inf_{t \in \mathbb{R}_+} \left((q_j)_{\ell k}(t, y(t))\right), \quad (\hat{q}_j)_{\ell k} = \sup_{t \in \mathbb{R}_+} \left((q_j)_{\ell k}(t, y(t))\right)$$

for $\ell, k = 1, \ldots, n$, $j = 1, \ldots, m$, where the infima and superia are taken over all solutions of the system (6.1.11) and for any $(q_j^*)_{\ell k} \in \left((\bar{q}_j)_{\ell k}, (\hat{q}_j)_{\ell k}\right)$ define $Q_j^* = \left((q_j^*)_{\ell k}\right)$, $j = 0, \ldots, m$.

Theorem 6.1.2. *i) If the conditions of Theorem 6.1.1 or Corollary 6.1.1 are satisfied, then system (6.1.11) is oscillatory.*

The proofs are essentially the same as Theorem 6.1.1, Corollary 6.1.1, and so we omit them here.

We remark that if we can give a priori upper and lower bounds for all solutions, then with some general assumptions we can easily show the uniform boundedness of the functions $(q_j)_{\ell k}(t, y(t))$ in (6.1.11) which are required by Theorem 6.1.2.

6.2. Application to Lotka-Volterra Models

As an application of our results, we derive oscillation properties for Lotka-Volterra models in the system (predator-prey and competition) cases.

We consider the system of delay logistic equations

$$\dot{N}_i(t) = N_i(t)[a_i - \sum_{j=1}^{m} b_{ij} N_j(t-\tau)], \quad i = 1, \ldots, m \tag{6.2.1}$$

with $N_i(t) = \varphi_i(t)$, $t \in [-\tau, 0]$, where

$$\tau \in (0, \infty), \quad a_i, b_{ij} \in \mathbb{R} \quad \text{for} \quad i, j = 1, \ldots, m, \quad \varphi_i \in C([-\tau, 0], \mathbb{R}_+) \quad \text{and}$$

$$\varphi_i(0) > 0, \quad i = 1, \ldots, m.$$

We assume that (6.2.1) has an equilibrium $N^* = (N_1^*, \ldots, N_m^*)^T$ with positive components. Set $N_i(t) = N_i^* e^{x_i(t)}$ for $t \geq 0$ and $i = 1, \ldots, m$, then the $x_i(t)$ satisfy

$$\dot{x}_i(t) + \sum_{j=1}^{m} p_{ij}(e^{x_j(t-\tau)} - 1) = 0, \quad i = 1, \ldots, m, \tag{6.2.2}$$

where $p_{ij} = b_{ij} N_j^*$, $i, j = 1, \ldots, m$. As shown in [5] we see that every solution of (6.2.2) satisfies

$$\lim_{t \to \infty} x_i(t) = 0, \quad i = 1, \ldots, m, \tag{6.2.3}$$

if the matrix $P = (p_{ij})$ satisfies

$$\nu_\infty(P) = \min_{1 \leq j \leq m} [N_j^*(b_{jj} - \sum_{i=1, i \neq j}^{m} |b_{ij}|)] > 0.$$

In [5] it is shown that if further

$$\nu_\infty(P) \tau e > 1, \tag{6.2.4}$$

then all solutions of (6.2.1) are oscillatory about N^*. Now we are ready to extend this result to more general criteria.

THEOREM 6.2.1. *Assume* $\nu_\infty(P) > 0$, *and for some* $i = 1, 2, \ldots,$ *or* ∞,

$$\nu_i(P) \tau e > 1. \tag{6.2.5}$$

Then (6.2.2) is oscillatory, and hence every solution of (6.2.1) is oscillatory about N^*.

PROOF: Rewrite (6.2.2) in the form

$$\dot{x}_i(t) + \sum_{j=1}^{m} p^*_{ij}(t) x_j(t-\tau) = 0, \quad i = 1, \ldots, m \tag{6.2.6}$$

where

$$p^*_{ij}(t) = p_{ij} \frac{e^{x_j(t-\tau)} - 1}{x_j(t-\tau)} \quad \text{for} \quad i, j = 1, \ldots, m,$$

and by (6.2.3)

$$\lim_{t \to \infty} p^*_{ij}(t) = p_{ij}, \quad i, j = 1, \ldots, m.$$

Define the matrix $P^*(t) = \left(p^*_{ij}(t)\right)$. Then

$$\lim_{t \to \infty} P^*(t) = P.$$

By (6.2.4) and the continuity of the Lozinskii measures we get that there is a $T_0 \geq 0$ such that

$$\nu_i\left(P^*(t)\right)\tau e > 1, \quad t \geq T_0.$$

Then by Theorem 6.1.1, we obtain the desired result.

REMARK 6.2.1. Obviously, Theorem 6.2.1 includes Theorem 1 in [5], and it gives more general criteria. For example, if we choose $i = 1$ or 2, then (6.2.5) becomes

$$\min_i \{p_{ii} - \sum_{j=1, j\neq i}^{m} |p_{ij}|\}\tau e > 1$$

or

$$\lambda_n\left(\frac{1}{2}(P + P^\tau)\right)\tau e > 1,$$

which are independent of (6.2.5).

□

References

[1] O. Arino, *Necessary and sufficient condition for oscillation of a neutral differential system with several delays*, J. Differential Equations. **81** (1989), 98-105.

[2] Y. Cao, L.-P. Fan and T.C. Gard, *Uniform persistence for population interaction models with time delay*, preprint.

[3] K. Gopalsamy, *Oscillations in linear systems of differential-difference equations*, Bull. Austral. Math. Soc., Series B **29** (1984), 377-387.

[4] K. Gopalsamy, *Oscillations in Lotka-Volterra systems with several delays*, J. Math. Anal. Appl. **159** (1991), 440-448.

[5] K. Gopalsamy, M.R.S. Kulenovic and G. Ladas, *Oscillation in a system of delay logistic equations*, J. Math. Anal. Appl. **146** (1990), 192-202.

[6] I. Györi, *Oscillations of nonlinear delay differential equations arising in population dynamics*, Differential Equations, Proceedings of the 1987 Equadiff Conference, Dekker, New York, (1989), 291-301.

[7] I. Györi and G. Ladas, "Oscillation Theory of Delay Differential Equations with Applications," Clarendon Press, Oxford, (1991).

[8] G. Ladas, *Linearized oscillation for neutral equations,* Differential Equations, Proceedings of the 1987 Equadiff Conference, Dekker, New York, (1989), 379-387.

[9] B.G. Zhang and K. Gopalsamy, *Global attractivity in the delay logistic equation with variable parameters*, Math. Proc. Camb. Phil. Soc. 107 (1990), 579-590.

CHAPTER 7

ASYMPTOTIC BEHAVIOR OF NONLINEAR EQUATIONS OF HIGHER ORDER

7.1. Introduction and Preliminaries

The asymptotic behavior of the nth order ordinary differential equation

$$y^{(n)} + f(t, y, y', \dots, y^{(n-1)}) = 0 \tag{7.1.0}$$

has been considered in [12] and a result about it has been established. For the case that $n = 2$, [1] obtains some further results. For $n = 1$ and 2, the above work generalizes and revises the results in [2,15].

In this chapter, we consider the neutral differential equation

$$[y(t) - py(t-r)]^{(n)} + f\big(t, y(t-\tau_0), y'(t-\tau_1), \dots, y^{(n-1)}(t-\tau_{n-1})\big) = 0 \tag{7.1.1}$$

where $f \in C(\mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R})$, $p, r, \tau_0, \dots, \tau_{n-1}$ are nonnegative constants, $0 < p < 1$, and we denote $\tau^* = \max\{r, \tau_0, \dots, \tau_{n-1}\}$.

Some qualitative properties of nonlinear neutral differential equations such as oscillation, nonoscillation and asymptotic behavior of nonoscillatory solutions have drawn much attention in the literature, see [3-6,8-11,13,14]. But there has not been much work done on the asymptotic behavior of all solutions of neutral equations, for that of first order equations, see [7,16]. Usually, the asymptotic behavior of NDDEs is much different from that of the corresponding ODEs. However, as we will see later, under some assumptions, there is a surprising similarity between the behavior of the two equations (7.1.0) and (7.1.1). Thus, this chapter gives an example in which time delays do not affect the asymptotic behavior, and hence, the nonoscillatory behavior of solutions of NDDEs, no matter how large the delays are.

In the following we will obtain some sufficient and/or necessary conditions for the following conclusion:

(C) Every solution $y(t)$ of Eq. (7.1.1) satisfies that

$$y^{(n-i)}(t)/t^{i-1} \to a_i \in \mathbb{R}, \quad i = 1, \ldots, n, \quad \text{as} \quad t \to \infty \tag{7.1.2}$$

where $y^{(0)}(t) = y(t)$, and a_i, $i = 1, \ldots, n$, satisfy the relations $(i-1)!\, a_i = a_1$.

Furthermore, Eq. (7.1.1) has some solutions such that $a_i > 0$ (resp. < 0), $i = 1, \ldots, n$.

Before stating the main results we give some hypotheses and lemmas.

Definition 7.1.1: A function $g(u)$ is said to belong to a function class $\mathcal{F}$

if $g(u) > 0$ is nondecreasing and continuous on $(0,\infty)$, and

$$g(u)/v \le g(u/v) \quad \text{for} \quad u > 0,\ v \ge 1.$$

We need the following hypotheses:

(H1) $|f(t, y_0, \ldots, y_{n-1})| \le \sum_{i=0}^{n-1} \varphi_i(t) g_i(|y_i|/(t-\tau_i)^{n-i-1})$

(H2) $|f(t, y_0, \ldots, y_{n-1})| \ge \sum_{i=0}^{n-1} \psi_i(t) h_i(|y_i|/(t-\tau_i)^{n-i-1})$

where $\varphi_i(t) > 0$, $\psi_i(t) > 0 \in C[t_0,\infty)$, and $g_i \in \mathcal{F}$, $h_i \in C[0,\infty)$ are increasing, $i = 0, \ldots, n-1$.

The following lemmas will be used laer.

LEMMA 7.1.1. *Let* *i)* $0 \le p < 1$, r, $\tau \ge 0$, $\nu = \max\{r, \tau\}$;

(ii) $f(t) > 0$ *is continuous and nondecreasing on* $[t_0, \infty)$,

$h(t) \ge 0 \in C[^t{}_0, \infty)$, $y(t) \ge 0 \in C[t_0 - \nu, \infty)$;

(iii) $w \in \mathcal{F}$;

(iv) $\eta(t) \ge 0 \in L[t_0 - \nu, t_0]$ *and* $\eta^* = \max_{t_0-\nu \le t \le t_0} \{\eta(t)\}$.

Assume

$$y(t) - py(t-r) \le f(t) + \int_{t_0}^{t} h(s) w(y(t-\tau)) ds, \quad t \ge t_0 \tag{7.1.3}$$

and

$$y(t) = \eta(t) \quad \textit{for} \quad t \in [t_0 - \nu, t_0].$$

Then

$$y(t) \le \eta^* + \frac{f(t)}{f(t_0)} G^{-1} \left\{ G\left(\frac{f(t_0)}{1-p}\right) + \int_{t_0}^{t} \frac{h(s)}{1-p}\, ds \right\}, \quad t \ge t_0 \tag{7.1.4}$$

where

$$G(v) = \int_{v_0}^{v} \frac{ds}{w(s)} \quad \text{for} \quad v_0 > 0, \quad v > 0.$$

PROOF: (7.1.4) is obviously true for the subset E of $[t_0, \infty)$ where $y(t) \le \eta^*$. So we only need to consider the inequality on $[t_0, \infty)\backslash E$. Choose $t \in [t_0, \infty)\backslash E$, and let $x(t) = \max_{t_0 - \nu \le s \le t} \{y(s)\}$. Then $x(t) = y(t^*)$ for some $t^* \in (t_0, t]$. From (7.1.3) we have

$$\begin{aligned} x(t) - py(t^* - r) &\le f(t) + \int_{t_0}^{t} h(s) w\big(x(s - \tau)\big) ds \\ &\le f(t) + \int_{t_0}^{t} h(s) w\big(x(s)\big) ds. \end{aligned}$$

Noting that

$$y(t^* - r) \le x(t^* - r) \le x(t),$$

we have

$$x(t) \le F(t) + \int_{t_0}^{t} H(s) w\big(x(s)\big) ds$$

where $F(t) = \frac{f(t)}{1-p}$, $H(t) = \frac{h(t)}{1-p}$.

Since F is nondecreasing and $w \in \mathcal{F}$ we get that

$$\begin{aligned}\frac{F(t_0)}{F(t)}x(t) &\leq F(t_0) + \int_{t_0}^{t} H(s)\,\frac{F(t_0)}{F(s)}\,w(x(s))ds \\ &\leq F(t_0) + \int_{t_0}^{t} H(s)w(\frac{F(t_0)}{F(s)}\,x(s))ds.\end{aligned}$$

By Bihari's inequality we obtain that

$$\frac{F(t_0)}{F(t)}x(t) \leq G^{-1}\{G(F(t_0)) + \int_{t_0}^{t} H(s)ds\},$$

i.e.,

$$\begin{aligned}x(t) &\leq \frac{F(t)}{F(t_0)}G^{-1}\{G(F(t_0)) + \int_{t_0}^{t} H(s)ds\} \\ &= \frac{f(t)}{f(t_0)}G^{-1}\{G(\frac{f(t_0)}{1-p}) + \int_{t_0}^{t} \frac{h(s)}{1-p}ds\}.\end{aligned}$$

In view of the definition of $x(t)$ we see (7.1.4) holds for $t \in [t_0, \infty)\backslash E$, and hence holds on $[t_0, \infty)$.

Lemma 7.1.2. *Let* $0 \leq p < 1$ *and* $r, k \geq 0$. *Assume a function* $y(t)$ *satisfies that*

$$\lim_{t\to\infty}[y(t) - py(t-r)]/t^k = a \tag{7.1.5}$$

for some $a \in \mathbb{R}$. *Then*

$$\lim_{t\to\infty} y(t)/t^k = a/(1-p).$$

PROOF: We only need to prove the lemma for the case that $0 < p < 1$. From (7.1.5) we see that

$$y(t) = py(t-r) + \big(a + o(1)\big)t^k \tag{7.1.6}$$

where $o(1) \to 0$ as $t \to \infty$. For any $\varepsilon > 0$, choose $t_0 \geq 0$ such that $|o(1)| < \varepsilon$ for the $o(1)$ in (7.1.6) and $t \geq t_0$. Then for any $t \geq t_0$ there exists a $t^* \in [t_0, t_0 + r]$ and a nonnegative integer n such that $t = t^* + nr$. In (7.1.6) setting $t = t^* + nr$ for $n = 1, 2, \ldots,$ and applying induction we get

$$y(t^* + nr) = p^n y(t^*) + \big(a + o(1)\big) \sum_{i=1}^{n} p^{n-i}(t^* + ir)^k$$

where $|o(1)| < \varepsilon$. Then by Stotz's Theorem we obtain that

$$\limsup_{n\to\infty} \frac{y(t^* + nr)}{(t^* + nr)^k} \leq (a + \varepsilon) \lim_{n\to\infty} \frac{\sum_{i=1}^{n} p^{-i}(t^* + ir)^k}{p^{-n}(t^* + nr)^k}$$

$$= (a + \varepsilon) \lim_{n\to\infty} \frac{p^{-n}(t^* + nr)^k}{p^{-n}(t^* + nr)^k - p^{-n+1}\big(t^* + (n-1)r\big)^k}$$
$$= \frac{a + \varepsilon}{1 - p}.$$

Similarly,

$$\liminf_{n\to\infty} \frac{y(t^* + nr)}{(t^* + nr)^k} \geq \frac{a - \varepsilon}{1 - p}.$$

Since $\varepsilon > 0$ is arbitrary, this means that

$$\lim_{t\to\infty} y(t)/t^k = \frac{a}{1-p}\,.$$

□

7.2. Asymptotic Analysis

THEOREM 7.2.1. *Under the hypothesis (H1), a sufficient condition for Conclusion (C) is that*

$$\int^{\infty} \varphi_i(t)dt < \infty, \quad i = 0,\dots,n-1. \tag{7.2.1}$$

PROOF: Denote

$$G_i(v) = \int_{v_0}^{v} \frac{1}{g_i(s)}\,ds, \quad i = 0,\dots,n-1.$$

Without loss of generality, we may assume that $y(t)$ is a solution of Eq. (7.1.1) having initial values on $[1-\tau^*,1] : u^{(i)}(t) = \eta_i(t)$, $t \in [1-\tau^*,1]$, $i = 0,\dots,n-1$, where t^* is defined in Section 7.1.

(i) At first we conclude that $y(t)$ satisfies that

$$|y^{(n-j)}(t)|/t^{j-1} \le c_{n-j}+d_{n-j}\int_1^t \sum_{i=0}^{n-j-1} \varphi_i(s)g_i(|y^{(i)}(s-\tau_i)|/(s-\tau_i)^{n-i-1})ds \tag{7.2.2}$$

for $t \ge 1$ and $j = 1, \ldots, n$, where $c_{n-j} > 0, d_{n-j} > 0$, and $\sum_{i=0}^{-1} = 0$;

and then $y(t)$ satisfies that

$$|y^{(i)}(t)|/t^{n-i-1} \le b_i, \quad i = 0, \ldots, n-1 \tag{7.2.3}$$

for $t \ge 1$, where b_i, $i = 0, \ldots, n-1$, are constants.

The proofs of (7.2.2) and (7.2.3) are by induction twice in the forward and backward ways, respectively, and using Lemma 7.1.1 $2n$ times. To see this, from Eq. (7.1.1) we have

$$y^{(n-1)}(t) - py^{(n-1)}(t-r) = \beta_{n-1} - \int_1^t f\big(s, y(s-\tau_0), y'(s-\tau_1), \ldots, y^{(n-1)}(s-\tau_{n-1})\big)ds.$$

Choose $c > 0$ such that $c > |\beta_{n-1}|$. Then by (H1) we get

$$\begin{aligned}|y^{(n-1)}(t)| - p|y^{(n-1)}(t-r)| &\le |y^{(n-1)}(t) - py^{(n-1)}(t-r)| \\ &\le [c + \int_1^t \sum_{i=0}^{n-2} \varphi_i(s) g_i\big(|y^{(i)}(s-\tau_i)|/(s-\tau_i)^{n-i-1}\big)ds] \\ &\quad + \int_1^t \varphi_{n-1}(s) g_{n-1}\big(|y^{(n-1)}(s-\tau_{n-1})|\big)ds.\end{aligned}$$

Using Lemma 7.1.1 we obtain for $t \ge 1$

$$\begin{aligned}|y^{(n-1)}(t)| &\le \eta_{n-1}^* + [1 + \frac{1}{c}\int_1^t \sum_{i=0}^{n-2} \varphi_i(s) g_i\big(|y^{(i)}(s-\tau_i)|/(s-\tau_i)^{n-i-1}\big)ds] \\ &\quad \times G^{-1}\{G(\frac{c}{1-p}) + \int_1^\infty \frac{\varphi_{n-1}(t)}{1-p}\,dt\} \\ &= c_{n-1} + d_{n-1}\int_1^t \sum_{i=0}^{n-2} \varphi_i(s) g_i\big(|y^{(i)}(s-\tau_i)|/(s-\tau_i)^{n-i-1}\big)ds\end{aligned} \tag{7.2.4}$$

where $\eta^*_{n-1} = \max_{1-\tau^* \le t \le 1} \{|\eta_{n-1}(t)|\}$, c_{n-1}, d_{n-1} are positive constants. This means that (7.2.2) is true for $j = 1$. Starting from (7.2.4), by induction and using Lemma 7.1.1 for the case where $p = 0$ we can show that (7.2.2) holds in general. Particularly, when $j = n$, (7.2.2) imples that

$$|y(t)|/t^{n-1} \le c_0 := b_0.$$

This means that (7.2.3) is true for $i = 0$. Furthermore, by a backward induction and using Lemma 7.1.1 for the case where $p = 0$ again we can show that (7.2.3) also holds. Since the process is similar to the parts i) and ii) of the proof of the Theorem in [12], we omit it here.

ii) From Eq. (7.1.1) we get that for $i = 1, \ldots, n$

$$y^{(n-i)}(t) - py^{(n-i)}(t-r) = \sum_{j=1}^{i} \frac{d_{ji}}{(i-j)!}\, t^{i-j} - \frac{1}{(i-j)!} \times \int_1^t (t-s)^{i-1} f\big(s, y(s-\tau_0), \ldots, y^{(n-1)}(s-\tau_{n-1})\big) ds \tag{7.2.5}$$

where $d_{1i} = \beta_{n-1} := y^{(n-1)}(1) - py^{(n-1)}(1-r)$. From (7.2.3) we know that when $t \ge 1$

$$\begin{aligned} &\int_1^t \Big(1 - \frac{s}{t}\Big)^{i-1} |f\big(s, y(s-\tau_0), \ldots, y^{(n-1)}(s-\tau_{n-1})\big)| ds \\ &\quad \le \sum_{i=0}^{n-1} \int_1^t \varphi_i(s) g_i\big(|y^{(i)}(s-\tau_i)|/(s-\tau_i)^{n-i-1}\big) ds \\ &\quad \le \sum_{i=0}^{n-1} g_i(b_i) \int_1^\infty \varphi_i(s) ds < \infty. \end{aligned}$$

This implies that

$$\lim_{t\to\infty}\int_1^t \left(1-\frac{s}{t}\right)^{i-1}|f(s,y(s-\tau_0),\ldots,y^{(n-1)}(s-\tau_{n-1}))|ds$$

exists, and then

$$\lim_{t\to\infty}\int_1^t \left(1-\frac{s}{t}\right)^{i-1}f(s,y(s-\tau_0),\ldots,y^{(n-1)}(s-\tau_{n-1}))ds := h_i$$

exists, and it is easy to show that

$$h_1 = h_2 = \cdots = h_n := h.$$

Dividing both sides of (7.2.5) by t^{i-1} and letting $t\to\infty$ we get

$$\frac{y^{(n-i)}(t)-py^{(n-i)}(t-r)}{t^{i-1}} \to \frac{d_{1i}-h_i}{(i-1)!} = \frac{\beta_{n-1}-h}{(i-1)!} := a_i(1-p). \qquad (7.2.6)$$

By Lemma 7.1.2, we see that (7.1.2) holds for $i=1,\ldots,n$.

iii) For any $\alpha>0$, because of (7.2.1) we can choose $T\geq 1$ so large that

$$\sum_{i=0}^{n-1} g_i\left(\frac{2\alpha}{(1-p)(n-i-1)!}\right)\int_T^\infty \varphi_i(s)ds < \frac{\alpha}{4}.$$

Let $y(t)$ be the solution of Eq. (7.1.1) satisfying $y^{(i)}(t)=0$ for $t\in[T-r^*,T]$, $i=0,\ldots,n-1$, $y(T)=y'(T)=\cdots=y^{(n-2)}(T)=0$, and

$y^{(n-1)}(T) = \alpha$. Then for $t > T$ and $i = 0, \ldots, n-1$

$$y^{(n-i)}(t) - py^{(n-i)}(t-r) = \frac{\alpha}{(i-1)!}(t-T)^{i-1} - \frac{1}{(i-1)!} \times \int_T^t (t-s)^{i-1} f\big(s, y(s-\tau_0), \ldots, y^{(n-1)}(s-\tau_{n-1})\big) ds. \tag{7.2.7}$$

So we conclude that there exists a maximal interval $(T, \widetilde{T})$ in which

$$0 < y^{(n-i)}(t)/t^{i-1} \le \frac{2\alpha}{(1-p)(i-1)!}, \quad i = 1, \ldots, n$$

or

$$0 < y^{(j)}(t)/t^{n-j-1} \le \frac{2\alpha}{(1-p)(n-j-1)!}, \quad j = 0, \ldots, n-1 \tag{7.2.8}$$

and

$$y^{(n-1)}(t) > \frac{\alpha}{2}.$$

However it follows that $\widetilde{T} = \infty$. For otherwise from (7.2.7)

$$\begin{aligned} 0 < \frac{y^{(n-i)}(\widetilde{T})}{\widetilde{T}^{i-1}} &\le p\,\frac{y^{(n-i)}(\widetilde{T}-r)}{(\widetilde{T}-r)^{i-1}} + \frac{\alpha}{(i-1)!} \\ &+ \frac{1}{(i-1)!}\int_T^{\widetilde{T}} \sum_{j=0}^{n-1} \varphi_j(s) g_j\big(|y^{(j)}(s-\tau_j)|/(s-\tau_j)^{n-j-1}\big) ds \\ &\le \frac{\alpha}{(i-1)!}\left(\frac{2p}{1-p} + 1 + \frac{1}{4}\right) \\ &= \frac{5+3p}{4(1-p)}\,\frac{\alpha}{(i-1)!} < \frac{2\alpha}{(1-p)(i-1)!} \end{aligned}$$

and

$$
\begin{aligned}
y^{(n-1)}(\widetilde{T}) &\geq py^{(n-1)}(\widetilde{T}-r)+\alpha-\int_{T}^{\widetilde{T}}\sum_{j=0}^{n-1}\varphi_j(s)g_j(|y^{(j)}(s-\tau_j)|(s-\tau_j)^{n-j-1})ds \\
&\geq \alpha-\sum_{j=0}^{n-1}g_j\Big(\frac{2\alpha}{(1-p)(n-j-1)!}\Big)\int_{T}^{\infty}\varphi_j(s)ds \\
&\geq \frac{3}{4}\,\alpha,
\end{aligned}
$$

which would contradict the definition of $\widetilde{T}$. (H1), (7.2.7) and (7.2.8) imply that for t sufficiently large

$$
y^{(n-1)}(t)-py^{(n-1)}(t-r)>\frac{\alpha}{2}\,.
$$

This and (7.2.6) imply that $a_1 \geq \frac{\alpha}{2(1-p)}$. From (7.2.6) we know that $a_i > 0$, $i = 1,\ldots,n$. This completes the proof of the theorem.

□

From Theorem 7.2.1 we obtain the following corollaries immediately.

COROLLARY 7.2.1. *Under assumption (H1), assume*

$$
\int^{\infty}\varphi_i(t)dt<\infty,\quad i=0,\ldots,n-1.
$$

Then Eq. (7.1.1) has unbounded or bounded nonoscillatory solutions if $n > 1$ *or* $n = 1$, *respectively.*

Corollary 7.2.2. *Under assumption (H1), assume*

$$\int^{\infty} \varphi_i(t)dt < \infty, \quad i = 0, \ldots, n-1.$$

Then all oscillatory solutions $y(t)$ *of Eq. (7.1.1) satisfy that*

$$y^{(n-i)}(t)/t^{i-1} \to 0, \quad i = 1, \ldots, n, \quad \text{as} \quad t \to \infty.$$

In particular, if $n = 1$, *then all oscillatory solutions* $y(t)$ *of Eq. (7.1.1) satisfy that* $y(t) \to 0$ *as* $t \to \infty$.

Theorem 7.2.2. *Under hypothesis (H2), a necessary condition for Conclusion(C) is that*

$$\int^{\infty} \psi_i(t)dt < \infty, \quad i = 0, \ldots, n-1.$$

Proof: Choose a solution $y(t)$ of Eq. (7.1.1) such that (7.1.2) holds for some $a_i > 0$, $i = 1, \ldots, n$. Let T be so large that

$$y^{(n-i)}(t)/t^{i-1} \geq \frac{a_i}{2}$$

for $t \geq T - \tau^*$ and $i = 1, \ldots, n$. Using hypothesis (H2) we see that for $t \geq T$

$$\begin{aligned}|f(t, y(t-\tau_0), \ldots, y^{(n-1)}(t-\tau_{n-1}))| &\geq \sum_{i=0}^{n-1} \psi_i(t) h_i(y^{(i)}(t-\tau_i)/(t-\tau_i)^{n-i-1}) \\ &\geq \sum_{i=0}^{n-1} h_i(a_i/2)\psi_i(t) > 0.\end{aligned}$$

Without loss of generality we may assume that for $t \geq T$

$$f(t, y(t-\tau_0), \ldots, y^{(n-1)}(t-\tau_{n-1})) > 0.$$

Integrating both sides of Eq. (7.1.1) from T to ∞ we have

$$\begin{aligned}y^{(n-1)}(T) - py^{(n-1)} - (1-p)a_1 &= \int_T^\infty f(s, y(s-\tau_0), \ldots, y^{(n-1)}(s-\tau_{n-1}))ds \\ &\geq \sum_{i=0}^{n-1} h_i(a_i/2) \int_T^\infty \psi_i(t)dt.\end{aligned}$$

This implies that

$$\int_T^\infty \psi_i(t)dt < \infty, \quad i = 0, \ldots, n-1$$

and completes the proof.

□

Combining Theorems 7.2.1 and 7.2.2 we obtain

COROLLARY 7.2.2. *Under hypothesis (H1) and (H2) with* $\varphi_i(t) = \psi_i(t)$, $i = 0, \ldots, n-1$, *a necessary and sufficient condition for Problem (P) is that*

$$\int^{\infty} \varphi_i(t)dt < \infty, \quad i = 0, \ldots, n-1.$$

References

[1] Shaozhu Chen, *Asymptoitc linearity of solutions of nonlinear differential equations,* Bull. Austral. Math. Soc. 35 (1987), 257-265.

[2] D.S. Cohen, *The asymptotic behavior of a class of nonlinear differential equations,* Proc. Amer. Math. Soc. 18 (1967), 607-609.

[3] L.H. Erbe and Q. Kong, *Oscillation results for second order neutral differential equations,* Funkcialaj Ekvacioj, to appear.

[4] L.H. Erbe and B.G. Zhang, *Oscillation of second order neutral differential equations,* Bull Austral. Math. Soc., 39 (1989), 71-80.

[5] S.R. Grace and B.S. Lalli, *Oscillation of nonlinear second neutral delay differential equations,* Rad. Mat. 3 (1987), 77-84.

[6] J.R. Graef, M.K. Grammatikopoulos and P.W. Spikes, *Asymptotic and oscillatory behavior of solutions of first order nonlinear neutral delay differential equations,* J. Math. Anal. Appl. 155 (1991), 562-571.

[7] J.R. Haddock, T. Krisztin and J. Wu, *Asymptotic equivalence of neutral equations and infinite delay equations,* Nonlinear Anal. TMA 14 (1990), 369-377.

[8] J. Jaroš and T. Kusano, *Asymptotic behavior of nonoscillatory solutions of nonlinear functional differential equations of neutral type,* Funkcial Ekvacioj. 32 (1989), 251-263.

[9] ———, *Oscillation properties of first order nonlinear functional differential equations of neutral type,* Diff. Integral Equations, 2 (1991), 425-436.

[10] G. Karakostas, *Convergence in a neutral delay differential equations,* Applicable Analysis, 40 (1991), 225-231.

[11] Y. Kitamura and T. Kusano, *Oscillation and asymptotic behavior of solutions of first-order functional differential equations of neutral type*, Funkcial. Ekvac., 33 (1990), 325-343.

[12] Q. Kong, *Asymptotic behavior of a class of nonlinear differential equations of nth order,* Proc. Amer. Math. Soc., 3 (1988), 831-838.

[13] G. Ladas and C. Qian, *Linearized oscillation for odd-order neutral delay differential equations*, J. Differential Equations. 88 (1990), 238-247.

[14] B.S. Lalli, *Oscillation of nonlinear second order neutral delay differential equations,* Bulletin of the Institute of Mathematics Academia Sinica, 3 (1990), 233-238.

[15] J. Tong, *The asymptotic behavior of a class of nonlinear differential equations of second order,* Proc. Amer. Math. Soc., 84 (1982), 235-236.

[16] Jianhong Wu, *Asymptotic periodicity of solutions to a class of neutral functional differential equations*, Proc. Amer. Math. Soc. 2 (1991), 355-363.

CHAPTER 8

CONCLUDING REMARKS

In this thesis we have discussed the oscillatory behavior of some kinds of NDDEs. The oscillation considered is mainly due to the time delays. That is, as the time delays grow larger, the equation may change from nonoscillatory to oscillatory.

In Chapter 2 we have investigated the first order linear NDDE with constant coefficients (2.1.3). New necessary and sufficient conditions for oscillation have been found based on the extrema of some functions defined on l^1-spaces. The results are easier to apply than those related to characteristic equations, and the explicit conditions derived there are sharp in the sense that they cover most known results for special cases.

In Chapter 3 we have investigated the first order linear NDDE with variable coefficients (3.1.1). Some necessary and/or sufficient conditions for oscillation have been found, and the results are analogues of the criteria related to the characteristic equations for equations with constant coefficients. Some comparison criteria for oscillation and nonoscillation are also obtained. The above work improves a number of recent results and for delay differential equations, it also improves the established conjecture by Hunt and York in [12, Chapter 3], by applying the "integral average" technique.

Chapter 4 is an extension of the work by Erbe and Zhang in [4, Chapter 4], where the second order superlinear NDDE (4.1.1) is considered. Some oscillation criteria have been obtained by investigating the oscillatory behavior of corresponding delay differential equations. In the mean time, some conjectures on the asymptotic behavior of the nonoscillatory solutions by Ladas and Sficas in [11, Chapter 4] have been established.

In Chapter 5 we have discussed the linear autonomous system of NDDEs (5.1.1). The first set of systematic explicit conditions for oscillation have been obtained by utilizing the Lozenskii measures. It solves the open problem 6.12.2 in [5, Chapter 1]. The conditions can be easily verified, and are sufficiently so that when the system is a scalar equation, they coincide with the results in Chapter 2.

Chapter 6 partially extends the results in Chapter 5 to the nonautonomous system of NDDEs (6.1.1). The oscillatory properties of (6.1.1) are determined by the oscillatory properties of the autonomous system (6.1.2). As a result, some criteria for oscillation of Lotka-Volterra population models are derived.

From Chapters 2–6 we observe that time delays usually have the tendency to cause oscillation. However, there are some exceptions. For example, if the corresponding ODE has a strong nonoscillatory behavior, the NDDE may preserve the nonoscillatory behavior even if the delays are sufficiently large.

Chapter 7 presents an example where the NDDE (7.1.1) and its corresponding ODE (7.1.0) exhibit the same polynomial-asymptotic behavior and hence exhibit the same nonoscillatory property.

Related to what has been done in this thesis, the following problems still remain unsolved:

1. Extend the results in Chapter 2 to higher order linear NDDEs with constant coefficients.

2. Extend the results in Chapter 3 to the first order NDDE (3.1.1) with arbitrary real p, or with variable p, or to higher order NDDEs with variable coefficients.

3. Find oscillation criteria for Eq. (4.1.1) where $f(y)$ is a sublinear function as $y \to 0$.

4. Obtain explicit conditions for oscillation of the following system

$$\frac{d}{dt}\left[y_i(t) - p_i y_i(t - r_i)\right] + \sum_{j=1}^{m} q_{ij} y_j(t - \tau_{ij}) = 0, \quad i = 1, \ldots, n,$$

   where p_i, r_i, q_{ij}, and τ_{ij} are constants, $i = 1, \ldots, n$; $j = 1, \ldots, m$.

5. Obtain some oscillation criteria for the system (6.1.1) according to Definition (1.2.3).

There are also some other open problems on oscillation of NDDEs, see §6.12 in [5, Chapter 1] for detail.

Finally, we would like to summarize some topics on oscillation theory which have been extensively discussed for ODEs, but rarely considered for NDDEs, or even for DDEs:

i) Oscillation caused by oscillatory coefficients.

ii) Nonoscillation of all solutions.

iii) Existence of an oscillatory solution.

iv) Distribution of zeros and the variability of amplitude of oscillatory solutions.

v) Monotonicity of the separation of the zeros of solutions.