UNIVERSITY OF ALBERTA

HEREDITARY PROPERTIES OF INFINITE-DIMENSIONAL BANACH SPACES

BY

PETR HABALA ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

The main results of the thesis are two constructions connected with hereditary properties of infinite-dimensional Banach spaces. We transfer an Asymptotic Biorthogonal System to general real interpolation spaces. We also construct a hereditarily indecomposable Banach space all of whose closed infinite-dimensional subspaces fail the Gordon–Lewis property.

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Introduction And Notation

1.a. Introduction

Recent progress in the theory of infinite-dimensional Banach spaces, which brought solutions of many problems open for decades, is based on new understanding of phenomena taking place in every infinite-dimensional subspace of a given Banach space. These phenomena are geometric by nature, but they also have a deep effect on hereditary structural properties of underlying Banach spaces (that is, structural properties shared by all infinite-dimensional subspaces of the Banach space).

Recall one of the fundamental notions in the Banach space theory—the notion of a Schauder basis. It provides us with a coordinate system in a Banach space. Not every separable Banach space has a basis, this deep result was proved only in 1973. However, it is a classical result that every Banach space contains an infinite-dimensional subspace with a basis.

Among important possible properties of a basis is unconditionality. A basis is unconditional if the norm of a vector essentially does not depend on the signs of its coordinates. For instance, an orthonormal basis of a Hilbert space is unconditional. An unconditional basis gives the space in question a certain symmetry, which greatly facilitates its analysis. However, in general it is not easy to find such a basis, and even some classical spaces (like $L_1[0,1]$ or $C[0,1]$) do not have it, although they have a Schauder basis. It was hoped for a long time that every Banach space at least admits a subspace with an unconditional basis. Only recently (1991) this turned out not to be true as demonstrated by an example by W. T. Gowers and B. Maurey ([GM]). They constructed a Banach space such that its all closed infinite-dimensional subspaces do not have an unconditional basis.

We remark that the Gowers–Maurey space has a stronger property, it is Hereditarily Indecomposable. This means that none of its subspaces can be written as a topological direct sum of two closed infinite-dimensional subspaces. Using similar methods, many examples with interesting properties were constructed; however, their discussion is beyond the scope of this introduction.

Another circle of problems solved recently is connected with behaviour of uniformly continuous functions on spheres of infinite-dimensional Banach spaces. We focus on the case of equivalent norms. A Banach space X is called distortable if there is an equivalent norm for which the maximal ratio of its values on the unit sphere of the original norm cannot be made arbitrarily close to 1 by passing to infinite-dimensional subspaces. That is, there exists $\lambda > 1$ so that for every infinite-dimensional subspace Y of X, the equivalent norm oscillates at least λ on the original unit sphere of Y. We will give the precise definition in the second chapter. In 1991, T. Schlumprecht built the first example of an arbitrarily distortable Banach space ([Sl]) by a construction related to the classical and well-known definition of Tsirelson's space ([T]). This result was the starting point for the Gowers–Maurey construction mentioned above and many other results. In particular, in 1992, E. Odell and T. Schlumprecht proved that the space ℓ_2 is arbitrarily distortable, thus solving another long-standing problem ([OS]). This shows that if a Banach space X is isomorphic to the Hilbert space ℓ_2, it does not necessarily contain subspaces almost isometric to ℓ_2.

The geometric concept underlying all these results is the notion of an Asymptotic Biorthogonal System introduced by Gowers and Maurey. An Asymptotic Biorthogonal System in a Banach space X provides an infinite sequence of subsets of the unit sphere of X which are well separated and each set almost intersects every infinite-dimensional subspace of X. The presence of such a system in a Banach space has many consequences for its geometry and hereditary properties. As one of preliminary results of [GM], Gowers and Maurey constructed an Asymptotic Biorthogonal System in Schlumprecht's space; this already implies that the space is arbitrarily distortable. They also proved that this implies poor unconditional properties of every infinite-dimensional subspace. We will give precise statements in the third chapter.

This brings us back to the unconditionality. It is usually difficult to determine whether a given Banach space has an unconditional basis. There are very few general conditions known which would ensure the existence of such a basis or prove that the space does not have one. An important step in this direction was made in the early 70's by Y. Gordon and D. R. Lewis, who introduced what is now called Gordon–Lewis property (usually referred to as the GL-property). This property is in an essential way weaker than the existence of an unconditional basis; however, in many situations it is easier to handle. It also behaves well with respect to passing to complemented subspaces. We remark that this is definitely not the case with an unconditional basis: it is still an open problem whether a complemented subspace of a space with an unconditional basis needs to have an unconditional basis. The

GL-property became a standard notion and an effective tool in the local theory of Banach spaces.

The GL-property was used by Gordon and Lewis ([GL]) to investigate unconditional properties of certain spaces of operators on ℓ_2. In particular, they showed that the classical Schatten classes do not have the GL-property. We recall that the spaces $C[0,1]$ and $L_1[0,1]$ have the GL-property but do not have an unconditional basis. It is not known whether the Gowers–Maurey space without an unconditional basic sequence has a closed infinite-dimensional subspace with the GL-property.

In this thesis we study the two notions we introduced last, in the context of hereditary properties. In the second chapter we prove that an Asymptotic Biorthogonal System can be transferred to some general real interpolation spaces from the space serving as a pattern space for our interpolation method. This generalizes an earlier construction by Maurey ([M]). In the third chapter we construct a Banach space all of whose closed infinite-dimensional subspaces fail the GL-property. Thus we answer in the negative the question raised by W. B. Johnson ([J]) whether every Banach space has a subspace verifying the GL-property.

1.b. Notation

We will use the standard Banach space notation as found e.g. in [LT] and [T-J]. Let us briefly recall the most important notions.

By B_X we mean the closed unit ball of a Banach space X, S_X stands for its unit sphere. By a basic sequence we mean a sequence $\{e_i\}_{i=1}^{\infty}$ in a Banach space X such that for some $\lambda > 0$ we have $\left\|\sum_{i=1}^{n} \alpha_i e_i\right\| \leq \lambda \left\|\sum_{i=1}^{m} \alpha_i e_i\right\|$ for all scalars $\alpha_1, \ldots, \alpha_m$ and $n < m \in I\!N$. The smallest such λ is the basic constant of $\{e_i\}$. A basic sequence is called bimonotone if $\left\|\sum_{i=k}^{n} \alpha_i e_i\right\| \leq \left\|\sum_{i=l}^{m} \alpha_i e_i\right\|$ for all scalars $\alpha_1, \ldots, \alpha_m$ and $1 \leq l \leq k < n \leq m \in I\!N$. A basic sequence $\{e_i\} \subset X$ is called a basis of X if $\overline{\text{span}}\{e_i\} = X$.

A sequence $\{e_i\}$ is called an unconditional basic sequence if there is $K > 0$ such that $\left\|\sum_{i=1}^{m} \varepsilon_i a_i e_i\right\| \leq K \left\|\sum_{i=1}^{m} a_i e_i\right\|$ for all scalars a_i and signs $\varepsilon_i = \pm 1$, $i = 1, \ldots, m$, $m \in I\!N$ arbitrary. The smallest such K is denoted by $\text{ubc}\{e_i\}$—the unconditional basic constant of $\{e_i\}$.

By an interval of natural numbers we mean a set $\{a, a+1, \ldots, b\}$ for some natural numbers $a < b$. Let X be a Banach space with a bimonotone basis $\{e_i\}$. For a vector

$x = (x_i)$ with a finite number of non-zero coordinates we write $\mathrm{ran}(x)$ to denote the range of x, that is, the smallest interval of natural numbers so that $i \in \mathrm{ran}(x)$ whenever $x_i \neq 0$. By $x < y$ we mean that ranges of x and y are successive in $I\!N$, $\max(\mathrm{ran}(x)) < \min(\mathrm{ran}(y))$. We say that vectors $x_1, x_2, \ldots$ are successive if $x_1 < x_2 < \ldots$.

By a block basic sequence we mean a sequence of successive non-zero vectors (which must be a basic sequence), the subspace spanned by such a sequence is called a block subspace. For an interval E of natural numbers we consider the corresponding projection in X and denote it by E again, that is, $E(e_i) = e_i$ if $i \in E$ and $E(e_i) = 0$ otherwise.

Let X, Y be Banach spaces. By an operator $A: X \mapsto Y$ we mean a continuous linear map from X to Y. Finally, unless specified otherwise, by a "subspace" we mean a closed infinite-dimensional subspace.

Chapter 2
Asymptotic Biorthogonal Systems

2.a. General Background on ABS

Let X be a Banach space and consider sets $A_k \subset S_X$, $A_k^* \subset B_{X^*}$ for $k \in \mathbb{N}$.

Definition 2.a-1.
The sets A_k, A_k^*, $k \in \mathbb{N}$, form an ABS—an Asymptotic Biorthogonal System—if there is $\delta > 0$ and a sequence $\varepsilon_k \searrow 0$ such that the sets satisfy:

(a) A_k are asymptotic, i.e. $\operatorname{dist}(Y, A_k) = 0$ *for every k and every infinite-dimensional subspace Y of X,*

(b) A_k^* is δ-norming for A_k for every $k \in \mathbb{N}$, that is, for every $x \in A_k$ there is $x^* \in A_k^*$ such that $X^*(x) \geq \delta$,

(c) for all $k \neq l$ and $x \in A_k$, $x^* \in A_l^*$ we have $|x^*(x)| < \varepsilon_{\min(k,l)}$.

This notion was introduced by W. T. Gowers and B. Maurey in [GM]. It turned out to be very important in recent developments of the Banach space theory because of its connection with unconditionality and distortions. Gowers and Maurey proved that if a separable space X contains an ABS, then given K, one can find an equivalent norm on X so that for every unconditional basic sequence $\{x_k\}$ in X we have $\operatorname{ubc}\{x_k\} \geq K$ with respect to the new norm ([GM]). E. Odell and T. Schlumprecht showed in [OS] that a separable space which admits an ABS can be also renormed, given $n \in \mathbb{N}$ and $\varepsilon > 0$, so that its every subspace contains $(1+\varepsilon)$-isomorphic copies of all n-dimensional Banach spaces with a monotone basis. N. Tomczak-Jaegermann observed that these copies are uniformly complemented.

Consider a numerical invariant $\alpha(\cdot)$ of Banach spaces that satisfies the following conditions:

—there is a function $f\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that $\alpha(X) \leq f\big(\mathrm{d}(X,Y)\big)\alpha(Y)$ for all isomorphic Banach spaces X, Y, where $\mathrm{d}(X,Y)$ denotes the Banach-Mazur distance,

—there is a function $g\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that $\alpha(Z) \leq g(C)\alpha(Y)$ whenever Z is a C-complemented subspace of Y,

—there is a sequence of finite-dimensional spaces E_n with bimonotone bases such that $\alpha(E_n) \to \infty$.

Many parameters used and investigated in the local theory of Banach spaces satisfy conditions of this kind. Representing the spaces E_n in a suitable renorming as above we obtain the following theorem by Tomczak-Jaegermann:

If X is a separable Banach space that admits an ABS, then given K, there is an equivalent norm $|\!|\!|\cdot|\!|\!|$ on X such that $\alpha(Y, |\!|\!|\cdot|\!|\!|) \geq K$ for every closed infinite-dimensional subspace Y of X.

Of equal importance is the connection between an ABS and distortions. Recall that given an equivalent norm $|\!|\!|\cdot|\!|\!|$ on a Banach space X, its level of distortion is defined by

$$\mathrm{d}(|\!|\!|\cdot|\!|\!|) = \inf \sup\{|\!|\!|x|\!|\!|/|\!|\!|y|\!|\!|;\ x, y \in S_Y\},$$

where the infimum runs over all closed infinite-dimensional subspaces Y of X. We are interested in the case when $\mathrm{d}(|\!|\!|\cdot|\!|\!|) > 1$, that is, $|\!|\!|\cdot|\!|\!|$ does not stabilize to a constant on unit spheres of infinite-dimensional subspaces of X. If there exists such an equivalent norm on X, we say that X is distortable. It was shown in the late 80's by Odell that the Tsirelson space is distortable. In 1991, Schlumprecht constructed an arbitrarily distortable space, that is, a space that admits equivalent norms whose levels of distortion go to infinity ([Sl]).

Notice that the existence of a distorted norm on a Banach space X implies the existence of two well separated asymptotic sets in a certain subspace Y of X; the existence of these sets in Y in turn implies that Y is distortable. Let us prove these facts.

If $|\!|\!|\cdot|\!|\!|$ is a distortion on X, for every $\delta \in (0,1)$ one can find a subspace Y of X, sets $A, B \subset S_Y$, $A^* \subset S_{Y^*}$, and $\varepsilon \leq (2-\delta)/\mathrm{d}(|\!|\!|\cdot|\!|\!|)$ such that

—A, B are asymptotic,

—A^* is δ-norming for A,

—$|x^*(x)| < \varepsilon$ for every $x^* \in A^*$ and $x \in B$.

Indeed, given $\delta > 0$, fix $\eta > 0$ to be determined later. Let λ be the infimum of numbers $\sup\{|\!|\!|y|\!|\!|;\ y \in S_Z\}$ taken over all infinite-dimensional subspaces Z of X. Since $|\!|\!|\cdot|\!|\!|$ is an equivalent norm, we have that $\lambda > 0$. Fix a subspace Y such that $\sup\{|\!|\!|y|\!|\!|;\ y \in S_Y\} < \lambda(1+\eta)$. Thus $|\!|\!|y|\!|\!| < \lambda(1+\eta)$ for every $y \in S_Y$. Note that for every infinite-dimensional subspace Z of Y there is $z \in S_Z$ such that $|\!|\!|z|\!|\!| > \lambda(1-\eta)$. Using the definition of the level of distortion we also find a vector $z \in S_Z$ such that

$|||z||| < (1+\eta)^2\lambda/\operatorname{d}(|||\cdot|||)$. We define

$$A = \{z \in S_Y;\ |||z||| > \lambda(1-\eta)\},$$
$$B = \{z \in S_Y;\ |||z||| < \lambda(1+\eta)^2/\operatorname{d}(|||\cdot|||)\}.$$

It follows from our observation above that both sets are asymptotic. Let

$$C = \{x^* \in B_{(X,|||\cdot|||)^*};\ \text{there is } x \in A \text{ such that } x^*(x) = |||x|||\},$$
$$A^* = \{y^* = x^*/\|x^*\|_* \in S_{Y^*};\ x^* \in C\},$$

where $\|\cdot\|_*$ denotes the dual norm of $(Y,\|\cdot\|)$, similarly we use $|||\cdot|||_*$ for the dual norm of $(Y,|||\cdot|||)$.

Let $x \in A$ and let $x^* \in C$ be such that $x^*(x) = |||x|||$. From $|||\cdot||| \leq \lambda(1+\eta)\|\cdot\|$ on Y we conclude that $\|\cdot\|_* \leq \lambda(1+\eta)|||\cdot|||_*$, that is, $\|x^*\|_* \leq \lambda(1+\eta)$. On the other hand, $\|x^*\|_* \geq x^*(x)/\|x\| = |||x||| > \lambda(1-\eta)$.

It follows that

$$y^*(x) = x^*(x)/\|x^*\|_* \geq \lambda(1-\eta)/\big(\lambda(1+\eta)\big) = \frac{1-\eta}{1+\eta}.$$

Also, for any $y \in B$,

$$|y^*(y)| \leq |||y||| \cdot |||x^*|||_*/\|x^*\|_* < \lambda(1+\eta^2)/\operatorname{d}(|||\cdot|||)/\lambda(1-\eta) = \frac{(1+\eta)^2}{(1-\eta)}\frac{1}{\operatorname{d}(|||\cdot|||)}.$$

Choosing $\eta > 0$ small enough we obtain the desired estimates.

Conversely, given a subspace Y, numbers $\delta, \varepsilon > 0$, and sets $A, B \subset S_Y$, $A^* \subset S_{Y^*}$ as above, we define

$$|||y||| = \|y\| \vee \sup\{x^*(y)/\varepsilon;\ x^* \in A^*\}.$$

One easily checks that $\|y\| \leq |||y||| \leq \|y\|/\varepsilon$ for $y \in Y$ and that $\operatorname{d}(Y,|||\cdot|||) \geq \delta/\varepsilon$. In particular, if a Banach space has an ABS, then it is arbitrarily distortable.

Gowers and Maurey observed that Schlumprecht's space actually contains an ABS, which is probably stronger than being arbitrarily distortable. This was an important observation as, unlike norms, the sets forming an ABS can be often transferred to other Banach spaces if we have a suitable map. This is how Odell and Schlumprecht showed that ℓ_2 is arbitrarily distortable; they transferred an ABS from Schlumprecht's space into ℓ_2.

An ABS can also be transferred if the connection between the spaces is not given by a map, but one space is determined in some sense by the other. This is the case when spaces are built by means of interpolation, which we address in the next section.

2.b. Interpolation Spaces And ABS

In this section we consider a class of general real interpolation spaces and prove the existence of ABS on such spaces. Since we work in a general context which is not covered by standard references, we first briefly prove some basic facts.

Let X_0, X_1 be Banach spaces that are subspaces of some linear space (so that $X_0 + X_1$ makes sense). For this couple, denoted by $\overline{X} = (X_0, X_1)$, let $\Sigma\overline{X}$ denote $X_0 + X_1$ endowed with the norm

$$\|x\| = \inf\left\{\|x_0\|_{X_0} + \|x_1\|_{X_1};\ x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\right\}.$$

Similarly, by $\Delta\overline{X}$ we denote the space $X_0 \cap X_1$ with the norm $\max(\|\cdot\|_{X_0}, \|\cdot\|_{X_1})$. We also write $\overline{X^*}$ for the couple (X_0^*, X_1^*).

For $t \in I\!R_+$, let

$$\begin{aligned} K(t,x) &= \inf\left\{\|x_0\|_{X_0} + t\|x_1\|_{X_1};\ x = x_0 + x_1, x_i \in X_i\right\} && \text{for } x \in \Sigma\overline{X}, \\ J(t,x^*) &= \max(\|x^*\|_{X_0}, t\|x^*\|_{X_1}) && \text{for } x^* \in \Delta\overline{X^*}. \end{aligned}$$

These are the standard functionals used in the interpolation theory. For a fixed x they define an increasing function of t; and for a fixed t they define equivalent norms on $\Sigma\overline{X}$ and $\Delta\overline{X^*}$, respectively. We also have $\left(\Sigma\overline{X}, K(t,\cdot)\right)^* = \left(\Delta\overline{X^*}, J(t^{-1},\cdot)\right)$ for every $t > 0$.

Fix $\theta \in (0,1)$. For every integer $n \in \mathbb{Z}$ we define the following functions:

$$\begin{aligned} \mathrm{k}_n(x) &= \inf\left\{2^{-\theta n}\|x_0\|_{X_0} + 2^{(1-\theta)n}\|x_1\|_{X_1};\ x = x_0 + x_1, x_i \in X_i\right\} && \text{for } x \in \Sigma\overline{X}, \\ \mathrm{j}_n(x^*) &= \max(2^{\theta n}\|x^*\|_{X_0^*}, 2^{(\theta-1)n}\|x^*\|_{X_1^*}) && \text{for } x \in \Delta\overline{X^*}. \end{aligned}$$

That is, $\mathrm{k}_n(x) = 2^{-\theta n}K(2^n, x)$ and $\mathrm{j}_n(x^*) = 2^{\theta n}J(2^{-n}, x^*)$. Therefore k_n and j_n are equivalent norms on $\Sigma\overline{X}$ and $\Delta\overline{X^*}$, respectively, and we have the duality result $(\Sigma\overline{X}, \mathrm{k}_n)^* = (\Delta\overline{X^*}, \mathrm{j}_n)$, in particular $|x^*(x)| \le k_n(x)j_n(x^*)$ for $x \in \Sigma\overline{X}$ and $x^* \in \Delta\overline{X^*}$.

Let E be a Banach space of two-sided sequences and let $\{e_n\}_{n\in\mathbb{Z}}$ denote the standard unit vector basis of E. Assume that $\{e_n\}$ is 1-unconditional, shrinking, and also C-translation invariant, i.e. $\|\{a_{n+m}\}_{n\in\mathbb{Z}}\|_E \le C\|\{a_n\}_{n\in\mathbb{Z}}\|_E$ for every $\{a_n\}_{n\in\mathbb{Z}} \in E$ and $m \in \mathbb{Z}$.

In particular, E is a discrete Banach lattice, and we will use the following standard notation. For $x = \{a_n\}_{n\in\mathbb{Z}} \in E$, we use $|x|$ to denote $\{|a_n|\}_{n\in\mathbb{Z}}$. Obviously, $|x| \in E$ and $\| \, |x| \, \|_E = \|x\|_E$. The analogous convention is used in E^* as well.

Definition 2.b-1.
Let $\overline{X}$ be a couple of Banach spaces. The K-real interpolation space $\overline{X}_{\theta,E;K}$ consists of all vectors $x \in \Sigma\overline{X}$ for which $\{k_n(x)\}_{n\in\mathbb{Z}}$ is in E; we define the interpolation norm by $\|x\|_{\theta,E;K} = \|\{k_n(x)\}_{n\in\mathbb{Z}}\|_E$.

Theorem 2.b-2.
Let $\theta \in (0,1)$ and E be as above. The functor $\overline{X} \mapsto \overline{X}_{\theta,E;K}$ is an interpolation method of exponent θ.

PROOF: Let $\overline{X}$ be a couple of Banach spaces. It is easy to show that $\overline{X}_{\theta,E;K}$ is a Banach space. Using $K(1,x) \leq \min(1,s)K(s,x)$ we readily obtain

$$\|x\|_{\Sigma\overline{X}} \leq \|\{2^{-n\theta}\min(1,2^n)\}_n\|_E \cdot \|x\|_{\theta,E;K}$$

and the norm of $\{2^{-n\theta}\min(1,2^n)\}_n$ is clearly finite. Thus we get that $\overline{X}_{\theta,E;K} \subset \Sigma\overline{X}$ and the inclusion is continuous. Using the inequality $K(t,x) \leq \min(1,t)\|x\|_{\Delta\overline{X}}$ we similarly see that $\Delta\overline{X} \subset \overline{X}_{\theta,E;K}$ and the inclusion is continuous.

In order to prove that $\overline{X}_{\theta,E;K}$ is, indeed, an interpolation space of exponent θ, we consider a linear mapping $T\colon \Sigma\overline{X} \mapsto \Sigma\overline{Y}$ such that $T|_{X_i}$ is a continuous operator into Y_i for $i = 0,1$. Let M_i be the operator norm of $T|_{X_i} \in \mathcal{B}(X_i, Y_i)$. We want to show that $T|_{\overline{X}_{\theta,E;K}}$ is an operator into $\overline{Y}_{\theta,E;K}$ and that $\|T\colon \overline{X}_{\theta,E;K} \mapsto \overline{Y}_{\theta,E;K}\| \leq cM_0^{1-\theta}M_1^{\theta}$ for some absolute constant c.

Let $x \in \overline{X}_{\theta,E;K}$. Using the boundedness of T restricted to X_0, X_1 we observe that $K(t,T(x)) \leq M_0K(tM_1/M_0, x)$. Since $K(s,x)$ is increasing in s, we may denote $m = [\log_2(M_1/M_0)] + 1$ and estimate $k_n(T(x)) \leq M_0 2^{m\theta} k_{n+m}(x)$. Since the basis of E is 1-unconditional and C-translation invariant, we get

$$\|T(x)\|_{\overline{Y}_{\theta,E;K}} \leq CM_0 2^{m\theta}\|x\|_{\overline{X}_{\theta,E;K}} \leq C2^{\theta}M_0^{1-\theta}M_1^{\theta}\|x\|_{\overline{X}_{\theta,E;K}}.$$

□

We will also require a duality result. For this, we define $\overline{X^*}_{\theta,E^*;J}$ as the space consisting of vectors $x^* \in \Sigma\overline{X^*}$ that can be written as a sum $x^* = \sum_{n\in\mathbb{Z}} x_n^*$ in $\Sigma\overline{X^*}$

such that $x_n^* \in \Delta\overline{X^*}$ and $\{j_n(x_n^*)\}_n \in E^*$. For such vectors we define $\|x^*\|_{\theta,E^*;J}$ as the infimum of $\|\{j_n(x_n^*)\}_n\|_{E^*}$ over all possible decompositions of x^* as above.

Theorem 2.b-3.
Let $\theta \in (0,1)$ and E be as above, let $\overline{X}$ be a couple of Banach spaces. If $\Delta\overline{X}$ is dense in $\overline{X}_{\theta,E;K}$, then every $u \in \overline{X^*}_{\theta,E^*;J}$ defines an element $x^* \in \left(\overline{X}_{\theta,E;K}\right)^*$ and $\|x^*\|^*_{\overline{X}_{\theta,E;K}} \leq \|u^*\|_{\theta,E^*;J}$.

PROOF: Consider $u \in \overline{X^*}_{\theta,E^*;J}$ and let $u = \sum x_n^*$ be a decomposition as in the definition. We will show that this vector defines a continuous functional on the space $\left(\Delta\overline{X}, \|\cdot\|_{\theta,E;K}\right)$, whose norm is bounded by $C\|\{\mathrm{j}_n(x_n^*)\}_n\|_{E^*}$.

Indeed, let $x \in \Delta\overline{X}$. Then using the duality of k_n and j_n we obtain

$$
\begin{aligned}
|u(x)| &\leq \sum_{n\in\mathbb{Z}} |x_n^*(x)| \leq \sum_{n\in\mathbb{Z}} \mathrm{j}_n(x_n^*)\,\mathrm{k}_n(x) \\
&\leq \|\{j_n(x_n^*)\}_n\|_{E^*}\,\|\{k_n(x)\}_n\|_E .
\end{aligned}
$$

□

Let A_k, A_k^* be an ABS in a Banach lattice X. It is called unconditional provided for every $k \in \mathbb{N}$ we have $x \in A_k$ iff $|x| \in A_k$ and $x^* \in A_k^*$ iff $|x^*| \in A_k^*$. Let A_k^+ denote $\{|x|;\ x \in A_k\}$. In the following lemma we identify properties of unconditional ABS needed in our next construction.

Lemma 2.b-4.
Let E be a Banach space with a 1-unconditional and shrinking basis $\{e_n\}_{n\in\mathbb{Z}}$. Let A_k, A_k^* be an unconditional ABS in E. Then for every $k \in \mathbb{N}$ the following is true:

- *given $\varepsilon > 0$ and vectors $u_1, u_2, \ldots$ with disjoint supports and non-negative coordinates, there are numbers $a_1, \ldots, a_M \geq 0$ such that $\operatorname{dist}(\sum a_i u_i, A_k^+) < \varepsilon$,*
- *let $\delta > 0$ be the constant from (b) in the definition of ABS; if $x \in A_k^+$, then there is $x^* \in A_k^*$ such that x^* has non-negative coordinates and $x^*(x) \geq \delta$.*

PROOF: Since $\operatorname{dist}(\overline{\operatorname{span}}\{u_i\}, A_k) = 0$, there are scalars $c_1, \ldots, c_M$ and $x \in A_k$ such that $\|\sum c_i u_i - x\|_E < \varepsilon$. As u_i have disjoint supports and non-negative coordinates, we have for every coordinate $\left|\left(\sum c_i u_i\right)_n\right| = \sum |c_i| \left(u_i\right)_n$, hence

$$
\left|\left(\sum |c_i| u_i - |x|\right)_n\right| \leq \left|\left(\sum c_i u_i - x\right)_n\right|.
$$

Therefore

$$\left\| \sum |c_i| u_i - |x| \right\|_E \le \left\| \sum c_i u_i - x \right\|_E < \varepsilon$$

and $|x| \in A_k^+$ as needed.

The second property follows easily from $x = |x|$ for $x \in A_k^+$ and

$$x^*(x) \le |x^*|(|x|).$$

□

We will now transfer an unconditional ABS from E into $\overline{X}_{\theta,E;K}$. The construction was used by Maurey in [M]. It is interesting to note that it depends only a little on the interpolation couple $\overline{X}$.

Theorem 2.b-5.

Let E be a Banach space of both-sided sequences such that its standard unit vector basis $\{e_n\}_{n\in\mathbb{Z}}$ is 1-unconditional, shrinking and C-translation invariant. Assume that A_k, A_k^* in an unconditional ABS in E. If $X = \overline{X}_{\theta,E;K}$ is an interpolation space for some $\theta \in (0,1)$ such that $\Delta\overline{X}$ is dense in X and X is totally incomparable with $\Sigma\overline{X}$, then X admits an unconditional ABS.

PROOF: Let $\delta' > 0$ and $\mu_n \searrow 0$ be constants associated with the ABS A_k, A_k^*, we may assume that $\mu = \max(\mu_n) < \delta'$. Define sets U_i, U_i^* by

$$U_i = \left\{ x \in \Sigma\overline{X};\ \operatorname{dist}(\{k_n(x)\}_n, A_i^+) < \mu_i \right\} \quad \text{(distance in } E\text{)}$$
$$U_i^* = \left\{ x^* \in \Sigma\overline{X^*};\ \text{ there is } \{x_n^*\}_{n\in\mathbb{Z}} \subset \Delta\overline{X^*} \text{ s.t. } x^* = \sum x_n^*, \{j_n(x_n^*)\}_n \in A_i^* \right\}.$$

We immediately see that sets U_i are uniformly bounded in X, and U_i^* are uniformly bounded subsets of X^* by Theorem 2.b-3. If we show the properties (a)–(c) of ABS for these sets, we get in particular that U_i are uniformly bounded from below, we therefore obtain an ABS by normalizing U_i and U_i^*. The sets are clearly unconditional.

We prove (c) with $\varepsilon_i = 2\mu_i$. Let $x \in U_i$, $x^* \in U_j^*$, $i \neq j$. Then there is $\{b_n\} \in A_i$ such that $\|\{k_n(x)\}_n - \{b_n\}_n\|_E < \mu_i \le \mu_{\min(i,j)}$ and a sequence $\{x_n^*\} \in \Delta\overline{X^*}$

satisfying $x^* = \sum x_n^*$ and $\{\mathrm{j}_n(x_n^*)\}_n \in A_j^*$. In particular, $\|\{\mathrm{j}_n(x_n^*)\}_n\|_{E^*} \leq 1$ and $\sum \mathrm{j}_n(x_n^*)b_n < \mu_{\min(i,j)}$. Using the duality of k_n and j_n we can estimate

$$\begin{aligned}|x^*(x)| &\leq \sum |x_n^*(x)| \leq \sum \mathrm{j}_n(x_n^*)\,\mathrm{k}_n(x) \\ &= \sum \mathrm{j}_n(x_n^*)b_n + \sum \mathrm{j}_n(x_n^*)(\mathrm{k}_n(x) - b_n) \\ &< \mu_{\min(i,j)} + \|\{j_n(x_n^*)\}_n\|_{E^*}\|\{k_n(x)\}_n - \{b_n\}_n\|_E < \varepsilon_{\min(i,j)}.\end{aligned}$$

The property (b) will be established with the constant $\delta = (\delta' - \mu)/2$. Let $x \in U_i$. By definition, there is $\{b_n\}_n \in A_i^+$ such that $\|\{\mathrm{k}_n(x)\}_n - \{b_n\}_n\|_E < \mu$. By Lemma 2.b-4, there is $\{c_n\}_n \in A_i^*$ such that $\sum c_n b_n > \delta'$ and $c_n \geq 0$. By the duality of k_n and j_n, for every $n \in \mathbb{Z}$ there is $x_n^* \in \Delta\overline{X^*}$ such that $x_n^*(x) \geq \frac{1}{2}\mathrm{k}_n(x)$ and $\mathrm{j}_n(x_n^*) = 1$. Set $x^* = \sum c_n x_n^*$. Then $\{\mathrm{j}_n(c_n x_n^*)\}_n = \{c_n\}_n \in A_i^*$, so $x^* \in U_i^*$. Also,

$$\begin{aligned}x^*(x) &= \sum c_n x_n^*(x) \geq \frac{1}{2}\sum c_n\,\mathrm{k}_n(x) \\ &= \frac{1}{2}\sum c_n b_n + \frac{1}{2}\sum c_n(\mathrm{k}_n(x) - b_n) \\ &\geq \frac{1}{2}\delta' - \frac{1}{2}\|\{c_n\}_n\|_{E^*}\|\{\mathrm{k}_n(x)\}_n - \{b_n\}_n\|_E \\ &\geq (\delta' - \mu)/2.\end{aligned}$$

It remains to show that (a) holds. Let Y be an arbitrary closed infinite-dimensional subspace of X and let $m \in \mathbb{N}$. Set $\nu_i = 2^{-i}\mu_m/4$. We will define inductively vectors $x_i \in S_Y$ for which the corresponding sequences $\{k_n(x_i)\}_{n\in\mathbb{Z}}$ in E have essentially disjoint supports. To be precise, we also construct a sequence of intervals $\emptyset = G_0 \subset G_1 \subset \ldots$ of $\mathbb{Z}$ such that for every $i \in \mathbb{N}$, $\|\{\mathrm{k}_n(x_i)\}|_{G_{i-1}}\|_E < \nu_i/2$ and $\|\{\mathrm{k}_n(x_i)\}|_{\mathbb{Z}\setminus G_i}\|_E < \nu_i/2$.

Indeed, assume that we have vectors $x_1, \ldots, x_{i-1} \in S_Y$ and intervals $G_0 \subset \cdots \subset G_{i-1}$. Consider the equivalent norm $\|x\|_i = \sum_{n\in G_{i-1}} \mathrm{k}_n(x)$ on $\Sigma\overline{X}$. Since this space is totally incomparable with X and, on the other hand, the injection $i\colon X \mapsto \Sigma\overline{X}$ is continuous (Theorem 2.b-2), there must be $x_i \in S_Y$ such that $\|x_i\|_i < \nu_i/2$. We then let $G_i \supset G_{i-1}$ to be an interval of $\mathbb{Z}$ such that $\|\{\mathrm{k}_n(x_i)\}|_{\mathbb{Z}\setminus G_i}\|_E < \nu_i/2$.

Let $F_i = G_i \setminus G_{i-1}$. Clearly, F_i are mutually disjoint subsets of $\mathbb{Z}$ and every $\{k_n(x_i)\}_n$ is essentially supported by F_i. For $i \in \mathbb{N}$ define $k_n^i = \mathrm{k}_n(x_i)$ if $n \in F_i$ and $k_n^i = 0$ otherwise. Then the vectors $u_i = \{k_n^i\}_{n\in\mathbb{Z}} \in E$ have disjoint supports in E and they span an infinite-dimensional subspace. By Lemma 2.b-4, there exist numbers $a_1, \dots, a_M \geq 0$ and $\{b_n\} \in A_m^+$ such that $\left\| \{b_n\}_n - \sum_{i=1}^{M} a_i \{k_n^i\}_n \right\|_E < \mu_m/2$. Note that $|a_i| \leq 1 + \mu \leq 2$.

We claim that $\sum_{i=1}^{M} a_i x_i \in U_i \cap Y$. It is enough to show that

$$\left\| \sum_{i=1}^{M} a_i \{k_n^i\}_n - \left\{ \mathrm{k}_n \left(\sum_{i=1}^{M} a_i x_i \right) \right\}_n \right\|_E < \mu_m/2,$$

that is,

$$\left\| \sum_{i=1}^{M} a_i \{k_n(x_i)\}|_{F_i} - \left\{ \mathrm{k}_n \left(\sum_{i=1}^{M} a_i x_i \right) \right\}_n \right\|_E < \mu_m/2.$$

Let $d_n^i = k_n(x_i) - k_n^i$ for $i \in I\!N$ and $n \in Z\!\!Z$. We have $\|\{d_n^i\}_n\|_E < \nu_i$. First, if $n \notin G_M$ then

$$\mathrm{k}_n \left(\sum_{i=1}^{M} a_i x_i \right) \leq \sum_{i=1}^{M} a_i \, \mathrm{k}_n(x_i) \leq 2 \sum_{i=1}^{M} \mathrm{k}_n(x_i) = 2 \sum_{i=1}^{M} d_n^i.$$

If $n \in F_j$ for some $1 \leq j \leq M$, then

$$\begin{aligned} \left| \mathrm{k}_n \left(\sum_{i=1}^{M} a_i x_i \right) - a_j \, \mathrm{k}_n(x_j) \right| &\leq \mathrm{k}_n \left(\sum_{i \neq j} a_i x_i \right) \\ &\leq \sum_{i \neq j} a_i \, \mathrm{k}_n(x_i) \\ &\leq 2 \sum_{i=1}^{M} d_n^i. \end{aligned}$$

Therefore the n-th coordinate of $\left\{ \mathrm{k}_n \left(\sum_{i=1}^{M} a_i x_i \right) \right\} - \sum_{i=1}^{M} a_i \{k_n(x_i)\}|_{F_i}$ is always bounded by $2 \sum_{i=1}^{M} d_n^i$. By the 1-unconditionality of $\{e_i\}$,

$$\begin{aligned} \left\| \sum_{i=1}^{M} a_i \{k_n(x_i)\}|_{F_i} - \left\{ \mathrm{k}_n \left(\sum_{i=1}^{M} a_i x_i \right) \right\}_n \right\|_E &\leq \left\| \left\{ 2 \sum_{i=1}^{M} d_n^i \right\}_n \right\|_E \\ &\leq 2 \sum_{i=1}^{M} \|\{d_n^i\}_n\|_E \\ &\leq 2 \sum_{i=1}^{M} \nu_i < \mu_m/2. \end{aligned}$$

This completes the proof.

□

Chapter 3
Banach Space without GL-Property Hereditarily

It is usually difficult to determine whether a given Banach space has an unconditional basis. There are very few general conditions known which would ensure the existence of such a basis—indeed, in most cases this is shown by actually constructing a concrete basis, and then proving that it has the desired unconditionality.

It is equally difficult to prove directly that a Banach space does not have an unconditional basis. An important step in this direction was made in the early 70's by Y. Gordon and D. R. Lewis, who related the unconditionality of a Banach space X to operator ideal norms of operators from X to ℓ_2. The property which is now called the GL-property is in essential way weaker than the existence of an unconditional basis, still it became a standard notion and an effective tool in the local theory of Banach spaces.

The GL-property was used by Gordon and Lewis ([GL]) to investigate unconditional properties of certain spaces of operators on ℓ_2. In particular, they showed that the classical Schatten classes do not have the GL-property. It is easy to show that if a Banach space has an unconditional basis, then it has the GL-property; Banach lattices also have this property. If X has the GL-property and Z is a complemented subspace of X, then Z has the GL-property as well. Since the Schatten classes fail the GL-property, they are not isomorphic to complemented subspaces of Banach lattices. We recall that the spaces $C[0,1]$ and $L_1[0,1]$ have the GL-property (as Banach lattices) but not an unconditional basis.

As noted in the introduction, it is not known whether the Gowers–Maurey space without an unconditional basic sequence has a closed infinite-dimensional subspace with the GL-property. The properties of the Gowers–Maurey space were proved directly by constructing, in every infinite-dimensional subspace, a sequence of badly unconditional vectors. This would not help in showing the failure of the GL-property that requires estimating certain operator ideal norms. We therefore need to find some connection between the structures similar to the ones found in the Gowers–Maurey space and the GL-property. In this chapter we find such relation and then construct a Banach space all of whose subspaces fail the GL-property. This space is also hereditarily indecomposable.

Defining the GL-property requires several preliminary definitions. Let X and Y

be Banach spaces and let $r \in [1, \infty)$. An operator $A\colon X \to Y$ is called r-summing if there is $C > 0$ such that for all vectors $x_1, \dots, x_n$, $n \in I\!N$, we have

$$\Big(\sum_{i=1}^{n} \|A(x_i)\|_Y^r\Big)^{1/r} \le C \sup\Big\{\Big(\sum_{i=1}^{n} |x^*(x_i)|^r\Big)^{1/r};\ x^* \in B_{X^*}\Big\}.$$

The smallest C with this property is called the r-summing norm of A and denoted by $\pi_r(A)$.

An operator $A\colon X \to Y$ is called L_1-factorable if there are operators $u\colon X \to L_1$ and $v\colon L_1 \to Y^{**}$ such that $i \circ A = v \circ u$, where i is the canonical embedding of Y into Y^{**}. By L_1-factorable norm $\gamma_1(A)$ we mean the infimum of $\|u\|\,\|v\|$, where u and v give a factorization as above. Similarly we define the notion of an L_∞-factorable operator and the norm $\gamma_\infty(A)$.

Definition 3-1.
A space X is said to have the Gordon–Lewis property (GL-property, in short) if every 1-summing operator $T\colon X \mapsto \ell_2$ is L_1-factorable. In that case we define the GL-constant of X, $\mathrm{gl}(X)$, as the smallest number K so that $\gamma_1(T) \le K\pi_1(T)$ for every 1-summing operator $T\colon X \mapsto \ell_2$.

Now we state precisely two facts discussed above. If a Banach space X has an unconditional basis $\{x_i\}$, then $\mathrm{gl}(X) \le \mathrm{unc}\{x_i\}$. If Z is a C-complemented subspace of a Banach space X possessing the GL-property, then $\mathrm{gl}(Z) \le C\,\mathrm{gl}(X)$.

One of the main advantages of the GL-property is that in many cases it is possible to determine the order of quantities involved. It usually involves identifying complemented finite-dimensional subspaces with dimensions growing to infinity, for which we can estimate asymptotic order of the appropriate operator norms. In particular, the GL-property became an important tool for ruling out the existence of an unconditional basis.

3.a. Lower Estimate for GL-constant

In this section we address the problem of estimating the GL-constant of a Banach space Y from below if we know that there are some badly unconditional vectors in Y. As an inspiration we mention a beautiful result by C. Schütt [Sü]. Let $y_{1,1}, \dots, y_{M,M}$ be an $M \times M$ matrix of vectors. We say that this matrix of vectors is K-tensor unconditional if $\left\| \sum_{k,l=1}^{M} \varepsilon_k \nu_l a_{k,l} y_{k,l} \right\| \leq K \left\| \sum_{k,l=1}^{M} a_{k,l} y_{k,l} \right\|$ for all scalars $a_{k,l}$ and signs $\varepsilon_k, \nu_l = \pm 1$. Assume moreover that these vectors form a basis of a Banach space Y when ordered lexicographically into a sequence $\{y_i\}$ (that is, $y_{(k-1)M+l} = y_{k,l}$). Schütt proved that we then have

$$\operatorname{unc}\{y_i\}_{i=1}^{M^2} \leq cK \operatorname{gl}(Y),$$

where c is an absolute constant.

However, these assumptions are too strong for our purposes. We will use an approach of G. Pisier from [P], which allows us to deduce a stronger estimate of a similar type.

First, let us recall several facts about Rademacher functions r_i on [0,1]. They are defined by $r_i(t) = \operatorname{sign}(\sin(2^i \pi t))$. We consider their products $r_k(s) r_l(t)$ acting on $[0,1] \times [0,1]$, and we write such a product simply as $r_k r_l$, that is, $(r_k r_l)(s,t) = r_k(s) r_l(t)$.

For $p \geq 1$ let L_p denote the space $L_p([0,1] \times [0,1])$ with the norm $\| \cdot \|_p = \| \cdot \|_{L_p}$. By R_p we denote the space $\overline{\operatorname{span}}\{r_k r_l\}$, where the completion is in the norm $\| \cdot \|_p$. It follows from the classical Khintchine inequality that all spaces R_p for $1 \leq p < \infty$ are isomorphic; moreover, it is easy to check that $\{r_k r_l\}$ is an orthonormal basis of the Hilbert space R_2. For $h \in \operatorname{span}\{r_k r_l\}$ we have $\|h\|_1 \leq \|h\|_2 \leq K(1,2)^2 \|h\|_1$ and $\|h\|_2 \leq \|h\|_r \leq K(2,r)^2 \|h\|_2$ for $r > 2$, where $K(1,2)$ and $K(2,r)$ denote the appropriate Khintchine constants.

Suppose that we have a Banach space Y and vectors $y^*_{1,1}, \dots, y^*_{1,M}, \dots, y^*_{M,M}$ in S_{Y^*}, let us denote $G = \left(\operatorname{span}\{y^*_{k,l}\}, \| \cdot \|_{Y^*}\right)$. We consider any renorming of G (we will call the renormed space G_Z) such that there exists a continuous linear retraction $J \colon Y^* \mapsto G_Z$, that is, there exists a linear map that is continuous as a map from $(Y^*, \| \cdot \|_{Y^*})$ to $(G, \| \cdot \|_{G_Z})$ and it is the identity when restricted to G as an algebraic map from Y^* to $G \subset Y^*$. For these two norms on $\operatorname{span}\{y^*_{k,l}\}$ we will consider the following constant: $\operatorname{unc}_{Rad}(y^*_{k,l}, G_Z, G)$ is defined as the smallest

$C > 0$ satisfying the following inequality for every choice of signs $\phi_{k,l} = \pm 1$:

$$\Big(\iint \Big\| \sum_{k,l=1}^{M} \phi_{k,l} r_k(s) r_l(t) y_{k,l}^* \Big\|_{G_Z}^2 \, ds\, dt\Big)^{\frac{1}{2}} \le C \Big(\iint \Big\| \sum_{k,l=1}^{M} r_k(s) r_l(t) y_{k,l}^* \Big\|_{G}^2 \, ds\, dt\Big)^{\frac{1}{2}}.$$

We then have the following estimate:

Theorem 3.a-1.
There is an absolute constant $C > 0$ such that

$$\mathrm{unc}_{Rad}(y_{k,l}^*, G_Z, G) \le C\|J\| \ln(M)\, \mathrm{gl}(Y).$$

PROOF: Set $q = 2\ln(M)$ and $r = 2q$. Fix arbitrary signs $\phi_{k,l} = \pm 1$. We consider an operator $w\colon L_{r'} \to G_Z$ defined by $w = \sum \phi_{k,l} r_k r_l \otimes y_{k,l}^*$, that is, for $f \in L_{r'}$ we let $w(f) = \sum \phi_{k,l} \big(\iint r_k(s) r_l(t) f(s,t)\, ds\, dt\big) y_{k,l}^*$. As in [P], using a well-known estimate for the r-summing norm and then Maurey's generalization of Grothendieck theorem on operators from L_∞ to a space of cotype q, we get

$$\Big(\iint \Big\| \sum \phi_{k,l} r_k r_l y_{k,l}^* \Big\|_{G_Z}^r \, ds\, dt\Big)^{1/r} \le \pi_r(w) \le C_q(G_Z) \tilde{K}(q,r) \gamma_\infty(w),$$

where $C_q(G_Z)$ is the cotype constant of the space G_Z and $\tilde{K}(q,r)$ can be taken equal to $c(1/q)(1/q - 1/r)^{-1/r'}$ with $c > 0$ being an absolute constant (*cf.* e.g. [T-J]). Substituting the value for r we get $\tilde{K}(q,r) = c(1/q)(1/2q)^{-1/r'} \le c(1/q)(1/2q)^{-1} = 2c$. Since $r \ge 2$, we can finally write

$$\Big(\iint \Big\| \sum \phi_{k,l} r_k r_l y_{k,l}^* \Big\|_{G_Z}^2 \, ds\, dt\Big)^{1/2} \le c\, C_q(G_Z) \gamma_\infty(w). \tag{1}$$

We denote by $\hat{w}$ the same algebraic operator as w, going into Y^*, $\hat{w}\colon L_{r'} \to Y^*$. Then $\hat{w}^*|_Y\colon Y \to L_r$ is given by the formula $\hat{w}^*|_Y = \sum \phi_{k,l} y_{k,l}^* \otimes r_k r_l$, that is, $\hat{w}^*(y)(s,t) = \sum \phi_{k,l} y_{k,l}^*(y) r_k(s) r_l(t)$ for $y \in Y$. Observe that $w = J \circ \hat{w}$, where J is the linear retraction, and also that $\big(\hat{w}^*|_Y\big)^* = \hat{w}$. Thus

$$\gamma_\infty(w) = \gamma_\infty(J \circ \hat{w}) \le \gamma_\infty(\hat{w})\|J\| \le \|J\| \gamma_1(\hat{w}^*|_Y).$$

Formally $\hat{w}^*|_Y$ acts from Y to L_r, but it takes values in R_r. Consider an operator $u\colon Y \to R_2$ which is algebraically the same as $\hat{w}^*|_Y$. Let $A\colon Y \to L_1$, $B\colon L_1 \to R_2$ be an arbitrary factorization $u = B \circ A$. By $B'\colon L_1 \to R_r$ denote the same algebraic operator as B but acting into R_r. By the Khintchine inequality discussed above we get $\|B'\| \leq K(2,r)^2\|B\|$. Since $\hat{w}^*|_Y = B' \circ A$, we have $\gamma_1(\hat{w}^*|_Y) \leq K(2,r)^2\|A\|\,\|B\|$. Since $u = B \circ A$ was an arbitrary factorization, this yields $\gamma_1(\hat{w}^*|_Y) \leq K(2,r)^2\gamma_1(u)$. By the definition of the GL-constant we obtain that $\gamma_1(u) \leq \mathrm{gl}(Y)\pi_1(u)$, which implies

$$\gamma_\infty(w) \leq K(2,r)^2\|J\|\ \mathrm{gl}(Y)\pi_1(u). \tag{2}$$

Finally, define $v\colon Y \to R_1$ by $v = \sum y^*_{k,l} \otimes r_k r_l$. For $y \in Y$ we have

$$\begin{aligned}\|u(y)\|_2 &= \Big(\iint \Big|\sum \phi_{k,l} y^*_{k,l}(y) r_k(s) r_l(t)\Big|^2 ds\,dt\Big)^{1/2}\\ &= \Big(\sum |\phi_{k,l} y^*_{k,l}(y)|^2\Big)^{1/2} = \Big(\sum |y^*_{k,l}(y)|^2\Big)^{1/2}\\ &\leq K(1,2)^2 \iint \Big|\sum y^*_{k,l}(y) r_k(s) r_l(t)\Big|\, ds\,dt = K(1,2)^2\|v(y)\|_1\end{aligned}$$

and therefore

$$\begin{aligned}\sum \|u(y_i)\|_2 &\leq K(1,2)^2 \sum \|v(y_i)\|_1\\ &\leq K(1,2)^2 \pi_1(v) \sup\{\textstyle\sum |y^*(y_i)|;\ y^* \in B_{Y^*}\}.\end{aligned}$$

This implies that $\pi_1(u) \leq K(1,2)^2\pi_1(v)$. Since v is taking values in a subspace of L_1, we can estimate its 1-summing norm (as in [P]) to get

$$\pi_1(v) \leq \iint \Big\|\sum r_k r_l y^*_{k,l}\Big\|_{Y^*} ds\,dt \leq \Big(\iint \Big\|\sum r_k r_l y^*_{k,l}\Big\|^2_{Y^*} ds\,dt\Big)^{1/2}.$$

Therefore

$$\pi_1(u) \leq K(1,2)^2 \Big(\iint \Big\|\sum r_k r_l y^*_{k,l}\Big\|^2_{Y^*} ds\,dt\Big)^{1/2}. \tag{3}$$

Now we put the estimates (1)–(3) together. First, we can incorporate the constant $K(1,2)^2$ into c. The space G_Z is M^2-dimensional, therefore we can use the classical estimate for the cotype constant $C_q(G_Z) \leq (M^2)^{1/q} = M^{2/2\ln(M)} = e$, we also use a classical estimate for the Khintchine constant, $K(2,r) \leq \sqrt{r} = 2\sqrt{\ln M}$ (*cf.* e.g. [T-J]), and finally we get that

$$\mathrm{unc}_{Rad}(y^*_{k,l}, G_Z, G) \leq C\|J\|\ln(M)\,\mathrm{gl}(Y).$$

□

The connection with unconditionality becomes clear when we observe the simple inequality

Observation 3.a-2.
For arbitrary signs $\phi_{k,l} = \pm 1$ we have

$$\min_{\varepsilon_k,\nu_l=\pm 1}\Big\|\sum_{k,l=1}^{M}\varepsilon_k\nu_l\phi_{k,l}y^*_{k,l}\Big\|_{G_Z} \le \mathrm{unc}_{Rad}(y^*_{k,l},G_Z,G)\max_{\varepsilon_k,\nu_l=\pm 1}\Big\|\sum_{k,l=1}^{M}\varepsilon_k\nu_l y^*_{k,l}\Big\|_{G}.$$

Thus, if vectors $y^*_{1,1}, y^*_{1,2}, \dots, y^*_{M,M}$ taken as a sequence are badly unconditional for some signs $\phi_{k,l} = \pm 1$, and vectors $y^*_{1,1}, \dots, y^*_{M,M}$, and $\phi_{1,1}y^*_{1,1}, \dots, \phi_{M,M}y^*_{M,M}$ are tensor sign-unconditional (i.e. the condition above with $a_{k,l} = 1$), then we get a lower estimate for $\mathrm{gl}(Y)$.

The interplay between the linear structure and the matrix structure of vectors will be also used in the next section. We will freely switch between the matrix notation $\{w_{k,l}\}_{k,l=1}^{M}$ and the linear notation $\{w_i\}_{i=1}^{M^2}$ for vectors and functionals, with the correspondence given by the lexicographic order, that is, $w_{k,l} = w_{(k-1)M+l}$. We will often write N for M^2.

3.b. Definition of the Space And Some Lemmas

We now restrict our attention to a class of spaces whose norms satisfy a lower estimate connected with a certain function f. First let us specify the set of functions we consider:

Definition 3.b-1.

By $\mathcal{F}$ we denote the family of all functions $f\colon [1,\infty) \to [1,\infty)$ satisfying the following properties:

(1) f is increasing and $f(1) = 1$,

(2) $\lim_{x\to\infty} \left(x^q/f(x)\right) = \infty$ for every $q > 0$,

(3) f is submultiplicative,

(4) $x/f(x)$ is concave and non-decreasing.

It is easy to observe that for instance $f(x) = \log_2(x+1)$ satisfies the above conditions.

Definition 3.b-2.

Let X be a space with a bimonotone basis and $f \in \mathcal{F}$. We say that X satisfies a lower f-estimate if for every vector x and successive intervals $E_1 < \cdots < E_n$ of $\mathbb{N}$ we have

$$\|x\| \geq \frac{1}{f(n)} \sum_{i=1}^{n} \|E_i x\|.$$

Let f be from $\mathcal{F}$. By $\mathcal{X}_f$ we denote the set of all Banach spaces with a bimonotone basis that satisfy the lower f-estimate. We also define $M_f(n) = f^{-1}(36n^2)$. Now we identify several structures that are important in this context.

Definition 3.b-3.

Let X be a Banach space with a basis. A vector x is called an ℓ_{1+}^n-vector with constant C if there are successive vectors $x_1 < \cdots < x_n$ such that $x = \sum_{i=1}^{n} x_i$ and $\|x_i\| \leq C\|x\|/n$. If moreover $\|x\| = 1$, then it is called an ℓ_{1+}^n-average. The number n is called the length of an ℓ_{1+}^n-vector, resp ℓ_{1+}^n-average.

Let $f \in \mathcal{F}$ and $X \in \mathcal{X}_f$. Let $x_1 < \cdots < x_N$ be norm-one vectors in X that are also $\ell_{1+}^{n_i}$-averages with constant $(1+\eta)$. We say that this sequence is a R.I.S. (Rapidly

Increasing Sequence) of length N with constant $(1+\eta)$ if the sequence $\{n_i\}$ satisfies the following conditions:

(1) $n_1 \geq 2\frac{1+\eta}{\eta}\frac{M_f(N/\eta)}{f'_+(1)}$,

(2) $\frac{\eta}{2}\sqrt{f(n_{i+1})} \geq |\operatorname{ran}(x_i)|$ *for* $i = 1, \ldots, N-1$.

A functional x^* in X^* is called an (M,f)-form if $\|x^*\| \leq 1$ and there are successive functionals $x_1^* < \cdots < x_M^*$ such that $x^* = \sum_{i=1}^{M} x_i^*$ and $\|x_i^*\| \leq 1/f(M)$. M is called its length.

Observe that if we take functions $f, g \in \mathcal{F}$ such that $g(M) \leq f(M)$, then every (M,f)-form is also an (M,g)-form. Consider $X \in \mathcal{X}_f$ for some $f \in \mathcal{F}$ and take successive functionals $x_1^* < \cdots < x_M^* \in B_{X^*}$. From the lower f-estimate it immediately follows that $x = \frac{1}{f(M)} \sum_{i=1}^{M} x_i^*$ is an (M,f)-form.

It was proved in [GM] (as Lemma 3, 4, 5, and 7) that the structures defined above have the following properties:

Lemma 3.b-4.
Let $X \in \mathcal{X}_f$ for some $f \in \mathcal{F}$. Then for every block subspace Y of X, every $\eta > 0$ and $n \in I\!N$ there exists $x \in S_Y$ that is an $\ell_{1+}^{n_i}$-average with constant $(1+\eta)$.

Lemma 3.b-5.
Let $X \in \mathcal{X}_f$ for some $f \in \mathcal{F}$. Let $x_1 < \cdots < x_N$ be vectors in X and $x = \sum_{i=1}^{N} x_i$, let $x^* \in X^*$ and E be an interval of $I\!N$. We have the following:

(i) If x is an ℓ_{1+}^N average with constant $(1+\eta)$ and x^* is an (M,f)-form, then

$$|x^*(Ex)| \leq \frac{1+\eta}{f(M)}(1+2M/N).$$

(ii) If $x_1 < \cdots < x_N$ is a R.I.S. with constant $(1+\eta)$, x^* is an $(M,\sqrt{f})$-form and $M \geq M_f(N/\eta)$, then

$$|x^*(Ex)| \leq (1+2\eta)$$

Note that we can use (ii) for (M,h)-forms whenever $h \in \mathcal{F}$ satisfies $h(M) \geq \sqrt{f(M)}$, since such a form is also an $(M,\sqrt{f})$-form as noted above.

Having a fixed R.I.S. $x_1 < \cdots < x_N$, where $x_i = \sum_{t=1}^{n_i} x_{it}$ are $\ell_{1+}^{n_i}$-averages, we define a function of fractional length of an interval $E \subset \mathbb{N}$. Set $j_E = \max\{i;\ Ex_i \neq 0\}$, $i_E = \min\{i;\ Ex_i \neq 0\}$, and $s_E = \max\{t;\ Ex_{j_E t} \neq 0\}$, $r_E = \min\{t;\ Ex_{i_E t} \neq 0\}$. We define $\lambda(E) = j_E - i_E + \frac{s_E}{n_{j_E}} - \frac{r_E}{n_{i_E}}$.

Lemma 3.b-6.
Let $X \in \mathcal{X}_f$ for some $f \in \mathcal{F}$. Let $h \in \mathcal{F}$ be such that $h \geq \sqrt{f}$. Let $x_1 < \cdots < x_N$ be a R.I.S. with constant $(1+\eta)$. Set $y = \sum_{i=1}^{N} \phi_i x_i$, where $\phi_i = \pm 1$. If for every interval E of $\mathbb{N}$ with $1 \leq \lambda(E)$ we have

$$\|Ey\| \leq \sup\{|x^*(Ey)|;\ x^* \text{ is an } (M,h)\text{-form},\ M \geq 2\},$$

then

$$\|y\| \leq (1+2\eta)\frac{N}{h(N)}.$$

This lemma was enough to establish an upper estimate in [GM]. Its proof went by induction on $\lambda(E)$. Unfortunately, we will see later that in our space we cannot hope to have its assumptions satisfied. We therefore need a lemma of more complicated form, however, it will be clear from its proof that it actually describes the same phenomenon:

Lemma 3.b-7.
Let $X \in \mathcal{X}_f$ for some $f \in \mathcal{F}$. Let g and h be functions from $\mathcal{F}$ such that $g, h \geq \sqrt{f}$ and $h = g$ on $[1, K_0]$ for some $K_0 > 1$. Let $x_1 < \cdots < x_N$ be a R.I.S. with constant $(1+\eta)$. Set $y = \sum_{i=1}^{N} \phi_i x_i$, where $\phi_i = \pm 1$.

If for every interval E of $\mathbb{N}$ with $1 \leq \lambda(E) < K_0$ we have

$$\|Ey\| \leq \sup\{|x^*(Ey)|;\ x^* \text{ is an } (M,g)\text{-form},\ M \geq 2\}$$

and for every interval E of $\mathbb{N}$ with $K_0 \leq \lambda(E)$ we have

$$\|Ey\| \leq \sup\{|x^*(Ey)|;\ x^* \text{ is an } (M,h)\text{-form},\ M \geq 2\},$$

then $\|Ey\| \leq (1+2\eta)\frac{\lambda(E)}{h(\lambda(E))}$ whenever $\lambda(E) \geq 1$, in particular

$$\|y\| \leq (1+2\eta)\frac{N}{h(N)}.$$

PROOF: It is easy to observe that $\phi_1 x_1 < \cdots < \phi_N x_N$ is a R.I.S. with constant $(1+\eta)$ again, so we can assume that $\phi_i = 1$. The proof goes by induction on $\lambda(E)$. First, if $\lambda(E) < K_0$, then only (M,g)-forms appear in our assumption. By Lemma 3.b-6 we therefore get $\|Ey\| \le (1+2\eta)\dfrac{\lambda(E)}{g(\lambda(E))}$, which is what we want as $g(\lambda(E)) = h(\lambda(E))$.

If $\lambda(E) \ge K_0$, we repeat the proof of Lemma 3.b-6, now with the function h.

□

This concludes the preliminary work. Fix numbers $p, q \in \mathbb{N}$ such that $0 < p/q < \frac{1}{6}$ and set $\varepsilon = \frac{p}{q}$. This ε will be fixed throughout the rest of our construction. We claim the following:

Proposition 3.b-8.
There exist functions $f, g, h_0 \in \mathcal{F}$, a number $K > 0$, an increasing sequence $\mathcal{J} = \{j_k\}_{k=1}^{\infty}$ of natural numbers, and functions $h_N \in \mathcal{F}$ for $N \in \mathcal{J}$ such that the following hold (we denote $\mathcal{K} = \{j_{2k-1}\}_{k=1}^{\infty}$, $\mathcal{L} = \{j_{2k}\}_{k=1}^{\infty}$):

(i) $f(x) = \log_2^4(x+1)$ *and* $g(x) = \log_2^2(x+1)$ *on* $[K, \infty)$, $g \ge \sqrt{f}$.

(ii) $g \le h_0 \le f$, $h_0 = f$ *on* $[n^{3/4}, n]$ *for all* $n \in \mathcal{L}$ *and* $h_0 = g$ *on* $\mathcal{K}$.

(iii) *for* $N = j_k \in \mathcal{J}$ *the function* h_N *satisfies* $h_N = g$ *on* $[1, N^{4/5}] \cup [j_{k+1}, \infty)$ *and* $g \le h_N \le f$. *The sequence* $\{h_N\}$ *satisfies* $\displaystyle\lim_{N \in \mathcal{J} \to \infty}\left(\frac{h_N(N)}{g(N)\ln(N)}\right) = \infty$.

(iv) $K^{64q} \le \min(\mathcal{J})$, $(2/\varepsilon)^{64q} \le \min(\mathcal{J})$, *and* $N^{\frac{1}{2}}$, $N^{1/64q}$ *are natural numbers for* $N \in \mathcal{J}$

(v) $2N^{3/4} + 4 < \dfrac{N^{4/5} - 2}{f(N^{4/5} - 2)}$ *for every* $N \in \mathcal{J}$

(vi) $(1+2\varepsilon)\dfrac{f(j_{2k})}{j_{2k}} \le \varepsilon/k^2$, $\dfrac{2}{f(j_{2k})} \le \varepsilon/k^2$, *and* $j_{2k}^{\varepsilon/64} \ge 2\dfrac{(1+\varepsilon)}{\varepsilon}\dfrac{M_f(k/\varepsilon)}{f'_+(1)}$ *for every* $k \in \mathbb{N}$

(vii) $2N/M^{\varepsilon/64} \le 1$ *and* $M \ge M_f(N/\varepsilon)$ *for* $M > N \in \mathcal{J}$

Note that (2) in the definition of $\mathcal{F}$ (Definition 3.b-1) shows that (v) is again a condition on $\min(\mathcal{J})$.

The proof will be postponed until the last section of this chapter, since it is a rather complicated but elementary real analysis. Let us shortly comment on why is the construction of h_N difficult. In [GM], functions similar to our h_0 and h_N are constructed. The main difference with condition (iii) above is that in the case of [GM], the functions corresponding to h_N for $N = j_k \in \mathcal{J}$ are required to be equal to g only on $[1, j_{k-1}] \cup [j_{k+1}, \infty)$ and then to be of higher order then g at j_k. If $\mathcal{J}$ is sufficiently lacunary, that is, the gaps between j_{k-1} and j_k are large enough, such functions can be constructed using relatively soft estimates. However, this requires the gap after j_{k-1} to be at least $e^{j_{k-1}}$. On the other hand, here we require that $h_N = g$ at $N^{4/5}$ and that $h_N(N)$ is already large enough to beat $g(N)$ by much more than $\ln(N)$. This forces us to analyze in detail the construction used in [GM] in order to get sharper estimates.

Now we define our space.

Let $\mathcal{Q}$ denote the set of all finitely supported scalar sequences with rational coordinates of magnitude less or equal to one. Fix an injection σ from the set of finite sequences of successive vectors from $\mathcal{Q}$ to $\mathcal{L}$ satisfying the following property:

If $\{z_i\}_{i=1}^n$ are successive in $\mathcal{Q}$, $z = \sum z_i$, then $\frac{\varepsilon}{2} f(\sigma(z_1, \ldots, z_n)^{\varepsilon/64}) \geq |\operatorname{ran}(z)|$.

In our next definition we will use the agreed upon identification of linear and matrix sets of vectors. For a normed space $X = (c_{00}, \|\cdot\|)$, $m \in I\!N$ and $n \in \mathcal{K}$ we set

$$A^*_m(X) = \Big\{ f(m)^{-1} \sum_{j=1}^m x^*_j;\ x^*_1 < \cdots < x^*_m,\ x^*_j \in B_{X^*} \Big\}$$

$$\Gamma^X_n = \{(x^*_1, \ldots, x^*_n);\ x^*_1 < \cdots < x^*_n \in \mathcal{Q},\ x^*_1 \in A^*_{j_{2n}}(X),\ x^*_{i+1} \in A^*_{\sigma(x^*_1, \ldots, x^*_i)}(X)\}$$

$$B^*_n(X) = \Big\{ \frac{1}{g(n)} \sum_{k,l=1}^{\sqrt{n}} \varepsilon_k \nu_l x^*_{k,l};\ (x^*_1, \ldots, x^*_n) \in \Gamma^X_n,\ \varepsilon_k, \nu_l = \pm 1 \Big\}.$$

Sequences from Γ^X_n are called *special sequences* of length n, functionals from $B^*_n(X)$ are called *special functionals* of length n.

We define implicitly

$$\|x\| = \|x\|_\infty \vee \sup\Big\{ \frac{1}{f(n)} \sum_{i=1}^n \|E_i x\|;\ E_1 < \cdots < E_n \text{ intervals and } n \geq 2 \Big\}$$
$$\vee \sup\{|x^*(Ex)|;\ x^* \in B^*_k(X),\ k \in \mathcal{K} \text{ and } E \text{ interval}\}.$$

The existence of such a norm is standard by now, see e.g. [GM]. Note that this definition gives a norm on c_{00} in which the canonical basis $\{e_i\}$ is bimonotone. Let X be the completion of $(c_{00}, \|\cdot\|)$. We immediately get that $X \in \mathcal{X}_f$.

Note that an expression $\frac{1}{f(n)}\sum_{i=1}^{n}\|E_ix\|$ from the definition of our norm can be obtained as $x^*(x)$, where x^* is an (n,f)-form. Indeed, we can take norming functionals x_i^* for $\|E_ix\|$ and set $x^* = \frac{1}{f(n)}\sum_{i=1}^{n}x_i^*$; by our observation, this is an (n,f)-functional. On the other hand, for every (n,f)-form x^* we have $|x^*(x)| \le \frac{1}{f(n)}\sum_{i=1}^{n}\|E_ix\|$. Thus

$$\begin{aligned}\|x\| = \|x\|_\infty \vee \sup\{|x^*(x)|;\ x^* \text{ an } (n,f)\text{-form}\}\\ \vee \sup\{|x^*(Ex)|;\ x^* \in B_k^*(X),\ k \in \mathcal{K} \text{ and } E \text{ interval}\}.\end{aligned}$$

Note that special functionals are (M,g)-forms. Consequently, in our space X the norm of every $x \in X$ is given by $\|x\|_\infty$, by some (n,f)-form or by some (M,g)-form. Since every (n,f)-form is also an (n,g)-form as $g \le f$, we have that the norm of every $x \in X$ is given either by $\|x\|_\infty$ or by (n,g)-forms.

This will give us an upper estimate for the norm using Lemma 3.b-6. Let $x_1 < \cdots < x_N$ be a R.I.S. with constant $(1+\eta)$, where $\eta < \frac{1}{6}$, set $x = \sum_{i=1}^{N} x_i$. Let E be an interval with $\lambda(E) \ge 1$. We observe that $\|Ex\|_\infty < \|Ex\|$. Indeed, since every x_i is an $\ell_{1+}^{n_i}$-average with constant $(1+\eta)$, we get

$$\|x_i\|_\infty \le \frac{1+\eta}{n_i} \le \frac{1+\eta}{n_1} \le \frac{(1+\eta)\eta}{2} \le \eta.$$

On the other hand, it is easy to see that $\|Ex\| > \eta$ if $\lambda(E) \ge 1$, because E covers big parts of some ℓ_{1+}^{n}-averages. We used here that $\eta < 1/6$.

This shows that when dealing with the sum of a R.I.S., we need not consider the supremum part of the definition of the norm. This immediately leads to the following estimates (we always assume $\eta < \frac{1}{6}$):

Corollary 3.b-9.

Let $x_1 < \cdots < x_N$ be a R.I.S. with constant $(1+\eta)$. Set $x = \sum_{i=1}^{N} x_i$. Then we have

$$\|x\| \le (1+2\eta)\frac{N}{g(N)}.$$

If there is $n \in \mathcal{L}$ such that $N \in [n^{3/4}, n]$, then

$$\|x\| \le (1+2\eta)\frac{N}{f(N)}.$$

PROOF: Assumptions of Lemma 3.b-6 are satisfied with (M, g)-forms, thus the first claim follows.

To prove the second part we use the function h_0 from Proposition 3.b-8 (ii). We claim that for an interval E with $\lambda(E) \geq 1$, the norm $\|Ex\|$ is done by (M, h_0)-forms. We know that it cannot be given by its supremum norm. Then there are the sums that can be identified with actions of (M, f)-forms, but these are also (M, h_0)-forms as $h_0 \leq f$. Finally, the special functionals are (M, g)-forms, but in this case $M \in \mathcal{K}$ and $h_0 = g$ on $\mathcal{K}$. Applying Lemma 3.b-6 we get the statement as $h_0 = f$ on $[n^{3/4}, n]$.

□

Corollary 3.b-10.
Let $x_1 < \cdots < x_N$ be a R.I.S. with constant $(1+\eta)$, where $N \in \mathcal{L}$. Let $M = N^{\eta}$. Then $\sum_{i=1}^{N} x_i$ is an ℓ_{1+}^{M} vector with constant $(1+64\eta)$.

PROOF: Set $m = N/M = N^{1-\eta}$. Let $y_j = \sum_{i=(j-1)m+1}^{jm} x_i$, denote $x = \sum_{i=1}^{N} x_i$. Then $x = \sum_{j=1}^{M} y_j$. Note that every y_j is a R.I.S. of length $m \in [N^{3/4}, N]$, therefore $\|y_j\| \leq (1+2\eta)\dfrac{m}{f(m)}$ by Corollary 3.b-9. On the other hand, by the lower f-estimate, $\|x\| \geq \dfrac{N}{f(N)}$. Thus we get

$$\|y_j\| \leq (1+2\eta)\frac{m}{f(m)}\frac{f(N)}{N}\|x\| = (1+2\eta)\frac{f(N)}{f(m)}\frac{\|x\|}{M}.$$

But for our f we have $f(m) \geq (1-\eta)^4 f(N)$, also $(1+2\eta)(1-\eta)^{-4} \leq (1+64\eta)$ using $\eta < \frac{1}{4}$. Thus we get $\|y_j\| \leq (1+64\eta)\dfrac{\|x\|}{M}$, that is, x is an ℓ_{1+}^{M}-vector with constant $(1+64\eta)$.

□

In the following lemma we find one of the crucial tools for our further computations.

Lemma 3.b-11.
Let $M, \widetilde{M} \in \mathcal{L}$ and $M, \widetilde{M} \geq j_{2N}$ for some $N \in I\!N$. Let $x_1 < \cdots < x_M$ be a R.I.S. of length M with constant $(1+\varepsilon/64)$ and x^* be an $(\widetilde{M}, f)$-form. Set $x = \sum_{i=1}^{M} x_i$ and $z = x/\|x\|$. If $\widetilde{M} \neq M$, then $|x^*(Ez)| < \varepsilon/N^2$ for every interval E of $I\!N$.

PROOF:

If $\widetilde{M} > M$ then from (vii) in Proposition 3.b-8 we get $\widetilde{M} \geq M_f(M/\varepsilon)$. By Lemma 3.b-5 (ii) it follows that $|x^*(Ex)| \leq (1+2\varepsilon)$. But from from the lower f-estimate we have that $\|x\| \geq \dfrac{M}{f(M)}$, so $|x^*(Ez)| \leq (1+2\varepsilon)\dfrac{f(M)}{M}$. The estimate then follows from $M \geq j_{2N}$ and (vi) in Proposition 3.b-8.

For the case $\widetilde{M} < M$ we use that z is also an ℓ_{1+}^N-average with constant $(1+\varepsilon)$, where $N = M^{\varepsilon/64}$ (Corollary 3.b-10). Therefore (vii) in Proposition 3.b-8 implies $\dfrac{2\widetilde{M}}{N} \leq 1$. Lemma 3.b-5 (i) shows that $|x^*(Ex)| \leq (1+\varepsilon)\dfrac{2}{f(\widetilde{M})}$. Again, $\widetilde{M} \geq j_{2N}$ and the claim follows from Proposition 3.b-8, (vi).

□

Next arguments require an additional "matrix-sequence" notational convention. For a subset $I \subset \{1,\ldots,M\} \times \{1,\ldots,M\}$, let $\tilde{I} \subset \{1,\ldots,M^2\}$ denote the same set under lexicographic ordering, that is, $\tilde{I} = \{(k-1)M + l;\ (k,l) \in I\}$. Note that $\tilde{I}$ is an interval if and only if I consists of several (or none) full rows and two possible rows that are not complete; from these incomplete rows, I contains an "end interval" of the first row and an "initial interval" of the second.

For $M \in I\!N$ we consider the set Φ_M of all sequences of signs $\{\phi_{k,l} = \pm 1\}_{k,l=1}^M$ satisfying

$$\Big|\sum_{(k,l)\in I} \varepsilon_k \nu_l \phi_{k,l}\Big| \leq 4M^{3/2}$$

for every choice of signs $\varepsilon_k, \nu_l = \pm 1$ and every set $I \subset \{1,\ldots,M\} \times \{1,\ldots,M\}$ such that $\tilde{I}$ is an interval.

Note that if a sequence of signs $\{\phi_{k,l}\}$ belongs to Φ_M, then for arbitrary signs $\varepsilon_k, \nu_l = \pm 1$ we have $\{\phi_{k,l}\varepsilon_k\nu_l\}_{k,l=1}^M \in \Phi_M$.

Lemma 3.b-12.
For every $M \in I\!N$ we have $\Phi_M \neq \emptyset$.

PROOF:

A simple probabilistic argument shows that there exist signs $\phi_{k,l} = \pm 1$ so that for arbitrary $\varepsilon_k, \nu_l = \pm 1$ we have

$$\Big|\sum \varepsilon_k \nu_l \phi_{k,l}\Big| \leq 2M^{3/2}.$$

Indeed, let $\phi_{k,l} = \pm 1$ be M^2 independent Bernoulli random variables, that is, every $\phi_{k,l}$ has the distribution $\mathbb{P}(\phi_{k,l} = 1) = \mathbb{P}(\phi_{k,l} = -1) = 1/2$. The well-known tail estimate states that for arbitrary scalars $\{\alpha_{k,l}\}$ and every $t > 0$ we have

$$\mathbb{P}\Big(\Big\{\big|\sum \alpha_{k,l}\phi_{k,l}\big| > t\Big\}\Big) \leq 2\exp(-t^2/2\sum \alpha_{k,l}^2).$$

It follows that for fixed signs $\varepsilon_k, \nu_l = \pm 1$, $k, l = 1\ldots, M$ we have

$$\mathbb{P}\Big(\Big\{\big|\sum \varepsilon_k \nu_l \phi_{k,l}\big| > tM\Big\}\Big) \leq 2\exp(-t^2/2),$$

we used $\alpha_{k,l} = \varepsilon_k \nu_l / M$. We may assume that $\varepsilon_1 = 1$, hence 2^{2M-1} possible choices of ε_k, ν_l remain to be considered. Thus we get

$$\mathbb{P}\Big(\Big\{\big|\sum \varepsilon_k \nu_l \phi_{k,l}\big| > tM \text{ for some } \varepsilon_k, \nu_l\Big\}\Big) \leq 2^{2M-1} 2\exp(-t^2/2).$$

Setting $t = 2M^{1/2}$ we get $2M\ln(2) - t^2/2 < 0$, hence the last probability is less than 1. This means that there exist a choice of signs $\phi_{k,l} = \pm 1$ which does not belong to the set considered, hence for all $\varepsilon_k, \nu_l = \pm 1$ we get $\left|\sum \varepsilon_k \nu_l \phi_{k,l}\right| \leq 2M^{3/2}$.

So now assume that $I = \{(k,l);\ k \in J,\ 1 \leq l \leq M\}$ for some $J \subset \{1, \ldots, M\}$. For any signs $\varepsilon_k, \nu_l = \pm 1$ let $\varepsilon'_k = \varepsilon_k$ for $k \in J$, $\varepsilon'_k = -\varepsilon_k$ for $k \notin J$. Then by the first estimate we get

$$\Big|\sum_{(k,l)\in I} \varepsilon_k \nu_l \phi_{k,l}\Big| = \frac{1}{2}\Big|\sum_{k,l=1}^{M} \varepsilon_k \nu_l \phi_{k,l} + \sum_{k,l=1}^{M} \varepsilon'_k \nu_l \phi_{k,l}\Big| \leq 2M^{3/2}.$$

Finally, let $I \subset \{1, \ldots, M\} \times \{1, \ldots, M\}$ be such that $\tilde{I}$ is an interval. Denote by J the set of all rows contained in I and let $I' = \{(k,l);\ k \in J,\ 1 \leq l \leq M\}$. As we observed, $|I \setminus I'| \leq 2M$. Thus

$$\Big|\sum_{(k,l)\in I} \varepsilon_k \nu_l \phi_{k,l}\Big| \leq \Big|\sum_{(k,l)\in I'} \varepsilon_k \nu_l \phi_{k,l}\Big| + \Big|\sum_{(k,l)\in I\setminus I'} \varepsilon_k \nu_l \phi_{k,l}\Big| \leq 2M^{3/2} + 2M \leq 4M^{3/2}.$$

□

As in [GM], to prove our main result we will consider N successive vectors of a complicated structure described in the next lemma, let z be their sum. In the space constructed in [GM], the norm of Ez for intervals E with $\lambda(E) \geq 1$ could not be given by special functionals of length N. This implied that $\|Ez\|$ was realized by

(M,h)-forms, where $h \in \mathcal{F}$ was a function satisfying $f \geq h \geq \sqrt{f}$, $h = g$ on $\mathcal{K}\setminus\{N\}$ and $h(N) = f(N)$. Thus an upper estimate for $\|z\|$ followed from Lemma 3.b-6.

In our space we can exclude special functionals of length N only for $\lambda(E)$ large. This difficulty is overcome by using the more complicated Lemma 3.b-7.

Lemma 3.b-13.
Let $N = M^2 \in \mathcal{K}$ and $y_1^*, \dots, y_N^*$ be a special sequence of length N, that is, each y_i^* is an (M_i, f)-form, $M_1 = j_{2N}$ and $M_{i+1} = \sigma(y_1^*, \dots, y_i^*)$. Let $z_1 < \dots < z_N$ be a sequence of vectors such that each z_i is a normalized R.I.S. of length M_i with constant $(1+\varepsilon/64)$. Assume that $|\frac{1}{2} - y_i^*(z_i)| < 1/N$ for every i. Then for every $\{\phi_{k,l}\} \in \Phi_M$ we have

$$\Big\|\sum_{k,l=1}^{M} \phi_{k,l} z_{k,l}\Big\| \leq (1+2\varepsilon)\frac{N}{h_N(N)}.$$

PROOF:

Set $z = \sum \phi_{k,l} z_{k,l}$. First we show that for any interval E with $\lambda(E) \geq N^{4/5}$, the norm $\|Ez\|$ cannot be given by special functionals of length N; more precisely, there is $\delta < \|Ez\|$ such that for every interval F and every $z^* \in B_N^*(X)$ we have $|z^*(F(Ez))| < \delta$.

Fix an interval E with $\lambda(E) \geq K_0$. Then $j_E - i_E - 1 \geq N^{4/5} - 2$ is the number of vectors $z_{k,l}$ fully covered by E and we know from the lower f-estimate that then

$$\|Ez\| \geq \frac{j_E - i_E - 1}{f(j_E - i_E - 1)} \geq \frac{N^{4/5} - 2}{f(N^{4/5} - 2)}.$$

On the other hand, let $z^* = g(N)^{-1} \sum \varepsilon_k \nu_l z_{k,l}^*$ be a special functional of length N and let F be an arbitrary interval. By the definition of Γ_N^X we have $z_1^* \in A_{j_{2N}}^*(X)$, and there is t so that $y_1^* = z_1^*, \dots, y_{t-1}^* = z_{t-1}^*$, but $z_t^* \neq y_t^*$. First note that $|1/2 - z_i^*(z_i)| < 1/N$ for $1 \leq i < t$. Next, every z_i^* is an $(\widetilde{M}_i, f)$-form, $\widetilde{M}_1 = j_{2N}$ and $\widetilde{M}_{i+1} = \sigma(z_1^*, \dots, z_i^*)$. Observe that $\widetilde{M}_i \geq j_{2N}$ for $1 < i \leq N$. Indeed, by the definition of σ we have $\sigma(z_1^*, \dots, z_{i-1}^*) \geq |\operatorname{ran}(z_1^*)| \geq j_{2N}$ as $z_1^* \in A_{j_{2N}}^*(X)$. Since σ is an injection, we get that $\widetilde{M}_i \neq M_j$ for $i \neq j$. Similarly, for $i > t$ we also have $\widetilde{M}_i \neq M_i$. Therefore $|z_i^*(FEz_j)| \leq \varepsilon/N^2$ for $i \neq j$ or $i = j > t$ by Lemma 3.b-11. Clearly $|z_i^*(FEz_i)| \leq 1$ for every i.

Set $\widetilde{J} = \{j;\ 1 \leq j < t,\ FEz_j = z_j\}$ and $\widetilde{I} = \{j;\ 1 \leq j \leq t,\ FEz_j \neq 0\}$. Let J and I be the corresponding subsets of $\{1, \dots, M\} \times \{1, \dots, M\}$. Since FE is an interval, $|I \setminus J| = |\tilde{I} \setminus \tilde{J}| \leq 2$. Thus

$$\begin{aligned}
|z^*(FEz)| &= \left|\left(g(N)^{-1}\sum \varepsilon_k \nu_l z^*_{k,l}\right)\left(\sum \phi_{k,l} FEz_{k,l}\right)\right| \\
&\leq \Big|\sum_{(k,l)\in J} \phi_{k,l}\varepsilon_k\nu_l z^*_{k,l}(FEz_{k,l})\Big| + \sum_{(k,l)\in I\setminus J} |z^*_{k,l}(FEz_{k,l})| + \sum \frac{\varepsilon}{N^2} \\
&\leq \frac{1}{2}\Big|\sum_{(k,l)\in J} \phi_{k,l}\varepsilon_k\nu_l\Big| + \sum_{(k,l)\in J}\frac{1}{N} + \sum_{(k,l)\in I\setminus J} |z^*_{k,l}(FEz_{k,l})| + \sum \frac{\varepsilon}{N^2} \\
&\leq \frac{1}{2}\Big|\sum_{(k,l)\in J} \phi_{k,l}\varepsilon_k\nu_l\Big| + 1 + 2 + \varepsilon \leq 2M^{3/2} + 4 = 2N^{3/4} + 4.
\end{aligned}$$

Since $2N^{3/4} + 4 < \dfrac{N^{4/5} - 2}{f(N^{4/5} - 2)}$ by (v) in Proposition 3.b-8, the estimates above yield that $|z^*(FEz)| < \delta < \|Ez\|$ for a suitable δ.

Now the required estimate will follow from Lemma 3.b-7. We know that the norm $\|Ez\|$ is not equal to the supremum norm. As we observed above, for an arbitrary interval E with $\lambda(E) \geq 1$, the norm $\|Ez\|$ is given by some (m, g)-form. Furthermore, the first part of the proof shows that if $\lambda(E) \geq N^{4/5} = K_0$, then $\|Ez\|$ has to be realized either by a special functional of length $m \neq N$ or by an (m, f)-form; it follows that in this case the norm $\|Ez\|$ is given by some (m, h_N)-form. All this shows that the assumptions of Lemma 3.b-7 are satisfied and so our conclusion follows.

□

Note that in the situation of Lemma 3.b-13, we can consider a special functional $x^* = \frac{1}{g(N)} \sum_{k,l=1}^{M} \varepsilon_k \nu_l y^*_{k,l}$. We estimate

$$\Big\|\sum_{k,l=1}^{M} \varepsilon_k \nu_l z_{k,l}\Big\| \geq x^*\Big(\sum_{k,l=1}^{M} \varepsilon_k \nu_l z_{k,l}\Big) \geq \Big(\frac{1}{2} - \frac{1}{N}\Big)\frac{N}{g(N)} - \varepsilon.$$

Together with Corollary 3.b-9 it shows that $\Big\|\sum_{k,l=1}^{M} \varepsilon_k \nu_l z_{k,l}\Big\| \approx \dfrac{N}{g(N)}$.

3.c. Subspaces of X Do Not Have GL-Property

Let Y be an infinite-dimensional subspace of X. We recall (Lemma 3.b-4) that if $X \in \mathcal{X}_f$ for some $f \in \mathcal{F}$, then in every infinite-dimensional block subspace Y of X we can find an ℓ_{1+}^n-average with constant C for any n and $C > 1$. In particular, in our space X, for an arbitrary block subspace Z of X, $\eta > 0$ and $N \in I\!N$, there are vectors $x_1 < \cdots < x_N \in Z$ forming a R.I.S. of length N with constant $(1+\eta)$.

Fix $N \in \mathcal{K}$. A standard approximation argument shows that for every $\eta > 0$ there is a block subspace Z of X such that for every $z \in S_Z$ we have $\inf \|z-y\| < \eta$, where the infimum runs over all $y \in S_Y$. Therefore, we can repeat the Gowers-Maurey construction inside Z and find by induction vectors $y_1, \ldots, y_N \in S_Y$, $z_1 < \cdots < z_N \in S_Z$, and functionals $y_1^*, \ldots, y_N^* \in X^*$ with the following properties:

- $y_i^* \in \mathcal{Q}$ is an (M_i, f)-form, $|y_i^*(z_i) - 1/2| < 1/N$; $M_1 = j_{2N}$ and $M_{i+1} = \sigma(y_1^*, \ldots, y_i^*)$
- z_i is a normalized R.I.S. of length $M_i \in \mathcal{L}$ with constant $(1+\varepsilon/64)$
- $\mathrm{ran}(y_i^*) = \mathrm{ran}(z_i)$
- $\|y_i - z_i\| < \varepsilon/(12N^2)$ and $(1+\varepsilon/3)^{-1} \le \left\|\sum \phi_i z_i\right\| / \left\|\sum \phi_i y_i\right\| \le (1+\varepsilon/3)$ for arbitrary $\phi_i = \pm 1$.

Note that from the conditions on M_i it immediately follows that $\{y_i^*\} \in \Gamma_N^X$, that is, it is a special sequence of length N. Also, each z_i is an $\ell_{1+}^{N_i}$ average with constant $(1+\varepsilon)$, where $N_i = M_i^{\varepsilon/64}$ (Corollary 3.b-10), and $\frac{\varepsilon}{2}\sqrt{f(N_{i+1})} \ge |\,\mathrm{ran}(z_i)|$ by the definition of σ. Thus the vectors $z_1 < \cdots < z_N$ form a R.I.S. of length N with constant $(1+\varepsilon)$. We can also see that y_i^* and z_i satisfy the assumptions of Lemma 3.b-13.

Let us define the following spaces:

$$F = \left(\mathrm{span}\{y_i\}_{i=1}^N, \|\cdot\|_X\right)$$
$$F^* = \left(\mathrm{span}\{y_i^*\}_{i=1}^N, \|y^*\|_{F^*} = \max\{y^*(y);\ y \in B_F\}\right)$$
$$G = \left(\mathrm{span}\{y_i^*\}_{i=1}^N, \|y^*\|_G = \max\{y^*(y);\ y \in B_Y\}\right)$$
$$G_Y = \left(\mathrm{span}\{y_i^*\}_{i=1}^N, \|y^*\|_{G_Y} = \max\left\{\frac{y^*(\sum \phi_i y_i)}{\|\sum \phi_i y_i\|};\ \phi_i = \pm 1\right\}\right)$$
$$G_Z = \left(\mathrm{span}\{y_i^*\}_{i=1}^N, \|y^*\|_{G_Z} = \max\left\{\frac{y^*(\sum \phi_i z_i)}{\|\sum \phi_i z_i\|};\ \phi_i = \pm 1\right\}\right).$$

F is a subspace of Y and G is a subspace of Y^*; the spaces G_Y and G_Z are two renormings of G. We claim that $\|\cdot\|_{G_Z} \le (1+\varepsilon)\|\cdot\|_{G_Y}$. Let us briefly outline the proof.

For simplicity, write $\nu = \varepsilon/(12N^2)$ and $\eta = \varepsilon/3$. Let $y^* = \sum b_i y_i^* \in G$. Clearly, $\|y^*\| \le N \max|b_i|$. Next, pick $1 \le k \le N$ such that $\max|b_i| = |b_k|$. Set $\varepsilon_i = \operatorname{sign}(y^*(y_i))$, and let $y = \sum \varepsilon_i y_i$. Then $\|y\| \le N$ and hence

$$\begin{aligned}\|y^*\|_{G_Y} &\ge |y^*(y)|/\|y\| \ge |y^*(y_k)|/N \ge \big(|y^*(z_k)| - \|y^*\|\|y_k - z_k\|\big)/N \\ &\ge |b_k|(1/2 - 1/N - N\nu)/N = |b_k|(1/2 - 1/N - \varepsilon/(12N))/N.\end{aligned}$$

Now pick $z = \sum \phi_i z_i$ with $\phi_i = \pm 1$, such that $\|y^*\|_{G_Z} = y^*(z)/\|z\|$. Setting $y = \sum \phi_i y_i$ we have $\|y\| \le (1+\eta)\|z\|$, also note that $\|y\| \ge 1$. Thus

$$\begin{aligned}\|y^*\|_{G_Z} &\le \frac{1+\eta}{\|y\|}\Big(|y^*(y)| + \|y^*\|\sum\|y_i - z_i\|\Big) \le (1+\eta)\big(\|y^*\|_{G_Y} + (N|b_k|)(N\nu)\big) \\ &\le (1+\eta)\Big(1 + \frac{N^2\nu}{1/2 - 1/N - \varepsilon/(12N)}\Big)\|y^*\|_{G_Y}.\end{aligned}$$

Since $N \in \mathcal{K}$ is large, we note that $1/2 - 1/N - \varepsilon/(12N) \ge 1/4$, also $4N^2\nu = \varepsilon/3$ and $(1+\varepsilon/3)^2 \le 1+\varepsilon$, so we conclude that $\|y^*\|_{G_Z} \le (1+\varepsilon)\|y^*\|_{G_Y}$.

Note that we only used the lower estimate for $\left\|\sum \phi_i z_i\right\| / \left\|\sum \phi_i y_i\right\|$ in this proof. Using the upper estimate we similarly show that $\|\cdot\|_{G_Y} \le (1+\varepsilon)\|\cdot\|_{G_Z}$, that is, these two norms are $(1+\varepsilon)$-equivalent. This is not needed to rule out the GL-property, but it will be used to give us some further properties of the space X.

The space F^* is isometric to the space $Y^*/F^\perp$, in particular the quotient map $\tilde{J}$ from Y^* to F^* satisfies $\tilde{J}(y_i^*) = y_i^*|_F$. From $\|\cdot\|_{G_Z} \le (1+\varepsilon)\|\cdot\|_{G_Y} \le (1+\varepsilon)\|\cdot\|_{F^*}$ we have that there exists a retraction $J\colon Y^* \to G_Z$ such that $\|J\| \le (1+\varepsilon)$. Thus we use Theorem 3.a-1 and Observation 3.a-2 to get for arbitrary $\phi_{k,l} = \pm 1$

$$\min_{\varepsilon_k,\nu_l=\pm1}\Big\|\sum_{k,l=1}^{M} \varepsilon_k\nu_l\phi_{k,l}y_{k,l}^*\Big\|_{G_Z} \le (1+\varepsilon)C\ln(N)\,\mathrm{gl}(Y)\max_{\varepsilon_k,\nu_l=\pm1}\Big\|\sum_{k,l=1}^{M}\varepsilon_k\nu_l y_{k,l}^*\Big\|_G.$$

On the other hand, fix any sequence $\{\phi_{k,l}\} \in \Phi_M$. For every $\varepsilon_k, \nu_l = \pm 1$ we see that z_i, y_i^*, and $\{\varepsilon_k\nu_l\phi_{k,l}\}_{k,l=1}^M$ satisfy assumptions of Lemma 3.b-13. Setting $z = \sum_{k,l=1}^{M} \varepsilon_k\nu_l\phi_{k,l}z_{k,l}$ we get $\|z\| \le (1+2\varepsilon)\dfrac{N}{h_N(N)}$. By our construction we have $\sum_{k,l=1}^{M} \varepsilon_k\nu_l\phi_{k,l}y_{k,l}^*(z) \ge N\big(\frac12 - \frac1N\big)$, so by definition

$$\Big\|\sum_{k,l=1}^{M}\varepsilon_k\nu_l\phi_{k,l}y_{k,l}^*\Big\|_{G_Z} \ge \frac{\sum\varepsilon_k\nu_l\phi_{k,l}y_{k,l}^*(z)}{\|z\|} \ge \frac{h_N(N)}{(1+2\varepsilon)}\Big(\frac12 - \frac1N\Big) \ge \frac{h_N(N)}{4(1+2\varepsilon)}.$$

Thus $\min\{\|\sum \varepsilon_k \nu_l \phi_{k,l} y^*_{k,l}\|_{G_Z};\ \varepsilon_k, \nu_l = \pm 1\} \geq \dfrac{h_N(N)}{4(1+2\varepsilon)}$.

We also have $\left\|\sum\limits_{k,l=1}^{M} \varepsilon_k \nu_l y^*_{k,l}\right\|_{X^*} \leq g(N)$ for every $\varepsilon_k, \nu_l = \pm 1$ as $\{y^*_i\}$ is a special sequence. Using $\|\cdot\|_G \leq \|\cdot\|_{X^*}$ we get $\max\{\|\sum \varepsilon_k \nu_l y^*_{k,l}\|_G;\ \varepsilon_k, \nu_l = \pm 1\} \leq g(N)$. Comparing the upper and lower estimate we get that

$$\min_{\varepsilon_k, \nu_l = \pm 1} \left\| \sum_{k,l=1}^{M} \varepsilon_k \nu_l \phi_{k,l} y^*_{k,l} \right\|_{G_Z} \geq \frac{h_N(N)}{4(1+2\varepsilon) g(N)} \max_{\varepsilon_k, \nu_l = \pm 1} \left\| \sum_{k,l=1}^{M} \varepsilon_k \nu_l y^*_{k,l} \right\|_G .$$

Consequently, there is a constant c depending on ε only so that

$$\mathrm{gl}(Y) \geq c \frac{h_N(N)}{g(N) \ln(N)}.$$

Since the expression on the right hand side tends to infinity as $N \in \mathcal{K} \to \infty$, it follows that Y does not have the GL-property.

□

We now show that our space X is hereditarily indecomposable, that is, no two closed infinite-dimensional subspaces add as a topological direct sum. Indeed, if Y_1 and Y_2 are two closed infinite-dimensional subspaces with $Y_1 \cap Y_2 = \{0\}$, we can fix $\phi_i \in \Phi_M$ and find a sequence $z_1 < \cdots < z_N$, vectors y_i and functionals y^*_i as in the previous construction, such that $y_i \in Y_1$ if $\phi_i = 1$ and $y_i \in Y_2$ if $\phi_i = -1$. Let v be the sum of those y_i that are in Y_1 and let w be the sum of y_i from Y_2. By a lower estimate for $\|v + w\|$ as in the remark after Lemma 3.b-13 and an upper estimate for $\|v - w\|$ as above we get

$$\frac{\|v+w\|}{\|v-w\|} \geq C(\varepsilon) \frac{N}{g(N)} \Big/ \frac{N}{h_N(N)} = C(\varepsilon) \frac{h_N(N)}{g(N)} \to \infty,$$

completing the proof.

¿From the estimate above it follows that vectors $y^*_{k,l}$ are badly unconditional, namely $\mathrm{unc}\{y^*_i\} \geq \dfrac{h_N(N)}{4(1+2\varepsilon) g(N)}$. On the other hand, we were free to change signs in a tensor fashion. Actually, it can be shown that for every $\varepsilon_k, \nu_l = \pm 1$ we have

$$\frac{g(N)}{4(1+4\varepsilon)} \leq \left\| \sum_{k,l=1}^{M} \varepsilon_k \nu_l y^*_{k,l} \right\|_{G_Y} \leq \left\| \sum_{k,l=1}^{M} \varepsilon_k \nu_l y^*_{k,l} \right\|_G \leq g(N)$$

and we also have analogous inequalities for the norm $\|\cdot\|_{G_Z}$.

We can achieve a similar behaviour for vectors $\{z_i\}$ and $\{y_i\}$. As shown after Lemma 3.b-13, we have

$$\left(\frac{1}{2}-\frac{1}{N}-\varepsilon\right)\frac{N}{g(N)} \leq \left\|\sum_{k,l=1}^{M} \varepsilon_k \nu_l z_{k,l}\right\| \leq (1+2\varepsilon)\frac{N}{g(N)},$$

whereas signs $\{\phi_i\} \in \Phi_M$ and Lemma 3.b-13 imply that $\mathrm{unc}\{z_i\} \geq \dfrac{h_N(N)}{4(1+2\varepsilon)g(N)}$. Using the closeness of norms of $\sum \phi_i y_i$ and $\sum \phi_i z_i$ we obtain analogous statements for $\{y_i\}$.

3.d. Proof of Proposition 3.b-8

Now we prove Proposition 3.b-8. By $l(x)$ we denote $\log_2(x+1) = \ln(2)^{-1}\ln(x+1)$. Let $p \geq 1$, an easy argument shows that the function $l(x)^p$ is submultiplicative. We have

$$\left(\frac{x}{l(x)^p}\right)' = \frac{\ln(2)l(x)(x+1) - p\,x}{l(x)^{p+1}\ln(2)(x+1)},$$

so $\frac{x}{l(x)^p}$ is increasing at x if $\ln(x+1)\frac{x+1}{x} \geq p$. The function on the left is increasing on $[1,\infty)$, hence there is $N(p) \geq 1$ such that $\frac{x}{l(x)^p}$ is decreasing on $[1, N(p)]$ and increasing on $[N(p),\infty)$. Note that $N(2) > 1$.

Also,

$$\left(\frac{x}{l(x)^p}\right)'' = \frac{p(p+1)x/\ln(2) - p(x+2)l(x)}{\ln(2)(x+1)^2 l(x)^{p+2}},$$

and similar argument with the condition $\ln(x+1)\frac{x+2}{x} \geq p+1$ shows that there is $M(p) > N(p)$ such that $\frac{x}{l(x)^p}$ is convex on $[1, M(p)]$ and concave on $[M(p),\infty)$. Note that if $q > p$, then $N(q) > N(p)$ and $M(q) > M(p)$. The function $x/l(x)$ is concave and increasing on $[1,\infty)$, that is, $N(1) = M(1) = 1$.

First, let us construct f and g. Consider $a(x) = l(x)^4$, $b(x) = l(x)^2$, $c(x) = l(x)$. Clearly $a(1) = b(1) = c(1)$ and $c \leq b \leq a$. Set $A(x) = \frac{x}{a(x)}$, $B(x) = \frac{x}{b(x)}$, $C(x) = \frac{x}{c(x)}$. We readily get that $A \leq B \leq C$, all three functions are supermultiplicative, and C is also concave.

We will need the equation of the tangent line T_y^A to A at y, which is

$$T_y^A(x) = \frac{x\,l(y)\ln(2) - x\frac{4y}{y+1} + \frac{4y^2}{y+1}}{l(y)^5\ln(2)}.$$

Note that such a tangent line is always increasing, moreover, $T_y^A(1) \to \infty$ as $y \to \infty$. Similar formula gives the tangent line T_y^B to B at y.

By the concave envelope of a function ϕ we mean a function $h \geq \phi$ that is the smallest concave function dominating ϕ. It is easy to observe that the concave envelope to a supermultiplicative function is again supermultiplicative (see [GM]).

Let F be the concave envelope of A and let G be the concave envelope of B. Since C itself is concave, we get that $F \leq G \leq C$, and F, G are supermultiplicative. Recall that functions A and B look in the following way: $A(1) = B(1) = 1$, they are first convex and decreasing, then convex and increasing, then concave and increasing. Thus F is a straight line from $(1,1)$ to $(P_f, A(P_f))$ for some P_f (precisely, P_f is such that $T^A_{P_f}(1) = 1$) and $F = A$ on $[P_f, \infty)$. Similarly we work with B, introducing the respective point P_g. We can observe that $P_g < P_f$. Note that the formula for T^A_y now gives the tangent line to F at y for $y \geq P_f$.

Define $f(x) = \dfrac{x}{F(x)}$, $g(x) = \dfrac{x}{G(x)}$. It follows that $f(x) = l(x)^4$ on $[P_f, \infty)$, $g(x) = l(x)^2$ on $[P_g, \infty)$, and both functions are in $\mathcal{F}$. The fact that f (resp. g) is increasing follows by a general argument using the concavity of $x/f(x)$. We now show that $g \geq \sqrt{f}$.

First, $g = \sqrt{f}$ on $[P_f, \infty)$. Also, $g(x) = \log_2^2(x+1)$ on $[P_g, P_f]$, while $f(x) < \log_2^4(x+1)$ there, so $g > \sqrt{f}$ there as well. Finally, let T_g be the line connecting $(1,1)$ with $(P_g, B(P_g))$ (i.e. $T_g = T^B_{P_g}$) and T_f be the line connecting $(1,1)$ with $(P_g, A(P_g))$. We have that $T_g = G$ and $T_f \leq F$ on $[1, P_g]$. Thus it is enough to show that $x/T_g(x) \geq \sqrt{x/T_f(x)}$.

Let us denote $P = P_g$ and $A = b(P)$. We can write $T_f(x) = 1 + (x-1)\dfrac{P/A^2 - 1}{P-1}$ and $T_g(x) = 1 + (x-1)\dfrac{P/A - 1}{P-1}$. We substitute it into the inequality, simplify and obtain the inequality $(1 - 1/A)^2 \alpha(x) \geq 0$, where $\alpha(x) = -Px^2 + P(P+1)x - P^2$. So we want $\alpha \geq 0$ on $[1, P]$. This follows from the fact that $\alpha(1) = \alpha(P) = 0$ as expected and $\alpha'(1) = P^2 - P \geq 0$.

Now we prove the existence of h_0, h_N and $\mathcal{J}$. First note that the conditions on $\mathcal{J}$ in Proposition 3.b-8 either require that $\mathcal{J}$ starts far enough, or have lacunarity character, that is, they are of this form:

There is a function $\varrho : \mathbb{N} \to \mathbb{N}$ such that if all $n < m \in \mathcal{J}$ satisfy $m \geq \varrho(n)$, then $\mathcal{J}$ is good for our purposes.

The construction of h_0 and h_N that follows will require two additional lacunarity type conditions on $\mathcal{J}$ and another lower bound for $\min(\mathcal{J})$. We will state them as we proceed with the constructions and at the end we choose for $\mathcal{J}$ any sequence satisfying these requirements.

Let $\mathcal{J} = \{j_k\}$ be an increasing sequence in $\mathbb{N}$ with $\min(\mathcal{J}) \geq P_f$, recall that

$\mathcal{K} = \{j_{2n-1}\}$ and $\mathcal{L} = \{j_{2n}\}$. We set

$$w(x) = \begin{cases} f(x) & x \notin \mathcal{K} \\ g(x) & x \in \mathcal{K}. \end{cases}$$

Then $g \leq w \leq f$. Let ϕ be the non-decreasing submultiplicative hull of w, that is, the largest non-decreasing submultiplicative function dominated by w. Such a function is given by the formula

$$\phi(x) = \inf\{w(x_1) \cdot \ldots \cdot w(x_n);\ x_i \geq 1,\ x_1 \cdots x_n \geq x\}.$$

Since g is non-decreasing, submultiplicative and dominated by w, we obtain that $g \leq \phi \leq w$, in particular $\phi = g$ on $\mathcal{K}$. Let $k < l$ be successive numbers from $\mathcal{K}$, that is, $(k, l) \cap \mathcal{K} = \emptyset$. Consider x such that $(k!)^4 \leq x$ and $f(x) \leq g(l)$. It was proved in [GM] that then $\phi(x) = f(x)$.

Set $W(x) = \dfrac{x}{w(x)}$, $\Phi(x) = \dfrac{x}{\phi(x)}$. Then $F \leq W \leq \Phi \leq G$, $\Phi = G$ on $\mathcal{K}$, and Φ is supermultiplicative. Also, for $k < l$ successive in $\mathcal{K}$ and x as above we have $\Phi(x) = F(x)$.

Let H be the concave envelope of Φ. Then H is concave and supermultiplicative and $\Phi \leq H \leq G$ as G is concave and dominates Φ. Define $h_0(x) = \dfrac{x}{H(x)}$. Then $h_0 \in \mathcal{F}$ and $g \leq h_0 \leq f$, we also have $h_0 = g$ on $\mathcal{K}$. We want to identify points between $\mathcal{K}$ where $h_0 = f$, that is, $H = F$. But this is rather easy.

Let $k \in \mathcal{K}$, $k = j_{2n-1}$ for some n. Recall that the tangent lines T_y^A to F at points y satisfy $T_y^A(1) \to \infty$ as $y \to \infty$ and they are increasing, therefore there exists $\alpha > (k!)^4$ such that tangent lines T_y^A for $y \geq \alpha$ dominate G on $[1, (k!)^4]$. Assuming that the next number l after k in $\mathcal{K}$ is very far, we know that $F = \Phi$ around α, so T_y^A are tangent lines to Φ as well. Take $j_{2n} \geq \alpha^{4/3}$. Since the function G grows slower than any line, there is a number $\beta > j_{2n}$ such that G is dominated by $T_{j_{2n}}^A$ on $[\beta, \infty)$. Finally, there is $l > \beta$ such that $f(\beta) \leq g(l)$. Thus, depending on $k = j_{2n-1}$ we chose the next j_{2n} and $j_{2n+1} \geq l$.

Now, since every x between $(k!)^4$ and β satisfies the condition above, we know that $\Phi = F$ on $[(k!)^4, \beta]$. Thus the tangent lines to Φ at points of this interval agree with tangent lines to F and as this function is concave, these tangent lines dominate $F = \Phi$ on $[(k!)^4, \beta]$. In particular this is true about points from the interval $[\alpha, j_{2n}]$, but the corresponding tangent lines dominate G and hence Φ on $[1, (k!)^4] \cup [\beta, \infty)$ by our construction. This says that for $y \in [j_{2n}^{3/4}, j_{2n}]$, the tangent line T_y^A to Φ

dominates Φ. Consequently, the concave envelope H of Φ is dominated by T_y^A as well, in particular $F(y) \le H(y) \le T_y^A(y) = F(y)$, that is, $f(y) = h_0(y)$. This completes the construction of h_0.

Now we construct the functions h_N. For computational reasons we start with a number N that is as big as needed later, for a start we require $N \ge P_f$ to avoid complications, and we will show that there exists a function $h_N \in \mathcal{F}$ such that $h_N = g$ on $[1, N]$, $g \le h_N \le f$, and for the resulting sequence $\{h_N\}$ we have $\lim\limits_{N\to\infty} \dfrac{h_N(N^{5/4})}{g(N^{5/4})\ln(N^{5/4})} = \infty$. In the end we will use a general argument to show that these functions can be constructed so that also $h_N = g$ on $[\varrho_N, \infty)$ for some $\varrho_N > N$. The construction will be similar to that of h_0.

Define $w_N(x) = b(x) = l(x)^2$ for $x \in [1, N]$ and $w_N(x) = a(x) = l(x)^4$ for $x > N$. This function satisfies $b \le w_N \le a$. Let ϕ_N be the non-decreasing submultiplicative hull of w_N. Since b is non-decreasing, submultiplicative and bounded by w_N, we get that $b \le \phi_N \le w_N$, in particular, $\phi_N = b$ on $[1, N]$.

Recall that ϕ_N is given by the formula

$$\phi_N(x) = \inf\{w_N(x_1) \cdot \ldots \cdot w_N(x_n);\ x_i \ge 1,\ x_1 \cdots x_n \ge x\}.$$

Let us consider the following functions:

$$\phi_{N1}(x) = \inf\{b(x_1) \cdots b(x_k);\ k \in \mathbb{N},\ 1 \le x_i \le N,\ x_1 \cdots x_k = x\}$$
$$\phi_{N2}(x) = \inf\{b(x_1) \cdots b(x_k)\, a(y);\ k \in \mathbb{N},\ 1 \le x_i \le N,\ y > N,\ x_1 \cdots x_k y = x\}.$$

Since w_N is increasing on $[1, \infty)$ and submultiplicative on (N, ∞), we observe that $\phi_N(x) = \min(\phi_{N1}(x), \phi_{N2}(x), w_N(x))$.

Consider the function $\alpha(x) = \ln(x+1)\ln(L/x+1)$ for some $L > 15$. We have $\alpha'(x) = \big((x+1)(1+x/L)\big)^{-1}(\beta(L/x) - \beta(x))$, where $\beta(x) = \ln(x+1)\dfrac{x+1}{x}$. Since β is increasing, the point $x = \sqrt{L}$ is the only local extrem of α. Since $\alpha'(1) > 0$, we have that α has a local maximum at $\sqrt{L}$. This means that for any interval $[\gamma, \delta]$ with $\gamma \ge 1$, the function $b(x)b(L/x)$ attains its minimum on $[\gamma, \delta]$ at γ or δ. The same can be shown about $b(x)a(L/x)$. Using this for functions ϕ_{N1}, ϕ_{N2} we get the following formulas:

Lemma 3.d-1.

Let $x > N$. Let $k \in \mathbb{N}$ satisfy $N^k \le x \le N^{k+1}$. Then $\phi_{N1}(x) = b^k(N)b(x/N^k)$.

Furthermore, $\phi_{N2}(x)$ is equal to the smallest of numbers $b^{k+1}(N)b(x/N^k)$ for k as above or $b^m(N)a(x/N^m)$ for $m \in I\!N$ such that $N^{m+1} < x$.

Thus we know how the functions ϕ_{N1} and ϕ_{N2} look like and we can estimate them. Set $R = \log_2(3)$. Note that for every $x \geq 2$ we now have $\log_2(x) \leq l(x) \leq R\log_2(x)$.

Lemma 3.d-2.
Let N_0 be such that for every $N \geq N_0$ we have $\log_2^k(N) \geq R^2(k+3)^2$ for every $k \in I\!N$. Fix $N \geq N_0$, let $x > N$.

(1) If $k \in I\!N$ satisfies $N^k \leq x \leq N^{k+1}$, then $b^{k+1}(N)b(x/N^k) > \phi_{N1}(x)$.

(2) If $m \in I\!N$ satisfies $N^{m+1} < x$, then $b^m(N)a(x/N^m) \geq w_N(x)$.

(3) For every $x \geq N^3$ we have $\phi_{N1}(x) \geq w_N(x)$.

PROOF: Since $b^k(N) < b^{k+1}(N)$, the first estimate in (1) follows.

To prove (2) we show that $\log_2^m(N)\log_2^2(x/N^m) \geq R^2\log_2^2(x)$. Denote $A = \log_2(x)$ and $B = \log_2(N)$. Then we want to show that the function $\alpha(A) = B^m(A-mB)^2 - R^2A^2$ is non-negative for $A \geq (m+1)B$. We have $\alpha'(A) = A(2B^m - 2R^2) - 2mB^{m+1}$. Since $\alpha''(x) = 2(B^m - R^2) \geq 0$, α' is an increasing function in A. Observing $\alpha'((m+1)B) = 2B(B^m - R^2(m+1)) \geq 0$ we get $\alpha' \geq 0$ for $A \geq (m+1)B$. Thus

$$\alpha(A) \geq \alpha((m+1)B) = B^2(B^m - R^2(m+1)^2) \geq 0.$$

Finally, (3) follows from $k \geq 3$ and

$$\begin{aligned} w_N(x) &= l(x)^4 \leq R^4\log_2^4(x) \leq R^4\log_2^4(N^{k+1}) \\ &= R^4(k+1)^4\log_2^4(N) \leq \log_2^{2k}(N) \leq l(N)^{2k}\log_2^2(x/N^k) \\ &\leq \phi_{N1}(x). \end{aligned}$$

□

Thus $\phi_{N2} \geq \min(\phi_{N1}, w_N)$ on $[N, \infty)$ if $N \geq N_0$, that is, $\phi_N = \min(\phi_{N1}, w_N)$ on $[1, \infty)$. We may assume that $N_0 \geq P_f$. We observe that $\phi_{N1} \leq w_N$ on $[N, N^2]$, so recalling (3) above and the definition of w_N we get

$$\phi_N(x) = \begin{cases} b(x) & x \leq N \\ \phi_{N1}(x) & N \leq x \leq N^2 \\ \min(\phi_{N1}(x), a(x)) & N^2 \leq x \leq N^3 \\ a(x) & N^3 \leq x \end{cases}.$$

We also know that for $x \in [N, N^2]$ we have $\phi_{N1}(x) = l(N)^2 l(x/N)^2$, and for $x \in [N^2, N^3]$ we have $\phi_{N1}(x) = l(N)^4 l(x/N^2)^2$.

Denote $\Phi_N(x) = \dfrac{x}{\phi_N(x)}$, $\Phi_{N1}(x) = \dfrac{x}{\phi_{N1}(x)}$ and $W_N(x) = \dfrac{x}{w_N(x)}$, recall that $A(x) = \dfrac{x}{a(x)}$ and $B(x) = \dfrac{x}{b(x)}$. Clearly $A \le W_N \le \Phi_N \le B$. The description of ϕ_N above transforms in the obvious way into the description of Φ_N. Note that $\Phi_{N1}(x) = \dfrac{N}{l(N)^2} B(x/N)$ for $x \in [N, N^2]$ and $\Phi_{N1}(x) = \dfrac{N^2}{l(N)^4} B(x/N^2)$ on $[N^2, N^3]$, that is, the "shape" of Φ_{N1} on each of these intervals is the same as the "shape" of B on $[1, N]$. In particular, since $N \ge P_f$, in each interval $[N, N^2]$ and $[N^2, N^3]$ the function Φ_{N1} is first decreasing and convex, then it becomes increasing and eventually concave.

Arguments in the part that follows say that something holds if N is large enough. We will use the notation $\alpha(N) \sim \beta(N)$ to say that $\alpha(N)/\beta(N) \to 1$ if $N \to \infty$. Recall that for the function l we have $l(N) \sim \log_2(N)$, hence $l(N^k) \sim k\, l(N)$. Using this we see that $\Phi_{N1}(N^2) = N^2 l(N)^{-4} \sim 16 W_N(N^2)$ and $\Phi_{N1}(N^3) = N^3 l(N)^{-6} \sim \dfrac{81}{l(N)^2} W_N(N^3)$.

Let H_N be the concave envelope of Φ_N, clearly $A \le \Phi_N \le H_N$ and $F \le H_N \le G$. From $N \ge P_f$ we get that $H_N = G$ on $[1, N]$. Also, if $N \ge N_0$, from $\Phi_N = A$ on $[N^3, \infty)$ and the fact that the tangent line to F considered at points going to infinity will eventually dominate G on $[1, N^3]$ (see above), we get that $H_N = F$ on $[M_N, \infty)$ for M_N large enough, clearly $M_N > N^3$.

Define $h_N(x) = \dfrac{x}{H_N(x)}$, then $h_N \in \mathcal{F}$ and $g \le h_N \le f$, we also have $h_N = g$ on $[1, N]$ and $h_N = f$ on $[M_N, \infty)$. We now prove that the functions $\{h_N\}$ satisfy the limit condition.

Consider functions Φ_N, Φ_{N1}, H_N and h_N constructed as above for $N \ge N_0$. Let S_N be the tangent line to $\Phi_N = \Phi_{N1}$ at $y_N = N^{4/5}$. We want to show that $\Phi_N \le S_N$. Fix $x_N = 2N^2$ and define "test functions" $U_N(x) = A(x)\dfrac{\Phi_{N1}(x_N)}{A(x_N)}$. Note that we can estimate $\dfrac{\Phi_{N1}(x_N)}{A(x_N)} \sim \dfrac{16}{R^2} \approx 6.4$, therefore $U_N \sim 6.4\, A$.

Lemma 3.d-3.
There is $N_1 > N_0$ such that for $N \ge N_1$ we have $\Phi_N \le U_N$ on $[x_N, \infty)$.

PROOF: Clearly $\Phi_{N1} \le A \le U_N$ on $[N^3, \infty)$. We will show that $\Phi_{N1} \le U_N$ on

$[x_N, N^3]$. Consider $R(x) = \Phi_{N1}(x)/A(x)$ on $[N^2, N^3]$. We see that

$$R'(x) = \frac{2l(x)^3}{\ln(2)l(N)^4 l(x/N^2)^3(x+1)(x+N^2)}\alpha(x),$$

where $\alpha(x) = 2l(x/N^2)(x+N^2) - l(x)(x+1)$. Then $\ln(2)\alpha'(x) = \ln\big(\varrho(x)\big) + 1$, where $\varrho(x) = \dfrac{(x+N^2)^2}{N^4(x+1)}$. ¿From $x \geq N^2$ we have

$$\varrho'(x) = \frac{(x+N^2)}{(x+1)^2}(x - N^2 + 2) \geq 0,$$

therefore ϱ is increasing and

$$\alpha'(x) \leq \alpha'(N^3) = \log_2\Big(\frac{(N+1)^2}{N^3+1}\Big) + \ln(2)^{-1}.$$

The fraction inside goes to zero, so if N is large enough, we will have $\alpha'(N^3) < 0$, that is, α will be decreasing on $[N^2, N^3]$. In this case we have

$$\alpha(x) \leq \alpha(N^2) = 4N^2 - l(N^2)(N^2+1) \leq N^2(4 - l(N^2)),$$

which is negative for N large.

Thus we find $N_1 \geq N_0$ such that for $N \geq N_1$, the ratio $R(x)$ is decreasing on $[N^2, N^3]$. In particular, for $N \geq N_1$,

$$\max\{\Phi_{N1}(x)/A(x);\ x \in [x_N, N^3]\} = \Phi_{N1}(x_N)/A(x_N).$$

That is, on $[x_N, N^3]$ we indeed have $\Phi_{N1} \leq U_N$. Put together, we know that $\Phi_{N1} \leq U_N$ on $[x_N, \infty)$, also $A \leq U_N$, hence $\Phi_N \leq U_N$ on $[x_N, \infty)$.

□

Note that Φ_{N1} on $[N^2, x_N]$ has the same behaviour as B on $[1, 2]$, therefore checking on B we see that Φ_{N1} is convex on $[N^2, x_N]$. Let us consider the left tangent S_{N1} to Φ_{N1} at N^2, its slope is ${\Phi_{N1}}'_-(N^2) = \big(l(N) - \frac{2}{\ln(2)}\frac{N}{N+1}\big)/l(N)^5$.

Lemma 3.d-4.
There is $N_2 \geq N_1$ such that for $N \geq N_2$ we have $\Phi_N \leq S_{N1}$ on $[N^2, \infty)$.

PROOF: Since ${\Phi_{N1}}'_-(N^2)/U_N'(N^2)$ goes to $R^2 > 1$ as $N \to \infty$, for large N we have $S_{N1}'(N^2) > U_N'(N^2)$; also,

$$U_N(N^2) \sim \frac{16}{R^2}A(N^2) < 16A(N^2) \sim \Phi_{N1}(N^2) = S_{N1}(N^2),$$

so by the concavity of U_N on $[N^2,\infty)$ we get that $U_N \le S_{N1}$ on $[N^2,\infty)$. Therefore also $\Phi_N \le S_{N1}$ on $[x_N,\infty)$.

On the other hand, for large N we get $\Phi_{N1}(x_N) < S_{N1}(x_N)$ by comparing their ratio, also $\Phi_{N1}(N^2) = S_{N1}(N^2)$ and Φ_{N1} is convex on $[N^2, x_N]$, hence $\Phi_{N1} \le S_{N1}$ there. We observe that $A \le U_N \le S_{N1}$ on $[N^2, x_N]$ as well, so $\Phi_N \le S_{N1}$ on $[N^2, x_N]$.

□

Recall that $y_N = N^{5/4}$ and S_N is the tangent line to $\Phi_N = \Phi_{N1}$ at y_N, it is given by

$$S_N(x) = \frac{x}{l(N)^2 l(N^{1/4})^2} + \frac{N^{1/4}}{N^{1/4}+1}\frac{2}{\ln(2) l(N)^2 l(N^{1/4})^3}(N^{5/4} - x).$$

Lemma 3.d-5.
There is $N_3 \ge N_2$ so that for $N \ge N_3$ we have $\Phi_N \le S_N$ on $[1,\infty)$.

PROOF: Recall that the behaviour of Φ_N on $[y_N, N^2]$ corresponds to that of B on $[N^{1/4}, N]$. Hence Φ_N is concave on $[y_N, N^2]$ for large N, then we have $\Phi_N \le S_N$ on $[y_N, N^2]$. By the concavity we also have $S_{N1} \le S_N$ on $[N^2,\infty)$, hence $\Phi_N \le S_N$ on $[N^2,\infty)$ by Lemma 14. We check that $\Phi_N(N) = B(N) \le S_N(N)$ for large N, $\Phi_N(y_N) = \Phi_{N1}(y_N) = S_N(y_N)$, therefore by first convexity and then concavity of $\Phi_N = \Phi_{N1}$ inbetween we get $\Phi_N \le S_N$ on $[N, y_N]$.

Finally, one can show that $S_N(1)/B(N) \to \infty$, therefore for large N we obtain that $S_N(1) > B(N)$. Since $\max\{B(x);\ x \in [1,N]\} = B(N)$ and S_N is increasing, $\Phi_N = B \le S_N$ on $[1,N]$ as well.

□

Corollary 3.d-6.
For $N \ge N_3$ we have $h_N(N^{5/4}) = \phi_N(N^{5/4}) = l(N)^2 l(N^{1/4})^2$.

In particular, $\lim\limits_{N\to\infty} \dfrac{h_N(N^{5/4})}{g(N^{5/4})\ln(N^{5/4})} = \infty.$

PROOF: Since $\Phi_N \le S_N$, it follows that $\Phi_N \le H_N \le S_N$. But $\Phi_N(y_N) = S_N(y_N)$, hence $H_N(y_N) = \Phi_N(y_N)$, that is, $h_N(N^{5/4}) = l(N)^2 l(N^{1/4})^2$. Consequently

$$\frac{h(N^{5/4})}{g(N^{5/4})\ln(N^{5/4})} = \frac{l(N)^2 l(N^{1/4})^2}{l(N^{5/4})^2 \ln(N^{5/4})} \sim \frac{4}{125\ln(2)}\frac{l(N)^4}{l(N)^3} \to \infty.$$

□

The functions we constructed still do not satisfy one condition of Proposition 3.b-8 (iii). This will be done using a general argument similar to the one used in the construction of h_0. Recall that given $M \geq 1$, there is a number $m(M) > M$ such that the tangent line T_M to F at M dominates G on $[m(M), \infty)$. Then there is $\varrho(M) > m(M)$ such that $f(m(M)) < g(\varrho(M))$. We have the following observation:

Lemma 3.d-7.
Let $h \in \mathcal{F}$ satisfies $g \leq h \leq f$ and $h = f$ on a neighbourhood of some $M \geq 1$. Then there is a function $\tilde{h} \in \mathcal{F}$ satisfying $g \leq \tilde{h} \leq f$, $\tilde{h} = h$ on $[1, M]$ and $\tilde{h} = g$ on $[\varrho(M), \infty)$.

The functions h_N satisfy these assumptions, in particular we found $M_N > N^3$ such that $h_N = f$ on $[M_N, \infty)$, therefore functions $\tilde{h}_N$ obtained by Lemma 3.d-7 will satisfy Proposition 3.b-8 (iii).

PROOF: Let us define $w(x) = h(x)$ for $x \in [1, \varrho(M)]$ and $w(x) = g(x)$ on $(\varrho(M), \infty)$. Let ϕ be its non-decreasing submultiplicative hull. We get that $g \leq \phi \leq w$, in particular $\phi = g$ on $[\varrho(M), \infty)$. Observe that if $x \leq m(M)$, then $w(x) \leq f(m(M)) \leq g(\varrho(M)) = w(\varrho(M))$, also, $w = h$ is submultiplicative and non-decreasing on $[1, m(M)]$, so it follows that $\phi = w = h$ on $[1, m(M)]$ by checking on the formula for ϕ.

Consider functions $F(x) = \dfrac{x}{f(x)}$, $G(x) = \dfrac{x}{g(x)}$, $H(x) = \dfrac{x}{h(x)}$, $\Phi = \dfrac{x}{\phi(x)}$; we have that $F \leq H \leq \Phi \leq G$, $\Phi = H$ on $[1, m(M)]$ and $\Phi = G$ on $[\varrho(M), \infty)$. Recall that F, G and H are supermultiplicative, non-decreasing, and concave. Let $\widetilde{H}$ be the concave envelope of Φ. It follows that $\Phi \leq \widetilde{H} \leq G$, in particular $\widetilde{H} = G$ on $[\varrho(M), \infty)$. Consider the tangent line T_N to F at M. By our assumption, it is also a tangent line to H at M, in particular $H \leq T_M$. Consequently, $\Phi \leq T_M$ on $[1, m(M)]$. On the other hand, by our choice of $m(M)$ we know that $\Phi \leq G \leq T_M$ on $[m(M), \infty)$. Consequently, $\Phi \leq T_M$, hence as in Corollary 16 we show that $\widetilde{H}(M) = \Phi(M) = H(M)$.

Using the concavity of H and the tangent line T_x to H at $x \in [1, M]$ we similarly show that $\widetilde{H} = H$ on $[1, M]$. Taking $\tilde{h}(x) = \dfrac{x}{\widetilde{H}(x)}$ we conclude the proof.

□

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