Essays On Proportional Reinsurance Under Random Horizon

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematical Finance

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Abstract

This thesis considers proportional reinsurance that insurance companies might employ in order to reduce risks and limit the impact of large claims. An insurance company is a typical example of a financial corporation, which can choose a production policy from an available set of control policies with different expected profit and risks. In this setting, this thesis considers two main cases depending whether the company pays dividends to the shareholders or not. For both cases, the main aim lies in measuring the impact of the liability and/or the random horizon. The liability payments could reflect the mortgage on the company's property or the amortization bond, while the random horizon could model the default of a firm or the death time of an agent. When the company pays dividends, its objective consists of maximizing the expected aggregate discounted dividend distributions. However, when there is no dividend payments, the objective resides in maximizing the expected total discounted cash reserve up to the bankruptcy time or the random horizon (whoever comes first). Thanks to the Bellman's principal, the control problems in all cases are reduced to Hamilton-Jacobi-Bellman equations. Smooth solutions to these equations, which take various forms depending on the interactions between the parameters of the corporation's model and the random horizon, are explicitly derived. Furthermore, the optimal policies for each control problem are explicitly derived.

Acknowledgments

First of all, I would like to express my very great appreciation and gratitude to my supervisor Dr. Tahir Choulli for his instructions and patience. Dr. Choulli had spend a significant amount of time on weekdays and weekends supporting me by his insightful suggestions and detailed reviews of the drafts of this thesis and by helping me with confronting the difficulties. His commitment towards his students is aspiring and working with him was a privilege. I learned a lot from him and gained valuable knowledge, and for that I am forever thankful. Without his support, this thesis would never have been completed in time.

I would also like to thank Dr. Alexander Melnikov, Dr. Abel Cadenillas, Dr. Peter Minev, Dr. Mike Kouritzin and Dr. Masahiro Watanabe for accepting to be members of my defense committee, and to all my professors at the Mathematics and Statistical Sciences department and at the School of Business for their encouragement and motivation. It was a great learning experience.

My sincere gratitude goes to the University of Alberta and the President Award for Refugees and Displaced Persons for making all this possible. This opportunity definitely changed my life and there are no words that could ever express my gratefulness for it.

Finally, my deepest thanks goes to my family for their everlasting love, care and support.

Contents

Chapter 1

Introduction

Insurance is a natural human response to the uncertainty that dominates our future, and when it comes to business, not taking uncertainty in consideration can lead to severe losses. This kind of protection is represented as a contract also known as policy that is written by an insurance provider to a company or the beneficial of this policy. The insured promise to pay periodical fixed amount of payment called premiums in return for the insurer to pay an agreed upon sum of money in case there was a claim by the insured. This kind of transaction, along with identifying, assessing and hedging risks is part of the risk management responsibilities in a firm. Thus, an insurance company can be seen as a typical example of a financial corporation in the problem of optimal risk control/dividend distribution. In order to face large claims or catastrophic events, insurance companies employ reinsurance, as a major risk-management tool that permits insurance companies to be protected against adverse fluctuations.

1.1 Reinsurance models

As insurance companies might get exposed to cover claims that are too large to be handled, they find the urge to lower this risk by distributing it to other insurance companies, in return for sharing the premium paid (profits). This process is called reinsurance. To elaborate on how this process works, through a broker, the company that wishes to purchase reinsurance, called the *cedent* and one or more other insurance companies (*the reinsurers*) enter a contract agreement that states the details of the transaction. This tool of risk hedging do not just increases the value of the cedent and contributes for it to remain solvent, but also increases its ability to withstand the financial burden in case unusual or catastrophic events occur. In general, the reinsurance process is considered as an important risk-management tool in the insurance world. Thus, naturally, one can ask why reinsurance is so important for insurance companies? It is important, because it allows them to perform the following protective tasks.

- Hedging adverse fluctuation that may incur in the course of business.
- Dealing with the appearance of excessively large claims such as catastrophic risks, or an unusually large number of claims. The most dangerous risk comes from the large claims and also a large number of claims can lead to a disastrous situation.
- Increase the capacity of the company by offering more services to its clients.
- Deal with financial distress due to unexpected changes in premium collection or profit.

Reinsurance can be classified into two major cases, proportion reinsurance and non-proportional reinsurance. In proportional reinsurance the sharing of risk between the ceding company and the reinsurer is determined at issue (i.e. a coverage against a fixed percentage of losses). However, in a non-proportional reinsurance scheme –such as excess-of-loss reinsurance– the sharing of the risk between the cedent and the reinsurer is done in a more sophisticated manner. To specify more this classification, we go back to the first mathematical model for cash reserve with and without reassurance, that was developed by Cramer-Lundberg. This model, when there is no risk sharing is given by

$$R(t) = R(0) + pt - \sum_{i=1}^{N_t} U_i.$$

Here $R(0)$ is the initial cash reserve, p is the premium rate, while U_i is the size of the i^{th} claim and N is the Poisson process that counts the claims happening. This model assumes that the company takes full risk. However, when the company considers reinsurance it bares the risk that is associated with the claim, and we denote that by $U^{(a)}$ where a is the retention level (i.e. the net amount of liability that results from an insurance claim or claims which is retained by a ceding company after reinsuring the balance amount of the liability) and the risk proportion that is left $U - U^{(a)}$ is diverted to the reinsurer. Thus, the model after reinsurance takes the following form

$$R^{(a)}(t) = R(0) + p^{(a)}t - \sum_{i=1}^{N_t} U_i^{(a)}.$$

For more details about the Cramer-Lundberg model we refer the reader to D. Dickson [9], Grandall [14], and A. Melnikov [23].

From this model, one can clearly define the **proportional reinsurance** (pro rata), as the case where $U_i^{(a)} = aU_i$ for $a \in [0,1]$. As its name suggests, under this type of coverage, an agreed upon percentage of the premiums and losses will be shared with the reinsurance company or companies. In other words, by a contractual agreement, the cedent pays a% of the claim and receives the same percentage of the premium and the reinsurance company would have the obligation for the rest of (1 - a)% of claim size and receives the same percentage of the premium.

Excess-of-loss reinsurance is an example of non proportional reinsurance , that consists of diverting losses excess of the ”retention” level a to the re-insurer, while the cedent will cover losses up to this retention level. Mathematically, this boils down to $U_i^{(a)} = \min(a, U_i)$, for a retention level $a \in (0, +\infty)$.

It is well known nowadays that the Cramer-Lundberg model do not really captures the activities of the insurance company very well, since a claim’s size is more likely to be insignificant compared to the reserve amount. Furthermore, as we scale the time to a continuous time the reserve process will become a stochastic process with drift and diffusion coefficients. Thus, the resulting diffusion models are more adequate for the case of big portfolios. For more details about these models, we refer the reader to [10, 12, 13], [14], [18], [19], [24], [28], [29]. In this thesis, we consider these diffusion models following the footsteps of [1, 6, 15, 17].

1.2 Dividend models

Dividend distribution form a very important business activity in any company that has shareholder looking forward to realize their investment in terms of profit and since dividend is the only form of income received by the shareholders of a company, it is the firm’s responsibility to account for optimal distribution of dividends as part of their financial activities. Dividends are paid from the liquid reserve of the company, therefore, the optimal distribution of dividend heavily depends on the level of the reserve corresponding to certain thresholds. When modeling for the optimal dividend distribution for a company, it is mainly about finding the policy that

maximizes the expected cumulative discounted dividend payouts up to the time of bankruptcy. During the last two decades there was a great interest in diffusion models for optimal dividend optimization and/or risk control techniques (see Jeanblanc Piqué and Shiryaev [20], Asmussen and Taksar [1], Radner and Shepp [27], Boyle, Elliott, and Yang [4], Højgaard and Taksar [15], [17], [16], Paulsen and Gjessing [25], and Taksar and Zhou [31], Choulli, Taksar and Zhou [6], [7], [8]. In those models the liquid assets of the company are modeled by a Brownian motion with constant drift and diffusion coefficients. The drift term corresponds to the expected profit per unit time, while the diffusion term is considered as risk. The larger the diffusion coefficient the greater the business risk the company takes on. If the company wants to decrease the risk from its business activities, it also faces a decrease in its profit. In other words, different business activities in this model correspond to changing simultaneously the drift and the diffusion coefficients of the underlying process.
In this setting, we consider the model developed by B. Højgaard and M. Taksar [15] and [17] , to which we add the random horizon and suppose that the company is facing a liability payments at a constant rate δ. This extends those existing models, and measures the impact of random horizon and its interplay with the liability.

1.3 Summary

This thesis has five Chapters including the introductory chapter. Chapter 2 (the next chapter) reviews the most important mathematical and statistical concepts that will be used throughout the thesis. Furthermore, it recalls some important results from the mathematical and statistical literature that plays crucial roles in the analysis of this thesis.

Chapter 3 is the first contribution of this thesis, and considers the model of the reserve process for a proportional reinsurance (the case of corporation model without dividend payouts). The main features of this model of reinsurance –in this thesis– consists of allowing the insurance company top pay liability at a constant rate δ, and being subject to a random horizon τ. This extends the results of B. Højgaard and M. Taksar (see [15]) to this interesting setting, and leads to challenging mathematical questions that are solved explicitly. In fact, the objective of the company in this case is to maximize the expected discounted total reserve up to the bankruptcy time or the random horizon, whoever comes first. This control problem, that depends on a risk control component which allows risk management at the company to assess the risk according to different levels of reserve, is transformed into solving the Hamilton-Jacobi-Bellman (HJB hereafter) equation. This equation is solved explicitly, and its solution depends on the interplay between the parameters of the cash reserve, the random horizon and the liability rate δ.
Chapter 4 is the second contribution of this thesis, and continues in the same spirit of Chapter 3 by considering the same model with random horizon, but without liability rate while adding constraints on risk control instead. Here again, the control problem with random horizon and constraints on risk is transformed into the HJB equation.This equation is solved explicitly and takes various forms depending on the interplay between all the parameters of the model.
In Chapter 5, we investigate the dividend distribution model, considered in B. Højgaard and M. Taksar [17], when the company faces liability payment are a constant rate δ and is subject to a random horizon. Similarly as in the previous chapters, the control problem is reduced to constructing the solution to an HJB equation. This equation is solved explicitly and the optimal policies are determined as well. This chapter not only extends B. Højgaard and M. Taksar [17] to this complex setting of random horizon and liability, but also presents a different approach compared to those papers.

Chapter 2

Mathematical Preliminaries

In this chapter we introduce the mathematical concepts and tools that will be used in the following chapters. Most of these definitions and theorems are used in many stochastic calculus books, see for instance A. Melnikov [23], R. Korn and E. Korn [22] and J. Michael Steele [30].

2.1 Stochastic basis

Our financial model is based on the probability space

$$(\Omega, \mathscr{F}, P).$$

Here Ω represents all of the possible outcomes ω, also called the sample space. Let $\mathscr{F}$ be a σ-algebra on Ω. (The elements of $\mathscr{F}$ are called events). In addition, we use a probability measure, that is, a (finite) measure P on $(\Omega, \mathscr{F})$ satisfying $P(\Omega) = 1$.

2.2 Filtration and stopping time

To model the evolution of the financial information over time, we introduce the following

Definition 2.2.1. *A filtration $(\mathscr{F}_t)_{t\geq 0}$ of $(\Omega, \mathscr{F})$ is a family of sub-σ-algebras of $\mathscr{F}$ satisfying*

$$\mathscr{F}_s \subseteq \mathscr{F}_t, \quad \textit{for any} \quad t \geq s \geq 0.$$

In the probabilistic literature, the quadruplet

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P) \tag{2.1}$$

is called a filtered probability space. Anther important probability tool we might use later in the thesis is the stopping time. Since our model is a stochastic model, then the time we use here is stochastic as well.

Definition 2.2.2. *Consider the filtered probability space given in (2.1), and let $\mathbb{T} \subset \mathbb{R}_+$.*
1) *A random variable $\tau : \Omega \to \mathbb{T} \cup \{+\infty\}$ is called a random time.*
2) *A random time is said to be a stopping time with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ if*

$$\{\tau \leq t\} = \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathscr{F}_t \quad \textit{for every} \quad t \in \mathbb{T}.$$

Stochastic processes are families of random variables indexed by time. The following are some important definitions that we need to introduce.

Definition 2.2.3. 1) *Suppose $\mathscr{F}_{t\in\mathbb{T}}$ is a filtration of the measurable space $(\Omega, \mathscr{F})$, and that X is a process defined on $(\Omega, \mathscr{L})$. Then X is said to be adapted to $\mathscr{F}_{t\in\mathbb{T}}$ if X_t is $\mathscr{F}_t$-measurable for each $t \in \mathbb{T}$.*
2) *A real-valued stochastic process $\{X_t\}_{t\in\mathbb{T}}$ is said to be a super-martingale with respect to the*

filtration $(\mathscr{F}_t)_{t\in\mathbb{T}}$ *if*

a) *each* X_t *is* $\mathscr{F}_t$*-measurable, i.e.* $(X_t)_{t\in\mathbb{T}}$ *is adapted to* $(\mathscr{F}_t)_{t\in\mathbb{T}}$*,*

b) $\mathbb{E}[|X_t|] < +\infty$*, for all* $t \in \mathbb{T}$*, and*

c) $X_t \geq \mathbb{E}[X_s|\mathscr{F}_t]$ *almost surely for* $s \geq t$*.*

If "$\geq$*" in property (3) is replaced by "*$\leq$*", then* X *is said to be a sub-martingale. If the process* X *is both a sub-martingale and a super-martingale, then it is called a martingale.*

3) *Let* $(X_t)_{t\geq 0}$ *be a stochastic process. This stochastic process is called measurable if the mapping*

$$[0,\infty)\times\Omega \longrightarrow \mathbb{R}$$

$$(s,\omega)\mapsto X_s(\omega)$$

is $\mathfrak{B}([0,\infty))\bigotimes\mathscr{F} - \mathfrak{B}(\mathbb{R})$*-measurable.*

4) *Let* $\{X_t\}$ *be a stochastic process. This stochastic process will be called progressively measurable if for all* $t \geq 0$ *the mapping*

$$[0,t]\times\Omega \longrightarrow \mathbb{R}$$

$$(s,\omega)\mapsto X_s(\omega)$$

is $\mathfrak{B}([0,t])\bigotimes\mathscr{G}_t - \mathfrak{B}(\mathbb{R})$*-measurable, where* $\mathscr{G}_t \subseteq \mathscr{F}$*.*

2.2.1 Brownian motion and Itô's formula

We start this subsection with introducing a popular example of martingales that is the Brownian motion.

Definition 2.2.4. *A continuous-time stochastic process* $\{W_t : 0 \leq t < T\}$ *is called a Standard Brownian Motion on* $[0,T)$ *if it has the following four properties:*

1) $W_0 = 0$.

2) *The increments of* W_t *are independent, i.e., for any finite set of times* $0 \leq t_1 < t_2 < \ldots < t_n < T$ *the random variables* $W_{t_2} - W_{t_1}$, $W_{t_3} - W_{t_2}, \ldots W_{t_n} - W_{t_{n-1}}$, *are independent.*

3) *For any* $0 \leq s \leq t < T$ *the increment* $W_t - W_s$, *has the Gaussian distribution with mean* 0 *and variance* $t - s$.

4) *For all* ω *in a set of probability one,* $W_t(\omega)$ *is a continuous function of* t.

A stochastic process $\{X_t\}$ is called an Itô's process if for all $t \geq 0$ it admits the following representation

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s,$$

where X_0 is $\mathscr{F}_0$-measurable and $\{K_t\}$ and $\{H_t\}$ are progressively measurable processes with

$$\int_0^t |K_s| ds < +\infty, \quad \int_0^t H_s^2 < +\infty \quad P - a.s.$$

Theorem 2.2.1. *Let* W_t *be a one dimensional Brownian motion, and* X_t *a real-valued Itô process with*

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s.$$

Let $f : \mathbb{R} \to \mathbb{R}$ *be twice continuously differentiable. Then , for all* $t \geq 0$ *we have*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X \rangle_s \\ &= f(X_0) + \int_0^t \left[f'(X_s) K_s + \frac{1}{2} f''(X_s) H_s^2 \right] ds + \int_0^t f'(X_s) H_s dW_s \quad P - a.s. \end{aligned}$$

2.3 Stochastic differential equations

The equation

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t, \quad t \geq 0, \quad \text{with} \quad X_0 = x,$$

is called a stochastic differential equation, where $X = (X_t)_{t\geq 0}$ is a stochastic process. In this thesis, we will encounter a reflecting SDE that we recall its definition below.

Definition 2.3.1. *A pair of continuous $\mathscr{F}_t$-adapted processes $((R_t, C_t); t \geq 0)$ is a solution of the SDE*

$$dR_t = \mu(t, R_t)dt + \sigma(t, R_t)dW_t + dC_t, \quad t \geq 0, \tag{2.2}$$

with reflection at 0 and initial condition $R_0 = x$ if

1) $R_t \geq 0$, $t \geq 0$.

2) *C is non-decreasing,* $C_0 = 0$.

3) $\int_0^t 1_{\{R_s>0\}} dC_s = 0$, $t \geq 0$.

The following theorems, that we borrow fro the literature, give sufficient conditions for the SDE to admit a solution.

Theorem 2.3.1. *Consider the stochastic differential equation*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t \quad X_0 = x_0, \quad 0 \leq t \leq T. \tag{2.3}$$

If the following space-variable Lipschitz condition

$$|\mu(t,x) - \mu(t,y)|^2 + |\sigma(t,x) - \sigma(t,y)|^2 \leq K|x-y|^2$$

and the spatial growth condition

$$|\mu(t,x)|^2 + |\sigma(t,x)|^2 \leq K(1+|x|^2),$$

then there exists a continuous adapted solution X_t for (2.3) satisfying

$$\sup_{0\leq t\leq T} \mathbb{E}[X_t^2] < +\infty.$$

Moreover, if X_t and Y_t are both continuous satisfying the above L^2-boundedness condition and are solutions to (2.3), then

$$P(X_t = Y_t \quad \textit{for all} \quad t \in [0,T]) = 1.$$

For the proof and more details about this existence and uniqueness result, we refer the reader to Theorem 9.1 in [30].

Theorem 2.3.2. *Let R_0 be a non-negative $\mathscr{F}_0$-measurable random variable. Assume that the measurable functions $\mu = \mu(t,x)$, $\sigma = \sigma(t,x)$ satisfy the*

1) *Lipschitz condition in x, uniformly in time:*

$$\exists K > 0 \; \forall t \geq 0 \; \forall x_1, x_2 \in \mathbb{R}_+ : |\mu(t,x_1) - \mu(t,x_2)| + |\sigma(t,x_1) - \sigma(t,x_2)| \leq K|x_1 - x_2|,$$

2) *and the linear growth condition in x, uniformly in time:*

$$\exists K > 0 \; \forall t \geq 0 \; \forall x \in \mathbb{R}_+ : |\mu(t,x)| + |\sigma(t,x)| \leq K(1 + |x|).$$

Then there exists a unique solution to the reflecting SDE (2.2).

We refer the reader to [26] for more details about this result and related topics.

Chapter 3

Impact of liability and random horizon

This chapter considers the case of a proportional reinsurance, where the insurance company (cedent) faces liability payments at a constant rate δ and is subject to a random horizon. This model extends the model discussed in [15], see also the references therein for related discussion on the model.

This chapter is divided into four sections. The first section introduces the model (mathematically and economically), as well as the objectives that are presented in terms of a control problem. Afterwards, it performs some preliminary analysis for this model. The second section discusses the solution to a Hamilton-Jacobi-Bellman (HJB) associated to the main control problem. The third section states the optimal policies for the main control problem, and the last section presents some graphical illustration of the results.

3.1 The mathematical model and its preliminary analysis

This section introduces the mathematical and economical model, defines our main objective, and presents our first analysis of the model afterwards.

3.1.1 The mathematical model and our objective

We first begin by introducing the model. Consider a firm with reserve R_t at any time t where $R_t = (R_t)_{t\geq 0}$ is a stochastic process that is a solution to the following stochastic differential equation:

$$dR_t = (\mu - \delta)dt + \sigma dW_t, \quad t \geq 0, \quad R_0 = x. \tag{3.1}$$

For a risk exposure a^π_t, at time t, and control policy $\pi = (a^\pi_t, t \geq 0)$, our reserve equation becomes:

$$dR^\pi_t = (a^\pi_t \mu - \delta)dt + \sigma dW_t, \quad R^\pi_0 = x, \tag{3.2}$$

and the bankruptcy time is given by

$$\tau_\pi = \inf\Big\{t \geq 0 : R^\pi_t = 0\Big\}.$$

Here, by convention, we put $\inf(\emptyset) = +\infty$.

<u>The model for random horizon.</u> Throughout this chapter, τ is a random time that is independent of the reserve process $R = (R_t,\, t \geq 0)$ (or equivalently $\mathscr{F}_\infty = \sigma(W_t, t \geq 0)$), and has an exponential distribution with mean λ^{-1}, i.e., $\mathbb{E}[\tau] = \lambda^{-1}$. Thus, the survival probability for this random time is given by

$$G_t := P(\tau \geq t | \mathscr{F}_t) = \exp(-\lambda t), \quad t \geq 0. \tag{3.3}$$

We define the return function of an initial reserve x under a control policy π as the total discounted cash reserve from time 0 to the default (random horizon) time τ or the bankruptcy time τ_π (whichever comes first).

The objectives. Our objectives consist of finding the optimal return function V, defined below, and describing the optimal policy $\pi^* \in \mathscr{A}$ such that

$$V(x) := \sup_{\pi \in \mathscr{A}} \mathbb{E}\left[\int_0^{\tau_\pi \wedge \tau} e^{-\gamma t} R_t^\pi dt\right] = \mathbb{E}\left[\int_0^{\tau \wedge \tau_{\pi^*}} e^{-\gamma t} R_t^{\pi^*} dt\right]. \tag{3.4}$$

where $\mathscr{A}$ denotes the set of all admissible control policies.

3.1.2 Properties of the optimal return function V

Proposition 3.1.1. *The following assertions hold.*

(a) *The function V, defined in (3.4), is concave and satisfies*

$$V(x) = \sup_{\pi \in \mathscr{A}(x)} \mathbb{E}\left[\int_0^{\tau_\pi} e^{-(\gamma+\lambda)t} R_t^\pi dt\right].$$

(b) *The optimal value function V, defined in (3.4), satisfies*

$$\frac{x}{\gamma+\lambda} \leq V(x) \leq \frac{x}{(\gamma+\lambda)} + \frac{|\mu-\delta|}{(\gamma+\lambda)^2}, \quad \forall x \geq 0. \tag{3.5}$$

Proof. This proof has two parts where we prove assertions (a) and (b) respectively.

Part 1. The equality in assertion (a) comes from the fact that

$$\mathbb{E}\left[\int_0^{\tau_\pi \wedge \tau} e^{-\gamma t} R_t^\pi dt\right] = \mathbb{E}\left[\int_0^{\tau_\pi} e^{-\gamma t} G_t R_t^\pi dt\right] = \mathbb{E}\left[\int_0^{\tau_\pi} e^{-(\gamma+\lambda)t} R_t^\pi dt\right],$$

see (3.3). The remaining of the proof proves concavity. Put $V_\pi(x) := \mathbb{E}\left[\int_0^{\tau_\pi} e^{-(\gamma+\lambda)t} R_t^\pi dt\right]$. Let x_1 and x_2 two positive initial reserves and $\eta \in (0,1)$. For any positive number ε there exist

$\pi_1 = (a^{\pi_1}(t),\ t \geq 0) \in \mathscr{A}(x_1)$ and $\pi_2 = (a^{\pi_2}(t),\ t \geq 0) \in \mathscr{A}(x_2)$ such that

$$V(x_1) - \varepsilon \leq V_{\pi_1}(x_1) \quad \text{and} \quad V(x_2) - \varepsilon \leq V_{\pi_2}(x_2). \tag{3.6}$$

Consider the policy $\pi_\eta = (a^{\pi_\eta}(t),\ t \geq 0)$ given by

$$a_{\pi_\eta}(t) := \eta a_{\pi_1}(t) + (1-\eta) a_{\pi_2}(t).$$

Then the reserve process for π_η is given by

$$R_t^{\pi_\eta} = \eta R_t^{\pi_1} + (1-\eta) R_t^{\pi_2}, \quad t \geq 0, \tag{3.7}$$

whose bankruptcy time is given by

$$\tau^{\pi_\eta} = \max(\tau^{\pi_1},\ \tau^{\pi_2}).$$

Thanks to (3.6) and (3.7), we derive

$$\begin{aligned} \eta V(x_1) + (1-\eta)V(x_2) - \varepsilon &\leq \eta V_{\pi_1}(x_1) + (1-\eta)V_{\pi_2}(x_2) = V_{\pi_\eta}(\eta x_1 + (1-\eta)x_2) \\ &\leq \sup_{\pi \in \mathscr{A}(x)} V_{\pi_\eta}(\eta x_1 + (1-\eta)x_2) = V(\eta x_1 + (1-\eta)x_2). \end{aligned}$$

Since both sides, right and left, don't depend on ε, we let ε go to zero and get concavity of V.

Part 2. Due $R^\pi_{\tau_\pi} = 0$ on $\{\tau_\pi \leq t\}$, it is clear that $R^\pi_t 1_{\{t \leq \tau_\pi\}} = R^\pi_{t \wedge \tau_\pi}$. Hence, on the one hand, we derive

$$\mathbb{E}\left[\int_0^{\tau^\pi} e^{-(\gamma+\lambda)t} R_t^\pi dt\right] = \mathbb{E}\left[\int_0^{+\infty} e^{-(\gamma+\lambda)t} R_t^\pi 1_{\{t \leq \tau_\pi\}} dt\right] = \int_0^{+\infty} e^{-(\gamma+\lambda)t} \mathbb{E}\left[R^\pi_{t \wedge \tau^\pi}\right] dt. \tag{3.8}$$

On the other hand, remark that

$$\begin{aligned} x \leq \mathbb{E}\left[R^\pi_{t\wedge\tau_\pi}\right] &= x + \mathbb{E}\left[\int_0^{t\wedge\tau_\pi} (a_u^\pi \mu - \delta) du\right] + \mathbb{E}\left[\sigma \int_0^{t\wedge\tau_\pi} a_u^\pi dW_u\right] \\ &= x + \mathbb{E}\int_0^{t\wedge\tau_\pi} (a_u^\pi \mu - \delta) du \leq x + |\mu - \delta| t. \end{aligned}$$

Thus, by inserting this inequality in (3.8), and using

$$\int_0^{+\infty} te^{-(\gamma+\lambda)t}dt = \frac{1}{(\gamma+\lambda)^2} \quad and \quad \int_0^{+\infty} e^{-(\gamma+\lambda)t}dt = \frac{1}{\gamma+\lambda},$$

we get

$$\frac{x}{(\gamma+\lambda)} \leq V(x) \leq \frac{x}{\gamma+\lambda} + \frac{|\mu-\delta|}{(\gamma+\lambda)^2}, \quad \forall x \geq 0.$$

This proves assertion (b) and ends the proof of the proposition. □

For the case when the liability rate is large enough, i.e. $\delta \geq \mu$, the solution to the HJB (3.9) takes the following form.

Lemma 3.1.1. *If $\delta \geq \mu$, then the optimal return function V -defined by (3.4)- is given by*

$$V(x) = \frac{x}{\gamma+\lambda} \quad \forall x \geq 0.$$

Proof. Due to $\delta \geq \mu$, for any $\pi \in \mathscr{A}$,

$$\mathbb{E}\left[\int_0^{\tau^\pi \wedge \tau} e^{-\gamma t} R_t^\pi dt\right] = \mathbb{E}\left[\int_0^{+\infty} e^{-(\gamma+\lambda)t} R_{t\wedge\tau_\pi}^\pi dt\right].$$

However,

$$\mathbb{E}\left[R_{t\wedge\tau_\pi}^\pi\right] \leq x + \mathbb{E}\left[\int_0^{\tau_\pi\wedge t} (a_t^\pi \mu - \delta) ds\right] + \mathbb{E}\left[\int_0^{\tau_\pi\wedge t} a_t^\pi \sigma dW_s\right] \leq x,$$

and hence, we get for any $\pi \in \mathscr{A}$

$$\mathbb{E}\left[\int_0^{\tau_\pi\wedge\tau} e^{-\gamma t} R_t^\pi dt\right] = \mathbb{E}\left[\int_0^{+\infty} e^{-(\gamma+\lambda)t} R_{t\wedge\tau_\pi}^\pi dt\right] \leq \int_0^{+\infty} e^{-(\gamma+\lambda)t} x dt = \frac{x}{\gamma+\lambda}.$$

By combining this with the left hand side of (3.5), the lemma follows. □

Proposition 3.1.2. *Suppose that V, defined in (3.4), is twice continuously differentiable on $(0,+\infty)$. Then V satisfies the following HJB equation*

$$\max_{0\le a\le 1}\left(\frac{1}{2}a^2\sigma^2V''(x)+(a\mu-\delta)V'(x)-(\gamma+\lambda)V(x)+x\right)=0,\quad V(0)=0. \tag{3.9}$$

Proof. Thanks to Proposition 3.1.1, the return function defined in (3.4) can be transformed to a function similar to that of B. Højgaard and M. Taksar [15] with the only difference is the increase of the discount factor by an amount of λ. Therefore, V satisfies (3.9) and the proof can be obtain from [15]. □

3.2 Construction of the smooth solution to the HJB (3.9)

The aim of this section resides in constructing the solution to this HJB equation (3.9). This construction depends heavily on the relationship between the parameters of the model. Throughout this section, we define the following operator, for any $a\in\mathbb{R}$,

$$\mathscr{L}^aV(x):=\frac{1}{2}a^2\sigma^2V''(x)+(a\mu-\delta)V'(x)-cV(x)+x;\quad c:=\gamma+\lambda. \tag{3.10}$$

Besides this operator, the maximizer function $a(x)$, that is defined and given by

$$a(x):=arg\max_{a\in\mathbb{R}}\mathscr{L}^aV(x)=-\frac{\mu V'(x)}{\sigma^2V''(x)}\ge 0, \tag{3.11}$$

plays a central role in our forthcoming analysis. Thus, we start by deriving some of its properties in the following.

Proposition 3.2.1. *Suppose that $\delta<\mu$. Then the following assertions hold.*

(a) $a(x)\ge\frac{2\delta}{\mu}$ *for all* $x\ge 0$.

(b) *If* $\frac{2\delta}{\mu} \geq 1$, *then* $a(0) = \frac{\mu}{2(\mu-\delta)}$ *(we always have* $\frac{\mu}{2(\mu-\delta)} \geq \frac{2\delta}{\mu}$*).*

(c) *If* $\frac{2\delta}{\mu} < 1$, *then* $a(0) = \frac{2\delta}{\mu}$.

Proof. Put for any $x \geq 0$,

$$\widetilde{a} := \widetilde{a}(x) := arg \max_{0 \leq a \leq 1} \mathscr{L}^a V(x)$$

Therefore, we get

$$\left(\frac{\widetilde{a}^2 \sigma^2}{2} V''(x) + (\widetilde{a}\mu - \delta) V'(x) - cV(x) + x \right) = 0.$$

The latter is equivalent to

$$\frac{\sigma^2}{2} V''(x) \left(\widetilde{a}^2 + (\widetilde{a}\mu - \delta) \frac{2V'(x)}{\sigma^2 V''(x)} - \frac{2\big(cV(x) - x\big)}{\sigma^2 V''(x)} \right) = 0.$$

By inserting (3.11) into the latter equation we get

$$\frac{\sigma^2}{2} V''(x) \left(\widetilde{a}^2 - 2(\widetilde{a}\mu - \delta) \frac{a(x)}{\mu} - \frac{2\big(cV(x) - x\big)}{\sigma^2 V''(x)} \right) = 0.$$

However, we know that by the definition of V, $V''(x) < 0$, thus

$$\left(\widetilde{a}^2 - 2(\widetilde{a}\mu - \delta) \frac{a(x)}{\mu} - \frac{2\big(cV(x) - x\big)}{\sigma^2 V''(x)} \right) = 0.$$

Or more conveniently,

$$\Big(\widetilde{a}(x) - a(x) \Big)^2 - a(x) \left(a(x) - \frac{2\delta}{\mu} \right) = \frac{2\big(cV(x) - x\big)}{\sigma^2 V''(x)}.$$

This equation implies that $a(x) \geq \frac{2\delta}{\mu}$ for all $x \geq 0$, and assertion (a) is proved. If $a(x) \leq 1$, then $\widetilde{a}(x) = a(x)$ and

$$a(x) \left(a(x) - \frac{2\delta}{\mu} \right) = -\frac{2\big(cV(x) - x\big)}{\sigma^2 V''(x)}.$$

Then either $a(0) = 0$ or $a(0) = \frac{2\delta}{\mu}$, since $V(0) = 0$. If $a(x) > 1$, then $\widetilde{a}(x) = 1$, and

$$1 - 2a(x)\big(1 - \frac{\delta}{\mu}\big) = \frac{2\big(cV(x) - x\big)}{\sigma^2 V''(x)}.$$

Or equivalently,

$$a(x) = \frac{\mu}{2(\mu-\delta)}\left(1 - \frac{2\big(cV(x)-x\big)}{\sigma^2 V''(x)}\right) \geq \frac{\mu}{2(\mu-\delta)}. \tag{3.12}$$

In the following, we distinguish two cases depending on whether, $\frac{2\delta}{\mu} \geq 1$ or $\frac{2\delta}{\mu} < 1$.

1) If $\frac{2\delta}{\mu} \geq 1$, then thanks to assertion (a), we have $a(x) \geq 1$ and hence, $\widetilde{a}(x) = 1$ for all $x \geq 0$ and (3.12) holds. By substituting $x = 0$ in (3.12) we get $a(0) = \frac{\mu}{2(\mu-\delta)}$. This proves assertion (b).

2) If $\frac{2\delta}{\mu} < 1$, then either $a(0) \leq 1$ or $a(0) > 1$. If $a(0) \leq 1$, then $\widetilde{a}(0) = a(0)$ and hence $a(0) = \frac{2\delta}{\mu}$. If $a(0) > 1$, then $\widetilde{a}(0) = 1$ and $a(0) = \frac{\mu}{2(\mu-\delta)}$, however, the latter implies that $2\delta > \mu$ and that contradicts the assumption $\frac{2\delta}{\mu} < 1$. This implies that, when $\frac{2\delta}{\mu} < 1$, we always have $a(0) = \frac{2\delta}{\mu}$. This ends the proof of the proposition. □

3.2.1 The case when $\frac{\mu}{2} \leq \delta < \mu$ (medium liability rate)

This subsection addresses the case where $\frac{\mu}{2} \leq \delta < \mu$, which corresponds to the case of medium liability rate.

Theorem 3.2.1. *Suppose that $\frac{\mu}{2} \leq \delta < \mu$. Then the smooth solution to (3.9) is given by*

$$V(x) = \frac{x}{(\gamma+\lambda)} + \frac{\mu-\delta}{(\gamma+\lambda)^2}\left(1 - e^{r_- x}\right), \quad x \geq 0, \tag{3.13}$$

where

$$r_\pm = \frac{-(\mu-\delta) \pm \sqrt{(\mu-\delta)^2 + 2\sigma^2(\gamma+\lambda)}}{\sigma^2}. \tag{3.14}$$

Proof. Put $c := (\gamma+\lambda)$. In the case of $\frac{2\delta}{\mu} \geq 1$, thanks to Proposition (3.2.1)-(a), we have $a(x) \geq 1$ for all $x \geq 0$. Thus, we get $0 = \max_{0\leq a \leq 1} \mathscr{L}^a V(x) = \mathscr{L}^1 V(x) = 0$. Or equivalently,

$$\frac{1}{2}\sigma^2 V''(x) + (\mu-\delta)V'(x) - cV(x) + x = 0.$$

The general solution for this linear second order ODE is given by

$$V(x) = \frac{x}{c} + \frac{\mu - \delta}{c^2} + C_1 e^{r_+ x} + C e^{r_- x}, \quad x \geq 0. \tag{3.15}$$

Here C_1 and C are free parameters to be determined below, while r_+ and r_- are given in (3.14). Thanks to Proposition 3.1.1-(b), $V(x)/x$ converges to $1/c$ when x goes to $+\infty$. This allows us to deduce that $C_1 = 0$, and hence our function becomes

$$V(x) = \frac{x}{c} + \frac{\mu - \delta}{c^2} + C e^{r_- x}, \quad x \geq 0.$$

By using the fact that $V(0) = 0$, we calculate C and get $C = -(\mu - \delta)/c^2$. This proves that the smooth (twice differentiable) solution to (3.9) in this case takes the form of (3.13). To prove the converse, we remark that V given by (3.13) is twice continuously differentiable, and satisfies

$$V'(x) = \frac{1}{c} - \frac{\mu - \delta}{c^2} r_- e^{r_- x} > 0, \quad \text{and} \quad V''(x) = -\frac{\mu - \delta}{c^2} r_-^2 e^{r_- x} < 0, \quad \forall\, x \geq 0.$$

This proves that V of (3.13) is twice continuously differentiable and concave. Furthermore,

$$a(x) = -\frac{\mu V'(x)}{\sigma^2 V''(x)} = \frac{\mu c e^{-r_- x}}{\sigma^2 (\mu - \delta) r_-^2} - \frac{\mu}{\sigma^2 r_-},$$

and hence, $a(x)$ is an increasing function with

$$a(0) = \frac{\mu c}{\sigma^2 (\mu - \delta) r_-^2} - \frac{\mu}{\sigma^2 r_-} = \frac{\mu}{2(\mu - \delta)} \geq \frac{2\delta}{\mu} \geq 1.$$

This guarantees the fact that V having the form (3.13) satisfies (3.9). □

3.2.2 The case $\delta < \frac{\mu}{2}$ (small liability rate)

This subsection deals with the case when the liability rate δ is small enough, i.e., $\delta < \frac{\mu}{2}$. In this case, the construction of the solution to the HJB (3.9), relies on the following proposition.

Proposition 3.2.2. *Suppose $\delta < \frac{\mu}{2}$, and let r_- be defined by (3.14) and g and Δ be given by*

$$g(y) := exp\left(\frac{2\sigma^2}{\mu^2}(\gamma+\lambda)\ln(y) - \frac{2\sigma^2}{\mu^2}y\right), \quad y > 0, \quad and \quad \Delta := (\gamma+\lambda)\left(1+\frac{\mu}{\sigma^2 r_-}\right). \tag{3.16}$$

Then the following assertions hold.

(a) *The equation $L(y) = 0$, where $L(y)$ is given by*

$$L(y) := H(y) - \frac{\mu}{2\delta}H(\Delta) \quad and \quad H(y) := \frac{1}{yg(y)} - \int_y^\Delta \frac{dt}{t^2 g(t)} \quad y > 0, \tag{3.17}$$

has a unique root $y_0 \in (0,\Delta)$, (i.e., $y_0 = H^{-1}(\frac{\mu}{2\delta}H(\Delta))$ exists).

(b) *The function $F(y)$ defined by*

$$F(y) := \frac{2\sigma^2\delta}{\mu^2}\int_{y_0}^{y}\left[\frac{1}{y_0 g(y_0)} - \left(\int_{y_0}^{t}\frac{du}{u^2 g(u)}\right)\right]g(t)dt, \tag{3.18}$$

is invertible on $[0,\Delta]$ and F^{-1} is well defined on $[0,F(\Delta)]$.

Proof. Throughout this proof, we consider the notation

$$b = \frac{2\sigma^2}{\mu^2}, \quad \text{and} \quad c = \gamma+\lambda.$$

This proof has two parts, part 1) and part 2), where we prove assertions (a) and (b) respectively.

Part 1. Here, we prove assertion (a). On the one hand, it is clear that H is continuous on $(0,+\infty)$, $H(\Delta) = 1/\Delta g(\Delta)$, and

$$H(0+) = \lim_{y \longrightarrow 0+}\left[\frac{1}{yg(y)}\left(1 - yg(y)\int_y^\Delta \frac{dt}{t^2 g(t)}\right)\right] = +\infty\left(1-\frac{1}{1+bc}\right) = +\infty.$$

On the other hand, H is differentiable and

$$H'(y) = -\frac{(1+bc-by)g(y)}{y^2\left[g(y)\right]^2} + \frac{1}{y^2 g(y)} = \frac{b(y-c)}{y^2 g(y)}.$$

Then, we deduce that L is strictly decreasing on $(0,\Delta)$, since $\Delta \in (0,c)$ and $L'(y) = H'(y) < 0$, for any $y \in (0,\Delta)$. Furthermore,

$$L(0+) = +\infty, \quad \textit{and} \quad L(\Delta) = H(\Delta) - \frac{\mu}{2\delta}H(\Delta) = H(\Delta)\left(1 - \frac{\mu}{2\delta}\right) < 0. \tag{3.19}$$

This proves that there exists a unique root $y_0 \in (0,\Delta)$ such that $L(y_0) = 0$.

Part 2. Here, we prove assertion (b). To this end, it is enough to prove that F is strictly monotone on $[0,\Delta]$, as it is clearly continuous on this interval. Thus, for any $y \in (0,\Delta)$, we have

$$F'(y) = b\delta\left[\frac{1}{y_0 g(y_0)} - \int_{y_0}^{y}\frac{du}{u^2 g(u)}\right]g(y) = b\delta\left[H(y_0) + \int_{y}^{\Delta}\frac{du}{u^2 g(u)}\right]g(y),$$

see (3.17) for the definition of $H(y)$. Hence, $F'(y) > 0$ for any $y \in (0,\Delta)$ due to $\int_y^{\Delta}\frac{du}{u^2 g(u)} > 0$ $H(y_0) > 0$ and as it is a consequence of $L(y_0) = 0$. In fact, this equation implies

$$H(y_0) = \frac{\mu}{2\delta}H(\Delta) = \frac{\mu}{2\delta\Delta g(\Delta)} > 0.$$

Therefore, F is strictly increasing on the interval $(0,\Delta)$. This proves that F is invertible on $[0,\Delta]$. Furthermore, the function F^{-1} is well defined on $[F(0),F(\Delta)]$ and

$$F(0) = -\int_0^{y_0}\left[\frac{b\delta}{y_0 g(y_0)} + b\delta\left(\int_t^{y_0}\frac{du}{u^2 g(u)}\right)\right]g(t)dt < 0.$$

This proves that F^{-1} is well defined on $[0,F(\Delta)]$, and the proof of the proposition is complete.

$\square$

Now, we state the solution for the HJB equation (3.9).

Theorem 3.2.2. *Suppose that $\delta < \frac{\mu}{2}$. Let r_- and F be given by (3.14) and (3.18) respectively. Then the smooth solution to (3.10) is given by*

$$V(x) = \begin{cases} \int_0^x \frac{dt}{F^{-1}(x)}, & \textit{if } 0 \le x \le x_{FR}, \\ \frac{x}{\gamma+\lambda} + \frac{\mu-\delta}{(\gamma+\lambda)^2} - \frac{\mu}{(\sigma^2 r_- + \mu)(\gamma+\lambda)r_-}e^{r_-(x - x_{FR})}, & \textit{if } x > x_{FR}, \end{cases} \tag{3.20}$$

where

$$x_{FR} = F\left[(\gamma+\lambda)\left(1+\frac{\mu}{\sigma^2 r_-}\right)\right]. \tag{3.21}$$

Proof. For the sake of simplicity, throughout this proof, we put

$$c := (\gamma+\lambda), \quad b := \frac{2\sigma^2}{\mu^2} \quad \textit{and} \quad \Delta := (\gamma+\lambda)\big(1+\frac{\mu}{\sigma^2 r_-}\big). \tag{3.22}$$

Since $a(0) = \frac{2\delta}{\mu}$, see Proposition 3.2.1-(c). Then, in a neighborhood of zero, we have $a(x) < 1$ and hence $\mathscr{L}^{a(x)}V(x) = 0$. Thus,

$$a(x) = \frac{2\delta}{\mu} + \frac{2}{\mu}\left(\frac{cV(x)-x}{V'(x)}\right).$$

By combining this equation with (3.11), we get

$$\frac{V'^2(x)}{V''(x)} + b\left(\delta V'(x) + cV(x) - x\right) = 0. \tag{3.23}$$

By performing the change of variable $V'\big(X(z)\big) = e^{-z}$, which implies that $V''\big(X(z)\big) = -e^{-z}/X'(z)$, then we get

$$-X'(z)e^{-z} + b\left(\delta e^{-z} + cV\big(X(z)\big) - X(z)\right) = 0.$$

By differentiating and applying the above change of variable again, we obtain

$$-X''(z)e^{-z} + X'(z)e^{-z} + b\left(-\delta e^{-z} + cX'(z)e^{-z} - X'(z)\right) = 0.$$

Or equivalently,

$$X''(z) - \left(1+bc-be^{z}\right)X'(z) + b\delta = 0. \tag{3.24}$$

The general solution to this second order linear ODE, takes the form of

$$X'(z) = k_1(z)e^{(1+bc)z-be^{z}}, \quad z \in \mathbb{R}, \tag{3.25}$$

where $k_1(z)$ is a positive function to be determined. By differentiating (3.25) and inserting it with its resulting derivative in (3.24) afterwards, we get

$$k_1'(z) = -b\delta exp\Big(-(1+bc)z + be^z\Big), \quad z \in \mathbb{R}.$$

Therefore, by integrating over $(z_0, z]$, where $z_0 \in \mathbb{R}$ is a free constant to be determined later on, we get

$$k_1(z) = k_1 - b\delta \int_{z_0}^{z} e^{-(1+bc)t+be^t} dt, \quad k_1 := k_1(z_0) > 0.$$

By using the change of variable $u = e^t$ in this equation, we get

$$k_1(z) = k_1 - b\delta \int_{e^{z_0}}^{e^z} t^{-(2+bc)} e^{bt} dt, \quad z \in \mathbb{R}.$$

Therefore, by inserting this in (3.25), we obtain

$$X'(z) = k_1 e^{(1+bc)z-be^z} - b\delta e^{(1+bc)z-be^z} \int_{e^{z_0}}^{e^z} t^{-(2+bc)} e^{bt} dt, \quad z \in \mathbb{R}.$$

By integrating this equation over $(z_0, z]$ where $z_0 \in \mathbb{R}$, we get

$$X(z) = k_1 \int_{z_0}^{z} e^{(1+bc)t-be^t} dt - b\delta \int_{z_0}^{z} e^{(1+bc)t-be^t} \left(\int_{z_0}^{t} u^{-(2+bc)} e^{bu} du \right) dt + X(z_0).$$

An application of the change of variable as before, leads to

$$X(z) = k_1 \int_{e^{z_0}}^{e^z} t^{bc} e^{-bt} dt - b\delta \int_{e^{z_0}}^{e^z} t^{bc} e^{-bt} \left(\int_{e^{z_0}}^{t} u^{-(2+bc)} e^{bu} du \right) dt + X(z_0), \quad z \in \mathbb{R}.$$

By using the notation in (3.16) and (3.22), the function $X(z)$ becomes

$$X(z) = k_1 \int_{e^{z_0}}^{e^z} g(t) dt - b\delta \int_{e^{z_0}}^{e^z} g(t) \left(\int_{e^{z_0}}^{t} \frac{du}{u^2 g(u)} \right) dt + X(z_0), \quad \forall z \in \mathbb{R}.$$

Consider the following function

$$F_0(y) := \int_{e^{z_0}}^{y} \left[k_1 - b\delta \left(\int_{e^{z_0}}^{t} \frac{du}{u^2 g(u)} \right) \right] g(t) dt, \quad y > 0. \tag{3.26}$$

Then $X(z) = F_0(e^z) + X(z_0)$, and by combining this with $V'\big(X(z)\big) = e^{-z}$, we obtain

$$V'(x) = \frac{1}{F_0^{-1}(x - X(z_0))}.$$

By integrating this equation and using $V(0) = 0$, we get

$$V(x) = \int_0^x \frac{dt}{F_0^{-1}(t - X(z_0))}, \quad \text{for } 0 \leq x \leq x_{FR}. \tag{3.27}$$

For $x > x_{FR}$, $\max_{0 \leq a \leq 1} \mathscr{L}^{a(x)} V(x) = \mathscr{L}^1 V(x) = 0$. Thus we solve the following

$$\frac{1}{2}\sigma^2 V''(x) + (\mu - \delta)V'(x) - cV(x) + x = 0.$$

Similar to the previous subsection, the solution to this ODE is given by

$$V(x) = \frac{x}{c} + \frac{\mu - \delta}{c^2} + k_2 e^{r_-(x - x_{FR})} + k_3 e^{r_+(x - x_{FR})}, \quad \text{for } x > x_{FR},$$

where $r_\pm$ are defined in (3.14) that we recall below

$$r_\pm = \frac{-(\mu - \delta) \pm \sqrt{(\mu - \delta)^2 - 2\sigma^2 c}}{\sigma^2}.$$

Thanks to Proposition 3.1.1-(a), we have $\lim\limits_{x \to +\infty} V(x)/x = 1/c$. Hence k_3 should be null, otherwise $V(x)/x$ will go to infinity when x goes to infinity. Thus, our function V becomes

$$V(x) = \frac{x}{c} + \frac{\mu - \delta}{c^2} + k_2 e^{r_-(x - x_{FR})}, \quad \text{for } x > x_{FR}. \tag{3.28}$$

The remaining part of the proof focuses on determining the remaining free parameters k_1, k_2, x_{FR} and $X(z_0)$ and shows that F_0 coincides to F that is defined in (3.18). By combining (3.28), (3.11) and the fact that $a(x_{FR}) = 1$, we get

$$1 = -\frac{\mu\left(\frac{1}{c} + k_2 r_-\right)}{\sigma^2\left(k_2 r_-^2\right)}.$$

This allows us to calculate completely k_2 as follows

$$k_2 = -\frac{\mu}{(\sigma^2 r_- + \mu) c r_-}. \tag{3.29}$$

This proves (3.20) on $(x_R, +\infty)$.

Since $V(x)$ is continuously differentiable, the equality $V'(x_{FR}-) = V'(x_{FR}+)$ implies that

$$F_0^{-1}(x_{FR} - X(z_0)) = \frac{c}{1 + k_2 c r_-} = \Delta, \quad \text{or equivalently} \quad x_{FR} = F_0(\Delta) + X(z_0). \tag{3.30}$$

On the one hand, by combining (3.11) and the fact $a(0) = 2\delta/\mu$, we get

$$\frac{2\delta}{\mu} = -\frac{\mu V'(0)}{\sigma^2 V''(0)} = \frac{\mu}{\sigma^2} F_0'\left[F_0^{-1}(-X(z_0))\right] F_0^{-1}(-X(z_0)).$$

By using (3.26), we get

$$g\left[F_0^{-1}(-X(z_0))\right] F_0^{-1}(-X(z_0)) \left[k_1 - b\delta \int_{e^{z_0}}^{F^{-1}(-X(z_0))} \frac{du}{u^2 g(u)}\right] = b\delta$$

Thus,

$$\frac{b\delta}{K_0 g(K_0)} + b\delta \int_{e^{z_0}}^{K_0} \frac{du}{u^2 g(u)} - k_1 = 0, \quad \text{where} \quad K_0 = F_0^{-1}(-X(z_0)) \tag{3.31}$$

By combining the fact that $a(x_{FR}) = 1$ and (3.11), we get

$$F_0'[F_0^{-1}(x_{FR} - X(z_0))] F_0^{-1}(x_{FR} - X(z_0)) = \frac{\sigma^2}{\mu},$$

and by using (3.30) and the derivative of (3.26), this becomes

$$g(\Delta)\Delta \left[k_1 - b\delta \int_{e^{z_0}}^{\Delta} \frac{du}{u^2 g(u)}\right] = \frac{\sigma^2}{\mu},$$

or equivalently

$$k_1 - b\delta \int_{e^{z_0}}^{\Delta} \frac{du}{u^2 g(u)} - \frac{\sigma^2}{\mu \Delta g(\Delta)} = 0. \tag{3.32}$$

By adding equations (3.31) and (3.32) to each other, we obtain

$$\frac{b\delta}{K_0 g(K_0)} - \frac{\sigma^2}{\mu\Delta g(\Delta)} + b\delta\int_{\Delta}^{K_0}\frac{du}{u^2 g(u)} = 0,$$

which is equivalent to

$$\frac{1}{K_0 g(K_0)} - \int_{K_0}^{\Delta}\frac{du}{u^2 g(u)} = \frac{\mu}{2\delta\Delta g(\Delta)}. \tag{3.33}$$

This equation has the same form as (3.17), therefore $H(K_0) = \mu H(\Delta)/2\delta$ and the solution to it (see Proposition 3.2.2) implies that $F_0^{-1}(-X(z_0)) = K_0 = y_0$. Hence

$$x_{FR} = F_0(\Delta) - F_0(y_0), \quad \text{where} \quad -X(z_0) = F_0(y_0). \tag{3.34}$$

By substituting $K_0 = y_0$ in (3.31), we get

$$k_1 = \frac{b\delta}{y_0 g(y_0)} + b\delta\int_{e^{z_0}}^{y_0}\frac{du}{u^2 g(u)}.$$

Then by using this equation, we derive

$$\begin{aligned}
F_0(y) - F_0(y_0) &= \int_{e^{z_0}}^{y}\left[k_1 - b\delta\left(\int_{e^{z_0}}^{t}\frac{du}{u^2 g(u)}\right)\right]g(t)dt - \int_{e^{z_0}}^{y_0}\left[k_1 - b\delta\left(\int_{e^{z_0}}^{t}\frac{du}{u^2 g(u)}\right)\right]g(t)dt \\
&= \int_{y_0}^{y}\left[k_1 - b\delta\left(\int_{e^{z_0}}^{t}\frac{du}{u^2 g(u)}\right)\right]g(t)dt \\
&= F(y)
\end{aligned}$$

This proves that

$$y \geq 0, \quad F(y) = F_0(y) - F_0(y_0) = x,$$

or equivalently

$$F_0^{-1}(x + F_0(y_0)) = F^{-1}(x) \quad \text{for all} \quad x \geq 0.$$

As a result, we get (3.21) (i.e. $x_{FR} = F_0(\Delta) - F_0(y_0) = F(\Delta)$), and

$$V(x) = \int_0^x \frac{dt}{F_0^{-1}\big(t - X(z_0)\big)} = \int_0^x \frac{dt}{F^{-1}(t)}, \quad \text{for} \quad x \in [0, x_{FR}].$$

This proves that the solution to (3.9), in this case of $\delta < \frac{\mu}{2}$, takes the form of (3.20). To prove the converse, it is enough to remark that V is twice differentiable, and satisfies

$$V'(x) = \begin{cases} \frac{1}{F^{-1}(x)} > 0, & \text{if} \quad 0 < x \leq x_{FR}, \\ \frac{1}{c} - \frac{\mu}{c(\sigma^2 r_- + \mu)} e^{r_-(x - x_{FR})} > 0, & \text{if} \quad x > x_{FR}, \end{cases}$$

since $\sigma^2 r_- + \mu < 0$, and

$$V''(x) = \begin{cases} -\frac{1}{F'\left[F^{-1}(x)\right]\left[F^{-1}(x)\right]^2} < 0 & \text{if} \quad x \leq x_{FR}, \\ -\frac{\mu r_-}{c(\sigma^2 r_- + \mu)} e^{r_-(x - x_{FR})} < 0 & \text{if} \quad x > x_{FR}. \end{cases}$$

Hence V given by (3.20) is strictly concave, strictly increasing and twice continuously differentiable. Furthermore, we derive

$$a(x) = \begin{cases} \frac{\mu}{\sigma^2} F^{-1}(x) F'\left[F^{-1}(x)\right] & \text{if} \quad x \leq x_{FR}, \\ \frac{1}{\sigma^2 r_-}\left[\mu(\sigma^2 r_- + \mu) e^{-r_-(x - x_{FR})}\right] & \text{if} \quad x > x_{FR}. \end{cases}$$

Hence $a(x)$ is increasing and $a(X_{FR}) = 1$. This guarantees that V given by (3.20) satisfies (3.9) when $\delta < \frac{\mu}{2}$. □

Remark 3.2.1. *The integral on the right hand side of (3.26) is not defined at $-\infty$, therefore, we take the integral from $z_0 > -\infty$ such that $\int_{z_0}^{z} e^{-(1+bc)t + be^t} dt$ is defined.*

3.3 Optimal policy and verification theorem

In this section, we construct the optimal control policy based on the solution of the HJB equation obtained in the previous sections. For each, $x \geq 0$ we define

$$a^*(x) := arg \max_{0 \leq a \leq 1} \left(\frac{1}{2} a^2 \sigma^2 V''(x) + (a\mu - \delta) V'(x) - (\gamma + \lambda) V(x) + x \right). \tag{3.35}$$

The function $a^*(x)$ represents the optimal feedback control function for the control component a_t^π, $t \geq 0$. More precisely, the value $a^*(x)$ is the optimal risk that one should take when the value of the current reserve is x. Thanks to the previous section, we get the following.

Proposition 3.3.1. *Suppose that $\delta < \mu$. Let Δ be given by (3.16), and the functions F and a^* be defined in (3.18) and (3.35) respectively. Then the following assertions hold.*
(a) *If $\delta \geq \frac{\mu}{2}$, then $a^*(x) = 1$, for all $x \geq 0$.*
(b) *If $\delta < \frac{\mu}{2}$, then*

$$a^*(x) = \begin{cases} \frac{\mu}{\sigma^2} F'\left[F^{-1}(x)\right] F^{-1}(x), & \textit{if} \quad x \leq F(\Delta), \\ 1, & \textit{if} \quad x > F(\Delta). \end{cases}$$

Proof. Since our V is twice continuously differentiable, $V'(x) > 0$, and $V''(x) < 0$ for all $x \geq 0$. Then for any $x \in [0, +\infty)$, as a function of $a \in \mathbb{R}$, the following

$$\mathscr{L}^a V(x) = \frac{\sigma^2 a^2}{2} V''(x) + (a\mu - \delta) V'(x) - (\gamma + \lambda) V(x) + x,$$

is a concave function with a maximum attained at $a(x)$, and it is increasing on $[0, a(x)]$ and decreasing on $[a(x), +\infty)$. Therefore, when $\frac{2\delta}{\mu} \geq 1$, that implies $a(x) \geq 1$ (see (3.2.1)), we get $a^*(x) = 1$, on the one hand. On the other hand, when $\frac{2\delta}{\mu} < 1$, we get $a(x) < 1$ for $x < F(\Delta)$ and $a(x) \geq 1$ for $x \geq F(\Delta)$. Thus, similarly as above, this implies

$$a^*(x) = a(x) = \frac{\mu}{\sigma^2} F'\left[F^{-1}(x)\right] F^{-1}(x), \quad \text{for} \quad x \leq F(\Delta),$$

and $a^*(x) = 1$ for $x > F(\Delta)$. This completes the proof. □

Theorem 3.3.1. *Let a^* be defined in (3.35). Then the following assertions hold.*

(a) *The following SDE*

$$dX_t = \Big(a^*(X_t)\mu - \delta\Big)dt + a^*(X_t)\sigma dW_t, \quad X_0 = x, \tag{3.36}$$

has a unique solution that we denote by $R^* = (R_t^*)_{t\geq 0}$.

(b) *R^* is the optimal cash reserve, and $\pi^* = \Big(a^*(R_t^*); t \geq 0\Big)$ is the optimal risk control satisfying*

$$\mathbb{E}\int_0^{\tau^*} e^{-(\gamma+\lambda)t} R_t^* dt = V(x), \quad \forall x \geq 0, \tag{3.37}$$

where

$$\tau^* = \inf\{t \geq 0 : R_t^* = 0\}.$$

Proof. The proof of this theorem has two parts, where we prove assertion (a) and (b).

Part 1. Here, we prove assertion (a). Since μ and σ are constants and not functions of x then the coefficients of our stochastic differential equation (3.36) is $a^*(x)$. Hence, thanks to Theorem 2.3.1, the existence and uniqueness of the solution to this SDE boils down to prove that the following Lipschitz and growth conditions hold

$$|a^*(x_1) - a^*(x_2)|^2 \leq K|x_1 - x_2|^2, \quad |a^*(x)|^2 \leq K(1+|x|^2) \quad \text{for all } x_1, x_2 \geq 0. \tag{3.38}$$

In order to prove these two conditions, it is enough to prove the following

$$|a^*(x_1) - a^*(x_2)| \leq K_1|x_1 - x_2|, \quad x_1, x_2 \geq 0. \tag{3.39}$$

In fact, under this condition, the Lipschitz condition (the first condition of (3.38)) becomes obvious, while for the growth condition (the second condition) we take $x_1 = x$ and $x_2 = 0$ and get

$$|a^*(x)| \le |a^*(0)| + K_1|x| \le \max(1, K_1)(1+|x|),$$

since $a^*(0) = 1$ for $\delta \ge \mu/2$ and $a^*(0) = \frac{2\delta}{\mu}$ for $\delta < \mu/2$. Hence, the remaining part of this part focuses on proving (3.39). For $x_1, x_2 > F(\Delta)$, we get $a^*(x_1) = a^*(x_2) = 1$, and hence (3.39) is satisfied. Remark that the cases when $x_1 > F(\Delta)$ and $x_2 \le F(\Delta)$, and when $x_1 \le F(\Delta)$ and $x_2 > F(\Delta)$ reduce to $x_1, x_2 \le F(\Delta)$, since a^* is a constant over $[F(\Delta), +\infty)$. For the case when x_1 and x_2 belong to $[0, F(\Delta)]$ we use Taylor's expansion and get

$$|a^*(x_1) - a^*(x_2)| = |(x_1 - x_2)\frac{d}{dy}a^*(y)|$$

where $y = \alpha x_1 + (1-\alpha)x_2$, and $\alpha \in (0,1)$. Therefore, it is enough to show that $\frac{d}{dy}a^*(y)$ is bounded over $[0, F(\Delta)]$. To this end, we calculate

$$\frac{d}{dy}a^*(y) = \frac{d}{dy}\Big[-\frac{\mu V'(y)}{\sigma^2 V''(y)}\Big] = \frac{\mu}{\sigma^2}\frac{d}{dy}\Big[F'\big(F^{-1}(y)\big)F^{-1}(y)\Big] = \frac{\mu}{\sigma^2} + \frac{\mu}{\sigma^2}\frac{F''\big(F^{-1}(y)\big)F^{-1}(y)}{F'\big(F^{-1}(y)\big)}.$$

By taking the supremum over $0 \le y \le F(\Delta)$ on both sides, we get

$$\sup_{0\le y\le F(\Delta)}|\frac{d}{dy}a^*(y)| \le \frac{\mu}{\sigma^2} + \frac{\mu}{\sigma^2}\sup_{0\le y\le F(\Delta)}|\frac{F''\big(F^{-1}(y)\big)F^{-1}(y)}{F'\big(F^{-1}(y)\big)}|.$$

Since $F^{-1}$is increasing on $[F(0), F(\Delta)]$ and $F^{-1}(0) = y_0$. This inequality becomes

$$\sup_{0\le y\le F(\Delta)}|\frac{d}{dy}a^*(y)| \le \frac{\mu}{\sigma^2} + \frac{\mu}{\sigma^2}\sup_{y_0\le z\le \Delta}|\frac{F''(z)z}{F'(z)}|. \tag{3.40}$$

Direct calculation of $|\frac{F''(z)z}{F'(z)}|$, leads to

$$\begin{aligned}|\frac{F''(z)z}{F'(z)}| &= |\frac{\Big[(\frac{bc}{z} - b)F'(z) - \frac{b\delta}{z^2}\Big]z}{F'(z)}| = |b(c-z) - \frac{b\delta}{zF'(z)}| \le b(c-\Delta) + \frac{b\delta}{|zF'(z)|}\\ &\le b(c-\Delta) + \big[y_0 g(y_0)\big]^{-1}\Big[|H(y_0) + \int_z^\Delta \frac{du}{u^2 g(u)}|\Big]^{-1}\end{aligned} \tag{3.41}$$

Remark that the function

$$K(z) := \left[H(y_0) + \int_z^{\Delta} \frac{du}{u^2 g(u)}\right],$$

is strictly decreasing (since $K'(z) = -1/z^2 g(z) < 0$),and hence we get,

$$\frac{1}{y_0 g(y_0)} \geq K(z) \geq K(\Delta) = H(y_0) = \frac{\mu}{2\delta} H(\Delta) = \frac{\mu}{2\delta \Delta g(\Delta)} > 0.$$

By inserting the lower inequality in (3.41), we get

$$|\frac{F''(z)z}{F'(z)}| \leq b(c-\Delta) + \left[y_0 g(y_0)\right]^{-1} \frac{2\delta\Delta g(\Delta)}{\mu}.$$

Then by combining this with (3.40), we deduce that $\frac{d}{dy} a^*(y)$ is bounded over $[0, F(\Delta)]$ by a constant K. This completes the proof of assertion (a).

Part 2. Here, we prove assertion (b). By applying Itô to $e^{-(\gamma+\lambda)t\wedge\tau^*} V(R^*_{t\wedge\tau^*})$, we deduce

$$\begin{aligned}
&e^{-(\gamma+\lambda)t\wedge\tau^*} V(R^*_{t\wedge\tau^*}) = \\
&V(x) - \int_0^{t\wedge\tau^*} (\gamma+\lambda) e^{-(\gamma+\lambda)s} V(R^*_s) ds + \int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s} \Big(a^*(R^*_s)\mu - \delta\Big) V'(R^*_s) ds \\
&+ \int_0^{t\wedge\tau^*} \frac{1}{2} e^{-(\gamma+\lambda)s} \sigma^2 \big(a^*(R^*_s)\big)^2 V''(R^*_s) ds + \int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s} \sigma a^*(R^*_s) V'(R^*_s) dW_s. \qquad (3.42)
\end{aligned}$$

Since $0 \leq V'(x) \leq K_1$, where $K_1 = \max\left(\frac{1}{y_0}, \frac{\sigma^2 r_-}{(\gamma+\lambda)(\sigma^2 r_- + \mu)}\right)$, we deduce that

$$\int_0^{t\wedge\tau^*} \sigma^2 \big(a^*(R^*_s)\big)^2 V'^2(R^*_s) ds \leq \sigma^2 K_1 t,$$

and hence, $\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s} \sigma a^*(R^*_s) V'(R^*_s) dW_s$ is a martingale which implies

$$\mathbb{E}\left[\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s} \sigma a^*(R^*_s) V'(R^*_s) dW_s\right] = 0.$$

Therefore, by using this equation and taking the expectation in both sides of (3.42), we obtain

$$\begin{aligned}\mathbb{E}\left[e^{-(\gamma+\lambda)t\wedge\tau^*}V(R^*_{t\wedge\tau^*})\right] &= V(x)-\mathbb{E}\left[\int_0^{t\wedge\tau^*}(\gamma+\lambda)e^{-(\gamma+\lambda)s}V(R^*_s)ds\right]\\ &\quad+\mathbb{E}\int_0^{t\wedge\tau^*}e^{-(\gamma+\lambda)s}\left[\frac{\sigma^2\big(a^*(R^*_s)\big)^2}{2}V''(R^*_s)+\Big(a^*(R^*_s)\mu-\delta\Big)V'(R^*_s)\right]ds.\\ &= V(x)+\mathbb{E}\left[\int_0^{t\wedge\tau^*}e^{-(\gamma+\lambda)s}\Big(\mathscr{L}^{a^*(R^*_s)}V(R^*_s)-R^*_s\Big)ds\right], \end{aligned} \tag{3.43}$$

where we recall

$$\mathscr{L}^aV(y)=\frac{\sigma^2a^2}{2}V''(y)+(a\mu-\delta)V'(y)-(\gamma+\lambda)V(y)+y,\quad a\geq 0, x\geq 0.$$

Since $\mathscr{L}^{a^*(y)}V(y)=0$ for all $y\geq 0$, and in particular for $y=R^*_s$, (3.43) becomes

$$\mathbb{E}\left[e^{-(\gamma+\lambda)(t\wedge\tau^*)}V(R^*_{t\wedge\tau^*})\right]=V(x)-\mathbb{E}\left[\int_0^{t\wedge\tau^*}e^{-(\gamma+\lambda)s}R^*_sds\right],$$

or equivalently

$$\mathbb{E}\left[e^{-(\gamma+\lambda)(t\wedge\tau^*)}V(R^*_{t\wedge\tau^*})\right]+\mathbb{E}\left[\int_0^{t\wedge\tau^*}e^{-(\gamma+\lambda)s}R^*_sds\right]=V(x). \tag{3.44}$$

The first term on the left hand side of this equation can be written as

$$\mathbb{E}\left[e^{-(\gamma+\lambda)\tau^*}V(R^*_{\tau^*})1_{\{\tau^*<t\}}+e^{-(\gamma+\lambda)t}V(R^*_t)1_{\{t\leq\tau^*\}}\right].$$

A combination of this with the fact that $V(0)=0$ and on $\{\tau^*<+\infty\}$, $R^*_{\tau^*}=0$ p-a.s, we get

$$0\leq\mathbb{E}\left[e^{-(\gamma+\lambda)(t\wedge\tau^*)}V(R^*_{t\wedge\tau^*})\right]=\mathbb{E}\left[e^{-(\gamma+\lambda)t}V(R^*_t)1_{\{\tau^*\geq t\}}\right]\leq e^{-(\gamma+\lambda)t}\mathbb{E}\left[V(R^*_t)\right].$$

By using the fact that $V(x)\leq\frac{x}{(\gamma+\lambda)}+\frac{\mu-\delta}{(\gamma+\lambda)^2}$ in the last inequality, we get

$$0\leq\mathbb{E}\left[e^{-(\gamma+\lambda)(t\wedge\tau^*)}V(R^*_{t\wedge\tau^*})\right]=\mathbb{E}\left[e^{-(\gamma+\lambda)t}V(R^*_t)1_{\{\tau^*\geq t\}}\right]\leq e^{-(\gamma+\lambda)t}\left[\frac{\mathbb{E}\left[R^*_t\right]}{\gamma+\lambda}+\frac{\mu-\delta}{(\gamma+\lambda)^2}\right]. \tag{3.45}$$

Remark that,

$$\mathbb{E}[R_t^*] = x + \mathbb{E}\int_0^t \big(a^*(R_s^*)\mu - \delta\big)ds \le x + (\mu - \delta)t, \quad t \ge 0.$$

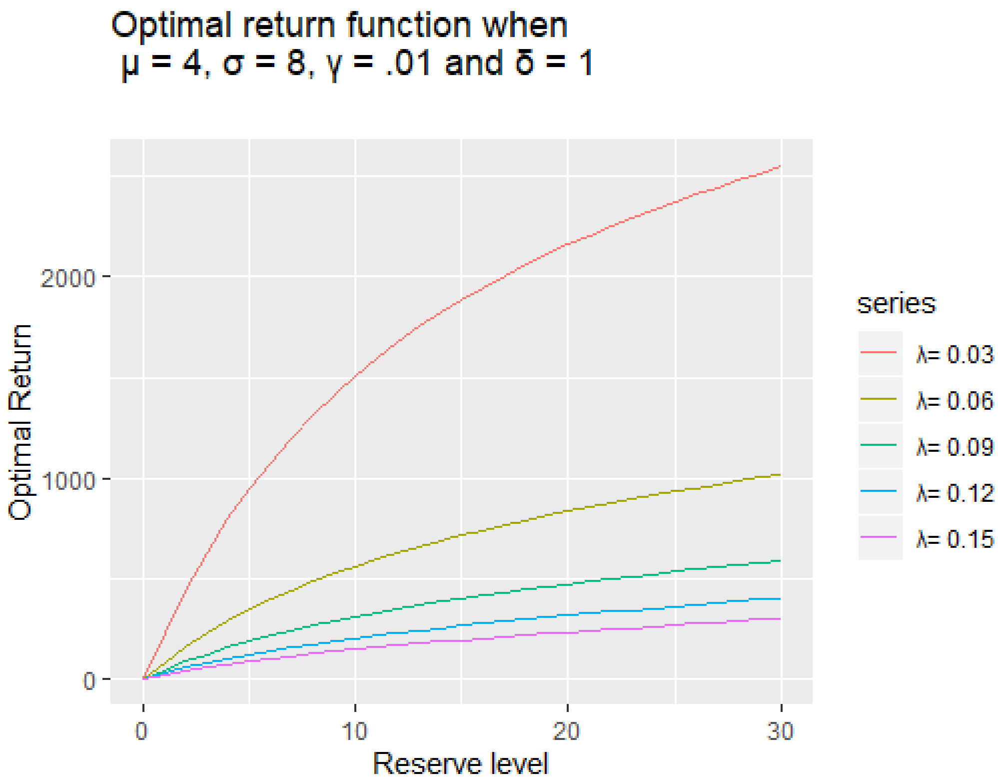
Then by inserting this in (3.45), we get

$$\mathbb{E}\left[e^{-(\gamma+\lambda)t}V(R_t^*)1_{\{\tau^*\ge t\}}\right] \le e^{-(\gamma+\lambda)t}\left[\frac{x}{(\gamma+\lambda)} + \frac{(\mu-\delta)t}{(\gamma+\lambda)} + \frac{\mu-\delta}{(\gamma+\lambda)^2}\right]. \tag{3.46}$$

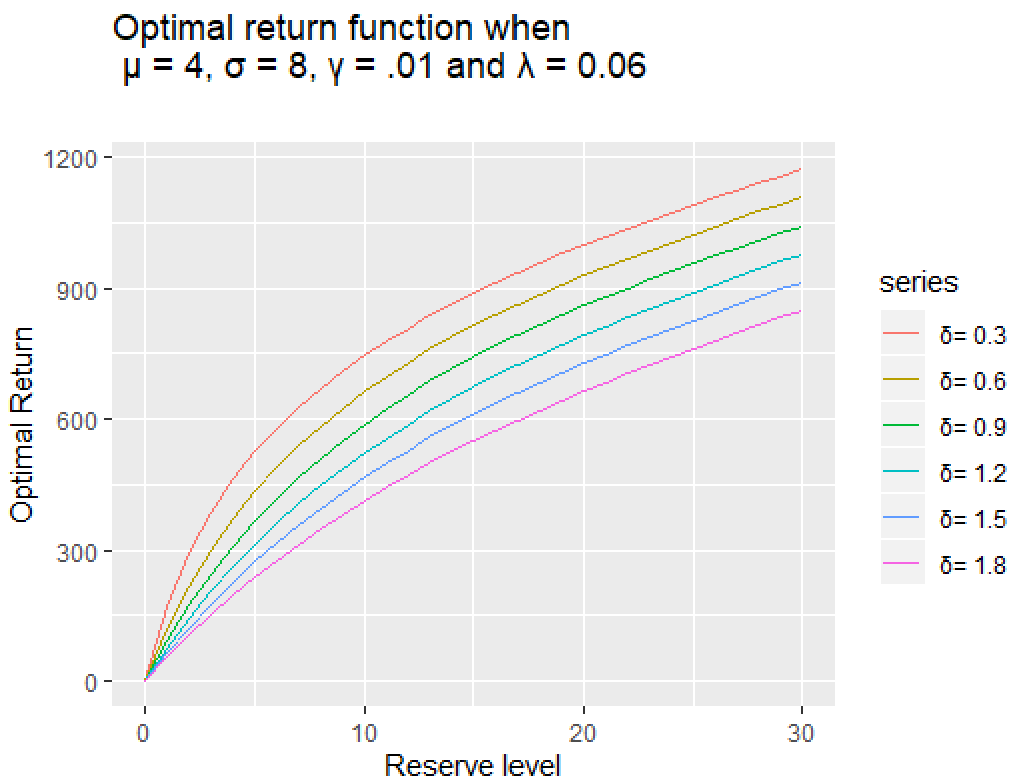
By combining (3.45) and (3.46) and taking the $t \longrightarrow +\infty$, we get (3.37). Thanks to (3.37) and (3.36) it is obvious that $\pi^* = (a^*(R_t^*), t \ge 0)$ is the optimal policy and R^* is the optimal cash reserve that corresponds to it. This ends the proof. □

3.4 Graphical illustrations

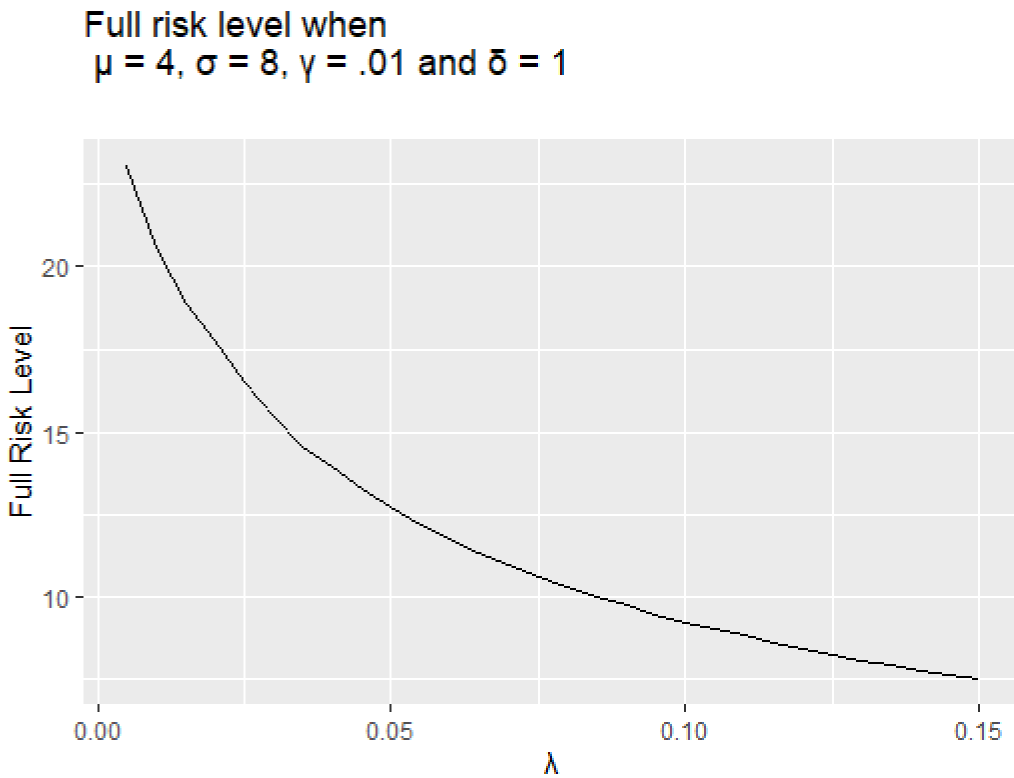
In this section we present some sensitivity analysis of the added parameters to the model λ and δ for the case of $\delta < \mu/2$. Here we do not illustrate the interplay of the other model parameters as similar work has been previously done (see for example B. Højgaard and M. Taksar [15]).



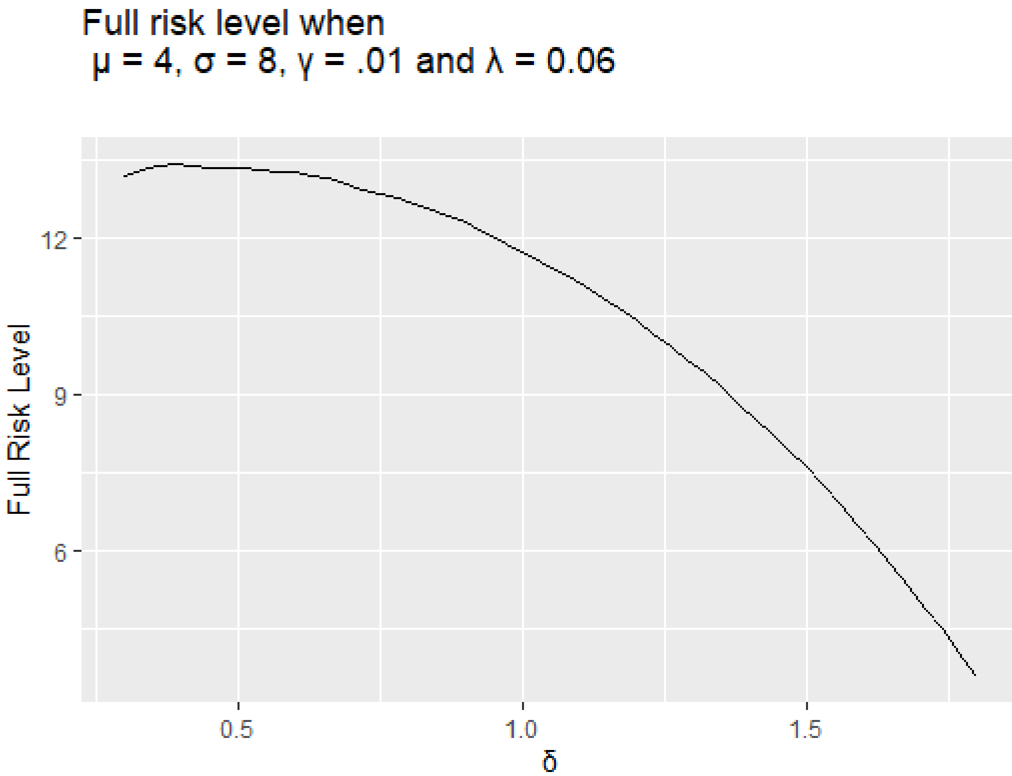
The above graph illustrates the interaction between the reserve level x and the optimal return function $V(x)$ for different insurance intensity rates λ. As we can see, the optimal return function decreases drastically in value as λ increases as well as it becomes flatter for a large enough intensity rate.



Similar to the previous Graph, The optimal return function V decreases in value as the liability increases. However, unlike the case for different λ, the optimal return function decreases the same amount that corresponds to the change of δ and does not change shape, it remains concave for larger liability rates.



This graph illustrates the full risk threshold as a function of λ. The threshold level x_R decreases as λ increase. In other words, as the intensity rate becomes bigger, the level of taking full risk becomes smaller.



Similar full risk level reaction to the increase in liability rate, x_R decreases when δ increases. However, the full risk level decreases much slower (concavely shaped) than that corresponding to an increase of intensity rate (convex shaped).

Chapter 4

Interplay between random horizon and constraints on risk control

This chapter alters the model of Chapter 3 by putting $\delta = 0$ and adding constraints on the risk control $(a_t^\pi)_{t\geq 0}$. These constraints consists of a allowing a_t^π to belong to $[\alpha, \beta]$ only, where $0 < \alpha < \beta$. The lower bound α might reflect the fact that the company should do minimum business α. This situation occurs when the company is public for instance.

Four sections are developed in this chapter. The first section defines the model (mathematically and economically) and the objectives in terms of a control problem. The second section explores the solution to a Hamilton-Jacobi-Bellman (HJB) associated to the main control problem. The third section determines the optimal policies for this control problem, while the last section presents some graphical illustration of the results.

4.1 The mathematical model and its preliminary analysis

As in the previous chapter, this section introduces the mathematical and economical model, defines our main objective, and presents our first analysis of the model afterwards.

4.1.1 The mathematical model and our objective

Similar to Chapter 3, we start by introducing the model. Consider a firm with reserve R_t at any time t where R_t is a stochastic process and a solution to the following stochastic differential equation:

$$dR_t = \mu dt + \sigma dW_t, \quad R_0 = x. \tag{4.1}$$

For a risk exposure a_t^π at time t and control policy $\pi = (a_t^\pi, t \geq 0)$, our reserve equation becomes:

$$dR_t^\pi = a_t^\pi(\mu dt + \sigma dW_t), \quad R_0^\pi = x, \tag{4.2}$$

and the bankruptcy time is given by

$$\tau_\pi = \inf\Big\{t \geq 0 : R_t^\pi = 0\Big\}.$$

Here, by convention, we put $\inf(\emptyset) = +\infty$.

<u>The model for random horizon.</u> Throughout this chapter, τ is a random time that is independent of the reserve process $R = (R_t,\, t \geq 0)$ (or equivalently $\mathscr{F}_\infty = \sigma(W_t, t \geq 0)$), and has an exponential distribution with mean λ^{-1}, i.e., $\mathbb{E}[\tau] = \lambda^{-1}$. Thus, the survival probability for this random time is given by

$$G_t := P(\tau > t | \mathscr{F}_t) = \exp(-\lambda t), \quad t \geq 0. \tag{4.3}$$

We define the return function of an initial reserve x under a control policy π as the total discounted reserve from time 0 to the default time or the random time τ (whichever comes first).

The objective. Our objective consists of finding the optimal return function and describing the optimal policy $\pi \in \mathscr{A}$ such that

$$V(x) := \sup_{\pi \in \mathscr{A}} \mathbb{E}\left[\int_0^{\tau_\pi \wedge \tau} e^{-\gamma t} R_t^\pi dt\right] = \mathbb{E}\left[\int_0^{\tau \wedge \tau_{\pi^*}} e^{-\gamma t} R_t^{\pi^*} dt\right]. \tag{4.4}$$

where $\mathscr{A}$ denotes the set of all admissible control policies.

4.1.2 Properties of the function V

Proposition 4.1.1. *The following assertions hold.*

(a) *The optimal value function V, defined in (4.4), satisfies*

$$\frac{x}{\gamma+\lambda} \le V(x) \le \frac{x}{\gamma+\lambda} + \frac{\mu\beta}{(\gamma+\lambda)^2}, \quad \forall x \ge 0. \tag{4.5}$$

(b) *The function V is concave and satisfies*

$$V(x) = \sup_{\pi \in \mathscr{A}} \mathbb{E}\left[\int_0^{\tau_\pi} e^{-(\gamma+\lambda)t} R_t^\pi dt\right].$$

Proof. The proof of this proposition is similar to Proposition (3.1.1) from Chapter 3. □

Proposition 4.1.2. *Suppose that V, defined in (4.4), is twice continuously differentiable on $(0,+\infty)$. Then V satisfies the following HJB equation*

$$\max_{\alpha \le a \le \beta} \left(\frac{1}{2}a^2\sigma^2 V''(x) + a\mu V'(x) - (\gamma+\lambda)V(x) + x\right) = 0, \quad V(0) = 0. \tag{4.6}$$

Proof. The proof follows directly from Proposition 3.1.2 by putting $\alpha = 0$ and $\beta = 1$. □

4.2 Construction of the smooth solution to the HJB (4.6)

The aim of this section lies in constructing the smooth solution to this HJB equation (4.6). Throughout this section, we define the following operator, for any $a \in \mathbb{R}$,

$$\mathscr{L}^a V(x) := \frac{1}{2}a^2\sigma^2 V''(x) + a\mu V'(x) - cV(x) + x; \quad c := \gamma + \lambda. \tag{4.7}$$

Besides this operator, the maximizer function $a(x)$, that is defined and is given by

$$a(x) := arg \max_{a\in\mathbb{R}} \mathscr{L}^a V(x) = -\frac{\mu V'(x)}{\sigma^2 V''(x)} \geq 0, \tag{4.8}$$

plays a central role in our forthcoming analysis. Thus, we start by deriving some of its properties in the following section.

Proposition 4.2.1. *It holds that*

$$a(0) = \frac{\alpha}{2}.$$

Proof. Put

$$\widetilde{a} = arg \max_{\alpha\leq a\leq\beta} \mathscr{L}^a V(0).$$

Thus, we get

$$\frac{\sigma^2\widetilde{a}^2}{2}V''(0) + \mu\widetilde{a}V'(0) = 0. \tag{4.9}$$

From (4.8), we get $a(0) = -\frac{\mu V'(0)}{\sigma^2 V''(0)}$, and by inserting it in (4.9), we deduce $\widetilde{a}^2/2 - \widetilde{a}a(0) = 0$. Since $\widetilde{a}$ cannot be zero as it belongs to $[\alpha,\beta]$, then $a(0) = \frac{\widetilde{a}}{2}$. Now, we discuss the following cases. If $a(0) \leq \alpha$, then $\widetilde{a} = \alpha$ and as a result, $a(0) = \frac{\alpha}{2}$. If $\alpha \leq a(0) \leq \beta$, then $a(0) = \widetilde{a}$ and this contradicts the fact that $a(0) = \frac{\widetilde{a}}{2}$. Thus, we certainly have $a(0) = \frac{\alpha}{2}$. This completes the proof of the proposition. $\square$

4.2.1 Solution of the HJB equation

We start this section by deriving an intermediate lemma. To this end, we define the following notation

$$G(y) := \int_0^y g(t)dt, \quad g(t) = \exp\left(\frac{2\sigma^2}{\mu^2}(\gamma+\lambda)\ln(t) - \frac{2\sigma^2}{\mu^2}t\right), \quad t > 0, \tag{4.10}$$

and

$$H(y) := yg(y), \quad K(y) := \frac{G(y)}{H(y)}, \quad y \geq 0, \quad c := \gamma+\lambda \quad \text{and} \quad \Delta := \sqrt{1+\frac{2\sigma^2}{\mu^2}c}. \tag{4.11}$$

Lemma 4.2.1. *Let K be the function defined by (4.11). If the condition*

$$K\left(H^{-1}\big(\alpha H(c\Delta/\Delta+1)/\beta\big)\right) > \frac{1}{\Delta+1}\ln\left(\frac{2\Delta}{\Delta-1}\right) \tag{4.12}$$

holds, then there exists $\tilde{y} \in (0, c\Delta/(\Delta+1))$ such that

$$L(\tilde{y}) = 0, \tag{4.13}$$

where

$$L(y) := (\Delta-1)^2\left(y - \frac{c\Delta}{\Delta-1}\right)e^{(\Delta-1)K(y)} + (\Delta+1)^2\left(\frac{c\Delta}{\Delta+1} - y\right)e^{-(\Delta+1)K(y)} + 4\Delta y e^{-2K(y)}. \tag{4.14}$$

Proof. Put

$$y_0 := \frac{c\Delta}{\Delta+1}.$$

Since

$$\lim_{y\to 0} K(y) = \frac{1}{1+bc} = \frac{1}{\Delta^2},$$

then

$$L(0+) = \lim_{y\to 0} L(y) = \left[-c\Delta(\Delta-1)e^{\frac{1}{\Delta}} + c\Delta(\Delta+1)e^{-\frac{1}{\Delta}} \right] e^{-\frac{1}{\Delta^2}}$$
$$= c\Delta[(\Delta+1)e^{-\frac{1}{\Delta}} - (\Delta-1)e^{\frac{1}{\Delta}}]e^{-\frac{1}{\Delta^2}}.$$

Notice that $L(0+) > 0$ if and only if $(\Delta+1)e^{-\frac{1}{\Delta}} - (\Delta-1)e^{\frac{1}{\Delta}} > 0$ which is equivalent to

$$\ln\left(\frac{\Delta+1}{\Delta-1}\right) - \frac{2}{\Delta} > 0.$$

Let $l(\Delta) = \ln\big((\Delta+1)/(\Delta-1)\big) - 2/\Delta$. Then

$$l'(\Delta) = \frac{\frac{-2}{(\Delta-1)^2}}{\frac{\Delta+1}{\Delta-1}} - \frac{2}{\Delta} = \frac{-2}{\Delta^2-1} + \frac{2}{\Delta} < 0.$$

This proves that the function l is strictly decreasing and $l(+\infty) = 0$ and $l(1+) = +\infty$. Hence, $l(\Delta) > 0$ which implies that $L(0+) > 0$. Further,

$$L(y_0-) = \left[\frac{-2c\Delta(\Delta-1)}{\Delta+1} e^{\Delta K(y_0)} + \frac{4c\Delta^2}{\Delta+1} e^{-K(y_0)} \right] e^{-K(y_0)}.$$

Since $L(y_0-) < 0$ if and only if $K(y_0) > \ln(2\Delta/(\Delta-1))/(\Delta+1)$ where $H^{-1}\big(\alpha H(y_0)/\beta\big) < H^{-1}\big(H(y_0)\big) = y_0$. Thus, to prove that $L(y) = 0$ has a root $\tilde{y}$, it is enough to prove that the function K is increasing. From the definition of $K(y)$ (4.11), we get

$$K'(y) = \frac{g(y)\Big(H(y) - G(y)(1+bc-by)\Big)}{H'^2(y)}.$$

$K'(y) > 0$ if and only if $k(y) = H(y) - G(y)(1+bc-by) > 0$. By differentiating this equation, we get $k'(y) = g(y)(1+bc-by) - g(y)(1+bc-by) + bG(y)$ which implies $k'(y) = bG(y) > 0$. Thus, K is increasing. This ends the proof. $\square$

The following theorem states the form of the smooth solution to the HJB equation (4.6), which has three different forms depending on the level of the cash reserve.

Theorem 4.2.1. *Let G, H and K be the functions defined in (4.10)-(4.11). Suppose (4.12) holds, and $\alpha = \beta H(\tilde{y})/H(c\Delta/(\Delta+1))$, where $\tilde{y}$ is a root to $L(y) = 0$ in Lemma 4.2.1.*
Then the smooth solution to (4.6) is given by

$$V(x) = \begin{cases} \dfrac{x}{\gamma+\lambda} + \dfrac{\alpha\mu}{(\gamma+\lambda)^2}(1 - e^{r_-(\alpha)x}) + C_1(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}), & if \quad 0 \le x \le x_\alpha, \\ \\ \displaystyle\int_{x_\alpha}^{x} \frac{dt}{G^{-1}\big(t/k_1\big)} + V(x_\alpha), & if \quad x_\alpha < x \le x_\beta, \\ \\ \dfrac{x}{\gamma+\lambda} + \dfrac{\beta\mu}{(\gamma+\lambda)^2} + \dfrac{\exp(r_-(\beta)(x - x_\beta))}{(\gamma+\lambda)\Delta r_-(\beta)}, & if \quad x > x_\beta. \end{cases}$$

Here

$$r_\pm(\kappa) = \frac{-\mu \pm \sqrt{\mu^2 + 2\sigma^2 c}}{\kappa\sigma^2} \tag{4.15}$$

$$k_1 = \frac{\beta\sigma^2}{\mu H(c\Delta/(\Delta+1))}, \tag{4.16}$$

$$C_1 = \frac{\alpha\sigma^2[-1 + 2\Delta(\Delta-1)^{-1}\exp(-(\Delta+1)K(\tilde{y}))]}{c\mu\Delta[(\Delta-1)\exp((\Delta-1)K(\tilde{y})) - (\Delta+1)\exp(-(\Delta+1)K(\tilde{y}))]}, \tag{4.17}$$

$$x_\alpha = \frac{\alpha\sigma^2}{\mu}K(\tilde{y}), \quad and \quad x_\beta = \frac{\beta\sigma^2}{\mu}K(c\Delta/(\Delta+1)). \tag{4.18}$$

Proof. This proof has three parts. Part 1 constructs the smooth solution to (4.6), and shows that it has the form described in the theorem. Part 2 determines the constants involved in the construction performed in part 1. Part 3 checks the concavity and the smoothness of the obtained function and argue that is in fact a solution to the HJB.

Part 1. This part constructs the smooth solution to the HJB. Throughout the proof, we put

$$c := \gamma + \lambda, \quad b := \frac{2\sigma^2}{\mu^2}, \quad y_0 := \frac{c\Delta}{\Delta + 1}, \quad \text{and} \quad \Delta := \sqrt{1 + bc}. \tag{4.19}$$

Put

$$x_\alpha := \inf\{x \geq 0 : \quad a(x) > \alpha\}.$$

Then for any $x \in [0, x_\alpha)$, $a(x) \leq \alpha$, and due to the continuity of $a(x)$ we have $a(x_\alpha) = \alpha$. This implies

$$\max_{\alpha \leq a \leq \beta} \mathscr{L}^a V(x) = \mathscr{L}^\alpha V(x) = 0,$$

or equivalently

$$\frac{1}{2}\alpha^2\sigma^2 V''(x) + \alpha\mu V'(x) - cV(x) + x = 0.$$

This is a second order linear non-homogeneous ODE that has a solution of the form

$$V(x) = \frac{x}{c} + \frac{\alpha\mu}{c^2} + C_1 e^{r_+(\alpha)x} + C_2 e^{r_-(\alpha)x}, \quad 0 \leq x \leq x_\alpha, \tag{4.20}$$

where $r_\pm(\alpha)$ are given by (4.15). Due to $V(0) = 0$, we get $C_2 = -C_1 - \alpha\mu/c^2$. By inserting this in (4.20), our function becomes

$$V(x) = \frac{x}{c} + \frac{\alpha\mu}{c^2}(1 - e^{r_-(\alpha)x}) + C_1(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}), \quad 0 \leq x \leq x_\alpha. \tag{4.21}$$

The first and second derivatives of the latter equation are given as

$$V'(x) = \frac{1}{c} - \frac{\alpha\mu r_-(\alpha)}{c^2} e^{r_-(\alpha)x} + C_1(r_+(\alpha)e^{r_+(\alpha)x} - r_-(\alpha)e^{r_-(\alpha)x}),$$

and

$$V''(x) = -\frac{\alpha\mu r_-^2(\alpha)}{c^2} e^{r_-(\alpha)x} + C_1(r_+^2(\alpha)e^{r_+(\alpha)x} - r_-^2(\alpha)e^{r_-(\alpha)x}).$$

By combining these two equations with $a(x_\alpha) = \alpha$ and $a(x) = -\frac{\mu V'(x)}{\sigma^2 V''(x)}$, we get

$$\begin{aligned} &-\frac{\alpha\sigma^2}{\mu}\left[-\frac{\alpha\mu r_-^2}{c^2}e^{r_-(\alpha)x_\alpha} + C_1(r_+^2 e^{r_+(\alpha)x_\alpha} - r_-^2 e^{r_-(\alpha)x_\alpha})\right] \\ &= \frac{1}{c} - \frac{\alpha\mu r_-}{c^2}e^{r_-(\alpha)x_\alpha} + C_1(r_+(\alpha)e^{r_+(\alpha)x_\alpha} - r_-(\alpha)e^{r_-(\alpha)x_\alpha}) \end{aligned}$$

or equivalently

$$\begin{aligned} 0 = &\frac{1}{c} - \frac{\alpha\mu r_-(\alpha)}{c^2}e^{r_-(\alpha)x_\alpha}\big(1 + \frac{\alpha\sigma^2 r_-(\alpha)}{\mu}\big) \\ &+ C_1\left[r_+(\alpha)e^{r_+(\alpha)x_\alpha}\big(1 + \frac{\alpha\sigma^2 r_+(\alpha)}{\mu}\big) - r_-(\alpha)e^{r_-(\alpha)x_\alpha}\big(1 + \frac{\alpha\sigma^2 r_-(\alpha)}{\mu}\big)\right]. \end{aligned}$$

By combining this equation with Δ from (4.19) and (4.15), we get

$$0 = \frac{1}{c} + \frac{\alpha\mu r_-}{c^2}e^{r_-(\alpha)x_\alpha}\Delta + C_1\left[r_+(\alpha)e^{r_+(\alpha)x_\alpha}\Delta + r_-(\alpha)e^{r_-(\alpha)x_\alpha}\Delta\right].$$

Hence, we obtain

$$C_1 = -\frac{\frac{1}{c\Delta} + \frac{\alpha\mu r_-}{c^2}e^{r_-(\alpha)x_\alpha}}{r_+ e^{r_+(\alpha)x_\alpha} + r_- e^{r_-(\alpha)x_\alpha}}. \tag{4.22}$$

Consider

$$x_\beta := \inf\{x \geq 0: \quad a(x) > \beta\},$$

and assume that,

$$\alpha \leq a(x) \leq \beta \quad \text{for} \quad x \in [x_\alpha, x_\beta]. \tag{4.23}$$

For $x_\alpha \leq x \leq x_\beta$, we assume that $\alpha \leq a(x) \leq \beta$, and hence $\mathscr{L}^{a(x)}V(x) = 0$. By substituting (4.8) in this equation, we get

$$-\frac{\mu^2 V'^2(x)}{2\sigma^2 V''(x)} - cV(x) + x = 0. \tag{4.24}$$

Consider the change of variable $V'(X(z)) = e^{-z}$, which implies that $V''(X(z)) = -e^{-z}/X'(z)$. By substituting these two equations in (4.24) and using (4.19), we get

$$X'(z)e^{-z} - bcV + bX(z) = 0.$$

By differentiating and performing the change of variable again, we obtain

$$X''(z) - (1 + bc - be^z)X'(z) = 0.$$

Since $X'(z) = -e^{-z}/V''\big(X(z)\big) > 0$, then

$$\frac{X''(z)}{X'(z)} = (1 + bc - be^z), \quad z \in \mathbb{R},$$

which results in

$$\ln\big(X'(z)\big) = (1 + bc)z - be^z + k, \quad \forall z \in \mathbb{R}.$$

By taking the exponent in both sides, we get

$$X'(z) = k_1 e^{(1+bc)z - be^z}, \quad \forall z \in \mathbb{R},$$

where k_1 is a positive free parameter to be determined later. By integrating the above equation, we get

$$X(z) = k_1 \int_{-\infty}^{z} e^{(1+bc)t - be^t} dt + k_2.$$

For the remaining part of the proof, we assume $k_2 = 0$. Then by performing the change of variable $u = e^t$, we get

$$X(z) = k_1 \int_0^{e^z} t^{bc} e^{-bt} dt.$$

Then, by using the functions g and G defined in (4.10), we get

$$X(z) = k_1 G(e^z).$$

In other words, this equation can be written as

$$x = k_1 G(e^z), \quad G(e^z) = \frac{x}{k_1}, \quad \text{for } \; x_\alpha \leq x \leq x_\beta.$$

Therefore,

$$\frac{1}{V'(x)} = e^z = G^{-1}\big(\frac{x}{k_1}\big), \quad \text{for } x_\alpha \le x \le x_\beta.$$

Hence, our solution to the HJB (4.6) on $[x_\alpha, x_\beta]$ takes the following form

$$V(x) = \int_{x_\alpha}^{x} \frac{1}{G^{-1}\big(\frac{t}{k_1}\big)} dt + V(x_\alpha), \quad x_\alpha \le x \le x_\beta.$$

We know that $a(x) = -\frac{\mu V'(x)}{\sigma^2 V''(x)}$, therefore,

$$a(x) = \frac{\mu k_1}{\sigma^2} g\left[G^{-1}\big(\frac{x}{k_1}\big)\right] G^{-1}\big(\frac{x}{k_1}\big).$$

By differentiating this equation, we deduce

$$a'(x) = \frac{\mu}{\sigma^2}\left[1 + bc - bG^{-1}\big(\frac{x}{k_1}\big)\right]. \tag{4.25}$$

This implies that $a(x)$ is increasing on $(0, k_1 G\big(c + \frac{1}{b}\big))$. Suppose that

$$a(x) \ge \beta, \quad \text{for} \quad x > x_\beta. \tag{4.26}$$

Then V is the solution to

$$\mathscr{L}^\beta V(x) = \frac{1}{2}\beta^2\sigma^2 V''(x) + \beta\mu V'(x) - cV + x = 0.$$

By using the same method as for the case $0 \le x \le x_\alpha$, the solution is given by

$$V(x) = \frac{x}{c} + \frac{\beta\mu}{c^2} + C_3 e^{r_-(\beta)(x - x_\beta)} + C_4 e^{r_+(\beta)(x - x_\beta)}, \tag{4.27}$$

where $r_\pm(\beta)$ are given in (4.15). Since, $V(x) \le x/c + \mu\beta/c^2$ (see (4.1.1)), then $V(x)/x$ converges to $1/c$ when $x \longrightarrow +\infty$. This forces C_4 to be null. Thus, our function becomes

$$V(x) = \frac{x}{c} + \frac{\beta\mu}{c^2} + C_3 e^{r_-(\beta)(x - x_\beta)}, \quad x > x_\beta. \tag{4.28}$$

Since $a(x_\beta) = \beta = -\frac{\mu V'(x_\beta)}{\sigma^2 V''(x_\beta)}$, we derive

$$\beta = -\frac{\mu\left(\frac{1}{c} + C_3 r_-(\beta)\right)}{\sigma^2\left(C_3 r_-^2(\beta)\right)}.$$

This implies that

$$C_3 = -\frac{\mu}{c\left(r_-^2(\beta)\beta\sigma^2 + \mu r_-(\beta)\right)} = \frac{1}{c\Delta r_-(\beta)}.$$

where $r_\pm(\beta)$ are given in (4.15).

Part 2. Here, we determine the remaining constants k_1, x_α and x_β. Since $V'(x_\beta-) = V'(x_\beta+)$, we get

$$\frac{1}{G^{-1}\left(\frac{x_\beta}{k_1}\right)} = \frac{1}{c} + \frac{1}{c\Delta}. \tag{4.29}$$

Or equivalently

$$\frac{x_\beta}{k_1} = G\left(\frac{c\Delta}{1+\Delta}\right). \tag{4.30}$$

Furthermore, by combining $V''(x_\beta+) = V''(x_\beta-)$ and (4.29), we get

$$k_1 = -\frac{c\Delta}{g\left(\frac{c\Delta}{\Delta+1}\right)\left(\frac{c\Delta}{\Delta+1}\right)^2 r_-(\beta)} = \frac{\beta\sigma^2}{g\left(\frac{c\Delta}{\Delta+1}\right)\left(\frac{c\Delta}{\Delta+1}\right)\mu},$$

and by inserting k_1 back in (4.30), we obtain x_β given by (4.18). Remark that due to (4.30), (4.25) and the fact that G is an increasing function, we deduce that $x_\beta < k_1 G\left(c + \frac{1}{b}\right)$ and hence, $a(x)$ is increasing over (x_α, x_β). This proves that assumption (4.23) holds. By using the fact that $a(x_\alpha) = \alpha = -\frac{\mu V'(x_\alpha)}{\sigma^2 V''(x_\alpha)}$, we derive

$$g\left(G^{-1}\left(\frac{x_\alpha}{k_1}\right)\right)G^{-1}\left(\frac{x_\alpha}{k_1}\right) = \frac{\alpha\sigma^2}{\mu k_1}.$$

A combination of this equation and (4.16), we get

$$g\left(G^{-1}\left(\frac{x_\alpha}{k_1}\right)\right)G^{-1}\left(\frac{x_\alpha}{k_1}\right) = -\frac{\alpha\sigma^2 r_-(\beta)}{\mu}\left[\frac{g\left(\frac{c\Delta}{1+\Delta}\right)c\Delta}{(1+\Delta)^2}\right] = \frac{\alpha}{\beta}g\left(\frac{c\Delta}{1+\Delta}\right)\left(\frac{c\Delta}{1+\Delta}\right). \tag{4.31}$$

Recall that g is given by (4.10), and put

$$F(y) := H(y) - \frac{\alpha}{\beta}H(y_0), \quad H(y) := yg(y), \quad y \geq 0, \quad \text{and} \quad y_0 := \frac{c\Delta}{1+\Delta}. \tag{4.32}$$

It is clear that $F'(y) = H'(y) = (1+bc-by)g(y)$ is increasing over the interval $(0, c+\frac{1}{b})$. Further, we have $F(0) = -\frac{\alpha}{\beta}H(y_0) = -\alpha y_0 g(y_0)/\beta < 0$ and $F(y_0) = \frac{\beta-\alpha}{\beta}y_0 g(y_0) > 0$. Thus, there exists a unique $y_\alpha \in (0, y_0)$, such that $F(y_\alpha) = 0$, or equivalently $H(y_\alpha) = \frac{\alpha}{\beta}H(y_0)$. By combining this and (4.31), we deduce that

$$G^{-1}\big(\frac{x_\alpha}{k_1}\big) = y_\alpha = H^{-1}\big(\frac{\alpha}{\beta}H(y_0)\big),$$

and we obtain (4.18). Furthermore, by combining (4.22) and (4.18), (4.17) follows.

For $x \in (x_\beta, +\infty)$, $a(x)$ takes the following form

$$a(x) = \frac{-\mu\left[1 + cC_3 r_-(\beta)e^{r_-(\beta)(x-x_\beta)}\right]}{\sigma^2 cC_3 r_-^2(\beta)e^{r_-(\beta)(x-x_\beta)}}.$$

Direct simplification of this equation, leads to

$$a(x) = \frac{\beta\Delta e^{-r_-(x-x_\beta)}}{1+\Delta} + \frac{\beta}{1+\Delta} > \frac{\beta\Delta}{1+\Delta} + \frac{\beta}{1+\Delta} = \beta, \quad \text{for} \quad x \geq x_\beta.$$

Hence, $a(x)$ is strictly increasing on $[x_\beta, +\infty]$ and $a(x_\beta) = \beta$, and hence assumption (4.26) is verified.

Part 3. Our next step is to check the concavity and the smoothness of V. It is clear that V is continuously twice differentiable on $(0, x_\alpha) \cup (x_\alpha, x_\beta) \cup (x_\beta, +\infty)$,

$$V''(x) = \begin{cases} -\frac{\alpha\mu r_-^2(\alpha)}{c^2}e^{r_-(\alpha)x} + C_1(r_+^2(\alpha)e^{r_+(\alpha)x} - r_-^2(\alpha)e^{r_-(\alpha)x}), & \text{if } 0 \leq x \leq x_\alpha, \\ \dfrac{-1}{k_1 g\left[G^{-1}\left(\frac{x}{k_1}\right)\right]\left[G^{-1}\left(\frac{x}{k_1}\right)\right]^2} < 0, & \text{if } x_\alpha \leq x \leq x_\beta, \\ C_3 r_-^2(\beta)e^{r_-(\beta)(x-x_\beta)} < 0, & \text{if } x \geq x_\beta. \end{cases}$$

It is obvious that V'' is always negative on $(x_\alpha, +\infty)$ which implies the concavity of V on this interval. Thus, in the following we focus on proving the concavity of V on $[0, x_\alpha]$. The remaining part is proving that $V'(x) > 0$ and $V''(x) < 0$ for $x \in [0, x_\alpha]$. Recall that

$$V(x) = \frac{x}{c} + \frac{\alpha\mu}{c^2}(1 - e^{r_-(\alpha)x}) + C_1(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}), \quad 0 \le x \le x_\alpha.$$

Hence,

$$V'(x) = \frac{1}{c} - \frac{\alpha\mu r_-(\alpha)}{c^2} e^{r_-(\alpha)x} + C_1(r_+(\alpha)e^{r_+(\alpha)x} - r_-(\alpha)e^{r_-(\alpha)x}),$$

and

$$V''(x) = -\frac{\alpha\mu r_-^2(\alpha)}{c^2} e^{r_-(\alpha)x} + C_1(r_+^2(\alpha)e^{r_+(\alpha)x} - r_-^2(\alpha)e^{r_-(\alpha)x}).$$

Since studying the sign of V' and V'' is related to knowing the sign of C_1, then we put

$$N(x) := \frac{2\Delta}{\Delta - 1} e^{r_- x} - 1, \quad D(x) := (\Delta - 1)e^{r_+ x} - (\Delta + 1)e^{r_- x} \tag{4.33}$$

and using this notation, we get

$$C_1 = \frac{\alpha\sigma^2}{\mu c\Delta}\left[\frac{N(x_\alpha)}{D(x_\alpha)}\right].$$

We start with studying this function in the neighborhood of zero where

$$V''(0) = -\frac{\alpha\mu r_-^2}{c^2} + C_1(r_+^2 - r_-^2), \quad V'(0) = \frac{1}{c} - \frac{\alpha\mu r_-}{c^2} + C_1(r_+ - r_-)$$

By applying the notation $r_\pm = \mu(-1 \pm \Delta)/\alpha\sigma^2$ and $\Delta^2 = 1 + 2\sigma^2 c/\mu$ on $V'(0)$, we get

$$V'(0) = \frac{(\Delta + 1)D(x_\alpha) + 2(\Delta - 1)N(x_\alpha)}{c(\Delta - 1)D(x_\alpha)},$$

which is equivalent to

$$V'(0) = \frac{(\Delta + 1)e^{r_+ x_\alpha} - (\Delta - 1)e^{r_- x_\alpha} - 2}{cD(x_\alpha)}.$$

The numerator in this equation is increasing for x_α, hence

$$(\Delta+1)e^{r_+ x_\alpha} - (\Delta-1)e^{r_- x_\alpha} - 2 \geq (\Delta+1) - (\Delta-1) - 2 = 0.$$

This proves that

$$V''(0) < 0, \quad \text{if and only if} \quad V'(0) > 0 \quad \text{if and only if} \quad D(x_\alpha) > 0.$$

Since $V''(x_\alpha+) = -1/k_1 g\left[G^{-1}(x_\alpha/k_1)\right]\left[G^{-1}(x_\alpha/k_1)\right]^2 < 0$, then by continuity of $V''(x)$ we get

$$V''(x_\alpha-) = -\frac{\alpha\mu r_-^2(\alpha)}{c^2} e^{r_-(\alpha)x} + C_1(r_+^2(\alpha)e^{r_+(\alpha)x} - r_-^2(\alpha)e^{r_-(\alpha)x}) < 0.$$

1) If $N(x_\alpha) > 0$, then $V''(x)$ is increasing and hence, $V''(x) \leq V''(x_\alpha) < 0$, for all $x \in [0, x_\alpha]$.

2) If $N(x_\alpha) < 0$, then on the one hand,

$$e^{-r_- x}V''(x) = -\frac{\alpha\mu r_-^2}{c^2} + C_1 r_-^2\left(\frac{r_+^2}{r_-^2} e^{(r_+ - r_-)x} - 1\right).$$

It is clear that the right hand side of this equation is decreasing for x and the highest value is when $x = 0$. By combining this fact with Proposition 4.2.1

$$e^{-r_- x}V''(x) < V''(0) = -\frac{2\mu}{\alpha\sigma^2}V'(0) < 0.$$

Put

$$Y := K(\tilde{y}).$$

On the other hand, $N(x_\alpha) < 0$ if and only if $Y > \ln(2\Delta/\Delta-1)/(\Delta+1) =: Y_1$, which implies that $Y > Y_0 = \ln(\Delta+1/\Delta-1)/(2\Delta)$, (since $Y_0 < Y_1$). Hence, $D(x_\alpha) > 0$, as this is equivalent to $Y > Y_0$. Therefore, in this case when $N(x_\alpha) < 0$, we always have $V'(0) > 0$ and hence $V''(0) < 0$. This ends the proof that V is strictly concave on the interval $[0, x_\alpha]$, and that V'' is continuous at x_α.

To conclude that V defined by (4.21) is a smooth solution to the HJB, it is enough to check the continuity of V' at x_α. Put

$$y_\alpha := G^{-1}\left(\frac{x_\alpha}{k_1}\right).$$

The continuity of V' at x_α is equivalent to $V'(x_\alpha-) = V'(x_\alpha+)$, or

$$\frac{1}{c} - \frac{\alpha\mu r_-}{c^2}e^{r_- x_\alpha} + C_1\left(r_+ e^{r_+ x_\alpha} - r_- e^{r_- x_\alpha}\right) = \frac{1}{y_\alpha}.$$

By applying similar notation as before, we deduce

$$\left[1 + \frac{2e^{-(\Delta+1)Y}}{\Delta-1}\right] + \frac{\left(\frac{2}{\Delta-1}e^{-(\Delta+1)} - \frac{1}{\Delta}\right)\left((\Delta-1)e^{(\Delta-1)Y} + (\Delta+1)e^{-(\Delta+1)Y}\right)}{(\Delta-1)e^{(\Delta-1)Y} - (\Delta+1)e^{-(\Delta+1)Y}} = \frac{c}{y_\alpha},$$

or equivalently

$$\frac{(\Delta-1)e^{(\Delta-1)Y} - (\Delta+1)e^{-(\Delta+1)Y} + 4e^{-2Y} - \frac{1}{\Delta}\left((\Delta-1)e^{(\Delta-1)Y} + (\Delta+1)e^{-(\Delta+1)Y}\right)}{(\Delta-1)e^{(\Delta-1)Y} - (\Delta+1)e^{-(\Delta+1)Y}} = \frac{c}{y_\alpha}$$

which can be reduced to

$$\frac{(\Delta-1)^2e^{(\Delta-1)Y} - (\Delta+1)^2e^{-(\Delta+1)Y} + 4\Delta e^{-2Y}}{(\Delta-1)e^{(\Delta-1)Y} - (\Delta+1)e^{-(\Delta+1)Y}} = \frac{c\Delta}{y_\alpha}, \quad \text{where} \quad 0 < y_\alpha < y_0 = \frac{c\Delta}{\Delta+1} < c.$$

This is equivalent to

$$(\Delta-1)^2\left(y_\alpha - \frac{c\Delta}{\Delta-1}\right)e^{(\Delta-1)Y} - (\Delta+1)^2\left(y_\alpha - \frac{c\Delta}{\Delta+1}\right)e^{-(\Delta+1)Y} + 4\Delta y_\alpha e^{-2Y} = 0,$$

which is the equation $L(Y) = 0$, where L is defined in (4.14). The solution to this latter equation, denoted by $\tilde{y}$, is guaranteed under the assumption (4.12) by Lemma 4.2.1 . This ends the proof of the theorem. $\square$

4.3 Optimal policy and verification theorem

In this section we construct the optimal control policy based on the solution of the HJB equation obtained in the previous sections. For each, $x \geq 0$ we define

$$a^*(x) := arg \max_{\alpha \leq a \leq \beta} \left(\frac{1}{2} a^2 \sigma^2 V''(x) + a\mu V'(x) - (\gamma + \lambda) V(x) + x \right). \tag{4.34}$$

The function $a^*(x)$ represents the optimal feedback control function for the control component a_t^π, $t \geq 0$. More precisely, the value $a^*(x)$ is the optimal risk that one should take when the value of the current reserve is x. Thanks to the previous section, we have the following proposition

Proposition 4.3.1. *Let G and g be the functions defined in (4.10), and k_1, x_α and x_β be given in (4.16), and (4.18) respectively. Then the function $a^*(x)$, defined in (4.34), is given by*

$$a^*(x) = \begin{cases} \alpha, & \text{if} \quad 0 \leq x \leq x_\alpha \\ \frac{\mu k_1}{\sigma^2} g\left[G^{-1}\left(\frac{x}{k_1}\right)\right] G^{-1}\left(\frac{x}{k_1}\right), & \text{if} \quad x_\alpha \leq x \leq x_\beta, \\ \beta & \text{if} \quad x > x_\beta. \end{cases}$$

Proof. Since our V is twice continuously differentiable, $V'(x) > 0$, and $V''(x) < 0$ for all $x \geq 0$. Then for any $x \in [0, +\infty)$, as a function of $a \in \mathbb{R}$, the following

$$\mathscr{L}^a V(x) = \frac{\sigma^2 a^2}{2} V''(x) + a\mu V'(x) - (\gamma + \lambda) + x,$$

is a concave function with a maximum attained at $a(x)$, and it is increasing on $[0, a(x)]$ and decreasing on $[a(x), +\infty)$. □

Theorem 4.3.1. *Let a^* be defined by Proposition 4.3.1. Then the following assertions hold.*

(a) *The following SDE*

$$dX_t = a^*(X_t)\mu dt + a^*(X_t)\sigma dW_t, \quad X_0 = x, \tag{4.35}$$

has a unique solution that we denote by $R^* = (R^*_t)_{t\geq 0}$.

(b) R^* *is the optimal cash reserve, and the optimal risk control is given by* $\pi^* = \left(a^*(R^*_t); t \geq 0\right)$ *which satisfies*

$$\mathbb{E}\int_0^{\tau^*} e^{-(\gamma+\lambda)t} R^*_t dt = V(x), \quad \forall x \geq 0,$$

where

$$\tau^* = \inf\{t \geq 0 : R^*_t = 0\}$$

Proof. This proof has part 1 and part 2, where we prove assertions a and b respectively.

Part 1. Thanks to Theorem 2.3.1 and similar arguments as the previous chapter, proving existence and uniqueness of the solution to the SDE by using Lipschitz and growth conditions can be reduced to proving the following inequality

$$|a^*(x_1) - a^*(x_2)| \leq K_1|x_1 - x_2|, \quad x_1, x_2 \geq 0. \tag{4.36}$$

By taking $x_1 = x$ and $x_2 = 0$, we obtain $|a^*(x)| \leq |a^*(0)| + K_1|x|$ where $a^*(0) = \alpha$. Hence the growth condition is satisfied as soon as (4.36) holds. For $x_1, x_2 > x_\beta$, we get $a^*(x_1) = a^*(x_2) = \beta$ and for $0 \leq x_1, x_2 \leq x_\alpha$, we get $a^*(x_1) = a^*(x_2) = \alpha$ and hence for both cases (4.36) is satisfied. For the case $x_\alpha \leq x_1, x_2 \leq x_\beta$, similarly to the previous chapter, it is enough to show that $\frac{d}{dy}a^*(y)$ is bounded. Hence

$$\frac{d}{dy}a^*(y) = \frac{\mu k_1}{\sigma^2}\frac{d}{dy}\left[g\big(G^{-1}(\frac{y}{k_1})\big)G^{-1}(\frac{y}{k_1})\right] = \frac{\mu}{\sigma^2}\left[1 + bc - bG^{-1}\big(\frac{x}{k_1}\big)\right]$$

Thus,

$$\begin{aligned}\left|\frac{d}{dy}a^*(y)\right| &\leq \frac{\mu}{\sigma^2}\left[1 + bc + bG^{-1}\big(\frac{x}{k_1}\big)\right] \leq \frac{\mu}{\sigma^2}\left[1 + bc + bG^{-1}\big(\frac{x_\beta}{k_1}\big)\right] = \frac{\mu}{\sigma^2}\left[1 + bc + b\big(\frac{c\Delta}{\Delta+1}\big)\right] \\ &\leq \frac{\mu}{\sigma^2}(1 + 2bc).\end{aligned}$$

Hence, $\frac{d}{dy}a^*(y)$ is bounded by a constant K. This completes the proof of assertion (a).

Part 2.. By applying Itô to $e^{-(\gamma+\lambda)t\wedge\tau^*}V(R^*_{t\wedge\tau^*})$, we deduce

$$\begin{aligned} e^{-(\gamma+\lambda)t\wedge\tau^*}V(R^*_{t\wedge\tau^*}) = V(x) - \int_0^{t\wedge\tau^*}(\gamma+\lambda)e^{-(\gamma+\lambda)s}V(R^*_s)ds + \int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}a^*(R^*_s)\mu V'(R^*_s)ds \\ + \int_0^{t\wedge\tau^*}\frac{1}{2}e^{-(\gamma+\lambda)s}\sigma^2\big(a^*(R^*_s)\big)^2V''(R^*_s)ds + \int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}\sigma a^*(R^*_s)V'(R^*_s)dW_s. \end{aligned} \tag{4.37}$$

Since $0 \le V'(x) \le K_1$, where $K_1 = \max\left(\frac{1}{y_\alpha}, \sup_{0\le x\le x_\beta}|V'(x)|, \frac{1}{y_0}\right)$, we deduce that

$$\int_0^{t\wedge\tau^*}\sigma^2\big(a^*(R^*_s)\big)^2V'^2(R^*_s)ds \le \sigma^2K_1t,$$

and hence, $\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}\sigma a^*(R^*_s)V'(R^*_s)dW_s$ is a martingale which implies

$$\mathbb{E}\left[\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}\sigma a^*(R^*_s)V'(R^*_s)dW_s\right] = 0.$$

Therefore, by using this equation and taking the expectation of (4.37), we obtain

$$\begin{aligned} \mathbb{E}\left[e^{-(\gamma+\lambda)t\wedge\tau^*}V(R^*_{t\wedge\tau^*})\right] &= V(x) - \mathbb{E}\left[\int_0^{t\wedge\tau^*}(\gamma+\lambda)e^{-(\gamma+\lambda)s}V(R^*_s)ds\right] + \mathbb{E}\left[\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}a^*(R^*_s)\mu V'(R^*_s)ds\right] \\ &\quad + \mathbb{E}\left[\int_0^{t\wedge\tau^*}\frac{1}{2}e^{-(\gamma+\lambda)s}\sigma^2\big(a^*(R^*_s)\big)^2V''(R^*_s)ds\right]. \\ &= V(x) + \mathbb{E}\left[\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}\left(\mathscr{L}^{a^*(R^*_s)}V(R^*_s) - R^*_s\right)ds\right], \end{aligned} \tag{4.38}$$

where $\mathscr{L}^{a(x)}V(x)$ is given by (4.7). Since, $\mathscr{L}^{a^*(y)}V(y) = 0$ for all $y \ge 0$, and in particular for $y = R^*_s$, equation (4.38) becomes

$$\mathbb{E}\left[e^{-(\gamma+\lambda)(t\wedge\tau^*)}V(R^*_{t\wedge\tau^*})\right] = V(x) - \mathbb{E}\left[\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}R^*_s ds\right],$$

or equivalently

$$\mathbb{E}\left[e^{-(\gamma+\lambda)(t\wedge\tau^*)}V(R^*_{t\wedge\tau^*})\right] + \mathbb{E}\left[\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}R^*_s ds\right] = V(x). \tag{4.39}$$

The first term on the left hand side of this equation can be written as

$$\mathbb{E}\left[e^{-(\gamma+\lambda)\tau^*}V(R^*_{\tau^*})1_{\{\tau^*<t\}}+e^{-(\gamma+\lambda)t}V(R^*_t)1_{\{\tau^*\geq t\}}\right].$$

Since $V(0)=0$, then $\{\tau^*<+\infty\}$ we get $R^*_{\tau^*}=0$. Hence

$$0\leq\mathbb{E}\left[e^{-(\gamma+\lambda)(t\wedge\tau^*)}V(R^*_{t\wedge\tau^*})\right]=\mathbb{E}\left[e^{-(\gamma+\lambda)(t)}V(R^*_t)1_{\{\tau^*\geq t\}}\right]\leq e^{-(\gamma+\lambda)t}\mathbb{E}V(R^*_t).$$

By applying the fact that $V(x)\leq\frac{x}{(\gamma+\lambda)}+\frac{\mu\beta}{(\gamma+\lambda)^2}$ on the last inequality, we get

$$\mathbb{E}\left[e^{-(\gamma+\lambda)(t)}V(R^*_t)1_{\{\tau^*\geq t\}}\right]\leq e^{-(\gamma+\lambda)t}\left[\frac{\mathbb{E}[R^*_t]}{(\gamma+\lambda)}+\frac{\mu\beta}{(\gamma+\lambda)^2}\right].\tag{4.40}$$

However,

$$\mathbb{E}[R^*_t]=x+\mathbb{E}\left[\int_0^t a^*(R^*_s)\mu ds\right]\leq x+\mu\beta t.$$

By inserting this in (4.40), we get

$$\mathbb{E}\left[e^{-(\gamma+\lambda)(t)}V(R^*_t)1_{\{\tau^*\geq t\}}\right]\leq e^{-(\gamma+\lambda)t}\left[\frac{x}{(\gamma+\lambda)}+\frac{\mu\beta t}{(\gamma+\lambda)}+\frac{\mu\beta}{(\gamma+\lambda)^2}\right].\tag{4.41}$$

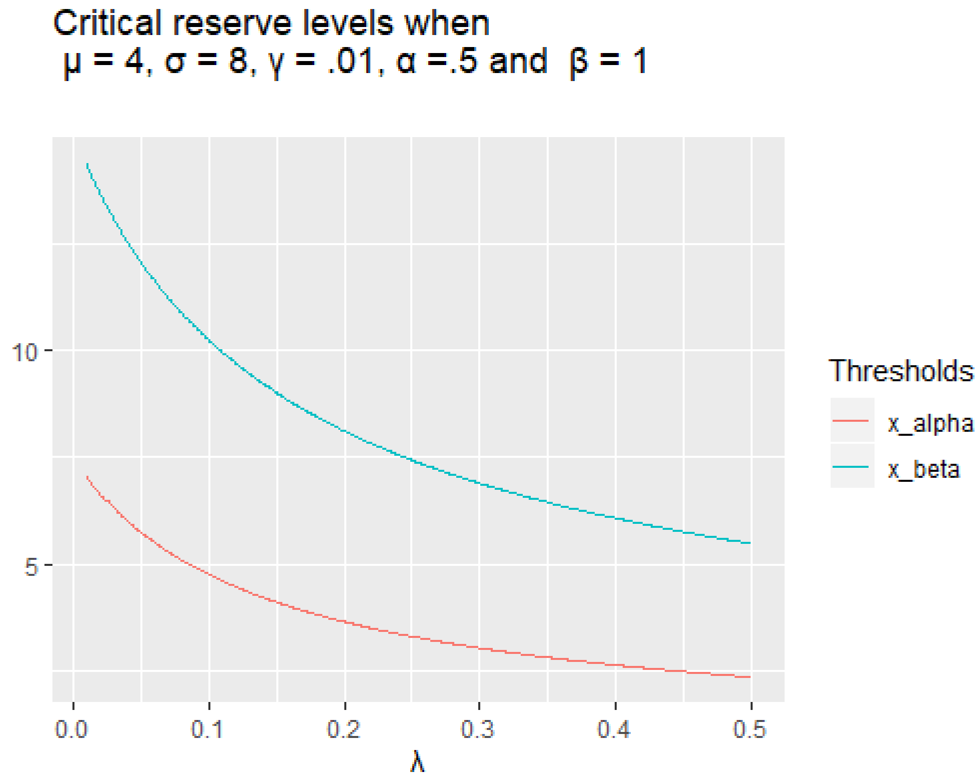
By combining (4.37), (4.3) and (4.41) and taking the $t\longrightarrow+\infty$, we get

$$V(x)=\mathbb{E}\left[\int_0^{\tau^*}e^{-(\gamma+\lambda)s}R^*_s ds\right].$$

This completes the proof. □

4.4 Graphical illustrations

This section demonstrates the impact of the intensity rate on the reserve critical levels x_α and x_β. These reserve thresholds are indicators that are crucial in estimating the amount of risk the company can handle along with its current activities.



It is clear from the graph that both x_α and x_β decrease as λ increases in value. This means that as the intensity rate increases, it is more likely that the company will reach bankruptcy.

Chapter 5

A dividend model under random horizon

In this chapter we look into the dividend distribution model, as we extend the model introduced in B. Højgaard and M. Taksar [17] by considering a company that is subject to liability payments of a constant rate δ along with adding a random horizon to the model. Afterwards, we move on to constructing the HJB equation corresponding to the control problem by adapting some of the techniques used in T. Choulli, M.Taksar, and X.Y Zhou [6]. Then we present the explicit solution to this equation along with the optimal policies.

This chapter has four sections. The first section presents the model (mathematically and economically), defines the objectives in terms of a control problem, and singles out some preliminary analysis for this model. The second section solves explicitly a Hamilton-Jacobi-Bellman (HJB) associated to the main control problem. The third section determines the optimal policies for the main control problem, while the last section presents some graphical illustration of the results.

5.1 The model, the objectives and preliminaries

This section has two subsections, where we introduce the model and derive the properties of the optimal value function respectively.

5.1.1 Mathematical and economic model

We consider a firm with reserve R_t at any time t, where R_t is a stochastic process satisfying the following stochastic differential equation.

$$dR_t = (\mu - \delta)dt + \sigma dW_t - dC_t, \quad R_0 = x. \tag{5.1}$$

Here the constants μ and σ are positive (i.e., they belong to $(0, +\infty)$) and $\delta \in [0, +\infty)$. The constant μ represents the reserve rate and σ is the reserve's volatility, while δ is the liability rate that the firm is paying. C is a non-decreasing process, that is right continuous with left limits, representing the aggregate dividends the firm distributes to its shareholders. For a risk exposure a_t^π at time t and control policy $\pi = (a_t^\pi, C_t^\pi, t \geq 0)$, our reserve equation (5.1) becomes

$$dR_t^\pi = (a_t^\pi \mu - \delta)dt + a_t^\pi \sigma dW_t - dC_t^\pi, \quad R_0^\pi = x, \tag{5.2}$$

and the bankruptcy time associated with this reserve is given by

$$\tau_\pi := \inf\left\{ t \geq 0 \;\middle|\; R_t^\pi = 0 \right\}. \tag{5.3}$$

Here, by convention, we put $\inf(\emptyset) = +\infty$.

<u>The model for random horizon.</u> Throughout this chapter, τ is a random time that is independent of the reserve process $R = (R_t,\, t \geq 0)$ (or equivalently $\mathscr{F}_\infty = \sigma(W_t, t \geq 0)$), and has an exponential distribution with mean λ^{-1}, i.e., $\mathbb{E}[\tau] = \lambda^{-1}$. Thus, the survival probability for this random time is given by

$$G_t := P(\tau \geq t | \mathscr{F}_t) = \exp(-\lambda t), \quad t \geq 0. \tag{5.4}$$

We define the return function of an initial reserve x under a control policy π as the total discounted accumulated dividend distribution from time 0 to the default (random horizon) time τ or the bankruptcy time τ_π (whichever comes first).

The objective. Our objectives consist of finding the optimal return function V, defined below, and describing the optimal policy $\pi \in \mathscr{A}$ such that

$$V(x) := \sup_{\pi \in \mathscr{A}} \mathbb{E}\left[\int_0^{\tau_\pi \wedge \tau} e^{-\gamma t} dC_t^\pi\right] = \mathbb{E}\left[\int_0^{\tau \wedge \tau_{\pi^*}} e^{-\gamma t} dC_t^{\pi^*}\right]. \tag{5.5}$$

where $\mathscr{A}$ denotes the set of all admissible control policies.

5.1.2 Properties of the value function

This subsection derives some properties of the value function V, that will play important role in our full construction of this function in the next section.

Proposition 5.1.1. *The function V, defined in (5.5), is concave and satisfies*

$$V(x) = \sup_{\pi \in \mathscr{A}} \mathbb{E}\left[\int_0^{\tau_\pi} e^{-(\gamma+\lambda)t} dC_t^\pi\right] \quad \textit{and} \quad V(0) = 0. \tag{5.6}$$

Proof. This proof has two parts, part 1 and part 2.

Part 1. The equality in the proposition statement is due to the fact that

$$\mathbb{E}\left[\int_0^{\tau_\pi \wedge \tau} e^{-\gamma t} dC_t^\pi\right] = \mathbb{E}\left[\int_0^{\tau_\pi} e^{-\gamma t} G_t dC_t^\pi\right] = \mathbb{E}\left[\int_0^{\tau_\pi} e^{-(\gamma+\lambda)t} dC_t^\pi\right].$$

Here we prove concavity. Put $V_\pi(x) := \mathbb{E}\left[\int_0^{\tau_\pi} e^{-(\gamma+\lambda)t} dC_t^\pi\right]$. Let x_1 and x_2 two positive initial reserves and $\eta \in (0,1)$. For any $\varepsilon > 0$ there exist $\pi_1 = (a^{\pi_1}, C^{\pi_1}) \in \mathscr{A}(x_1)$ and $\pi_2 = (a^{\pi_2}, C^{\pi_2}) \in \mathscr{A}(x_2)$ such that

$$V(x_1) - \varepsilon \leq V_{\pi_1}(x_1) \quad \text{and} \quad V(x_2) - \varepsilon \leq V_{\pi_2}(x_2). \tag{5.7}$$

Consider the policy $\pi_\eta = (a^{\pi_\eta}, C^{\pi_\eta})$ given by

$$a_{\pi_\eta}(t) := \eta a_{\pi_1}(t) + (1-\eta) a_{\pi_2}(t) \quad \text{and} \quad C_t^{\pi_\eta} := \eta C_t^{\pi_1} + (1-\eta) C_t^{\pi_2}. \tag{5.8}$$

Then the reserve process for π_η is given by

$$R^{\pi_\eta} = \eta R^{\pi_1} + (1-\eta) R^{\pi_2},$$

whose bankruptcy time is given by

$$\tau^{\pi_\eta} = \max(\tau^{\pi_1}, \tau^{\pi_2}).$$

Now we calculate

$$\begin{aligned}\eta V(x_1) + (1-\eta)V(x_2) - \varepsilon &\leq \eta V_{\pi_1}(x_1) + (1-\eta) V_{\pi_2}(x_2) = V_{\pi_\eta}(\eta x_1 + (1-\eta)x_2) \\ &\leq V(\eta x_1 + (1-\eta)x_2).\end{aligned}$$

Since both sides right and left don't depend on ε, we let ε go to zero and get concavity of V.

Part 2. Here we prove the rest of the statements of the proposition. Remark that for any $\pi \in \mathscr{A}$, we have the following

$$\mathbb{E}\left[\int_0^{\tau \wedge \tau_\pi} e^{-\gamma t} dC_t^\pi\right] = \mathbb{E}\left[\int_0^{\tau_\pi} e^{-\gamma t} G_t dC_t^\pi\right] = \mathbb{E}\left[\int_0^{\tau_\pi} e^{-(\gamma+\lambda)t} dC_t^\pi\right].$$

This proves the first equality in (5.6). If $x = 0$, then for any $\pi \in \mathscr{A}$, $R_0^\pi = 0$, and hence $\tau_\pi = 0$ (see its definition in (5.3)). Therefore, in this case of $x = 0$, we have

$$V(0) = \sup_{\pi \in \mathscr{A}} \mathbb{E}\left[\int_0^{\tau_\pi} e^{-(\gamma+\lambda)t} dC_t^\pi\right] = 0.$$

□

Proposition 5.1.2. *If the function V, defined by (5.5), is twice continuously differentiable, then it satisfies the following HJB equation*

$$0 = \max\left[\max_{0\le a\le 1}\left(\frac{a^2\sigma^2}{2}V''(x)+(a\mu-\delta)V'(x)-(\gamma+\lambda)V(x)\right),1-V'(x)\right],\quad x\ge 0. \tag{5.9}$$

Proof. Similar to Chapter 3, the function V, defined by (5.5) can be transformed into (5.6) by using Proposition 5.1.1. Hence, the resulting value function is the same as the one introduced in T. Choulli, M.Taksar, and X.Y Zhou [6] with the only difference is a larger discount factor that results from adding λ to γ. Therefore, the proof of the HJB follows by adapting the same method as in [6]. This ends the proof of the proposition. □

Throughout this chapter, we consider the following reserve threshold

$$x_D := \inf\{x\ge 0:\quad V'(x)\le 1\}, \tag{5.10}$$

where by convention we put $\inf(\phi) = +\infty$.

Proposition 5.1.3. *If $\delta\ge\mu$, then*

$$V(x) = x,\quad \textit{for all } x\ge 0.$$

Proof. Suppose that $\delta\ge\mu$. Then we have

$$\frac{a^2\sigma^2}{2}V''(x)+(a\mu-\delta)V'(x)-(\gamma+\lambda)V(x)<0,$$

for any $a\in[0,1]$ and any $x>0$. Hence, the HJB equation (5.9) reduces to

$$V'(x) = 1,\quad \text{for all } x>0.$$

This implies that $x_D = 0$, and by integrating the above equation and using $V(0) = 0$, we deduce that $V(x) = x$, $\forall x \geq 0$. This proves the lemma. □

Then, for the case when $\delta < \mu$, we state the following proposition.

Proposition 5.1.4. *Suppose $\delta < \mu$. Then $x_D > 0$, and the function V solution to (5.9) satisfies*

$$V(x) = x - x_D + V(x_D), \quad \text{if} \quad x \geq x_D, \tag{5.11}$$

and

$$0 = \max_{0 \leq a \leq 1} \left(\frac{a^2\sigma^2}{2} V''(x) + (a\mu - \delta)V'(x) - (\gamma + \lambda)V(x) \right), \quad \text{if} \quad 0 \leq x < x_D. \tag{5.12}$$

Proof. Due to (5.9), it is clear that $V'(x) \geq 1$, for all $x \geq 0$. Since V is concave (see Proposition 5.1.1), which implies that V' is decreasing, we get

$$1 \leq V'(x) \leq V'(x_D) = 1, \quad \text{for all} \quad x \geq x_D.$$

This implies that $V'(x) = 1$, for all $x \geq x_D$, and (5.11) follows immediately.
The rest of this proof deals with proving (5.12). From the definition of x_D (see (5.10)), we get

$$V'(x) > 1, \quad \text{for all} \quad 0 \leq x < x_D.$$

Hence, for $x \in [0, x_D)$, (5.9) becomes

$$\max_{0 \leq a \leq 1} \left(\frac{a^2\sigma^2}{2} V''(x) + (a\mu - \delta)V'(x) - (\gamma + \lambda)V(x) \right) = 0, \quad 0 \leq x < x_D.$$

This proves the lemma. □

5.2 Construction of the smooth solution to the HJB (5.9)

Consider the maximizer function, $a(x)$, defined by

$$a(x) := \arg\max_{a \in \mathbb{R}} \mathscr{L}^a V(x), \tag{5.13}$$

where

$$\mathscr{L}^a V(x) := \frac{1}{2} a^2 \sigma^2 V''(x) + (a\mu - \delta) V'(x) - (\gamma + \lambda) V(x), \quad \text{for all} \quad a \in \mathbb{R}. \tag{5.14}$$

Since $a(x)$ will play a key role in our analysis, we start deriving some of its properties in the following lemma.

Lemma 5.2.1. *Suppose $\delta < \mu$. Then the following assertions hold.*

(a) *For all $x \geq 0$,*

$$a(x) = -\frac{\mu V'(x)}{\sigma^2 V''(x)} \geq 0. \tag{5.15}$$

(b) *For any $0 \leq x < x_D$,*

$$a(x) \geq \frac{2\delta}{\mu} + \frac{2(\gamma + \lambda) V(x)}{\mu V'(x)}.$$

(c) *For $x \geq 0$ such that $a(x) \leq 1$, we have*

$$a(x) = \frac{2\delta}{\mu} + \frac{2(\gamma + \lambda) V(x)}{\mu V'(x)}, \quad 0 \leq x < x_D.$$

(d) *It holds that*

$$a(0) = \begin{cases} \frac{2\delta}{\mu} & \text{if} \quad \delta < \frac{\mu}{2}, \\ \frac{\mu}{2(\mu - \delta)} & \text{if} \quad \delta \geq \frac{\mu}{2}. \end{cases}$$

Proof. This proof is divided into four parts, where we prove the four assertions of the lemma separately.

Part 1 Since $a(x)$ is the maximizer of $\mathscr{L}^a V(x)$, then it is the root to

$$\frac{\partial}{\partial a}(\mathscr{L}^a V(x)) = a\sigma^2 V''(x) + \mu V'(x) = 0.$$

This together with $V''(x) < 0$ and $V'(x) \geq 1$, implies (5.15).

Part 2. By using (5.15), we derive

$$\begin{aligned}
\max_{a\in\mathbb{R}} \mathscr{L}^a V(x) &= \mathscr{L}^{a(x)} V(x) \\
&= \frac{1}{2}\left[-\frac{\mu V'(x)}{\sigma^2 V''(x)}\right]^2 \sigma^2 V''(x) + \left[-\frac{\mu V'(x)}{\sigma^2 V''(x)}\right]\mu V'(x) - \delta V'(x) - (\gamma+\lambda)V(x) \\
&= -\frac{\mu^2 V'^2(x)}{2\sigma^2 V''(x)} - \delta V'(x) - (\gamma+\lambda)V(x) \\
&= \frac{\mu}{2} V'(x) a(x) - \delta V'(x) - (\gamma+\lambda)V(x) \\
&= \frac{\mu V'(x)}{2}\left(a(x) - \frac{2\delta}{\mu} - \frac{2(\gamma+\lambda)V(x)}{\mu V'(x)}\right).
\end{aligned} \tag{5.16}$$

By combining this equation with the facts that

$$\max_{a\in\mathbb{R}} \mathscr{L}^a V(x) \geq \max_{0\leq a\leq 1} \mathscr{L}^a V(x) = 0, \quad \text{for} \quad 0 \leq x \leq x_D,$$

and $V'(x) \geq 1 > 0$, assertion (b) follows immediately.

Part 3. Due to (5.12), for $x \geq 0$ such that $a(x) \leq 1$, we derive

$$\max_{a\in\mathbb{R}} \mathscr{L}^a V(x) = \max_{0\leq a\leq 1} \mathscr{L}^a V(x) = 0, \quad \text{for all} \quad 0 \leq x < x_D.$$

Then a combination of this with (5.16) implies assertion (c).

Part 4. This part of the proof focuses on calculating $a(0)$. To this end, we put

$$\widetilde{a} := arg \max_{0\leq a\leq 1} \mathscr{L}^a V(0),$$

and derive

$$0 = \mathscr{L}^{\widetilde{a}}V(0) = \frac{\widetilde{a}^2\sigma^2}{2}V''(0) + (\widetilde{a}\mu - \delta)V'(0) = \sigma^2 V''(0)\left(\frac{\widetilde{a}^2}{2} - \widetilde{a}a(0) + \frac{\delta}{\mu}a(0)\right). \quad (5.17)$$

This is equivalent to

$$\widetilde{a}^2 - 2\widetilde{a}a(0) + \frac{2\delta}{\mu}a(0) = 0, \quad (5.18)$$

since $V''(0) < 0$, and hence

$$(\widetilde{a} - a(0))^2 = a(0)(a(0) - \frac{2\delta}{\mu}).$$

1) Suppose that $a(0) \leq 1$. Then, $\widetilde{a} = a(0)$, and hence, either $\widetilde{a} = a(0) = 0$ or $\widetilde{a} = a(0) = \frac{2\delta}{\mu}$. From (5.16) and $V'(0) \geq 1$, we deduce that $a(0) > 0$ when $\delta > 0$ and hence $\widetilde{a} = a(0) = \frac{2\delta}{\mu}$ as soon as $a(0) \leq 1$.

2) If $a(0) > 1$, then $\widetilde{a} = 1$, and hence we get $1 - 2a(0) + \frac{2\delta}{\mu}a(0) = 0$. This is equivalent to $a(0) = \mu/2(\mu - \delta)$. However, this is possible only if $\delta > \mu/2$. Therefore, we conclude that when $\delta < \mu/2$, we have $a(0) \leq 1$ and hence, $a(0) = 2\delta/\mu$ on one hand. On the other hand, when $\delta > \mu/2$, we get $a(0) = \mu/2(\mu - \delta) \geq 1$. This ends the proof of the lemma. □

This lemma obviously tells us that the solution to the HJB equation (5.9) depends heavily on the ratio $2\delta/\mu$. This leads to two subsections where the two cases whether $2\delta/\mu$ is less that one or not will be discussed in details.

5.2.1 The case when $\frac{\mu}{2} \leq \delta < \mu$

Thanks to Lemma 5.2.1-(b), in this case, we get $a(x) \geq \frac{2\delta}{\mu} \geq 1$, and hence

$$\max_{0 \leq a \leq 1} \mathscr{L}^a V(x) = \mathscr{L}^1 V(x) = 0, \quad \text{if} \quad 0 \leq x \leq x_D.$$

By solving this equation, we obtain

$$V(x) = C_2 e^{r_+(x-x_D)} + C_3 e^{r_-(x-x_D)}, \quad \text{if} \quad 0 \le x \le x_D, \tag{5.19}$$

where C_2 and C_3 are free parameters to be determined, and $r_\pm$ are given by

$$r_\pm = \frac{-(\mu-\delta) \pm \sqrt{(\mu-\delta)^2 + 2(\gamma+\lambda\sigma^2)}}{\sigma^2}.$$

To calculate the constants C_2 and C_3, we use the relations $V'(x_D) = 1$ and $V''(x_D) = 0$, which are equivalent to

$$C_2 r_+ + C_3 r_- = 1 \quad \text{and} \quad C_2 r_+^2 + C_3 r_-^2 = 0.$$

By solving these two equations, we get

$$C_2 = -\frac{r_-}{r_+(r_+ - r_-)}, \quad \text{and} \quad C_3 = \frac{r_+}{r_-(r_+ - r_-)}. \tag{5.20}$$

To calculate x_D, we use the fact that $V(0) = 0$, which implies that

$$C_2 e^{-r_+ x_D} + C_3 e^{-r_- x_D} = 0.$$

This is equivalent to

$$e^{(r_+ - r_-)x_D} = -\frac{C_2}{C_3} = (\frac{r_-}{r_+})^2,$$

and hence, we get

$$x_D = \frac{1}{r_+ - r_-} \ln\left((\frac{r_-}{r_+})^2 \right).$$

This proves the following theorem.

Theorem 5.2.1. *If $\mu/2 \le \delta < \mu$, then the smooth solution to (5.9) is given by*

$$V(x) = \begin{cases} -\frac{r_-}{r_+(r_+-r_-)} e^{r_+(x-x_D)} + \frac{r_+}{r_-(r_+-r_-)} e^{r_-(x-x_D)}, & \textit{if} \quad 0 \le x \le x_D, \\ x - x_D + \frac{1}{r_-} + \frac{1}{r_+}, & \textit{if} \quad x > x_D, \end{cases}$$

where

$$x_D := \frac{1}{r_+ - r_-} \ln\left((\frac{r_-}{r_+})^2 \right) \tag{5.21}$$

and

$$r_\pm = \frac{-(\mu-\delta) \pm \sqrt{(\mu-\delta)^2 + 2(\gamma+\lambda)\sigma^2}}{\sigma^2}. \tag{5.22}$$

5.2.2 The case of small liability rate such that $\delta < \frac{\mu}{2}$

In this case, we start analyzing the maximizer function $a(x)$ defined in (5.13). To this end, we need the following notation. Consider the following function

$$F(u) = u + c\ln(u-c), \quad u > c. \tag{5.23}$$

It is clear that F is strictly increasing (since $F'(u) = 1 + c/(u-c) > 0$, for $u > c$), is continuous on $(c, +\infty)$, $F(c+) = -\infty$ and $F(+\infty) = +\infty$. This proves that $F : (c, +\infty) \to F(c, +\infty) = (-\infty, +\infty)$ is invertible.

Lemma 5.2.2. *Suppose that $\delta < \frac{\mu}{2}$ and let F be given in (5.23). Then , we have*

$$a(x) = F^{-1}\left(\frac{\mu^2 + 2(\gamma+\lambda)\sigma^2}{\mu\sigma^2} x + F\left(\frac{2\delta}{\mu}\right)\right), \quad \textit{if} \quad 0 \le x \le x_R, \tag{5.24}$$

where

$$x_R := \frac{\mu\sigma^2}{\mu^2 + 2(\gamma+\lambda)\sigma^2}\left[F\left(1\right) - F\left(\frac{2\delta}{\mu}\right)\right] < x_D. \tag{5.25}$$

Proof. Thanks to Lemma 5.2.1-(a) we have

$$V''(x) = -\frac{\mu V'(x)}{\sigma^2 a(x)}. \tag{5.26}$$

By substituting this equation back in $\mathscr{L}^{a(x)}V(x) = 0$, we get

$$\frac{1}{2}a^2(x)\sigma^2\left[-\frac{\mu V'(x)}{\sigma^2 a(x)}\right] + (a(x)\mu - \delta)V'(x) - (\gamma+\lambda)V(x) = 0.$$

Since our case here is the case when $\delta < \mu/2$, we have $a(0) = 2\delta/\mu$ (see Lemma 5.2.1) and $a(x)$ is continuous, then in a neighborhood of zero, we have

$$\frac{a(x)\mu}{2}V'(x) = \delta V'(x) + (\gamma+\lambda)V(x).$$

By differentiating both sides of the above equation, we get

$$\frac{\mu}{2}\left[a'(x)V'(x) + a(x)V''(x)\right] = \delta V''(x) + (\gamma+\lambda)V'(x).$$

By inserting $a(x) = -\mu V'(x)/(\sigma^2 V''(x))$ in this equation, we obtain

$$a'(x)V'(x) + a(x)\left(-\frac{\mu V'(x)}{\sigma^2 a(x)}\right) = \frac{2\delta}{\mu}\left(-\frac{\mu V'(x)}{\sigma^2 a(x)}\right) + \frac{2(\gamma+\lambda)}{\mu}V'(x),$$

which implies

$$a'(x) = \frac{\mu^2 + 2(\gamma+\lambda)\sigma^2}{\mu\sigma^2} - \frac{2\delta}{\sigma^2 a(x)}.$$

This is equivalent to

$$a'(x) = \frac{\mu^2 + 2(\gamma+\lambda)\sigma^2}{\mu\sigma^2}\left(1 - \frac{c}{a(x)}\right) \quad \text{where} \quad c := \frac{2\delta\mu}{\mu^2 + 2(\gamma+\lambda)\sigma^2}.$$

Since $a(x) \geq 2\delta/\mu > c$, for all $x \in [0, x_D]$ (see Lemma 5.2.1-(b) for details), by integrating the above equation, we get

$$a(x) + c\ln(a(x) - c) - a(0) - c\ln(a(0) - c) = \frac{\mu^2 + 2(\gamma+\lambda)\sigma^2}{\mu\sigma^2}x. \tag{5.27}$$

By using the notation in (5.23), this equation becomes

$$F\left(a(x)\right) = \frac{\mu^2 + 2(\gamma+\lambda)\sigma^2}{\mu\sigma^2}x + F\left(\frac{2\delta}{\mu}\right).$$

It is clear that $a(x)$ is increasing (since F^{-1} is increasing) and takes the value 1 at x_R, which solution to the equation $a(x_R) = 1$. This leads to (5.25) and (5.24). This proves the lemma. $\square$

Now, we state our solution to the HJB (5.9), when $\delta < \mu/2$, in the following.

Theorem 5.2.2. *Suppose that $\delta < \mu/2$. Then the smooth solution to the HJB (5.9) is given by*

$$V(x) = \begin{cases} C_1\left(\frac{2\delta}{\mu}-c\right)^{\frac{\mu}{\sigma^2 K}} \int_0^x \left(F^{-1}(Kt+\Delta)-c\right)^{-\frac{\mu}{\sigma^2 K}} dt, & \text{if} \quad 0 \le x \le x_R \\ -\frac{r_-}{r_+(r_+-r_-)} e^{r_+(x-x_D)} + \frac{r_+}{r_-(r_+-r_-)} e^{r_-(x-x_D)} & \text{if} \quad x_R < x \le x_D \\ x - x_D - \frac{r_-}{r_+(r_+-r_-)} + \frac{r_+}{r_-(r_+-r_-)} & \text{if} \quad x > x_D \end{cases}$$

where $r_\pm$ are given by (5.22),

$$C_1 = \frac{1-c}{(\frac{2\delta}{\mu}-c)(r_+-r_-)}\left(r_+ e^{r_-(x_R-x_D)} - r_- e^{r_+(x_R-x_D)}\right), \tag{5.28}$$

$$K := \frac{\mu^2+2(\gamma+\lambda)\sigma^2}{\mu\sigma^2}, \quad \Delta := \frac{2\delta}{\mu} + c\ \ln\left(\frac{2\delta}{\mu}-c\right), \quad c := \frac{2\delta\mu}{\mu^2+2(\gamma+\lambda)\sigma^2}, \tag{5.29}$$

and

$$x_D := x_R - \frac{1}{r_+-r_-}\ln\left(\frac{r_+(r_-+\frac{\mu}{\sigma^2})}{r_-(r_++\frac{\mu}{\sigma^2})}\right), \quad x_R := \frac{1}{K}\left[F\left(1\right)-\Delta\right]. \tag{5.30}$$

Proof. From (5.15), we have

$$\frac{V''(x)}{V'(x)} = -\frac{\mu}{\sigma^2}\frac{1}{a(x)}.$$

Then, by inserting (5.24) in this equation and integrating the resulting equation afterwards, we obtain

$$\ln\left(V'(x)\right) = -\frac{\mu}{\sigma^2}\int_0^x \frac{1}{F^{-1}\left(Kt+\Delta\right)} dt + C.$$

Consider the change of variable $F^{-1}\left(Kt+\Delta\right) = y$. Then $Kt+\Delta = F(y)$ and $Kdt = F'(y)dy$. By substituting back in the above equation, we get

$$\ln\left(V'(x)\right) = -\frac{\mu}{\sigma^2 K}\int_{F^{-1}(\Delta)}^{F^{-1}(Kx+\Delta)} \frac{dy}{y-c} + C.$$

This implies that

$$\ln\big(V'(x)\big) = \ln\left[\frac{F^{-1}(Kx+\Delta)-c}{F^{-1}(\Delta)-c}\right]^{-\frac{\mu}{\sigma^2 K}} + C,$$

or equivalently

$$V'(x) = C_1\left[\frac{F^{-1}(\Delta)-c}{F^{-1}(Kx+\Delta)-c}\right]^{\frac{\mu}{\sigma^2 K}},$$

where C_1 is a positive constant to be determined. By integrating again, we obtain

$$V(x) = C_1\Big(F^{-1}(\Delta)-c\Big)^p \int_0^x \frac{dt}{\Big[F^{-1}(Kt+\Delta)-c\Big]^p} \quad \text{where} \quad p := \frac{\mu}{\sigma^2 K}.$$

This proves the theorem for $x \in [0, x_R]$. If $x_R < x \leq x_D$, then $a(x) \geq 1$ and hence the maximum of $\mathscr{L}^a V(x)$ is attained at $a = 1$. Thus, the HJB (5.9), in this case reduces to the following ODE

$$\frac{\sigma^2}{2}V''(x) + (\mu-\delta)V'(x) - (\gamma+\lambda)V(x) = 0.$$

The solution to this ODE is given by

$$V(x) = C_2 e^{r_+(x-x_D)} + C_3 e^{r_-(x-x_D)}, \quad \text{if} \quad x_R < x \leq x_D,$$

where $r_\pm$ are given by (5.22). To calculate the constants C_2 and C_3, we use $V'(x_D) = 1$ and $V''(x_D) = 0$. These are equivalent to

$$C_2 r_+ + C_3 r_- = 1 \quad , \quad C_2 r_+^2 + C_3 r_-^2 = 0.$$

By solving these two equations, we get

$$C_2 = -\frac{r_-}{r_+(r_+-r_-)} \quad , \quad C_3 = \frac{r_+}{r_-(r_+-r_-)}. \tag{5.31}$$

To calculate x_D we use the continuity of V' and V'' at x_R, in other words, we use

$$V'(x_R-) = V'(x_R+), \quad \text{and} \quad V''(x_R-) = V''(x_R+).$$

These equations with the fact that $a(x_R) = -\mu V'(x_R-)/\sigma^2 V''(x_R-) = 1$, leads to

$$C_2 r_+ e^{r_+(x_R - x_D)} + C_3 r_- e^{r_-(x_R - x_D)} = -\frac{\sigma^2}{\mu}\big(C_2 r_+^2 e^{r_+(x_R - x_D)} + C_3 r_-^2 e^{r_-(x_R - x_D)}\big).$$

This is equivalent to

$$C_2 r_+ (1 + \frac{\sigma^2 r_+}{\mu}) e^{(r_+ - r_-)(x_R - x_D)} = -C_3 r_- (1 + \frac{\sigma^2 r_-}{\mu}),$$

and by combining this equation with (5.31), the first equation in (5.30) follows. The constant C_1 can be obtained by using the continuity of V' at x_R (or equivalently $V'(x_R-) = V'(x_R+)$). Hence

$$\frac{C_1 \left(F^{-1}(\Delta) - c\right)^p}{(F^{-1}(Kx_R + \Delta) - c)^p} = -\frac{r_- e^{r_+(x_R - x_D)}}{(r_+ - r_-)} + \frac{r_+ e^{r_-(x_R - x_D)}}{(r_+ - r_-)}.$$

By combining the value of x_R from (5.30) and $F^{-1}(\Delta) = \frac{2\delta}{\mu}$ we obtain $F^{-1}(Kx_R + \Delta) = 1$ and hence we get (5.28). This ends the theorem. □

5.3 Optimal policy and the verification theorem

In this section, we construct the optimal control policy based on the solution of the HJB equation obtained in the previous section. For each $x \geq 0$, we define

$$a^*(x) := arg \max_{0 \leq a \leq 1} \left(\frac{1}{2} a^2 \sigma^2 V''(x) + (a\mu - \delta)V'(x) - (\gamma + \lambda)V(x)\right). \tag{5.32}$$

The function $a^*(x)$ represents the optimal feedback control function for the control component a_t^π, $t \geq 0$. More precisely, the value $a^*(x)$ is the optimal risk that one should take when the value of the current reserve is x. Thanks to the previous section, we get the following.

Proposition 5.3.1. *Suppose that $\delta < \mu$. Let the functions F and a^* be defined in (5.23) and (5.32) respectively. Then the following assertions hold.*

(a) *If $\delta \geq \frac{\mu}{2}$, then $a^*(x) = 1$, for all $x \geq 0$.*

(b) *If $\delta < \frac{\mu}{2}$, then*

$$a^*(x) = \begin{cases} F^{-1}\left(\frac{\mu^2+2(\gamma+\lambda)\sigma^2}{\mu\sigma^2}x + F\left(\frac{2\delta}{\mu}\right)\right), & \text{if} \quad x \leq x_R, \\ 1, & \text{if} \quad x > x_R. \end{cases}$$

where

$$x_R = \frac{\mu\sigma^2}{\mu^2+2(\gamma+\lambda)\sigma^2}\left[F\left(1\right) - F\left(\frac{2\delta}{\mu}\right)\right].$$

Proof. Since our V is twice continuously differentiable, $V'(x) > 0$, and $V''(x) < 0$ for all $x \geq 0$. Then for any $x \in [0, +\infty)$, as a function of $a \in \mathbb{R}$, the following

$$\mathscr{L}^a V(x) = \frac{\sigma^2 a^2}{2}V''(x) + (a\mu - \delta)V'(x) - (\gamma+\lambda)V(x),$$

is a concave function with a maximum attained at $a(x)$. Furthermore, it is increasing on $[0, a(x)]$ and decreasing on $[a(x), +\infty)$. Therefore, when $\frac{2\delta}{\mu} \geq 1$, that implies $a(x) \geq 1$, we get $a^*(x) = 1$, on the one hand. On the other hand, when $\frac{2\delta}{\mu} < 1$, we get $a(x) < 1$ for $x < x_R$ and $a(x) \geq 1$ for $x \geq x_R$. Thus, similarly as above, this implies

$$a^*(x) = a(x) = F^{-1}\left(\frac{\mu^2+2(\gamma+\lambda)\sigma^2}{\mu\sigma^2}x + F\left(\frac{2\delta}{\mu}\right)\right), \quad x \leq x_R$$

and $a^*(x) = 1$ for $x > x_R$. This completes the proof. $\square$

Theorem 5.3.1. *The reflecting stochastic differential equation*

$$\begin{aligned} &dR = \big(a^*(R_t)\mu - \delta\big)dt + a^*(R_t)\sigma dW_t - C_t, \quad R_0 = x, \\ &R_t \leq x_D, \quad \int_0^{+\infty} 1_{\{R_s < x_D\}} dC_s = 0, \end{aligned} \tag{5.33}$$

has a unique solution that we denote by (R^*, C^*).

Proof. Remark that by putting $X_t := x_D - R_t$ for $t \geq 0$, (5.33) becomes

$$dX_t = -\big(a^*(x_D - X_t)\mu - \delta\big)dt - a^*(x_D - X_t)\sigma dW_t + C_t, \quad X_0 = x$$
$$X_t \geq 0, \quad \int_0^{+\infty} 1_{\{X_s > 0\}} dC_s = 0.$$

Thanks to Theorem 2.3.2, to prove that the reflecting SDE has a unique solution, it is enough to prove that the coefficients of the SDE satisfy the following Lipschitz and growth conditions

$$|a^*(x_1) - a^*(x_2)| \leq K|x_1 - x_2|, \quad |a^*(x)| \leq K(1 + |x|). \tag{5.34}$$

The growth condition is a consequence of the Lipschitz condition and by taking $x_1 = x$ and $x_2 = 0$. Hence

$$|a^*(x)| \leq |a^*(0)| + K_1|x| \leq \max(1, K_1)(1 + |x|).$$

In the remaining part of the proof, we focus on proving that the Lipschitz condition hold for different levels of x_1 and x_2 compared with the threshold x_R. For $x_1, x_2 > x_R$, we get $a^*(x_1) = a^*(x_2) = 1$ and hence (5.34) is satisfied.

Remark that the cases when $x_1 > x_R$ and $x_2 \leq x_R$, and when $x_1 \leq x_R$ and $x_2 > x_R$ reduce to $x_1, x_2 \leq x_R$ since a^* is a constant over $[x_R, +\infty)$. For the case when x_1 and x_2 belong to $[0, x_R]$ we use Taylor's expansion, and derive

$$|a^*(x_1) - a^*(x_2)| = |(x_1 - x_2)\frac{d}{dy}a^*(y)|,$$

where $y = \alpha x_1 + (1 - \alpha)x_2$ for some $\alpha \in (0, 1)$. Therefore, it is enough to show that $\frac{d}{dy}a^*(y)$ is bounded over $[0, x_R]$. To this end, we calculate $\frac{d}{dy}a^*(y)$

$$|\frac{da^*(y)}{dy}| = \frac{2\delta}{\sigma^2 c}\left|1 - \frac{c}{a^*(y)}\right|, \quad \text{where} \quad c = \frac{2\delta\mu}{\mu^2 + 2(\gamma + \lambda)\sigma^2}.$$

Since $a^*(y)$ is an increasing function and $a^*(y) \geq 2\delta/\mu$, we deduce that $a^*(y) - c \geq 2\delta/\mu - c$. Hence,

$$|\frac{da^*(y)}{dy}| = \frac{2\delta}{\sigma^2 c}\left|1 - \frac{c}{a^*(y)}\right| \leq \frac{2\delta}{\sigma^2 c}\left(1 + |\frac{c}{a^*(y)}|\right) \leq \frac{2\delta}{\sigma^2 c}\left(1 + \frac{c\mu}{2\delta}\right).$$

This ends the theorem. $\square$

Theorem 5.3.2. *Let V be the concave and twice continuously differentiable solution of the HJB equation (5.9), and $(R_t^*, C_t^*; t \geq 0)$ be a solution to the Skorokhod problem (5.33). Then $\pi^* = \left(a^*(R_t^*), C_t^*; t \geq 0\right)$ is the optimal policy and*

$$\mathbb{E}\left[\int_0^{\tau^*} e^{-(\gamma+\lambda)t} dC_t^*\right] = V(x), \quad \forall x \geq 0, \tag{5.35}$$

where

$$\tau^* = \inf\{t \geq 0 : R_t^* = 0\}.$$

Proof. By applying Itô to $e^{-(\gamma+\lambda)t\wedge\tau^*} V(R^*_{t\wedge\tau^*})$, we deduce

$$\begin{aligned} e^{-(\gamma+\lambda)t\wedge\tau^*} V(R^*_{t\wedge\tau^*}) = V(x) &- \int_0^{t\wedge\tau^*} (\gamma+\lambda) e^{-(\gamma+\lambda)s} V(R_s^*) ds + \int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}\left(a^*(R_s^*)\mu - \delta\right) V'(R_s^*) ds \\ &+ \int_0^{t\wedge\tau^*} \frac{1}{2} e^{-(\gamma+\lambda)s} \sigma^2 \big(a^*(R_s^*)\big)^2 V''(R_s^*) ds + \int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s} \sigma a^*(R_s^*) V'(R_s^*) dW_s \\ &- \int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s} V'(R^*_{s_-}) dC_s^* + \sum_{0<s\leq t\wedge\tau^*} e^{-(\gamma+\lambda)s}\big[V(R_s^*) - V(R^*_{s_-}) - V'(R^*_{s_-})\Delta R_s^*\big]. \end{aligned} \tag{5.36}$$

Since $0 \leq V'(x) \leq K_1$, where $K_1 = \max\left(C_1, r_+\left(\frac{1}{r_+ + \frac{\mu}{\sigma^2}}\right)\left(\frac{r_+(r_- + \mu/\sigma^2)}{r_-(r_+ + \mu/\sigma^2)}\right)^{r_-/(r_+ - r_-)}, 1\right)$, we deduce that

$$\int_0^{t\wedge\tau^*} \sigma^2 \big(a^*(R_s^*)\big)^2 V'^2(R_s^*) ds \leq \sigma^2 K_1 t,$$

and hence, $\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}\sigma a^*(R_s^*)V'(R_s^*)dW_s$ is a martingale which implies

$$\mathbb{E}\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}\sigma a^*(R_s^*)V'(R_s^*)dW_s = 0.$$

Therefore, by using this equation and taking the expectation in both sides of (5.36), we obtain

$$\begin{aligned}
\mathbb{E}\Big[e^{-(\gamma+\lambda)t\wedge\tau^*}V(R^*_{t\wedge\tau^*})\Big] &= V(x)\\
&+\mathbb{E}\Big[e^{-(\gamma+\lambda)s}\Big(\frac{1}{2}\sigma^2\big(a^*(R_s^*)\big)^2V''(R_s^*)+\big(a^*(R_s^*)\mu-\delta\big)V'(R_s^*)-(\gamma+\lambda)V(R_s^*)\Big)ds\Big]\\
&+\mathbb{E}\Big[-\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}V'(R^*_{s_-})dC_s^* + \sum_{0<s\le t\wedge\tau^*} e^{-(\gamma+\lambda)s}\Big(V(R_s^*)-V(R^*_{s_-})-V'(R^*_{s_-})\Delta R_s^*\Big)\Big],
\end{aligned} \tag{5.37}$$

where we recall

$$\mathscr{L}^a V(y) = \frac{\sigma^2 a^2}{2}V''(y)+(a\mu-\delta)V'(y)-(\gamma+\lambda)V(y), \quad a\ge 0, x\ge 0.$$

Since, $\mathscr{L}^{a^*(y)}V(y)=0$ for all $y\ge 0$, and in particular for $y=R_s^*$, this implies that

$$\begin{aligned}
\mathbb{E}\Big[e^{-(\gamma+\lambda)(t\wedge\tau^*)}V(R^*_{t\wedge\tau^*})\Big] &= V(x)\\
&+\mathbb{E}\Big[-\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}V'(R^*_{s_-})dC_s^* + \sum_{0<s\le t\wedge\tau^*} e^{-(\gamma+\lambda)s}\Big(V(R_s^*)-V(R^*_{s_-})-V'(R^*_{s_-})\Delta R_s^*\Big)\Big],
\end{aligned} \tag{5.38}$$

Since $V'(x)=1$ for all $x\ge x_D$ and $dC_s^* = 1_{\{R_s^*=x_D\}}dC_s^*$, then

$$V(R_s^*)-V(R^*_{s_-}) = V(R_s^*)-V(R_s^*+\Delta C_s^*) = V(x_D)-V(x_D+\Delta C_s^*) = x_D-(x_D+\Delta C_s^*) = -\Delta C_s^*. \tag{5.39}$$

Using a similar argument, we get

$$V'(R^*_{s_-})\Delta R_s^* = V'(R_s^*-\Delta R_s^*)\Delta R_s^* = -V'(R_s^*+\Delta C_s^*)\Delta C_s^* = -V'(x_D+\Delta C_s^*) = -\Delta C_s^*. \tag{5.40}$$

Further,

$$\begin{aligned}\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}V'(R^*_{s_-})dC^*_s &= \int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}V'(R^*_s+\Delta C^*_s)dC^*_s \\ &= \int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}V'(x_D+\Delta C^*_s)dC^*_s \\ &= \int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}dC^*_s. \end{aligned} \tag{5.41}$$

By combining (5.38), (5.39), (5.40) and (5.41), we deduce

$$\mathbb{E}\left[e^{-(\gamma+\lambda)t\wedge\tau^*}V(R^*_{t\wedge\tau^*})\right]+\mathbb{E}\left[\int_0^{t\wedge\tau^*} e^{-(\gamma+\lambda)s}dC^*_s\right]=V(x). \tag{5.42}$$

The first term on the left hand side of this equation can be written as

$$\mathbb{E}\left[e^{-(\gamma+\lambda)\tau^*}V(R^*_{\tau^*})1_{\{\tau^*<t\}}+e^{-(\gamma+\lambda)t}V(R^*_t)1_{\{t\leq\tau^*\}}\right].$$

A combination of this with the fact that $V(0)=0$ and on $\{\tau^*<+\infty\}$, $R^*_{\tau^*}=0$ P-a.s, we get

$$0\leq\mathbb{E}\left[e^{-(\gamma+\lambda)(t\wedge\tau^*)}V(R^*_{t\wedge\tau^*})\right]=\mathbb{E}\left[e^{-(\gamma+\lambda)t}V(R^*_t)1_{\{\tau^*\geq t\}}\right]\leq e^{-(\gamma+\lambda)t}\mathbb{E}\left[V(R^*_t)\right].$$

By applying the fact that $V(x)\leq K_1x$ on the last inequality (where $V(x)=xV'(\eta_x)$ by using Taylor, and $V'(x)\leq K_1$), we get

$$0\leq\mathbb{E}\left[e^{-(\gamma+\lambda)(t\wedge\tau^*)}V(R^*_{t\wedge\tau^*})\right]=\mathbb{E}\left[e^{-(\gamma+\lambda)t}V(R^*_t)1_{\{\tau^*\geq t\}}\right]\leq e^{-(\gamma+\lambda)t}K_1\mathbb{E}\big[R^*_t\big]. \tag{5.43}$$

However,

$$\mathbb{E}[R^*_t]=x+\mathbb{E}\left[\int_0^t\big(a^*(R^*_s)\mu-\delta\big)ds\right]-\mathbb{E}\left[\int_0^t dC^*_s\right]\leq x+\mathbb{E}\left[\int_0^t\big(a^*(R^*_s)\mu-\delta\big)ds\right]\leq x+(\mu-\delta)t.$$

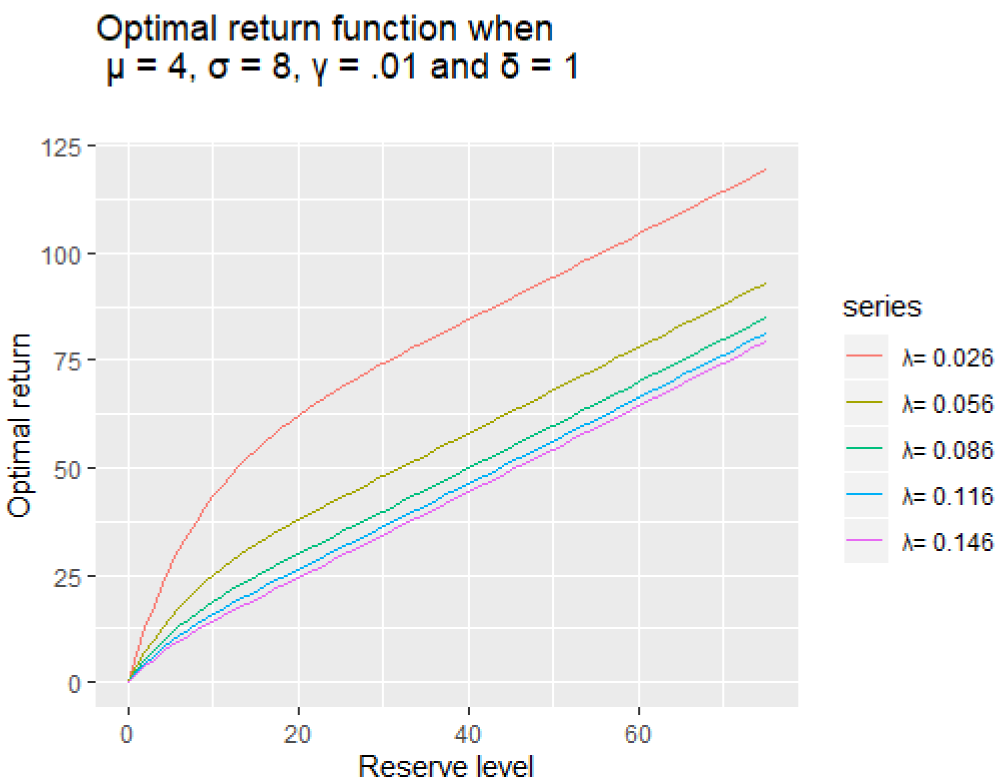
By inserting this in (5.43), we get

$$\mathbb{E}\left[e^{-(\gamma+\lambda)t}V(R^*_t)1_{\{\tau^*\geq t\}}\right]\leq e^{-(\gamma+\lambda)t}K_1\left[x+(\mu-\delta)t\right]. \tag{5.44}$$

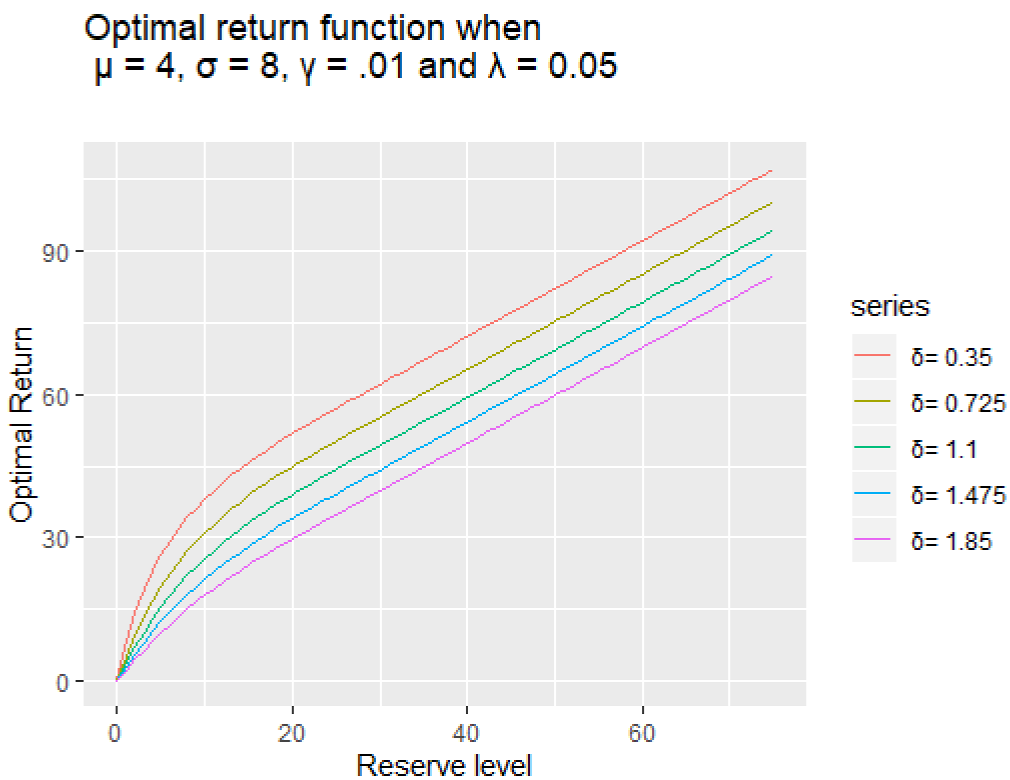
By combining (5.43) and (5.44) and taking the $t\longrightarrow+\infty$, we get (5.35). Thanks to (5.35) and (5.33) it is obvious that $\pi^*=(a^*(R^*_t),t\geq0)$ is the optimal policy and R^* is the optimal cash reserve that corresponds to it. $\square$

5.4 Graphical illustrations

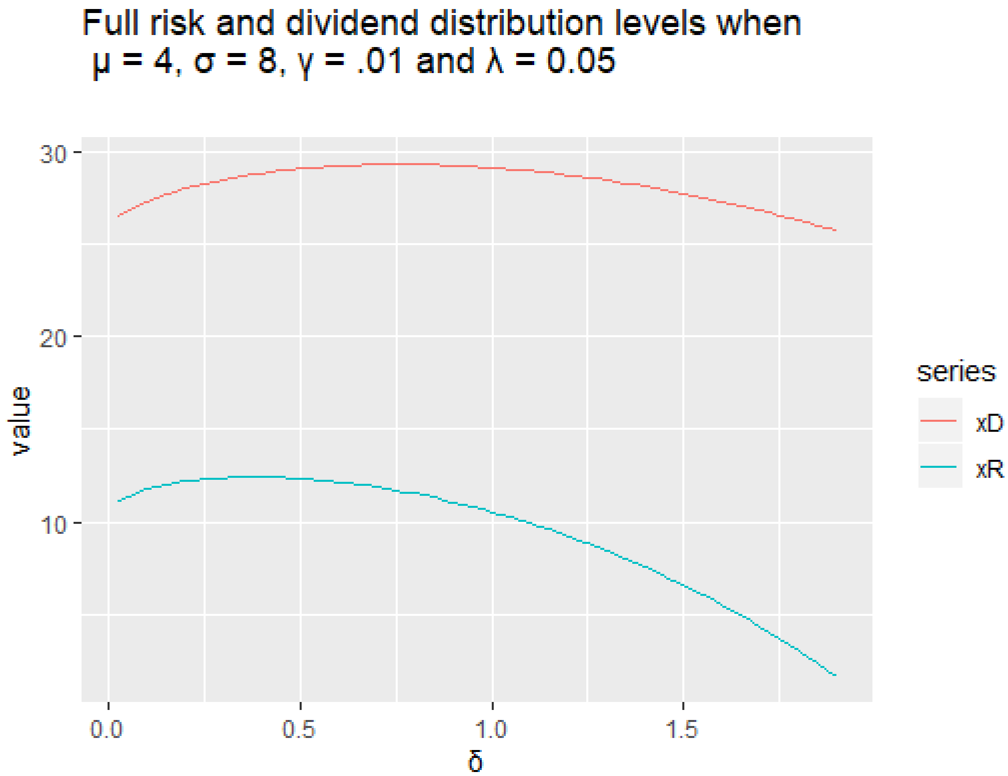
In this section we present visualization of the sensitivity analysis of the optimal dividend distribution model discussed in this chapter. Here we illustrate the affects of the parameters λ and δ on the optimal return function and the threshold levels of taking full risk and distribution of dividends x_R and x_D respectively. For more details on the interplay of the other parameters, we refer the reader to [15] and [17].



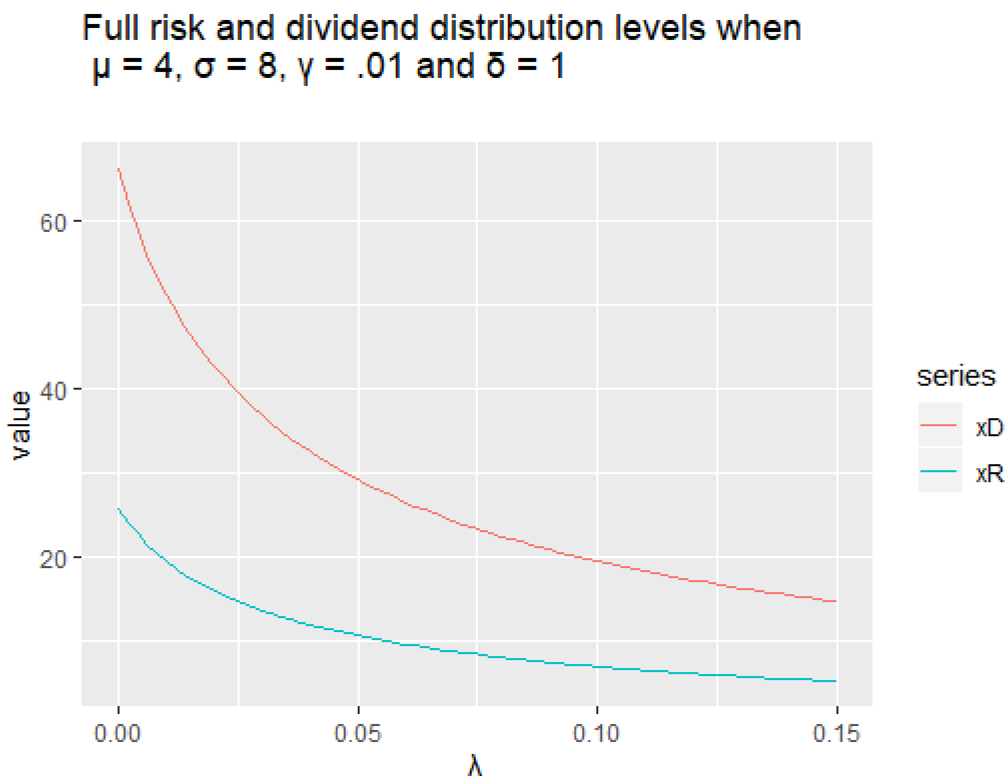
As the graph illustrates, the optimal return function has a sharp reaction to the change of the intensity rate. It decreases and becomes less concave as the intensity rate λ increases.



Similar to the model in Chapter 3, the increase in liability rate has the affects of decreasing the optimal the optimal return function as well as making it looks less concave.



As it is shown in this graph, both threshold levels of full risk and dividend distribution x_R and x_D respectively decrease when δ increase.



Presented above, it is clear that the threshold levels x_R and x_D sharply decrease as the intensity rate λ increases.

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