UNIVERSITY OF ALBERTA

EMERGENCE AND EVOLUTION OF MATHEMATICS CLASSROOM DISCOURSES

BY

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ABSTRACT

From the enactivist point of view, this study explores the dynamics of mathematics classroom discourses. The thesis is composed of interpretations of some of the discourses that occurred in a Namibian mathematics classroom, during one school term. In particular it investigates how those discourses emerged and evolved as the students acted and interacted with one another and with the teacher.

The focus of the study was not only on the ideas that were taken up for discourse but also those that occurred as possibilities. Episodes from a grade-11 mathematics classroom are used to discuss how the participants talked about mathematics. The data created consisted of field notes from classroom observations and audiotape records from discussion groups of students working on assigned activities.

The interpretations and discussions around the emergent themes may provide mathematics educators and teachers with learning opportunities to understand differently students' engagement in mathematical discourses. I also focused on the case of teaching mathematics in Namibian classrooms where English (as a second language) is the medium of instruction. In this study I observed that many generative possibilities for discourse emerge in a mathematics classroom and both the teacher and students contribute to that emergence.

DEDICATION

This work is dedicated to my late grandmother, Petrina Shikongo Shaanika, whose parental education guided me in growing up a responsible woman.

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INTRODUCTION

Reflecting on My Teaching of Mathematics

This research resulted from reflecting on my own past teaching experience and from thinking of ways in which I could improve my understanding of teaching and learning mathematics. I began my teaching career immediately after completing my Bachelor's Degree in Education, with mathematics and science as my school subjects, at the University of Namibia. I taught mathematics and sciences at the secondary school level for two years, before I took up a two-year Master of Education program at the University of Alberta, Canada.

Reading the literature and research reports in mathematics education, while at the University of Alberta, has exposed me to innovative and interesting ideas about teaching and learning mathematics. Learning theories that I became familiar with include radical constructivism, social constructivism, socio-cultural discourses, and theories based on critical theory, complexity science and many others. One current idea supported by advocates of some of these theories is the emphasis on learning and teaching mathematics through discussions and conversations. This prompted me to reflect on and think about my own mathematics teaching that I did back in Namibia.

Did I ever engage my students in mathematics conversations or encourage discussions in my mathematics classrooms? Of course, I often assigned group work to the students. Perhaps I assigned these students into groups, believing I was using the so-called learner-centered approach that the Namibian Ministry of Education (1993) introduced to the school system after independence. This approach to teaching and learning calls for students' active and physical participation in teaching and learning processes. The 'Namibian Education for All' policy stipulates that in addition to the old teacher-centred approach (TCA) teachers should employ a more learner-centered *approach (LCA) in their classrooms (Ministry of Basic Education and Culture, 1993). Learner-centered teaching methods include projects, small group or whole-class discussions, dramatization, experimenting, journal writing, debates, argumentations, and any other activities that actively engage students in the teaching-learning process (Ministry of Basic Education and Culture, 1993; Sibuku, 1996). However, like other teachers who might have misapplied the learner-centered approach, I might have taken group work to be plainly equivalent to such a teaching approach, which is actually not the case (van Graan, 1997).*

I do not remember noticing much talk amongst my students, let alone between the students and myself. I did most of the talking and students had very little to say and they only talked when I asked them a question. It is this problem that led me to the need to go back into the mathematics classroom to look for possibilities of involving students in mathematical discourses; possibilities that I might have missed, ignored or taken for granted during those first two years of my teaching. My inquisitiveness here lies in how learning possibilities arise in conversations and discussions and how we, as mathematics teachers and students, may listen to them and possibly engage in conversation with them.

With the narrative above I intended to present a picture of what this study is all about; that is, "talk in mathematics classrooms". Even though this study will benefit me, in that I will learn how to listen and pay attention to possibilities for emerging mathematical "talk" among students or between the students and the teacher, I believe that it has potential to benefit other mathematics teachers and educators, too, who are interested in teaching mathematics through dialogue.

Statement of the Problem

In contrast to the more traditional practices of teaching mathematics where students work in isolation and take their individually completed exercises to the teacher's desk for marking and corrections, educators have noted the value of learners working together and interacting. This allows learners to make meaning of and determine the appropriateness of their solutions to the mathematics tasks and problems they work on. Some studies in mathematics educational research have investigated classroom activities with the purpose to explore the social and cultural situatedness of learning and teaching mathematics (Cobb, 1999, Yackel 2000). Such research has emphasized not only the individual but also the social processes of sense making of the mathematics being taught and learnt (Yackel, 2000). Learning is at once individual and social (Simmt, 2000). Lampert, Rittenhouse and Crumbaugh (1996) write, knowing mathematics "is not thought to be a private interaction between knower and subject matter or a one-on-one interaction between teacher and individual student, but it is understood as a broadly social practice engaged in with peers and more knowledgeable others" (p. 738). In addition, there has been an indication of the need for research to pay attention to classroom situations in which teachers "are attempting to help students develop their own understanding through the social negotiation of meaning as they work together and talk with each other and with the teacher" (Kysh, 1999, p. 283).

It is believed that learning mathematics through dialogue is important and plays a role in students' mathematical development and should therefore be encouraged in mathematics classrooms (Cobb, et al., in Sfard, Nesher, Streefland, Cobb & Mason, 1998). However, as Sfard (1998) argues, a mere belief that learning through dialogue is important may not be convincing enough. The question should therefore be "not whether to teach through conversation, but rather how" (p. 50). Even though students' talk about mathematics amongst themselves is widely considered vital to learning mathematics, "the mechanisms that make it effective are not well understood" (Stacey & Gooding, 1998). Questions have also been raised about how communication can effectively take place in a mathematics classroom (Borasi, Siegel, Fonzi, & Smith, 1998). Lesh, Lovitts and Kelly (2000) also write: "One of the most important current needs in basic research on student learning processes is the need for insight explanatory models of these processes" (p. 23).

My interest in this study was to look at the possibilities in a mathematics classroom for engaging students in mathematical dialogue and to explore how learning can be enhanced. It was, specifically, to investigate discourses that arose in a Namibian mathematics classroom. I sought insight into the dynamical patterns of the emerging discourses that arose as the learners and the teacher acted and interacted with one another.

This study was guided by the following questions:

- How do discourses emerge and evolve in a mathematics classroom?
- What teacher or student actions and/or interactions emerge into mathematics classroom discourses (whether small-group or whole-class discourses)?

In this study I make sense, not only of how students contribute to mathematical classroom discourses, but also how they initiate these discourses with or without the explicit instructions of the teacher.

The Research Purpose

What can be learnt from the complex classroom interactions about mathematics classroom discourses? As the title of my thesis suggests, this study, *Emergence and evolution of mathematics classroom discourses*, investigates the dynamical patterns through which discourses emerge and evolve in a mathematics classroom. I sought a deep understanding of how the discourses arise especially from the interactions within the classroom. I was mostly interested in paying attention to both the discourses that are deliberately planned and directed by the teacher and those that spontaneously arise as the students talk about mathematics or about talking about mathematics (Yackel, 2000) with one another and with the teacher. The purpose of my research was to look for features that one an observer or participant might pay attention to when engaging in mathematical discourses.

Insights from this study might help improve student's participation in and initiation of mathematics classroom discourses, and hence encourage students' mathematical thinking. For if a teacher is aware of classroom characteristics that emerge into meaningful discourses, she or he will be able to recognize emerging projects that are either likely or unlikely to result in rich learning and hence behave appropriately (Stacey and Gooding, 1998). This is, however, not to suggest that the teacher or students are able to predict the outcome of the emerging discourses, but rather that they will be able to shape their future behaviors should such discourses arise. It also helps Namibian mathematics teachers and educators in identifying ways in which learner-centred instruction could be enhanced in Namibian classroom; and engaging students in open classroom discourses is just one of those different ways.

Definitions of Terms

Discourse

The key concept used in this thesis is *"discourse"*. According to the *Oxford English Dictionary*, a discourse can be defined as a "mutual intercourse of language", a talk or conversation between human beings. Any "spoken or written treatment of a subject, in which it is handled or discussed at *length*" can be treated as a discourse. I added emphasis to the word length for it is significant to my working definition of the term discourse. For instance, a dialogue limited to less than two utterances and hence to a very short of time may not be considered as a discourses but a whole network of utterances that occur over a period of time. Simmt, Glanfield and Sookochoff (2000) define a mathematics classroom discourse as a "talk among teachers and students in mathematics classes…[It is a] social interaction in language that supports the construction of mathematics in the microculture of the classroom" (p. 46). While I observed for any type of discourse (spoken or written) that emerged in the mathematics classroom, I paid specific focus on the "talk" that shaped the instruction and learning processes in this classroom.

Discussion

The second term that I used in this work is "discussion". The term discussion is interpreted and can be understood in different ways by different people. The Oxford English Dictionary, defines a discussion as "an examination or investigation (of a matter) by argument for and[/or] against." A discussion can also be an "argument or debate with a view to elicit truth or establish a point [or] a distinction in which a subject is treated from different sides". However, a discussion "is triggered not by constantly talking about the subject matter…but more appropriately by suitably presenting" (Abele, 1998) the issue under concern.

Davis (1996) defines a discussion as a "coordinated action in which the respective speakers are attempting to impose their perspectives on the other" (p. 39). In a discussion, asserts Davis, a speaker is more concerned with explaining and defending his or her point of view. When discussing something, discussants serve as audience for one another, but it appears to matter little who the audience is or what the audience does or does not understand. In some interactions…the listener continues to speak regardless of to whom he or she is speaking (Gordon-Calvert, 1999).

Conversation

On the other hand, a conversation is an "interchange of thoughts and words; familiar discourse or talk" that occurs between humans (Oxford English Dictionary). In contrast to discussion, a conversation is "more concerned with arriving at a shared understanding" (Davis, 1996, p. 39). According to Davis, unlike in a discussion, there is "true" listening in a conversation. "A conversation is more than an intertwining of separate voices" (p. 40). Davis also views a conversation as an unconscious act in which the participants are never aware that they are conversing but they can be aware once the conversation has taken place. In Gordon-Calvert's (2001) view, a conversation is a more intimate interaction compared to a discussion. In a conversation, those who are involved "feel the need to discuss events, to exchange opinions, or to tell others about their own experiences and knowledge gained" (Abele, 1998, p. 143).

Having made the above distinctions—between discussion, conversation and discourse—I am not suggesting, however, that one way of communicating in a mathematics classroom is better than the other. Rather, all three can enhance learning if they are effectively used. For example, it is through discussion (arguments and debates) that a class can investigate a given mathematical concept with the purpose to try to verify or refute it. On the other hand, a conversation can be used in investigating a certain phenomenon where two or more persons are trying to arrive at a common perception of such a phenomenon. In this work I use the term "discourse" to mean either discussion or conversation or both, because both conversation and discussion are part of discourse.

Emergence and Evolution

Emergence and evolution are other terms that will keep showing up in my interpretation and are as well part of the title to this work. The proposed study looks at how mathematics classroom discourses emerge and evolve and it is therefore necessary for me to define what I mean by 'emerging' or 'evolving' discourses. To emerge means to appear, to show up or to arise. According to the Oxford English Dictionary, the term emergence refers to "the process of coming forth, issuing from concealing, obscuring, or confinement, which may also be understood in terms of an "evolutionary process". By evolving, I simply mean the way in which ideas that arise in the classroom may transform into different discourses as they bump up against one another (Davis and Simmt, 2003). Evolution, according to Varela, Thompson and Rosch (2001), is not a matter of optimizing, as many theories say, but that of satisficing, a process that allows "any structure that has sufficient integrity to persist" (p. 196). In terms of mathematics classroom, I take the discourses that evolve to be those that have the potential to do so as long as they enable for the emergence of newer ideas which may or may not be taken up for discussions and conversations. I view these discourses as part of the living beings that generate them but not as entities separate from them.

Significance of the Study

The Namibian Context

Since independence in1990, Namibia has been advocating for a new educational system (see chapter 1). The learner-centred education (LCE) system was introduced together with new school curricula during the transition period from the colonial era to the independent and democratic era[1]. The new system replaced the old school curricula and started afresh with what were considered to be relevant curricula of benefit to all Namibian people (Tjukuua, 2000). This change had several implications not only to the political and societal dimensions but also to classroom practices, especially the strategies taken toward teaching and learning. In particular, the LCE requires a more "communicative and language-rich mathematics classroom" (Adler, 1998, p. 24) as compared to the teacher-centered approach in which most talk is done by the teacher alone.

The North American Context

Over the past two decades, mathematics educational research in North America emphasized the learning possibilities that arise in mathematics classrooms when active student-participation in the teaching and learning of mathematics is enhanced (Richards, 1991; Confrey, 1991; Kysh, 1998). As well, these researchers have particularly emphasized the place of classroom discourses in enhancing student's active participation in learning mathematics (Gordon-Calvert, 2001).

The ongoing classroom-based research in North America offers a good framework for studying the dynamics of mathematical discourses in Namibian classrooms. Whereas in North America mathematics teaching is increasingly oriented towards student-participation through talking about mathematics, in Namibia despite

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the move from TCE to LCE, very little is known about how the role of classroom discourse is central in enhancing students' learning of mathematics.

Educational Research in Namibia

The Namibian Institute for Educational Development (NIED) conducted a couple of descriptive studies with the aim to investigate what happens in Namibian elementary and secondary classrooms, in terms of teaching and learning. One of the findings was that most of the classrooms visited lacked the sharing of knowledge and collaborative learning (van Graan, 1997). In most of the classrooms observed, teachers seemed to have interpreted learner-centered education to mean students' group work. However, in these working groups very little discussion among students occurred—individual students continue to work independently. This may be due to the fact that we, Namibian teachers, continue to teach mathematics that is restricted to exercises and tasks done through pencil and paper seatwork.

Based on the findings by the NIED studies, van Graan (1997) and Crebbin (1997) suggest that educational research in Namibia must be aimed at investigating classroom environments in which the learners and teachers share knowledge and learn collaboratively and revealing insight into the ways in which these environments might be improved to enhance learning.

The ESL[2] Issue in Namibian Classrooms

A crucial area of study in the Namibian context is to understand how students express their mathematical ideas and thoughts in English, which is the official language of mathematics instruction. Unlike most North American classrooms in

[2] English as a second language.

which English is the students' first language or at least one in which the students communicate in English daily, English is a second language to most Namibian students and third to some (Tjukuua, 2000). Students rarely speak English outside the classroom context for it is common practice for them to switch to their mother tongues after school. In a mathematics classroom where the teacher is supposed to encourage students to speak English, such switching from language to language rarely if ever occurs (Adler, 1998). It is therefore important that educational research investigates Namibian classroom discourses with the aim to find ways in which students' learning of mathematics through dialogue can be encouraged.

Given the need for Namibian mathematics education to move to a more learner-centered approach in teaching, I am well convinced that my study will inform not only my own teaching of mathematics but also that of other Namibian mathematics teachers as they try to make their classrooms more learner-centered and dialogical. It will contribute to improvements in the use of classroom discourses in mathematics by presenting a detailed explanation of how mathematical discourses emerge and how they may evolve. More importantly, I hope to present explanations of how students' participation in mathematical discourses—in which English as a second language is and must be used—can be improved in Namibian classrooms. The study may also add to prevailing and ongoing international research and on issues of current views of mathematics learning as a collective, social activity (Davis, Sumara & Luce-Kapler, 2000).

Reflexively, my study may contribute to the North American and especially Canadian education research for improving the learning situations in the increasingly diverse classrooms (Blades, Johnston, Simmt, Mgombelo, Wiltse, & Leard, 2000).

CHAPTER ONE: REVIEW OF RELATED LITERATURE

Types of Mathematical Discourses

Richards (1991) discusses four types of discourses that arise in different communities of mathematical practice. For example, he calls the mathematics that is communicated in the community of mathematicians and scientists "research math". "Inquiry math" constitutes the mathematical discourses that arise among mathematically literate adults, who may not necessarily be mathematicians or mathematics teachers or students. According to Richards, the mathematics that is found in publications, such as books and journals, is "journal math". Lastly, he calls the mathematical communications in the mathematics classroom among students and between the students and the teacher, "school math". According to Richards, learning and teaching mathematics should move from school math to inquiry math, for school math has negative effects on students' learning of mathematics.

I find Richard's characterizing of mathematical discourses problematic because it is based on where the discourse takes place but not on the kind of mathematics that the community is discoursing. Yet, at the same time, he suggests that teacher and mathematics be engaged in "inquiry math", a kind of discourse, which most likely takes place among individuals outside the mathematics classroom, who may or may be not mathematically literate. For me this sounds more like suggesting that a fresh-water fish should try to behave like a marine water fish, but such behaviors are dependent on the type of water, in which that fish dwells. No matter what we choose to call the type of mathematics the students engage in, the students' experience with that mathematics "is not pregiven, but it emerges in the classroom interactions" (Voigt, 1995, p. 192).

Yackel (2000) discusses two types of "talk" that may occur in a mathematics classroom. Since she situates her discussions within the mathematics classroom, we may call it school math using Richards' categorization. The first type of discussion, Yackel observes, is that of talking about mathematics itself—the subject being studied. The second type of discussions arises when students and the teacher talk about talking about mathematics.

Another study that may be connected to Richards' characterization of mathematical discourses is by Kysh (1998). Kysh sought a "better" understanding of what and how students, engaged in small group work, learn through discussing mathematics. In her observations, Kysh encountered four types of working relationships among students that she calls: cooperative, competitive, businesslike and teacher/student working relations.

In cooperative group discussions, Kysh observed, students work out a given problem together with each participant building on what others have contributed. However, in the competitive and businesslike relations, Kysh noticed that students tended to first work individually before they share their ideas with other group members. Whereas in the teacher/student working groups, there is at least one student explaining the problem to other students while the rest of the group listens and rephrases the problem in order to make sense of it and be able to solve it together.

Another type of mathematical discourse found in the literature is argumentation. People engage in argumentation privately or publicly both inside and outside school classrooms (Lampert, Rittenhouse & Crumbaugh, 1996). In such argumentation chances are high that the discussants will disagree at one point another. Not many people favour this type of discourse for it is understood in terms of fighting or war and teachers also find it difficult to implement it in their classrooms as a type of teaching strategy (Gordon-Calvert, 2001). One of the risks students fear in argumentative lessons is the chance of losing the battle, which usually results in the student feeling defeated. For them, argumentation seems more like an unfriendly way or quarrelling (Lampert, et al, 1996). Argumentation also allows a speaker to separate mathematical knowledge from emotions (Gordon-Calvert, 2001), which, even though, is viewed as good characteristic of a mathematics classroom, is quite difficult to implement. Gordon Calvert also views an argumentation as a situation in which the individuals feel obliged to take sides.

> [They] *address* one another with the intent of attempting to *convince* the other of the truth of their own argument. This form of discourse demands a response from the listener. The response expected and listened for in the other's gesture is either *agreement* or *disagreement* with what has been proposed. (p. 343)

Others who favour argumentation in a mathematics classroom are for example, Lampert (1990), Lampert (1996) and Cobb and Yackel (1996). In an inquiry mathematics classroom, such as proposed by Richard (1991), claims Cobb and Yackel, argumentation provides opportunities for the students and teachers to negotiate taken-as-shared social norms and socio-mathematical norms.

Why Discourses in Mathematics Classrooms?

Engaging students in mathematical discourses is one way of enhancing social learning in the classroom and providing an opportunity for learner-centered teaching and learning that most teachers and educators are hoping for. Therefore paying attention to the emerging classroom discourses may contribute to students' development of mathematical argumentation and reasoning (Yackel, 2000). Several researchers and authors have discussed how small group and whole-class discussions can enhance the learning and teaching of mathematics (Davis 1990; Davidson 1990; Yackel, Cobb, Wood, Wheatly, & Merkel, 1990). Not only whole-class discussions are important but small group discussions are of value too. According to Tobin and Tippins (1993),

> ...group discussions can play a significant role in the learning of students by providing time for interaction with peers to answer student-generated questions, clarify understandings of specific [subject] content, identify and resolve problems. Group interactions also provide a milieu in which students can negotiate differences of opinion and seek consensus. (p. 11)

Involving students in discourses also challenges the notion of the teacher's role as the sole owner of authority, information provider and answer verifier in the mathematics classroom (Cobb, 1999; Richards, 1991; Yackel, 2000). To a certain extent, authority is and should be distributed and shared by all participants across the classroom. However, as Davis and Simmt (2003) advise, this distribution and sharing of authority should not be taken as a matter of "anything goes". Rather it is worth understanding that, "with the emergence of any complex collective, standards of acceptable activity—of rightness and wrongness—inevitably arise" (p. 10). Having said that, it is worth noticing that authority in the mathematics classroom does not only reside within the actions of deciding what and who is right or wrong but also with the notion of who creates the knowledge and ideas that become part of the collective. Therefore allowing students to talk about mathematics and about their talks about mathematics can be one of many ways in which students are invited to the community of school mathematicians and hence experience the sense of mathematical authorship (Whitin & Whitin, 1997).

Situated cognition theorists view learning as a "social co-participation" in a community of practices that is directed toward expertise and determined by the cultural and social context in which it is enmeshed (Lave & Wenger, 1991). They call this process of co-participation "apprenticeship[3]" under the supervision of an expert. In a classroom discourse, for example, a teacher or other (but advanced) student(s) may act as the expert to an individual student or a group of students who do not yet know much about a topic under discussion. From this social, mutual interaction both the apprentice and the expert learn. This co-participation also plays a role in students' understanding (Confrey, 1995), for their cognitive processes are more effective in the presence of others (teacher or other students), especially "the others whose competences are more developed" (Davis, et al. 2000, p. 67) than their own. By saying this, however, I do not view students as apprentices pretending or expected to pretend to be mathematicians (Davis & Sumara, 2002). Rather, I argue, instead of looking at learning as a preparation for future practices in the field of mathematics learners may be allowed to practice inquiry mathematics by engaging in mathematical discourses rather than school mathematics. The aim to teach mathematics is not "to prepare students to be mathematicians by mimicking" (Gordon-Calvert, 2001, p. 13) the practices of mathematicians, but to have students engage in personally authentic mathematics activity. Speaking from a Lacanian and enactivist points of view, Mgombelo (2002) discusses dialogue and conversation as the necessary conditions for mathematical cognition.

Insight from the Reviewed Studies

There is an extensive research done on mathematics classroom discourses, including the ones that I outlined above and any others that I have cited in this thesis. However, most of these studies (e.g. Richards, 1991; Gordon-Calvert, 2001; Yackel, 2000) have been carried out in either elementary mathematics classrooms or

[3] Another term that these authors use to describe their theory of learning, especially in recent work, is legitimate peripheral participation, which for several reasons, I choose not to use in my writing.

conducted through the use of elementary or junior high school students. An exception to this is Kysh's study, which was done in a post-secondary environment, which only looked at the interactions among students in small groups. None of the literature I reviewed had investigated the nature of discourses in a senior high or secondary mathematics classroom and therefore a need to explore mathematical discourses in these classrooms remains.

Also considering the case in Namibian mathematics education, there is a serious need for classroom-based research in order to make mathematics education more productive and meaningful to both the students and the teachers.

In the first two chapters, I introduced the study of investigating mathematics discourses and defined what I meant by the "emergence and evolution of" these discourses. I also presented arguments for teaching and learning mathematics through discussions and conversations. In the following chapter, I present the theoretical frameworks from which I draw my understanding and interpretations of teaching and learning mathematics.

CHAPTER TWO: THEORETICAL FRAMEWORKS[4]

Vygotskian Perspectives

> [T]*he most significant moment in the course of intellectual development, which gives birth to the purely human forms of practical and abstract intelligence, occurs when speech and practical activity, two previously completely independent lines of development, converge.*
>
> (Vygotsky, 1978, p. 24)

When exploring the roles played by social activities in learning, I see no better place to begin than by exploring Vygotsky's work, for I believe he was the first, among the modern psychologists, to talk about the mechanisms through which the social culture and language become part of human nature, and hence of cognition (Cole & Scribner, 1978). The social constructivist perspective on learning, which draws nearly on Vygotsky's psychology also views learning as a socially constructed activity (Geelan, 1997; Davis & Sumara, 2002). From this point of view, it is asserted, individual cognition is largely influenced by the cultural, social and linguistic contexts in which it occurs (Vygotsky, 1981). This notion goes beyond the representationist view on individual learning, which is solely concerned with individuals making sense of the external worlds around them without the help of others (Cobb & Bauersfeld, 1995).

Also important is Vygotsky's discussion on how a child's learning proceeds on a social plane in a zone of proximal development (Confrey, 1995). Even though most of Vygotsky's work on language is referred to as a view of child's cognitive development, we can still apply it to adult students' situations for as Vygotsky asserts "any analysis of the origin of uniquely human form of behavior [should be based on the] unity of perception, speech, and action" (1978, p. 26).

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[4] I use the plural term "theoretical frameworks" because my work draws from different perspectives, not with the aim of combining these perspectives but using each of them while I formulate my understanding of teaching, learning and knowing.

Looking for Complementary Perspectives

Every theory has its strengths and weaknesses. While I use the Vygotsky's perspective as one of my theoretical frameworks, using it as the sole framework may limit my interpretations and explanations of the phenomenon that I explored in this project. One of the reasons is that it (the Vygotskian approach) mainly looks at the social and cultural aspects only as aids to individual learning. It does not consider the culture and society as agents that learn as well. Also, Vygotsky's view—at least his interpreters—mainly emphasize the influence of adults and already existing cultural artefacts on an individual child's learning. Almost nothing is said about the influence of the children's actions and interactions on each other plus on the emergent artefacts and projects on learning. For example, in a mathematics classroom discussion: How do student-generated questions and ideas direct other students', teacher or whole-class' thinking as well as future discourses? Because of this challenge I therefore adopt another perspective that I discuss in the following paragraphs—enactivism. From an enactivist perspective, Simmt (2000) suggests, "If we assume that mathematics knowing arises in the interaction between a person and his or her environment, then we must acknowledge the implications this has for others (because they are a significant part of the environment)" (p. 92).

An Enactivist Perspective

Enactivism is a theory of cognition that is based on the work of Varela, Thompson and Rosch (2000) who view learning as an embodied perceptually guided action. This notion is also visible in Varela's (1999) work. I adopt the enactivist theory for it helps me overcome the challenge I discussed above that I would have faced if I were to use Vygotsky's perspectives only in discussing my interpretations. I attempt this task without necessarily having to "deny, minimize, or contradict the core assertions" (Davis & Sumara, 2002, p. 12) that each of these two perspectives (Vygotsky's and enactivism) holds. Neither do I wish to collapse[5] any of these perspectives together as one grand theory, for they might not be compatible in all circumstances. Rather, I explore how each of them relate and contribute to my study and how they help me to better understand the nature of discourses in a mathematics classroom.

Maturana (1978) observes that knowing, and hence knowledge, arises from the social interactions between the learning individual and its environment. This environment may include essential aspects as, what Vygotsky (1978) already called, the "tools for cognitive development". In a classroom for example, such tools can include the shared culture, the teacher, other students, books, and, most importantly, the language that arises and co-emerges with these complex interactions. Learning is grounded in this phenomenon of language (or the linguistic domain, in Maturana's terms). As Wales (1984) puts it, language "though not the only tool, is a powerful tool to comprehend and learn in order to make sense of one's world" (p. 16). Therefore for discussions on discussions and other discourse projects in a mathematics classroom, the issue of language—both formal and emergent—is crucial. Language plays an important role in social interactions in creating what Maturana (1978) and Maturana and Varela (Capra, 1996) termed *structural coupling*, among the learning systems as they recursively interact. Maturana (1978) writes: "When two or more organisms interact recursively… each becoming a medium for the…other, the result is mutual ontogenic structural coupling" (p. 47).

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[5] This tendency has been identified and discouraged among researchers who attempt to reconcile between two or more differing theories on learning. For further discussions on this, see Geelan (1997), Davis & Simmt (2003) and Davis & Sumara (in press).

Davis (1996) explains this further saying that when two or more organisms interact, each of them emerges a different organism and hence they all together "co-emerge". It is during this structural coupling that the classroom or discussion group may shape its future discussions and ways of decoding arising or emerging discourses. This ability of linguistic usage is one of the conditions that provide a collective psychological milieu in which both the individual and social mind is immersed (Hutton, 1988).

Structural Coupling and Discussion Groups

In what ways could the notion of structural coupling inform our view on deliberately assigning students to groups in the classroom? Drawing from such a notion of the readily occurring processes of structural coupling, some authors (e.g. Halliday, 1978; Winograd & Flores, 1986; Richards, 1991) have argued against the traditional classroom practices of "throwing" students into groups for discussions/conversations. The argument they make is that students should be allowed to work with individuals or groups with whom they have naturally formed (through structural coupling) a consensual domain.

With allowing students to work in coupled groups, however might be necessary to, sometimes, allow "divorce" between the already existing couples and to create room for many new and deliberate "couples" to form[6]. This may also result in more complex possibilities for discourses and hence provide us with a better understanding of the dynamics of how students take part in interactive learning. Some researchers too have realised that when students are allowed to fully practice the freedom to work with only those that they know well, the tendency to discuss off-task/task-unrelated issues appears to be high (Kempa & Ayob, 1991). Gordon-Calvert

[6] The terms "couples", "coupling" or "divorce", that I use here, should not be interpreted as limited to groups of two but rather referring to any group size.

(2001) asserts that those who hold such a belief are those who view mathematics learning as a finite game in which the rules are fixed and goals are determined. In mathematics classrooms where mathematics is treated as a finite game,

> Changing the boundaries or manipulating the rules to accommodate certain actions or solutions may appear as cheating; or activity outside the boundaries may be viewed as a diversion or as off-track behavior because it is at odds with the purpose for engaging in mathematics. (p. 126)

Hence the question of when to assign groups or when to allow students to work in groups, which naturally emerge from structural coupling, remains undecided. On the other hand, criticism has been offered to those actions of assigning students distinct roles within working groups (Davis & Simmt, 2003) for there appears to be no predetermined roles in human societies (Maturana & Varela, 1992). Flexibility to allow for diversity in a classroom may therefore be allowed to be demonstrated in such a way that any roles picked up by the participants spontaneously occur. Such naturally occurring roles are easily observable through the organisms' linguistic expressions as they define their relationships with others.

Understanding a Mathematics Classroom as an Autopoietic System

In the Santiago Theory of learning[7], Maturana and Varela view cognition—i.e. the way of knowing—as a process of life. Cognition, according to them, involves such processes as self-organizing, self-updating and self-perpetuating which all have their roots in Maturana's notion of living organisms as autopoietic systems. The word autopoietic refers to systems that are capable of continually creating themselves (Reid, 1995). Autopoietic systems learn by making distinctions and learning mathematics is about making distinctions and making distinctions from these distinctions (Simmt, 2000).

[7] This theory is well explained in Capra (1996 & 2002).

Mathematics classrooms as well as groups that arise within them can be viewed as complex, self-organizing and self-updating, autopoietic systems (Davis & Simmt, 2003). Even though the teacher comes to the classroom with deliberately prepared issues or topics for discussion, the emerging discourses might take a different direction to what she or he anticipated. This is an example of the autopoietic nature of the learning collective as a living unit on its own, for which the teacher like his or her students is just an "organ", albeit a more specialized organ. Paying attention to not only the deliberate interactions among learning individuals but also to the spontaneous, structural behavior of the social learning unity may therefore allow room for the observation of ignored or unrecognized/taken-for-granted learning behaviors in a mathematics classroom (Namukasa, 2002).

Understanding the classroom as an autopoietic unity might give the researcher as well as the teacher and the students a chance to pay attention to the emerging properties of the collective to which they are just organs that can in many ways increase the emergence of artefacts, ideas and projects that transcend possibilities of them as individuals. The notion of learning as doing also seems to allow us as teachers and students to adopt the stance that cognizing systems (including the individuals, sub-collectives and co-collectives) learn by doing the mathematics. Discourse may therefore not mean just talking about the subject but also doing or making it. Moreover it is not only doing but also living—knowing is doing is being (Maturana & Varela, 1992). Learning systems only learn to ensure their fit with a co-evolving world through structural coupling.

In Maturana's own words: "Human beings talk about things because they generate the things they talk about" (1978, p. 56). This quote can be interpreted in possibly many ways of which I outline two here due to my current capability in doing so as well as to the scope of this thesis. First, from a constructivist point of view, human beings are able to talk about ideas that they generate or construct and that consequently grow into potentially meaningful discussions. It is this ability that allows the collective to keep the discussions going and to generate more and new ideas that may condition future related discussions. For instance, in mathematics classroom discourses, students ought to discuss ideas, thoughts, and doubts that they generate and not only to talk about ideas that someone else—i.e. a mathematician—has developed for them.

Second, a more radical interpretation of the same quote above can be interpreted from views of those who look at classroom discourses and the mathematics itself as evolving features (Cobb, Stephen, McClain & Gravemeijer, 2001; Gordon-Calvert, 2001) of the learning system[8]. When one talks of students interacting with one another or with the teacher, one not only means interactions between students as talking bodies, but also the interactions between the generated ideas as they bump into one another (Davis & Simmt, 2003) and taken up for discussions, conversations or argumentations. Each emergent idea, therefore, has a potential to generate or evolve into other ideas, which may also unfold into further discourse that may create new dynamic structures that are able to self-reproduce through autopoiesis.

[8] I owe part of this second interpretation to Dr. D. Sumara (personal communication, December 5, 2002) for he pointed out to me how much sense it would make if one was to understand Maturana's quote above to mean that the ideas that are generated have a potential to generate further ideas for discussions.

CHAPTER THREE: RESEARCH DESIGNS

Learning to Re-Search

For some people research may simply imply the collection of data or what is generally called information in order to answer a question or solve a problem. This research focuses on a question of how classroom mathematical discourses emerge and evolve. But what do I mean by doing research or researching the topic of *Emergence and Evolution of Mathematics Classroom Discourses*?

I am writing this work for a broader audience, which is of two different cultures. First, is the Canadian culture, ranging from mathematics educators to mathematics teachers and students. The second audience is the Namibian culture, also ranging from mathematics educators to mathematics teachers and students. The understanding of and familiarity with educational research varies within and among these three groups (educators, teachers and students) and across the two mentioned cultures, with the educators being better informed than the teachers who are in turn better informed than students. It is, therefore, useful for me to briefly explain what I mean by researching my topic and to describe the types of research designs and methodologies used in my work.

Doing research is more than just gathering, analyzing, interpreting, and reporting data, as most people take it to be (Lesh, Lovits & Kelly, 2000). It is much more than the mere application of research methods (Tobin & Tippins, 1993). However, most of the research books that I read on research (Creswell, 1994; Punch, 2000 & Creswell, 2002), hardly discuss what research is, but rather concentrate on the types of research methods there are and how to conduct research. As the prefix may suggest, then to research implies to re-search. If the word "search" is literally taken to mean look then to re-search means to re-look something. To re-look suggests looking again. Understood in these terms to research, then, means to look at something again. In this study I am re-looking at the discourses that arise in a mathematics classroom. This may suggest that I have been to a mathematics classroom before and might have looked at the discourses before. But because I did not look well or there is something that my looking has missed—i.e., how the discourses emerge and evolve, I needed to go back to the mathematics classroom and attentively re-look again at this phenomenon in order to gain an understanding the dynamics of discourses in a mathematics classroom.

Assumption and Rationale for Qualitative Research

Research has been traditionally divided into two main types: qualitative research and quantitative research. However, there are many other ways in which research may be classified. For example, some research can be post-modern or positivistic or art-based research, etc. It is not my intention to present an extensive analysis of these different kinds of research but, rather, I will just make a distinction between the former two.

Qualitative research can be defined as an inquiry process of exploring a phenomenon with an aim to understand a social or human problem or any topic that one is interested in (Stake, 1995; Creswell, 2002). In a qualitative study the researcher seeks to both find out and understand a particular phenomenon especially from the participants' point of view (Wilson, 1998). A qualitative researcher asks questions of the kind: with what happens or when does this happen? How does it happen? Or in what ways does it occur? And so on. Qualitative reports are mostly textual or pictorial data representations of a particular situation that conveys the research findings.

On the other hand, and equally important, quantitative research is an inquiry approach for describing trends and explaining the relationship(s) among variables. Quantitative studies are based on testing theories "composed of variables, measured with numbers, and analyzed with statistical procedures, in order to determine whether the predictive generalizations of the theory hold true" (Creswell, 1994, p. 2). A quantitative researcher asks questions of: How is **A** related to **B**? What happens to **Y** when this happens to **X**? What causes this? And so on. Quantitative reports are mostly numerical data presentation of a particular situation that conveys the researcher's findings.

Considering the distinctions I made between qualitative and quantitative research, it appears that my research questions and interests and capabilities greatly lent themselves to the qualitative research than to quantitative. This study neither involves collections of any statistical quantitative data nor does it propose any hypothesis or test a predetermined theory. Therefore, my decision to do a qualitative study was not entirely that I favour the qualitative approach over the quantitative one. It is rather determined by the nature of my research topic and problem, for it sought not to find relationships among variables, but to explore and understanding a particular phenomenon, i.e. the emergence and evolution of classroom discourses in mathematics. Had I sought to investigate questions such as "In which classrooms do emergent discourses occur most" or "What is the correlation between emergent classroom discourses and, say, student performance on provincial or regional mathematics examinations" a quantitative approach, or the combination of the two approaches, would have been appropriate.

Another difference is that quantitative studies involve large numbers of research participants (traditionally called subjects), while qualitative research involves small numbers of participants. In my study, for example, I worked with a single mathematics classroom that is composed of 37 participants: one mathematics teacher, one visiting (in-service) teacher and 35 students. For a quantitative study such a research sample might be considered relatively a small size; yet for a qualitative study it is of great advantage that one limits the population size to a number although not too small to limit the usefulness of the data but not too big to limit the possibility of any textual and qualitative data.

Finding a Research Design for my Study

The first time I became exposed to the literature in complexity science, and hence enactivism, was when I took a course on cognition and curriculum taught by Dr. Dennis Sumara, Dr. Brent Davis and Dr. Elaine Simmt, at the University of Alberta. I took this course for two consecutive terms—the fall 2002 and the winter 2003. It was at that same time that I was developing a proposal for this study and also being attracted to reading more about complexity theories and began to explore how it relates to my own work. As a result I then ended up using the enactivist perspective to understand how mathematics classroom discourses emerge and evolve.

One of the challenges I faced in my research study while using enactivism as my theoretical perspective was finding a research design that is commensurable with the research in general and with enactivism in particular. Despite the use of enactivism as a theoretical framework in mathematics education research and the field of education in general, very little is offered as to what research methodologies or designs could be possibly used for studies holding the same complexity views. Of some of the research designs commonly used in mathematics education research, I have considered *action research, case study, ethnography, grounded theory, and* *phenomenology*. What perturbed me was the fact that none of these research designs would really work well for my research project. None could work for at least two reasons. One; it could be too simplistic—unable to offer an adequate explanation of how to observe and analyze the complex behaviors of the learning systems. Two, it could be too prescriptive—i.e. it does not allow me to do certain things that are not allowed in its paradigm. Below I briefly discuss the reasons why the five research methods mentioned above were not compatible with this study.

Action Research

In action research, the researcher seeks to understand a certain phenomenon with a purpose of improving their behaviours (McNiff, 1998; Creswell, 2002). Similarly, an educational researcher conducts action research when he or she wants to improve his or her educational practices or to solve a particular pertaining problem in the school or classrooms. Even though my research study may greatly contribute to the teacher-participants' and my understanding of classroom discourses, and hence help us learn better how to motivate and manage discourses when teaching mathematics, this learning and change will only be consequential and I am not reporting on it.

Case Study

Case study is any research study that looks, in depth, at a particular individual (e.g. a student, a parent, a teacher or a principal, etc.), program, a class, phenomenon or an event, with the purpose to understand the dynamics within singular settings at a particular moment (Merriam, 1998; Leedy & Ormrod, 2001; Eisenhardt, 2002). Central to case study is that, there has to be a case—e.g. something that had happened and the researcher wants to find out more about that happening. Since there was no case I went to follow up in this study, it is not a case study.

Ethnography

In ethnography, the researcher looks at an entire group—more specifically, a group that shares a common culture—in depth. Ethnographic studies are for "gaining an understanding of a particular, intact culture" (Leedy, et al., 2001, p. 151). Creswell, 2002 observes that the hallmark for ethnographic research designs is to contextually explore and interpret cultural patterns of behaviors, beliefs, and language that a certain cultural group shared for a long period of time. Although I sought to mainly focus on the culture of one mathematics classroom my research study did not seem fundamentally ethnographic. I would be lying to myself if I claim that social and cultural aspects of the participants had no impact on the path that this study took as well as its results. However, even though the participants in this study were of the same cultural group, and that they might have many norms and social behaviors in common, I did not consider studying their culture to be central to my investigations. For my research question on understanding the dynamics of the emergent classroom discourses, the culture of the students appeared not to matter, as long as they were grade 11 mathematics students plus their teacher. I simply chose to work with grade 11 because students in grade 12 were going to be busy preparing for their final examinations during the time that my study happened. This was not a convenient time for either the participants or the researcher (myself), had I conducted the study in a grade 12 class.

Secondly, even though the participants were from the same cultural group, the study did not seek an understanding of that culture but of the dynamics of classroom discourses that the participants generated. Such discourses might and might not be related to their cultures.

Grounded Theory (GT)

Glaser (2002) describes grounded theory as the "generation of conceptualizations into integrated patterns, which are denoted by categories and their properties" (p. 2). In grounded theory, the researcher systematically investigates a number of individuals to generate a theory or a model that explains a process, an action or interactions about a substantive topic in which the individuals have been involved (Creswell, 2002). According to Creswell (2002), a grounded theory-based study is a systematic procedure for generating a theory that explains a particular phenomenon. Since I did not plan to and did not formulate any theory in the study, this research was not a grounded theory either.

Narrative Inquiry

When a researcher is interested in exploring and reporting on people's histories and their current lives while focusing on a central event or process, the study is called a narrative inquiry. The researcher collects detailed stories and narrates about the experiences and lives of the participants with an object to understand these experiences (Creswell, 2002; Clandinin, 2002). Connely and Clandinin (1990) describe narrative inquiry as a study of how people experience the world. Since I investigated interactions between the teacher and students and tried to understand these behaviors, I am actually studying their at-a-moment life styles in a school context during specified classroom activities. However, I did not focus onto their wider life experiences; nor did I generate stories about their lives. Narrative inquiry researches both the inside and outside self (Clandinin, 2002) of the human experiences and mine just touches on the outside experience with a singular phenomenon—classroom discourses. Although I might draw from narrative inquiry as writing to weave the research report when I write my thesis, this is not entirely a narrative inquiry study.

Phenomenology

The last but not least type of research method, is phenomenology. Phenomenology is a term used to describe a "person's perception of the meaning of an event, as opposed to the event as it exists external to the person". A phenomenological study seeks to get access to "people's perceptions, perspectives, and understandings of a particular situation" (Leedy, et al., 2001, p. 153). It is an inquiry into "lived experience". A phenomenological researcher seeks a deep and meaningful understanding of "the nature or meaning of the [participant's] everyday experiences" with the world (van Manen, 1988). Classroom discourse is just one of the experiences that students have in the classroom situation in a day and at some moments for a limited time. Looking at such a phenomenon did not tell me much about the students' or teacher's everyday experiences. This is simply an empirical study attempting to explain the process of learning and teaching mathematics, as it occurs when discourses emerge and evolve. It therefore did not seem phenomenological enough for me to say I am doing phenomenology.

Finding none of the above qualitative research methods suitable for my research, I looked for something more appropriate. I needed a proscriptive method in my research, which would not limit how I was to do my research. In prescriptive situations—particular things are allowed all else is prohibited whereas in proscriptive situations, "what is not prohibited is allowed" (Davis, Simmt & Sumara, 2003). This is not to suggest that prescriptive designs should be considered valueless (Davis, 1996), but that they may be inadequate in explaining complex learning behaviors that arise from the interactions that I observed. Similarly I am not suggesting a total ignorance of prescriptive designs. Rather, I see them limiting the dynamistic nature of my study, especially when I take an enactivist stand, which asserts that learning beings are complex, self-determining and self-organizing systems, that is nothing, but their structures, determine how they live.

Enactivism Research Methods

The lack of a research method for my research study prompted me to review some of the related studies already done in mathematics education that have used the enactivist perspective as their theoretical framework Some of these studies are such as Davis (1996), Simmt (2000), Gordon-Calvert (2001), Mgombelo (2002), and Glanfield (2003).

Using hermeneutics and enactivism, Davis calls for an alternative way of teaching of mathematics as a type of listening. Simmt used enactivism to explore mathematics knowing in action. Gordon-Calvert searched for a place for conversation in the practice of mathematics. Speaking from both enactivism and psychoanalysis, Mgombelo called for a consideration of mathematics' teacher content knowledge in addition to their pedagogical knowledge. Glanfield discussed teachers' mathematics understanding as an evolving phenomenon.

Looking at these studies I had hoped to find a succient and common research method or design that I would also be able to use in my research study. That was not the case. This led me to questioning a possibility for a research design (Miranda, 2003b) for not only my research study but also any other study of the same nature—studies that use enactivism as their theoretical framework.

Setting the Conditions for Enactivism Research Methods

Just at the time when I was in the middle of my research, I met with some of authors of the reviewed studies I mentioned above in an ad hoc discussion at a small conference, "Complexity Sciences and Education" (2003c) hosted by the University of Alberta. We talked about the need to identify and document research methods for enactivist studies in mathematics education. I wished to know what commonalities I could find in the way they carried out their research studies and writing, in order to help me figure out how such a research method would look like if it existed. The four of the researchers that I met with were Simmt, Mgombelo, Glanfield and Pilane. Even though I had not read Pilane's (2003) work, prior to this meeting, she was also part of the meeting and had equally contributed to the setting of the conditions I discussed below. Pilane's work was around physical embodiment and how students approach open entry lab activities to gain freedom to explore science.

As we held our discussions, these researchers expressed that they did not follow a specific method, and were amazed at how they never thought about the need to name a research method. On suggesting that we look at how each of them went about their work, we decided to note down a few things that would set the conditions for the research method that I had proposed for—and that was thereafter called the enactivism research method.

These conditions arose from the commonalities identified from how these researchers had done their studies in terms of data collection, data analysis, data interpretation and writing the research reports (theses). It was quite amazing that even there was no pre-given that even though there was not a documented research method to follow, these research studies had quite a lot in common. This helped us set some of the desired, but not sufficient, conditions for enactivism research methods.

- An enactivism research is task-oriented. In other words, behaviors of the participants are articulated as the participants engage in particular activities: mathematical activity or whatever subject content the researcher is interested in.

- The observer is a participant observer; that is, he or she cannot place him or herself outside the participants' world and explain their behaviors as a completely objective observer.
- Having multiple observers in data creation is an ideal for this type of research. Such observers could be other people (the participants, colleagues, or technological tools such as audio tapes or video cameras) and the participants themselves if possible.
- There is also a need for multiple interpreters who can be the same as or different from the observers.
- More stress is put here on multiplicity of data collection, analysis and interpretation because enactivist research is not a personal action but a community-based study. One cannot just tell any story, but one has to tell stories that matter to the community. And that is how having multiple beings involved in the study helps, for they might have common interests in the community and would know what matters to the larger community.
- Another condition in this type of research is the responsibility to explain. The observer explains not only the procedures of observation but also the phenomenon that she or he had observed (Maturana, 1978).
- What matters, in enactivist research, is not so much as WHAT one looks at (the researched) but HOW one looks and how one understands what he or she is looking for.
- Writing up the research report of an enactivist research study is recursive writing; that is, one may come back to a similar scene and retell a story as he or she presents his or her explanations. The explanation is always incomplete and hence each time the data seems new as one comes back to the

data to retell the story but telling it in a different way. Such an approach is visible in Simmt's (2000) work.

It was not long after I met with Simmt, Mgombelo, Glanfield and Pilane that I came across Reid's (1995) doctoral theses, which also used enactivism as a theoretical framework. According to Reid, enactivism is not only a theory of knowing but could also be treated as a "basis of a methodology research" (p. 119). This is similar to the notion that there should be coherence between a theoretical framework and the methodologies that are used in both data collection and data analysis, as was agreed on in my meeting with Simmt and others.

How Enactivism Shapes My Research

Concerns among mathematics educators about the role of 'talk' in mathematics teaching and learning have a long history (Wales, 1984). However, even though the role of language in a mathematics classroom had been the main focus for a few decades, the focus has recently shifted from language and mere discussion to discourse (Sierpinska, 1998)—that is, any interactive learning situations in a classroom either written or verbal mathematical communication. My research was therefore in line with this current shift in interest where I go beyond talking about the importance of teaching and learning mathematics through discourse to investigating how possibilities for such pedagogy present themselves in a mathematics classroom. Therefore the stories that I present here are not just any stories but stories of things that matter not only to the Namibian community but also to the international community of mathematics educators and teachers.

This research was task oriented, in that I carried out the observation as the classroom participants were engaged with mathematical activities assigned by the teacher or that emerged from their interactive communication. The discussions I present in this thesis are therefore my interpretations of the participants' interactions as they do those activities and as they interpret them—interpretations of interpretations. As will be discussed in the following chapter, five, under the section of the role of researcher, I was part of this study not as an external observer but as a participant observer. I participated in this study not only through making observations but also through interacting with the participants (teacher/in-service teacher or individual students and groups) as they make sense of the mathematics they were learning.

On the multiplicity of interpretation of data, I had several interpreters apart from myself who also took part in making sense of and interpreting the data. From the notes that I took during classroom observations and the audiotapes that I recorded, I made transcript excerpts that I shared with the class teacher, my supervisor and fellow graduate students. This helped me create a community with which I could discuss and make sense of the data and in deciding what counts as useful and generative to the mathematics education research community.

Central to my actions in this work was the notion of viewing learning individuals or collectives as complex systems. That implies that their happenings do not totally depend on the environment but these individuals and their environments are co-dependent on each other. The teacher's teaching, materials used and the utterances made acted as triggers to the students' thinking, and understanding in this mathematics classroom.

CHAPTER FOUR: RE-SEARCHING

Research Site

This research project took place in a Namibian secondary school, situated in the northern part of the country. Even though this is an urban school, it is located in one of the educational regions that are nationally characterized as formerly disadvantaged.

The school was a large community, by Namibian standards. Table 1 below represents the information about the size of the school.

Teachers	27		
Learners	766		
Boys	346		
Girls	420		
Per Grade	Boys	Girls	Total
8	27	43	70
9	40	68	108
10	43	53	96
11	105	105	210
12	131	151	282

Table 1: Statistical information about the size of the research site

This particular school follows the general curriculum that is used in all other public schools in the country. In grade 11, the students take up a two-year curriculum that they may complete by the end of grade 12. This curriculum has three different options. The most difficult is the High International General Certificate of Secondary Education (HIGCSE). The second one is the International General Certificate of Secondary Education (IGCSE), and is made up of two options: Extended and Core curricula. Extended has a higher level of difficulty than the Core level, but lower than that of HIGCSE. A student could have one or more or all subjects at any of these three curricular levels. The class in which I collected data consisted of students who were taking all of their subjects at the HIGCSE level. This had nothing to do with my choosing this particular class, for I only learned about it when I had already began inquiring with them. Moreover, the teacher, herself, made the decision which class she would allow me to do the research with, after I communicated to her the nature and purpose of my study.

Classroom Set-up[9]

The grade-11 mathematics class, involved in my research, consisted of 33 students: 13 girls and 20 boys. The ages of these students ranged between 16 and 20 years. There was no fixed and uniform seat arrangement in this classroom. Some students sat in groups all through the term while others sat alone or in pairs. One could say the classroom was disorganized, since the groups did not sit in any order. For example, a student may be sitting alone facing the front of the classroom but when it is time for seatwork, he or she may move closer to a neighboring individual or group to interact with them. This—the messy seating arrangement— is probably the preferred seating arrangement in many Namibian classrooms since learner-centered education came into force. It is believed that students should be allowed to interact with each other as they learn.

The types of activities that arose in this classroom were typical exercises for practicing mathematics. The discussions and conversations that emerged were also about those exercises. The class talked about things such as solution methods,

[9] See appendix A for a diagrammatic presentation of the classroom where I did this research study.

difference in obtained answers and possibilities for solving problems that students generated. There were no assigned projects or assignments for either individual students or group of students. Such kind of activities rarely or never occur in Namibian mathematics classrooms, since the teaching is shaped around what is prescribed in the mathematics curriculum. So, what is outside the curriculum is considered as extra work or waste of time.

Data Creation

In this section, I outline the methods through which I created and analyzed the data for my study. Even though I know that this part of research is generally known as "data collection" I am hesitating to use that term for this part of my research. From most recent views on research, the researcher's role is understood as a process of making personal sense of experience with the world in order to construct knowledge (Tobin & Tippins, 1993). Tobin and Tippins view the term "data collection" as a problematic one because it

> ...implies that data are out there to be gathered up. As is often the case, the use of the collection metaphor can constrain thinking about actions associated with the process of data creation. From a constructivist perspective data are not collected, but are constructed from experience using personal theoretical frameworks that have greatest salience to the goals of the individual conducting the research. (p. 15)

The data were created in two forms. The primary data consisted of field notes that I took each day I sat in and observed the mathematics grade-11 classroom. Originally I intended to sit in this class at least once a week and especially during the lessons where the teacher was going to introduce a new topic or use group or whole-class discussions as her teaching strategy for the day. It turned out that Ndapewa, the grade 11 mathematics teacher, allowed me to sit in her class as often as I could and I managed to do that more than once a week, provided that the lessons I was to observe did not clash with my own lessons, for I was also teaching as a full time grade 12 Mathematics teacher, at the time of data creation. Each lesson that I observed was 45 minutes long or 90 minutes in case of double periods. After observing each lesson, I took the field notes with me and transcribed them as soon as I could, mostly in the afternoon of the same day that I did the observation. This helped me remember the moments, which I was transcribing, and fill in details of the class from memory.

The second form of data is in form of audiotapes that I recorded when the students were assigned group work or participated in discussions. Out of the seven weeks that I created data, only one lesson was used for group discussion/work, which was a double period. However, I was able to observe some discourse in ordinary lessons that were not designed for group work. Below is a chart (Table 2.) recording the number and types of lessons that I observed in this class. It also shows the length of the lessons in terms of minutes.

Week	# Observations	Topics covered	Nature of lesson
1	4	• Algebraic representation • Simplifying algebraic expressions • Solving linear equations • Making subject of a formula	Whole-class
2	1	• Making subject of a formula	Whole-class
3	2	• Factorizing quadratic equations • Difference of two squares	Whole-class
4	3	• Simplifying quadratic fractions • Variation • Solving equations	Whole-class
5	2	• Solving equations	Small groups
6	2	• Simultaneous equations	Whole-class
7	2	• Solving quadratic equations by the formula	Whole-class

Table 2: A record of the lessons observed

Role of the Researcher

I entered this classroom with permission and the intention to act as a participant-observer, but not an observer-participant. Moschkovic and Brenner

(2000) discussed differences between a participant-observer and an observer-participant. A participant-observer, which I was in this study, is a researcher who "observes and participates in activities without being identified as belonging to one of the social categories of the community being observed" (p. 476). In this sense, I was not present in this class to be identified as the teacher or one of the students, but rather, just a participant whose role, other than interacting with other participants, was also to make observations. The latter type of observer, which I was not—observer-participant—is "an observing participant, such as a teacher-researcher, [and] is a part of the classroom community" (p. 476).

Even though my participation was not to be as crucial as that of the teacher or the students, I believe that my presence played a role in Ndapewa's lessons each day I sat in her class. However, as will be revealed by the following discussions (of my interpretations) of some of these lessons, the circumstances of most of these lessons did not allow much verbal participation from me, let alone from the students, especially due to the fact that few small-group discussions/work or whole-class "talk" moments occurred.

Model for Observation

As I made these observation and tried to make sense of them, I used Simmt's (2000) model of how one may observe knowing in action. This model presents an extensive explanation of Simmt and Kieren's (1999) model of observing how the "co-recursive social interaction can lead to an expanded cognitive domain for individuals and the community in which they are observed" (p. 298).

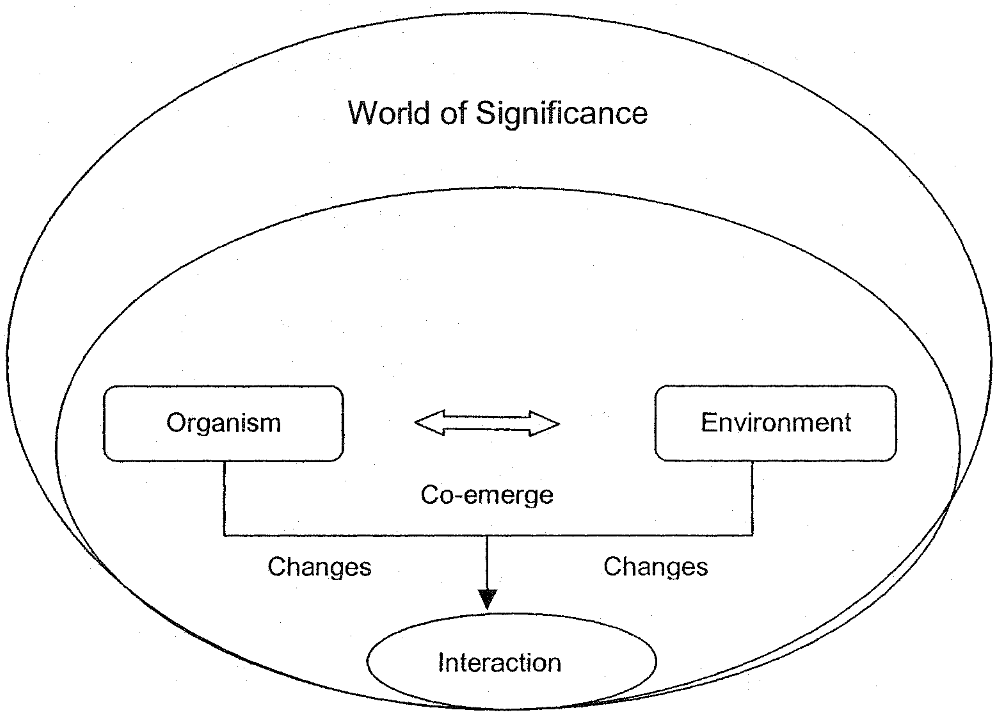

Figure 1: Simmt's (2000) model for observing human knowing in a mathematical community

During the interactions between a living being and its environment, both the individual and its environment bring forth a world of significance (Simmt, 2000). In this brought forth world of significance, the being and the environment co-emerge into different entities that can be distinguished by an observer (Maturana, 1978). This environment might include other living beings or non-living parts with which the individual act and interact.

In the classroom that was involved in my study, the worlds of significance brought forth by individual students—i.e. the interactions among individual students and between individual groups of students and the teacher—are the mathematical worlds that they had individually and socially constructed. Individuals interacted with each other, with the teacher, with the mathematics they brought forth, with the mathematics books, with the chalk and the utterances on the chalkboard, etc. All these together formed the environment in which the students and the teacher operated.

The environment or world with which the individuals interact is not an object or a process but rather it is more a "setting of and field for all of our experience, but one that cannot be found apart from our structure, behavior, and cognition" (Varela, Thompson & Rosch, 2001, p. 142). Therefore what an observer sees in or says about this environment, assert Varela et al, say as much about the observed as it does about its environment.

This is what Maturana and Varela (1992) call "structural coupling". Understood in these terms both the person and the individual or a group of individuals and the environment co-specify each other's worlds of significance. The individual is changed by his or her environment and at the same time he or she impacts that environment. This implies that the environment does not completely prescribe what happens to the individual being or a group of beings but both the individual and the environment determine what happens in and results from those interactions.

According to Simmt (2000) the stimuli in the environment act "as perturbations to trigger potential changes in both the organism and its environment" (117). When this happens, the organism and its environment are said to be structurally coupled for as long as the interactions between them persist (Reid, 1995). Once these interactions cease to exist then the structural coupling is no longer visible. Following Simmt's model of observing mathematics knowing in action, one is able to identify three sites for interactive knowing: "a person interacting with his or her own thoughts; two (or more) people interacting with each other, and a person interacting with the interaction of others" (2000, p. 135). This, Simmt advises, helps one to understand cognition in terms of the person-in-an-environment as promoted by the enactivist perspective.

Data Analyses and Interpretations

In the following section, I present the analyses and interpretations of the created data by going back to the transcripts that I created from the field notes and the audiotapes recorded during the classroom observations. I present discussions around several moments that emerge during these observed lessons—moments that mostly caught my interest because of "talk" that arose and/or the silent moments that communicated in this classroom.

In these analyses I look for different themes that emerged in the discourse and then offer my interpretations of those themes. These themes include Brainstorming, Discussions, Conversations, Explanations and Argumentations. I also present the pieces of transcripts taken from the classroom and discuss discourses that arose from the students' actions and interactions with one another and with the teacher and those that emerged into discussions and conversations as the students talked about the assigned tasks. To have multiple interpretations I have included others in interpreting the transcripts emerging from the study, such as my supervisor and fellow graduate students especially those whose work is also guided by enactivism.

Many moments arose in this classroom. I selected only a few. I picked upon some but not all moments. I picked up on the moments that emerged into discussions, conversations or arguments or those that presented possibilities, but for unknown reasons were not taken up, for discussions, conversations or debates. Therefore my discussions of such moments may not imply that they are of any relevance to the reader who might read this work, but to myself. I then leave it up to the reader to judge the relevance to them of these discussions (Moschkovic & Brenner, 2000). As I present each episode, I connect it to the literature that I reviewed as well as to my theoretical framework—enactivism.

In some of the cases that I present in this writing, I did not necessarily give credit to individuals who made particular statements since I view the whole classroom as one learning body, in which all its members are equally accountable for its success or/and failure. However, at some points I mentioned the names of the interacting persons, just for cohesive purposes.

Taken-as-shared Norms

While illustrating their views in constructivism, Cobb and Yackel (1996) discuss two types of norms that may be socially and interactively constituted by the teachers and the student in a mathematics classroom. These are the social norms and socio-mathematical norms. They also provide a distinction between these two types of norms. Social norms are norms that do not necessarily carry mathematical characters with them and they could emerge in any classroom despite of what subject content is being looked at, at the moment the norms are constituted. Whereas socio-mathematical norms are specific to the mathematical aspects of the activities done in class. All these norms do not necessarily have to be spelt out by the teacher or by the students. They may be observed in the social interactions from which they emerge, for they are not distinct from those interactions (Yackel, 2000).

A good example of social norms formed in my research was, for example, the taken-as-shared routines through which the lessons proceeded. The teacher introduced a mathematical idea and made some explanatory notes or examples on the chalkboard. As she did this, the students copied what she wrote into their notebooks. She would then follow this up with two or more exercises to be done on the chalkboard as examples through whole-class discussions. The teacher wrote the answers on the board as the students suggested what steps to be taken. A few more exercises were put up on the board for seatwork. When given time for answering had passed, individuals were called up onto the chalkboard to offer their answers, solution methods or solutions, which were then open to criticism in terms of being right or wrong. Extra work was then assigned as homework and this was usually reviewed at the beginning of the following day's lesson. As we can see here, even though this a general routine that lessons in Ndapewa class followed, it could as well take place in almost any classroom, let be it a social science class, chemistry class, language arts class or physics class.

According to Cobb and Yackel, socio-mathematical norms have to do with the decisions of "what counts as mathematically different, mathematically sophisticated, mathematically efficient, mathematically elegant in a mathematics classroom (1996, para. 11). One example of a socio-mathematical norm in terms of mathematical difference emerged in one of the lessons I observed where a boy called Toivo expressed that even though he had an answer to the homework question similar to everybody else's, his solution method was different from other students' and the teacher's method. Even though there were voices in the background saying that Toivo's method was no different, once the teacher confirmed that Toivo's method was quite different, these voices could not longer be heard.

Social and socio-mathematical norms not only enable interactive learning in a mathematics classroom, but also constrain it (Yackel, 2000). That is, they may lead to as well as inhibit learning possibilities for both the teacher and the students (Cobb & Yackel, 1996). In the discussions to follow the students and teacher negotiate many socio-mathematical norms among them as they discuss the mathematics of the lesson.

CHAPTER FIVE: FORMS OF WHOLE-CLASS DISCOURSES

This class interacted around a number of mathematical tasks that were put up on the chalkboard as examples, exercises and home work questions. In this chapter, I present, interpret and discuss the activities that arose in the class as I look at the interactions among the participants. These activities were structured around and structured whole-class discourses that provided for emergent discussions. From those interactions I created a few themes under which I discuss the discussions that rose in the class. These are brainstorming, discussions, explanations, evolving discourses, classroom rituals and preferring teacher-led whole-class exercises to group work.

Brainstorming

Even though I knew that I wanted to look into how mathematics classroom discourses emerge and evolve, I was not sure exactly what to look for in this mathematics classroom, especially during the very first lesson[10]. I therefore prepared myself to pay attention to any interactive features that might arise among the students and between Ndapewa—the teacher—and her students. It was not until I sat in Ndapewa's first class that I realized that apart from argumentation, debate, discussion, and conversation, another type of discourse that could arise in a mathematics classroom is brainstorming. According to the Merriam-Webster

[10] I am calling this the first lesson, because even if it is not the very first day of the school term, it is the first day that Ndapewa is teaching—presenting a planned lesson. The class spent the whole of the first week revising the previous exam papers. Even though I could sit in the class then, I could not collect any data, because I had not yet obtained written consent from the participants.

Online Dictionary, brainstorming is "a group problem-solving technique that involves the spontaneous contribution of ideas from all members of the group".

Some educators view brainstorming as one of the most effective teaching strategies because it enhances the students' creative and critical thinking skills as well as their social development (Church, 2001). One of the cognitive benefits of brainstorming is that students tend to come up with creative ideas that they would not have thought of on their own (Brown & Paulus, 2002). Brainstorming is considered as "one problem solving technique that has proven to be highly successful on all levels of learning". It can provide "a student the means by which early contact with peers becomes a stimulating and challenging experience" (Wood, 1970, p. 160). Having said this, the notion of brainstorming and its relationship to classroom discourse is not common in the discourse-related literature in mathematics education research.

Ndapewa began her first lesson of the term by posing a question to her class after writing the topic of the day (*Algebraic Representations*) on the chalkboard. This question initiated a brainstorming activity.[11]

> *[1]* *"Reading this topic, what do you think we will be doing?" (The teacher asked while pointing at the topic algebraic representation, that she wrote on the board)*
> *"I think we will do something about equations." One student suggested.*

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[11] In some of the cases that I present in this writing, I did not necessarily give credit to each individual who made particular statements since I view the whole classroom as one learning body, whose all its members are equally accountable for its success or/and its failure. However, at some points I mention the names of the interacting person, just for cohesive purposes. Therefore the discussions might look as if the teacher was having conversations and discussions with one or another learner, i.e. her class or individual pairs or groups of students.

"Algebra!" John shouted.

"We will be working with letters but not with numbers." Joel added.
"Right or wrong?[12]" Asked the teacher.
"Wrong!" The class responded.
"We will use both letters and...?"
"Numbers!"

"We will do substitution!" Steven called out.
"What do you mean?" The teacher asked him.
"I mean replacement."
"Replacement of what?"
"Of numbers by letters."

"I think we will do factorization." Asser had another idea.
"What does factorization have to do with?" The teacher probed.
"Finding the factor of the equation."
"Not necessarily the factor but...?"
"Common factor!"

"Transferring of like terms!" Olavi added.
"To where?"
"From one side of the equation to the other."

"Simplifying!" A girl who I could not get her name suggested.

In the above brainstorm, the discourse proceeded "according to student ideas and understanding" (Haroutunian-Gordon & Tartakoff, 1996, p.3) as their thoughts opened up possibilities for ideas to emerge and bump against one another (Davis & Simmt, 2003). Unlike lessons where the teacher may prepare a series of questions or ask question after question, here the teacher had only one planned question, "Reading this topic, what do you think we will be doing?" all other questions and comments that followed arose in response to what the students said. In a brainstorm, the nature and direction of a lesson is not pre-

[12] In a classroom where right and wrong answers are distinguishable, creative thoughts can be simply lost because a student might think that others would say that his or her ideas are of no significance (Wood, 1970).

determined by the teacher. Rather, ideas emerge "from the actions and language and interaction of the students and the subsequent actions and interactions of the teacher" (Kieren, 1999, p. 111). In lessons like this, where the students are allowed to demonstrate their mathematical beliefs we are able to observe "the relationship between the knower and what is known, [as students] put themselves in the position of the authors of ideas…— the teacher or the textbook [is not] the primary source of an idea's legitimacy" (Lampert, 1990, p. 34).

Emerging from this brainstorming lesson was an interaction around the topic of "algebra". As we can see from the transcript above [1]-[31], students' ideas kept popping up one after another until the teacher stopped taking further contributions and moved to the next part of the lesson. As interesting as the brainstorm was, rich discussions, conversations or arguments could have also emerged from these explorative interactions, had both the teacher and the students allowed so, at that moment. We can see for instance when John suggested that the topic is going to be all about "algebra". The question about what algebra is could have been posed here. This would have opened the possibility for John or other students to explain what they meant by algebra. Joel thought that the class would be dealing with "letters but not with numbers" [4]. Could there be a reason why Joel thought that emphasis should be given to the use of letters only but not numbers? Or what did he mean? And why is Joel's answer perceived to be wrong, not only by the teacher but also by the rest of the class? Of course in algebra we use both numbers and letters, but could Joel's statement "letters but not numbers" mean something? Here was another possibility for a mathematics conversation or discussion on considering: in what ways Joel's statement was right or wrong.

A discussion or conversation with relevant examples could have also emerged from Steven's suggestion that algebraic representation is about "substitution", which he defined as "the replacement…of numbers by letters". This idea takes me back to Joel's thought of "working with letters but not with numbers". Could Joel have understood the replacement of numbers by letters to mean that we use letters, instead of using numbers, in algebraic manipulation, as opposed to not using numbers at all?

After Asser [16] defined what he meant by factorization, the teacher made a distinction between finding just any factor and finding a common factor. For Asser, factorizing means "finding the factor of the equation". The teacher [17-18] added that the class was going to work in situations of finding "not necessarily the factor" of equations, as suggested by Asser, but finding the "common factor" of equations or rather expressions. Is there a difference between finding a factor and finding a common factor of two or more algebraic expressions? This could have also been discussed.

Discussions

Developing Language for Doing Mathematics

After the short brainstorm (still in the first lesson), the teacher pointed out to the class that it is important that they get used to algebraic terminology in order to be able to use the "correct mathematical language". The teacher wrote three terms on the chalkboard that she asked the class to discuss the similarities and differences between them. The terms were equation, numerical expression and algebraic expression.

"Are these the same?" The teacher asked after writing the terms—equation, numerical expression and algebraic expression, on the board.

"They are different." The majority of the class responded.
[25] *"How different are they? Efron?" Efron stood up but did not say anything, he smiled and sat back down, and the discourse continued.*

"In equation, we have the equal sign between the left side and the right side, but not in the expressions." Moses suggested.

"In numerical expressions, the terms are separated by the plus or minus signs." John added.
"They may also contain the multiplication (×) and the division (÷) signs". The teacher confirmed while writing the differences on the board. "And this also applies to algebraic expressions." She continued.

"Numerical expressions contain only numbers." Sara said.
[30] *"And what about algebraic expressions?" The teacher asked.*
"In algebraic expressions we have both letters and numbers."

In this talk about talking about mathematics (Yackel, 2000), the teacher and the students discussed and agreed on the different terminologies that they would be using in communicating their algebraic understanding during the course. In this way, a discussion emerged where they were being explicit about the vocabulary they would be using and their understandings of it: equation, numerical expression and algebraic expression. The teacher guided this sharing, through posing relevant questions that elicited students' ideas; this grew into a discussion. Ndapewa wanted not only to know how much her students knew about algebraic terms and to test if they could make sense of the mathematical language used in algebra but also to convey to them the "correct mathematical language" that they would use in algebra.

Even though Efron was one of the students who suggested that the three terms were different, he did not offer any difference among them when he was pointed at, rather he just stood up quietly, smiling at the teacher—perhaps indicating that he could not think of any difference at that moment. One interpretation of this exchange could be that Efron did not participate in this part of the lesson, since he only stood up but did not say anything. There is yet another way to interpret Efron's behavior—being in the class, shouting out that the three terms were different, standing up and smiling at the teacher before sitting back down—could have counted for as much as we can account for Moses' participation in the activity. Efron's silent action might have also motivated or permitted other students to share their ideas with the rest of the class by providing opportunities. Let us say for example that Efron had said something that might have been seen to be wrong, like in [5], either by the teacher or other students, how would this have affected other students' further participation?

Moses came to Efron's rescue [26] and mentioned that equations "have the equal sign between the left and right sides", which algebraic expressions do not have. John indicated that he too knew something about numerical expressions, i.e. "in numerical expressions, the terms are separated by the plus or minus sign" [27]. The teacher confirmed his idea and added that the terms may also be separated by the multiplication or the division signs. The fact that so far only boys had stood up to give their ideas disturbed me, but I felt better[13] when Sara joined the conversation by suggesting that "numerical expressions contain numbers only" whereas "in algebraic expressions we have both letters and numbers".

The discussion above shows how this class seemed to treat algebra as a set of "equations" and "expressions" that are separated by mathematical operations such as "plus", "minus", "multiplication" and "division" signs. Nowhere in this discourse did a discussion or conversation on the usefulness of algebra emerge, or even the question of what algebra is. Where does it originate? What is algebra? Who invented algebra, etc.?

Several authors (e.g. Fauvel, 1991, Russ, 1991, Bauersfeld, 1992) have argued that there is a need to discuss with the students some historical issues in mathematics. This becomes handy, especially when the class is beginning a new topic. This is not, however, to say that the whole session of Ndapewa's class could have been used up exploring the history of algebra but to point out the possibilities. Including the historical aspects in the lesson not only informs the students of where the concepts originate but also helps them "become aware of how the mathematics they study has developed" (Russ, 1991, p. 8). Being highly dependent on what the curriculum prescribes most teachers are reluctant to raise and discuss the content that is not mentioned in the curricular documents; in my experience this is especially true with respect to historical facts. It is therefore not

[13] I was somewhat concerned, that up to this point girls had not been verbally participating in the activity.

surprising that we do not see historical discussions on algebra emerging in this lesson.

Some teachers and educators think that it is an extra burden to consider developments in mathematics but what they do not realize is that the mathematical concepts and their applications that they teach their students are as much a product of historical processes (Russ, 1991). In fact, asserts, Russ, "[i]t can come as a surprise to realize that these ideas are no more 'fixed' now than they have been [in the past], and that we and our pupils are in the midst of their continuing evolution" (p. 7).

Doing Exercises Together

Ndapewa had her own way of doing mathematics together with her students. In fact, as she informed me in some of the conversations that I had with her, she prefers doing mathematics on the board with the whole class instead of assigning and facilitating group work or having lengthy whole-class discussions. From Ndapewa's point of view, group work/discussions require more time and they usually slow down the rate at which the curriculum is covered. Ndapewa gave her students something to do at home or at the hostel (homework), almost everyday. This, according to her, keeps them engaged with mathematics and helps them learn mathematics through problem solving rather than memorizing the notes and examples that she gives them in class. I was impressed not only because Ndapewa gave homework everyday but also to learn that, giving homework to students meant something to her. Some teachers hardly give homework or know why they should give homework to their mathematics students.

Solve:

$7(y-3)=4(3-y)$

Figure 2: An exercise on solving linear equations

The equation, $7(y-3)=4(3-y)$ (Fig. 2), was one of the exercises given in the homework the previous day. To describe the activities that this class worked on, I use the word exercise instead of "problem", unless if the participants themselves called it a problem. Simmt (2000) suggests that one does not, in advance, define an activity assigned to students as a problem. Rather, she suggests, we call them tasks or prompts that have "the potential to occasion [the students'] mathematical understanding [and] can trigger their actions" (p. 27).

Ndapewa re-wrote an exercise question from a previous activity on the chalkboard the following day so that the class could solve it "together".

> *"The first thing we could do here is...expand. What do we mean by expanding?" Ndapewa began the discussion with a question. "Opening the brackets." The majority of the class responded.*
>
> *April volunteered to suggest what would be obtained after expanding.*
> *"Seven y minus twenty one is equal to twelve minus four y."*
> *The teacher wrote on the board* $7y-21=12-4y$
>
> [35] *"Sonia, what do we do next?"*
> *"We collect like terms. Seven y plus four y is equal to twelve plus twenty one* ($7y+4y=12+21$)*."*

"Foibe, what do we do from here?"
"You have to add them together."
"We have to simplify. Let's do that."
[40] *"Eleven y equals to..."*
"Thirty three." About four students said out almost at the same time, while the teacher writes $11y = 33$.

"Hiskiel, what is y?"
"y is three."
"y is equal to thirty three divided by eleven. So, y is equal to three." The teacher confirmed as she wrote on the chalkboard:
$y = \frac{33}{11} \Rightarrow y = 3$.

As we can see from the above transcript, instead of providing the answer to the students, to the previous homework exercise, the teacher let the class work on it by engaging almost the whole class. However, there is a difference between this activity and the brainstorming lesson [1] – [31]. In this case, a student had to be first pointed at or called on by the teacher in order to take part in this activity, whereas in the case of brainstorming any student was given a chance to give his or her opinion.

Right after the majority of the class defined what was meant by expanding an equation as "opening the brackets", the teacher began selecting who should respond to particular questions. Sonia was the first to be given a chance to suggest what should be done next, after April volunteered to give the first step of "opening the brackets" of the given equation. Collecting like terms was, according to Sonia, the next step. One of the rules in solving any algebraic equation is that, whatever one does to the left side of the equation, one does the same thing to the right side of the same equation. This rule of equivalence (O'Rode, 2002), I observed not only in the students in Ndapewa's class but in my own classes too with new students from grade 11, is not paid attention to. Solving algebraic equations is simply taken as a routine that involves collecting like terms together (O'Rode, 2002). It makes me wonder if students really understand what they are supposed to be doing. It was interesting for me to observe that Ndapewa, the teacher, did not seem to have any problem with such a statement, "collect the like terms" made by these students, without challenging Sonia's and the class' understanding of solving algebraic equations.

In contrast to brainstorming where students offered ideas at their own whim, in this activity where the teacher chose respondents by pointing to students, more girls participated. On one hand this pointing at the students who should participate is an advantage for it allows the quiet students take part including girls who are generally believed to be not willing to participate. On the other hand, it is disadvantageous because many students may not freely participate because they would be waiting for the teacher to point at them. As the students gave suggestions of what could be done, the teacher put these up on the chalkboard. In this case, even though the students seemed to be playing the "boss" here in that they told the teacher what to do, the teacher still had the power over who should suggest what to be done. In Cobb's (1999) terms, there was lack of distributed authority across the class.

While trying to involve almost all the student, unlike in the brainstorming, the teacher practiced total control over when and to whom the authority should be given. In the brainstorming lesson, authority was evenly distributed across the whole class, since a student did not need to wait to be pointed at before he or she gave his or her ideas. Rather, they would just call out whatever they had in mind, and then that would be taken as either right or wrong [5]. As we could see the ideas kept popping up after the teacher posed a question of what they thought would be studied just by looking at the topic written on the board. "I think we will do something about equations", "algebra", "we will be working with letters but not with numbers", "we will do substitution", "transferring like terms", "simplifying", were some of the ideas that kept recurrently coming up as the class proceeded.

On the other hand, in this current episode, it is not the question that is posed first, but rather, an individual is first selected and then told what she was expected to contribute to the activity. A teacher's choice to shape a classroom discourse in such ways could be either or both mathematical and social (Lampert, Rittenhouse & Crumbaugh, 1996). I found this strategy interesting in that I think on one hand it is probably a better way for the teacher to ensure that the students who have not said anything for the day could also take part in the activity. Only some individuals would have participated in the discourse, should the lesson have been an open brainstorm. It also makes me realize that one needs not give total control to the students all the time for some students might dominate the discourses while other students' opinions are neglected. It is therefore important that there is a balance between teacher and student authority in a mathematics classroom (Cobb, Yackel, Wood, 1992). But on the other hand it could also serve as an inhibitor of students' willing to verbally participate in the lesson. Those students who had ideas to share could not do so for they were not pointed at or prompted to give their thoughts.

How different is different?

After it was agreed that "y is three" and everyone agreed that they got the same answer in their homework, Toivo spoke up and expressed that even though he got the same answer, his solution took a route different from what the teacher put up on the chalkboard.

"I have a different method from that one."

[45] *"Please come and show us how you did yours."*

Without explaining, Toivo wrote his solution on the chalkboard as follows:

$$y(7+4)=12+21$$

$$y=\frac{12+21}{7+4}$$

$$y=\frac{33}{11}=3$$

While Toivo was writing his solution on the board, I could hear voices in the background saying that:

"It is just the same. There is no difference."

"Well done. It is quite a different method but it leads us to the same correct answer."

In the above excerpt we see Toivo perceiving his solution method as different from what was previously done on the board. It was not necessarily clear whether everyone else, apart from Toivo, followed the same procedures as presented by the teacher on the board, but he was the only one who announced that his method was different from the rest of the class'. Upon invitation to go to the chalkboard and show the class how he went about his solution, he just picked a piece of chalk and began writing his solution on the board, without saying anything. For Toivo, it was probably not necessary for him to explain what he was doing. But how different is his method from the one already on the chalkboard? "It is the same. There is no difference", was what some students reacted with as soon as Toivo's work became visible to them. Yet the teacher congratulated Toivo and assured him that his method was "quite different but it leads us to the same question".

Here is another possibility of discourse, which could have evolved into a debate of how mathematically different Toivo's solution method was (Cobb, Yackel, Wood, 1992). This concern emerged among the students whose voices could be heard in the background who did not see a difference between Toivo's method and their methods or the one that the teacher wrote on the chalkboard. Were they satisfied when the teacher confirmed that the two methods were different? Toivo could have perceived the difference in what a person had to do when solving this particular problem. If we look at Foibe's method, she had to add the terms together first in order to obtain $11y = 33$. On the other hand, Toivo factored the left side of the equation into $y(7+4) = 12 + 21$. Understanding the rules of solving equations, he divided both sides by $7+4$, which finally led him to the answer. But if we look at method done on the chalkboard, $7y$ was added to $4y$ resulting in $11y$. Hiskiel must have then divided both sides by 11 to get "y is three".

Funnelling for a strategy

Another case where different methods of solving a similar problem emerged was on an occasion when the class was busy with "making the subject of a formula". The class was at the moment working with the formula $T = 2\pi\sqrt{\frac{d}{y}}$ (Fig. 3), to make y the subject of the formula.

Make y a subject of the formula

$$T = 2\pi\sqrt{\frac{d}{y}}$$

Figure 3: An exercise on making y the subject of a formula

"We are interested in y, isn't? What do we do next? Penny!"
"Square both sides."

[50] *"We have to get rid off something first. What is it?"*
"2π!" Many students responded at one time.
The teacher did this (getting rid off 2π) on the board by dividing both sides by 2π and obtained $\frac{T}{2\pi} = \sqrt{\frac{d}{y}}$.

Following the common instructional pattern of the class—that is, to work out an example and put up some exercise questions to be done by the class as a whole with one person (mostly the teacher) writing the suggestions on the board—the teacher asked Penny what could be done first in order to solve the given equation. Penny suggested that we "square both sides" first. Since this did not conform to the teacher's expectation, the teacher discarded Penny's suggestion and called for something different: “We have to get rid of something first”. In this sense the teacher indicates to the class that she knows what should be done first but it is not what Penny says. Such a response shuts down the possibilities for further genuine discussions or conversation where Penny’s proposition and other student-led work would have taken the class.

Even though the teacher knew that “we must first get rid of something” she did not tell the class it what was—the thing that should be gotten rid off. She, instead, asked the class to figure out what it was. Here, it appears, the teacher was not listening *to* Penny but rather *for* a particular response (Davis, 1996).

Davis (1996) talks about two kinds of listening in a mathematics classroom: evaluative listening and interpretive listening. In evaluative listening, the teacher does not “seem to be listening *to* learners, but listening *for* something in particular” (p. 106). The latter type of listening is when the teacher is “compelled to listen *to* the students…and not merely *for* particular responses” (p. 109). An example of interpretive listening is when the teacher listens to the students as they try to answer a question that she or he or the class posed and that she does not know the answer to. This kind of listening demands active participation from both the speaker and the listener.

Should the teacher have listened to Penny, what route would have the class taken in solving this equation? In the next transcript, one student brought Penny’s method back to the discussion, which eventually the whole class preferred. After both sides were squared and the result was written down, the teacher asked the class what should be done next.

"What do we do next?"
"Square both sides." The class, in a chorus, said repeating what Penny suggested in the beginning.
Just after the teacher wrote the result of squaring "both sides", which was $\left(\frac{T}{2\pi}\right)^2 = \frac{d}{y}$, *Asser raised his hand.*

"Asser, are you okay?" The teacher asked him.
"I don't know how to get $\frac{d}{y}$*."*
"Do you remember the common rule of indices? What do we do when we square a square root of something? The square root of d over y [writing $\sqrt{\frac{d}{y}}$*] means d over y [writing* $\frac{d}{y}$ *on a separate part of the chalkboard] to the power..?"*
"d over y to the power a half." The class, including Asser, responded at once.
"Therefore the square root of d over y squared is the same as..."
"d over y to the power half squared". The teacher wrote on the board $\left(\frac{d}{y}\right)^{\frac{1}{2}} \times \left(\frac{d}{y}\right)^{\frac{1}{2}}$ *and continued explaining to Asser in a form of a question:*
"When the bases are the same, what do we do with the powers?"
"We add them together." Here the whole class responded instead. I didn't quite hear whether Asser took part in the choir this time. However, after the teacher wrote the final steps that led to $\frac{d}{y}$*, Asser confirmed,*
"It is clear now", in response to the teacher's direct look at him.

Many interactive patterns of this kind occur occurred in Ndapewa's class. This type of pattern matches with what Bauersfeld (1988) called funnelling. In "funnel pattern of interaction", he asserts, the teacher guides the students through the activity toward his or her desired answers.

> First the teacher recognizes a student with difficulties...The teacher opens with a short question in order to stimulate self correction...[When the teacher] receives an unsatisfactory reaction [she or her] goes further back

> to collect clear and prerequisites for the insight aiming at an adequate reaction from the student…[After] deterioration has come down to a simplest exacting recitation…the pattern terminates with the presentation of the solution, independently from who produces it. (p. 36)

In the example above, the teacher funnelled the students' responses right from the beginning to the end of the interaction. The first example is when the teacher rejected Penny's suggestion and directed the class to a different line of thinking when she suggested that they rather "get rid off something" instead of "squaring both sides" of the equation. The second example is when Asser raised a concern about how to obtain $\frac{d}{y}$. The teacher then guided Asser through the activity by posing leading questions that had finally led to the final answer and he confirmed, "It is okay now". The teacher's question "Do you remember the rules of indices?" enabled Asser to connect the current problem to the previous lessons on "indices". The funnelling went on to reminding Asser what it means to square a square root of something.

After the funnelling Asser's understanding, the class then returned to the stage it was working on before Asser intervened.

The teacher multiplied both sides of the equation by y and obtained: $\left(\frac{T}{2\pi}\right)y = d$. Toivo had the question of difference again.

> *"Would my answer be still the same if I put* y *on the left side of the term?" Meaning if he wrote his equation this way:* $y\left(\frac{T}{2\pi}\right) = d$.
> *"It will still give you the same thing." The teacher replied and carried on with what she was explaining. "We want to make* y *the subject of the formula, what do we do next?"*

[65] *"We multiply both sides by the reciprocal." The majority suggested.*
"The reciprocal of what?"
"Of (t over two pi) squared."

Since I was already solving the problem in my field notebook differently from squaring both sides by the reciprocal, I decided to carry on while listening and following what the class was doing. This is what I had written down, after squaring both sides.

$$T = 2\pi\sqrt{\frac{d}{y}} \Rightarrow \frac{T}{2\pi} = \sqrt{\frac{d}{y}} \Rightarrow \left(\frac{T}{2\pi}\right)^2 = \frac{d}{y} \Rightarrow y\left(\frac{T}{2\pi}\right)^2 = d \Rightarrow y = d \div \left(\frac{T}{2\pi}\right)^2 \Rightarrow y = \frac{4d\pi^2}{T^2}$$

From Toivo's perspective (see [44]), one could say that the teacher's method was quite different from the one I had. The solution put on the board was as follows:

$$\left(\frac{2\pi}{T}\right)^2 \times \left(\frac{T}{2\pi}\right)^2 y = d\left(\frac{2\pi}{T}\right)^2 \Rightarrow y = d\left(\frac{2\pi}{T}\right)^2$$

"Some of you went ahead and simplified to $y = d \times \frac{4\pi^2}{T^2} = \frac{d4\pi^2}{T^2}$*, which is also okay." The teacher confirmed with class.*
Just when the class was ready to move to the next problem, Steven raised his hand up and suggested something similar to what I did.
"What if I divided both sides by t over two pi (squared) instead of finding the reciprocal?"
[70] *"Does that mean you would leave your answer like this [writing* $\frac{d}{\left(T/2\pi\right)^2}$*]?"*
"Yes." Steven affirmed and the teacher assured him that it is also okay as long as it gave him the same answer that the class got.
"Whoever finds the first method difficult or too long, you can follow Steven's method."

Better Methods

In most of my classes that I taught mathematics, I had always experienced cases where students asked me to provide them with the easiest or the shortest methods of solving mathematical problems. Ndapewa's class was no exception. There were also those who liked to have shortcuts or quick recipes to solving problems. Before the class moved on to the exercise after the discovery of Steven's method for making y the subject of the formula $T = 2\pi\sqrt{\frac{d}{y}}$, Olavi expressed that

"There is no need to get rid of the 2π [referring to the step taken in [50-51]".
"What do you suggest we do, Olavi?"
[75] *"We square both sides".*
"Let's do that and see what we get."
The teacher did what Olavi instructed, on the chalkboard as follows.

$$\left(T^2\right)=\left(2\pi\sqrt{\frac{d}{y}}\right)^2=\frac{4\pi^2 d}{y}\Rightarrow T^2 y=4\pi^2 d\Rightarrow y=\frac{4\pi^2 d}{T^2}$$

"Ahaa! That one is better." The majority of the students exclaimed.
"So, do you prefer Olavi's method?" The teacher asked them
"Yes!" The teacher labeled the three methods A (whole-class' method), B (Steven's method) and C (Olavi's method) respectively and told the class that they could follow any of the methods that they think is easier or shorter for them.

This same solution method was provided by Penny in the beginning of solving the formula, but was rejected or rather put on hold by the teacher when she said that class should first get rid of something—before they square both sides. After the teacher's route was followed, the class, through Olavi, realized that the teacher's method was rather long. Olavi proposed that there was no need to get rid of anything as the teacher prescribed. Rather, Olavi thought that we could just "square both sides" right away. Here we can see the consequence that may result in a teacher's rejection or shutting down students' thoughts and just focus on what she or he expects to hear. After all, it may turn out that what the teacher offers does not present better chances for learning than the students' own ideas.

Explanations

When people in any conversational interaction try to offer explanations to one another as they try to understand the mathematics they are doing, such explanations may be referred as to explanations in action (Gordon-Calvert, 2001). This could also be interpreted as what Simmt (2000) calls "mathematics knowing in action". In the classroom for example, as the class tried to work out the exercises, the individuals provided explanations to each other as they made sense of that mathematics. Not only do we explain situations because we are obliged to do so but because it is of human nature to live by explanations. Human beings, asserts Maturana (1988), "like to explain and to ask questions that demand an explanation for their answer. Furthermore, if we are in the mood to ask a question that demands an explanation, we become pacified only when we find an explanatory answer to our questions" (par. 8). A mathematics classroom discourse therefore is not only important in talking about mathematics but also in facilitating mathematical explanations that the participants offer to one another (Simmt, Glanfield & Sookochoff, 2000).

In one of the observed lessons, the class was still dealing with 'making a subject of the formula as the main topic and the teacher wrote two of the exercise questions that were explored on a previous day. The class could only work on one of them before the bell rang. This, $a\left(\sqrt{\frac{y^2-n}{m}}\right)=\frac{a^2}{b}$, was the problem of the day. Again we had to make y the subject of the formula. In collaboration with the students, the teacher completed the first step on the chalkboard by squaring both sides of the formula $\left(a\sqrt{\left(\frac{y^2-n}{m}\right)}\right)^2=\left(\frac{a^2}{b}\right)^2$ (Fig. 4).

Make y a subject of the formula

$$\left(a\sqrt{\left(\frac{y^2-n}{m}\right)}\right)^2=\left(\frac{a^2}{b}\right)^2$$

Figure 4: The second exercise on making y the subject of the a formula

That resulted in $a^2\left(\frac{y^2-n}{m}\right)=\frac{a^4}{b^2}$.

[80] *"From here now, we want to get rid of m. How do we do that?"*
"Multiply both sides by m." The students replied.
The teacher multiplied both sides by m and obtained

$$a^2y^2-a^2n=\frac{a^2m}{b^2}.$$

"What is next?"
"You have to remove b squared."
"Do I have to?"
[85] *"No"*

The teacher put up an exercise on the board that the class was to work out together as she wrote the students' suggested answers on the chalkboard. In this case, she began by informing the class what the goal was: "From here now, we want to get rid of m" [80]. This kind of communication, where the conveyor uses the word "we" to convey something that he or she personally prefers, is common in mathematical text (written or oral). When the word "we" is used in such tones, the speaker is actually telling the listener of his or her personal opinion and is imposing that onto the listeners into agreeing with the speaker so that the goal or intention as viewed as mutual one (Rowland, 1999).

The Teacher's Need to Explain

After it was agreed that "we" did not have to remove b^2 as one student suggested, a brief conversation transpired between the teacher and Lisa.

"I am a bit confused." Lisa said.
"Eh heh?"
"I don't know where the other (a) came from (meaning the second (a) in $a^2y^2 - a^2n$ of part [81])."

"Lets look at that again (writing $a^2(y^2 - n)$). How would you simplify that one?"
[90] "Expand."
"Meaning?"
"Open the brackets. Oh! Now I got it."

"What do we do next?"
"Collect like terms."
[95] "What do you mean?"
"I mean, take everything without y to the other side."
The teacher did that and wrote the result on the board:

$$a^2y^2 = \frac{a^4m}{b^2} + a^2n.$$

"Next?"
"We remove a squared." The whole class responded.

"How?"

[100] *"By dividing by a squared."*

Writing on the chalkboard, the teacher worked out this step as follows: $\frac{1}{a^2} \times a^2 y^2 = \frac{1}{a^2} \times \frac{a^4 m}{b^2} + \frac{1}{a^2} \times a^2 n \Rightarrow y^2 = \frac{a^2 m}{b^2} + n$.

In this conversation, the teacher performed what Davis (1996) calls attentive listening. The teacher attentively listed to Lisa's difficulty in following what was going on at that moment. She was confused and the teacher showed an interest in listening to where the confusion was. "Eh heh?" she responded to Lisa. And when Lisa pointed out what was confusing her, the teacher politely took Lisa back to the genesis of the solution method in order to show her how "the other (a)" was obtained. After rewriting the expression ($a^2(y^2 - n)$) the teacher asked Lisa how one would simplify it. Lisa suggested she could "expand". The teacher asked her what the "meaning" of expanding was. She meant, "opening the brackets". The conversation continued with this idea through as the teacher prompted Lisa with a couple of questions, Lisa was quick to announce that "now [she] got it".

Just after Lisa "got it", another student raised a concern about how $+a^2 n$ ended up on the right hand side of the equation.

> *"How come we have positive (a) squared (n) [$+a^2 n$] on the right side [of the equation]?"*
>
> *"When we took negative (a) squared (n) [$-a^2 n$] from the left side to the right side, the negative sign changed to the positive sign." The teacher explained to the student and went ahead continuing from [100].*
>
> *"That is the same as (writing* $y^2 = \frac{1}{a^2} \times \frac{a^4 m}{b^2} + \frac{1}{a^2} \times \frac{a^2 n}{1}$*). If we divide a squared into a to the power four, what do we get?"*
>
> *"a squared." The class answered.*

[105] *"So, (writing* $y^2 = \frac{a^2m}{b^2} +$*). This one (pointing at* $\frac{1}{a^2} \times \frac{a^2}{1}$*) cancels and I get...?"*

"Only n*." The following is the few steps written down after this was agreed upon.*

$$y^2 = \frac{a^2m}{b^2} + n \Rightarrow \sqrt{y^2} = \sqrt{\frac{a^2m}{b^2} + n} \Rightarrow y = \pm\sqrt{\frac{a^2m+n}{b^2}} or \pm\sqrt{\frac{a^2m}{b^2} + \frac{n}{1}}$$

Quickly Brian raised his hand and asked:

"If I am making y the subject, can I not leave it with the square (meaning to leave the answer as $y^2 = \frac{a^2m}{b^2} + n$*)?"*

"No. The highest power you can leave on your subject is 1."

"Ms., it is no use to put one below en (meaning $\frac{n}{1}$ *part of the final answer in [106])." Suggested Toivo.*

[110] *"But sometimes you may be asked to leave your answer as a fraction. Then in that case, you have to leave it as* $\frac{n}{1}$.

These student-generated questions kept the conversations going hand in hand between the teacher and at least one student. The students posed questions about "how" the exercises were being done and the teacher felt a responsibility to explain to the students. Interesting here is that the students were not only concerned about the right answer but also about how to do mathematics. For example Lisa became confused and the teacher attentively went back through the activity and assisted her in understanding how a in the second part of the expression $a^2y^2 - a^2n$, was obtained. She also explained to another student who did not know how $(+a^2n)$ ended up on the right side of the equation. However, this explanation was provided in such a way that $\left(-a^2n\right)$ was simply taken from one side to the other side of the equation. These illustrate ways in which mathematics was treated as a set of fixed procedures. Taking symbols from one side of the equation to the other might be viewed as a procedural explanation and would likely not be considered as insightful or profound mathematical discussion or conversation.

Counter Examples

In the preceding discussion, the teacher offered explanations to the students' questions. In attempting to show the students why the methods worked the way they did, the teacher often connected the activities to the content covered before, which led them to understand the emergent explanations. In the following section, the teacher offers examples and counter examples in order to explain to the students figure out how to obtain the solutions.

After the class worked on the given exercise and obtained an answer, $y^2 = \frac{a^2m}{b^2} + n \Rightarrow \sqrt{y^2} = \sqrt{\frac{a^2m}{b^2} + n} \Rightarrow y = \pm\sqrt{\frac{a^2m+n}{b^2}} or \pm\sqrt{\frac{a^2m}{b^2} + \frac{n}{1}}$, Patrick was interested in knowing why $y = \pm\sqrt{\frac{a^2m}{b^2} + \frac{n}{1}}$ could not be simplified further by squaring both sides of the equation.

> *"Ms., why did you not square the other side?" Asked Patrick*
> *"Which side?"*
>
> *Lisa tried to rephrase Patrick's question and said*
> *"Why is (y) not equal to (am) over (b) plus (n), since the square root of (a) squared is (a) and that of (b) squared is (b)?"*
>
> *"Let's have an example of that and see (writing* $\sqrt{\frac{2^2}{9} + 5} = \sqrt{y^2}$ *).*
> *Can we find the square root of that?"*

[115] *"What if it was like this?" Asked Simon, while writing on the board* $y = \sqrt{\frac{a^2m^2}{b^2} + n^2}$.

"Still we cannot find it."

"Why?" Asked Steven.

"Only if it was like this (writing $\sqrt{\left(\frac{a^2m^2 + n^2}{b^2}\right)^2}$ *). Then we can find the square root."*

"The answer to that would be $\frac{am}{b} + n$*." One student suggested.*

[120] *"Are you sure? The teacher asked. Lets give* a, m, b *and* n *some value and see how that works. Let (writing* $a = 2, b = 4, m = 6$ *and* $n = 9$*) for example...lets substitute and see whether it gives us the same answer. That would then be (writing* $\sqrt{\frac{a^2m^2}{b^2} + n^2} = \frac{am}{b} + n$*). So, we are saying that...(while writing and saying out the following on the board.)"*

$$\sqrt{\frac{2^2 \times 6^2}{4^2} + 9^2} \stackrel{?}{=} \frac{2 \times 6}{4} + 9$$

$$\sqrt{\frac{4 \times 36}{16} + 81} \stackrel{?}{=} \frac{12}{4} + 9$$

$$\sqrt{\frac{144}{16} + 81} \stackrel{?}{=} 3 + 9$$

$$\sqrt{9 + 81} \stackrel{?}{=} 12$$

$$\sqrt{90} \stackrel{?}{=} 12$$

$$9.49 \stackrel{?}{=} 12$$

"Does that work?"

"No! Hahaha!" The class responded.

"Lets see if the second one works." Said the teacher as she began to write the following on the chalkboard.

LHS (short for Left hand side) $\stackrel{?}{=}$ *RHS (for right hand side)*

$$\sqrt{\left(\frac{a^2m^2}{b^2}+n^2\right)^2} \stackrel{?}{=} \frac{a^2m^2}{b^2}+n^2$$

$$\sqrt{(90)^2} \stackrel{?}{=} 90$$

$$90 = 90$$

[125] *"Ahaa! Hehehe!" The class responded with the Aha! expression. "Now we can see that the other suggestion does not work and this one does."*

Even though only one problem was dealt with during this lesson, until the bell rang, I appreciated this class activity. I just liked the discussions that emerged from this single exercise. This was quite a great moment where students behaved mathematically. Even though it began with students answering the teacher's intended questions as usual, it became more interactive and generative especially after the teacher wrote down the solution and Patrick posed a question. Patrick wanted to know why it was not necessary or possible to find the square root of the expression $\left(\frac{a^2m}{b^2}+\frac{n}{1}\right)$. Instead of explaining why so, the teacher gave a related example where she assigned to constant values to the variables, $\sqrt{\frac{2^2}{9}+5}=\sqrt{y^2}$ and asked if it was possible to determine $\sqrt{\frac{2^2}{9}+5}$. This was not discussed and Simon suggested another example. Was the teacher's counterexample not sufficient in explaining to Simon and the rest of the class why one could not find a square root of $\left(\frac{a^2m}{b^2}+\frac{n}{1}\right)$?

Building on Maturana's notion of validations of explanations, Gordon Calvert (2001) discusses how a community of mathematics practice may not take up an offered explanation, should it be insufficient in meeting that particular community's criteria for "acceptability". "Acceptance of an explanation is not made through judgments of right or wrong, correct or incorrect. Instead, explanations at the time they are offered are accepted if they are viable and informative given the possibilities and constraints of the immediate situation" (p. 64). "What if it was like this" (Simon suggested instead, while writing $y = \sqrt{\frac{a^2m^2}{b^2} + n^2}$ on the chalkboard)? Would it be possible to find that square root? Even thought the counterexample provided by the teacher could be clear enough in indicating how it was impossible to show a square root of the concerned fraction, it appeared not to have been sufficient or viable in explaining to the students why that is so.

When Simon gave a familiar example and asked if one could find its square root, again, the teacher assured him that "still" it is not possible to find that square root? But, why…Steven wanted to know too. Again, the teacher did not explain why it would not be possible to find the square root of the expression but rather she gave another example, which would be possible to factorize, from her point of view: "Only if it was like this". The teacher asserted while writing $\sqrt{\left(\frac{a^2m^2 + n^2}{b^2}\right)^2}$ on the board. One student thought the answer to that would obviously be $\frac{am}{b} + n$. To check if this is so, the teacher took another example by giving the variables involved some values and plugging them into the equation suggested by this student to see if the right and left sides of the equation would weigh the same, which they did not [120]. When she used the rules of squares and roots, applying the same values of a, b, m and n, the two sides of the equation equalled [124]. Interesting in this episode, is the way the students were interpreting the notion of finding a square root of an algebraic fraction. For example where one student suggested that the square root of $\left(\frac{a^2m^2+n^2}{b^2}\right)^2$ would be $\frac{am}{b}+n$, she seem to be finding the square root of every term separately.

In this episode, we can see a combination of different teacher-student interactions, from posing questions to one another to challenging each other's actions. At the beginning of the interaction, after writing the problem on the board the teacher suggested that "we want to get rid of (m)." The teacher is stating a goal not as her own goal but as that of the whole class by beginning her statement with the word "we". But the question was "how do we do that?" How do we get rid off that (m)? The class told her to "multiply both sides by (m)", which she did and then asked what was next. Here we can see how the teacher involved the whole class in making a decision about what the interest of the group was and about what could be done before she wrote anything on the board. With the majority following what is going on, Lisa got "a bit confused" because she did not know how a appeared in the second part of the left side of the equation ([81]-87). It did not take Lisa too long to work her understanding out when the teacher took her back to the "rules of indices", as we can hear in her expression "Oh! Now I got it."

The exercise went on with almost everyone taking part in a choir-like response, until one student shared with class his problem of understanding how (positive) $+an$ ended up on the right hand side of the equation. The teacher explained this action as taking negative an or $(-an)$ from the left side of the equation to the right side and changing its sign to a positive. No attention was paid to the logic of adding an to both sides of the equation, for example, as one of the rules of solving algebraic equations. Brian also wanted to know whether he could not leave his final answer in the form of y^2 equals to something. The teacher pointed out to him and the rest of the class that the highest power one could leave on his or her subject of a formula is 1.

I was just reminded of a student (Toivo) who likes closely analyzing differences among mathematical methods when he stood up and suggested that is rather useless to represent n as $\frac{n}{1}$. "Ms., it is no use to put one below en", was his statement. However, at this point, the teacher made a connection to situations that students might find themselves, for example answering tests or exam questions, that "sometimes you may be asked to leave your answer as a fraction", then it would probably be useful "to put one below an" in such cases.

Student as Teacher

During the fourth week of the first month into the trimester, an in-service teacher, Kauna, doing her teaching practice, took over the teaching of my project class[14]. The episode below is from the third day after she began working with this class. Kauna just began a new topic because Ndapewa had already completed the topic that the class was recently busy with, i.e. solving linear equations. The class was now dealing with 'solving quadratic equations'. Below are the exercises that the class, as a whole, factorized: (Fig. 5)

1. $x^2 - 8x + 15$
2. $x^2 + x - 6$
3. $3x^2 + 11x + 6$
4. $5x^2 - 9x - 2$

Figure 5: Whole-class exercises on factorization

Even thought the title written on the board was "Solving quadratic equations", the exercises that the teacher put on the board were not equations but just quadratic expression. I will however use the term equations to indicate what the participants where communicating.

As an example, the in-service teacher started solving the first exercise ($x^2 - 8x + 15$) on the chalkboard and posed a question to the whole class.

> *"What do we use when we factorize quadratic equations?"*
> *"Trial and error!" The class responded.*

[14] Kauna is a mathematics teacher, who, was upgrading her teaching qualification through a particular adult learning programme and was visiting the school for her teaching practice. Written consent was first obtained from this teacher before I went back to the class to collect data from her lessons. Throughout this document, I referred to her as the in-service teacher to make a distinction between her and Ndapewa, the classroom teacher.

"We have to find the factors of what?"

[130] *"x squared and fifteen."*

"The factors of x squared are x and x (writing $x^2 \to \begin{matrix} x \\ x \end{matrix}$*) and the factors of fifteen are three and five or one and fifteen (writing* $15 \to \begin{matrix} 3 \\ 5 \end{matrix} \to \begin{matrix} 1 \\ 15 \end{matrix}$*)".*

"Which factors do we take?"

"Three and five." The in-service teacher wrote as the final answer $(x-3)(x-5)$*, while assuring the class that the two constants have to be both negative so that their product is positive and the sum of their cross multiplied products will be* $-8x$*. She then asked for a volunteer to work out the next exercise.*

Having not been in the class for the first two days of Kauna's teaching, I was not aware of what was done in class previously. Her beginning question in this excerpt shows that the class had already discussed the methods that are used in solving quadratic equations. Her question "What do we use to solve quadratic equations?" was confidently answered that we use "trial and error." I could also not expect anyone in class to ask what trial error was, because this might have been discussed the day before. Even the way the class began doing the activities, was such that they already knew what to do next. Kauna, the in-service teacher, asked, "We have to find the factors of what?" and the class responded "x squared and fifteen". Even the way the class negotiated over which factors should be used did not raise any confused concerned, which is an indication that the students knew what was going on—that is, taking factors that are both negative so that their product yields to positive fifteen and their sum to negative eight.

I was amazed by the way the students were participating in this discourse. It has only been two days since Kauna had been teaching them and they were already comfortable with doing the exercises with her. The responses were even loud and clear. What I forgot to ask Ndapewa, the usual teacher, was whether this was the first time Kauna did her teaching practice at the school or whether it was a continuous process that she had been doing with the same class from before.

Next, the in-service teacher asked for volunteers to work on the subsequent exercises.

> *Simon volunteered to do the second exercise ($x^2 + x - 6$) (See Fig. 5). Simon began with his plan and an explanation of how is he is going to execute the plan.*
>
> *"I have to find two factors of -6. One must be negative and the other one must be positive so that their sum can give us $+x$."*
>
> [135] *"Show us". Responded the class.*
> *"I have 3 and -2, so (writing $(x+3)(x-2)$), this is the answer."*
> *"Do you all agree?"*
> *"Yes!"*
>
> *The next person to volunteer for number three ($3x^2 + 11x + 6$) was Efron. He presented his explanation in a manner similar to Simon's.*
>
> *"We have to find the factors of this (pointing at 6) and that (pointing at $3x^2$). We can have x and three x for three x squared (writing $3x^2 \rightarrow \begin{matrix} x \\ 3x \end{matrix}$). And in this case (pointing at 6) we have three and two, or six and one for six (writing $6 \rightarrow \begin{matrix} 3 \\ 2 \end{matrix} \rightarrow \begin{matrix} 6 \\ 1 \end{matrix}$)".*
>
> *Next Efron wrote $3x \times 3 = 9x$ and $x \times 2 = 2x$, as a result of a cross-multiplication among the factors.*
>
> [140] *"And their sum will be (writing $9x + 2x = 11x$) eleven x. So this one is correct (pointing at part $6 \rightarrow \begin{matrix} 3 \\ 2 \end{matrix}$ of the set of his factors). Therefore this (writing) $(3x+2)(x+3)$ is the correct answer. Isn't [it]" (He asked jokingly)?*
>
> *"Yes!" The class responded while laughing.*

Students were even volunteering to be the "teacher of the day" so they could go to the chalkboard to offer answers to the given exercises. They also knew that they had to explain their methods and their thinking by giving reasons why they did what they were doing. This can be seen in Simon's explanation. "I have to find two factors of -6. One must be negative and the other one must be positive so that their sum can give us $+x$." That was his first step, finding the two factors of the number -6. He also understood that he must consider the signs that these two factors should have in order for him to get the sum of $+x$ when he cross-multiplied and added the products together.

The students also seemed to have understood the notion of doing mathematics together as the whole class, the way their teacher and the in-service teacher do it. This could be heard in their statements beginning with the word "we". Look at for example in line [136] when Efron suggested, "we have to find the factors of this…and that…. In this case we have three and two, or six and one for six". He, like the teacher, tried to get everyone to conform everyone to his thinking and assume that they all agree to what he is doing.

Efron also seemed to know how to act as a teacher when he jokingly asked the class "therefore this… is the correct answer. Isn't [it]?" The students also seemed to know how to treat Efron as if he was their teacher when they responded with a "Yes". However, Simon did not receive this treatment when he went to the board to offer his solution. First he told the class that he was going to "to find two factors of -6. One must be negative and the other one must be positive so that their sum can give us $+x$." While laughing some students treated Simon with some intimidating voices saying "show us". This did not discourage him though. He went on and provided an answer to his exercise, which, the class agreed, was correct. The class carried on with the activities when the teacher asked for another volunteer to work out the last exercise (in Fig. 5).

> *"Now we need a volunteer to do number four (Maria shot her arm in the air.). Maria!" The teacher requested.*
>
> *Maria got up and walked toward the chalkboard offering to factorize ($5x^2 - 9x - 2$).*
>
> *She began*
> *"We are looking for factors of $5x^2$ and -2."*
> *Next, without saying anything, she wrote $5x^2 \rightarrow \begin{matrix} 5x \\ x \end{matrix}$ and $-2 \rightarrow \begin{matrix} 2 \\ 1 \end{matrix}$.*
> *"And we cross-multiply".*
> *Maria cross-multiplied the factors ($5x$ by 1 and x by 2) and got $5x$ and $2x$.*
> *"First I will make two x negative because it is the smaller one."*
> *She wrote $5x - 2x = 3x$.*
> [145] *"This is not correct because their sum must be negative nine x."*
> *Maria decided to change her multiplication order by multiplying $5x$ by 2 and then x by 1, which gave her $10x$ and x.*
> *"Now I will make ten x negative so that I can get a negative number when I add them together (meaning $-10x + x$)." She wrote down her addition process as $x - 10x = -9x$ and concluded that $(5x+1)(x-2)$ is the correct answer. Just as Maria was moving away from the chalkboard back to her seat, Paulus raised a question.*
>
> *"Does it matter how one arranges the terms? For example, will it be different if we change the solution to problem number three to (he got up to the board and wrote $(3x+3)(x+2)$)?" The solution previously given in number three, by Efron, was $(3x+2)(x+3)$.*
>
> *The in-service teacher multiplied out what Paulus suggested by opening the brackets and got $3x^2 + 9x + 6$.*
> *"You can see that this is not the same as the original expression (pointing at the expression in number three, which is $3x^2 + 11x + 6x$). There is a big difference".*

Okay, do you want to do more exercises or do you want to continue with something new?"
"Do more exercises."

Before the end of this lesson, I was already making conclusions by comparing Kauna's class to Ndapewa's, even though it was the same group of students. In Kauna's lessons students were actively engaged in the activities. Most of them volunteered but not pointed to. Also, the girls seemed courageous enough to participate. However, the teaching is as much a procedural act as it is in Ndapewa's lessons. The in-service teacher provided an example and put up a series of exercises to be worked on, while following the familiar example. Another way that the in-service teacher took the students into consideration, as Ndapewa also does, was when she asked them if they wanted to do more exercises of if they wanted to "continue with something new". The students wanted to "do more exercises".

The following exercises were put up on the chalkboard for individual work. Individual students were granted five minutes to do the activity as seatwork.

Factorize the following:

1. $2x^2 + 5x + 3$
2. $120x^2 + 67x - 5$
3. $12x^2 + 4x - 14$

Figure 6: A set of individual exercises on factorization

When the given time had passed the in-service teacher called for a volunteer to give an answer to the first exercise. Ruben offered an opportunity to go to the board and present his solution. He wrote down $\begin{matrix} 2x & 3 \\ x & 1 \end{matrix}$.

"These are the factors of $2x^2$ *and* 3*". He said pointing at what he wrote. He then went ahead to write* $(2x+3)(x+1)$ *as his solution.*
"Is he correct?"
"Yes!"
"Do we have a volunteer for number two?"

Nobody responded to the in-service teacher's question, so she granted two more minutes for the class to finish, for they still looked busy working. A minute passed and the bell rang. Despite the bell, the in-service teacher went ahead and offered a solution to the second exercise.

"What are the factors of one-twenty?"
"Eight and fifteen."
"And for 5?"
"Five and one."

By this time many students were already making movements showing that they have to leave the class. This was another type of non-verbal communication that occurred in this classroom. Students' movement and noise making was a clear alert to themselves and to the teacher that it was time to drop mathematics and move onto the next lesson. The teacher continued with the solution with very few students responding and the majority picking up their bags.

"Therefore, the right answer is (writing $(8x-5)(15x+11)$*). I will see you tomorrow. Please take problem number three for homework."*

"I don't think that is correct." Simon said as he picked up his school bag. Probably the in-service teacher could not hear him because of the noise in the class; and even if other students could hear him, everybody was rushing to be the first at the biology laboratory and reserve a good seat for him or herself.

Student Explanation of Methods

The need to explain is part of the human nature (Maturana, 1988). In a mathematics classroom, students' curiosity calls for explanations from the teacher. The WHY and HOW questions posed by the students to either the teacher or other students are not simply for intimidating purposes but for the students' effort to understand various mathematics concepts and processes.

Current views on mathematical explanations have shifted the responsibility for explanation from the teacher to the student (Gordon-Calvert, 2001). When a student, for example, offers a solution to a given problem, it is not the teacher that is bound to explaining what the student has done and why she or he has done so. Rather, the student has the responsibility of explaining the solution methods that she or he used.

A systematic explanation can be seen in Maria's work as she solved problem number four above. She first explained what was needed in order to solve the assigned problem. "We are looking for factors of $5x^2$ and -2", was her suggestion, and she wrote down the factors she had in mind. She was also happy to accept that the answer she got with her first choice of factors was wrong and was brave enough to try another strategy, we can see in her utterances in line [142] "This is not correct because their sum must be -9", which then gave her $3x$.

Evolving Discourses

After solving the problem that was given for homework the previous day, the in-service teacher wrote a new topic to explore on the chalkboard, "Removing the brackets", and put up an example $(a-b)(a+b)$.

> [160] *"We all know how to remove[15] brackets. Who can do this for us?" Rehabiam volunteered to go to the board and offer the answer, which he wrote as follows.*
> $(a-b)(a+b)=a^2+ab-ab-b^2$.
> *"Thank you. Did you all follow? Is that correct?"*
> *"Yes." The class responded.*
> *"We call this answer the difference between two squares. If you have two brackets with the same terms but different signs the answer is called...the difference of two squares (pointing at Rehabiam's answer on the chalkboard.)"*

"Factorizing a difference of two squares" was the next subtopic the in-service teacher wrote on the chalkboard.

> *"Factorizing the difference of two squares is just a reverse of removing the brackets. Let's say for example reverse what Rehabiam gave us. We have a squared minus b squared. So, to reverse this we will have...?"*
> [165] *"a minus b times a plus b."*
>
> *"Let's have another example."*
>
> *The in-service teacher wrote on the chalkboard,* $9a^2-25b^2=(3a-5b)(3a+5b)$.
>
> *"It's easy, isn't?"*
> *"Yes! It is."*

[15] In Ndapewa's lessons, this was termed "opening the brackets" or "expanding". The students were probably used to both terms, opening brackets and removing the brackets because they did not show an indication of not knowing what the teacher meant when she wrote, "removing the brackets".

Even though there is not much to say based on the transcript above, it provides the basis for the next discussion on the debate that arose in the class about the factorability of an algebraic expression.

One of the foci of my study was to look at how mathematics classroom discourses evolve. Mathematical discourses emerge and evolve. In fact mathematics itself may be viewed as an evolving phenomenon (Gordon-Calvert, 2001). Any discourse first emerges with the interactions of the participants in the classroom—that is, it arises or occurs—and then it may or may not be taken up for a discussion or conversation. Some of these emerging discourses may be taken up for discussions and others may not—we may say they die. Analyzing this feature of the mathematics classroom enables me to interact with the students' thoughts since the "evolution of mathematical themes…seems to correspond to students' cognitive development" (Yackel, 1995). The following episode shows how a discourse on factorizing emerged and how it evolved.

The class just finished factoring a difference of two squares, $9a^2 - 25b^2$. The discourse did not stop just there it evolved. This evolution took place when Nakale expressed how he wondered what would happen if the expression to be factorized was like this $9a^2 + 25b^2$.

The freedom in this class for students to change the mathematical constraints or parameters in the activities allowed for explorative learning. More than two cases had occurred so far where students changed the original equation or algebraic expression and posed the "what if questions". For example when Simon was concerned about finding the square root of $\frac{a^2m^2}{b^2} + n^2$, which originated from the solution obtained when the class made y the subject of the formula $a\left(\sqrt{\frac{y^2-n}{m}}=\frac{a^2b}{\,}\right)$ whose solution was $y=\pm\sqrt{\frac{a^2m}{b^2}+\frac{n}{1}}$. Such a learning environment that allows for the emergence of open talks can help the students begin related discourses and hence be able to "unveil ascriptions and interpretations" (Bauersfeld, 1992, p. 25).

> *"What if there is a positive sign?" Asked Nakale "I mean if we have* $9a^2+25b^2$ *?"*
> *"Is it possible to factorize that?" The in-service teacher posed the question back to the class.*
> *"No!" Exclaimed the majority of the students, but Simon had a different thought.*
> *"Yes." He said.*
>
> *"Come and show us on the board." The in-service teacher suggested.*
> *Simon wrote,* $3a(3a)+5a(5a)$.
> *"Do you call that factorizing? It is impossible."*
> *Simon went back to his seat, smiling. But still Ruben insisted that this expression was factorizable. He was called to the chalkboard to provide his solution. As Ruben walked to the board, Steven made an interesting comment "Under this topic it is not possible, but in reality it is possible."*
> *Ruben went ahead and wrote his suggested solution on the board,* $(3a)^2+(5b)^2$. *He stood still staring at his work.*
>
> *"You better stop." Some of the boys in the background said to Ruben.*
> *"Uh, you don't know anything". He responded back.*
> *As an indication that he really better stop, the in-service teacher extended her hand out to Simon taking the chalk back from him, even thought he was still indicating that he could factorize this expression* $(9a^2+25b^2)$ *further.*

The action of taking away the chalk from Ruben—a nonverbal but loud and clear communication from the teacher to Ruben was also of some significance in this activity. This action was a way of shutting down Ruben's attempt to factorize the expression and to make his argument clear why he says that what he was doing was factoring. Instead of giving the student the opportunity to explain and justify his claim, the teacher asked him to stop. On the one hand, he was given the opportunity and with the other hand it was taken away by the teacher. The in-service teacher's action of extending out her arm to take away the chalk from Ruben was an indication that he "better stop" what he was doing. After taking the chalk away from Ruben, the in-service teacher announced that,

> *"We can't factorize this, can we?"*
> *"No." The class answered.*

As Simon walked back to his seat, I could still hear Steven insisting that

> *"Under this topic, it is not possible but in reality it is possible to factorize that".*

What is factorizing?

In the above episode, the in-service teacher and a couple of students seem to have different conceptions of what factorizing is—at least under the topic of factorizing a difference of two squares. Nakale asked how one would factorize the polynomial $9a^2 - 25b^2$, if the sign between the two monomials were positive. "What if there is a positive sign? I mean if we have $9a^2 + 25b^2$?" was his question. The in-service teacher wondered "is it possible to factorize that?" Some students felt that it was not possible, but Simon believed that it was possible to factorize the expression. He proudly went to the chalkboard and wrote down the factors as $3a(3a) + 5a(5a)$.

The in-service teacher's question "do you call that factorizing" was of significance here. On one hand it was a question intended to shut down this line of thinking—the thinking that one could factorize $9a^2 + 25b^2$—where the class could argue about the factorizability of this expression offered by Nakale. On the other hand, it could be (and was) a trigger to us (class participants) to consider the question of what factorizing means. Ruben was also convinced that one can factorize the expression, $9a^2 + 25b^2$, and he wrote the factors $(3a)^2 + (5b)^2$ on the board, as his solution. Could Simon and Ruben have understood factorizing (or to factor) an expression to mean resolving such an expression into its simpler factors? It seems so, for they partially factorized each term of the expression— $9a^2$ into $3a \times 3a$ and $25b^2$ into $5a \times 5a$ (Simon's solution) or $(3a)^2 + (5a)^2$ (Ruben's solution). This also took me back to the very first lesson in Ndapewa's class where the class was brainstormed to discuss their ideas about algebra. When Asser suggested that algebra was about factorizing, he was also prompted to define what he meant by the term factorizing. He said it means, "finding the factor", and the teacher added that it means, "finding a common factor". Therefore in this episode, Simon and Ruben were acting a world that was currently brought forth in the previous lessons where the meaning of factorization was taken as common understanding. It was this world of significance that shaped their understanding into arguing that what they were doing was factorizing (Kieren, 1999, Simmt, 2000).

This case presented one opportunity from which the teacher could learn something from the students, something not only about how to define factorizing but also about the students' way of thinking and reasoning. If we are to speak of collective understanding in terms of co-emergence and self-determinism, then "the teacher may be prompted to probe her role as both a learning member of [the collective] and as a special catalyst in it" (Kieren & Simmt, 2002, p. 873). In Maturana's view this was a possibility for the teacher to conform to the students' understanding instead of just expecting students to conform to her thinking. Maturana (1988) writes:

> If we attend to what we do in daily life where we answer a question with a discourse that is accepted by a listener as an explanation, we may notice two things: a) that what we do is to propose a reformulation of a particular situation of our praxis of living; and b) that our reformulation of our praxis of living is accepted by the listener as a reformulation of his or her praxis of living. (par. 9)

In the example above, the teacher expects the students especially the two boys (Ruben and Simon) to reformulate their conception of factorization to hers but she would not reformulate hers to theirs. While Ruben and Simon might have understood factorizing in these terms—that to factor a term or a number is to reduce it into its simplest terms—, the in-service teacher seemed to have wanted the factorization of the whole binomial, not a partial one. It also appears that the teacher needed something that would work under the discussion of "a difference of two squares". So, since this was not the case, the conclusion was that "it is impossible", to factorize the expression. One could have also factorized this differently if one were to argue for Simon and Ruben's interactions. For example, would one call this,

$$9a^2 + 25b^2 = 3 \times 3 \times a \times a + 5 \times 5 \times b \times b = 3(3 \times a \times a) + 5(5 \times b \times b) = \ldots,$$

factorizing, from the in-service teacher's point of view? May be the in-service teacher would only be satisfied with this factorization if both the two terms ended up with a common factor.

Another question that occurred to me was that of the reality of what we do in the mathematics classroom. Steven, on the other hand, despite Simon's and Ruben's fall in trying to factorize the above expression, kept insisting that in the case the class was dealing with at the moment, it was not possible to factorize the expression $9a^2 + 25b^2$ "but in real life it is possible". Here Steven chose to disagree (Lampert, Rittenhouse & Crumbaugh, 1996) to what the class had concluded and yet he was ready to argue why he would say one could factorize the given expression. To Steven there was more to factorizing than what they were doing in this lesson. There was real life, apart from this kind of factorization. If we were in a real life situation, according to him, one would be able to factorize the expression. Is real life what Simon and Ruben were doing then? Or is there another real life apart from that? Not hearing what Steven suggested or probably ignoring it, the in-service teacher and the rest of the class agreed that the expression is not factorizable.

Classroom Rituals and Routines

The following episode is from one of Ndapewa's lessons after taking back her class after the in-service teacher, Kauna, had left. The core of this week's lessons was to simplify algebraic fractions that the class had been dealing with for about two days, after Kauna's departure. These activities involved finding the lowest common multiples of algebraic fractions before the students were able to simplify them.

To begin the lesson, the teacher wrote an exercise (Fig. 6) on the chalkboard that, as usual, she asked individual volunteers to work out on the chalkboard.

Simplify

$$\frac{2(x+2)}{x^2+4x-5}-\frac{1}{x+5}$$

Figure 7: An exercise on simplifying algebraic fractions

"You always have to check both the numerator and denominator and see if you can factorize them further. Who can factorize this for us (pointing to the give problem)? Elena!"

Elena decided to factorize the denominator of the first term, x^2+4x-5 *to* $(x+5)(x-1)$. *This gave her* $\frac{2(x+2)}{(x+5)(x-1)}-\frac{1}{(x+5)}$ *and the teacher thanked her, taking back the chalk, as an indication that she could stop there.*

[190] *"Who can do the next step? (David volunteered to) David, how do we find the lowest common multiple of the two fractions?"* *"Since they both have x+5 underneath, x+5 should be there and then add the rest". (He said while writing* $\frac{\quad}{(x+5)}$*).*

He wrote $\frac{2(x+2)-(x-1)}{(x+5)(x-1)}$, *and continued explaining.*

"So, if we simplify the top, we're gonna have (writing $2x+4-x+1$*). So we end up..." (writing* $\frac{(x+5)}{(x+5)(x-1)}$*).*

David went back to his seat.
"Is there anything else we can do?" The teacher asked.
"Yes!" Responded the class.

"What can we do?"
"Cancel."

The teacher cancelled and obtained $\frac{\cancel{(x+5)}}{\cancel{(x+5)}(x-1)}=\frac{1}{x-1}$.

She then wrote six more problems on the chalkboard and asked the students choose what to do with them.

"Choose any three that we can do together and you do the rest alone."
Class chose 5, 6 and 7, which were as follows:

5. $\frac{2x+1}{5}-\frac{x+2}{2}$, *6.* $\frac{1}{8x}+\frac{1}{4x}$ *and* *7.* $\frac{5}{x-2}+\frac{3}{x+3}$.

For exercise number 5, Brian was asked to mention the lowest common multiple of 5 and 2, which he said was
"Ten."

The teacher then wrote this down and simplified
$\frac{2(2x+1)-5(x+2)}{10}=\frac{(4x+2)-5x-10}{10}=\frac{-x-8}{10}=\frac{-(x+8)}{10}$.

Nakale intervened quickly:
"I have a question. How do I know where to put the brackets?"
"Because we are multiplying the whole term with a number, so we put it in brackets."
The class moved to exercise number 6, $\frac{1}{8x}+\frac{1}{4x}$.
"What is the lowest common multiple of eight x *and four* x*?"*

"Eight x*."*
"Are you sure?"
"Yes."
"I think you are wrong."
"No!" The teacher smiled as she as she teased the students and wrote $\frac{1+2}{8x}$ *as the next step, and one student, Aletta, contributed by saying:*
"I think that is $2x$*." Meaning the 2 in numerator 1+2.*

The teacher simplified the fraction on the board to $\frac{\overset{2}{\cancel{8}}x}{\underset{1}{\cancel{4}}x} = 2$ *and confirmed*

"It is 2 only." The final answer obtained was then $\frac{3}{8x}$.

When the class moved on to number 7, $\frac{5}{x-2}+\frac{3}{x+3}$, *the teacher asked:*

"Kawana, what is the lowest common multiple of $x-2$ *and* $x+3$*?"*

"$x-2$ *times* $x+3$*."*

The teacher then wrote the lowest common multiple given by Kawana as a denominator, $\frac{\quad}{(x-2)(x+3)}$.

"Joel, what is this lowest common factor (pointing at the denominator $(x-2)(x+3)$*) divided by* $x-2$ *" (Meaning* $\frac{(x-2)(x+3)}{x-2}$*)?*

[210] *"*$x+1$*." Responded Kawana.*

"Where did you get $x+1$*? If I have* x *minus two, times* x *plus three, divided by* x *minus two...?" (The teacher reminded him as while writing* $\frac{(x-2)(x+3)}{x-2}$ *on the sides of the chalkboard). "What do I do?" She asked.*

"Cancel $x-2$ *[on top] with* $x-2$ *below."*

"What do we get?"

"$x+3$*."*

Teacher finally wrote $\frac{5(x+3)+3(x-2)}{(x-2)(x+3)}$ *as the result and continued her questioning.*

In this discussion, it is actually the teacher who was doing the activity. She told the class what steps were to be taken and then asked them how such steps could be taken. For example when it was time to do exercise number 6. $\frac{1}{8x}+\frac{1}{4x}$,

it was not the students' idea to find the lowest common multiple of $8x$ and $4x$ but the teacher's. This was suggested by her question that was directed to the class: "What is the lowest common multiple of eight x and four x?" The students then responded by giving the lowest common multiple as "Eight x." This generated a brief playful conversation in which the teacher probed the students by asking if they were "sure?". With enough confidence they responded, with a loud "Yes!". The teacher then teased them further with a doubt: "I think you are wrong." The students insisted that "No!" they were not wrong. This same procedure was

[215] *"April, can I cancel this and that?" (pointing at the terms on the denominator and the numerator respectively).*
"No."

"Why not?"
"Because the top is addition and below is a multiplication."
"What do we do?"

The whole class responded that we should "expand the brackets."

After 'expanding the brackets' the teacher obtained $\frac{5x+15+3x-6}{(x-2)(x+3)}$.

[220] *"Olavi, can we simplify the top?"*
"$8x$ plus 9."

The fraction was then written as $\frac{8x+9}{(x-2)(x+3)}$, *and the teacher went ahead asking:*
"Ruben, can we simplify further?"
"Yes."

"Are you sure?"

[225] "No."

"Why can we not simplify further?"

To this the teacher answered her own question that it could not be simplified further "because the numerator and the denominator do not have anything in common."

Many of this classroom's lessons took place through almost a fixed routine. The excerpt above is one of the characteristic examples of daily routines that were visible in most of Ndapewa's classes. The class begins with a problem that is solved as an example, by involving the whole class, and then assigning more problems as individual work. Even though the students were engaged in the problem-solving processes, they only do so because the teacher demanded their involvement. The teacher puts a question on the chalkboard, she poses a question on how the problem should be tackled, and an individual either volunteers to offer a suggestion or is pointed at whether he or she rose a hand or not. The individual student who offers to contribute does so in a time specified by the teacher too. Once the teacher is satisfied by what the student had offered she takes away the chalk and calls on the next student. In such traditional classes, few opportunities, for students to actively engage in generating the mathematical knowledge, occur. Students were, for example, involved by suggesting to the teacher what should be done in order to solve particular problems, but this was done due to the teacher's invitation. Also, student generated questions about mathematics rarely occurred. And when they did, they were only taken up if those questions supported the intended outcomes of the teacher's lesson. Classroom rituals can "become much more full and satisfying when there are occasional moments of complete and full attention, producing moments which can be re-entered, savoured, and used to inform future practice" (Mason, 2002, p. 28).

In *A Mind So Rare,* Donald (2001) discusses how rituals and routines of the learning process play a role in human development. The key step in learning, he asserts, is the interlocking of the students' attentive behaviors with those of other students as well as that of the teacher. Physical movement, mental and verbal participation are some of the activities that frequently occur in a mathematics classroom, for example, and all these may be implicated in recognizable recursive routines that the teacher and the students may engage in. Furthermore, these routines need to be mutual in such a way that the students are the most dynamic members of the classroom culture. If we look at most of the excerpt transcript presented in this work, the teacher's actions are much more prevalent than that those of the students.

Eliminating Inspection

After the class has completed the topic of algebraic fractions, the students were given 10 minutes to work out a couple of exercise questions that were put on the chalkboard for individual work. While students were busy working on the exercise, the teacher wrote notes (Fig. 7) on the chalkboard, to introduce the next topic, 'variation'.

VARIATION
Direct Variation
- x varies as y
- x varies directly as y
- x is proportional to y
All of the above means the same and are written in symbols as:
$x \alpha y$, where "α" sign can be replaced by "$=k$", and k is a constant: x =

Figure 8: Notes on Variation

After the students were done with the exercise and the teacher reread the notes above to them she gave them an example of how to use variation.

> *y varies with z, and y = 2 when z = 5, find:*
> *the value of y when z = 6*
> *the value of z when y = 5.*
>
> *Lisa was the one who volunteered to solve this on the board. While she wrote on the board* $\begin{matrix} z=5 \\ y=6 \end{matrix}$, *she murmured:*
> *"You cross multiply". Immediately the teacher asked*
> *"Is she correct?"*
> *"No". Responded the class.*
>
> *"Kaleb, tell us." Kaleb did not say anything and the teacher asked Paulus.*

Just while Lisa was still trying to make sense of the question in order to solve for the two variables, the teacher stopped her with the question she posed to the class. "Is she correct?" In most of the exercise done so far in this class, the students were given a chance to explain their solution methods and final answers before they were announced right or wrong. In this case, Lisa did not get such a chance. Her work was declared wrong before she could even explain why she did what she was doing. The class' decision that Lisa's work was wrong called for another volunteer when the teacher asked Kaleb for an idea. Kaleb could not provide anything and Paulus was pointed at to say what he was thinking.

"Paulus, what do you think?"

[230] *"May be when the value of y is 2, the value of z is 5, then may be when the value of z is 6 then the value of y is 2.4".*

"Let us not use the method of inspection here." The teacher said while pointing at Nakale in order to give him a chance to contribute.

"You have to put $y = kx$ first"
Teacher wrote on the board as she agreed with Nakale: $y = kz$ which she changed into $2 = 5k$, after substituting the values for y and z.

"So, what is k?" The teacher asked Nakale and asked him to keep talking.
"Go ahead".

[235] *"k is 2 divided by 5"*
The teacher wrote the equation on the board to show the next step by substituting the value for k into the equation. This resulted in

$y = kz = \frac{2}{5} \times z$, *and z is 6.*

i.e. $y = \frac{12}{5} = 2\frac{2}{5}$ *or 2.4.*

It is not clear here why the teacher chose to change the way she allowed the students take part in this activity. Yet I find what I observed in this interaction to be interesting. First Lisa (a girl) volunteered to do the exercise. Her work was perceived to be wrong and she went back to her seat. This time the teacher did not ask for volunteers but she pointed at individual students. The teacher asked Kaleb to take part even though he did not show the willingness to. Kaleb did not have anything to say and Paulus was pointed at next. Paulus suspected that "maybe when the value of y is 2, the value of z is 5, then maybe when the value of z is 6 then the value of y is 2.4". The teacher declined this "method of inspection" and called on another student to enlighten the class on what could be done. Nakale had his hand up and he was given a chance to give an idea. Notice here that since the first girl volunteered to go to the board to work out the exercise the teacher stopped allowing volunteering students to go onto the board. As well, no more girls took part. Three male students were pointed at; only one of which was able to provide not only the "right" answer but also suggested an answer that the teacher was satisfied with.

The answer provided by Nakale, $y = 2.4$, was exactly the same as the one that was suggested by Paulus. Not considering in what ways Paulus' answer was correct the teacher suggested that no "method of inspection" was to be used. I am not sure if the teacher or any of the students realised that the answers are all the same because no connection was made after Nakale's answer was obtained. Should Paulus have been given a chance to explain his answer and how he obtained it, it might have generated an interesting discussion especially about different ways of solving the problem.

Sometimes, even though students' solution methods lead to the answer that is expected, we, as teachers restrict them to believing that there is only one fixed method that is right. In such cases the teacher uses his or her power to reduce the students' authority in deciding what methods they would use in doing mathematics.

Preferring Teacher-led Whole-class exercises to Group Work

In one of our conversations Ndapewa and I discussed how each of us enjoyed the lesson on algebraic representations. She specifically pointed out that she likes using brainstorming especially when beginning a new topic. For Ndapewa, brainstorming was perhaps only for opening new topics. However, as suggested in the literature, brainstorming can be used anytime during the lesson even if it is not of a new topic. It could also be used as a problem solving strategy (Wood, 1970). As well, brainstorming is another way teachers and students can openly talk about a mathematics concept, even though they are not solving any problem, but just talking about mathematics and their experiences with mathematics.

Brainstorming, according to Ndapewa, helps her students gain self-confidence in participating in future classroom discussions. It also "allows me to know how much they already know about the new topic, so that I know where to go from there and how fast or slow I must carry on", she said. Ndapewa also mentioned that she does not like telling the students what will be done in every lesson because she wants them to "find out things for themselves", or remember things that they have done in the previous grades so that they are able to connect that content to the new content.

As much as Ndapewa claimed to like brainstorming in her lessons, especially when she is beginning a new topic, there was only once brainstorming clearly occurred in her class—or at least for the lessons that I observed. This is not to say students' ideas did not emerge in subsequent lessons, but that they did not do so as in brainstorming but as responses to the teacher's questions. This is not to blame Ndapewa that she did not use brainstorming. It is not only the teacher who is responsible for motivating discussions to occur but the whole class as an organ is. Yet, the teacher has some degree of power and authority of inviting issues to discussions that are potential for brainstorming. On the other hand, speaking from a teacher's point of view, I know that sometimes a teacher may ask a question with the intention to have discussions about it, but she or he may not get desirable responses from the class. In this case, then, Ndapewa could not have possibly forced brainstorming to occur.

This chapter was around the interactive discussions that emerged from the interactions between individual students or the class as a whole and the teacher as they did activities together. Questions put up on the chalkboard provided for opportunities and possibilities for whole-class discussions. As evident in the transcript most of the discussions were shaped, enabled and constrained by the teacher or the in-service teacher's responses to the students' actions and reactions toward the activities.

The following chapter presents another way in which the students engaged with the mathematical activities by talking about the work amongst themselves in small groups. The students' interactions and the interactions between individual groups and the teacher helped me create the themes under which I make sense of those interactions. Those themes are such as student mathematical interpretations, what goes on in a small group, better method, students' problems in algebra, on-task off-task conversations, ignoring some members of the group and group dynamics and common understanding.

CHAPTER SIX: SMALL GROUP DISCOURSES

Conversations

The whole-class discourses discussed in the preceding chapter emerged from the teacher's interactions with the class as she answered given questions on the chalkboard together with the students. Some activities were also done by having individual students volunteer to go to the board to offer their solutions and explanations. In these discourses, conversations, or what Gordon-Calvert (2001) would call genuine conversations rarely occurred. Most of the interactive 'talk' behaviors that emerged in those lessons are better referred to as discussions. We might also wonder to what extent we might classify emergent understanding as mathematical or problem solving because most of the talk was more about the symbols, signs and steps that were required to work though algebraic algorithms and procedures. And almost all of those activities were in response to teacher demands and questions. The whole class looked at one or more mathematical concepts with the teacher deciding most of the directions that were taken as well as judging the correctness and wrongness of the questions. In the current chapter, I look at the interactive talk that emerged among some students as they worked on mathematical activities in small groups and to a large extent without the direction or questions of the teacher.

One day the teacher, Ndapewa, decided to assign group work to the class. She made this decision in response to believing that I was not satisfied by what I got from her lessons so far. I assured her that she must not feel obliged to change her teaching plans because of my project, since I wanted the project to be as natural as possible—i.e. taking place in natural classroom activities as the teacher planned without any pressure from me—as it would have been should I have not been present. In spite of my assurance for her not to feel obliged to assign group work, she insisted that from reading the transcripts that I had been sharing with her, my project "was not working".

As it turned out, the next mathematics period was a double one, that is, 90 minutes long. This was ample time for students to do exercises together in small group and so the teacher took advantage of it and assigned group work (Fig. 9).

Find x, when the area of the rectangle shown exceeds the area of the square by $2cm^2$.

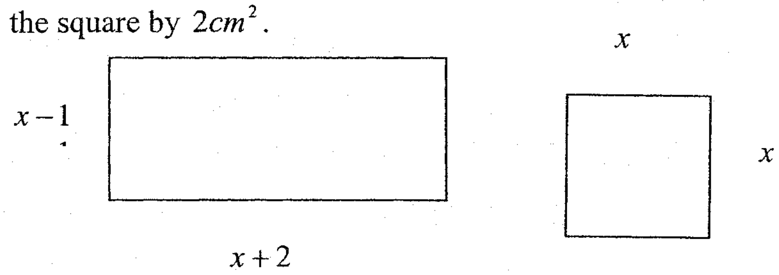


Figure 9: Applying Geometry to Algebra

The following episode is from the group at the table that I sat during group work—the group of students F, G, H and I (see appendix A), while I have audiotapes recording two more groups—groups of students J, K, L, M and N and of students AE, AF, AG and AH (see appendix A). These groups were selected randomly. Groups to be tape-recorded had to be as far apart from one another as possible in such a way that the voices would be clear on the tapes.

Even though I had two groups with an audiotape each, I only discuss the happenings of one of the groups, which stood out for me because the four students held constructive conversations over a considerably long period of time compared to the other group. These conversations from this group were also loud and clear while I had difficulty catching all the words that the other group had uttered for they spoke so low.

Students' Mathematical Interpretations

The group I sat with began the activity with Johanna's comment on the easiness of the first exercise.

> *"This one is easy. Let's do this one." Said Johanna, as they all move closer. Johanna was also the one who recorded the group's discussions in her notebook before they were copied into individual workbooks.*

I did not hear any verbal negotiations among these students, over who should write the down the conversations they were having. Yet I wondered whether Johanna offered to keep such a responsibility or whether it was just another way in which roles are naturally distributed through a system of learning beings (Maturana & Varela, 1992).

> *"That will be x minus one...times x plus two...is equal to x squared...minus two ($(x-1)(x+2)=x^2-2$)." Rehabiam offered the first step. As I observed the group members working their way through the exercise, I noticed a look of disappointment on their faces when they got a solution for x. I especially saw Rehabiam frown.*

> *"That can't be. Johanna doubted.*
> [240] *"What is wrong?" I intervened.*
> *"We got x equals to zero."*
> *"But is that possible? I mean x is supposed to be the length of the square, right?" (I asked pointing at the square drawn in Johanna's book).*

"Yes." They all replied.

Johanna thought that the given problem was pretty easy, and that encouraged the students to get together and work it out. This was done progressively and the solution was obtained. Only when the solution suggested that the length of the given square is zero, that the group experienced a crisis. This could be heard in Johanna's expression "that can't be" and by the look on Rehabiam's face—a frown. Rehabiam's expression suggested that there was something wrong with the solution they arrived at.

Even though I knew this group was going a wrong path right from the beginning, I did not want to intervene. I did not want to be the cause of any behavior but needed to let them determine what is a wrong or right answer. I wanted them to refute their own method when they realize that it was not a wise one. However, when they indicated that something was wrong, I thought I decided to come in with a question: "What's wrong?"

> *Just when I suggested we go through their solution to see where they went wrong, the teacher passed by.*
>
> *"How far are you? What did you get?" She asked.*
> [245] *"Zero." Olavi replied.*
> *"I think that is wrong. Let me see."*
> *The teacher looked at their solution. She then reread the statement out loud and told them to think again as she shook her head and left.*

Ndapewa kept in contact with the students all through the class by visiting their desks as they worked in these small groups. Ndapewa visited the group I sat with for the first time since the class began working on the problem. To find out how far they went the teacher asked, "What did you get?" "Zero" was the solution they had obtained. Without saying how wrong this solution was the teacher assured the group that that must be a wrong answer. The teacher read the question loud to the group and told them to think again about it. Without much guidance from the teacher, the group went ahead with the exercises.

In these group interactions, we may see how active all group members are in the discourse. It is such situations that we begin to observer a collective. The problems were discussed collectively by taking into consideration of each member involved and recording the results of the conversations in one book. Failure to find the desired solution for x in the first exercise was collectively expressed by their actions such as frowning, saying out loud that "that can't be" and that "we got zero".

I, together with the four students, went through their solution step by step. When I asked why they subtracted two from x^2—that is $(x-1)(x+2)=x^2-2$, Johanna explained that it is

> *"Because it says the area of the rectangle exceeds the area of the square by two centimetre squared."*
> *"What do you mean it exceeds?" I asked.*
> *"It is more than" Olavi offered.*
> *[250] "Bigger" (Rehabia and Johanna replied concurrently)."*

It was at this point that I noticed that Shaanika, a student that Ndapewa described as one of the slower students in the class, was busy solving the problem on his own. I turned to him to ask what he obtained. Without saying anything, he showed me his writing. He was not done yet, but he began the solution the same way that others in the group have. Knowing that he would end up with the same problematic situation I invited him to join us in the examination of the group solution.

> *"You said the word exceeds means bigger than, right?" I continued with a question.*
> *"Yes." Johanna confirmed. I then offered an example.*
>
> *"Okay. Let's say, for example, that the area of the rectangle represents my age and the area of the square represents yours (I pointed to Shaanika as an attempt to engage him in the conversation). So, how would this problem explain the difference in our ages?"*
>
> *"Your age exceeds the age of Shaanika by two." Rehabiam suggested.*
> *"Yes. Two years?" Olavi confirmed*
>
> *"Okay, so, if you knew my age, how would you work out Shaanika's age? Or if you knew Shaanika's age, how would you work mine out?"*
> *"Oooohhhh, Add two." Exclaimed Olavi.*
>
> *"Add to what?" I interrogated.*
> *"To Shaanika's age."*
>
> ***[260]*** *"Yes, or you can minus two from your age." Johanna added.*

This activity was probably easy as Johanna suggested. It was just another way of solving algebraic equations. It was given in form of what many call 'word problems', with a help of a diagram. I like Namukasa's interpretation of word problems that they are math problems that are "dressed up; they are dressed up in words" (Namukasa, personal communication, November 20, 2002). It sounds more like the teacher had an algebraic equation that she took words to wrap around this problem in order to make it more interesting and to situate it in a context related to the area of polygons. However, the grouped seemed not to have interpreted the problem carefully. This lack of or limited interpretation led the group to interpret the question wrong. Since the work "exceeds" means "more than", "larger", the students decided to subtract 2 from the area of the square. The instruction says, "The area of the rectangle exceeds the area of the square by two squared centimetres". It seems that the group made sense of this in such a way that they have to reduce the area of the square by 2.

One of the requests I had from this class' teacher was that I should try not to give the students direct answers as I interacted with them. This might be interpreted as the teacher not wanting me to be the cause of the students' behaviors. However, from an enactivist perspective these students are self-determined learning individuals and what I had to offer is better understood as a trigger to the student's thinking rather than the cause of their behaviors (Capra, 2002). Yet I had to contribute my part to the group's activity since I was there as a participant observer who had to interact with other participants.

Only when I offered a practical example that the collective began looking at the problem differently. "Let's say, for example that the area of the rectangle represents my age." I also let the area of the square to represent Shaanika's age, and asked the group to us this information to solve our dilemma. Soon they deduced that I would then be two years older than Shaanika. Olavi suggested that we then have "to add two". I asked him to be clearer on what we add two to. He further said we add two to Shaanika's age in order to work out my age. Another idea from Johanna was to subtract two from my age in order to work out Shaanika's age. One could call what I did here as occasioning the students' understanding (Simmt, 2003)

What Goes on in a Small Group

After I offered an example of my age and that of Shaanika's, as a trigger to the group's interpretation of the problem, they began making sense of the statement question in a different way. This enabled the group to formulate the equation this way: $(x-1)(x+2)=x^2+2$. Again Shaanika moved away from the desk and began solving the problem in his own book. When I probed the group why they were not involving Shaanika in their discussion, Olavi responded that

> *"Shaanika is too quiet, and he does want to participate."*
> *"Oh he doesn't?" I asked looking at Shaanika, who is just innocently smiling at me.*
> *"Even if we force him he will just be quite." Rehabiam continued.*
> *"x is equal to four" was the conclusion that the group has made.*

> *[265] "But now you only have x. You are supposed to find the area of both the rectangle and the square*[16]*. How do you go about that?"*
> *"Substitute!"*
> *"Substitute what?"*
> *"x into these" (pointing at the expressions that the collective recorded in Johanna's note book:*
> *Area (rectangle)=* $(x-1)(x+2)$, *Area (square)=* (x^2)*).*

Collectives do emerge whether we want them to or not (Davis & Simmt, 2003). Davis and Simmt sustain the idea that even if we plan or not, collectives may and will occur. In this class for example, there were already collectives that were originally formed groups in the class. The same would also, then, I argue, apply to the break up of a collective, whether we like it or not. I am not here talking about the collectives only in terms of physical bodies getting together to do mathematics. Rather I talk of collectives that may emerge through languaging

[16] I do not know why I suggested that we find the area of both the area and the rectangle here. The statement only says, "Find x". Even though my actions here had diverted the group outside the intended activity, it seems to have helped us verify our solution for x by substituting its value in the equations so that we could relate the two areas.

(Maturana, 1998) and other discursive interactions. Let's for example look at the collective that initially emerged from Olavi, Johanna, Rehabiam, Shaanika and my interactions. Even though Shaanika participated both verbally and physically in this group's activity, he kept pulling out of and away from it. He listened to and looked at other members when the group was deciding to form the equation of solving the equation that they formed from the given word problem, $(x-1)(x+2)=x^2$. Then from there he worked out the solution by himself.

All that time I was thinking of how one could get a person like Shaanika to verbally participate in this group work, moreover this was supposed to be group work—individuals working together as one group, a collective. Upon my attempt to help out the collective when it got stuck with the solution they obtained for x, I offered an example. I formulated my example in such a way that Shaanika could also take physical part in the activity with other members. I suggested we imagine "that the area of the rectangle represents my age and the area of the square represents yours (I pointed to Shaanika as an attempt to engage him in the collective conversation)" [260]. So, how would this problem explain the difference in our ages?"

Even my prompting this particular collective to involve Shaanika in what we were doing, my prompt did not seem to have worked. Olavi's statement "Shaanika is too quiet" indicates that perhaps even if one wanted Shaanika to take part in the collective, it might not make a difference because he is just "too quiet". Rehabiam confirmed, "even if we force him he will just be quiet". Sometimes collectives may experience frustration when other members do not contribute to the decision-making and knowledge generation at hand. General views have it that, such members benefit but they do not give to the collective—they only take away something from the collective (Miranda, 2003a). A key aspect in observing collective knowing and understanding is through the observation of the ideas that the collective generates. Even though Shaanika was part of this collective physically and probably mentally, it was not easy for me to observe what he had to offer to the collective. Even though from complex views we may argue that his being there might have made a difference to the collective's success, it is difficult to interpret his participation in terms of generating ideas and keeping the group discussion continue. Also, since my thesis is that we enable students to share their mathematical thinking with one another I find myself wondering if it is satisfactory that students do not talk at all during a mathematical activity.

The second word problem that the group was supposed to solve was as follows. Find the three consecutive numbers whose sum is 78. (I sat with the same group for the whole lesson.)

> *"Three consecutive numbers. The sum is 78." Olavi rephrased the statement.*
>
> *Rhabiam suggested that these numbers could be* $x + x + x$.
> *Johanna wrote the equation in her book as* $x + x + x = 78$.
>
> [270] *"So, three* x *is equal to seventy-eight. (writing* $3x = 78$*)."*
> *"Then* x *is seventy-eight divided by three (writing* $x = 78/3 = 26$*)."*
>
> *"So, what are the other two numbers?" I probed.*
> *"We can work those out now." Olavi said confidently.*
>
> *While Johanna was writing down their final answer I intervened again.*
> *"Are all the three numbers the same?"*

[275] *"No!" Responded Johanna.*
"Why did you name all of them x? Because if they are all x, then whatever x is, say 2, then all the three numbers would be 2. Is the sum seventy-eight?"
"Aha!" All the three students (Olavi, Johanna and Rehabiam) expressed in a surprise.
Olavi had a different idea though.

"But if I find one x, I will find the others by subtracting. For example, if x is twenty-six, then the other two umbers must be $26-1$ and $26-2$."
"So, what do you get?"
[280] *"Twenty-five and twenty-four."*
"If you add twenty-six and twenty-five and twenty-four together, will you get seventy-eight?" I asked.

Rehabiam took a calculator to check this out.
"Oh no. Its gives me seventy-five."

One concern that mathematics teachers, including myself, have is that the way students interpret given questions affect the solutions that they come up with. Here the students took the three consecutive numbers to be all represented by the letter x. Not realizing that the use of one letter will result in the three consecutive numbers to be equal. But even when the difficulty was pointed out to them the students expressed their comfort with answering the question that way.

Gordon-Calvert (2001) discusses how a community's satisfaction with a given explanation or understanding may help such a community continue as long as their understanding allows. For example, this collective of Johanna, Rehabaim and Olavi was satisfied with using one letter to represent the three consecutive numbers until I problematized their choice by asking whether all the three numbers were the same. Johanna announced that they were not the same and that had allowed the students to realize it, as could be heard in their "Aha!"

expression. Even though my prompt was viewed to trigger new thoughts about the three consecutive numbers, Olavi was still convinced that he would be able to find the remaining two numbers just using that one value of x. "But if I find one x, I will find the others by subtracting. For example, if x is twenty-six, then the other two umbers must be $26-1$ and $26-2$." We, for that moment, accepted his explanation not yet complete but plausible enough to allow us use numbers and see whether they would give us 78 if we added them to 26. We waited while Rehabiam found the sum on his calculator, which gave him 75 instead of 78. Olavi's explanation was then perceived as not adequate enough for it did not satisfy the group's expectation of obtaining a sum of 78. Not trying to easily lead them to the right question I wanted to question more in such a way that I could occasion (Simmt, 2002) their thinking of how to figure out a method of finding the three consecutive numbers.

Better method

Again I tried to offer the group a method that I considered, from my point of view a better one. Since I did not want to lead them, I let them struggle with the method they went with until they have realised that their answer was wrong. I asked,

"But the three numbers are consecutive, aren't they? "
"Yes they are. Confirmed Olavi. "
[285] *"What does that mean?" I asked.*
"It means that they follow one another. For example, one, two and three."
"Are those consecutive?"
"Yes."
"So, if my first number is one? What is the next?" I insisted on asking further.
[290] *"Two." Olavi answered.*

"How do I get that?"
"By adding one." All three students answered.
"And the next?"
"Add one again."
[295] *"Two plus one is three."*
"Okay. Lets look at the numbers you had before. If our first number is x, what is the second one?"
"x plus one."
"And the next one?"
"x plus one plus one. That's x plus two"
[300] *"Next one?"*
"x plus three."
"Um huh! There you go. Can we take any three of those as our consecutive numbers and find out?"
Johanna wrote in her book $(x)(x+1)(x+2)=78$.

My actions in this interaction served as an eliciting of the students' conception of the term consecutive. The word consecutive was central to this particular activity. When I asked the students what it meant, they readily suggested that it meant numbers that followed each other. Despite their clear understanding of what the term 'consecutive words' meant, the students were taking the variable x to represent all the three consecutive numbers. In order to enable them to determine how being consecutive made in a difference in the exercise, I prompted the students by asking how one would know that certain numbers are consecutive. Their construal of one, two, three, etc as consecutive numbers was backed by the understanding that you keep "adding one" each time in order to get the next number. I therefore suggested that since we have one of the numbers, which was x, how we could figure out the terms to represent the other two numbers. The students' response generated a sequence of consecutive terms; x, $x+1$, $x+2$, $x+3$, etc. Since the sequence was long—that is consisted of more than three terms, I did not know which of those the group would choose to work with as their three consecutive numbers. I asked if we could take any three of those and they decided to use x, $x+1$ and $x+2$.

The group then went ahead to solve what Johanna wrote down. I noticed that Johanna, as she recorded the discussion in her book, was multiplying the three consecutive terms that they chose to use, $(x)(x+1)(x+2)=78$. It is also interesting that it is not only Johanna who happens to be thinking that they are to multiply but so too Rehabiam, who answers my questioning in the following part. I am not sure if they forgot to put the addition signs between the terms or if the brackets have led them to making that mistake. Immediately I took the group back to the instruction, which says

> *"Three consecutive numbers whose sum is 78. How come you are multiplying?"*
> *"We want to form an equation and solve for x." Rehabiam responded.*
> *"But what do we get when we multiply?"*
> *"A product." Johanna confirmed.*
> *"And what did the question ask us to find?"*
> *"The sum of seventy-eight."*
> *"So, what should we do to get a sum."*
> *"We add."*
> *"Is there still a need to multiply?"*
> *"No! We just add." Johanna said while re-writing the equation by inserting the plus signs between the terms. She wrote* $x+(x+1)+(x+2)=78$.

The solution found for x was 25. Together the students then proceeded to find the three numbers by substituting x into the terms. And the three consecutive numbers whose sum was to be seventy-eight were twenty-five, twenty-six and twenty-seven. Only the first two were similar to what the collective was suggesting when they used the first method. However, they could have gone on to find the third number through the method of speculation. My behaviour in this group seemed to be like that of the teacher when she would not allow the method of inspection for solving the problems. These students acted comfortable with finding the first number, which was 26 and had planned to find the rest of the two numbers through inspecting.

Not being conscious that I was diverting the students' collective knowing and funnelling them towards the method that I thought would lead to the right answer, I questioned the students' actions. My question "How come you are multiplying?" seemed to be suggesting that the students were supposed to be doing something other than multiplying. They multiplied because they wanted to "for an equation and solve for x". This was probably just a computational mistake that allowed them to see nothing wrong with the equation. This they realized when they all agreed that when one adds they obtain a sum but when they multiply they get a product. And in this exercise we were talking of a sum.

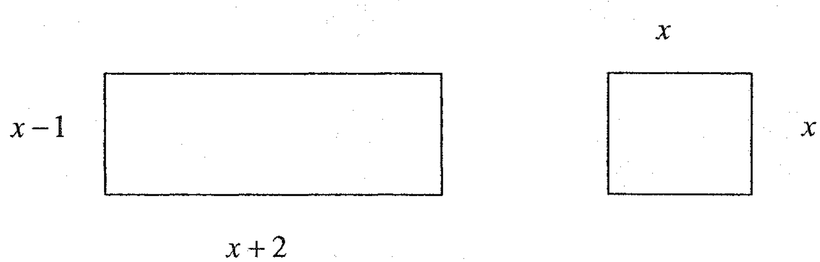
Students' Problems in Algebra

> In the process of justifying what makes sense to themselves and their peers, students struggle with constructing a mathematical language and a set of relationships among its terms that is internally consistent. (Lampert, Rittenhouse & Crumbaugh, 1996, p. 738)

Algebra is generally viewed as a bridge between arithmetic and geometry. It is also one of the area that mathematics students have been observed to have difficulty with, when solving mathematical problems. These difficulties partly originate from students' lack of understanding of arithmetic, since arithmetic serves as the basis for algebra (Hiciomeroglu, 2003). The part of algebra in which students experience such difficulties is when they work on word problems, perhaps because they first have to interpret what the statement says and then be able to represent it symbolically. According to (Hiciomeroglu, 2003), students need to understand a problem first before they are able to represent it symbolically. In both the discussion groups that I present below, evidence shows how students were experiencing the word problems assigned by the teacher.

The following is a transcript of another group that was busy solving the area word problem (See Fig. 9):

Find x, when the area of the rectangle shown exceeds the area of the square by $2cm^2$.



Noisily, students moved their chairs closer to the desks. The teacher's voice could be clearly heard in the background as she reminded the students when she walked around the classroom checking the students' work over their shoulders.

"I don't want anybody doing their own individual work."

I picked upon this brief moment as they teacher made it explicit to the class that she did not want the students "doing their own individual work". In all the seatwork before this day, students could work either individually, in pairs or small groups, whichever worked better for them. But this day they were not allowed such freedom. Could Ndapewa have said that to make sure that students were forced to work with one another so that I was able to get discussions for my research data? As I mentioned in the beginning, she had decided to assign group work today because she "noticed that I was not really getting what I was looking for". Even though I never told Ndapewa that this was the case, she however, concluded that discussions were not taking place as I, the researcher, expected to. This had probably resulted from my sharing with Ndapewa, the transcript that I made out of each observation. She could have done some reflections as she read those transcript and that might have given her an idea that not much talk among the students was occurring.

The group of two boys (Paulus and Simon) and two girls (April and Elena) was busy solving the first assigned problem. Since I was not physically present at this group I am only able to make sense of what I could hear on the audiotape and connect it to what was generally going on in the class during that lesson. Simon kept insisting—

> *"You can use this one...You can use this one...You can use this one."*
>
> [315] *"We can use it or not?" asked April who read loud what Simon was suggesting. "x minus one times x plus two is equal to two."*
>
> *Elena re-read the given statement loud,*
> *"Find x, when the area of the rectangle shown exceeds the area of the square by two centimetre squared."*
>
> *"I don't think we are using it." Paulus disagreed with what Simon had suggested.*
> *"Eh?" Concerned Elena.*
> *"Because they didn't say the area of the rectangle is equal to two. Now you are saying that..."*

[320] "This two is for what?" Elena asked.
"Is telling you that the area of the rectangle will go up with two. It's like I am twenty and Simon is twenty-five. Simon is older than me by five years."
"Yeah like that!" Exclaimed Simon.
"Like that." Paulus confirmed. "Then you don't..."
"Okay...[inaudible]." Agreed Elena.
[325] "Yeah. Remove the brackets, and this..."

The students in this group began making sense of the exercise collectively. While Simon and April were deciding whether they could use the equation they constructed or not, Elena read the statement out loud. Elena's reading seemed to have allowed Paulus to conclude that they could use the equation they had constructed in the way they were doing so: $(x-1)(x+2)=2$. Understanding that the left side of this equation represents the area of the rectangle, Paulus' thought that the group could not use this equation "Because they didn't say the area of the rectangle is equal to two".

The problem of making meaning of this algebraic word problem seemed a common in this classroom. Just like to any other students in this class, this problem seemed easy to Simon who thought that they could just use the equation $(x-1)(x+2)=2$. Yet interesting, interpreting the problem and expressing it algebraically was the most difficult part of this task. This could be heard in his repeated suggestion that "you can use this one", and when April said out loud what Simon was suggesting. Here Simon seems to have just took the area of the rectangle in consideration but did not give any thought to the area of the square. Paulus' understanding of the questions seems more articulate than Simon's when he thought that they should not use the equation $(x-1)(x+2)=2$, "because they didn't say that the area of the rectangle is equal to two." In trying to make sense of the question, Elena asked what the two was for; "this two is for what?". Paulus explained this that it supposed to tell them "that the area of the rectangle will go up with two." Here Paulus made sense of the word exceeds as to meat that 'to go up'.

I was amazed to hear Paulus using an example of his age and that of Simon to make the question clearer. This was at the same time I was interacting with the group of Johanna, Rehabiam, Olavi and Shaanika, while solving the same problem. These two groups sat in two opposite corners of the classroom and there is one group between them. I can therefore not claim that this group had heard the use of age example in our group.

Just before Paulus completed his suggestion the teacher passed by their group and Elena called for the teacher's attention.

> *"Ms., we are confused."*
> *"Get stuck." Paulus added.*
> *"Confused?" Asked the teacher.*
> *"Yeah."*
> *"What is happening?"*
> *"Here." Paulus indicated to the teacher where the problem was.*
> *"Eh heh?" The teacher responded willing to listen.*
> *"The statement is saying the area of the rectangle must go up by two."*
> *"Eh heh. By what?"*
> *"Two centimetre squared."*
> *"Let's read the statement, read the statement." The teacher suggested.*
> *Paulus read the statement out loud...Elena read along as Paulus finishes.*
> *"What does that statement say?"*
> *"Go up. The area of a rectangle has to go up by two. Is that what they mean?"*
> *Teacher read the second part of the statement loud, "...shown, exceeds the area of the square by two centimetre squared."*
> *"Meaning that..., it's like I am saying Simon is older than me by five years."*

"Eh heh?" The teacher seemed to be still listening.
"The formula..." Paulus continued and the teacher interrupted.
"So, you know the formula for the area of the rectangle. Isn't it?"
"Yeah." The whole group replied.
"Do you know the area of the square?"
"Yes!" "Yeah!" Paulus and Elena responded simultaneously.
"So, you have to make something out of that. The statement..., we understand what you are saying. But you have to make something out of the area of the square and...I mean the area of the rectangle and the area of the square."

Could the statement "Ms., we are confused" be meant to say that the group did not understand what it was expected to? One of the roles of the teacher during group work/discussions is to keep an eye on the students and check how they are progressing and to help them out when they "get stuck." This group called for help from Nadapewa when she passed by to ask what the question is really asking for. When asked to define what the statement was saying, Paulus explained what he had been offering to his group before. "Go up. The area of the rectangle has to go up." Paulus' suggestion "go up" is not as clear but the teacher seemed to have made sense of what he was saying when she asked him, "Eh heh, by what?" should the area go up. Not so sure anymore if that is what it really means, Paulus asked the teacher for confirmation, "Is that what they mean?" The teacher did not have much to offer to this group, for it was still stuck even after she suggested that they, "Make something out of the area of the rectangle and the area of the square". After she left, the group struggled further with the question.

"Let's, let's, let's form an equation like this." Paulus suggested.
"Yes, we have to form an equation." Simon confirmed.
"Like I told you." Elena responded.
"That minus this..." Simon began.
"It gives you two." [i.e. the area of the rectangle minus the area of the square gives two].

"Yeah! That's the perfect answer."
"Okay, what can I write here?" Asked Elena.
"Subtract the area of the square."
[355] *"I subtract it from where?"*
"From the area of the rectangle. Then the answer is two."
"What is the area of the rectangle again?"
"The area..." Simon began offering.
"Of the rectangle is this one." Paulus continued.
[360] *"Just do like this...then you subtract from..."*
"Is it this, from this line or this one?" Elena asked.
"Like this, neh? You just..." Simon said.
"x squared or what?" [i.e. the area of the square]
"x plus two is what we must start with. x plus two, blanket [bracket], x minus one, blanket [bracket] then you subtract...You subtract x squared...is equal to..." Paulus explained and paused.
[365] *"Two!" The three of them shouted simultaneously.*
[$(x+2)(x-1)-x^2=2$].
"Then from here we solve this one."
"Why not squared?" Simon tried to take the group back, suggesting that the two must be squared too, since the statement says that the area of the rectangle exceeds the area of the square by two centimetre squared.
"It is just because it is the unity [unit]. The unity [unit] of the area is just like..." Elena tried to explain to Simon.
"Does that mean that the unit is the only thing which is squared?" Simom asked.
[370] *"Yes. Yeah." Paulus and Elena responded to Simon respectively.*
"Always when you are calculating the area, this is the unity [unit] you have to use."
"Ooooh!" Exclaimed Simon to Elena's explanation.
"Okay. And then from there we solve this one. By multiplying the term in blanket [bracket] by this tem in the blanket [bracket.]" Paulus suggested again, talking about the expression $(x-1)(x+2)$.

After the teacher left the group, the students decided to "make something out of the area of the rectangle and the area of the square like the teacher advised them to." Paulus suggested that they form an equation. Elena pointed out that that is what she has been urging other students to do, "like I told you." Using the conclusion they made that the area of the rectangle has to "go up by two", as an understanding of "the area of the rectangle exceeds the area of the square by

$2cm^2$", the group decided to "subtract the area of the square from the area of the rectangle…and that gives two." This was what Paulus was convinced would work. Elena played a role of the recorder, almost similar to what the teacher does on the chalkboard when working out a problem with the whole class. Elena asked for a reminder, "What is the area of the rectangle again?" Both Paulus and Simon pointed Elena to the area of the rectangle, and she asked for directions on how she could begin writing the equation down. She wanted to know "Is it from this line or this one? x squared or what?" Paulus, who sounded pretty much in control of the group, told her that "x plus two is what we must start with. x plus two, blanket [bracket], x minus one, blanket [bracket] then you subtract…You subtract x squared…is equal to…"

The group then went ahead to open the bracket in order to solve the equation $(x+2)(x-1)-x^2=2$, for x. Since Elena was the one who was recoding the answers on the paper, she kept track of all the steps, while collaborating with other members of the group. She said out loud what would be obtained if they expanded the brackets:

"x squared…We have minus x".
[375] *"Minus?" asked April.*
"Yep. Positive with negative."
"Yeah!" Paulus exclaimed.
"Oh? This times this…mmhhh." Simon wondered.
"Plus two x, and then we have minus…two and then minus x squared. And then it is equal to…" Elena continued.
[380] *"Two." Elena, Paulus and Simon concluded simultaneously.*

Their conclusion would, from what they said in the tape, look like $x^2-x+2x-2-x^2=2$.

"Now we are collecting the like terms. This one and this one are alike."

"But we should consider the signs. Positive and negative."
"Just zero."
"Zero?"
"Yeah. Positive and negative."
"You know what to do neh?" Elena asked while Paulus and Simon exchanged ideas about considering the signs.

"We just need to do like this. x squared plus two minus two..."
"You don't need to write this again. You just..." Simon interrupted Elena while she explains. But Elena went ahead.

"Minus x squared. They cancel each other."
"And then from there, these will all give us zero."
"They will cancel again?"
"Yeah!"
"And then we have two or what? Two x here neh? Is it two x?" Elena asked for confirmation.
"Here we got x. Not two." Paulus corrected her.
"Negative x plus two x minus two..."
"Is equal to two." Paulus completed Elena's sentence.

"Then is equal to two." Elena agreed.
"Then this one and this one...they give us x. Negative x and two x ...they give us x." Paulus concluded.
"One? One x?" Elena asked.
"Yeah."

"It will be like neh? x minus two x. Like this." Simon said, again confused about the signs.
"The answer will be just like...because negative x plus two x ...the answer will be?" Paulus tried to explain to Simon who sounded confused in disagreement.
"Negative x."
"No. Will be...will be positive x because the bigger number is positive."

"Oh yeah!" Simon seemed to get what was going on now.
"Here we got x." Elena pointed out.
"Yeah." Confirmed Paulus.
"No man, negative." Despite Paulus' effort to make it clear that $-x+2x=x$, Simon still insisted that it should be $-x$.
"Negative? This one is negative, and this one is positive..."
"You are subtracting." Said Simon.
"And the bigger...the bigger one is positive." Paulus tried again.
"Look! You are subtracting a bigger number from a smaller number."

"No, no! Listen! Listen! What we need to do...we just need to write x here and then it's equal to two and then plus two, first, like this."
[415] *"Yeah, first." Paulus agreed.*
"And then after that we got x here and the is equal to...four. That's the answer."
"Yeah. That's the correct answer."
"But this neh, look here. You are subtracting." Simon still sounded disagreeing. Insisting that $-x+2x=-x$.
"Yes it is the way of solving." Elena said to him and Paulus offered to explain to him further when he said
[420] *"Simon, Simon, let me tell you."*
"Eh?"
"x is negative and two x is positive. Between x and two x, which one is the bigger one?"
"Two x."
"Two x. Then from...when we subtract, the answer will be what?"
[425] *"Positive."*
"Yeah. It's positive x. The answer is four."

In this group, Simon always seemed to have a question or not agree with what the rest of the group was proceeding with. His understanding of adding and subtracting integers did not seem adequate to help him understand why $-x+2x=x$. For him, it sounded like, since the negative sign is in front of the x then one is supposed to be subtracting $2x$ from x. This could be heard from his statement "Look, you are subtracting a bigger number from a smaller number". In his view since $2x$ is the bigger number and it is being subtracted from a smaller number, that is, x, then the difference should be $-x$. Elena did not sound so much enthusiastic in offering an explanation to Simon. She just called him to "listen". "Listen, listen. What we have to do...we just need to write x here and then it's equal to two and then plus two..." Elena's concern was not to make Simon understand, but rather to figure out what should be done and how the solution method was to be written down. However, Paulus offered to help out

Simon when he called out "Simon, Simon, let me tell you." He used the same explanation method that he and Elena have been trying to make sense to Simon but this time he put it in a form of a question to Simon. "x is negative and two x is positive. Between x and two x, which one is the bigger one?" When Simon agreed that the answer must be positive, they together concluded that x is equal to four.

Without going back to the question statement to check what they were looking for, the collective made a conclusion based on their plausible but elusive (Gordon-Calvert, 2001) understanding, which helped believe that the value of x that they found represent the area of the rectangle shown in the diagram (Figure 1.).

"That's the area of this neh? Of the rectangle?" Elena asked.
"Yeah of the rectangle." Paulus affirmed.
"And of the square?"
"Of the square, we can just subtract two." Paulus offered.
[430] "Then we get two."
"Oh yeah. We can just subtract this two from four. Because they said this one is bigger than that...is greater than by two. Like that. Then we go to problem number two."
"We proceed to problem number two." Agreed Elena.
"Or talk about something else. Let's talk about the Radio."
"Aah! Leave the Radio...My name is Elena." Elena teased speaking right into tape recorder.
"You're supposed to speak like an American." Simon suggested to Elena.
"Yeah. Yeah. Its true."
[435] "What?" April who seemed to be quiet for almost the rest of the lesson asked unknowing what was happening.
"Anyway, we are speaking like that." Paulus added and all of them grinned, and laughed.

On-task Off-task Conversations

In any discussion group, issues that may be considered as off-task discussions may occur. Such an experience may differ from observer to observer.

Yet, an observer who finds a conversation to be off-task may be prompted to say how off-task such a conversation is. For example, in one of the groups the students talked about several things apart from the equations that they were supposed to solve. In the conversation above, Simon could clearly be heard changing a topic that Elena brought up and suggesting that they "talk about the radio". Simon also made fun of Elena's accent when she was reading out loud the second question statement [433]-[435]. He told her that she was "supposed to be speaking like an American". She agreed and the whole group laughed. Paulus on the other hand felt that they have already been speaking "like an American". This puzzled me as I asked myself how the discussion of "speaking like an American" emerged? Does it have to do with these students' knowing that their voices were being recorded? Did it have to do with the fact that the researcher who was recording their voices is a student from Canada? And if so, what does Canada have to do with speaking "American?" Is it probably because many people perceive Canada and the U.S.A. as America altogether?

One could call this conversation an off-task one since the students were no longer talking about the activity. But the conversation emerged from the students' attempt to decide what they should do next. One student suggested they go to the next exercise, "problem number two". But Simon suggested they talk about something else, maybe the Radio. But Elena did not want the group to tamper with the Radio so she told Simon to "leave the Radio" and she spoke, not about the Radio, but 'to the radio', when she said, "My name is Elena". However, the group managed to take the decision of going to "problem number two" instead of talking.

Ignoring Some Members of the Group

There are times when particular individuals try to be part of or belong to a certain group, but the other members may not seem to be willing to adopt such individuals. This could be heard, for example, in one of the discussion groups where the conversations between Elena and Paulus dominated the activity. Simon and April's voices could only be heard in the background trying to offer something to the group. Sometimes their views were taken up and sometimes they were not.

For instance, the group was concerned about the word "consecutive" in the question statement of the second exercise, below. While the groups were still busy with the exercise, the teacher asked the whole class how far they have gone as far as the first question was concerned.

> *"Anyone who solved problem number one yet?"*
> *"Yes, we got the answer for number one." Elena said out loud answering the teacher.*
> *"Let me see." Said the teacher.*
> *[440] "Let's wait. She's coming" Elena said, and she began reading out loud the statement for the next question as they waited for the teacher to come to look at their solution. The second exercise was:*

Find three consecutive whole numbers whose sum is 78.

Solving this particular problem was also perceived as easy. Paulus thought that the term "consecutive" was the only problematic part of the question.

> *"The word that we need to understand...the word that is creating a problem in the equation...consecutive...if we could understand the word consecutive, then nothing will be difficult."*

"I think it's the numbers which are..." April tried to define what the term consecutive numbers means.

"The sum. You know? The sum of three...we can just say x plus x plus x." Elena suggested.
"Yeah." Affirmed Paulus.
"We got x plus x plus x ..." Elena repeated.
"Is equal to 78." All the three of them added.
"And then we start collecting..." Elena suggested.
"Collecting like terms." Added Paulus.
"We bring x on that side."
"Yeah, on the variable side, the left side. And then the other side...the non-variable side." Paulus finished.
"The sum of consecutive whole numbers...whole numbers...whole numbers." Elena repeated the phrase stressing on the part that says "whole numbers." Simon's voice appeared again.
"Yeah. It's like neh...we take x plus y or plus two...like two."
"No, no. We can only choose one letter." Elena disagreed with Simon.
"How? We are not told that they are..." Simon tried to explain his point, which he could not finish because Paulus and Elena went ahead with the solution, despite what Simon was about to offer.
"Just do like this x plus x plus x." Paulus told Elena.
"Then we will have three x."
"Then we divide both sides by three...then we get 26."

"I am not so sure if this is correct or not." Simon doubted.
"I am sure that we are going to be a hundred percent correct." Paulus assured him. But April, too, doubted when she said "I don't think so."
Twenty-six plus twenty-six plus twenty-six is equal to seventy-eight. It means that we are correct, hundred percent." Paulus assured them again.

I cannot help it but to compare this group's activity to the group that I sat with at the same time, Shaanika was one student who would not verbally take part in the group activities. I tried involving him but the group assured me that it would not change him. I also saw very little effort from Shaanika himself, as a sign of willingness to offer anything to the group other than his silence.

In the above episode, however, there are two students who tried to contribute to the work. It could be clearly heard from the voices on the audiotape that Simon made a larger effort to be part of the group, as compared to April, but she too tried to engage herself into the discussion. For example, when Paulus and Elena concluded that all the three consecutive whole numbers should be x, a brief conversational argument occurred. Simon expressed that Elena and Paulus' decision was not correct. April doubted too. Simon suggested using different letters for the three consecutive integers. He seems to have understood the term consecutive than Paulus and Elena. Knowing that these three consecutive numbers could not be just one constant he (Simon) suggested to call these numbers x, y, etc. But Elena insisted that that is not the case. Rather, she thought, they had "to choose only one letter". In a disagreeing voice, Simon asked how that was possible when they are "not told that they are…" Was he just about to ask how someone would call all the three consecutive numbers x when the statement did not say that all the three numbers were equal? Moreover, if numbers are consecutive…it is impossible for them to be equal. Even though Simon offered his argument and reason, Elena shut him down. She objected to Simon's suggestion to use different letters. According to her, the group could "only use one letter". Simon's question of how this is possible was ignored and Paulus ordered Elena to "Just do like this, x plus x plus x".

Group Dynamics and Common Understanding

Listening to these group discussions presented good examples of what Gordon-Calvert (2001) considers to be "genuine conversations". They (group discussions) allowed the students to talk about mathematics with one another. Unlike in facilitated whole-class discussions, the students, themselves, had total control of what should be done and how it should be done as far as practicing the assigned exercises was concerned.

I see commonalities between this group and the group that I sat with that day and the features that emerged with these two groups. The two groups' use of examples with the age when solving for x involving the area of the rectangle and the square, and their action of taking $3x$ as the sum of three consecutive whole numbers comes to my mind when I compare these two groups. Further, it seems that in both these groups, there is at least an individual who tried to quide others when the group was not doing well. For example in my group, I pointed out to the students that it was problematic to name all the three consecutive numbers since these three numbers are not necessarily equal. Similarly, in Simon's group, Simon tried to point this case to the group but he was not actually heard. One could hear this in Simon's suggestion that the collective should use letters like x, y and so on. This was turned down by Elena who believed that only one letter should be chosen to represent the value of the three consecutive numbers. April also expressed her worries similar to that of Simon's. Simon said, "I am not so sure if this is correct or not". When Paulus assured him that everything they were doing was "a hundred percent correct", April said, "I don't think so".

After having a conversation about "American English" and about the Radio, Elena told the group that they should not touch or talk about the radio and suggested an alternative:

"Let's just proceed with our mathematics."
"Yeah man." Simon agreed. But Paulus could not see what more to speak about since they were done with the last exercise and were just waiting for the teacher to check their solution. Rather, he said

"Unless may be you can bring another problem from your own mind. Apart from this one you can just bring another one. And then we can solve it."

[465] "The product of three consecutive numbers is..." Elena began offering a problem.
"Two!" Paulus completed it.
"...Is twice as much as two. Solve that equation. Solve the problem." Elena said.
"What?" Simon asked for a pardon and Elena repeated the question but this time she changed it.
"The product of two consecutive numbers is twice as much as two."

To act mathematically, sometimes students need to generate mathematics problems that they may develop methods for solving them. In the case above, the students seemed to be getting bored or running out of activities. They suggested several things that they could do, for example, talking about the radio, and about the accent that they should be speaking as they discuss in their group. Paulus suggested that the collective comes up with another problem that they could solve it. It is interesting that Elena, in formulating the problem, used the word "consecutive". "The product of three consecutive numbers is...as twice as two." The word consecutive is the same word that the group was having trouble with in one of the problems the teacher posed for them. Paulus expressed that the word consecutive was problematic, as soon as the collective began looking at that particular problem (Find three consecutive whole numbers whose sum is 78).

Just before the group began solving Elena's problem, the teacher showed up and asked

[470] "Done?"

"Yes!"
"Okay."
"This is the first problem." They informed the teacher.
"Eh heh. The first one?"
[475] *"Yes."*
"Oooh, why would you put the two here?"
"Because we have subtracted the square...I mean the area of the square from the area of the rectangle." Paulus explained.
"In order to?" The teacher asked.
"To get two."
[480] *"In order to get the difference, right?"*
"Yes."
"Then from there we collect the like terms. Then we got negative x ..."
"x squared and x squared will give us zero." The teacher said following what the group did.
"Yeah!" They all agreed.

[485] *"This one and this one will give us?" The teacher prompted.*
"Positive."
"How did you get minus x here?"
"No, first we have just cancelled x squared and x squared, then we remove the brackets."
"Oh first of all you removed the brackets?"
[490] *"Yes. Yeah." The group responded.*
"Minus x plus two x minus two minus x squared, right?"
"Yeah."
"Then the x squared and the x squared will give us..."
"Zero."
[495] *"Then we got that and that. This one moves there. This one we've got x. And that's a hundred percent correct. You take another dimension of finding the difference. You are the only ones who used this method so far. There are other two ways we could follow. Now if you substitute it in here, you see that the difference is exactly two."*

In this conversation between the group and the teacher, solving the equation appeared as mere manipulation and transferring of symbols and numbers. The words: area, rectangle, square and greater than, which would keep the algebra connected to the problem and contribute to a mathematical conversation were hardly present in the utterances made. The teacher's question:

Why would you put a two here? Implies that the two was just put there as without any meaning. Was the two added to or subtracted from something? This does not come clear in the word 'put'. The group, according to Paulus, subtracted the area of the square from the area of the rectangle in order to get two, which the teacher referred to as the difference. Other language forms used were such as the cancellation of terms, and moving terms from one place to the other. The interaction continued as follows below.

> *" What is this now?"*
> *"This is problem number two."*
> *"Two consecutive numbers?"*
> *"Yes."*
> [500] *"Before you solve, I said, I want a list of what to follow because every group will give us that list of what to follow how to solve this. So, you know...find the numbers. They didn't say find the number, not only one number you have to find. You have to find three different whole numbers. So, there is something wrong there."*
> *"Oooh...That's what I told you." Simon blamed the others.*
> *"First of all, deduce the reasoning of...Let's map out step number one...what should I do to solve the problem? Number two, number three, number four, number five, or if there is any...depending on how many you would take."*

Here the teacher clarifies to the group what they are supposed to do. She also tells them what she wants them to do: "Before you solve, I said, I want a list of what to follow". Simon now realizes that he had been right from the beginning that they should have used "different terms." He goes on blaming the group in the next line.

> *"Yeah! I told you that we must use different terms."*
> *"Okay. Let's do that." Elena agreed. "Which numbers can we take?"*
> [505] *"You know what we can do now? What we can do now...we can...let's go for step number one." Suggested Paulus.*
> *"I think here we can put 26." Elena still insisted on taking the value of x that they have already found.*

"How?" Simon asked, still in doubts.
"No. Let's just write step number one…we find three different numbers so that when we add them together, they give us seventy-eight."
"Not different [numbers] but three different terms." Simon said.
[510] *"Three different terms?" Asked Paulus.*
"Yeah. Like what and what?" Elena agreed with Paulus' question.
"May be…let's use letters instead of numbers… x plus y plus z." Simon further suggested.
"x plus y plus z? But how will you find the numbers if you got different letters? How will you find the numbers?" Elena asked.
"Numbers are also different."
[515] *"You know, these things are just relating to finding x." Elena tried to convince Simon.*

The group is still working its way out to finding the three consecutive but until now, nothing promising had happened other than falling back to the answer obtained previously. It is interesting to notice how this group's meaning making of the term consecutive was limiting their ability to solve the word equation. Even though Simon knew and insisted that the three numbers had to be different, this did not help much because his suggestion left the group with three unknown variables, x, y and z, which the students were not able to solve for. Elena wondered how one would fine the other letters if they used all three unknown letters. Paulus was still concerned about the word consecutive. He turned to ask some other students what the word meant.

"You know where the problem is neh?"
"Mmhh."
"What is the word consecutive? Anybody may be who has got an idea what consecutive means?" Paulus continued.
"Consecutive means a number which is divisible by two." April, suggested.
[520] *"By two neh?" Asked Paulus.*
"Yes."
"And is twenty-six divisible by two?"
"Yes."
"Then we are okay."

[525] *"Let's see in a dictionary." Elena called for a dictionary.*
"They said we have to write the steps we need to follow"
"Yes, we need to deduce steps." Simon confirmed, Paulus' statement.
"Our first step is: We need to find three numbers that when we add them together, they give us seventy-eight. And then from there...we check."
"Consecutive means neh..." Simon tried to say what the word consecutive means, probably after finding the dictionary. But Paulus went ahead with articulating the steps that should be noted down in Elena's book.
[530] *"And then from there we solve the non-variable number. For example if we use x, we solve for x. We substitute to check or to prove whether our answer is correct or to confirm that our answer is correct."*

The steps were written down and read out loud by Elena.
Find the three numbers, that when we add them together they give us seventy-eight.
Solve the equation.
Substitute to check whether what we got is correct.

"What does consecutive mean?" Elena followed up Simons prompt of defining the word consecutive.
"That follows one after another. That happens or follows one after another. Like this. Oh I know."
"I just need a dictionary." Elena insisted.
"What I am thinking...Let's not get confused by the word consecutive. Let's just set out steps." Paulus suggested. At this moment, the first side of the tape got finished and the students called me to turn it.

The ESL (English as a Second Language) Issue

In mathematics classrooms where English is used as a the medium of instruction, but it is the second or third language to almost all students, students might experience difficulty in understanding the mathematics they are learning. Since the language use in the classroom is different from language use at home, students may not receive sufficient clarification of the rules of the game for classroom interaction, which can contribute to their difficulty to learn these rules. For example in the above episode, this group and seemingly all the groups in the class were experiencing a problem with the word "consecutive" that was in the second group exercise. Most of the group were not able to readily solve this particular problem because they do not understand what the word consecutive means. Paulus a student in one of the groups even told his fellow group members that if only they would understand what consecutive meant, then "nothing will be difficult". Even though he was worried about their understanding of the word, he was still confident that they were doing the right thing in solving the problem and was sure that their answer was "a hundred percent correct".

April who has been trying right from the beginning to offer a definition of the term "consecutive" finally got a chance to do so. Her suggestion was that consecutive numbers "are numbers which are divisible by two". This suggestion made the group feel even more confident with the answer that they obtained when they solved for x --i.e. $x = 26$. "Is twenty-six divisible by two?" was Paulus' concern and he indicated that if that was the case "then we are okay".

Other second-language-problem related situations were when students misspelled words; for example "blanket" for bracket, "unity" for unit, etc. This problem is not only common in mathematics classrooms or in schools but also to many other people who's English is a second language.

Structural Determinism

The conversations around the three consecutive numbers continued, with a debate between Simon on one side and Elena and Paulus on the other. Even after the teacher told them that they were not supposed to be doing what they had done—taking one variable to represent the values of all the three consecutive numbers—this had not changed the behaviors of how to solve this particular exercise (Maturana, 1978). Here I bring in the notion of structural-determinism of living learners. It is not the environment or other entities in the environment that determines which perturbations organism responds to, but the organism itself specifies those responses (Capra, 2002).

> *Despite what the teacher suggested to the group, Paulus went on giving the same equation that the group had been working with.*
> *" x plus x plus x is equal to seventy-eight".*
> *"But you know neh? They will give us the same number." Simon offered his doubts.*
> *"Yes they will give us the same number".*
> *"But we were asked for different numbers".*
>
> *[365] "The problem is..."*
> *Before Paulus could finish his explanation of the problem, Elena mediated: "No...the thing is, you need not to have different letters".*
> *"Yeah"! "Paulus verified".*
> *"You will never find the others if you have different letters. You will never". Elena tried to convince Simon why it is not wise to pick different variables for three numbers.*
>
> *"But we are looking for different numbers". Simon insisted.*
> *"No, we are not looking...is the statement state different numbers? It said just three numbers". Elena presented her reasoning why she would not go for different numbers, and Paulus backed her argument by saying,*
>
> *[370] "This word doesn't mean different. We can have the same number, which can give us seventy-eight. Example, twenty-six. We can have twenty-six three times, then..."*
> *"...It will give us seventy-eight". Both Elena and Paulus completed Paulus' statement.*
> *"Therefore, let's go proceed on our step three". Paulus suggested and then he continued: "three x is equal to seventy-eight. Then we divide both sides by three. Then we substitute..."*

In the above conversation, an argument emerged, in which Simon tries to offer his understanding of what the question is asking for—"three different numbers". But his colleagues would not take his suggestions, because they are convinced that is they take three different variable for the values of the three consecutive numbers, they would "never find the other" two terms, after finding the first one. "You will never", Elena stressed. Simon kept on insisting "but we are supposed to find different numbers" while Paulus and Elena kept insisting that there is no need to have three different letters.

This was an example of how an entity or rather its structure determines what happens to it or with it (Maturana, 1978). The teacher, on making her turn to this group's desk, informed these students that they were not supposed to take on one value of x for the numbers are supposed to be consecutive. Even Simon co-emerging with and within this group offered his doubts that if they used only one letter, the solution would give them one number and that might be problematic. Despite these two individuals' commends, the "emergent organism" that resulted from the interactions between Paulus and Elena was resistant to those commends and therefore, they could not allow either Simon or the teacher's interactions with them change that structural coupling.

Unproductive group work

Just when one of the groups was still struggling finding the three consecutive numbers whose sum was to be 78, the teacher passed by and the group called out for her assistance.

"Ms! Ms!"
"Halloo...our time is up." The teacher responded and reminded the group of time.

The teacher came and read the group's written work.

[375] *"Okay, let three letters be three numbers. So, what are those letters could be three numbers? Can you write for me the three letters you think should be numbers?" The teacher asked.*

"x, ... $x+x+x$" Paulus began.
"But this is the same number you are adding on. Isn't it?" The teacher probed.
"Yes, but here we are not asked for different numbers."

It is interesting here to hear Simon defending Paulus' move on using x as the only variable, when he challenged the teacher that they "were not asked for different number". In the previous episodes, it was Simon against the rest of the group who insisted that it was not wise to use only x because they have to look for different numbers. Here Simon now seems to take Paulus and Elena's side to saying that they "were not asked for different numbers". The teacher went ahead and convinced him otherwise.

"But the question is...find the numbers. They have to be different, if you have numbers in plural. Isn't it? If you find x, that is only one number. Is that a number or numbers? x is equal to twenty six. It is a number."

[380] *"Yeah! It is a number." Paulus seemed to agree with the teacher.*
"But I said find numbers in plural." The teacher stressed on the plurality of the numbers to be found and Simon tried to point out where the misunderstanding is.

"Our mistake is...um. We are supposed to make write it like this. x, x ..."
"Though you can write it three times and they can be twenty six. But that is still the same number you are talking about. And that number is x. You understand what I am saying?"
"Yeah!"

[385] *"The idea here is a hundred percent correct…let three letters…not necessarily three letters but let something be…so we can say let three numbers be…"*
"Letters!" Simon completed the teacher's statement. And the teacher continued…

"The first number can be x*. What is the meaning of consecutive?"*
"That the is the word which is creating the problem." Paulus reported to the teacher. And the teacher confirmed with him…
"That is the word which is creating the problem in the statement. Right?"
[390] *"Yeah." The whole group agreed.*

"What does it mean? What do you think it means? Consecutive?"
"It means numbers following each other."
"They have to follow one another. In a way we calculate. If I have four, five, six, these are consecutive numbers. If I got ten, eleven, twelve, these are three consecutive numbers. If I got a hundred, a hundred and one, a hundred and two…consecutive numbers. So they have to be consecutive. If the first number is x*, what is the second number will be?"*
" y *!* y *!" Both Simon and Paulus responded.*

[395] *"No, we have to use the…exactly the same unknown for us to be able to solve." The teacher suggested in disagreeing with the two boys.*
"Yeah." Affirmed Paulus.
"We have to use exactly the same unknown. The second number will be?" The teacher repeated and asked again.

"Two x*." Suggested Simon.*
"Not two x*." The teacher disagreed.*
[400] *"* x *!* x *!" Both Simon and Elena said.*
"It won't be x*. If you say…let my* x *be one, isn't it?"*
"Okay." Paulus seemed to be following.
"So, the next number will be still one. Isn't it? That's what you are telling me. The next number will still be one, is that true?"
"Not really." Paulus said.
[405] *"Not really? So, find those three numbers. So, if the first number is* x*, the next number will be? The third number will be? But make sure you use the…exactly the same unknown." Just after the teacher walked away, the bell rang and students were already moving to go to the hostel or home. I stopped the tapes and packed everything.*

> There are many ways to turn classroom discussion or group work into a great supplier of learning opportunities; there are even more ways to turn them into a waste of time, or worse than that—into a barriers to learning. (Sfard, 1998, p. 50)

This group does not seem to have achieved much as far as the second activity was concerned. Fumbling around the word consecutive and trying to solve the equation that the group formulated, they were unable to produce a result desired by the teacher. Even though they kept calling the teacher for help, this did not make much difference because the group easily slipped back to the way they were formulating and solving the equation.

Paulus who was concerned about the word that was creating a problem in the statement—consecutive, did not care much about figuring out what it meant by the term consecutive. This could be heard in his suggestion in the first second part of the transcript that the group should not get confused about the word consecutive. Unlike Paulus, Elena, Simon and April took an effort to define the term consecutive, by finding the dictionary and guessing what the term meant. When the teacher came around for the last time, Paulus who was not so concerned with defining the word consecutive indicated that he did know what it meant. The teacher asked: "What does consecutive mean?" Paulus reported to the teacher that "that is the word which is creating the problem" in the statement and the teacher agreed with him. The teacher asked him what consecutive was and without hesitation Paulus said, "it means numbers following each other". The bell rang before this group found a solution to the problem. More to that, the class did not return to these activities the next lesson where everyone could learn about how other groups or individuals had gone about doing the activity.

In this chapter, I observed students work on mathematical activities in small group with very little teacher intervention. This was an opportunity where students talked about the exercises not only with the teacher but also mostly with other students. As evident in the transcripts discussed, genuine conversations among students emerged which were not present in the case of whole-class discussions. This shows how students' own meaning making of mathematics is of value to a large extent, when they work in groups independent of the teacher. As compared to whole-class discussions, there were also some students who would shut down other students' ideas wither explicitly or implicitly. Participation from every member of the groups was not quite available. The size of the groups might be one of the factors that played a role in limiting students' verbal participation in the group work. Also, as it appeared in these group interactions, students took the work to be independent in such a way that they struggled with the activities for long before they asked the teacher for help.

The following chapter is an overview of the whole study in terms of research questions and the theoretical framework. It also looks at future research possibilities and improvements on teaching and learning mathematics through discourse.

CHAPTER SEVEN: CONCLUSIONS AND RECOMMENDATIONS

It is now time for me to reflect on the research questions I posed at the beginning of the thesis and to look at how I have or have not responded to them. Did I gain insight into the discourses that emerge in a mathematics classroom? What, teacher or student, actions and/or interactions emerge into mathematics classroom discourses? In what ways did my research methods and theoretical frameworks play a role in this study? And now that I looked at mathematics classroom discourses, if I were to extend or improve this area of study, what would I do next? For the purpose of doing research is not only to find answers to questions but also to pose more questions related or unrelated to the current questions.

This study arose out of my experience as a high school mathematics teacher working with the context of a new approach (leaner-centered approach) in Namibian education and my exposure to the North American research literature focused on teaching and learning. Interested in how possibilities for discussions and conversations emerge and evolve in a mathematics classroom, I conducted a qualitative research study in which I used the enactivist perspectives to make sense of the interactions that arose in the class and the possibilities that these interactions had provided. The study involved a one mathematics classroom consisting of 33 students, one visiting in-service teacher and one mathematics teacher.

A total number of 16 observations were carried out over a one-term period. Data were created from these observations as participants interactively worked on mathematics activities as a whole class and in small groups. One type of discourse observed in the emerging discourses was brainstorming; mathematical ideas emerged and hence gave rise to new ideas as the participants' ideas kept bumping against one other. In whole-class discussions, several possibilities for talking about mathematics had emerged; some of which where taken up for further discussions and the discussions themselves. Teacher response to students' behaviours served as one of the effects on the life span of the emerging ideas that were taken up for discussions. Also valuable was the students' engagement with mathematics as they worked on and talked about assigned activities in small groups. With very little or no intervention from the teacher, the students showed willingness to make meaning of mathematics, which to some extend was not only allowed but also constrained by the social interactions among the students and by their language capabilities.

Emergence and Evolution of Mathematics Classroom Discourses

Using enactivism as both my theoretical framework and research method helped me observe this mathematics class—the teacher, in-service teacher and the students—as a learning system. In this research as I observed the participants interacting with one another, I was also able to reflect on my observations as I observed myself interacting with my own thoughts, the participants and their thoughts.

In the two chapters of the discussion of the data, the study shows how discourses emerged and evolved from student interactions with one another, with the teacher and with the mathematics that they together brought forth in those interactions. The discussions and conversations that emerged with the participants' interactions were not just the verbal communications but included non-verbal ones, such as gestures. Such gestures included signs as smiling, standing up, silence, taking away a chalk, noisy movements towards the end of the lesson, and the utterances written on the chalkboard and in the students' workbooks. However, due to the scope of my research, I kept my attention on the verbal interactions.

The study also shows that there still is room in a mathematics classroom for student to make meaning of mathematics if they are allowed to freely engage in mathematical discourse by questioning the teacher and other students' mathematical thinking. At the same time, many episodes discussed herein present evidence on how the teacher's actions and responses to students' mathematical behaviors may shut down the possibilities of potential discourses from which students can learn by talking about mathematics and their experiences with that mathematics.

The main driving force encouraging the involvement of students in lessons in Namibian schools, as discussed in the beginning is the notion of the learner-centered approaches to learning. Despite the theoretical claims and the government mandates that supports the use of such approaches these were not well understood or practiced in the mathematics classroom I studied. The way that students participated in the class studied included a scattered arrangement of desks (rather than rows); students going to the chalkboard to answer the teacher's questions; and students standing up to respond to the teacher's questions.

Useful to my study is Nkoma's (2002) research that looked at South African teacher and "black" students' views of what counts as success in learning mathematics as far as learner-centered education is concerned. Namibia and South Africa have a lot in common and share a history back from the colonial era to their independence moments. The political situations in both countries had had a large impact on the educational systems. A systematic analysis of these commonalities is well discussed in O'Callaghan (1977), Mbamba (1981; 1982) and Cohen (1994). Considering these common backgrounds between the two educational systems, it is therefore appropriate for me to adopt Nkhoma's interpretations of what he calls "learner-centered instruction" and what I refer here to as learner-centered teaching and learning. As I mentioned in the beginning, the Namibian educational system at large has been working toward a more learner-centered educational environment in all the schools. This is, however not to mean that teacher-centered education is being rejected, but that a healthy balance between the two approaches is looked forward to. Moreover, the notion of teacher-centered education and learner-centered education are not different approaches to school curriculum, but rather two sides of the same coin (Fleener, 2002).

Since my study looked at students' involvement in the generation of discourses, and knowing that teaching methods have an impact on how students participate in the classroom, I find it helpful to look at the mathematics the class engaged in my study in terms of teacher-centeredness and learner-centeredness. Below are some of the observable features of both teacher-centered and learner-centered approaches to teaching that Nkhoma pointed out. Here I discuss them in relation to the mathematics classroom that I had observed.

- In teacher-centered "instruction", according to Nkhoma, the degree of teacher talk is greater than that of the students, while in learner-centered "instruction", student talk is as much or greater that that of the teacher.

As can be seen from most of the episodes presented from the classroom I observed, students did not frequently engaged in talk. Students talked when they were either called upon or volunteered a response to the teacher's questions. However, there were a few lessons in which evidence could be seen that students tried to communicate their mathematics understanding in this class. For example in the brainstorming lesson, students actively engaged in bringing up their ideas occasioned and bumped up against other students' ideas.

One example of a case when a student acted independently of a teacher's request was when the class was answering a previous homework exercise, and Toivo told the class that he had a solution method different from the method that was used by the teacher. In this case, Toivo was not obliged to answer any pre-specified question but he acted mathematically to point out how he did his homework differently from everybody else.

The second case was when the class had already finished factorizing a difference between the two squares, $9a^2$ and $25b^2$. Just after the teacher finished writing the final answer on the chalkboard, Nakale spontaneously suggested that the class tries to factorize the sum of those same squares. Even though when the teacher and many other students expressed that it was not possible to factorize such an expression. Nakale did not succumb to their "No!" statement. He insisted that he could factorize that expression and even offered up to the chalkboard to show he could do that. However, Nakale's mathematical behavior was finally shut down when the teacher made a final statement that what Nakale was doing was not factorizing and therefore the expression was not factorizable. This instance suggests that there is still a need for teachers, especially in Namibian mathematics classrooms to allow students act out their mathematical behaviors and present their reasoning instead of giving students an attitude that inhibit their mathematical thinking.

- The second feature of a teacher-centered class is that teaching is directed more to the whole class and less to individuals or small groups. In learner-centered cases, focus is on individual or small-group instruction.

Visible in the evidence from the class I studied, most of the teaching took place class-wide within the routine: the teacher introduces a topic, offers an example, solves one or two related problems and gives students seat work to work on, which is followed by a series of homework exercises. When stress is put on such a routine, the notion of learner-centered teaching and learning fades away. In my conversations with Ndapewa, she communicated to me some of the reasons why, she personally, and perhaps other mathematics teachers in similar situations, tend to use this more teacher-centered teaching method. One of the reasons was the pressure that the mathematics curriculum puts on the mathematics teacher. "Classrooms are over crowded. You might get a class of 35 to 38 students". Even though a teacher might be willing to individualize her teaching or focus it on small groups, she or he might not be able to effectively do that due to the large number of students that she or he has to cater for. Until something is done to improve the teacher-student ratio in Namibian classrooms, improving mathematics teaching and learning by involving students in discourses will not be easy to obtain.

- Teacher-centered teachers "rely heavily upon text book" (Nkhoma, 2002, p.) in their teaching, while learner-centered teachers consider a variety of teaching resources and materials to be used either by the teacher or the students.

In the observed classroom, all the exercises and notes that were used in the classroom were directly taken from a textbook. The only other materials that were available in this class were: the chalkboard, the chalks and the students note books. It is therefore the teacher's responsibility to make sure that he or she keeps a variety of resources from which students could learn. Some of these resources include the mathematical ideas that emerged within the class' interactions. Instead of shutting down such ideas, the teacher together with the students may take them up for discussions or conversations, enabling providing students more opportunities to learn how to behave mathematically and develop their mathematical understanding rather than simply focusing on the manipulation of numbers and symbols.

- In classrooms where a teacher-centered approach is frequently used, there is a fixed arrangement of student desks into rows with students "facing the chalkboard with a teacher's desk nearby (Nkhoma, 2002, p.). In learner-centered classrooms, student desks may be rearranged differently to enhance the possibilities for fruitful student interaction.

As discussed in chapter four under the discussion of the classroom set-up, there was no fixed seat arrangement in this classroom. Students moved freely around the classroom and had choices on whether to sit alone, in pairs or small groups.

Recommendations

It is evident from the discussions presented in the two chapters of data that students in this mathematics classroom were willing to talk about mathematics through the activities they did either in whole-class discussions or small groups. While trying to implement the learner-centered approach in their teaching, the two teachers involved in this study also seemed to have allowed, to a certain degree, students to take part in some of the activities. However, the students' participation was restricted to answering teacher's questions and less on taking up students' own ideas for discussions and conversations.

Also there were no individual or group projects or assignments to allow students to actively engage with mathematics, other than the homework questions they copied from the chalkboard. As evident in some of the conversations I had with the teacher, when emphasis is put on completing the curriculum in time having too many discussions or assigning projects outside the curricular documents takes up valuable time and may result in not completing the program.

The problems above imply two things for future research to focus on. There is a need to address Namibian mathematics teachers' philosophy of or/and beliefs about actively engaging students in teaching mathematics through discourse. Their beliefs may play a role in the teacher's actions and responses towards students' behaviors. Teacher training programs and in-service training through workshops need to point out to help teachers the importance of letting students talk about mathematics among themselves or with the teacher, without having students feel intimidated when they are shown to be wrong. Research studies need to be undertaken to closely evaluate the curriculum and look for possibilities for improvement and the ways that enable teachers to actively engage their students in doing mathematics without the pressure and limitations from the curriculum.

Following from observations discussed in this thesis, was also the emergent question that allows me to extend on this study in future: the question of teacher and student mathematical interpretations. The teacher, the in-service teacher and the students engaged in this research appeared to have different perceptions of different mathematical concepts that were explored in the activities done. These interpretations play a big role in the students' understanding and hence in the way they learn mathematics. As Confrey and Smith (1994) suggest, it is important that research pays "close attention to how a mathematics problem is conceptualized, worked on and evaluated by students" (p. 31). There is a need for analyses that take students' interpretations of mathematical activities into consideration. I therefore find it useful to study this phenomenon through further observations and interpretations of individuals or groups engaged in mathematics activities in or outside a mathematics classroom.

One of the discrepancies I had observed in the lessons of this class was the lack of social sharing of knowledge with the whole class especially from group discussions. What emerged in individual groups ended at the group level and did not go beyond that to the whole-class level as a collective. This would have been a learning opportunity, especially that, as the teacher herself observed, different working groups came up with different solution methods of working out the assigned activities.

Mathematics education in Namibia is in the process of evolving from teacher-centered instruction to leaner-centered instruction. In my study I have illustrated how one particular teacher and her students enacted this mandate. The teacher working from a position of experience included the learners in ways that made sense to her lessons. Students demonstrated their willingness to participate in the lessons but as I observed the mathematics was focused on procedure and not problem creating or/and solving. To a great extent discourse even in this limited circumstance contributed to interactive learning and opened up possibilities for deeper questions on mathematics. Clearly opportunities are there but it is a question of whether we, as teachers listen or pay attention *to* them in order for those possibilities to be recognized, valued and taken up for discussions, conversations, debates or argumentations.

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APPENDICES

Appendix A

A diagrammatic representation of the classroom in which this research study took place.