University of Alberta

ASYMPTOTIC STRUCTURES IN BANACH SPACES

by

Bünyamin Sarı ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

Mathematics

Department of Mathematical and Statistical Sciences

Edmonton, Alberta

Fall 2003

Your file Votre référence
ISBN: 0-612-88044-3
Our file Notre référence
ISBN: 0-612-88044-3

University of Alberta

Release Form

Name of Author: Bünyamin Sarı

Title of Thesis: Asymptotic Structures in Banach Spaces

Degree: Doctor of Philosophy

Year this Degree Granted: 2003

(Signed) ....

.ematical

ces

a

Canada, T6G 2G1

Date: ..2 May 2003.....

UNIVERSITY OF ALBERTA

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Asymptotic Structures in Banach Spaces** submitted by **Bünyamin Sarı** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematics.

Dr. Edward Odell (University of Texas at Austin)

May 1, 2003

To my family

Abstract

This thesis is composed of two independent parts:

Part I studies the asymptotic structures of Banach spaces through the notion of envelope functions. Analogous to the original ones, a new notion of disjoint-envelope functions is introduced and the properties of these functions in connection to the asymptotic structures are studied. One of the central result obtained using these functions is a new characterization for asymptotic-ℓ_p spaces. One application of this result yields a solution to a conjecture on the structure of so-called Tirilman spaces. Apart from some other applications of the envelope functions, the finite representability of these functions are investigated.

Part II is on the structure of the set of spreading models of Orlicz sequence spaces. In the case when an Orlicz sequence space admits few (countable) spreading models, a description of this set is established.

Acknowledgments

First and foremost, I sincerely thank my advisor Nicole Tomczak-Jaegermann. I am truly fortunate to have such an exceptional teacher introducing me to the beauty of Banach Space Geometry and also teaching me how to approach mathematical research. I will always be indebted to her for her constant help and generous support at various stages of my graduate study, and particularly grateful for her endless patience with me.

I thank Cuixia Hao for fruitful discussions on the structure of Orlicz sequence spaces. I have also benefited from discussions with Steve Dilworth and Ted Odell. In particular, the idea of the proof of Theorem 4.3.9 evolved from a remark of S. Dilworth. T. Odell have made numerous suggestions on all parts of the thesis. I sincerely thank them both for sharing their ideas with me.

I also thank all members of the Functional Analysis Group with whom I had the opportunity to interact freely. I thank Tony Lau and Laurent Marcoux for the wonderful courses they taught which help build intuition in early years of my graduate study.

Ali Ülger got me interested in functional analysis in my undergrad years and motivated me to come to Alberta for the graduate school. I thank him deeply.

My colleague and officemate Razvan Anisca has been a wonderful friend both in and out of Math Department, I would like to thank him and Monica Ilie for their friendship. I also met many nice people in Edmonton. I thank Julia Chu for all we have shared and for her friendship. I also thank Cengiz and his family, Murat, Fatih, Faruk and all my friends for their pleasant company.

Contents

Chapter 1

Introduction

This thesis is devoted to study some problems in Asymptotic Geometric Analysis of infinite-dimensional Banach spaces.

In general, asymptotic methods in the theory of infinite-dimensional Banach spaces rely on stabilizing information of finite nature "at infinity". This way we discard properties which may appear sporadically in the space and could be removed by passing to appropriately chosen subspaces or some other substructures. First methods of this kind began to develop in the 1970's, in connection with Ramsey theorems and the notion of a spreading model (to be described later in this introduction). The ideas behind what is now called an asymptotic theory of (infinite-dimensional) Banach spaces were crystallized in the early nineties in connection with spectacular developments of the infinite-dimensional Banach spaces.

From the very beginning of Functional Analysis initiated by the work of Banach in the 1920's the objective of the classical theory of infinite-dimensional spaces have been mainly to establish a structure theory for Banach spaces. Besides isomorphism type questions, primarily, the problems were centered around seeking subspaces with 'nice' structural properties in all Banach spaces. Must every infinite-dimensional Banach space which is isomorphic to all its infinite-dimensional closed subspaces be isomorphic to a Hilbert space? Does every infinite-dimensional Banach space contain one of the classical spaces c_0 or ℓ_p for some $1 \leq p < \infty$? Even these simply stated questions raised by Banach turned out to be not so trivial. In fact, the first problem, called the homogeneous Banach space problem, was solved only in the nineties. We will briefly discuss the developments in this decade shortly. The latter question was answered in the early seventies, when Tsirelson [Ts] discovered a counterexample. Tsirelson's space is now referred to as "the first truly non-classical Banach space". The definition of this space involves a clever inductive procedure which enables a certain geometric property to pass to every infinite-dimensional subspace, and the *saturation* property achieved this way prevents the space from containing c_0 or any ℓ_p. Soon afterwards, Figiel and Johnson [FJ] gave an analytic description of the norm in the space now denoted by T, which is the dual of Tsirelson's original example. The norm in T appears as a solution to an implicit equation, contrary to the definitions of the classical spaces for which the norms are given by explicit formulas. This idea of defining a norm implicitly has become relatively commonplace. Many new Tsirelson-like spaces have been engineered since then to solve a good many problems in Banach space theory (cf. [CS]).

The deep developments of the nineties have shed light on the ideas initiated with Tsirelson's example. It turned out that the Tsirelson-like spaces are not just a collection of pathological examples but, in fact, they hold the key to a deeper understanding of the infinite-dimensional phenomena. The idea of saturating spaces with a desired geometric property was re-vitalized when Schlumprecht [S] defined a space which initiated a series of results answering fundamental and long standing problems of Banach space theory. For Gowers and Maurey [GM], Schlumprecht's space was a starting point which lead them to their ground-breaking construction of a space with no unconditional basic sequences. Their space, in fact, has a stronger property, namely, it is hereditary indecomposable (H.I.), which means that none of its closed subspaces can be written as a topological direct sum of two infinite-dimensional closed subspaces.

Gowers then showed that H.I. spaces arise *naturally* among Banach spaces. His famous dichotomy theorem states that every Banach space has either a subspace with an unconditional basis or an H.I. subspace [G1]. It is remarkable to note that these new spaces, despite their 'unnatural' definitions, played important part in the solutions of fundamental problems about the classical spaces. The dichotomy theorem of Gowers combined with a result of Komorowski and Tomczak-Jaegermann [KT] gave a positive solution to the homogeneous space problem: a Hilbert space is the only infinite-dimensional Banach space, up to isomorphism, which is isomorphic to every infinite-dimensional closed subspace of itself. Also Schlumprecht's space played an important role in the solution of another famous problem known as the distortion of Hilbert (and the classical ℓ_p) spaces [OS3]. These developments and the discovery of new spaces had a great impact on the classical understanding of 'nice' subspaces. Quoting from Maurey, Milman and Tomczak-Jaegermann [MMT], "it has been realized recently that such a nice and elegant structural theory (of infinite-dimensional Banach spaces) does not exist. Recent examples (or counterexamples to classical problems) due to Gowers and Maurey [GM] and Gowers [G2], [G3] showed much more diversity in the structure of infinite-dimensional Banach spaces than was expected."

At the other end of the spectrum, in the last three decades, having employed new powerful techniques from other areas of mathematics such as probability and combinatorics, there have been deep developments in the local theory of Banach spaces (cf. e.g [MS]). This theory is asymptotic in nature; striking regularities of finite-dimensional spaces are observed when dimension increases to infinity.

In the light of the developments of the nineties, the dichotomic nature of finite vs. infinite-dimensional theory naturally invited the formulation of a similar asymptotic approach for infinite-dimensional spaces.

Again the first ideas come from the seventies with the notion of a spreading model, which involves stabilization of norms at infinity. In 1974, Brunel and Sucheston [BS] gave a simple but unexpected application of Ramsey theorem to Banach spaces. Roughly speaking, Ramsey type theorems are of the following form. Given any finite coloring of some mathematical structure (such as graphs or a set of n-tuples from a sequence of vectors in a Banach space), there is a substructure (hence a subgraph or a set of n-tuples from a subsequence) which is monochromatic. In other words, any function defined on the structure into a finite set can be stabilized (becomes constant) on a substructure. As a direct application with obvious approximation and diagonalization arguments, Brunel and Sucheston showed that in every Banach space every normalized basic sequence $\{x_i\}$ has a subsequence $\{y_i\}$ on which the norm of any linear combination of n vectors of $\{y_i\}$ stabilizes (they span the same finite dimensional space) provided that they are sufficiently far along $\{y_i\}$. Consequently, the iterated limit

$$\lim_{i_1\to\infty}\dots\lim_{i_k\to\infty}\|\sum_k a_k y_{i_k}\|$$

exists and it defines a norm on the linear space of finite scalars c_{00}. (The reason for the iterated limit is that keeping the order of scalars is important in this definition.) The space c_{00} with this new norm is called a *spreading model* (generated by $\{y_i\}$). This new object we obtain behaves relatively 'better' than the original sequence $\{y_i\}$. For instance, the unit vector basis $\{e_i\}$ of a spreading model has the 'spreading' property, which means that $\|\sum_{i=1}^k a_i e_{n_i}\| = \|\sum_{i=1}^k a_i e_{m_i}\|$ for all scalars $(a_i)_{i=1}^k$, $n_1 < \ldots < n_k$ and $m_1 < \ldots < m_k$. Moreover, the basis is often unconditional. Thus starting with an arbitrary basic sequence, a spreading model provides subsequences of finite (but of arbitrary) length with 'nice' properties.

Spreading models are proven to be very useful in Banach space theory (cf. e.g. [BL]). When considering questions about finding nice finite-dimensional subspaces, we can simply assume that the space has a spreading (and even unconditional [R]) basis by passing to a spreading model. Since any space finitely representable in a spreading model is finitely representable in the generating sequence (this fact is immediate from the definition of a spreading model), we can then transfer these finite spaces into the space. The proof of the classical Krivine's theorem, for instance, follows this scheme (cf. e.g. [MS]).

Despite their usefulness, spreading models do not reflect the intrinsic properties of a space. To access information about subspace structures of a Banach space, one has to look at the *blocks* of, rather than subsequences of, a basis. This is the Bessaga and Pelczynski principle, which states that every subspace Y of a space X with a basis has a further subspace Z isomorphic to a block subspace. This reduces many problems about subspaces of Banach spaces to ones about block subspaces.

Is there a *block Ramsey* theorem which could provide stronger stabilization results than that of spreading models? The answer turned out to have an interesting twist. Gowers [G1] indeed proved such an infinite block Ramsey theorem which lead to his famous dichotomy theorem mentioned above. On the other hand, as the solution of the distortion problem [OS3] showed, a truly infinite-dimensional phenomena in general may not stabilize.

In the light of these results, Maurey, Milman and Tomczak-Jaegermann [MMT] have introduced a new type of stabilization which gave rise to a new notion of asymptotic structures. This theory of asymptotic structures essentially introduced to study the structure of infinite-dimensional spaces, and yet it involves stabilization of finite-dimensional subspaces which appear *everywhere far away* in the space. The main idea is to bridge finite-dimensional and infinite-dimensional theories. Such finite-dimensional spaces which appear everywhere far away in the space are called *asymptotic spaces.* The notion of asymptotic spaces generalizes the notion of spreading models; but it has essential differences.

To explain this notion, we first recall some basic notations. The precise definitions and some other aspects of the asymptotic theory will be given in

Chapter 2. For subsets I and J of the natural numbers $\mathbb{N}$, we write $I < J$ if $\max I < \min J$. For simplicity, we consider a Banach space with a basis $\{x_i\}$. For a vector $y = \sum_i a_i x_i$ in X, the support of y, supp$y$, is just the set of i for which a_i is non-zero. A block vector is a vector with a finite support, and blocks are successive, $y_1 < y_2$, if supp$y_1 <$ suppy_2.

An n-dimensional space E with a monotone normalized basis $\{e_i\}$ is an *asymptotic space* of X (we denote by $E \in \{X\}_n$), if there exist successive blocks $y_1, \ldots, y_n$ in X as close to $\{e_i\}$ as we wish, and arbitrarily far and spread out with respect to the basis $\{x_i\}$. Precisely, given $\varepsilon > 0$, for arbitrarily large m_1 there is a block y_1 with $\{m_1\} <$ suppy_1 such that for an arbitrarily large m_2 there is a block y_2 with $\{m_2\} <$ suppy_2, etc, such that $y_1, \ldots, y_n$ obtained after n steps are successive and $(1+\varepsilon)$-equivalent to the basis $\{e_i\}$. The successive blocks $y_1, \ldots, y_n$ are called *permissible* vectors. The asymptotic structure of X consists of all asymptotic spaces of X. Now it is clear that if $\{e_i\}$ is the natural basis of any spreading model generated by a subsequence of the basis $\{x_i\}$ of X, then for all n, the span of the first n vectors $\{e_i\}_{i=1}^n$ is an asymptotic space of X. In fact, one can always find better asymptotic spaces; it is a consequence of the classical Krivine's theorem that for every X there is $1 \leq p \leq \infty$ such that $\ell_p^n \in \{X\}_n$ for all n. Thus $\{X\}_n$ is never empty.

The first general problem in this context is to describe the set of asymptotic spaces of a given Banach space X. The definition of an asymptotic space already hints that this might not be an easy task. As it is common practice in analysis, a starting point would be then to define some relevant functions on the asymptotic structure of X and hope to get information through studying these functions.

This, in fact, is the main project of the first part of this thesis. The functions we consider are called envelope functions; they have been introduced by Milman and Tomczak-Jaegermann [MT1] and used to discover a new class of Banach spaces, called asymptotic-ℓ_p spaces.

For any finite sequence of scalars $a = (a_i)$ the *upper envelope* is a function

$r(a) = \sup \| \sum_i a_i e_i \|$, where the supremum is taken over all natural bases $\{e_i\}$ of asymptotic spaces $E \in \{X\}_n$ and all n. Similarly, the *lower envelope* is a function $g(a) = \inf \| \sum_i a_i e_i \|$, where the infimum is taken over the same set. These functions clearly define upper and lower bounds for the 'spectrum' of the set of asymptotic spaces of X. The useful fact about the envelopes is that they have nice properties, for instance, they are always close to some ℓ_p-norms (see section 2.4).

A Banach space X is called *asymptotic-ℓ_p* $(1 \leq p \leq \infty)$ if there exists C such that for all n and $E \in \{X\}_n$, the basis in E is C-equivalent to the unit vector basis of ℓ_p^n. Clearly this happens if and only if both g and r are equivalent to the ℓ_p-norm. As examples, the original Tsirelson's space is an asymptotic-ℓ_∞ and its dual T is an asymptotic-ℓ_1 space. Asymptotic-ℓ_p spaces appear naturally in connection with the distortion problem [MT1]. Thus the structure of these spaces is of particular interest. An interesting result proved in [MMT] says that for $1 < p < \infty$, if X is a Banach space such that there exists C such that for every n, every asymptotic space $E \in \{X\}_n$ is C-isomorphic to ℓ_p^n, then X is an asymptotic-ℓ_p space. This means that in the asymptotic setting isomorphisms imply the equivalence of bases (at least for ℓ_p spaces for $1 < p < \infty$). This is truly an asymptotic phenomenon; its classical analogue requires a strong additional assumption on bases (cf. [LT] and also [JMST]).

In Part I, we prove another result of this type; a characterization of asymptotic-ℓ_p spaces in terms of the 'ℓ_p-behavior' of disjoint-permissible vectors with constant coefficients. To describe this result, we consider a Banach space X with *an asymptotic unconditional structure*, which means that there exists C such that for every n and $E \in \{X\}_n$, the basis $\{e_i\}$ in E is C-unconditional. In such a space X, consider the set (we denote by $\{X\}^d$) of all sequences of normalized vectors $\{x_j\}$ which are disjointly supported with respect to the basis of some asymptotic space for X. These are called *disjoint-permissible vectors.* Then we have the following characterization for asymptotic-ℓ_p spaces. Let $1 \leq p \leq \infty$.

For a Banach space X with asymptotic unconditional structure if there exists K such that for every n and $\{x_j\}_{j=1}^n \in \{X\}^d$,

$$\frac{n^{1/p}}{K} \leq \|\sum_{j=1}^{n} x_j\| \leq K n^{1/p},$$

then X is asymptotic-ℓ_p.

For the proof, we introduce the notion of disjoint-envelope functions; these are the natural analogues of the envelope functions defined on the set $\{X\}^d$. For any finite sequence of scalars $a = (a_i)$, the *upper disjoint-envelope* is a function $r^d(a) = \sup \|\sum_i a_i x_i\|$, where the supremum is taken over all $\{x_i\} \in \{X\}^d$. Similarly, the *lower disjoint-envelope* is a function $g^d(a) = \inf \|\sum_i a_i x_i\|$ defined over the same set. The disjoint envelopes share the nice properties as similar to the original envelopes.

The new results of Part I are contained in Chapter 3. The first section of the chapter is about the notion of asymptotic unconditionality, which is a notion of great importance for the thesis. This notion was first introduced by Milman and Sharir [MiS] and, in particular, they gave a characterization of asymptotic unconditionality in terms of norming permissible functionals. Section 3.1 is mainly devoted to the proof of a reformulation of this result in the context of asymptotic structures. The proof is rather complicated, so it is divided into several parts to emphasize several facts involved, which are of independent interest.

The disjoint-envelope functions are introduced in Section 3.2, which also develops some properties of these functions analogous to those of original envelopes. In [MT2], it is shown that the envelope functions on a reflexive space X and on its dual X^* are in natural duality. In Section 3.3 we show that the analogous result holds for the disjoint-envelopes as well.

The characterization for asymptotic-ℓ_p spaces mentioned above is given in Section 3.4. We also show by presenting suitable examples that this result cannot be improved by replacing disjoint-permissible vectors (i.e., vectors in the set $\{X\}^d$) in the assumption with permissible vectors. These examples turn out to be a class of Banach spaces already in the literature, called the Tirilman spaces. Incidentally, as a byproduct of the results proved in Section 3.4, we obtained a solution to a conjecture of Casazza and Shura [CS] about the structure of the Tirilman spaces. These are presented in Section 3.5.

Section 3.6 contains another application of the envelope functions. Using the (original) envelopes and a stabilization result due to Milman and Tomczak-Jaegermann [MT2], we prove an asymptotic analog of a classical result of James concerning reflexivity: A Banach space X with an asymptotic unconditional structure must either have ℓ_1^n or ℓ_∞^n as asymptotic spaces for all n, or it contains a reflexive subspace.

The final section of Chapter 3 deals with a finite representability problem for the envelope functions. As remarked earlier, for every Banach space X the envelope functions r and g are always close to some ℓ_p (and ℓ_q) norms, where p and q depend on the asymptotic structure of X only. A similar fact holds also for the disjoint-envelope functions. We refer to these p and q as the *power types* of the envelopes. The problem we are concerned with here is whether these ℓ_p and ℓ_q spaces are finitely representable in X asymptotically. Namely, is it true that for all n, $\ell_p^n, \ell_q^n \in \{X\}_n$ (resp. $\ell_p^n, \ell_q^n \in \{X\}^d$), where p and q are the power types of r and g (resp. r^d and g^d)? We show in Section 3.7 that the answer is affirmative for the disjoint-envelopes and yet it is negative for the original envelopes.

In Part II we study the structure of the set of spreading models of Orlicz sequence spaces.

As we remarked earlier, a spreading model involves a stabilization on subsequences of a basic sequence, and hence it may not provide intrinsic properties of the space. However, one can consider the set of all spreading models (which will be denoted by $SP(X)$) of a Banach space X. In some instances, from the information about the set $SP(X)$ one can get infinite-dimensional information of X (cf. e.g. [OS5], [OS4]).

Our particular object of attention is the following novel approach to the spreading model theory due to Androulakis, Odell, Schlumprecht and Tomczak-Jaegermann [AOST]. Defining a partial order on the set $SP(X)$, they have studied the structure of the partially ordered set $SP(X)$, and, for instance, they showed that every countable set of spreading models (generated by weakly null sequences) of a space X admits an upper bound with respect to this partial order. In some cases, using the results about the structure of the set $SP(X)$ they have obtained interesting applications concerning the existence of certain operators on X.

Following this direction, in Chapter 4 we study the structure of this partially ordered set for Orlicz sequence spaces. We showed that if the set of spreading models of an Orlicz sequence space is countable, then it contains both the upper and the lower bounds, and the upper bound is the space itself and the lower bound is some ℓ_p space. This and some other results are given in Section 4.3. Section 4.2 reviews some basic facts about Orlicz sequence spaces. The precise definition of a spreading model and a discussion of the results of [AOST] are presented in the introduction of Chapter 4.

1.1 Notations and Basic Concepts in the Geometry of Banach Spaces

In this thesis, all spaces are real *separable Banach* spaces and all subspaces are closed subspaces. By X, $Y, \dots$ we usually denote infinite-dimensional Banach spaces; we reserve E, $F, \dots$ to denote finite-dimensional Banach spaces.

The norm in X is denoted by $\|.\|_X$, or simply by $\|.\|$ if there is no ambiguity. By B_X we denote the closed unit ball $\{x \in X : \|x\| \leq 1\}$, and by S_X the unit sphere $\{x \in X : \|x\| = 1\}$ of X.

Linear continuous maps between two Banach spaces X and Y are called *operators* and denoted by $T : X \to Y$. If T is an isomorphism between X and Y, the *isomorphism constant* C is defined by $C = \|T\|\|T^{-1}\|$ and in this case we write $X \overset{C}{\simeq} Y$, or simply $X \simeq Y$ if we do not want to specify the isomorphism constant. We will say that X and Y are C-isomorphic or simply isomorphic.

For a set E in X, $\overline{\text{span}}[E]$ denotes the closed *linear span* of E in X and $\overline{\text{conv}}[E]$ the closed *convex hull* of E.

As examples of Banach spaces we shall often use the classical sequence spaces c_0, and ℓ_p $(1 \leq p \leq \infty)$. c_0 is the space of all real sequences $x = (a_n)$ with $\lim_{n\to\infty} a_n = 0$ with the norm $\|x\|_\infty = \sup_n |a_n|$. For any $1 \leq p < \infty$, ℓ_p is the space of real sequences $x = (a_n)$ with $\sum_n |a_n|^p < \infty$, and the norm $\|x\|_p = (\sum_{n=1}^\infty |a_n|^p)^{1/p}$. ℓ_∞ is the space of all bounded real sequences $x = (a_n)$ with the norm $\|x\|_\infty = \sup_n |a_n|$. For each $n \in \mathbb{N}$, ℓ_p^n $(1 \leq p \leq \infty)$ denotes the n-dimensional space $\mathbb{R}^n$ with ℓ_p-norm.

Perhaps the most fundamental notion we use throughout the thesis is the notion of a basis. A *Schauder basis* or simply a *basis* for a Banach space X is a sequence $\{x_n\}$ of vectors in X such that every vector x in X has a unique representation of the form $x = \sum_n a_n x_n$ where each a_n is a scalar and the sum converges in the norm topology. For each n, the mapping $x \to a_n$ then defines a continuous linear functional x_n^* on X. A sequence $\{x_n\}$ in X is a *basic sequence* if $\{x_n\}$ is a basis for its closed linear span in X. The *basis projections* of a basis $\{x_i\}$, defined by $P_n(\sum_{i=1}^\infty a_i x_i) = \sum_{i=1}^n a_i x_i$ for $n = 1, 2, \ldots$, are (necessarily) uniformly bounded linear operators, and the supremum of the norms of these basis projections is called the *basis constant.* A basis is called *monotone* provided that its basis constant is one. In most of the applications we will consider the normalized monotone bases. A sequence $\{x_n\}$ is called *normalized* if for each n, $\|x_n\| = 1$.

A pair $\{u_n, u_n^*\}$ of sequences is a *biorthogonal system* on X if $u_n \in X$ and $u_n^* \in X^*$ for all n with the property that $u_n^*(u_m) = \delta_{nm}$, i.e., $u_n^*(u_n) = 1$

and $u_n^*(u_m) = 0$ for $n \neq m$. If $\{x_n\}$ is a basis for X, then the sequence of biorthogonal functionals $\{x_n^*\}$ is a basic sequence in X^*.

In studying the structure of Banach spaces with a basis, it is desirable to have some additional properties of the basis which essentially provide a more 'computable' environment in the space. Among the most important ones is the property of unconditionality.

A basis $\{x_n\}$ is said to be *unconditional* (or *C-unconditional*) if there exists a constant $C > 0$ such that for all scalars $\{a_n\}$ and signs $\theta_n = \pm 1$, we have

$$\|\sum_n \theta_n a_n x_n\| \leq C\|\sum_n a_n x_n\|.$$

That is, inserting plus-minus signs into the sum does not increase the norm more than C. The smallest such C is called the *unconditional basis constant* of $\{x_n\}$. Being unconditional for a basis $\{x_n\}$ is equivalent to the fact that every permutation of $\{x_n\}$ is also a basis.

We frequently use the following observation concerning the unconditional constant. Let $\{x_n\}$ be an unconditional basis for X with constant C. Then for every $x = \sum_n a_n x_n$ in X and every bounded sequence of reals $\{\lambda_n\}$, we have

$$\|\sum_n \lambda_n a_n x_n\| \leq C \sup_n |\lambda_n| \|\sum_n a_n x_n\|.$$

A *block basis* $\{y_j\}$ of the basis $\{x_i\}$ is a sequence of non-zero vectors of the form $y_j = \sum_{i=n_j+1}^{n_{j+1}} a_i x_i$ for some sequence $n_1 < n_2 < \ldots$. Every block basis $\{y_j\}$ of the basis $\{x_i\}$ is a basic sequence and the basis constant of a block basis is no larger than the basis constant of $\{x_i\}$. Recall that for a vector $y = \sum_i a_i x_i$, the support of y is $\mathrm{supp} y = \{i : a_i \neq 0\}$. A sequence $\{y_j\}$ is *disjointly supported* if for every j, $\mathrm{supp} y_j$ is finite and $\mathrm{supp} y_j \cap \mathrm{supp} y_k = \emptyset$ whenever $j \neq k$. A block basis $\{y_j\}$ is not only disjointly supported but also it is successive, i.e., $\max \mathrm{supp} y_j < \min \mathrm{supp} y_{j+1}$, for all j. Unlike a block basis, a disjointly supported sequence might not be a basic sequence in general.

However, it is a (unconditional) basic sequence if the basis $\{x_i\}$ is unconditional. In the literature, the disjointly supported sequences are sometimes referred as 'blocks', however this distinction is important to us, and it will be emphasized throughout the thesis.

We say that a basic sequence $\{x_n\}$ *dominates* the basic sequence $\{y_n\}$ if there exists a constant A such that for all scalars $\{a_n\}$, we have

$$\|\sum_n a_n y_n\| \leq A\|\sum_n a_n x_n\|.$$

If $\{x_n\}$ dominates $\{y_n\}$ and $\{y_n\}$ dominates $\{x_n\}$, then we say that $\{x_n\}$ is *equivalent* to $\{y_n\}$. That is, there exist constants A and B such that for all scalars $\{a_n\}$, we have

$$\frac{1}{B}\|\sum_n a_n x_n\| \leq \|\sum_n a_n y_n\| \leq A\|\sum_n a_n x_n\|.$$

The smallest constant $K = AB$ of this form is called the *equivalence constant.* In this case we say that $\{x_n\}$ is K-equivalent to $\{y_n\}$ to emphasize the constant, and denote this by $\{x_n\} \overset{K}{\sim} \{y_n\}$.

A stronger property than unconditionality is symmetry of a basis. A basis $\{x_n\}$ is *symmetric* provided that every permutation of $\{x_n\}$ is equivalent to $\{x_n\}$. In particular, every permutation of $\{x_n\}$ is a basis, so a symmetric basis is unconditional. A basis $\{x_n\}$ is called *subsymmetric* provided that it is unconditional and equivalent to each subsequence of itself. A symmetric basis is subsymmetric (cf. 3.a.3 of [LT]). The unit vector bases for c_0 and ℓ_p $(1 \leq p < \infty)$ are symmetric. Not every subsymmetric basis is symmetric, and some examples of such bases are, for instance, considered in section 3.5.

Part I

Envelope Functions and Asymptotic Structures of Banach Spaces

Chapter 2

Asymptotic Structures of Banach spaces : A General View

In this chapter we shall describe basic general notions of asymptotic infinite dimensional theory of Banach spaces. Our purpose here is not to give an all-inclusive review of this more recent and fast growing theory, but rather to motivate the problems we study in this thesis as well as to recall the necessary fundamental notions and notations we use throughout.

2.1 Basic Concepts

We have recalled some of the standard Banach space notation in section 1.1, for the unexplained terms we refer to the standard textbook of Lindenstrauss and Tzafriri [LT]. For the notation and basic concepts of asymptotic structure, we shall follow [MMT].

Here we would like to start with some fundamental notation which is essential in the sequel. Let X be a Banach space with a fixed basis (or a minimal system, which will be introduced shortly) $\{x_i\}$. Recall that the support of a vector $x = \sum_i a_i x_i$, denoted by supp$x$, is the set of all i such that $a_i \neq 0$. The set of natural numbers is denoted by $\mathbb{N}$. For non-empty subsets I and J of $\mathbb{N}$

we write $I < J$ if $\max I < \min J$. For $n \in \mathbb{N}$ and $x \in X$ we write $n < x$ if $n < \min \operatorname{supp} x$. For $x, y \in X$ we write $x < y$ if $\operatorname{supp} x < \operatorname{supp} y$. We call $(x_1, \ldots, x_n)$ an n-tuple of successive blocks if $x_1 < x_2 < \ldots < x_n$. If Y is a set of block vectors, (for instance, a tail subspace) we write $n < Y$ if $n < y$ for all $y \in Y$.

Let $\{x_n\}$ be a basis (or a minimal system) for X, and let $I \subset \mathbb{N}$. By X_I we will denote the set of all vectors $x \in X$ such that $\operatorname{supp} x \subset I$, and by $S(X_I)$ we will denote the set of all normalized vectors in X_I.

2.1.1 Games and Asymptotic Spaces

Asymptotic structures of a Banach space X are defined with respect to a fixed family $\mathcal{B} = \mathcal{B}(X)$ of infinite-dimensional subspaces of X, which satisfies two conditions.

The filtration condition says that

$$\text{For every } X_1, X_2 \in \mathcal{B} \text{ there exists } X_3 \in \mathcal{B} \text{ such that } X_3 \subset X_1 \cap X_2.$$

The norming condition says that there exists $C < \infty$ such that

$$\|x\| \leq C \sup \|x\|_{X/X_0} \qquad \text{for all } x \in X,$$

where the supremum is taken over all subspaces $X_0 \in \mathcal{B}$ and $\|\cdot\|_{X/X_0}$ denotes the norm in the quotient space X/X_0.

Natural examples of such families are the family $\mathcal{B}^0(X)$ of all subspaces of X of finite-codimension, and the families of all tail subspaces with respect to a fixed basis or a fixed minimal system in X. These families will be denoted by $\mathcal{B}^t(X)$, if the reference system is clear in the context.

Recall that $\{u_i\}$ is called a *minimal* system in X, if there exists a sequence $\{u_i^*\}$ in X^* such that $\{u_i, u_i^*\}$ is a biorthogonal system. Unless otherwise stated, we shall assume that $\{u_i\}$ is *fundamental* (i.e., $\overline{\operatorname{span}}[u_i]_{i\geq 1} = X$) and that $\{u_i^*\}$

is *total* (i.e., if $u_i^*(x) = 0$ for all i then $x = 0$) and *norming* (i.e., there exists $C < \infty$ such that $\|x\| \leq C \sup\{|x^*(x)| \mid \|x^*\| \leq 1, x^* \in \overline{\text{span}}[u_i^*]_{i\geq 1}\}$, for all $x \in X$) (we sometimes say C-norming, to emphasize the constant C). It is well known that every Banach space contains a minimal (and fundamental and 1-norming) minimal system. This was first proved by Markushevich and minimal systems are sometimes called Markushevich bases. Moreover, if X^* is separable the system can be chosen so that, in addition, $\overline{\text{span}}[u_i^*]_{i\geq 1} = X^*$ (Theorem 1.f.4, [LT]). If $\{u_i\}$ is such a minimal system in X, a tail subspace is a subspace of the form $X^n = \overline{\text{span}}[u_i]_{i>n}$, for some $n \in \mathbb{N}$.

If $\mathcal{B}$ is a family satisfying the filtration condition, we may introduce an equivalent norm on X in such a way that $\mathcal{B}$ is 1-norming in the new norm. Therefore unless otherwise stated, we shall assume that the family $\mathcal{B}$ is 1-norming. Then by a compactness argument it is easy to see that the following condition holds:

for every finite-dimensional subspace $W \subset X$ and every $\varepsilon > 0$ there is $Z \in \mathcal{B}$ such that $\|x\| \leq (1+\varepsilon)\|x+z\|$, for all $x \in W$ and $z \in Z$.

By $\mathcal{M}_n$ we denote the space of all n-dimensional Banach spaces with fixed normalized monotone bases, and the distance given by (the logarithm of) the equivalence constant of the bases (see section 1.1). Recall that $\mathcal{M}_n$ is a compact metric space.

Let us recall the language of *asymptotic games* [MMT] that is convenient for describing asymptotic structures. In such a game (with respect to a fixed family $\mathcal{B}$) there are two players **S** and **V**. Rules of the moves are the same for all games. Set $Y_0 = X$. For $k \geq 1$, in the kth move, player **S** chooses a subspace $Y_k \in \mathcal{B}(X)$, $Y_k \subset Y_{k-1}$, and then player **V** chooses a vector $x_k \in S(Y_k)$ in such a way that the vectors $x_1, \ldots, x_k$ form a basic sequence with the basis constant smaller than or equal to 2. Further rules will ensure that the games considered here will stop after a finite number of steps specified in advance.

A space $E \in \mathcal{M}_n$ with the basis $\{e_i\}$ is called an *asymptotic space* for X

(with respect to $\mathcal{B}$) if for every $\varepsilon > 0$ we have

$$\begin{aligned} &\forall Y_1 \in \mathcal{B} \;\; \exists y_1 \in S(Y_1) \;\; \forall Y_2 \in \mathcal{B}, Y_2 \subset Y_1 \;\; \exists y_2 \in S(Y_2) \quad \ldots \\ &\ldots \;\; \forall Y_n \in \mathcal{B}, Y_n \subset Y_{n-1} \;\; \exists y_n \in S(Y_n) \\ &\{y_1, \ldots, y_n\} \overset{1+\varepsilon}{\sim} \{e_i\}_{i=1}^n. \end{aligned}$$

(By abuse of notation, we will write $\{y_1, \ldots, y_n\} \overset{1+\varepsilon}{\sim} E$ instead.) Any n-tuple $(y_1, \ldots, y_n)$ obtained as above is called *permissible*. We say that **V** has a *winning strategy* in a *vector game* for E and $\varepsilon > 0$, if he can choose vectors $\{y_1, \ldots, y_n\} \overset{1+\varepsilon}{\sim} E$. In particular, E is an asymptotic space for X if **V** has a winning strategy for E and $\varepsilon > 0$. The vector y_i is called an ith *winning* move of **V** in a *vector game* for E and ε.

The set of all n-dimensional asymptotic spaces for X is denoted by $\{X\}_n$. It is easy to see that the set $\{X\}_n$ is closed in $\mathcal{M}_n$.

It was proved in [MMT], 1.4 and 1.5, that this set can be also characterized in terms of a different asymptotic game called a *subspace game.* Given a family $\mathcal{F} \subset \mathcal{M}_n$ and $\varepsilon > 0$, we say that **S** has a winning strategy in a subspace game for $\mathcal{F}$ and $\varepsilon > 0$ if

$$\begin{aligned} &\exists Y_1 \in \mathcal{B} \;\; \forall y_1 \in S(Y_1) \;\; \exists Y_2 \in \mathcal{B}, Y_2 \subset Y_1 \;\; \forall y_2 \in S(Y_2) \quad \ldots \\ &\ldots \;\; \exists Y_n \in \mathcal{B}, Y_n \subset Y_{n-1} \;\; \forall y_n \in S(Y_n) \\ &\exists F \in \mathcal{F} \quad \{y_1, \ldots, y_n\} \overset{1+\varepsilon}{\sim} F. \end{aligned}$$

It is shown there that $\{X\}_n$ coincides with the smallest subset $\mathcal{F} \subset \mathcal{M}_n$ such that for every $\varepsilon > 0$, **S** has a winning strategy for $\mathcal{F}$ and $\varepsilon > 0$.

We refer to any such subspace Y_i as an ith *winning* move of **S** in a *subspace game* for $\{X\}_n$ and ε, and to vectors $\{y_1, \ldots, y_i\}$ (with $1 \le i \le n$) as the first i moves of **V** in the same subspace game. Note that the basis constant of $\{y_1, \ldots, y_i\}$ is less than or equal to $1 + \varepsilon$.

Asymptotic spaces can be also described in terms of countably branching trees (cf. [MiS], [KOS], [OS1]). We will use a tree which describes the moves of the player **V** in a vector game.

For $n \in \mathbb{N}$, let T_n be a countably branching tree of length n on $\mathbb{N}$. This means that $T_n = \{(s_1, \ldots, s_j) \mid s_i \in \mathbb{N} \quad \text{for} \quad 1 \leq i \leq j, \quad 1 \leq j \leq n\}$, ordered by the relation $(s_1, \ldots, s_j) \prec (t_1, \ldots, t_k)$ if $j \leq k$ and $s_i = t_i$ for all $1 \leq i \leq j$. For each $E \in \{X\}_n$ and $\varepsilon > 0$ we can build an asymptotic tree $\mathcal{T}(E, \varepsilon)$ on $S(X)$ indexed by T_n and consisting of winning moves of **V** in a vector game for E and ε. That is, $\mathcal{T}(E, \varepsilon) = \{x(s_1, \ldots, s_i) \in S(X) \mid (s_1, \ldots, s_i) \in T_n\}$, with the order on $\mathcal{T}$ induced by T_n.

Denoting by $\{e_i\}$ the natural basis in E we get that any branch $x(s_1) \prec x(s_1, s_2) \prec \ldots \prec x(s_1, \ldots, s_j)$ of $\mathcal{T}(E, \varepsilon)$ is $(1+\varepsilon)$-equivalent to $\{e_1, \ldots, e_j\}$, for $1 \leq j \leq n$. Moreover, for any node $\sigma = x(s_1, \ldots, s_i) \in \mathcal{T}(E, \varepsilon)$ with $1 \leq i < n$ and any subspace $Y \in \mathcal{B}$ there is a successor $\sigma' = x(s_1, \ldots, s_i, s'_{i+1}) \in \mathcal{T}(E, \varepsilon)$ with $\sigma \prec \sigma'$ and $\sigma' \in Y$.

If $\tau \in T_n$ and $x(\tau) \in \mathcal{T}(E, \varepsilon)$, we refer to $x(\tau)$ as the τ'th *winning* move of **V** in a vector game determined by $\mathcal{T}(E, \varepsilon)$.

Let us now recall some of the immediate properties of asymptotic spaces. Let $E \in \{X\}_n$ with the basis $\{e_i\}_{i=1}^n$. If $\{f_i\}_{i=1}^k$ is a successive block basis of $\{e_i\}_{i=1}^n$ and $F = \overline{\text{span}}\{f_i\}$, a block subspace of E, then $F \in \{X\}_k$. Moreover, given $n \in \mathbb{N}$ and $\varepsilon > 0$, the subspace player **S** has a strategy in an asymptotic game such that after n moves, all normalized successive blocks of the n-tuple resulting from the game, are all permissible, i.e., each of them is $(1+\varepsilon)$-equivalent to the basis of some asymptotic space (1.8.3, [MMT]).

The 'juxtaposition' of finitely many asymptotic spaces is also an asymptotic space. Namely, let $n_1, \ldots, n_k \in \mathbb{N}$ and let $E_j \in \{X\}_{n_j}$ for $j = 1, \ldots, k$. For every $N \geq \sum_j n_j$ and any disjoint subsets I_j of $\{1, \ldots, N\}$ with $|I_j| = n_j$, there exists an asymptotic space $F \in \{X\}_N$ with the basis $\{f_i\}$ such that $\{f_i\}_{i \in I_j} \overset{1+\varepsilon}{\sim} E_j$ for $j = 1, \ldots, k$.

Finally, it follows from Krivine's theorem that for every Banach space X, there exists $1 \leq p \leq \infty$ so that for all n, $\ell_p^n \in \{X\}_n$, hence $\{X\}_n$ is non-empty.

2.2 Asymptotic-ℓ_p Spaces

Banach spaces X with a 'simple' asymptotic structure in the sense that there are no asymptotic spaces $E \in \{X\}_n$ other than $E \sim \ell_p^n$ (whose existence follows from Krivine's theorem as remarked above) are of special interest and are called asymptotic-ℓ_p spaces.

Precisely, a Banach space X is an *asymptotic-ℓ_p space* (with respect to a family $\mathcal{B}$) for $1 \leq p \leq \infty$ if there is a constant C such that for all n and $E \in \{X\}_n$, the basis $\{e_i\}$ in E is C-equivalent to the unit vector basis of ℓ_p^n. That is, for some k, l such that $kl \leq C$ and $(1/k)\|a\|_p \leq \|\sum_{i=1}^n a_i e_i\| \leq l\|a\|_p$, for all scalars (a_i). We denote this by $E \overset{C}{\sim} \ell_p^n$.

The class of asymptotic-ℓ_p spaces is quite rich, and it clearly contains ℓ_p spaces (for $p = \infty$ we take c_0). Trivial examples which are not always isomorphic to ℓ_p spaces are obtained, for instance, by taking ℓ_p-direct sum $(\sum \bigoplus F_n)_{\ell_p}$ of arbitrary finite dimensional spaces F_n, with $\sup \dim F_n = \infty$.

More interesting examples of asymptotic-ℓ_p spaces which do not contain isomorphic copies of ℓ_p spaces are p-convexified Tsirelson spaces $T_{(p)}$ [FJ], and the most famous of all is the Tsirelson space T [Ts], an asymptotic-ℓ_1 space. Although in this thesis we shall not use the Tsirelson space, the existence and the construction of this space is behind most of new phenomena and developments of the asymptotic theory of infinite-dimensional Banach spaces, and we would like to recall the definition of this space.

The main feature of the Tsirelson space T is that the norm does not have an explicit formula and it is given as the solution to an implicit equation. We follow Figiel and Johnson's [FJ] description of the (dual of the original) Tsirelson space. Let c_{00} be the linear space of finitely supported sequences. T is the completion of $(c_{00}, \|.\|)$, where $\|.\|$ is given by the implicit equation

$$\|x\| = \max\Big\{\|x\|_\infty, \sup \frac{1}{2}\sum_{i=1}^{n} \|E_i x\|\Big\},$$

where the inner supremum is taken over all n and $n \leq E_1 < \ldots < E_n$, successive finite intervals of $\mathbb{N}$ supported after n. Here $E_i x$ denotes the restriction of x to the set E_i. Of course, the existence of such a norm must be verified, and although it is not difficult to show this we refer to [CS] for the proof (and for more information on Tsirelson space).

The unit vectors (e_i) form a 1-unconditional basis for T, and it is easy to see from the definition that whenever a block sequence $(x_i)_{i=1}^n$ satisfies $n \leq x_1 < \ldots < x_n$, then $\|\sum_{i=1}^n x_i\| \geq (1/2)\sum_{i=1}^n \|x_i\|$. (To see this, simply take the smallest interval E_i containing suppx_i for each $1 \leq i \leq n$.) Thus, every sequence of n successive normalized blocks supported after n is 2-equivalent to the unit vector basis of ℓ_1^n (the other inequality is trivially obtained by the triangle inequality). Hence T is an asymptotic-ℓ_1 space. In fact, it has a stronger property, namely, that for some $C < \infty$ any normalized n-blocks supported after n is C-equivalent to ℓ_1^n. In spite of such a rich 'ℓ_1'-structure, T does not contain any subspace isomorphic to ℓ_1 and it is reflexive.

Idea of defining Tsirelson-like spaces have developed intensively over the last decade, and in particular, more examples of asymptotic-ℓ_1 spaces with some more interesting properties are constructed (cf. [AD] and [ADKM]).

A rather interesting result proved in [MMT] says that the equivalence condition in the definition of asymptotic-ℓ_p spaces can be relaxed considerably. A Banach space X is asymptotic-ℓ_p (for $1 < p < \infty$) if (and only if) for all n and all $E \in \{X\}_n$, E is C-isomorphic to ℓ_p^n. This result suggests further possibilities in this direction and motivated by this result, we prove another characterization for asymptotic-ℓ_p spaces in section 3.4.

2.3 Duality and Permissible Norming

Whenever we have a property for a Banach space X, it is natural to consider if the dual space X^* possesses the same or a dual property. A priori it is not obvious how to link the asymptotic structures in X and in X^*. In general terms, to establish a connection one requires the norming functionals in X^* of permissible vectors to be permissible (in X^*) as well.

A natural setting to seek duality relations is that we assume that X has a *shrinking* minimal system $\{u_i, u_i^*\}$ (this means that $\{u_i^*\}$ is fundamental in X^*) and asymptotic structures in X and X^* are determined by the tail families $\mathcal{B}^t(X)$ and $\mathcal{B}^t(X^*)$ with respect to $\{u_i\}$ and $\{u_i^*\}$ respectively. Then we have the following *permissible norming lemmas* (cf. Theorem 2.2 of [MiS] and 4.5 of [MMT]).

Lemma 2.3.1 *Let X be a Banach space with a shrinking minimal system. There exists an equivalent norm which is 2-equivalent to the original norm such that*

(i) for every $\delta > 0$ and every tail subspace $\tilde{V} \in \mathcal{B}^t(X^*)$ there exists a tail subspace $\tilde{X} \in \mathcal{B}^t(X)$ such that for every $x \in S(\tilde{X})$ there is $f \in S(\tilde{V})$ with $f(x) \geq 1 - \delta$,

(ii) for every $\{e_i\} \in \{X\}_n$ and $\varepsilon > 0$ the following holds: for all scalars (a_i) there exists a permissible n-tuple $\{x_i\}$ in X satisfying $\{x_i\} \overset{1+\varepsilon}{\sim} \{e_i\}$, and a permissible n-tuple $\{g_i\}$ in X^* and non-zero scalars (b_i) such that $g_i(x_j) = 0$ for $i \neq j$ and

$$\|\sum_{i=1}^n a_i x_i\| \|\sum_{i=1}^n b_i g_i\| \leq (1+\varepsilon)\Big(\sum_{i=1}^n a_i x_i\Big)\Big(\sum_{i=1}^n b_i g_i\Big).$$

A somewhat stronger version of part (ii) has been proved in Proposition 2.1, [MT2]. Namely, one can choose the n-tuple $\{x_i\}$ independent of the scalar sequence (a_i) if X has a bimonotone basis. In section 3.1 we will prove another stronger version for Banach spaces with asymptotic unconditional structure, which turns out to characterize asymptotic unconditionality (Theorem 3.1.2).

To avoid repetitions, we will always assume without loss of generality that X is re-normed so that the conclusion of Lemma 2.3.1 holds. Let us also remark that if the dual space X^* is also separable then the shrinking assumption can be dropped. In that case one can find a minimal system $\{u_i, u_i^*\}$ so that $\{u_i^*\}$ is also fundamental in X^* and hence 1-norming (cf. Proposition 1.f.4, [LT] and the remarks following Theorem 2.2, [MiS]).

2.4 Envelopes

The most important tool used in this first part of the thesis is the notion of envelope functions. Envelopes are defined on the asymptotic structures of Banach spaces and some properties of families of asymptotic spaces $\{X\}_n$ can be demonstrated through these functions.

For any sequence with finite support $a = (a_i) \in c_{00}$ the *upper envelope* is a function $r_X(a) = \sup \|\sum_i a_i e_i\|$, where the supremum is taken over all natural bases $\{e_i\}$ of asymptotic spaces $E \in \{X\}_n$ and all n. Similarly, the *lower envelope* is a function $g_X(a) = \inf \|\sum_i a_i e_i\|$, where the infimum is taken over the same set.

It follows immediately from the properties of the set $\{X\}_n$ that the functions r_X and g_X are 1-unconditional and 1-subsymmetric. And it is easy to see that r_X is a norm on c_{00} and that g_X satisfies triangle inequality on disjointly supported vectors. That is, if $a = (a_i)$ and $b = (b_i) \in c_{00}$ such that supp$a$ is disjoint from supp$b$ (with respect to the unit vector basis in c_{00}), then $g_X(a+b) \leq g_X(a) + g_X(b)$.

The upper envelope function r_X is sub-multiplicative. i.e., for any finite number of successive vectors $b^i = (b^i_j) \in c_{00}$ such that $r_X(b^i) \leq 1$, and for any $a = (a_i) \in c_{00}$, we have

$$r_X\Big(\sum_i a_i b^i\Big) \leq r_X(a).$$

Similarly, the lower envelope g_X is super-multiplicative. I.e., if $g_X(b^i) = 1$ for all i, then

$$g_X\Big(\sum_i a_i b^i\Big) \geq g_X(a).$$

Note that in terms of bases, the multiplicativity properties simply mean that in the 'space' (c_{00}, r_X) (resp. in (c_{00}, g_X)) the unit vector basis dominates (resp. is dominated by) every block basis in the space. The proof of these multiplicativity properties can be found in [MT1], where the envelope functions are first defined and used.

As examples, the c_0 and ℓ_p $(1 \leq p < \infty)$ norms are both sub- and super-multiplicative. In fact, these are the typical examples of the envelope functions in view of the following result, which says that the envelope functions are always 'close' to some ℓ_p-norms (1.9.3, [MMT]).

Proposition 2.4.1 *Let X be a Banach space. There exists $1 \leq p, q \leq \infty$ such that for all $\varepsilon > 0$ there exist $C_\varepsilon, c_\varepsilon > 0$ such that for all $a \in c_{00}$ we have*

$$c_\varepsilon \|a\|_{q+\varepsilon} \leq g_X(a) \leq \|a\|_q \quad \text{and} \quad \|a\|_p \leq r_X(a) \leq C_\varepsilon \|a\|_{p-\varepsilon}.$$

Here it is understood that if $q = \infty$ (resp. $p = 1$), then g_X is equivalent to $\|.\|_\infty$ (resp. r_X is equivalent to $\|.\|_1$). Note that a Banach space X is an asymptotic-ℓ_p space if and only if both g_X and r_X are equivalent to $\|.\|_p$, and by the above inequalities, if and only if g_X is equivalent to r_X.

In section 3.2 we will introduce the notion of *disjoint-envelope functions* for Banach spaces with asymptotic unconditional structure, and establish the similar properties as above. The envelope and the disjoint-envelope functions then will be our main tools in studying the asymptotic structures of Banach spaces.

Chapter 3

Envelope Functions and Asymptotic Structure

We start this Chapter with an important notion of asymptotic unconditionality. Let us emphasize that, unless otherwise stated, we will always consider the asymptotic structure of Banach spaces with respect to tail families $\mathcal{B}^t$ of a minimal system $\{u_i\}$. Moreover, $\mathcal{B}^t$ will be assumed to be 1-norming.

3.1 Asymptotic Unconditionality

A Banach space X has *asymptotic unconditional structure* if there exists a constant $C \geq 1$ such that for all $n \in \mathbb{N}$ and for every asymptotic space $E \in \{X\}_n$ the natural basis $\{e_i\}_{i=1}^n$ in E is C-unconditional.

As an easy consequence of permissible norming lemmas, Lemma 2.3.1, we observe that this property is self-dual for dual Banach spaces.

Proposition 3.1.1 *Let X be a Banach space with a shrinking, fundamental and 1-norming minimal system. If X^* has asymptotic unconditional structure with constant C, then X also has asymptotic unconditional structure with the same constant.*

Proof Let $\{e_i\}_{i=1}^n \in \{X\}_n$ and $\varepsilon > 0$. Fix an arbitrary scalar sequence $\{a_i\}$ and a sequence $\{\epsilon_i\}$ of signs. By Lemma 2.3.1, there exist a permissible n-tuple $\{x_i\}$ in X such that $\{e_i\} \overset{1+\varepsilon}{\sim} \{x_i\}$ and a permissible n-tuple $\{g_i\}$ in X^* and scalars $\{b_i\}$ such that $g_i(x_j) = 0$ for $i \neq j$ and

$$\|\sum_{i=1}^n \epsilon_i a_i x_i\| \|\sum_{i=1}^n b_i g_i\| \leq (1+\varepsilon)\Big(\sum_{i=1}^n \epsilon_i a_i x_i\Big)\Big(\sum_{i=1}^n b_i g_i\Big).$$

By interchanging the signs between scalars $\{a_i\}$ and $\{b_i\}$ and using the assumption, the latter term is less than or equal to

$$(1+\varepsilon)\|\sum_{i=1}^n a_i x_i\| \|\sum_{i=1}^n \epsilon_i b_i g_i\| \leq C(1+\varepsilon)\|\sum_{i=1}^n a_i x_i\| \|\sum_{i=1}^n b_i g_i\|.$$

That is, $\|\sum_{i=1}^n \epsilon_i a_i x_i\| \leq C(1+\varepsilon)\|\sum_{i=1}^n a_i x_i\|$. And since $\varepsilon > 0$ was arbitrary, it follows that

$$\|\sum_{i=1}^n \epsilon_i a_i e_i\| \leq C\|\sum_{i=1}^n a_i e_i\|,$$

as desired. □

Our main aim in this section is to prove the following characterization of asymptotic unconditionality in terms of norming permissible functionals. This is a reformulation of Theorem 2.3 of [MiS].

Theorem 3.1.2 *Let X be a Banach space. X has asymptotic unconditional structure (with respect to $\mathcal{B}^t(X)$) if and only if the following holds.*

There exists a constant $C \geq 1$ such that for all $n \in \mathbb{N}$ and $\{e_i\}_{i=1}^n \in \{X\}_n$ and any partition $\{A_1, A_2, \ldots, A_l\}$ of $\{1, \ldots, n\}$ and $\varepsilon > 0$, there exists a permissible n-tuple $\{x_i\}_{i=1}^n$ in X satisfying $\{x_i\}_{i=1}^n \overset{1+\varepsilon}{\sim} \{e_i\}_{i=1}^n$ such that for all $\{a_i\}_{i=1}^n$ there exists a permissible n-tuple $\{g_i\}_{i=1}^n$ of functionals in X^* and scalars $\{b_i\}_{i=1}^n$ such that $g_i(x_j) = 0$ whenever $i \neq j$ and for all $1 \leq j \leq l$, $\|\sum_{i\in A_j} b_i g_i\| \leq 1+\varepsilon$ and

$$\|\sum_{i\in A_j} a_i x_i\| \leq C\Big(\sum_{i\in A_j} a_i x_i\Big)\Big(\sum_{i\in A_j} b_i g_i\Big).$$

The property described in the theorem is called Property A in [MiS] and it is stated in terms of trees. Note that Lemma 2.3.1 says that such a property is always satisfied for trivial partitions, i.e., there is only one set $A_1 = \{1, \dots, n\}$ or each A_j is a singleton.

It is easy to see that the property described above implies asymptotic unconditionality of X. Indeed, let $\{e_i\}_{i=1}^n \in \{X\}_n$ and $\varepsilon > 0$ be arbitrary, and let $\{a_i\}_{i=1}^n$ and $\{\epsilon_i\}_{i=1}^n$ be arbitrary sequence of scalars and signs respectively. We apply the property to the partition $\{A_1, A_2\}$ of $\{1, \dots, n\}$, where $A_1 \subset \{1, \dots, n\}$ is the set of all i such that $\epsilon_i = 1$; and A_2 is the complement of A_1 in $\{1, \dots, n\}$. Then there exist $\{x_i\}_{i=1}^n \in X$ such that $\{x_i\}_{i=1}^n \overset{1+\varepsilon}{\sim} \{e_i\}_{i=1}^n$ and $\{g_i\}_{i=1}^n \in X^*$ such that $\| \sum_{i \in A_j} b_i g_i \| \leq 1 + \varepsilon$ and

$$\| \sum_{i \in A_j} a_i x_i \| \leq C \Big(\sum_{i \in A_j} a_i x_i \Big) \Big(\sum_{i \in A_j} b_i g_i \Big),$$

for $j = 1, 2$.

But this implies, by the triangle inequality, that

$$\| \sum_{i=1}^n \epsilon_i a_i e_i \| \leq (1 + \varepsilon) \Big(\| \sum_{i \in A_1} a_i x_i \| + \| \sum_{i \in A_2} a_i x_i \| \Big) \leq 2C(1 + \varepsilon)^2 \| \sum_{i=1}^n a_i e_i \|.$$

The converse is more complicated and we will split the proof into several parts. We believe that each step is of independent interest.

We have defined the notion of permissibility for the sequence of successive (normalized) vectors. We now extend this definition in a natural way to a sequence of successive intervals in $\mathbb{N}$.

For a fixed n and $\varepsilon > 0$, we say that a sequence $\{I_1, \dots, I_n\}$ of successive intervals in $\mathbb{N}$ is permissible if for all normalized vectors $\{x_i\}_{i=1}^n$ such that $\operatorname{supp} x_i \subset I_i$, $\{x_i\}_{i=1}^n$ is permissible. i.e., $\{x_i\} \overset{1+\varepsilon}{\sim} E$ for some $E \in \{X\}_n$.

For a tail subspace $Y \in \mathcal{B}^t(X)$, if $M \in \mathbb{N}$ is such that $Y = \overline{\operatorname{span}}[u_i]_{i \geq M}$ and for a finite interval $I \subset \mathbb{N}$, by $I \geq Y$ we will mean that $\min I \geq M$.

Now we describe a winning strategy for **S** in a special subspace game, which we will call *a subspace game for intervals.* The rules of a subspace game for intervals are the same as of a subspace game, except that in the kth step the vector player $\mathbf{V}$ chooses a finite interval I_k (rather than a vector x_k) such that $I_k \geq Y_k$, where $Y_k \in \mathcal{B}^t$ being kth move of S.

Let us observe that given n and $\varepsilon > 0$, the subspace player $\mathbf{S}$ has a winning strategy for $\{X\}_n$ in a subspace game for intervals such that after n moves, if $\{I_1, \ldots, I_n\}$ is any sequence of successive intervals played by $\mathbf{V}$, then $\{I_1, \ldots, I_n\}$ is permissible.

Indeed, let $Y_1 \in \mathcal{B}^t$ be the first move of $\mathbf{S}$ playing the winning strategy in the subspace game for $\{X\}_n$ and $\varepsilon > 0$. Fix a $\delta > 0$ to be defined later. Let I_1, the first move of $\mathbf{V}$, be an arbitrary finite interval such that $I_1 \geq Y_1$. Let $\mathcal{N}_1$ be a finite δ-net in $S(X_{I_1})$. Considering any vector from $\mathcal{N}_1$ as a first move of $\mathbf{V}$ and a winning move $\tilde{Y}_2 \in \mathcal{B}^t$ for player S for $\{X\}_n$ and $\varepsilon > 0$; take the intersection $Y_2 \in \mathcal{B}^t$ of all these (finitely many) tail subspaces $\tilde{Y}_2$. Y_2 is the second winning move for $\mathbf{S}$. Thus Y_2 has the following property which is valid for all $y_1 \in \mathcal{N}_1$: treating y_1 as the first move of $\mathbf{V}$, Y_2 is a winning move of $\mathbf{S}$ in a subspace game for $\{X\}_n$ and $\varepsilon > 0$. Given an arbitrary finite interval I_2 such that $I_2 \geq Y_2$, take a finite δ-net in $S(X_{I_2})$ and choose Y_3 such that for all $y_2 \in \mathcal{N}_2$, treated as a second move of $\mathbf{V}$, Y_3 is a winning move of $\mathbf{S}$ and so on. After n steps, given any $\{x_i\}_{i=1}^n$ such that $x_i \in S(X_{I_i})$, i.e., $\text{supp} x_i \subset I_i$, $\{x_i\}_{i=1}^n$ is, by a standard perturbation argument, $(1+\varepsilon)(1+n\delta)$-equivalent to some $E \in \{X\}_n$. Consequently, $\{I_1, \ldots, I_n\}$ is permissible.

The above strategy can be elaborated further to get the following.

Lemma 3.1.3 *The subspace player* $\mathbf{S}$ *has a winning strategy in a subspace game for* $\{X\}_{2n}$ *and* $\varepsilon > 0$ *such that the following holds.*

There exists an integer K_1 *such that for each finite interval* $I_1 > K_1$ *there is an integer* $M_1 > I_1$ *such that for each integer* $L_2 > M_1$ *there is an integer* $K_2 > L_2$ *such that for each finite interval* $I_2 > K_2$ *there is* $M_2 > I_2$ *and so on such that for each finite interval* $I_n > K_n$ *there is* $M_n > I_n$*, and let* $L_{n+1} > M_n$

be arbitrary.

Then, denoting $\Delta_{(i,j)} = (M_i, L_j)$ *for* $i < j$ *and* $\Delta_{(0,j)} = (K_1, L_j)$, *any sequence of intervals of the form*

$$\{I_1, \Delta_{(1,k_2)}, I_{k_2}, \Delta_{(k_2,k_3)} \ldots, I_{k_m}, \Delta_{(k_m,n+1)}\}$$

or

$$\{\Delta_{(0,k_1)}, I_{k_1}, \Delta_{(k_1,k_2)}, I_{k_2}, \ldots, I_{k_m}, \Delta_{(k_m,n+1)}\},$$

where $1 \leq k_1 < \ldots < k_m \leq n$, *is permissible.*

Proof The subspace player **S** will follow a winning strategy for $\{X\}_{2n}$ and $\varepsilon > 0$ in a subspace game for intervals such that all the intervals resulting from this game will be permissible with an additional trick described below.

The numbers $M_1, \ldots, M_n$ and $K_1, \ldots, K_n$ will denote $\min \operatorname{supp} Y_l$, where Y_l is the lth winning move of **S** for $1 \leq l \leq 2n$. Recall that if $Y = \overline{\operatorname{span}}[u_i]_{i \geq K} \in \mathcal{B}^t$, then $\min \operatorname{supp} Y = K$

Let $Y_1 = \overline{\operatorname{span}}[u_i]_{i \geq K_1}$ be the first winning move of **S** for $\{X\}_{2n}$ and $\varepsilon > 0$. Let I_1 be an arbitrary finite interval such that $I_1 > K_1$. **S** chooses $Y_2 = \overline{\operatorname{span}}[u_i]_{i \geq M_1} \in \mathcal{B}^t$ as a winning move for $\{X\}_{2n}$ and $\varepsilon > 0$ as described in the subspace game for intervals. (In precise terms this has been explained before the statement of the lemma).

Let $L_2 > M_1$ be arbitrary. The next move of **S** depends on two considerations.

Considering the interval $\Delta_{(1,2)} = (M_1, L_2)$ as a second move of **V**, **S** chooses $Y_3^1 \in \mathcal{B}^t$ so that $\{I_1, \Delta_{(1,2)}\}$ is a permissible pair of intervals.

Secondly, pretending that $\Delta_{(0,2)} = (K_1, L_2)$ as a first move of **V** in a subspace game for intervals with the length less than $2n$, **S** chooses $Y_3 \subset Y_3^1$ as his winning move. $Y_3 = \overline{\operatorname{span}}[u_i]_{i \geq K_2}$ is the actual choice of **S**. This ends the first step of the construction. Note that $K_1 < I_1 < M_1 < L_2 < K_2$.

In next steps of the game the strategy for **S** is similar to the above but gets slightly more complicated due to increasing additional considerations after every step. Hence we explain the move of **S** in an arbitrary step (inductively) in detail.

Suppose that for $j \geq 1$, $I_1, \ldots, I_j$, $M_1, \ldots, M_j$ and $K_1, \ldots, K_j$ have been already obtained so that the following holds.

Suppose that any sequence of the form

$$\{I_1, \Delta_{(1,k_2)}, I_{k_2}, \Delta_{(k_2,k_3)} \ldots, I_{k_m}, \Delta_{(k_m,j+1)}\}$$

or

$$\{\Delta_{(0,k_1)}, I_{k_1}, \Delta_{(k_1,k_2)}, I_{k_2}, \ldots, I_{k_m}, \Delta_{(k_m,j+1)}\},$$

where $1 \leq k_1 < \ldots < k_m \leq j$ and $L_{j+1} > M_j$ is arbitrary, is permissible.

Fix $L_{j+1} > M_j$. Then, by assumption, any sequence of intervals of the above forms is permissible. Hence for each $1 \leq k_1 < \ldots < k_m \leq j$, **S** has a next move Y^k. Take the intersection Y of all these (finitely many) tail subspaces Y^k. Then $Y = \overline{\text{span}}[u_i]_{i \geq K_{j+1}}$ is the next move of **S**, and $K_{j+1} > L_{j+1}$.

Now let I_{j+1} be given arbitrary interval such that $I_{j+1} > K_{j+1}$. For each $1 \leq k_1 < \ldots < k_m \leq j$ and a sequence of permissible intervals as above, **S** has next move Y^k such that $\{I_1, \Delta_{(1,k_2)}, I_{k_2}, \Delta_{(k_2,k_3)} \ldots, I_{k_m}, \Delta_{(k_m,j+1)}, I_{j+1}\}$ or $\{\Delta_{(0,k_1)}, I_{k_1}, \Delta_{(k_1,k_2)}, I_{k_2}, \ldots, I_{k_m}, \Delta_{(k_m,j+1)}, I_{j+1}\}$ is permissible. Again, the actual choice of **S** is the intersection Y of all these Y^k's, so that all the sequences of intervals as above are permissible. If $Y = \overline{\text{span}}[u_i]_{i \geq M_{j+1}}$, then $M_{j+1} > K_{j+1}$ and it is easily seen that the property assumed for jth step holds for $(j+1)$th step as well.

S repeats this strategy until obtaining I_n and M_n, and hence the permissibility properties are satisfied for the claimed intervals. □

Lemma 3.1.4 *Let X be a Banach space with an asymptotic unconditional minimal system with constant $C \geq 1$. Let $\{e_i\}_{i=1}^n \in \{X\}_n$ be arbitrary. Then there exists a permissible n-tuple satisfying $\{x_i\}_{i=1}^n \overset{1+\varepsilon}{\sim} \{e_i\}_{i=1}^n$ such that for any subset*

$A \subset \{1, \ldots, n\}$ and any scalars $\{a_i\}$ there exists a finitely supported $\phi_A \in X^*$ with $\|\phi_A\| = 1$ such that

$$\langle \phi_A, \sum_{i \in A} a_i x_i \rangle \geq (1/C) \| \sum_{i \in A} a_i x_i \|$$

and $\langle \phi_A, x_j \rangle = 0$ for all $j \notin A$.

Proof Let $\varepsilon > 0$. Consider an asymptotic game in which **S** follows his winning strategy for $\{X\}_{2n}$ and $\varepsilon > 0$ as described in the previous lemma, and **V** follows his winning strategy for $\{e_i\}_{i=1}^n$.

We use the same notation as in Lemma 3.1.3, and suppose without loss of generality that the first move of **S** is $Y_1 = X$. i.e., $K_1 = 1$. Let $x_1 \in S(Y_1)$ be the first move of **V** for $\{e_i\}_{i=1}^n$. Let I_1 be any finite interval containing suppx_1, and let M_1 be as in Lemma 3.1.3. Let $L_2 > M_1$ be arbitrary and K_2 be as in Lemma 3.1.3. Let x_2 be the second move of **V** for $\{e_i\}_{i=1}^n$ such that supp$x_2 > K_2$, and let I_2 be an interval such that supp$x_2 \subset I_2$ and so on.

At the end, the resulting vectors $\{x_i\}_{i=1}^n$ are $(1+\varepsilon)$-equivalent to $\{e_i\}_{i=1}^n$ and moreover all the resulting sequence of intervals are permissible as described in Lemma 3.1.3.

Let $1 \leq k_1 < k_2 < \ldots < k_m \leq n$ be the elements of A. By Lemma 3.1.3, in particular, the sequence of intervals

$$\{I_1, \Delta_{(1,k_2)}, I_{k_2}, \Delta_{(k_2,k_3)} \ldots, I_{k_m}, \Delta_{(k_m,n+1)}\}$$

and

$$\{\Delta_{(0,k_1)}, I_{k_1}, \Delta_{(k_1,k_2)}, I_{k_2}, \ldots, I_{k_m}, \Delta_{(k_m,n+1)}\}$$

are permissible. If $k_1 = 1$, we use the first the sequence above, otherwise we use the second. Moreover, since L_{n+1} is arbitrary, we take $\Delta_{(k_m,n+1)}$ to be the interval (M_{k_m}, ∞).

Now set $k_0 = 0$ and $L_1 = \min I_1$ (which can be 1, of course). Note also that $\Delta_{(0,k_1)} = [1, L_{k_1})$ since $K_1 = 1$. Consider $\Delta = \bigcup_{i=0}^{m} \Delta_{(k_i,k_{i+1})}$ (if $k_1 = 1$ take $\Delta_{(0,k_1)} = \emptyset$). Put $X_\Delta = \overline{\text{span}}\{u_i\}_{i\in\Delta}$ and consider the *finite* dimensional quotient space X/X_Δ.

Let $\{a_i\}_{i=1}^n$ be an arbitrary sequence of scalars and $\phi'_A \in (X/X_\Delta)^*$ be a norm-one functional such that $\langle \phi'_A, \sum_{i\in A} a_i x_i + X_\Delta \rangle = \| \sum_{i\in A} a_i x_i \|_{X/X_\Delta}$.

We can identify ϕ'_A isometrically with $\phi_A \in X_\Delta^\perp$ such that $\langle \phi'_A, y + X_\Delta\rangle = \langle \phi_A, y\rangle$ for all $y \in X$.

Now since,

$$\| \sum_{i\in A} a_i x_i \|_{X/X_\Delta} := \inf\{ \| \sum_{i\in A} a_i x_i + \sum_{i=0}^{m} z_i \| : \; z_i \in X_{\Delta_{(k_i,k_{i+1})}}, \; 0 \le i \le m\},$$

and by Lemma 3.1.3, the sequence $\{\Delta_{(0,k_1)}, I_{k_1}, \Delta_{(k_1,k_2)}, I_{k_2}, \ldots, I_{k_m}, \Delta_{(k_m,n+1)}\}$ is permissible, the sequence $\{\frac{z_1}{\|z_1\|}, x_{n_1}, \frac{z_2}{\|z_2\|}, x_{n_2}, \ldots, x_{n_s}, \frac{z_{s+1}}{\|z_{s+1}\|}\}$ is permissible and therefore C-unconditional, by the assumption.

Hence it follows that

$$\langle \phi_A, \sum_{i\in A} a_i x_i \rangle = \| \sum_{i\in A} a_i x_i \|_{X/X_\Delta} \ge (1/C) \| \sum_{i\in A} a_i x_i \|.$$

Moreover, $\text{supp}\phi_A = \text{supp}\phi'_A \subset [1, M_{k_1}) \cup (L_{k_2}, M_{k_2}) \cup \ldots \cup (L_{k_m}, M_{k_m})$. i.e., ϕ_A is finitely supported. □

Remark In the above lemma, we were able to choose $\{x_i\}_{i=1}^n$ independent of scalars $\{a_i\}$. This is already stronger than and should be compared with Lemma 2.3.1.

Proof of Theorem 3.1.2 We have already shown one implication. For the converse, assume that X has asymptotic unconditional structure with constant $C \ge 1$, and let $\{e_i\}_{i=1}^n \in \{X\}_n$ be arbitrary. Let $\mathcal{T}(\{e_i\}_{i=1}^n, \varepsilon) = \{x(s_1, \ldots, s_n)\}$ be an asymptotic tree for $\{e_i\}_{i=1}^n$ (see 2.1.1). Following the winning strategy for **S** in a subspace game described in Lemma 3.1.3, we construct a subtree $\mathcal{T}'(\{e_i\}_{i=1}^n, \varepsilon) = \{x(s_1, \ldots, s_n)\}$ of this tree such that every branch of this subtree is obtained by Lemma 3.1.3.

Precisely, let I_1 be the smallest interval such that $I_1 \supset \text{supp}x(s_1)$, and let M_1 be as in Lemma 3.1.3. For each $L_2 > M_1$, let K_2 be as in Lemma 3.1.3 and choose s_2 (a node in $\mathcal{T}'$) such that $\min \text{supp}x(s_1, s_2) \geq K_2$ (hence $x(s_1, s_2)$ depends on given L_2). Let I_2 be the smallest interval such that $I_2 \supset x(s_1, s_2)$ and so on.

Let $x(\bar{s}_1)$ be the first winning move of $\mathbf{V}$ determined by $\mathcal{T}'(\{e_i\}_{i=1}^n, \varepsilon) = \{x(s_1, \ldots, s_n)\}$.

Let $\{A_1, \ldots, A_l\}$ be a disjoint partition of $\{1, \ldots, n\}$. Fix a small $\delta > 0$ to be defined later, and let $\mathcal{N}$ be a finite δ-net in the unit ball of l_∞^n. Fix an arbitrary $\{a_i\}_{i=1}^n \in \mathcal{N}$. For every branch $\gamma = (\bar{s}_1, s_2, \ldots, s_n)$ of $\mathcal{T}'(\{e_i\}_{i=1}^n, \varepsilon) = \{x(s_1, \ldots, s_n)\}$, let

$$w_\gamma^j = w^j(\bar{s}_1, s_2, \ldots, s_n) = \sum_{n_i \in A_j} a_i x(\bar{s}_1, \ldots, s_{n_i}),$$

where $1 \leq n_1 < n_2 < \ldots < n_s \leq n$ are elements of A_j.

And let $\phi_\gamma^j = \phi^j(\bar{s}_1, \ldots, s_n)$ be the norm-one functional obtained by Lemma 3.1.4 applied to subset A_j , that is, $\phi_\gamma^j(w_\gamma^j) \geq (1/C)\|w_\gamma^j\|$.

For each $1 \leq j \leq l$, let $\phi^j(\bar{s}_1, s_2, \ldots, s_{n-1})$ be a w^*-cluster point of the sequence $\{\phi^j(\bar{s}_1, \ldots, s_n)\}$ (with $s_n \to \infty$); then let $\phi^j(\bar{s}_1, s_2, \ldots, s_{n-2})$ be a w^*-cluster point of $\{\phi^j(\bar{s}_1, s_2, \ldots, s_{n-1})\}$ (with $s_{n-1} \to \infty$), and so on, let $\phi^j(\bar{s}_1)$ be a w^*-cluster point of $\{\phi^j(s_1, s_2)\}$ (with $s_2 \to \infty$).

We repeat this construction for all $\{a_i\}_{i=1}^n \in \mathcal{N}$ and $1 \leq j \leq l$ and eventually obtain l-fixed limits $\phi^1, \ldots, \phi^l$ corresponding to each A_j (and for all $\{a_i\} \in \mathcal{N}$).

First we note the following important property of these limit points obtained at each step. For each $1 \leq j \leq l$, let $1 \leq n_1 < n_2 < \ldots < n_s \leq n$ be the elements of A_j. Let r be an integer such that $n_t < r < n_{t+1}$ (if such an r exists). Then $\max \text{supp}\phi^j(\bar{s}_1, s_2, \ldots, s_{r-1}) \leq M_{n_t}$ and therefore depends only on $s_1, s_2, \ldots s_{n_t - 1}$. And if $r < n_1$ then $\phi^j(\bar{s}_1, \ldots, s_{r-1}) = 0$.

Indeed, by Lemma 3.1.4, $\phi^j(\bar{s}_1, \ldots, s_r)$ vanishes on $[M_{n_t}, L_{n_{t+1}}]$. Hence if

$s_1, s_2, \ldots, s_r \to \infty$, then $L_{n_{t+1}} \to \infty$ as well. Therefore $\phi^j(\bar{s}_1, \ldots, s_{r-1})$ vanishes on $[M_{n_t}, \infty)$. For a similar reason, when $r < n_1$, we get $\phi^j(\bar{s}_1, \ldots, s_{r-1}) = 0$.

We now construct the desired permissible vectors $\{x_i\}_{i=1}^n$ and functionals $\{g_i\}_{i=1}^n$. Note that while the choice of $\{x_i\}_{i=1}^n$ will be independent of scalars $\{a_i\}_{i=1}^n$, the choice of functionals will depend on the particular sequence of scalars.

We have already fixed $x_1 = x(\bar{s}_1)$. Let $h_1 = \phi^{j_1}$ and $g_1 = h_1/\|h_1\|$, where $1 \in A_{j_1}$. Then supp$h_1 \subset [1, M_1)$ and moreover, $\phi^j = 0$ for all $j \neq j_1$ by above remark.

For $k \in \mathbb{N}$ by Q_k we denote the natural projection of X^* onto the kth tail subspace.

Now choose $\bar{s}_2$ (from the index tree of $\mathcal{T}'$) such that

$$\|Q_{M_2}(\phi^j(\bar{s}_1, \bar{s}_2) - \phi^j(\bar{s}_1))\| \leq \varepsilon/n, \quad \text{for all } 1 \leq j \leq l.$$

Let $x_2 = x(\bar{s}_1, \bar{s}_2)$ (The second winning move of $\mathbf{V}$).

And put $h_2 = \phi^{j_2}(\bar{s}_1, \bar{s}_2) - Q_{M_2}(\phi^{j_2}(\bar{s}_1, \bar{s}_2))$, where $2 \in A_{j_2}$ and, let $g_2 = h_2/\|h_2\|$.

Now it is easy to check the following. If $j_1 = j_2$ (that is, $1, 2 \in A_{j_1}$), then $h_1 + h_2$ well approximates $\phi^{j_1}(\bar{s}_1, \bar{s}_2)$ in norm. Indeed, in this case, $Q_{M_2}\phi^{j_1}(\bar{s}_1) = \phi^{j_1}(\bar{s}_1)$ and

$$\begin{aligned} \|(h_1 + h_2) - \phi^{j_1}(\bar{s}_1, \bar{s}_2)\| &= \|\phi^{j_1}(\bar{s}_1) - Q_{M_2}(\phi^{j_1}(\bar{s}_1, \bar{s}_2))\| \\ &= \|Q_{M_2}(\phi^{j_1}(\bar{s}_1) - \phi^{j_1}(\bar{s}_1, \bar{s}_2)\| \leq \varepsilon/n. \end{aligned}$$

If $j_1 \neq j_2$, then, since $\phi^{j_2}(\bar{s}_1) = 0$, h_2 well approximates $\phi^{j_2}(\bar{s}_1, \bar{s}_2)$. And for all $j \neq j_1, j_2$, $\phi^j(\bar{s}_1, \bar{s}_2) = 0$.

In next steps, the construction of x_i's and g_i's are carried out inductively in a similar fashion.

Suppose that $x_1 = x(\bar{s}_1)$, $x_2 = x(\bar{s}_1, \bar{s}_2)$, $\ldots, x_r = x(\bar{s}_1, \ldots, \bar{s}_r)$ and $h_1, h_2, \ldots, h_r$

have been already obtained so that

$$\| \sum_{t \in A_j, t \leq r} h_t - \phi^j(\bar{s}_1, \ldots, \bar{s}_r) \| \leq \varepsilon r/n,$$

for all $1 \leq j \leq l$ (we have already shown this for $r = 1$ and $r = 2$).

Choose $\bar{s}_{r+1}$ such that

$$\| Q_{M_{r+1}}(\phi^j(\bar{s}_1, \ldots, \bar{s}_{r+1}) - \phi^j(\bar{s}_1, \ldots, \bar{s}_r) \| \leq \varepsilon/n$$

for all $1 \leq j \leq l$.

And let $x_{r+1} = x(\bar{s}_1, \ldots, \bar{s}_{r+1})$ (($r+1$)th winning move of **V**) and put

$$h_{r+1} = \phi^{j_{r+1}}(\bar{s}_1, \ldots, \bar{s}_{r+1}) - Q_{M_{r+1}}(\phi^{j_{r+1}}(\bar{s}_1, \ldots, \bar{s}_{r+1})),$$

where $r + 1 \in A_{j_{r+1}}$.

Now let us check that the property assumed for r is still satisfied with these choices for $r + 1$ as well.

Indeed, if $j = j_{r+1}$, (i.e., $r + 1 \in A_j$) then

$$\begin{aligned}
\| \sum_{t \in A_j, t \leq r+1} h_t - \phi^j(\bar{s}_1, \ldots, \bar{s}_{r+1}) \| &\leq \| \sum_{t \in A_j, t \leq r} h_t - \phi^j(\bar{s}_1, \ldots, \bar{s}_r) \| \\
&+ \| h_{r+1} - (\phi^j(\bar{s}_1, \ldots, \bar{s}_{r+1}) - \phi^j(\bar{s}_1, \ldots, \bar{s}_r)) \| \\
&\leq \varepsilon r/n + \| Q_{M_{r+1}}(\phi^j(\bar{s}_1, \ldots, \bar{s}_{r+1}) - \phi^j(\bar{s}_1, \ldots, \bar{s}_r) \| \\
&\leq \varepsilon r/n + \varepsilon/n = \varepsilon(r+1)/n,
\end{aligned}$$

since $Q_{M_{r+1}}\phi^j(\bar{s}_1, \ldots, \bar{s}_r) = \phi^j(\bar{s}_1, \ldots, \bar{s}_r)$ (for $j = j_{r+1}$).

If $j \neq j_{r+1}$ (i.e., $r + 1 \notin A_j$), then $Q_{M_{r+1}}\phi^j(\bar{s}_1, \ldots, \bar{s}_{r+1}) = \phi^j(\bar{s}_1, \ldots, \bar{s}_{r+1})$. And hence

$$\begin{aligned}
\| \sum_{t \in A_j, t \leq r+1} h_t - \phi^j(\bar{s}_1, \ldots, \bar{s}_{r+1}) \| &\leq \| \sum_{t \in A_j, t \leq r} h_t - \phi^j(\bar{s}_1, \ldots, \bar{s}_r) \| \\
&+ \| Q_{M_{r+1}}(\phi^j(\bar{s}_1, \ldots, \bar{s}_{r+1}) - \phi^j(\bar{s}_1, \ldots, \bar{s}_r) \|
\end{aligned}$$

$$\leq \varepsilon r/n + \varepsilon/n = \varepsilon(r+1)/n.$$

Repeating this construction, at the end, we obtain $x_1 = x(\bar{s}_1), \ldots, x_n = x(\bar{s}_1, \ldots, \bar{s}_n)$ and functionals $\{h_1, h_2, \ldots, h_n\}$ so that for all $1 \leq j \leq l$, $\| \sum_{t \in A_j} h_t - \phi^j \| \leq \varepsilon$, $\phi^j(w^j) \geq (1/C)\|w^j\|$ and $\phi^j(w^j) = 0$ for $i \neq j$. Now let $g_i = h_i/\|h_i\|$, $b_i = \|h_i\|$ and let $\delta > 0$ small enough so that $\| \sum_{i \in A_j} a_i x_i - w^j \| \leq \varepsilon$ for all scalars $\{a_i\}$ and for all $1 \leq j \leq l$. Then it follows that, for all scalars $\{a_i\}$,

$$\| \sum_{i \in A_j} a_i x_i \| \leq (1+\varepsilon)^2 C \Big(\sum_{i \in A_j} a_i x_i \Big) \Big(\sum_{i \in A_j} b_i g_i \Big),$$

for all $1 \leq j \leq l$, as desired.

Finally note that the permissibility of $\{g_i\}_{i=1}^n$ follows from the construction. Indeed, since supp$g_i \subset (L_i, M_i)$ for all $1 \leq i \leq n$, and since the choices of L_i's were arbitrary, we can choose those to be the winning moves of **S** in a subspace game for $\{X^*\}_n$.

□

3.2 Disjoint Envelope Functions

Let X be a Banach space with an asymptotic unconditional structure (with a constant $C \geq 1$). We define the set $\{X\}^d$ of all normalized *disjoint-permissible vectors* in X as follows. For $n \in \mathbb{N}$, $\{x_i\}_{i=1}^n \in \{X\}^d$ if there exist $\{e_j\}_{j=1}^m \in \{X\}_m$ for some $m \geq n$ and a disjoint partition $\{A_1, A_2, \ldots, A_n\}$ of $\{1, 2, \ldots, m\}$ such that for each $1 \leq i \leq n$, $x_i = \sum_{j \in A_i} \alpha_j e_j$ for some scalars $\alpha = (\alpha_j)$ such that $\|x_i\| = 1$.

First we make a few remarks about the set $\{X\}^d$ (where superscript d stands for 'disjoint'). Clearly, for all $n \in \mathbb{N}$ and $\{e_i\}_{i=1}^n \in \{X\}_n$ we have that $\{e_i\}_{i=1}^n \in \{X\}^d$. i.e., $\bigcup_n \{X\}_n \subset \{X\}^d$. If $\{x_i\} \in \{X\}^d$, then $\{x_i\}$ is an unconditional basic sequence (with constant C). It is also clear that if $\{u_j\}$ is a block (successive or just disjoint) basis of some $\{x_i\} \in \{X\}^d$, then $\{u_j\} \in \{X\}^d$ as well. Finally, if $\{x_i\}_{i=1}^n \in \{X\}^d$ then $\{x_{\pi(i)}\}_{i=1}^n \in \{X\}^d$, where π is a permutation of $\{1, \ldots, n\}$. This property, obviously, is not shared, in general, by the bases of asymptotic spaces.

We also have the following property of $\{X\}^d$ which is inherited from $\{X\}_n$. If $\{x_i\}_{i=1}^{n_1}$ and $\{y_i\}_{i=1}^{n_2}$ are in $\{X\}^d$, then there exists $\{z_i\}_{i=1}^{n_1+n_2} \in \{X\}^d$ such that $\{z_i\}_{i=1}^{n_1} \overset{1}{\sim} \{x_i\}_{i=1}^{n_1}$ and $\{z_i\}_{i=n_1+1}^{n_1+n_2} \overset{1}{\sim} \{y_i\}_{i=1}^{n_2}$.

Indeed, if $\{x_i\}_{i=1}^{n_1}$ and $\{y_i\}_{i=1}^{n_2}$ are disjoint blocks of the bases $\{e_i\}_{i=1}^{m_1}$ and $\{f_i\}_{i=1}^{m_2}$ of some asymptotic spaces respectively, then we can find an asymptotic space $\{g_i\}_{i=1}^{m_1+m_2}$ such that $\{e_i\}_{i=1}^{m_1} \overset{1}{\sim} \{g_i\}_{i=1}^{m_1}$ and $\{f_i\}_{i=1}^{k} \overset{1}{\sim} \{g_i\}_{i=m_1+1}^{m_1+m_2}$ (1.8.2, [MMT]). Hence the corresponding disjoint blocks $\{z_i\}_{i=1}^{n_1+n_2}$ of $\{g_i\}_{i=1}^{m_1+m_2}$ have the desired property. When $\{x_i\}_{i=1}^{n_1}$ and $\{y_i\}_{i=1}^{n_2}$ are in $\{X\}^d$, to avoid repetitions, we will simply say that $\{x_i, y_i\} \in \{X\}^d$ without referring to $\{z_i\}$.

We define now the natural analogs of envelope functions on $\{X\}^d$.

Definition 3.2.1 *Let X be a Banach space with an asymptotic unconditional structure. For $a = (a_i) \in c_{00}$, let $g_X^d(a) = \inf\|\sum_i a_i x_i\|$ and $r_X^d(a) = \sup\|\sum_i a_i x_i\|$, where the inf and the sup is taken over all $\{x_i\} \in \{X\}^d$. We call g_X^d and r_X^d the disjoint-lower and disjoint-upper-envelope functions respectively.*

It is easy to see that both functions g_X^d and r_X^d are 1-symmetric and 1-unconditional. Moreover, while r_X^d defines a norm on c_{00}, g_X^d satisfies triangle inequality on disjointly supported vectors (of c_{00}).

Indeed, let $a = (a_i)$ and $b = (b_i)$ be two disjoint vectors in c_{00} and let $\varepsilon > 0$ be arbitrary. Pick $\{x_i\}$ and $\{y_i\}$ in $\{X\}^d$ such that $g_X^d(a) + \varepsilon/2 \geq \|\sum_i a_i x_i\|$ and $g_X^d(b) + \varepsilon/2 \geq \|\sum_i b_i y_i\|$. Then, by the above remark, $\{x_i, y_i\} \in \{X\}^d$ and hence

$$
\begin{aligned}
g_X^d(a+b) &\leq \|a_1x_1 + b_1y_1 + a_2x_2 + b_2y_2 + \dots\| \\
&\leq \|a_1x_1 + a_2x_2 + \dots\| + \|b_1y_1 + b_2y_2 + \dots\| \\
&\leq g_X^d(a) + g_X^d(a) + \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, it follows that $g_X^d(a+b) \leq g_X^d(a) + g_X^d(b)$, whenever $a, b \in c_{00}$ are disjointly supported.

To compare the disjoint-envelope functions with the original envelopes, note that for all $a \in c_{00}$, by the definition of these functions we have, $g_X^d(a) \leq g_X(a) \leq r_X(a) \leq r_X^d(a)$.

We will use the following convenient notation. Let (e_i) be the unit vector basis of c_{00}. For $a = (a_i) \in c_{00}$, occasionally we will write $g_X^d\left(\sum_i a_i e_i\right)$ instead of $g_X^d(a)$. Moreover, for any finite number of successive vectors $b^i = (b^i_j) \in c_{00}$ such that $g_X^d(b^i) = 1$ for $i = 1, 2, \dots$ and for any vector $a = (a_i) \in c_{00}$, we write $g_X^d(\sum_i a_i x_i)$ instead of $g_X^d(\sum_i a_i b^i)$, where $x_i = \sum_j {b^i}_j e_j$ are blocks of the basis (e_i) of c_{00} normalized with respect to g_X^d. We'll use similar notation for r_X^d as well.

Next we establish some of the properties of the disjoint-envelope functions similar to the original ones.

Lemma 3.2.2 *Let X be a Banach space with asymptotic unconditional structure.*

(i) The upper envelope function r_X^d is sub-multiplicative. i.e., for all $(a_i) \in$

c_{00}, we have

$$r_X^d\Big(\sum_i a_i x_i\Big) \le r_X^d\Big(\sum_i a_i e_i\Big),$$

for any sequence of successive blocks (x_i) in c_{00} with $r_X^d(x_i) \le 1$ for all $i = 1, 2, \ldots$.

(ii) The lower envelope function g_X^d is super-multiplicative. i.e., for all $(a_i) \in c_{00}$, we have

$$g_X^d\Big(\sum_i a_i x_i\Big) \ge g_X^d\Big(\sum_i a_i e_i\Big),$$

for any sequence of successive blocks (x_i) in c_{00} with $g_X^d(x_i) = 1$ for all $i = 1, 2, \ldots$.

Proof (i) Let $x_i = \sum_{j=k_i+1}^{k_{i+1}} b_j e_j$ for some $1 \le k_1 < k_2 < \ldots$ be a block basis of c_{00} with $r_X^d(x_i) \le 1$ for all $i = 1, 2 \ldots$, and let $a = (a_1, a_2, \ldots, a_l) \in c_{00}$ and $\varepsilon > 0$ be arbitrary. Then there exists $\{u_j\}_{j=1}^{k_l+1} \in \{X\}^d$ such that

$$\begin{aligned} r_X^d\Big(\sum_{i=1}^{l} a_i x_i\Big) - \varepsilon &= r_X^d\Big(\sum_{i=1}^{l} a_i\Big(\sum_{j=k_i+1}^{k_{i+1}} b_j e_j\Big)\Big) - \varepsilon \\ &\le \Big\|\sum_{i=1}^{l} a_i\Big(\sum_{j=k_i+1}^{k_{i+1}} b_j u_j\Big)\Big\|. \end{aligned}$$

Set $c_i = \| \sum_{j=k_i+1}^{k_{i+1}} b_j u_j \|$, then $c_i \le r_X^d(x_i) \le 1$. Let $w_i = (1/c_i) \sum_{j=k_i+1}^{k_{i+1}} b_j u_j$, for $i = 1, \ldots l$, then $\{w_i\} \in \{X\}^d$. Thus the latter term in above is equal to

$$\begin{aligned} \Big\|\sum_{i=1}^{l} a_i c_i w_i\Big\| &\le r_X^d(a_1 c_1, a_2 c_2, \ldots, a_l c_l) \\ &\le r_X^d(a_1, a_2, \ldots, a_l) \end{aligned}$$

The last inequality is due to unconditionality of r_X^d and the fact that $c_i \le 1$ for

$i = 1, \dots, l$. Since $\varepsilon > 0$ was arbitrary, we have obtained that

$$r_X^d\Big(\sum_{i=1}^l a_i x_i\Big) \le r_X^d\Big(\sum_{i=1}^l a_i e_i\Big),$$

as desired.

(ii) The proof of this part follows similar lines and hence we skip it.

□

Lemma 3.2.3 *Let X be a Banach space with an asymptotic unconditional structure. Then, for all $(a_i) \in c_{00}$, we have*

(i)

$$g_X^d\Big(\sum_i a_i x_i\Big) \le r_X^d\Big(\sum_i a_i e_i\Big),$$

where (x_i) is a sequence of successive blocks in c_{00} with $g_X^d(x_i) \le 1$ for all $i = 1, 2, \dots$.

(ii)

$$g_X^d\Big(\sum_i a_i e_i\Big) \le r_X^d\Big(\sum_i a_i x_i\Big),$$

where (x_i) is a sequence of successive blocks in c_{00} with $r_X^d(x_i) = 1$ for all $i = 1, 2, \dots$.

Proof (i) Let $x_i = \sum_{j=k_i+1}^{k_{i+1}} b_j e_j$ for some $1 \le k_1 < k_2 < \dots$ be a block basis of c_{00} with $g_X^d(x_i) \le 1$ for all $i = 1, 2 \dots$ let $a = (a_1, a_2, \dots, a_l)$ be arbitrary scalars and let $\varepsilon > 0$. For each i, pick $\{u_j^i\}_j \in \{X\}^d$ such that

$$\Big\|\sum_j b_j u_j^i\Big\| \le g_X^d(x_i) + \varepsilon \le 1 + \varepsilon.$$

Then $\{u_j^i\}_{i,j} \in \{X\}^d$. Let $c_i = \|\sum_j b_j u_j^i\|$ and $w_i = (1/c_i)\sum_j b_j u_j^i$ for $i = 1, 2, \dots$. Then we have,

$$g_X^d\Big(\sum_i a_i x_i\Big) \;=\; g_X^d\Big(\sum_i a_i\Big(\sum_j b_j e_j\Big)\Big)$$

$$\leq \Big\|\sum_i a_i\Big(\sum_j b_j u_j^i\Big)\Big\|$$
$$= \Big\|\sum_i a_i c_i w_i\Big\|,$$

and since $\{w_i\} \in \{X\}^d$, the latter term above is less than or equal to

$$r_X^d(a_1c_1, a_2c_2, \ldots, a_lc_l) \leq (1+\varepsilon) r_X^d\Big(\sum_i a_i e_i\Big),$$

where the last inequality follows from the unconditionality of r_X^d. Finally, since $\varepsilon > 0$ was arbitrary, the desired inequality follows.

(ii) The proof of this part is similar. □

The most interesting fact about disjoint-envelope functions is that they are always close to some l_p-norms. This is similar as for the original envelope functions (see Proposition 2.4.1). As remarked in [MMT], this is a general fact for multiplicative functions which satisfy a weaker triangle inequality as g_X does.

Proposition 3.2.4 *Let X be a Banach space with asymptotic unconditional structure. Then there exist $1 \leq p, q \leq \infty$ such that for all $\varepsilon > 0$ there exist $C_\varepsilon, c_\varepsilon > 0$ such that for all $a \in c_{00}$ we have*

$$c_\varepsilon \|a\|_{q+\varepsilon} \leq g_X^d(a) \leq \|a\|_q \quad \text{and} \quad \|a\|_p \leq r_X^d(a) \leq C_\varepsilon \|a\|_{p-\varepsilon}.$$

Here it is understood that if $q = \infty$ (resp. $p = 1$), then g_X^d is equivalent to $\|.\|_\infty$ (resp. r_X^d is equivalent to $\|.\|_1$). If $p = \infty$, then for all $r < \infty$ there exists $C_r < \infty$ such that $r_X^d(a) \leq C_r \|a\|_r$.

Remark The proof of these inequalities follows from well known standard arguments as in the case of the original envelope functions. As we show below for r_X^d case, the proofs make use of the classical theorem of Krivine (for a statement of Krivine's theorem see section 3.6.3). However, since the lower disjoint-envelope g_X^d is not necessarily a norm, to be able to use Krivine's theorem, one needs to check that the theorem holds in a more general setting, namely for functions which satisfy the triangle inequality for disjointly supported vectors. To avoid this cumbersome work, we postpone the proof of g_X^d case to the section 3.6.3, where we give a different and a self-contained proof.

Proof (r_X^d *Case*) For $n, m \in \mathbb{N}$, the sub-multiplicativity of r_X^d implies that

$$r_X^d\Big(\sum_{i=1}^{nm} e_i\Big) \le r_X^d\Big(\sum_{i=1}^{n} e_i\Big) r_X^d\Big(\sum_{i=1}^{m} e_i\Big).$$

Hence, by induction, we get that $r_X^d\Big(\sum_{i=1}^{n^k} e_i\Big) \le r_X^d\Big(\sum_{i=1}^{n} e_i\Big)^k$, for all $n, k \in \mathbb{N}$. Let

$$1/p = \inf \ln r_X^d\Big(\sum_{i=1}^{n} e_i\Big) / \ln n.$$

Then, clearly, $r_X^d\Big(\sum_{i=1}^{n} e_i\Big) \ge n^{1/p}$, for all $n \in \mathbb{N}$. Moreover, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $n \in \mathbb{N}$, we have $r_X^d\Big(\sum_{i=1}^{n} e_i\Big) \le C_\varepsilon n^{1/p-\varepsilon}$.

Now consider the space (c_{00}, r_X^d). The unit vector basis $\{e_i\}$ is symmetric and since $r_X^d\Big(\sum_{i=1}^{n} e_i\Big) \ge n^{1/p}$ for all n, it follows from Krivine's theorem (see section 3.6.3) that there exists $r \le p$ such that ℓ_r is block finitely representable in the space (c_{00}, r_X^d), i.e., for all $\delta > 0$ and $n \in \mathbb{N}$, there exists a sequence of successive blocks $\{x_i\}_{i=1}^n$ in (c_{00}, r_X^d) such that $\{x_i\}_{i=1}^n \overset{1+\delta}{\sim} \ell_r^n$. Moreover, by the sub-multiplicativity of r_X^d, for all n, we have

$$\frac{n^{1/r}}{1+\delta} \le r_X^d\Big(\sum_{i=1}^{n} x_i\Big) \le r_X^d\Big(\sum_{i=1}^{n} e_i\Big) \le C_\varepsilon n^{1/p-\varepsilon}.$$

Since this is true for all ε and n, it follows that $r \ge p - \varepsilon$ for all $\varepsilon > 0$, and thus it follows that $r = p$.

Finally, by sub-multiplicativity of r_X^d and that $\delta > 0$ can be chosen arbitrarily, it follows that $r_X^d(a) \ge \|a\|_p$ for all $a \in c_{00}$.

To prove the upper $\ell_{p-\varepsilon}$ estimate for r_X^d we make use of an auxiliary norm $\sigma_{p-\varepsilon}$. For $1 \le s \le \infty$, the unit ball of the norm σ_s is the convex hull of all vectors

$\alpha = (\sum_i |\alpha_i|)^{-1/s}(\alpha_i)_{i=1}^n$, where $\alpha_i = \pm 1$ or 0. (The norm σ_s is equivalent to the norm of Lorentz sequence space $d(w,1)$, where the weight $w = (w_i)$ satisfies $\sum_{i=1}^n w_i = n^{1/s}$ [LT].) A direct estimate shows that for all $s' < s$ there exists $C_{s'} < \infty$ and $C_{s'}$ does not depend on n so that $\sigma_s(a) \leq C_{s'}||a||_{s'}$ for all $a \in c_{00}$. Now fix $\varepsilon > \delta > 0$. As we remarked earlier, there exists $C_\delta > 0$ such that $r_X^d\left(\sum_{i=1}^n e_i\right) \leq C_\delta n^{1/p-\delta}$. Put $p - \delta = s$, and hence $r_X^d(a) \leq C_\delta \sigma_s(a) \leq C_\delta C_{p-\varepsilon}||a||_{p-\varepsilon}$ for all $a \in c_{00}$. Hence, for all $a \in c_{00}$, $r_X^d(a) \leq C_\varepsilon ||a||_{p-\varepsilon}$, where $C_\varepsilon = C_\delta C_{p-\varepsilon}$.

□

Using the super-multiplicativity of g_X^d, as in the first part of the above proof, we easily obtain the following.

There exists $1 \leq q \leq \infty$ such that for all $\varepsilon > 0$ there exists a constant c_ε such that for all n,

$$c_\varepsilon n^{1/q+\varepsilon} \leq g_X^d\left(\sum_{i=1}^n e_i\right) \leq n^{1/q}. \tag{3.1}$$

Definition 3.2.5 *Let p be as in Proposition 3.2.4 and let q be as in (3.1). We say that the lower-disjoint envelope g_X^d has power type-q and the upper-disjoint envelope r_X^d has power type-p.*

We define similarly the power types of the original envelope functions. We say that p and q are the power types of r_X and g_X respectively, when $1 \leq p, q \leq \infty$ as in Proposition 2.4.1.

As we remarked earlier, it is clear from the definitions that $g_X^d(a) \leq g_X(a) \leq r_X(a) \leq r_X^d(a)$ for all $a \in c_{00}$. Also note that for asymptotic-ℓ_p spaces all these functions are equivalent. However, in general they might be very different as the next example shows.

Example 3.2.6 *There exists a Banach space X with an unconditional basis such that while the power type of g_X is 1, g_X^d is equivalent to $||.||_\infty$.*

Proof The Schlumprecht space S has this property. Recall that the space S is defined on c_{00} as follows [S]. For $a \in c_{00}$, put

$$\|a\| = \max\Big\{\|a\|_\infty, \sup_{l\geq 2} \frac{1}{\log_2(l+1)} \sum_{i=1}^{l} \|E_i(a)\|\Big\},$$

where the inner sup runs through all subsets E_i of $\mathbb{N}$ such that $\max E_i < \min E_{i+1}$. Here $\|a\|_\infty = \sup_i |a_i|$ and $E_i(a) = \sum_{j\in E_i} a_j e_j$ for $a = \sum_i a_i e_i \in c_{00}$.

The space S is well known and much studied in recent years, and we shall use some of these results here.

It is easy to see that the unit vector basis $\{e_i\}$ is 1-subsymmetric and 1-unconditional. Subsymmetry implies that for any successive normalized blocks $\{x_i\}_{i=1}^n$ of $\{e_i\}$, we have $\{x_i\}_{i=1}^n \in \{S\}_n$. Also from the definition of the norm, $\|\sum_{i=1}^n x_i\| \geq \frac{n}{\log_2(n+1)}$ for all $\{x_i\}_{i=1}^n \in \{S\}_n$. This implies that the power type of g_S is 1.

On the other hand, it is shown in [KL] by a much more delicate calculation that c_0 is disjointly finitely representable in S. i.e., for all $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists a sequence $\{x_i\}_{i=1}^n$ of disjointly supported vectors such that $\{x_i\}_{i=1}^n \overset{1+\varepsilon}{\sim} \ell_\infty^n$. The subsymmetry of the basis implies that for all disjointly supported sequences $\{x_i\}_{i=1}^n$, we have $\{x_i\}_{i=1}^n \in \{S\}^d$. i.e., $g_S^d \sim \|.\|_\infty$. Therefore while g_S is close to the ℓ_1-norm, g_S^d is equivalent to $\|.\|_\infty$.

Moreover, it can be deduced from the proof of [KL] that one can find disjoint permissible vectors $\{x_i\}_{i=1}^n$ such that $\{x_i\}_{i=1}^n \overset{1+\varepsilon}{\sim} \ell_\infty^n$ in *every* block subspace Y of S. Thus, for every block subspace Y of S we have that g_Y is close to the ℓ_1-norm and g_Y^d is equivalent to $\|.\|_\infty$.

Also it is easy to verify that for the dual space S^*, we have that r_{S^*} has power type-∞ and $r_{S^*}^d$ is equivalent to the ℓ_1-norm (see Proposition 3.3.1 below). □

3.3 Duality for Disjoint-Envelope Functions

Recall that the lower envelopes g_X and g_X^d satisfy the triangle inequality only for disjoint vectors, they do not necessarily define a norm on c_{00}. For this reason, we introduce, as in [MT2], the following norms on c_{00} which are 'close' to the lower envelopes.

Denote by $\bar{g}_X$ (resp. $\bar{g}_X^d$) the largest norm on c_{00} which is less than or equal to g_X (resp. g_X^d). Also by $r_X^*, \bar{g}_X^*, r_X^{d*}$ and $\bar{g}_X^{d*}$, we denote the norms on c_{00} which are dual to $r_X, \bar{g}_X, r_X^d$ and $\bar{g}_X^d$ respectively.

The following duality relations for the original envelope functions have been established in [MT2]. Let X be a reflexive Banach space with a minimal system, then

$$(1/4)r_{X^*}^*(a) \leq \bar{g}_X(a) \leq 4r_{X^*}^*(a) \quad \text{and} \quad (1/4)\bar{g}_{X^*}^*(a) \leq r_X(a) \leq 4\bar{g}_{X^*}^*(a),$$

for all $a \in c_{00}$. The proof of these inequalities makes use of Lemma 2.3.1.

Equipped with Theorem 3.1.2 we show next that the similar duality relations hold for disjoint-envelope functions for reflexive Banach spaces with asymptotic unconditional structure.

Proposition 3.3.1 *Let X be a reflexive Banach space with a minimal fundamental and 1-norming system $\{u_i, u_i^*\}$. If X has an asymptotic unconditional structure with constant $C \geq 1$, then*

$$(1/C)r_{X^*}^{d*}(a) \leq \bar{g}_X^d(a) \leq r_{X^*}^{d*}(a) \quad \text{and} \quad (1/C)\bar{g}_{X^*}^{d*}(a) \leq r_X^d(a) \leq \bar{g}_{X^*}^{d*}(a),$$

for all $a \in c_{00}$.

Proof By Proposition 3.1.1, X^* also has an asymptotic unconditional structure with the same constant C, hence the functions $r_{X^*}^{d*}$ and $\bar{g}_{X^*}^{d*}$ are well-defined.

Let $\{x_i\} \in \{X\}^d$ and $\varepsilon > 0$. Theorem 3.1.2 implies that there exist $\{x_i'\}$ in X satisfying $\{x_i\} \overset{1+\varepsilon}{\sim} \{x_i'\}$ and $\{g_i'\}$ in X^* satisfying $\{g_i'\} \overset{1+\varepsilon}{\sim} \{g_i\}$ for some

$\{g_i\} \in \{X^*\}^d$ such that $g_i'(x_i') \geq 1/C$ and $g_i'(x_j') = 0$ for $i \neq j$.

Fix $a = (a_i) \in c_{00}$. Then for any $b = (b_i) \in c_{00}$, we have

$$\begin{aligned} 1/C \sum_i |a_i b_i| &\leq \Big(\sum_i a_i x_i'\Big)\Big(\sum_i b_i g_i'\Big) \\ &\leq \Big\|\sum_i a_i x_i'\Big\|_X \Big\|\sum_i b_i g_i'\Big\|_{X^*} \leq r^d_{X^*}(b)\Big\|\sum_i a_i x_i'\Big\|_X. \end{aligned}$$

Taking the supremum over all $b \in c_{00}$ with $r^d_{X^*}(b) \leq 1$ and then the infimum over all $\{x_i\} \in \{X\}^d$ we get, by the definition of the norm $r^{d*}_{X^*}$, that $r^{d*}_{X^*}(a) \leq C g^d_X(a)$ for all $a \in c_{00}$. Also, since $\bar{g}^d_X$ is the largest norm which is less than or equal to g^d_X, it follows that $r^{d*}_{X^*}(a) \leq C\bar{g}^d_X(a)$ for all $a \in c_{00}$. Moreover, applying the same argument to X^*, we also get $r^{d*}_X(a) \leq C\bar{g}^d_{X^*}(a)$, and hence by duality, $r^d_X(a) \geq (1/C)\bar{g}^{d*}_{X^*}(a)$. These estimates prove the left hand inequalities in each statement.

To prove the remaining inequalities, let $a = (a_i) \in c_{00}$ and $\{x_i\} \in \{X\}^d$ be arbitrary. Then there exists an asymptotic space $\{e_j\} \in \{X\}_m$ for some m such that for all $i = 1, 2, \ldots$, $x_i = \sum_{j \in A_i} \alpha_j e_j$ for some partition $\{A_i\}$ of $\{1, 2, \ldots, m\}$. Now by Lemma 2.3.1, there exist permissible vectors $\{y_j\}_{j=1}^m \in X$ satisfying $\{y_j\}_{j=1}^m \overset{1+\varepsilon}{\sim} \{e_j\}_{j=1}^m$ and permissible functionals $\{g_j\}_{j=1}^m \in X^*$ and scalars $\{b_j\}$ such that $\|\sum_{j=1}^m b_j g_j\| \leq 1 + \varepsilon$ with $g_j(x_i) = 0$ for $i \neq j$ and

$$\Big\|\sum_i a_i\Big(\sum_{j \in A_i} \alpha_j y_j\Big)\Big\| \leq (1+\varepsilon)\Big(\sum_i a_i\Big(\sum_{j \in A_i} \alpha_j y_j\Big)\Big)\Big(\sum_j b_j g_j\Big).$$

(Note that we have applied Lemma 2.3.1 to $\{e_j\}_{j=1}^m$ and the sequence of scalars $(d_j)_{j=1}^m$ where $d_j = a_i \alpha_j$ for all $j \in A_i$ and $i = 1, 2, \ldots$.)

For each $i = 1, 2, \ldots$, let $w_i = \sum_{j \in A_i} b_j g_j$ and $\|w_i\| = c_i$. Then note that

$$\begin{aligned} \frac{w_i}{c_i}\Big(\sum_{j \in A_i} \alpha_j y_j\Big) &\leq \Big\|\sum_{j \in A_i} \alpha_j y_j\Big\| \\ &\leq (1+\varepsilon)\|x_i\| \leq 1 + \varepsilon, \end{aligned}$$

and clearly $w_i(\sum_{j\in A_k} \alpha_j^k y_j) = 0$ for $i \neq k$.

Now it follows from above that

$$\begin{aligned}\Big\|\sum_i a_i\Big(\sum_{j\in A_i}\alpha_j y_j\Big)\Big\| &\le (1+\varepsilon)\sum_i |a_i c_i|\Big|\frac{w_i}{c_i}\Big(\sum_{j\in A_i}\alpha_j y_j\Big)\Big| \\ &\le (1+\varepsilon)^2\sum_i |a_i c_i| \\ &\le (1+\varepsilon)^2 \bar{g}^d_{X^*}(c)\bar{g}^{d^*}_{X^*}(a),\end{aligned}$$

where we set $c = (c_i)$.

But

$$\bar{g}^d_{X^*}(c) \le g^d_{X^*}(c) \le \|\sum_i c_i\frac{w_i}{c_i}\| \le \|\sum_{j=1}^m b_j g_j\| \le 1+\varepsilon.$$

Thus, we have obtained that

$$\|\sum_i a_i x_i\| \le (1+\varepsilon)^4 \bar{g}^{d^*}_{X^*}(a).$$

Now taking the supremum over all $\{x_i\} \in \{X\}^d$ we get, since $\varepsilon > 0$ was arbitrary, that $r^d_X(a) \le \bar{g}^{d^*}_{X^*}(a)$ for all $a \in c_{00}$.

Again applying the same argument to X^*, we obtain that $r^d_{X^*}(a) \le \bar{g}^{d^*}_X(a)$, and by duality $r^{d*}_{X^*}(a) \ge \bar{g}^d_X(a)$, which finishes the proof of the all inequalities.

□

3.4 A Characterization of Asymptotic-ℓ_p Spaces

In this section, we prove a characterization for asymptotic-ℓ_p spaces. Our starting point is the following consequence of the results proved in [KOS].

Suppose that for a Banach space X there exists a constant $K > 0$ such that for all n and permissible vectors $\{x_i\}_{i=1}^n$ in X we have

$$\Big\|\sum_{i=1}^n x_i\Big\| \ge n/K.$$

Then X is an asymptotic-ℓ_1 space.

Let us note that although this result is not stated in [KOS] as we formulated above, this is a consequence of Propositions 6.7 and 6.8 proved there, and we will provide a proof of this statement (see Corollary 3.4.3).

The above result shows that asymptotic-ℓ_1 spaces can be fully characterized by the ℓ_1-behavior of its normalized permissible vectors on constant coefficients. A natural question in this context is whether this remains true in general. We formulate the question for $1 < p < \infty$.

Problem Let $1 < p < \infty$. Suppose that there exists a constant C so that for all n and for all permissible vectors $\{x_i\}_{i=1}^n$ in a Banach space X we have

$$\frac{n^{1/p}}{K} \leq \|\sum_{i=1}^{n} x_i\| \leq K n^{1/p}.$$

Is X an asymptotic-ℓ_p space?

We can also restate this problem in terms of the envelope functions. Let X be a Banach space and suppose that there is $1 < p < \infty$ and a constant K such that for all n, $g_X(1,1,\ldots,1) \overset{K}{\sim} n^{1/p} \overset{K}{\sim} r_X(1,1,\ldots,1)$. Does it follow that $g_X(a) \sim \|a\|_p \sim r_X(a)$ for all $a \in c_{00}$?

We will see in the next section that the answer to this problem is negative in general, even for spaces with an unconditional basis. However, we will show here that this is true if we replace the envelope functions g_X and r_X in the assumption with the disjoint- envelope functions g_X^d and r_X^d. This was also our initial reason to introduce the disjoint-envelope functions.

The main result of this section is the following characterization for asymptotic-ℓ_p spaces.

Theorem 3.4.1 *Let X be a Banach space with asymptotic unconditional structure. Suppose that there exist $1 \leq p \leq \infty$ and constant $K > 0$ such that for all*

$n \in \mathbb{N}$ and for all $\{x_i\}_{i=1}^n \in \{X\}^d$, we have

$$\frac{n^{1/p}}{K} \leq \Big\|\sum_{i=1}^n x_i\Big\| \leq K n^{1/p}.$$

Then X is an asymptotic-ℓ_p space.

For the proof, we will require the following characterization of the unit vector basis of ℓ_p, which is of independent interest. The idea of the proof of the next theorem is inspired by the proof of the Proposition 6.9 in [KOS].

Theorem 3.4.2 *Let X be a Banach space with a subsymmetric and unconditional basis (x_i). Suppose that there exist $1 \leq p \leq \infty$ and a constant $K > 0$ such that for all $n \in \mathbb{N}$ and for all disjointly supported normalized vectors $\{y_i\}_{i=1}^n$ in X, we have*

$$\frac{n^{1/p}}{K} \leq \Big\|\sum_{i=1}^n y_i\Big\| \leq K n^{1/p}.$$

Then (x_i) is equivalent to the unit vector basis of l_p.

Proof Without loss of generality, we may (and will) assume that the basis is 1-subsymmetric and 1-unconditional.

We need to show that there is a constant $C > 0$ so that for all $a = (a_i) \in c_{00}$, we have

$$\frac{1}{C}\Big(\sum_i |a_i|^p\Big)^{1/p} \leq \Big\|\sum_i a_i x_i\Big\| \leq C\Big(\sum_i |a_i|^p\Big)^{1/p}.$$

First we give the proof of the left hand inequality. Suppose to the contrary that the lower ℓ_p-estimate fails. That is, for all $\varepsilon > 0$ there exists $a = (a_i)_{i=1}^k$ such that $\|\sum_{i=1}^k a_i x_i\| < \varepsilon$ while $\sum_{i=1}^k |a_i|^p = 1$. By unconditionality of the basis (x_i) we can assume that all a_i's are positive.

Fix $\varepsilon < \frac{1}{K^3 2^{1/p}}$, and let (a_i) be as above. Since the a_i's are positive, we normalize by taking the pth root and rewrite our assumption in the form, $\sum_{i=1}^k a_i = 1$ while $\|\sum_{i=1}^k a_i^{1/p} x_i\| < \varepsilon < \frac{1}{K^3 2^{1/p}}$.

By a slight perturbation, if necessary, we assume that a_i's are positive rationals and we write $a_i = \frac{n_i}{N}$ for $1 \le i \le k$, where n_i, N are natural numbers. Put also $N = n_i m_i + k_i$, $0 \le k_i < n_i$, $1 \le i \le k$. Now consider the vector $x = \sum_{i=1}^{k} {a_i}^{1/p} \sum_{j=1}^{N} x_j^i$, where $x_j^i = x_{(i-1)N+j}$ for $1 \le i \le k$ and $1 \le j \le N$. i.e., x is of the form $x = (a_1^{1/p}, \dots, a_1^{1/p}, a_2^{1/p}, \dots, a_2^{1/p}, \dots, \dots, a_k^{1/p}, \dots, a_k^{1/p})$ with respect to $(x_1, x_2, \dots, x_{kN})$, where each block consists of N constant coefficients $a_i^{1/p}$. First, we estimate the norm of x from below.

For each $1 \le i \le k$, since $N = k_i(m_i+1)+(n_i-k_i)m_i$, we may fix a partition

$$\{1, \dots, N\} = \bigcup_{\mu=1}^{k_i} A_{\mu,i} \cup \bigcup_{v=1}^{n_i-k_i} B_{v,i},$$

where $|A_{\mu,i}| = m_i + 1$ for each $\mu = 1, \dots, k_i$ and $|B_{v,i}| = m_i$ for each $v = 1, \dots, n_i - k_i$. Then we have

$$\begin{aligned} \|x\| &= \Big\|\sum_{i=1}^{k} {a_i}^{1/p} \sum_{j=1}^{N} x_j^i\Big\| = \Big\|\sum_{i=1}^{k} (\frac{n_i}{N})^{1/p} \sum_{j=1}^{N} x_j^i\Big\| \\ &= \Big\|\sum_{i=1}^{k} \Big(\sum_{\mu=1}^{k_i} (\frac{n_i}{N})^{1/p} \sum_{j \in A_{\mu,i}} x_j^i + \sum_{v=1}^{n_i-k_i} (\frac{n_i}{N})^{1/p} \sum_{j \in B_{v,i}} x_j^i\Big)\Big\|. \end{aligned} \tag{3.2}$$

Now, using the assumption, we estimate the norm of each of the disjoint blocks appearing in (3.2).

For each $\mu = 1, \dots, k_i$, since $|A_{\mu,i}| = m_i + 1$, we have

$$\begin{aligned} \Big\|(\frac{n_i}{N})^{1/p} \sum_{j \in A_{\mu,i}} x_j^i\Big\| &\ge \frac{(m_i+1)^{1/p}}{K} (\frac{n_i}{N})^{1/p} \\ &\ge \frac{(n_i m_i + n_i)^{1/p}}{N^{1/p} K} \ge \frac{1}{K}. \end{aligned}$$

For each $v = 1, \dots, n_i - k_i$, since $|B_{v,i}| = m_i$, we have

$$\Big\|(\frac{n_i}{N})^{1/p} \sum_{j \in B_{v,i}} x_j^i\Big\| \ge \frac{{n_i}^{1/p}}{N^{1/p}} \frac{{m_i}^{1/P}}{K} \ge \frac{1}{2^{1/p} K}.$$

Let

$$u_\mu^i = \frac{\sum_{j\in A_{\mu,i}} x_j^i}{\|\sum_{j\in A_{\mu,i}} x_j^i\|} \quad \text{and} \quad w_v^i = \frac{\sum_{j\in B_{v,i}} x_j^i}{\|\sum_{j\in B_{v,i}} x_j^i\|}.$$

By the unconditionality of the basis and by the above estimates for the blocks u_μ^i and w_v^i appearing in (3.2), we obtain that the expression (3.2) is greater than or equal to

$$\frac{1}{2^{1/p}K}\Big\|\sum_{i=1}^{k}\Big(\sum_{\mu=1}^{k_i} u_\mu^i + \sum_{v=1}^{n_i-k_i} w_v^i\Big)\Big\|. \tag{3.3}$$

The blocks u_μ^i and w_v^i are disjointly supported (in fact, note that the partition can be chosen so that they become successive) and normalized, therefore by the assumption, (3.3) is greater than or equal to

$$\frac{1}{2^{1/p}K^2}\Big(\sum_{i=1}^{k} n_i\Big)^{1/p} = \frac{N^{1/p}}{2^{1/p}K^2}.$$

We have used that $1 = \sum_{i=1}^{k} a_i = \sum_{i=1}^{k} \frac{n_i}{N}$. Thus, we have obtained that

$$\|x\| \geq \frac{N^{1/p}}{2^{1/p}K^2}. \tag{3.4}$$

On the other hand, letting $y_j = \sum_{i=1}^{k} {a_i}^{1/p} x_j^i$ for $1 \leq j \leq N$, by subsymmetry of the basis (x_i) and the assumption, we have $\|y_j\| < \varepsilon$. Thus, since $\{y_j\}$ are disjointly supported, by the assumption we have

$$\|x\| = \Big\|\sum_{i=1}^{k} {a_i}^{1/p} \sum_{j=1}^{N} x_j^i\Big\| = \Big\|\sum_{j=1}^{N} y_j\Big\| \leq \varepsilon K N^{1/p}. \tag{3.5}$$

Comparing two estimates in (3.4) and (3.5), we arrive at a contradiction by taking ε small enough. Hence there exists $C < \infty$ ($C \geq K^3 2^{1/p}$) so that $\|\sum_i a_i x_i\| \geq (1/C)(\sum_i |a_i|^p)^{1/p}$, for all (a_i).

The proof of the upper ℓ_p estimate is similar. Suppose to the contrary that for all $M > 1$ there exists a positive scalar sequence $({a_i}^{1/p})_{i=1}^{k}$ such that

$\|\sum_{i=1}^{k} a_i{}^{1/p} x_i\| > M$ while $\sum_{i=1}^{k} a_i = 1$. Fix $M > 2^{1/p}K^3$ and find $(a_i{}^{1/p})_{i=1}^{k}$ satisfying the above. With the same set up as in the first part of the proof, we estimate the norm of the vector x in (3.2) from above. Thus,

$$\Big\|\sum_{i=1}^{k} a_i{}^{1/p} \sum_{j=1}^{N} x_j^i\Big\| \leq K^2 2^{1/p} N^{1/p}.$$

On the other hand, as in (3.5), using the assumption again we have

$$\Big\|\sum_{i=1}^{k} a_i{}^{1/p} \sum_{j=1}^{N} x_j^i\Big\| \geq \frac{MN^{1/p}}{K}.$$

But this is a contradiction by the choice of M. Hence there exists an absolute constant $C \geq 1$ ($C \geq 2^{1/p}K^3$) such that

$$\Big\|\sum_{i} a_i x_i\Big\| \leq C\Big(\sum_{i} |a_i|^p\Big)^{1/p},$$

for all scalars (a_i). The proof is now complete. □

Let us remark that the above proof uses only the disjointly supported vectors. In particular, the conclusion of the theorem holds for more general functions (e.g. for the lower disjoint-envelope g_X^d) than the norms, which satisfy the conditions.

Proof of Theorem 3.4.1 Clearly it is sufficient to show that both g_X^d and r_X^d are equivalent to $\|.\|_p$. The assumption already implies that

$$\frac{n^{1/p}}{K} \leq g_X^d\Big(\sum_{i=1}^{n} e_i\Big) \leq r_X^d\Big(\sum_{i=1}^{n} e_i\Big) \leq Kn^{1/p},$$

where (e_i) is the unit vector basis of c_{00}.

Let $\{u_i\}$ and $\{w_i\}$ be arbitrary successive blocks of c_{00} with $g_X^d(u_i) = 1$ and $r_X^d(w_i) = 1$ for all $i = 1, 2, \ldots$ respectively. From the multiplicativity properties of the disjoint envelopes, Lemma 3.2.2, and Lemma 3.2.3, it follows that

$$\frac{n^{1/p}}{K} \le g_X^d\Big(\sum_{i=1}^n e_i\Big) \le g_X^d\Big(\sum_{i=1}^n u_i\Big) \le r_X^d\Big(\sum_{i=1}^n e_i\Big) \le Kn^{1/p}$$

and

$$\frac{n^{1/p}}{K} \le g_X^d\Big(\sum_{i=1}^n e_i\Big) \le r_X^d\Big(\sum_{i=1}^n w_i\Big) \le r_X^d\Big(\sum_{i=1}^n e_i\Big) \le Kn^{1/p}.$$

That is, $g_X(\sum_{i=1}^n u_i) \overset{K^2}{\sim} n^{1/p}$ and $r_X(\sum_{i=1}^n w_i) \overset{K^2}{\sim} n^{1/p}$ for all successive normalized block bases (u_i) and (w_i) of (c_{00}, g_X^d) and (c_{00}, r_X^d) respectively. Moreover, since both g_X^d and r_X^d are also 1-symmetric, the same estimates hold for all disjointly supported normalized sequences in (c_{00}, g_X^d) and (c_{00}, r_X^d) respectively. Thus, by Theorem 3.4.2, both g_X^d and r_X^d are equivalent to $\|.\|_p$ with a constant which depends only on K. Therefore X is an asymptotic-ℓ_p space.

□

As we remarked at the beginning of the section, for $p = 1$ this result can be considerably improved.

Corollary 3.4.3 *Suppose that for a Banach space X there exists a constant $K > 0$ such that for all n and permissible vectors $\{x_i\}_{i=1}^n$ in X we have*

$$\Big\|\sum_{i=1}^n x_i\Big\| \ge n/K.$$

Then X is an asymptotic-ℓ_1 space.

Proof (*Sketch*) It is sufficient to show that the lower (original) envelope function satisfies $g_X(a) \ge c\|a\|_1$, for some constant c. (The upper estimate trivially follows from the triangle inequality.)

The proof follows the same lines in the first part of the proof of Theorem 3.4.1, so we only indicate the few differences.

With the same setup as in the first part of the proof of Theorem 3.4.1, assume that the above estimate fails and consider the vector x in (3.2). Then the estimate, $g_X(x) \geq \frac{N}{2K^2}$ in (3.4) holds because , as we remarked there, the blocks appearing in (3.3) can be arranged to be successive. On the other hand, the upper estimate $g_X(x) < \varepsilon KN$, in (3.5) simply follows from the triangle inequality for g_X on disjointly supported vectors. Thus we arrive at a contradiction. □

3.5 Tirilman Spaces

In this section, we deviate somewhat from the general theme of the thesis, and turn our attention to a particular class of Banach spaces, called the Tirilman spaces.

To complement the main result of the previous section, we show here that the characterization of asymptotic-ℓ_p spaces given in Theorem 3.4.1 cannot be strengthened further as stated in the problem discussed there. Namely, we show that for all $1 < p < \infty$, there is a Tirilman space X with the property that for all n and permissible vectors $\{x_i\}_{i=1}^n$ in X, we have $\|\sum_{i=1}^n x_i\| \overset{K}{\sim} n^{1/p}$ for some constant independent of n, and yet X is not an asymptotic-ℓ_p space.

Additionally, as a consequence of Theorem 3.4.2 of the previous section, we also obtain a solution to a conjecture of Casazza and Shura on the Tirilman spaces.

The Tirilman spaces are introduced and studied by Casazza and Shura [CS]. Their definitions depend on a slight modification of the original spaces constructed by L. Tzafriri [T] (The name 'Tirilman' comes from the Romanian surname of L. Tzafriri.)

We recall now the definition and few properties of these spaces, which we shall use.

Let $1 < p < \infty$. Fix $0 < \gamma < 1$. As in the case of Tsirelson's and Schlumprecht's spaces, the norm is defined on c_{00} implicitly. For all $a = (a_i) \in c_{00}$, let

$$\|a\| = \max\Big\{\|a\|_\infty, \gamma\sup\frac{\sum_{j=1}^k \|E_j a\|}{k^{1/q}}\Big\},$$

where the inner supremum is taken over all finite successive intervals of natural numbers $1 \leq E_1 < E_2 < \ldots < E_k$ and all k, and $1/p + 1/q = 1$.

The Banach space $(c_{00}, \|.\|)$, which is defined with the parameters p and γ, is called a Tirilman space and denoted by $Ti(\gamma, p)$.

It is not difficult to show that such a norm indeed exists. Also, it is immediate from the definition that the unit vectors $\{e_i\}_{i=1}^{\infty}$ form a normalized 1-subsymmetric 1-unconditional basis for $Ti(\gamma, p)$.

We shall also use the following results proved in [CS].

Theorem 3.5.1 *Let $1 < p < \infty$. There exists $0 < \gamma < 1$ such that the following hold for $Ti(p, \gamma)$.*

(1) For any normalized successive blocks $\{x_j\}_{j=1}^{n}$ of the basis $\{e_i\}_i$, we have

$$\gamma n^{1/p} \leq \|\sum_{j=1}^{n} x_j\| \leq 3^{1/q} n^{1/p}.$$

(2) $Ti(p, \gamma)$ does not contain isomorphs of any l_p $(1 \leq p < \infty)$ or of c_0. In particular, $Ti(p, \gamma)$ is a reflexive space.

Observe that the left hand side inequality in the first part of the Theorem easily follows from the definition of the norm. Indeed, let $\{x_j\}_{j=1}^{n}$ be a normalized successive block sequence of the basis. For each $1 \leq j \leq n$, let E_j be the smallest interval containing the support of x_j. Then,

$$\|\sum_{j=1}^{n} x_j\| \geq \gamma \frac{\sum_{j=1}^{n} \|E_j x_j\|}{n^{1/q}} \geq \gamma \frac{n}{n^{1/q}} = \gamma n^{1/p}.$$

For the proof of other statements see Lemma X.d.4 and Theorem X.d.6 of [CS] (Note that in [CS] the proof of these statements are given for $p = 2$ only, appropriate modifications are necessary for the general case).

Example 3.5.2 *Let $1 < p < \infty$. Then there exists $0 < \gamma < 1$ such that the Tirilman space $Ti(p, \gamma)$ has the property that for all n and all permissible vectors*

$\{x_j\}_{j=1}^n$, *we have* $\|\sum_{i=1}^n x_i\| \overset{K}{\sim} n^{1/p}$, *where* K *depends on* γ *and* p *only, and yet* $Ti(p,\gamma)$ *is not an asymptotic-*ℓ_p *space.*

Proof By Theorem 3.5.1, there exists $0 < \gamma < 1$ such that the Tirilman space $Ti(p,\gamma)$ has the property that for all n and successive blocks $\{x_j\}_{j=1}^n$ of the basis, we have $\gamma n^{1/p} \leq \|\sum_{j=1}^n x_j\| \leq 3^{1/q} n^{1/p}$. In particular, the same estimates hold for all permissible vectors. On the other hand, since the basis $\{e_i\}$ is subsymmetric, if $Ti(p,\gamma)$ was asymptotic-ℓ_p, this would imply that the basis $\{e_i\}$ is equivalent to the unit vector basis of ℓ_p. However, this contradicts the part (2) of Theorem 3.5.1. □

Moreover, Casazza and Shura conjectured that $Ti(2,\gamma)$, where $0 < \gamma < 10^{-6}$, has a symmetric basis (Conjecture X.d.9, [CS]). (As it is shown in [CS], for $0 < \gamma < 10^{-6}$ the conclusion of Theorem 3.5.1 holds.) However, this is not the case, as the next theorem shows.

Theorem 3.5.3 *Let* $1 < p < \infty$ *and let* $0 < \gamma < 1$ *be as in Theorem 3.5.1. Then* $Ti(p,\gamma)$ *contains no symmetric basic sequences.*

Proof To the contrary, suppose that there is a symmetric basic sequence $\{x_i\}_{i=1}^\infty$ in $Ti(p,\gamma)$. By Theorem 3.5.1, $Ti(p,\gamma)$ is reflexive, thus $\{x_i\}$ is weakly null and by so-called sliding hump argument there exists a subsequence which is equivalent to a block basis of the unit vector basis (e_i) of $Ti(p,\gamma)$ (cf. Proposition 1.a.12 of [LT]). Since the sequence $\{x_i\}$ is symmetric, it is equivalent to all of its subsequences, in particular, $\{x_i\}$ itself is equivalent to a block basis of $\{e_i\}$. Now it follows from the first part of Theorem 3.5.1 that for all n and all normalized successive blocks $\{u_i\}_{i=1}^n$ of $\{x_i\}$, we have

$$\gamma n^{1/p} \leq \|\sum_{i=1}^n u_i\| \leq 3^{1/q} n^{1/p}.$$

By symmetry of $\{x_i\}$, the same estimates hold for all disjointly supported normalized vectors $\{u_i\}_{i=1}^n$. Thus by Theorem 3.4.2, $\{x_i\}$ must be equivalent to the unit vector basis of l_p, which contradicts the second part of Theorem 3.5.1. □

The definition of $Ti(p,\gamma)$ in [CS] has been modelled on spaces constructed by Tzafriri in [Tz]. This definition was fully analogous to that of the Tirilman spaces, except that in the implicit equation of the norm the inner supremum is taken over all *disjoint* subsets E_j of the natural numbers (rather than successive ones) [Tz]. In this case, as it is easily seen, the unit vectors form a symmetric basis for the space. In the literature of the Tsirelson-like spaces, the Tzafriri spaces are the *modified* Tirilman spaces (cf. [CS]).

It is well known, for instance, that the modified Tsirelson space is canonically isomorphic to the Tsirelson space, i.e., the unit vector bases are equivalent.

A natural question then was raised in [CS] (see X.D. Notes and Remarks 3) whether the same holds for the Tirilman spaces. It follows immediately from Theorem 3.5.3 that the answer is negative. In fact, Theorem 3.5.3 implies the following

Corollary 3.5.4 *Let $1 < p < \infty$ and let $0 < \gamma < 1$ be as in Theorem 3.5.1. Then the Tzafriri space with these parameters p and γ does not imbed into $Ti(p,\gamma)$.*

3.6 Envelopes and Reflexivity

In this section, we return to the original envelope functions and use them to give an application relating the asymptotic structure of a Banach space to its infinite-dimensional subspace structure.

Recall that a classical result of James (cf. [LT]) asserts that a Banach space with an unconditional basis is either reflexive or has a subspace isomorphic to c_0 or ℓ_1. Using the envelope functions we prove the following asymptotic analog of this result for Banach spaces with asymptotic unconditional structure.

Theorem 3.6.1 *Let X be an infinite-dimensional Banach space with asymptotic unconditional structure. Then either $\ell_1^n \in \{X\}_n$ or $\ell_\infty^n \in \{X\}_n$ for all n or X contains an infinite-dimensional reflexive subspace.*

This result is an immediate consequence of Propositions 3.6.2 and 3.6.3 proved below. Also observe that the Proposition 2.4.1 implies that for every Banach space X the lower envelope g_X is equivalent to $\|.\|_\infty$ if and only if its power type is $q = \infty$, and the upper envelope r_X is equivalent to $\|.\|_1$ if and only if its power type is $p = 1$.

Recall that a Banach space with a basis $\{x_i\}$ is reflexive if and only if $\{x_i\}$ is both shrinking and boundedly complete. The property of shrinking is equivalent to the fact that for every $x^* \in X^*$, the norm of the restriction of x^* to $\text{span}[x_i]_{i=n}^\infty$ tends to zero as n tends to infinity. A basis $\{x_i\}$ is boundedly complete provided that whenever the sequence $\{\sum_{i=1}^n a_i x_i\}_{n=1}^\infty$ is bounded, then it is convergent (cf. [LT]).

The next proposition which is stated in terms of the power types of the envelope functions is a simple generalization of 4.2 of [MMT], where it was proved for stabilized asymptotic-ℓ_p spaces. In the proof, we shall make use of the stabilization result due to Milman and Tomczak-Jaegermann [MT1], which we recall now.

Let X be a Banach space and let $\mathcal{B}$ be a family satisfying the filtration conditions. There exists a subspace $Z \subset X$ with a basis $\{z_i\}$ such that

(i) for all $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $N = N(n, \varepsilon)$ such that for any normalized successive blocks $z_N < w_1 < \ldots < w_n$ of $\{z_i\}$ there exists $E \in \{X\}_n$ such that $\{w_1, \ldots, w_n\} \overset{1+\varepsilon}{\sim} E$.

(ii) for every $n \in \mathbb{N}$ and every space $E \in \{X\}_n$ the following is true for every $\varepsilon > 0$:

$$\forall M_1 \in \mathbb{N}\ \exists m_1 > M_1\ \ \forall M_2 > M_1\ \exists m_2 > M_2 \quad \ldots$$
$$\ldots\ \forall M_n > M_{n-1}\ \ \exists m_n > M_n \quad \{z_{m_1}, \ldots, z_{m_n}\} \overset{1+\varepsilon}{\sim} E.$$

Moreover, for $k \in \mathbb{N}$, the basis constant of $\{z_i\}_{i\geq k}$ is less than or equal to $1+\varepsilon_k$, for some sequence $\varepsilon_k \downarrow 0$.

We will call this subspace Z a *stabilizing subspace* of X. In the language of games, (i) means that the winning strategy of the subspace player **S** in a subspace game in Z is very simple. For every $\varepsilon > 0$ and $n \in \mathbb{N}$, **S** may always choose the same tail subspace $Z_N = \overline{\text{span}}\{z_i\}_{i\geq N}$, where $N = N(n, \varepsilon)$, regardless of the moves of **V**. And the part (ii) says that the winning strategy for the vector player **V** is also simple. For every $E \in \{X\}_n$ and $\varepsilon > 0$, the vector player **V** can restrict its moves only to basis vectors $\{z_i\}$ as his winning strategy.

Proposition 3.6.2 *Let X be a Banach space and let q and p be power types of g_X and r_X respectively. If $1 < p, q < \infty$, then X contains an infinite-dimensional reflexive subspace.*

Proof Let $Z \subset X$ be a stabilizing subspace with basis $\{z_i\}$. Fix $\varepsilon > 0$ for the rest of the proof. Then for all $n \in \mathbb{N}$ there exists $N = N(n, \varepsilon)$ such that any n successive blocks supported (with respect to $\{z_i\}$) after N are $(1+\varepsilon)$-equivalent to some asymptotic space $E \in \{X\}_n$.

We will show that $\{z_i\}$ is both shrinking and boundedly complete, hence Z is reflexive.

Since the power type p of r_X satisfies $p > 1$, by Proposition 2.4.1, for any fixed $p > r > 1$ there exists a constant C_r (which depends on r only) such that $r_X(a) \leq C_r\|a\|_r$ for all $a = (a_i) \in c_{00}$. This, in particular, implies that for all $n \in \mathbb{N}$ and $\{e_i\}_{i=1}^n \in \{X\}_n$, we have $\|\sum_{i=1}^n a_i e_i\| \leq C_r\|a\|_r$. Now if $\{z_i\}$ was not shrinking, then there exists $z^* \in Z^*$ with $\|z^*\| = 1$ and $\delta > 0$ for which there is a normalized block sequence $\{u_i\}_i$ of $\{z_i\}$ such that $z^*(u_i) \geq \delta$ for all $i = 1, 2, \ldots$. But for all $n \in \mathbb{N}$, there exists $N = N(n, \varepsilon)$ such that whenever $N < u_1 < \ldots u_n$, then $\{u_i\}_{i=1}^n$ is a permissible sequence. Hence,

$$C_r(1+\varepsilon)n^{1/r} \geq \|\sum_{i=1}^n u_i\| \geq z^*(\sum_{i=1}^n u_i) \geq n\delta.$$

Since $r > 1$, this is a contradiction for a large enough n. Therefore $\{z_i\}$ must be shrinking.

Also $\{z_i\}$ is boundedly complete. Indeed, since the power type q of g_X satisfies $q < \infty$, by Proposition 2.4.1, for any fixed $q < s < \infty$, there exists c_s such that $g_X(a) \geq c_s\|a\|_s$ for all $a \in c_{00}$. In particular, for all $n \in \mathbb{N}$ and $\{e_i\}_{i=1}^n \in \{X\}_n$, we have that $\|\sum_{i=1}^n a_i e_i\| \geq c_s\|a\|_s$. Suppose to the contrary that $\{z_i\}$ is not boundedly complete. Then there exists a normalized block basis $\{u_i\}$ of $\{z_i\}$ such that $\sup_n \|\sum_{i=1}^n u_i\| = M < \infty$. But for all n, there exists $N = N(n, \varepsilon)$ such that if k is such that $N < u_k$, then the sequence $N(n, \varepsilon) < u_{k+1} < \ldots < u_{k+n}$ is permissible. Hence,

$$2M \geq 2\|\sum_{i=1}^{k+n} u_i\| \geq \|\sum_{i=k+1}^{k+n} u_i\| \geq c_s(1+\varepsilon)n^{1/s}.$$

Since $s < \infty$, this is a contradiction for large enough n. Thus $\{z_i\}$ is boundedly complete, and the proof is completed. □

Proposition 3.6.3 *Let X be a Banach space with an asymptotic unconditional structure. Then*

(i) $g_X(.)$ is equivalent to $\|.\|_\infty$ if and only if for all n, $\ell_\infty^n \in \{X\}_n$,

(ii) $r_X(.)$ is equivalent to $\|.\|_1$ if and only if for all n, $\ell_1^n \in \{X\}_n$.

Proof The proof of part (i) is easy. If g_X is C-equivalent to $\|.\|_\infty$ for some $C > 0$, then for all $n \in \mathbb{N}$, there exists $\{e_i\}_{i=1}^n \in \{X\}_n$ such that $\|\sum_{i=1}^n e_i\| \leq 2g_X(1, 1, \ldots, 1) \leq 2C$. The unconditionality of the basis $\{e_i\}_{i=1}^n$ then implies that $\{e_i\}_{i=1}^n$ is $4C^2$-equivalent to the unit vector basis of ℓ_∞^n. The constant $4C^2$ is independent of n, and now by well known standard blocking argument of James, for all n and $\varepsilon > 0$, one can find blocks $\{x_i\}_{i=1}^n$ of $\{e_i\}_{i=1}^m$ for large enough m such that $\{x_i\}_{i=1}^n \overset{1+\varepsilon}{\sim} \ell_\infty^n$. Since $\{x_i\}_{i=1}^n \in \{X\}_n$, the result follows. The converse implication is trivial.

To prove the second part, assume that r_X is equivalent to $\|.\|_1$. By Proposition 2.4.1, necessarily $r_X(.) \overset{1}{\sim} \|.\|_1$. (However this is not important for the proof.) Fix $n \in \mathbb{N}$ and pick an asymptotic space $E \in \{X\}_n$ with the natural basis $\{e_i\}_{i=1}^n$ such that $\|\sum_{i=1}^n e_i\| \geq (1/2)r_X(1,\ldots,1) \geq n/2$. Pick $x^* \in E^*$ with $\|x^*\| = 1$ and $x^*(\sum_{i=1}^n e_i) = \|\sum_{i=1}^n e_i\|$. Consider the set $I = \{i : |x^*(e_i)| \geq 1/4\}$. Since $|x^*(e_i)| \leq 1$ for all i, a standard argument shows that the cardinality k of I satisfies $k = |I| \geq n/3$. For an arbitrary scalar sequence $a = (a_i)$, let $\varepsilon_i = \text{sgn} a_i x^*(e_i)$ for $i \in I$. Then

$$\|\sum_{i\in I} \varepsilon_i a_i e_i\| \geq x^*(\sum_{i\in I} \varepsilon_i a_i e_i) = \sum_{i\in I} |a_i||x^*(e_i)| \geq (1/4)\sum_{i\in I} |a_i|.$$

This shows that $\{e_i\}_{i\in I}$ is $4C$-equivalent to the unit vector basis in ℓ_1^k, by the unconditionality of the basis (with constant C). Clearly, a subsequence of the basis in asymptotic space spans an asymptotic space and again by James blocking argument we reduce the constant to $1+\varepsilon$. i.e., $\ell_1^k \in \{X\}_k$ and the result follows. □

It was a famous open problem in Banach space theory whether every infinite-dimensional Banach space contains a subspace which is either reflexive or isomorphic to c_o or ℓ_1. Gowers [G] solved this in the negative by constructing a counterexample. Gowers' example contains ℓ_1^n's uniformly. i.e., for all $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists a sequence $\{x_i\}_{i=1}^n$ in the space such that $\{x_i\}_{i=1}^n \overset{1+\varepsilon}{\sim} \ell_1^n$. Gowers then suggested the existence of infinite-dimensional Banach spaces which does not contain ℓ_1^n's uniformly and without infinite-dimensional reflexive subspaces (see also Q5 in [O]). The following consequence of Theorem 3.6.1 implies that if there is such a space X, then X cannot contain a subspace with asymptotic unconditional structure.

Corollary 3.6.4 *Let X be an infinite-dimensional Banach space which does not contain ℓ_1^n's uniformly. If X has asymptotic unconditional structure, then X contains an infinite-dimensional reflexive subspace.*

Indeed, it is well known and easy to see that if X does not contain ℓ_1^n's uniformly, it cannot contain ℓ_∞^n's uniformly either. In particular, X does not have asymptotic spaces isomorphic to ℓ_1^n or ℓ_∞^n for all n, so the conclusion follows from Theorem 3.6.1.

3.7 Finite Representability of Envelopes

Recall that we have shown in Proposition 3.6.3 that for a Banach space X with asymptotic unconditional structure, if g_X is equivalent to $\|.\|_\infty$ (respectively $r_X \sim \|.\|_1$), then $\ell_\infty^n \in \{X\}_n$ (respectively $\ell_1^n \in \{X\}_n$) for all $n \in \mathbb{N}$. An identical proof shows that the same remains true if we replace the envelope functions with the disjoint envelopes g_X^d and r_X^d (in this case of course $\ell_1^n, \ell_\infty^n \in \{X\}^d$).

A natural question, which we consider in this section, is whether for every Banach space X with asymptotic unconditional structure, $l_q^n, l_p^n \in \{X\}_n$ ($l_q^n, l_p^n \in \{X\}^d$) for all $n \in \mathbb{N}$, where q and p are the power types of g_X and r_X (g_X^d and r_X^d) respectively.

Quite remarkably, the disjoint envelopes case of the problem has an affirmative answer. Namely, we will prove the following theorem.

Theorem 3.7.1 *Let X be a Banach space with asymptotic unconditional structure. Let $1 \leq p \leq q \leq \infty$ be the power types of r_X^d and g_X^d respectively. Then $l_p^n, l_q^n \in \{X\}^d$ for all $n \in \mathbb{N}$.*

The proof of this theorem is non-trivial but the ideas were already known since the 70's. This theorem can be viewed as a 'disjoint-block' version of the classical Maurey-Pisier Theorem ([MP], see also [MS]). Such a 'disjoint-block' version was already proved by Milman and Sharir [MiS] in a different formulation. They have defined the notion of 'asymptotic block type and cotype' and showed, analogously to the Maurey-Pisier Theorem, that if q is the infimum of asymptotic block cotype and p is the supremum of asymptotic block type of the space X with an asymptotic unconditional structure, then ℓ_q and ℓ_p are 'disjointly' block finitely representable in X.

Although they make use of different notions, Theorem 3.7.1 in spirit is equivalent to Milman-Sharir's result. However, our proof here, which is based on a recent presentation of the proof of the Maurey-Pisier Theorem given by Maurey [M], is somewhat shorter than that of [MiS].

An important ingredient of the proof, as in the proof of the Maurey-Pisier theorem, is Krivine's theorem. We will use the following statement which is actually a corollary of Krivine's theorem, as stated in [M].

Krivine's Theorem *Let $r, s \geq 1$, let X be a Banach space. Suppose that for some $\kappa > 0$ and for every $n \geq 2$, X contains a normalized (suppression) unconditional sequence $\mathbf{y}^{(n)} = (y_1^{(n)}, \ldots, y_n^{(n)})$ such that*

$$\|\sum_{i \in C} y_i^{(n)}\| \geq \kappa |C|^{1/r}$$

for every subset $C \subset \{1, \ldots, n\}$, or such that

$$\|\sum_{i \in C} y_i^{(n)}\| \leq \kappa |C|^{1/s}$$

for every subset $C \subset \{1, \ldots, n\}$. Then for some $p \leq r$ (or $p \geq s$) and for every $k \geq 1, \varepsilon > 0$, there is $N(k, \varepsilon)$ such that whenever $n \geq N(k, \varepsilon)$, it is possible to form k successive blocks of $\mathbf{y}^{(n)}$ that are $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^k.

Proof of Theorem 3.7.1 Without loss of generality we will assume that the asymptotic unconditionality constant is $C = 1$ (in the general case the estimates in the proof should be multiplied by C).

g_X^d Case. Let q be the power type of g_X^d. If $q = 1$, since $g_X^d \leq g_X$, then the power type of g_X is also equal to 1. Thus it follows immediately from Krivine's theorem that $\ell_1^n \in \{X\}_n$ for all n. (This fact does not require the asymptotic unconditionality assumption.)

Now suppose that $q > 1$. Let $1 < s < q$ and for all $n \in \mathbb{N}$, let $\phi(n)$ be the smallest real number for which

$$\sum_{i=1}^{n} |a_i|^s \le \phi(n)^s \| \sum_{i=1}^{n} a_i x_i \|^s$$

for all $\{x_i\}_{i=1}^n \in \{X\}^d$ and scalars $\{a_i\}$.

Since the power type of g_X^d is q and $s < q$, it follows that ϕ is not bounded as a function of n, and it is easy to see that it is increasing.

We will refer to the following argument as 'the exhaustion' argument.

Fix $0 < \varepsilon < 1/2$ and pick $\{x_i\}_{i=1}^n \in \{X\}^d$ and scalars $\{a_i\}$ such that $\sum_{i=1}^n |a_i|^s = 1$ and

$$1 > (1-\varepsilon)\phi(n)^s \| \sum_{i=1}^{n} a_i x_i \|^s. \tag{3.6}$$

Let $(B_\alpha)_{\alpha \in I}$ be a maximal family of mutually disjoint subsets of $\{1, 2, \ldots, n\}$, possibly empty, such that

$$\sum_{i \in B_\alpha} |a_i|^s \le \varepsilon \| \sum_{i \in B_\alpha} a_i x_i \|^s. \tag{3.7}$$

Let B be the union of the sets B_α, $B = \bigcup_{\alpha \in I} B_\alpha$, and let m be the cardinality of the index set I (note that $m < n$ because $|B_\alpha| > 1$). Then

$$\begin{aligned} \sum_{i \in B} |a_i|^s &= \sum_{\alpha \in I} \sum_{i \in B_\alpha} |a_i|^s \le \sum_{\alpha \in I} \varepsilon \| \sum_{i \in B_\alpha} a_i x_i \|^s \\ &\le \varepsilon \phi(m)^s \| \sum_{\alpha \in I} \sum_{i \in B_\alpha} a_i x_i \|^s \le \varepsilon \phi(n)^s \| \sum_{i=1}^{n} a_i x_i \|^s, \end{aligned} \tag{3.8}$$

here the second inequality uses the definition of $\phi(m)$ applied to vectors $\{u_\alpha\}_{\alpha=1}^m \in \{X\}^d$, where $u_\alpha = \frac{\sum_{i \in B_\alpha} a_i x_i}{\| \sum_{i \in B_\alpha} a_i x_i \|}$ for all $\alpha \in I$, and the last inequality uses the unconditionality of $\{x_i\}$ and the fact that $\phi(m) \le \phi(n)$.

Let A denote the complement of B and for every $j \geq 0$ let

$$A_j = \{i \in A : 2^{-j-1} < |a_i| \leq 2^{-j}\}.$$

Then from (3.6) and (3.8) it follows that

$$\sum_{i\in A} |a_i|^s > (1-2\varepsilon)\phi(n)^s \|\sum_{i=1}^{n} a_i x_i\|^s. \tag{3.9}$$

Let j_1 be the smallest $j \geq 0$ such that A_j is non-empty, and let $k = |A_{j_0}|$ be the cardinality of the largest set A_{j_0} among all A_j's. Then by (3.9),

$$\begin{aligned} k\sum_{j=j_1}^{\infty} 2^{-js} &\geq \sum_{j=j_1}^{\infty} 2^{-js}|A_j| \geq \sum_{i\in A} |a_i|^s \\ &> (1-2\varepsilon)\phi(n)^s \|\sum_{i=1}^{n} a_i x_i\|^s \geq (1-2\varepsilon)\phi(n)^s 2^{-j_1 s - s}. \end{aligned}$$

This shows that k is large when $\phi(n)$ is large. i.e., since $\phi(n)$ increases to infinity with n, so does k.

Now by maximality of B,

$$\sum_{i\in C} |a_i|^s > \varepsilon \|\sum_{i\in C} a_i x_i\|^s,$$

for every non-empty subset $C \subset A_{j_0}$. Since $2^{-j_0-1} < |a_i| \leq 2^{-j_0}$ for every $i \in A_{j_0}$, it follows that

$$\|\sum_{i\in C} x_i\| \leq 2(1/\varepsilon)^{1/s}|C|^{1/s} \leq 2(1/\varepsilon)|C|^{1/s},$$

for all $C \subset A_{j_0}$.

Therefore we have obtained that there exists a constant $\kappa = 2(1/\varepsilon)$ such that for all $k \in \mathbb{N}$ there exists $\{x_i\}_{i=1}^{k} \in \{X\}^d$ such that $\|\sum_{i\in C} x_i\| \leq \kappa |C|^{1/s}$ for all $C \subset \{1, \ldots, k\}$ and $s < q$.

Now by Krivine's theorem, there is $q' \geq s$ such that $\ell_{q'}^n \in \{X\}^d$ for all n. But since $s < q$ was arbitrary and q is the power type of g_X^d, it follows that $q' = q$, hence the proof of this case is completed.

r_X^d *Case.* The proof of this case is similar but there are slight differences.

Let p be the power type of r_X^d. If $p = \infty$, since $r_X \leq r_X^d$, then the power type of r_X is also equal to infinity. Again it follows immediately from Krivine's theorem that $\ell_\infty^n \in \{X\}_n$ for all n. (This does not require the asymptotic unconditionality assumption.)

Now suppose that $p < \infty$, and fix $p < r$. For each $n \geq 1$, let $\psi(n)$ be the smallest constant such that

$$\|\sum_{i=1}^n a_i x_i\|^r \leq \psi(n)^r \sum_{i=1}^n |a_i|^r,$$

for all $\{x_i\}_{i=1}^n \in \{X\}^d$ and scalars $\{a_i\}$.

Since the power type of r_X^d is p and $p < r$, it follows that $\psi(n)$ increases to infinity.

Fix $0 < \varepsilon < 1/2$ and pick $\{x_i\}_{i=1}^n \in \{X\}^d$ and scalars $\{a_i\}$ such that $\sum_{i=1}^n |a_i|^r = 1$ and

$$\|\sum_{i=1}^n a_i x_i\|^r > (1-\varepsilon)\psi(n)^r. \tag{3.10}$$

Let $(B_\alpha)_{\alpha \in I}$ be a maximal family of mutually disjoint subsets of $\{1, 2, \ldots, n\}$ such that

$$\|\sum_{i=1}^n a_i x_i\|^r \leq \varepsilon \sum_{i \in B_\alpha} |a_i|^r. \tag{3.11}$$

Let B be the union of the sets B_α, $B = \dot{\bigcup}_{\alpha \in I} B_\alpha$, and m be the cardinality of the index set I. Then

$$\begin{aligned}
\|\sum_{i\in B} a_i x_i\|^r &= \|\sum_{\alpha\in I}\sum_{i\in B_\alpha} a_i x_i\|^r \\
&\leq \psi(m)^r \sum_{\alpha\in I} \|\sum_{i\in B_\alpha} a_i x_i\|^r \qquad (3.12) \\
&\leq \varepsilon\psi(m)^r \sum_{\alpha\in I}\sum_{i\in B_\alpha} |a_i|^r \leq \varepsilon\psi(n)^r.
\end{aligned}$$

Let A denote the complement of B and for every $j \geq 0$ let

$$A_j = \{i \in A : 2^{-j-1} < |a_i| \leq 2^{-j}\}.$$

Then $A = \bigcup_{j=0}^{\infty} A_j$ because $\sum_{i=1}^{n} |a_i|^r = 1$. Let $k = \max_j |A_j|$ denote the maximal cardinality of the sets $\{A_j\}_{j\geq 0}$. Then,

$$\begin{aligned}
\|\sum_{i\in A} a_i x_i\| &= \|\sum_{j=0}^{\infty}\sum_{i\in A_j} a_i x_i\| \\
&\leq \sum_{j=0}^{\infty} \|\sum_{i\in A_j} a_i x_i\| \leq k\sum_{j=0}^{\infty} 2^{-j} = 2k. \qquad (3.13)
\end{aligned}$$

Hence, using (3.10), (3.12) and (3.13), we obtain

$$\begin{aligned}
(1-\varepsilon)^{1/r}\psi(n) &< \|\sum_{i=1}^{n} a_i x_i\| \\
&\leq \|\sum_{i\in B} a_i x_i\| + \|\sum_{i\in A} a_i x_i\| \\
&\leq \varepsilon^{1/r}\psi(n) + 2k,
\end{aligned}$$

which shows that k is big when $\psi(n)$ is big. Let j_0 be such that $|A_{j_0}| = k$. By maximality of B we obtain that for every non-empty subset C of A_{j_0}, we have

$$\|\sum_{i\in C} a_i x_i\|^r > \varepsilon \sum_{i\in C} |a_i|^r \geq \varepsilon 2^{-(j_0+1)r}|C|.$$

It follows that

$$\|\sum_{i\in C} x_i\| \geq (1/2)\varepsilon^{1/r}|C|^{1/r}.$$

Since we can find such vectors $\{x_i\}_{i=1}^k \in \{X\}^d$ for all $k \in \mathbb{N}$, the result follows again from Krivine's theorem. i.e., $\ell_p^n \in \{X\}^d$ for all $n \in \mathbb{N}$.

□

We now give the proof the remaining part of Proposition 3.2.4, as was promised in section 3.2.

Proof of Proposition 3.2.4 (g_X^d *Case*) We have already shown using the super-multiplicativity of g_X^d that there exists $1 \leq q \leq \infty$ such that for all $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that for all n, we have

$$c_\varepsilon n^{1/q+\varepsilon} \leq g_X^d\Big(\sum_{i=1}^n e_i\Big) \leq n^{1/q}.$$

We first show the lower estimate for the envelope, i.e., for all $\varepsilon > 0$ there exists c'_ε such that $g_X^d(a) \geq c'_\varepsilon \|a\|_{q+\varepsilon}$.

For every $\varepsilon > 0$ and $n \in \mathbb{N}$, let $\phi_\varepsilon(n)$ be the smallest constant such that

$$\|a\|_{q+\varepsilon} \leq \phi_\varepsilon(n) g_X^d\Big(\sum_{i=1}^n a_i x_i\Big),$$

for all disjointly supported vectors $\{x_i\}_{i=1}^n$ such that $g_X^d(x_i) = 1$ for all i, and scalars $a \in c_{00}$.

If for every $\varepsilon > 0$, $\sup_n \phi_\varepsilon(n) < \infty$, then there is nothing to prove.

Suppose that for some $\varepsilon_0 > 0$, $\sup_n \phi_{\varepsilon_0}(n) = \infty$. Then, it follows from the exhaustion argument as in the proof of Theorem 3.7.1 (g_X^d Case) that there exists a constant $\kappa > 0$ such that for all n, there exists disjointly supported vectors $\{x_i\}_{i=1}^n$ such that $g_X^d(x_i) = 1$ for all i, and

$$g_X^d\Big(\sum_{i=1}^n x_i\Big) \leq \kappa n^{1/q+\varepsilon_0}.$$

Now fix $\varepsilon_1 < \varepsilon_0$. Then, there exists c_{ε_1} such that for all n, we have

$$c_{\varepsilon_1} n^{1/q+\varepsilon_1} \le g_X^d\Big(\sum_{i=1}^n e_i\Big) \le g_X^d\Big(\sum_{i=1}^n x_i\Big) \le \kappa n^{1/q+\varepsilon_0}.$$

When n is large enough, this is a contradiction. Therefore, for every $\varepsilon > 0$, there exists $1/c'_\varepsilon = \sup_n \phi_\varepsilon(n) < \infty$ such that $g_X^d(a) \ge c'_\varepsilon \|a\|_{q+\varepsilon}$, as desired.

For the upper estimate, note that by Theorem 3.7.1, $\ell_q^n \in \{X\}^d$ for all $n \in \mathbb{N}$. This immediately implies that $g_X^d(a) \le \|a\|_q$, for all $a \in c_{00}$. The proof is now completed. $\square$

We end this section with a few remarks concerning the finite representability problem for the (original) envelope functions case.

First, we observe that the answer to this problem is negative in general. For instance, if (e_i) is the summing basis for $X = c_0$, then r_X is equivalent to $\|.\|_1$, where the asymptotic structure is with respect to the summing basis (e_i), but $\ell_1^n \notin \{X\}_n$ for all n. Moreover, a non-reflexive Banach space X constructed in [KOS], Example 6.4, has the property that for all n, there exists $\{e_i\}_{i=1}^n \in \{X\}_n$ such that $\|\sum_{i=1}^n e_i\| = 1$, in particular, $g_X \sim \|.\|_\infty$, and yet c_0 is not block finitely representable in X, in particular, $\ell_\infty^n \notin \{X\}_n$ for all n.

In these examples, the asymptotic structures are (necessarily) not unconditional. As it has been shown in Proposition 3.6.3, for a Banach space X with asymptotic unconditional structure, if the power type of g_X is $q = \infty$ (respectively the power type of r_X is $p = 1$), then $\ell_\infty^n \in \{X\}_n$ (resp. $\ell_1^n \in \{X\}_n$) for all n. Also, if $q = 1$ (resp. $p = \infty$), then regardless of the asymptotic unconditionality assumption, Krivine's theorem yields that $\ell_1^n \in \{X\}_n$ (resp. $\ell_\infty^n \in \{X\}_n$) for all n.

It is likely that there are also examples of Banach spaces with asymptotic unconditional structure with power types of the envelopes satisfying $1 < p, q < \infty$ and yet $\ell_p^n, \ell_q^n \notin \{X\}_n$. However we do not know how to construct such examples. We do not even know if there are such spaces without asymptotic unconditional structure. For instance, for every $1 < p < \infty$, one can define a new norm in a natural way on the space X of [KOS] mentioned above to obtain reflexive X_p spaces without asymptotic unconditional structure. However, these natural 'p-versions' of X do not seem to provide such examples. Finally, we do not know if reflexivity plays a role in this problem. This question was raised in [KOS], Problem 6.5.

Part II

Spreading Models of Orlicz Sequence Spaces

Chapter 4

The Structure of The Set of Spreading Models of Orlicz Sequence Spaces

4.1 Introduction

It is a well known consequence of Ramsey theorem that for every normalized basic sequence (y_i) in a Banach space X and for every $(\varepsilon_n) \searrow 0$ there exists a subsequence (x_i) of (y_i) and a normalized basic sequence $(\tilde{x}_i)$ in some Banach space $\tilde{X}$ such that: For all $n \in \mathbb{N}$, $(a_i)_{i=1}^n \in [-1,1]^n$ and $n \leq k_1 < \ldots < k_n$

$$\left| \|\sum_{i=1}^n a_i x_{k_i}\| - \|\sum_{i=1}^n a_i \tilde{x}_i\| \right| < \varepsilon_n.$$

The sequence $(\tilde{x}_i)$ is called *the spreading model* of (x_i) (or a spreading model of X) and it is a suppression 1-unconditional basic sequence if (y_i) is weakly null. The subsequence (x_i) is called *a good subsequence* of (y_i) which generates the spreading model $(\tilde{x}_i)$ and it has the property that every further subsequence of (x_i) generates the same spreading model $(\tilde{x}_i)$. However (y_i) might have many good subsequences, each generating a different spreading model.

In general, a spreading model of a Banach space X behaves better than X. For example, it is shown in [R] that one can always find a 1-unconditional spreading model of X. It is well known, however, that unconditional basic sequences need not exist in X [GM]. On the other hand, some of the classical conjectures which have been proved to fail for an arbitrary Banach space X fail for spreading models as well. There are examples of Banach spaces for which no spreading model $(\tilde{x}_i)$ is equivalent to unit vector basis of c_0 or ℓ_p for some $1 \leq p < \infty$ [OS2]. It is not true even that every Banach space X admits a spreading model which is either isomorphic to c_0 or ℓ_1 or is reflexive [AOST].

In a more general context, given a Banach space X, it is also of interest to study the set (or a particular subset) of all spreading models of X. One such approach due to Androulakis, Odell, Schlumprecht and Tomczak-Jaegermann [AOST] is following. Consider the set $SP_\omega(X)$, the partially ordered set of all spreading models $(\tilde{x}_i)$ generated by normalized weakly null sequences in X. The partial order is defined by domination: $(\tilde{x}_i) \geq (\tilde{y}_i)$ if for some $K < \infty$, $K\|\sum_i a_i\tilde{x}_i\| \geq \|\sum_i a_i\tilde{y}_i\|$ for all scalars (a_i). And identify $(\tilde{x}_i)$ and $(\tilde{y}_i)$ in $SP_\omega(X)$ if $(\tilde{x}_i) \geq (\tilde{y}_i)$ and $(\tilde{y}_i) \geq (\tilde{x}_i)$. What can be said about the structure of the partially ordered set $SP_\omega(X)$?

The following theorem proved in [AOST] says that every countable subset of $SP_\omega(X)$ admits an upper bound in $SP_\omega(X)$.

Theorem 4.1.1 (AOST) *Let $(C_n) \subset (0, \infty)$ such that $\sum_n C_n^{-1} < \infty$ and let X be a Banach space. For all $n \in \mathbb{N}$, let $(x_i^n)_i$ be a normalized weakly null sequence in X having spreading model $(\tilde{x}_i^n)_i$. Then there exists a semi-normalized weakly null basic sequence (y_i) in X such that $(\tilde{y}_i)$ C_n-dominates $(\tilde{x}_i^n)_i$ for all $n \in \mathbb{N}$.*

The purpose of this chapter is to study the structure of the set $SP_\omega(X)$ when X is an Orlicz sequence space. In this case the above quoted theorem takes a simple form and it is particularly well illustrated. One of our main observation is the following. If an Orlicz sequence space X admits a spreading model $(\tilde{x}_i)$ which dominates (but is not equivalent to) the (symmetric) unit vector basis of X, then $SP_\omega(X)$ contains an uncountable increasing chain. As a consequence, we give a description of the structure of the set of spreading models of reflexive Orlicz sequence spaces X which have only countably many mutually non-equivalent spreading models. We show that in this case the set $SP_\omega(X)$ has a very special form: it contains both the upper and the lower bounds and moreover the upper bound is the space X itself and the lower bound is some ℓ_p space.

4.2 Preliminaries in Orlicz Sequence Spaces

We recall the basics of Orlicz sequence spaces following the book [LT], with which our notation is consistent.

An Orlicz function M is a real valued continuous non-decreasing and convex function defined for $t \geq 0$ such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. If $M(t) = 0$ for some $t > 0$, M is said to be a degenerate function.

To any Orlicz function M we associate the space ℓ_M of all sequences of scalars $x = (a_1, a_2, \ldots)$ such that $\sum_{n=1}^{\infty} M(|a_n|/\rho) < \infty$ for some $\rho > 0$. The space ℓ_M is equipped with the norm

$$\|x\| = \inf\{\rho > 0 : \sum_{n=1}^{\infty} M(|a_n|/\rho) \leq 1\},$$

which makes ℓ_M into a Banach space called an Orlicz sequence space.

The subspace h_M of ℓ_M consisting of those sequences $x = (a_1, a_2, \ldots) \in \ell_M$ for which $\sum_{n=1}^{\infty} M(|a_n|/\rho) < \infty$ *for every* $\rho > 0$ is closed and the unit vectors $\{e_n\}_{n=1}^{\infty}$ form a symmetric basis of h_M.

It is easy to verify that if M is a degenerate Orlicz function then $\ell_M \simeq \ell_\infty$ and $h_M \simeq c_0$. Since we will not be interested in these spaces, all the Orlicz functions appearing in this chapter will be assumed to be non-degenerate, unless otherwise stated.

An Orlicz function M is said to satisfy Δ_2-condition at zero if

$$\limsup_{t\to 0} \frac{M(2t)}{M(t)} < \infty.$$

It is easily checked that the Δ_2-condition at zero implies that, for every positive number Q, $\lim_{t\to 0}\sup \frac{M(Qt)}{M(t)} < \infty$ (this condition is called the Δ_Q-condition).

Some other conditions, each of which is equivalent to Δ_2-condition (Proposition 4.a.4, [LT]), are :

a) $\ell_M = h_M$

b) ℓ_M does not contain a subspace isomorphic to ℓ_∞

c) The unit vectors form a boundedly complete symmetric basis of ℓ_M.

In particular, if ℓ_M (or h_M) is reflexive, then M satisfies Δ_2-condition.

Two Orlicz functions M_1 and M_2 are *equivalent at zero* if there exist positive constants K, k, t_0 such that

$$K^{-1}M_2(k^{-1}t) \le M_1(t) \le KM_2(kt)$$

for all $0 < t \le t_0$. When M_1 or M_2 satisfies Δ_2-condition then they are equivalent (at zero) if there exist constants $K > 0$ and $t_0 > 0$ such that $K^{-1} \le M_1(t)/M_2(t) \le K$ for all $0 < t \le t_0$. This is the case if and only if ℓ_{M_1} and ℓ_{M_2} consist of the same sequences i.e. the unit vector bases in ℓ_{M_1} and ℓ_{M_2} are equivalent.

For an Orlicz function M consider the following subsets of the Banach space $C(0, \frac{1}{2})$ of all real valued continuous functions on $(0, \frac{1}{2})$;

$$\begin{aligned} E_{M,\Lambda} &= \overline{\left\{\frac{M(\lambda t)}{M(\lambda)};\ 0<\lambda<\Lambda\right\}}, \quad E_M = \bigcap_{0<\Lambda} E_{M,\Lambda} \\ C_{M,1} &= \overline{\text{conv}}E_{M,1} \ \text{ and } \quad C_M = \overline{\text{conv}}E_M \end{aligned}$$

where the closure is taken in the norm topology of $C(0, \frac{1}{2})$. Then $E_{M,1}$, E_M,

$C_{M,1}$ and C_M are non-empty norm compact subsets of $C(0, \frac{1}{2})$ consisting entirely of Orlicz functions (Lemma 4.a.6, [LT]).

The importance of these sets is due to the following result (Proposition 4.a.7 and Theorem 4.a.8, [LT]).

Theorem 4.2.1 *For every Orlicz function M the following assertions are true.*

i) Every infinite-dimensional subspace Y of h_M contains a closed subspace Z which is isomorphic to some Orlicz sequence space h_N.

ii) Let X be a subspace of h_M which has a subsymmetric basis $\{x_i\}$. Then X is isomorphic to some Orlicz sequence space h_N and $\{x_i\}$ is equivalent to the unit vector basis of h_N. Moreover the function N belongs to the set $C_{M,1}$.

iii) An Orlicz sequence space h_N is isomorphic to a subspace of h_M if and only if N is equivalent to some function in $C_{M,1}$.

By (ii) of the above theorem, every subsymmetric basic sequence in an Orlicz sequence space is symmetric.

Finally we recall that every Orlicz sequence space h_M contains isomorphic copies of some ℓ_p or c_0. Moreover the set of p's for which ℓ_p is contained in h_M is a closed interval (Theorem 4.a.9, [LT]).

4.3 Spreading Models of Orlicz Sequence Spaces

By Theorem 4.2.1, the set $C_{M,1}$ 'coincides' (i.e. there is a one-to-one correspondence) with the collection of all subspaces of h_M which have a subsymmetric (or a symmetric) basis. The following proposition shows that the collection $SP_\omega(h_M)$ of all spreading models of h_M generated by weakly null basic sequences is also 'contained' in the set $C_{M,1}$. The proof is a simple generalization of the argument given in [LT] (Proposition 4.a.7).

Proposition 4.3.1 *Let M be an Orlicz function. Let $(\tilde{x}_i)$ be a spreading model generated by a weakly null sequence (x_i) in h_M. Then there exists $N \in C_{M,1}$*

such that $(\tilde{x}_i)$ is equivalent to the unit vector basis of h_N. Moreover, $(\tilde{x}_i)$ is equivalent to a subsequence of (x_i).

Proof Let (y_i) be the good subsequence of (x_i) which generates $(\tilde{x}_i)$. Since (x_i) (and hence (y_i)) is weakly null by passing to a further subsequence if necessary we can assume that (y_i) is a block basic sequence of the unit vector basis of h_M.

For each $i = 1, 2, \ldots$ let $y_i = \sum_{l=n_{i-1}+1}^{n_i} c_l e_l$. To every vector y_i we associate the function $M_i(t) = \sum_{l=n_{i-1}+1}^{n_i} M(|c_l|t)$. Since y_i is normalized, $\sum_{l=n_{i-1}+1}^{n_i} M(|c_l|) = 1$ and hence the functions $\{M_i\}_{i=1}^{\infty}$, as elements of $C(0, \frac{1}{2})$, belong to the set $C_{M,1}$.

Now by the norm compactness of $C_{M,1}$ (in $C(0, \frac{1}{2})$), there exists a subsequence $\{M_{i_n}\}_{n=1}^{\infty}$ of $\{M_i\}$ and an Orlicz function $N \in C_{M,1}$, which might be degenerate, so that $|M_{i_n}(t) - N(t)| \leq 2^{-n}$ for $0 \leq t \leq 1/2$ and $n = 1, 2, \ldots$. Assume for simplicity of notation that the subsequence $\{M_{i_n}\}_{n=1}^{\infty}$ coincides with the whole sequence $\{M_i\}$.

Now for any $a = (a_i)_{i=1}^{m} \in c_{00}$, we have

$$\begin{aligned}
\Big\|\sum_{i=1}^{m} a_i \tilde{x}_i\Big\| &= \lim_{k_1\to\infty} \cdots \lim_{k_m\to\infty} \Big\|\sum_{i=1}^{m} a_i y_{k_i}\Big\| \\
&= \lim_{k_1\to\infty} \cdots \lim_{k_m\to\infty} \inf\Big\{\rho : \sum_{i=1}^{m} M_{k_i}(|a_i|/\rho) \leq 1\Big\} \\
&= \inf\Big\{\rho : \sum_{i=1}^{m} N(|a_i|/\rho) \leq 1\Big\} \\
&= \Big\|\sum_{i=1}^{m} a_i e_i\Big\|_{h_N}
\end{aligned}$$

Moreover, the above argument yields that $(\tilde{x}_i)$ is actually equivalent to a subsequence of (x_i). Indeed, since $|M_{i_n}(t) - N(t)| \leq 2^{-n}$ for $0 \leq t \leq 1/2$ and $n = 1, 2, \ldots$, it follows that $\sum_{n=1}^{\infty} M_{i_n}(|a_n|) < \infty$ if and only if $\sum_{n=1}^{\infty} N(|a_n|) < \infty$, provided that N is non-degenerate. Hence the corresponding subsequence (y_{i_n}) is equivalent to unit vector basis of h_N (Proposition 4.a.7, [LT]). If $N(t) = 0$ for some $t > 0$, then (z_n) is equivalent to unit vector basis of c_0 which, in this case, is isomorphic to h_N. □

Obviously, by Theorem 4.2.1, for every $N \in C_{M,1}$, h_N is a spreading model of h_M. Hence, with some abuse of notation, we can write

$$SP_\omega(h_M) \subset C_{M,1} \subset SP(h_M),$$

where $SP(h_M)$ denotes the set of all spreading models of h_M.

Proposition 4.3.2 *Let M_1 and M_2 be two Orlicz functions. Then the unit vector basis of h_{M_1} dominates the unit vector basis of h_{M_2} if and only if there exist constants $K > 0, k > 0$ and $t_0 > 0$ such that $M_2(t) \leq KM_1(kt)$ for all $0 < t \leq t_0$.*

Proof Suppose that the unit vector basis of h_{M_1} dominates the unit vector basis of h_{M_2}. Let $\|\sum_{i=1}^n e_i\|_{h_{M_1}} = \rho_n$. We may assume that $\rho_n \nearrow \infty$. Indeed, otherwise both h_{M_1} and h_{M_2} are isomorphic to c_0 and the conclusion is trivial. By assumption, in particular, there exists a constant $K \geq 1$ such that $\|\sum_{i=1}^n e_i\|_{h_{M_2}} \leq K\|\sum_{i=1}^n e_i\|_{h_{M_1}}$ for all $n \in N$. Then by definition of the norms, $M_2(1/\rho_n) \leq KM_1(1/\rho_n)$ for all $n \in N$. Now let $0 < t \leq 1$ be arbitrary and suppose that for some n we have $1/\rho_{n+1} < t \leq 1/\rho_n$. Also let $k = \sup_n \rho_{n+1}/\rho_n$ (note that $k \leq 2$). Hence $M_2(t) \leq M_2(1/\rho_n) \leq KM_1(1/\rho_n) \leq KM_1(kt)$.

Conversely, suppose that $M_2(t) \leq KM_1(kt)$ for all $0 < t \leq t_0$. Let $(a_1, a_2, \ldots)$ be an arbitrary scalar sequence such that $|a_i| \leq t_0$. Then

$$\begin{aligned} k\|\sum_i a_i e_i\|_{h_{M_2}} &= \inf\{\rho : \sum_i M_2(k|a_i|/\rho) \leq 1\} \\ &\leq \inf\{\rho : \sum_i M_1(|a_i|/\rho) \leq 1/K\} \\ &= K\|\sum_i a_i e_i\|_{h_{M_1}} \end{aligned}$$

Moreover the assumption $|a_i| \leq t_0$ is not a restriction. Since the inequality we have just proved is homogeneous, by rescaling we can always assume that $|a_i| \leq t_0$. □

Definition 4.3.3 *Let N_1 and N_2 be two Orlicz functions. We say that N_1 dominates N_2 and denote by $N_2 \leq N_1$ if there exist constants $K > 0, k > 0$ and $t_0 > 0$ such that $N_2(t) \leq KN_1(kt)$ for all $0 < t \leq t_0$. We write $N_2 < N_1$ if $N_2 \leq N_1$ but $N_1 \not\leq N_2$.*

We shall occasionally also write $N_1 \geq N_2$ instead of $N_2 \leq N_1$. Obviously, $N_2 \leq N_1$ and $N_1 \leq N_2$ means that N_1 is equivalent to N_2. Hence by Proposition 4.3.2, we have

$$N_2 \leq N_1 \text{ if and only if } h_{N_2} \leq h_{N_1},$$

where the latter relation means that the unit vector basis of h_{N_1} dominates the unit vector basis of h_{N_2}.

As mentioned earlier, it is shown in [AOST] that for an arbitrary Banach space X every countable subset of $SP_\omega(X)$ admits an upper bound in $SP_\omega(X)$. When X is an Orlicz sequence space, the corresponding result becomes an easy observation. Before stating this result we need the following easy lemma which will be used in the sequel.

Lemma 4.3.4 *Let M be an Orlicz function. The unit vector basis (e_i) of h_M is weakly null if and only if h_M is not isomorphic to ℓ_1 if and only if $\lim_{t\to 0} M(t)/t = 0$. In particular, $h_N \in SP_\omega(h_M)$ if and only if $N \in C_{M,1}$ and $\lim_{t\to 0} N(t)/t = 0$.*

Proof The first equivalence follows from standard known results; if h_M is isomorphic to ℓ_1, since ℓ_1 has unique symmetric basis, then the unit vector basis (e_i) of h_M is equivalent to the unit vector basis of ℓ_1 and hence it is not weakly null. Moreover, if (e_i) is not weakly null, since it is symmetric, it is equivalent to the unit vector basis of ℓ_1 (cf. Proposition 3.b.5, [LT]).

For the second equivalence, first we note that for every Orlicz function M, $\lim_{t\to 0} M(t)/t$ exists. This follows from the fact that the function $M(t)/t$ is monotone. Indeed, by convexity of M, for all $0 < t < s$, we have $M(t) \leq (t/s)M(s) + (1 - t/s)M(0) = (t/s)M(s)$. i.e., $M(t)/t \leq M(s)/s$.

Moreover, for all n, by definition of the norm of h_M, we have

$$\frac{\|\sum_{i=1}^n e_i\|_{h_M}}{n} = \frac{1}{nM^{-1}(1/n)} = \frac{M(t_n)}{t_n},$$

where M^{-1} is the inverse function of M and for all n, $M^{-1}(1/n) = t_n$. (Note also that t_n tends to zero.) It follows that, $\lim_{n\to\infty} \|\sum_{i=1}^n e_i\|_{h_M}/n$ exists as well. Now recall the well known fact that a subsymmetric unconditional basis (y_i) is equivalent to the unit vector basis of ℓ_1 if and only if $\lim_{n\to\infty} \|\sum_{i=1}^n y_i\|/n > 0$ (cf. [BL]). Since the unit vector basis (e_i) of h_M is symmetric, in particular, it is subsymmetric, consequently it follows that the unit vector basis (e_i) of h_M is not equivalent to the unit vector basis of ℓ_1 if and only if $\lim_{n\to\infty} \|\sum_{i=1}^n e_i\|_{h_M}/n = 0$ if and only if $\lim_{t\to 0} M(t)/t = 0$.

Finally, if $h_N \in SP_\omega(h_M)$, then by the remark following Proposition 4.3.1 the unit vector basis of h_N is equivalent to a subsequence of the generating weakly null basic sequence in h_M, therefore it is weakly null and by above $\lim_{t\to 0} N(t)/t = 0$.

□

Remark It follows from the above Lemma and the remark following Proposition 4.3.1 that if an Orlicz sequence space h_M does not contain an isomorphic copy of ℓ_1, then the sets $SP_\omega(h_M)$ and $C_{M,1}$ coincide. i.e., $SP_\omega(h_M) = C_{M,1}$.

Proposition 4.3.5 *Let M be an Orlicz function. Suppose that $h_{N_1}, h_{N_2}, \ldots \in SP_\omega(h_M)$. Then there exists $h_{N_0} \in SP_\omega(h_M)$ such that h_{N_0} dominates h_{N_i} for every $i \in \mathbb{N}$.*

Proof By Lemma 4.3.4, $N_1, N_2, \ldots \in C_{M,1}$ and $\lim_{t\to 0} N_i(t)/t = 0$ for all i. Define $N_0(t) = \sum_{i=1}^\infty 2^{-i} N_i(t)$, then clearly $N_0 \in C_{M,1}$. For every $i \in \mathbb{N}$, $N_0(t) \geq 2^{-i} N_i(t)$ for all $t > 0$. Hence h_{N_0} dominates h_{N_i} for every $i \in \mathbb{N}$. It remains to show that $\lim_{t\to 0} N_0(t)/t = 0$.

By uniform boundedness of $C_{M,1}$ in $C(0, \frac{1}{2})$, the sequence $\{N_i\} \subset C_{M,1}$ is uniformly bounded on $(0, 1/2)$ (in fact it is bounded by 1, cf. Lemma 4.a.6

of [LT]). i.e., $\sup_i N_i(t) \leq 1$ on $(0,1/2)$. In particular, since, as in Lemma 4.3.4, $N_i(t)/t$ non-decreasing, we have $N_i(t)/t \leq 2N_i(1/2) \leq 2$ for all $i \in \mathbb{N}$ and $0 < t \leq 1/2$.

Let $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $2^{-m} < \varepsilon/4$. Since $\lim_{t\to 0} N_i(t)/t = 0$ for all i, there exists $t_\varepsilon > 0$ such that for all $0 < t < t_\varepsilon$, $\sum_{i=1}^m 2^{-i}\frac{N_i(t)}{t} < \varepsilon/2$. Then for all $0 < t < t_\varepsilon$,

$$\begin{aligned}\frac{N_0(t)}{t} &= \sum_{i=1}^{m} 2^{-i}\frac{N_i(t)}{t} + \sum_{i=m+1}^{\infty} 2^{-i}\frac{N_i(t)}{t} \\ &< \varepsilon/2 + 2\sum_{i=m+1}^{\infty} 2^{-i} < \varepsilon.\end{aligned}$$

Consequently, $\lim_{t\to 0} N_0(t)/t = 0$, as desired.

□

We have seen by Proposition 4.3.1 that every spreading model $(\tilde{x}_i)$ of an Orlicz sequence space h_M generated by a weakly null sequence in h_M corresponds to a function N in $C_{M,1}$. This reduces the study of the partially ordered set $SP_\omega(h_M)$ to the study of the partially ordered set $C_{M,1}$. Hence our next results are on the structure of the set $C_{M,1}$.

We start with an easy observation which will be used frequently in the sequel.

Lemma 4.3.6 *Let M be an Orlicz function satisfying the Δ_2-condition. Then for all $N \in C_{M,1}$, there exists a sequence (G_n) of Orlicz functions which belong to the equivalence class of M in $C_{M,1}$ such that (G_n) converges uniformly in the norm topology of $C(0,\frac{1}{2})$ to N.*

Proof By the definition of the set $C_{M,1}$, clearly, for every $N \in C_{M,1}$ there exists a sequence (G_n), where $G_n = \sum_{i\in\sigma_n} \alpha_i^{(n)} \frac{M(\lambda_i^{(n)}t)}{M(\lambda_i^{(n)})}$ for some finite subset $\sigma_n \in \mathbb{N}$ and scalars $\alpha_i^{(n)}$ with $\sum_{i\in\sigma_n} \alpha_i^{(n)} = 1$ and $0 < \lambda_i \leq 1/2$ such that (G_n) converges uniformly to N in norm topology of $C(0,\frac{1}{2})$, due to the norm compactness of $C_{M,1}$ in $C(0,\frac{1}{2})$.

To show that G_n is equivalent to M for every $n \in \mathbb{N}$, it is sufficient to show that the functions $\frac{M(\lambda t)}{M(\lambda)}$ $(0 < \lambda \leq 1/2)$ are equivalent to M. Since M satisfies Δ_2-condition, so does every function in $C_{M,1}$ (with the same constant). Hence,

$$\lim_{t\to 0} \frac{M(t)}{M(\lambda t)/M(\lambda)} = \lim_{t\to 0} \frac{M(\lambda)M(t)}{M(\lambda t)} = KM(\lambda),$$

where K is the Δ_2-condition constant. Also due to the Δ_2-condition, M is not degenerate, hence $M(\lambda) \neq 0$. Hence it follows that the functions $\frac{M(\lambda t)}{M(\lambda)}$ and hence G_n are equivalent to M, for every $n \in \mathbb{N}$. Note that, if N is not equivalent to M, then the equivalence constants grow to infinity as n increases.

□

Before stating an important result on the structure of the set $C_{M,1}$, first we need also the following lemma which is a reformulation in our context of Proposition 3.7 of [AOST].

Lemma 4.3.7 *Let $C \subset C_{M,1}$ be a non-empty subset satisfying the following two conditions:*

(i) C does not have a maximal element with respect to domination.

(ii) For every $(N_i) \subset C$ there exists $N \in C$ such that $N_i \leq N$ for every $i \in \mathbb{N}$.

Then for all ordinals $\alpha < \omega_1$ there exists $N^\alpha \in C$ such that if $\alpha < \beta < \omega_1$ then $N^\alpha < N^\beta$.

Sketch of the Proof We use transfinite induction. Suppose that N^α have been constructed for $\alpha < \beta < \omega_1$. Then N^β is chosen using (i) if β is a successor ordinal and (ii) if β is a limit ordinal. □

The following theorem gives an important criterium on the structure of the set $C_{M,1}$.

Theorem 4.3.8 *Let M be an Orlicz function satisfying Δ_2-condition. Suppose that there exists $N_0 \in C_{M,1}$ such that $N_0 \not\lesssim M$. Then the set $C_{M,1}$ contains an uncountable increasing chain of mutually non-equivalent Orlicz functions.*

Proof We will show that there exists a subset C of $C_{M,1}$ which satisfies the conditions (i) and (ii) of Lemma 4.3.7.

First, we observe that the assumption implies that there exists $N_0' \in C_{M,1}$ satisfying $N_0' \not\precsim M$ which is, additionally, of the form

$$\sum_{i=1}^{\infty} c_i \frac{M(\lambda_i t)}{M(\lambda_i)},$$

for some $c_i > 0$ with $\sum_i c_i = 1$, and for $0 < \lambda_i < 1$.

Indeed, let (G_n) be a sequence in the equivalence class of M which converges uniformly to N_0 (Lemma 4.3.6). Since $N_0 \not\precsim M$, there exists a sequence $(t_k) \searrow 0$ such that for all $k \in \mathbb{N}$,

$$\frac{M(t_k)}{N_0(t_k)} < \frac{1}{k2^k}.$$

For every k, let n_k be such that $G_{n_k}(t_k) \geq (1/2)N_0(t_k)$, and put $N_0'(t) = \sum_{k=1}^{\infty} 2^{-k} G_{n_k}(t) \in C_{M,1}$. Then,

$$N_0'(t_k) \geq 2^{-k} G_{n_k}(t_k) \geq 2^{-(k+1)} N_0(t_k) \geq (k/2)M(t_k).$$

That is, $\limsup_{t\to 0} \frac{N_0'(t)}{M(t)} = \infty$ and hence $N_0' \not\precsim M$. And clearly,

$$\begin{aligned} N_0'(t) = \sum_{k=1}^{\infty} 2^{-k} G_{n_k}(t) &= \sum_k 2^{-k} \sum_i \alpha_i^{(n_k)} \frac{M(\lambda_i^{(n_k)} t)}{M(\lambda_i^{(n_k)})} \\ &= \sum_i c_i \frac{M(\lambda_i t)}{M(\lambda_i)}, \end{aligned}$$

for some c_i such that $\sum_i c_i = 1$ and $0 < \lambda_i < 1$.

For convenience of notation we denote N_0' by N_0 again. So suppose that $N_0(t) = \sum_i c_i \frac{M(\lambda_i t)}{M(\lambda_i)}$. Observe that $c_i \neq 0$ for infinitely many i's, due to the assumption that $N_0 \not\precsim M$.

For all n, let s_n be the normalized partial sum,

$$s_n(t) = \frac{1}{\sum_{i=1}^{n} c_i} \sum_{i=1}^{n} c_i \frac{M(\lambda_i t)}{M(\lambda_i)}.$$

Then $s_n \in C_{M,1}$. Let $k_0 \in \mathbb{N}$ such that $\sum_{i=1}^{k_0} c_i \geq 1/2$. Then for all $n \geq k_0$, we have $s_n(t) \leq 2N_0(t)$ for all $0 \leq t \leq 1$. Let us relabel the sequence $\{s_n\}_{n=k_0}^{\infty}$ and denote it again by $\{s_n\}_{n=1}^{\infty}$.

Now let

$$C = \Big\{ \mathcal{N} \in C_{M,1} : \ \mathcal{N}(t) = \sum_{n=1}^{\infty} b_n s_n(t), \text{ for some } b_n \geq 0 \text{ and } \sum_n b_n = 1 \Big\}.$$

First, we remark that for all $\mathcal{N} \in C$, we have $N_0 \not\precsim \mathcal{N}$. Indeed, let $\mathcal{N} = \sum_{n=1}^{\infty} b_n s_n(t) \in C$ for some $b_n \geq 0$ with $\sum_n b_n = 1$ and let $\varepsilon > 0$ be arbitrary. Let $m \in \mathbb{N}$ such that $\sum_{n=m+1}^{\infty} b_n < \varepsilon/4$. Using the fact that $\sum_{n=1}^{m} b_n s_n(t)$ is equivalent to M and $N_0 \not\precsim M$, we pick $t_\varepsilon > 0$ such that $\sum_{n=1}^{m} b_n \frac{s_n(t_\varepsilon)}{N_0(t_\varepsilon)} < \varepsilon/2$. Then, since $s_n(t) \leq 2N_0(t)$ for all n and t, we have

$$\begin{aligned} \frac{\mathcal{N}(t_\varepsilon)}{N_0(t_\varepsilon)} &= \sum_{n=1}^{m} b_n \frac{s_n(t_\varepsilon)}{N_0(t_\varepsilon)} + \sum_{n=m+1}^{\infty} b_n \frac{s_n(t_\varepsilon)}{N_0(t_\varepsilon)} \\ &< \frac{\varepsilon}{2} + 2 \sum_{n=m+1}^{\infty} b_n < \varepsilon. \end{aligned}$$

i.e., $\liminf_{t \to 0} \frac{\mathcal{N}(t)}{N_0(t)} = 0$, and $N_0 \not\precsim \mathcal{N}$.

Now we check the conditions (ii) and (i) of Lemma 4.3.7 for the set C.

(ii) If $\mathcal{N}_i(t) = \sum_n b_n^{(i)} s_n(t) \in C$ for some $b_n^{(i)} \geq 0$ with $\sum_n b_n^{(i)} = 1$ and $i = 1, 2, \ldots$, then we put $\mathcal{N}(t) = \sum_{i=1}^{\infty} 2^{-i} \mathcal{N}_i(t)$. Then,

$$\mathcal{N}(t) = \sum_i 2^{-i} \sum_n b_n^{(i)} s_n(t) = \sum_n c_n s_n(t),$$

where $c_n \geq 0$ with $\sum_n c_n = 1$. i.e., $\mathcal{N} \in C$. Moreover, for all i, we have $\mathcal{N} \geq \mathcal{N}_i$.

(i) Suppose that there is a maximal element $\mathcal{M} \in C$. Then $\mathcal{M}(t) = \sum_n b_n s_n(t)$ for some $b_n \geq 0$ such that $\sum_n b_n = 1$. By the above remark, $N_0 \not\lesssim \mathcal{M}$, and hence there exists a sequence $(t_k) \searrow 0$ such that for all k,

$$\frac{\mathcal{M}(t_k)}{N_0(t_k)} < \frac{1}{k2^k}.$$

Since the partial sums s_n converge to N_0, for all k we may choose (n_k) such that $s_{n_k}(t_k) \geq (1/2)N_0(t_k)$. Let $\mathcal{M}_0(t) = \sum_k 2^{-k} s_{n_k}(t) \in C$. Then for all k,

$$\mathcal{M}_0(t_k) \geq 2^{-k} s_{n_k}(t_k) \geq 2^{-(k+1)} N_0(t_k) \geq (k/2)\mathcal{M}(t_k).$$

i.e., $\limsup_{t\to 0} \frac{\mathcal{M}_0(t)}{\mathcal{M}(t)} = \infty$ and $\mathcal{M}_0 \not\lesssim \mathcal{M}$, a contradiction. Therefore, C does not contain a maximal element.

The proof is now complete by Lemma 4.3.7. □

Remark. Recently, it has been shown in [FPR] that the set $SP(X)$ of all spreading models of a Banach space X is either countable (up to equivalence) or has cardinality continuum. Using this together with Theorem 4.3.8 we immediately obtain the following

Corollary 4.3.9 *Let M be an Orlicz function which satisfies Δ_2-condition. Suppose that h_M admits a spreading model h_N generated by a normalized weakly null sequence such that the unit vector basis of h_M does not dominate the unit vector basis of h_N.*

Then the set $SP(h_M)$ has (up to equivalence) cardinality continuum.

Finally, we end this chapter with the following consequence of Theorem 4.3.8, which gives a description of the set of spreading models of reflexive Orlicz sequence spaces with only countably many spreading models.

Corollary 4.3.10 *Let h_M be reflexive Orlicz sequence space. Suppose that $SP_\omega(h_M)$ is countable. i.e., the number of mutually non-equivalent spreading models generated by weakly null sequences in h_M is countable. Then*

(i) h_M is the upper bound of $SP_\omega(h_M)$,

(ii) ℓ_p for some $1 < p < \infty$ is the lower bound of $SP_\omega(h_M)$.

Proof Since h_M is reflexive, M satisfies Δ_2-condition (see the remark following the definition of Δ_2-condition in section 4.2 and also Proposition 4.b.2, [LT]). Also by the remark following Lemma 4.3.4, we have $SP_\omega(h_M) = C_{M,1}$. (Note that by reflexivity, $SP_\omega(h_M)$ also coincides with the set $SP(h_M)$ of all spreading models of h_N.)

(i) By Proposition 4.3.5, the upper bound exists. Suppose that there exists $N \in C_{M,1}$ such that N is not equivalent to M and h_N is the upper bound for $SP_\omega(h_M)$. It follows that $N \not\lesssim M$ and by Theorem 4.3.8, $C_{M,1}$ contains uncountable mutually non-equivalent Orlicz functions, and thus $SP_\omega(h_M)$ is uncountable, a contradiction. Therefore h_M must be the upper bound.

(ii) Since the set of p's for which ℓ_p embeds into h_M is a closed interval (Theorem 4.a.9, [LT]), it follows from the assumption that this set is singleton. Hence there exists a unique $1 < p < \infty$ such that $\ell_p \in SP_\omega(h_M)$. Moreover it follows from Theorem 4.2.1 that h_M is ℓ_p-saturated. i.e., every subspace of h_M has a further subspace which contains an isomorphic copy of ℓ_p. For Orlicz sequence spaces, by Theorem 4.2.1, ℓ_p embeds into h_M if and only if $t^p \in C_{M,1}$. In particular, for all $N \in C_{M,1}$, the function t^p belongs to $C_{N,1}$. Moreover, the assumption that M satisfies Δ_2-condition implies that N also satisfies Δ_2-condition for all $N \in C_{M,1}$.

If ℓ_p is not the lower bound of $SP_\omega(h_M)$, then there exists $N \in C_{M,1}$ such that $t^p \not\lesssim N$. But, by the above, $t^p \in C_{N,1}$, hence it follows from Theorem 4.3.8 that $C_{N,1} \subset C_{M,1}$ is uncountable. This implies that $SP_\omega(h_M)$ is uncountable, a contradiction. Therefore ℓ_p must be the lower bound of $SP_\omega(h_M)$.

□

Bibliography

[AD] A.S. Argyros and I. Deliyanni, *Examples of asymptotic ℓ_1 Banach spaces*, Trans. Amer. Math. Soc. **349** (1997), no. 3, 973–995.

[ADKM] A.S. Argyros, I. Deliyanni, D. Kutzarova and A. Manoussakis, *Modified mixed Tsirelson spaces*, J. Funct. Anal. **159** (1998), no. 1, 43–109.

[AOST] G. Androulakis, E. Odell, Th. Schlumprecht and N. Tomczak-Jaegermann, *On the structure of the spreading models of a Banach space*, Preprint.

[BL] B. Beauzamy and J.-T. Lapreste, *Modeles etales des espaces de Banach* [Spreading models of Banach spaces] Travaux en Cours. [Works in Progress] Hermann, Paris, 1984. iv+210 pp.

[BS] A. Brunel and L. Sucheston, *On B-convex Banach spaces*, Math. Systems Theory **7** (1974), no. 4, 294–299.

[CS] P.G. Casazza and T.J. Shura, *Tsirelson's space*, Lecture Notes in Math., vol. 1363, Springer-Verlag, Berlin and New York, 1989.

[FJ] T. Figiel and W. B. Johnson, *A uniformly convex Banach space which contains no ℓ_p*, Composito Math. **29** (1974), 179-190.

[FPR] V. Ferenczi, A. Pelczar and C. Rosendal, *On a question of Haskell P. Rosenthal*, preprint.

[G1] W. T. Gowers, *An infinite Ramsey theorem and some Banach-space dichotomies*, Annals of Mathematics, **156** no. 3 (2002), 797-833.

[G2] W.T. Gowers, *A space not containing c_0, ℓ_1 or a reflexive subspace*, Trans. Amer. Math. Soc. **344** (1994), 407-420.

[G3] W. T. Gowers,*A hereditarily indecomposable space with an asymptotic unconditional basis*, Geometric aspects of functional analysis (Israel, 1992–1994), 112–120, Oper. Theory Adv. Appl., **77**, Birkhuser, Basel, 1995.

[GM] W.T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. **6** (1993), 851-874.

[JMST] W. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, *Symmetric structures in Banach spaces*, Mem. Amer. Math. Soc. **19** (1979), no. 217, v+298 pp.

[KOS] H. Knaust, E. Odell and Th. Schlumprecht, *On asymptotic structure, the Szlenk index and UKK properties in Banach spaces*, Positivity **3** (1999), 173-199.

[KT] R. Komorowski and N. Tomczak-Jaegermann, *Banach spaces without local unconditional structure*, Israel J. Math. **89** (1995), no. 1-3, 205–226.

[KL] D. Kutzarova and P. Lin, *Remarks about Schlumprecht space*, Proc. Amer. Math. Soc., **128** (2000), no.7, 2059-2068.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, New York, 1977.

[M] B. Maurey, *Type, cotype and K-convexity*, Handbook of the Geometry of Banach Spaces (W.B. Johnson and J. Lindenstrauss, eds.), vol. 2, Elsevier, Amsterdam, to be published.

[MMT] B. Maurey, V.D. Milman and N. Tomczak-Jaegermann, *Asymptotic infinite-dimensional theory of Banach spaces*, Oper. Theory: Adv. Appl. **77** (1994), 143-175.

[MP] B. Maurey and G. Pisier, *Series de variables aleatoires vectorielles independantes et proprietes geometriques des espaces de Banach*, Studia Math. **58** (1976), no. 1, 45–90.

[MiS] V.D. Milman and M. Sharir, *Shrinking minimal systems and complementation of ℓ_p^n-spaces in reflexive Banach spaces*, Proc. London Math. Soc. **39** (1979), 1-29.

[MS] V.D. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, lecture Notes in Math., vol. 1200, Springer-Verlag, Berlin and New York, 1986, 156 pp.

[MT1] V.D. Milman and N. Tomczak-Jaegermann, *Asymptotic-ℓ_p spaces and bounded distortions*, (Bor-Luh Lin and W.B. Johnson, eds.), Contemp. Math. **144** (1993), 173-195.

[MT2] V.D. Milman and N. Tomczak-Jaegermann, *Stabilized asymptotic structures and envelopes in Banach spaces*, Geometric Aspects of Functional Analysis (Israel Seminar 1996-2000), Lecture Notes in Math., 1745, Springer-Verlag, Berlin and New York, 2000, 223-237.

[O] E. Odell, *On Subspaces, Asymptotic Structure, and Distortion of Banach Spaces; Connections with Logic*, (C. Finet and C. Michaux, eds.), Analysis and Logic, Cambridge University Pr., 2003.

[OS1] E. Odell and Th. Schlumprecht, *Trees and branches in Banach spaces*, Trans. Amer. Math. Soc. **354** (2002), no.10, 4085-4108.

[OS2] E. Odell and Th. Schlumprecht, *On the richness of the set of p's in Krivine's theorem*, Oper. Theory: Adv. Appl. **77** (1995), 177-198.

[OS3] E. Odell and Th. Schlumprecht, *The distortion problem*, Acta Math. **173** (1994), no. 2, 259–281.

[OS4] E. Odell and Th. Schlumprecht, *A problem on spreading models*, J. Funct. Anal. **153** (1998), no. 2, 249–261.

[OS5] E. Odell and Th. Schlumprecht, *Asymptotic properties of Banach spaces under renormings*, J. Amer. Math. Soc. **11** (1998), no. 1, 175–188.

[R] H. Rosenthall, *Some remarks concerning unconditional basic sequences*, Longhorn Notes: Texas Functional Analysis Seminar 1982-83, University of Texas, Austin, 15-48.

[S] Th. Schlumprecht, *An arbitrarily distortable Banach space*, Israel J. Math. **76** (1991), 81-95.

[Ts] B. S. Tsirelson, *Not every Banach space contains ℓ_p or c_0*, Func. Anal. Appl. **8** (1974), 139-141.

[Tz] L. Tzafriri, *On the type and cotype of Banach spaces*, Israel J. Math., **32**, 1979, 32-38.