HYPERINTERPOLATIONS ON THE SPHERE

by

Xingming Chu

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematics

Department of Mathematical and Statistical Sciences

University of Alberta

Abstract

Hyperinterpolation on the unit sphere of the Euclidean space was proposed by S-loan in 1995. But pointwise convergence in the uniform form can not be achieved by hyperinterpolation. Reimer later proposed the generalized hyperinterpolation, which has the advantages that uniform convergence for all continuous function-s can be achieved and it requires positive cubature formulas of less precision and hence significantly reduces the cost of computations. However, Reimers technique only allows him to obtain best approximation order result for C^1 functions. The main purpose of this thesis is to consider generalized hyperinterpolation of higher order and a best approximation order is obtained. The smoothness of functions is measured in terms of a new K-functional we introduced. Finally, the thesis also collects several useful positive cubature formulas and discusses briefly the construction methods.

Acknowledgement

Pursuing a master degree in pure mathematics at the department of mathematical and statistical sciences of the University of Alberta is an exciting journey for me. During the past two years, I made great progress in the mathematical knowledge by taking different mathematical courses and doing mathematical research.

Firstly, I want to thank for my supervisor, Professor Feng Dai, who constantly inspired and encouraged me throughout the past two years of my master studies. I learned lots of methods and concepts about the approximation theory. In addition to the research, his valuable life advices make me more clear about my future academic career and life. He encouraged me to participate in various academic activities. I had the extremely good fortune to be his graduate student and shall always hold him in the highest regard.

My appreciation also goes to my teachers, professor Tony Lau, Bin Han, Tong Zhang etc.. They not only taught me analysis courses but also gave me many valuable suggestions on how to do the research on pure and applied mathematics.

Finally, I am especially grateful to my parents who has always supported my studies in Canada.

Contents

Chapter 1

Preliminaries

In this chapter we will introduce some basic facts and concepts on spherical harmonics, Fourier-Laplace series and approximation theory on the unit sphere. Most of the materials in this chapter can be found in [1]-[31].

1.1 Spherical Harmonics

Let $\mathbb{N}$ denote the set of all natural numbers and $\mathbb{R}^d$ be the d-dimensional Euclidean space with norm $\|x\| = \sqrt{x_1^2 + x_2^2 + ... + x_d^2}$ for $x = (x_1, x_2, ...x_d) \in \mathbb{R}^d$. As usual, we denote by Δ_d the Laplace operator on $\mathbb{R}^d$

$$\Delta_d = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_d^2}.$$

Suppose f is a function defined on $\mathbb{R}^d$. If f satisfies the Laplace equation, we say that f is a harmonic function. If for all r in the complex field and all $x \in \mathbb{R}^d$, $f(rx) = r^k f(x)$ for some constant $k \in 0 \cup \mathbb{N}$, then we say that f is a homogeneous function of degree k.

Definition 1.1.1: (See [2, p.2]) The set of all homogeneous polynomials of d variables with degree k is denoted by P_k^d. We use A_k^d to denote the subset of harmonic functions in P_k^d. When $d \geq 2$, we denote by $\mathbb{S}^{d-1}$ the unit sphere of $\mathbb{R}^d$,

$$\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}.$$

Definition 1.1.2: (See [2, p.2]) Suppose f is a harmonic function in P_k^d. The restriction of f on $\mathbb{S}^{d-1}$ is called spherical harmonic of d variables of degree k. The set of all such functions is denoted by H_k^d.

We use d_k^d to denote the dimension of P_k^d. Clearly, H_k^d and A_k^d are all linear spaces over the complex filed $\mathbb{C}$. As is well known, they have the same dimension which is denoted by a_k^d. Let $\rho(x, y) = \arccos(x, y)$ denote the distance between $x, y \in \mathbb{S}^{d-1}$. And let $B(x, r) = \{y \in \mathbb{S}^{d-1} : \rho(x, y) \leq r\}$ denote the spherical cap centered at $x \in \mathbb{S}^{d-1}$ and of radius $r > 0$. Throughout this thesis, every set and every function f are assumed to be Lebesgue measurable. We denote by $\sigma(x)$ the surface element of $\mathbb{S}^{d-1}$ and the letter x in connection with $d\sigma(x)$ means that the integration is carried out with respect to x. We shall use the notation $A \sim B$ to mean that there exists an inessential constant $c > 0$, called the constant of equivalence, such that

$$c^{-1}A \leq B \leq cA.$$

We denote by $\prod_N^d$ the space of all spherical polynomials of degree at most N on $\mathbb{S}^{d-1}$. The spaces $H_k^d, k = 0, 1, 2....$ of spherical harmonics are mutually orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_{\mathbb{S}^{d-1}} f(x)\overline{g(x)}d\sigma(x).$$

Now we will introduce a very important theorem in spherical harmonics, which is a well known result in spherical harmonic analysis and it plays crucial roles for analysis on the sphere. For the proof, we refer to [2, p.5].

Theorem 1.1.3: If $k, l \in \mathbb{Z}_+, k \neq l$ Then $H_k^d \perp H_l^d$ with respect to the inner product $\langle f, g\rangle = \int_{\mathbb{S}^{d-1}} f(x)\overline{g(x)}d\sigma(x)$.

Theorem 1.1.4: (See [2, p.5]) The space $\prod_N^d$ can be written as a direct sum

$$\prod_N^d = \bigoplus_{k=0}^N H_k^d.$$

The dimension of H_k^d is given by

$$a_k^d = \dim H_k^d = \frac{(2k+d-2)\Gamma(k+d-1)}{(k+d-2)\Gamma(k+1)\Gamma(d-1)} \asymp k^{d-2}, \text{ as } k \to \infty,$$

which also implies

$$\dim \prod_N^d = \sum_{k=0}^N a_k^d = C_{N+d}^N \asymp (N+1)^{d-1}.$$

Corollary 1.1.5: (See [2, p.4]) The restriction on $\mathbb{S}^{d-1}$ of any polynomial is just a finite sum of spherical harmonics.

Theorem 1.1.6: (See [2, p.5]) Let $d \geq 2$. $L^2(\mathbb{S}^{d-1}) = \bigoplus \sum_{k=0}^\infty H_k^d$ where $\bigoplus \sum$ denotes the orthogonal direct sum.

By the theorem 1.1.6, for every $f \in L^2(\mathbb{S}^{d-1})$ there holds the following expansion in L^2-metric

$$f = \sum_{k=0}^\infty Y_k(f),$$

where $Y_k(f)$ denotes the orthogonal projection of f onto H_k^d.

1.2 Zonal Harmonics

Let $d \geq 2$, we know that H_k^d is a subspace of $L^2(\mathbb{S}^{d-1})$. Let $(H_k^d)^*$ be the dual space of H_k^d. Then by Riesz Representation Theorem, we know that there is a unique $g \in H_k^d$ such that

$$L_g(f) =< f, g >,$$

holds for all $f \in H_k^d$.

Definition 1.2.1: (See [2, p.7]) Let $\xi \in \mathbb{S}^{d-1}$ and let L be defined as above. The unique $z \in H_k^d$ which satisfies

$$L_z(f) =< f, z >= f(\xi), \text{ for any } f \in H_k^d,$$

is called Zonal Harmonics with pole ξ of d variables of degree k. And is denoted by $Z_\xi^{d,k}$. We have the following characterizations of the Zonal Harmonics:

Lemma 1.2.2: (See [2, p.7]) Let $Z_\xi^{d,k}$ be Zonal Harmonic. Then

(1) For any orthonormal base $(y_1, y_2, ...y_{a_k^d})$ of H_k^d,

$$Z_\xi^{d,k}(\eta) = \sum_{j=1}^{a_k^d} \overline{y_j(\xi)} y_j(\eta), \eta \in \mathbb{S}^{d-1}.$$

(2) For any ξ and η in $\mathbb{S}^{d-1}$,

$$\overline{Z_\xi^{d,k}(\eta)} = Z_\xi^{d,k}(\eta) = Z_\eta^{d,k}(\xi)$$

(3) For any rotation ρ on $\mathbb{R}^d$, and for any ξ and η in $\mathbb{S}^{d-1}$,

$$Z_{\rho\xi}^{d,k}(\rho\eta) = Z_\xi^{d,k}(\eta).$$

Hence, the equation $Z_{\rho\xi}^{d,k}(\rho\eta) = Z_\xi^{d,k}(\eta)$ shows that the value of $Z_\xi^{d,k}(\eta)$ is only dependent of the scalar product $\xi\eta$(of course, also dependent of d and k). This means that there is a function $P_k^d(t)(-1 \leq t \leq 1)$ such that

$$Z_\xi^{d,k}(\eta) = c_{d,k} P_k^d(\xi\eta).$$

Theorem 1.2.3: (See[2, p.13]) Let $d \geq 3$, $k \in \mathbb{Z}_+$. For any orthonormal base $(y_1, y_2, ...y_{a_k^d})$ of H_k^d and any $\xi, \eta \in \mathbb{S}^{d-1}$,

$$P_k^d(\xi\eta) = \frac{1}{c_{d,k}} \sum_{j=1}^{a_k^d} \overline{y_j(\xi)} y_j(\eta).$$

This conclusion is called addition formula of spherical harmonics.

1.3 Laplace-Beltrami Operator

The theory of spherical harmonics is closely related to the Laplace-Beltrami operator Δ_0 on $\mathbb{S}^{d-1}$.

Definition 1.3.1: (See [2, p.16]) For a function $f \in C^2(\mathbb{S}^{d-1})$, let

$$F(x) = f(\tfrac{x}{||x||}), x \neq 0, x \in \mathbb{R}^d.$$

The Laplace-Beltrami operator Δ_0 is defined by

$$\Delta_0 f(x) = \Delta_d F(x) = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} F(x), f \in C^2(\mathbb{S}^{d-1}).$$

More importantly, each space H_k^d can be seen as the space of eigenfunctions of Δ_0 corresponding to the eigenvalue $t_k = -k(k+d-2)$.

Theorem 1.3.2: (See [2, p.17]) Every element of H_k^d is an eigenfunction corresponding to the eigenvalue $-k(k+d-2)$ of the operator Δ_0,

$$H_k^d = \{f \in C^2(\mathbb{S}^{d-1}) : \Delta_0 f = -k(k+d-2)f\}, k = 0, 1, 2......$$

Furthermore, H_k^d is the eigenspace of Δ_0 corresponding to the eigenvalue $-k(k+d-2)$. For a proof, we refer to [20, p.600].

Definition 1.3.3: (See [2, p.17]) Given $r \in \mathbb{R}$, we define the fractional Laplace-Beltrami operator $(-\Delta_0)^r$ in a distributional sense by

$$\mathrm{Y}_k(-\Delta_0)^r f = (-k(k+d-2))^r \mathrm{Y}_k f, k = 0, 1, 2.....$$

Clearly, $(-\Delta_0)^r$ coincides with $(-\Delta_0)$ when $r = 1$.

1.4 The Fourier-Laplace Series

We have defined spherical harmonic expansions for L^2 functions, now we are going to extend this concept to L^1. Let $(y_1, y_2, ...y_{a_k^d})$ be an orthonormal base of $H_k^d (d \geq 3, k \geq 0)$. Let $f \in L^1(\mathbb{S}^{d-1})$, we define

$$c(f, y_j) = \int_{\mathbb{S}^{d-1}} f(\xi)\overline{y_j(\xi)} d\sigma(\xi)$$

the Fourier coefficients of f with respect to y_j. We consider the sum

$$Y_k(f) = \sum_{j=1}^{a_k^d} c(f, y_j) y_j.$$

Obviously, $Y_k(f) \in H_k^d$. And we have

$$Y_k(f)(\xi) = \int_{\mathbb{S}^{d-1}} f(\eta)(\sum_{j=1}^{a_k^d} \overline{y_j(\eta)} y_j(\xi)) d\sigma(\eta)$$

$$= c_{d,k} \int_{\mathbb{S}^{d-1}} f(\eta) P_k^d(\xi\eta) d\sigma(\eta),$$

where

$$c_{d,k} = \frac{(2k+d-2)\Gamma(k+d-2)\Gamma(\frac{d-1}{2})}{\Gamma(d-1)\Gamma(k+\frac{d-1}{2})|\mathbb{S}^{d-1}|}.$$

Note that it extends the definition $Y_k f$ to all $f \in L^1(\mathbb{S}^{d-1})$.

Definition 1.4.1: (See [2, p.43]) For $f \in L(\mathbb{S}^{d-1})$, the space of Lebesgue integrable functions on $\mathbb{S}^{d-1}$, we call the function $Y_k(f)$ its projection on H_k^d and the series $\sum_{k=0}^{\infty} Y_k(f)$ its expansion in spherical harmonic or Fourier-Laplace series.

We will treat Banach Space $L^p(\mathbb{S}^{d-1})$, $1 \leq p \leq \infty$, when $1 \leq p < \infty$ the norm of $L^p(\mathbb{S}^{d-1})$ is defined by

$$||f||_p = \{\int_{\mathbb{S}^{d-1}} |f(\xi)|^p d\sigma(\xi)\}^{\frac{1}{p}}.$$

When $p = \infty$, we use $C(\mathbb{S}^{d-1})$ to denote $L^p(\mathbb{S}^{d-1})$. Then the norm $||\cdot||_\infty$ is actually the maximum norm, i.e

$$||f||_\infty = \max\{|f(\xi)| : \xi \in \mathbb{S}^{d-1}\}.$$

We know if $f \in L^2(\mathbb{S}^{d-1})$ then its F-L series converges to f in L^2-metric certainly. Only this case is simple, in more general case the convergence problem is quite complicated. Let $f \in L^1(\mathbb{S}^{d-1})(d \geq 3)$, the F-L series of f is $\displaystyle\sum_{k=0}^{\infty} Y_k(f)$, where

$$Y_k(f)(\xi) = \int_{\mathbb{S}^{d-1}} f(\eta)(\sum_{j=1}^{a_k^d} \overline{y_j(\eta)} y_j(\xi)) d\sigma(\eta)$$

$$= c_{d,k} \int_{\mathbb{S}^{d-1}} f(\eta) P_k^d(\xi\eta) d\sigma(\eta).$$

Then the partial sums of $Y_k(f)(\xi)$ is

$$S_N f(\xi) = \sum_{k=0}^{N} Y_k(f)(\xi)$$

$$= \int_{\mathbb{S}^{d-1}} f(\eta)[\sum_{k=0}^{N} c_{d,k} P_k^d(\xi\eta) d\sigma(\eta)].$$

We write

$$D_N^d(t) = \sum_{k=0}^{N} c_{d,k} P_k^d(t), -1 \le t \le 1,$$

and consider the operator S_N on $C(\mathbb{S}^{d-1})$. It is easy to see that the norm of the operator S_N as a linear operator from $C(\mathbb{S}^{d-1})$ to $C(\mathbb{S}^{d-1})$ is just

$$l_N^d = \int_{\mathbb{S}^{d-1}} |D_N^d(\xi\eta)| d\sigma(\eta),$$

which is independent of ξ. We call it Lebesgue Constant of S_N.

Theorem 1.4.2: (See [2, p.48]) Let $d \ge 3$ and let S_N be the partial sum operator. Then the Lebesgue Constant of S_N has the order $O(N^{\frac{d-2}{2}})$, that is

$$B_1 N^{\frac{d-2}{2}} \le l_N^d \le B_2 N^{\frac{d-2}{2}},$$

where B_1, B_2 are positive constants.

Since $\frac{d-2}{2} > 0$ the operator S_N are not uniformly bounded. Clearly, if $f \in L^2(\mathbb{S}^{d-1})$, then the partial sums of $\sigma(f)$ converge to f in the L^2-norm. Furthermore, It is known that if $d \ge 3$, $1 \le p \le \infty$ and $p \ne 2$, then there exists a function $f \in L^p(\mathbb{S}^{d-1})$ whose Fourier-Laplace series diverges in the norm of $L^p(\mathbb{S}^{d-1})$.

As a matter of fact, one normally uses the Cesaro means of $\sigma(f)$ to study the summability of the Fourier-Laplace series because the convergence property of the partial sums of Fourier-Laplace series is not generally good when $d \ge 3$.

1.5 Cesàro Means

We know that the convergence property of the partial sums of Fourier-Laplace series is not generally good. The Lebesgue Constant of S_N has the order $O(N^{\frac{d-2}{2}})$. Thus, it is naturally important to investigate the linear summation of F-L series. The most important linear means are the Cesàro means.

Definition 1.5.1: (See [2, p.48]) The Cesàro means (C, δ) of $\sigma(f)$ or simply $f \in L(\mathbb{S}^{d-1})$ are defined by

$$\sigma_N^\delta f(\xi) = \sum_{k=0}^{N} Y_k f(\xi) \frac{A_{N-k}^\delta}{A_N^\delta}, N = 0, 1, ...$$

where

$$A_N^\delta = \frac{\Gamma(N+\delta+1)}{\Gamma(N+1)\Gamma(\delta+1)}$$

are Cesàro numbers and $Y_k(f)$ is the orthogonal projection of f to H_k^d.

We see that the partial sums of S_N are just the $(C, 0)$ means since

$$S_N(f)(\xi) = \sum_{k=0}^{N} Y_k(f)(\xi) = \sigma_N^0(f)(\xi).$$

1.6 Jacobi, Gegenbauer and Legendre Polynomials

Jacobi polynomials (occasionally called hypergeometric polynomials) $P_k^{(\alpha,\beta)}(t)$ are a class of classical orthogonal polynomials. They are orthogonal with respect to the weight function $(1-t)^\alpha(1+t)^\beta$ on the interval [-1, 1]. In this section we are going to make a general discussion on the classical orthogonal polynomials. Let $\alpha, \beta > -1$, we define a measure $\sigma^{(\alpha,\beta)}$ on the class of all Lebesgue measurable sets on the real line by

$$\sigma^{(\alpha,\beta)}(t) = (1-t)^\alpha(1+t)^\beta dt, t \in [-1, 1].$$

Write the L^p space, $1 \leq p < \infty$, on $[-1, 1]$ with respect to the measure $\sigma^{\alpha,\beta}$ as $L_{\alpha,\beta}^p$ which has the norm defined by

$$\|f\|_p = (\int_{-1}^{1} |f(t)|^p (1-t)^\alpha(1+t)^\beta dt)^{\frac{1}{p}}.$$

There is a unique orthogonal system $\{P_k^{(\alpha,\beta)} : k \in \mathbb{Z}\}$ in $L_{(\alpha,\beta)}^2$ which is complete and satisfies the following conditions

(1) $P_k^{(\alpha,\beta)} \in \prod_k$,

(2) $\int_{-1}^{1} P_k^{(\alpha,\beta)}(t) P_j^{(\alpha,\beta)}(t)(1-t)^\alpha(1+t)^\beta dt = 0, k \neq j$,

(3) $P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1)}$.

Definition 1.6.1: (See [2, p.33]) The polynomials satisfying the above three conditions are called Jacobi Polynomials and the Jacobi Polynomials of degree k are denoted by $P_k^{(\alpha,\beta)}$. The expression for $P_k^{(\alpha,\beta)}(t)$ is given by

$$P_k^{(\alpha,\beta)}(t) = \frac{\Gamma(\alpha+k+1)}{k!\Gamma(\alpha+\beta+k+1)} \sum_{m=0}^{n} C_n^m \frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)} (\frac{t-1}{2})^m,$$

where $C_n^m = \frac{n!}{(m!)(n-m)!}$.

Gegenbauer polynomials or ultraspherical polynomials $C_k^\lambda(t)$ are orthogonal polynomials on the interval [-1,1] with respect to the weight function $(1-t^2)^{\lambda-\frac{1}{2}}$.

Definition 1.6.2: (See [2, p.33]) The Gegenbauer polynomials $C_k^\lambda(t)$ are orthogonal polynomials on the interval $[-1,1]$ with respect to the weight function $(1-t^2)^{\lambda-\frac{1}{2}}$. They are defined by

$$(1-2xt+x^2)^{-\lambda} = \sum_{k=0}^{\infty} C_k^\lambda(t) x^k.$$

An important property of the Gegenbauer polynomials is that they are special cases of the Jacobi polynomials,

$$C_k^\lambda(t) = \frac{\Gamma(k+2\lambda)\Gamma(\lambda+1/2)}{\Gamma(2\lambda)\Gamma(k+\lambda+1/2)} P_k^{(\lambda-1/2,\lambda-1/2)}(t).$$

Legendre functions are solutions to Legendre's differential equation:

$$\frac{d}{dt}[(1-t^2)\frac{d}{dt}P_k(t)] + k(k+1)P_k(t) = 0.$$

These solutions for $k = 0, 1, 2, ...$(with the normalization $P_k(1) = 1$) form a polynomial sequence of orthogonal polynomials called the Legendre polynomials.

Definition 1.6.3: The Legendre polynomials may be expressed as the following:

$$(1-2xt+x^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} P_k(t) x^k.$$

We can easily see that $P_k(t) = C_k^{\frac{1}{2}}(t)$. In addition, an important property of the Legendre polynomials is that they are orthogonal with respect to the L^2 inner product on the interval $-1 \le t \le 1$:

$$\int_{-1}^{1} P_n(t) P_m(t) dt = \frac{2}{2n+1} \delta_{mn},$$

where δ_{mn} denotes the Kronecker delta, equal to 1 if $m = n$ and to 0 otherwise.

1.7 Translation Operators

In this section we will use S_θ to denote the translation operator on $\mathbb{S}^{d-1}$ and then we will also introduce some important properties of the translation operator.

Definition 1.7.1: (See [2, p.57]) The translation operator S_θ on $\mathbb{S}^{d-1}$ with step $\theta \in [0, \pi]$ is defined by

$$S_\theta(f)(x) = \frac{1}{|\mathbb{S}^{d-2}| \sin^{d-2}(\theta)} \int_{\{y \in \mathbb{S}^{d-1}: x \cdot y = \cos\theta\}} f(y) d\sigma(y),$$

where $f \in L^1(\mathbb{S}^{d-1})$, $x \in \mathbb{S}^{d-1}$. The significance of the operator S_θ lies in the following fact: for $f \in L^1(\mathbb{S}^{d-1})$, and $K \in L^1([-1,1], (1-t^2)^{\frac{d-3}{2}})$,

$$\int_{\mathbb{S}^{d-1}} f(y) K(xy) d\sigma(y) = |\mathbb{S}^{d-2}| \int_0^\pi S_\theta(f)(x) K(\cos\theta) \sin^{d-2}(\theta) d\theta, x \in \mathbb{S}^{d-1}.$$

The integral on the left hand side of this last equation is called the spherical convolution of f and K. The following are some useful theorems on the operator S_θ.

Theorem 1.7.2: (See [2, p.57]) For each θ, S_θ is a multiplier operator on $\mathbb{S}^{d-1}$ in the sense that

$$Y_k(S_\theta(f)) = \frac{P_k^{(\frac{d-3}{2}, \frac{d-3}{2})}(\cos\theta)}{P_k^{(\frac{d-3}{2}, \frac{d-3}{2})}(1)} Y_k f, k = 0, 1, \cdots.$$

Theorem 1.7.3: (See [2, p.57]) Each S_θ is a positive operator and of strong type (p, p) with norm 1.

$$||S_\theta||_{(p,p)} = \sup\{\|S_\theta f\|_p : \|f\|_p = 1\} = 1,$$

for all $1 \leq p \leq \infty$.

Proof: (See [2, p.57-p.58]).

1.8 Approximation on the sphere

A central problem for approximation theory is to characterize the best approximation of a function by polynomials, or other classes of simple functions, in terms of the smoothness of the function. In this section we study the characterization of the best approximation by polynomials on the sphere. In the classical setting of one variable, the smoothness of a function on $\mathbb{S}^1$ is described by the modulus of smoothness defined via the forward difference of the function. A main challenge for the sphere $\mathbb{S}^{d-1}$ with $d \geq 3$ is how to define a modulus of smoothness that will characterize the smoothness, as multiplication on the higher dimensional sphere is not commutative. It becomes clear only recently that a satisfactory modulus of smoothness can be defined as the maximum of the moduli of smoothness of one variable in $\theta_{i,j}$, the angle of the polar coordinate on the (x_i, x_j), over all possible choices of (i, j). Conspicuously, the number of such angles, $d(d-1)/2$, is the dimension of $SO(d)$(In the following sections, we use $SO(d)$ to denote the group of rotations on $\mathbb{S}^{d-1}$). This modulus of smoothness allows us to tap into the rich resources of trigonometric approximation theory for ideas and tools, and effectively reduces a large part of problems in approximation theory on $\mathbb{S}^{d-1}$ to those of trigonometric approximation.

Definition 1.8.1: (See [1, p.76]) For $f \in L^p(\mathbb{S}^1)$ if $1 \leq p < \infty$ or $f \in C(\mathbb{S}^1)$ if $p = \infty$, the error of best approximation by trigonometric polynomials of degree at most N is defined by

$$E_N(f)_p = \inf_{g \in \prod_N(\mathbb{S}^1)} \|f - g\|_p.$$

The central problem in trigonometric approximation theory is to characterize $E_N(f)_p$ in terms of the smoothness of the function f. For this purpose we need the notion of modulus of smoothness, usually defined through the forward difference of f.

Let I denote the identity operator and S_θ be the translation operator defined by $S_\theta f(x) = f(x+\theta)$. For $r = 1, 2, ...$, the forward difference operator $\overrightarrow{\Delta}_\theta^r$ is defined by

$$\overrightarrow{\Delta}_\theta = S_\theta - I \text{ and } \overrightarrow{\Delta}_\theta^r = (S_\theta - I)^r.$$

The binomial theorem implies that

$$\overrightarrow{\Delta}_\theta^r f(x) = \sum_{j=0}^{r} (-1)^{r-j} C_r^j f(x + \theta j).$$

Definition 1.8.2: (See [1, p.76]) For $f \in L^p(\mathbb{S}^1)$ if $1 \leq p < \infty$ or $f \in C(\mathbb{S}^1)$ if $p = \infty$, $r = 1, 2, ...$ and $t > 0$,

$$w_r(f, t)_p = \sup_{|\theta| \leq t} \|\overrightarrow{\Delta}_\theta^r f\|_p.$$

The modulus of smoothness $w_r(f, t)_p$ is a continuous and increasing function of t with $w_r(f, t)_p \to 0$, $t \to 0$. Furthermore it satisfies the following properties:

(i) $w_r(f+g,t)_p \le w_r(f,t)_p + w_r(g,t)_p$,

(ii) $w_r(f,\lambda t)_p \le (\lambda+1)^r w_r(f,t)_p, \lambda \ge 0$,

(iii) $w_r(f,t)_p \le ct^r w_r(f^{(r)},t)_p$, if $f^{(r)} \in L^p(S^1)$.

Theorem 1.8.3: (See [1, p.77]) For $f \in L^p(\mathbb{S}^1)$ if $1 \le p < \infty$ or $f \in C(\mathbb{S}^1)$ if $p = \infty$, $r = 1, 2, \cdots$,

$$E_N(f)_p \le c_{p,r} w_r(f, \tfrac{1}{N})_p,$$

$$w_r(f, \tfrac{1}{N})_p \le c_{p,r} N^{-r} \sum_{k=1}^{N} (k)^{r-1} E_{k-1}(f)_p.$$

The first inequality in theorem 1.8.3 is usually called the Jackson estimate, its proof requires constructing a trigonometric polynomial whose error of approximation is bounded by the modulus of smoothness. The second inequality in theorem 1.8.3 is often called the Bernstein estimate as its proof relies on the Bernstein inequality in the theorem below. In both cases, a proof will come out as a special case of our approximation on $\mathbb{S}^{d-1}$.

Theorem 1.8.4 (See [1, p.77]) (*Bernstein inequality*) For $1 \le p \le \infty$ and $T_N \in \prod_N(\mathbb{S}^1)$,

$$\|T_N^{(k)}\|_p \le N^{k-r} \|T_N^{(r)}\|_p, k > r.$$

There are many ways to define a modulus of smoothness on the sphere $\mathbb{S}^{d-1}$. The one we define in this section has the advantage that it relies on the modulus of smoothness on $\mathbb{S}^1$, even though $\mathbb{S}^{d-1}$ is not in itself a product space of $\mathbb{S}^1$, which allows us to utilize the results in the previous section. For $1 \le i \ne j \le d$ and $t \in \mathbb{R}$. Recall that $Q_{i,j,\theta}$ denotes a rotation by the angle θ in the (x_i, x_j)-plane. As an example, for $(i,j) = (1,2)$, the action of the rotation $Q_{1,2,\theta} \in SO(d)$ is given by

$$Q_{1,2,\theta}(x_1, ...x_d) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta, x_3, ...x_d)$$

There are $d(d-1)/2$ distinct angles $\theta_{i,j}$. For $r = 1, 2, ...$, we use $Q_{i,j,\theta}$ to define the difference operator

$$\Delta^r_{i,j,\theta} = (I - T(Q_{i,j,\theta}))^r, 1 \le i \ne j \le d,$$

where $T(Q)$ denotes the rotation operator $T(Q)f(x) = f(Q^{-1}x)$. Since $Q_{i,j,\theta} = Q_{j,i,-\theta}$, we have $\Delta^r_{i,j,\theta} = \Delta^r_{j,i,-\theta}$. Because of

$$Q_{1,2,\theta}(x_1, ...x_d) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta, x_3, ...x_d).$$

$\Delta^r_{i,j,\theta}$ can be expressed in the forward difference. Our modulus of smoothness on the sphere is defined in terms of these differences.

Definition 1.8.5: (See [1, p.81]) Given $r \in \mathbb{N}$, $t > 0$, and $f \in L^p(\mathbb{S}^{d-1})$, $1 \leq p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ for $p = \infty$, the modulus of smoothness of order r of f in the L^p metric is defined by

$$w_r(f,t)_p = \sup\{||\Delta^r_{i,j,\theta} f||_p, 1 \leq i < j \leq d, |\theta| \leq t\},$$

for $r = 1$ we write $w_r(f,t)_p = w(f,t)_p$. Our modulus of smoothness $w_r(f,t)_p$ is the maximum among all possible choices of (i,j). Just as in the case of $\mathbb{S}^1$, the modulus of smoothness is a continuous and increasing function of t and it satisfies the properties

(i) For $s < r$, $w_r(f,t)_p < 2^{r-s} w_s(f,t)_p$,

(ii) For $\lambda > 0, w_r(f, \lambda t)_p \leq (\lambda + 1)^r w_r(f,t)_p$.

Recall the quantity $E_N(f)_p$ of best approximation by polynomials defined in $\mathbb{S}^1$. Our main result in this part is a characterization of the best approximation by polynomials in terms of the modulus of smoothness on $\mathbb{S}^{d-1}$.

Theorem 1.8.6: (See [1, p.88]) For $f \in L^p(\mathbb{S}^{d-1})$, $1 \leq p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ when $p = \infty$,

$$E_N(f)_p \leq c_{p,r} w_r(f, \tfrac{1}{N})_p,$$

on the other hand,

$$w_r(f, \tfrac{1}{N})_p \leq c_{p,r} N^{-r} \sum_{k=1}^{N} k^{r-1} E_{k-1}(f)_p.$$

Besides modulus of smoothness, the smoothness of a function can also be described by the K-functional. We define the K-functional via the differential operators $D_{i,j} = x_i \partial_j - x_j \partial_i$ which turns out to be equivalent to $w_r(f,t)_p$, as it is often the case in approximation theory. The K-functional is defined via the Sobolev space and is often easier to apply when the function is known to be differentiable. First Let us also state a Bernstin type inequality for the differential operator $D_{i,j}$.

Lemma 1.8.7: (See [1, p.83]) Let f be a polynomial in $\prod_N(\mathbb{S}^{d-1})$, for $1 \leq i < j \leq d, r \in \mathbb{N}$,

$$||D_{i,j}^r f||_p \leq N^r ||f||_p, 1 \leq p \leq \infty.$$

Definition 1.8.8: (See [1, p.90]) For $r \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W_p^r(\mathbb{S}^{d-1})$ consists of functions $f \in L^p(\mathbb{S}^{d-1})$ with distributional derivatives $D_{i,j}^r f, 1 \leq i < j \leq d$, all belong to $L^p(\mathbb{S}^{d-1})$, where $L^p(\mathbb{S}^{d-1})$ is replaced by $C(\mathbb{S}^{d-1})$ when $p = \infty$. The norm of the space is defined by

$$||f||_{W_p^r(\mathbb{S}^{d-1})} = ||f||_p + \sum ||D_{i,j}^r f||_p.$$

Definition 1.8.9: For $r \in \mathbb{N}$ and $t \geq 0$,

$$K_r(f,t)_p = \inf_{g \in W_p^r(\mathbb{S}^{d-1})} \{||f-g||_p + t^r \max_{1 \leq i < j \leq d} ||D_{i,j}^r g||_p\}.$$

It describes how well the function can be approximated by smooth functions in certain sense.

Theorem 1.8.10: (See [1, p.91]) Let $r \in \mathbb{N}$,$f \in L^p(\mathbb{S}^{d-1})$ if $1 \leq p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. For $0 < t < 1$,

$$w_r(f,t)_p \approx K_r(f,t)_p, 1 \leq p \leq \infty.$$

Proof: See [1, p.91].

In particular, this shows that the best approximation $E_N(f)_p$ can be characterized by the k functional. Furthermore, we can now consider approximation in the Sobolev space.

Corollary 1.8.11: (See [1, p.92]) If $r \in \mathbb{N}$ and $f \in W_p^r(\mathbb{S}^{d-1}), 1 \leq p \leq \infty$, then

$$E_N(f)_p \leq cN^{-r}||f||_{W_p^r}.$$

Proof: See [1, p.92].

Next we discuss briefly two other moduli of smoothness on the sphere. Historically, the first modulus of smoothness is defined in terms of the spherical means, or the translation operator, $T_\theta f$, which we recall as

$$T_\theta f(x) = \tfrac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f(x\cos\theta + u\sin\theta) d\sigma(u).$$

We denote this modulus of smoothness by $w_r^*(f,t)_p$. It is defined by, for $r = 1, 2 \cdots$,

$$w_r^*(f,t)_p = \sup_{|\theta| \leq t} ||(I - T_\theta)^{\frac{r}{2}} f||_p,$$

where $(I - T_\theta)^{\frac{r}{2}}$ is defined in terms of infinite series when $\frac{r}{2}$ is not an integer. The characterization of the best polynomial approximation, both direct and inverse theorems, can be established in terms of $w_r^*(f,t)_p$. Furthermore, this modulus of smoothness is equivalent to the K-functional defined by

$$K_r^*(f,t)_p = \inf_g\{||f-g||_p + t^r||(-\Delta_0)^{\frac{r}{2}}g||_p\},$$

where $(-\Delta_0)^{\frac{r}{2}}$ is the Laplace-Beltrami operator on the sphere and the infimum is taken over all g for which $(-\Delta_0)^{\frac{r}{2}}g \in L^p(\mathbb{S}^{d-1})$.

Theorem 1.8.12: (See [1, p.91]) Let $r \in \mathbb{N}$,$f \in L^p(\mathbb{S}^{d-1})$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. For $0 < t < 1$,

$$w_r^*(f,t)_p \approx K_r^*(f,t)_p, 1 \le p \le \infty.$$

The fact that both moduli of smoothness, $w_r(f,t)_p$ and $w_r^*(f,t)_p$, characterize the best approximation by polynomials does not imply that the two are equivalent, since the inverse theorem of the characterization is of weak type. Only a partial result is known in this regard.

Theorem 1.8.13: (See [1, p.95]) Let $f \in L^p(\mathbb{S}^{d-1})$, $1 < p < \infty$. For $0 < t < 1$, $w_r(f,t)_p \le cw_r^*(f,t)_p$ if $r \in \mathbb{N}$ and $w_r(f,t)_p \sim w_r^*(f,t)_p$ if $r = 1, 2$.

Proof: (See [1, p.95]).

Recall that $T(Q)f(x) = f(Q^{-1}x)$ for $Q \in SO(d)$. For $t > 0$, define

$$Q_t = \{Q \in SO(d) : \max d(x, Qx) \le t\},$$

where $d(x,y)$ is the geodesic distance on $\mathbb{S}^{d-1}$, for $t > o$ and $r > 0$ define

$$\widetilde{w_r}(f,t)_p = \sup_{Q \in Q_t} ||\Delta_Q^r f||_p,$$

where $\Delta_Q^r = (I - T(Q))^r$. The main results on this modulus of smoothness are summarized as follows.

Theorem 1.8.14: For $1 < p < \infty$,

$$\widetilde{w_r}(f,t)_p \sim w_r^*(f,t)_p.$$

Theorem 1.8.15: (See [1, p.95]) For $1 < p < \infty$,

$$\widetilde{w_r}(f,t)_p \sim K_r^*(f,t)_p,$$

whereas the equivalence fails for $p = 1$ and $p = \infty$. From the equivalence of $K_r^*(f,t)_p$ and $w_r^*(f,t)_p$, we have the following theorem.

Theorem 1.8.16: (See [1, p.95]) For $f \in L^p(\mathbb{S}^{d-1})$ if $1 \leq p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ when $p = \infty$,

$$w_r(f,t)_p \leq \widetilde{w_r}(f,t)_p, 1 \leq p \leq \infty, r \in \mathbb{N}.$$

Proof: See [1, p.95].

Theorem 1.8.17: For $1 < p < \infty$, $r = 1, 2$,

$$\widetilde{w_r}(f,t)_p \sim w_r^*(f,t)_p \sim w_r(f,t)_p.$$

1.9 Weighted Polynomial Inequalities

Polynomial inequalities have been playing crucial roles in approximation theory and related fields. Several such inequalities on the unit sphere will be established in this section.

Definition 1.9.1:(See [1, p.100]) A weight function on $\mathbb{S}^{d-1}$ is a nonnegative integrable function on $\mathbb{S}^{d-1}$. Let w be a fixed weight function on $\mathbb{S}^{d-1}$ normalized by

$$\int_{\mathbb{S}^{d-1}} w(y) d\sigma(y) = 1.$$

We denote by $L_{p,w}$ the weighted Lebesgue space of functions on $\mathbb{S}^{d-1}$ with quasi-norm

$$||f||_{p,w} = \left(\int_{\mathbb{S}^{d-1}} |f(y)|^p w(y) d\sigma(y)\right)^{\frac{1}{p}},$$

and for $p = \infty$ we assume that L^∞ is replaced by $C(\mathbb{S}^{d-1})$, the space of continuous functions on $\mathbb{S}^{d-1}$ with the usual uniform norm $||f||_\infty$. Given a set $E \in \mathbb{S}^{d-1}$, we write

$$w(E) = \int_E w(x) d\sigma(x).$$

Definition 1.9.2: (See [1, p.100]) A weight function w on $\mathbb{S}^{d-1}$ is a doubling weight if there exists a constant $L > 0$ such that for any $x \in \mathbb{S}^{d-1}$ and $t > 0$

$$w(B(x, 2t)) \leq L w(B(x,t)),$$

the least constant L is called the doubling constant of w and is denoted by L_w. For the rest of this section, unless otherwise stated, the letter w always denotes a doubling weight on $\mathbb{S}^{d-1}$ with doubling constants L_w and with $S_w = \frac{\log L_w}{\log 2}$. Many of the weights on $\mathbb{S}^{d-1}$ that appear in analysis satisfy the doubling condition. Given $\epsilon \in (0, \pi)$, a finite subset Λ is called $\epsilon-$separated if $\min_{w,w' \in \Lambda} \rho(w, w') \geq \epsilon$, and it is called maximal ϵ-separated if it is ϵ-separated and satisfies $\max_{x \in \mathbb{S}^{d-1}} \min_{w \in \Lambda} \rho(x, w) < \epsilon$.

Definition 1.9.3: (See [1, p.105]) For $\beta > 0$ and $f \in C(\mathbb{S}^{d-1})$, we define the maximal function

$$f^*_{\beta,N}(x) = \max |f(y)|(1 + N\rho(x, y))^{-\beta}, x, y \in \mathbb{S}^{d-1}, N = 0, 1, 2 \cdots .$$

Recall that the weighted Hardy-Littlewood maximal function M_w is defined by

$$M_w g(x) = \sup_{0<r\leq\pi} \frac{1}{w(B(x, r))} \int_{B(x,r)} |g(y)|w(y)d\sigma(y).$$

Since w has the doubling property, it follows that for $1 < p \leq \infty$,

$$||M_w g||_{p,w} \leq C(\tfrac{p}{p-1})^{\frac{1}{p}} ||g||_{p,w}.$$

Then, we have the following theorem.

Theorem 1.9.4: (See [1, p.105]) If $f \in \prod_N^d$ and $\beta > 0$, then for any $x \in \mathbb{S}^{d-1}$ we have

$$f^*_{\beta,N}(x) \leq C_1(M_w(|f|^{\frac{s_w}{\beta}})(x))^{\frac{\beta}{s_w}}.$$

Proof: See [1, p.105].

Corollary 1.9.5: (See [1, p.107]) For $0 < p \leq \infty$, $f \in \prod_N^d$ and $\beta > \frac{1}{p}$, we have

$$||f||_{p,w} \leq ||f^*_{\beta,N}||_{p,w} \leq C||f||_{p,w},$$

where $C > 0$ depends only on d, L and β when β is big or close to $\frac{1}{p}$. As consequences of the above theorem, we have the following useful corollary,

Corollary 1.9.6: (See [1, p.107]) If $0 < p \leq \infty$, $\theta \in (0, \pi)$, $\Lambda \in \mathbb{S}^{d-1}$ is $\epsilon - separated$, then for any $f \in \prod_N^d$ and $\beta \geq 1$,

$$(\sum_{w\subset\Lambda} (Osc(f; w, \beta))^p w(B(x, \theta)))^{\frac{1}{p}} \leq C(N\theta)||f||_{p,w},$$

where

$$Osc(f; w, \beta) = \max_{x,y \in B(w,\theta)} |f(x) - f(y)|,$$

$C > 0$ depends only on d, β, L and p when p is small.

In many applications, we need to deal with a finite sums of function evaluations instead of integral. The Marcinkiewicz-Zygmund inequality shows that these sums can often be bounded by the integrals, if the points on which function evaluations take place are well separated. We start with a definition that quantifies the separation of points.

Theorem 1.9.7: (See [1, p.109]) If $\beta \geq 1$, $\delta \in (0, \pi)$, $\Lambda \subset \mathbb{S}^{d-1}$ is $\frac{\epsilon}{N}$-separated, and $f \in \prod_M^d$ with $M \geq N$, then

(i) $\sum_{\eta \subset \Lambda} (\max_{x \in B(\eta, \frac{\delta}{N})} |f(x)|^p) w(B(\eta, \frac{\delta}{N})) \leq c_{w,p} (\frac{M}{N})^{s_w} \|f\|_{p,w}^p$, where c depends only on d, p, L and β.

(ii) There exists a constant $\delta_0 > 0$ depending only on the dimension d and the doubling constant L such that for any $f \in \prod_N^d$ and any maximal $\frac{\delta}{N}$-separated subset $\Lambda \subset \mathbb{S}^{d-1}$ with $\delta \in (0, \delta_0]$ we have

$$||f||_{p,w}^p \sim \sum_{\eta \in \Lambda} (\min_{x \in B(\eta, \frac{\delta}{N})} |f(x)|^p) w(B(\eta, N^{-1}\delta))$$

$$\sim \sum_{\eta \in \Lambda} (\max_{x \in B(\eta, \frac{\delta}{N})} |f(x)|^p) w(B(\eta, N^{-1}\delta)),$$

where the constant of equivalence depends only on d, L and p.

1.10 Cubature formulas on the sphere

Cubature formulas, synonym of numerical integration formulas, are essential tools for discretizing integrals. A cubature formula is a finite linear sum of function evaluations that approximates an integral. The strength of a cubature formula is often measured by the number of polynomials that it preserves.

Definition 1.10.1: (See [1, p.121]) A cubature formula

$$Q_N(f) = \sum_{k=1}^{N} \lambda_k f(x_k), \lambda_k \in R, x_k \in \mathbb{S}^{d-1},$$

is of degree $N+1$ for the measure on $\mathbb{S}^{d-1}$ if

$$\int_{\mathbb{S}^{d-1}} f(x)d\sigma(x) = Q_N(f), \forall f \in \prod_{N+1}^{d},$$

a cubature formula is positive if $\lambda_k > 0$ for $0 \leq k \leq N$. The points x_k in $Q_N(f)$ are called nodes and the coefficients λ_k in $Q_N(f)$ are called weights of the cubature formula. In particular, $\sum_{k=1}^{N} \lambda_k = 1$, Since

$$Q_N(1) = \sum_{k=1}^{N} \lambda_k = 1.$$

We are particularly interested in positive cubature formulas since they are numerically stable and there will not be wild oscillation in their weights as $0 \leq \lambda_k \leq 1$ if $Q_N(f)$ is positive.

The strength of a cubature formula is measured by its degree. For a fixed number of nodes, N, higher the N is, stronger is the $Q_N(f)$. The one with the highest degree is called the Gaussian type, a tribute to the Gaussian quadrature formula of one variable. This correlation between the number of points N and the degree of precision N is often considered by asking how many points are needed for a fixed degree.

Theorem 1.10.2: (See [1, p.123]) If a positive cubature formula for the integral $\int_{\mathbb{S}^{d-1}} f(x)d\sigma(x)$ is of degree $2m+1$, then its number of nodes N satisfies

$$N \geq 2C_{m+d-1}^{m}.$$

Proof: see [1, p.123].

Chapter 2

Hyperinterpolations on the sphere

In this chapter, we will first briefly review some basic facts about hyperinterpolations, and then prove several new results.

2.1 Hyperinterpolation operators on $\mathbb{S}^{d-1}$: Definitions and basic facts

Hyperinterpolation on the sphere was introduced by I. Sloan [34] in 1995. It provides a constructive approximation method that is much more favorable in comparison with the regular spherical polynomial interpolation. The main idea of the hyperinterpolation is to use positive cubature formulas to discretize spherical convolution operators. It turns out that this discretization technique is easier to implement in many applications.

To be precise, assume that $\{Q_N\}$ is a sequence of positive cubature formulas on $\mathbb{S}^{d-1}$:

$$\int_{\mathbb{S}^{d-1}} f(x)d\sigma(x) = Q_N f := \sum_{w\in\Lambda_N} \lambda_{N,w} f(w), \quad \forall f \in \prod_{N+1}^d,$$

where Λ_N is a finite subset of $\mathbb{S}^{d-1}$, and $\lambda_{N,w} > 0$ for all $w \in \Lambda_N$. A cubature formula like Q_N is said to be of degree $N+1$. Hyperinterpolation now arises if cubature formulas are used in the evaluation of the orthogonal projections of the spherical harmonic expansions. It is known that for $f \in H_k^d$, by the use of the reproducing kernel function, the orthogonal projections

$$Y_k : C(\mathbb{S}^{d-1}) \to H_k^d$$

can be represented in the form

$$(Y_k f)(x) = \int_{\mathbb{S}^{d-1}} f(y) G_k(xy) d\sigma(y), x, y \in \mathbb{S}^{d-1},$$

where here and throughout,

$$G_k(t) = \frac{2k+d-2}{(d-2)|\mathbb{S}^{d-1}|} C_k^{\frac{d-2}{2}}(t), t \in [-1, 1],$$

$|\mathbb{S}^{d-1}|$ is the surface area of $\mathbb{S}^{d-1}$, and $C_k^{\frac{d-2}{2}}(t)$ is the usual ultraspherical polynomial of degree k and index $\frac{d-2}{2}$. Thus, we have, for $f \in \prod_N^d$, by the use of the reproducing kernel function, the orthogonal projections

$$Y_k : C(\mathbb{S}^{d-1}) \to \prod_N^d$$

can be represented in the form

$$(Y_k f)(x) = \int_{\mathbb{S}^{d-1}} f(y) E_N(xy) d\sigma(y), x, y \in \mathbb{S}^{d-1},$$

where $E_N(xy)$ is the reproducing kernel of $\prod_N^d$ and $E_N(xy) = \sum_{k=0}^{N} G_k(xy)$.

Definition 2.1.1: [See 34] The hyperinterpolation operator $L_N : C(\mathbb{S}^{d-1}) \to \prod_N^d$, $N = 1, 2, \cdots$ are defined by

$$L_N f(x) = \sum_{w \in \Lambda_{2N-1}} \lambda_{2N-1,w} f(w) E_N(xw),$$

where $x \in \mathbb{S}^{d-1}$, $f \in C(\mathbb{S}^{d-1})$ and

$$E_N(x \cdot y) = \frac{\lambda+N}{\lambda} C_N^{\frac{d-2}{2}}(x \cdot y).$$

To investigate the convergence of the operators L_N, one need to consider the following uniform norm: which is defined for arbitrary bounded linear operators $L_N : C(\mathbb{S}^{d-1}) \to C(\mathbb{S}^{d-1})$ by

$$||L_N||_\infty = \sup\{||L_N f||_\infty : f \in C(\mathbb{S}^{d-1})\}.$$

It was shown in [35] that under some regularity condition on the distribution of the nodes of the cubature formulas Q_{2N-1} for $d = 3$ and in [36] for an arbitrary d without any additional assumptions on the cubature formulas that

$$a_d N^{\frac{d-2}{2}} \le ||L_N||_\infty \le b_d N^{\frac{d-2}{2}},$$

where a_d and b_d are constants depending only on the dimension d. As a result, in general, for $f \in C(\mathbb{S}^{d-1})$, $L_N f$ diverges as $N \to \infty$.

Of crucial importance in many applications is the degree of the cubature formulas used in hyperinterpolation. As is well known, a positive cubature formula of degree N on $\mathbb{S}^{d-1}$ will require at least $O(N^{d-1})$ distinct points on the sphere, and even a slight reduction of the degree of cubature formulas will significantly reduce the cost of computations. For this reason, the concept of generalized hyperinterpolations will be considered next section. Compared with the hyperinterpolation, a generalized hyperinterpolation requires less degree of precision in cubature formulas, but guarantees uniform convergence combining the ideas of hyperinterpolation and summation.

2.2 Generalized hyperinterpolation on the unit sphere

In spite of the best-order result, pointwise convergence cannot be attained by hyperinterpolation. For this reason, Reimer [39] introduced the concept of generalized hyperinterpolation, whose definition is given as follows.

Recall that for each positive integer N, Q_N is a positive cubature formula on $\mathbb{S}^{d-1}$, as given in last section. Next, assume that $(a_{N,k})_{N,k=0}^{\infty}$ is some infinite real matrix which satisfies the following conditions:

(i) $a_{N,k} = 0$ for $N < k$;

(ii) $\lim\limits_{N\to\infty} a_{N,k} = 1$ for $k = 0, 1$;

(iii) $D_N(t) := \sum_{k=0}^{N} a_{N,k} G_k(t) \geq 0$ for $-1 \leq t \leq 1$, and $N = 1, 2, \cdots$.

Then we define the partial-sum operators

$$\Lambda_N = \sum_{k=0}^{N} a_{N,k} Y_k.$$

with the representation

$$(\Lambda_N f)(x) = \int_{\mathbb{S}^{d-1}} f(y) D_N(xy) d\sigma(y).$$

Definition 2.2.1: (See [39]) With the above assumptions, the generalized hyperinterpolation operators associated to $\{D_N(t)\}_{N=0}^{\infty}$ are defined as

$$GL_{N,D_N} f(x) = \sum_{w \in \Lambda_N} \lambda_{N,w} f(w) D_N(xw),$$

where $f \in C(\mathbb{S}^{d-1}), x \in \mathbb{S}^{d-1}$. We will write GL_N for GL_{N,D_N} when D_N is understood. We see that Generalized hyperinterpolation arises if a quadrature rule, which satisfies the Weak Assumptions on Q, is used in the evaluation of the integral $\int_{\mathbb{S}^{d-1}} f(y)D_N(xy)d\sigma(y)$.

Theorem 2.2.2: (See [39]) With the above assumptions, the generalized hyperinterpolation operators GL_N converge to the identity operator in the uniform norm as $N \to \infty$.

Note that the hyperinterpolation operator L_N requires a positive cubature formula of degree $2N$ on $\mathbb{S}^{d-1}$, whereas the generalized hyperinterpolation GL_N requires a positive cubature formula of degree $N+1$ only. Thus, generalized hyperinterpolation significantly reduces the cost of evaluation. Furthermore, it was shown by Reimer [39, Theorem 2] that generalized hyperinterpolation can achieve uniform convergence result for an arbitrary $f \in C(\mathbb{S}^{d-1})$.

2.3 Approximation error of the generalized hyperinterpolation

Theorem 2.2.2 implies that

$$\lim_{N\to\infty} \|GL_N f - f\|_\infty = 0, \;\; f \in C(\mathbb{S}^{d-1}),$$

but it does not give the rate of the convergence of $GL_N f$. In this section, we will establish a direct theorem for the approximation.

Given a function $f : \mathbb{S}^{d-1} \to \mathbb{R}$, and $u, v \in \mathbb{S}^{d-1}$ with $u \perp v$, we set

$$f_{u,v}(\varphi) = f(u\cos\varphi + v\sin\varphi), \;\; \varphi \in \mathbb{R}.$$

It is easily seen that $f \in C^j(\mathbb{S}^{d-1})$ if and only if $f_{u,v} \in C^j(\mathbb{R})$ for all $u, v \in \mathbb{S}^{d-1}$ with $u \cdot v = 0$. Motivated by this fact, we define the the first and the second order moduli of smoothness of $f \in C(\mathbb{S}^{d-1})$ by

$$\begin{aligned}\omega_1^*(f,t) &= \sup\{|f(x) - f(y)| : \;\; x, y \in \mathbb{S}^{d-1}, \;\; \arccos(x \cdot y) \le t\} \\ &= \sup\{|f_{u,v}(\theta_1) - f_{u,v}(\theta_2)| : \;\; |\theta_1 - \theta_2| \le t, \;\; u, v \in \mathbb{S}^{d-1}, \;\; u \cdot v = 0\},\end{aligned}$$

and

$$\begin{gathered}\omega_2^*(f,t) = \sup\{|f_{u,v}(\alpha+\theta) - 2f_{u,v}(\alpha) + f_{u,v}(\alpha-\theta)| : \\ \alpha \in \mathbb{R}, \;\; |\theta| \le t, \;\; u, v \in \mathbb{S}^{d-1}, \;\; u \cdot v = 0\},\end{gathered}$$

respectively.

Theorem 2.3.1: For the first order moduli of smoothness, Reimer [39, Theorem 6] proved that the inequality

$$||f - GL_{N,D_N} f||_\infty \leq C_d \omega_1^*(f, N^{-1})$$

holds for arbitrary N and $f \in C(\mathbb{S}^{d-1})$, where the constant C_d is independent of N, f and the cubature formulas Q_N. Reimer [39] further asked whether second-order results can be attained by generalized hyperinterpolation operators; that is, whether a similar inequality holds for a second order moduli of smoothness. An affirmative answer to this question was given by Feng Dai [41].

Theorem 2.3.2: For the second order moduli of smoothness, Feng Dai [41] proved that the inequality

$$||f - GL_{N,D_N} f||_\infty \leq C_d \omega_2^*(f, N^{-1})$$

holds for arbitrary N and $f \in C(\mathbb{S}^{d-1})$, where the constant C_d is independent of N, f and the cubature formulas Q_N. Indeed, a more general result was obtained in [41].

Theorem 2.3.3: Suppose that

$$D_N(t) = \sum_{k=0}^{N} a_{N,k} G_k(t), N = 1, 2......$$

is a sequence of polynomials on $[-1, 1]$ satisfying

$$\textstyle\int_{\mathbb{S}^{d-1}} D_N(e \cdot y) d\sigma(y) = 1, \;\; e = (1, 0, \cdots, 0) \in \mathbb{S}^{d-1}, \;\; N = 1, 2.....$$

and

$$\textstyle\sup_N \int_0^\pi (1 + N\theta)^2 |D_N \cos\theta| \sin^{d-2}\theta d\theta \leq K < \infty.$$

Let GL_{N,D_N} denote the generalized hyperinterpolation operators associated with the kernel D_N. Under these assumptions, it was proved in [41] that for all $f \in C(\mathbb{S}^{d-1})$,

$$||f - GL_{N,D_N}||_\infty \leq CK \omega_2^*(f, N^{-1}).$$

where $C > 0$ depends only on d. Note that in this result, the assumption that D_N is positive is no longer needed, whereas positivity plays a crucial role in the original proof of Reimer [39]. In fact, a necessary and sufficient condition was determined in [1] for a sequence of positive kernels D_N to satisfy the above second order estimate.

2.4 New results for higher order measures of smoothness

In this section, we will further extend the results of [41] to the case of higher order smoothness. Under certain conditions, we will show that the generalized hyperinterpolation guarantees a faster convergence rate when $f \in C^\ell(\mathbb{S}^{d-1})$ with $\ell \geq 3$. Let $\{D_N\}_{N=1}^\infty$ be a sequence of algebraic polynomials on $[-1,1]$ satisfying the following conditions for some positive integer r:

(i) Each D_N is an algebraic polynomial of degree at most N, and $\int_{\mathbb{S}^{d-1}} p(y) D_N(x \cdot y)\, d\sigma(y) = p(x)$ for all $p \in \Pi_{r-1}^d$;

(ii) $\sup_N \int_0^\pi (1+N\theta)^r |D_N(\cos\theta)| \sin^{d-2}\theta \leq K < \infty$.

Definition 2.4.1: For $f \in C(\mathbb{S}^{d-1})$, define the r-th order generalized hyperinterpolation operator by

$$\widetilde{GL}_{N,r,D_N} f(x) = \sum_{w \in \Lambda_{N+r-1}} \lambda_{N+r-1,w} f(w) D_N(x \cdot w), x \in \mathbb{S}^{d-1}.$$

For $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$, set $|\alpha| = \sum_{j=1}^d \alpha_j$ and $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$. For $\ell \in \mathbb{N}$, set

$$\nabla^\ell f = \{D^\alpha f : \ |\alpha| = \ell\}.$$

Let $\varphi \in C^\infty[0,\infty)$ be such that $\varphi(t) = 1$ for $\frac{3}{4} \leq t \leq \frac{5}{4}$ and $\varphi(t) = 0$ for $0 \leq t \leq \frac{1}{2}$ or $t \geq 2$.

Definition 2.4.2: For $f \in C^\ell(\mathbb{S}^{d-1})$ with $\ell \in \mathbb{N}$, the ℓ-th order tangential operator is defined by

$$\nabla_0^\ell f(x) = \nabla^\ell \widetilde{f}(x), \ \ x \in \mathbb{R}^d,$$

where $\widetilde{f} \in C^\ell(\mathbb{R}^d)$ is given by

$$\widetilde{f}(x) = f(\frac{x}{||x||})\varphi(\|x\|).$$

For $g \in C^\ell(\mathbb{S}^{d-1})$, let

$$\widetilde{g}(x) = g(\tfrac{x}{||x||})\varphi(\|x\|).$$

Definition 2.4.3: The ℓ-th order K-functional of $f \in C(\mathbb{S}^{d-1})$ is defined by

$$K_\ell(f,t) = \inf\Big\{\|f-g\|_{L^\infty(\mathbb{S}^{d-1})} + t^\ell \|\nabla_0^\ell g\|_{L^\infty(B^d)} : \ \ g \in C^\ell(\mathbb{S}^{d-1})\Big\}, \ \ t>0,$$

where $B^d := \{x \in \mathbb{R}^d : \ \ \|x\| \le 1\}$ is the unit ball of $\mathbb{R}^d$. It can be shown that for the moduli of smoothness defined in the last section,

$$K_\ell(f,t) \approx \omega_\ell^*(f,t), \ \ t \in (0,1), \ \ \ell = 1,2.$$

In this section, we will show the following result.

Theorem 2.4.4: With the above notation and assumptions, for all $f \in C(\mathbb{S}^{d-1})$,

$$||\widetilde{GL}_{N,r,D_N} f - f||_\infty \le C K_r(f, N^{-1}).$$

The proof of Theorem relies on several lemmas.

Lemma 2.4.5: Suppose that Ω is a finite subset of $\mathbb{S}^{d-1}$, $\{\mu_\omega : \ \ \omega \in \Omega\}$ is a set of positive numbers, and N is a positive integer. If for some $0 < p_0 < \infty$ the inequality

$$\sum_{\omega\in\Omega} \mu_\omega |f(\omega)|^{p_0} \le C_1 \int_{\mathbb{S}^{d-1}} |f(x)|^{p_0}\, d\sigma(x) \tag{2.4.1}$$

with C_1 independent of f, holds for all $f \in \Pi_N^d$, then the following regularity condition must be satisfied:

$$\sup_{x\in\mathbb{S}^{d-1}} \sum_{\omega\in\Omega\cap B(x,\frac{1}{N})} \mu_\omega \le C_2 N^{-(d-1)}, \tag{2.4.2}$$

where $C_2 = CC_1$ and $C>0$ depends only on d and p_0. Conversely, if the regularity condition (2.4.2) is satisfied for some constant $C_2 > 0$, then for any $0 < p < \infty$ and any $f \in \Pi_m^d$ with $m \ge n$,

$$\sum_{\omega\in\Omega} \mu_\omega |f(\omega)|^p \le CC_2 \Big(\frac{m}{n}\Big)^{d-1} \int_{\mathbb{S}^{d-1}} |f(y)|^p\, d\sigma(y), \tag{2.4.3}$$

where $C>0$ depends only on d and p. The following lemma is due to Reimer.

Lemma 2.4.6: Assume that the following positive cubature formula of degree $N+1$ holds on $\mathbb{S}^{d-1}$.

$$\textstyle\int_{\mathbb{S}^{d-1}} f(x) d\sigma(x) = \sum\limits_{w\in\Lambda_N} \lambda_{N,w} f(w),$$

where Λ_N is a finite subset of $\mathbb{S}^{d-1}$,$\lambda_{N,w} > 0$ for $w \in \Lambda_N$. Then

$$\max_{x\in\mathbb{S}^{d-1}} \sum_{w\in\Lambda_N \bigcap B(x,N^{-1})} \lambda_{N,w} \le CN^{-(d-1)},$$

where $C > 0$ depends only on d.

The following lemma was proved in [35]:

Lemma 2.4.7: Suppose that α is a fixed nonnegative number, n is a positive integer and f is a nonnegative function on $\mathbb{S}^{d-1}$ satisfying

$$f(x) \le C_1(1 + Nd(x, y))^\alpha f(y), x, y \in \mathbb{S}^{d-1},$$

then for any $0 < p < 1$, there exists a nonnegative spherical polynomial $g \in \prod_N^d$ such that

$$C^{-1}f(x) \le g(x)^p \le Cf(x), \text{ for any } x \in \mathbb{S}^{d-1}$$

where $C > 0$ depends only on d, C_1, p and α.

Now we are in a position to prove the main theorem

Proof: First, we show that for $g \in C^r(\mathbb{S}^{d-1})$,

$$||\widetilde{GL}_{N,r,D_N} g - g||_\infty \le CN^{-r}\|\nabla_0^r g\|_{L^\infty(B^d)}. \tag{2.4.4}$$

Let

$$\widetilde{g}(x) = g\Big(\frac{x}{\|x\|}\Big)\varphi(\|x\|),$$

where $\varphi \in C^\infty[0, \infty)$ satisfies that $\varphi(t) = 1$ for $\frac{3}{4} \le t \le \frac{5}{4}$ and $\varphi(t) = 0$ for $0 \le t \le \frac{1}{2}$ or $t \ge 2$. We denote by $T_x^{r-1}\widetilde{g}(y)$ the Taylor polynomial of $\widetilde{g}$ of degree $r - 1$ at the point $x \in \mathbb{S}^{d-1}$; that is,

$$T_x^{r-1}\widetilde{g}(y) = \sum_{|\alpha|\le r-1} \frac{D^\alpha\widetilde{g}(x)}{\alpha!}(y - x)^\alpha,$$

where $\alpha! = \alpha_1! \cdots \alpha_d!$, and $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ for $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$. Let $p_x = T_x^{r-1}\widetilde{g}\Big|_{\mathbb{S}^{d-1}}$ for each fixed $x \in \mathbb{S}^{d-1}$. Clearly, $p_x \in \Pi_{r-1}^d$, and $p_x(x) = g(x)$. Furthermore, by Taylor's theorem, we have that for $x, y \in \mathbb{S}^{d-1}$,

$$\begin{aligned}
|g(y) - p_x(y)| = |\widetilde{g}(y) - T_x^{r-1}\widetilde{g}(y)| &\le C||x - y||^r \sup_{z\in B^d} | \bigtriangledown^r \widetilde{g}(z)| \\
&= C||x - y||^r \sup_{z\in B^d} | \bigtriangledown_0^r g(z)| \\
&= C||x - y||^r \| \bigtriangledown_0^r g\|_{L^\infty(B^d)}.
\end{aligned}$$

Since $p_x D_N \in \Pi^d_{N+r-1}$, it follows that

$$\begin{aligned} g(x) = p_x(x) &= \int_{\mathbb{S}^{d-1}} p_x(y) D_N(x \cdot y)\, d\sigma(y) \\ &= \sum_{w \in \Lambda_{N+r-1}} \lambda_{N+r-1,w} D_N(x \cdot w) p_x(w) = \widetilde{GL}_N p(x), \end{aligned}$$

here and elsewhere in the proof, for simplicity we write λ_w for $\lambda_{N+r-1,w}$, and $\widetilde{GL}_N$ for $\widetilde{GL}_{N,r,D_N}$. Thus,

$$\begin{aligned} |\widetilde{GL}_N g(x) - g(x)| = |\widetilde{GL}_N g(x) - p_x(x)| &\le \sum_{w \in \Lambda_{N+r-1}} \lambda_w |D_N(x \cdot w)||g(w) - p_x(w)| \\ &\le C\| \bigtriangledown_0^r g\|_{L^\infty(B^d)} \sum_{w \in \Lambda_{N+r-1}} \lambda_w |D_N(x \cdot w)| \|x - w\|^r \end{aligned}$$

By Lemma, there exists a positive algebraic polynomial q_N on $[-1, 1]$ of degree at most N such that

$$q_N(x \cdot y) \sim (1 + N\|x - y\|)^r, \quad x, y \in \mathbb{S}^{d-1}.$$

Thus,

$$\begin{aligned} |\widetilde{GL}_N g(x) - g(x)| &\le CN^{-r}\| \bigtriangledown_0^r g\|_{L^\infty(B^d)} \sum_{w \in \Lambda_{N+r-1}} \lambda_w |D_N(x \cdot w)| q_N(x \cdot w) \\ &\le CN^{-r}\| \bigtriangledown_0^r g\|_{L^\infty(B^d)} \int_{\mathbb{S}^{d-1}} |D_N(x \cdot y)|(1 + Nd(x, y))^r\, d\sigma(y) \\ &\le CN^{-r}\| \bigtriangledown_0^r g\|_{L^\infty(B^d)}. \end{aligned}$$

This proves (2.4.4). Next, we show that

$$\sup_N \|\widetilde{GL}_N f\|_\infty \le K\|f\|_\infty. \tag{2.4.5}$$

Indeed, for any $f \in C(\mathbb{S}^{d-1})$, we have

$$\|\widetilde{GL}_N f\|_\infty \le L_N \|f\|_\infty,$$

where $L_N = \sup_{x\in\mathbb{S}^{d-1}} \sum_{w\in\Lambda_{N+r-1}} \lambda_w |D_N(x\cdot w)|$.
However, by Lemma, we have

$$L_N \le \sup_{x\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |D_N(x\cdot y)|\, d\sigma(y) \le K < \infty.$$

Finally, we show (). Indeed, by definition of the K-functional, for $f \in C(\mathbb{S}^{d-1})$, we may find a function $g \in C^r(\mathbb{S}^{d-1})$ such that

$$\|f-g\|_{L^\infty(\mathbb{S}^{d-1})} + N^{-r}\| \bigtriangledown_0^r g\|_{L^\infty(B^d)} \le 2K_r(f, N^{-1}).$$

Thus,

$$\begin{aligned} \|f - \widetilde{GL}_N f\|_\infty &\le \|f-g\|_\infty + \|g - \widetilde{GL}_N g\|_\infty + \|\widetilde{GL}_N g - \widetilde{GL}_N f\|_\infty \\ &\le C\|f-g\|_\infty + \|g - \widetilde{GL}_N g\|_\infty \\ &\le C\|f-g\|_\infty + CN^{-r}\| \bigtriangledown_0^r g\|_{L^\infty(B^d)} \\ &\le CK_r(f, N^{-1}). \end{aligned}$$

As a simple consequence of Theorem 2.4.1, we have

Corollary 2.4.8: If $f \in C^r(\mathbb{S}^{d-1})$, then

$$||\widetilde{G}L_{N,D_N} f - f||_\infty \le CN^{-r}.$$

2.5 The Newman-Shapiro summation method

We combine the idea of hyperinterpolation with the concept of summation in what we call generalized hyperinterpolation. An important example arises in generalized hyperinterpolation when one considers the Newman-Shapiro operators on $\mathbb{S}^{d-1}$, they are based on some particular singular integrals, introduced by Newman and Shapiro in 1964 [40]. We proved that they inherit from the original Newman-Shapiro operators the important property that the approximation error can be estimated uniformly by means of the modulus of continuity of the first order, such that a Jackson-type inequality on the sphere is realized by discrete operators.

Following Newman and Shapiro [40], we define the univariate kernel polynomials

$$B_{2N+1}(t) = B_{2N}(t) = \gamma_{N+1}\left(\tfrac{G_{N+1}(t)}{t-\xi_{N+1}(t)}\right)^2, t \in [-1.1].$$

where $\xi_{N+1}(t)$ is the largest root of $G_{N+1}(t)$, and the constant γ_{N+1} is chosen so that

$$\int_{\mathbb{S}^{d-1}} B_{2N}(xy)d\sigma(y) = 1, x \in \mathbb{S}^{d-1}.$$

Definition 2.5.1: The Newman-Shapiro operators are then defined by

$$T_{2N}f(x) = \int_{\mathbb{S}^{d-1}} f(y)B_{2N}(xy)d\sigma(y), x \in \mathbb{S}^{d-1}, N = 1, 2....$$

For the Newman-Shapiro operators T_N, Reimer [39] proved that

$$||f - T_N f|| \leq Cw_1^*(f, N^{-1}).$$

And Reimer [39, Theorem 7] also proved that

$$||f - T_N f|| \leq Cw_2^*(f, N^{-1}).$$

It was shown in [39, p.197-199]] that the kernels satisfy the conditions:
(i) $\int_{\mathbb{S}^{d-1}} B_N(xy)d\sigma(y) = 1.$
(ii) $\int_0^\pi (1 + N\theta)^r (|B_N(\cos\theta)| \sin^{d-2}\theta)d\theta \leq K < \infty.$

We denote by GL_{N,B_N} the generalized hyperinterpolation operators associated to the kernels B_N. While for the discrete operators GL_{N,B_N} , he proved [39, Theorem 6] that

$$||f - GL_{N,B_N}||_\infty \leq Cw_1^*(f, N^{-1}).$$

F.Dai [41] proved that

$$||f - GL_{N,B_N}||_\infty \leq Cw_2^*(f, N^{-1}).$$

We don't have the similar high order result for Newman-Shapiro operators, maybe very special choices of the quadrature rules could make the following inequality holds,

$$||f - T_N f||\infty \leq CK_r(f, N^{-1}).$$

Chapter 3

Cubature formulas on the sphere

In problems that deal with data, as frequently encountered in applied mathematics, it is often necessary to discretize integrals to obtain discrete processes of approximation. Cubature formulas, synonym of numerical integration formulas, are essential tools for discretizing integrals. Unlike one variable, fundamental problems of cubature formulas in several variables are still open, including those on the sphere. In this chapter we discuss several aspects of cubature formulas on the sphere.

A cubature formula is a finite linear sum of function evaluations that approximates an integral. The strength of a cubature formula is often measured by the number of polynomials that it preserves. The correlation between the number of points N and the degree of precision n is often considered by asking how many points are needed for a fixed degree. If a cubature formula on the sphere is of degree n, then its number of nodes N satisfies

$$N \geq \dim(\prod_{[\frac{n}{2}]}(\mathbb{S}^{d-1})) = C^m_{m+d-1} + C^{m-1}_{m+d-2}, m = [\tfrac{n}{2}].$$

3.1 Gaussian cubature formulas

Recall the Gegenbauer weight function $w_\lambda(t) = (1-t^2)^{\lambda-\frac{1}{2}}$, the Gegenbauer polynomials are orthogonal on [-1, 1] with respect to the weight function. For $n \neq m$,

$$\int_{-1}^{1} C_n^\lambda(t) C_m^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}} dt = 0.$$

where $C_n^\lambda(t)$ is defined in 1.6.2. It is well known that such quadrature formulas exist and they are based on the zeros of the Gegenbauer polynomials. The polynomial $C_n^\lambda(t)$ has n distinct real zeros in [-1,1], which we denote by $t_{k,n}^{(\lambda)}$

$$-1 < t_{1,n}^{(\lambda)} < t_{2,n}^{(\lambda)} < \cdots < t_{n,n}^{(\lambda)} < 1.$$

Furthermore, we define $\theta_{k,n}^{(\lambda)}$ by

$$t_{k,n}^{(\lambda)} = \cos\theta_{k,n}^{(\lambda)}, \theta_{k,n}^{(\lambda)} \in (0,\pi), 1 \leq k \leq n.$$

Then we can give the definition of the Gaussian quadrature formula.

Definition 3.1.1: (See [1, p.126]) The Gaussian quadrature of degree $2n-1$ for w_λ is given by

$$\int_{-1}^{1} f(x)w_\lambda(x)dx = \sum_{k=1}^{n} \mu_{k,n}^{(\lambda)} f(t_{k,n}), f \in \prod_{2n-1}^{d},$$

where the quadrature weights $\mu_{k,n}^{(\lambda)}$ is given by

$$\mu_{k,n}^{(\lambda)} = \frac{\pi\Gamma(n+2\lambda)}{2^{2\lambda}[\Gamma(\lambda+1)^2](1-t_{k,n}^2)[C_{n-1}^{\lambda+1}(t_{k,n})]^2}.$$

The formula for the weights are given in [42, p.352], which we have simplified by applying (4.7.27) of [42]. Changing variable $x = \cos\theta$,

$$\int_0^\pi f(\cos\theta)(\sin\theta)^{2\lambda}d\theta = \sum_{k=1}^{n} \mu_{k,n}^{(\lambda)} f(\cos\theta_{k,n}^{\lambda}).$$

The Gaussian quadrature is known to have the highest degree of precision among all quadratures with the same number of nodes.

3.2 Product type cubature formulas on the sphere

These cubature formulas are constructed by parametrizing the integral over the sphere in polar coordinates. These cubature formulas are useful and are essentially the only family of formulas on the sphere that are positive and explicitly constructed. We are now ready to construct product type cubature formulas on the sphere. First we consider $\mathbb{S}^2$, In spherical coordinates, $x_1 = r\sin\theta\sin\phi, x_2 = r\sin\theta\cos\phi, x_3 = r\cos\theta$. Set

$$g(\theta,\phi) = f(\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta), 0 \le \theta \le \pi, 0 \le \phi \le 2\pi.$$

Recall that the Gegenbauer polynomial $C_n^{1/2}(t) = P_n(t)$(when $\lambda = \frac{1}{2}$), the Legendre polynomial.

Theorem 3.2.1: (See [1, p.127]) Let $\phi_{k,n} = \pi k/n$, $\theta_{j,n} = \theta_{j,n}^{\frac{1}{2}}$ be associated with the zeros of the Legendre polynomial and $\mu_{j,n} = \mu_{j,n}^{\frac{1}{2}}$. Then the cubature formula

$$\int_{\mathbb{S}^2} f(x)d\sigma(x) = \frac{\pi}{n}\sum_{k=0}^{2n-1}\sum_{j=1}^{n} \mu_{j,n} g(\phi_{k,n}, \theta_{j,n})$$

is of degree $2n-1$. This cubature formula is called a product type. The number of nodes is $2n^3$ and the nodes are heavily clustered at the north and south poles $(0,0,\pm 1)$, instead of more evenly distributed over the sphere. The product cubature rule on $\mathbb{S}^{d-1}$ has more or less the same structure and can be constructed by induction. In spherical coordinates, let

$$g(\theta_1,\theta_2,,,,,,\theta_{d-1}) = f(\sin\theta_{d-1}...\sin\theta_1, \sin\theta_{d-1}...\sin\theta_2\cos\theta_1, \cos\theta_{d-1}).$$

Then we have,

Theorem 3.2.2: (See [1, p.128]) Let $\phi_{k,n} = \pi k/n$, $0 \le k \le 2n-1$. Then the cubature formula

$$\int_{\mathbb{S}^{d-1}} f(x)d\sigma(x) = \tfrac{\pi}{n}\sum_{k=0}^{2n-1}\sum_{j_2=1}^{n}\cdot\cdot\sum_{j_{d-1}=1}^{n}\prod_{i=2}^{d-1}\mu_{i,n}^{\frac{i-1}{2}} g(\phi_{k,n},\theta_{j_2,n}^{1/2},,,\theta_{j_{d-1},n}^{(d-2)/2})$$

is of degree $2n-1$. The number of nodes of this cubature formula is $2n^d$.

3.3 Positive Cubature Formulas

We are now ready to introduce the positive cubature formulas, which we can establish for a doubling weight. In the following theorem, δ_0 denotes a sufficiently small positive constant depending only on the doubling constant of w.

Theorem 3.3.1: (See [1, p.131]) Let w be a doubling weight on $\mathbb{S}^{d-1}$. Given a maximal $\frac{\delta}{n}$-separated subset $\Lambda \in \mathbb{S}^{d-1}$ with $\delta \in (0,\delta_0)$. There exist positive numbers λ_η for all $\eta \in \Lambda$ and

$$\int_{\mathbb{S}^{d-1}} f(x)w(x)d\sigma(x) = \sum \lambda_\eta f(\eta), f \in \prod_n^d.$$

Let N be the number of nodes of the cubature formula in Theorem 3.3.1. Then the degree of precision of the cubature formula is in the order of $N^{\frac{1}{d-1}}$. The desirable features of this cubature formula make it an important tool for theoretical studies. Constructing the cubature formula in theorem 3.3.1, since the nodes are given, amounts to determine the weights λ_η for each $\eta \in \Lambda$. However, since the weights satisfy an under-determined linear system of more variables than equations, it is a difficult task to identify a positive solution as specified in the theorem among infinite solutions of the system. In fact, at this moment, no practical method for constructing such a cubature formula, when n is moderately large, is known.

The next theorem reveals a close relationship between Marcinkiewicz-Zygmund inequalities and positive cubature formula.

Theorem 3.3.2: Suppose we have a positive cubature formula of degree $2n$ on $\mathbb{S}^{d-1}$:

$$\int_{\mathbb{S}^{d-1}} f(y)d\sigma(y) = \sum_{w\in\Lambda} \lambda_w f(w),\ f \in \prod_{2n}^d,$$

where $\lambda_w \geq 0$. Then for any $f \in \prod_{[\frac{n}{2}]}^d$,

$$||f||_p \sim (\sum_{w\in\Lambda} \lambda_w |f(w)|^p)^{\frac{1}{p}},\ \text{if } p \in (0,\infty),$$

$$||f||_p \sim \sum_{w\in\Lambda} |f(w)|,\ \text{if } p = \infty,$$

Conversely, suppose we have the following Marcinkiewicz-Zygmund inequalities for some $0 < p < \infty$ and large positive integer n

$$||f||_p^p \sim \frac{1}{n^{d-1}} \sum_{w\in\Lambda} |f(w)|^p,\ f \in \prod_n^d,$$

where Λ is a finite subset of $\mathbb{S}^{d-1}$, then there exist positive $\lambda_w \sim \frac{1}{n^{d-1}}$ for each $w \in \Lambda$, and a number $\gamma \in (0,1)$ depending only on d and p, for which

$$\int_{\mathbb{S}^{d-1}} f(y)d\sigma(y) = \sum_{w\in\Lambda} \lambda_w f(w),\ f \in \prod_{\gamma n}^d.$$

3.4 Spherical t-designs

In this section we will introduce some facts and concepts of the spherical t-designs. The concept of spherical t-designs was introduced by Delsarte, Goethals and Seidel in 1977. Since then, spherical t-designs have been studied extensively. In this section we will discuss a spherical t-designs which was introduced by Sloan and Robert S.Womersley.

The additional constraint of equal weights means that only the nodes are in our disposal, which makes it harder to construct such cubature formulas. The lower bound for the number of nodes clearly applies to the equal weights cubature formulas. The additional constraint suggests that the nodes of such cubature formulas are in general more than ordinary cubature formulas of the same degree. It has long been conjectured, however, that a spherical t-designs containing $o(t^{d-1})$ nodes on $\mathbb{S}^{d-1}$ exists. Recently this conjecture was confirmed and the following is the result.

Theorem 3.4.1: There exists a positive constant c_d, depending only on d, such that for each positive integer $N \geq c_d t^{d-1}$, there exists a set of N points $x_1, ...x_N \in \mathbb{S}^{d-1}$ for which the spherical t- designs holds for all $f \in \prod_t^d$.

Definition 3.4.2: Let χ_N be a set of N points on the unit sphere $\mathbb{S}^{d-1}$ and let $\prod_t^d$ be the linear space of restrictions of polynomials of degree at most t. The set χ_N is a spherical t-design if

$$\tfrac{1}{N}\sum_{j=1}^{N} g(x_j) = \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} g(x)d\sigma(x)$$

holds for all spherical polynomials $g \in \prod_t(\mathbb{S}^{d-1})$, where $d\sigma(x)$ denotes the surface measure on $\mathbb{S}^{d-1}$ and $|\mathbb{S}^{d-1}|$ is the surface area of the unit sphere $\mathbb{S}^{d-1}$. In other words, we see that χ_N is a spherical t-design if the average value over χ_N of any polynomial of degree at most t is equal to the average value of the polynomial over the sphere.

It is well known that, for $d = 2$, the dimension of $\prod_t^d$ is equal to $(t+1)^2$. Our claim is that the constructed well conditioned spherical t-designs with $N \geq (t+1)^2$ are valuable for numerical integration and if $N = (t+1)^2$ also for polynomial interpolation. When $N = (t+1)^2$, the quadrature rule and the interpolant are consistent, in that the quadrature rule for a given function f is the exact integral of the interpolation of f.

Let $\{y_{l,k} : k = 1, 2, ...2l+1, l = 0, 1, ...t\}$ be a set of spherical harmonics orthonormal with respect to the L_2 inner product,

$$< f, g >= \int_{\mathbb{S}^2} f(x)g(x)d\sigma(x),$$

where $y_{l,k}$ is a spherical harmonic of degree l. The addition theorem for spherical harmonics on $\mathbb{S}^2$ gives

$$\sum_{k=1}^{2l+1} y_{l,k}(x)y_{l,k}(y) = \frac{2l+1}{4\pi}P_l(xy) \text{ for all } x, y \in \mathbb{S}^2,$$

where $x \cdot y$ is the inner product and P_l is the Legendre polynomial of degree l normalized so that $P_l(1) = 1$. For $t \geq 1$ and $N \geq (t+1)^2$, let $\mathbb{Y}_t^0$ be the $((t+1)^2 - 1) \times N$ matrix defined by

$$\mathbb{Y}_t^0(\chi_N) = [y_{l,k}(x_j)], k = 1, 2, , , 2l+1, l = 1, 2 \cdots t, j = 1, 2 \cdots N,$$

and let $\mathbb{Y}_t(\chi_N)$ be a $((t+1)^2 \times N$ matrix with an added leading row consisting of the degree 0 spherical harmonic, that is,

$$\begin{bmatrix} \frac{1}{\sqrt{4\pi}} e^T \\ \mathbb{Y}_t^0(\chi_N) \end{bmatrix} \tag{3.4.1}$$

where $e = [1, ...1]^T$. It is well known that there are many equivalent conditions for a set $\chi_N \subset \mathbb{S}^2$ to be a spherical t-design. Among these equivalent statements, one that plays an important role in the subsequent discussion is the following theorem.

Theorem 3.4.3: (See [43, p.2138]) A finite set χ_N is a spherical t-design if and only if

$$\sum_{j=1}^{N} y_{l,k}(x_j) = 0.$$

Chen and Womersley [45] reformulated the problem of finding a spherical t-design with $N = (t+1)^2$ points as a system of underdetermined nonlinear equations. The nonlinear function $\mathbb{C}_t$ is defined by

$$\mathbb{C}_t(\chi_N) = \mathbb{E}\mathbb{G}_t(\chi_N)e,$$

where

$$\mathbb{E} = [1, -\mathbb{I}_{N-1}] \in \mathbb{R}^{(N-1)*N} \text{ and } \mathbb{G}_t(\chi_N) = \mathbb{Y}_t(\chi_N)^T \mathbb{Y}_t(\chi_N).$$

[43] gives some conditions for which χ_N is a spherical t-design.

Theorem 3.4.4: (See [43, p.219]) Let $N = (t+1)^2$. Suppose the Gram matrix $\mathbb{G}_t(\chi_N)$ is nonsingular. Then χ_N is a spherical t-design if and only if $\mathbb{C}_t(\chi_N) = 0$.

Example 3.4.5: If $t = 1$, χ_4 consists of the following four points:

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \tag{3.4.2}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \tag{3.4.3}$$

The Gram matrix $\mathbb{G}_1$ for these points is

$$\begin{bmatrix} \frac{4}{4\pi} & \frac{1}{4\pi} & \frac{1}{4\pi} & \frac{1}{4\pi}-\frac{1}{2} \\ \frac{1}{4\pi} & \frac{4}{4\pi} & \frac{1}{4\pi} & \frac{1}{4\pi}+\frac{\sqrt{2}}{2} \\ \frac{1}{4\pi} & \frac{1}{4\pi} & \frac{4}{4\pi} & \frac{1}{4\pi}+\frac{1}{2} \\ \frac{1}{4\pi}-\frac{1}{2} & \frac{1}{4\pi}+\frac{\sqrt{2}}{2} & \frac{1}{4\pi}+\frac{1}{2} & \frac{4}{4\pi} \end{bmatrix} \tag{3.4.4}$$

we can show that χ_4 is not a spherical design since $|\mathbb{G}_1(\chi_4)|$ is equal to 0.

Definition 3.4.6: (See [43, p.2140]) The set $\chi_N \subset \mathbb{S}^2$ is a fundamental system for $\prod_t$ if the zero polynomial is the only element of $\prod_t$ that vanishes at each point in χ_N, that is, if

$$p \in \prod\nolimits_t, p(x_i) = 0, i = 1, 2, \cdots N$$

implies $p(x) = 0$ for all $x \in \mathbb{S}^2$.

Lemma 3.4.7: (See [43, p.2130]) A set $\chi_N \subset \mathbb{S}^2$ is a fundamental system if and only if $\mathbb{Y}_t(\chi_N)$ is of full row rank $(t+1)^2$. For $N \geq (t+1)^2$, we will use both of the matrices

$$\mathbb{H}_t(\chi_N) = \mathbb{Y}_t(\chi_N)\mathbb{Y}_t(\chi_N)^T \text{ and } \mathbb{G}_t(\chi_N) = \mathbb{Y}_t(\chi_N)^T\mathbb{Y}_t(\chi_N).$$

Corollary 3.4.8: (See [43, p.2130]) A set χ_N is a fundamental system if and only if $\mathbb{H}_t(\chi_N)$ is nonsingular.

As pointed out in [44], extremal systems are good for polynomial interpolation and have good geometrical properties. Chen and Womersley [45] and then Chen, Frommer, and Lang [46] verified that a spherical t-design exists in a neighborhood of an extremal system.

Definition 3.4.9: (See [43, p.2141]) A set χ_N of $N \geq (t+1)^2$ points is an extremal spherical t-design if the determinant of the matrix $\mathbb{H}_t(\chi_N)$ is maximal subject to the constraint that χ_N is a spherical t-design.

It needs to be emphasized that we can never know if a computed set of points is a global rather than a local maximizer. Thus, in practice, we prefer to say that the computed sets are well conditioned spherical designs rather than to claim that they are truly extremal spherical designs.

Example 3.4.10: (See [43, p.2142]) We consider $\chi_4 = x_1, x_2, x_3, x_4 \subset \mathbb{S}^2$, which is

$$\begin{bmatrix} 0 & \frac{\sqrt{8}}{3} & \frac{-\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ 0 & 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{3} \\ 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \tag{3.4.5}$$

this is a extremal spherical design because $\mathbb{G}_1(\chi_4)$ is equal to the following

$$\begin{bmatrix} \frac{1}{\pi} & 0 & 0 & 0 \\ 0 & \frac{1}{\pi} & 0 & 0 \\ 0 & 0 & \frac{1}{\pi} & 0 \\ 0 & 0 & 0 & \frac{1}{\pi} \end{bmatrix} \tag{3.4.6}$$

and $\mathbb{C}_1(\chi_4)$ is equal to the following:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.4.7}$$

$|\mathbb{G}_1(\chi_4)| = (\frac{1}{\pi})^4$. The set χ_4 is also an extremal spherical design because $|\mathbb{H}_1(\chi_4)|$ is maximal($|\mathbb{H}_1(\chi_4)| = (\frac{1}{\pi})^4$) subject to the constraint $\mathbb{C}_1(\chi_4) = 0$.

We now discuss the computational construction of well conditioned spherical t-design for $N(t+1)^2$. Interval methods [46, 47, 48] are then used to prove the existence of a well conditioned true spherical t-design in a narrow interval and to place relatively close upper and lower bounds on the determinant of the matrix $\mathbb{H}_t(\chi_N)$ over the interval. We consider the following optimization problem:

$$\max \ln |\mathbb{H}_t(\chi_N)|$$

subject to $\mathbb{C}_t(\chi_N) = 0$. The following strategy is adopted. Choose a nonnegative integer t, $N \geq (t+1)^2$, and a fundamental system χ_N^0 as a starting point set.

(i) Use the Gauss Newton method (see [49], page 256) to find an approximate solution χ_N of $\mathbb{C}_t(\chi_N)$ starting from χ_N^0.

(ii) Use a nonlinear programming method to find

$$\max \ln |\mathbb{H}_t(\chi_N)| \text{ starting from } \chi_N.$$

Next we use the computed well conditioned spherical t-designs with $N = (t+1)^2$ points to evaluate integration and interpolation for a function on $\mathbb{S}^2$.

Example 3.4.11: If $f(x) = 6x$, then we can easily compute the following integral

$$\int_{\mathbb{S}^2} f(x) d\sigma(x) = 0.$$

We know that for $N = (t+1)^2$, the quadrature rule for a given function f is the exact integral of the interpolant of f. Now we use the extremal spherical design to estimate this integral, we consider $\chi_4 = x_1, x_2, x_3, x_4 \subset \mathbb{S}^2$,

$$\begin{bmatrix} 0 & \frac{\sqrt{8}}{3} & \frac{-\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ 0 & 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{3} \\ 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \tag{3.4.8}$$

This is a extremal spherical design. And we have the following:

$$\frac{4\pi}{4}(0 + 6 * \frac{\sqrt{8}}{3} + 6 * (-\frac{\sqrt{2}}{3}) + 6 * (-\frac{\sqrt{2}}{3})) = 0.$$

Example 3.4.12: (See [43, p.2154]) Now we consider the well-known Franke functions as adapted by Renka:

$f(x) = 0.75 \exp(-\frac{(9x-2)^2}{4} - \frac{(9y-2)^2}{4} - \frac{(9z-2)^2}{4} + 0.75 \exp(-\frac{9x+1}{49} - \frac{9y-1}{10} - \frac{9z+1}{10}) + 0.5 \exp(-\frac{(9x-7)^2}{4} - \frac{(9y-3)^2}{4} - \frac{(9z-5)^2}{4}) - 0.2 \exp(-(9x-4)^2 - (9y-7)^2 - (9z-5)^2)$

$$\int_{\mathbb{S}^2} f(x) d\sigma(x) = 6.6961822200736179523.$$

In [43, p.2154] it shows that the absolute error decreases dramatically to around 10^{-9} at $t = 60$,the high degree spherical t-designs deal successfully with a complicated function as long as it is smooth. As expected, the high degree spherical t-designs deals successfully with a complicated function as it is smooth.

3.5 Other Types of Cubature Formulas

Now we consider the quadrature formula

$$\int_{-1}^{1} f(t)p(t)dt = \sum_{k=1}^{N} p_k f(t_k),$$

which is exact for every algebraic polynomial $f(t)$ of degree q for equal weights $p_1 = p_2 = \cdots p_N$ and $N = q$, was stated in [50]. We shall consider only the following special case:

$$p(t)dt = (1-t^2)^{\frac{d-3}{2}}dt / \int_{-1}^{1}(1-\tau^2)^{\frac{d-3}{2}}d\tau, d = 3, 4...$$

We say that the set of nodes $\chi_N = \{t_k\}_1^N$ belongs to the class $V_{q,n}$ if the formula

$$\int_{-1}^{1} f(t)p(t)dt = \frac{1}{N}\sum_{k=1}^{N} f(t_k)$$

is exact for every polynomial of degree q.

Example 3.5.1: (See [51, p,1068]) Let $\chi \subset \mathbb{S}^{d-1}$, consist of the $2d$ points. $\{1, 0, ..., 0\}, ..., \{0, ..., 0, 1\}$. χ is a minimal 3-design [52]. Therefore, for any $a \in \mathbb{S}^{d-1}$ the $2d$ points $\pm a_1, , , , , \pm a_d$ belong to $V_{3,d}$. Putting $a = \{1, 0, 0, , , \}, a = \{\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}, , , , , \}$. We obtain the well-known [53] quadrature formula

$$\int_{-1}^{1} f(t)p(t)dt = \frac{1}{2d}\{f(-1) + (2d-2)f(0) + f(1)\},$$

$$2\int_{-1}^{1} f(t)p(t)dt = f(\tfrac{-1}{\sqrt{d}}) + f(\tfrac{1}{\sqrt{d}}),$$

which are exact for every algebraic polynomial $f(x)$ of degree 3.

Example 3.5.2: (See [51, p,1069]) Let $d = 3$ and let χ consist of the 12 vertices of an icosahedron situated on $\mathbb{S}^2$. χ is a minimal 5-design [52]. In the case when a is one of the vertices of χ, we obtain the quadrature formula

$$\int_{-1}^{1} f(t)p(t)dt = \frac{1}{6}\{f(-1) + 5f(\tfrac{-1}{\sqrt{5}}) + 5f(\tfrac{1}{\sqrt{5}}) + f(1)\}.$$

Example 3.5.3: (See [51, p,1069]) In the case when a is the unit normal to one of the 20 faces of the icosahedron, we obtain the following version of formula

$$\int_{-1}^{1} f(t)dt = \frac{1}{2}\{f(-\sqrt{\tfrac{1}{3}(1+\tfrac{2}{\sqrt{5}})}) + f(-\sqrt{\tfrac{1}{3}(1-\tfrac{2}{\sqrt{5}})})$$

$$+f(\sqrt{\tfrac{1}{3}(1-\tfrac{2}{\sqrt{5}})}) + f(\sqrt{\tfrac{1}{3}(1+\tfrac{2}{\sqrt{5}})})\}.$$

Example 3.5.4: (See [51, p,1069]) Let χ be the set of $d+1$ vertices of a regular simplex inscribed in $\mathbb{S}^{d-1}$. χ is a minimal 2-design. Taking one of its vertices as a, we obtain the quadrature formula

$$\int_{-1}^{1} f(t)p(t)dt = \frac{1}{d+1}\{f(1) + df(-\tfrac{1}{d})\},$$

which are exact for every second-order polynomial $f(t)$.

Bibliography

[1] Feng Dai and Yuan Xu, Approximation Theory and Harmonic Analysis on Spheres and Balls, Springer Monographs in Mathematics, Springer, 2013.

[2] Kunyang Wang and Luoqing Li, Harmonic Analysis and Approximation on the unit Sphere, Science press, Beijing, 2000.

[3] Sheng.Wang, Generalized Ul'yanov type inequality on the sphere $\mathbb{S}^{d-1}$, Acta Math.Sinica (Chin.Ser.) 54 (2011), no. 1, 115-124.

[4] E.M.Stein and G.Weiss, Introduction to Fourier analysis on Euclidean space, Princeton university press, Princeton New Jersey,1971.

[5] G.Szego, orthogonal polynomials, Amercian Mathematics Society Colloquium Publication, Vol, XXIII, 1939.

[6] C.Muller, Spherical Harmonics, Lecture notes in Mathematics, 17, 1966.

[7] S.M.Nikolskii and P.I.Lizorkin, Approximation of functions on the sphere,Izv. An SSSR, Ser.Mat, 51(3)(1987), 635-651.

[8] S.M.Nikolskii and P.I.Lizorkin, Approximation on the sphere, survey, Translated from the Russian by Jerzy Trzeciak, Banach Center Publ., 22, Approximation and function spaces(Warsaw, 1986), 281-292, PWN, Warsaw, 1989.

[9] Feng Dai, Multivariate polynomial inequalities with respect to doubling weights, J.Funct. Analysis 235 (2006), 137-170.

[10] Feng Dai, Jackson-type inequality for doubling weights on the sphere, Constr. Approx. 24(2006), 91-112.

[11] Feng Dai, Characterizations of function spaces on the sphere using spherical frames, Trans.of the Amer.Math.Soc. 359 (2007), 567-589.

[12] Feng Dai and Z.Ditzian, Jackson inequality for Banach spaces on the sphere, Acta Math.Hungar.118 (2008), 171-195.

[13] Feng Dai and Z.Ditzian, Littlewood-Paley theory and a sharp Marchaud inequality, Acta Sci.Math.(Szeged) 71 (2005), 65-90.

[14] Feng Dai and Z.Ditzian, Combinations of multivariate averages, J. Approx. Theory 131 (2004), 268-283.

[15] Feng Dai, Z.Ditzian and H.W.Huang, Equivalence of measures of smoothness in $L^p(\mathbb{S}^{d-1}), 1 < p < \infty$, Studia Math. 196 (2010), 179-205.

[16] Feng Dai and H.P.Wang, Positive cubature formulas and Marcinkiewicz-Zygmund inequalities on spherical caps, Constr. Approx. 31 (2010), 1-36.
[17] Feng Dai and Yuan Xu, Cesàro means of orthogonal expansions in several variables, Const. Approx.29 (2009), 129-155.
[18] Feng Dai and Yuan Xu, Moduli of smoothness and approximation on the unit sphere and the unit ball, Advances in Math. 224 (2010), no. 4, 1233-1310.
[19] Feng Dai and Y.Xu, Polynomial approximation in Sobolev spaces on the unit sphere and the unit ball, J. Approx. Theory 163 (2011), 1400-1418.
[20] I.K.Daugavet(1974), Some applications of the Marcinkiewicz Berman identity, Vestnik Leningrad university, Math., 1:321-327.
[21] B.Bojanov and I.K.Georgieva, Interpolation by bivariate polynomials based on Radon projections, Studia Math 162 (2004), 141-160.
[22] R.Askey, and S.Wainger, On the behavior of special classes of Ultra spherical expansions.I, J.Analyse Math. 15(1965), 193-220.
[23] Feng Dai, On generalized hyperinterpolation on the sphere, Proceedings of the American Mathematical Society 134, (2006), 2931-2941.
[24] Feng Dai(2006), Multivariate polynomial inequalities with respect to doubling weights and A1 weights, J.Funct. Analysis 235, 137-170.
[25] Feng Dai, Z.Ditzian, Jackson inequality for Banach spaces on the sphere, Acta Math.Hungar. 118 (2008), no. 1-2, 171-195.
[26] Feng Dai and Xu.Y.(2009), Cesàro means of orthogonal expansions in several variables, Const.Approx. 29 (2009), 129-155.
[27] Feng Dai, Z.Ditzian and Hongwei.Huang, Equivalence of measures of smooth-ness in $L^p(\mathbb{S}^{d-1}), 1 < p < 1$. Studia Mathematica 196, 179-205.
[28] Feng Dai.and Xu.Y,(2009) Boundedness of projection operators and Cesàro means in weighted L^p space on the unit sphere. Trans. Amer. Math. Soc.361, 3189-3221.
[29] Stein, E.Stein,E. M.(1993). Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatroy Integrals, Princeton Univ. Press, Princeton,NJ, 1993.
[30] Stein, E.M. Topics in harmonic analysis related to the LittlewoodPaley theory, Annals of Mathematical Studies, Vol.63.Princeton, NJ:Princeton Univ.Press, 1970.
[31] Stein, E.M., Singular Integrals and Dikerentiability Properties of Function-

s, Princeton University Press (February 1, 1971).
[32] W.Trebels, Multipliers for (C,a)-Bounded fourier expansions in Banach spaces And Approximation Theory, Lecture notes in Math.Springer Verlag, Berlin, 1973.
[33] C.D.Sogge, Oscillatory integrals and spherical harmonics, Duke Math.J.53 (1986), 43-65.
[34] I.H.Sloan (1995), Polynomial interpolation and hyperinterpolation over general regions.J.Approx.Theory, 83:238-254.
[35] I.H.Sloan, R.S.Womersley(2000), Constructive polynomial approximation on the sphere.J.Approx.Theory, 103:91-98.
[36] M.Reimer(2000), Hyperinterpolation on the sphere at the minimal projection order.J.Approx.Theory, 104:272-286.
[37] M.Reimer, Constructive Theory of Multivariate Functions, Mannheim: BI-Wissenschaftsverlag(1990).
[38] Gunter Felbecker and Walter Schempp, A Generalization of Bohman-Korovkin Theorem, Math.Z.122,63-70, Springer-Verlag 1971.
[39] M.Reimer, Generalized hyperinterpolation on the sphere and Newman Shapiro operators, Constr.Approx.18 (2002), no.2, 183-203.
[40] D.J.Newman, H.S.Shapiro, Jacksons theorem in higher dimensions.In:U'ber Approximation stheorie.Basel: Birkhauser, pp.208-219(1964).
[41] Feng Dai, On generalized Hyperinterpolation on the sphere, Proceedings of the American Mathematical Society 134, (2006), 2931-2941.
[42] G.Szego, Orthogonal polynomials, Amer.Math.Soc.Colloq.Vol.23, Providence, 1975.
[43] Congpei An, Xiaojun Chen, Ian H.Sloan, and Robert S.Womersley, Well Conditioned Spherical Designs For Integration And Interpolation On The Two-Sphere. Siam J.Numer.Anal.Vol.48,No.6, pp.2135-2157.
[44] I.H.Sloan and R.S.Womersley, Extremal systems of points and numerical integration on the sphere, Adv.Comput.Math., (2004),pp.107-125.
[45] X.Chen and R.S.Womersley, Existence of solutions to systems of undetermined equations and spherical designs, SIAM J. Numerical.Anal., 44(2006), pp.2326-2341.
[46] X.Chen, A.Frommer, and B.Lang,Computational existence proof for spherical t-designs, Numerical Math., to appear.

[47] G.Alefeld and J.Herzberger, Introduction to Interval Computations, Computer Science Applied Mathematics, Academic Press, New York, 1983.
[48] S.M.Rump, Intlab Interval Laboratory, in Developments in Reliable Computing, T.Csendes, ed., Kluwer Academic Publishers, Dordrecht, 1999, pp.77-104.
[49] J.Nocedal and S.J.Wright, Numerical Optimization, 2nd ed., Springer, Berlin, 2006.
[50] P.L.Chebyshev, On quadratures, Collected papers, vol. III, Akad. Nauk SSSR, Moscow 1948, pp.49-62.
[51] V.A.Yudin, Distribution of the points of a design on the sphere.Izvestiya: Mathematics 69:5 1061-1079.
[52] P.Delsarte, G. M.Goethals, and J.J.Seidel, Spherical codes and designs, Geom.Dedicata 6 (1979), 363-388.
[53] V.I.Krylov, Approximate calculation of integrals, Fizmatgiz, Moscow 1959; English translation., Macmillan, New York 1962.