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CONVERGENCE OF MARKOV CHAIN APPROXIMATIONS TO STOCHASTIC REACTION–DIFFUSION EQUATIONS[1]

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In the context of simulating the transport of a chemical or bacterial contaminant through a moving sheet of water, we extend a well-established method of approximating reaction–diffusion equations with Markov chains by allowing convection, certain Poisson measure driving sources and a larger class of reaction functions. Our alterations also feature dramatically slower Markov chain state change rates often yielding a ten to one-hundred-fold simulation speed increase over the previous version of the method as evidenced in our computer implementations. On a weighted L^2 Hilbert space chosen to symmetrize the elliptic operator, we consider existence of and convergence to pathwise unique mild solutions of our stochastic reaction–diffusion equation. Our main convergence result, a quenched law of large numbers, establishes convergence in probability of our Markov chain approximations for each fixed path of our driving Poisson measure source. As a consequence, we also obtain the annealed law of large numbers establishing convergence in probability of our Markov chains to the solution of the stochastic reaction–diffusion equation while considering the Poisson source as a random medium for the Markov chains.

1. Introduction and notation. Recently, the problem of assessing water pollution has become a matter of considerable concern. For proper groundwater management, it is necessary to model the contamination mathematically in order to assess the effects of contamination and predict the transport of contaminants. A large number of models in the deterministic case have been developed and solved analytically and numerically [see Jennings, Kirkner and Theis (1982); Marchuk (1986); Celia, Kindred and Herrera (1989); Kindred and Celia (1989); Van der Zee (1990); Xin (1994); Barrett and Knabner (1997, 1998); Chen and Ewing (1997); Dawson (1998); Hossain and Miah (1999) and Hossain and Yonge (1999)]. Based upon Kallianpur and Xiong (1994), we consider a more realistic model by introducing some randomness in a meaningful way. We assume that the undesired (chemical or biological) contaminants are released by different factories along the groundwater system (or river). There are r such factories located at different sites $\kappa_1, \dots, \kappa_r$ in the region $E = [0, L_1] \times [0, L_2]$. Each of the

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factories releases contaminants at the jump times of independent Poisson processes $N_1(t), \dots, N_r(t)$ with random magnitudes $\{A_i^j, j = 1, 2, \dots\}$ which are i.i.d. with common distribution $F_i(da)$. Upon release, the contaminants are distributed in the area $B(\kappa_i, \varepsilon) = \{x : |x - \kappa_i| < \varepsilon\} \subset (0, L_1) \times (0, L_2)$ according to a proportional function $\theta_i(x)$ satisfying

$$\theta_i(x) \geq 0, \quad \operatorname{supp} \theta_i \subseteq B(\kappa_i, \varepsilon) \quad \text{and} \quad \int_{B(\kappa_i, \varepsilon)} \theta_i(x)\, dx = 1.$$

We assume that θ_i is bounded and continuous on $B(\kappa_i, \varepsilon)$ $(i = 1, 2, \dots, r)$. For example, we can take

$$\theta_i(x) = \frac{1}{\pi \varepsilon^2} 1_{B(\kappa_i, \varepsilon)}(x),$$

which is the uniformly distributed function in $B(\kappa_i, \varepsilon)$ as used in Kallianpur and Xiong (1994), or (letting $|\cdot|$ denote Euclidean distance)

$$\theta_i(x) = \left(\int_{B(\kappa_i, \varepsilon)} \exp\left\{ -\frac{1}{\varepsilon^2 - |z - \kappa_i|^2} \right\} dz \right)^{-1} \exp\left\{ -\frac{1}{\varepsilon^2 - |x - \kappa_i|^2} \right\}, \qquad x \in E,$$

which is a smooth function with decay along radial lines in $B(\kappa_i, \varepsilon)$. Once released, the contaminants diffuse and drift through the sheet of water largely due to the movement of the water itself. Also, there is the possibility of nonlinear reaction of the contaminants due to births and deaths of bacteria or adsorption of chemicals, which refers to adherence of a substance to the surface of the porous medium in groundwater systems.

We define and abbreviate

$$\partial_1 f(x_1, x_2) := \frac{\partial}{\partial x_1} f(x_1, x_2) = \lim_{\substack{h \to 0 \\ (x_1 + h, x_2) \in E}} \frac{f(x_1 + h, x_2) - f(x_1, x_2)}{h},$$

$$\partial_2 := \frac{\partial}{\partial x_2}, \qquad \Delta := \partial_1^2 + \partial_2^2, \qquad \nabla := (\partial_1\, \partial_2)^T.$$

The stochastic model described as above can be written formally as follows:

$$\begin{aligned} \frac{\partial}{\partial t} u(t, x) &= D \Delta u(t, x) - V \cdot \nabla u(t, x) + R(u(t, x)) \\ &\quad + \sum_{i=1}^{r} \sum_{j=1}^{\infty} A_i^j(\omega) \theta_i(x) 1_{t = \tau_i^j(\omega)}, \qquad x \in [0, L_1] \times [0, L_2], \end{aligned} \tag{1.1}$$

subject to

$$\partial_1 u(t, L_1, x_2) = \partial_1 u(t, 0, x_2) = 0, \qquad \partial_2 u(t, x_1, L_2) = \partial_2 u(t, x_1, 0) = 0,$$

$$u(0, x) = u_0(x),$$

where $u(t,x)$ denotes the concentration of a dissolved or suspended substance, $D>0$ denotes the dispersion coefficient, $V=(V_1,V_2)$ with $V_1>0$, $V_2=0$ denotes the water velocity, $R(\cdot)$ denotes the nonlinear reaction term, $\{\tau_i^j, j\in\mathbb{Z}_+\}$ are the jump times of independent Poisson processes $N_i(t)$ $(i=1,2,\dots,r)$ with parameters η_i, and $u_0(x)$ denotes the initial concentration of the contaminants in the region $[0,L_1]\times[0,L_2]$. Here, we adopt the Neumann boundary condition which means that the contaminant concentration is constant across the boundary of the region $[0,L_1]\times[0,L_2]$. All the random variables A_i^j and τ_i^j (or $N_i(t)$) are defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$. Moreover, we assume $R:[0,\infty)\to\mathbb{R}$ is continuous with

$$R(0)\ge 0 \quad\text{and}\quad \sup_{u>0}\frac{R(u)}{1+u}<\infty,$$

and for some $q\ge 1$, $K>0$ as well as all $u,v\in\mathbb{R}_+$,

$$|R(u)-R(v)|\le K|u-v|(1+u^{q-1}+v^{q-1}),\qquad |R(u)|\le K(1+u^q). \tag{1.2}$$

These assumptions amount to nonnegativity at 0, linear growth for the positive part of R, a local Lipschitz condition and polynomial growth. We will interpret solutions to (1.1) as mild solutions defined below (see Definition 1.3).

REMARK 1.1. Kurtz (1971) introduced the stochastic particle Markov chain method of approximating differential equations. Arnold and Theodosopulu (1980), Kotelenez (1986, 1988) and Blount (1991, 1994, 1996) studied Markov chain approximation for a chemical reaction with diffusion provided that the nonlinear reaction term is a polynomial with a negative leading coefficient. Our assumptions on R are much weaker.

Let us define a differential operator $\mathcal{A}=D\Delta-V\cdot\nabla$ with Neumann boundary conditions in both variables. We take the initial domain $\mathcal{D}_0(\mathcal{A})$ of $\mathcal{A}$ to be $\{f\in\mathcal{C}^2(E):\partial_1 f(0,x_2)=\partial_1 f(L_1,x_2)=\partial_2 f(x_1,0)=\partial_2 f(x_1,L_2)=0\}$, where $\mathcal{C}^2(E)$ denotes the twice continuously differentiable functions on E. Letting $\rho(x)=e^{-2cx_1}$ and $c=\frac{V_1}{2D}$, we can rewrite $\mathcal{A}$ as

$$\mathcal{A}=D\left[\frac{1}{\rho(x)}\frac{\partial}{\partial x_1}\left(\rho(x)\frac{\partial}{\partial x_1}\right)+\frac{\partial^2}{\partial x_2^2}\right].$$

For convenience, we define a Hilbert space H as follows.

DEFINITION 1.2. $(H,\langle\cdot,\cdot\rangle)$ is the Hilbert space $L^2(E,\rho(x)\,dx)$ with norm

$$\|f\|=\left\{\int_E f^2(x)\rho(x)\,dx\right\}^{1/2}.$$

$(\mathcal{A}, \mathcal{D}_0(\mathcal{A}))$ is symmetric on H and admits a unique self-adjoint extension with domain $\mathcal{D}(\mathcal{A}) = \{f \in H : |\nabla f|, \Delta f \in H$ and $\partial_1 f(0, x_2) = \partial_1 f(L_1, x_2) = 0$, $\partial_2 f(x_1, 0) = \partial_2 f(x_1, L_2) = 0\}$. We define a random process $\Theta(t, x)$ by

$$\Theta(t, x) = \sum_{i=1}^{r} \theta_i(x) \sum_{j=1}^{N_i(t)} A_i^j(\omega),$$

and find (1.1) can be rewritten as

$$du(t, x) = [\mathcal{A}u(t, x) + R(u(t, x))]\, dt + d\Theta(t, x), \qquad u(0) = u_0. \tag{1.3}$$

We consider a pathwise mild solution of our stochastic partial differential equation (SPDE) (1.3). Let $T(t)$ be the C_0-semigroup generated by $\mathcal{A}$.

DEFINITION 1.3. A process $u(t)$, $t \geq 0$ is a mild solution to (1.3) in H if it satisfies

$$u(t) = T(t)u_0 + \int_0^t T(t-s)R(u(s))\, ds + \int_0^t T(t-s)\, d\Theta(s). \tag{1.4}$$

For any separable Hilbert space V, $C_V[0, T]$ and $D_V[0, T]$ denote, respectively, the V-valued continuous and càdlàg functions h such that $h(t) \in V$ for all $0 \leq t \leq T$. For càdlàg functions h, we define

$$h(\tau_-) \doteq \begin{cases} 0, & \tau = 0, \\ \lim_{s \nearrow \tau} h(s), & 0 < \tau \leq T. \end{cases}$$

We shall use the notation $C, C(\omega), C(N, l), C(T)$ and so on, for finite constants (depending on ω, resp. N, l, etc.), which may be different at various steps in the proofs of our results in the paper.

In this paper, we discuss unique pathwise $D_H[0, T]$-valued solutions and Markov chain approximations (i.e., distribution convergence) to SPDE (1.3). These results are vital for application of filtering theory to pollution dispersion tracking problems in the sense that the original signal can be replaced with a tractable Markov chain approximation. [The reader is referred to Kushner (1977), Di Masi and Runggaldier (1981) or Bhatt, Kallianpur and Karandikar (1999) for justification about this substitution of signal for calculation purposes.] In this manner, Monte Carlo and Kallianpur–Striebel based methods of filtering become more feasible. Our Markov chain approximations employ improved rate schemes over previous works of Kotelenez (1986, 1988) and Blount (1991, 1994, 1996), resulting in far more efficient computer implementation of approximate solutions to (1.3) and even a more general allowable class of reaction functions R in (1.3).

The contents of this paper are organized as follows. In Section 2, we shall construct the Markov chain approximations to our pollution model (1.3) via the stochastic particle method and the random time changes approach. In Section 3, we shall state and prove our main results establishing that there exists a pathwise unique solution to (1.3) as well as a Markov chain approximation that converges in probability to the solution of (1.3) for each fixed path of our Poisson source. This later part is our quenched law of large numbers. As a corollary, we also establish the annealed law of large numbers while considering the Poisson source as a random medium of the Markov chains.

2. Construction of Markov chain via stochastic particle method. The Markov chain approximation discussed in this paper is motivated by the stochastic particle models of chemical reaction with diffusion studied by Arnold and Theodosopulu (1980), Kotelenez (1986, 1988) and Blount (1991, 1994, 1996). In their models, the operator $\mathcal{A}$ is replaced by the Laplacian and only the internal fluctuation caused by reaction and diffusion was considered. They proved that a sequence of Markov chain approximations converges to the solution of deterministic models weakly (in the distribution convergence sense). In our models, we have two kinds of randomness, which are the external fluctuation coming from the Poisson sources and the internal fluctuation in implementing the reaction and diffusion. We also feature a new method of forming the Markov chain approximations that is more efficient for computer implementation. Before defining the stochastic particle models, we prepare some preliminaries concerning the differential operator $\mathcal{A}$ and its discretization. Basic calculations will bear out the following lemma whose proof is omitted.

LEMMA 2.1. *The eigenvalues and eigenfunctions* $\{(\lambda_p, \phi_p)\}_{p=(p_1,p_2)\in(\mathbb{N}_0)^2}$ *of* $\mathcal{A}$ *are given by*

$$\lambda_p = \lambda^1_{p_1} + \lambda^2_{p_2}, \quad \phi_p(x) = \phi^1_{p_1}(x_1)\phi^2_{p_2}(x_2), \qquad p_1, p_2 \in \mathbb{N}_0,$$

$$\lambda^1_0 = 0, \quad \lambda^1_{p_1} = -D\left(\frac{p_1\pi}{L_1}\right)^2 - Dc^2, \qquad p_1 \in \mathbb{N},$$

$$\lambda^2_0 = 0, \quad \lambda^2_{p_2} = -D\left(\frac{p_2\pi}{L_2}\right)^2, \qquad p_2 \in \mathbb{N};$$

$$\phi^1_0(x_1) = \sqrt{\frac{2c}{1-e^{-2cL_1}}}, \qquad \phi^2_0(x_2) = \sqrt{\frac{1}{L_2}},$$

$$\phi^1_{p_1}(x_1) = \sqrt{\frac{2}{L_1}}\sin\left\{\frac{p_1\pi x_1}{L_1} + \alpha_{p_1}\right\}\exp\{cx_1\}, \qquad p_1 \in \mathbb{N},$$

$$\phi^2_{p_2}(x_2) = \sqrt{\frac{2}{L_2}}\cos\left\{\frac{p_2\pi x_2}{L_2}\right\}, \qquad p_2 \in \mathbb{N},$$

where $\alpha_{p_1} = \tan^{-1}(-\frac{p_1\pi}{L_1 c})$, $c = \frac{V_1}{2D}$.

Now, we divide $[0, L_1) \times [0, L_2)$ into $L_1 N \times L_2 N$ cells of size $\frac{1}{N} \times \frac{1}{N}$:

$$I_k \doteq \left[\frac{k_1 - 1}{N}, \frac{k_1}{N}\right) \times \left[\frac{k_2 - 1}{N}, \frac{k_2}{N}\right), \qquad k = (k_1, k_2),$$
$$k_1 = 1, 2, \ldots, L_1 N, \quad k_2 = 1, 2, \ldots, L_2 N.$$

Let $H^N = \{\varphi \in H : \varphi \text{ is constant on each } I_k\}$. To facilitate the removal of the discrete gradient as we did in the continuous limit case, we define the uncommon discrete gradient in the first variable

$$\nabla_N^{V_1} f(x) = DN^2(1 - e^{-c/N})\left[f\left(x + \frac{e_1}{N}\right) - f(x)\right]$$
$$+ DN^2(e^{c/N} - 1)\left[f(x) - f\left(x - \frac{e_1}{N}\right)\right]$$

and the usual discrete Laplacian

$$\Delta_N f(x) = N^2\left[f\left(x + \frac{e_1}{N}\right) + f\left(x - \frac{e_1}{N}\right) - 2f(x)\right]$$
$$+ N^2\left[f\left(x + \frac{e_2}{N}\right) + f\left(x - \frac{e_2}{N}\right) - 2f(x)\right]$$
$$\doteq \Delta_{Nx_1} f(x) + \Delta_{Nx_2} f(x),$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Now, we look at the discretized approximation: $\mathcal{A}^N \doteq D\Delta_N - \nabla_N^{V_1}$. We define the following discrete gradients:

$$\widetilde{\nabla}_{Nx_i} f(x) = N\left[f\left(x + \frac{e_i}{2N}\right) - f\left(x - \frac{e_i}{2N}\right)\right],$$

$$\nabla_{Nx_i}^{+} f(x) = N\left[f\left(x + \frac{e_i}{N}\right) - f(x)\right]$$

and

$$\nabla_{Nx_i}^{-} f(x) = N\left[f\left(x - \frac{e_i}{N}\right) - f(x)\right], \qquad i = 1, 2.$$

In order to take the boundary conditions into account for the discretized approximation scheme, we extend all function $f \in H^N$ to the region $[-\frac{1}{N}, L_1 + \frac{1}{N}] \times [-\frac{1}{N}, L_2 + \frac{1}{N}]$ by letting

$$f(x_1, x_2) = f\left(x_1 + \frac{1}{N}, x_2\right), \qquad x_1 \in \left[-\frac{1}{N}, 0\right), \ x_2 \in [0, L_2];$$

$$f(x_1, x_2) = f\left(x_1 - \frac{1}{N}, x_2\right), \qquad x_1 \in \left[L_1, L_1 + \frac{1}{N}\right), \ x_2 \in [0, L_2];$$

$$f(x_1, x_2) = f\left(x_1, x_2 + \frac{1}{N}\right), \qquad x_1 \in [0, L_1], \ x_2 \in \left[-\frac{1}{N}, 0\right);$$

$$f(x_1,x_2)=f\Big(x_1,x_2-\frac{1}{N}\Big),\qquad x_1\in[0,L_1],\ x_2\in\Big[L_2,L_2+\frac{1}{N}\Big)$$

and denote this class of functions by H_{bc}^N. Then, H_{bc}^N is the domain of $\mathcal{A}^N$. Basic calculations will give the following lemma whose proof is omitted.

LEMMA 2.2. (i) *$\mathcal{A}^N$ with domain H_{bc}^N is self-adjoint and can be represented as*

$$\begin{aligned}\mathcal{A}^N f(x) &:= D\Big[\frac{1}{\rho}\widetilde{\nabla}_{Nx_1}(\rho\widetilde{\nabla}_{Nx_1})+\Delta_{Nx_2}\Big]f(x)\\ &= -D\bigg\{\frac{1}{2\rho(x)}\Big[\nabla_{Nx_1}^-\Big(\rho\Big(\cdot+\frac{e_1}{2N}\Big)\nabla_{Nx_1}^+ f\Big)(x)\\ &\qquad + \nabla_{Nx_1}^+\Big(\rho\Big(\cdot-\frac{e_1}{2N}\Big)\nabla_{Nx_1}^- f\Big)(x)\Big]\\ &\qquad +\frac{1}{2}[\nabla_{Nx_2}^-(\nabla_{Nx_2}^+ f)(x)+\nabla_{Nx_2}^+(\nabla_{Nx_2}^- f)(x)]\bigg\}.\end{aligned}\tag{2.1}$$

(ii) *The eigenvalues and eigenfuntions $\{\lambda_p^N,\phi_p^N\}_{p=(p_1,p_2)=(0,0)}^{(L_1N-1,L_2N-1)}$ for $\mathcal{A}^N$ are given by*

$$\lambda_p^N=\lambda_{p_1}^{1,N}+\lambda_{p_2}^{2,N},\qquad \phi_p^N(x)=\phi_{p_1}^{1,N}(x_1)\phi_{p_2}^{2,N}(x_2),$$

$$\lambda_0^{1,N}=0,\qquad \lambda_{p_1}^{1,N}=2DN^2\cos\frac{p_1\pi}{L_1N}-DN^2(e^{c/N}+e^{-c/N})\quad (p_1\neq 0),$$

$$\lambda_0^{2,N}=0,\qquad \lambda_{p_2}^{2,N}=2DN^2\Big(\cos\frac{p_2\pi}{L_2N}-1\Big)\quad (p_2\neq 0),$$

$$\phi_0^{1,N}(x_1)=\sqrt{\frac{2c}{1-e^{-2cL_1}}},\qquad \phi_0^{2,N}(x_2)=\sqrt{\frac{1}{L_2}},$$

$$\phi_{p_1}^{1,N}(x_1)=\sum_{k_1=0}^{L_1N-1}\sqrt{\frac{4c}{(1-e^{-2c/N})L_1N}}\sin\Big(\frac{p_1\pi k_1}{L_1N}+\alpha_{p_1}^N\Big)e^{ck_1/N}1_{k_1}(x_1),$$

$$\begin{aligned}\phi_{p_2}^{2,N}(x_2)=\sum_{k_2=0}^{L_2N-1}\bigg(&-\sqrt{\frac{1-\cos(p_2\pi/(L_2N))}{L_2}}\sin\frac{p_2\pi k_2}{L_2N}\\ &+\frac{\sin(p_2\pi/(L_2N))}{\sqrt{L_2(1-\cos(p_2\pi/(L_2N)))}}\cos\frac{p_2\pi k_2}{L_2N}\bigg)1_{k_2}(x_2),\end{aligned}$$

where $c=\frac{V_1}{2D}$, $\alpha_{p_1}^N\in(-\frac{\pi}{2},0)$ is given by

$$\alpha_{p_1}^N=\tan^{-1}\Big(-\frac{e^{-c/N}\cos(p_1\pi/(L_1N))}{1-e^{-c/N}\cos(p_1\pi/(L_1N))}\tan\frac{p_1\pi}{L_1N}\Big),$$

and $1_{k_1}(x_1)$, $1_{k_2}(x_2)$ *are the indicator functions on* $[\frac{k_1}{N}, \frac{k_1+1}{N})$, $[\frac{k_2}{N}, \frac{k_2+1}{N})$, *respectively.*

REMARK 2.3. Substituting $\cos(x) \approx 1 - \frac{x^2}{2}$ for small $|x|$ and $e^{c/N} + e^{-c/N} - 2 \approx \frac{c^2}{N^2}$ for large N into the formula for λ_p^N, we find that $\lambda_p^N \approx \lambda_p$ for large N and $\frac{p_1}{N}, \frac{p_2}{N}$ small. Applications of Taylor's theorem yield $\frac{11}{12}|\lambda_p| \le |\lambda_p^N| \le |\lambda_p|$ for $N > \pi$, which will be used in proving Lemma 3.6 and Theorem 3.1. Moreover, one finds that $\lim_{N\to\infty} \lambda_p^N = \lambda_p$.

Let $T^N(t) = \exp(\mathcal{A}^N t)$. Then, ϕ_p^N are eigenfunctions of $T^N(t)$ with eigenvalues $\exp\{\lambda_p^N t\}$. Now we describe the stochastic particle systems. Let $l = l(N)$ be a function such that $l(N) \to \infty$ as $N \to \infty$. l^{-1} can loosely be thought of as the "mass" or the "amount of concentration" of one particle. We let $n_k(t)$ denote the number of particles in cell k at time t for $k = (k_1, k_2) \in \{1, \dots, L_1N\} \times \{1, \dots, L_2N\}$ and also, to account for our Neumann boundary conditions, we set

$$n_{0,k_2}(t) = n_{1,k_2}(t), \quad n_{L_1N+1,k_2}(t) = n_{L_1N,k_2}(t), \qquad k_2 = 1, \dots, L_2N,$$
$$n_{k_1,0}(t) = n_{k_1,1}(t), \quad n_{k_1,L_2N+1}(t) = n_{k_1,L_2N}(t), \qquad k_1 = 1, \dots, L_1N.$$

Then $\{n_k(t)\}$ is modeled as a Markov chain with transition rates defined below. First,

$$n_k \to n_k \pm 1 \text{ at rate } lR^{\pm}(n_k l^{-1}) \qquad \text{for } k \in \{1, \dots, L_1N\} \times \{1, \dots, L_2N\},$$

where $n_k \to n_k + 1$ if $R(n_k l^{-1}) > 0$ and $n_k \to n_k - 1$ if $R(n_k l^{-1}) < 0$, $R^+ = R \vee 0$ and $R^- = -(R \wedge 0)$. Next, we recall $c = \frac{V_1}{2D}$ and define the following drift–diffusion mechanism:

$$(n_k, n_{k+e_1}) \to (n_k - 1, N_{k+e_1} + 1) \text{ at rate } (DN^2 e^{-c/N} n_{k+e_1} - DN^2 e^{c/N} n_k)^-,$$
$$(n_k, n_{k+e_1}) \to (n_k + 1, n_{k+e_1} - 1) \text{ at rate } (DN^2 e^{-c/N} n_{k+e_1} - DN^2 ec/N n_k)^+$$

for all $k = (k_1, k_2)$ with $k_1 \in \{0, 1, \dots, L_1N\}$, $k_2 \in \{0, 1, \dots, L_2N + 1\}$,

$$(n_k, n_{k+e_2}) \to (n_k - 1, n_{k+e_2} + 1) \text{ at rate } (DN^2 n_{k+e_2} - DN^2 n_k)^-,$$
$$(n_k, n_{k+e_2}) \to (n_k + 1, n_{k+e_2} - 1) \text{ at rate } (DN^2 n_{k+e_2} - DN^2 n_k)^+$$

for all $k = (k_1, k_2)$ with $k_1 \in \{0, 1, \dots, L_1N + 1\}$, $k_2 \in \{0, 1, \dots, L_2N\}$.

We shall write $\delta_{1,N}(n_k) = DN^2 e^{-c/N} n_{k+e_1} - DN^2 e^{c/N} n_k$ and $\delta_{2,N}(n_k) = DN^2 n_{k+e_2} - DN^2 n_k$.

REMARK 2.4. Suppose $R(x) = b(x) - d(x) = \sum_{i=0}^m c_i x^i$ be a polynomial for $x \in \mathbb{R}$, with $c_m < 0$ and $b(x), d(x)$ being polynomials of degree less than or equal to m with nonnegative coefficients satisfying $d(0) = 0$. Then, the previous works apply to the case $V \equiv 0$, $r = 0$ and the usual diffusion mechanism as used in Arnold and Theodosopulu (1980), Kotelenez (1986, 1988) and Blount (1991, 1994, 1996) would be

$$\begin{aligned} n_k &\to n_k + 1 \text{ at rate } lb(n_k l^{-1}), \\ n_k &\to n_k - 1 \text{ at rate } ld(n_k l^{-1}), \\ (n_k, n_{k\pm e_i}) &\to (n_k - 1, n_{k\pm e_i} + 1) \text{ at rate } DN^2 n_k, \qquad i = 1, 2 \end{aligned}$$

for all k in the ranges indicated above. In our new scheme we slow these rates down significantly by comparing the number of particles in adjacent cells and birth to death rates. This makes computation far more efficient and simplifies implementation.

Finally, we incorporate the Poisson sources into the approximations. Let

$$K_i^N \doteq \left\{k : \left[\frac{k_1 - 1}{N}, \frac{k_1}{N}\right) \times \left[\frac{k_2 - 1}{N}, \frac{k_2}{N}\right) \subset B(\kappa_i, \varepsilon)\right\}, \qquad i = 1, 2, \ldots, r.$$

Then, we add source contamination according to

$$\{n_k\}_{k \in K_i^N} \to \{n_k + \lfloor l\theta_i(k) A_i^j(\omega) + 0.5 \rfloor\}_{k \in K_i^N}$$
$$\text{at time } \tau_i^j,\ i = 1, 2, \ldots, r,\ j \in \mathbb{Z}_+.$$

Now we use the aforementioned transition rates to construct our model in the probabilistic setting. However, rather than immersing ourselves immediately in the mathematics of model building we note that the same random numbers would be supplied by the computer for the Markov chain approximation regardless of the values of l and N. Naturally, more numbers would be utilized for large l, N, but the most salient point is that any realistic modelling scheme should exhibit a dependence between models with different values of l, N. We provide one such scheme and note that different schemes will yield different implementation algorithms and different precise rate of convergence results such as central limit theorems and laws of the iterated logarithm. We let $\{N_k\}_{k=0}^\infty$ be an increasing sequence in $\mathbb{N}$ such that $N_k \to \infty$ as $k \to \infty$. For any $N \in \{N_k\}_{k=0}^\infty$, there exists a unique $n \in \mathbb{N}$ such that $2^{n-1} < N \le 2^n$. We recall that the A_i^j, τ_i^j are defined on $(\Omega, \mathcal{F}, \mathbb{P})$, note that the Poisson processes in our Markov chain mechanism should be independent of $\{A_i^j, \tau_i^j\}$ and let $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ be another probability space on which is defined independent standard Poisson processes $\{X_{+,N}^{k,R}, X_{-,N}^{k,R}, X_{+,N}^{k,1}, X_{-,N}^{k,1}, X_{+,N}^{k,2}, X_{-,N}^{k,2}\}_{k=(k_1,k_2)=(1,1)}^{(L_1 N, L_2 N)}$, $\{X_{+,N}^{k,1}, X_{-,N}^{k,1}, k = (0, k_2)\}_{k_2=1}^{L_2 N}$ and $\{X_{+,N}^{k,2}, X_{-,N}^{k,2}, k = (k_1, 0)\}_{k_1=1}^{L_1 N}$ (see the Appendix for a computer-workable construction). From the two probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, we define the product space

$$(\Omega_0, \mathcal{F}_0, \mathbb{P}_0) = (\Omega \times \overline{\Omega}, \mathcal{F} \otimes \overline{\mathcal{F}}, \mathbb{P} \times \overline{\mathbb{P}}).$$

In the sequel, $\lfloor r \rfloor$ denotes the greatest integer not more than a real number r. We let

$$n_k^N(0) = \left\lfloor l \left(\int_{I_k} \rho(x)\,dx \right)^{-1} \int_{I_k} u(0,x)\rho(x)\,dx \right\rfloor, \tag{2.2}$$
$$k \in \{(1,1), \ldots, (L_1 N, L_2 N)\}.$$

Then, following Ethier and Kurtz [(1986), pages 326–328], for $k \in \{(1,1), \ldots, (L_1 N, L_2 N)\}$, we let

$$\begin{aligned}
n_k^N(t) &= n_k^N(0) + X_{+,N}^{k,R}\left(l \int_0^t R^+(n_k^N(s) l^{-1})\,ds \right) \\
&\quad - X_{-,N}^{k,R}\left(l \int_0^t R^-(n_k^N(s) l^{-1})\,ds \right) \\
&\quad + \sum_{i=1}^{2} \left[X_{+,N}^{k,i}\left(\int_0^t \delta_{i,N}^+(n_k^N(s))\,ds \right) - X_{-,N}^{k,i}\left(\int_0^t \delta_{i,N}^-(n_k^N(s))\,ds \right) \right] \\
&\quad - \sum_{i=1}^{2} \left[X_{+,N}^{k-e_i,i}\left(\int_0^t \delta_{i,N}^+(n_{k-e_i}^N(s))\,ds \right) \right. \\
&\qquad \left. - X_{-,N}^{k-e_i,i}\left(\int_0^t \delta_{i,N}^-(n_{k-e_i}^N(s))\,ds \right) \right] \\
&\quad + \sum_{i=1}^{r} \sum_{j=1}^{\infty} \lfloor l\theta_i(k) A_i^j + 0.5 \rfloor 1_{t \ge \tau_i^j} 1_{k \in K_i^N}.
\end{aligned} \tag{2.3}$$

Equation (2.3) provides a very explicit and powerful construction of our Markov chain approximations to (1.3). Equation (2.3) can be implemented directly on a computer. However, to exploit the mathematical richness of our representation, we avail ourselves of the following lemma. In preparation for the statement of this lemma, we define $\widetilde{\Omega} = \prod_{m=0}^{\infty} \widetilde{\Omega}_m$, where $\widetilde{\Omega}_m = D_{\mathbb{R}^{L_1 N_m \times L_2 N_m} \cup \{\Delta\}}[0,\infty)$ and $\mathbb{R}^{L_1 N_m \times L_2 N_m} \cup \{\Delta\}$ is the one-point compactification of $\mathbb{R}^{L_1 N_m \times L_2 N_m}$ [see page 165 of Ethier and Kurtz (1986)]. Set $\widetilde{\mathcal{F}} = \bigotimes_{m=0}^{\infty} \mathcal{B}(\widetilde{\Omega}_m)$, which is the σ-algebra generated by open sets under Skorohod J_1 topology and countable products. For each $\omega \in \Omega$, we let $\{\mathcal{G}_t^{N,\omega}\}_{t \ge 0}$ be the smallest right continuous filtration such that

$$\begin{aligned}
&\Bigg\{ X_{\sigma,N}^{k,R}\left(l \int_0^t R^{\sigma}(n_k^N(s) l^{-1})\,ds \right), \\
&\quad X_{\sigma,N}^{k,i}\left(\int_0^t \delta_{i,N}^{\sigma}(n_k^N(s))\,ds \right),\ \sigma = +,-,\ i = 1,2 \Bigg\}_{k=(1,1)}^{(L_1 N, L_2 N)}, \\
&\Bigg\{ X_{\sigma,N}^{k,1}\left(\int_0^t \delta_{1,N}^{\sigma}(n_k^N(s))\,ds \right),\ \sigma = +,-,\ k = (0,k_2) \Bigg\}_{k_2=1}^{L_2 N}
\end{aligned}$$

and

$$\left\{X^{k,2}_{\sigma,N}\left(\int_0^t \delta^{\sigma}_{2,N}(n^N_k(s))\,ds\right),\ \sigma=+,-,\ k=(k_1,0)\right\}_{k_1=1}^{L_1N}$$

are adapted to $\{\mathcal{G}_t^{N,\omega}\}\subset\overline{\mathcal{F}}$.

LEMMA 2.5. (i) *$n^N(t)=\{n^N_k(t)\}_{k=(1,1)}^{(L_1N,L_2N)}$ is well defined up to (possible) explosion time $\tau_\infty=\inf\{t:n^N(t-)=\Delta\}$, and for each $\omega\in\Omega$ there exists a unique probability measure $\widetilde{\mathbb{P}}^\omega$ on $(\widetilde{\Omega},\widetilde{\mathcal{F}})$ such that*

$$\begin{aligned}\bar{\mathbb{P}}(\bar\omega\in\bar\Omega:n^{N_{m_1}}(\bar\omega,\omega)\in A_1,\dots,n^{N_{m_j}}(\bar\omega,\omega)\in A_j)\\=\widetilde{\mathbb{P}}^\omega(\widetilde\omega\in\widetilde\Omega:\widetilde\omega_{m_1}\in A_1,\dots,\widetilde\omega_{m_j}\in A_j)\end{aligned}\tag{2.4}$$

for all $A_i\in\mathcal{B}(D_{\mathbb{R}^{L_1N_{m_i}\times L_2N_{m_i}}\cup\{\Delta\}}[0,\infty))$, $i=1,\dots,j$; $j=1,2,\dots$. Moreover, we have that for each $B\in\widetilde{\mathcal{F}}$, $\omega\to\widetilde{\mathbb{P}}^\omega(B)$ is $(\Omega,\mathcal{F})$-measurable, and $\omega\to\int_{\widetilde\Omega}f(\omega,\widetilde\omega)\widetilde{\mathbb{P}}^\omega(\widetilde\omega)$ is $\mathcal{F}$-measurable for each bounded measurable function f.

(ii) *We have $\tau_\infty=\infty$ and for $t\geq 0$,*

$$\begin{aligned}n^N(t,x)&=n^N(0,x)+\int_0^t\mathcal{A}^Nn^N(s,x)\,ds+l\int_0^tR(n^N(s,x)l^{-1})\,ds\\&\quad+\sum_{k=(1,1)}^{(L_1N,L_2N)}\bar\Theta^N_k(t)1_k(x)+\sum_{k=(1,1)}^{(L_1N,L_2N)}(Z^N_{k,R,+}(t)+Z^N_{k,R,-}(t))1_k(x)\\&\quad+\sum_{k=(1,1)}^{(L_1N,L_2N)}\sum_{i=1}^{2}[Z^N_{k,i}(t)-Z^N_{k-e_i,i}(t)]1_k(x),\end{aligned}\tag{2.5}$$

where $n^N(t,x):=n^N_k(t)$, $\forall\,x\in I_k$, $1_k(\cdot)$ denotes the indicator function on I_k,

$$\bar\Theta^N_k(t)=\sum_{i=1}^r\sum_{j=1}^\infty\lfloor l\theta_i(k)A_i^j+0.5\rfloor 1_{t\geq\tau_i^j}1_{k\in K_i^N},$$

and

$$\begin{aligned}Z^N_{k,R,+}(t)&=X^{k,R}_{+,N}\left(l\int_0^tR^+(n^N_k(s)l^{-1})\,ds\right)-l\int_0^tR^+(n^N_k(s)l^{-1})\,ds,\\Z^N_{k,R,-}(t)&=-X^{k,R}_{-,N}\left(l\int_0^tR^-(n^N_k(s)l^{-1})\,ds\right)+l\int_0^tR^-(n^N_k(s)l^{-1})\,ds,\\Z^N_{k,i}(t)&=X^{k,i}_{+,N}\left(\int_0^t\delta^+_{i,N}(n^N_k(s))\,ds\right)-X^{k,i}_{-,N}\left(\int_0^t\delta^-_{i,N}(n^N_k(s))\,ds\right)\\&\quad-\int_0^t\delta_{i,N}(n^N_k(s))\,ds,\qquad i=1,2,\end{aligned}$$

are $\mathcal{L}^2$-martingales with respect to $\{\mathcal{G}_t^{N,\omega}\}$ under probability measure $\widetilde{\mathbb{P}}^\omega$.

REMARK 2.6. Since the proof of Lemma 2.5 is largely standard but technical, we just sketch the basic ideas here. For the proof of part (i) of Lemma 2.5, we can use Fubini's theorem, Theorem D of Halmos (1950) and monotone convergence theorem to show that $\widetilde{\mathbb{P}}^{\omega}$ is σ-additive and $\omega \to \widetilde{\mathbb{P}}^{\omega}(B)$ is measurable for each $B \in \widetilde{\mathcal{F}}$. Then, the monotone class theorem gives us the final claim. For the proof of part (ii) of Lemma 2.5, we can employ a stopping argument, Hölder's inequality, the linear growth of R^+, Burkholder's inequality and Gronwall's inequality to show that for $p \geq 1$,

$$\widetilde{\mathbb{E}}^{\omega}\Big[\sup_{t\leq T}\langle n^N(t), 1\rangle^p\Big] \leq C(N, T, l, \omega).$$

This is enough to justify all the statements.

Note that $\widetilde{\mathbb{P}}^{\omega}$ is the probability measure for the quenched results. However, to use the quenched results within the annealed ones we need to know that $\omega \to \widetilde{\mathbb{P}}^{\omega}(B)$ is measurable for each $B \in \widetilde{\mathcal{F}}$. We can write

$$\mathbb{P}_0(d\omega_0) = \widetilde{\mathbb{P}}^{\omega}(d\widetilde{\omega})\mathbb{P}(d\omega), \qquad \omega_0 = (\omega, \widetilde{\omega}).$$

To get the density in each cell, we divide $n_k^N(t)$ by l and consequently the description of the stochastic particle model can be given by

$$u^{l,N}(t,x) = \sum_{k_1=1}^{L_1N}\sum_{k_2=1}^{L_2N} \frac{n_k^N(t)}{l} 1_k(x). \tag{2.6}$$

Now we set

$$Z_{R^+}^{l,N}(t) \doteq \sum_{k=(1,1)}^{(L_1N,L_2N)} l^{-1} Z_{k,R,+}^N(t) 1_k, \qquad Z_{R^-}^{l,N}(t) \doteq \sum_{k=(1,1)}^{(L_1N,L_2N)} l^{-1} Z_{k,R,-}^N(t) 1_k,$$

$$Z_R^{l,N}(t) = Z_{R^+}^{l,N}(t) + Z_{R^-}^{l,N}(t),$$

$$Z_D^{l,N}(t) \doteq \sum_{k=(1,1)}^{(L_1N,L_2N)} \sum_{i=1}^{2} l^{-1}(Z_{k,i}^N(t) - Z_{k-e_i,i}^N(t)) 1_k$$

and

$$\Theta^{l,N}(t,\cdot) = \sum_{i=1}^{r}\sum_{j=1}^{N_i(t)}\sum_{k\in K_i^N} l^{-1}\lfloor l\theta_i(k)A_i^j(\omega) + 0.5\rfloor 1_k(\cdot).$$

Then from (2.5), it follows that

$$\begin{aligned} u^{l,N}(t) = u^{l,N}(0) &+ \int_0^t \mathcal{A}^N u^{l,N}(s)\,ds + \int_0^t R(u^{l,N}(s))\,ds \\ &+ Z_{R^+}^{l,N}(t) + Z_{R^-}^{l,N}(t) + Z_D^{l,N}(t) + \Theta^{l,N}(t). \end{aligned} \tag{2.7}$$

By variation of constants and (2.7), it follows that $u^{l,N}(t) = u^{l,N}(t, \omega_0)$ satisfies

$$\begin{aligned} u^{l,N}(t) &= T^N(t)u^{l,N}(0) + \int_0^t T^N(t-s)R(u^{l,N}(s))\,ds \\ &\quad + \int_0^t T^N(t-s)\,dZ^{l,N}_{R^+}(s) + \int_0^t T^N(t-s)\,dZ^{l,N}_{R^-}(s) \\ &\quad + \int_0^t T^N(t-s)\,dZ^{l,N}_D(s) + \int_0^t T^N(t-s)\,d\Theta^{l,N}(s). \end{aligned} \tag{2.8}$$

In this section, we have constructed the Markov chain via stochastic particle model. In the next section, we shall prove the laws of large numbers for $u^{l,N}$.

3. Laws of large numbers. For $f : E \to \mathbb{R}$, let $\|f\|_\infty = \sup_{x\in E}|f(x)|$. We need the following.

HYPOTHESES. *For each fixed $\omega \in \Omega$ and q as defined in* (1.2), *we suppose that*:

(i) $\|\widetilde{\mathbb{E}}^\omega(u^{l,N}(0))^{2q}\|_\infty \le C(\omega) < \infty$.
(ii) $(N, l(N))$ *is any sequence satisfying* $l(N) \to \infty$ *as* $N \to \infty$.
(iii) $\|u^{l,N}(0) - u_0\| \to 0$ *in probability* $\widetilde{\mathbb{P}}^\omega$.
(iv) $\|u^{l,N}(0)\|_\infty \le C(N, l, \omega) < \infty$.
(v) $\|u_0\|_\infty \le c_0 < \infty$.

We note that $u^{l,N}(0)$ defined by (2.2) and (2.6) satisfies (i), (iii) and (iv) in the Hypotheses. However, we do not necessarily assume that $u^{l,N}(0)$ is given in this way and any $u^{l,N}(0)$ satisfying the Hypotheses will be fine. Through Hypothesis(ii) our dependence on (l, N) is reduced to dependence only on N and we will write u^N for $u^{l(N),N}$. Now we have the following *quenched* law of large numbers.

THEOREM 3.1. *Under the Hypotheses, there exists a pathwise unique solution u to* (1.3) *and*

$$\sup_{t\le T}\|u^N(t, \omega, \cdot) - u(t, \omega)\| \to 0 \qquad \textit{in probability } \widetilde{\mathbb{P}}^\omega \textit{ as } N \to \infty. \tag{3.1}$$

When $N_i(t)$ and A_i^j are considered to be random variable (i.e., ω is no longer fixed), the Markov chain $u^{l,N}$ evolves in this random medium. We can show that there exists a unique $D_H[0, T]$-valued mild solution to (1.3) by reducing our local Lipschitz condition to a global one (through temporary modification of R), using Picard's successive approximation, and stopping. Consequently, $(\bar{\omega}, \omega) \to \sup_{t\le T}\|u^N(t, \bar{\omega}, \omega) - u(t, \omega)\|$ is jointly measurable. As a corollary of Theorem 3.1, we have the following *annealed* law of large numbers.

COROLLARY 3.2. *Under the Hypotheses, there exist a unique mild solution* u *to* (1.3) *and*

$$\sup_{t\leq T}\|u^N(t)-u(t)\|\to 0$$

in probability $\mathbb{P}_0$ *as* $N\to\infty$.

PROOF. Applying the quenched result in Theorem 3.1 we have

$$\widetilde{\mathbb{E}}^\omega f\Big(\sup_{t\leq T}\|u^N(t,\omega)-u(t,\omega)\|\Big)\to f(0),$$

for any bounded, continuous function f. Now, by dominated convergence theorem, we obtain

$$\mathbb{E}_0 f\Big(\sup_{t\leq T}\|u^N(t)-u(t)\|\Big)\to f(0).$$

This implies that $\sup_{t\leq T}\|u^N(t)-u(t)\|\to 0$ in distribution or equivalently in probability $\mathbb{P}_0$. $\square$

Before proving Theorem 3.1, we prepare some preliminary lemmas. For convenience, we introduce the projective mapping $P^N: H\to H^N$,

$$\widetilde{f_N}=P^N f=\sum_k\Big(\int_{I_k}\rho(x)\,dx\Big)^{-1}\int_{I_k}f(x)\rho(x)\,dx\cdot 1_k(\cdot) \tag{3.2}$$

and set $\rho_+^N(\cdot)=e^{-c/N}\rho_N(\cdot)$, $\rho_-^N(\cdot)=e^{c/N}\rho_N(\cdot)$, where $\rho_N(\cdot)=\sum_k N^2\int_{I_k}\rho(x)\times dx\cdot 1_k(\cdot)$. The following lemma is used in Lemma 3.4 and Lemma 3.5.

LEMMA 3.3. *Suppose* $\|u^N(0)\|_\infty\leq C(N,l,\omega)<\infty$ *and* $f\in H$, *then*

$$\widetilde{\mathbb{E}}^\omega[\langle Z_{R^+}^N(t),f\rangle^2]=\frac{1}{N^2 l}\widetilde{\mathbb{E}}^\omega\int_0^t\langle R^+(u^N(s)),\widetilde{f_N}^2\cdot\rho_N\rangle\,ds,$$

$$\widetilde{\mathbb{E}}^\omega[\langle Z_R^N(t),f\rangle^2]=\frac{1}{N^2 l}\widetilde{\mathbb{E}}^\omega\int_0^t\langle |R|(u^N(s)),\widetilde{f_N}^2\cdot\rho_N\rangle\,ds$$

and

$$\widetilde{\mathbb{E}}^\omega[\langle Z_D^N(t),f\rangle^2]\leq\frac{1}{N^2 l}\widetilde{\mathbb{E}}^\omega\int_0^t\sum_{i=1}^4\alpha_i(f,u^N(s))\,ds,$$

where for $f\in H$,

$$\begin{aligned}\alpha_1(f,u^N(s))&=\langle N^2(e^{-2c/N}-1)^2e^{2c/N}\widetilde{f_N}^2\rho_+^N,Du^N(s)\rangle\\&\quad+2\langle N(e^{-2c/N}-1)(\nabla_{Nx_1}^+\widetilde{f_N})\widetilde{f_N}\rho_+^N,Du^N(s)\rangle\end{aligned}$$

$$
\begin{aligned}
&+\langle e^{-2c/N}(\nabla^+_{Nx_1}\widetilde{f}_N)^2\cdot\rho^N_+(\cdot), Du^N(s)\rangle,\\
\alpha_2(f,u^N(s)) &= \langle N^2(e^{2c/N}-1)^2e^{-2c/N}\widetilde{f}_N^{\,2}\rho^N_-, Du^N(s)\rangle\\
&\quad+2\langle N(e^{2c/N}-1)(\nabla^-_{Nx_1}\widetilde{f}_N)\widetilde{f}_N\rho^N_-, Du^N(s)\rangle\\
&\quad+\langle e^{2c/N}(\nabla^-_{Nx_1}\widetilde{f}_N)^2\cdot\rho^N_-(\cdot), Du^N(s)\rangle,\\
\alpha_3(f,u^N(s)) &= \langle(\nabla^+_{Nx_2}\widetilde{f}_N)^2\rho_N(\cdot), Du^N(s)\rangle
\end{aligned}
$$

and

$$
\alpha_4(f,u^N(s)) = \langle(\nabla^-_{Nx_2}\widetilde{f}_N)^2\rho_N(\cdot), Du^N(s)\rangle.
$$

PROOF. In as much as the proofs of the three parts follows the same steps, we just show the first part. Now by the independence we have that the quadratic covariation $[X^{k^1,R}_{+,N}, X^{k^2,R}_{+,N}]=0$ for $k^1\neq k^2\doteq(k^2_1,k^2_2)$. Moreover, $s\to n^N_{ki}(s)$ is cadlag and hence [cf. Billingsley (1968), page 110] almost surely bounded on $[0,T]$, so $\int_0^T R^+(n^N_{ki}(s)l^{-1})\,ds<\infty$ almost surely. Therefore, by two applications of Theorem II.22 in Protter (1990), we find that

$$
\begin{aligned}
&\left[X^{k^1,R}_{+,N}\left(l\int_0^t R^+(n^N_{k1}(s)l^{-1})\,ds\wedge\cdot\right), X^{k^2,R}_{+,N}\left(l\int_0^t R^+(n^N_{k2}(s)l^{-1})\,ds\wedge\cdot\right)\right]_v\\
&\quad=[X^{k^1,R}_{+,N}, X^{k^2,R}_{+,N}]_{(l\int_0^t R^+(n^N_{k1}(s)l^{-1})\,ds)\wedge(l\int_0^t R^+(n^N_{k2}(s)l^{-1})\,ds)\wedge v}=0
\end{aligned}
\tag{3.3}
$$

and by the Kunita–Watanabe inequality,

$$
\begin{aligned}
&\left|\left[X^{k^1,R}_{+,N}\left(l\int_0^t R^+(n^N_{k1}(s)l^{-1})\,ds\wedge\cdot\right), X^{k^2,R}_{+,N}\left(l\int_0^t R^+(n^N_{k2}(s)l^{-1})\,ds\wedge\cdot\right)\right]_v\right|\\
&\quad\le\left([X^{k^1,R}_{+,N}]_{l\int_0^t R^+(n^N_{k1}(s)l^{-1})\,ds}\right)^{1/2}\cdot\left([X^{k^2,R}_{+,N}]_{l\int_0^t R^+(n^N_{k2}(s)l^{-1})\,ds}\right)^{1/2},
\end{aligned}
\tag{3.4}
$$

which is $\widetilde{\mathbb{P}}^\omega$-integrable by Cauchy–Schwarz inequality and Lemma 2.5. Hence, letting $v\to\infty$, and using (3.3), (3.4) and dominated convergence, we have that

$$
\widetilde{\mathbb{E}}^\omega([Z^N_{k^1,R,+}, Z^N_{k^2,R,+}]_t)=0\qquad\forall\,k^1\neq k^2,\ t\ge 0.
$$

Therefore, by the bilinear property of quadratic variation and the fact that $\langle Z^N_{R^+}(t), f\rangle$ is a $\mathcal{L}^2$-martingale, one has that

$$
\begin{aligned}
\widetilde{\mathbb{E}}^\omega(\langle Z^N_{R^+}(t), f\rangle^2) &= \widetilde{\mathbb{E}}^\omega\left\{\left[\sum_k l^{-1}Z^N_{k,R,+}\langle 1_k, f\rangle\right]_t\right\}\\
&=\sum_k l^{-2}\langle 1_k, f\rangle^2\widetilde{\mathbb{E}}^\omega[Z^N_{k,R,+}]_t.
\end{aligned}
\tag{3.5}
$$

We let $\tau_k(t)=l\int_0^t R^+(n^N_k(s)l^{-1})\,ds$. By Lemma 2.5, we know that $\tau_k(t)$ is nondecreasing in t and $\{X^{k,R}_{+,N}(\tau_k(t))\}$ is a pure-jump $\{\mathcal{G}^{N,\omega}_t\}$-semimartingale with jump size 1. It follows that

$$\widetilde{\mathbb{E}}^{\omega}[Z^N_{k,R,+}]_t = \widetilde{\mathbb{E}}^{\omega}[X^{k,R}_{+,N}(\tau_k(\cdot))]_t = \widetilde{\mathbb{E}}^{\omega}\{X^{k,R}_{+,N}(\tau_k(t))\}$$
$$(3.6)\qquad = \widetilde{\mathbb{E}}^{\omega}\left\{l \int_0^t R^+(n_k^N(s)l^{-1})\,ds\right\}.$$

Now, by (3.5) and (3.6), we have

$$\widetilde{\mathbb{E}}^{\omega}[\langle Z^N_{R^+}(t), f\rangle^2] = \sum_k l^{-2}\langle 1_k, f\rangle^2 \widetilde{\mathbb{E}}^{\omega}\left\{l \int_0^t R^+(n_k^N(s)l^{-1})\,ds\right\}$$
$$= \frac{1}{N^2 l}\widetilde{\mathbb{E}}^{\omega}\int_0^t \langle R^+(u^N(s)), \widetilde{f_N}^2 \cdot \rho_N\rangle\,ds. \qquad \square$$

For convenience, we put

$$(3.7)\qquad Y_{R^+}(t) = \int_0^t T^N(t-s)\,dZ^N_{R^+}(s), \qquad Y_R(t) = \int_0^t T^N(t-s)\,dZ^N_R(s)$$

and

$$(3.8)\qquad Y_D(t) = \int_0^t T^N(t-s)\,dZ^N_D(s), \qquad Y(t) = Y^N(t) = Y_R(t) + Y_D(t).$$

If $J \in \{D, R\}$, then by variation of constants we have

$$Y_J(t) = \int_0^t \mathcal{A}^N Y_J(s)\,ds + Z^N_J(t).$$

We let $Y_{J,p}, Z_{J,p}$ denote $\langle Y_J, \phi_p^N\rangle$, $\langle Z_J, \phi_p^N\rangle$ and use (3.7)–(3.8) to conclude that $\mathcal{A}^N Y_J(s), \phi_p^N \in H^N$, so it follows trivially that

$$\left\langle \int_0^t \mathcal{A}^N Y_J(s)\,ds, \phi_p^N\right\rangle = \int_0^t \langle \mathcal{A}^N Y_J(s), \phi_p^N\rangle\,ds.$$

Indeed, we have by Lemma 2.2, the previous equation and Itô's formula, respectively,

$$(3.9)\qquad Y_{J,p}(t) = \int_0^t \lambda_p^N Y_{J,p}(s)\,ds + Z_{J,p}(t),$$

$$(3.10)\qquad Y_{J,p}^2(t) = 2\lambda_p^N \int_0^t Y_{J,p}^2(s)\,ds + 2\int_0^t Y_{J,p}(s-)\,dZ_{J,p}(s) + \sum_{s\le t}(\delta Z_{J,p}(s))^2.$$

Using (3.9), (3.10) and Lemma 3.3 with $f = \phi_p^N$, stopping (3.10) to reduce the local martingale and utilizing monotone convergence, Fatou's lemma and Gronwall's inequality with an interchange of integration, one gets the following lemma.

LEMMA 3.4. *Assume that* $\|u^N(0)\|_\infty \le C(N,l,\omega) < \infty$. *Then*:

(a) $\widetilde{\mathbb{E}}^\omega \langle Y_D(t), \phi_p^N\rangle^2 \le (N^2 l)^{-1}\widetilde{\mathbb{E}}^\omega \int_0^t \sum_{i=1}^4 \alpha_i(\phi_p^N, u^N(s)) \exp\{2\lambda_p^N(t-s)\}\,ds$.

(b) $\widetilde{\mathbb{E}}^\omega \langle Y_R(t), \phi_p^N\rangle^2 = (N^2 l)^{-1}\widetilde{\mathbb{E}}^\omega \int_0^t \langle |R|(u^N(s)), (\phi_p^N)^2\rho_N\rangle \exp\{2\lambda_p^N(t-s)\}\,ds$.

(c) $\langle Y_D(t), \phi_p^N\rangle^2 \le A(\phi_p^N)(t)$, *where* $A(\phi_p^N)(t) \doteq 2\int_0^t Y_{D,p}(s-)\,dZ_{D,p}(s) + \sum_{s\le t}(\delta Z_{D,p}(s))^2$ *is a submartingale satisfying*

$$\widetilde{\mathbb{E}}^\omega A(\phi_p^N)(t) \le (N^2 l)^{-1}\widetilde{\mathbb{E}}^\omega \int_0^t \sum_{i=1}^4 \alpha_i(\phi_p^N, u^N(s))\,ds.$$

(d) $\langle Y_R(t), \phi_p^N\rangle^2 \le B(\phi_p^N)(t)$, *where* $B(\phi_p^N)(t)$ *is a submartingale satisfying*

$$\widetilde{\mathbb{E}}^\omega B(\phi_p^N)(t) = (N^2 l)^{-1}\widetilde{\mathbb{E}}^\omega \int_0^t \langle |R|(u^N(s)), (\phi_p^N)^2\rho_N\rangle\,ds.$$

Next, we need to estimate the moments of $u^N(t)$. Motivated by Lemma 3.2 of Kotelenez (1988), we have the following lemma.

LEMMA 3.5. *For each fixed* $\omega \in \Omega$ *and* $2\beta \ge 1$,

$$\sup_{s\le t}\|\widetilde{\mathbb{E}}^\omega (u^N(s))^{2\beta}\|_\infty \le C(t, l, \|\widetilde{\mathbb{E}}^\omega (u^N(0))^{2\beta}\|_\infty, \omega) < \infty,$$

where C *is decreasing in* l.

PROOF. Setting $\xi_k = (\sqrt{\sigma_N(k)})^{-1} 1_k(\cdot)$ with $\sigma_N(k) = \int_{I_k}\rho(x)\,dx$, from (2.8) and the fact that $\int_0^t T^N(t-s)\,dZ_{R^-}^N(s) - \int_0^t T^N(t-s) R^-(u^N(s))\,ds \le 0$, we obtain that

$$\begin{aligned} u^N(t,x) &\le \langle T^N(t)u^N(0), \xi_k\rangle \frac{1}{\sqrt{\sigma_N(k)}} \\ &\quad + \left\langle \int_0^t T^N(t-s)R^+(u^N(s))\,ds, \xi_k\right\rangle \frac{1}{\sqrt{\sigma_N(k)}} \\ &\quad + \left\langle \int_0^t T^N(t-s)\,dZ_{R^+}^N(s), \xi_k\right\rangle \frac{1}{\sqrt{\sigma_N(k)}} \\ &\quad + \left\langle \int_0^t T^N(t-s)\,dZ_D^N(s), \xi_k\right\rangle \frac{1}{\sqrt{\sigma_N(k)}} \\ &\quad + \left\langle \int_0^t T^N(t-s)\,d\Theta^N(s,\omega), (\sigma_N(k))^{-1}1_k\right\rangle \end{aligned} \tag{3.11}$$

for $x \in I_k$. Therefore, for $2\beta \geq 1$ and $x \in I_k$, one has that

$$
\begin{aligned}
(u^N(t,x))^{2\beta} \leq 5^{2\beta-1}\Bigg\{ & |\langle T^N(t)u^N(0), \xi_k\rangle (\sigma_N(k))^{-1/2}|^{2\beta} \\
& + \left|\left\langle \int_0^t T^N(t-s)R^+(u^N(s))\,ds, \xi_k\right\rangle \frac{1}{\sqrt{\sigma_N(k)}}\right|^{2\beta} \\
& + |\langle Y_{R^+}(t), \xi_k\rangle|^{2\beta}(\sigma_N(k))^{-\beta} + |\langle Y_D(t), \xi_k\rangle|^{2\beta}(\sigma_N(k))^{-\beta} \\
& + \left|\left\langle \int_0^t T^N(t-s)\,d\Theta^N(s,\omega), (\sigma_N(k))^{-1}1_k\right\rangle\right|^{2\beta}\Bigg\}.
\end{aligned} \tag{3.12}
$$

Using Tonelli's theorem, Hölder's inequality, the linear growth of $R^+(\cdot)$ and Minkowski's integral inequality, we find that

$$
\begin{aligned}
&\widetilde{\mathbb{E}}^\omega \left|\left\langle \int_0^t T^N(t-s)R^+(u^N(s))\,ds, \xi_k\right\rangle \frac{1}{\sqrt{\sigma_N(k)}}\right|^{2\beta} \\
&\quad \leq t^{2\beta-1}\int_0^t \widetilde{\mathbb{E}}^\omega |\langle R^+(u^N(s)), T^N(t-s)\xi_k\rangle (\sigma_N(k))^{-1/2}|^{2\beta}\,ds \\
&\quad \leq t^{2\beta-1}\int_0^t \bigg\{\int_E \big(\widetilde{\mathbb{E}}^\omega |R^+(u^N(s,x))\cdot T^N(t-s)\xi_k(x) \\
&\qquad\qquad \times (\sigma_N(k))^{-1/2}|^{2\beta}\big)^{1/2\beta}\rho(x)\,dx\bigg\}^{2\beta}\,ds \\
&\quad \leq Ct^{2\beta-1}\int_0^t \bigg\{\int_E \big(\widetilde{\mathbb{E}}^\omega |1+u^N(s,x)|^{2\beta}\big)^{1/2\beta} \\
&\qquad\qquad \times \big(T^N(t-s)\xi_k(x)(\sigma_N(k))^{-1/2}\big)\rho(x)\,dx\bigg\}^{2\beta}\,ds \qquad (3.13) \\
&\quad \leq Ct^{2\beta-1}\int_0^t \Big(1+\sup_{v\leq s}\|\widetilde{\mathbb{E}}^\omega(u^N(v))^{2\beta}\|_\infty\Big) \\
&\qquad\qquad \times \big(\langle T^N(t-s)1, \xi_k\rangle(\sigma_N(k))^{-1/2}\big)^{2\beta}\,ds \\
&\quad \leq Ct^{2\beta} + Ct^{2\beta-1}\int_0^t \sup_{v\leq s}\|\widetilde{\mathbb{E}}^\omega(u^N(v))^{2\beta}\|_\infty\,ds.
\end{aligned}
$$

Similarly, we can show that

$$
\widetilde{\mathbb{E}}^\omega|\langle T^N(t)u^N(0), \xi_k\rangle(\sigma_N(k))^{-1/2}|^{2\beta} \leq \|\widetilde{\mathbb{E}}^\omega(u^N(0))^{2\beta}\|_\infty. \tag{3.14}
$$

Now, following the arguments in the proof of Lemma 3.2 in Kotelenez (1988), for fixed $t > 0$ and $J \in \{D, R^+\}$, we define $\mathcal{L}^2$-martingales by

$$L_J(s,k) = \begin{cases} \left\langle \int_0^s T^N(t-v)\, dZ_J^N(v), \xi_k \right\rangle (\sigma_N(k))^{-1/2}, & s \le t, \\ L_J(t,k), & s > t. \end{cases}$$

Then, by Lemma 3.3, the predictable quadratic variations of $L_{R^+}(s,k)$ and $L_D(s,k)$ are given by

$$\langle\langle L_{R^+}(\cdot,k)\rangle\rangle_s = \frac{1}{lN^2\sigma_N(k)} \int_0^s \langle R^+(u^N(v)), (T^N(t-v)\xi_k)^2 \rho_N \rangle\, dv \tag{3.15}$$

and

$$\langle\langle L_D(\cdot,k)\rangle\rangle_s \le \frac{1}{lN^2\sigma_N(k)} \int_0^s \sum_{i=1}^4 \alpha_i(T^N(t-v)\xi_k, u^N(v))\, dv. \tag{3.16}$$

Note that by (2.3), the maximal jump size of $L_J(s,k)$ is $\frac{1}{l}$. Then, by Burkholder's inequality, we have

$$\begin{aligned} \widetilde{\mathbb{E}}^\omega |L_J(t,k)|^{2\beta} &\le C\widetilde{\mathbb{E}}^\omega [L_J(\cdot,k)]_t^\beta \\ &\le C\widetilde{\mathbb{E}}^\omega [\langle\langle L_J(\cdot,k)\rangle\rangle_t + l^{-2}]^\beta \\ &\le C\widetilde{\mathbb{E}}^\omega [\langle\langle L_J(\cdot,k)\rangle\rangle_t^\beta] + Cl^{-2\beta}. \end{aligned} \tag{3.17}$$

By (3.15) and (3.13), we find that

$$\begin{aligned} &\widetilde{\mathbb{E}}^\omega [\langle\langle L_{R^+}(\cdot,k)\rangle\rangle_t^\beta] \\ &\quad \le Cl^{-\beta}\widetilde{\mathbb{E}}^\omega \left| \int_0^t \langle T^N(t-s)R^+(u^N(s)), \xi_k\rangle (\sigma_N(k))^{-1/2}\, ds \right|^\beta \\ &\quad \le Cl^{-\beta}\left(1 + t^{2\beta} + t^{2\beta-1}\int_0^t \sup_{v\le s} \|\widetilde{\mathbb{E}}^\omega (u^N(v))^{2\beta}\|_\infty\, ds\right). \end{aligned} \tag{3.18}$$

Setting $\Gamma_N(f) = D[e^{-c/N}(\nabla_{Nx_1}^+ f)^2 + e^{c/N}(\nabla_{Nx_1}^- f)^2 + (\nabla_{Nx_2}^+ f)^2 + (\nabla_{Nx_2}^- f)^2]$ for $f \in H_{bc}^N$, one finds that

$$\sum_{i=1}^4 \alpha_i(f, u^N(s)) \le C\langle u^N(s), f^2\rangle + C\langle u^N(s), \Gamma_N(f)\rangle.$$

Therefore, by (3.16), it follows that

$$\begin{aligned} &\widetilde{\mathbb{E}}^\omega [\langle\langle L_D(\cdot,k)\rangle\rangle_t^\beta] \\ &\quad \le Cl^{-\beta}\widetilde{\mathbb{E}}^\omega \left(\int_0^t \langle u^N(s), (T^N(t-s)\xi_k)^2\rangle\, ds\right)^\beta \\ &\qquad + Cl^{-\beta}\widetilde{\mathbb{E}}^\omega \left(\int_0^t \langle u^N(s), \Gamma_N(T^N(t-s)\xi_k)\rangle\, ds\right)^\beta. \end{aligned} \tag{3.19}$$

Obviously, the first term on the right-hand side of (3.19) is dominated by the same bound in (3.18) (up to some constant). For the second term on the right-hand side of (3.19), by two applications of Minkowski's inequality, and noting that $\langle \Gamma_N(f), 1\rangle = \langle -2\mathcal{A}^N f, f\rangle$, $f \in H^N_{bc}$, and $\frac{d\langle T^N(t-s)f, T^N(t-s)f\rangle}{ds} = \langle -2\mathcal{A}^N \times T^N(t-s)f, T^N(t-s)f\rangle$, $f \in H^N$, we find that for $\beta \geq 1$,

$$\begin{aligned}
&\left\{\widetilde{\mathbb{E}}^\omega\left(\int_0^t \langle u^N(s), \Gamma_N(T^N(t-s)\xi_k)\rangle\, ds\right)^\beta\right\}^{1/\beta} \\
&\quad \leq \int_0^t [\widetilde{\mathbb{E}}^\omega(\langle u^N(s), \Gamma_N(T^N(t-s)\xi_k)\rangle^\beta)]^{1/\beta}\, ds \\
&\quad \leq \int_0^t \left[\int_E \left(\widetilde{\mathbb{E}}^\omega |u^N(s,x)|^\beta \cdot |\Gamma_N(T^N(t-s)\xi_k)(x)|^\beta\right)^{1/\beta} \rho(x)\, dx\right] ds \\
&\quad \leq \left\{\sup_{s\leq t} \|\widetilde{\mathbb{E}}^\omega(u^N(s))^\beta\|_\infty\right\}^{1/\beta} \cdot \int_0^t \langle \Gamma_N(T^N(t-s)\xi_k), 1\rangle\, ds \\
&\quad \leq \left\{\sup_{s\leq t} \|\widetilde{\mathbb{E}}^\omega(u^N(s))^{2\beta}\|_\infty\right\}^{1/2\beta} \cdot \int_0^t \langle -2\mathcal{A}^N T^N(2(t-s))\xi_k, \xi_k\rangle\, ds \\
&\quad \leq \left\{\sup_{s\leq t} \|\widetilde{\mathbb{E}}^\omega(u^N(s))^{2\beta}\|_\infty\right\}^{1/2\beta},
\end{aligned}$$

that is,

$$\widetilde{\mathbb{E}}^\omega\left(\int_0^t \langle u^N(s), \Gamma_N(T^N(t-s)\xi_k)\rangle\, ds\right)^\beta \leq \left\{\sup_{s\leq t} \|\widetilde{\mathbb{E}}^\omega(u^N(s))^{2\beta}\|_\infty\right\}^{1/2}. \tag{3.20}$$

Combining (3.15)–(3.20), we obtain that

$$\begin{aligned}
&\widetilde{\mathbb{E}}^\omega[|\langle Y_{R^+}(t), \xi_k\rangle|^{2\beta}(\sigma_N(k))^{-\beta} + |\langle Y_D(t), \xi_k\rangle|^{2\beta}(\sigma_N(k))^{-\beta}] \\
&\quad \leq C\left(l^{-\beta}\left[\sup_{v\leq t} \|\widetilde{\mathbb{E}}^\omega(u^N(v))^{2\beta}\|_\infty\right]^{1/2} + l^{-2\beta}\right) \\
&\qquad + Cl^{-\beta}\left(1 + t^{2\beta} + t^{2\beta-1}\int_0^t \sup_{v\leq s} \|\widetilde{\mathbb{E}}^\omega(u^N(v))^{2\beta}\|_\infty\, ds\right).
\end{aligned} \tag{3.21}$$

Next, the contraction property of T_t^N yields

$$\begin{aligned}
&\left|\left\langle \int_0^t T^N(t-s)\,d\Theta^N(\omega,s), (\sigma_N(k))^{-1}1_k \right\rangle\right| \\
&\quad = \sum_{i=1}^{r}\sum_{j=1}^{N_i(t,\omega)} \Big\langle T^N(t-\tau_i^j(\omega)) \sum_{k\in K_i^N} l^{-1}\lfloor l\theta_i(k)A_i^j(\omega)+0.5\rfloor 1_k, \\
&\qquad\qquad (\sigma_N(k))^{-1}1_k \Big\rangle \qquad (3.22)\\
&\quad \le \sum_{i=1}^{r}\sum_{j=1}^{N_i(t,\omega)} (\|\theta_i\|_\infty A_i^j(\omega)+l^{-1}) \doteq c(t,l,\omega).
\end{aligned}$$

Combining (3.12)–(3.14), (3.21) and (3.22), we find that

$$\begin{aligned}
&\sup_{s\le t}\|\widetilde{\mathbb{E}}^\omega(u^N(s))^{2\beta}\|_\infty \\
&\quad \le 5^{2\beta-1}\Big\{\|\widetilde{\mathbb{E}}^\omega(u^N(0))^{2\beta}\|_\infty + Ct^{2\beta} \\
&\qquad + Ct^{2\beta-1}(1+l^{-\beta})\int_0^t \sup_{v\le s}\|\widetilde{\mathbb{E}}^\omega(u^N(v))^{2\beta}\|_\infty\,ds \\
&\qquad + Cl^{-\beta}\Big(\sup_{s\le t}\|\widetilde{\mathbb{E}}^\omega(u^N(s))^{2\beta}\|_\infty\Big)^{1/2} + Cl^{-2\beta} \\
&\qquad + Cl^{-\beta}(1+t^{2\beta}) + c(t,l,\omega)\Big\}.
\end{aligned}$$

Therefore, by Gronwall's inequality and $Cl^{-\beta}a^{1/2}\le \frac{a}{2}+C^2l^{-2\beta}$, we conclude that

$$\sup_{s\le t}\|\widetilde{\mathbb{E}}^\omega(u^N(s))^{2\beta}\|_\infty \le C(t,l,\|\widetilde{\mathbb{E}}^\omega(u^N(0))^{2\beta}\|_\infty,\omega),$$

where $C(\cdot)$ is obviously decreasing in l and measurable in ω. $\square$

Next, we employ the technique of Blount (1991, 1994) to derive some crucial estimates. Let $M=(\log N)^2$ and consider $0\le n\le \sqrt{2}N/M$. For an index $p\in\{0,1,2,\dots,L_1N-1\}\otimes\{0,1,\dots,L_2N-1\}$, let $|p|=(p_1^2+p_2^2)^{1/2}$ and let $B_n=\{p:nM\le|p|\le(n+1)M\}$. For $n\ge1$, $\max_{p\in B_n}|p|/\min_{p\in B_n}|p|\le (n+1)/n\le 2$. Thus by Remark 2.3, there exists $C>0$ such that

$$\frac{\max_{p\in B_n}\lambda_p^N}{\min_{p\in B_n}\lambda_p^N}\le C$$

for $n,N\ge1$. If $|B_n|$ denotes the cardinality of B_n, then $|B_n|\le\beta_n$, where $\beta_n=CM^2(n+1)$. Thus $\beta_n/N^2\le C(\log N)^2/N\to0$ as $N\to\infty$ and $\sum_{n=1}^{[\sqrt{2}N/M]}\beta_n\le CN^2$.

LEMMA 3.6. (i) *Let* τ^b *be an* $\{\mathcal{G}_t^{N,\omega}\}$ *stopping time such that* $\sup_{t\le T}\|u^N(t\wedge\tau^b-)\|\le b<\infty$. *Then there exist* $l_0, N_0, a>0$ *such that for* $n\ge 1$, $l\ge l_0$, $N\ge N_0$ *and* $d\in(0,1)$,

$$\widetilde{\mathbb{P}}^\omega\left(\sup_{t\le T}\left(\sum_{p\in B_n}\langle Y_D(t\wedge\tau^b),\phi_p^N\rangle^2\right)\ge d^2\beta_n/N^2\right)\le c(T)N^2\beta_n^{1/2}(ad^2l/b)^{-\beta_n^{1/2}}.$$

(ii) $\sup_{t\le T}\|Y_D(t\wedge\tau^b)\|\to 0$ *in probability* $\widetilde{\mathbb{P}}^\omega$ *as* $N\to\infty$ *for any* $b>0$, *where* τ^b *is as in* (i).

(iii) *Assume that* $\sup_N\|\widetilde{\mathbb{E}}^\omega(u^N(0))^q\|_\infty<\infty$. *Then* $\sup_{t\le T}\|Y_R(t)\|\to 0$ *in probability* $\widetilde{\mathbb{P}}^\omega$ *as* $N\to\infty$.

(iv) $\sup_{t\le T}\|Y^N(t)\|\to 0$ *in probability* $\widetilde{\mathbb{P}}^\omega$ *as* $N\to\infty$.

(v) *Assume that* $\sup_N\|\widetilde{\mathbb{E}}^\omega(u^N(0))^{2q}\|_\infty<\infty$. *Then the distributions of* $\{\int_0^{\cdot} T^N(\cdot-s)R(u^N(s))\,ds\}$ *on* $C_H[0,T]$ *are relatively compact.*

PROOF. The proof of (i) is almost the same as that of Lemma 3.21(b) in Blount (1991). The only difference is the covariance structure of $Z_D^N(t)$ as determined in Lemma 3.3, but all the estimates in the proof of Lemma 3.21 in Blount (1991) are still valid by changing some notation and constants. We omit the details here. The proofs of (ii)–(v) are similar to those of Lemma 3.5, Lemma 3.6, Lemma 4.1 and Lemma 3.7 in Blount (1994). We refer to Blount (1994) for details. Here we only point out that for the proof of (iv), although we do not assume that $R(x)<0$ for large x as Blount (1994) did, we can still use (3.11), the linear growth of R^+ and Gronwall's inequality to prove that

$$\begin{aligned}&\sup_{t\le T}\|u^N(t\wedge\sigma)\|\\ &\quad\le C(T)\biggl(\|u^N(0)\|+\sup_{t\le T}\biggl\|\int_0^t T^N(t-s)\,d\Theta^N(s,\omega)\biggr\|\\ &\qquad\qquad+CT+a+1+\sup_{t\le T}\|Y_{R^+}(t)\|\biggr),\end{aligned}\tag{3.23}$$

where $\sigma=\inf\{t:\|Y_D(t)\|\ge a>0\}$. The first two terms in (3.23) are bounded by Hypothesis (i) and Lemma 3.7 (to follow), so we can apply (iii) and the argument of Blount (1994) to establish (iv) here. □

LEMMA 3.7. *For each fixed* $\omega\in\Omega$,

$$\sup_{t\le T}\left\|\int_0^t T^N(t-s)\,d\Theta^N(s,\omega)-\int_0^t T(t-s)\,d\Theta(s,\omega)\right\|\to 0\qquad \text{as } N\to\infty.$$

PROOF. Basic calculation yields

$$\left\|\int_0^t T^N(t-s)\,d\Theta^N(s,\omega) - \int_0^t T(t-s)\,d\Theta(s,\omega)\right\|$$

$$\begin{aligned}
&\le \sum_{i=1}^{r}\sum_{j=1}^{N_i(t,\omega)} l^{-1}\|1_{B(\kappa_i,\varepsilon)}\| \qquad (3.24)\\
&\quad + \sum_{i=1}^{r}\sum_{j=1}^{N_i(t,\omega)} A_i^j(\omega)\|T^N(t-\tau_i^j(\omega))\theta_i^N - T(t-\tau_i^j(\omega))\theta_i\|,
\end{aligned}$$

where

$$\theta_i^N(\cdot) = \sum_{k\in K_i^N} \theta_i(k)1_k(\cdot), \qquad i = 1,2,\dots,r.$$

By using the projection mapping P^N defined in (3.2) and the contraction of $T^N(t)$, we find that

$$\begin{aligned}
&\|T^N(t-\tau_i^j(\omega))\theta_i^N - T(t-\tau_i^j(\omega))\theta_i\|\\
&\quad = \|T^N(t-\tau_i^j(\omega))\theta_i^N - T^N(t-\tau_i^j(\omega))P^N\theta_i\|\\
&\qquad + \|T^N(t-\tau_i^j(\omega))P^N\theta_i - T(t-\tau_i^j(\omega))\theta_i\| \qquad (3.25)\\
&\quad \le \|\theta_i^N - P^N\theta_i\| + \|T^N(t-\tau_i^j(\omega))P^N\theta_i - T(t-\tau_i^j(\omega))\theta_i\|\\
&\quad := \Phi_1^N + \Phi_2^N(t).
\end{aligned}$$

For Φ_1^N, it is easy to see that

$$\Phi_1^N \le \|\theta_i^N - \theta_i\| + \|P^N\theta_i - \theta_i\|, \qquad (3.26)$$

which tends to zero as $N\to\infty$. On the other hand, by Taylor's theorem, it is easily seen that $\mathcal{A}^N P^N f \to \mathcal{A}f$ strongly in H for $f\in\mathcal{D}_0(\mathcal{A})$ (the dense subset of H defined in Section 1). Thus, by the Trotter–Kato theorem, we find that $\Phi_2^N(t)\to 0$ uniformly in $[0,T]$. Therefore, we have proved that

$$\lim_{N\to\infty}\sup_{t\le T}\|T^N(t-\tau_i^j(\omega))\theta_i^N - T(t-\tau_i^j(\omega))\theta_i\| = 0. \qquad (3.27)$$

Now (3.24) completes the proof. □

In the sequel, we always consider the Skorohod metric d on $D_H[0,T]$ so that $(D_H[0,T],d)$ is a complete separable metric space [cf. Ethier and Kurtz (1986), pages 116–118]. For convenience, we let

$$v^N(t) = T^N(t)u^N(0) + \int_0^t T^N(t-s)R(u^N(s))\,ds + Y^N(t),$$

and $\gamma^N(t) = \int_0^t T^N(t-s)\,d\Theta^N(s)$. Then $u^N(t) = v^N(t) + \gamma^N(t)$.

LEMMA 3.8. (i) *For each fixed* ω, *the distributions of* $\{(u^N, v^N)\}$ *are relatively compact in* $(D_H[0,T], d)^2$.

(ii) *If* $\{(u^{N_m}, v^{N_m})\} \subset \{(u^N, v^N)\}$ *and* $(u^{N_m}, v^{N_m}) \to (\varphi, v)$ *in distribution on* $(D_H[0,T], d)^2$ *as* $N_m \to \infty$, *and* (φ, v) *is defined on some probability space* $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, *then for* $1 \le \beta \le 2q$,

$$\sup_{t\le T} \mathbb{E}^* \langle \varphi^\beta(t,\omega), 1\rangle \le C(T,\omega) < \infty. \tag{3.28}$$

PROOF. (i) follows from Lemma 3.6(iv), (v), Lemma 3.7, (2.8) and the fact that $\sup_{t\le T} \|T^N(t)u^N(0) - T(t)u_0\| \to 0$ in probability $\widetilde{\mathbb{P}}^\omega$ by the Trotter–Kato theorem and a subsequence argument.

(ii) We first consider $v^N(t)$ and notice $\sup_{0\le t\le T} \|v^{N_m}(t) - v^{N_m}(t-)\| = \sup_{0\le t\le T} \|Y^{N_m}(t) - Y^{N_m}(t-)\| \to 0$ in probability as $m \to \infty$ by Lemma 3.6(iv). Therefore, by Theorem 3.10.2 of Ethier and Kurtz (1986), we find that $v \in C_H[0,T]$. Next, by Theorem 5.1 of Billingsley (1968) and Skorohod representation, there exist $\{\hat{v}^{N_m}(t)\}$, $\hat{v}(t)$ on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ such that $\hat{v}^{N_m}(t) = v^{N_m}(t)$, $\hat{v}(t) = v(t)$ in distribution, and $\hat{v}^{N_m}(t) \to \hat{v}(t)$ in H a.s. for each $t \in [0,T]$. Let $\gamma(t) = \int_0^t T(t-s)\, d\Theta(s)$. By Lemma 3.7, γ^{N_m} is deterministic when ω is fixed and $\gamma^{N_m}(t) \to \gamma(t)$ in H. Therefore, we have $\hat{u}^{N_m}(t) = \hat{v}^{N_m}(t) + \gamma^{N_m}(t) \to \hat{\varphi}(t) = \hat{v}(t) + \gamma(t)$ in H almost surely. However, this implies that there exists a subsequence $\{N_j\} \subset \{N_m\}$ such that $(\hat{u}^{N_j}(t,x))^\beta \to (\hat{\varphi}(t,x))^\beta$ a.e. $x \in E$ almost surely. Then, we can use Fatou's lemma, Tonelli's theorem and Lemma 3.5 to conclude that

$$\begin{aligned}
\mathbb{E}^* \int_E \varphi^\beta(t,x)\rho(x)\,dx &= \hat{\mathbb{E}} \int_E \hat{\varphi}^\beta(t,x)\rho(x)\,dx \\
&= \hat{\mathbb{E}} \int_E \liminf_{j\to\infty} (\hat{u}^{N_j}(t,x))^\beta \rho(x)\,dx \\
&\le \liminf_{j\to\infty} \int_E \hat{\mathbb{E}} (\hat{u}^{N_j}(t,x))^\beta \rho(x)\,dx \\
&\le L_1 L_2 \sup_m \sup_{t\le T} \|\widetilde{\mathbb{E}}^\omega (u^{N_m}(t))^\beta\|_\infty \le C(T,\omega). \qquad \square
\end{aligned}$$

Finally we are in a position to prove Theorem 3.1.

PROOF OF THEOREM 3.1. We use the notation directly above Lemma 3.8 and find from the proof of Lemma 3.8 that $v \in C_H[0,T]$. Then we can use Skorohod representation followed by Lemma 3.10.1 in Ethier and Kurtz (1986) to find $D_H[0,T]$-valued random elements $\{\hat{v}^{N_m}\}$, $\hat{v}$ on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ such that $\hat{v}^{N_m} = v^{N_m}$, $\hat{v} = v$ in distribution and

$$\sup_{t\le T} \|\hat{v}^{N_m}(t) - \hat{v}(t)\| \to 0 \quad \text{a.s. } \hat{\mathbb{P}} \text{ as } m \to \infty. \tag{3.29}$$

Then, it follows by Lemma 19 of Dawson and Kouritzin (1997) that there are $D_H[0,T]$-valued processes $\{\breve{v}^{N_m}, m=1,2,\ldots\}$, $\breve{v}$ and $\{\breve{Y}^{N_m}, m=1,2,\ldots\}$ on some probability space $(\breve{\Omega}, \breve{\mathcal{F}}, \breve{\mathbb{P}})$ such that

$$\mathcal{L}(\breve{v}, \breve{v}^{N_1}, \breve{v}^{N_2}, \ldots) = \mathcal{L}(\hat{v}, \hat{v}^{N_1}, \hat{v}^{N_2}, \ldots) \quad \text{on } \prod_{m\in\mathbb{N}_0} \mathcal{B}(D_H[0,T]), \tag{3.30}$$

$$\mathcal{L}\begin{pmatrix}\breve{v}^{N_m}\\ \breve{Y}^{N_m}\end{pmatrix} = \mathcal{L}\begin{pmatrix}v^{N_m}\\ Y^{N_m}\end{pmatrix} \qquad \text{for all } m=1,2,\ldots. \tag{3.31}$$

Here, $\mathcal{L}(X)$ denotes the law of random variable X on a complete separable metric space S. We define a measurable mapping $G_N : D_H[0,T]\times D_H[0,T] \to D_H[0,T]$ by

$$\begin{aligned} G_N(\phi,\psi)(t) &= P^N\phi(t) - T^N(t)(P^N\phi(0)+\gamma^N(0)) \\ &\quad - \int_0^t T^N(t-s)R(P^N\phi(s)+\gamma^N(s))\,ds - P^N\psi(t). \end{aligned}$$

Thus, from $\tilde{\mathbb{P}}^{\omega}(G_{N_m}(v^{N_m}, Y^{N_m})=0)=1$ and (3.31), it follows that

$$\begin{aligned} G_{N_m}(\breve{v}^{N_m}, \breve{Y}^{N_m}) &= \breve{v}^{N_m} - T^{N_m}(\breve{v}^{N_m}(0)+\gamma^{N_m}(0)) \\ &\quad - \int_0^t T^{N_m}(t-s)R(\breve{v}^{N_m}(s)+\gamma^{N_m}(s))\,ds - \breve{Y}^{N_m}(t) \\ &= 0 \quad \text{a.s. } \breve{\mathbb{P}}. \end{aligned}$$

Then $\breve{u}^{N_m} = \breve{v}^{N_m} + \gamma^{N_m}$ satisfies

$$\begin{aligned} \breve{u}^{N_m}(t) &= T^{N_m}(t)\breve{u}^{N_m}(0) + \int_0^t T^{N_m}(t-s)R(\breve{u}^{N_m}(s))\,ds \\ &\quad + \breve{Y}^{N_m}(t) + \gamma^{N_m}(t) \quad \text{a.s. } \breve{\mathbb{P}}. \end{aligned} \tag{3.32}$$

Using Lemma 3.6(iv), (3.29), (3.30) and (3.31), we find a subsequence $\{N_j\} \subset \{N_m\}$ such that

$$\sup_{t\le T} \|\breve{v}^{N_j}(t) - \breve{v}(t)\| \to 0 \quad \text{a.s. } \breve{\mathbb{P}} \text{ as } j\to\infty \tag{3.33}$$

and

$$\sup_{t\le T} \|\breve{Y}^{N_j}(t)\| \to 0 \quad \text{a.s. } \breve{\mathbb{P}} \text{ as } j\to\infty. \tag{3.34}$$

Recalling $\sup_{t\le T}\|\gamma^{N_j}(t)-\gamma(t)\| \to 0$ surely from Lemma 3.7, one finds

$$\sup_{t\le T} \|\breve{u}^{N_j}(t) - \breve{\varphi}(t)\| \to 0 \quad \text{a.s. } \breve{\mathbb{P}} \text{ as } j\to\infty, \tag{3.35}$$

where $\breve{\varphi}(t) \doteq \breve{v}(t) + \gamma(t)$. Now, we identify $\breve{\varphi}$. By (3.32), we have with $\breve{\varphi}(0) = u_0$,

$$\begin{aligned} \breve{\varphi}(t) &= T(t)\breve{\varphi}(0) + \int_0^t T(t-s)R(\breve{\varphi}(s))\,ds + \int_0^t T(t-s)\,d\Theta(s,\omega) \\ &\quad + \breve{\varepsilon}^1_{N_j}(t) + \breve{\varepsilon}^2_{N_j}(t) + \breve{\varepsilon}^3_{N_j}(t), \end{aligned} \tag{3.36}$$

where

$$\breve{\varepsilon}^1_{N_j}(t) = \breve{\varphi}(t) - \int_0^t T(t-s)\,d\Theta(\omega,s) - \left(\breve{u}^{N_j}(t) - \int_0^t T^{N_j}(t-s)\,d\Theta^{N_j}(s,\omega)\right),$$

$$\breve{\varepsilon}^2_{N_j}(t) = \left(T^{N_j}(t)\breve{u}^{N_j}(0) - T(t)\breve{\varphi}(0)\right) + \breve{Y}^{N_j}(t),$$

and

$$\breve{\varepsilon}^3_{N_j}(t) = \int_0^t T^{N_j}(t-s)R(\breve{u}^{N_j}(s))\,ds - \int_0^t T(t-s)R(\breve{\varphi}(s))\,ds.$$

By (3.35) and Lemma 3.7, it follows that

$$\sup_{t\le T} \|\breve{\varepsilon}^1_{N_j}(t)\| \to 0 \quad \text{a.s. } \breve{\mathbb{P}} \text{ as } j \to \infty. \tag{3.37}$$

By the Trotter–Kato theorem and (3.34), we have

$$\sup_{t\le T} \|\breve{\varepsilon}^2_{N_j}(t)\| \to 0 \quad \text{a.s. } \breve{\mathbb{P}} \text{ as } j \to \infty. \tag{3.38}$$

We let

$$\breve{g}^{N_j}(t) = \int_0^t T^{N_j}(t-s)R(\breve{u}^{N_j}(s))\,ds, \qquad \breve{g}(t) = \int_0^t T(t-s)R(\breve{\varphi}(s))\,ds$$

and consider

$$\begin{aligned} \breve{\varepsilon}^3_{N_j}(t) &= \sum_{|p|\le n} \left[\langle \breve{g}^{N_j}(t), \phi_p^{N_j}\rangle \phi_p^{N_j} - \langle \breve{g}(t), \phi_p\rangle\phi_p\right] \\ &\quad + \sum_{|p|>n} \langle \breve{g}^{N_j}(t), \phi_p^{N_j}\rangle\phi_p^{N_j} \\ &\quad - \sum_{|p|>n} \langle \breve{g}(t), \phi_p\rangle\phi_p. \end{aligned}$$

By applying the Cauchy–Schwarz inequality and Remark 2.3, we have, for $|p| \neq 0$,

$$\begin{aligned} &\|\langle \breve{g}^{N_j}(t), \phi_p^{N_j}\rangle\phi_p^{N_j}\|^2 \\ &\quad = \left|\int_0^t \exp(\lambda_p^{N_j}(t-s))\langle R(\breve{u}^{N_j}(s)), \phi_p^{N_j}\rangle\,ds\right|^2 \\ &\quad \le \int_0^t \exp(2\lambda_p^{N_j}(t-s))\,ds \cdot \int_0^t \langle R(\breve{u}^{N_j}(s)), \phi_p^{N_j}\rangle^2\,ds \\ &\quad \le \frac{C}{|p|^2}\int_0^t \langle R(\breve{u}^{N_j}(s)), \phi_p^{N_j}\rangle^2 ds. \end{aligned}$$

Thus,

$$\sum_{|p|>n} |\langle \breve{g}^{N_j}(t), \phi_p^{N_j} \rangle|^2 \leq \frac{C}{n^2} \int_0^t \sum_p \langle R(\breve{u}^{N_j}(s)), \phi_p^{N_j} \rangle^2 ds \leq \frac{C}{n^2} \int_0^t \langle 1, R^2(\breve{u}^{N_j}(s)) \rangle ds.$$

Therefore, by Hypothesis (i), (1.2) and Lemma 3.5, it follows that for some constant $C(T,\omega) < \infty$,

$$\breve{\mathbb{E}} \left[\sup_{t \leq T} \left\| \sum_{|p|>n} \langle \breve{g}^{N_j}(t), \phi_p^{N_j} \rangle \phi_p^{N_j} \right\|^2 \right] \leq \frac{C(T,\omega)}{n^2}. \tag{3.39}$$

Similarly, by Lemma 3.8(ii), we find that

$$\breve{\mathbb{E}} \left[\sup_{t \leq T} \left\| \sum_{|p|>n} \langle \breve{g}(t), \phi_p \rangle \phi_p \right\|^2 \right] \leq \frac{C(T,\omega)}{n^2}. \tag{3.40}$$

It is easy to see that

$$\begin{aligned}
&\langle \breve{g}^{N_j}(t), \phi_p^{N_j} \rangle \phi_p^{N_j} - \langle \breve{g}(t), \phi_p \rangle \phi_p \\
&\quad = \int_0^t \exp(\lambda_p^{N_j}(t-s)) \langle R(\breve{u}^{N_j}(s)), \phi_p^{N_j} \rangle ds \phi_p^{N_j} \\
&\qquad - \int_0^t \exp(\lambda_p(t-s)) \langle R(\breve{\varphi}(s)), \phi_p \rangle ds \phi_p \\
&\quad = \int_0^t \exp(\lambda_p^{N_j}(t-s)) \langle R(\breve{u}^{N_j}(s)), \phi_p^{N_j} \rangle ds (\phi_p^{N_j} - \phi_p) \\
&\qquad + \int_0^t \exp(\lambda_p^{N_j}(t-s)) \langle R(\breve{u}^{N_j}(s)), \phi_p^{N_j} - \phi_p \rangle ds \phi_p \\
&\qquad + \int_0^t \exp(\lambda_p^{N_j}(t-s)) \langle R(\breve{u}^{N_j}(s)) - R(\breve{\varphi}(s)), \phi_p \rangle ds \phi_p \\
&\qquad + \int_0^t (\exp(\lambda_p^{N_j}(t-s)) - \exp(\lambda_p(t-s))) \langle R(\breve{\varphi}(s)), \phi_p \rangle ds \, \phi_p \\
&\quad := \sum_{i=1}^4 \breve{\Gamma}_i^{N_j}(t).
\end{aligned}$$

Note that for fixed p,

$$|\lambda_p^{N_j} - \lambda_p| + \|\phi_p^{N_j} - \phi_p\|_\infty \to 0 \qquad \text{as } j \to \infty$$

and

$$\sup_{j,p}(\|\phi_p^{N_j}\|_\infty + \|\phi_p\|_\infty) < \infty.$$

Therefore, by Lemma 3.5 and Lemma 3.8(ii), it follows that

$$\breve{\mathbb{E}}\Big[\sup_{t\le T}\|\breve{\Gamma}_i^{N_j}(t)\|_\infty\Big] \to 0 \qquad \text{as } j\to\infty, i=1,2,4.$$

For $\breve{\Gamma}_3^{N_j}(t)$, we have by (1.2) and the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\sup_{t\le T}\|\breve{\Gamma}_3^{N_j}(t)\|_\infty \\
&\quad= \sup_{t\le T}\left|\int_0^t \exp(\lambda_p^{N_j}(t-s))\langle R(\breve{u}^{N_j}(s)) - R(\breve{\varphi}(s)), \phi_p\rangle\, ds\right| \cdot \|\phi_p\|_\infty \\
&\quad\le \int_0^T |\langle R(\breve{u}^{N_j}(s)) - R(\breve{\varphi}(s)), \phi_p\rangle|\, ds \cdot \|\phi_p\|_\infty \\
&\quad\le \|\phi_p\|_\infty^2 \int_0^T \langle 1, |R(\breve{u}^{N_j}(s)) - R(\breve{\varphi}(s))|\rangle\, ds \\
&\quad\le \sqrt{3}K\|\phi_p\|_\infty^2 \int_0^T \|\breve{u}^{N_j}(s) - \breve{\varphi}(s)\| \\
&\qquad\qquad \times \langle 1, 1 + (\breve{u}^{N_j}(s))^{2(q-1)} + \breve{\varphi}^{2(q-1)}(s)\rangle^{1/2}\, ds \\
&\quad\le \sqrt{3}K\|\phi_p\|_\infty^2 \left(\int_0^T \|\breve{u}^{N_j}(s) - \breve{\varphi}(s)\|^2\, ds\right)^{1/2} \\
&\qquad \times \left(\int_0^T \langle 1, 1 + (\breve{u}^{N_j}(s))^{2(q-1)} + \breve{\varphi}^{2(q-1)}(s)\rangle\, ds\right)^{1/2},
\end{aligned}$$

which tends to zero in probability by (3.35), Lemma 3.5 and Lemma 3.8(ii). Thus, we have

$$\sup_{t\le T}\|\breve{\varepsilon}_{N_j}^3\| \to 0 \tag{3.41}$$

in probability $\breve{\mathbb{P}}$. Combining (3.37), (3.38) and (3.41), we obtain

$$\sup_{t\le T}\|\breve{\varepsilon}_{N_j}^1(t) + \breve{\varepsilon}_{N_j}^2(t) + \breve{\varepsilon}_{N_j}^3(t)\| \to 0 \qquad \text{in probability } \breve{\mathbb{P}} \text{ as } j\to\infty.$$

It follows by (3.36) that

$$\breve{\varphi}(t) = T(t)\breve{\varphi}(0) + \int_0^t T(t-s)R(\breve{\varphi}(s))\, ds + \int_0^t T(t-s)\, d\Theta(s,\omega) \qquad \text{a.s. } \breve{\mathbb{P}}.$$

Therefore, almost sure convergence of $\breve{u}^{N_j}$ to a pathwise solution of (1.3) follows from (3.35). We now show that the solution is unique. Let $u(t)$ be a pathwise mild solution of (1.3). Then we have

$$\begin{aligned} u(t,x) &= T(t)u(0,x) + \int_0^t T(t-s)R(u(s,x))\,ds + \int_0^t T(t-s)\,d\Theta(s,x) \\ &\le T(t)u(0,x) + \int_0^t T(t-s)R^+(u(s,x))\,ds + \int_0^t T(t-s)\,d\Theta(s,x) \\ &\le \|u(0)\|_\infty + Ct + C\int_0^t \|u(s)\|_\infty\,ds + \sum_{i=1}^r \sum_{j=1}^{N_i(t,\omega)} \|\theta_i\|_\infty A_i^j(\omega). \end{aligned}$$

By Gronwall's inequality, it follows that $\sup_{t\le T}\|u(t)\|_\infty \le c(T,\omega) < \infty$. Now let u_1, u_2 be two solutions of (1.3) such that $u_1(0) = u_2(0) = u_0$. Then

$$u_1(t) - u_2(t) = \int_0^t T(t-s)\big[R(u_1(s)) - R(u_2(s))\big]\,ds. \tag{3.42}$$

By (1.2) and the above estimate, we find that there exists $C(T,\omega) < \infty$ such that

$$\|u_1(t) - u_2(t)\| \le C(T,\omega)\int_0^t \|u_1(s) - u_2(s)\|\,ds.$$

By Gronwall's inequality, it follows that $u_1(t) = u_2(t)$ for any $t \in [0,T]$. But T is arbitrary, so $u_1(t) = u_2(t)$ for any $t > 0$. Convergence in probability for u^N then follows from (3.31), the fact that $\varphi = \breve{\varphi} = u$ is deterministic, and the arbitrariness of the original $\{N_m\}_{m=1}^\infty$. □

APPENDIX

Here, we give a computer-workable construction for the collection of independent Poisson processes used in Section 2.

Assume that $\{(X^R_{+,j}, X^R_{-,j})\}_{j=(j_1,j_2)=(1,1)}^{(L_1,L_2)}$, $\{(X^1_{+,j}, X^1_{-,j}), j = (j_1,j_2), j_1 = 0, 1,\dots,L_1, j_2 = 1,\dots,L_2\}$ and $\{(X^2_{+,j}, X^2_{-,j}), j = (j_1,j_2), j_1 = 1,\dots,L_1; j_2 = 0,1,\dots,L_2\}$ are independent standard Poisson processes on $(\Omega,\mathcal{F},\mathbb{P})$. Let $\{(\xi^{R,l,j}_{+,m}, \xi^{R,l,j}_{-,m}, \zeta^{R,l,j}_{+,m}, \zeta^{R,l,j}_{-,m}), l = 1,2,\dots,n; m = 1,2,\dots\}_{j=(j_1,j_2)=(1,1)}^{(L_1,L_2)}$, $\{(\xi^{1,l,j}_{+,m}, \xi^{1,l,j}_{-,m}, \zeta^{1,l,j}_{+,m}, \zeta^{1,l,j}_{-,m}), j \in \{(j_1,j_2), j_1 = 0,1,\dots,L_1, j_2 = 1,\dots,L_2\}, l = 1,\dots,n, m = 1,2,\dots\}$ and $\{(\xi^{2,l,j}_{+,m}, \xi^{2,l,j}_{-,m}, \zeta^{2,l,j}_{+,m}, \zeta^{2,l,j}_{-,m}), j \in \{(j_1,j_2), j_1 = 1,\dots,L_1, j_2 = 0,1,\dots,L_2\}, l = 1,\dots,n, m = 1,2,\dots\}$ be independent Bernoulli trials with $p = \frac{1}{2}$ on $(\Omega,\mathcal{F},\mathbb{P})$. Now, we construct the Poisson processes that are used to build our model in (2.3). For convenience, we let $\bar{\xi}^{R,l,j}_{+,m} = 1 - \xi^{R,l,j}_{+,m}$, etc. Then we will think of $\xi^{R,l,j}_{+,m}$ as a one in the lth position and $\bar{\xi}^{R,l,j}_{+,m}$ as a zero. Thus, we have one-to-one correspondence using the binary expansion of cell $k-(1,1)$ $(1 \le k_1 \le N, 1 \le k_2 \le N)$, for example [for each $j \in \{(1,1), \dots, (L_1, L_2)\}$],

$$\begin{aligned} k_1 - 1 \quad &\leftrightarrow \quad (0,1,\dots,1,1) \quad \leftrightarrow \quad \bar{\xi}^{R,n,j}_{+,m}\xi^{R,n-1,j}_{+,m}\cdots\xi^{R,2,j}_{+,m}\xi^{R,1,j}_{+,m}, \\ k_2 - 1 \quad &\leftrightarrow \quad (1,0,\dots,0,1) \quad \leftrightarrow \quad \zeta^{R,n,j}_{+,m}\bar{\zeta}^{R,n-1,j}_{+,m}\cdots\bar{\zeta}^{R,2,j}_{+,m}\zeta^{R,1,j}_{+,m} \end{aligned}$$

and we define the standard Poisson processes

$$X^{R,j,N}_{+,k}(t) = \sum_{m=1}^{X^R_{+,j}(4^n t)} \bar{\xi}^{R,n,j}_{+,m}\xi^{R,n-1,j}_{+,m}\cdots\xi^{R,2,j}_{+,m}\xi^{R,1,j}_{+,m}\zeta^{R,n,j}_{+,m}\bar{\zeta}^{R,n-1,j}_{+,m}\cdots\bar{\zeta}^{R,2,j}_{+,m}\zeta^{R,1,j}_{+,m},$$

$$X^{R,j,N}_{-,k}(t) = \sum_{m=1}^{X^R_{-,j}(4^n t)} \bar{\xi}^{R,n,j}_{-,m}\xi^{R,n-1,j}_{-,m}\cdots\xi^{R,2,j}_{-,m}\xi^{R,1,j}_{-,m}\zeta^{R,n,j}_{-,m}\bar{\zeta}^{R,n-1,j}_{-,m}\cdots\bar{\zeta}^{R,2,j}_{-,m}\zeta^{R,1,j}_{-,m}$$

and so on. If $j \in \{(1,1), \dots, (L_1, L_2)\}$, we construct $\{X^{1,j,N}_{+,k}, X^{1,j,N}_{-,k}, X^{2,j,N}_{+,k}, X^{2,j,N}_{-,k}\}$ for $k \in \{(1,1), \dots, (N,N)\}$ by using the same procedure as above. If $j \in \{(0,1), \dots, (0, L_2)\}$, we only construct $\{X^{1,j,N}_{+,k}, X^{1,j,N}_{-,k}\}$ for $k \in \{(1,1), (1,2), \dots, (1,N)\}$. If $j \in \{(1,0), \dots, (L_1, 0)\}$, we only construct $\{X^{2,j,N}_{+,k}, X^{2,j,N}_{-,k}\}$ for $k \in \{(1,1), \dots, (N,1)\}$. Then, $\{X^{R,j,N}_{+,k}, X^{R,j,N}_{-,k}, \dots, X^{2,j,N}_{-,k}, k_1, k_2 = 1, 2, \dots, N, j_1 = 1, \dots, L_1, j_2 = 1, 2, \dots, L_2\}$, $\{X^{1,j,N}_{+,k}, X^{1,j,N}_{-,k}, k_1 = 1, k_2 = 1, \dots, N, j_1 = 0, j_2 = 1, \dots, L_2\}$ and $\{X^{2,j,N}_{+,k}, X^{2,j,N}_{-,k}, k_1 = 1, \dots, N, k_2 = 1, j_1 = 1, \dots, L_1, j_2 = 0\}$ are independent Poisson processes for fixed N. Next, to simplify notation, we write $X^{k,\sharp}_{+,N}(t)$ for $X^{\sharp,j,N}_{+,i}$, $X^{k,\sharp}_{-,N}(t)$ for $X^{\sharp,j,N}_{-,i}$, where $\sharp = R, 1, 2$ and $k = (k_1, k_2) := ((i_1-1)L_1 + j_1, (i_2-1)L_2 + j_2) \in \{(1,1), \dots, (L_1 N, L_2 N)\}$, $i_1, i_2 = 1, \dots, N$, $j = (j_1, j_2) \in \{(1,1), \dots, (L_1, L_2)\}$, and write $X^{k,1}_{+,N}(t)$ for $X^{1,j,N}_{+,i}$, $X^{k,1}_{-,N}(t)$ for $X^{1,j,N}_{-,i}$, where $k = (k_1, k_2) := ((i_1-1)L_1 + j_1, (i_2-1)L_2 + j_2) \in \{(0,1), \dots, (0, L_2 N)\}$, $i_1 = 1, i_2 = 1, \dots, N$, $j = (j_1, j_2) \in \{(0,1), \dots, (0, L_2)\}$ and $X^{k,2}_{+,N}(t)$ for $X^{2,j,N}_{+,i}$, $X^{k,2}_{-,N}(t)$ for $X^{2,j,N}_{-,i}$, where $k = (k_1, k_2) := ((i_1-1) L_1 + j_1, (i_2-1)L_2 + j_2) \in \{(1,0), \dots, (L_1 N, 0)\}$, $i_1 = 1, \dots, N, i_2 = 1$, $j = (j_1, j_2) \in \{(1,0), \dots, (L_1, 0)\}$. In this manner, we have constructed the collection of independent Poisson processes as used in (2.3).

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