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NAME OF AUTHOR ..GILLES LAPONTE..................

TITLE OF THESIS ..SOME ASPECTS OF THE DESCRIPTION..
..OF RELATIVISTIC PARTICLES IN.....
..EXTERNAL FIELDS...................

UNIVERSITY ..OF ALBERTA..........................

DEGREE FOR WHICH THESIS WAS PRESENTED ....Ph.D. in THEORETICAL PHYSICS

YEAR THIS DEGREE GRANTED ....1973.....................

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DATED ..August 13th.......1973

NL-91 (10-68)

THE UNIVERSITY OF ALBERTA

SOME ASPECTS OF THE DESCRIPTION OF RELATIVISTIC PARTICLES IN EXTERNAL FIELDS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

IN

THEORETICAL PHYSICS

DEPARTMENT OF PHYSICS

EDMONTON, ALBERTA

FALL 1973

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled SOME ASPECTS OF THE DESCRIPTION OF RELATIVISTIC PARTICLES IN EXTERNAL FIELDS submitted by Gilles Labonté in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Theoretical Physics.

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TO MY PARENTS

ABSTRACT

We study the description of the motion of relativistic particles in both time dependent and time independent potentials. The differential equations of motion considered are the standard linear spin zero and one half equations. They are always meaningful in the sense that, at all times, unique well defined operator valued distributions in the three space variables are determined.

We discuss the problem of determining which set of creation and annihilation operators, among the infinite possible number of such sets, is relevant in a given problem. We examine the implementation of certain simple requirements which seem to be obviously necessary in order for the mathematical formalism to be able to describe satisfactorily a physical system. (The present study can thus be considered as being done from a point of view somewhat similar to the one of the so called "axiomatists" in completely quantized field theory.) We show that whenever the equation of motion is homogeneous, (which is the case in most problems of interest), the study of all physical requirements reduces to studying Bogoliubov transformations between creation and annihilation operators.

We study such transformations in the third chapter where we obtain some new important results concerning their general properties. We also discuss the possibility of non-conservation of the total charge.

We examine in details the following particular problems: a quantized field in presence of an external source, electrons and positrons acted upon by a plane electromagnetic wave, Dirac fields acted upon by potentials of the form $\underset{\sim}{A}(x)\delta(t)$ and $\underset{\sim}{A}(x)\theta(t-t_o)$.

We study also Dirac fields in presence of potentials which have time dependences which can be represented by sequences of step functions. We then discuss the limiting case where the time dependence is continuous.

We prove that the requirements that there exists a unitary evolution operator or that physical particles can be described are exactly equivalent. We show that a satisfactory physical interpretation can be obtained with potentials of quite general time dependences if and only if at all times these potentials are such that the necessary and sufficient conditions obtained on $\underset{\sim}{A}(x)$ in the problem $\underset{\sim}{A}(x)\theta(t-t_o)$ are satisfied. We derive some new estimates on potentials for which these conditions are satisfied.

ACKNOWLEDGMENTS

I want to thank my supervisor, Dr. A.Z. Capri, for his advice and constant encouragement which have been of precious help to me. I have learned much from him during the course of this work and I have greatly appreciated his patience and willingness to discuss any aspect of a physical problem.

I thank also the Department of Physics as a whole for giving me the opportunity to study at the University of Alberta.

I am very grateful to all the institutions which have given me financial support during the realization of this work. These are the Department of Physics of the University of Alberta, the National Research Council of Canada and the Department of Advanced Education of the Government of Alberta.

I also want to thank Mrs. M. Yiu for her expert and fast typing of my thesis.

TABLE OF CONTENTS

CHAPTER PAGE

PRELIMINARIES AND NOTATION

The theory of ordinary quantum mechanics is already firmly established as giving a good description of physical phenomena at the atomic level. The essential difference between this and classical mechanics is that now the observable quantities are eigenvalues of self adjoint operators which are such that canonically conjugate variables do not commute (for example: the momentum $\underset{\sim}{p}$ and position $\underset{\sim}{x}$ are such that $[p_i, x_j] = -i\hbar\delta_{ij}$ where $\hbar = h/2\pi$, h being Planck's constant).

At the subatomic level i.e. in elementary particles physics, it is clear that processes occur where particles are destroyed or created. Such processes are in accordance with the theory of relativity in which mass is a form of energy so that it becomes possible to create or absorb particles whenever the interactions give rise to changes in energy greater or equal to the rest energy (i.e. mass) of the particles in question.

Since elementary particles have properties of both: relativistic systems and quantum mechanical systems, one must try to make a theory incorporating the fundamental hypotheses of both of these theories.

The theory so obtained however is so complicated that up to now nobody has even been able to prove that it is consistent and non-trivial (in the sense that not only free particles could be described). The main reason for trying to keep this framework is that perturbation calculations (in power of the coupling constants) in certain cases gave very good agreement with experiments.

There is a class of simple problems which can help us gain some understanding of the situation. These deal with idealized systems where the particles do not interact between themselves but are only acted upon by an external force. This is the type of systems that we will be considering in the present work.

Such systems have the advantage of being described by a simpler mathematical formalism that is used for those where the influence of the particles on one another is taken into account. They nevertheless have the property that particles can be created or destroyed when the energy of the system is changed.

The finding of a relativistic wave equation for elementary particles is complicated by the existence of the spin. For a spinless particle, such an equation is obtained by using the same rule as for the Schrödinger equation: the energy operator is $E = i\hbar \frac{\partial}{\partial t}$ and the momentum operator $\underset{\sim}{p} = -i\hbar \underset{\sim}{\nabla}$. The classical relation giving the energy of a free particle:

$$E^2 = p^2c^2 + m^2c^4$$

where m is the mass of the particle, thus leads to the (Klein-Gorkon) wave equation

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \phi(t,\underset{\sim}{x}) = (-\hbar^2c^2 \underset{\sim}{\nabla}^2 + m^2c^4) \phi(t,\underset{\sim}{x}).$$

As customary, we shall use the natural units where $\hbar = 1$ and $c = 1$. We shall use the following relativistic notation:

$$x^\mu = (x^0,x^1,x^2,x^3) = (x^0,\underset{\sim}{x})$$

where $x^0 = t$, the time parameter and $x^1 = x$, $x^2 = y$, $x^3 = z$, the space coordinates. The metric tensor $g_{\mu\upsilon}$ is such that

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{\mu\upsilon} = 0 \ (\mu \neq \upsilon).$$

The product of two 4-vectors is then

$$x\cdot y = x^0y^0 - \underset{\sim}{x}\cdot\underset{\sim}{y} = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3.$$

The Klein-Gordon equation can then be written as

$$(\Box + m^2)\ \phi(x) = 0$$

where $\Box = \frac{\partial^2}{\partial t^2} - \underset{\sim}{\nabla}^2$.

For spin ½ particles, as electrons for example, the wave equation is the Dirac equation:

$$(-i\gamma\cdot\partial + m)\ \psi(x) = 0$$

where $\gamma\cdot\partial = \gamma^0 \frac{\partial}{\partial t} - \underset{\sim}{\gamma}\ \underset{\sim}{\nabla}$;

$\gamma^\mu = \gamma^0, \gamma^1, \gamma^2, \gamma^3$ are 4×4 constant matrices such that $\{\gamma^\mu, \gamma^\upsilon\} = \gamma^\mu\gamma^\upsilon + \gamma^\upsilon\gamma^\mu = 2g^{\mu\upsilon}$.

Spin 0 and ½ particles are the only two types of particles which will be considered explicitly in the present work. The equations used are essentially the above free equations, except for an additional term representing the potential energy of the particles (as these will be acted upon by a given force). The example which will be most discussed is that of electrons in presence of an electromagnetic potential. Systems of electrons are those for which the complete theory is most succesful.

The quantum mechanical states of such systems must now obviously include states in which there is any arbitrary number of particles present, as the number of these particles can change. Operators are then needed to describe the creation and annihilation of particles. These are defined simply as the operators which when applied on a state of n particles will give either a state of n+1 particles or n-1 particles. The space of such states is called Fock-Hilbert space.

Although we recalled the main properties of such a formalism in the text, a more detailed discussion of it can be found in the chapters six to ten of S.S. Schweber [1961].

CHAPTER I

TIME INDEPENDENT EXTERNAL FIELD PROBLEM

In the first part of this chapter, we present the formalism used to describe a system consisting of relativistic particles acted upon by a time independent potential.

Our main purpose in doing so is to introduce the notations and concepts which will be needed in the other chapters. We will also stress the fact that time independent external field systems can be quite rigorously treated, exactly as is the case for free field systems.

In the second part, we show that there are indeed many different ways to associate particles with such systems. A commonly used quantization method will be studied in more details. We will then discuss the physical considerations which lead one to regard the operators used in the first section as giving the proper description.

1) Quantized Field Interacting with Time Independent Potentials

Introduction

In this section, we present a quantum field theoretical treatment of the time independent external field interaction.

The field will be quantized in the manner described by, for example, A.I. Akhiezer and V.B. Berestetsky [1953] and S.S. Schweber [1961]. Our presentation differs however in that we make explicit use of wave packets. This is done in order to understand clearly what happens in light of the formalism for the c-number scattering theory as presented by J.M. Jauch [1958], S.T. Kuroda [1959] (also by T. Kato [1966], R.G. Newton [1966]).

In this formalism, it is clear that problems with time independent potentials can be treated rigorously without recourse to the procedure of adiabatic switching on and off of the potential. (According to this procedure, a factor $e^{-\varepsilon|t|}$ is introduced in the potential to facilitate the evaluation of certain quantities by increasing their convergence. At the end of all calculations, the limit $\varepsilon \to +0$ is taken.) This, obviously, is justified only when a system has a certain stability (or continuity) of properties under variation of the time dependence of the potential. Although this might be desirable for most physical situations, it has not yet been proven true in the description of relativistic particles in external fields. This is why we avoid this procedure here.

We will examine in detail the properties of the creation and annihilation operators introduced in the quantization of the field. We will prove that these are in fact the operators associated with the observable particles either initially or finally.

We will see that the total Hilbert space can be separated in a sum of subspaces denoted $\mathcal{F}_{in}^{(n,m)}$ which are the spaces containing all possible states where there are n bound particles and m bound antiparticles (when bound states are possible).

Similar scattering experiments can be done in any one of these particular subspaces $\mathcal{F}_{in}^{(n,m)}$ and will yield exactly the same answers to the scattering problem. The presence of bound particles has no influence whatsoever on a scattering experiment.

The only effect taking place is ordinary elastic scattering with many particles. Such a situation is expected here since there is no interaction between the particles. Their charge allows them only to be acted upon by the external field; they are not themselves the cause of any electromagnetic field.

Although the bound particles never appear in a scattering experiment when the external field is time independent, it is useful to have the necessary formalism to describe them. This is used in calculating the energy levels of atoms for example.

In subsequent sections, we will study cases where the external field depends on time. We will then be able to describe, among other things, the freeing of bound particles (and the binding of free particles) in an external field. This should happen when, for example, the external field goes from $A(\underset{\sim}{x})$ (which allows bound states) to $A = 0$ (or vice versa).

Let us consider for the moment the Dirac electron-positron field in interaction with a time independent electromagnetic potential $A(\underset{\sim}{x})$. The differential equation of motion is

$$\{-i\gamma.\partial + m\}\psi(x) = e\gamma .A(\underset{\sim}{x})\psi(x) \qquad (1.1)$$

i.e. $i \frac{\partial}{\partial t} \psi(t,\underset{\sim}{x}) = h\psi(t,\underset{\sim}{x})$

where

$$h = \gamma^{o}[i\underset{\sim}{\gamma}.\underset{\sim}{\partial} + m] - e\gamma^{o}\gamma.A(\underset{\sim}{x}) \quad .$$

<u>Solution</u>

The quantized field operator solution is

$$\psi(t,\underset{\sim}{x}) = e^{-iht} \psi(0,\underset{\sim}{x}) \qquad (1.2)$$

where

$$\psi(0,\underset{\sim}{x}) = \sum_{E} b_E f_E(\underset{\sim}{x}) + \sum_{E'} d^{\dagger}_{E'} f_{E'}(\underset{\sim}{x})$$

$$+ \sum_{\beta} \{b_\beta f_{+\beta}(\underset{\sim}{x}) + d^{\dagger}_{\beta} f_{-\beta}(\underset{\sim}{x})\} \quad . \qquad (1.3)$$

The "coefficients" in the expansion (1.3): b_E, $d^{\dagger}_{E'}$, b_β and $d^{\dagger}_{\beta}$ are here operators. They will be defined properly in a moment. We first explain what are the functions $f(\underset{\sim}{x})$ in (1.3) and what their labels mean.

i) The functions $f_E(\underset{\sim}{x})$ and $f_{E'}(\underset{\sim}{x})$ are eigenfunctions of h corresponding to the discrete eigenvalues $+E > 0$, $-E' < 0$. They are square integrable and orthonormal. A single index E or E' is used to characterize these functions but it implicitly means rather a set of indices containing also any other good quantum number which would come from a conserved quantity due to a symmetry in the system (like the spin index s in the present case).

The functions $f_{\varepsilon\beta}(\underset{\sim}{x})$ are square integrable functions which contain only eigenmodes of h with the same sign ε of energy. (β denotes also implicitly the spin index s). Such functions can be obtained simply through the action of $\Omega_{\pm}$ (the Møller wave operator) on the corresponging free functions.

Let us then define free wave packets as

$$f^{o}_{+\beta}(\underset{\sim}{x}) = \int d\underset{\sim}{p} h_{\beta'}(\underset{\sim}{p}) \frac{1}{(2\pi)^{3/2}} \left(\frac{m}{\omega(\underset{\sim}{p})}\right)^{\frac{1}{2}} w^{s}(-\underset{\sim}{p}) e^{-i\underset{\sim}{p}.\underset{\sim}{x}}$$

$$f^{o}_{-\beta}(\underset{\sim}{x}) = \int d\underset{\sim}{p} h_{\beta'}(\underset{\sim}{p}) \frac{1}{(2\pi)^{3/2}} \left(\frac{m}{\omega(\underset{\sim}{p})}\right)^{\frac{1}{2}} v^{s}(\underset{\sim}{p}) e^{-i\underset{\sim}{p}.\underset{\sim}{x}} \qquad (1.4)$$

where the functions $h_{\beta'}(\underset{\sim}{p})$ for $\beta' = 1,2,3,\ldots$ form an arbitrary basis in L^2. The other terms

$$\frac{1}{(2\pi)^{3/2}} \left(\frac{m}{\omega(\underset{\sim}{p})}\right)^{\frac{1}{2}} w^{s}(-\underset{\sim}{p}) e^{-i\underset{\sim}{p}.\underset{\sim}{x}} \quad \text{and} \quad \frac{1}{(2\pi)^{3/2}} \left(\frac{m}{\omega(\underset{\sim}{p})}\right)^{\frac{1}{2}} v^{s}(\underset{\sim}{p}) e^{-i\underset{\sim}{p}.\underset{\sim}{x}}$$

are the eigenmodes of h_o corresponding respectively to eigenvalues $\omega(\underset{\sim}{p})$ and $-\omega(\underset{\sim}{p})$ where $\omega(\underset{\sim}{p}) = \sqrt{\underset{\sim}{p}^2 + m^2}$. They are in the continuous spectrum of h_o so that they are normalizable only to Dirac delta functions.

Because of the intertwining properties of $\Omega_{\pm}$, the functions defined as

$$f_{\varepsilon\beta}(\underset{\sim}{x}) = \Omega_{\pm} f^{o}_{\varepsilon\beta}(\underset{\sim}{x})$$

will have the same eigenvalue composition as $f^{o}_{\varepsilon\beta}(\underset{\sim}{x})$. By this we mean the following: since

$$h_o f^{o}_{+\beta}(\underset{\sim}{x}) = \int d\underset{\sim}{p} h_{\beta'}(\underset{\sim}{p}) \omega(\underset{\sim}{p}) \left[\frac{1}{(2\pi)^{3/2}} \left(\frac{m}{\omega(\underset{\sim}{p})}\right)^{\frac{1}{2}} w^{s}(-\underset{\sim}{p}) e^{-i\underset{\sim}{p}.\underset{\sim}{x}}\right] \qquad (1.5)$$

when $\Omega_\pm$ is applied on each side of this equation and $\Omega_\pm h_o = h\Omega_\pm$ is used, we obtain the similar expression

$$hf_{+\beta}(\underset{\sim}{x}) = \int d\underset{\sim}{p}h_{\beta'}(\underset{\sim}{p})\omega(\underset{\sim}{p})\left(\Omega_\pm \frac{1}{(2\pi)^{3/2}}\left(\frac{m}{\omega(\underset{\sim}{p})}\right)^{\frac{1}{2}} w^s(-\underset{\sim}{p})e^{-i\underset{\sim}{p}.\underset{\sim}{x}}\right).$$

The same obviously holds for $f_{-\beta}(\underset{\sim}{x})$.

If we were not using wave packets, we could have said simply that from each eigenfunction of h_o with eigenvalue $\pm\omega(\underset{\sim}{p})$, we obtain a corresponding eigenfunction of h also with eigenvalue $\pm\omega(\underset{\sim}{p})$. These would be

$$f_{(\varepsilon,\underset{\sim}{p},s)}(\underset{\sim}{x}) = \Omega_\pm f^o_{(\varepsilon,\underset{\sim}{p},s)}(\underset{\sim}{x})$$

where $f^o_{(\varepsilon,p,s)}(\underset{\sim}{x})$ is the term in bracket in (1.5).

The set of functions $f^o_{+\beta}(\underset{\sim}{x})$ and $f^o_{-\beta}(\underset{\sim}{x})$ form a basis in $(L^2)^4$. Similarly, $f_{+\beta}(\underset{\sim}{x})$, $f_{-\beta}(\underset{\sim}{x})$ together with all the eigenfunctions corresponding to the discrete spectrum of h form a complete orthonormal set in $(L^2)^4$.

Hereafter, we will use the set of functions $f_{\varepsilon\beta}(\underset{\sim}{x})$ obtained through Ω_+ from $f^o_{\varepsilon\beta}(\underset{\sim}{x})$. One could indifferently use those obtained through Ω_- or in fact any complete set of such functions with different asymptotic boundary conditions. The physical results would be the same; the above choice is made arbitrarily, for the moment.

ii) We now come back to the definition of the operators (b,d). These operators and their adjoints have the following anticommutation relations.

All the operators b and $b^{\dagger}$ anticommute with all the operators d and $d^{\dagger}$.

The "d" type operators and the "b" type operators have the same anticommutation relations.

These are

$$\{b_E, b^{\dagger}_{\bar{E}}\} = \delta_{E\bar{E}} \qquad \{b_E, b_{\bar{E}}\} = 0$$

$$\{b_E, b_\beta\} = 0 \qquad \{b_E, b^{\dagger}_\beta\} = 0$$

$$\{b_\beta, b^{\dagger}_{\beta'}\} = \delta_{\beta\beta'}, \quad \{b_\beta, b_{\beta'}\} = 0 \qquad (1.6)$$

We note that the notation b(β) or b(E) will sometimes be used to facilitate the writing of certain expressions. The notation with an index is however more appropriate here since the variables β and E are discrete.

They operate on the Hilbert space ($\mathcal{H}$) with the Fock space structure for which the vacuum state $|0\rangle$ is defined as

$$b_E|0\rangle = 0 \qquad d_{E'}|0\rangle = 0 \ \forall\ E \text{ and } E'$$

and

$$b_\beta|0\rangle = 0 \qquad d_\beta|0\rangle = 0 \ \forall\ \beta \qquad (1.7)$$

The other states of this space are obtained (as in the free field case) by the action of the creation operators $b^\dagger$ and $d^\dagger$ on the vacuum state. The general state is then a normalizable linear combination with different values of k, ℓ, n and m of states of the type

$$|\underline{\alpha}_\ell \underline{E}_k ; \underline{\beta}_m \underline{E}'_n> = b(\alpha_1)^\dagger b(\alpha_2)^\dagger \ldots b(\alpha_\ell)^\dagger \times$$

$$\times\, b(E_1)^\dagger \ldots b(E_k)^\dagger d(\beta_1)^\dagger \ldots d(\beta_m)^\dagger \times$$

$$\times\, d(E'_1)^\dagger \ldots d(E'_n)^\dagger |0> .$$

The relation between the creation and annihilation operators discussed here and those used in the treatments which do not use wave packets is simply

$$b_\beta = \int d\underset{\sim}{p} h^*_{\beta'}(\underset{\sim}{p}) b_s(\underset{\sim}{p})$$

$$d^\dagger_\beta = \int d\underset{\sim}{p} h^*_{\beta'}(\underset{\sim}{p}) d^\dagger_s(\underset{\sim}{p}) \quad , \qquad (1.8)$$

the operators b_E and $d^\dagger_{E'}$ being the same. The operators $b_s(\underset{\sim}{p})$ and $d_s(\underset{\sim}{p})$ satisfy the anticommutation relations similar to those of b_β and d_β except that the Kronecker delta functions $\delta_{\beta\beta'}$ are now replaced by Dirac delta functions $\delta(\underset{\sim}{p}-\underset{\sim}{p}')$.

We note also that, similarly to the definition (1.8), one could define more general operators as

$$b(g) = \int d\underset{\sim}{p} \sum_{s=1}^{2} g_s^*(\underset{\sim}{p}) b_s(\underset{\sim}{p})$$

$$d(g)^\dagger = \int d\underset{\sim}{p} \sum_{s=1}^{2} g_s^*(\underset{\sim}{p}) d_s^\dagger(\underset{\sim}{p}) \qquad (1.9)$$

where $g_s(\underset{\sim}{p})$ is a completely arbitrary L^2 function of $\underset{\sim}{p}$ and s.

In what follows, we will on occasions use any one of the above mentioned forms for the creation and annihilation operators (whenever we find that the meaning of certain expressions is made clearer by using it).

iii) The quantum field theoretical energy operator for this system is simply

$$H = \sum_E E b_E^\dagger b_E + \sum_{E'} E' d_{E'}^\dagger d_{E'} + \int d\underset{\sim}{p} \sum_{s=1}^{2} \omega(\underset{\sim}{p}) \{ b_s^\dagger(\underset{\sim}{p}) b_s(\underset{\sim}{p}) + d_s^\dagger(\underset{\sim}{p}) d_s(\underset{\sim}{p}) \} . \qquad (1.10)$$

It can easily be checked, using the anticommutation relations, that the field operator $\psi(\underset{\sim}{x})$ has its time development ruled by the Heisenberg equation for the time variation of operators:

$$i \frac{\partial}{\partial t} \psi(t,\underset{\sim}{x}) = [\psi(t,\underset{\sim}{x}), H] \quad . \qquad (1.11)$$

H can formally be written in terms of the field operator as

$$H = \int d\underset{\sim}{x}\ \psi^{\dagger}(t,\underset{\sim}{x})h\psi(t,\underset{\sim}{x}) + \left(\sum_{E'} E' + \int d\underset{\sim}{p} \sum_{s=1}^{2} \omega(\underset{\sim}{p})\right) \quad . \qquad (1.12)$$

The term in bracket in this expression is the equivalent here of

$$\left(\int d\underset{\sim}{p} \sum_{s=1}^{2} \omega(\underset{\sim}{p}) \right)$$

for the free field case where

$$H_{free} = \int d\underset{\sim}{x}\ \psi^{\dagger}_{free}(t,\underset{\sim}{x})h_o\psi_{free}(t,\underset{\sim}{x}) + \left(\int d\underset{\sim}{p} \sum_{s=1}^{2} \omega(\underset{\sim}{p})\right) \quad .$$

For expressions written in terms of free field operators, this sort of additive infinite constant is usually taken into account by introducing the simplifying notation:

$$H_{free} = \ :\int d\underset{\sim}{x}\ \psi^{\dagger}_{free}(t,\underset{\sim}{x})h_o\psi_{free}(t,\underset{\sim}{x}): \quad .$$

This is referred to as normal ordering of the expression. It corresponds to reordering it such that all the annihilation operators are to the right of the creation operators. In the case of expression (1.12), we can define a similar normal ordering symbol which refers to ordering with respect to the operators $b, b^{\dagger}$, $d, d^{\dagger}$. We can denote this simply by adding a subscript

A to the usual symbol as follows:

$$H = :\int d\underset{\sim}{x}\ \psi^{\dagger}(t,\underset{\sim}{x})h\psi(t,\underset{\sim}{x}):_A \quad . \tag{1.13}$$

The Hamiltonian H on the previously defined Fock-Hilbert space acts just like a free field Hamiltonian in the corresponding free field space. It is then self adjoint and has a dense domain in $\mathcal{H}$ so that the operator e^{-iHt} is unitary for all t. This is the time translation operator for the problem; we have

$$\psi(t,\underset{\sim}{x}) = e^{iH(t-t_o)}\ \psi(t_o,\underset{\sim}{x})e^{-iH(t-t_o)} \quad . \tag{1.14}$$

The following operators

$$Q = e\sum_E b_E^{\dagger}b_E - e\sum_{E'} d_{E'}^{\dagger}d_{E'} + e\int d\underset{\sim}{p}\sum_{s=1}^{2}\{b_s^{\dagger}(\underset{\sim}{p})b_s(\underset{\sim}{p}) - d_s^{\dagger}(\underset{\sim}{p})d_s(\underset{\sim}{p})\} \tag{1.15}$$

$$N = \sum_E b_E^{\dagger}b_E + \sum_{E'} d_{E'}^{\dagger}d_{E'} + \int d\underset{\sim}{p}\sum_{s=1}^{2}\{b_s^{\dagger}(\underset{\sim}{p})b_s(\underset{\sim}{p}) + d_s^{\dagger}(\underset{\sim}{p})d_s(\underset{\sim}{p})\} \tag{1.16}$$

commute with the energy operator H. They are then left invariant by the time translation operator e^{-iHt}, i.e. they are constants of the motion. As we will see later, they can be identified respectively as the total charge and the total number of particles operators.

From their anticommutation relations, it can be seen that the operators $b_E^\dagger$, $b_s^\dagger(\underset{\sim}{p})$, $d_E^\dagger$, and $d_s^\dagger(\underset{\sim}{p})$ each then create one particle of respective energy E, $\omega(\underset{\sim}{p})$, E' and $\omega(\underset{\sim}{p})$ and charge e, e, -e, -e. Similarly, their respective adjoints b_E, $b_s(\underset{\sim}{p})$, d_E, and $d_s(\underset{\sim}{p})$ annihilate these particles.

We will justify in the subsequent section the identification of the objects created and annihilated by the operators b, $b^\dagger$, d, $d^\dagger$ as the physical particles.

Physical Interpretation

In this section we want to show how the previous formalism is used to describe the scattering particles and the bound particles.

We will be working within the Heisenberg representation of quantum mechanics. Accordingly, a vector in $\mathcal{H}$ which represents the physical state of the system will not change in time. The dynamical evolution is described by the operators corresponding to physical quantities which evolve according to the Heisenberg equation of motion (1.11).

Let us start by examining the properties of the field operator for large times. We will see that in the far past and far future, it can describe some freely moving particles.

We recall that free particles are associated with free fields as follows. An arbitrary quantized free field is (we will use the index "o" to characterize free field operators)

$$\psi_o(t,\underset{\sim}{x}) = e^{-ih_o t} \psi_o(0,\underset{\sim}{x})$$

where

$$\psi_o(0,\underset{\sim}{x}) = \sum_\beta \{b^o_\beta f^o_{+\beta}(\underset{\sim}{x}) + d^{o\dagger}_\beta f^o_{-\beta}(\underset{\sim}{x})\} \quad . \tag{1.17}$$

The creation and annihilation operators for the free particles and antiparticles are gotten as

$$b^o_\beta = \int d\underset{\sim}{x} [e^{-ih_o t} f^o_{+\beta}(\underset{\sim}{x})]^* \psi_o(t,\underset{\sim}{x}) \tag{1.18}$$

$$d^{o\dagger}_\beta = \int d\underset{\sim}{x} [e^{-ih_o t} f^o_{-\beta}(\underset{\sim}{x})]^* \psi_o(t,\underset{\sim}{x}) \quad . \tag{1.19}$$

They are defined on the Fock-Hilbert space where the vacuum state is the state $|0\rangle$ such that

$$b^o_f|0\rangle_o = 0 \quad \text{and} \quad d^o_f|0\rangle_o = 0 \quad \forall f \in L^2 ; \tag{1.20}$$

it is the zero particle and antiparticle state. The operators $b_f^{o\dagger}$ and $d_f^{o\dagger}$ create respectively free particles and antiparticles which have a momentum and spin distribution given by the wave packet $f_s(\underset{\sim}{p})$.

Their total energy, charge and number operators are

$$H_o = \int d\underset{\sim}{p} \sum_{s=1}^{2} \omega(\underset{\sim}{p}) \{b_s^o(\underset{\sim}{p})^\dagger b_s^o(\underset{\sim}{p}) + d_s^o(\underset{\sim}{p})^\dagger d_s^o(\underset{\sim}{p})\} \qquad (1.21)$$

$$Q_o = e\int d\underset{\sim}{p} \sum_{s=1}^{2} \{b_s^o(\underset{\sim}{p})^\dagger b_s^o(\underset{\sim}{p}) - d_s^o(\underset{\sim}{p})^\dagger d_s^o(\underset{\sim}{p})\} \qquad (1.22)$$

$$N_o = \int d\underset{\sim}{p} \sum_{s=1}^{2} \{b_s^o(\underset{\sim}{p})^\dagger b_s^o(\underset{\sim}{p}) + d_s^o(\underset{\sim}{p})^\dagger d_s^o(\underset{\sim}{p})\} \quad . \qquad (1.23)$$

The energy operator H_o rules the time evolution of the system through the Heisenberg equation:

$$i \frac{\partial}{\partial t} \psi_o(t,\underset{\sim}{x}) = [\psi_o(t,\underset{\sim}{x}), H_o]$$

or equivalently

$$\psi_o(t,\underset{\sim}{x}) = e^{iH_o(t-t')} \psi_o(t',\underset{\sim}{x}) e^{-iH_o(t-t')} \quad .$$

Given the relations (1.18) and (1.19), we can say that we will have here asymptotically freely moving particles if creation annihilation operators defined through

$$\lim_{t\to\mp\infty}\int d\underset{\sim}{x}\,[e^{-ih_o t} f^o_{\varepsilon\beta}(\underset{\sim}{x})]^* \psi(t,\underset{\sim}{x}) \qquad (1.24)$$

exist. The limit $t \to -\infty$ will give us the initial operators (those in the far past) and the limit $t \to +\infty$ the final operators (those in the far future).

These definitions, we remark, are the same ones as used in the Lehmann Symanzik Zimmermann formulation of field theoretical scattering theory (cf. for example, P. Roman [1969]).

The limits (1.24) are easily calculated here. Since the meaning of (1.2) is that

$$\int d\underset{\sim}{x}\,[g(\underset{\sim}{x})]^*\psi(t,\underset{\sim}{x}) = \int d\underset{\sim}{x}\,[e^{iht}g(\underset{\sim}{x})]^*\psi(0,\underset{\sim}{x}) \qquad \forall\; g \in (L^2)^4$$

we have

$$\left(\int d\underset{\sim}{x}\,[e^{-ih_o t} f(\underset{\sim}{x})]^*\psi(t,\underset{\sim}{x}) - \int d\underset{\sim}{x}\,[\Omega_\pm f(\underset{\sim}{x})]^*\psi(0,\underset{\sim}{x})\right)|\phi\rangle$$

$$= \left(\int d\underset{\sim}{x}\,[e^{iht}e^{-ih_o t} f(\underset{\sim}{x}) - \Omega_\pm f(\underset{\sim}{x})]^*\psi(0,\underset{\sim}{x})\right)|\phi\rangle \quad \forall\; |\phi\rangle \in \mathcal{H}\,. \qquad (1.25)$$

From the anticommutator

$$\{\psi_\mu(0,\underset{\sim}{x}),\ \psi^\dagger_\nu(0,\underset{\sim}{x}')\} = \delta_{\mu\nu}\,\delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

which follows from the anticommutation relations of the creation and annihilation operators, we have that

$$\left\| \left(\int d\underset{\sim}{x}\ g(\underset{\sim}{x})^{*}\psi(0,\underset{\sim}{x}) \right) |\phi\rangle \right\|^2 \leq (g,g)\ \langle\phi|\phi\rangle \ \forall\ g \in (L^2)^4 \text{ and}$$

$$\forall\ |\phi\rangle \in \mathcal{H} . \tag{1.26}$$

Combining equations (1.25) and (1.26), we obtain that

$$\lim_{t\to\mp\infty} \left\| \left(\int d\underset{\sim}{x}[e^{-ih_o t} f(\underset{\sim}{x})]^{*}\psi(t,\underset{\sim}{x}) - \int d\underset{\sim}{x}[\Omega_{\pm} f(\underset{\sim}{x})]^{*}\psi(0,\underset{\sim}{x}) \right) |\phi\rangle \right\|$$

$$\leq \lim_{t\to\mp\infty} \| e^{iht} e^{-ih_o t} f(\underset{\sim}{x}) - \Omega_{\pm} f(\underset{\sim}{x}) \|$$

and this by the definition of $\Omega_{\pm}$ is zero. We then obtain

$$\lim_{t\to\mp\infty} \int d\underset{\sim}{x}[e^{-ih_o t} f^{o}_{\epsilon\beta}(\underset{\sim}{x})]^{*}\psi(t,\underset{\sim}{x}) = \int d\underset{\sim}{x}[\Omega_{\pm} f^{o}_{\epsilon\beta}(\underset{\sim}{x})]^{*}\psi(0,\underset{\sim}{x}) .$$

(this being a strong operator limit).

The initial operators are therefore

$$b^{in}_{\beta} = \int d\underset{\sim}{x}[\Omega_{+} f^{o}_{+\beta}(\underset{\sim}{x})]^{*}\psi(0,\underset{\sim}{x})$$

$$= \int d\underset{\sim}{x}[f_{+\beta}(\underset{\sim}{x})]^{*}\psi(0,\underset{\sim}{x})$$

and

$$d^{in\dagger}_{\beta} = \int d\underset{\sim}{x}[f_{-\beta}(\underset{\sim}{x})]^{*}\psi(0,\underset{\sim}{x}) .$$

Using equation (1.3), we see that

$$b^{in}_{\beta} = b_{\beta} \quad \text{and} \quad d^{in}_{\beta} = d_{\beta} . \tag{1.27}$$

Similarly, for the final operators, we obtain

$$b_\beta^{out} = \int d\underset{\sim}{x} [\Omega_- f^o_{+\beta}(\underset{\sim}{x})]^* \psi(0,\underset{\sim}{x})$$

$$= \sum_{\beta'} \{ b_{\beta'} (\Omega_- f^o_{+\beta}, \Omega_+ f^o_{+\beta'}) + d^\dagger_{\beta'} (\Omega_- f^o_{+\beta}, \Omega_+ f^o_{-\beta'}) \}$$

and by the definition of $S = (\Omega_-)^* \Omega_+$, this is

$$= \sum_{\beta'} \{ b_{\beta'} (f^o_{+\beta}, Sf^o_{+\beta'}) + d^\dagger_{\beta'} (f^o_{+\beta}, Sf^o_{-\beta'}) \} \quad .$$

We recall that $[S, H_o] = 0$ so that $[S, H_o/|H_o|] = 0$. It is then easy to see that this implies

$$(f^o_{+\beta}, Sf^o_{-\beta'}) = 0 \quad . \tag{1.28}$$

We then get

$$b_\beta^{out} = \sum_{\beta'} b_{\beta'} (f^o_{+\beta}, Sf^o_{+\beta'}) \equiv \sum_{\beta'} b_{\beta'} S_{+\beta,+\beta'} \tag{1.29}$$

and similarly,

$$d_\beta^{out\,\dagger} = \sum_{\beta'} d^\dagger_{\beta'} (f^o_{-\beta}, Sf^o_{-\beta'}) \equiv \sum_{\beta'} d^\dagger_{\beta'} S_{-\beta,-\beta'} \quad . \tag{1.30}$$

By using the unitarity property of S, we can invert these relations to obtain

$$b_\beta = \sum_{\beta'} b^{out}_{\beta'} (S^\dagger)_{+\beta,+\beta'}$$

$$d^\dagger_\beta = \sum_{\beta'} d^{out\,\dagger}_{\beta'} (S^\dagger)_{-\beta,-\beta'} \quad . \tag{1.31}$$

We thus have the important result that the three sets of operators

$$\{b_f^{in}, d_f^{in} : f \in L^2\},\ \{b_f, d_f : f \in L^2\} \text{ and } \{b_f^{out}, d_f^{out} : f \in L^2\} \tag{1.32}$$

are identical. This is obvious for the first two sets and comes from the fact that S is unitary in L^2 for the third.

Physically, this means that if we identify the set of initial operators $\{b_f^{in}, d_f^{in} : f \in L^2\}$ and their adjoints as annihilation and creation operators for physical particles and antiparticles of a certain type, it follows that the other two sets must also be associated with exactly these same objects.

It is also evident that all these operators are well defined in the Fock-Hilbert space $\mathcal{H}$.

We remark that the operators b_E, $b_E^\dagger$, $d_{E'}$ and $d_{E'}^\dagger$ which were associated with the discrete eigenvalues +E, -E' of h do not appear at all in the asymptotic forms which are used to identify what is created and annihilated by the $b_\beta^\dagger$, b_β, $d_\beta^\dagger$ and d_β. Their meaning is then not directly determined but we know however that they were really defined in exactly the same manner as the b_β's and d_β's. They should therefore have the same nature except that they are associated with the discrete spectrum of h while the latters are associated with its continuous spectrum.

We will then admit that the operators b_E, $b_E^\dagger$, $d_{E'}$ and $d_{E'}^\dagger$ also create and annihilate physical particles but that these are bound in the external potential whereas the b_β's and d_β's are associated with scattering particles.

It is interesting to look at the form of the energy, charge and number operators. For the asymptotically free particles, we have

$$H_{in} = H_{out} = \int d\underset{\sim}{p} \sum_{s=1}^{2} \omega(\underset{\sim}{p}) \{ b_s^\dagger(\underset{\sim}{p}) b_s(\underset{\sim}{p}) + d_s^\dagger(\underset{\sim}{p}) d_s(\underset{\sim}{p}) \}$$

$$Q_{in} = Q_{out} = e\int d\underset{\sim}{p} \sum_{s=1}^{2} \{ b_s^\dagger(\underset{\sim}{p}) b_s(\underset{\sim}{p}) - d_s^\dagger(\underset{\sim}{p}) d_s(\underset{\sim}{p}) \}$$

$$N_{in} = N_{out} = \int d\underset{\sim}{p} \sum_{s=1}^{2} \{ b_s^\dagger(\underset{\sim}{p}) b_s(\underset{\sim}{p}) + d_s^\dagger(\underset{\sim}{p}) d_s(\underset{\sim}{p}) \}$$

whereas the operators for the complete system are, as given in equations (1.21), (1.22) and (1.23):

$$H = H_{\substack{in\\out}} + \sum_E E b_E^\dagger b_E + \sum_{E'} E' d_{E'}^\dagger d_{E'} \tag{1.33}$$

$$Q = Q_{\substack{in\\out}} + e \sum_E b_E^\dagger b_E - e \sum_{E'} d_{E'}^\dagger d_{E'} \tag{1.34}$$

$$N = N_{\substack{in\\out}} + \sum_E b_E^\dagger b_E + \sum_{E'} d_{E'}^\dagger d_{E'} \tag{1.35}$$

We can see that these are naturally separated into a scattering particles part and a bound particles part.

Let us call respectively H_b, Q_b and N_b the last terms of the sums appearing in (1.33), (1.34) and (1.35). These operators commute with the total Hamiltonian H so that they represent constants of the motion. The energy, charge and number of the corresponding bound particles and those of the scattering particles are then separately conserved in time.

Corresponding to this, there is a degeneracy of the asymptotic free particles vacuum states. According to the equations (1.20), these are defined respectively by the equations

$$b_f^{in}|0\rangle_{in} = 0 \qquad \text{and} \qquad b_f^{out}|0\rangle_{out} = 0$$

$$d_f^{in}|0\rangle_{in} = 0 \quad \forall\, f \in L^2 \qquad d_f^{out}|0\rangle_{out} = 0 \quad \forall\, f \in L^2. \tag{1.36}$$

These states can be any normalizable linear combination of states

$$|(0 \text{ scattering particles})\underline{E}_n;(0 \text{ scattering antiparticles})\underline{E}'_m\rangle \tag{1.37}$$

with all possible values of $\underline{E}_n$ and $\underline{E}'_m$. They are not, therefore, unique whenever h has a discrete spectrum. There is in fact a whole subspace of asymptotic free particles vacuum states. We will denote by M this subspace which contains $|0\rangle$ and all vectors of the form (1.37) as a basis.

Accordingly, every vector in M will generate a Fock-Hilbert subspace through $b^{in\dagger}$ and $d^{in\dagger}$ in the usual way. We denote by $\mathcal{F}_{in}^{(n,m)}$ the Fock-Hilbert subspace so gotten from the basis vector (1.37) of M.

Fock-Hilbert subspaces $\mathcal{F}_{out}^{(n,m)}$ can similarly be defined with the final creation operators $b^{out\dagger}$ and $d^{out\dagger}$ and from the identity

$$\{b_f^{in}, d_f^{in} : f \in L^2\} \equiv \{b_f^{out}, d_f^{out} : f \in L^2\} ,$$

we have

$$\mathcal{F}_{out}^{(n,m)} = \mathcal{F}_{in}^{(n,m)} .$$

The total Hilbert space $\mathcal{H}$ can then be written as

$$\mathcal{H} = \bigoplus_{\text{all } n,m} \mathcal{F}_{in}^{(n,m)} . \tag{1.38}$$

Description of Scattering

The question asked in the field theoretical description of a scattering experiment is the same one as asked in the c-number scattering theory except for the necessary modification due to the possible change in the number of particles.

If the system is in a state where initially there were n freely moving particles characterized by the indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ and m free anti-particles labelled $\beta_1, \beta_2, \ldots, \beta_m$, we want to know what is the probability that in the far future there are, in this state, n' freely moving particles and m' antiparticles characterized respectively by $\alpha'_1, \alpha'_2, \ldots, \alpha'_{n'}$ and $\beta'_1, \beta'_2 \ldots, \beta'_{m'}$.

Let us then suppose that the system is in the state (we use the notation $\underline{\alpha}_n = \alpha_1, \alpha_2, \ldots, \alpha_n$)

$$|\underline{\alpha}_n \underline{E}_\ell ; \underline{\beta}_m \underline{E}'_{\ell'}> = (-1)^{\ell m} b(\alpha_1)^\dagger b(\alpha_2)^\dagger \ldots b(\alpha_n)^\dagger \times$$

$$\times\ d(\beta_1)^\dagger \ldots d(\beta_m)^\dagger |0\underline{E}_\ell ; 0\underline{E}'_{\ell'}> . \qquad (1.39)$$

We recall that according to the Heisenberg representation, the description of the motion of the system is given by the change in time of the operators representing the physical quantities whereas the state vector does not change. Therefore, to see what are the properties of the system for large times, we have here to find what are the properties of the state (1.39) in terms of asymptotic operators. This is easily done by using the equation (1.27), i.e.

$$b_\beta^\dagger = b_\beta^{in^\dagger} \quad \text{and} \quad d_\beta^\dagger = d_\beta^{in^\dagger} \quad .$$

Obviously, the system is then in a state where in the far past there were n freely moving particles and m antiparticles characterized exactly as we wanted. The vacuum state $|0>_{in}$, for the scattering experiment where the numbers of initial particles n and m would be varied is here $|0\ \underline{E}_\ell\ ;\ 0\ \underline{E}'_{\ell'}>$.

To see what is happening in the system in the far future, we use the equation (1.31) which implies

$$b_\beta^\dagger \ [illegible]\ _{\beta',+\beta} \quad \text{and} \quad d_\beta^\dagger = \sum_{\beta'} d_{\beta'}^{out^\dagger} (S^\dagger)_{-\beta,-\beta'}$$

to re[illegible] state of the system (1.39) in terms of $b^{out^\dagger}$ [illegible] †. It is evident that in the far future there a[illegible]gain n free particles and m free antiparticles. The amp[illegible]de of the probability that these be characterize[illegible] the indices $\underline{\alpha}'_n$ and $\underline{\beta}'_m$ is easily seen to be

$$S(+\alpha_1',+\alpha_1)S(+\alpha_2',+\alpha_2)\ldots S(+\alpha_n',+\alpha_n)(S^\dagger)(-\beta_1,-\beta_1')\ldots$$
$$\times (S^\dagger)(-\beta_m,-\beta_m') . \quad (1.40)$$

The expression (1.40) for all values of n, n', m, m' and the indices gives the complete answer to the question of scattering. The set of all these elements, as in ord[illegible]y scattering theory, forms the scattering matrix. This will be denoted by S and, from now on, the c-number scattering matrix will be denoted by S_c. S can also be defined through the equations:

$$b_\beta^{in} = S\, b_\beta^{out}\, S^\dagger \qquad d_\beta^{in} = S\, d_\beta^{out}\, S^\dagger$$

$$b_E = S\, b_E\, S^\dagger \qquad d_{E'} = S\, d_{E'}\, S^\dagger$$

and
$$S^\dagger = S^{-1} . \tag{1.41}$$

The field theoretical scattering matrix can be straightforwardly written in terms of the creation and annihilation operators. It has then a more concise and useful form. This form is easily found as it is a trivial case of generalized Bogoliubov transformation (discussed in Chapter III). It is simply

$$S = \exp i \sum_\beta \sum_{\beta'} \{ b_\beta^\dagger (\ln S_c)(+\beta,+\beta') b_{\beta'} - d_\beta^\dagger (\ln S_c)(-\beta',-\beta) d_{\beta'} \} . \tag{1.42}$$

2) Possibility of Different Field Quantizations

In this section, we show that with a given field operator it is possible to associate as many sets of creation and annihilation operators as there are bases in $(L^2)^4$. We give the general relation between different such sets of operators; this is a generalized Bogoliubov transformation. We then discuss some properties that such operators must have in order for them to be able to represent physical particles.

i) We saw in the preceding section that the field operator was a linear superposition of creation and annihilation operators (cf. equations (1.2) and (1.3)) where the functions

$$e^{-iht}\, f_E(\underset{\sim}{x}), \quad e^{-iht}\, f_{+\beta}(\underset{\sim}{x}),$$

$$e^{-iht}\, f_{E'}(\underset{\sim}{x}), \quad e^{-iht}\, f_{-\beta}(\underset{\sim}{x})$$

formed a basis in the space $(L^2)^4$ (i.e. the 4-component L^2 space). As we will see later, this will also be the case in most external field problems; even when the potential is time dependent. The field operator can then generally be written as

$$\psi(\underset{\sim}{x}) = \sum_k \sum_{\mu=1}^{4} \psi_{\mu k}\, f_{\mu k}(\underset{\sim}{x}) \tag{2.1}$$

where $f_{\mu k}(x)$ for $\mu = 1,2,3,4$ and $k = 1,\ldots\infty$ form a basis in $(L^2)^4$. That is

$$(f_{\mu k}, f_{\nu \ell}) = \int d\underset{\sim}{x}\ f_{\mu k}(\underset{\sim}{x})^* f_{\nu\ell}(\underset{\sim}{x}) = \delta_{\mu\nu}\delta_{k\ell} \tag{2.2}$$

and

$$\sum_k \sum_{\mu=1}^{4} f_{\mu k}(\underset{\sim}{x})\ f_{\mu k}(\underset{\sim}{x}')^* = \delta(\underset{\sim}{x}-\underset{\sim}{x}') \ . \tag{2.3}$$

The $\psi_{\mu k}$'s satisfy the anticommutation relations of creation and annihilation operators:

$$\{\psi_{\mu k}, \psi_{\nu\ell}\} = 0 \qquad \{\psi_{\mu k}, \psi^\dagger_{\nu\ell}\} = \delta_{\mu\nu}\delta_{k\ell} \tag{2.4}$$

and are well defined bounded operators on some Hilbert space $\mathcal{H}$.

Since the operators $\psi_{\mu k}$ are well defined bounded operators, any operator

$$\psi(f) \equiv \int d\underset{\sim}{x}\ f^*(\underset{\sim}{x})\psi(\underset{\sim}{x}) \qquad \text{where} \qquad f \in (L^2)^4 \tag{2.5}$$

is also a well defined bounded operator in $\mathcal{H}$ as is evident from the following argument.

$$||\psi(f)|\phi>||^2 = ||\sum_k \sum_{\mu=1}^{4} (f,f_{\mu k})\psi_{\mu k}|\phi>||^2$$

$$= \sum_k \sum_{\mu=1} \sum_{k'} \sum_{\mu'=1} (f,f_{\mu k})(f,f_{\mu' k'})^* <\phi|\psi^\dagger_{\mu' k'}\psi_{\mu k}|\phi>$$

and upon using (2.4), this is

$$\leq \sum_{k} \sum_{\mu=1}^{4} |(f,f_{\mu k})|^2 = (f,f) \quad \forall |\phi> \in \mathcal{H} .$$

The adjoint $\psi^\dagger(f)$ is also well defined and we always have

$$\{\psi(f),\psi(g)\} = 0 \qquad \{\psi(f),\psi^\dagger(g)\} = (f,g) \quad \forall \; f \text{ and } g \in (L^2)^4 . \tag{2.6}$$

Such a field operator $\psi(\underset{\sim}{x})$ having the property that $\psi(f)$ is well defined on $\mathcal{H}$ for all $f \in (L^2)^4$ will be said to be an operator valued distribution on the test function space $(L^2)^4$.

To any basis $f'_{\mu k} \in (L^2)^4$, it is then possible to associate a set of well defined operators

$$\psi'_{\mu k} \equiv \int d\underset{\sim}{x} \; f'_{\mu k}(\underset{\sim}{x})^* \, \psi(\underset{\sim}{x}) \tag{2.7}$$

having the anticommutation relations of creation and annihilation operators in $\mathcal{H}$:

$$\{\psi'_{\mu k},\psi'_{\nu \ell}\} = 0 \qquad \{\psi'_{\mu k},\psi'^{\dagger}_{\nu \ell}\} = \delta_{\mu\nu}\delta_{k\ell} \quad . \tag{2.8}$$

With a given field operator $\psi(\underset{\sim}{x})$ one can then associate as many such sets as there are bases in $(L^2)^4$.

We note that because of the symmetry in (2.8) or (2.4) between $\psi_{\mu k}$ and $\psi^{\dagger}_{\mu k}$ for a given μ and k, any one of these could be called a creation (or annihilation) operator. Considerations on the non-negativeness of the energy operator will help decide which one should create or annihilate physical particles. We will then consider (as this will be the usual case) that $\psi_{\mu k}$ for $\mu = 1,2$ and $\psi^{\dagger}_{\mu k}$ for $\mu = 3,4$ are the annihilation operators.

In order to be able to identify more clearly which are the creation operators or the annihilation operators, the following notation is used.

$$B_{\mu k} = \psi_{\mu k} \quad \text{for} \quad \mu = 1,2 \quad \text{and} \quad D^{\dagger}_{\mu k} = \psi_{\mu+2,k} \quad \text{for} \ \mu = 1,2 \ .$$

ii) It is easy to find the relations between two different sets of creation and annihilation operators associated with a field $\psi(\underset{\sim}{x})$. By definition, we have

$$\psi(\underset{\sim}{x}) = \sum_{k} \sum_{\mu=1}^{2} \{B_{\mu k} \ f_{\mu k}(\underset{\sim}{x}) + D^{\dagger}_{\mu k} \ f_{\mu+2,k}(\underset{\sim}{x})\} \tag{2.9}$$

and also

$$\psi(\underset{\sim}{x}) = \sum_{k} \sum_{\mu=1}^{2} \{B'_{\mu k} \ f'_{\mu k}(\underset{\sim}{x}) + D'^{\dagger}_{\mu k} \ f'_{\mu+2,k}(\underset{\sim}{x})\} \tag{2.10}$$

where $\{f_{\mu k}(\underset{\sim}{x})\}$ and $\{f'_{\mu k}(\underset{\sim}{x})\}$ are two different bases in $(L^2)^4$. From comparing (2.9) and (2.10), one can see that we must have

$$B'_{\mu k} = \sum_{\ell} \sum_{\nu=1}^{2} \{B_{\nu\ell}(f'_{\mu k}, f_{\nu\ell}) + D^{\dagger}_{\nu\ell}(f'_{\mu k}, f_{\nu+2,\ell})\} \quad (2.11)$$

$$D'^{\dagger}_{\mu k} = \sum_{\ell} \sum_{\nu=1}^{2} \{B_{\nu\ell}(f'_{\mu+2,k}, f_{\nu\ell}) + D^{\dagger}_{\nu\ell}(f'_{\mu+2,k}, f_{\nu+2,\ell})\}. \quad (2.12)$$

This sort of linear transformation between two sets of creation and annihilation operators is called a generalized Bogoliubov transformation by analogy with the case where the index k runs only over a finite set of numbers.

iii) We will say that a set of creation and annihilation operators $\{B_{\mu k}, D_{\mu k}\}$ can represent physical particles if, in the Hilbert space considered, there is a vector which can represent the physically possible state of there being none of these particles in the system. That is there exists a vector $|0\rangle \in \mathcal{H}$ such that $B_{\mu k}|0\rangle = 0$ and $D_{\mu k}|0\rangle = 0$ for $\mu = 1,2$ and $\forall$ k.

Furthermore, once a given set of such operators is chosen to represent physical particles, the whole relevant Hilbert space for the system is determined.

If a field operator $\psi(t,\underset{\sim}{x})$ is the only field operator variable for a given problem, it (with the external field) must give a complete description of the system. If then at some time say "t_o", $\psi(t_o,\underset{\sim}{x})$ describes certain particles, all the possible states of the system at this time will consist of zero, one, two ... etc. particles states. That is states of the Fock-Hilbert space $\mathcal{F}_{t_o}$ associated to the vacuum $|0>$ through the particles creation operators in the usual way. If $\mathcal{F}_{t_o}$ describes all physically possible states at some time "t_o", it must also describe all possible states at all times. This is necessary, for example, in order to be able to use the Heisenberg representation formalism in which the motion is described by a state vector which does not change in time while the operators representing the dynamical variables evolve according to the Heisenberg equation of motion. We must therefore consider $\mathcal{F}_{t_o}$ as the total relevant Hilbert space $\mathcal{H}$.

We remark that for any other arbitrary set of operators $\{B'_{\mu k}, D'_{\mu k}\}$ there will not always be such a state in $\mathcal{H}$ (as we will see later) even if states $|\phi>$

$$B'_{\mu k}|\phi> = 0$$

$$D'_{\mu k}|\phi> = 0 \qquad \text{for } \mu = 1,2 \qquad (2.13)$$

and a given fixed value of k can always be found in $\mathcal{H}$.

This last fact is easily shown by constructing states

$$|\phi\rangle = B'_{1k}B'_{2k}D'_{1k}D'_{2k}|\tilde{\phi}\rangle \quad \text{where } |\tilde{\phi}\rangle \in \mathcal{H} .$$

These cannot be zero $\forall\ |\tilde{\phi}\rangle \in \mathcal{H}$ since this would imply $B'_{1k}B'_{2k}D'_{1k}D'_{2k} \equiv 0$. This is inconsistent with the anti-commutation relations since we must have

$$\{D'^{\dagger}_{2k},[D'^{\dagger}_{1k},\{B'^{\dagger}_{2k},[B'^{\dagger}_{1k},B'_{1k}B'_{2k}D'_{1k}D'_{2k}]\}]\} = 1 \neq 0 .$$

All such states $|\phi\rangle$ are easily seen to satisfy (2.12) since $\psi'_{\mu k}\psi'_{\mu k} = 0 \ \forall\ \mu, k$ follows from $\{\psi'_{\mu k},\psi'_{\mu k}\} = 0$.

The necessary and sufficient conditions for two sets of creation and annihilation operators to define each a vacuum state in the same space together with the general properties of Bogoliubov transformations will be discussed at length in Chapter III.

Physical Requirements.

As is now evident from the preceding section a well-defined operator valued distribution $\psi(t,\underset{\sim}{x})$ is determined at all times t by

$$\psi(t,\underset{\sim}{x}) = e^{-iht}\ \psi(0,\underset{\sim}{x}) \quad , \tag{2.14}$$

whenever $\psi(0,\underset{\sim}{x})$ is itself an operator valued distribution. This follows from the fact that $e^{-iht}\ \forall\ t$ is a unitary operator on $(L^2)^4$.

Since the operator $\psi(0,\underset{\sim}{x})$ can be defined on many different Fock-Hilbert spaces, one must have certain general criteria which will help determine which one among these spaces should be chosen. This is what we discuss in this section.

We will examine in particular the possibility of quantization through the free field functions $f^{o}_{\varepsilon\beta}(\underset{\sim}{x})$ since this is the most frequently encountered one (for example: H.E. Moses [1954], K.O. Friedrichs [1953], P.J.M. Bongaarts [1970], [illegible] Seiler [1972]).

The decomposition of the field operator $\psi(0,\underset{\sim}{x})$ would then be

$$\psi(0,\underset{\sim}{x}) = \sum_{\beta} \{b^{o}_{\beta} f^{o}_{+\beta}(\underset{\sim}{x}) + d^{o\dagger}_{\beta} f^{o}_{-\beta}(\underset{\sim}{x})\} \tag{2.15}$$

and the Fock-Hilbert space $\mathcal{H}_o$ defined by the action of $b^{o\dagger}$ and $d^{o\dagger}$ on the vacuum $|0>^{o}$ defined by

$$b^{o}_{\beta}|0>^{o} = 0 \;, \quad d^{o}_{\beta}|0>^{o} = 0 \quad \forall \; \beta \;. \tag{2.16}$$

i) In order to have a meaningful physical theory, either one of the following requirements are usually expected to be satisfied in any quantum field theory. These requirements are in fact what should fix the quantization.

a) A well defined non-negative energy operator H exists on $\mathcal{H}$ such that the solution of the differential equation of motion satisfies the Heisenberg equation

$$i \frac{\partial}{\partial t} \psi(t,\underset{\sim}{x}) = [\psi(t,\underset{\sim}{x}),H] \ .$$

This means that Planck's relation that energy is proportional to the frequency of vibration of the field modes is satisfied (Y. Takahashi [1969]).

b) There is a unitary operator $U(t,t_o)$ relating the fields at times t and t_o such that

$$\psi(t,\underset{\sim}{x}) = U^{\dagger}(t)\psi(0,\underset{\sim}{x})U(t) \ .$$

c) We do not care about giving a physical interpretation to the fields for finite times. We only ask that they always be well defined operator valued distributions on a Hilbert space $\mathcal{H}$ and that they describe freely moving particles for very large times.

These last two alternatives are those presented by A.S. Wightman at the Coral Gables Conference [1971] for systems where the c-number problem is well defined, as is the case here.

ii) We now examine the implications of the previous requirements. We start with b); let us suppose that the Hilbert space chosen is $\mathcal{H}_o$.* We require that there exists a unitary operator $U(t)$ on $\mathcal{H}_0$ such that

$$\psi(t,\underset{\sim}{x}) = U^\dagger(t)\psi(0,\underset{\sim}{x})U(t) \ . \tag{2.17}$$

From the decomposition (2.15) for $\psi(0,\underset{\sim}{x})$ and the relation (2.14), one can see that the above equation is

$$\sum_\beta \{b^o_\beta e^{-iht} f^o_{+\beta}(\underset{\sim}{x}) + d^{o\dagger}_\beta e^{-iht} f^o_{-\beta}(\underset{\sim}{x})\}$$

$$= \sum_\beta \{U^\dagger(t) b^o_\beta U(t) f^o_{+\beta}(\underset{\sim}{x}) + U^\dagger(t) d^{o\dagger}_\beta U(t) f^o_{-\beta}(\underset{\sim}{x})\} . \tag{2.18}$$

Upon integrating each side of this equation with $[f_{\epsilon\beta}(\underset{\sim}{x})]^*$, one obtains that the above requirement implies that there must exist a unitary operator $U(t)$ such that

$$b^o_\beta(t) = \sum_{\beta'} \{b^o_{\beta'}(f^o_{+\beta}, e^{-iht} f^o_{+\beta'}) + d^{o\dagger}_{\beta'}(f^o_{+\beta}, e^{-iht} f^o_{-\beta'})\}$$

$$d^o_\beta(t)^\dagger = \sum_{\beta'} \{b^o_{\beta'}(f^o_{-\beta}, e^{-iht} f^o_{-\beta'}) + d^{o\dagger}_{\beta'}(f^o_{-\beta}, e^{-iht} f^o_{-\beta'})\}$$

(2.19)

where

$$b^o_\beta(t) = U^\dagger(t) b^o_\beta U(t) \quad \text{and} \quad d^o_\beta(t) = U^\dagger(t) d^o_\beta U(t) \ .$$

That is: the Bogoliubov transformation on the right hand side of equation (2.19) must be unitarily implementable.

We will prove in Chapter IV that this is in fact possible if and only if the Hilbert space $\mathcal{H}_o$ coincide exactly with the Fock-Hilbert space $\mathcal{H}$ defined in the previous section 1). That is: if and only if the set of operators $\{b^o_\beta, d^o_\beta\}$ is unitarily equivalent to the set $\{b_E, b_\beta, d_E, d_\beta\}$ introduced in section 1). We are then brought back to considering the Hilbert space $\mathcal{H}$ as the physically relevant one. In this space, the time translation operator is already known to exist; according to equation (1.14), it is

$$U(t) = e^{-iHt}$$

where the energy operator H is given by equation (1.10).

In the case of time independent potentials, it is trivial that the two requirements a) and b) are exactly equivalent. We have just seen that when there exists a U(t), it is in fact e^{-iHt} so that a well defined energy operator H exists. Conversely, whenever there exists a well defined energy operator H, e^{-iHt} is a unitary operator having the property (2.17).

iv) It can however happen that two (or more) sets of particle creation and annihilation operators both have a vacuum in the same space $\mathcal{H}$. It is then easy to see that different phenomena are predicted, according to which set we interpret as representing the physical particles. For example, we have seen that the number of particles operator is here

$$N = \sum_{E} b_{E}^{\dagger} b_{E} + \sum_{E'} d_{E'}^{\dagger} d_{E'} + \sum_{\beta} (b_{\beta}^{\dagger} b_{\beta} + d_{\beta}^{\dagger} d_{\beta})$$

and is time independent. This comes from the fact that it commutes with H so that the number of these particles is conserved.

The number operator for the other particles at time zero is

$$N^{o} = \sum_{\beta} \{b_{\beta}^{o\dagger} b_{\beta}^{o} + d_{\beta}^{o\dagger} d_{3}^{o}\} .$$

At time t, this becomes

$$e^{+iHt} N^{o} e^{-iHt}$$

which in general will be different from N_{o}. The use of the operators $\{b_{\beta}^{o}, d_{\beta}^{o}\}$ in the case of time independent potentials is responsible for such interpretations as are given in the references mentioned at the beginning of this section. One finds, for example, in the article by Bongaarts [1970]: Pairs of particles and antiparticles are created and annihilated continuously .... . However, for the limit $t \to \pm\infty$, the theory reduces to a one-particle situation ..... . This is an asymptotic result, for $t \to \pm\infty$ and comes from a dynamical situation for finite times that [illegible]plicated .....".

[illegible] consideration can help us decide between [illegible]ts of operators. This will be related to [illegible] requirement.

iii) In[illegible] to examine the third requirement, let us consid[illegible] completely general operator valued distribution [illegible]4).

We d[illegible]t require anything for finite times. In particula[illegible] this means that we do not know at the beginning of the treatment on what space $\psi(0,\underset{\sim}{x})$ is defined.

We as[illegible]ver, that the asymptotic creation and annihila[illegible]on operators correpond to the observable physical particles.

The relevant physical particles operators must then be those defined as

$$\lim_{t\to\mp\infty} \int d\underset{\sim}{x}\,[e^{-ih_o t} f_{\varepsilon\beta}(\underset{\sim}{x})]^{*} e^{-iht}\,\psi(0,\underset{\sim}{x})$$

They must then be related to $\psi(0,\underset{\sim}{x})$ as follows: When $t\to-\infty$, the operators are

$$b_{\beta}^{in} = \int d\underset{\sim}{x}\,[\Omega_{+} f^{o}_{+\beta}(\underset{\sim}{x})]^{*}\,\psi(0,\underset{\sim}{x})$$

$$d_{\beta}^{in\dagger} = \int d\underset{\sim}{x}\,[\Omega_{+} f^{o}_{-\beta}(\underset{\sim}{x})]^{*}\,\psi(0,\underset{\sim}{x}) \qquad (2.20)$$

and when $t \to +\infty$,

$$b_\beta^{out} = \int d\underset{\sim}{x} [\Omega_- f^o_{+\beta}(\underset{\sim}{x})]^* \; \psi(0,\underset{\sim}{x})$$

$$b_\beta^{out\dagger} = \int d\underset{\sim}{x} [\Omega_- f^o_{-\beta}(\underset{\sim}{x})]^* \; \psi(0,\underset{\sim}{x}) \quad . \qquad (2.21)$$

In order for these to represent physical particles, we need that the physical space contains vacuum states $|0\rangle_{in}$ and $|0\rangle_{out}$ for them.

We recall that when there are eigenfunctions of h associated with discrete eigenvalues, the operators Ω_+ and Ω_- are not unitary. The equation (2.20) (or (2.21)) will not then be invertible in the sense that the operator valued distribution $\psi(0,\underset{\sim}{x})$ cannot be written completely in terms of the operators b^{in}, d^{in} (or b^{out}, d^{out}). That is: these operators do not form an irreducible set. In this case, one must define other creation and annihilation operators as

$$b_E = \int d\underset{\sim}{x} \; f_E(\underset{\sim}{x})^* \; \psi(0,\underset{\sim}{x})$$

$$d^\dagger_{E'} = \int d\underset{\sim}{x} \; f_{E'}(\underset{\sim}{x})^* \; \psi(0,\underset{\sim}{x}) \quad . \qquad (2.22)$$

A set of functions $\Omega f^o_{\varepsilon\beta}$ together with f_E and $f_{E'}$ form a basis in $(L^2)^4$ so that we can invert the equations (2.20) and obtain:

$$\psi(0,\underset{\sim}{x}) = \sum_E b_E f_E(\underset{\sim}{x}) + \sum_{E'} d^\dagger_{E'} f_{E'}(\underset{\sim}{x})$$

$$+\sum_\beta \{b^{in}_\beta [\Omega_+ f^o_{+\beta}(\underset{\sim}{x})] + d^{in^\dagger}_\beta [\Omega_+ f^o_{-\beta}(\underset{\sim}{x})]\} \qquad (2.23)$$

or from (2.21):

$$\psi(0,\underset{\sim}{x}) = \sum_E b_E f_E(\underset{\sim}{x}) + \sum_{E'} d^\dagger_{E'} f_{E'}(\underset{\sim}{x})$$

$$+\sum_\beta \{b^{out}_\beta [\Omega_- f^o_{+\beta}(\underset{\sim}{x})] + d^{out^\dagger}_\beta [\Omega_- f^o_{-\beta}(\underset{\sim}{x})]\} \qquad (2.24)$$

The proper Hilbert space for the problem is then the Fock-Hilbert space defined through the set of operators b_E, b^{in}_β, $d_{E'}$, d^{in}_β and the vacuum $|0\rangle$ such that

$$b_E|0\rangle = 0 \;\forall E \qquad d_{E'}|0\rangle = 0 \;\forall E'$$

$$b^{in}_\beta|0\rangle = 0 \qquad d^{in}_\beta|0\rangle = 0 \;\forall \beta \qquad (2.25)$$

Equations (2.23) and (2.25) are easily recognized as being exactly those used in the initial treatment of this problem. As was shown, there is always a unitary operator on $\mathcal{H}$ relating the set of operators b^{out}_β, d^{out}_β to b^{in}_β, d^{in}_β.

If, however, we had started with a given different Hilbert space, say $\mathcal{H}_o$ as before, we would again evidently have obtained that it is necessary for the operators $\{b^o_\beta, d^o_\beta\}$ to be unitarily related to $\{b_E, b_\beta, d_E, , d_\beta\}$.

iv) In the time independent problem, all three requirements a), b) and c) are equivalent in favoring the Hilbert space initially considered. In this space, we have a well defined self adjoint energy operator H; a unitary time translation operator $U(t,t_o) = e^{-iH(t-t_o)}$ exists $\forall$ t and t_o and the operators representing the asymptotic free particles have a vacuum state.

Furthermore, in this space, the creation and annihilation operators defined by

$$b_E(t) = \int d\underset{\sim}{x}\; f_E(\underset{\sim}{x})^* \; \psi(t,\underset{\sim}{x}) \qquad d^\dagger_{E'}(t) = \int d\underset{\sim}{x}\; f_{E'}(\underset{\sim}{x})^* \, \psi(t,\underset{\sim}{x})$$

$$b_\beta(t) = \int d\underset{\sim}{x}\; f_{+\beta}(\underset{\sim}{x})^* \, \psi(t,\underset{\sim}{x}) \qquad d^\dagger_\beta(t) = \int d\underset{\sim}{x}\; f_{-\beta}(\underset{\sim}{x})^* \, \psi(t,\underset{\sim}{x})$$

have a vacuum state at all times (it is simply $|0\rangle$). The particles associated with them are the physical particles i.e. those observable as asymptotic free particles.

When the necessary conditions are satisfied for a different set of creation and annihilation operators to also have a vacuum in $\mathcal{H}$, we will say that these represent pseudo-particles since they are not observable.

It is interesting and it can help much in understanding the concept of these pseudo-particles to look at an "inverse" situation. Let us consider the free field system described by the differential equation

$$\{-i\gamma.\partial + m\}\psi(x) = 0.$$

There is no doubt in this case about which Fock-Hilbert space should be taken as the physically relevant one. It can be found in any textbook on quantum field theory and indeed agrees with the discussion of this section that the proper creation and annihilation operators are those defined in

$$\psi(t,\underset{\sim}{x}) = e^{-ih_o t} \sum_{\beta}\{b_\beta f^o_{+\beta}(\underset{\sim}{x}) + d^\dagger_\beta f^o_{-\beta}(\underset{\sim}{x})\} \quad .$$

The particles always behave as free particles and there is never any creation nor annihilation of particles.

However, here also, one can define other creation and annihilation operators and it can happen that these have a vacuum state in the Hilbert space considered. (In fact, this will happen for the operators defined through the basis associated with the operator h considered before whenever the transformation (2.19) is unitarily implementable.) The other particles so defined will obviously behave differently from free particles.

No importance is nevertheless given to these since they will not be observable in the system considered.

CHAPTER II

TIME DEPENDENT PROBLEM

Introduction

We study in this chapter some general properties of systems with time dependent external fields. We will start by recalling an important result shown by A.Z. Capri [1969] mainly that the linear differential equations of motion for such external field systems are meaningful in the sense that they determine unique, well defined, causal operator valued distributions in the four space-time variables on some Hilbert space.

We will then prove that it follows from these results that all times, the fields are well defined operator valued distributions in the three space variables. This is done by showing that the field operator-distributions at different times can be related through a c-number unitary operator u(t,t') (which we give explicitly in terms of the fundamental solutions of Capri) as

$$\psi(t,\underset{\sim}{x}) = u(t,t')\psi(t',\underset{\sim}{x}) \ .$$

This, together with all the properties derived by Capri, establishes the properties of the operator u(t,t') which are more or less assumed in other treatments as for example those by B. Schroer, R. Seiler and J.A. Swieca [1970] and R. Seiler [1972] where one starts directly from the above equation.

We then examine the requirements which need be imposed because of physical considerations like those discussed in the first chapter. We show that this reduces to examining certain Bogoliubov transformations between creation and annihilation operators.

The question as to whether or not the "interpolating" (i.e. at finite times) quantized field in external field systems should have physical interest is discussed as in the previous article by G. Labonté and A.Z. Capri [1972]. This is resolved by considering that at any given time it is possible to keep the external field constant to the value reached and that then one can examine the system and observe what particles are present. From this, it follows that it is necessary that physical particles be associated with the system at any given time.

We explain how these are associated with the field operator. The method used to do this also appears very useful in defining directly the energy operator simply as the sum of the energy of all particles (without recourse to the Lagrangian formalism). The operator so obtained will evidently be self-adjoint, bounded below and have a dense domain whenever there exists a zero particle state. This method can also be used to find on which coefficients of the general c-number solution of a problem one should impose the (anti-) commutation relations of creation and annihilation operators (i.e. as a method of quantization of time dependent systems).

This general formalism will then be applied to a first simple model describing a boson field in the presence of an external source-field. The proper definition of the operators to be associated with physical particles will be seen to shed much light on the nature of this system.

A second model is studied where charged particles interact with an external electromagnetic wave. We examine, in particular, a commonly used quantization procedure for this model and show that it is not satisfactory. We finally suggest how this problem could be treated within the present formalism.

1) Reduction to a c-number Problem

We recall some of the results of A.Z. Capri [1969] which show that the Dirac equation for quantized fields acted upon by smooth time dependent external fields determines well-defined operator valued distributions in the four variables $(t,\underset{\sim}{x})$. This is illustrated by the study of the equation

$$(-i\gamma.\partial + m)\psi(t,\underset{\sim}{x}) = e\gamma.A(t,\underset{\sim}{x})\psi(t,\underset{\sim}{x}) \quad . \qquad (1.1)$$

Let the external field $A^{\mu}(t,\underset{\sim}{x})$ be a smooth function of t and $\underset{\sim}{x}$: in particular, let it be in the space $\mathcal{S}$. This is the space of infinitely differentiable functions which go to zero at infinity, in all directions of space time, faster than any inverse polynomial. We also suppose that $A(t,\underset{\sim}{x})$ has a bounded support (t_s,t_f) in time i.e. $A(t,\underset{\sim}{x}) = 0$ for all times which are not in the interval (t_s,t_f). This here simplifies the analysis since we can talk of times at which there is no external field in action.

Instead of working directly with the differential equation (1.1), one studies the following equivalent integral equation:

$$\psi(x) = \psi_o(x) + \int G(x-y)e\gamma.A(y)\psi(y)dy \qquad (1.2)$$

where $\psi_o(x)$ is a free solution of the Dirac equation and $G(x-y)$ is a fundamental solution of the free equation i.e. satisfies

$$\{-i\gamma.\partial_x + m\}G(x-y) = \delta(x-y) \quad . \tag{1.3}$$

Since the external field $A(t,\underset{\sim}{x}) = 0 \;\forall\; t \not\in (t_s,t_f)$, the solution $\psi(x)$ should be a free solution for these times. One incorporates the corresponding initial or final condition in the integral equation (1.2) by writing

$$\psi(x) = \psi_{in}(x) + \int S_r(x-y)e\gamma.A(y)\psi(y)dy \tag{1.4}$$

or

$$\psi(x) = \psi_{out}(x) + \int S_a(x-y)e\gamma.A(y)\psi(y)dy \tag{1.5}$$

where $S_r(x-y)$ and $S_a(x-y)$ satisfy equation (1.3) and

$$S_r(x-y) = 0 \quad \text{when} \quad (x_o-y_o) < 0 \quad \text{or} \quad (x-y)^2 < 0 \tag{1.6}$$

$$S_a(x-y) = 0 \quad \text{when} \quad (x_o-y_o) > 0 \quad \text{or} \quad (x-y)^2 < 0 \quad . \tag{1.7}$$

The equations (1.4) and (1.5) are the Källen-Yang-Feldman equations. $\psi_{in}(x)$ and $\psi_{out}(x)$ are the free fields to which $\psi(x)$ reduces for $t < t_s$ and $t > t_f$.

It is to be noted here that, for these time dependent problems, one must always consider that the Hilbert space of physical states is the Fock-Hilbert space of the initial free particles states. These are in fact the only physical states possible up to time t_s since it is only then that the external field starts to act on the particles. The Hilbert space will thus be here the Fock-Hilbert space associated in the usual manner with the free field $\psi_{in}(x)$.

In order to show that the equations (1.4) and (1.5) establish meaningful relations between field operator valued distributions, one smears these equations with an arbitrary function $f \in \mathcal{S}$. Then, equation (1.4), for example, can be written as

$$\psi(T_r f) = \psi_{in}(f) \qquad (1.8)$$

where $\psi(f) = \int dx\, f^*(x)\psi(x)$ and $T_r f$ is defined as

$$(T_r f)(x) = f(x) - e[\gamma.A(x)]^\dagger \int dy\; S_r^\dagger(y-x) f(y) \quad . \qquad (1.9)$$

Capri showed that the operator T_r maps any function of $\mathcal{S}$ into a function of $\mathcal{S}$ and has an inverse (in $\mathcal{S}$) such that

$$\psi(f) = \psi_{in}(T_r^{-1} f) \quad . \qquad (1.10)$$

This shows that the field $\psi(x)$ is a well defined operator valued distribution on the space $\mathcal{S}$ of functions since the free field $\psi_{in}(x)$ is known to have this property.

From the properties of T_r and those of the operator T_a similarly defined through equation (1.5) instead of (1.4), one can show that there are fundamental solutions defined by

$$S_r^A(f,g) = S_r(T_r^{-1}f,\ g)$$

$$S_a^A(f,g) = S_a(T_a^{-1}f,\ g) \quad .$$

These both satisfy the two equations:

$$\{-i\gamma^\mu \frac{\partial}{\partial x^\mu} + m - e\gamma.A(x)\}S^A(x,y) = \delta(x-y)$$

$$i \frac{\partial}{\partial y^\mu} S^A(x,y)\gamma^\mu + mS^A(x,y) - S^A(x,y)e\gamma.A(y) = \delta(x-y) \quad . \tag{1.11}$$

As the free fundamental solutions, they have the properties

$$S_r^A(x,y) = 0 \quad \text{when} \quad (x_o-y_o) < 0 \quad \text{or} \quad (x-y)^2 < 0$$

$$S_a^A(x,y) = 0 \quad \text{when} \quad (x_o-y_o) > 0 \quad \text{or} \quad (x-y)^2 < 0 \quad . \tag{1.12}$$

In terms of these, it is easy to show that the operators T_r^{-1} and T_a^{-1} are just

$$(T_r^{-1} f)(x) = f(x) + [e\gamma.A(x)]^\dagger \int dy [S_r^A(y,x)]^\dagger f(y) \qquad (1.13)$$

$$(T_a^{-1} f)(x) = f(x) + [e\gamma.A(x)]^\dagger \int dy [S_a^A(y,x)]^\dagger f(y) \ . \qquad (1.14)$$

Upon replacing (1.13) in equation (1.10), we then obtain

$$\psi(x) = \psi_{in}(x) + \int S_r^A(x,y) e\gamma.A(y) \psi_{in}(y) dy \qquad (1.15)$$

and similarly,

$$\psi(x) = \psi_{out}(x) + \int S_a^A(x,y) e\gamma.A(y) \psi_{out}(y) dy \ . \qquad (1.16)$$

The Fields as Operator Valued Distributions on $[L^2(E^3)]^4$

Using the form (1.15) and the properties of the distribution S_r^A, we shall now show that the field operator $\psi(t,\underset{\sim}{x})$ can be meaningfully written as

$$\psi(t,\underset{\sim}{x}) = u(t,t')\psi(t',\underset{\sim}{x})$$

where $u(t,t')$ is a c-number unitary operator on $[L^2(E^3)]^4$ and that ψ satisfies the canonical equal time anti-commutation relations

$$\{\psi_\mu(t,\underset{\sim}{x}), \psi_\nu^\dagger(t,\underset{\sim}{x}')\} = \delta_{\mu\nu}\, \delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

at all times.

i) From equation (1.15) and the following relation for a free field:

$$\psi_{in}(y_o,\underset{\sim}{y}) = -i\int S(y_o-z_o,\underset{\sim}{y}-\underset{\sim}{z})\gamma^o\psi_{in}(z_o,\underset{\sim}{z})d\underset{\sim}{z} \tag{1.17}$$

(where $-iS(y_o-z_o,\underset{\sim}{y}-\underset{\sim}{z})$ is the integral kernel for the operator $e^{-ih_o(y_o-z_o)}$ in 3-space variables), one has

$$\psi(x_o,\underset{\sim}{x}) = \int\{\delta(x-y)+S_r^A(x,y)e\gamma A(y)\}(-i)S(y_o-z_o,\underset{\sim}{y}-\underset{\sim}{z})\gamma^o dy \times$$

$$\times\ \psi_{in}(z_o,\underset{\sim}{z})d\underset{\sim}{z} \quad . \tag{1.18}$$

where z_o is arbitrary. Let us consider here $z_o < t_s$ and define $u(x_o,z_o)$ as the c-number operator corresponding to the integral kernel

$$K_{(x_o,z_o)}(\underset{\sim}{x},\underset{\sim}{y}) = -i\int\{\delta(x-y)+S_r^A(x,y)e\gamma.A(y)\}S(y_o-z_o,\underset{\sim}{y}-\underset{\sim}{z})\gamma^o dy. \tag{1.19}$$

The equation (1.18) can then be written as

$$\psi(x_o,\underset{\sim}{x}) = u(x_o,z_o)\psi_{in}(z_o,\underset{\sim}{x}) \ .$$

We now prove that $u(x_o,z_o)$ is unitary on $[L^2(E^3)]^4$ i.e.

$$(u(x_o,z_o)f,u(x_o,z_o)g) = (f,g) \ \forall\ f \text{ and } g \in [L^2(E^3)]^4 \ .$$

Let us consider an arbitrary function $f(\underset{\sim}{x}) \in \mathcal{S}(E^3)$ which has bounded support. From the results of the previous section, it follows that

$$u(x_o,z_o)f(\underset{\sim}{x}) = \int K_{(x_o,z_o)}(\underset{\sim}{x},\underset{\sim}{z})f(\underset{\sim}{z})d\underset{\sim}{z} \qquad (1.20)$$

is a distribution in the four variables $(x_o,\underset{\sim}{x})$ on $\mathcal{S}$. (This is easily checked by integrating (1.20) with an arbitrary function of $\mathcal{S}$). It satisfies the Dirac equation

$$i\frac{\partial}{\partial x_o} u(x_o,z_o)f(\underset{\sim}{x}) = \gamma^o[(i\underset{\sim}{\gamma}\cdot\underset{\sim}{\partial}_x + m) - e\gamma.A(x)]u(x_o,z_o)f(\underset{\sim}{x}) \qquad (1.21)$$

with initial condition at $x_o = z_o$:

$$u(z_o,z_o)f(\underset{\sim}{x}) = f(\underset{\sim}{x}) . \qquad (1.22)$$

With the help of equation (1.21), one gets

$$i\frac{\partial}{\partial x_o}(u(x_o,z_o)f,u(x_o,z_o)g) = i\int d\underset{\sim}{x}\frac{\partial}{\partial x^k}\{[u(x_o,z_o)f(\underset{\sim}{x})]^* \times$$

$$\times\ \gamma^o\gamma_k[u(x_o,z_o)g(\underset{\sim}{x})]\} . \qquad (1.23)$$

Because of the causal propagation of the solution $u(x_o,z_o)f(\underset{\sim}{x})$, its support in space-time is the light cone subtended by the bounded support of $f(\underset{\sim}{x})$. For any given "surface" x_σ = constant, this is then a bounded region of space. We apply Green's theorem:

$$\int_V dx \, \underset{\sim}{\partial} . \underset{\sim}{J}(\underset{\sim}{x}) = \int_{S_V} d\underset{\sim}{s} \, . \, \underset{\sim}{J}(\underset{\sim}{x})$$

to evaluate (1.23), with V = all space so that S_V is a surface at spatial infinity. Because of the previously mentioned property of the support of uf, $\underset{\sim}{J}(\underset{\sim}{x}) = 0$ on S_V so that (1.23) is

$$i \frac{\partial}{\partial x_o} (u(x_o,z_o)f, \, u(x_o,z_o)g) = 0 \quad \forall \; f \text{ and } g$$

i.e. (uf, ug) is independent of x_o. Using then $x_o = z_o$, we obtain from (1.22):

$$(u(x_o,z_o)f, \, u(x_o,z_o)g) = (f,g) \; .$$

Since $\mathcal{S}(E^3)$ is dense in $[L^2(E^3)]^4$, this shows that $u(x_o,z_o)$ is unitary on the space $[L^2(E^3)]^4$.

We now define a unitary operator

$$u(t,t') = u(t,z_o)u^\dagger(t',z_o) \, .$$

It is easily checked, by differentiating $u(t,z_o)u^\dagger(t',z_o)$ and using the fact that

$$\frac{\partial}{\partial z_o} u(t,z_o) = -iu(t,z_o)h_o \; ,$$

that it is independent of the arbitrarily chosen $z_o < t_s$. It also relates the field operators at different times as,

$$\psi(t,\underset{\sim}{x}) = u(t,t')\psi(t',\underset{\sim}{x}) .$$

ii) The anticommutator $\{\psi_\mu(t,\underset{\sim}{x}),\ \psi_\nu^\dagger(t,\underset{\sim}{x}')\}$ can now be easily evaluated. For $t' < t_s$, we had

$$\psi(t,\underset{\sim}{x}) = u(t,t')\psi_{in}(t',\underset{\sim}{x}) .$$

When smeared with a $[L^2(E^3)]^4$ function, this is

$$\int d\underset{\sim}{x}\ f^*(\underset{\sim}{x})\psi(t,\underset{\sim}{x}) = \int d\underset{\sim}{x}[u^\dagger(t,t')f(\underset{\sim}{x})]^*\ \psi_{in}(t',\underset{\sim}{x})$$

so that

$$\{\psi(t,f),\psi^\dagger(t,g)\} = \{\int d\underset{\sim}{x}[u^\dagger(t,t')f(\underset{\sim}{x})]^*\psi_{in}(t',\underset{\sim}{x}),$$
$$\int d\underset{\sim}{x}'\psi_{in}^\dagger(t',\underset{\sim}{x}')\ [u^\dagger(t,t')g(\underset{\sim}{x}')]\}.$$

Using the free field anticommutation relation

$$\{\psi_\mu^{in}(t',\underset{\sim}{x}),\psi_\nu^{in\dagger}(t',\underset{\sim}{x}')\} = \delta_{\mu\nu}\,\delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

and the unitarity of the c-number operator $u(t,t')$, we obtain

$$\{\psi(t,f),\psi^\dagger(t,g)\} = (f,g) \quad \forall\ f \text{ and } g \in [L^2(E^3)]^4$$

or equivalently

$$\{\psi_\mu(t,\underset{\sim}{x}),\psi_\nu^\dagger(t,\underset{\sim}{x}')\} = \delta_{\mu\nu}\delta(\underset{\sim}{x}-\underset{\sim}{x}') \ .$$

iii) These results establishing the relation

$$\psi(t,\underset{\sim}{x}) = u(t,t')\psi(t',\underset{\sim}{x})$$

could have been obtained also by studying directly the c-number operator u(t,t') defined by

$$i\frac{\partial}{\partial t}u(t,t') = h(t)u(t,t')$$

as is done by B. Schroer, R. Seiler and J.A. Swieca [1970] and by R. Seiler [1972]. One then shows that the c-number operator u(t,t') is unitary on the space $[L^2(E^3)]^4$ so that

$$\psi(t,\underset{\sim}{x}) = u(t,t')\psi_{in}(t',\underset{\sim}{x}) \quad \text{for} \quad t' < t_s \ .$$

is meaningful in the sense that $\int d\underset{\sim}{x}\ f^*(\underset{\sim}{x})\psi(t,\underset{\sim}{x})$ is a well defined operator on the Fock-Hilbert space of the free field ψ_{in}.

The above mentioned authors give the result that there exists such c-number operators for the Dirac equation with external potentials $A(t,\underset{\sim}{x})$ having the same properties as those considered here and for a spin zero field interacting with a potential $V(t,\underset{\sim}{x})$ of the same type. In this last case, the equation of motion is

$$(\Box + m^2)\phi(t,\underset{\sim}{x}) = V(t,\underset{\sim}{x})\phi(t,\underset{\sim}{x}) \quad .$$

Although, up to now, are mentioned only cases where the external fields are in the space of smooth functions $\mathcal{S}$, the results can certainly be extended to more general external fields. In fact, in later sections, we will treat explicitly certain cases where $A(t,\underset{\sim}{x})$ is far from being a smooth function of time.

We remark that if instead of perturbing the motion of free particles by introducing the smooth time dependent external field, one considered particles moving in a time independent potential (as we treated in the first chapter) and perturb their motion with the smooth external field, one could very probably use the same proofs with little modifications to obtain similar results. One would simply replace the distribution $-iS(x-x')\gamma^o$ (from which are defined

$$S_r(x-x') = \theta(t-t')S(x-x') \text{ and } S_a(x-x') = -\theta(t'-t)S(x-x'))$$

which is the 3-space kernel of the operator $e^{-ih_o(t-t')}$ by the corresponding kernel for the operator $e^{-ih(t-t')}$; $h = h_o - e\gamma^o\gamma \cdot A(\underset{\sim}{x})$. Now, $A(\underset{\sim}{x})$ would be the initial and final time independent potential. The field $\psi(t,\underset{\sim}{x})$ would thus always be a well defined operator valued distribution on $[L^2(E^3)]^4$ acting in the Fock-Hilbert space associated with the particles in the constant potential $A(\underset{\sim}{x})$.

2) The Particles and the Energy Operator

We now examine the physical requirement concerning the existence of an energy operator. This operator needs to be bounded below i.e. have a lowest energy state and must govern the time development of the system through the Heisenberg equation of motion. Since here the strength and (or) shape of the external field acting on the quantum system change in time, the energy of this system and thus its lowest eigenstate will also change in time. The energy operator then depends on time; we will denote it by H(t) and we must have for the field variable $\psi(t,\underset{\sim}{x})$:

$$i \frac{\partial}{\partial t} \psi(t,\underset{\sim}{x}) = [\psi(t,\underset{\sim}{x}), H(t)] \ . \tag{2.1}$$

i) It is to be remarked that the independent field variables for external field problems are $\psi(t,\underset{\sim}{x})$ and $A(t,\underset{\sim}{x})$ and that in fact the Heisenberg equation of motion for a general operator O(t) which is a function of ψ and A is:

$$i \frac{\partial}{\partial t} O(t) = [O(t), H(t)] + i\left[\frac{\partial}{\partial t} O(t)\right]_{\text{explicit}} \tag{2.2}$$

where $\partial O(t)/\partial t$ is the total variation in time of the operator $O(t)$ and $[\partial O(t)/\partial t]_{explicit}$ is the partial variation in time of $O(t)$ due to its explicit time dependence. For example, the Hamiltonian operator $H(t)$ itself will be a function of both $\psi(t,\underset{\sim}{x})$ and $A(t,\underset{\sim}{x})$ so that

$$i \frac{\partial}{\partial t} H(t) = 0 + i[\frac{\partial}{\partial t} H(t)]_{explicit} .$$

For the external field $A(t,\underset{\sim}{x})$, which is the operator $A(t,\underset{\sim}{x})I$, I being the identity operator in the Hilbert space considered, one has

$$i \frac{\partial}{\partial t} A(t,\underset{\sim}{x})I = i[\frac{\partial}{\partial t} A(t,\underset{\sim}{x})]_{explicit} I$$

since $[A(t,\underset{\sim}{x})I,H(t)] = A(t,\underset{\sim}{x})[I,H(t)] = 0$.

ii) We recall that a time independent external field does not create or annihilate physical particles and that the total quantum energy of the system is conserved. If then the external field $A(t,\underset{\sim}{x})$ were to be kept to its value $A(t_o,\underset{\sim}{x})$ for all times after t_o, there should be no more creation or annihilation of particles nor change in the energy of the system after t_o. That is: the number of particles and the energy associated with the field $\forall\, t > t_o$ and in particular for asymptotic times will be the same as at time t_o. This fact, as we discussed in the reference by G. Labonté and A.Z. Capri [1972], can be used to define the physical particles, the energy operator and the vacuum state at any time t_o for time dependent systems.

Let us then consider an auxiliary field $\psi_{t_o}(t,\underset{\sim}{x})$ defined such that it satisfies

$$(-i\gamma.\partial + m)\psi_{t_o}(t,\underset{\sim}{x}) = e\gamma.A(t_o,\underset{\sim}{x})\psi_{t_o}(t,\underset{\sim}{x}), \tag{2.3}$$

and

$$\psi_{t_o}(t_o,\underset{\sim}{x}) = \psi(t_o,\underset{\sim}{x}) . \tag{2.4}$$

This auxiliary field is uniquely defined since (2.3) and (2.4) combined form a well defined (initial value) Cauchy problem. Manifestly, it would be the solution of the true equation of motion if $A(t,\underset{\sim}{x})$ were to be kept constant after time t_o.

As we have seen previously, the solution of equation (2.3) can be written as

$$\psi_{t_o}(t,\underset{\sim}{x}) = e^{-ih(t_o)[t-t']}\psi_{t_o}(t',\underset{\sim}{x})$$

where $h(t_o) = \gamma^o(i\underset{\sim}{\gamma}.\underset{\sim}{\partial} + m) - e\gamma^o\gamma.A(t_o,\underset{\sim}{x})$ is self adjoint for quite general t-independent potentials $A(t_o,\underset{\sim}{x})$. Using the initial condition (2.4), we then have

$$\psi_{t_o}(t,\underset{\sim}{x}) = e^{-ih(t_o)[t-t_o]} \psi(t_o,\underset{\sim}{x}) . \qquad (2.5)$$

Since

$$\psi(t,\underset{\sim}{x}) = u(t,t')\psi_{in}(t',\underset{\sim}{x}) \text{ with } t' < t_s ,$$

we have

$$\psi_{t_o}(t,\underset{\sim}{x}) = e^{-ih(t_o)[t-t_o]} u(t_o,t')\psi_{in}(t',x) . \quad t' < t_s \qquad (2.6)$$

This shows that for all times the auxiliary field is a well defined operator valued distribution on $[L^2(E^3)]^4$ since the c-number operator $e^{-ih(t_o)[t-t_o]}u(t_o,t')$ is unitary on this space. It will satisfy, for all times t, the canonical anticommutation relations since $\psi(t_o,\underset{\sim}{x})$ was previously shown to satisfy them.

For field motion in a time independent potential, we know how to extract from the field operator the creation and annihilation operators for physical particles. The particles so defined are animated of individual motion; after a long time, if their energy is sufficient, they will be outside the region where the external field is and move as free particles while, if they do not have enough energy to escape its attraction, they will remain bound in the potential.

We recall that for the field $\psi_A(t,\underset{\sim}{x})$ moving in the potential $A(\underset{\sim}{x})$, these operators can be obtained as follows:

$$b_E(t) = b_E e^{-iEt} = \int d\underset{\sim}{x}\,[f_E(\underset{\sim}{x})]^* \psi_A(t,\underset{\sim}{x})$$

$$d^\dagger_{E'}(t) = d^\dagger_{E'} e^{iE't} = \int d\underset{\sim}{x}\,[f_{E'}(\underset{\sim}{x})]^* \psi_A(t,\underset{\sim}{x})$$

$$b_\beta(t) = \int d\underset{\sim}{p}\; h^*_{\beta'}(\underset{\sim}{p}) b_s(\underset{\sim}{p}) e^{-i\omega(\underset{\sim}{p})t} = \int d\underset{\sim}{x}\,[f_{+\beta}(\underset{\sim}{x})]^* \psi_A(t,\underset{\sim}{x})$$

$$d^\dagger_\beta(t) = \int d\underset{\sim}{p}\; h^*_{\beta'}(\underset{\sim}{p}) d^\dagger_s(\underset{\sim}{p}) e^{i\omega(\underset{\sim}{p})t} = \int d\underset{\sim}{x}\,[f_{-\beta}(\underset{\sim}{x})]^* \psi_A(t,\underset{\sim}{x})$$

(2.7)

where $f_E(\underset{\sim}{x})$, $(f_{E'}(\underset{\sim}{x}))$ are eigenfunctions of energy $+E$, $(-E')$ of $h = h_o - e\gamma^o\gamma.A(\underset{\sim}{x})$. $f_{\pm\beta}(\underset{\sim}{x}) = \Omega_- f^o_{\pm\beta}(\underset{\sim}{x})$, where here Ω_-, instead of Ω_+, is used such that $h_{\beta'}(\underset{\sim}{p})$ will be the momentum distribution of the scattered freely moving particles.

The particles creation and annihilation operators for the auxiliary field at any time t are thus obtained by taking $A(t_o,\underset{\sim}{x})$ as $A(\underset{\sim}{x})$ and $\psi_{t_o}(t,\underset{\sim}{x})$ as $\psi_A(t,x)$. They are

$$b_E(t_o,t) = \int d\underset{\sim}{x}\,[f_E^{t_o}(\underset{\sim}{x})]^* \psi_{t_o}(t,\underset{\sim}{x})$$

$$d^\dagger_{E'}(t_o,t) = \int d\underset{\sim}{x}\,[f_{E'}^{t_o}(\underset{\sim}{x})]^* \psi_{t_o}(t,\underset{\sim}{x})$$

$$b_\beta(t_o,t) = \int d\underset{\sim}{x}\,[f_{+\beta}^{t_o}(\underset{\sim}{x})]^* \psi_{t_o}(t,\underset{\sim}{x})$$

$$d^\dagger_\beta(t_o,t) = \int d\underset{\sim}{x}\,[f_{-\beta}^{t_o}(\underset{\sim}{x})]^* \psi_{t_o}(t,\underset{\sim}{x}) \qquad (2.8)$$

where the functions f are defined as before and carry the index t_o of the operator $h(t_o) = h_o - e\gamma^o\gamma.A(t_o,\underset{\sim}{x})$. The energy, charge and number of particles operators for the auxiliary field are known to be independent of t so that if we define $b(t_o) = b(t_o,t_o)$ and $d(t_o) = d(t_o,t_o)$, these can be written as

$$H_{t_o} = \int d\underset{\sim}{p} \sum_{s=1}^{2} \omega(p)\{b_s^\dagger(\underset{\sim}{p},t_o)b_s(\underset{\sim}{p},t_o) + d_s^\dagger(\underset{\sim}{p},t_o)d_s(\underset{\sim}{p},t_o)\}$$

$$+ \sum_{E(t_o)} E(t_o)b_E^\dagger(t_o)b_E(t_o) + \sum_{E'(t_o)} E'(t_o)d_{E'}^\dagger(t_o)d_{E'}(t_o) \tag{2.9}$$

$$Q_{t_o} = e\int d\underset{\sim}{p} \sum_{s=1}^{2} \{b_s^\dagger(\underset{\sim}{p},t_o)b_s(\underset{\sim}{p},t_o) - d_s^\dagger(\underset{\sim}{p},t_o)d_s(\underset{\sim}{p},t_o)\}$$

$$+ e\sum_{E(t_o)} b_E^\dagger(t_o)b_E(t_o) - e\sum_{E'(t_o)} d_{E'}^\dagger(t_o)d_{E'}(t_o) \tag{2.10}$$

$$N_{t_o} = \int d\underset{\sim}{p} \sum_{s=1}^{2} \{b_s^\dagger(\underset{\sim}{p},t_o)b_s(\underset{\sim}{p},t_o) + d_s^\dagger(\underset{\sim}{p},t_o)d_s(\underset{\sim}{p},t_o)\}$$

$$+ \sum_{E(t_o)} b_E^\dagger(t_o)b_E(t_o) + \sum_{E'(t_o)} d_{E'}^\dagger(t_o)d_{E'}(t_o) \ . \tag{2.11}$$

Since $\psi_{t_o}(t,\underset{\sim}{x})$ at $t = t_o$ is equal to $\psi(t_o,\underset{\sim}{x})$, these operators will also correspond to the energy charge and number of particles associated with $\psi(t,\underset{\sim}{x})$ at time

t_o i.e.

$$H_{t_o} = H(t_o) \ , \quad Q_{t_o} = Q(t_o) \quad \text{and} \quad N_{t_o} = N(t_o) \ .$$

The physical particles creation and annihilation operators at time t_o can be defined directly from the field $\psi(t,\underset{\sim}{x})$ as

$$b_\gamma(t_o) = \int d\underset{\sim}{x}\,[f_{+\gamma}^{t_o}(\underset{\sim}{x})]^* \ \psi(t_o,\underset{\sim}{x})$$

$$d_\lambda^\dagger(t_o) = \int d\underset{\sim}{x}\,[f_{-\lambda}^{t_o}(\underset{\sim}{x})]^* \ \psi(t_o,\underset{\sim}{x}) \ . \qquad (2.12)$$

The less explicit but more succinct notation where

$$\{\gamma\} \equiv \{E\}U\{\beta\} \quad \text{and} \quad \{\lambda\} = \{E'\}U\{\beta\}$$

has been used since now it is clear what the meaning of the indices and the properties of the corresponding functions are.

We remark that this result is essentially the same as the one obtained by M.I. Shirkov [1968], [1968]'. However, his method to obtain these operators is based on the requirement that the creation and annihilation operators diagonalize the Hamiltonian at time t_o. Ours stresses the fact that if one does not change the external field after time t_o (such that a stable situation is obtained) no more particles are created nor annihilated and the particles then observable are the physical particles which were in the system at time t_o. This justifies more clearly why the operators (2.12) should be associated with the physical particles and specifies which set of positive and negative energy functions one should use in smearing the field operator to obtain them.

iii) Since

$$\psi(t_o,\underset{\sim}{x}) = u(t_o,t')\psi_{in}(t',\underset{\sim}{x}) \quad \text{for} \quad t' < t_s \ ,$$

upon using the decomposition of ψ_{in} in terms of initial creation and annihilation operators, one obtains that (2.12) is the following Bogoliubov transformation

$$b_\gamma(t_o) = \sum_\beta \{ (f^{t_o}_{+\gamma}, u(t_o,t') f^o_{+\beta}) b^{in}_\beta(t') + (f^{t_o}_{+\gamma}, u(t_o,t') f^o_{-\beta}) d^{in}_\beta(t')^\dagger \}$$

$$d^\dagger_\lambda(t_o) = \sum_\beta \{ (f^{t_o}_{-\lambda}, u(t_o,t') f^o_{+\beta}) b^{in}_\beta(t') + (f^{t_o}_{-\lambda}, u(t_o,t') f^o_{-\beta}) d^{in}_\beta(t')^\dagger \}$$

(2.13)

These operators will satisfactorily describe the physical particles, of which the system is composed at time t_o, when the Bogoliubov transformation (2.13) is unitarily implementable or, equivalently, when there exists a vector $|0\rangle_{t_o} \in \mathcal{H}_{in}$ such that

$$b_E(t_o)|0\rangle_{t_o} = 0 \quad \forall\, E \qquad d_{E'}(t_o)|0\rangle_{t_o} = 0 \quad \forall\, E'$$

$$b_s(\underset{\sim}{p},t_o)|0\rangle_{t_o} = 0 \quad \forall\, \underset{\sim}{p},s \qquad d_s(\underset{\sim}{p},t_o)|0\rangle_{t_o} = 0 \quad \forall\, \underset{\sim}{p},s\ . \tag{2.14}$$

$|0\rangle_{t_o}$ will represent the zero particle (therefore the lowest energy and zero charge) state. When it exists the operators $H'(t_o)$, $Q(t_o)$ and $N(t_o)$ will be well defined on dense domains in $\mathcal{H}_{in}$.

iv) We now prove that the operator $H(t_o)$ as defined above gives the proper (i.e. according to the Heisenberg equation) rate of change of the field operator $\psi(t,\underset{\sim}{x})$ at time t_o. Having

$$i\frac{\partial}{\partial t}\psi_{t_o}(t,\underset{\sim}{x}) = \gamma^o[i\underset{\sim}{\gamma}.\underset{\sim}{\partial} + m - e\gamma.A(t_o,\underset{\sim}{x})]\psi_{t_o}(t,\underset{\sim}{x})$$

and

$$i\frac{\partial}{\partial t}\psi(t,\underset{\sim}{x}) = \gamma^o[i\underset{\sim}{\gamma}.\underset{\sim}{\partial} + m - e\gamma.A(t,\underset{\sim}{x})]\psi(t,\underset{\sim}{x})\ ,$$

since the field operators coincide at time t_o, one has

$$i\frac{\partial}{\partial t}\psi_{t_o}(t,\underset{\sim}{x})\Big|_{t=t_o} = i\frac{\partial}{\partial t}\psi(t,\underset{\sim}{x})\Big|_{t=t_o}\ . \tag{2.15}$$

Since, furthermore, the auxiliary field is known to satisfy the Heisenberg equation:

$$i \frac{\partial}{\partial t} \psi_{t_o}(t,\underset{\sim}{x}) = [\psi_{t_o}(t,\underset{\sim}{x}), H_{t_o}]$$

for all times t, one has at $t = t_o$

$$i \frac{\partial}{\partial t} \psi_{t_o}(t,\underset{\sim}{x})\Big|_{t=t_o} = [\psi_{t_o}(t_o,\underset{\sim}{x}), H_{t_o}] \, .$$

From equation (2.15) and $\psi_{t_o}(t_o,\underset{\sim}{x}) = \psi(t_o,\underset{\sim}{x})$, it then follows that

$$i \frac{\partial}{\partial t} \psi(t,\underset{\sim}{x})\Big|_{t=t_o} = [\psi(t_o,\underset{\sim}{x}), H(t_o)]$$

i.e. the Heisenberg equation is satisfied.

v) When the potential is time independent, the particles annihilation operators appear in the expansion for the field $\psi(t,\underset{\sim}{x})$ and $\bar{\psi}(t,\underset{\sim}{x})$ as "coefficients" in front of the positive frequency (or equivalently: "positive energy" since then the Heisenberg equation clearly means $\omega = E$) c-number solutions of the field equation. The annihilation operator part of the field can then be obtained by extracting the positive frequency part of the field as

$$\psi^{(+)}(t,\underset{\sim}{x}) = \frac{1}{(2\pi)} \int_0^{+\infty} dk_o \, e^{-ik_o t} \int dt' \, e^{ik_o t'} \, \psi(t',\underset{\sim}{x}) \tag{2.16}$$

and correspondingly, the vacuum state can be defined by

$$\psi^{(+)}(t,\underset{\sim}{x})|0\rangle = 0 \qquad \bar{\psi}^{(+)}(t,\underset{\sim}{x})|0\rangle = 0 \quad \forall\, \underset{\sim}{x} . \tag{2.17}$$

Although this method could be formally applied also to time dependent external field problems, it would be difficult to interpret the physical meaning of the "vacuum" $|\phi_o(t)\rangle$ so defined. This difficulty comes from the definition of $\psi^{(+)}$ by equation (2.16). Using the Yang-Feldman equation, one can see that

$$\psi^{(+)}(t,\underset{\sim}{x}) = \psi_{in}^{(+)}(t,\underset{\sim}{x}) + \int S_r^{(+)}(t-y_o,\underset{\sim}{x}-\underset{\sim}{y})\, e\gamma . A(y)\psi(y)dy . \tag{2.18}$$

For times $t < t_s$ the field is known to be $\psi_{in}(t,\underset{\sim}{x})$ and there are only free particles in the system since the external field has not been "switched on" yet. However, since $S_r^{(+)}$, unlike S_r, does not have support only in the forward light cone, the second term in (2.18) is not null for $t < t_s$ so that

$$\psi^{(+)}(t,\underset{\sim}{x}) \neq \psi_{in}^{(+)}(t,\underset{\sim}{x}) \quad \text{for} \quad t < t_s .$$

This shows that $\psi^{(+)}(t,\underset{\sim}{x})$ cannot be the part of the field $\psi(t,\underset{\sim}{x})$ corresponding to physical particles annihilation operators.

The method of decomposition into positive and negative frequencies can however be used to obtain the creation and annihilation operators parts from the auxiliary field since this field is the solution of an equation with a time independent potential. Thus, at an arbitrary time t, the part of the field $\psi(t,\underset{\sim}{x})$ containing only annihilation operators is

$$\frac{1}{(2\pi)} \int_0^{+\infty} dk_o \, e^{-ik_o t} \int dt' \, e^{ik_o t'} \, \psi_t(t',\underset{\sim}{x})$$

where $\psi_t(t',\underset{\sim}{x})$ is the auxiliary field.

3) Time Translation Operator

One could consider as more basic the requirement that there exists a unitary time translation operator $U(t,t')$ such that

$$\psi(t,\underset{\sim}{x}) = U^{\dagger}(t,t')\psi(t'\underset{\sim}{x})U(t,t') \quad \forall \quad t \text{ and } t' . \quad (3.1)$$

We will prove later the more or less expected result that this is exactly equivalent to the previous requirement concerning the existence of a well defined energy operator at all times. For the moment, we just show again how this problem reduces to the study of a Bogoliubov transformation.

When $A(t,\underset{\sim}{x})$ is initially zero, it is easy to see that the above $U(t,t')$ will exist if and only if $U(t,t')$ for $t' < t_s$ exists such that

$$\psi(t,\underset{\sim}{x}) = U^{\dagger}(t,t')\psi_{in}(t',\underset{\sim}{x})U(t,t') \quad . \qquad (3.2)$$

Since

$$\psi(t,\underset{\sim}{x}) = u(t,t')\psi_{in}(t',\underset{\sim}{x}) ,$$

(3.2) can be written as

$$U^{\dagger}(t,t')\psi_{in}(t',\underset{\sim}{x})U(t,t') = u(t,t')\psi_{in}(t',\underset{\sim}{x}) \quad . \qquad (3.3)$$

Upon integrating each side of this equation with $f^o_{+\alpha}(\underset{\sim}{x})^*$ and then $f^o_{-\alpha}(\underset{\sim}{x})^*$, one obtains that $U(t,t')$ must be such that

$$U^\dagger(t,t')b^{in}_\alpha(t')U(t,t') = \sum_\beta \{(f^o_{+\alpha},u(t,t')f^o_{+\beta})b^{in}_\beta(t') + (f^o_{+\alpha},u(t,t')f^o_{-\beta})d^{in}_\beta(t')^\dagger\}$$

$$U^\dagger(t,t')d^{in}_\alpha(t')^\dagger U(t,t') = \sum_\beta \{(f^o_{-\alpha},u(t,t')f^o_{+\beta})b^{in}_\beta(t') + (f^o_{-\alpha},u(t,t')f^o_{-\beta})d^{in}_\beta(t')^\dagger\}. \tag{3.4}$$

There will therefore exist a unitary time translation operator $U(t,t')$ if and only if the Bogoliubov transformation defined by the right hand side terms of (3.4) is unitarily implementable. We note that the operators on the left hand side of (3.4) are generally not the same as the physical particles creation and annihilation operators; they coincide only when $A(t_o,\underset{\sim}{x})=0$.

<u>The S-Matrix</u>

The third possible requirement, as mentioned in Chapter I, is that one would not care about giving a physical interpretation to the theory for finite times but ask only that a) the field always be a well defined operator valued distribution on a Hilbert space, b) it describes freely moving particles for very large times (i.e. only the S-matrix should exist).

i) The condition a) has already been shown to be satisfied.

The condition b) obviously can be satisfied only if the external field remains constant to a certain value $A(t_f, \underset{\sim}{x})$ after some time t_f. Otherwise, particles would keep on being created and annihilated in the region of the potential and these would still be non freely moving scattering particles. One must then suppose that

$$A(t,x) = A(t_f, \underset{\sim}{x}) \quad \forall \quad t \geq t_f \quad .$$

As we saw in Chapter I, $A(t_f, \underset{\sim}{x})$ should be a good scattering potential if the system is to describe freely moving particles for very large times.

The creation and annihilation operators for these particles, as is obvious from the discussion of section 2), will in fact be the operators for physical particles associated with the auxiliary field $\psi_{t_f}(t, \underset{\sim}{x})$ (which is here the actual field $\forall$ $t \geq t_f$). At any time $t_o \geq t_f$, these are given by the equation (2.13) with $u(t,t') = e^{-h_f(t-t_f)} u(t_f, t')$ where $h_f = h_o - e\gamma^o \gamma . A(t_f, \underset{\sim}{x})$.

There will exist a vacuum $|0>_T$ defined by the operators $b(T \geq t_f)$ and $d(T \geq t_f)$ at one arbitrary time $T > t_f$ if and only if there exists a vacuum $|0>_t$ for all $t \geq t_f$. This is so since $|0>_t$ is in fact time independent for all $t \geq t_f$ due to the time independence of $A(t,\underset{\sim}{x})$ during this time. The present condition will then be satisfied if and only if the Bogoliubov transformation (2.13) with $t_o = t_f$ (or equivalently $\forall\ t \geq t_f$) is unitarily implementable. There will then exist also an energy operator $H(t \geq t_f) = H(t_f)\ \forall\ t \geq t_f$ such that

$$\psi(t,\underset{\sim}{x}) = e^{iH(t_f)[t-t_f]}\,\psi(t_f,\underset{\sim}{x})e^{-iH(t_f)[t-t_f]} \quad . \qquad (3.5)$$

ii) As above, for the vacuum state, the properties of the time independent problem imply that there will exist, at an arbitrary time $T > t_f$, a unitary operator $U(T,t')$ such that

$$\psi(T,\underset{\sim}{x}) = U^{\dagger}(T,t')\psi_{in}(t',\underset{\sim}{x})U(T,t') \qquad T \geq t_f, \quad t' \leq t_s$$

if and only if $U(t_f,t')$ exists. As is obvious from the previously mentioned property:

$$U(t_f,t') = U(T,t')e^{iH(t_f)[T-t_f]} \quad .$$

Therefore, whenever $U(T,t')$ exists, we have

$$e^{iH(t_f)[t-t_f]} U^{\dagger}(t_f,t')\psi_{in}(t',\underset{\sim}{x})U(t_f,t')e^{-iH(t_f)[t-t_f]}$$

$$= \psi(t,\underset{\sim}{x})$$

$$= e^{-ih_f(t-t_f)} U^{\dagger}(t_f,t')\psi_{in}(t',x)U(t_f,t') \ .$$

Upon multiplying each side of this equation by $U(t_f,t')$ from the left and $U^{\dagger}(t_f,t')$ from the right, one obtains

$$V^{\dagger}\psi_{in}(t',\underset{\sim}{x})V = e^{-ih_f(t-t_f)} \psi_{in}(t',\underset{\sim}{x}) \qquad (3.6)$$

where

$$V = U(t_f,t')e^{-iH(t_f)(t-t_f)} U^{\dagger}(t_f,t') \quad .$$

This shows that it is always a necessary condition that the transformation $e^{-ih_f(t-t_f)}$ on the free field $\psi_{in}(t',\underset{\sim}{x})$ be unitarily implementable.

This means that the Bogoliubov transformation defined by the right hand side of equation (3.4) where $u(t,t')$ is replaced by $e^{-ih_f(t-t_f)}$ must be unitarily implementable. It is important to remark here that the conditions under which this will be so are completely independent of how the external field $A(t_f,\underset{\sim}{x})$ was obtained (i.e. "switched on"). The differences in switching on $A(t_f,\underset{\sim}{x})$ are taken into account completely in $U(t_f,t')$.

Such a transformation of a free field as $e^{iha}\psi_{in}$ with an arbitrary "a" will be studied in detail when we solve the problem where the external field is $A(t,\underset{\sim}{x}) = \theta(t-t_s)A(\underset{\sim}{x})$.

iii) The requirement that only the final creation and annihilation operators represent physical particles seems to be weaker than the others mentioned up to now since the Bogoliubov transformation (2.13) then needs to be unitarily implementable only for $t_o \geq t_f$.

However, it seems very reasonable to accept that, during an experiment in which an external field $A(t,\underset{\sim}{x})$ is "switched on" from $A(t_s,\underset{\sim}{x}) = 0$, one could decide to keep the potential constant at any time $t_o > t_s$ and that then physical particles should be observable in the system. If this is accepted, the present requirement becomes exactly equivalent to the one discussed in section 2).

4) Example: The Scalar Field with a c-number Source

The model we will now study does not describe a scattering system in the usual sense but is useful as an example to illustrate how the method previously discussed can be generally applied to obtain the physical particles creation and annihilation operators. The field operator solution to this model is well known (see for example P. Roman [1969]). Although it was also treated as an example by Shirkov [1968], our treatment with the auxiliary field method can be seen to be much more direct. In his case, one starts with the Hamiltonian as given from the Lagrangian formalism and, more or less guesses which creation and annihilation operators will be such that the Hamiltonian is diagonal. In our case, the creation and annihilation operators for the physical particles are directly determined from the field operator solution. The energy operator is then defined such that the energy is simply the sum of the energies of all the particles.

We also find the unitary operator relating the creation and annihilation operators at different times; this was not obtained by Shirkov.

Our additional final discussion of the case when the source is kept constant after a certain time also sheds much light on the nature of the physical system so described.

i) In this section, we give the solution to the equation of motion. We will be considering a system of bosons; the corresponding problem with fermions can be treated in much the same way.

The differential equation of motion for the field operator is

$$(\Box + m^2)\phi(t,\underset{\sim}{x}) = \rho(t,\underset{\sim}{x}) \tag{4.1}$$

where the external source $\rho(t,\underset{\sim}{x})$ is real and taken such that

$$\rho(t,\underset{\sim}{x}) = 0 \quad \forall\, t < t_s .$$

Under this condition the system at times $t < t_s$ is described by a free boson field $\phi_{in}(x)$. It can be in any state where there is any number of free particles present. For ϕ_{in}, we use the standard decomposition

$$\phi_{in}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\underset{\sim}{k}}{\sqrt{2\omega(\underset{\sim}{k})}} \{a(\underset{\sim}{k})e^{-ik.x} + a^\dagger(\underset{\sim}{k})e^{ik.x}\} \tag{4.2}$$

where $k.x = \omega(\underset{\sim}{k})t - \underset{\sim}{k}.\underset{\sim}{x}$; $\omega(\underset{\sim}{k}) = \sqrt{\underset{\sim}{k}^2 + m^2}$. The operators "a" and "$a^\dagger$" satisfy the usual commutation relations for bosons creation and annihilation operators.

The solution of equation (4.1) with the above initial condition is easily seen to be

$$\phi(x) = \phi_{in}(x) + \int \Delta_r(x-y)\rho(y)dy \qquad (4.3)$$

where $\Delta_r(x)$ is the fundamental solution of the Klein-Gordon equation defined as

$$\Delta_r(x) = -\theta(x_o)\Delta(x) \qquad (4.4)$$

with

$$\Delta(x) = -\frac{1}{(2\pi)^3}\int \frac{d\underset{\sim}{k}}{\omega(\underset{\sim}{k})} e^{i\underset{\sim}{k}.\underset{\sim}{x}} \sin \omega(k)t \;. \qquad (4.5)$$

ii) We now define the physical particles creation and annihilation operators associated with this field at an arbitrary time t_o. This is done by using the auxiliary field method discussed in section 2).

The auxiliary field $\phi_{t_o}(x)$ must be a solution of the equation

$$(\Box + m^2)\phi_{t_o}(t,x) = \rho(t_o,x) \qquad (4.6)$$

with, here, the Cauchy conditions

$$\phi_{t_o}(t_o,\underset{\sim}{x}) = \phi(t_o,\underset{\sim}{x})$$

$$(\partial_o\phi_{t_o})(t_o,\underset{\sim}{x}) = (\partial_o\phi)(t_o,\underset{\sim}{x}) \;. \qquad (4.7)$$

This field is easily gotten for times $t \geq t_o$, it will simply be given by (4.3) where $\rho(t,\underset{\sim}{x})$ is kept at the value $\rho(t_o, x)$ for all $t \geq t_o$. It is then

$$\phi(x) = \phi_{in}(x) + \int_{-\infty}^{t_o} dy_o \int dy \; \Delta_r(x-y)\rho(y)$$

$$+ \int_{t_o}^{+\infty} dy_o \int dy \; \Delta_r(x-y)\rho(t_o, y) \quad . \tag{4.8}$$

For $t \geq t_o$, using equation (4.4), one can rewrite this as

$$\phi(x) = \phi_{in}(x) - \int_{-\infty}^{t_o} dy_o \int dy \; \Delta(x-y)\rho(y) - \int_{t_o}^{t} dy_o \int dy \, \Delta(x-y)\rho(t_o, y) . \tag{4.9}$$

By generalizing this time dependence to all values of t, one obtains the field which satisfies (4.6) and (4.7) i.e. the auxiliary field $\phi_{t_o}(x)$. Its Fourier decomposition is easily obtained and one can then see that it can be written simply as

$$\phi_{t_o}(t, x) = \frac{1}{(2\pi)^{3/2}} \int \frac{dk}{\sqrt{2\omega(k)}} \{A_{t_o}(k) \, e^{i[\omega(k)t - k.x]}$$

$$+ A^{\dagger}_{t_o}(k) \, e^{i[\omega(k)t - k.x]} \}$$

$$+ \frac{1}{(2\pi)^{3/2}} \int \frac{dk}{\omega^2(k)} \, \tilde{\rho}(t_o, k) \, e^{ik.x} \tag{4.10}$$

where the particles creation and annihilation operators are defined by

$$A_{t_o}(\underset{\sim}{k}) = a(\underset{\sim}{k}) + \frac{i}{\sqrt{2\omega(\underset{\sim}{k})}} \int_{-\infty}^{t_o} dy_o \; e^{i\omega y_o} \; \tilde{\rho}(y_o,\underset{\sim}{k})$$

$$- \frac{e^{i\omega t_o}}{\sqrt{2}\;\omega(k)^{3/2}} \; \tilde{\rho}(t_o,\underset{\sim}{k}) \qquad (4.11)$$

and

$$\tilde{\rho}(t_o,k) = \frac{1}{(2\pi)^{3/2}} \int d\underset{\sim}{x} \; e^{-i\underset{\sim}{k}.\underset{\sim}{x}} \; \tilde{\rho}(t_o,\underset{\sim}{x}) \; .$$

From the time dependence of the auxiliary field (4.10), it is obvious that the generator of time-translations i.e. the energy operator is simply

$$H_{t_o} = \int d\underset{\sim}{k} \; \omega(\underset{\sim}{k}) \; A^{\dagger}_{t_o}(\underset{\sim}{k}) \; A_{t_o}(\underset{\sim}{k}) \, , \qquad (4.12)$$

and the Heisenberg equations:

$$i \frac{\partial}{\partial t} \phi_{t_o}(t,\underset{\sim}{x}) = [\phi_{t_o}(t,\underset{\sim}{x}),H_{t_o}]$$

$$i \frac{\partial}{\partial t} \{\frac{\partial}{\partial t} \phi_{t_o}\} (t,\underset{\sim}{x}) = [\{\frac{\partial}{\partial t} \phi_{t_o}\}(t,\underset{\sim}{x}),H_{t_o}]$$

are satisfied.

In terms of the field operator variables, the energy operator can be formally written as

$$H_{t_o} = \frac{1}{2} \int d\underset{\sim}{x}\{[\partial_{\mu}\phi(x)]^2 + m^2\phi(x)^2\} - \int d\underset{\sim}{x} \; \rho(x)\phi(x) - c_{t_o} \qquad (4.13)$$

where

$$c_{t_o} = \frac{1}{2}\int d\underset{\sim}{k}\ \omega(\underset{\sim}{k}) - \frac{1}{2}\int d\underset{\sim}{k}\ \frac{|\tilde{\rho}(t_o,\underset{\sim}{k})|^2}{\omega(\underset{\sim}{k})^2} .$$

The first term: $\frac{1}{2}\int d\underset{\sim}{k}\ \omega(\underset{\sim}{k})$ of c_{t_o} is the normal infinite term which has to be subtracted in the expression corresponding to (4.13) in the case of a free boson field (i.e. when $\rho(x) = 0$).

iii) The above results hold for any arbitrary time t_o so that one can write in general the following decomposition for the field $\phi(x)$ into particle creation and annihilation operators:

$$\phi(t,\underset{\sim}{x}) = \frac{1}{(2\pi)^{3/2}}\int \frac{d\underset{\sim}{k}}{\sqrt{2\omega(\underset{\sim}{k})}}\{A(\underset{\sim}{k},t)e^{-i[\omega t-\underset{\sim}{k}.\underset{\sim}{x}]} + A^{\dagger}(\underset{\sim}{k},t)e^{i[\omega t-\underset{\sim}{k}.\underset{\sim}{x}]}\} + \frac{1}{(2\pi)^{3/2}}\int \frac{d\underset{\sim}{k}}{\omega(\underset{\sim}{k})^2}\ \rho(t,\underset{\sim}{k})\ e^{i\underset{\sim}{k}.\underset{\sim}{x}} , \qquad (4.14)$$

where

$$A(\underset{\sim}{k},t) \equiv A_t(\underset{\sim}{k}) = a(\underset{\sim}{k}) + \frac{i}{\sqrt{2\omega(\underset{\sim}{k})}}\int_{-\infty}^{t} dy_o e^{i\omega y_o}\ \tilde{\rho}(y_o,\underset{\sim}{k}) - \frac{e^{i\omega t}}{\sqrt{2}\ \omega(\underset{\sim}{k})^{3/2}}\ \tilde{\rho}(t,\underset{\sim}{k}) . \qquad (4.15)$$

The operators A and $A^{\dagger}$ satisfy:

$$[A(\underset{\sim}{k},t),\ A^{\dagger}(\underset{\sim}{k}',t)] = \delta(\underset{\sim}{k}-\underset{\sim}{k}')$$

$$[A(\underset{\sim}{k},t),\ A(\underset{\sim}{k}',t)] = 0 \ . \tag{4.16}$$

The energy operator is

$$H(t) = \int d\underset{\sim}{k}\ \omega(\underset{\sim}{k})A^{\dagger}(\underset{\sim}{k},t)A(\underset{\sim}{k},t) \ . \tag{4.17}$$

The energy spectrum at any time is exactly the same as for the initial free particles. The lowest non-zero eigenvalue is m which is the lower bound of the continuous spectrum. The operators $A^{\dagger}(\underset{\sim}{k},t)$ $(A(\underset{\sim}{k},t))$ create (annihilate) quanta of energy of mass m (and spin 0) so that these quanta are particles of the same nature as those which can be initially present in the system.

The vacuum state at time t is defined by

$$A(k,t)|0\rangle_t = \quad \forall\ k \ . \tag{4.18}$$

Since this state must be in the initial Hilbert space, one can write it as a linear superposition of initial basis states:

$$|0\rangle_t = \alpha^{o}(t)|0\rangle_{in} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \int \frac{d\underset{\sim}{k}_1}{\sqrt{2\omega_1}} \cdots \frac{d\underset{\sim}{k}_n}{\sqrt{2\omega_n}} \times$$

$$\alpha^{n}(\underset{\sim}{k}_1,\underset{\sim}{k}_2 \ldots \underset{\sim}{k}_n,t)\, a^{\dagger}(\underset{\sim}{k}_1)\ldots a^{\dagger}(\underset{\sim}{k}_n)|0\rangle_{in} \ .$$

Replacing this and the definition (4.15) of $A(\underset{\sim}{k},t)$ in (4.18), one can equate the coefficients of 0-particle, 1-particle, etc...... states separately and obtain that

$$\alpha^n(\underset{\sim}{k}_1,\ldots,\underset{\sim}{k}_n,t) = \frac{(-1)^n}{\sqrt{n!}} \prod_{i=1}^{n} \sqrt{2\omega(\underset{\sim}{k}_i)}\, S(\underset{\sim}{k}_i,t)\,\alpha^o(t) \qquad (4.19)$$

where

$$S(\underset{\sim}{k},t) = \frac{i}{\sqrt{2\omega(\underset{\sim}{k})}} \int_{-\infty}^{t} dy_o\, e^{i\omega y_o}\, \tilde{\rho}(y_o,\underset{\sim}{k}) - \frac{e^{i\omega t}}{\sqrt{2}\,\omega^{3/2}}\, \tilde{\rho}(t,\underset{\sim}{k}) .$$

One now can determine $\alpha^o(t)$ by demanding that $|0>_t$ be normalized to 1 and with the solution (4.19), one can write $|0>_t$ simply as

$$|0>_t = \exp[-\tfrac{1}{2}\int d\underset{\sim}{k}\,|S(\underset{\sim}{k},t)|^2]\,\exp[-\int d\underset{\sim}{k}\; S(\underset{\sim}{k},t)\,a^\dagger(\underset{\sim}{k})]\,|0>_{in} . \qquad (4.20)$$

Since the two sets of creation and annihilation operators $A(\underset{\sim}{k},t)$, $A^\dagger(\underset{\sim}{k},t)$ and $a(\underset{\sim}{k})$, $a^\dagger(\underset{\sim}{k})$ have both a vacuum state in $\mathcal{H}_{in}$ associated with them and are irreducible, they must be related by a unitary transformation. This can be gotten here simply by symmetrizing the operator on the left hand side of the above equation. One obtains the unitary operator

$$T(t) = \exp\int d\underset{\sim}{k}[S(\underset{\sim}{k},t)a^{\dagger}(\underset{\sim}{k}) - S^{*}(\underset{\sim}{k},t)a(\underset{\sim}{k})] \ . \qquad (4.21)$$

With the help of the formula

$$Ae^{B} = e^{B}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}\,\Omega_{n}(B,A)$$

where

$$\Omega_{n+1}(B,A) = [B,\Omega_{n}(B,A)] \quad \text{and} \quad \Omega_{o}(B,A) = A \ ,$$

one can check that

$$T^{\dagger}(t)a(\underset{\sim}{k})T(t) = A(\underset{\sim}{k},t) \ . \qquad (4.22)$$

One has also

$$H(t) = T^{\dagger}(t)\ H_{in}\ T(t)$$

where H_{in} is the initial energy operator:

$$H_{in} = \int d\underset{\sim}{k}\ \omega(\underset{\sim}{k})a^{\dagger}(\underset{\sim}{k})a(\underset{\sim}{k}) \ .$$

Similarly the number of particles operator is

$$N(t) = T^{\dagger}(t)\ N_{in}\ T(t)$$

where

$$N_{in} = \int d\underset{\sim}{k}\ a^{\dagger}(\underset{\sim}{k})a(\underset{\sim}{k}) \ .$$

T(t) is the unitary time evolution operator for the particles creation and annihilation operators describing the system. It is not however the time evolution operator for the field variables; this is due to the fact that the creation and annihilation operators have an explicit time dependence and satisfy the Heisenberg equation (2.2).

iv) The following remark sheds much light on the physical nature of the systems described by the equation considered.

As soon as the external source is kept constant in time, say after a certain time t_f, a stable situation is obtained, as it should. The particles creation and annihilation operators are given by

$$A(\underset{\sim}{k},t) = a(\underset{\sim}{k}) + \frac{i}{\sqrt{2\omega}} \int_{-\infty}^{t_f} dy_o \, e^{i\omega y_o} \, \tilde{\rho}(y_o,\underset{\sim}{k}) - \frac{e^{i\omega t_f}}{\sqrt{2}\,\omega^{3/2}} \tilde{\rho}(t_f,\underset{\sim}{k}) \quad \forall\, t \geq t_f$$

These are time independent; no particles are therefore created nor annihilated after t_f. Moreover the field operator becomes

$$\phi(t,\underset{\sim}{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\underset{\sim}{k}}{\sqrt{2\omega}} \{A(\underset{\sim}{k},t_f) e^{-i[\omega t-\underset{\sim}{k}.\underset{\sim}{x}]} + A^{\dagger}(\underset{\sim}{k},t_f) \, e^{i[\omega t-\underset{\sim}{k}.\underset{\sim}{x}]}\} + \frac{1}{(2\pi)^{3/2}} \int \frac{d\underset{\sim}{k}}{\omega^2} \tilde{\rho}(t_f,\underset{\sim}{k}) e^{i\underset{\sim}{k}.\underset{\sim}{x}} . \tag{4.23}$$

The first term containing the operators $A(\underset{\sim}{k},t_f)$, $A^\dagger(\underset{\sim}{k},t_f)$ is simply a quantized free boson field. Since it describes all the particles which could be present in the system at time t, it is clear that the system will contain only free particles; this will be so for all times $t \geq t_f$. The second term of (4.23) is the stable constant c-number field established around the source $\rho(t_f,\underset{\sim}{x})$; it is such that

$$\frac{\partial}{\partial t} B(t_f,\underset{\sim}{x}) = 0$$

and satisfies the equation

$$(\Box + m^2)B(t_f,\underset{\sim}{x}) = \rho(t_f,\underset{\sim}{x}) .$$

A long time after t_f, the particles in the system will all have drifted away from the region of the source where only the classical field due to the c-number source will remain.

The particles are here moving freely in the field of the source because the two fields, quantized and classical, are of the same nature and the two do not interact.

5) Example: The Volkov Problem

The problem of describing charged particles interacting with a monochromatic wave has aroused considerable interest as can be judged from the partial list (16) of references on this subject mentioned by T.W.B. Kibble [1966].

The Klein-Gordon and Dirac equations are exactly soluble with such an external electromagnetic field. Their solutions were first obtained by D.M. Volkov [1935]. Since then, there has been a controversy over whether or not the formalism which we will describe below should be valid and what is the proper use of the Volkov solutions.

i) Let us consider the system consisting of a charged boson field interacting with a classical electromagnetic wave. The external potential is $A(n.x)$ where $n = (n_o, \underset{\sim}{n})$ is a light-like 4-vector i.e. $n.n = 0$. The orientation of the space axes is here chosen such that the x_1 axis is parallel to $\underset{\sim}{n}$ so that $n = (1,1,0,0)$ and $n.x = (t-x_1)$. $A(n.x)$ satisfies the equation

$$\Box A^{\mu}(n.x) = 0$$

together with the Lorentz condition

$$\partial_\mu A^\mu(n.x) = 0$$

which reduces to

$$n.A(n.x) = \text{constant} .$$

It is also taken to be a "wave packet" in the sense that

$$\lim_{|n.x|\to\infty} A^\mu(n.x) = 0 ;$$

this implies in particular that

$$n.A(n.x) = 0$$

i.e. $A_1(n.x) = A_o(n.x)$.

The differential equation of motion for a charged spin zero boson field in this potential is

$$\{[\partial^\mu - ieA^\mu(n.x)][\partial_\mu - ieA_\mu(n.x)] + m^2\}\phi(t,x) = 0 .$$

Upon using

$$\partial_\mu A^\mu(n.x) = 0,$$

this becomes

$$[\Box + m^2]\phi(x) = [2ieA_\mu(n.x)\partial^\mu + e^2 A_\mu(n.x)A^\mu(n.x)]\phi(x) \tag{5.1}$$

and the field $\phi^\dagger(x)$ satisfies the complex conjugate of this equation.

The c-number equation (5.1) is easily solved by finding an operator T(n.x) such that if f(x) is a solution of the free Klein-Gordon equation, Tf(x) will be a solution of (5.1).

Let us suppose that f(x) is such that

$$\Box f(x) = -m^2 f(x) .$$

When this equation is transformed by T, it becomes

$$T\Box f(x) = -m^2 Tf(x) .$$

This will correspond to equation (5.1) when

$$T\Box f(x) = (\Box - 2ieA.\partial - e^2 A.A)\, T\, f(x)$$

for all f(x). One must therefore have

$$[T,\Box] = -(\partial^\mu \partial_\mu T) - 2(\partial^\mu T)\partial_\mu$$

$$= (-2ieA.\partial - e^2 A.A)T . \qquad (5.2)$$

The coefficients of this differential equation depend only on (n.x); ~~correspondingly~~, there exists a solution T depending only on (n.x). Upon using

$$\partial_\mu T(n.x) = n_\mu \frac{dT}{d(n.x)} ,$$

it becomes:

$$2\left(\frac{dT}{d(n.x)}\right)(n.\partial) = [2ieA(n.x).\partial + e^2A(n.x).A(n.x)]T \;. \tag{5.3}$$

The operators $(n.\partial)$, $A(n.x).\partial$ and $A(n.x).A(n.x)$ all intercommute. $(n.\partial)$ has an inverse on the space of solutions of the Klein-Gordon equation so that (5.3) can be written as

$$\frac{dT}{d(n.x)} = \frac{1}{2(n.\partial)}(2ieA.\partial + e^2A.A)T \;.$$

The solution with the condition $T \to I$ as $(n.x) \to -\infty$ is simply

$$T = \exp \frac{1}{2(n.\partial)}\left\{2ie\left[\int_{-\infty}^{n.x} dy\ A(y)\right].\partial + e^2\left[\int_{-\infty}^{n.x} dy\ A(y).A(y)\right]\right\} \tag{5.4}$$

and this operator has an inverse.

If we consider for example the particular solution of the free equation $e^{-i[\omega(\underset{\sim}{k})t - \underset{\sim}{k}.\underset{\sim}{x}]}$; the corresponding solution of (5.1) is simply

$$T\,e^{-ik.x} = e^{-ik.x}\,e^{-iJ(k,n.x)} \tag{5.5}$$

where $J(k,n.x)$ is defined as

$$J(k,n.x) = -\frac{1}{2[\omega(k)-k_1]}\left\{2e\left[\int_{-\infty}^{n.x} A(y)dy\right].k + e^2\left[\int_{-\infty}^{n.x} A(y).A(y)dy\right]\right\}$$

$$= \int_{-\infty}^{n.x} I(k,y)dy \;. \tag{5.6}$$

The differential operators $i(n.\partial)$, $i\partial_2$, $i\partial_3$ all commute with the external field $A(n.x)$ so that the Volkov solutions can be characterized by their eigenvalues. The solution (5.5), for example, is an eigenfunction of $(n.\partial)$ with eigenvalue $[\omega(\underset{\sim}{k})-k_1]$ and of $i\partial_2$ and $i\partial_3$ with eigenvalue k_2, k_3.

ii) To obtain an operator valued distribution which would satisfy equation (7.1), a method commonly used (see, for example: L.S. Brown and T.W.B. Kibble [1964], J.W. Meyer [1970], R.A. Neville and F. Rohrlich [1971]) is essentially the following.

One knows that the solutions of (5.1) can be related in a one to one fashion to free Klein-Gordon solutions. A quantized field satisfying (5.1) will then be obtained through the action of T on a quantized free field distribution $\phi_o(x)$. This is

$$\phi(x) = T(\underset{\sim}{n}.x)\phi_o(x) \tag{5.7}$$

where the free charged boson field has the usual decomposition in creation and annihilation operators:

$$\phi_o(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\underset{\sim}{k}}{\sqrt{2\omega(\underset{\sim}{k})}} \{a(k)e^{-ik.x} + b^\dagger(k)e^{ik.x}\}. \tag{5.8}$$

The two sets of operators $\{a(\underset{\sim}{k}), a^\dagger(\underset{\sim}{k})\}$ and $\{b(\underset{\sim}{k}), b^\dagger(\underset{\sim}{k})\}$ intercommute and both satisfy similar commutation relations:

$$[a(\underset{\sim}{k}),\ a^{\dagger}(\underset{\sim}{k}')] = \delta(\underset{\sim}{k}-\underset{\sim}{k}') \qquad [a(\underset{\sim}{k}),a(\underset{\sim}{k}')] = 0\ . \quad (5.9)$$

These are defined on the usual Fock-Hilbert space for free bosons systems.

Upon replacing (5.8) in (5.7), the field $\phi(x)$ is obtained as:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\underset{\sim}{k}}{\sqrt{2\omega(\underset{\sim}{k})}} \{a(\underset{\sim}{k})e^{-ik.x}\, e^{-iJ(k,n.x)}$$

$$+ b^{\dagger}(\underset{\sim}{k})e^{ik.x}e^{-iJ(-k,n.x)}\} \qquad (5.10)$$

and its adjoint is

$$\phi^{\dagger}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\underset{\sim}{k}}{\sqrt{2\omega(\underset{\sim}{k})}} \{a^{\dagger}(\underset{\sim}{k})e^{ik.x}e^{iJ(k,n.x)}$$

$$+ b(\underset{\sim}{k})e^{-ik.x}e^{iJ(-k,n.x)}\}\ . \qquad (5.11)$$

In the work of Neville and Rohrlich [1971] a different, however equivalent, set of variables is used; we shall describe it briefly. Instead of using the variables $\underset{\sim}{k}$, one can use variables which we shall call $\underset{\sim}{\alpha}$ defined as

$$\alpha_1 = \omega(\underset{\sim}{k}) - k_1 \quad (>0)\ , \quad \alpha_2 = k_2\ , \quad \alpha_3 = k_3\ . \qquad (5.12)$$

The notation $\underline{\alpha}$ is used for (α_2, α_3). The k's are obtained from the $\underset{\sim}{\alpha}$'s as:

$$k_o = \omega(\underset{\sim}{k}) = \frac{1}{2\alpha_1}(\underset{\sim}{\alpha}^2 + m^2)$$

$$k_1 = \frac{1}{2\alpha_1}(\underline{\alpha}^2 + m^2 - \alpha_1^2)$$

$$k_2 = \alpha_2 \quad , \quad k_3 = \alpha_3 \quad . \tag{5.13}$$

The free Klein-Gordon solution $e^{-i[\omega t - \underset{\sim}{k}.\underset{\sim}{x}]}$ can be written as

$$e^{-i[\omega t-\underset{\sim}{k}.\underset{\sim}{x}]} = \exp i[\frac{1}{2\alpha_1}(\underline{\alpha}^2 + m^2)(t-x_1) + \frac{\alpha_1}{2}(t+x_1) + \underline{\alpha}.\underline{x}] \ .$$

This solution when multiplied by the factor $1/(2\pi)^{3/2}\sqrt{2\alpha_1}$ for the proper normalization, will be denoted by $f(\underset{\sim}{\alpha},x)$. $I(k,n.x)$ can be written, in terms of these variables as

$$I(k,n.x) = -\frac{1}{2\alpha_1}\left[2e[A_o(n.x)\alpha_1 - \underline{A}(n.x).\underline{\alpha}] + e^2A(n.x).A(n.x)\right] .$$

and the Volkov solution (5.5) as

$$f(\underset{\sim}{\alpha},x)\ e^{-iJ(\underset{\sim}{\alpha},n.x)} \ .$$

The free field operator (5.8) becomes

$$\phi_o(x) = \int d\underset{\sim}{\alpha}\{a(\underset{\sim}{\alpha})f(\underset{\sim}{\alpha},x) + b^\dagger(\underset{\sim}{\alpha})f(-\underset{\sim}{\alpha},x)\} \tag{5.14}$$

with

$$[a(\underset{\sim}{\alpha}),a^\dagger(\underset{\sim}{\alpha}')] = \delta(\underset{\sim}{\alpha}-\underset{\sim}{\alpha}') = [b(\underset{\sim}{\alpha}),b^\dagger(\underset{\sim}{\alpha}')]$$

$$[a(\underset{\sim}{\alpha}),\ a(\underset{\sim}{\alpha}')] = 0 = [b(\underset{\sim}{\alpha}),b(\underset{\sim}{\alpha}')] \ . \tag{5.15}$$

The field $\phi(x)$ defined in equation (5.10) is equivalently written as

$$\phi(x) = \int d\underset{\sim}{\alpha} \left[a(\underset{\sim}{\alpha}) f(\underset{\sim}{\alpha}, x) e^{-iJ(\underset{\sim}{\alpha}, n.x)} + b^{\dagger}(\underset{\sim}{\alpha}) f(-\underset{\sim}{\alpha}, x) e^{-iJ(-\underset{\sim}{\alpha}, n.x)} \right]. \tag{5.16}$$

iii) What leads to the controversy over this problem is that certain authors (for example, I.I. Goldman [1964], A.I. Nikishov and V.I. Ritus [1964], and the previously mentioned authors) effectively interpret the Volkov solutions as the wave functions of the electron in the sense of ordinary quantum mechanics. This is considered explicitly by the two first authors mentioned while in the treatment of the others, this comes about when the physical particles creation and annihilation operators are taken to be those appearing in the decomposition (5.10) or (5.16) of the field operator.

We want to show explicitly in the next sections that this formalism (i.e. the quantization procedure described above) does not in fact agree with the usual ways of quantization. That is: the field operators do not satisfy the canonical equal time commutation relations and, what is more difficult to accept from our point of view, the Heisenberg equation of motion is not satisfied.

iv) In the following calculations of commutators, either expressions for the field (5.10) or (5.16) can be used. If one uses (5.16), the commutators can be calculated using the commutation relations (5.15) and then the variables of integration $\underset{\sim}{\alpha}$ in the integrals can be changed to $\underset{\sim}{k}$ according to equation (5.12). Exactly the same results are obtained as when using (5.10).

The commutator $[\phi(x), \phi^\dagger(x')]$ has already been calculated in closed form by L.S. Brown and T.W.B. Kibble [1964]. The following result has been obtained:

$$[\phi(x), \phi^\dagger(x')] = ie^{ie(x-x').\bar{A}} \Delta(x-x'; M^2) \qquad (5.17)$$

where

$$\bar{A} = \frac{1}{(n.x - n.x')} \int_{n.x'}^{n.x} A(y)\, dy$$

and

$$M^2 = M^2(n.x, n.x', m) = m^2 - e^2\, \overline{A.A} + e^2\, \bar{A}.\bar{A} \geq m^2 \; . \qquad (5.18)$$

The distribution $\Delta(x-x'; M^2)$ has the same form as $\Delta(x-x'; m^2)$ which is the corresponding result of this commutator for free fields. (The mass m is replaced here by M). In particular, if $x_o = x_o'$, the commutator vanishes.

The Lagrangian density corresponding to the equation (5.1) is

$$L(x) = (\partial_\mu \phi^\dagger + ieA_\mu \phi^\dagger).(\partial^\mu \phi - ieA^\mu \phi) - m^2 \phi^\dagger.\phi$$

where $B.C = \frac{1}{2}(BC + CB)$. The canonically conjugate fields are then

$$\Pi_\phi(x) = \frac{\partial L}{\partial(\partial_o \phi)} = (\partial_o \phi^\dagger + ieA_o \phi^\dagger)$$

$$\Pi_{(\phi^\dagger)}(x) = \frac{\partial L}{\partial(\partial_o \phi^\dagger)} = (\partial_o \phi - ieA_o \phi) \qquad (5.19)$$

and the Hamiltonian density is

$$H(x) = (\partial_o \phi^\dagger).(\partial_o \phi) + (\nabla \phi^\dagger + ie\underset{\sim}{A}\phi^\dagger).(\nabla \phi - ie\underset{\sim}{A}\phi)$$

$$+ m^2 \phi^\dagger.\phi - e^2 A_o^2 \phi^\dagger.\phi \qquad (5.20)$$

We now calculate the equal time canonical commutator.

$$[\phi(x), \Pi_\phi(x')]_{x_o = x'_o}$$

Since we know that

$$[\phi(x), \phi^\dagger(x')]_{x_o = x'_o} = 0 ,$$

all we have to calculate is

$$[\phi(x), \frac{\partial \phi^\dagger}{\partial x_o}(x')]_{x_o = x'_o}$$

From the expression (5.17), one obtains:

$$[\phi(x), \frac{\partial}{\partial x_o'} \phi^{\dagger}(x')]_{x_o=x_o'} = \left\{\frac{\partial}{\partial x_o'}[e^{ie(x-x').\bar{A}} \; i\Delta(x-x';M^2)]\right\}_{x_o=x_o'} . \tag{5.21}$$

The right hand side of this equation is

$$\left\{[\frac{\partial}{\partial x_o'} e^{ie(x-x').\bar{A}}] \; ie\,\Delta(x-x';M^2)\right\}_{x_o=x_o'}$$

$$+ \left\{e^{ie(x-x').\bar{A}} \; \frac{\partial}{\partial x_o'}\Big|_{M^2 \text{ fixed}} \; i\Delta(x-x';M^2)\right\}_{x_o=x_o'}$$

$$+ \left\{e^{ie(x-x').\bar{A}} \; [\frac{\partial}{\partial M^2}\Big|_{\substack{\text{other}\\ \text{variables}\\ \text{fixed}}} \; i\Delta(x-x';M^2)]\frac{\partial M^2}{\partial x_o'}\right\}_{x_o=x_o'} \tag{5.22}$$

Upon using for $\Delta(x)$ the representation:

$$\Delta(x-x';M^2) = \frac{1}{(2\pi)^3}\int \frac{d\underset{\sim}{k}^2}{\sqrt{\underset{\sim}{k}^2+M^2}} \; e^{i\underset{\sim}{k}.(\underset{\sim}{x}-\underset{\sim}{x}')} \; \sin\sqrt{k^2+M^2}\,(x_o-x_o') , \tag{5.23}$$

one obtains:

$$\frac{\partial}{\partial x_o'}\Big|_{M^2 \text{fixed}} \Delta(x-x';M^2) = \frac{1}{(2\pi)^3}\int d\underset{\sim}{k} \; e^{i\underset{\sim}{k}.(\underset{\sim}{x}-\underset{\sim}{x}')} \times$$

$$\times \cos\sqrt{\underset{\sim}{k}^2+M^2}\,(x_o-x_o') \tag{5.24}$$

$$\left.\frac{\partial}{\partial M^2}\right|_{\substack{\text{other}\\\text{variables}\\\text{fixed}}} \Delta(x-x';M^2) = -\frac{1}{2(2\pi)^3}\int \frac{d\underset{\sim}{k}}{\sqrt{\underset{\sim}{k}^2+M^2}}\, e^{i\underset{\sim}{k}.(\underset{\sim}{x}-\underset{\sim}{x}')} \times$$

$$\times\left\{ [(x_o-x_o')\cos\sqrt{\underset{\sim}{k}^2+M^2}\,(x_o-x_o') - \frac{\sin\sqrt{\underset{\sim}{k}^2+M^2}\,(x_o-x_o')}{\sqrt{\underset{\sim}{k}^2+M^2}}\right\} .$$

(5.25)

The following identity will also be useful:

$$\frac{\partial}{\partial x_o'}\bar{C} = \frac{\partial}{\partial x_o'}\ \frac{1}{(n.x - n.x)}\int_{n.x'}^{n.x} C(y)\, dy$$

$$= \frac{1}{(n.x - n.x')}\{\bar{C} - C(n.x')\} \quad . \tag{5.26}$$

The terms (5.23), (5.24), (5.25) are easily evaluated when $x_o = x_o'$; (5.23) and (5.25) are null and (5.24) is $\delta(\underset{\sim}{x}-\underset{\sim}{x}')$. The terms (5.23) and (5.25) appear in (5.22) multiplied respectively by

$$\frac{\partial}{\partial x_o'}\, e^{ie(x-x').\bar{A}}$$

and $\partial M^2/\partial x_o'$ which are finite when $x_o = x_o'$ and $x_1 \neq x_1'$ and can be infinite when $x_1 = x_1'$, as can be seen from equation (5.26). We thus obtain that

$$[\phi(x), \frac{\partial}{\partial x_o'}\phi^\dagger(x')]_{x_o=x_o'} = 0 \quad \text{when} \quad x_1 \neq x_1' \quad .$$

To obtain what this commutator is for $x_1 = x_1'$, one could use L' Hospital's rule but it is simpler here to calculate it explicitly.

From equation (5.11) for $\phi^\dagger(x)$ and upon using

$$[\frac{\partial}{\partial x_o'} + ieA_o(n.x')]\, e^{ik.x'}\, e^{iJ(k,n.x')}$$

$$= iW[k_1, \underline{k}, A(n.x')]\, e^{ik.x'}\, e^{iJ(k,n.x')}$$

where

$$W[k_1,\underline{k},A(n.x')] = \left\{\omega(\tilde{k}) + \frac{[\omega(\tilde{k})+k_1]}{2[\underline{k}^2+m^2]}[2e\underline{k}.\underline{A}(n.x')+e^2\underline{A}^2(n.x')]\right\},$$

we obtain

$$\Pi_\phi(x') = \frac{i}{(2\pi)^{3/2}}\int\frac{d\tilde{k}}{\sqrt{2\omega(\tilde{k})}}\,\{W[k_1,\underline{k},A(n.x')]e^{ik.x'}e^{iJ(k,n.x')}a^\dagger(\tilde{k})$$

$$- W[k_1,-\underline{k},A(n.x')]e^{-ik.x'}e^{iJ(-k,n.x')}b(\tilde{k})\}.$$

One then has

$$[\phi(x),\Pi_\phi(x')] = \frac{i}{2(2\pi)^3}\int\frac{d\tilde{k}}{\omega(\tilde{k})}\,\{W[k_1,\underline{k},A(n.x')]\,\times$$

$$\times\, e^{-ik(x-x')}\, e^{-i[J_k(n.x)-J_k(n.x')]}$$

$$+ W[k_1,-\underline{k},A(n.x')]e^{ik.(x-x')}e^{-i[J_{-k}(n.x)-J_{-k}(n.x')]}\}.$$

When $x_o = x_o'$ and $x_1 = x_1'$, this becomes, upon changing the variables of integration $\underline{k}$ to $-\underline{k}$ in the second term,

$$= \frac{i}{(2\pi)^3} \int \frac{d\underset{\sim}{k}}{\omega(\underset{\sim}{k})} W[k_1,\underline{k},A(n.x)] e^{i\underline{k}.(\underline{x}-\underline{x}')}$$

$$= \frac{i}{(2\pi)^3} \int \frac{d\underset{\sim}{k}}{\omega(\underset{\sim}{k})} \{\omega(\underset{\sim}{k}) + \frac{[\omega(\underset{\sim}{k})+k_1]}{2[\underline{k}^2+m^2]}[2e\underline{k}.\underline{A}(n.x)+e^2\underline{A}^2(n.x)]\} \times$$

$$\times e^{i\underline{k}.(\underline{x}-\underline{x}')} \quad (5.27)$$

The term containing k_1 as factor does not contribute to the integral since it is an odd function of k_1 and $\int_{-\infty}^{+\infty} dk_1$ odd function $(k_1) = 0$. (5.27) is therefore

$$\frac{i}{(2\pi)^2} [\frac{1}{2\pi}\int dk_1] \int d\underline{k}\{1 + \frac{1}{2(\underline{k}^2+m^2)}[2e\underline{k}.\underline{A}(n.x)+e^2\underline{A}^2(n.x)]\} \times$$

$$\times e^{i\underline{k}.(\underline{x}-\underline{x}')} .$$

The first term of the remaining integral is

$$\int d\underline{k}\, e^{i\underline{k}.(\underline{x}-\underline{x}')} = (2\pi)^2 \delta(\underline{x}-\underline{x}') .$$

The second term

$$\int d\underline{k} \frac{2e\underline{k}.\underline{A}(n.x)}{2(\underline{k}^2+m^2)} e^{i\underline{k}.(\underline{x}-\underline{x}')}$$

can be evaluated with the help of standard mathematical tables and one obtains

$$i(2\pi)em(\underline{x}-\underline{x}').\underline{A}(n.x) \frac{K_1[m||\underline{x}-\underline{x}'||]}{||\underline{x}-\underline{x}'||}$$

where $||\underline{x}|| = [x_2^2+x_3^2]^{\frac{1}{2}}$ and $K_1(y)$ is a modified Bessel function the behaviour of which at small y is

$$K_1(y) \approx \frac{1}{y} \qquad\qquad 0 < y << 1$$

and which for $y >> 1$, decreases exponentially.

The third term

$$\frac{e^2}{2}\underline{A}^2(n.x)\int d\underline{k}\,\frac{e^{i\underline{k}.(\underline{x}-\underline{x}')}}{\underline{k}^2+m^2}$$

can also be evaluated exactly from standard tables and one obtains for it

$$\pi e^2\underline{A}^2(n.x)\;K_o[m||\underline{x}-\underline{x}'||]\;.$$

$K_o(y)$ is again a modified Bessel function which for small y is as

$$K_o(y) \approx \ln\frac{2}{y} \qquad\qquad 0 < y << 1$$

and for large y decreases exponentially.

We then obtain that

$$[\phi(x),\Pi_\phi(x')]_{x_o=x_o'} = 0 \qquad \text{when } x_1 \neq x_1'$$

and $$= \frac{i}{(2\pi)^2}\{\frac{1}{2\pi}\int dk_1\}\{(2\pi)^2\,\delta(\underline{x}-\underline{x}') +$$

$$+\,i(2\pi)em(\underline{x}-\underline{x}').\underline{A}(n.x)\,\frac{K_1[m||\underline{x}-\underline{x}'||]}{||\underline{x}-\underline{x}'||}$$

$$+\,\pi e^2\underline{A}^2(n.x)K_o[m||\underline{x}-\underline{x}'||]\} \qquad \text{when } x_1 = x_1'\;.$$

This can be written as

$$[\phi(x),\Pi_\phi(x')]_{x_o=x_o'} = i\delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

$$+ i\,\frac{e^2}{2(2\pi)}\,\delta(x_1-x_1')\underline{A}^2(n.x)K_o[m||\underline{x}-\underline{x}'||]$$

$$- \frac{em}{2\pi}\,\delta(x_1-x_1')\,(\underline{x}-\underline{x}').\underline{A}(n.x)\,\frac{K_1[m||\underline{x}-\underline{x}'||]}{||\underline{x}-\underline{x}'||} \tag{5.28}$$

$$= i\delta(\underset{\sim}{x}-\underset{\sim}{x}') + F(x_o,\underset{\sim}{x},\underset{\sim}{x}') \quad . \tag{5.29}$$

Since F is non-zero this is obviously different from the equal time canonical commutation relation.

v) We now show that the Heisenberg equation of motion is not satisfied.

It is generally admitted and has been verified with all other models considered in the present work that the energy operator can always be written formally as

$$H(t) = \int d\underset{\sim}{x}\; H(x) + c$$

where c is some constant. (The knowledge of c is not important here since we will only be calculating commutators with H(t)). The time evolution of the field operators should then be governed by the equation

$$i\frac{\partial}{\partial x_0}\phi(x) = [\phi(x_0,\underset{\sim}{x}), \int d\underset{\sim}{x}'\; H(x_0,\underset{\sim}{x}')] \; . \tag{5.30}$$

In order to obtain the commutator on the right hand side of this equation, we still have to evaluate

$$[\phi(x), \underset{\sim}{\partial}\phi^{\dagger}(x')]_{x_0=x_0'} \; .$$

Upon using again the expression (5.17) for the commutator $[\phi,\phi^{\dagger}]$ and the representation (5.23) for Δ, it is easily seen that for i = 2,3

$$\begin{aligned}[\phi(x), \frac{\partial}{\partial x_i'}\phi^{\dagger}(x')]_{x_0=x_0'} &= \{-e\bar{A}_i e^{ie(x-x').\bar{A}}\,\Delta(x-x';M^2) \\ &\quad + e^{ie(x-x').\bar{A}}\, i\frac{\partial}{\partial x_i'}\Delta(x-x';M^2)\}_{x_0=x_0'} \\ &= 0 \; .\end{aligned} \tag{5.31}$$

We now calculate $[\phi,\partial^1\phi^{\dagger}]_{x_0=x_0'}$:

$$\begin{aligned}[\phi(x), \frac{\partial}{\partial x_1'}\phi^{\dagger}(x')]_{x_0=x_0'} &= \Big([\frac{\partial}{\partial x_1'} e^{ie(x-x').\bar{A}}]\; i\Delta(x-x';M^2) \\ &+ e^{ie(x-x').\bar{A}}\,[\frac{\partial}{\partial x_1'}\Big|_{M^2\text{fixed}} i\Delta(x-x';M^2)] \\ &+ e^{ie(x-x').\bar{A}}\,[\frac{\partial}{\partial M^2}\Big|_{\substack{\text{all other}\\ \text{variables}\\ \text{fixed}}} i\Delta(x-x';M^2]\,\frac{\partial M^2}{\partial x_1'}\Big)_{x_0=x_0'}\end{aligned} \tag{5.32}$$

We now show that the first and third terms of this are the same as those in equation (5.22) except for the sign which is the opposite. We have

$$\frac{\partial}{\partial x_1^T} e^{ie(x-x').\bar{A}} = [ie\bar{A}_1 + ie(x-x')\cdot\frac{\partial \bar{A}}{\partial x_1^T}]e^{ie(x-x').\bar{A}} \;;$$

since $\frac{\partial}{\partial x_1^T}\bar{C} = -\frac{\partial}{\partial x_o^T}\bar{C}$ and $A_1 = A_o$, this is

$$= [ie\bar{A}_o + ie(x-x')\cdot\frac{\partial \bar{A}}{\partial x_o^T}]e^{ie(x-x').\bar{A}}$$

$$= -\frac{\partial}{\partial x_o^T} e^{ie(x-x').\bar{A}}$$

so that the first term of (5.32) is the same as for (5.22) with a minus sign. The same holds for the third terms because $\partial M^2/\partial x_1' = -\partial M^2/\partial x_o'$.

The second term of (5.32) is easily evaluated with the help of (5.23);

$$\left(\left.\frac{\partial}{\partial x_1^T}\right|_{M^2\text{fixed}} i\Delta(x-x';M^2)\right)_{x_o=x_o'} = 0 \;.$$

Since in (5.22), it was the second term which gave $\delta(\underset{\sim}{x}-\underset{\sim}{x}')$ in $[\phi(x),\Pi_\phi(x')]_{x_o=x_o'}$ and the first and third terms contributed to $F(x_o,\underset{\sim}{x},\underset{\sim}{x}')$; the second term being absent here (i.e. in (5.32)), we obtain:

$$[\phi(x), \frac{\partial}{\partial x_1^T}\phi(x')]_{x_o=x_o'} = -F(x_o,\underset{\sim}{x},\underset{\sim}{x}') \;. \qquad (5.33)$$

It is now easy with $H(x)$ as given in equation (5.20) and the commutators that we have calculated to obtain that

$$[\phi(x_o,\underset{\sim}{x}), \int d\underset{\sim}{x}' \; H(x_o,\underset{\sim}{x}')] = i \frac{\partial}{\partial x_o} \phi(x) +$$

$$+ \int d\underset{\sim}{x}' F(x_o,\underset{\sim}{x},\underset{\sim}{x}') \left(\frac{\partial}{\partial x_o} - \frac{\partial}{\partial x_1'} + ieA_1(n.x)\right) \phi(x_o,\underset{\sim}{x}'). \quad (5.34)$$

By comparing with the Heisenberg equation (5.30), it is obvious that it is not the same.

vi) We have thus proved that the quantization of the Volkov problem according to equations (5.10) and (5.16) is not in agreement with the present general theory of external fields. Until further study, we therefore agree with Z. Fried and J.H. Eberly [1964], Z. Fried, A. Baker, D. Korff [1966], M.I. Shirkov [1968]', in that the Volkov solutions, even though they are exact c-number solutions to the time dependent equations, might not be so useful in treating the quantized field problem (at least not in the way which has been examined here).

A difficulty with this problem lies in that the quantization is not fixed by an initial condition like when the external field is null at some initial time. In order to obtain a satisfactory quantization of the field, one could start from the most general c-number solution of the equation of motion and find, with the help of the auxiliary field method, on which coefficients of the general solution the commutation relations of creation and annihilation operators should be imposed. This should give the proper physical particles creation and annihilation operators and the Heisenberg equation would be satisfied.

With regard to the use of the Volkov solutions as the wave functions of electrons in the sense of non-relativistic quantum mechanics, we make the following remark. The use of a c-number solution of the time dependent Klein-Gordon or Dirac equation to calculate transition probabilities between two states is justified only when the creation and annihilation of particles in these states can be neglected. (This will be seen in more details in the part on the S-matrix in section 2) of Chapter IV). In the present case of a sustained radiation field, it is far from obvious that particles will not be continually created and annihilated along the path of the wave so that this would give a more and more important contribution to any transition amplitude.

CHAPTER III

GENERALIZED BOGOLIUBOV TRANSFORMATIONS

Introduction

In many interesting problems where a quantized field interacts with an external field, the differential equation of motion for the quantized field operator-valued distributions are linear and homogeneous. This was the case, for example, in the previously discussed problems where charged particles are acted upon by a classical electromagnetic field.

As we have seen, examining the physical requirements that there exists a unitary time evolution operator or that new creation and annihilation operators are related to physical objects, reduces in both cases to studying Bogoliubov transformations. In other words, when appropriately written in terms of particle creation and annihilation operator variables, the equations of motion become generalized Bogoliubov transformations.

The present chapter, where we discuss the general properties of such transformations, is therefore very important since it is really a study of the general solution of the operator equation of motion for all external field problems of this class.

Particular attention will be given to the question: when is there a vacuum (zero particle) state associated with the transformed creation and annihilation operators in the Fock-Hilbert space determined by the initial creation and annihilation operators? This question, as we will see, is equivalent to asking: when is there a unitary operator relating the two sets of creation and annihilation operators in the initial Fock-Hilbert space?

In the case of boson fields, the conditions under which this is so, together with the explicit form of the unitary operator implementing the transformation were derived by K.O. Friedrichs [1953]. These conditions are also mentioned by B. Schroer, R. Seiler and A. Swieca [1970], who refer to a different proof in the work of D. Shale [1962]. The main result is that the Bogoliubov transformation will be unitarily implementable if and only if

$$\int d\alpha d\beta \left| \int d\gamma M_1^{-1}(\gamma,\alpha) M_2(\gamma,\beta) \right|^2 < \infty$$

for either a transformation like

$$A(\gamma) = \int d\alpha M_1(\gamma,\alpha) a(\alpha) + \int d\beta M_2(\gamma,\beta) a^\dagger(\beta)$$

or with two different operator variables as (1.1). This is also the necessary and sufficient condition for the existence of a new vacuum.

In the case of fermion fields, when M_1 has an inverse, the above condition (which is equivalent to $\int d\gamma d\beta |M_2(\gamma,\beta)|^2 < \infty$) is also necessary and sufficient for the existence of a unitary operator implementing the transformation. This was studied together with boson systems by Friedrichs. A peculiarity of fermions however is that the relations,(1.7) and (1.9) below,between the M_i's are not sufficient to guarantee the existence of an inverse to M_1 (or M_4^*); contrary to the boson case where a minus sign would appear in front of $M_2M_2^*$ and $M_3M_3^*$ in the corresponding relations. An additional physical restriction on the external field would then probably be imposed by requiring M_1^{-1} to exist.

This is why more caution seems to be required when dealing with fermions, as already remarked by B. Schroer, R. Seiler and A. Swieca. However, in the article by R. Seiler [1972], the following result is mentioned with reference to a theorem by Shale and Stinespring [1965]: it is sufficient for a unitary operator to exist that M_2 is such that $\int d\gamma d\beta |M_2(\gamma,\beta)|^2 < \infty$ (and M_3 such that $\int d\alpha d\lambda |M_3(\lambda,\alpha)|^2 < \infty$).

This result is nevertheless incomplete in that the nature of the new vacuum states obtained and of the unitary operators implementing the "strong" Bogoliubov transformations (i.e. those where M_I^{-1} or or $(M_4^*)^{-1}$ do not exist) were not known (cf. remark at the end of the talk given by R. Seiler [1972]'). It is one of our major results to have found the explicit form of these (see G. Labonté [1973]).

In the first sections of this chapter, we give the definition of generalized Bogoliubov transformations and obtain in a similar fashion as Capri [1969] the equations which must be satisfied by the components of the new vacuum state when it is expanded in terms of initial Fock-basis states. Necessary conditions for the existence of solutions are then found and in the subsequent sections 3)-7) we solve the equations explicitly.

While the main result of these sections is the finding of the exact form of the new vacuum state, another important result is obtained. This consists of a simple proof, using only elementary mathematical concepts, of the necessity and sufficience of the above mentioned conditions for the existence of the new vacuum.

The results obtained in section 3) concerning the solutions of the one-variable integral equations are more complete, even in the case of "weak" transformations, than previous treatments in that a series solution is obtained instead of simply a symbolic or formal solution.

In section 4), we prove a theorem about a general property of linear equations for antisymmetric functions. This theorem is used later mainly to prove the uniqueness of the solutions found but it is undoubtably of more general interest.

As we discuss in section 8), once the form of the new vacuum is obtained, it is easy to see what is the general effect of the Bogoliubov transformation. We will show that it can be decomposed in a product of simpler transformations and from this we obtain the unitary operator implementing it.

This also proves straightforwardly that the conditions obtained are necessary and sufficient for the existence of a unitary operator implementing the Bogoliubov transformation.

Finally, in section 9), we discuss some of the implications of having "strong" Bogoliubov transformations with particular attention given to the non-conservation of the charge.

1) Definition

We consider transformations of the type

$$B(\gamma) = \int d\alpha M_1(\gamma,\alpha) b(\alpha) + \int d\beta M_2(\gamma,\beta) d^\dagger(\beta)$$

$$D^\dagger(\lambda) = \int d\alpha M_3(\lambda,\alpha) b(\alpha) + \int d\beta M_4(\lambda,\beta) d^\dagger(\beta) \qquad (1.1)$$

where the four infinite sets of indices $\{\alpha\}$, $\{\beta\}$, $\{\gamma\}$ and $\{\lambda\}$ are not necessarily the same and each one can be discrete or continuous or have both continuous and discrete elements. Integrals, like

$$\int d\alpha, \quad \text{mean} \quad \underset{\text{(over continuous } \alpha\text{'s)}}{\int d\alpha} + \underset{\text{(over discrete } \alpha\text{'s)}}{\sum_\alpha}$$

In equation (1.1), $b(\alpha)$ and $d^\dagger(\beta)$ are operators such that with their adjoints they satisfy the anticommutation relations

$$\{b(\alpha), b^\dagger(\alpha')\} = \delta_{\alpha\alpha'}, \qquad \{d(\beta), d^\dagger(\beta')\} = \delta_{\beta\beta'}, \qquad (1.2)$$

with all other anticommutators vanishing. It is understood that for continuous values of the indices, these are operator-valued distributions i.e. they are well defined operators only when smeared with a square integrable function of the index.

They are defined on the Hilbert space $\mathcal{H}$ consisting of states

$$|\psi\rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\sqrt{n!m!}} \int d\underline{\alpha}_n d\underline{\beta}_m \; \psi^{nm}(\underline{\alpha}_n;\underline{\beta}_m) \; |\underline{\alpha}_n;\underline{\beta}_m\rangle \qquad (1.3)$$

such that

$$<\psi|\psi> = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int d\underline{\alpha}_n \, d\underline{\beta}_m |\psi^{nm}(\underline{\alpha}_n;\underline{\beta}_m)|^2 < \infty \qquad (1.4)$$

and where by definition

$$|\underline{\alpha}_n;\underline{\beta}_m> = |\alpha_1,\alpha_2,\ldots,\alpha_n;\ \beta_1,\beta_2,\ldots,\beta_m>$$

$$= b^\dagger(\alpha_1)b^\dagger(\alpha_2)\ldots b^\dagger(\alpha_n)d^\dagger(\beta_1)d^\dagger(\beta_2)$$

$$\ldots d^\dagger(\beta_m)|0> \qquad (1.5)$$

and the vacuum state $|0>$ is such that

$$b(\alpha)|0> = 0 \ \forall\ \alpha \qquad d(\beta)|0> = 0 \ \forall\ \beta \ . \qquad (1.6)$$

We will use the notation $(\underline{\alpha}_n \hat{i}) = (\alpha_1,\alpha_2,\ldots\alpha_{i-1},\alpha_{i+1},\ldots\alpha_n)$ and $(\underline{\alpha}_n \overset{\ell}{\underset{i}{\times}}) = (\alpha_1,\alpha_2,\ldots\alpha_{i-1},\alpha_\ell,\alpha_{i+1},\ldots\alpha_n)$.

The anticommutation relations of the operators B, D will be of the same type as for the operators b, d when the following relations hold between the M_i's :

$$\int d\alpha M_1(\gamma,\alpha)M_1^*(\gamma',\alpha) + \int d\beta M_2(\gamma,\beta)M_2^*(\gamma',\beta) = \delta_{\gamma\gamma'} \qquad (1.7a)$$

$$\int d\alpha M_3(\lambda,\alpha)M_3^*(\lambda',\alpha) + \int d\beta M_4(\lambda,\beta)M_4^*(\lambda',\beta) = \delta_{\lambda\lambda'} \qquad (1.7b)$$

$$\int d\alpha M_1(\gamma,\alpha)M_3^*(\lambda,\alpha) + \int d\beta M_2(\gamma,\beta)M_4^*(\lambda,\beta) = 0 \ . \qquad (1.7c)$$

By demanding that the transformation (1.1) had an inverse and using the relations (1.7) together with the similar relations for the inverse transformation, we can see that the inverse is the adjoint of the transformation (1.1), i.e.

$$b(\alpha) = \int d\gamma M_1^*(\gamma,\alpha)B(\gamma) + \int d\lambda M_3^*(\lambda,\alpha)D^\dagger(\lambda)$$

$$d^\dagger(\beta) = \int d\gamma M_2^*(\gamma,\beta)B(\gamma) + \int d\lambda M_4^*(\lambda,\beta)D^\dagger(\lambda) \qquad (1.8)$$

with

$$\int d\gamma M_1(\gamma,\alpha)M_1^*(\gamma,\alpha') + \int d\lambda M_3(\lambda,\alpha)M_3^*(\lambda,\alpha') = \delta_{\alpha\alpha'} \qquad (1.9a)$$

$$\int d\gamma M_2(\gamma,\beta)M_2^*(\gamma,\beta') + \int d\lambda M_4(\lambda,\beta)M_4^*(\lambda,\beta') = \delta_{\beta\beta'} \qquad (1.9b)$$

$$\int d\gamma M_1(\gamma,\alpha)M_2^*(\gamma,\beta) + \int d\lambda M_3(\lambda,\alpha)M_4^*(\lambda,\beta) = 0 \, . \qquad (1.9c)$$

It is easily seen that the requirements (1.7) and (1.9) are meaningful if and only if the M_i's are all integral kernels of transformations of square integrable functions of the indices into other square integrable functions. This is also sufficient for the B, $B^\dagger$, D, $D^\dagger$ to be well defined operator valued distributions on all $\mathcal{H}$.

2) Necessary Conditions for Existence of a New Vacuum

We want to examine under what conditions a state $|\phi_o\rangle \in \mathcal{H}$ will exist such that

$$B(\gamma)|\phi_o\rangle = 0 \;\forall\; \gamma \quad \text{and} \quad D(\lambda)|\phi_o\rangle = 0 \;\forall\; \lambda\,. \tag{2.1}$$

Writing

$$|\phi_o\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{\sqrt{n!m!}} \int d\underline{\alpha}_n\, d\underline{\beta}_m\, \psi^{nm}(\underline{\alpha}_n;\underline{\beta}_m)\, |\underline{\alpha}_n;\underline{\beta}_m\rangle \tag{2.2}$$

and using $B(\gamma)$ as given in equation (1.1), we obtain that the following relations between the ψ^{nm}'s must exist in order for $B(\gamma)|\phi_o\rangle$ to be null.

$$\int d\alpha_j M_1(\gamma,\alpha_j)\psi^{nm}(\underline{\alpha}_n;\underline{\beta}_m) = \frac{1}{\sqrt{mn}}\sum_{i=1}^{m} (-1)^{n+i+j}\, M_2(\gamma,\beta_i) \times$$

$$\times\; \psi^{n-1\;m-1}(\underset{\hat{j}}{\underline{\alpha}}n;\; \underset{\hat{i}}{\underline{\beta}}m) \tag{2.3}$$

for $n=1,2,\ldots,$ $m=1,2,\ldots,$ α_j being any one of the $\underline{\alpha}_n$ and

$$\int d\alpha_j M_1(\gamma,\alpha_j)\psi^{no}(\underline{\alpha}_n;-) = 0 \quad \text{for} \quad n=1,2,\ldots\,. \tag{2.4}$$

Similarly, $D(\lambda)|\phi_o\rangle = 0$ leads to

$$\int d\beta_j M_4^*(\lambda,\beta_j)\psi^{nm}(\underline{\alpha}_n;\underline{\beta}_m) = \frac{1}{\sqrt{mn}}\sum_{i=1}^{n} (-1)^{n+i+j+1}\, M_3^*(\lambda,\alpha_i) \times$$

$$\times\; \psi^{n-1\;m-1}(\underset{\hat{i}}{\underline{\alpha}}n;\; \underset{\hat{j}}{\underline{\beta}}m) \tag{2.5}$$

for $n=1,2,\ldots,$ $m=1,2,\ldots,$ β_j being any one of the $\underline{\beta}_m$ and

$$\int d\beta_j M_4^*(\lambda;\beta_j)\psi^{om}(-;\underline{\beta}_m) = 0 \quad \text{for} \quad m=1,2,\ldots \quad (2.6)$$

We can find immediately a necessary condition for the equations (2.3) and (2.5) to have square integrable solutions ψ^{nm}. (Square integrability of ψ^{nm} for each n and m being required by relation (1.4)).

From the equations (1.9), we have

$$\int d\gamma \left| \int d\alpha M_1(\gamma,\alpha)\psi(\alpha)\right|^2 \leq \int d\alpha |\psi(\alpha)|^2 \quad (2.7)$$

$$\int d\gamma \left| \int d\beta M_2(\gamma,\beta)\psi(\beta)\right|^2 \leq \int d\beta |\psi(\beta)|^2 \quad . \quad (2.8)$$

Equation (2.7) implies that ψ^{nm} can be square integrable only if the right hand side of (2.3) is square integrable, i.e. we need

$$\int d\gamma \; \underset{\hat{j}}{d\underline{\alpha}_n} \; d\underline{\beta}_m \left| \frac{1}{\sqrt{mn}} \sum_{i=1}^{m} (-1)^{n+i+j} M_2(\gamma,\beta_i) \times \right.$$

$$\left. \times \; \psi^{n-1\; m-1}(\underset{\hat{j}}{\underline{\alpha}_n} \; ; \; \underset{\hat{i}}{\underline{\beta}_m}) \right|^2 < \infty \; .$$

This is

$$\frac{1}{mn} \int d\gamma \; \underset{\hat{j}}{d\underline{\alpha}_n} \; d\underline{\beta}_m \sum_{i=1}^{m} |M_2(\gamma,\beta_i)|^2 \; |\psi^{n-1\; m-1}(\underset{\hat{j}}{\underline{\alpha}_n} \; ; \; \underset{\hat{i}}{\underline{\beta}_m})|^2$$

$$+ \frac{1}{mn} \sum_{i' \neq i=1}^{m} \sum_{i'=1}^{m} \int d\gamma \; \underset{\hat{j}}{d\underline{\alpha}_n} \; \underset{\hat{i}\hat{i}'}{d\underline{\beta}_m} \left(\int d\beta_i \psi^{n-1\; m-1}(\underset{\hat{j}}{\underline{\alpha}_n}; \underset{\hat{i}'}{\underline{\beta}_m})^* M_2(\gamma,\beta_i) \right) \times$$

$$\times \left(\int d\beta_{i'} M_2^*(\gamma,\beta_{i'})\psi^{n-1\; m-1}(\underset{\hat{j}}{\underline{\alpha}_n}; \underset{\hat{i}}{\underline{\beta}_m}) \right) < \infty$$

and supposing that $\psi^{n-1\ m-1}$ was non-zero and square integrable; because of equation (2.8), the second integral above is finite so that in fact we need the first term to be finite, i.e.

$$\int d\gamma d\beta |M_2(\gamma,\beta)|^2 < \infty \quad . \tag{2.9}$$

Repeating the same argument with equation (2.5), we can see that we need also

$$\int d\lambda d\alpha |M_3(\lambda,\alpha)|^2 < \infty \quad . \tag{2.10}$$

3) Properties of the Kernels of the Integral Equations Obtained

In the usual treatments, it is assumed that the integral operators M_1 and M_4^* have inverses so that a unique solution to the system of equations (2.3)-(2.6) is immediately determined by the action of the inverse operators. For example, if we consider the equation of type (2.3):

$$\int d\alpha M_1(\gamma,\alpha)\psi(\alpha) = \phi(\gamma) \quad \text{with} \quad \phi(\gamma) \in L^2 , \qquad (3.1)$$

the solution is written simply (formally) as $\psi = M_1^{-1}\phi$. In the present case however, we do not assume that M_1^{-1} nor $(M_4^*)^{-1}$ exist so that we need to study these equations more carefully.

We will derive in this section various general properties of the integral operators involved in the transformation (1.1). The conditions under which there will exist solutions to the one variable integral equations (as (3.1)) will be found. We will give also explicitly these solutions in the form of series the properties of which will be discussed in detail. Even when M_1 and M_4^* have inverses, the solutions in this form are also more useful than the symbolic $M_1^{-1}\phi$, for example.

Let us start by examining equation (3.1).

i) If $\int d\gamma M_1^*(\gamma,\alpha)\phi(\gamma) = 0$, then

$$\int d\gamma \left| \int d\alpha M_1(\gamma,\alpha)\psi(\alpha) \right|^2 = \int d\alpha'\psi^*(\alpha') \int d\gamma M_1^*(\gamma,\alpha') \int d\alpha M_1(\gamma,\alpha)\psi(\alpha)$$

$$= \int d\alpha'\psi^*(\alpha') \int d\gamma M_1^*(\gamma,\alpha')\phi(\gamma) = 0$$

so that there could exist a solution to (3.1) only if $\phi(\gamma) \equiv 0$.

ii) Let us now consider the case where $M_1^*\phi \neq 0$. Equation (3.1) implies

$$\int d\gamma M_1^*(\gamma,\alpha') \int d\alpha M_1(\gamma,\alpha)\psi(\alpha) = \int d\gamma M_1^*(\gamma,\alpha')\phi(\gamma)$$

which by equation (1.9a) is

$$\int d\alpha [\delta_{\alpha'\alpha} - \int d\lambda M_3^*(\lambda,\alpha')M_3(\lambda,\alpha)]\psi(\alpha) = \int d\gamma M_1^*(\gamma,\alpha')\phi(\gamma) .$$

We will write this simply as

$$(I - M_3^*M_3)\psi = M_1^*\phi \quad . \tag{3.2}$$

This integral equation has a simple nature and standard techniques can be used to deal with it (cf. for example Smithies [1962]). The kernel $(M_3^*M_3)(\alpha',\alpha) = \int d\lambda M_3^*(\lambda,\alpha')M_3(\lambda,\alpha)$ is a non-null hermitian L^2 kernel. It therefore admits the spectral decomposition

$$(M_3^*M_3)(\alpha',\alpha) = \sum_{n=1}^{\infty} \frac{1}{\Lambda_n} X_n(\alpha')X_n^*(\alpha) \tag{3.3}$$

where (Λ_n, X_n) is the characteristic system defined by

$$\int d\alpha\,(M_3^* M_3)(\alpha',\alpha) X_n(\alpha) = \frac{1}{\Lambda_n} X_n(\alpha') \tag{3.4}$$

and the series (3.3) is relatively uniformly absolutely convergent. We have

$$\int d\alpha d\alpha' \,|(M_3^* M_3)(\alpha',\alpha)|^2 = \sum_{n=1}^{\infty} \left(\frac{1}{\Lambda_n}\right)^2 < \infty \quad . \tag{3.5}$$

The equation (1.9a) implies $1/\Lambda_n = 1 - ||M_1 X_n||^2$ and equation (3.4): $1/\Lambda_n = ||M_3 X_n||^2$ so that

$$0 \leq \frac{1}{\Lambda_n} \leq 1 \quad . \tag{3.6}$$

Because

$$\sum_{n=1}^{\infty} \left(\frac{1}{\Lambda_n}\right)^2 < \infty \quad ,$$

where each characteristic value in the sequence is included a number of times equal to its rank, there is necessarily only a finite number of X_n corresponding to the same given Λ_n. In particular, we shall call N the number of eigenfunctions with eigenvalue 1 and denote these by $G_n'(\alpha) = X_n(\alpha)$, n=1,2,...N. These satisfy the equation

$$\int d\alpha [\delta_{\alpha'\alpha} - (M_3^* M_3)(\alpha',\alpha)] G_n'(\alpha) = 0$$

or equivalently,

$$\int d\alpha M_1(\gamma,\alpha)G_n'(\alpha) = 0 \; . \tag{3.7}$$

iii) We shall now prove that the series

$$\psi_s = \sum_{r=0}^{\infty} (M_3^* M_3)^r M_1^* \phi \tag{3.8}$$

converges relatively uniformly absolutely. We have

$$(M_3^* M_3)M_1^*\phi = \sum_{n=1}^{\infty} \frac{1}{\Lambda_n} X_n(\alpha')\int d\alpha X_n^*(\alpha)\int d\gamma M_1^*(\gamma,\alpha)\phi(\gamma) \; ;$$

using the property (3.7) for X_n, $n=1,2,\ldots N$, this is

[illegible] $\frac{1}{\Lambda_n} X_n(\alpha')\int d\alpha X_n^*(\alpha)\int d\gamma M_1^*(\gamma,\alpha)\phi(\gamma)$. (3.9)

Upon de[illegible]

$$R([illegible]) = \sum_{n=N+1}^{\infty} \frac{1}{\Lambda_n} X_n(\alpha')X_n^*(\alpha) \; , \tag{3.10}$$

we can w[illegible] (3.9) as

$$(M_3^* M_3)M_1^*\phi = R\, M_1^*\phi$$

and

$$(M_3^* M_3)^r M_1^*\phi = R^r M_1^*\phi \; . \tag{3.11}$$

We therefore examine the series

$$\sum_{r=0}^{\infty} R^r M_1^*\phi \; .$$

By definition, it will be relatively uniformly absolutely convergent if the sequence

$$\{a_n(\alpha)\} = \{\sum_{r=0}^{n} |(R^r M_1^* \phi)(\alpha)|\}$$

is relatively uniformly convergent. We have here

$$|a_{n+1}(\alpha) - a_n(\alpha)| = |(R^{n+1} M_1^* \phi)(\alpha)|$$

$$= |\int d\alpha' R(\alpha,\alpha')(R^n M_1^* \phi)(\alpha')|.$$

and using the inequality $(f,h) \leq ||f|| \; ||h||$, we obtain

$$\leq [\int d\alpha' |R(\alpha,\alpha')|^2]^{\frac{1}{2}} ||R^n M_1^* \phi|| \quad . \quad (3.12)$$

Since for all f,

$$||Rf||^2 \leq \frac{1}{\Lambda_o^2} \; ||f||^2 \quad (3.13)$$

where $1/\Lambda_o$ is the biggest eigenvalue of R, we have

$$||R^n M_1^* \phi|| \leq (\frac{1}{\Lambda_o})^n \; ||M_1^* \phi|| \quad .$$

Equation (3.12) therefore implies (defining $r(\alpha) \equiv [\int d\alpha' |R(\alpha,\alpha')|^2]^{\frac{1}{2}}$)

$$|a_{n+1}(\alpha) - a_n(\alpha)| \leq (\frac{1}{\Lambda_o})^n \; ||M_1^* \phi|| r(\alpha) \quad . \quad (3.14)$$

For any given $\varepsilon > 0$, however small, we can find $n(\varepsilon)$, mainly

$$n(\varepsilon) = \frac{\ln \varepsilon}{\ln \frac{1}{\Lambda_o}} ,$$

such that for all $n \geq n(\varepsilon)$,

$$|a_{n+1}(\alpha) - a_n(\alpha)| \leq \varepsilon \, r'(\alpha)$$

where $r'(\alpha) = ||M_1^* \phi|| r(\alpha)$ is a positive definite L^2 function. This shows the relative uniform convergence of $\{a_n\}$ and thus

$$\sum_{r=0}^{\infty} R^r M_1^* \phi = \sum_{r=0}^{\infty} (M_3^* M_3)^r M_1^* \phi$$

converges relatively uniformly absolutely.

iv) The series ψ_s satisfies equation (3.2) and is orthogonal to all $G_i'(\alpha)$, $i=1,2,\ldots N$.

If $M_3 M_1^* \phi = 0$, the series ψ_s reduces to $M_1^* \phi$ which is then obviously a solution of (3.2).

If $M_3 M_1^* \phi \neq 0$, the series ψ_s is infinite since if we suppose $(M_3^* M_3)^r M_1^* \phi = 0$ for some r, then

$$(M_3^* M_3)^{2^r} M_1^* \phi = 0 . \qquad (3.15)$$

and

$$||(M_3^*M_3)^{2^{r-1}} M_1^*\phi||^2 = (M_1^*\phi, (M_3^*M_3)^{2^r} M_1^*\phi) = 0$$

$$\text{i.e.} \quad (M_3^*M_3)^{2^{r-1}} M_1^*\phi = 0 \ .$$

By repeating the argument starting at equation (3.15) r times, we obtain that we must have

$$(M_3^*M_3)M_1^*\phi = 0 \quad \text{i.e.} \quad M_3M_1^*\phi = 0$$

which is not the case here so that the series ψ_s is infinite. We therefore have

$$(I-M_3^*M_3)\psi_s = \sum_{r=0}^{\infty} (M_3^*M_3)^r M_1^*\phi - \sum_{r=1}^{\infty} (M_3^*M_3)^r M_1^*\phi$$

$$= M_1^*\phi \ .$$

The series ψ_s is then always a solution of equation (3.2). It is orthogonal to all $G_i'(\alpha)$, $i=1,2,\ldots N$ since because of the previously demonstrated relative uniform absolute convergence of the series ψ_s:

$$(G_i', \psi_s) = \sum_{r=0}^{\infty} (G_i', (M_3^*M_3)^r M_1^*\phi)$$

$$= \sum_{r=0}^{\infty} (G_i', R^r M_1^*\phi) = 0 \ . \qquad (3.16)$$

The most general solution of equation (3.2) is

$$\psi_g = \sum_{r=0}^{\infty} (M_3^* M_3)^r \, M_1^* \phi + \phi_o \tag{3.17}$$

where ϕ_o is such that $(I - M_3^* M_3)\phi_o = 0$.

v) We now examine when ψ_g will also satisfy equation (3.1). This is equivalent to ψ_s satisfying

$$\int d\alpha M_1(\gamma,\alpha)\psi_s(\alpha) = \phi(\gamma) \tag{3.18}$$

since $M_1\phi_o = 0$.

If ϕ is such that $M_3 M_1^* \phi = 0$, we know that $\psi_s = M_1^*\phi$ so that we must have

$$\int d\alpha M_1(\gamma,\alpha)\int d\gamma' M_1^*(\gamma',\alpha)\phi(\gamma') = \phi(\gamma)$$

or equivalently

$$\int d\gamma M_2^*(\gamma,\beta)\phi(\gamma) = 0 \ .$$

For the case where $M_3 M_1^*\phi = 0$, we start by proving that the identity

$$M_1(M_3^* M_3) = (M_2 M_2^*) M_1 \ . \tag{3.19}$$

For all ψ, we have by equation (1.7c)

$$\int d\alpha M_1(\gamma,\alpha)\int d\lambda M_3^*(\lambda,\alpha)\int d\alpha' M_3(\lambda,\alpha')\psi(\alpha') =$$

$$-\int d\beta M_2(\gamma,\beta)\int d\lambda M_4^*(\lambda,\beta)\int d\alpha' M_3(\lambda,\alpha')\psi(\alpha')$$

and by (1.9c)

$$= \int d\beta M_2(\gamma,\beta) \int d\gamma' M_2^*(\gamma',\beta) \int d\alpha' M_1(\gamma',\alpha')\psi(\alpha') \quad ;$$

this shows that (3.19) is true.

We therefore have, because of the relative uniform absolute convergence of ψ_s:

$$M_1\psi_s = \sum_{r=0}^{\infty} M_1(M_3^*M_3)^r M_1^*\phi = \sum_{r=0}^{\infty} (M_2M_2^*)^r M_1M_1^*\phi \quad . \qquad (3.20)$$

The kernel $(M_2M_2^*)(\gamma,\gamma') = \int d\beta M_2(\gamma,\beta)M_2^*(\gamma',\beta)$ has essentially the same properties as $(M_3^*M_3)$. In particular, it has only a finite number M of characteristic functions $F_n(\gamma)$, n=1,2,...M with eigenvalue 1. These satisfy

$$\int d\gamma' [\delta_{\gamma\gamma'} - (M_2M_2^*)(\gamma,\gamma')]F(\gamma') = 0$$

or equivalently

$$\int d\gamma' M_1^*(\gamma',\alpha)F(\gamma') = 0 \ .$$

$M_1M_1^*\phi$ being orthogonal to all such homogeneous solutions, the series (3.20) converges relatively uniformly absolutely. Using equation (1.7a), we have

$$M_1\psi_s = \sum_{r=0}^{\infty} (M_2M_2^*)^r \ (I - M_2M_2^*)\phi \quad . \qquad (3.21)$$

Using the identity

$$(I - M_2M_2^*)\phi = (I - M_2M_2^*)[\phi - \sum_{n=1}^{M} F_n(F_n,\phi)] , \qquad (3.22)$$

the above series is equal to

$$\sum_{r=0}^{\infty} (M_2M_2^*)^r [I - M_2M_2^*][\phi - \sum_{n=1}^{M} F_n(F_n,\phi)]$$

$$= \sum_{r=0}^{\infty} (M_2M_2^*)^r [\phi - \sum_{n=1}^{M} F_n(F_n,\phi)] - \sum_{r=1}^{\infty} (M_2M_2^*)^r [\phi - \sum_{n=1}^{M} F_n(F_n,\phi)] .$$

These two series converge relatively uniformly absolutely since

$$[\phi - \sum_{n=1}^{M} F_n(F_n,\phi)]$$

is orthogonal to all F_n, n=1,2,...M so that we finally obtain

$$M_1\psi_s = \phi - \sum_{n=1}^{M} F_n(F_n,\phi) . \qquad (3.23)$$

Evidently, equation (3.18) is satisfied if and only if

$$(F_n,\phi) = 0 \quad \text{for all } n=1,2,...M .$$

vi) In the case of equation (2.3) where ϕ has the form

$$\phi(\gamma) = \sum_{i=1}^{m} M_2(\gamma,\beta_i)\phi_i ,$$

we show that $M_3M_1^*\phi = 0$ can happen only when $M_1^*\phi = 0$. Suppose

$$\int d\alpha M_3(\lambda,\alpha)\int d\gamma M_1^*(\gamma,\alpha)\sum_{i=1}^{m} M_2(\gamma,\beta_i)\phi_i = 0 ,$$

then, by equation (1.9c),

$$\int d\alpha M_3(\lambda,\alpha)\int d\lambda' M_3^*(\lambda',\alpha)\sum_{i=1}^{m} M_4(\lambda',\beta_i)\phi_i = 0$$

i.e.

$$\int d\lambda' M_3^*(\lambda',\alpha)\sum_{i=1}^{m} M_4(\lambda',\beta_i)\phi_i = 0 .$$

Using again equation (1.9c), this means

$$\int d\gamma M_1^*(\gamma,\alpha)\sum_{i=1}^{m} M_2(\gamma,\beta_i)\phi_i = 0$$

i.e. $M_1^*\phi = 0$.

vii) We have shown the following results concerning equation (2.3). Calling the term on its right hand side $\phi(\gamma)$, if $\int d\gamma M_1^*(\gamma,\alpha)\phi(\gamma) = 0$, there can exist a solution only if $\phi(\gamma) = 0$. If $\int d\gamma M_1^*(\gamma,\alpha)\phi(\gamma) \neq 0$, there exists a solution if and only if $\phi(\gamma)$ is orthogonal to all $F_n(\gamma)$, $n=1,2,\ldots M$ (if these exist) such that $\int d\beta M_2(\gamma,\beta)\int d\gamma' M_2^*(\gamma',\beta)F(\gamma') = F(\gamma)$ or equivalently $\int d\gamma M_1^*(\gamma,\alpha)F(\gamma) = 0$. It then has the form (3.17).

viii) The homogeneous solutions of the two equations

$$\int d\beta M_4^*(\lambda,\beta)G(\beta) = 0 \qquad (3.24)$$

and

$$\int d\gamma M_1^*(\gamma,\alpha)F(\gamma) = 0 \qquad (3.25)$$

are in one to one correspondence.

Suppose a non zero $G(\beta)$ satisfies equation (3.24), then

$$\int d\lambda M_4(\lambda,\beta')\int d\beta M_4^*(\lambda,\beta)G(\beta) = 0$$

and using equation (1.9b),

$$\int d\gamma M_2(\gamma,\beta')\int d\beta M_2^*(\gamma,\beta)G(\beta) = G(\beta')$$

so that $\int d\beta M_2^*(\gamma,\beta)G(\beta) \neq 0$ and by equation (1.9c)

$$\int d\gamma M_1^*(\gamma,\alpha)\left[\int d\beta M_2(\gamma,\beta)G^*(\beta)\right] = -\int d\lambda M_3^*(\lambda,\alpha)\int d\beta M_4(\lambda,\beta)G^*(\beta)$$

which is null by hypothesis. For each $G(\beta)$ satisfying the equation (3.24), the function $\int d\beta M_2(\gamma,\beta)G^*(\beta)$ is non zero and satisfies the equation (3.25).

Similarly, if $F(\gamma) \neq 0$ satisfies equation (3.25), using (1.7a) we can see that $\int d\gamma M_2^*(\gamma,\beta)F(\gamma) \neq 0$ and, because of (1.7c), this satisfies equation (3.24).

Since there are M $F_n(\gamma)$, n=1,2,...M satisfying $M_1^*F = 0$, there are M $G_n(\beta)$, n=1,2,...M satisfying $M_4^*G = 0$.

A similar one to one correspondence can be established between the N solutions $G_n'(\alpha)$, n=1,2,...N of $\int d\alpha M_1(\gamma,\alpha)G'(\alpha) = 0$ and the solutions $F'(\lambda)$ of $\int d\lambda M_4(\lambda,\beta)F'(\lambda) = 0$.

ix) The following results concerning equation (2.5) can be obtained in a similar manner as those for (2.3). Writing the right hand side of (2.5) as $\chi(\lambda)$, if $\int d\lambda M_4(\lambda,\beta)\chi(\lambda) = 0$, there can exist a solution only if $\chi(\lambda) = 0$. If $\int d\lambda M_4(\lambda,\beta)\chi(\lambda) \neq 0$, there exists a solution if and only if $\chi(\lambda)$ is orthogonal to all N $F'(\lambda)$ (if these exist) such that

$$\int d\alpha M_3^*(\lambda,\alpha)\int d\lambda' M_3(\lambda',\alpha)F'(\lambda') = F'(\lambda)$$

or equivalently

$$\int d\lambda M_4(\lambda,\beta)F'(\lambda) = 0 .$$

It then has the form

$$\psi_g = \sum_{r=0}^{\infty} (M_2M_2^*)^r M_4\chi + \chi_o$$

where χ_o satisfies $(I - M_2M_2^*)\chi_o = 0$, i.e.

$$\int d\gamma M_2(\gamma,\beta)\int d\beta' M_2^*(\gamma,\beta')\chi_o(\beta') = \chi_o(\beta)$$

or equivalently

$$\int d\beta M_4^*(\lambda,\beta)\chi_o(\beta) = 0 .$$

4) A General Property of Antisymmetric Functions

We prove the following theorem. There exists an antisymmetric function G^n of n variables $x_1, x_2, \ldots, x_n$ satisfying the equation

$$0(x_j)\ G^n(\underline{x}_n) = 0 \quad \forall\ j = 1,2,\ldots,n \qquad (4.1)$$

where $0(x_j)$ is a linear operator depending only on the jth variable x_j if and only if there exist n linearly independent functions of one variable: $G_i(x)$, $i=1,2,\ldots,n$ satisfying

$$0(x)\ G_i(x) = 0. \qquad (4.2)$$

Trivially, if there exist n $G_i(x)$ satisfying equation (4.2), the function

$$F(\underline{x}_n) = \det \begin{vmatrix} G_1(x_1)G_1(x_2) & \ldots & G_1(x_n) \\ G_2(x_1)G_2(x_2) & \ldots & G_2(x_n) \\ \vdots & & \\ G_n(x_1)G_n(x_2) & \ldots & G_n(x_n) \end{vmatrix} \qquad (4.3)$$

is an antisymmetric function of the $\underline{x}_n$ and satisfies equation (4.1).

Let us now suppose that a function $G^n(\underline{x}_n)$ satisfies (4.1); we want to show that we can generate from it n linearly independent $G_i(x)$.

$G^n(\underline{x}_n)$ being non-zero, there always exists a square integrable function $f(x_n)$ such that $\int dx_n f^*(x_n) G^n(\underline{x}_n) \neq 0$; we call $f_{n-1}(x_n)$ one such function and define

$$\int dx_n f^*_{n-1}(x_n) G^n(\underline{x}_n) = G^{n-1}(\underline{x}_{n-1}) \neq 0 \ . \tag{4.4}$$

We repeat the same argument with G^{n-1}, G^{n-2} etc. and define a sequence of non-zero functions

$$\int dx_{n-1} f^*_{n-2}(x_{n-1}) G^{n-1}(\underline{x}_{n-1}) = G^{n-2}(\underline{x}_{n-2}),$$

$$\vdots \qquad \qquad \vdots$$

$$\int dx_{n-i+1} f^*_{n-i}(x_{n-i+1}) G^{n-i+1}(\underline{x}_{n-i+1}) = G^{n-i}(\underline{x}_{n-i}) \tag{4.5}$$

$$\vdots \qquad \qquad \vdots$$

$$\int dx_2 \ f^*_1(x_2) \ G^2(x_1, x_2) = G^1(x_1) \ . \tag{4.6}$$

All these functions are square integrable, antisymmetric and satisfy equation (4.1).

$G^1(x_1)$ is the first of the functions we wanted. We will use $G_1(x_1) = G^1(x_1)$.

Another such functions is G_2 defined as

$$\int dx_1 G_1(x_1)^* \ G^2(x_1, x_2) = G_2(x_2) \ .$$

It is non-zero since using the definition of $G_1(x)$, we have

$$(f_1, G_2) = \int dx_1 |G_1(x_1)|^2 \neq 0 \ .$$

It is also orthogonal to G_1 since

$$(G_1,G_2) = \int dx_1 dx_2 G_1(x_1)^* G_1(x_2)^* G^2(x_1,x_2) \quad .$$

By relabelling the variables of integration, this must be equal to $\int dx_1 dx_2 G_1(x_1)^* G_1(x_2)^* G^2(x_2,x_1)$ but because of the antisymmetry of G^2, this is equal to $-\int dx_1 dx_2 G_1(x_1)^* G_1(x_2)^* \times G^2(x_1,x_2)$ so that $(G_1,G_2) = 0$. We then have two linearly independent solutions.

Let us suppose that we have successfully constructed i such non-zero orthogonal functions $G_1(x), G_2(x), \ldots G_i(x)$ with the first $G^i(\underline{x}_i)$ as

$$G_i(x_j) = (-1)^{i+j} \int d\underset{\hat{j}}{x_i}\; F^{i-1}(\underset{\hat{j}}{\underline{x}_i})^*\; G^i(\underline{x}_i) \qquad (4.7)$$

where

$$F^{i-1}(\underline{x}_{i-1}) = \det \begin{pmatrix} G_1(x_1) G_1(x_2) & \ldots & G_1(x_{i-1}) \\ G_2(x_1) G_2(x_2) & \ldots & G_2(x_{i-1}) \\ \vdots & & \vdots \\ G_{i-1}(x_1) G_{i-1}(x_2) & \ldots & G_{i-1}(x_{i-1}) \end{pmatrix} \qquad (4.8)$$

We shall prove that a non-zero independent G_{i+1} can be defined in the same way.

We define

$$G_{i+1}(x_j) = (-1)^{i+j+1} \int d\underset{\hat{j}}{x_{i+1}}\; F^i\; (\underset{\hat{j}}{\underline{x}_{i+1}})^*\; G^{i+1}(\underline{x}_{i+1}) \qquad (4.9)$$

where $F^i(\underline{x}_i)$ is a determinant similar to (4.8).

$$F^{i}(\underline{x}_i) = \sum_{j=1}^{i} (-1)^{i+j} G_i(x_j) F^{i-1}(\underline{x}_{\hat{j}}i) . \qquad (4.10)$$

We first prove that G_{i+1} cannot be zero since

$$\int \underline{dx}_i F^{i}(\underline{x}_i)^* G^{i+1}(\underline{x}_{i+1}) = 0$$

implies

$$\int \underline{dx}_{i+1} f_i^*(x_{i+1}) F^{i}(\underline{x}_i)^* G^{i+1}(\underline{x}_{i+1}) = 0$$

which by definition of $G^{i}(\underline{x}_i)$ is

$$\int \underline{dx}_i F^{i}(\underline{x}_i)^* G^{i}(\underline{x}_i) = 0 .$$

Upon replacing F^i by its explicit expression (4.10), we obtain

$$\sum_{j=1}^{i} (-1)^{i+j} \int \underline{dx}_i G_i(x_j)^* F^{i-1}(\underline{x}_{\hat{j}}i)^* G^{i}(\underline{x}_i) = 0$$

which by equation (4.7) is simply

$$\sum_{j=1}^{i} \int dx_j G_i(x_j)^* G_i(x_j) = 0$$

i.e. $\quad i \, ||G_i||^2 = 0$

which we know does not hold. We therefore have $G_{i+1} \neq 0$.

We now show that G_{i+1} is orthogonal to all G_ℓ, $\ell = 1,2,\ldots,i$. We have

$$(G_\ell, G_{i+1}) = \int dx_j G_\ell(x_j)^* G_{i+1}(x_j) = (-1)^{i+j+1} \int \underline{dx}_{i+1} \times$$
$$G_\ell(x_j)^* F^{i}(\underline{x}_{\hat{j}}i+1)^* G^{i+1}(\underline{x}_{i+1}) .$$

We now use [illegible] id[illegible]

$$\sum_{j=1}^{i} \text{[illegible]}^{+1} \text{[illegible]}_j) F^i(\underline{x}_{\hat{j}}\, i+1) = 0 \;,$$

which holds s[illegible] left hand side of this is a determinant of the [illegible].8) where the two rows with G_ℓ are repeated; we o[illegible]n

$$\sum_{j=1}^{i} (G_\ell, \text{[illegible]}) = 0 \qquad \text{i.e.} \quad (G_\ell, G_{i+1}) = 0.$$

We have thus bee[illegible]le to define a new function $G_{i+1}(x)$ which satisfies equ[illegible]ion (4.2).

Since the a[illegible] procedure can be repeated as many times as we have [illegible]ns $G^i(\underline{x}_i)$, we can obtain n orthogonal solutions t[illegible] (4.2): $G_i(x)$, $i=1,2,\ldots,n$; one corresponding to each $G^i(\underline{x}_i)$.

5) Vanishing Vacuum Amplitudes

In this section we show that the only possible non-zero solutions to the system of equations (2.3)-(2.6) are the ψ^{nm} generated from $\psi^{NM} = c\,\phi'^{N}(\underline{\alpha}_N)\,\phi^{M}(\underline{\beta}_M)$ where c is an arbitrary constant. If N=0, $\phi'^{N} = 1$, otherwise

$$\phi'^{N}(\underline{\alpha}_N) = \det \begin{pmatrix} G_1'(\alpha_1) G_2'(\alpha_1) & \cdots & G_N'(\alpha_1) \\ G_1'(\alpha_2) G_2'(\alpha_2) & \cdots & G_N'(\alpha_2) \\ \vdots & & \vdots \\ G_1'(\alpha_N) G_2'(\alpha_N) & \cdots & G_N'(\alpha_N) \end{pmatrix} \equiv \det G'(\underline{\alpha}_N)$$

where $G_k'(\alpha)$, k=1,2,... N are the N linearly independent homogeneous solutions of

$$\int d\alpha\; M_1(\gamma,\alpha)\; G'(\alpha) = 0 \;.$$

If M = 0, $\phi^{M} = 1$, otherwise,

$$\phi^{M}(\underline{\beta}_M) = \det \begin{pmatrix} G_1(\beta_1) G_1(\beta_2) & \cdots & G_1(\beta_M) \\ G_2(\beta_1) G_2(\beta_2) & \cdots & G_2(\beta_M) \\ \vdots & & \vdots \\ G_M(\beta_1) G_M(\beta_2) & \cdots & G_M(\beta_M) \end{pmatrix} \equiv \det G(\underline{\beta}_M)$$

where $G_k(\beta)$, k=1,2,...M are the M linearly independent solutions of

$$\int d\beta\; M_4^*(\lambda,\beta)\; G(\beta) = 0 \;.$$

i) Using the theorem proved in section 4), we can immediately deduce that $\psi^{nm} = 0$ is the only possible solution when $(n-m) > N$, $m=0,1,2,\ldots$ or $(m-n) > M$, $n=0,1,\ldots$

In order to show this, let us consider ψ^{nm} where $(n-m) > N$. Because of the recursion type equations (2.3) and (2.5) which they must satisfy, these are generated from $\psi^{n-m,o}(\underline{\alpha}_{n-m};-)$ such that

$$\int d\alpha_j \, M_1(\gamma,\alpha_j)\psi^{n-m,o}(\underline{\alpha}_{n-m};-) = 0 \, .$$

The only possible solution to this equation however is $\psi^{n-m,o} = 0$. The next amplitude $\psi^{n-m+1,1}$ must then, by equation (2.3) satisfy

$$\int d\alpha_j \; M_1(\gamma,\alpha_j)\psi^{n-m+1,1}(\underline{\alpha}_{n-m+1};\beta_1) = 0$$

so that it also can only be null. By repeating the same argument, we easily obtain by induction that all ψ^{nm} where $(n-m) > N$, $m=0,1,2,\ldots$ must vanish.

The proof for $\psi^{nm} = 0$ when $(m-n) > M$, $n=0,1,2,\ldots$ can be done in a similar manner, using the equations (2.5) and (2.6).

ii) Let us examine the equations for the amplitudes ψ^{nm} with $(m-n) \leq M$ and $m > n$, i.e. $n < m \leq M+n$. Obviously, this is necessary only if $M \neq 0$ since otherwise, there exist no m such that $n < m \leq n$.

These amplitudes ψ^{nm} are generated from ψ^{or}, where $r = m-n$, which satisfies

$$\int d\beta_j \cdot M_4^*(\lambda,\beta_j)\,\psi^{or}(-;\underline{\beta}_r) = 0 \qquad (5.1)$$

and must themselves satisfy the equations (2.3) and (2.5).

The equation (2.3) for $\psi^{1,r+1}$ is

$$\int d\alpha_1 M_1(\gamma,\alpha_1)\psi^{1,r+1}(\alpha_1;\underline{\beta}_{r+1}) = \frac{1}{\sqrt{r+1}} \sum_{i=1}^{r+1} (-1)^{1+i+1} \times$$

$$M_2(\gamma,\beta_i)\psi^{or}(-;\underset{\hat{i}}{\underline{\beta}}{}_{r+1}) \quad . \qquad (5.2)$$

We now use the results of section 3) to examine when solutions $\psi^{1,r+1}$ to (5.2) will exist.

We first have to see whether

$$\int d\gamma M_1^*(\gamma,\alpha) \left(\sum_{i=1}^{r+1} (-1)^i \; M_2(\gamma,\beta_i)\psi^{or}(-;\underset{\hat{i}}{\underline{\beta}}{}_{r+1})\right) = 0 \;. \quad (5.3)$$

If this is so, then

$$\int d\beta_j M_4^*(\lambda,\beta_j) \sum_{i=1}^{r+1} (-1)^i \int d\gamma M_1^*(\gamma,\alpha) M_2(\gamma,\beta_i)\psi^{or}(-;\underset{\hat{i}}{\underline{\beta}}{}_{r+1}) = 0 \;\forall\; j \;;$$

all terms in the sum with $i \neq j$ vanish because of equation (5.1) so that we must have

$$\left(\int d\gamma M_1^*(\gamma,\alpha)\int d\beta_j M_2(\gamma,\beta_j) M_4^*(\lambda,\beta_j)\right)\psi^{or}(-;\underset{\hat{j}}{\underline{\beta}}{}_{r+1}) = 0 \;. \quad (5.4)$$

It can easily be seen from the equations (1.7) and (1.9) that $[\int d\gamma M_1^*(\gamma,\alpha)\int d\beta_j M_2(\gamma,\beta_j) M_4^*(\lambda,\beta_j)] = 0$ implies either $\int d\lambda M_3^*(\lambda,\alpha) M_3(\lambda,\alpha') = \delta_{\alpha\alpha'}$, which is not possible if $||M_3|| < \infty$, or $M_3(\lambda,\alpha) = 0$ and $M_2(\gamma,\beta) = 0$, which is a trivial case for which we know the unique solution will be $|\phi_o\rangle \doteq |0\rangle$. This is then not null for the general case where M_3 and $M_2 \neq 0$, (5.3) holds then if and only if ψ^{or} is taken to be null.

If this is the case, i.e. ψ^{or} is taken zero, we must examine the equation for $\psi^{2,r+2}$. These are (2.3) and (2.5) with $\psi^{1,r+1}$ satisfying

$$\int d\alpha_1 M_1(\gamma,\alpha_1)\psi^{1,r+1}(\alpha_1;\underline{\beta}_{r+1}) = 0 \qquad (5.5)$$

$$\int d\beta_j M_4^*(\lambda,\beta_j)\psi^{1,r+1}(\alpha_1;\underline{\beta}_{r+1}) = 0 \quad \forall\, j \qquad (5.6)$$

We see that whether ψ^{or} is taken null or not, we still have to examine equations of the type

$$\int d\alpha_j M_1(\gamma,\alpha_j)\psi^{\ell+1,r+\ell+1}(\underline{\alpha}_{\ell+1};\underline{\beta}_{r+\ell+1}) = \frac{1}{\sqrt{(\ell+1)(r+\ell+1)}} \times \sum_{i=1}^{r+\ell+1} (-1)^{\ell+1+i+j} M_2(\gamma,\beta_i)\psi^{\ell,r+\ell}(\underset{\hat{j}}{\alpha}\ell+1;\underset{\hat{i}}{\beta}r+\ell+1) \qquad (5.7)$$

$$\int d\beta_j M_4^*(\lambda,\beta_j)\psi^{\ell+1,r+\ell+1}(\underline{\alpha}_{\ell+1};\underline{\beta}_{r+\ell+1}) = \frac{1}{\sqrt{(\ell+1)(r+\ell+1)}} \times \sum_{i=1}^{\ell+1} (-1)^{\ell+i+j} M_3^*(\lambda,\alpha_i)\psi^{\ell,r+\ell}(\underset{\hat{i}}{\alpha}\ell+1;\underset{\hat{j}}{\beta}r+\ell+1) \qquad (5.8)$$

where $\psi^{\ell,r+\ell}$ satisfies

$$\int d\beta_j M_4^*(\lambda,\beta_j)\psi^{\ell,r+\ell}(\underline{\alpha}_\ell;\underline{\beta}_{r+\ell}) = 0 \quad \forall\, j \qquad (5.9)$$

and also, if $\psi^{\ell,r+\ell}$ are taken zero up to $(\ell-1)$, $(\ell>0)$:

$$\int d\alpha_j M_1(\gamma,\alpha_j)\psi^{\ell,r+\ell}(\underline{\alpha}_\ell;\underline{\beta}_{r+\ell}) = 0 \;\forall\; j \quad . \tag{5.10}$$

We note that if for all $\ell=0,1,2,\ldots,M-r$, $\psi^{\ell,r+\ell}$ are taken zero, the only possibility for all subsequent $\psi^{\ell,r+\ell}$, $\ell > M-r$ is also zero since (5.9) does not have non zero antisymmetric solutions with more than M variables β.

If on the other hand, for some $\ell \leq M-r$, $\psi^{\ell,r+\ell}$ is taken non zero, $\int d\gamma M_1^*(\gamma,\alpha)$ r.h.s. (5.7) $\neq 0$ by the same argument used above starting at equation (5.3).

We now have to examine whether the r.h.s.(5.7) is orthogonal to all M $F_k(\gamma)$'s such that $\int d\gamma M_1^*(\gamma,\alpha)F_k(\gamma) = 0$ (or equivalently $\int d\beta M_2(\gamma',\beta)\int d\gamma M_2^*(\gamma,\beta)F_k(\gamma) = F_k(\gamma')$) since we know from section 3) that this is the necessary and sufficient condition for the existence of a solution $\psi^{\ell+1,r+\ell+1}$. That is, we need

$$\sum_{i=1}^{r+\ell+1} \int d\gamma F_k^*(\gamma)M_2(\gamma,\beta_i)\psi^{\ell,r+\ell}(\underline{\alpha}_\ell;\; \underset{\hat{i}}{\underline{\beta}}{}_{r+\ell+1}) = 0 \;\forall\; F_k \tag{5.11}$$

We have previously shown that $\int d\gamma F_k^*(\gamma)M_2(\gamma,\beta_i) = G_k(\beta_i)$ is a solution of $\int d\beta_i M_4^*(\lambda,\beta_i)G(\beta_i) = 0$ and if the F_k are taken orthonormal, so will be the $G_k(\beta_i)$. Equation (5.11) is then

$$\sum_{i=1}^{r+\ell+1} (-1)^i\; G_k(\beta_i)\psi^{\ell,r+\ell}(\underline{\alpha}_\ell;\; \underset{\hat{i}}{\underline{\beta}}{}_{r+\ell+1}) = 0 \;\forall\; G_k \tag{5.12}$$

which written in matrix form (we get one row for each G_k) is

$$\begin{pmatrix} G_1(\beta_1)G_1(\beta_2) & \cdots & G_1(\beta_{r+\ell+1}) \\ G_2(\beta_1)G_2(\beta_2) & \cdots & G_2(\beta_{r+\ell+1}) \\ \vdots & & \vdots \\ G_M(\beta_1)G_M(\beta_2) & \cdots & G_M(\beta_{r+\ell+1}) \end{pmatrix} \begin{pmatrix} (-1)^1 \psi^{\ell,r+\ell}(\underline{\alpha}_\ell;\ \underline{\beta}_{\hat{1}\,r+\ell+1}) \\ (-1)^2 \psi^{\ell,r+\ell}(\underline{\alpha}_\ell;\ \underline{\beta}_{\hat{2}\,r+\ell+1}) \\ \vdots \\ (-1)^{r+\ell+1}\psi^{\ell,r+\ell}(\underline{\alpha}_\ell;\underline{\beta}_{r+\ell}) \end{pmatrix} = 0 \ . \tag{5.13}$$

Obviously, if $M \geq r+\ell+1$, we can extract a system of $r+\ell+1$ linear equations from (5.13) and since the determinant of the resulting $(r+\ell+1)\times(r+\ell+1)$ matrix is not null, the functions G_k being independent, there is no non-zero $\psi^{\ell,r+\ell}$ satisfying this condition. Such $\psi^{\ell,r+\ell}$ then cannot generate other amplitudes satisfying (5.7) and (5.8). Thus, the only case for which (5.11) can be satisfied is when $r+\ell+1 > M$. Since we already had the restriction $r+\ell \leq M$, the only possibility is $r+\ell = M$. The system of equations (5.13) in this case becomes

$$\begin{pmatrix} G_1(\beta_1)G_1(\beta_2)\cdots G_1(\beta_M) \\ G_2(\beta_1)G_2(\beta_2)\cdots G_2(\beta_M) \\ \vdots \qquad\qquad \vdots \\ G_M(\beta_1)G_M(\beta_2)\cdots G_M(\beta_M) \end{pmatrix} \begin{pmatrix} (-1)^1 \psi^{\ell M}(\underline{\alpha}_\ell;\underline{\beta}_{\hat{1}\,M+1}) \\ (-1)^2 \psi^{\ell M}(\underline{\alpha}_\ell;\ \underline{\beta}_{\hat{2}\,M+1}) \\ \vdots \\ (-1)^M \psi^{\ell M}(\underline{\alpha}_\ell;\ \underline{\beta}_{\hat{M}\,M+1}) \end{pmatrix} = \begin{pmatrix} G_1(\beta_{M+1}) \\ G_2(\beta_{M+1}) \\ \\ G_M(\beta_{M+1}) \end{pmatrix} (-1)^M \psi^{\ell M}(\underline{\alpha}_\ell;\underline{\beta}_M) \tag{5.14}$$

We define as $\phi^M(\underline{\beta}_M)$ the determinant of the matrix on the r.h.s. (5.14) and denote as $\phi^M_{\hat{G}_i}(\underline{\beta}_{\hat{j}\,M})$ the subdeterminant of this same matrix when the row $G_i(\beta)$ and the column of functions of β_j are left out. Inverting equation (5.14) we obtain (since the determinant $\phi^M \neq 0$) the unique solution

$$(-1)^i \psi^{\ell M}(\underline{\alpha}_\ell; \underset{\hat{i}}{\underline{\beta}}M+1) = \frac{1}{\phi^M(\underline{\beta}_M)} \left[\sum_{j=1}^{M} (-1)^{i+j} \phi^M_{\hat{G}_j} (\underset{\hat{i}}{\underline{\beta}}M) G_j(\beta_{M+1}) \right] \times$$

$$(-1)^M \psi^{\ell M}(\underline{\alpha}_\ell; \underline{\beta}_M) \quad ,$$

which by using the definition of ϕ^M is simply

$$\psi^{\ell M}(\underline{\alpha}_\ell; \underset{\hat{i}}{\underline{\beta}}M+1) = \frac{\phi^M(\underset{\hat{i}}{\underline{\beta}}M+1)}{\phi^M(\underline{\beta}_M)} \psi^{\ell M}(\underline{\alpha}_\ell; \underline{\beta}_M) \tag{5.15}$$

i.e.

$$\frac{\psi^{\ell M}(\underline{\alpha}_\ell; \underset{\hat{i}}{\underline{\beta}}M+1)}{\phi^M(\underset{\hat{i}}{\underline{\beta}}M+1)} = \frac{\psi^{\ell M}(\underline{\alpha}_\ell; \underline{\beta}_M)}{\phi^M(\underline{\beta}_M)} \quad \forall \; i \quad . \tag{5.16}$$

This is easily seen to imply

$$\psi^{\ell M}(\underline{\alpha}_\ell; \underline{\beta}_M) = \psi_\ell(\underline{\alpha}_\ell)\phi^M(\underline{\beta}_M) \tag{5.17}$$

where $\psi_\ell(\underline{\alpha}_\ell)$ is a constant if $\ell = 0$ whereas, for $\ell > 0$, it must be an antisymmetric function of the α's which satisfies (5.10). Because of this last property, $\ell \leq N$ is needed in order to have $\psi_\ell \neq 0$.

We have thus gotten the necessary and sufficient conditions for equation (5.7) to have solutions $\psi^{\ell+1,r+\ell+1}$.

iii) We now have to examine whether (5.8) can also be satisfied.

We have $\int d\lambda M_4(\lambda,\beta)$ r.h.s. (5.8) $\neq 0$ since

$$\sum_{i=1}^{\ell+1} (-1)^{\ell+i+j} \int d\lambda M_4(\lambda,\beta) M_3^*(\lambda,\alpha_i) \psi_\ell (\underset{i}{\hat{\alpha}} \ell+1) \phi^M (\underset{j}{\hat{\beta}} M+1) = 0$$

implies

$$[\int d\alpha_k M_1(\gamma,\alpha_k)(-1)^{\ell+k+j} \int d\lambda M_4(\lambda,\beta) M_3^*(\lambda,\alpha_k)] \psi_\ell (\underset{i}{\hat{\alpha}} \ell+1) \phi^M (\underset{j}{\hat{\beta}} M+1) = 0$$

(where (5.10) is used when $\ell > 0$). Since the factor in parenthesis is not zero when M_3 and $M_2 \neq 0$ and we know $\phi^N \neq 0$, this equation holds if and only if ψ_ℓ is taken zero. However, if this were the case, $\psi^{\ell+1,M+1}$ would have to satisfy $\int d\beta_j M_4^*(\lambda,\beta_j) \psi^{\ell+1,M+1}(\underline{\alpha}_{\ell+1};\underline{\beta}_{M+1}) = 0 \;\forall j$ so that it also could only be null together with all other $\psi^{\ell+r,M+r}$. A non-zero solution is then possible only if we take $\psi_\ell \neq 0$ so that $\int d\lambda M_4(\lambda,\beta)$ r.h.s. (5.8) $\neq 0$.

It is now necessary to examine whether the r.h.s. (5.8) is orthogonal to all N $F_k'(\lambda)$ such that $\int d\lambda M_4(\lambda,\beta) F_k'(\lambda) = 0$ since we know that this is a necessary and sufficient condition for an equation of the type (5.8) to have a solution. We must have

$$\sum_{i=1}^{\ell+1} (-1)^{\ell+i+j} \int d\lambda F_k'(\lambda)^* M_3^*(\lambda,\alpha_i) \psi_\ell (\underset{i}{\hat{\alpha}} \ell+1) \phi^M (\underset{j}{\hat{\beta}} M+1) = 0 \;\forall F_k$$

i.e.

$$\sum_{i=1}^{\ell+1} (-1)^i \; G_k'(\alpha_i) \psi_\ell (\underset{i}{\hat{\alpha}} \ell+1) = 0 \;\forall G_k' \qquad (5.18)$$

where $G_k'(\alpha_i) = \int d\lambda F_k'(\lambda)^* M_3^*(\lambda,\alpha_i)$, $k=1,2,\ldots N$ are linearly independent solutions of $\int d\alpha_i M_1(\gamma,\alpha_i)G'(\alpha_i) = 0$. We note that the system of equations (5.18) is of exactly the same type as the system (5.12) a discussion similar to the one used in dealing with the previous equations is easily seen to give the following results. If $N \geq \ell+1$, (5,18) can only be satisfied if $\psi_\ell = 0$ which, as we saw, implies all $\psi^{\ell+r,M+r}$ also are zero. $N < \ell+1$ is therefore needed and since already $\ell \leq N$ was necessary, we see that only $\ell = N$ is possible for a non zero ψ_ℓ to satisfy (5.18).

This together with $\ell < M$ (c.f. remark after equation (5.10)) means that when $N \geq M$, $\ell = N$ is not possible so that the only solution will be $\psi^{nm} = 0$ when $m > n$. If $N < M$, (5.18) is satisfied if and only if

$$\psi_N(\underline{\alpha}_N) = c\ \phi'^N(\underline{\alpha}_N) = c \det \begin{pmatrix} G_1'(\alpha_1) G_1'(\alpha_2) & \cdots & G_1'(\alpha_N) \\ G_2'(\alpha_1) G_2'(\alpha_2) & \cdots & G_2'(\alpha_N) \\ \vdots & & \vdots \\ G_N'(\alpha_1) G_N'(\alpha_2) & \cdots & G_N'(\alpha_N) \end{pmatrix} \quad (5.19)$$

We therefore have shown the following results concerning all ψ^{nm} such that $n < m$. If $M = 0$, the only possible solution is all $\psi^{nm} = 0$. If $M \neq 0$ and $M \leq N$, the unique possibility is again all $\psi^{nm} = 0$. If $M \neq 0$ and $N < M$, the only possibly non-zero family of ψ^{nm} consists of those generated from $\psi^{NM}(\underline{\alpha}_N;\underline{\beta}_M)$ which is

$$\psi^{NM}(\underline{\alpha}_N;\underline{\beta}_M) = c\ \phi'^{N}(\underline{\alpha}_N)\phi^{M}(\underline{\beta}_M) \qquad (5.20)$$

if $N \neq 0$ and $\psi^{0M}(-;\underline{\beta}_M) = c\ \phi^{M}(\underline{\beta}_M)$ if $N = 0$.

iv) The results concerning the ψ^{nm} with $n > m$ can be deduced in the same manner. If $N = 0$ or $N \neq 0$ and $N \leq M$, the only solutions for all ψ^{nm} will be $\psi^{nm} = 0$. If $N \neq 0$ and $N > M$, the only possibly non zero family of such ψ^{nm} will consist of those generated from $\psi^{NM}(\underline{\alpha}_N;\underline{\beta}_M)$ which is again

$$\psi^{NM}(\underline{\alpha}_N;\underline{\beta}_M) = c\ \phi'^{N}(\underline{\alpha}_N)\phi^{M}(\underline{\beta}_M) \quad \text{if} \quad M \neq 0$$

and $$\psi^{N0}(\underline{\alpha}_N;-) = c\ \phi'^{N}(\underline{\alpha}_N) \qquad \text{if} \quad M = 0 .$$

v) Similarly, also, we obtain that a non-zero family of amplitudes ψ^{nm} with $n = m$ can exist only if $N = M$ and it is generated from $\psi^{NM} = c\ \phi'^{N}\phi^{N}$.

6) The Solution of the Equations

A unique non-zero set of amplitudes is always determined by the system of equations (2.3)-(2.6). It is the family of amplitudes generated from ψ^{NM} which, as determined in the previous section is

$$\psi^{NM}(\underline{\alpha}_N;\underline{\beta}_M) = c \det \begin{pmatrix} G'(\underline{\alpha}_N) & 0 \\ 0 & G(\underline{\beta}_M) \end{pmatrix} . \qquad (6.1)$$

The amplitudes $\psi^{N+\ell,M+\ell}$ for $\ell > 0$ are

$$\psi^{N+\ell,M+\ell}(\underline{\alpha}_{N+\ell};\underline{\beta}_{M+\ell}) = \frac{c(-1)^{\frac{\ell(\ell+1)}{2}}}{\sqrt{(N+\ell)!(M+\ell)!}} \times$$

$$\times \det \left[\begin{array}{c|cccc} & \chi(\alpha_1,\beta_1) & \chi(\alpha_1,\beta_2) & \cdots & \chi(\alpha_1,\beta_{M+\ell}) \\ & \chi(\alpha_2,\beta_1) & & & \\ G'(\underline{\alpha}_{N+\ell}) & \vdots & & & \vdots \\ & \chi(\alpha_{N+\ell},\beta_1) & \chi(\alpha_{N+\ell},\beta_2) & \cdots & \chi(\alpha_{N+\ell},\beta_{M+\ell}) \\ \hline 0 & & G(\underline{\beta}_{M+\ell}) & & \end{array} \right]$$

(6.2)

where $\chi(\alpha_j,\beta_i)$ is defined as

$$\chi(\alpha_j,\beta_i) = \sum_{r=0}^{\infty}\int d\alpha\,(M_3^*M_3)^r(\alpha_j;\alpha)\int d\gamma M_1^*(\gamma,\alpha)M_2(\gamma,\beta_i) \quad . \tag{6.3}$$

When $N = 0$, the left hand side upper matrix of G_k' is not there; when $M = 0$, the lower right hand side matrix of G_k is not there and when $N = M = 0$, $\psi^{NM} = \psi^{OO}$ = constant and $\psi^{N+\ell,M+\ell} = \psi^{\ell\ell}$ is the determinant of the upper right hand side matrix of χ's.

In order to see that $\psi^{N+\ell,M+\ell}$ satisfies (2.3)-(2.6) for any given j, we simply expand the determinant for $\psi^{N+\ell,M+\ell}$ along the j^{th} row and obtain

$$\psi^{N+\ell,M+\ell}(\underline{\alpha}_{N+\ell};\underline{\beta}_{M+\ell}) = \left(\frac{c(-1)^{\frac{\ell(\ell+1)}{2}}}{\sqrt{(N+\ell)!\,(M+\ell)!}}\right)\left(\sum_{i=1}^{N} G_i'(\alpha_j)\,[\text{its cofactor}]\right.$$

$$+\sum_{i=1}^{M+\ell}(-1)^{j+N+i}\chi(\alpha_j,\beta_i)\left(\frac{\sqrt{(N+\ell-1)!\,(M+\ell-1)!}}{c(-1)^{\frac{(\ell-1)\ell}{2}}}\right)\times$$

$$\left.\times\,\psi^{N+\ell-1,M+\ell-1}(\underline{\alpha}_{N+\ell}^{\hat{j}};\ \underline{\beta}_{M+\ell}^{\hat{i}})\right) \quad . \tag{6.4}$$

Since $\int d\alpha_j M_1(\gamma,\alpha_j)G_i'(\alpha_j) = 0 \ \forall\, i$, we have

$$\int d\alpha_j M_1(\gamma,\alpha_j)\psi^{N+\ell,M+\ell}(\underline{\alpha}_{N+\ell};\underline{\beta}_{M+\ell}) = \frac{1}{\sqrt{(N+\ell)(M+\ell)}}\sum_{i=1}^{M+\ell}(-1)^{N+\ell+i+j}\times$$

$$\left[\int d\alpha_j M_1(\gamma,\alpha_j)\chi(\alpha_j,\beta_i)\right]\psi^{N+\ell-1,M+\ell-1}(\underline{\alpha}_{N+\ell}^{\hat{j}};\underline{\beta}_{M+\ell}^{\hat{i}}) \quad . \tag{6.5}$$

We have already shown in section 3) that

$$\int d\alpha_j M_1(\gamma,\alpha_j)\chi(\alpha_j,\beta_i) = M_2(\gamma,\beta_i) - \sum_{k=1}^{M} F_k(\gamma)\int d\gamma' F_k(\gamma')\overset{*}{M}_2(\gamma',\beta_i)$$

$$= M_2(\gamma,\beta_i) - \sum_{k=1}^{M} F_k(\gamma)G_k(\beta_i) \; .$$

The right hand side of (6.5) is then

$$\frac{1}{\sqrt{(N+\ell)(M+\ell)}}\sum_{i=1}^{M+\ell}(-1)^{N+\ell+i+j}\, M_2(\gamma,\beta_i)\;\times$$

$$\times\;\psi^{N+\ell-1,M+\ell-1}\,(\underset{\hat{j}}{\alpha}N+\ell;\underset{\hat{i}}{\beta}M+\ell)$$

$$-\sum_{k=1}^{M}\frac{F_k(\gamma)}{\sqrt{(N+\ell)(M+\ell)}}\sum_{i=1}^{M+\ell}(-1)^{N+\ell+i+j}\, G_k(\beta_i)\;\times$$

$$\times\;\psi^{N+\ell-1,M+\ell-1}\,(\underset{\hat{j}}{\alpha}N+\ell;\underset{\hat{i}}{\beta}M+\ell) \; . \qquad (6.6)$$

For any given k,

$$\sum_{i=1}^{M+\ell}(-1)^{i}\, G_k(\beta_i)\psi^{N+\ell-1,M+\ell-1}\,(\underset{\hat{j}}{\alpha}N+\ell;\underset{\hat{i}}{\beta}M+\ell) = 0$$

since this is proportional to a determinant like (6.2) where one of the first $N+\ell$ is deleted and a row: $[0,0\ldots0,G_k(\beta_1),G_k(\beta_2),\ldots G_k(\beta_{M+\ell})]$ is added after the last one so that this row is repeated twice. We are therefore left with only the first term of (6.6) and (2.3) is satisfied.

Equation (2.5) is similarly verified for any j by expanding the determinant (6.2) along the $(N+j)^{th}$ column. $\psi^{N+\ell,M+\ell}$ as given by (6.1) and (6.2) therefore satisfies the system of equations (2.3)-(2.6).

This set of amplitudes is furthermore the unique non-zero solution. In the preceding section, we saw that a non-zero family of ψ^{nm} could only be generated from ψ^{NM}, as given by (6.1), which is unique up to a multiplicative constant. Another family $\bar{\psi}^{nm} \neq \psi^{nm}$ for some or all $n > N$ $m > M$, generated also from ψ^{NM}, cannot exist since we know, from the section 4) that $\psi_o^{N+1,M+1} = (\bar{\psi}^{N+1,M+1} - \psi^{N+1,M+1})$ which would then satisfy the homogeneous equations

$$\int d\alpha_j M_1(\gamma,\alpha_j)\psi_o^{N+1,M+1}(\underline{\alpha}_{N+1};\underline{\beta}_{M+1}) = 0 \quad \forall\, j \qquad (6.7)$$

$$\int d\beta_j M_4^*(\lambda,\beta_j)\psi_o^{N+1,M+1}(\underline{\alpha}_{N+1};\underline{\beta}_{M+1}) = 0 \quad \forall\, j \qquad (6.8)$$

can only be = 0 i.e. $\bar{\psi}^{N+1,M+1} = \psi^{N+1,M+1}$. By repeating the same argument, we show that we need $\bar{\psi}^{N+2,M+2} = \psi^{N+2,M+2}$, etc. .... The fact that there exists no anti-symmetric solution to (6.7) with more than N variables α and to (6.8) with more than M variables β thus ensures uniqueness of the non-zero amplitudes ψ^{nm} which satisfy (2.3)-(2.6).

7) Proof that the New Vacuum has Finite Norm

In the previous sections, we have proved that the equations (2.3)-(2.6) determine uniquely all amplitudes ψ^{nm} of the expansion (2.2) for $|\phi_o\rangle$. We now show that $\langle\phi_o|\phi_o\rangle = \sum_{\ell=0}^{\infty} ||\psi^{N+\ell,M+\ell}||^2 < \infty$.

In order to evaluate $||\psi^{N+\ell,M+\ell}||$, let us use the expansion (6.4) for the determinant (6.2) along the first row (i.e. j = 1). Since χ has been shown, in section 3), to be orthogonal to all $G'(\alpha)$, we have:

$$||\psi^{N+\ell,M+\ell}(\underline{\alpha}_{N+\ell};\underline{\beta}_{M+\ell})||^2 = \frac{c^2}{(N+\ell)!(M+\ell)!} \times$$

$$||\sum_{i=1}^{n} G_i'(\alpha_1)\,[\text{its cofactor}]||^2$$

$$+ \frac{1}{(N+\ell)(M+\ell)}||\sum_{i=1}^{M+\ell} (-1)^i \chi(\alpha_1,\beta_i) \times$$

$$\times \psi^{N+\ell-1,M+\ell-1}(\underset{\hat{1}}{\underline{\alpha}_{N+\ell}}, \underset{\hat{i}}{\underline{\beta}_{M+\ell}})||^2 . \qquad (7.1)$$

i) The term $\sum_{i=1}^{N} G_i'(\alpha_1)$[its cofactor] can be seen, by examining the determinant (6.2) and always expanding cofactors along a column G_c', to consist of $N.(N+\ell-1)!/\ell!$ terms of the following type: each term contains a product of N G_c', with different variables α, multiplied by a determinant of the type $D^{M+\ell}$, defined below, which depends on the remaining $(N+\ell)-N$ variables

α and the $M+\ell$ variables β. We can write symbolically each such term as

$$\pm\left(\prod_{k=1}^{N} G_k'\right)(\underline{\alpha}_N')\, D^{M+\ell}\left(\frac{\underline{\alpha}}{\hat{\underline{\alpha}}_N'}N+\ell;\underline{\beta}_{M+\ell}\right) \tag{7.2}$$

where $\underline{\alpha}_N'$ designates a set of N α's (among which, there is always α_1) and $D^{M+\ell}$ is defined as,

$$D^{M+\ell}(\underline{\alpha}_\ell;\underline{\beta}_{M+\ell}) = \det\begin{pmatrix} \chi(\alpha_1,\beta_1) & \chi(\alpha_1,\beta_2) & \cdots & \chi(\alpha_1,\beta_{M+\ell}) \\ \chi(\alpha_2,\beta_1) & & & \\ \vdots & & & \vdots \\ \chi(\alpha_\ell,\beta_1) & \chi(\alpha_\ell,\beta_2) & & \chi(\alpha_\ell,\beta_{M+\ell}) \\ G_1(\beta_1) & G_1(\beta_2) & & G_1(\beta_{M+\ell}) \\ G_2(\beta_1) & & & \\ \vdots & & & \vdots \\ G_M(\beta_1) & G_M(\beta_2) & & G_M(\beta_{M+\ell}) \end{pmatrix} \tag{7.3}$$

We remark that two terms of the type (7.2) will differ in having different sets $\underline{\alpha}_N'$ or in having the same sets $\underline{\alpha}_N'$ but ordered differently in the product $G_1', G_2', \ldots G_N'$ (or a mixture of these two possibilities). The terms are therefore orthogonal as functions of $\underline{\alpha}_{M+\ell}$. If they have the same sets $\underline{\alpha}_N'$

but placed in a different order, there will be products $\int d\alpha_j G_k'(\alpha_j)^* G_{k'}'(\alpha_j)$ where $k \neq k'$, which are null. If they have different sets $\underline{\alpha}_N'$ and say α_j is in one of the sets and not in the other, the product will contain the integral $\int d\alpha_j G_k'(\alpha_j)^* D^{M+\ell}(\ldots\alpha_j\ldots;\underline{\beta}_{M+\ell})$ which is null since $\int d\alpha_j G_k'(\alpha_j)^* \chi(\alpha_j,\beta_i) = 0$. We therefore have

$$\left|\left|\sum_{i=1}^{N} G_i'(\alpha_1)\,[\text{its cofactor}]\right|\right|^2 = \sum \left|\left|\text{each individual term}\right|\right|^2;$$

the norm of all terms being equal, this is

$$= \frac{N(N+\ell-1)!}{\ell!} \left|\left|\left(\prod_{k=1}^{N} G_k'\right) D^{M+\ell}\right|\right|^2$$

and if the G_k''s are taken normalized to 1,

$$= \frac{N(N+\ell-1)!}{\ell!} \left|\left|D^{M+\ell}\right|\right|^2 \tag{7.4}$$

where

$$\left|\left|D^{M+\ell}\right|\right|^2 = \int \underline{d\alpha}_\ell \, \underline{d\beta}_{M+\ell} \left|D^{M+\ell}(\underline{\alpha}_\ell;\underline{\beta}_{M+\ell})\right|^2 . \tag{7.5}$$

This term will now be dealt with together with the second term of (7.1). We simply remark for the moment that when developed along its first row, $D^{M+\ell}$ can be written as

$$D^{M+\ell}(\underline{\alpha}_\ell;\underline{\beta}_{M+\ell}) = \sum_{i=1}^{M+\ell} (-1)^{i+1} \chi(\alpha_1,\beta_i) D^{M+\ell-1}(\underset{\hat{1}}{\underline{\alpha}}\ell;\underset{\hat{i}}{\underline{\beta}}M+\ell) . \tag{7.6}$$

ii) Let us then consider the general expression

$$\int d\beta_{M+\ell} \; \Big| \sum_{i=1}^{M+\ell} (-1)^{i} \; \chi(\alpha,\beta_i) F^{M+\ell-1}(\beta_{\hat{i}} {}_{M+\ell}) \Big|^2 \qquad (7.7)$$

where $F^{M+\ell-1}$ is an antisymmetric function of $(M+\ell-1)$ variables. This is equal to

$$\sum_{i=1}^{M+\ell} \int d\beta_{M+\ell} |\chi(\alpha,\beta_i)|^2 \; |F^{M+\ell-1}(\beta_{\hat{i}} {}_{M+\ell})|^2$$

$$+ \sum_{i'\neq i=1}^{M+\ell} \sum_{i'=1}^{M+\ell} \int d\beta_{M+\ell} \; (-1)^{i+i'} \chi^*(\alpha,\beta_i) F^{M+\ell-1}(\beta_{\hat{i}} {}_{M+\ell})^* \times$$

$$\times \; \chi(\alpha,\beta_{i'}) F^{M+\ell-1}(\beta_{\hat{i}'} {}_{M+\ell}) \quad . \qquad (7.8)$$

We will show that the last term above is always negative or null. Every integral of the sum can be written as

$$\int d\beta_{\hat{i}\,\hat{i}'} {}_{M+\ell} \; (-1)^{i+i'} \Big[\int d\beta_i \chi^*(\alpha,\beta_i) F^{M+\ell-1}(\beta_{\hat{i}'} {}_{M+\ell})\Big] \times$$

$$\times \; \Big[\int d\beta_{i'} \chi^*(\alpha,\beta_{i'}) F^{M+\ell-1}(\beta_{\hat{i}} {}_{M+\ell})\Big]^* . \quad (7.9)$$

The term $\int d\beta_{i'} \chi^*(\alpha,\beta_{i'}) F^{M+\ell-1}(\beta_{\hat{i}} {}_{M+\ell})$, upon relabelling the variable of integration $\beta_{i'} : \beta_i$, is equal to

$$\int d\beta_i \chi^*(\alpha,\beta_i) F^{M+\ell-1}(\beta_1,\beta_2,\ldots\hat{\beta}_i\ldots\beta_{M+\ell} \not{\tfrac{i}{i'}})$$

$$= \int d\beta_i \chi^*(\alpha,\beta_i)(-1)^{i+i'+1} F^{M+\ell-1}(\underset{\hat{i}'}{\beta}_{M+\ell})$$

so that (7.9) is equal to

$$- \int \underset{\hat{i}\ \hat{i}'}{d\beta}_{M+\ell} \left| \int d\beta_i \chi^*(\alpha,\beta_i) F^{M+\ell-1}(\underset{\hat{i}'}{\beta}_{M+\ell}) \right|^2 .$$

The last term of (7.8) is then always negative or null and we have

$$\int \underline{d\beta}_{M+\ell} \left| \sum_{i=1}^{M+\ell} (-1)^i \chi(\alpha,\beta_i) F^{M+\ell-1}(\underset{\hat{i}}{\beta}_{M+\ell}) \right|^2 \leq$$

first term of (7.8)

that is

$$\leq (M+\ell)\left[\int d\beta |\chi(\alpha,\beta)|^2\right]\left[\int \underline{d\beta}_{M+\ell-1} |F^{M+\ell-1}(\underline{\beta}_{M+\ell-1})|^2\right]. \tag{7.10}$$

iii) We can use this result to evaluate (7.5) and obtain

$$||D^{M+\ell}||^2 \leq (M+\ell)\,||\chi||^2\,||D^{M+\ell-1}||^2$$

and repeating for $||D^{M+\ell-1}||^2$, etc. ... we obtain

$$||D^{M+\ell}||^2 \leq (M+\ell)(M+\ell-1)\ldots(M+1)||\chi||^{2\ell}\ ||D^M||^2$$

where $D^M = \det G(\underline{\beta}_M)$, so that $||D^M||^2 = M!$ and

$$||D^{M+\ell}||^2 \leq (M+\ell)!\ ||\chi||^{2\ell} \quad . \qquad (7.11)$$

Similarly, for the second term on the right hand side of (7.1), we obtain that it is always

$$\leq \frac{1}{(N+\ell)} ||\chi||^2\ ||\psi^{N+\ell-1,M+\ell-1}||^2 \quad . \qquad (7.12)$$

We thus get the estimate

$$||\psi^{N+\ell,M+\ell}||^2 \leq \frac{c^2 N||\chi||^{2\ell}}{(N+\ell)\ell!} + \frac{||\chi||^2}{(N+\ell)} ||\psi^{N+\ell-1,M+\ell-1}||^2 \ . \qquad (7.13)$$

By using the similar estimate on $||\psi^{N+\ell-1,M+\ell-1}||^2$ etc. ... and $||\psi^{NM}||^2 = c^2\ N!\ M!$ we obtain that the right hand side of (7.13) is equal to

$$c^2 N||\chi||^{2\ell} \sum_{i=0}^{\ell-1} \frac{1}{(N+\ell)(N+\ell-1)\ldots(N+\ell-i)(\ell-i)!} +$$

$$+ \frac{c^2 N!M!||\chi||^{2\ell}}{(N+\ell)(N+\ell-1)\ldots(N+1)} \quad . \qquad (7.14)$$

When $N = 0$, the first term vanishes and when $N \geq 1$, $(N+\ell-i) \geq (\ell+1-i)$ for all ℓ's so that the first sum is always smaller or equal to $\ell/(\ell+1)! < 1/\ell!$; also $(N+\ell)(N+\ell-1)\ldots(N+1) \geq \ell!$ so that

$$||\psi^{N+\ell,M+\ell}||^2 \leq c^2(N + N!M!)\,\frac{||\chi||^{2\ell}}{\ell!} \quad . \tag{7.15}$$

Since this holds for all ℓ's, we have that the series for $|\phi_o>$ converges strongly to a state of $\mathcal{H}$ and

$$<\phi_o|\phi_o> = \sum_{\ell=0}^{\infty} ||\psi^{N+\ell,M+\ell}||^2 \leq c^2(N+N!M!)\sum_{\ell=0}^{\infty} \frac{||\chi||^{2\ell}}{\ell!}$$

i.e.

$$<\phi_o|\phi_o> \leq c^2(N+N!M!)e^{||\chi||^2} < \infty \quad . \tag{7.16}$$

The arbitrary multiplicative constant c can therefore be chosen such that $<\phi_o|\phi_o> = 1$.

8) The Unitary Operator

In the previous sections, we have proved that the conditions $||M_2|| < \infty$ and $||M_3|| < \infty$ are necessary and sufficient for the existence of a unique vacuum state $|\phi_o\rangle$ associated with the "new" creation and annihilation operators (B,D) in the Fock-Hilbert space defined by the operators (b,d).

In this section, we will prove that, under the above conditions, there exists a unitary operator implementing the transformation (1.1) and we will give an explicit form for it.

i) In order to see more clearly what happens in the transformation (1.1), we decompose it in a product of simpler transformations.

Before doing this, we introduce the following unified notation for simplicity. We call $X_n^3(\alpha)$ and $X_n^1(\lambda)$, n=1,2,... the respective eigenfunctions (defined as in equations (3.3)-(3.4)) of $(M_3^*M_3)(\alpha',\alpha)$ an[illegible] $(M_3^*M_3)(\lambda',\lambda)$ where the first N correspond to the e[illegible] value 1 (i.e. for n=1,2,...N, $\int d\alpha M_1(\gamma,\alpha)X_n^3(\alpha) = 0$ and $\int d\lambda M_4(\lambda,\beta)X_n^1(\lambda) = 0$). Similarly, $X_n^2(\beta)$ and $X_n^4(\gamma)$, n=1,2,... are the respective eigenfunctions of $(M_2M_2^*)(\beta',\beta)$ and $(M_2M_2^*)(\gamma',\gamma)$ where the first M correspond to the eigenvalue 1 (i.e. for n=1,2,...M,

$\int d\beta M_4^*(\lambda,\beta)X_n^2(\beta) = 0$ and $\int d\gamma M_1^*(\gamma,\alpha)X_n^4(\gamma) = 0$).

We define the smeared operator variables:

$$b_n = \int d\alpha X_n^3(\alpha)^* b(\alpha) \qquad d_n^\dagger = \int d\beta X_n^2(\beta) d^\dagger(\beta)$$

$$B_n = \int d\gamma X_n^4(\gamma)^* B(\gamma) \qquad D_n^\dagger = \int d\lambda X_n^1(\lambda) D^\dagger(\lambda) \quad . \qquad (8.1)$$

We recall that the functions X_n^i for each i=1,2,3,4 form a basis in L^2 so that

$$b(\alpha) = \sum_{n=1}^{\infty} X_n^3(\alpha) b_n \qquad d^\dagger(\beta) = \sum_{n=1}^{\infty} X_n^2(\beta)^* d_n^\dagger$$

$$B(\gamma) = \sum_{n=1}^{\infty} X_n^4(\gamma) B_n \qquad D^\dagger(\lambda) = \sum_{n=1}^{\infty} X_n^1(\lambda)^* D_n^\dagger \quad . \qquad (8.2)$$

Because of the properties of the functions X_n^i, we see that the Bogoliubov transformation (1.1) is in fact

$$B_n = d_n^\dagger \quad \text{for } n=1,2,..M \qquad D_n^\dagger = b_n \quad \text{for } n=1,2,..N \qquad (8.3)$$

$$B_{M+n'} = \sum_{n=1}^{\infty} \{ (X_{M+n'}^4, M_1 X_{N+n}^3) b_{N+n} + (X_{M+n'}^4, M_2 X_{M+n}^2) d_{M+n}^\dagger \}$$

$$D_{N+n'}^\dagger = \sum_{n=1}^{\infty} \{ (X_{N+n'}^1, M_3 X_{N+n}^3) b_{N+n} + (X_{N+n'}^1, M_4 X_{M+n}^2) d_{M+n}^\dagger \} \quad (8.4)$$

for n'=1,2,...∞ .

One can immediately notice that the equation (8.4) itself is a Bogoliubov transformation between the two sets of operators $\tilde{b}_n = b_{N+n}$, $\tilde{d}_n = d_{M+n}$; $n=1,2,\ldots\infty$ and $\tilde{B}_n = B_{M+n}$, $\tilde{D}_n = D_{N+n}$; $n=1,2,\ldots\infty$. The kernels $\tilde{M}_1(n',n) \equiv (X^4_{M+n'}, M_1 X^3_{N+n})$ and $\tilde{M}_4(n',n)^* = (X^2_{M+n}, M_4^* X^1_{N+n'})$ do not have any non-zero homogeneous solutions associated with them as is easily deduced from the properties of the X^i_n's. (8.4) is therefore a "weak" Bogoliubov transformation and together with equation (8.3), this gives a simple decomposition of the transformation (1.1).

ii) Let us now look more particularly at "weak" Bogoliubov transformations. Different forms for the unitary operator implementing such transformations are given by K.O. Friedrichs [1953]. One such form is easily obtained by making the following ansatz which is suggested by a close examination of the nature of the new vacuum state:

$$T = e^{-dF_4 b}\, e^{-d^\dagger F_3 d}\, e^{-b^\dagger F_2 b}\, e^{-b^\dagger F_1 d^\dagger} \qquad (8.5)$$

where expressions like $e^{-b^\dagger F_1 d^\dagger}$ stand for

$$\exp\{-\sum_{n'=1}^{\infty}\sum_{n=1}^{\infty} b^\dagger_{n'}\, F_1(n',n)\, d^\dagger_n\} \; .$$

The requirements that

$$T^{-1} \begin{matrix} b_n \\ b_n^\dagger \\ d_n \\ d_n^\dagger \end{matrix} \; T = \begin{matrix} B_n \\ B_n^\dagger \\ D_n \\ D_n^\dagger \end{matrix} \qquad n=1,2,\ldots \tag{8.6}$$

ensure that T is unitary and when using the expression for the operators (B,D) in terms of (b,d) and the formula

$$Ae^C = e^C \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Omega_n(C,A) \quad ,$$

where

$$\Omega_o(C,A) = A, \quad \Omega_{n+1}(C,A) = [C, \Omega_n(C,A)] \quad , \tag{8.7}$$

(8.6) determines completely the functions F_i, $i=1,2,3,4$. A straightforward calculation shows that these are given by

$$(e^{-F_2})(n,m) = M_1(n,m)$$

$$(e^{-F_3})(n,m) = [M_4(n,m)]^* \; .$$

$F_1(n,m)$ is the solution of

$$\sum_n M_1(n',n)F_1(n,m) = -M_2(n',n)$$

that is

$$F_1(n,m) = \frac{1}{\psi^{oo}} \psi^{11}(n,m)$$

where ψ^{11} and ψ^{oo} are the first two amplitudes in the expansion of the new vacuum in terms of original Fock-basis states. $F_4(n,m)$ is the solution of

$$\sum_{n'} M_1(n',n)F_4(m,n') = -M_3(m,n) .$$

There then exists a unitary operator T of the form (8.5) implementing the "weak" Bogoliubov transformation corresponding to a set of kernels $M_i(n,m)$; i=1,2,3,4.

iii) In the case of the general transformation (8.3)-(8.4), we can then define a unitary operator T_o such that

$$T_o^\dagger\, b_{N+n}\, T_o = B_{M+n}$$

$$T_o^\dagger\, d_{M+n}\, T_o = D_{N+n} \qquad \forall\, n = 1,2,\ldots\infty \tag{8.8}$$

and

$$T_o^\dagger\, b_i\, T_o = b_i \qquad \text{for } i=1,2,\ldots N ,$$

$$T_o^\dagger\, d_i\, T_o = d_i \qquad \text{for } i=1,2,\ldots N . \tag{8.9}$$

That is: T_o implements the "weak" Bogoliubov transformation (8.4) so that it exists as we saw in the previous section. It does not contain the operator variables not entering in (8.4) because of the requirement (8.9). It can be written in the form (8.5) for example, where the operators (b,d) are $\tilde{b}_n$, $\tilde{d}_n$; n=1,2,... and the M_i's are the $\tilde{M}_i$'s.

It is easy to check that the unitary operator

$$U = \{\exp -i(M+N)\pi \sum_{n=1}^{\infty} [b_n^\dagger b_n + d_n^\dagger d_n]\}(d_M^\dagger - d_M) \ldots$$

$$(d_2^\dagger - d_2)(d_1^\dagger - d_1)(b_N^\dagger - b_N)(b_2^\dagger - b_2)(b_1^\dagger - b_1) \quad . \tag{8.10}$$

commutes with b_{N+n} and d_{M+n} for $n=1,2,\ldots\infty$ and

$$U^\dagger b_i U = D_i \text{ for } i=1,2,..N, \quad U^\dagger d_i U = B_i \text{ for } i=1,2,\ldots M \,. \tag{8.11}$$

The unitary operator

$$T = T_o U = U T_o$$

will then transform the creation and annihilation operators as

$$T^\dagger b_n T = D_n \quad n=1,2,\ldots N \qquad T^\dagger b_n T = B_n \quad n=N+1,\ldots\infty$$

$$T^\dagger d_n T = B_n \quad n=1,2,\ldots M \qquad T^\dagger d_n T = D_n \quad n=M+1,\ldots\infty \,.$$

The vacuum $|\phi_o\rangle$ (i.e. the new zero particle state) which we have calculated is then simply

$$|\phi_o\rangle = T^\dagger|0\rangle = T_o^\dagger b_1^\dagger b_2^\dagger \ldots b_N^\dagger d_1^\dagger \ldots d_M^\dagger|0\rangle \quad . \tag{8.12}$$

9) Conclusion

i) Let us consider a quantum mechanical system where the operators b, $b^\dagger$, d, $d^\dagger$ are associated with particles at some time t_1 and B, $B^\dagger$, D, $D^\dagger$ are associated with the same particles at a later time t_2 according to an equation of motion like (1.1) in the Heisenberg representation.

If the system is in the state $|0\rangle$; initially, there are no particles present. The state with zero particles at the later time t_2 is $|\phi_o\rangle$. The state $T_o|\phi_o\rangle$ is one in which there is a non-zero probability of finding pairs of particles and antiparticles created by $B^\dagger_{M+n}$ and $D^\dagger_{M+n}$, $n=1,2,\ldots\infty$. $|0\rangle$, which is

$$|0\rangle = T|\phi_o\rangle = B^\dagger_M\ldots B^\dagger_2 B^\dagger_1 D^\dagger_N\ldots D^\dagger_1\ T_o|\phi_o\rangle\ ,$$

is then a state in which M particles and N anti-particles have certainly been created and in which there is a non-zero probability of finding also pairs of particles and antiparticles.

If, on the other hand, the system is in the state $|\phi_o\rangle$;

$$|\phi_o\rangle = b^\dagger_1 b^\dagger_2\ldots b^\dagger_N\ d^\dagger_1\ldots d^\dagger_M\ T^\dagger_o|0\rangle\ ,$$

initially there are at least N particles and M antiparticles plus possibly some pairs of particles and antiparticles. After the interaction, there are no particles present.

We see that there are states of the system in which particles and antiparticles will be created or annihilated separately with certainty. If these are assigned different charges, obviously, the total charge is not conserved unless N = M.

This can also be seen by examining the charge operators; the initial charge operator is

$$Q(t_1) = e \sum_{n=1}^{\infty} \{b_n^{\dagger} b_n - d_n^{\dagger} d_n\}$$

and the new charge operator is

$$Q(t_2) = e \sum_{n=1}^{\infty} \{B_n^{\dagger} B_n - D_n^{\dagger} D_n\} \; .$$

By their action on one of the two corresponding Fock-bases, one can see that the following relation holds between these two operators:

$$Q(t_2) = Q(t_1) + (M - N) \; .$$

This gives therefore a rigorous adequate treatment for the phenomena of isolated particles complete absorption or emission by the external field which is discussed, for example, by A.I. Akhiezer and V.B. Berestetsky [1953].

ii) It is easy to see that the results obtained apply also to the case of one operator variable transformations like

$$B(\gamma) = \int d\alpha M_1(\gamma,\alpha) b(\alpha) + \int d\alpha M_2(\gamma,\alpha) b^\dagger(\alpha) .$$

In terms of smeared operators defined similarly as in the case discussed here, this transformation reduces to

$$B_n = b_n^\dagger \quad , \quad n=1,2,\ldots M$$

and an ordinary Bogoliubov transformation between the other B_{M+n}'s and b_{M+n}'s. The unitary operator relating them has the form $T = T_o U$ where T_o is the usual one for the "weak" transformation and

$$U = \{\exp -iM\pi \sum_{n=1}^{\infty} b_n^\dagger b_n\}(b_M^\dagger - b_M)\ldots(b_2^\dagger - b_2)(b_1^\dagger - b_1) .$$

iii) We finally remark that the case of a system of bosons is different since then N and M are always zero. The particles are always created and annihilated in pairs (and thus the total charge conserved for charged bosons). This is not so here, when $N \neq M$, even though this could seem to hold in all cases due to the aspect of formal algebraic relations between the field operators.

CHAPTER IV

SOLUTION OF SOME TIME DEPENDENT EXTERNAL FIELD PROBLEMS

Introduction

In the first section of this chapter, we examine the simple problem where a free Dirac field is disturbed by a pulse of electromagnetic field. The time dependence of the potential will be considered to be a Dirac δ-function. We will see that such a system can be described by a free field which has a simple discontinuity at the time of the interaction. Such a model has not been studied before even though it leads one to consider what is probably the simplest form of Bogoliubov transformation on a field operator.

We obtain the necessary and sufficient conditions that the potential must satisfy in order for the final field to describe satisfactorily the particles. The condition on the electric potential is not too restrictive but we find that no magnetic field is possible. We will also prove that, for such a system, the total charge is conserved.

In the second section, we examine a system where the external field is suddenly "switched on" to $A(\underset{\sim}{x})$ at some time and kept constant to this value after this time. The potential will then have the form $\theta(t-t_o)A(\underset{\sim}{x})$.

This problem also has not been studied before.

We will see however that the Bogoliubov transformations to be examined are the same as those obtained in certain treatments of the time independent problem (by H.E. Moses [1953], [1954], K.O. Friedrichs [1953], P.J.M. Bongaarts [1970], for example). These transformations have already been mentioned in Chapter I; they correspond to the following requirements. (As Moses and Friedrichs). One requires the unitary equivalence of the two sets of creation and annihilation operators associated with ψ_A with the help of the two sets of functions defined respectively by the spectrum of h and of h_o. (As Bongaarts) One demands the existence of a unitary evolution operator for the field ψ_A in the Fock-Hilbert space associated with the creation and annihilation operators defined by integrating ψ_A with the "free" functions $f^o_{\pm\beta}$.

These transformations are given much importance also by the fact that their unitary implementability appears, in the general time dependent problem, as a necessary condition for the satisfactory description of physical particles. This was mentioned in Chapter II (part on the S-matrix of section 2)) in connection with the transformation e^{-iha} on a free field where "a" is an arbitrary parameter.

We will prove the following crucial results concerning the two transformations mentioned above. At any fixed time, any one of the two is unitarily implementable if and only if the other one is also unitarily implementable. Furthermore, if they are unitarily implementable at any one given time, they will be unitarily implementable at all times.

Some of the implications of this have been discussed in the second section of Chapter I where this result has been mentioned already. Probably the most important one of them is that whenever a unitary evolution operator for the field exists, there exists also a well defined generator of the time translations which always has the physically expected form of the sum over the energies of all individual particles.

We then discuss the possibility of non-conservation of the charge. This possibility of individual particles creation or annihilation had been mentioned previously, for example, by A.I. Akhiezer and V.S. Berestetsky [1953] in their discussion of the properties of the eigenvalues of h as functions of the depth of the potential. By using the results which we have obtained (in Chapter III) on the properties of strong Bogoliubov transformations, we obtain here the new result that such a "phenomenon" is quite rigorously described within the external field formalism.

We next show how all the information about the physical system is obtained from the formalism. We will calculate as an example the probability amplitude that there be finally an electron in a given state when initially there was only a free electron in the system. We also discuss the possibility of defining isometric operators relating the two energy operators H_A and H_{in} which are defined on the same space.

In the final part of this section, we review briefly the known results concerning the conditions on $A(\underset{\sim}{x})$ for the relevant Bogoliubov transformations to be unitarily implementable. We then derive a different condition than that of Bongaarts [1970], which was the best one up to now. Although our conditions are somewhat comparable to his in that they still do not allow for any magnetic field, they are more simply derived.

We also obtain similar conditions in the case of a Dirac field coupled to a pseudo-vector field. There are, to our knowledge, no previous results for this problem.

In the third section, we study a family of more complicated models for which the time dependence of the potential is approximated by finite series of step functions. We prove that the requirement that either there exists a unitary evolution operator or that physical particles are satisfactorily described can be satisfied if and only if each potential of the series satisfies the conditions of the previous model. That is: if and only if each potential allows for a satisfactory treatment when used in the "one step function" problem.

We then discuss the limiting case of the above systems when the number of steps becomes arbitrarily large over a finite time interval. We show that the previous results can thus be generalized to include arbitrary piecewise continuous time dependences. This follows from the fact that in the many step functions case, all our proofs are independent of the number of steps and of the lengths of the time intervals.

This is an important result since it shows clearly that the way the external field varies in time is not really so crucial but it is much more its spatial properties which will determine whether or not the system can be satisfactorily described.

We finally note that the conditions we obtain on $A(t,\underset{\sim}{x})$ are more general than those of R. Seiler [1972]. He gives the sufficient conditions: $A_o(t,\underset{\sim}{x})$ is a test function in $\mathcal{S}(t,\underset{\sim}{x})$ and $\underset{\sim}{A}(t,\underset{\sim}{x}) = 0$. These, we believe, were the only previous results concerning such time dependent problems.

1) Electrons in a Potential $A(t,\underset{\sim}{x}) = A(\underset{\sim}{x})\delta(t)$

i) The differential equation of motion is

$$(-i\gamma.\partial + m)\psi(x) = e\gamma.A(\underset{\sim}{x})\delta(t)\psi(x) \quad . \tag{1.1}$$

For all $t < 0$, the term on the right hand side of this equation is null; ψ is therefore a free field. By definition, it is $\psi_{in}(x)$ i.e.

$$\psi(t,\underset{\sim}{x}) = \psi_{in}(t,\underset{\sim}{x}) \quad \forall\ t < 0 \ . \tag{1.2}$$

Similarly, for all $t > 0$, the external field is not acting on the quantized field which again is free and is $\psi_{out}(x)$ i.e.

$$\psi(t,\underset{\sim}{x}) = \psi_{out}(t,\underset{\sim}{x}) \quad \forall\ t > 0 \ . \tag{1.3}$$

We note that at $t = 0$ the field is not defined yet; all we know from (1.2) and (1.3) is that there is a discontinuity in the field at this time. (Otherwise $\psi_{out}(0,\underset{\sim}{x}) = \psi_{in}(0,\underset{\sim}{x})$ which implies $\psi_{out}(x) = \psi_{in}(x)$ and there is no interaction).

From the Källen-Yang-Feldman equations corresponding to (1.1), one gets

$$\psi_{out}(0,\underset{\sim}{x}) = \psi_{in}(0,\underset{\sim}{x}) + \int S(0-y_o,\underset{\sim}{x}-\underset{\sim}{y})e\gamma.A(\underset{\sim}{y})\delta(y_o)\psi(y_o,\underset{\sim}{y})d\underset{\sim}{y}dy_o$$

$$= \psi_{in}(0,\underset{\sim}{x}) + ie\gamma^o\gamma.A(\underset{\sim}{x})\psi(0,\underset{\sim}{x}) \ . \tag{1.4}$$

One can see that ψ_{out} depends very much on what is taken for $\psi(0,\underset{\sim}{x})$.

If $\psi(0,\underset{\sim}{x})$ is taken to be $\psi_{in}(0,\underset{\sim}{x})$ (or $\psi_{out}(0,\underset{\sim}{x})$), one gets what could be called a late (or early) interaction since this means that at time zero the field is still ψ_{in} (or already ψ_{out}). For late interaction,

$$\psi_{out}(0,\underset{\sim}{x}) = [1 + ie\gamma^{o}\gamma.A(\underset{\sim}{x})]\psi_{in}(0,\underset{\sim}{x}) \tag{1.5}$$

and for early interaction

$$\psi_{out}(0,\underset{\sim}{x}) = \frac{1}{[1 - ie\gamma^{o}\gamma.A(\underset{\sim}{x})]}\psi_{in}(0,\underset{\sim}{x}) \quad . \tag{1.6}$$

We will show later that these two solutions have to be rejected if one wants the interaction process to be unitary.

Let us consider the field

$$\psi(x) = \psi_{in}(x) + \theta(t)[\psi_{out}(x) - \psi_{in}(x)] \quad . \tag{1.7}$$

It satisfies the previously mentioned conditions (1.2) and (1.3) and has a "simple jump" at $t = 0$. One has

$$\delta(t)\psi(x) = \delta(t)\psi_{in}(x) + \delta(t)\theta(t)[\psi_{out}(x) - \psi_{in}(x)] \;;$$

the term $\delta(t)\theta(t)$ is undetermined since the function $\theta(t)$ is not defined at $t = 0$. We then consider for the moment the general form:

$$\delta(t)\psi(x) = \delta(t)\psi_{in}(x) + \alpha\delta(t)[\psi_{out}(x) - \psi_{in}(x)] \quad (1.8)$$

where α is a completely arbitrary real number (i.e., we take $\theta(t=0) = \alpha$). One can see that this solution (1.8) contains (1.5) and (1.6) as particular cases.

Upon replacing the field (1.7) in equation (1.1) (or equivalently in equation (1.4)), one gets

$$\psi_{out}(0,\underset{\sim}{x}) = \psi_{in}(0,\underset{\sim}{x}) + ie\gamma^{o}\gamma.A(\underset{\sim}{x})[(1-\alpha)\psi_{in}(0,\underset{\sim}{x})+\alpha\psi_{out}(0,\underset{\sim}{x})]$$

so that for

$$\psi_{out}(0,\underset{\sim}{x}) = \frac{[1 + ie(1-\alpha)\gamma^{o}\gamma.A(\underset{\sim}{x})]}{[1 - ie\alpha\gamma^{o}\gamma.A(\underset{\sim}{x})]}\psi_{in}(0,\underset{\sim}{x}), \quad (1.9)$$

the field (1.7) is a solution of the field equation. We note that

$$\frac{1}{[1-ie\alpha\gamma^{o}\gamma.A(\underset{\sim}{x})]} \equiv \frac{[(I-ie\alpha A_{o}(\underset{\sim}{x})) - ie\alpha\gamma^{o}\gamma.A(\underset{\sim}{x})]}{[1- 2ie\alpha A_{o}(\underset{\sim}{x}) - e^{2}\alpha^{2}A(\underset{\sim}{x}).A(\underset{\sim}{x})]}$$

and this term is always well defined.

ii) We will show in this section that requiring $\psi_{out}(x)$ and $\psi_{in}(x)$ to have the same anticommutation relations, which is necessary for them to be unitarily related, leads to a particular value of α which is $\alpha = \frac{1}{2}$.

The free fields $\psi_{out}(x)$ and $\psi_{in}(x)$ satisfy the same free Dirac equation so that they will have the same anticommutation relations for arbitrary space time points if and only if they have the same anticommutation relations at equal times. We then require that their t = 0 anticommutation relations be the same.

For $\psi_{in}(x)$, we have

$$\{\psi_{\mu}^{in}(0,\underset{\sim}{x}),\ \psi_{\nu}^{in}(0,\underset{\sim}{x}')^{\dagger}\} = \delta_{\mu\nu}\delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad (1.10)$$

From the equation (1.5), one knows that

$$\psi^{out}(0,\underset{\sim}{x}) = M(\underset{\sim}{x})\psi^{in}(0,\underset{\sim}{x}) \quad (1.11)$$

where $M(\underset{\sim}{x})$ is a 4 × 4 matrix. Therefore,

$$\{\psi_{\mu}^{out}(0,\underset{\sim}{x}),\psi_{\nu}^{out}(0,\underset{\sim}{x}')^{\dagger}\} = [M(\underset{\sim}{x})M^{\dagger}(\underset{\sim}{x})]_{\mu\nu}\ \delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad (1.12)$$

so that

$$M(\underset{\sim}{x})M^{\dagger}(\underset{\sim}{x}) = I = M^{\dagger}(\underset{\sim}{x})M(\underset{\sim}{x}) \quad (1.13)$$

is needed in order for (1.12) to be equal to (1.10).

In the present case,

$$M(x) = \frac{[1 + ie(1-\alpha)\gamma^{0}\gamma.A(\underset{\sim}{x})]}{[1 - ie\alpha\gamma^{0}\gamma.A(\underset{\sim}{x})]} ; \quad (1.14)$$

using the property $[e\gamma^{0}\gamma.A(\underset{\sim}{x})]^{\dagger} = [\overline{e\gamma^{0}\gamma.A(\underset{\sim}{x})}]$, it is easy to calculate that

$$M(x)M^{\dagger}(x) = \frac{\{1 + [e(1-\alpha)\gamma^{o}\gamma.A(x)]^{2}\}}{\{1 + [e\alpha\gamma^{o}\gamma.A(x)]^{2}\}} .$$

This will be equal to 1 if and only if $\alpha = \frac{1}{2}$.

Therefore a necessary condition to have ψ_{out} unitarily related to ψ_{in} is that $\psi(0,x)$ be taken as

$$\psi(0,x) = \frac{1}{2}\,[\lim_{t\to+0}\psi + \lim_{t\to-0}\psi] \tag{1.15}$$

so that

$$\psi_{out}(0,x) = \frac{[1 + i\,\frac{e}{2}\,\gamma^{o}\gamma.A(x)]}{[1 - i\,\frac{e}{2}\,\gamma^{o}\gamma.A(x)]}\,\psi_{in}(0,x) . \tag{1.16}$$

For general times t, the free field ψ_{out} is given by

$$\psi_{out}(t,x) = e^{-ih_{o}t}\,\psi_{out}(0,x)$$

$$= -i\int S(t,x-y)\gamma^{o}\,\psi_{out}(0,y)dy$$

so that

$$\psi_{out}(t,x) = -i\int S(t,x-y)\gamma^{o}M(y)\psi_{in}(0,y)dy . \tag{1.17}$$

iii) Let us use for the free field $\psi_{in}(x)$ the standard decomposition

$$\psi_{in}(x) = \frac{1}{(2\pi)^{3/2}}\int dp\,(\frac{m}{\omega(p)})^{\frac{1}{2}}\sum_{s=1}^{2}\{b_{s}(p)w^{s}(p)e^{-i[\omega(p)t-p.x]}$$

$$+ d_{s}^{\dagger}(p)v^{s}(p)e^{+i[\omega(p)t-p.x]}\} \tag{1.18}$$

where w and v are such that

$$[\gamma^{o}\omega(\underset{\sim}{p})-\underset{\sim}{\gamma}.\underset{\sim}{p}-m]w(\underset{\sim}{p}) = 0 \quad \text{and} \quad [\gamma^{o}\omega(\underset{\sim}{p})-\underset{\sim}{\gamma}.\underset{\sim}{p}+m]v(\underset{\sim}{p}) = 0 \ .$$

We shall need also the Fourier transform of $M(x)$ defined as

$$\tilde{M}(\underset{\sim}{k}) = \frac{1}{(2\pi)^{3/2}} \int d\underset{\sim}{x}\ e^{-i\underset{\sim}{k}.\underset{\sim}{x}}\ M(\underset{\sim}{x}) \ . \qquad (1.19)$$

The final free field has a similar decomposition as (1.18) in terms of the final creation and annihilation operators for the particles. These can then be obtained from $\psi_{out}(0,\underset{\sim}{x})$ as

$$b_s^{out}(\underset{\sim}{k}) = \left(\frac{m}{\omega(\underset{\sim}{k})}\right)^{\frac{1}{2}} w^s(\underset{\sim}{k})^{\dagger} \frac{1}{(2\pi)^{3/2}} \int d\underset{\sim}{x}\ e^{-i\underset{\sim}{k}.\underset{\sim}{x}}\ \psi_{out}(0,\underset{\sim}{x})$$

$$d_s^{out}(\underset{\sim}{k})^{\dagger} = \left(\frac{m}{\omega(\underset{\sim}{k})}\right)^{\frac{1}{2}} v^s(\underset{\sim}{k}) \frac{1}{(2\pi)^{3/2}} \int d\underset{\sim}{x}\ e^{i\underset{\sim}{k}.\underset{\sim}{x}}\ \psi_{out}(0,\underset{\sim}{x}) \ . \qquad (1.20)$$

Using the relation (1.11) between $\psi_{out}(0,\underset{\sim}{x})$ and $\psi_{in}(0,\underset{\sim}{x})$, the decomposition (1.18) and the definition (1.19), one obtains the following Bogoliubov transformation:

$$b_s^{out}(\underset{\sim}{k}) = \frac{1}{(2\pi)^{3/2}} \int d\underset{\sim}{p} \left(\frac{m}{\omega(\underset{\sim}{p})}\right)^{\frac{1}{2}} \left(\frac{m}{\omega(\underset{\sim}{k})}\right)^{\frac{1}{2}} \sum_{r=1}^{2} \{ [w^s(\underset{\sim}{k})^{\dagger} \tilde{M}(\underset{\sim}{k}-\underset{\sim}{p}) w^r(\underset{\sim}{p})] \times$$

$$\times\, b_r(\underset{\sim}{p}) + [w^s(\underset{\sim}{k})^{\dagger} \tilde{M}(\underset{\sim}{k}+\underset{\sim}{p}) v^r(\underset{\sim}{p})] d_r^{\dagger}(\underset{\sim}{p}) \}$$

$$d_s^{out}(\mathbf{k})^\dagger = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{p}\left(\frac{m}{\omega(\mathbf{p})}\right)^{\frac{1}{2}}\left(\frac{m}{\omega(\mathbf{k})}\right)^{\frac{1}{2}}\sum_{r=1}^{2}\{[v^s(\mathbf{k})^\dagger\tilde{M}(-\mathbf{k}-\mathbf{p})w^r(\mathbf{p})]b_r(\mathbf{p})$$

$$+\ [v^s(\mathbf{k})^\dagger\tilde{M}(-\mathbf{k}+\mathbf{p})v^r(\mathbf{p})]d_r^\dagger(\mathbf{p})\}\ . \qquad (1.21)$$

As seen in Chapter III on Bogoliubov transformations, there will exist an S-matrix relating the "out" Fock basis to the "in" Fock basis or, equivalently, (1.21) will be unitarily implementable, if and only if

$$\int d\mathbf{p}\, d\mathbf{k} \sum_{r=1}^{2}\sum_{s=1}^{2}\left(\frac{m}{\omega(\mathbf{p})}\right)\left(\frac{m}{\omega(\mathbf{k})}\right)|w^s(\mathbf{k})^\dagger\,\tilde{M}(\mathbf{k}+\mathbf{p})v^r(\mathbf{p})|^2 < \infty \qquad (1.22a)$$

$$\int d\mathbf{p}\, d\mathbf{k} \sum_{r=1}^{2}\sum_{s=1}^{2}\left(\frac{m}{\omega(\mathbf{p})}\right)\left(\frac{m}{\omega(\mathbf{k})}\right)|v^s(\mathbf{k})^\dagger\,\tilde{M}(-\mathbf{k}-\mathbf{p})w^r(\mathbf{p})|^2 < \infty\ . \qquad (1.22b)$$

In order to study (1.22a), we will use the identity

$$\sum_{r=1}^{2}\sum_{s=1}^{2}\left(\frac{m}{\omega(\mathbf{p})}\right)\left(\frac{m}{\omega(\mathbf{k})}\right)|w^s(\mathbf{k})^\dagger\,\tilde{M}(\mathbf{k}+\mathbf{p})v^r(\mathbf{p})|^2$$

$$= \frac{1}{4\omega(\mathbf{p})\omega(\mathbf{k})}\,\text{Trace}\ \tilde{M}^\dagger(\mathbf{k}+\mathbf{p})[\gamma.k+m]\gamma^0\tilde{M}(\mathbf{k}+\mathbf{p})[\gamma.p-m]\gamma^0 \qquad (1.23)$$

and write $M(\mathbf{x})$ as

$$M(\mathbf{x}) = c_0(\mathbf{x}) - \gamma^0\boldsymbol{\gamma}.\mathbf{c}(\mathbf{x}) \qquad (1.24)$$

where

$$c_0(\mathbf{x}) = \frac{1+\frac{e^2}{4}\,\mathbf{A}(\mathbf{x}).\mathbf{A}(\mathbf{x})}{[1-ieA_0(\mathbf{x}) - \frac{e^2}{4}\,\mathbf{A}(\mathbf{x}).\mathbf{A}(\mathbf{x})]}$$

$$c(x) = \frac{-ieA(x)}{[1 - ieA_0(x) - \frac{e^2}{4} A(x).A(x)]}$$

The condition (1.22a) can then be written as

$$\int dp\, dk\, \frac{1}{4\omega(p)\omega(k)} \text{Trace}[\gamma.\tilde{c}^*(k+p)][\gamma.k+m] \times$$

$$\times [\gamma.\tilde{c}(k+p)][\gamma.p-m] < \infty .$$

The trace is easily calculated by using the properties of the γ matrices and one obtains

$$\int \frac{dp\, dk}{\omega(p)\omega(k)}\{(p.\tilde{c}^*)(k.\tilde{c})+(k.\tilde{c}^*)(p.\tilde{c})-(k.p)(\tilde{c}^*.c)-m^2(\tilde{c}^*.c)\}<\infty \qquad (1.25)$$

where we have used $\tilde{c}$ for $\tilde{c}(p+k)$.

The integrand in (1.25) is always positive, as is obvious from the left hand side of (1.23). It must therefore go to zero at infinity in any direction of the six-dimensional space (p,k) if (1.25) is to be satisfied. However, upon evaluating its limit, for any constant $(p+k) = a$, as $p_1 \to \epsilon\infty$; $(k_1 \to -\epsilon\infty)$; ϵ being "+" or "-", one obtains that it goes to

$$2|\tilde{c}_2(a)|^2 + 2|\tilde{c}_3(a)|^2 .$$

The integral can then be finite only if $\tilde{c}_2(a)$ and $\tilde{c}_3(a)$ are null for all "a". Calculating similarly the limit of the integrand as $p_2 \to \epsilon\infty$, one obtains that $\tilde{c}_1(\underset{\sim}{a})$ must also be null for all "$\underset{\sim}{a}$". Since $\tilde{c}(\underset{\sim}{a})$ is the Fourier transform of $c(\underset{\sim}{x})$, one must then have $c(\underset{\sim}{x}) = 0$ for all $\underset{\sim}{x}$. From the definition of $c(\underset{\sim}{x})$, one can see that this is so if and only if $A(\underset{\sim}{x}) = 0$ for all $\underset{\sim}{x}$.

The only case to be considered now is then

$$M(\underset{\sim}{x}) = c_0(\underset{\sim}{x}) = \frac{1 + i\frac{e}{2}A_0(\underset{\sim}{x})}{1 - i\frac{e}{2}A_0(\underset{\sim}{x})} . \tag{1.26}$$

This can be written as

$$c_0(\underset{\sim}{x}) = 1 + F(\underset{\sim}{x})$$

with

$$F(\underset{\sim}{x}) = \frac{ieA_0(\underset{\sim}{x})}{1 - i\frac{e}{2}A_0(\underset{\sim}{x})} \tag{1.27}$$

so that

$$\tilde{c}_0(\underset{\sim}{k}) = (2\pi)^{3/2}\,\delta(\underset{\sim}{k}) + \tilde{F}(\underset{\sim}{k}) .$$

Both conditions (1.22) in this case can be seen to reduce to

$$\int \frac{d\underset{\sim}{p}\,d\underset{\sim}{k}}{\omega(\underset{\sim}{p})\omega(\underset{\sim}{k})}\,[\omega(\underset{\sim}{k})\omega(\underset{\sim}{p}) + \underset{\sim}{p}\cdot\underset{\sim}{k} - m^2]\,|\tilde{F}(\underset{\sim}{p}+\underset{\sim}{k})|^2 < \infty . \tag{1.28}$$

Upon changing the variables of integration to

$$\underset{\sim}{q} = \frac{1}{2}(\underset{\sim}{k}+\underset{\sim}{p}) \qquad \underset{\sim}{\ell} = \frac{1}{2}(\underset{\sim}{k}-\underset{\sim}{p}) ,$$

condition (1.28) can be written as

$$\int d\underset{\sim}{q}\,|\tilde{F}(2\underset{\sim}{q})|^2 \int d\underset{\sim}{\ell}\left(1 - \frac{m^2 + \underset{\sim}{\ell}^2 - \underset{\sim}{q}^2}{\omega(\underset{\sim}{\ell}+\underset{\sim}{q})\,\omega(\underset{\sim}{\ell}-\underset{\sim}{q})}\right) < \infty . \tag{1.29}$$

The $\underset{\sim}{\ell}$ integration can be performed explicitly; after some manipulations, with the help of standard tables of integrals, one obtains that the above condition is

$$\int d\underset{\sim}{q} |\tilde{F}(2\underset{\sim}{q})|^2 |\underset{\sim}{q}| \omega(\underset{\sim}{q}) \; P_{\frac{1}{2}}^{-1} \left(\frac{m^2 - \underset{\sim}{q}^2}{m^2 + \underset{\sim}{q}^2}\right) < \infty . \qquad (1.30)$$

where $P_{\frac{1}{2}}^{-1}$ is an associate Legendre polynomial. It has the following property:

$$P_{\frac{1}{2}}^{-1}(\cos\phi) \leq \frac{8}{\sqrt{\pi}} \sin^{\frac{1}{2}}\phi$$

so that

$$P_{\frac{1}{2}}^{-1}\left(\frac{m^2 - \underset{\sim}{q}^2}{m^2 + \underset{\sim}{q}^2}\right) < \frac{8}{\sqrt{\pi}} \sqrt{2m} \; \frac{|\underset{\sim}{q}|^{\frac{1}{2}}}{\omega(\underset{\sim}{q})}$$

and (1.30) is certainly satisfied if

$$\int d\underset{\sim}{q} |\underset{\sim}{q}|^{3/2} |\tilde{F}(\underset{\sim}{q})|^2 < \infty . \qquad (1.31)$$

This shows that (1.30) is not a too strong condition on $F(\underset{\sim}{q})$. Furthermore, since

$$F(\underset{\sim}{x}) = \frac{i}{\frac{1}{eA_0(\underset{\sim}{x})} - \frac{i}{2}}$$

$A_0(\underset{\sim}{x})$ could have a very bad behaviour as a function of $\underset{\sim}{x}$ while (1.30) could be satisfied. For example, if at some point $\bar{x}$, $A_0(\bar{x})$ is infinite, $F(\bar{x}) = -2$. In fact, $F(\underset{\sim}{x})$ is always bounded; whatever $A_0(\underset{\sim}{x})$, we have $|F(\underset{\sim}{x})|^2 \leq 4$.

iv) We now prove that, for the system considered, no isolated particles can be created with certainty; the Bogoliubov transformation (1.21) is a "weak" one and the total charge is conserved.

This is shown by proving that there are no non-trivial solutions ψ and ϕ to the equations

$$\frac{1}{(2\pi)^{3/2}}\int d\underset{\sim}{p} \sum_{r=1}^{2} \left(\frac{m}{\omega(\underset{\sim}{p})}\right)^{\frac{1}{2}} w^s(\underset{\sim}{k})^\dagger \tilde{c}_o(\underset{\sim}{k}-\underset{\sim}{p}) w^r(\underset{\sim}{p})\psi(\underset{\sim}{p},r)=0 \quad \forall\ \underset{\sim}{k},s \tag{1.32}$$

$$\frac{1}{(2\pi)^{3/2}}\int d\underset{\sim}{p} \sum_{r=1}^{2} \left(\frac{m}{\omega(\underset{\sim}{p})}\right)^{\frac{1}{2}} v^r(\underset{\sim}{p})^\dagger \tilde{c}_o^*(\underset{\sim}{p}-\underset{\sim}{k}) v^s(\underset{\sim}{k})\phi(\underset{\sim}{p},r)=0 \quad \forall\ \underset{\sim}{k},s \tag{1.33}$$

One can see that ψ is a solution of (1.32) if and only if there exists a non-zero function $a(\underset{\sim}{k},s)$ such that

$$\frac{1}{(2\pi)^{3/2}}\int d\underset{\sim}{p} \sum_{r=1}^{2} \tilde{c}_o(\underset{\sim}{k}-\underset{\sim}{p}) w^r(\underset{\sim}{p})\psi(\underset{\sim}{p},r) = \sum_{s=1}^{2} \left(\frac{m}{\omega(\underset{\sim}{k})}\right)^{\frac{1}{2}} v^s(-\underset{\sim}{k})a(\underset{\sim}{k},s) \tag{1.34}$$

since the 4-vectors w^s and v^s, $s = 1,2$, form a basis in E^4. Let us denote as $\Psi(\underset{\sim}{k})$ the right hand side of (1.34). Since it is a property of the 4-vectors v^s that $v^s(\underset{\sim}{k})\gamma^o v^{s'}(\underset{\sim}{k}) = -\delta_{ss'}$, we have

$$\int d\underset{\sim}{k}\ \Psi(\underset{\sim}{k})^\dagger \gamma^o \Psi(\underset{\sim}{k}) = -\int d\underset{\sim}{k} \sum_{s=1}^{2} \left(\frac{m}{\omega(\underset{\sim}{k})}\right) |a(\underset{\sim}{k},s)|^2 . \tag{1.35}$$

Using now the left hand side of equation (1.34) together with the property

$$\frac{1}{(2\pi)^3}\int d\underset{\sim}{k}\ \tilde{c}_o^*(\underset{\sim}{k}-\underset{\sim}{p}')\tilde{c}_o(\underset{\sim}{k}-\underset{\sim}{p}) = \delta(\underset{\sim}{p}-\underset{\sim}{p}') \qquad (1.36)$$

which follows from $c_o^*(\underset{\sim}{x})c_o(\underset{\sim}{x}) = 1$, the following is obtained:

$$\int d\underset{\sim}{k}\ \Psi(\underset{\sim}{k})^\dagger \gamma^o \Psi(\underset{\sim}{k}) = \int d\underset{\sim}{p} \sum_{r=1}^{2}\sum_{r'=1}^{2}\left(\frac{m}{\omega(\underset{\sim}{p})}\right)\psi^*(\underset{\sim}{p},r)\psi(\underset{\sim}{p},r')w^r(\underset{\sim}{p})^\dagger \times$$

$$\times\ \gamma^o w^{r'}(\underset{\sim}{p}) \quad .$$

Since the 4-vectors w^r have the property $w^r(\underset{\sim}{p})^\dagger\gamma^o w^{r'}(\underset{\sim}{p}) = \delta_{rr'}$,

$$\int d\underset{\sim}{k}\ \Psi(\underset{\sim}{k})^\dagger\gamma^o\Psi(\underset{\sim}{k}) = \int d\underset{\sim}{p}\sum_{r=1}^{2}\left(\frac{m}{\omega(\underset{\sim}{p})}\right)|\psi(\underset{\sim}{p},r)|^2 \ . \qquad (1.37)$$

Clearly, (1.37) is compatible with (1.35) if and only if $\psi(\underset{\sim}{p},r) = 0 = a(\underset{\sim}{k},s)$, since one is always negative while the other is always positive. The only possible solution to equation (1.32) is then $\psi \equiv 0$.

We can show, in a similar manner, that the only possible solution to equation (1.33) is $\phi \equiv 0$. We thus obtain the announced result.

2) Electrons in a Potential $A(t,\underset{\sim}{x}) = A(\underset{\sim}{x})\theta(t-t_o)$

The equation of motion is

$$(-i\gamma.\partial + m)\psi(t,\underset{\sim}{x}) = \theta(t-t_o)e\gamma.A(\underset{\sim}{x})\psi(t,\underset{\sim}{x}) \ . \qquad (2.1)$$

We see that $\forall\ t < t_o$, the solution of (2.1) is the free field $\psi_{in}(x)$. For times $t > t_o$, the field must satisfy the equation

$$(-i\gamma.\partial + m)\psi(t,\underset{\sim}{x}) = e\gamma.A(\underset{\sim}{x})\psi(t,\underset{\sim}{x}) \qquad (2.2)$$

$\psi_A(x)$ will denote the fields satisfying this equation.

One can see by integrating (2.1) about t_o, that, as a distribution, the field must be continuous in t at time t_o. We then have ψ_A determined by the initial condition

$$\psi_A(t_o,\underset{\sim}{x}) = \psi_{in}(t_o,\underset{\sim}{x})$$

and the differential equation (2.1) is equivalent to the relations

$$\psi(t,\underset{\sim}{x}) = \psi_{in}(t,\underset{\sim}{x}) \quad \forall\ t \le t_o$$

$$= \psi_A(t,\underset{\sim}{x}) = e^{-ih(t-t_o)}\psi_{in}(t_o,\underset{\sim}{x}) \quad \forall\ t \ge t_o \qquad (2.3)$$

where $h = \gamma^o(i\underset{\sim}{\gamma}.\underset{\sim}{\partial}+m) - e\gamma^o\gamma.A(\underset{\sim}{x})$.

The c-number operator e^{-iha} is unitary on $(L^2)^4$ for all "a". As we have seen in the Chapters I and II, $\psi(t,\underset{\sim}{x})$ is then at all times a well defined operator valued distribution in $(L^2)^4$ acting on the Fock-Hilbert space $\mathcal{H}_{in}$ of the free field ψ_{in}.

All the previous detailed general discussion of Chapter II can be applied here upon using $u(t,t') = e^{-ih(t-t_o)} e^{-ih_o(t_o-t')}$ for $t > t_o$, $t' < t_o$. The time at which the potential is "switched on" (which was t_s in Chapter II) is here t_o.

We recall that there will exist a state $|\phi_o\rangle$ consisting of zero final particles if and only if the following Bogoliubov transformation is unitarily implementable.

$$b_\gamma(t) = \sum_\beta \{ (f_{+\gamma}, e^{-ih(t-t_o)} f^o_{+\beta}) b^{in}_\beta(t_o) + (f_{+\gamma}, e^{-ih(t-t_o)} f^o_{-\beta}) \times$$
$$\times\, d^{in}_\beta(t_o)^\dagger \}$$

$$d^\dagger_\lambda(t) = \sum_\beta \{ (f_{-\lambda}, e^{-ih(t-t_o)} f^o_{+\beta}) b^{in}_\beta(t_o) + (f_{-\lambda}, e^{-ih(t-t_o)} f^o_{-\beta}) \times$$
$$\times\, d^{in}_\beta(t_o)^\dagger \} \quad . \qquad (2.4)$$

This is the transformation corresponding to (2.13) in Chapter II. We remark that, after t_o, the vacuum will be here time independent exactly as the initial vacuum

$|0\rangle$ was up to time t_o. This is due to the fact that the external field does not change after t_o.

It is easy to see that examining the existence of a unitary operator U(t,t') relating the field operators at any two times t and t', reduces here to examining the existence of $U(t,t_o)$ for any time $t > t_o$. The Bogoliubov transformation to be studied in this case is the one corresponding to (3.4) of Chapter II, i.e.

$$b^o_\alpha(t)=\sum_\beta\{(f^o_{+\alpha},e^{-ih(t-t_o)}f^o_{+\beta})b^{in}_\beta(t_o)+(f^o_{+\alpha},e^{-ih(t-t_o)}f^o_{-\beta})\times$$
$$\times\, d^{in}_\beta(t_o)^\dagger\}$$

$$d^o_\alpha(t)^\dagger=\sum_\beta\{(f^o_{-\alpha},e^{-ih(t-t_o)}f^o_{+\beta})b^{in}_\beta(t_o)+(f^o_{-\alpha},e^{-ih(t-t_o)}f^o_{-\beta})\times$$
$$\times\, d^{in}_\beta(t_o)^\dagger\} \qquad (2.5)$$

As we have seen in Chapter III, the conditions under which (2.4) is unitarily implementable are

$$\sum_\gamma\sum_\beta|(f_{+\gamma},e^{-ih(t-t_o)}f^o_{-\beta})|^2<\infty \qquad (2.6)$$

$$\sum_\lambda\sum_\beta|(f_{-\lambda},e^{-ih(t-t_o)}f^o_{+\beta})|^2<\infty \qquad (2.7)$$

For the transformation (2.5), these are

$$\sum_\alpha\sum_\beta|(f^o_{+\alpha},e^{-ih(t-t_o)}f^o_{-\beta})|^2<\infty \qquad (2.8)$$

$$\sum_\alpha\sum_\beta|(f^o_{-\alpha},e^{-ih(t-t_o)}f^o_{+\beta})|^2<\infty \qquad (2.9)$$

The transformation (2.5), with $t = 0$, is exactly the same as (2.19) of Chapter I. In this last case, this was the transformation to be studied if one required that there exist a unitary evolution operator on the Fock-Hilbert space associated with the solution of the time independent problem exactly as if it were a free field. The conditions (2.8) and (2.9) were examined by P.J.M. Bongaarts [1970] in this context.

The conditions (2.6) and (2.7) are the same as those examined by H.E. Moses [1953], [1954] and K.O. Friedrichs [1953]. In their treatment of the time independent problem, a transformation like (2.5) was to be studied if one required the following. The two sets of creation and annihilation operators defined respectively with $\{f_{+\gamma}(\underset{\sim}{x}), f_{-\lambda}(\underset{\sim}{x})\}$ and $\{f^{o}_{\varepsilon\beta}(\underset{\sim}{x})\}$ both have a vacuum state in the same space.

Proof of the Equivalence of all the Physical Requirements

We now show that, at any fixed time $t \neq t_o$, the conditions (2.8) and (2.9) are satisfied if and only if the conditions (2.6) and (2.7) are satisfied. Furthermore, if (2.6) and (2.7) or (2.8) and (2.9) hold at any one time $t \neq t_o$, they will also hold for all times t.

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The transformation (2.5), with $t_o = 0$, is exactly the same as (2.19) of Chapter I. In this last case, this was the transformation to be studied if one required that there exist a unitary evolution operator on the Fock-Hilbert space associated with the solution of the time independent problem exactly as if it were a free field. The conditions (2.8) and (2.9) were examined by P.J.M. Bongaarts [1970] in this context.

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We remark that the proof will be completely independent of the type of interaction and is valid for any problem where a self-adjoint c-number energy operator h exists.

i) We first show that for any given time $t \neq t_s$, the conditions (2.6) and (2.7) imply (2.8) and (2.9).

Since $\{f_{+\gamma}\}$ and $\{f_{-\lambda}\}$ form a basis in $(L^2)^4$, we have

$$(f^o_{+\alpha}, e^{-ih(t-t_o)} f^o_{-\beta}) = (f^o_{+\alpha}, \sum_\gamma f_{+\gamma})(f_{+\gamma}, e^{-ih(t-t_o)} f^o_{-\beta})$$

$$+ (f^o_{+\alpha}, \sum_\lambda f_{-\lambda})(f_{-\lambda}, e^{-ih(t-t_o)} f^o_{-\beta}) \qquad (2.10)$$

Defining

$$||\phi_{\alpha\beta}|| = \{\sum_{\alpha\beta}|\phi_{\alpha\beta}|^2\}^{\frac{1}{2}} ,$$

and using the inequality

$$||\phi_{\alpha\beta} + \phi'_{\alpha\beta}|| \leq ||\phi_{\alpha\beta}|| + ||\phi'_{\alpha\beta}|| ,$$

we see that if the norms of the two terms on the right hand side of (2.10) are finite, the right hand side of (2.10) will have a finite norm.

Upon using the general property

$$\sum_\alpha |(f^o_{+\alpha}, \psi)|^2 \leq (\psi, \psi) , \qquad (2.11)$$

which comes from

$$\sum_\alpha |(f^o_{+\alpha},\psi)|^2 + \sum_\alpha |(f^o_{-\alpha},\psi)|^2 = (\psi,\psi) \ ,$$

with $\psi) = \sum_\gamma f_{+\gamma})(f_{+\gamma}, e^{-ih(t-t_o)} f^o_{-\beta})$, we obtain

$$||(f^o_{+\alpha}, \sum_\gamma f_{+\gamma})(f_{+\gamma}, e^{-ih(t-t_o)} f^o_{-\beta})||^2 \le \sum_{\beta\gamma} |(f_{+\gamma}, e^{-ih(t-t_o)} f^o_{-\beta})|^2 \tag{2.12}$$

and this is finite by hypothesis. Similarly, upon using

$\sum_\beta |(\psi, f^o_{-\beta})|^2 \le (\psi,\psi)$, with $(\psi = (f^o_{+\alpha}, \sum_\lambda f_{-\lambda})(e^{-ih(t-t_o)} f_{-\lambda};$

we obtain

$$||(f^o_{+\alpha}, \sum_\lambda f_{-\lambda})(f_{-\lambda}, e^{-ih(t-t_o)} f^o_{-\beta})||^2 \le \sum_{\alpha\lambda} |(f_{-\lambda}, f^o_{+\alpha})|^2 \ . \tag{2.13}$$

We have

$$\sum_{\alpha\lambda} |(f_{-\lambda}, e^{-ih(t-t_o)} f^o_{+\alpha})|^2 \equiv \sum_{\alpha\lambda} |(e^{ih(t-t_o)} f_{-\lambda}, f^o_{+\alpha})|^2 \tag{2.14}$$

and because of the way the set of functions $\{f_{-\lambda}\}$ is defined with respect to the spectrum of h, the unitary operator $e^{ih(t-t_o)}$ maps it back into itself in a one to one fashion. Therefore

$$\sum_{\lambda}|(e^{ih(t-t_o)}f_{-\lambda},f^o_{+\alpha})|^2 = \sum_{\lambda}|(f_{-\lambda},f^o_{+\alpha})|^2 \qquad (2.15)$$

since the sums run over all values of λ. Having (2.14) finite by hypothesis, (2.13) will then also be finite.

This shows that $||(f^o_{+\alpha},e^{-ih(t-t_o)}f^o_{-\beta})|| < \infty$; by similar arguments, we can also show that $||(f^o_{-\alpha},e^{-ih(t-t_o)}f^o_{+\beta})|| < \infty$.

ii) We now prove that if at some time $t \neq t_o$ the conditions (2.8) and (2.9) are satisfied, the conditions (2.6) and (2.7) are also satisfied. We have the identity

$$(f_{+\gamma},e^{-ih(t-t_o)}f^o_{-\beta})-(f_{+\alpha},P^o_-e^{-h(t-t_o)}f^o_{-\beta})$$
$$=(f_{+\gamma},P^o_+e^{-ih(t-t_o)}f^o_{-\beta}) \qquad (2.16)$$

where

$$P^o_- = \sum_{\alpha} f^o_{-\alpha})(f^o_{-\alpha} \quad \text{and} \quad P^o_+ = \sum_{\alpha} f^o_{+\alpha})(f^o_{+\alpha} .$$

Using $||A|| - ||B|| \leq ||A - B||$, we obtain

$$||(f_{+\gamma},e^{-ih(t-t_o)}f^o_{-\beta})||-||(f_{+\gamma},P^o_-e^{-ih(t-t_o)}f^o_{-\beta})|| \leq$$
$$||(f_{+\gamma},P^o_+e^{-ih(t-t_o)}f^o_{-\beta})|| . \qquad (2.17)$$

Since

$$||(f_{+\gamma}, P_{-}^{o} e^{-ih(t-t_o)} f_{-\beta}^{o})||^2 = ||(f_{+\gamma}, f_{-\alpha}^{o})||^2 - ||(f_{+\gamma}, P_{-}^{o} e^{-ih(t-t_o)} f_{+\beta}^{o})||^2 ,$$

if $(f_{+\gamma}, P_{-}^{o} e^{-ih(t-t_o)} f_{+\beta}^{o}) \neq 0$, we can find a $c < 1$ such that

$$||(f_{+\gamma}, P_{-}^{o} e^{-ih(t-t_o)} f_{-\beta}^{o})||^2 = c^2 ||(f_{+\gamma}, f_{-\alpha}^{o})||^2 . \quad (2.18)$$

By the same argument used to show that

$$||(f_{-\lambda}, e^{-ih(t-t_o)} f_{+\alpha}^{o})|| = ||(f_{-\lambda}, f_{+\alpha}^{o})|| ,$$

we obtain

$$||(f_{+\gamma}, f_{-\alpha}^{o})|| = ||(f_{+\gamma}, e^{-ih\tau} f_{-\alpha}^{o})|| \quad \forall \tau \qquad (2.19)$$

so that (2.18) implies

$$||(f_{+\gamma}, P_{-}^{o} e^{-ih(t-t_o)} f_{-\beta}^{o})|| = c\, ||(f_{+\gamma}, e^{-ih(t-t_o)} f_{-\alpha}^{o})||$$

which when replaced in (2.17) gives

$$(1-c)||(f_{+\gamma}, E^{-ih(t-t_o)} f_{-\beta}^{o})|| \leq ||(f_{+\gamma}, P_{+}^{o} e^{-ih(t-t_o)} f_{-\beta}^{o})|| . \qquad (2.20)$$

Since

$$\sum_{\gamma} |(f_{+\gamma},\psi)|^2 = (\psi,\psi) - \sum_{\lambda} |(f_{\lambda},\psi)|^2 \leq (\psi,\psi) ;$$

using

$$\psi = \sum_{\alpha} f^{o}_{+\alpha}(f^{o}_{+\alpha}, e^{-ih(t-t_o)} f^{o}_{-\beta}) ,$$

we obtain from (2.20):

$$(1-c)||(f_{+\gamma}, e^{-ih(t-t_o)} f^{o}_{-\beta})|| \leq ||(f^{o}_{+\alpha}, e^{-ih(t-t_o)} f^{o}_{-\beta})||$$

and since $(1-c) > 0$ and

$$||(f^{o}_{+\alpha}, e^{-ih(t-t_o)} f^{o}_{-\beta})|| < \infty ,$$

this implies that

$$||(f_{+\gamma}, e^{-ih(t-t_o)} f^{o}_{-\beta})|| < \infty.$$

Let us now consider the case where

$$(f_{+\gamma}, P^{o}_{-} e^{-ih(t-t_o)} f^{o}_{+\beta}) = 0 . \qquad (2.21)$$

We remark that if $(f_{+\gamma}, P^{o}_{-} e^{+ih(t-t_o)} f^{o}_{+\beta})$ was not also null, we could as above prove that

$$||(f_{+\gamma}, e^{ih(t-t_o)} f^{o}_{-\beta})|| < \infty ,$$

using

$$||(f^{o}_{+\alpha}, e^{ih(t-t_o)} f^{o}_{-\beta})|| < \infty$$

which holds by hypothesis. Since

$$||(f_{+\gamma}, e^{ih(t-t_o)} f^o_{-\beta})|| = ||(f_{+\gamma}, e^{-ih(t-t_o)} f^o_{-\beta})||$$

by equation (2.19), this would imply that

$$||(f_{+\gamma}, e^{-ih(t-t_o)} f^o_{-\beta})|| < \infty ;$$

we therefore have to consider only the case where also

$$(f_{+\gamma}, P^o_-, e^{ih(t-t_o)} f^o_{+\beta}) = 0 .$$

Equation (2.21) implies that

$$||(f^o_{\pm\beta}, \sum_\gamma f_{+\gamma})(f_{+\gamma}, f^o_{-\alpha}) + (f^o_{+\beta}, e^{ih(t-t_o)} f^o_{-\alpha})||^2$$

$$= ||(f^o_{\pm\beta}, \sum_\gamma f_{+\gamma})(f_{+\gamma}, f^o_{-\alpha})||^2 + ||(f^o_{+\beta}, e^{ih(t-t_o)} f^o_{-\alpha})||^2 .$$

Upon using $||A|| - ||B|| \leq ||A + B||$, we should therefore have

$$||(f^o_{\pm\beta}, \sum_\gamma f_{+\gamma})(f_{+\gamma}, f^o_{-\alpha})|| - ||(f^o_{+\beta}, e^{ih(t-t_o)} f^o_{-\alpha})|| \leq$$

$$\left[||(f^o_{\pm\beta}, \sum_\gamma f_{+\gamma})(f_{+\gamma}, f^o_{-\alpha})||^2 + ||(f^o_{+\beta}, e^{+ih(t-t_o)} f^o_{-\alpha})||^2\right]^{1/2} . \tag{2.22}$$

Since $\sqrt{a^2+b^2} < |a| + |b|$ if $|a||b| \neq 0$ and $\sqrt{a^2+b^2} = |a|+|b|$ if $|a||b| = 0$, supposing that $(f^o_{\pm\beta}, \sum_\gamma f_{+\gamma})(f_{+\gamma}, f^o_{-\alpha}) \neq 0$ and $(f^o_{+\beta}, e^{ih(t-t_o)} f^o_{-\alpha}) \neq 0$, we can then always find a $c < 1$ such that (2.22) implies

$$||(f^{o}_{\pm\beta},\sum_{\gamma} f_{+\gamma})(f_{+\gamma},f^{o}_{-\alpha})|| - ||(f^{o}_{+\beta},e^{ih)t-t_{o})} f^{o}_{-\alpha})|| \leq$$

$$c\{||(f^{o}_{\pm\beta},\sum_{\gamma} f_{+\gamma})(f_{+\gamma},f^{o}_{-\alpha})|| + ||(f^{o}_{+\beta},e^{ih(t-t_{o})} f^{o}_{-\alpha})||\}$$

so that

$$||(f^{o}_{\pm\beta},\sum_{\gamma} f_{+\gamma})(f_{+\gamma},f^{o}_{-\alpha})|| \leq \frac{(1+c)}{(1-c)}||(f^{o}_{+\beta},e^{ih(t-t_{o})} f^{o}_{-\alpha})||<\infty \quad (2.23)$$

Since

$$||(f_{+\gamma},f^{o}_{-\alpha})||^{2} = ||(f^{o}_{+\beta},\sum_{\gamma} f_{+\gamma})(f_{+\gamma},f^{o}_{-\alpha})||^{2} + $$

$$+ ||(f^{o}_{-\beta},\sum_{\gamma} f_{+\gamma})(f_{+\gamma},f^{o}_{-\alpha})||^{2} , \quad (2.24)$$

equation (2.23) implies $||(f_{+\gamma},f^{o}_{-\alpha})|| < \infty$.

When one of the two terms $(f^{o}_{\pm\beta},\sum_{\gamma} f_{+\gamma})(f_{+\gamma},f^{o}_{-\alpha})$ is null, it is easy to see that the above conclusion nevertheless holds, since the remaining term on the right hand side of (2.24) always satisfies (2.23).

Let us now consider the case where

$$(f^{o}_{+\beta},e^{ih(t-t_{o})} f^{o}_{-\alpha}) = 0 \quad (2.25)$$

As before, if $(f^{o}_{+\beta},e^{-ih(t-t_{o})} f^{o}_{-\alpha}) \neq 0$, we can show that $||(f_{+\gamma},f^{o}_{-\alpha})|| < \infty$ so that we only have to consider the case where also

$$(f^o_{+\beta}, e^{-ih(t-t_o)} f^o_{-\alpha}) = 0 \quad . \tag{2.26}$$

Equation (2.25) implies that $e^{ih(t-t_o)} f^o_{-\alpha} = \sum_\beta a^\alpha_\beta f^o_{-\beta}$ and since $e^{ih(t-t_o)}$ is unitary, this is a one to one mapping so that $e^{ih(t-t_o)}$ maps the whole set $\{f^o_{-\alpha}\}$ into itself. As can be seen by (2.26), the same holds for the set $\{f^o_{+\alpha}\}$. It therefore follows that

$$e^{ih(t-t_o)} P^o_+ = P^o_+ e^{ih(t-t_o)}$$

and

$$e^{ih(t-t_o)} P^o_- = P^o_- e^{ih(t-t_o)} \quad .$$

This can be seen to imply

$$[h, P^o_+] = 0 \quad \text{and} \quad [h, P^o_-] = 0 \; .$$

which holds if and only if $A(\underset{\sim}{x}) = 0$. In this case, all conditions are trivially satisfied.

In a similar manner, we can prove that

$$||(f_{-\lambda}, e^{-ih(t-t_o)} f^o_{+\beta})|| < \infty \; .$$

We have then shown that for any given time $t \neq t_o$, the conditions (2.6), (2.7) and (2.8), (2.9) are exactly equivalent. Since

$$||(f_{+\gamma}, e^{-ih\tau} f^{o}_{-\beta})|| = ||(f_{+\gamma}, f^{o}_{-\beta})||$$

and

$$||(f_{-\lambda}, e^{-ih\tau} f^{o}_{+\beta})|| = ||(f_{-\lambda}, f^{o}_{+\beta})|| \quad \forall \tau \quad ,$$

we see that if at some time t

$$||(f_{+\gamma}, e^{-ih(t-t_o)} f^{o}_{-\beta})|| \quad \text{and} \quad ||(f_{-\lambda}, e^{-ih(t-t_o)} f^{o}_{+\beta})||$$

are finite, they will be finite at all times so that also

$$||(f^{o}_{-\alpha}, e^{-ih(t-t_o)} f^{o}_{+\beta})|| \quad \text{and} \quad ||(f^{o}_{+\alpha}, e^{-ih(t-t_o)} f^{o}_{-\beta})||$$

will be finite at all times.

The Energy Operator

It will be useful here for clarity not to use the wave packet notation. We will call $f_E(\underset{\sim}{x})$ and $f_{E'}(\underset{\sim}{x})$ the eigenfunctions of h defined by

$$hf_E(\underset{\sim}{x}) = Ef_E(\underset{\sim}{x}) \qquad E \geq 0$$

$$hf_{E'}(\underset{\sim}{x}) = -E'f_{E'}(\underset{\sim}{x}) \qquad E' > 0 \quad , \tag{2.27}$$

where E and -E' can be in either the discrete of continuous spectrum of h; $-E' \epsilon S(-)$, the negative part of the spectrum S of h and $E \in \overline{S(-)}$ the complement of $S(-)$. Corresponding to these, creation and annihilation operators are defined as

$$b(E,t) = b(E)e^{-iEt} = \int d\underset{\sim}{x}\ f_E(\underset{\sim}{x})^* \psi_A(t,\underset{\sim}{x})$$

$$d^\dagger(E',t) = d^\dagger(E')e^{iE't} = \int d\underset{\sim}{x}\ f_{E'}(\underset{\sim}{x})^* \psi_A(t,\underset{\sim}{x}) \quad , \qquad (2.28)$$

so that the field operator $\psi_A(t,\underset{\sim}{x})$ can be written as

$$\psi_A(t,\underset{\sim}{x}) = \int dE\ b(E)e^{-iEt} f_E(\underset{\sim}{x}) + \int dE' d^\dagger(E')e^{iE't} f_{E'}(\underset{\sim}{x}) \qquad (2.29)$$

where

$$\int dE = \int_{\text{over all } E} dE \quad \text{and} \quad \int dE' = \int_{\text{over all } E'} dE' \ .$$

The operators $b_\gamma(t)$ and $d_\lambda(t)$ were just $b(E)e^{-iEt}$ and $d(E')e^{iE't}$ smeared with L^2 functions of E and E'. The new vacuum $|\phi_o\rangle$ defined by the b_γ and d_λ is consequently also such that

$$b(E)|\phi_o\rangle = 0 \ \forall E \text{ and } d(E')|\phi_o\rangle = 0 \ \forall E' \ . \qquad (2.30)$$

When the conditions (2.6) and (2.7) are satisfied the vacuum $|\phi_o\rangle$ exists and the set of states generated from it by the action of the creation operators $b^\dagger_\gamma(t)$, $d^\dagger_\lambda(t)$ forms a basis in $\mathcal{H}_{in}$. It is then clear that the energy operator

$$H = \int dE\, b^{\dagger}(E)b(E) + \int dE'\, d^{\dagger}(E')d(E') \qquad (2.31)$$

is well defined with a dense domain in $\mathcal{H}_{in}$.

Since (2.8) and (2.9) are equivalent to (2.6) and (2.7), there exists a unitary evolution operator for the field if and only if H is well defined. It then always has the form

$$U(t,t_o) = e^{-iH(t-t_o)} \qquad (2.32)$$

where H is as given by (2.31).

We remark that all the general results of Bongaarts [1970] have been obtained here in a much simpler fashion. Furthermore, whereas he demonstrated only the existence of H, we have gotten here its explicit form. It always has the physically expected form of the sum of individual particles energies which was proposed by K.O. Friedrichs [1953].

We finally point out that the domains D(H) and $D(H_o)$ of H and H_o will be, in general, different. In particular, the initial vacuum $|0\rangle$ will not necessarily be in D(H). This can easily be seen as follows: if $|0\rangle \in D(H)$, then $H|0\rangle \in \mathcal{H}$ and can be written as a linear superposition of initial Fock basis states where all "coefficients" are square integrable so that in particular we must have $|\langle\sigma|H|0\rangle|^2 < \infty$. Here, however, using

(2.31) and the expression (2.4) corresponding to b(E) and $d^{\dagger}(E')$, we find

$$<0|H|0> = \int dE \sum_{\beta} E|(f_E, f^o_{-\beta})|^2 + \int dE' \sum_{\beta} E'|(f_{E'}, f^o_{+\beta})|^2 .$$

Requiring that this be finite is therefore more restrictive than simply asking (2.6) and (2.7) which are equivalent to

$$\int dE \sum_{\beta} |(f_E, f^o_{-\beta})|^2 < \infty \quad \text{and} \quad \int dE' \sum_{\beta} |(f_{E'}, f^o_{+\beta})|^2 < \infty . \tag{2.33}$$

On Charge Conservation and Individual Particles Creation

i) As we saw, in the chapter on general Bogoliubov transformations, the total charge is not necessarily conserved in transformations as (2.4) and (2.5). In fact, after time t_o, the charge operator for physical particles is

$$Q = Q_{in} + M - N . \tag{2.34}$$

where N and M are respectively the number and non-zero solutions to the equation

$$\sum_{\beta} (f_{+\gamma}, e^{-ih(t-t_o)} f^o_{+\beta}) \phi_{\beta} = 0 \quad \forall \gamma \tag{2.35}$$

and to

$$\sum_{\beta} (f_{-\lambda}, e^{-ih(t-t_o)} f^o_{-\beta}) \psi_{\beta} = 0 \quad \forall \lambda \quad . \tag{2.36}$$

Since, $\forall$ t, $e^{+ih(t-t_o)}$ maps the sets $\{f_{+\gamma}\}$ and $\{f_{-\lambda}\}$ onto themselves in a one to one fashion, N and M are time independent. The equations (2.35) and (2.36) are always equivalent to

$$\sum_{\beta} (f_{+\gamma}, f^o_{+\beta}) \phi_{\beta} = 0 \quad \forall \gamma \tag{2.37}$$

and

$$\sum_{\beta} (f_{-\lambda}, f^o_{-\beta}) \psi_{\beta} = 0 \quad \forall \lambda \quad . \tag{2.38}$$

ii) Let us examine in particular a case where the potential is purely "attractive" in the sense that

$$(\psi, [-e\gamma^o \gamma . A] \psi) \leq 0 \; \forall \psi \quad . \tag{2.39}$$

In order for equation (2.38) to be satisfied, we need that there exists $\{a_{\gamma}\} \neq \{0\}$ such that

$$\sum_{\beta} f^o_{-\beta}) \psi_{\beta} = \sum_{\gamma} f_{+\gamma}) a_{\gamma} \quad . \tag{2.40}$$

Defining $\psi^o_-) = \sum_{\beta} f^o_{-\beta}) \psi_{\beta}$, we have

$$(\psi^o_-, h \psi^o_-) = (\psi^o_-, h_o \psi^o_-) + (\psi^o_-, [-e\gamma^o \gamma . A] \psi^o_-)$$

and since ψ^o_- contains only negative eigenvalues of h_o, this is

$$= -(\psi_-^o, |h_o| \psi_-^o) + (\psi_-^o, [-e\gamma^o\gamma . A]\psi_-^o) \quad . \qquad (2.41)$$

Upon using (2.39) and

$$-(\psi_-^o, |h_o| \psi_-^o) \leq -m(\psi_-^o, \psi_-^o) \quad ,$$

we see that

$$(\psi_-^o, h \; \psi_-^o) < 0 \quad . \qquad (2.42)$$

However, from the right hand side of (2.40), since by definition the $f_{+\gamma}$'s contain only non-negative eigenvalues of h, we should have

$$(\psi_-^o, h \, \psi_-^o) \geq 0 \quad . \qquad (2.43)$$

This is in contradiction with (2.42) so that equation (2.40) cannot hold and M must be null. There can however be solutions to equation (2.35) in general. The number N of these solutions is the number of positrons created with certainty when the system is in the state $|0\rangle$. Since M = 0, there is however no certain creation of electrons in this state.

iii) An example of this is discussed by A.I. Akhiezer and V.B. Berestetsky [1953]. They consider the following spherically symmetric potential:

$$\underset{\sim}{A}(\underset{\sim}{x}) = 0 \ ; \quad eA_o(\underset{\sim}{x}) = V_o > 0 \quad \forall\ r < r_o$$

$$= 0 \quad \forall\ r > r_o \ .$$

This potential clearly satisfies condition (2.39).

They solve for the c-number solutions $f_E(\underset{\sim}{x})$ and $f_{E'}(\underset{\sim}{x})$ and find the following. When V_o is smaller than a certain value, there is no bound state and the energy spectrum is the same as for free particles. As V_o is increased, bound electron states i.e. levels with energies $0 \le E_i < m$ start appearing and the energy of any given level decreases as the depth of the potential is increased. Eventually, for V_o greater than a certain value, the bound states energies will become negative. If the potential strength is further increased, there will be a point at which the levels will reach the continuum of negative energies.

As soon as an electron energy level becomes negative, the state must be considered a positron state just as all other negative energy levels since otherwise the total energy operator for the system would not be non-negative definite as it should. If for example, we consider for simplicity only one such bound electron state of energy $-E$, we would have

$$H = \int d\underset{\sim}{p} \sum_{s=1}^{2} \omega(\underset{\sim}{p}) \{ b_s^{\dagger}(\underset{\sim}{p}) b_s(\underset{\sim}{p}) + d_s^{\dagger}(\underset{\sim}{p}) d_s(\underset{\sim}{p}) \} - E b_E^{\dagger} b_E \ .$$

We see that $b_E^\dagger|\phi_o\rangle$ is an eigenstate with energy $-E < 0$ so that H is not non-negative definite.

A free electron energy level is associated with the wave function

$$\Psi(t,\underset{\sim}{x}) = e^{-ih_o t} \sum_\beta a_\beta f^o_{+\beta}(\underset{\sim}{x}) .$$

We have seen that the c-number evolution operator for the system considered here is

$$u(t,t') = e^{-ih(t-t_o)} e^{-ih_o(t_o-t')} \quad \text{for } t \geq t_o \text{ and } t' \leq t_o .$$

The above wave function for times $t > t_o$ then becomes

$$\Psi(t,\underset{\sim}{x}) = e^{-ih(t-t_o)} e^{-ih_o t_o} \sum_\beta a_\beta f^o_{+\beta}(\underset{\sim}{x}) .$$

If equation (2.37) has N non-zero solutions $\{\phi_\beta\}$, there will exist N non-zero $\{a_\beta\}$ given by

$$e^{-ih_o t_o} \sum_\beta a_\beta f^o_{+\beta}(\underset{\sim}{x}) = \sum_\beta \phi_\beta f^o_{+\beta}(\underset{\sim}{x})$$

such that $\Psi(t,\underset{\sim}{x})$ at times $t > t_o$ will contain only negative energy eigenfunctions of h (i.e. be a linear superposition of the $f_{-\lambda}$ only). In this sense, we see that N simply correspond to the number of free electron energy levels which, under the "switching on" of the potential V_o at time t_o, are transformed into purely negative energy levels.

iv) Let us now examine what happens with "weak" external fields; by "weak", we mean fields which satisfy the condition

$$|(\psi, e\gamma^{o}\gamma . A\psi)| < (\psi, |h_{o}|\psi) \quad \forall\ \psi \neq 0 \tag{2.44}$$

where $|h_o| = (h_o^2)^{\frac{1}{2}}$. This condition, we note, can be used (see for example the Chapters V and VI of T. Kato [1966]) to ensure that the c-number operator $h = h_o - e\gamma^{o}\gamma . A$ is essentially self-adjoint with nice domain properties with respect to h_o. It is shown in the above reference that this condition gives, for example, the restriction $Z \leq 87$ for the Coulomb potential Ze/r.

Under this condition, equation (2.40) could not hold since from its left hand side,

$$(\psi_-^{o}, h\ \psi_-^{o}) = -(\psi_-^{o}, |h_o|\psi_-^{o}) - (\psi_-^{o}, e\gamma^{o}\gamma . A\psi_-^{o})$$

$$\leq -(\psi_-^{o}, |h_o|\psi_-^{o}) + |(\psi_-^{o}, e\gamma^{o}\gamma . A\psi_-^{o})|$$

and by (2.44) this is <0 which, as before, contradicts the inequality (2.43) obtained from its right hand side. There are then no non-zero solutions to equation (2.38). In a similar manner as above, it is easy to show also that there will not be any non-zero solution to (2.37). The condition (2.44) therefore ensures that the field is too weak to create with certainty any individual electron or positron. There is only normal pair creation of electrons and positrons.

v) The lepton number (or pseudo-charge) operator associated with the pseudo-particles is

$$Q^o = Q_{in} + M^o - N^o \quad . \tag{2.45}$$

N^o and M^o are respectively the number of non-zero solutions, at time t, to the equation

$$\sum_\beta (f^o_{+\alpha}, e^{-ih(t-t_o)} f^o_{+\beta})\phi_\beta = 0 \quad \forall \alpha \tag{2.46}$$

and to

$$\sum_\beta (f^o_{-\alpha}, e^{-ih(t-t_o)} f^o_{-\beta})\psi_\beta = 0 \quad \forall \alpha \quad . \tag{2.47}$$

Since here

$$e^{-ih(t-t_o)}\psi_{in}(t_o,\underset{\sim}{x}) = e^{iH(t-t_o)}\psi_{in}(t_o,\underset{\sim}{x})e^{-iH(t-t_o)} \quad \forall t,$$

we have

$$b^o_\alpha(t) = e^{iH(t-t_o)} b^{in}_\alpha(t_o) e^{-iH(t-t_o)}$$

$$d^o_\alpha(t) = e^{iH(t-t_o)} d^{in}_\alpha(t_o) e^{-iH(t-t_o)} \quad , \tag{2.48}$$

so that in fact

$$Q^o(t) = e^{iH(t-t_o)} Q_{in} \, e^{-iH(t-t_o)} \quad . \tag{2.49}$$

Using equation (2.34) and the fact that $[Q,H] = 0$, we obtain $[Q_{in},H] = 0$ so that

$$Q^{o}(t) = Q_{in} \quad \forall \; t \; . \tag{2.50}$$

This implies that we always have $M^{o} = N^{o}$ and the pseudo-particles will then always be created or annihilated by pairs.

In the case of "weak" external fields, we will always have $0 = M^{o} = N^{o}$, as is shown below. If there exists $\{\phi_{\beta}\} \neq \{0\}$ satisfying equation (2.46), there exists $\{a_{\alpha}\} \neq \{0\}$ such that

$$e^{-ih(t-t_{o})} \sum_{\beta} f^{o}_{+\beta})\phi_{\beta} = \sum_{\alpha} f^{o}_{-\alpha})a_{\alpha} \; . \tag{2.51}$$

Defining $\phi^{o}_{+} = \sum_{\beta} f^{o}_{+\beta})\phi_{\beta}$ and $\phi^{o}_{-} = \sum_{\alpha} f^{o}_{-\alpha})a_{\alpha}$, we have

$$(e^{-ih(t-t_{o})}\phi^{o}_{+}, h\, e^{-ih(t-t_{o})}\phi^{o}_{+}) = (\phi^{o}_{+}, h\, \phi^{o}_{+})$$

$$= (\phi^{o}_{+}, |h_{o}|\phi^{o}_{+}) - (\phi^{o}_{+}, e\gamma^{o}\gamma.A\phi^{o}_{+})$$

$$\geq (\phi^{o}_{+}, |h_{o}|\phi^{o}_{+}) - |(\phi^{o}_{+}, e\gamma^{o}\gamma.A\phi^{o}_{+})|$$

$$> 0$$

by condition (2.44). Upon using the right hand side of equation (2.51) however, we must have

$$
\begin{aligned}
(\phi_-^o, h\,\phi_-^o) &= -(\phi_-^o, h_o\,\phi_-^o) - (\phi_-^o, e\gamma^o\gamma . A\phi_-^o) \\
&\leq -(\phi_-^o, h_o\,\phi_-^o) + |(\phi_-^o, e\gamma^o\gamma . A\phi_-^o)| \\
&< 0
\end{aligned}
$$

again by condition (2.44). We see that the equation (2.51) is then not compatible with the "weak" field condition i.e. equation (2.46) cannot have any non-zero solution.

Description of Scattering (S-matrix)

Let us now go back to the more explicit notation where, when h defines a good scattering system, the meaning of the general indices E and E' is

$$
\begin{aligned}
\{f_E(\underset{\sim}{x})\} &= \{f_E(\underset{\sim}{x})\}U\{f_{(+,\underset{\sim}{p},s)}(\underset{\sim}{x})\} \\
\{f_{E'}(\underset{\sim}{x})\} &= \{f_{E'}(\underset{\sim}{x})\}U\{f_{(-,\underset{\sim}{p},s)}(\underset{\sim}{x})\} \quad . \qquad (2.52)
\end{aligned}
$$

As we have seen in the Chapters I and II, the operators $b_s(\underset{\sim}{p})$ and $d_s(\underset{\sim}{p})$ defined by equation (2.28) correspond to the particles in the external field which after a long time will be moving freely, i.e.

$$b_s^{out}(\underset{\sim}{p}) = b_s(\underset{\sim}{p}) \qquad d_s^{out}(\underset{\sim}{p}) = d_s(\underset{\sim}{p}) \ . \tag{2.53}$$

The operators $b(E)$ and $d(E')$ are the bound particles annihilation operators and the energy operator can be written as

$$H = \int d\underset{\sim}{p} \sum_{s=1}^{2} \omega(\underset{\sim}{p})\{b_s^\dagger(\underset{\sim}{p})b_s(\underset{\sim}{p}) + d_s^\dagger(\underset{\sim}{p})d_s(\underset{\sim}{p})\}$$

$$+ \sum_E E\, b^\dagger(E)b(E) + \sum_{E'} E'\, d^\dagger(E')d(E') \ . \tag{2.54}$$

i) Initially, we know that there are only free particles in the system; let us suppose then that the system is in the state where there are n particles and m antiparticles respectively characterized by the wave packet indices $(\alpha_1,\alpha_2,\ldots\alpha_n)$ and $(\beta_1,\beta_2,\ldots\beta_m)$ present before time t_o. This state of the system is represented by

$$|\underline{\alpha}_n;\underline{\beta}_m\rangle = b^{in}(\alpha_1)^\dagger\, b^{in}(\alpha_2)^\dagger \ldots b^{in}(\alpha_n)^\dagger\, d^{in}(\beta_1)^\dagger \ldots$$

$$d^{in}(\beta_m)^\dagger\, |0\rangle \ . \tag{2.55}$$

(We note that the time used for the operators $b^{in}(t)^\dagger$ and $d^{in}(t)^\dagger$ does not matter since if different times are used, the resulting difference in the above vector is only a phase so that the same physical state is represented. That is so because a physical state is in fact represented by a ray in $\mathcal{H}$ and not by a vector. We will then use here t = 0.)

For all times before t_o, there are these same n freely moving particles and m free antiparticles in the system.

In order to see what the situation is at times t after time t_o, we simply rewrite the state (2.55) in terms of the new particle variables. We use the inverse of the transformation (2.4) to write the operators b^{in}, d^{in} in terms of b(t), d(t) and also write $|0\rangle$ in terms of new Fock basis states for the particles. Equivalently, we can calculate all probability amplitudes that the system be in the configuration represented by any state of the type

$$|\underline{\alpha}'_{n'}, \underline{E}_\ell; \underline{\beta}'_{m'}, \underline{E}'_{\ell'}\rangle'_t .$$

These are states where at time $t \to t_o$ there are n' scattering particles characterized by the wave packet indices $(\underline{\alpha}'_{n'})$ moving in the potential $A(\underline{x})$, ℓ bound particles occupying the energy levels $E_1, E_2, \ldots E_\ell$ and m' scattering antiparticles and ℓ' bound antiparticles. This probability amplitude is simply

$$_t'\langle\underline{\alpha}'_{n'}, \underline{E}_\ell ; \underline{\beta}'_{m'}, \underline{E}'_{\ell'}|\underline{\alpha}_n ; \underline{\beta}_m\rangle . \qquad (2.56)$$

These amplitudes for all n, m and n', ℓ, m', ℓ' are in fact just the elements of the unitary transformation relating the free particle Fock basis and the Fock basis obtained from the application of $b^{\dagger}(t)$ and $d^{\dagger}(t)$ on $|\phi_o\rangle$. This unitary operator is simply related to the one implementing the Bogoliubov transformation (2.4), the general form of which was given in Chapter III. Since b(t = 0) and d(t = 0) are the final operators for the particles, as seen in Chapter I, this unitary operator with the parameter t = 0 is the field theoretical S matrix (if this is defined as the operator relating all final states to all initial states).

It is important to remark that even when bound states are possible the unitary operator relating the two sets of creation and annihilation operators b^{in}, d^{in} and b, d can exist. In fact, this operator would exist even when the spectra of h_o and h are not "similar" (in the sense that h defines a good scattering system; this case is considered more explicitly because of its importance). Its existence is simply linked to that of a well defined second quantized energy operator H and the fact that physical particles (i.e. quanta of energy) can be appropriately described after time t_o.

ii) We calculate, as an example, the probability amplitude for the following interesting phenomenon. There is finally in the system only a bound electron with energy E when initially there was only one freely moving electron. The state of the system is

$$|\alpha;0\rangle = b_{\alpha}^{in^{\dagger}}|0\rangle \quad ,$$

α being the wave packet index of the initial free electron. The probability amplitude we want is given by

$$'\langle E;0|\alpha,0\rangle = \langle\phi_o|b(E)b_{\alpha}^{in^{\dagger}}|0\rangle \quad .$$

Using the inverse of the relation (2.4), we obtain

$$'\langle E;0|\alpha,0\rangle = (f_E, e^{iht_o}e^{-ih_o t_o}f_{+\alpha}^{o})\langle\phi_o|0\rangle$$

$$- \sum_{\lambda}(f_{-\lambda}, e^{iht_o}e^{-ih_o t_o}f_{+\alpha}^{o})\,'\langle E;\lambda|0\rangle \quad .$$

This is generally non-zero if there is no certain creation of particles and antiparticles ($\langle\phi_o|0\rangle$ and $'\langle E;\lambda|0\rangle$ are non-zero) so that the binding of a free electron can be rigorously described.

It is interesting to note that the first term of this expression gives the contribution to the process which would be obtained from ordinary c-number quantum mechanics i.e. one particle theory where the possibility of creation and annihilation of particles would not be considered. It would be simply the probability amplitude that the initial electron becomes bound.

From this point of view, the second term would be interpreted as giving the probability amplitude that: an electron in the state E and a positron in an arbitrary state characterized by λ are created ($\langle E,\lambda|0\rangle$); the positron annihilates with the initial electron which under the influence of the external field has "dropped" to the state described by the wave function $f_{-\lambda}$ (according to the probability amplitude

$$(f_{-\lambda},\ e^{iht_o}\ e^{-ih_o t_o}\ f^o_{+\alpha})\ ).$$

When there is no certain creation nor annihilation of particles, there are generally two such contributions to the single particle "scattering" probabilities between any two states when the potential is time dependent. The ordinary quantum mechanical interpretation of the Dirac equation as a single particle equation will then give correct results only insofar as there is no certain creation or annihilation of particles and the second "process" described above can be neglected.

iii) In c-number scattering theory, isometric (unitary when there are no bound states) operators Ω are defined such that h is associated with a good scattering system when

$$h\Omega = \Omega h_o .$$

This is what is meant by h is "similar" to h_o. Here, in the second quantized theory, as is obvious from the equation (2.54) giving the form of the Hamiltonian H, there is the same kind of similarity between the spectrum of H and that of the initial free field Hamiltonian H_{in}:

$$H_{in} = \int d\underset{\sim}{p} \sum_{s=1}^{2} \omega(\underset{\sim}{p})\{b_s^{in}(\underset{\sim}{p})^{\dagger} b_s^{in}(\underset{\sim}{p}) + d_s^{in}(\underset{\sim}{p})^{\dagger} d_s^{in}(\underset{\sim}{p})\} .$$

The two operators have the same continuous spectrum.

This similarity can here also be expressed by the existence of an isometric operator which, whenever $|\phi_o\rangle$ exists can be defined as

$$W = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\sqrt{n!m!}} \int \underline{dps}_n \; \underline{dqt}_m \; |\underline{ps}_n;\underline{qt}_m\rangle' \; \langle\underline{ps}_n;\underline{qt}_m| \quad (2.57)$$

It has the properties:

$$W^{\dagger}W = I \qquad WW^{\dagger} = I - P_b \qquad (2.58)$$

where

$$P_b = \sum_E b^{\dagger}(E)b(E) + \sum_{E'} d^{\dagger}(E')d(E')$$

is the projection operator on the subspace of states in which there are bound particles. We have

$$b_s(\underset{\sim}{p})\ W = W\ b_s^{in}(\underset{\sim}{p}) \qquad \text{i.e. } b_s^{out}(\underset{\sim}{p})\ W = W\ b_s^{in}(\underset{\sim}{p})$$

$$d_s(\underset{\sim}{p})\ W = W\ d_s^{in}(\underset{\sim}{p}) \qquad d_s^{out}(\underset{\sim}{p})\ W = W\ d_s^{in}(\underset{\sim}{p}) \qquad (2.59)$$

$$b(E)\ W = 0$$

$$d(E')\ W = 0 \qquad (2.60)$$

and consequently, it relates H and H_{in} as

$$H\ W = W\ H_{in}\ . \qquad (2.61)$$

If there are no bound states after t_o, W is unitary and as we can see from (2.59), it is then the adjoint of the unitary operator implementing the Bogoliubov transformation (2.4) (i.e. $W = S^{\dagger}$).

When bound states are possible, W relates only final scattering particles to the initial scattering particles creation and annihilation operator, so in this sense it also could be called a "scattering" operator. We remark that there are many isometric operators with the property (2.59); others than W can be obtained for example by using states $|\underline{ps}_n\ \underline{E}_\ell;\underline{qt}_m\ \underline{E'}_{\ell'}>'$ where there are ℓ bound electrons and ℓ' positrons instead of only scattering particles as with $|\underline{ps}_n;\underline{qt}_m>'$. However, requiring the property (2.58) or (2.60) or (2.61) defines W uniquely.

Fields Which Satisfy the Existence Conditions

We now give examples of external electromagnetic fields which satisfy the conditions (2.6) and (2.7) or (2.8) and (2.9).

i) Moses [1954] and Friedrichs [1953] have studied in particular good scattering systems where there are no bound states. Using a perturbation expansion for the functions $f_{\pm\alpha} = \Omega f^{o}_{\pm\alpha}$, they examined the conditions (2.6) and (2.7). In the first approximation, they have shown that no external magnetic field $\underset{\sim}{A}(\underset{\sim}{x})$ could satisfy them and that for $|\tilde{A}_o(\underset{\sim}{p})|$ (where $\tilde{A}_o(\underset{\sim}{p})$ is the Fourier transform of $A_o(\underset{\sim}{x})$) bounded and integrable they would be satisfied. From this, they deduce that (2.6) and (2.7) will be satisfied with all orders of the perturbation series for $\underset{\sim}{A}(\underset{\sim}{x}) = 0$ and $A_o(\underset{\sim}{x})$ sufficiently weak.

Bongaarts [1970] gives the following example of external electromagnetic field which satisfies the conditions (2.8) and (2.9).

$\underset{\sim}{A}(\underset{\sim}{x}) = 0$

$A_o(\underset{\sim}{x})$ is a bounded L^2 function whose Fourier transform $\tilde{A}_o(\underset{\sim}{p})$ is continuous except possibly at $\underset{\sim}{p} = 0$ and has the following behaviour at 0, ∞: there exist positive numbers c_1, c_2, ε_1, ε_2, δ, N such that

$$|\tilde{A}_0(p)| \le c_1 |p|^{-2+\varepsilon_1} \qquad \text{for } 0 < |p| < \delta$$

$$|\tilde{A}_0(p)| \le c_2 |p|^{-(5/2+\varepsilon_2)} \qquad \text{for } |p| > N. \qquad (2.62)$$

ii) We will derive here the following sufficient conditions on external electromagnetic fields to ensure that the conditions (2.6) and (2.7) are satisfied:

$$A(x) = 0$$

$$\int dx\, e^4 [A_0(x)]^4 < \infty$$

$$\int dp\, |p|^2 e^2 |A_0(p)|^2 < \infty$$

(or equiva[illegible] $e^2[\partial A_0(x)]^2 < \infty$) (2.63)

Our derivat[illegible]ot use perturbation series and is much simple[illegible]at of Bongaarts while yielding more elegant resu[illegible]

We exa[illegible] the inequalities:

$$\sum_{\gamma}\sum_{\beta} |(f_{+} \ [illegible] \ _{\beta})|^2 < \infty \qquad (2.64)$$

and

$$\sum_{\lambda}\sum_{\beta} |(f_{-\lambda}, f^{o}_{+\beta})|^2 < \infty \qquad (2.65)$$

which we have already shown to be equivalent to (2.6) and (2.7). (2.64) can be written as

$$\text{Trace } P_+ P_-^o < \infty \qquad (2.66)$$

where $P_+ = \sum_\gamma f_{+\gamma}) (f_{+\gamma}$ is the projection operator on the space of functions associated with the non-negative spectrum of h and $P_-^o = \sum_\beta f_{-\beta}^o) (f_{-\beta}^o$ projects on the space of functions associated with the negative spectrum of h_o. These operators can be written in terms of unnormalizable eigenfunctions (of h or h_o) as

$$P_+ = \int dE\, f_E)(f_E \quad \text{and} \quad P_-^o = \int d\underset{\sim}{p} \sum_{s=1}^{2} f_{(-,\underset{\sim}{p},s)}^o)(f_{(-,\underset{\sim}{p},s)}^o$$

Using these forms the condition (2.64) or (2.66) can be written as

$$\int dE \int d\underset{\sim}{p} \sum_{s=1}^{2} \left| \int d\underset{\sim}{x}\, v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_E(\underset{\sim}{x}) \right|^2 < \infty \qquad (2.67)$$

where $v^s(\underset{\sim}{p})$ is the usual spinor such that

$$(\gamma^o\omega(\underset{\sim}{p}) - \underset{\sim}{\gamma}\cdot\underset{\sim}{p} + m)v^s(\underset{\sim}{p}) = 0 \quad .$$

We have

$$(f_{(-,\underset{\sim}{p},s)}^o, h\, f_E) = \int d\underset{\sim}{x}\, v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} \{h_o - e\gamma^o\underset{\sim}{\gamma}\cdot\underset{\sim}{A}(\underset{\sim}{x})\} f_E(\underset{\sim}{x})$$

$$= E\int d\underset{\sim}{x}\, v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_E(\underset{\sim}{x})$$

and using $(f_{(-,\underset{\sim}{p},s)}^o, h\, f_E) = (h\, f_{(-,\underset{\sim}{p},s)}^o, f_E)$, this is also

$$= -\omega(\underset{\sim}{p})\int d\underset{\sim}{x}\; v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} f_E(\underset{\sim}{x}) - \int d\underset{\sim}{x}\; v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} e\gamma^o\gamma.A(\underset{\sim}{x}) f_E(\underset{\sim}{x})$$

so that

$$\int d\underset{\sim}{x}\; v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} f_E(\underset{\sim}{x}) = \frac{-1}{[E+\omega(\underset{\sim}{p})]}\int d\underset{\sim}{x}\; v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} e\gamma^o\gamma.A(\underset{\sim}{x}) f_E(\underset{\sim}{x}) \tag{2.68}$$

We note that even if the functions used are not normalizable, the above manipulations are rigorous. This can be seen by using any wave packet made up of these functions; equation (2.68) then has the meaning of an equality between two distributions in $\underset{\sim}{p}$ and E over the space of L^2 functions of $\underset{\sim}{p}$ and E. Let us now take $A(\underset{\sim}{x}) = 0$; we have again:

$$\int d\underset{\sim}{x}\; v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} eA_o(\underset{\sim}{x})\{h_o - eA_o(\underset{\sim}{x})\} f_E(\underset{\sim}{x})$$

$$= E\int d\underset{\sim}{x}\; v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} eA_o(\underset{\sim}{x}) f_E(\underset{\sim}{x}) \tag{2.69}$$

$$= \int d\underset{\sim}{x}\; v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} \{h_o eA_o(\underset{\sim}{x}) + [eA_o(\underset{\sim}{x}), h_o] - e^2A_o^2(\underset{\sim}{x})\} f_E(\underset{\sim}{x})$$

$$= -\omega(\underset{\sim}{p})\int d\underset{\sim}{x}\; v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} eA_o(\underset{\sim}{x}) f_E(\underset{\sim}{x})$$

$$- \int d\underset{\sim}{x}\; v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} \{ie\gamma^o\underset{\sim}{\gamma}.[\underset{\sim}{\partial} A_o(\underset{\sim}{x})] + e^2A_o^2(\underset{\sim}{x})\} f_E(\underset{\sim}{x}) \tag{2.70}$$

so that (2.69) and (2.70) imply:

$$\int d\underset{\sim}{x}\ v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} eA_o(\underset{\sim}{x}) f_E(\underset{\sim}{x})$$

$$= \frac{-1}{[E+\omega(\underset{\sim}{p})]} \int d\underset{\sim}{x}\ v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} \{ie\gamma^o\underset{\sim}{\gamma}.[\underset{\sim}{\partial}A_o(\underset{\sim}{x})]+e^2A_o^2(\underset{\sim}{x})\} f_E(\underset{\sim}{x}).$$

This replaced in equation (2.68) gives:

$$\int d\underset{\sim}{x}\ v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} f_E(\underset{\sim}{x})$$

$$= \frac{1}{[E+\omega(\underset{\sim}{p})]^2} \int d\underset{\sim}{x}\ v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} \{ie\gamma^o\underset{\sim}{\gamma}.[\underset{\sim}{\partial}A_o(\underset{\sim}{x})]+e^2A_o^2(\underset{\sim}{x})\} f_E(\underset{\sim}{x}) \ . \tag{2.71}$$

Since always

$$\frac{1}{[E+\omega(\underset{\sim}{p})]^2} \le \frac{1}{[\omega(\underset{\sim}{p})]^2}$$

and

$$\int dE \left| \int d\underset{\sim}{x}\ g(\underset{\sim}{x})^* f_E(\underset{\sim}{x}) \right|^2 = \int d\underset{\sim}{x}\ g^*(\underset{\sim}{x}) g(\underset{\sim}{x}) - \int dE' \left| \int d\underset{\sim}{x}\ g^*(\underset{\sim}{x}) f_{E'}(\underset{\sim}{x}) \right|^2$$
$$\le \int d\underset{\sim}{x}\ g^*(\underset{\sim}{x}) g(\underset{\sim}{x}) \ ,$$

upon using these inequalities with (2.71), we obtain

$$\int d\underset{\sim}{p} \sum_{s=1}^{2} \int dE \left| \int d\underset{\sim}{x}\ v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} f_E(\underset{\sim}{x}) \right|^2 \le \int d\underset{\sim}{p} \sum_{s=1}^{2} \frac{1}{\omega(\underset{\sim}{p})^4} \times$$

$$\int d\underset{\sim}{x}\ v^s(\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} \{ie\gamma^o\underset{\sim}{\gamma}.[\underset{\sim}{\partial}A_o]+e^2A_o^2\}\{ie\gamma^o\underset{\sim}{\gamma}.[\underset{\sim}{\partial}A_o]+e^2A_o^2\}^{\dagger} e^{-i\underset{\sim}{p}.\underset{\sim}{x}} v^s(\underset{\sim}{p})$$

$$= \int d\underset{\sim}{p} \sum_{s=1}^{2} \frac{1}{\omega(\underset{\sim}{p})^4} \int d\underset{\sim}{x} \{e^2[\underset{\sim}{\partial}A_o(\underset{\sim}{x})]^2 + e^4A_o^4(\underset{\sim}{x})\} \ . \tag{2.72}$$

Since $\int d\underset{\sim}{p} \frac{1}{\omega(\underset{\sim}{p})^4} < \infty$, the condition (2.67) will be satisfied when

$$\int d\underset{\sim}{x}\, e^2 [\partial A_o(\underset{\sim}{x})]^2 < \infty \quad \text{and} \quad \int d\underset{\sim}{x}\, e^4 A_o^4(\underset{\sim}{x}) < \infty \quad . \qquad (2.73)$$

Similar calculations can be done for the term

$$\sum_\lambda \sum_\beta |(f_{-\lambda}, f^o_{+\beta})|^2 = \int dE' \int d\underset{\sim}{p} \sum_{s=1}^{2} \left| \int d\underset{\sim}{x}\; w^s(-\underset{\sim}{p})^* e^{i\underset{\sim}{p}.\underset{\sim}{x}} f_{E'}(\underset{\sim}{x}) \right|^2$$

and the condition $\underset{\sim}{A}(\underset{\sim}{x}) = 0$ together with (2.73) again ensures that it is finite.

iii) The above method allows us also to obtain some results for the case of Dirac particles interacting with an external pseudo-vector field as

$$(-i\gamma.\partial + m)\psi(x) = -i\, G\, \gamma_5 \gamma . \phi(x) \psi(x) \quad .$$

We note that no results were given previously for this case. Taking again $\phi(x) = \theta(t-t_o)\phi(\underset{\sim}{x})$, this problem can be treated exactly as for the previous system with now

$$h = h_o + i\, G\, \gamma^o \gamma_5 \gamma . \phi(\underset{\sim}{x}) \quad . \qquad (2.74)$$

For the conditions (2.6) and (2.7), we derive as before;

$$(f^{o}_{(-,\underset{\sim}{p},s)}, f_E) = \frac{iG}{[E+\omega(\underset{\sim}{p})]} (f^{o}_{(-,\underset{\sim}{p},s)}, \gamma^{o}\gamma_5\gamma\cdot\phi f_E)$$

and taking $\underset{\sim}{\phi}(\underset{\sim}{x}) = 0$,

$$= \frac{-iG}{[E+\omega(\underset{\sim}{p})]} (f^{o}_{(-,\underset{\sim}{p},s)}, \gamma_5\phi_o f_E) \quad . \tag{2.75}$$

We now repeat a calculation similar to that done in equations (2.69), (2.70) for $(f^{o}_{(-,\underset{\sim}{p},s)}, \gamma_5\phi_o h\ f_E)$, using

$$[\gamma_5\phi_o, h_o] = i\gamma^{o}\gamma_5\underset{\sim}{\gamma}\cdot(\underset{\sim}{\partial}\phi_o) - 2m\gamma^{o}\gamma_5\phi_o \ .$$

Replacing the result obtained in (2.75), we get

$$(f^{o}_{(-,\underset{\sim}{p},s)}, f_E) = \frac{-iG}{[E+\omega(\underset{\sim}{p})]^2} (f^{o}_{(-,\underset{\sim}{p},s)}, \{i\gamma^{o}\gamma_5\underset{\sim}{\gamma}\cdot(\underset{\sim}{\partial}\phi_o)$$

$$- 2m\gamma^{o}\gamma_5\phi_o + iG(\gamma_5\phi_o)^2\} f_E) \ .$$

As in the previous case, we can see that when

$$\int d\underset{\sim}{x}\ \phi_o^2(\underset{\sim}{x}) < \infty \quad ,$$

$$\int d\underset{\sim}{x}\ \phi_o^4(\underset{\sim}{x}) < \infty$$

and $$\int d\underset{\sim}{x}[\underset{\sim}{\partial}\phi_o(\underset{\sim}{x})]^2 < \infty \ , \tag{2.76}$$

the condition (2.6) is satisfied and so also will be (2.7).

We note that the key to the successful application of the above proof resides in the fact that $h = h_o + \gamma^o M(\underset{\sim}{x})$ and $M(\underset{\sim}{x})$ is such that the commutator $[\gamma^o M(\underset{\sim}{x}), h_o]$ does not contain any derivative term. It can be checked that the most general 4×4 matrix $M(\underset{\sim}{x})$ which has this property is of the form $M(\underset{\sim}{x}) = \alpha_o(\underset{\sim}{x})\gamma^o + \alpha_5(\underset{\sim}{x})\gamma^o\gamma_5$. In particular, with the magnetic potential, $M(\underset{\sim}{x}) = e\underset{\sim}{\gamma}.\underset{\sim}{A}(\underset{\sim}{x})$ and

$$[e\gamma^o\underset{\sim}{\gamma}.\underset{\sim}{A}(\underset{\sim}{x}), h_o] = -2m\gamma_i A_i + i\gamma_k\gamma_i(\partial_k A_i) + 2i(\gamma_k\gamma_i)A_i\partial_k \quad . \tag{2.77}$$

A term like $(\gamma_k\gamma_i)A_i\partial_k$ appearing on the right hand side of equation (2.70) becomes $(\gamma_k\gamma_i)A_i p_k$ and we can see that we then run into trouble with the integration $\int d\underset{\sim}{p}$ in equation (2.72). Terms in $\underset{\sim}{p}^2$ in the numerator will prevent the convergence of the $\underset{\sim}{p}$-integral so that an alternate proof must be found.

3) Potential with Many Step Functions in Time and Continuous Time Dependence

Sequence of Step Functions in Time

i) Let us start by considering the following external field: $A(t,\underset{\sim}{x}) = \theta(t_1-t)\theta(t-t_o)A(\underset{\sim}{x})$, $t_1 > t_o$. The external field is null before t_o and is null again after t_1. The solution of the field equation is therefore the free field ψ_{in} before t_o, the field ψ_A, which satisfies equation (2.2), for $t_o < t < t_1$ and again a free field: ψ_{out} after t_1. As in the first case discussed, the solution must be continuous so that

$$\begin{aligned}
\psi(t,\underset{\sim}{x}) &= \psi_{in}(t,\underset{\sim}{x}) && t \le t_o \\
&= \psi_A(t,\underset{\sim}{x}) = e^{-ih(t-t_o)}\psi_{in}(t_o,\underset{\sim}{x}) && t_o \le t \le t_1 \\
&= \psi_{out}(t,\underset{\sim}{x}) = e^{-ih_o(t-t_1)}\psi_A(t_1,\underset{\sim}{x}) \\
&= e^{-ih_o(t-t_1)}e^{-ih(t_1-t_o)}\psi_{in}(t_o,\underset{\sim}{x}) && t \ge t_o \ .
\end{aligned} \tag{3.1}$$

Since the system up to time t_1 must behave exactly as in the previous problem, we just have to examine the time interval $t > t_1$. There is then no external field so that the functions to be used in associating physical particles with the field are simply those associated with the positive and negative parts of the spectrum of h_o. This means that, now, the particles creation and annihilation operators $b_\gamma(t > t_1)$, $d_\lambda(t > t_1)$ and their adjoints are the same as the $b^o_\alpha(t > t_1)$, $d^o_\alpha(t > t_1)$ and their adjoints. These create and annihilate the physical particles which one would observe after time t_1. There is now just one Bogoliubov transformation to be examined instead of two (as previously the equations (2.4) and (2.5)); here in terms of initial free particles operators we have:

$$b_\alpha(t)=b^o_\alpha(t)=\sum_\beta\{(f^o_{+\alpha},u(t,t_o)f^o_{+\beta})b^{in}_\beta(t_o)+(f^o_{+\alpha},u(t,t_o)f^o_{-\beta})\times \times d^{in}_\beta(t_o)^\dagger\}$$

$$d^\dagger_\alpha(t)=d^o_\alpha(t)^\dagger=\sum_\beta\{(f^o_{-\alpha},u(t,t_o)f^o_{+\beta})b^{in}_\beta(t_o)+(f^o_{-\alpha},u(t,t_o)f^o_{-\beta})\times \times d^{in}_\beta(t_o)^\dagger\} \qquad (3.2)$$

where $u(t,t_o) = e^{-ih_o(t-t_1)}e^{-ih(t_1-t_o)}$.

There will be a unique vacuum state in $\mathcal{H}$ defined by these operators if and only if

$$\sum_\alpha\sum_\beta|(f^o_{+\alpha},e^{-ih_o(t-t_1)}e^{-ih(t_1-t_o)}f^o_{-\beta})|^2 < \infty$$

$$\sum_\alpha\sum_\beta|(f^o_{-\alpha},e^{-ih_o(t-t_1)}e^{-ih(t_1-t_o)}f^o_{+\beta})|^2 < \infty .$$

Since $e^{-ih_o(t-t_1)}$ t maps the set of functions $\{f^o_{+\alpha}\}$ and $\{f^o_{-\alpha}\}$ back onto themselves in a one to one fashion, the above conditions are equivalent to

$$\sum_\alpha \sum_\beta |(f^o_{+\alpha}, e^{-ih(t_1-t_o)} f^o_{-\beta})|^2 < \infty$$

$$\sum_\alpha \sum_\beta |(f^o_{-\alpha}, e^{-ih(t_1-t_o)} f^o_{+\beta})|^2 < \infty \quad . \qquad (3.3)$$

These are exactly the conditions (2.8) and (2.9) at $t = t_1$. These were shown to hold for all times if they were satisfied at any time $t \neq t_o$ so that if (2.8) and (2.9) were satisfied (3.3) will be satisfied and vice versa.

We define the operators $b^1_s(p)$ and $d^1_s(p)$ as

$$b^1_s(\underset{\sim}{p}; t > t_1) = b^1_s(\underset{\sim}{p})\, e^{-i\omega(\underset{\sim}{p})t}$$

$$d^1_s(\underset{\sim}{p}; t > t_1) = d^1_s(\underset{\sim}{p})\, e^{-i\omega(\underset{\sim}{p})t} \quad .$$

In the present problem, b^1 and d^1 are the annihilation operators for the freely moving particles after time t_1, i.e.

$$b^1_s(\underset{\sim}{p}) = b^{out}_s(\underset{\sim}{p}) \qquad\qquad d^1_s(\underset{\sim}{p}) = d^{out}_s(\underset{\sim}{p}) \quad .$$

The energy operator for $t > t_1$ is

$$H^{out} = H(t>t_1) = \int d\underset{\sim}{p} \sum_{s=1}^{2} \omega(\underset{\sim}{p}) \{ b^1_s(\underset{\sim}{p})^\dagger\, b^1_s(\underset{\sim}{p}) + d^1_s(\underset{\sim}{p})^\dagger\, d^1_s(\underset{\sim}{p}) \} \quad .$$

The time translation operator $U(t,t')$ for $t > t_1$ and $t' < t_o$ is simply

$$U(t,t') = e^{-iH^{in}(t_o-t')} e^{-iH(t_1-t_o)} e^{-iH^{out}(t-t_1)} . \quad (3.4)$$

Since

$$b_s^{out}(\underset{\sim}{p}) e^{-i\omega(\underset{\sim}{p})t} = U^{\dagger}(t,t') b_s^{in}(\underset{\sim}{p}) e^{-i\omega(\underset{\sim}{p})t'} U(t,t')$$

$$d_s^{out}(\underset{\sim}{p}) e^{-i\omega(\underset{\sim}{p})t} = U^{\dagger}(t,t') d_s^{in}(\underset{\sim}{p}) e^{-i\omega(\underset{\sim}{p})t'} U(t,t') ,$$

the S-matrix is

$$S = U(0,0) = e^{-iH^{in}t_o} e^{-iH(t_1-t_o)} e^{iH^{out}t_1} . \quad (3.5)$$

Having

$$H^{out} = e^{iH(t_1-t_o)} H^{in} e^{-iH(t_1-t_o)} ,$$

(3.4) and (3.5) can be rewritten as

$$U(t,t') = e^{-iH^{in}(t-t')} e^{+iH^{in}(t_1-t_o)} e^{-iH(t_1-t_o)}$$

$$S = e^{iH^{in}(t_1-t_o)} e^{-iH(t_1-t_o)} .$$

ii) Let us now consider cases where the external field changes in time according to a sequence of N step functions:

$$A(t,\underset{\sim}{x}) = 0 \qquad \forall\ t < t_o$$
$$= A_i(\underset{\sim}{x}) \qquad t_{i-1} < t < t_i \qquad i=1,2,\ldots N$$
$$= A_{N+1}(\underset{\sim}{x}) \ \forall\ t > t_N \ . \tag{3.6}$$

As in the previous examples, the field operator solution for $t_{j-1} \leq t \leq t_j$ is simply

$$\psi(t,\underset{\sim}{x}) = e^{-ih_j(t-t_{j-1})}\, e^{-ih_{j-1}(t_{j-1}-t_{j-2})} \ldots$$
$$\ldots\, e^{-ih_1(t_1-t_o)}\, e^{-ih_o(t_o-t')}\, \psi_{in}(t',\underset{\sim}{x}) \tag{3.7}$$

where h_j is defined as

$$h_j = h_o - e\gamma^o\gamma.A_j(\underset{\sim}{x}) \ . \tag{3.8}$$

We are going to show that

$$\psi(t,\underset{\sim}{x}) = u(t,t_o)\psi_{in}(t_o,\underset{\sim}{x}) \tag{3.9}$$

is unitarily implementable for all $t > t_o$ if and only if all h_j are such that the conditions (2.8) and (2.9) are satisfied. Since these were shown to be equivalent to (2.6) and (2.7) there will then exist also well defined energy operators H_j of the form

$$H_j = \int_{S_j(-)} dE\ E\ b^j(E)^{\dagger} b^j(E) + \int_{S_j(-)} dE'\ E'\ d^j(E')^{\dagger} d^j(E') \tag{3.10}$$

where $S_j(-)$ is the negative part of the spectrum of h_j and $\overline{S_j(-)}$ its complement.

Let us firstly examine the case $t_o \leq t \leq t_1$; (3.9) is then simply

$$\psi(t,\underset{\sim}{x}) = e^{-ih_1(t-t_o)}\,\psi_{in}(t_o,\underset{\sim}{x}) \qquad (3.11)$$

and we have already shown that this is unitarily implementable if and only if h_1 is such that (2.6) and (2.7) are satisfied and that there exists an energy operator H_1 of the form (3.10). We have

$$\psi(t,\underset{\sim}{x}) = U_1^{\dagger}(t-t_o)\psi_{in}(t_o,\underset{\sim}{x})U_1(t-t_o) \qquad (3.12)$$

where

$$U_1(t-t_o) = e^{-iH_1(t-t_o)} .$$

Let us now consider $t_1 \leq t \leq t_2$;

$$\psi(t,\underset{\sim}{x}) = e^{-ih_2(t-t_1)}\,e^{-ih_1(t_1-t_o)}\,\psi_{in}(t_o,\underset{\sim}{x}) . \qquad (3.13)$$

Using (3.12) this is equivalent to

$$U_1(t_1-t_o)\psi(t,\underset{\sim}{x})U_1^{\dagger}(t_1-t_o) = e^{-ih_2(t-t_1)}\,\psi_{in}(t_o,\underset{\sim}{x}). \qquad (3.14)$$

We then see that (3.13) is unitarily implementable if and only if (3.14) is. As before, this is so if and only if h_2 is such that (2.6) and (2.7) are satisfied.

There will then exist a unitary operator

$$\bar{U}_2(t-t_1) = e^{-i\bar{H}_2(t-t_1)}$$

such that

$$e^{-ih_2(t-t_1)}\psi_{in}(t_o,x) = \bar{U}_2^\dagger(t-t_1)\psi_{in}(t_o,x)\bar{U}_2(t-t_1) \ .$$

$\bar{H}_2$ is of the form (3.10). The creation and annihilation operators $\bar{b}^2(E)$, $\bar{d}^2(E)$ and their adjoints are also related to $b^{in}(t_o)$, $d^{in}(t_o)$ according to equation (2.4). That is $\bar{H}_2$ is the Hamiltonian defined in the first problem when $A(\underset{\sim}{x})$ is $A_2(\underset{\sim}{x})$. We have

$$\psi(t,\underset{\sim}{x}) = U_1^\dagger(t_1-t_o)\bar{U}_2^\dagger(t-t_1)\psi_{in}(t_o,\underset{\sim}{x})\bar{U}_2(t-t_1)U_1(t_1-t_o) \ . \tag{3.15}$$

There is no difficulty in extending this to any number of steps. By induction, using the same argument as used in going from equation (3.13) to (3.15), we thus prove the required result. We obtain that the unitary operator implementing (3.9) is simply, for $t_{j-1} \leq t \leq t_j$,

$$U(t,t_o) = \bar{U}_j(t-t_{j-1})\bar{U}_{j-1}(t_{j-1}-t_{j-2})\ldots\bar{U}_2(t_2-t_1)U_1(t_1-t_o) \tag{3.16}$$

where

$$\bar{U}_\ell(t-t') = \text{[illegible]}\ t-t') \quad . \qquad (3.17)$$

$\bar{H}_\ell$ is the energy [illegible] defined in section 2) with $A(x) =$ [illegible]). [illegible]ticular, if $A_\ell(x) = 0$ for some ℓ, [illegible] H[illegible]

The e[illegible]ator H_j for times $t_{j-1} \le t \le t_j$ is the generat[illegible]me translations such that

$$\psi(t,x) = e^{iH_j(\text{[illegible]}}\ \text{[illegible]}\ \psi(t_{j-1},x)e^{-iH_j(t-t_{j-1})}$$

i.e.,

$$\psi(t,x) = e^{iH_j(t-t_j\text{[illegible]}}\ \text{[illegible]}^\dagger(t_{j-1},t_o)\psi_{in}(t_o,x)U(t_{j-1},t_o) \times$$

$$\times\, e^{-iH_j(t-t_{j-1})} \ .$$

Comparing this wit[illegible]

$$\psi(t,x) = U^\dagger(t,t_o)\psi_{\text{[illegible]}}(t_o,x)U(t,t_o)$$

$$= U^\dagger(t_{j-1},t_o)e^{i\bar{H}_j(t-t_{j-1})}\psi_{in}(t_o,x)e^{-i\bar{H}_j(t-t_{j-1})} \times$$

$$\times\, U(t_{j-1},t_o) \ ,$$

we see that

$$e^{iH_j(t-t_{j-1})} = U^\dagger(t_{j-1},t_o)e^{-i\bar{H}_j(t-t_{j-1})}U(t_{j-1},t_o)$$

so that

$$H_j = U^\dagger(t_{j-1},t_o)\ \bar{H}_j\ U(t_{j-1},t_o) \ . \qquad (3.18)$$

The energy operator H_j will then have the form (3.10). In fact, the particles creation and annihilation operators b^j and d^j at time t_{j-1} are given by

$$b^j(E)e^{-iEt_{j-1}} = U^{\dagger}(t_{j-1},t_o)\bar{b}^j(E)U(t_{j-1},t_o)$$

$$d^j(E')e^{-iE't_{j-1}} = U^{\dagger}(t_{j-1},t_o)\bar{d}^j(E')U(t_{j-1},t_o). \quad (3.19)$$

If $A_{N+1}(\underset{\sim}{x}) \neq 0$, the field theoretical S-matrix, relating the operations b^j, d^j to the initial operators b^{in}, d^{in}, is made up of the unitary transformation relating $\bar{b}^j(E)$, $\bar{d}^j(E')$ to the initial operators b^{in}, d^{in} and $U(t_N,t_o)$.

If $A_{N+1}(\underset{\sim}{x}) = 0$, $\psi(t,\underset{\sim}{x})$ for $t > t_N$ is the free field $\psi_{out}(t,\underset{\sim}{x})$ and

$$\psi_{out}(t,\underset{\sim}{x}) = [e^{-iH_o(t-t_N)}U(t_N,t_o)]^{\dagger}\,\psi_{in}(t_o,\underset{\sim}{x})[e^{-iH_o(t-t_N)} \times U(t_N,t_o)]$$

so that the S matrix which in this case $(A_{N+1} = 0)$ is such that $\psi_{out}(0,\underset{\sim}{x}) = S^{\dagger}\psi_{in}(0,\underset{\sim}{x})S$ is simply

$$S = e^{iH^{in}(t_N-t_o)}\,U(t_N,t_o) \quad . \quad (3.20)$$

Potentials with Continuous Time Dependence

Let us now consider an external field of the form

$$A(t,\underset{\sim}{x}) = 0 \qquad t < t_o$$

$$\in L^p(t_o,t_f) \qquad t_o < t < t_f \qquad (1 \leq p < \infty)$$

$$= A(t_f,\underset{\sim}{x}) \qquad t > t_f \qquad (3.21)$$

where $A(t,\underset{\sim}{x}) \in L^p(t_o,t_f)$ means that for all $t \in (t_o,t_f)$

$$\int_{t_o}^{t} dt' |A^\mu(t',\underset{\sim}{x})|^p < \infty \, .$$

It is known (see for example T. Kato [1966]; example 1.13) that any such function of time can be approximated as closely as one wants, by a finite series of step functions in time (i.e. like those discussed in the preceding section).

Let us then divide the time interval (t_o,t_f) in N equal sub-intervals $\Delta t = (t_f-t_o)/N$ and call $u_N(t,t_o)$ the c-number unitary operator defined as in equation (3.7) in which we take

$$h_j = h_o - e\gamma^o\gamma A_j(\underset{\sim}{x}) \qquad (3.22)$$

$$A_j(\underset{\sim}{x}) = A(t_j,\underset{\sim}{x}) = A(t_o + j\Delta t,\underset{\sim}{x}) \, . \qquad (3.23)$$

We will suppose that $A(t,\underset{\sim}{x})$ is such that the following limit defining the c-number unitary operator $u(t,t_o)$ exists:

$$\lim_{N\to\infty} u_N(t,t_o) = u(t,t_o) \quad . \tag{3.24}$$

We now want to see what is implied when, $\forall$ t such that $t_o \leq t \leq t_f$, the conditions (2.6) and (2.7) are satisfied i.e. when

$$\text{Trace } P_+(t)\ P_-^o < \infty$$

$$\text{Trace } P_-(t)\ P_+^o < \infty \qquad \forall\ t \ . \tag{3.25}$$

$P_+(t)$, $P_-(t)$ are the projection operators on the functions corresponding respectively to non-negative and negative eigenvalues of $h(t) = h_o - e\gamma^o\gamma A(t,\underset{\sim}{x})$.

Whatever N the conditions (2.6), (2.7) will obviously be satisfied $\forall\ t_j$ i.e. we have

$$\text{Trace } P_+^j\ P_-^o < \infty$$

$$\text{Trace } P_-^j\ P_+^o < \infty \qquad \forall\ t_j \ . \tag{3.26}$$

As was shown in the previous section ii), the Bogoliubov transformation corresponding to

$$\psi(t,\underset{\sim}{x}) = u_N(t,t_o)\psi_{in}(t_o,\underset{\sim}{x}) \tag{3.27}$$

will then be unitarily implementable at all times whatever N. In particular, when N becomes very large, (3.27) becomes (by hypothesis)

$$\psi(t,\underset{\sim}{x}) = u(t,t_o)\psi_{in}(t_o,\underset{\sim}{x}) \qquad (3.28)$$

so that this should be unitarily implementable for all times when (3.25) is satisfied.

Conversely, if (3.28) is assumed unitarily implementable (by an operator $U(t,t_o)$) for all times $t_o \leq t \leq t_f$, then

$$\psi(t-dt,\underset{\sim}{x}) = u(t-dt,t_o)\psi_{in}(t_o,\underset{\sim}{x}) \qquad (3.29)$$

is unitarily implementable (by $U(t-dt,t_o)$). Since by definition

$$u(t,t_o) = e^{-ih(t)dt}\, u(t-dt,t_o) \ , \qquad (3.30)$$

one has

$$U^\dagger(t,t_o)\psi_{in}(t_o,\underset{\sim}{x})U(t,t_o) = u(t,t_o)\psi_{in}(t_o,\underset{\sim}{x})$$

$$= e^{-ih(t)dt}\, u(t-dt,t_o)\psi_{in}(t_o,\underset{\sim}{x})$$

$$= e^{-ih(t)dt}\, U^\dagger(t-dt,t_o)\psi_{in}(t_o,\underset{\sim}{x})U(t-dt,t_o) \ .$$

Therefore

$$e^{-ih(t)dt}\, \psi_{in}(t_o,\underset{\sim}{x}) = U(t-dt,t_o)U^\dagger(t,t_o)\psi_{in}(t_o,\underset{\sim}{x}) \times$$

$$\times\, U(t,t_o)U^\dagger(t-dt,t_o) \ , \qquad (3.31)$$

the transformation $e^{-ih(t)dt}$ on $\psi_{in}(t_o, \underset{\sim}{x})$ must then be unitarily implementable. From section 2), we know that this is true if and only if h(t) is such that the conditions (2.6) and (2.7) hold. Since (3.31) must be true for all t, (2.6) and (2.7) must hold for all t (i.e. (3.25) must hold) and then a well defined energy operator of the usual form must exist at all times.

We thus obtain that when the external field $A(t,\underset{\sim}{x})$ is such that a c-number unitary operator can be defined by equation (3.24),

$$\psi(t,\underset{\sim}{x}) = u(t,t_o)\psi_{in}(t_o,\underset{\sim}{x})$$

is unitarily implementable for all $t \geq t_o$ if and only if for all t, h(t) is such that conditions (2.6), (2.7) are satisfied. There will then exist also, at all times, a well defined energy operator H(t) of the form

$$H(t) = \int_{\overline{S_t(-)}} dE\ E\ b(E,t)^{\dagger} b(E,t) + \int_{S_t(-)} dE'\, E'\ d(E',t)^{\dagger} d(E',t).$$

$S_t(-)$ is the negative part of the spectrum of h(t) and $\overline{S_t(-)}$ its complement.

This is the same property as the one demonstrated in the previous section with a finite sequence of step functions in time.

Although a more detailed proof than the one we gave above could be desirable, the important results obtained are certainly expected to be true due to the following facts. In the case of the sequence of step functions, the results are completely independent of the number (N) and lengths of the time intervals. In section 2), we have proved that the conditions (2.6), (2.7) (and (2.8), (2.9)) also are independent of the length of the time intervals "a" $\neq 0$ appearing in e^{-iha}.

The results obtained in this section confirm the great importance of the conditions (2.6) and (2.7) (or the equivalent (2.8) and (2.9)). These now appear as really crucial for having a satisfactory physical interpretation of the present formalism.

The most urgent problem to be resolved at the moment is whether or not these conditions can be satisfied with magnetic potentials. We recall that in the case of the potential $\delta(t)A(\underset{\sim}{x})$, we were able to prove explicitly that no magnetic fields were allowed. This however was a very particular example; in the general case, no such negative answer has been obtained even though, up to now, nobody has been able to obtain any magnetic potential with which (2.6) and (2.7) are satisfied.

It would certainly be worthwhile also to try and get better estimates on $A_o(\underset{\sim}{x})$. It is reasonable to expect that, in the final version of the external field theory, there should be almost no restrictions on the potentials. The only restrictions should come from the usual c-number requirements (like demanding that A is such that h(t) is self-adjoint and u(t,t') exists for example) just as in the treatment of systems with time independent potentials.

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