TOPOLOGICAL INVARIANT MEANS AND ACTION OF LOCALLY COMPACT SEMITOPOLOGICAL SEMIGROUPS

by

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Abstract

Let a locally compact semitopological semigroup S have a separately continuous left action on a locally compact Hausdorff X. We define a jointly continuous left action of the measure algebra $M(S)$ on the bounded Borel measure space $M(X)$ which is an analogue of the convolution of measure algebras $M(S)$. We further introduce a separately continuous left action of $M(S)$ on the dual of a $M(S)$-invariant subspace A of $M(X)^*$ in analogue with Arens product. We consider the fixed point of this action on the set of means on A (topological S-invariant mean on A) and characterize its existence in analogue with topological right stationary, ergodic properties, Dixmier condition etc. A notion of topological (S, A)-lumpy is introduced and its relation with topological S-invariant mean on A is studied. The relation of existence of topological invariant means on a subspace of X and on X itself is also studied.

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Table of Contents

Chapter 1

Introduction

Let S be a locally compact semitopological semigroup, X be a locally compact space under separately continuous left action of S. We denote by $M(S)$ and $M(X)$ the Banach space of bounded Borel measures respectively on S and X. The main purpose of this thesis is to build a separately continuous left action of $M(S)$ on $M(X)^*$ through the convolution of measures in $M(S)$ and $M(X)$. Hence we construct a separately continuous left action of $M(S)$ on the continuous dual of $M(S)$-invariant subspaces A in $M(X)^*$. Then we characterize the existence of $M(S)$-invariant means on A. We use topological S-invariant for $M(S)$-invariant in this thesis.

In Chapter 2, we introduce some known results on locally compact semitopological semigroup relating to this thesis.

In Chapter 3, we first show that we can properly build an jointly continuous left action of $M(S)$ on $M(X)$ in section 3.1. Then we introduce the notion of topology S-invariant means on an $M(S)$ invariant subspace A of $M(X)^*$ which is a generalization of topological left invariant means in the case where $X = S$. We generalize some of the well-known characterizations of the existence of topological S-invariant means from $X = S$ case in section 3.2, e.g., Dixmier condition, topological right stationary, ergodic property, etc.. In section 3.3, we generalize results in [32] and characterize the existence of topological S-invariant mean on $M(X)^*$ by identifying $M(X)^*$ as a Banach subspace of

$\prod_{\mu \in M(X)} \mathcal{L}_\infty(|\mu|)$. In section 3.4, we study a subset T of X for a topological S-invariant subspace A in $M(X)^*$, such that there exists a mean M on A and M is topological S-invariant on the set $\{F \in A;\ \chi_T \leq F \leq 1\}$, where χ_T is the characteristic functional of T in $M(X)^*$. Such set is said to be topological (S, A)-lumpy. We prove that for each $(S, M(X)^*)$-lumpy subset, we can always find a topological S-invariant subspace A of $M(X)^*$ such that it has a topological S-invariant mean M with $M(\chi_T) = 1$. Then in section 3.4, we study locally compact Borel subspace T of X which is closed under the left action of locally compact Borel subsemigroup R in S (in particular, we may let $R = S$). We prove that there exists a topological R-invariant mean on $M(T)^*$ if and only if there exists a topological R-invariant mean M on $M(X)^*$ with $M(\chi_T) = 1$. If we assume further that R is topological S-lumpy, we prove that the existence of topological R-invariant mean on $M(T)^*$ is equivalent with the existence of topological S-invariant mean on $M(X)^*$.

In chapter 4, we generalize the results in [18] and [23] to show that the existence of S-invariant means in the convex hull of multiplicative means on $LUC(S, X)$, where S is a semitopological semigroup and X is a topological space under separately continuous left action of S, reflect the structure of S, i.e., we can decompose S into finitely many open and closed disjoint subsets.

Chapter 2

Preliminaries

2.1 Definition and Notations

Throughout this thesis, a topological space is assumed to be Hausdorff.

Definition 2.1.1. A semigroup S which is also a topological space is a *topological [resp. semitopological] semigroup* if multiplication in S is a jointly [resp. separately] continuous.

A group G is a *topological [resp. semitopological] group* if it is a topological [resp. semitopological] semigroup and further the inverse map $G \to G$ by $g \mapsto g^{-1}$ is continuous.

Remark 2.1.2. Note that when S is a locally compact semitopological group, S is also a topological group (see [12]). However in general, a semitopological semigroup is not necessarily a topological semigroup.

Example 2.1.3. ([21, P133]) Let $\mathbb{R}$ be equipped with the usual topology. Let $S = \mathbb{R} \cup \{\infty\}$ be the one point compactification of $\mathbb{R}$. Let $s, t \in S$, define $s \cdot t = \begin{cases} s+t & s,t \in \mathbb{R} \\ \infty & s = \infty \text{ or } t = \infty \end{cases}$. Then S is a semigroup under "$\cdot$". In addition, the multiplication of S defined by "$\cdot$" is separately continuous. Thus, S is a compact semitopological semigroup. Assume the multiplication defined by $\cdot$ is jointly continuous, let $s_n = n+1$, $t_n = -n$ be two sequences in S, we have that $\infty = \lim_n s_n + \lim_n t_n = \lim_n(s_n + t_n) = 1$, which is impossible. Therefore, S is compact semitopological semigroup that is not a topological semigroup.

Let S be a locally compact space. We denote by $M(S)$ the Banach space of bounded Borel measures on S equipped with total variation norm and by $P(S)$ the set of probability measures on S. By Riesz representation theorem, $M(S)$ is identified with continuous functionals on $C_0(S)$, where $C_0(S)$ is the space of continuous functions on S that vanish at infinity. It is shown in [20] that when S is a topological semigroup, we may define the convolution of measures in $M(S)$ using Fubini's theorem, i.e., let $\nu, \mu \in M(S)$, define

$$\int f d(\nu * \mu) = \iint f(st) d\nu(s) d\mu(t) = \iint f(st) d\mu(t) d\nu(s) \qquad (2.1)$$

for all $f \in C_0(S)$. The new measure $\nu * \mu$ in $M(S)$ is uniquely identified by Riesz representation theorem. Further, the above equation can be extended to the case where $f \in \mathcal{L}_1(|\nu| * |\mu|)$ (details see [20]).

Wong generalized (2.1) to the case where S is a locally compact semitopological semigroup in [33] using Glicksberg's result in [15].

Theorem 2.1.4 ([15]). *Let f be a separately continuously and bounded function on $S \times T$, where S and T are locally compact spaces. Then for each $\mu \in M(S)$, the map*

$$t \mapsto \int f(s,t) d\mu(s)$$

is continuous.

This theorem enables the convolution of measures in $M(S)$ to be well-defined. Wong [33] manage to extend (2.1) to the case where $f \in \mathcal{L}_1(|\nu| * |\mu|)$.

From now on, we will let S be a locally compact semitopological semigroup. With the convolution of measures in $M(S)$ being defined, we may construct the action of $M(S)$ on its second dual using Arens product. Let $\nu \in M(S)$,

$F \in M(S)^*$, N, $M \in M(S)^{**}$, we define

$$\begin{aligned} \nu \odot F(\mu) &= F(\nu * \mu) \\ M \odot F(\mu) &= M(\mu \odot F) \\ N \odot M(G) &= N(M \odot G) \end{aligned} \tag{2.2}$$

for all $\mu \in M(S)$, $G \in M(S)^*$.

Similarly, let $\nu \in M(S)$, $F \in M(S)^*$, we define $F \odot \nu(\mu) = F(\mu * \nu)$ for all $\mu \in M(S)$.

Definition 2.1.5. Let A be a subspace of $M(S)^*$. We say that A is *topological left invariant*, if $\{\nu \odot F;\ \nu \in M(S), F \in A\} \subset A$. We say A is *topological left introverted*, if A is topological left invariant and $\{M \odot F;\ F \in A, M \in A^*\} \subset A$.

The property of A being topological left introverted allows us to define $N \odot M$ in A^* for all $N, M \in A^*$, while A being topological left invariant enables us to define $\nu \odot F$ in A for all $\nu \in M(S)$, $F \in A$ as (2.2).

Definition 2.1.6. Let A be a topological left invariant subspace of $M(S)^*$ containing $\mathbb{1}$, where $\mathbb{1} \in M(S)^*$ and $\mathbb{1}(\mu) = \mu(S)$ for any $\mu \in M(S)$. We say $M \in A^*$ is a mean on A if $M(\mathbb{1}) = 1 = \|M\|$. We denote by $\mathfrak{M}(A)$ the set of means on A. We say $M \in \mathfrak{M}(A)$ is topological left invariant if we have $M(\nu \odot F) = M(F)$ holds for all $\nu \in P(S)$ (or equivalently for all $\nu \in M(S)$). We abbreviate $\mathfrak{M}(M(S)^*)$ as $\mathfrak{M}(S)$.

2.2 Topological Left Invariant Means

Throughout this section S shall be a locally compact semitopological semigroup. We shall give some characterizations of the existence of topological left invariant means on $M(S)^*$ in this section that have been shown in literature.

Definition 2.2.1. We say S is *topological right stationary* if for any $F \in M(S)^*$, there exists a net μ_α in $M(S)$ such that $F \odot \mu_\alpha \xrightarrow{weak^*} c\mathbb{1}$ for some scalar c.

Theorem 2.2.2 (Wong [29]). *The following are equivalent:*

1. *$M(S)^*$ has a topological left invariant mean.*

2. *$|\nu(S)| = \inf\{\|\nu * \mu\|\,;\ \mu \in P(S)\}$ holds for all $\nu \in M(S)$.*

3. *There is a net μ_α in $P(S)$, such that $\|\nu * \mu_\alpha - \mu_\alpha\| \to 0$.*

4. *S is topological right stationary.*

Wong [32] introduces another characterization of the existence of topological left invariant means on $M(S)^*$ by identifying $M(S)^*$ with a subspace of $\prod_{\mu \in M(S)} \mathcal{L}_\infty(|\mu|)$.

Definition 2.2.3. We say $f \in \prod_{\mu \in M(S)} \mathcal{L}_\infty(|\mu|)$ is a generalised function on S, if it satisfies

i) $\|f\| = \sup_{\mu \in M(S)} \|f_\mu\|_{\mu,\infty} < \infty$.

ii) $f_\nu = f_\mu$ $|\nu|$-a.e., if $|\nu| \ll |\mu|$, where $\nu, \mu \in M(S)$.

We denote by $GL(S)$ the set of all generalised functions. The set $GL(S)$ with the norm $\|\cdot\|$ defined in i) is a Banach space (see [32]). We say $f \in GL(S)$ is non-negative if $f_\mu \geq 0$ μ-a.e., for all $\mu \in M(S)$. It is shown in [32] that an action of $M(S)$ on $GL(S)$ can be properly defined by letting

$$(\nu \odot f)_\mu = \nu \odot f_{\nu*\mu} = \iint f_{\nu*\mu}(st)d\nu(s)d\mu(t)$$

for any $\nu, \mu \in M(S)$, $f \in GL(S)$.

Definition 2.2.4. A functional $m \in GL(S)^*$ is a *mean* on $GL(S)$ if $\|m\| = m(\mathbb{1}) = 1$, where $\mathbb{1} \in GL(S)$, $\mathbb{1}_\mu = 1$ for all $\mu \in M(S)$. Suppose, further, $m(\nu \odot f) = m(f)$ for all $\nu \in P(S), f \in GL(S)$, then m is said to be a *topological left invariant mean on $GL(S)$*.

Theorem 2.2.5 (Wong [32]). *The linear map $T : (GL(S), \|\cdot\|) \to (M(S)^*, \|\cdot\|)$ by $Tf(\mu) = \int f_\mu d\mu$ for any $\mu \in M(S)$ is an isometric order preserving isomorphism. Moreover, T commute with the action of $M(S)$ on $GL(S)$. Therefore, $GL(S)$ has a topological left invariant mean if and only if $M(S)^*$ has one.*

2.3 Support of Topological Left Invariant Means

In the case where S is a discrete semigroup, subsets of S that support left invariant mean are studied by Mitchell [25]. For locally compact semitopological semigroup, [31] and Day [9] introduce topological left thickness and topological left lumpy respectively to generalize the notion of left thickness in discrete cases. Throughout this section, we let S be a locally compact semitopological semigroup.

Definition 2.3.1. (Wong [31]) A Borel subset $T \subset S$ is said to be *topological left thick*, if for each $0 < \epsilon \leq 1$ and each compact subset $K \subset S$, there exists $\mu \in P(S)$, such that $\nu * \mu(T) > 1 - \epsilon$ holds for all $\nu \in P(S)$ with $\nu(S - K) = 0$.

Definition 2.3.2. (Day [9]) A Borel subset $T \subset S$ is called *topological left lumpy*, if for each $0 < \epsilon \leq 1$ and $\nu \in P(S)$, there exists $\mu \in P(S)$, such that $\nu * \mu(T) > 1 - \epsilon$.

Comparison of both definitions as supports of topological left invariant means are shown in the following.

Theorem 2.3.3 (Wong [31]). *Let $T \subset S$ be a Borel subset. Suppose there exists a net μ_α in $P(S)$, such that for each compact subset $K \subset S$, $\|\nu * \mu_\alpha - \mu_\alpha\| \to 0$ uniformly for all $\nu \in P(S)$ with $\nu(S - K) = 0$, then the following are equivalent:*

1. *T is topological left thick.*

2. *There exists a topological left invariant mean* $M \in \mathfrak{M}(S)$ *such that* $M(\chi_T) = 1$*, where* χ_T *is the characteristic functional of* T *in* $M(S)^*$*, i.e.,* $\chi_T(\mu) = \mu(T)$ *for all* $\mu \in M(S)$*.*

Theorem 2.3.4 (Day [9]). *Let* $T \subset S$ *be a Borel subset. If* $M(S)^*$ *has a topological left invariant mean, then the following are equivalent:*

1. T *is topological left lumpy.*

2. *There exists a topological left invariant mean* $M \in \mathfrak{M}(S)$*, such that* $M(\chi_T) = 1$*.*

As it is shown in Theorem 2.2.2, the requirement in Theorem 2.3.3 is stricter than the existence of topological left invariant mean on $M(S)^*$. Hence topological left lumpy is a better characterization of the support of a topological left invariant mean.

However, the two definitions are equivalent when $M_a^l(S) \cap P(S)$ is not empty, where

$$M_a^l(S) = \{\mu \in M(S);\ s \mapsto \delta_s * \mu \text{ is continuous for all } s \in S\}$$

Measures in $M_a^l(S)$ are said to be left absolutely continuous. It is well known that in the case where S is a locally compact group, $M_a^l(S)$ is identified with $L_1(S)$. The equivalence of topological left thickness and topological left lumpiness is implied by the following theorem.

Theorem 2.3.5 (Ghaffari [13]). *Assume that* $M_a^l(S) \cap P(S)$ *is not empty. Then the following are equivalent:*

1. $M(S)^*$ *has a topological left invariant mean.*

2. *There exists a net* $\mu_\alpha \in P(S)$*, such that for each compact subset* K *of* S*, we have* $\|\nu * \mu_\alpha - \mu_\alpha\| \to 0$ *uniformly for any* $\nu \in P(S)$ *with* $\nu(S - K) = 0$*.*

An example of a locally compact semitopological semigroup that does not have a absolutely continuous probability measure is given in the following.

Example 2.3.6 (Wong [30]). Let $S = [0, 1]$ be equipped with the usual topology. Multiplication on S is defined by $st = s$ for any $s, t \in S$. Then S is a compact semitopological semigroup. We denote by $C(S)$ the Banach space of continuous functions on S with supremum norm. Let $\nu \in M(S)$ we have $\delta_s * \nu = \delta_s$ for all $s \in S$, where δ_s is the Dirac measure of s, i.e., $\delta_s(E) = \begin{cases} 1 & s \in E \\ 0 & \text{otherwise} \end{cases}$ for any Borel subset E of X. Suppose there exists $\nu \in M_a^l(S) \cap P(S)$, then the closed unit ball of $C(S)$ is equicontinuous, since for each $f \in C(S)$, we have

$$|f(s) - f(t)| = \left| \int f d\delta_s * \nu - \int f d\delta_t * \nu \right| \leq \|\delta_s * \nu - \delta_t * \nu\|$$

Hence by Arzela-Ascoli theorem (see [11, IV. 6.7]), the closed unit ball of $C(S)$ is compact, which is impossible.

Wong generalizes the definition of topological left lumpy to topological left A-lumpy in [35], where A is an topological left invariant subspace of $M(S)^*$.

Definition 2.3.7 ([35]). Let A be a topological left invariant subspace of $M(S)^*$, $T \subset S$ be a Borel subset. We say T is *topological left A-lumpy*, if for each triple (ϵ, ν, F) where $0 < \epsilon \leq 1$, $\nu \in P(S)$, and $F \in A$ with $\chi_T \leq F \leq \mathbb{1}$, there exists $\mu \in P(S)$, satisfying $\nu * \mu(F) > 1 - \epsilon$.

It is clear that if A contains χ_T, then T being topological left A-lumpy is equivalent with T being topological left lumpy. It is shown in this paper that T "supports" a topological left invariant mean on A when A is a topological left introverted subspace of $M(S)^*$.

Theorem 2.3.8 (Wong [35]). *Let $T \subset S$ be a Borel subset and A be a topological left introverted subspace of $M(S)^*$ containing $\mathbb{1}$. Assume that A has a topological left invariant mean, then the following are equivalent:*

1. T is topological left A-lumpy.

2. There exists a topological left invariant mean $M \in \mathfrak{M}(A)$, such that $M(F) = 1$ for any $F \in A$ with $\chi_T \leq F \leq \mathbb{1}$.

When T is a topological A-lumpy Borel subsemigroup of S for a topological introverted subspace A of $M(S)^*$, Wong proves the following theorem in the same paper.

Theorem 2.3.9 (Wong [35]). *Let $T \subset S$ be a topological A-lumpy Borel subsemigroup of S and A be a topological left introverted subspace of $M(S)^*$ containing $\mathbb{1}$. Then A has a topological left invariant mean if and only if $A|_T = \{F|_T;\ F \in A\} \subset M(T)^*$ has one, where $F|_T(\mu_T) = F(\mu)$ for all $\mu \in M(S)$ and $\mu_T(E) = \mu(E)$ for all Borel subset E of T.*

Chapter 3

Topological S-invariant means on Locally Compact Space

In this chapter, we shall look at the set $M(X)$ of all bounded Borel measures on a locally compact space X which is under separately continuous left action of a locally compact semitopological semigroup S. We shall properly build a separately continuous left action of the measure algebra $M(S)$ on $M(X)$. Further we introduce the left action of $M(S)$ on the second dual of $M(X)$ using Arens product and defined the notion of topological S-invariant mean on a proper subspace A of $M(X)^*$ which is a generalized notion of topological left invariant mean when $X = S$. We shall first give some characterizations of the existence of such means. Then we shall look at a Borel subset T in X that will potentially supports a topological S-invariant mean M on A. We say a subset T of X support a mean M on $M(X)^*$ if $M(\chi_T) = 1$, where $\chi_T(\mu) = \mu(T)$ for all $\mu \in M(X)$. In the last part of this chapter, the relation between the existence of topological R-invariant mean on $M(T)^*$ and on $M(X)^*$ itself will be studied, where R is a locally compact Borel subsemigroup of S (in particular, $R = S$) and T is a locally compact Borel subspace of X which is closed under the left action of R.

3.1 Convolutions of Measures

Let X be an locally compact space. We denote by $BM(X)$ the space of bounded Borel measurable functions on X, by $CB(X)$ the space of continuous bounded functions on X, by $C_0(X)$ the subspace of $CB(X)$ of functions that vanish at infinity and by $C_c(X)$ the subspace of $CB(X)$ of functions that have compact supports. The supremum norm on $CB(X)$ is denoted by $\|\cdot\|$. It is well known that the spaces $(CB(X), \|\cdot\|)$ and $(C_0(X), \|\cdot\|)$ are Banach spaces (see [3]).

We denoted by $M(X)$ the Banach space of all bounded regular Borel measure with total variation norm $\|\cdot\|$, i.e. for $\mu \in M(X)$, $\|\mu\| = |\mu|(X)$. By Riesz representation theorem, the Banach space $(M(X), \|\cdot\|)$ is isomeric isomorphic with the continuous dual of $C_0(S)$ via $<\mu, f>= \int f d\mu$, where $f \in C_0(X)$, $\mu \in M(X)$.

We say $\mu \in M(S)$ is positive ($\mu \geq 0$), if $\mu(E) \geq 0$ for any Borel subset E in X. The cone of positive measures on X is denoted by $M^+(X)$. We denote by $P(X) := \{\mu \in M^+(X);\ \ \|\mu\| = 1\}$ the set of all probability measure on X.

Let $\nu, \mu \in M(X)$, we say that ν is absolutely continuous with respect to μ ($\nu \ll \mu$), if $\nu(F) = 0$ for any compact $|\mu|$-null set F.

Now let S be a locally compact semitopological semigroup, X be a locally compact space, The left action $S \times X \to X$ by $(s, x) \mapsto sx$ is separately continuous. Let $f \in BM(X)$, define $l_s f(x) := f(sx)$, $r_x f(s) = f(sx)$, for any $s \in S$, $x \in X$.

For each Borel subset E of X, $s \in S$ and $x \in X$, we define $s^{-1}E = \{x \in X;\ sx \in E\}$, $Ex^{-1} = \{s \in S;\ sx \in E\}$. We denote by ξ_E the characteristic function of E on X, i.e., $\xi_E = \begin{cases} 1 & s \in E \\ 0 & \text{otherwise} \end{cases}$; by χ_E the characteristic functional of E on $M(X)$, i.e., $\chi_E(\mu) = \mu(E)$ for all $\mu \in M(X)$.

Remark 3.1.1. If $f \in BM(X)$, let $s \in S$, there is no guarantee that $l_s f$ stays in $BM(X)$. Let $f \in CB(X)$. Let x_α be a net converging to x in X and $s \in S$, then $sx_\alpha \to sx$. Hence $|l_s f(x_\alpha) - l_s f(x)| = |f(sx_\alpha) - f(sx)| \to 0$.

Therefore $l_s f \in CB(X) \subset BM(X)$. Similarly, $r_x f \in CB(X)$ for all $x \in X$, $f \in CB(X)$.

It is shown in Glicksberg [15, 1.2] that whenever f is a bounded separately continuous function on $S \times X$, we have $\iint f(sx)d\nu(s)d\mu(x) = \iint f(sx)d\mu(x)d\nu(s)$ for all $\nu \in M(S)$, $\mu \in M(X)$. This result allows us to define convolution of $M(S)$ and $M(X)$. Let $\nu \in M(S)$, $\mu \in M(X)$, we define $\nu * \mu$ by setting

$$\int f d\nu * \mu = \iint f(sx)d\nu(s)d\mu(x) = \iint f(sx)d\mu(x)d\nu(s) \tag{3.1}$$

for all $f \in C_0(X)$. The convolution $\nu * \mu$ is uniquely defined in $M(X)$ by Riesz representation Theorem.

Let $\nu_1, \nu_2 \in M(S)$, $\mu \in M(X)$. By the convolution we constructed above, we have $\int f d\nu_1 * (\nu_2 * \mu) = \iint l_s f(x) d\nu_2 * \mu(x) d\nu_1(s)$, for any $f \in C_0(X)$. However $l_s f$ may not stay in $C_0(X)$ even in the case where $X = S$. An example is given in the following.

Example 3.1.2. Let $S = X = \mathbb{R}$ be equipped with usual topology and the multiplication be defined by $a \cdot b = \max\{a, b\}$ for all $a, b \in S$. Then S is a semitopological semigroup. Let f be a continuous function that is supported on $[a, b]$, we have $(-\infty, b) \subset \text{supp } (l_b f)$. Thus $l_b f \notin C_0(S)$.

In order to define an action of $M(S)$ on $M(X)$, we want to extend (3.1) further to the case where $f \in \mathcal{L}_1(|\nu| * |\mu|)$. Similar to the construction shown in [20, 19.10], we approach our desired result by a series of lemmas.

Definition 3.1.3. Let f be a non-negative real valued function on X. We say f is lower semicontinuous, if for each $x \in X$, $\alpha \in \mathbb{R}$ such that $f(x) > \alpha$, there exists a neighborhood U of x such that $f(y) > \alpha$ for any $y \in U$.

Remark 3.1.4. 1. Note that if f is lower semicontinuous on X, then it is measurable. Since $\{x \in X;\ f(x) > \alpha\}$ is open for any $\alpha \in \mathbb{R}$.

2. Note that for each open subset V of X, its characteristic function ξ_V is

lower semicontinuous.

3. For each lower semicontinuous function f on X, we have $f(x) = \sup\{g(x);\ g \in C_c(X),\ 0 \le g \le f\}$ (see [20, 11.8]).

Lemma 3.1.5. *Let V be an open subset of X, let $\nu \in M^+(S)$, $\mu \in M^+(X)$. Then $x \to \nu(Vx^{-1})$ and $s \to \mu(s^{-1}V)$ are defined everywhere and Borel measurable. Moreover*

$$\int \xi_V d\nu * \mu = \iint \xi_V(sx) d\nu(s) d\mu(x) = \iint \xi_V(sx) d\mu(x) d\nu(s)$$

Proof. Since V is open, by the remark above, ξ_V is lower semicontinuous and

$$\xi_V(x) = \sup\{f(x);\ f \in C_c(X),\ 0 \le f \le \xi_V\}$$

Hence $\xi_{Vx^{-1}}(s) = \xi_V(sx) = \sup\{r_x f(s);\ f \in C_c(X),\ 0 \le f \le \xi_V\}$ holds for all $s \in S$. Since the action of S on X is separately continuous, the set Vx^{-1} is open in S. By [20, 11.13],

$$\nu(Vx^{-1}) = \int \xi_{Vx^{-1}} d\nu = \sup\{\int r_x f d\nu;\ f \in C_c(X),\ 0 \le f \le \xi_V\} \quad (3.2)$$

By Glicksberg [15, 1.2], the function $x \to \int r_x f d\nu$ is defined everywhere and continuous. Hence it is Borel measurable on X. Then by Monotone convergence theorem, $x \mapsto \nu(Vx^{-1})$ is Borel measurable on X. Similarly, we have $s \mapsto \mu(s^{-1}V)$ is Borel measurable on S. Furthermore, by [20, 11.13] again, we have

$$\begin{aligned}
\int \xi_V d\nu * \mu &= \sup\{\int f d\nu * \mu;\ f \in C_c(X),\ 0 \le f \le \xi_V\} \\
&= \sup\{\iint f(sx) d\nu(s) d\mu(x);\ f \in C_c(X),\ 0 \le f \le \xi_V\} \\
&= \iint \sup\{f(sx);\ f \in C_c(X),\ 0 \le f \le \xi_V\}\ d\nu(s) d\mu(x) \\
&= \iint \xi_V(sx)\ d\nu(s) d\mu(x)
\end{aligned}$$

The proof of the other equivalence is similar. □

Lemma 3.1.6. *For each compact subset K of X, the functions $x \to \nu(Kx^{-1})$ and $s \to \mu(s^{-1}K)$ are defined everywhere and Borel measurable for all $\nu \in M(S)^+$, $\mu \in M^+(X)$. Furthermore,*

$$\int \xi_K d\nu * \mu = \iint \xi_K(sx)d\nu(s)d\mu(x) = \iint \xi_K(sx)d\mu(x)d\nu(s)$$

Proof. Since X is Hausdorff, K is closed in X. Hence Kx^{-1} is also closed in X since the action of S on X is separately continuous. Then we have

$$\begin{aligned}\nu(Kx^{-1}) &= \int \xi_K(sx)d\nu(s) = \int (1 - \xi_{X-K})(sx)d\nu(s)\\ &= \nu(S) - \nu((X-K)x^{-1})\end{aligned}$$

Therefore, by Lemma 3.1.5, the function $x \to \nu(Kx^{-1})$ is defined everywhere and Borel measurable. Similarly, we have that $s \to \mu(s^{-1}K)$ is defined everywhere and Borel measurable. Moreover,

$$\begin{aligned}\int \xi_K d\nu * \mu &= \int (\mathbb{1} - \xi_{X-K})d\nu * \mu\\ &= \iint d\nu(s)d\mu(x) - \iint \xi_{X-K}(sx)d\mu(s)d\nu(x) \quad \text{By Lemma 3.1.5}\\ &= \iint \xi_K(sx)d\nu(s)d\mu(x)\end{aligned}$$

We can prove the other equivalence similarly. □

Lemma 3.1.7. *For each σ-compact subset Λ of X, let $\nu \in M^+(S)$, $\mu \in M^+(X)$. The functions $x \to \nu(\Lambda x^{-1})$ and $s \to \mu(s^{-1}\Lambda)$ are defined everywhere and Borel measurable. Furthermore,*

$$\int \xi_\Lambda d\nu * \mu = \iint \xi_\Lambda(sx)d\mu(s)d\nu(x) = \iint \xi_\Lambda(sx)d\nu(x)d\mu(s)$$

Proof. Since Λ is σ-compact, there is a sequence $\{K_n\}$ of compact sets in X, such that $K_{n-1} \subset K_n$ and $\Lambda = \cup K_n$. Then $\Lambda x^{-1} = \cup K_n x^{-1}$ is Borel in S since it is a union of closed sets. Since $\xi_{K_n} \uparrow \xi_\Lambda$, $\xi_{K_n x^{-1}} \uparrow \xi_{\Lambda x^{-1}}$. Then by monotone convergence theorem and Lemma 3.1.6, we have that $x \to \nu(\Lambda x^{-1})$ is Borel measurable. Further,

$$\begin{aligned}\int \xi_\Lambda d\nu * \mu = \lim \int \xi_{K_n} d\nu * \mu = \lim \iint \xi_{K_n}(sx) d\nu(s) d\mu(x) \\ = \iint \xi_\Lambda(sx) d\nu(s) d\mu(x)\end{aligned}$$

The proof of the other equivalence is similar. □

Lemma 3.1.8. *Let $\nu \in M^+(S)$, $\mu \in M^+(X)$. Let N be a $\nu * \mu$-null subset of X, i.e., $\nu * \mu(N) = 0$. Then $\nu(Nx^{-1}) = 0$ for μ-almost all x. and $\mu(s^{-1}N) = 0$ ν-a.e..*

Proof. By the regularity of $\nu * \mu$ in $M(X)$ and Lemma 3.1.5,

$$\begin{aligned}0 = \nu * \mu(N) &= \inf\{\nu * \mu(V);\ N \subset V, V \text{ is open in } X\} \\ &= \inf\{\iint \xi_V(sx) d\nu(s) d\mu(x);\ N \subset V, V \text{ is open in } X\} \\ &= \inf\{\int \nu(Vx^{-1}) d\mu(x);\ N \subset V, V \text{ is open in } X\} \\ &\geq \int \nu(Nx^{-1}) d\mu(x)\end{aligned}$$

Since $\mu \in M^+(X)$, $\nu(Nx^{-1}) = 0$ μ-a.e.. Similarly we can prove that $\mu(s^{-1}N) = 0$ ν-a.e.. □

Lemma 3.1.9. *Let $\nu \in M^+(S)$ $\mu \in M^+(X)$ and A be a $\nu * \mu$-measurable subset of X. Then $x \to \nu(Ax^{-1}) \in \mathcal{L}_1(\mu)$ is defined μ-almost everywhere while $s \to \mu(s^{-1}A) \in \mathcal{L}_1(\nu)$ is defined ν-almost everywhere. Moreover,*

$$\int \xi_A d\nu * \mu = \iint \xi_A(sx) d\nu(s) d\mu(x) = \iint \xi_A(sx) d\mu(x) d\nu(s)$$

Proof. By [20, 11.32], there exists a σ-compact subset Λ of X and a $\nu * \mu$-null subset N of X, such that $\Lambda \cap N = \emptyset$ and $A = \Lambda \cup N$. By Lemma 3.1.7, we have that Λx^{-1} is Borel for all $x \in X$. By Lemma 3.1.8, Nx^{-1} is defined

ν-almost everywhere and is ν-measurable. Thus Ax^{-1} is defined ν-almost everywhere and is ν-measurable. Moreover, we have $\nu(Ax^{-1}) = \nu(\Lambda x^{-1}) + \nu(Nx^{-1}) = \nu(\Lambda x^{-1}) \in \mathcal{L}_1(\mu)$, since $\int \nu(\Lambda x^{-1})d\mu = \iint \xi_\Lambda d\nu * \mu \leq \infty$ by Lemma 3.1.7.

Furthermore, by Lemma 3.1.7 and 3.1.8, we have

$$\begin{aligned}\int \xi_A d\nu * \mu &= \iint \xi_\Lambda(sx)d\nu(s)d\mu(x) + \nu * \mu(N)\\ &= \int \nu(\Lambda x^{-1})d\mu(x) = \int \nu(Ax^{-1})d\mu(x)\\ &= \iint \xi_A(sx)d\nu(s)d\mu(x)\end{aligned}$$

The rest can be proved similarly. □

Lemma 3.1.10. *Let $\nu \in M^+(S)$, $\mu \in M^+(X)$. Then for each $f \in \mathcal{L}_1(\nu * \mu)$, $x \to \int f(sx)d\nu(s) \in \mathcal{L}_1(\mu)$ is defined μ-almost everywhere while $s \to \int f(sx)d\mu(x) \in \mathcal{L}_1(\nu)$ is defined ν-almost everywhere. Moreover,*

$$\int f d\nu * \mu = \iint f(sx)d\nu(s)d\mu(x) = \iint f(sx)d\mu(x)d\nu(s)$$

Proof. Define $f_+(x) := max\{f(x), 0\}$, $f_-(x) := -min\{f(x), 0\}$. Since $f \in \mathcal{L}_1(\nu * \mu)$, both f_+ and f_- are nonnegative and $f_+, f_- \in \mathcal{L}_1(\nu * \mu)$. Without loss of generality, it suffices to prove the result for the case where $f \in \mathcal{L}_1(\nu * \mu)$ and f is non-negative.

Let $A_{k,n} := \{x \in X;\ \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\}$ $(k, n \in \mathbb{N})$. Then $A_{k,n}$ is $\nu * \mu$-measurable since f is $\nu * \mu$-measurable. Thus $f_n := \sum_{k=1}^{n2^n-1} \frac{k}{2^n} \xi_{A_{n,k}}$, $n \in \mathbb{N}$ is an increasing sequence of $\nu * \mu$-measurable functions and $f_n \uparrow f$ pointwisely. Since

$$\int f_n(sx)d\nu(s) = \sum_{k=1}^{n2^n-1} \frac{k}{2^n} \int \xi_{A_{k,n}}(sx)d\nu(s) = \sum_{k=1}^{n2^n-1} \frac{k}{2^n} \nu(A_{k,n}x^{-1})$$

by monotone convergence theorem and Lemma 3.1.9, $\int f_n(sx)d\nu(s) \uparrow \int f(sx)d\nu(s)$ is defined μ-almost everywhere and is μ-measurable.

Moreover, by Lemma 3.1.9

$$\begin{aligned}
\int f d\nu * \mu &= \lim_n \int f_n d\nu * \mu \\
&= \lim_n \sum_{k=1}^{n2^n-1} \frac{k}{2^n} \int \xi_{A_{k,n}} d\nu * \mu \\
&= \lim_n \sum_{i=k=1}^{n2^n-1} \frac{k}{2^n} \iint \xi_{A_{n,k}}(sx) d\nu(s)\mu(x) \\
&= \lim_n \iint f_n(sx) d\nu(s) d\mu(x) = \iint f(sx) d\nu(s) d\mu(x)
\end{aligned}$$

Since $f \in \mathcal{L}_1(\nu * \mu)$, we have that $\int f(sx)d\nu(s) \in \mathcal{L}_1(\mu)$. The rest of this lemma can then be proved similarly. □

Lemma 3.1.11. *Let $\mu, \sigma \in M^+(X)$ such that $\mu \ll \sigma$, then $\nu * \mu \ll \nu * \sigma$ for all $\nu \in M^+(S)$.*

Proof. Let F be a compact subset of X such that $\nu*\sigma(F) = 0$. By Lemma 3.1.8, we have $\sigma(s^{-1}F) = 0$ ν-a.e. Since $\mu \ll \sigma$, we have $\mu(s^{-1}F) = 0$ ν-a.e. Hence by Lemma 3.1.9, $\nu * \mu(F) = \int \mu(s^{-1}F)d\nu(s) = 0$. Therefore, by [20, 14.19], $\nu * \mu \ll \nu * \sigma$. □

Corollary 3.1.12. *Let $\mu, \sigma \in M(X)$ such that $\mu \ll \sigma$, then $|\nu|*|\mu| \ll |\nu|*|\sigma|$ for any $\nu \in M(S)$.*

Lemma 3.1.13. *Let $\nu, \theta \in M(S)$ such that $\nu \ll \theta$, then $|\nu| * |\mu| \ll |\sigma| * |\mu|$ for any $\mu \in M(X)$.*

Proof. Similar to Lemma 3.1.11 □

Below is an interesting remark on the relation between absolute continuity of measures and measurability of functions on X.

Proposition 3.1.14. *Let X be a locally compact space. Let $\sigma, \mu \in M(X)$, such that $\sigma \ll \mu$. Then any μ-measurable set is σ-measurable. In particular, any μ-measurable function is σ-measurable*

Proof. Let E be a μ-measurable subset of X, then it is also $|\mu|$-measurable. By [20, 11.32], we have $E = \Lambda \cup N$, where Λ is a σ-compact subset of X and $|\mu|(N) = 0$. Since $\sigma \ll \mu$, we have $\sigma(N) = 0$ which implies N is σ-measurable. Therefore E is σ-measurable.

Let f be a μ-measurable function on X, then $\{x \in X;\ f(x) < a\}$ is μ-measurable for all $a \in \mathbb{R}$. Consequently, $\{x \in X;\ f(x) < a\}$ is σ-measurable for all $a \in \mathbb{R}$ from the above argument. Therefore f is σ-measurable. $\square$

For each $\mu \in M(X)$, let $\mu^+ = (|\mu| + \mu)/2$, $\mu^- = (|\mu| - \mu)/2$. Then $\mu^+, \mu^- \geq 0$, $\mu^+, \mu^- \ll \mu$.

We now come to prove our first main theorem.

Theorem 3.1.15. *Let S be a locally compact group, X be a locally compact space. Assume the left action of S acting on X is separately continuous. Let $\nu \in M(S)$, $\mu \in M(X)$, $f \in \mathcal{L}_1(|\nu| * |\mu|)$. Then $x \to \int f(sx)d\nu(s) \in \mathcal{L}_1(|\mu|)$, $s \to \int f(sx)d\mu(x) \in \mathcal{L}_1(|\nu|)$. Moreover,*

$$\int f d\nu * \mu = \iint f(sx)d\nu(s)d\mu(x) = \iint f(sx)d\mu(x)d\nu(s)$$

Proof. Since $|\nu| * |\mu| = (\nu^+ + \nu^-) * (\mu^+ + \mu^-)$, by Lemma 3.1.11 and 3.1.13, we have $\nu^i * \mu^j \ll |\nu| * \mu^j \ll |\nu| * |\mu|$, where $i, j \in \{+, -\}$. Then by Proposition 3.1.14, f is $\nu^i * \mu^j$-measurable ($i, j \in \{+, -\}$). In addition,

$$\int |f|\, d\nu^i * \mu^j \leq \sum_{i,j \in \{+,-\}} \int |f|\, d\nu^i * \mu^j = \int |f|\, d\,|\nu| * |\mu| < \infty$$

Thus $f \in \mathcal{L}_1(\nu^i * \mu^j)$. Hence $x \to \int f(sx)d\nu^i(s) \in \mathcal{L}_1(\mu^j)$ by Lemma 3.1.10 for all $i, j \in \{+, -\}$. Therefore, $\int f(sx)d\nu(s) \in \mathcal{L}_1(|\mu|)$.

Furthermore, by Lemma 3.1.10

$$
\begin{aligned}
\int f d\mu * d\nu &= \int f d(\mu^+ - \mu^-) * (\mu^+ - \mu^-) \\
&= \int f d\nu^+ * \mu^+ - \int f d\nu^+ * \mu^- - \int f d\nu^- * \mu^+ + \int f d\nu^- * \mu^- \\
&= \iint f(sx) d\nu^+(s) d\mu^+(x) - \iint f(sx) d\nu^+(s) * \mu^-(x) \\
&- \iint f(sx)\nu^-(s) d\mu^+(x) + \int f(sx)\nu^-(s) d\mu^-(x) \\
&= \iint f(sx) d\nu(s) d\mu(x)
\end{aligned}
$$

□

Thus we have extended (3.1) to $\mathcal{L}_1(|\nu| * |\mu|)$ where $\nu \in M(S)$, $\mu \in M(X)$. Even in the case where $X = S$, our result is more general than Wong [33], in which (3.1) is only shown for $\nu, \mu \in M^+(S)$.

The following corollary of Theorem 3.1.15 shows that the action of $M(S)$ on $M(X)$ defined by convolution is proper.

Corollary 3.1.16. *For any $\nu_1, \nu_2 \in M(S)$, $\mu \in M(X)$, we have $\nu_1 * (\nu_2 * \mu) = (\nu_1 * \nu_2) * \mu$.*

Proof. By Riesz representation theorem, it suffices to prove $\int f d\nu_1 * (\nu_2 * \mu) = \int f d(\nu_1 * \nu_2) * \mu$ for any $f \in C_0(X)$.

Let $f \in C_0(X)$, $s \in S$, $x \in X$. Then $r_x f \in CB(S) \subset \mathcal{L}_1(|\theta|)$ for all $\theta \in M(S)$ and $l_s f \in CB(X) \subset \mathcal{L}_1(|\sigma|)$ for all $\sigma \in M(X)$. Thus

$$
\begin{aligned}
\int f d(\nu_1 * \nu_2) * \mu &= \iint r_x f(s) d\nu_1 * \nu_2(s) d\mu(x) \\
&= \iiint r_x f(s_1 s_2) d\nu_1(s_1) d\nu_2(s_2) d\mu(x)
\end{aligned}
$$

On the other hand, by Theorem 3.1.15, we have $x \to \int f(s_1 x) d\nu_1(s_1) \in$

$\mathcal{L}_1(|\nu_2| * |\mu|)$. Hence,

$$\int f d\nu_1 * (\nu_2 * \mu) = \int \Big(\int f(s_1 x) d\nu_1(s_1) \Big) d\nu_2 * \mu(x)$$
$$= \iiint f(s_1 s_2 x) d\nu_1(s_1) d\nu_2(s_2) d\mu(x)$$

Therefore, $\int f d\nu_1 * (\nu_2 * \mu) = \int f d(\nu_1 * \nu_2) * \mu$ for any $f \in C_0(X)$. $\square$

Further, it is clear that the action of $M(S)$ on $M(X)$ is jointly continuous when $M(S)$ and $M(X)$ are equipped with the norm topology.

Before we go on to the next section, we shall give an interesting corollary of Lemma 3.1.9 in the following. Let $\mu \in M(X)$, we denote by supp (μ) the support of μ, i.e.,

$$\text{supp } (\mu) = \{x \in X;\ |\mu|(N_x) > 0, \text{ for any open neighborhood } N_x \text{ of } x\}$$

Corollary 3.1.17. *For any $\nu \in P(S)$, $\mu \in P(X)$, supp $(\nu * \mu) = \overline{\text{supp } (\nu) \text{supp } (\mu)}$.*

Proof. Let $A =$ supp (ν), $B =$ supp (μ) and $C = \overline{AB}$. $C \subset$ supp $(\nu * \mu)$ is clear.

Since C is closed, C is $\nu * \mu$-measurable. Then by Lemma 3.1.9, we have

$$1 \geq \nu * \mu(C) = \iint \xi_C(sx) d\nu(s) d\mu(x) \geq \int \xi_{AB}(sx) d\nu(s) d\mu(x)$$
$$\geq \int \xi_A(s) \xi_B(x) d\nu(s) d\mu(x) = \nu(A)\mu(B) = 1$$

This implies supp $(\nu * \mu) \subset C$. Hence $C =$ supp $(\nu * \mu)$. $\square$

3.2 Topological S-invariant Means

In this section, we let S be a locally compact semitopological semigroup, X be a locally compact space and we assume the left action of S on X is separately continuous.

As we have shown in the end of last section, the action of $M(S)$ on $M(X)$ is jointly continuous which naturally introduces an separately continuous action of $M(S)$ on $M(X)^*$. Let $\nu \in M(S)$, $F \in M(X)^*$, define $\nu \odot F(\mu) = F(\nu * \mu)$ for all $\mu \in M(X)$. Then $\nu \odot F \in M(X)^*$ and $\|\nu \odot F\| \leq \|\nu\| \, \|F\|$. Similarly, let $F \in M(X)^*$, $\mu \in M(X)$, define $F \odot \mu(\nu) = F(\nu * \mu)$ for all $\nu \in M(S)$. Then $F \odot \mu \in M(S)^*$ and $\|F \odot \mu\| \leq \|F\| \, \|\mu\|$.

Let A be a subspace of $M(X)^*$, we say A is *topological S-invariant*, if $\{\nu \odot F;\ \nu \in M(S), F \in A\} \subset A$. We say that A is *S-invariant*, if $\{\delta_s \odot F;\ s \in S, F \in A\} \subset A$ where δ_s is the Dirac measure of s.

Remark 3.2.1. The definition of topological S-invariant subspace A of $M(X)^*$ is equivalent as requiring $\nu \odot F \in A$ for any $\nu \in P(S)$, $F \in A$, since $P(S)$ spans $M(S)$.

Let A be a topological S-invariant subspace of $M(X)^*$. For each $M \in A^*$, $F \in A^*$, we define

$$M \odot F(\nu) = M(\nu \odot F) \qquad (\nu \in M(S))$$

We will write $M_L F$ for $M \odot F$ in this thesis. It is easy to check that $\|M_L F\| \leq \|M\| \, \|F\|$ and $M_L F \in M(S)^*$. Further, using Arens product, for each $N \in M(S)^{**}$, $M \in A^*$, we define

$$N \odot M(F) = N(M_L F) \quad (F \in A)$$

Thus $N \odot M \in M(X)^{**}$, $\|N \odot M\| \leq \|N\| \, \|M\|$.

Let $N_\alpha \xrightarrow{weak^*} N \in M(S)^{**}$ be a net in $M(S)^{**}$. Then we have

$$N_\alpha \odot M(F) = N_\alpha(M_L F) \to N(M_L F) = N \odot M(F) \qquad (M \in A^*, F \in A)$$

Thus the convolution of $M(S)^{**}$ and A^* is weak* continuous on the first variable.

Let $M_\beta \xrightarrow{weak^*} M \in A^*$ be a net in A^*. We have

$$M_{\beta_L}F(\nu) = M_\beta(\nu \odot F) \to M(\nu \odot F) = M_L F(\nu) \quad (\nu \in M(S), F \in A)$$

which is equivalent as $Q\nu \odot M_\beta \xrightarrow{weak^*} Q\nu \odot M$, where Q is the natural embedding of $M(S)$ into $M(S)^{**}$

Hence the action of $M(S)$ on $M(X)^*$ is separately continuous. However, it is not true in general that $N \odot M_\beta \xrightarrow{weak^*} N \odot M$ for any $N \in M(S)^{**}$ which is equivalent as requiring $M_{\beta_L}F \xrightarrow{weak} M_L F$ for any $F \in A$. This further requires that weak and weak* topology coincide on the set $\{M_L F;\ M \in A^*, F \in A\}$. In the end of this chapter, we will introduce a topological S-invariant subspace A of $M(X)^*$, such that the action of $M(S)^{**}$ on A^* is separately continuous.

Now let A be a subspace of $M(X)^*$ containing $\mathbb{1}$, where $\mathbb{1} \in M(X)^*$ and $\mathbb{1}(\mu) = \mu(S)$ for any $\mu \in M(X)$. A mean M on A is an element M in A^* such that $M(\mathbb{1}) = \|M\| = 1$. We denote by $\mathfrak{M}(A)$ the set of means on A. In the case where $A = M(X)^*$, we will abbreviate $\mathfrak{M}(M(X)^*)$ as $\mathfrak{M}(X)$.

Remark 3.2.2. Let N be a mean on $M(S)^*$, M be a mean on A. Then $M_L\mathbb{1}(\nu) = M(\nu \odot \mathbb{1}) = M(\mathbb{1}) = 1$ for all $\nu \in M(S)$. Hence

$$N \odot M(\mathbb{1}) = N(M_L\mathbb{1}) = N(\mathbb{1}) = 1 \leq \|N \odot M\| \leq \|N\|\,\|M\| = 1$$

Thus $N \odot M$ is a mean on A.

Proposition 3.2.3. *Let A be a subspace of $M(X)^*$ containing $\mathbb{1}$. Then $QP(X)$ is weak* dense in $\mathfrak{M}(A)$.*

Proof. Note that $QP(X)$ is convex, hence $\overline{QP(X)}^{weak^*}$ is convex. Assume $\overline{QP(X)}^{weak^*} \subsetneq \mathfrak{M}(A)$, let $M \in \mathfrak{M}(A) - \overline{QP(X)}^{weak^*}$. Since $(A^*, weak^*)$ is locally convex, by Hahn Banach separation theorem, there exists $F \in (A^*, weak^*)^* = A$, such that $M(F) > a$ and $N(F) \leq a$ whenever $N \in \overline{QP(X)}^{weak^*}$. Let $b = \sup\{F(\mu);\ \mu \in P(X)\}$. Then there exists a net $\{\mu_\alpha\} \in P(X)$, such that $Q\mu_\alpha(F) = F(\mu_\alpha) \to b$. Thus we have $b \leq a$. Hence

$M(F) > a \geq b = \sup\{F(\mu);\ \mu \in P(X)\}$, which implies $\|M\| > 1$. This contradicts with the fact that M is a mean on A. Therefore, $\overline{QP(X)}^{weak^*} = \mathfrak{M}(A)$. $\square$

Let A be a [topological] S-invariant subspace of $M(X)^*$ containing $\mathbb{1}$. We say $M \in \mathfrak{M}(A)$ is [topological] S-invariant if $M(\delta_s \odot F) = M(F)$, for any $s \in S$, $F \in A$ [$M(\nu \odot F) = M(F)$, for any $\nu \in P(S)$, $F \in A$].

Theorem 3.2.4. *Let A be a [topological] S-invariant subspace of $M(X)^*$ containing $\mathbb{1}$. If $M(S)^*$ has a [topological] left invariant mean, then A also has a [topological] S-invariant mean.*

Proof. We only prove for the case where A is a topological S-invariant subspace of $M(X)^*$ containing $\mathbb{1}$. The S-invariant case is similar.

Let $M \in \mathfrak{M}(A)$, N be the topological left invariant mean on $M(S)^*$, then $N \odot M$ is a mean on A by the above remark. Moreover, let $\nu \in P(S)$, we have $N \odot M(\nu \odot F) = N(M_L(\nu \odot F)) = N(\nu \odot M_L F) = N \odot M(F)$. Therefore, $N \odot M$ is a topological S-invariant mean on A. $\square$

In the following, we let A be a topological S-invariant subspace of $M(X)^*$ containing $\mathbb{1}$. We shall show a few characterizations of A having a topological S-invariant mean.

Definition 3.2.5. Let $F \in A$, $Z(F) := \overline{\{F \odot \mu;\ \mu \in P(X)\}}^{weak^*}$, $K(F) := \{c \in \mathbb{R};\ c\mathbb{1} \in Z(F)\}$. We say A is *topological X-stationary* if $K(F)$ is not empty for all $F \in A$.

Lemma 3.2.6. *For each F in A, we have*

i) *$Z(F)$ is weak* compact and $\|G\| \leq \|F\|$ for any $G \in Z(F)$.*

ii) *$Z(F) = \{M_L F;\ M \in \mathfrak{M}(A)\}$.*

iii) *$Z(cF) = cZ(F)$, $K(cF) = cK(F)$, for any $c \in \mathbb{R}$.*

iv) *$Z(c\mathbb{1} + F) = c + Z(F)$, $K(c\mathbb{1} + F) = c + K(F)$ for any $c \in \mathbb{R}$.*

v) $Z(F+G) \subset Z(F) + Z(G)$ for any $F, G \in A$.

vi) If A is topological X-stationary, then $K(F - \nu \odot F) = \{0\}$ for any $\nu \in P(S)$.

vii) If M is topological S-invariant mean on A, then $M(F) \in K(F)$.

Proof. i) Let U be the norm closed unit ball of $M(S)^*$. By Banach-Alaoglu theorem, U is weak* compact. Hence it is weak* closed. Since $\|F \odot \mu\| \leq \|F\| \|\mu\| = \|F\|$ for any $\mu \in P(X)$, we have $Z(F) \subset \overline{\|F\| \, U}^{weak^*} = \|F\| \, U$. Thus for each $G \in Z(F)$, $\|G\| \leq \|F\|$ and $Z(F)$ is weak* compact since it is weak* closed in $\|F\| \, U$.

ii) Let M_α be a net in $\mathfrak{M}(A)$. Since $\mathfrak{M}(A)$ is weak* compact, passing through a subnet if necessary, there exists $M \in \mathfrak{M}(A)$, such that $M_\alpha \xrightarrow{weak^*} M$. Then $M_{\alpha L} F(\nu) = M_\alpha(\nu \odot F) \to M(\nu \odot F) = M_L F(\nu)$ for all $\nu \in M(S)$. Thus $M_{\alpha L} F \xrightarrow{weak^*} M_L F$. Therefore $\{M_L F;\ M \in \mathfrak{M}(A)\}$ is weak* compact. Hence,

$$
\begin{aligned}
Z(F) &= \overline{\{F \odot \mu;\ \mu \in P(X)\}}^{weak^*} = \overline{\{Q\mu_L F;\ \mu \in P(X)\}}^{weak^*} \\
&\subset \overline{\{M_L F;\ M \in \mathfrak{M}(A)\}}^{weak^*} = \{M_L F;\ M \in \mathfrak{M}(A)\}
\end{aligned}
$$

Conversely, since $QP(X)$ is weak* dense in $\mathfrak{M}(A)$, then for each $M \in \mathfrak{M}(A)$, there exists a net σ_β in $P(X)$, such that $Q\sigma_\beta \xrightarrow{weak^*} M$. Hence $F \odot \sigma_\beta(\nu) = Q\sigma_\beta(\nu \odot F) \to M_L F(\nu)$ for any $\nu \in M(S)$ which implies $M_L F \in Z(F)$. Therefore, $Z(F) = \{M_L F;\ M \in \mathfrak{M}(A)\}$.

iii) By definition.

iv)$Z(c\mathbb{1} + F) = \{M_L(c\mathbb{1} + F);\ M \in \mathfrak{M}(A)\} = \{c + M_L(F);\ M \in \mathfrak{M}(A)\} = c + Z(F)$. In particular, $K(c\mathbb{1} + F) = c + K(F)$.

v) Let $F, G \in A$, $\mu \in P(S)$. Since $(F+G) \odot \mu = F \odot \mu + G \odot \mu$

$$
Z(F+G) \subset \overline{\{F \odot \mu;\ \mu \in P(X)\} + \{G \odot \mu;\ \mu \in P(X)\}}^{weak^*}
$$

By [11, 415, Lemma 4], since $Z(F)$ is weak* compact and convex

$$\overline{\{F\odot\mu;\ \mu\in P(X)\}+\{G\odot\mu;\ \mu\in P(X)\}}^{weak^*}=Z(F)+Z(G)$$

Hence $Z(F+G)\subset Z(F)+Z(G)$.

vi) Since A is topological X-stationary, by ii) if there exists $c\neq 0$, $M\in\mathfrak{M}(A)$, such that $M_L(F-\nu\odot F)=c\mathbb{1}$. Then $kc=\sum_{i=1}^{k}M_LF(\nu^i)-\nu\odot M_LF(\nu^i)=M_LF(\nu)-M_LF(\nu^{k+1})\leq 2\|F\|$ holds for any $k\in\mathbb{N}$, which contradict the fact that $\|F\|\leq\infty$.

vii) If M is a topological S-invariant mean on A, then $M_LF=M(F)\mathbb{1}\in Z(F)$, thus $M(F)\in K(F)$. □

Theorem 3.2.7. *Let $H=\{\nu\odot F-F;\ F\in A,\nu\in M(S)\}$. Then the following are equivalent:*

i) A has a topological S-invariant mean.

ii) There exist a net $\{\mu_\alpha\}$ in $P(X)$, such that $\nu*\mu_\alpha-\mu_\alpha\xrightarrow{\sigma(M(X),A)}0$ for all $\nu\in P(S)$.

iii) A is topological X-stationary and there exists a sublinear functional p on A, such that $p(F)\in K(F)$, for all $F\in A$.

iv) For any $G\in H$, $0\in K(G)$

v) For any $G\in H$, $\sup\{G(\mu);\ \mu\in M(X)\}\geq 0$.

vi) $\inf\{\|\mathbb{1}-G\|;\ G\in H\}=1$.

Proof. i)⇒ ii) Let M be a topological S-invariant mean on A. Since $QP(X)$ is weak* dense in $\mathfrak{M}(A)$, there exists a net $\{\mu_\alpha\}$ in $P(X)$, such that $Q\mu_\alpha\xrightarrow{weak^*}M$. Thus

$$F(\nu*\mu_\alpha)-F(\mu_\alpha)=Q\mu_\alpha(\nu\odot F-F)\to M(\nu\odot F-F)=0$$

holds for all $\nu\in P(S)$, $F\in A$. Therefore, $\nu*\mu_\alpha-\mu_\alpha\xrightarrow{\sigma(M(X),A)}0$ for any $\nu\in P(S)$.

ii)$\Rightarrow$ iii) Since $\mathfrak{M}(A)$ is weak* compact, passing through a subnet if necessary, there exists $M \in \mathfrak{M}(A)$, such that $Q\mu_\alpha \xrightarrow{weak^*} M$. Then for any $F \in A$, $\nu \in P(S)$, we have

$$F(\nu * \mu_\alpha - \mu_\alpha) = Q\mu_\alpha(\nu \odot F - F) \to M(\nu \odot F - F) = 0$$

Hence M is a topological S-invariant mean on A. Let $p = M$, p is clearly a sublinear functional on A. Moreover, by Lemma 3.2.6 vii) $p(F) = M(F) \in K(F)$ for all $F \in A$.

iii) $\Rightarrow$ i) Since $p(0) \in K(0) = \{0\}$

$$0 = p(F - F) \leq p(F) + p(-F) \Rightarrow -p(-F) \leq p(F)$$

holds for all $F \in A$.

Fix $G \in A$, by [3, Corollary 6.6], there exists $M' \in (\text{span}\{G\})^*$, such that $-p(-G) \leq M'(G) \leq p(G)$. Let $c \in \mathbb{R}$, if $c \geq 0$, we have

$$-cp(-G) \leq M'(G) \leq cp(G) = p(cG)$$

If $c < 0$, we have

$$cp(G) \leq M'(G) \leq -cp(-G) = p(cG)$$

Thus M' is dominated by p on the linear span of G. Therefore by Hahn-Banach extension theorem, there exists $M \in A^*$, such that $M(F) \leq p(F)$ for any $F \in A$. Hence

$$-\|F\| \leq -p(-F) \leq -M(-F) = M(F) \leq p(F) \leq \|F\|$$

for any $F \in A$. In particular, we have

$$1 = -p(-\mathbb{1}) \leq M(\mathbb{1}) \leq p(\mathbb{1}) = 1$$

By Lemma 3.2.6 vi), we have

$$0 = -p(\nu \odot F - F) \leq M(F - \nu \odot F) \leq p(F - \nu \odot F) = 0$$

holds for all $\nu \in p(S)$, $F \in A$. Thus M is a topological S-invariant mean on A.

iii)$\Rightarrow$ iv) Let $G \in H$, then $G = \sum_{i=1}^{n} a_i(F_i - \nu_i \odot F_i)$, where $\nu_i \in P(S)$, $F_i \in A$, $a_i \in \mathbb{R}$. By Lemma 3.2.6 iii), we have

$$p(a_i(F_i - \nu_i \odot F_i)) \in K(a_i(F_i - \nu_i \odot F_i)) = a_i K(F_i - \nu_i \odot F_i) = \{0\}$$

Hence

$$0 = -\sum_{i=1}^{n} p(-a_i(F_i - \nu_i \odot F_i)) \leq -p(-G) \leq p(G) \leq \sum_{i=1}^{n} p(a_i(F_i - \nu_i \odot F_i)) = 0$$

Thus $0 = p(G) \in K(G)$.

iv)$\Rightarrow$ v) By Lemma 3.2.6 ii), there exists $M \in \mathfrak{M}(A)$, such that $M_L F$ vanishes on $M(S)$. Then let $\nu \in P(S)$, we have

$$\begin{aligned} 0 = M(\nu \odot G) &\leq \sup\{\nu \odot G(\mu);\ \mu \in P(X)\} \\ &\leq \sup\{G(\mu);\ \mu \in P(X)\} \end{aligned}$$

v) $\Rightarrow$ vi) For each $G \in H$, $-G \in H$. Then by ii)

$$\sup\{-G(\mu);\ \mu \in P(X)\} \geq 0 \Leftrightarrow \inf\{G(\mu);\ \mu \in P(X)\} \leq 0$$

Thus for any $\epsilon > 0$, there exists $\mu \in M(X)$, such that $G(\mu) \leq \epsilon$. Hence $1 - \epsilon \leq 1 - G(\mu) \leq \|\mathbb{1} - G\|$. Therefore

$$1 \leq \inf\{\|\mathbb{1} - G\|\,;\ G \in H\} \leq \|1 - 0\| = 1$$

vi) $\Rightarrow$ i) Since $\inf\{\|\mathbb{1} - G\|\,;\ G \in H\} = 1$, then $\mathbb{1} \notin \overline{H}$. By [20, B.15], there exists $M \in A^*$, such that $M(G) = 0$ for any $G \in H$, $M(\mathbb{1}) = 1$ and $\|M\| = 1$. In particular, $M(F - \nu \odot F) = 0$ holds for all $\nu \in P(S), F \in A$. Therefore M is a topological S-invariant mean on A. $\square$

A local characterization of the existence of topological S-invariant means is given in the following.

Theorem 3.2.8. *Let A be a topological S-invariant subspace of $M(X)^*$ containing $\mathbb{1}$. Then A has a topological S-invariant mean if and only if for any finite subset $\mathcal{F}$ in A and finite subset Θ in $P(S)$, there exists a mean M on A, such that $M(\nu \odot F) = M(F)$ for any $\nu \in \Theta$, $F \in \mathcal{F}$.*

Proof. Let $\alpha = \{\mathcal{F}, \Theta\}$ be a directed set, where $\mathcal{F}$ is a finite subset in A, Θ is a finite subset in $P(S)$. We say $\{\mathcal{F}_0, \Theta_0\} = \alpha_0 \leq \alpha = \{\mathcal{F}, \Theta\}$, if $\mathcal{F}_0 \subset \mathcal{F}$ and $\Theta_0 \subset \Theta$. Let $\{M_\alpha\}$ be a net in $\mathfrak{M}(A)$, such that for each $\alpha = \{\Theta, \mathcal{F}\}$, $M_\alpha(F - \nu \odot F) = 0$ whenever $F \in \mathcal{F}$, $\nu \in \Theta$. Since $\mathfrak{M}(A)$ is weak* compact, passing through its subnet if necessary, there exists $M \in \mathfrak{M}(A)$, such that $M_\alpha \xrightarrow{weak^*} M$. Consequently, $M(\nu \odot F) = M(F)$, for all $F \in A$, $\nu \in P(S)$. $\square$

An interesting consequence of $M(X)^*$ being topological X-stationary is proved in the following.

Proposition 3.2.9. *Assume $M(X)^*$ is topological X-stationary, then for each $\nu \in M(S)$, $|\nu(S)| = \inf\{\|\nu * \mu\|;\ \mu \in P(X)\}$.*

Proof. Let $\nu \in M(S)$, $\mu \in P(X)$, then $|\nu(S)| = |\nu(S)|\,|\mu(X)| = |\nu * \mu(X)| \leq \|\nu * \mu\|$. Let $a = \inf\{\|\nu * \mu\|;\ \mu \in P(X)\}$, then $|\nu(S)| \leq a$. Let $I_\nu := \overline{\{\nu * \mu;\ \mu \in P(X)\}}^{\|\cdot\|}$. Since I_ν is convex, by Hahn Banach extension theorem, there exists $F \in M(X)^*$, such that $\|F\| = 1$, $|F(\sigma)| \geq a$ for any $\sigma \in I_\nu$. In particular $|F \odot \mu(\nu)| \geq a$ for any $\mu \in P(X)$. Hence $|M_L F(\nu)| \geq a$ for all $M \in \mathfrak{M}(X)$. Since $M(X)^*$ is topological X-stationary, let $c \in K(F)$, then

$$a \leq |c\mathbb{1}(\nu)| = |c|\,|\nu(S)| \leq |c|\,a \leq \|F\|\,a = a$$

Therefore $|c| = 1$ and $|\nu(S)| = a$. $\square$

We denote by $mwp(S, X)$ the subspace of all F in $M(X)^*$ such that $\{F \odot \mu, \mu \in P(X)\}$ is relatively compact in weak topology of $M(S)^*$. Such functions are called almost periodic functions on $M(X)$.

Proposition 3.2.10. *Let $A = mwp(S, X)$, then*

i) A is topological S-invariant subspace of $M(X)^*$ containing $\mathbb{1}$.

ii) $\overline{\{F \odot \mu, \mu \in P(X)\}}^{weak} = Z(F)$ for any $F \in A$.

iii) $M_L F \in mwp(S)$ for any $F \in A, M \in \mathfrak{M}(A)$.

Proof. i) Let $F \in A$, $\{\mu_\alpha\}$ be any net in $P(X)$. Since $\overline{\{F \odot \mu, \mu \in P(X)\}}^{weak}$ is weakly compact, passing through a subnet if necessary, we have $F \odot \mu_\alpha \xrightarrow{weak} G \in M(S)^*$. Therefore, let $\nu \in P(S)$, we have

$$M((\nu \odot F) \odot \mu_\alpha) = Q\nu \odot M(F \odot \mu_\alpha) \to Q\nu \odot M(G) = M(\nu \odot G)$$

holds for all $M \in A^*$. Thus $(\nu \odot F) \odot \mu_\alpha \xrightarrow{weak} \nu \odot G \in M(S)^*$ which implies that $\overline{\{(\nu \odot F) \odot \mu, \mu \in P(X)\}}^{weak}$ is weakly relatively compact. Thus $\nu \odot F \in A$, A is a topological S-invariant subspace of $M(X)^*$ containing $\mathbb{1}$.

ii) Let $G \in \overline{\{F \odot \mu, \mu \in P(X)\}}^{weak}$, then there exists a net $\{\mu_\alpha\}$ in $P(X)$, such that $F \odot \mu_\alpha \xrightarrow{weak} G$. Since $\mathfrak{M}(A)$ is weak* compact, passing through a subnet if necessary, there exists $M \in \mathfrak{M}(A)$ such that $Q\mu_\alpha \xrightarrow{weak^*} M \in \mathfrak{M}(A)$. Hence

$$G(\nu) = Q\nu(G) = \lim_\alpha Q\nu(F \odot \mu_\alpha) = \lim_\alpha Q\mu_\alpha(\nu \odot F) = M_L F(\nu)$$

for all $\nu \in M(S)$. Therefore $\overline{\{F \odot \mu, \mu \in P(X)\}}^{weak} \subset Z(F)$.

Conversely, let $M \in \mathfrak{M}(A)$. Since $QP(X)$ is weak* dense in $\mathfrak{M}(A)$, there exist a net μ_α in $P(X)$, such that $Q\mu_\alpha \xrightarrow{weak^*} M$. Thus

$$F \odot \mu_\alpha = (Q\mu_\alpha)_L F \xrightarrow{weak^*} M_L F$$

for all $F \in A$. On the other hand, $\overline{\{F \odot \mu, \mu \in P(X)\}}^{weak}$ is weakly compact, passing though a subnet if necessary, $F \odot \mu_\alpha$ converge weakly. Therefore $F \odot \mu_\alpha \xrightarrow{weak} M_L F$.

iii) Let $\{\theta_\beta\}$ be a net in $P(S)$, let $M \in \mathfrak{M}(A)$, $F \in A$. Then

$$M_L F \odot \theta_\beta(\nu) = M((\nu * \theta_\beta) \odot F) = M(\theta_\beta \odot \nu \odot F) = (Q\theta_\beta \odot M)_L F(\nu)$$

for all $\nu \in M(S)$.

Since from ii) we know that $\{M_L F;\ M \in \mathfrak{M}(A)\}$ is weakly compact. Passing through a subnet if necessary, there exists $G \in M(S)^*$ such that

$$M_L F \odot \theta_\beta = (Q\theta_\beta \odot M)_L(F) \xrightarrow{weak} G \in M(S)^*$$

which implies $\{M_L F \odot \theta, \theta \in P(S)\}$ is weakly relatively compact. Thus $M_L F \in mwp(S)$. $\square$

Theorem 3.2.11. *Let $A = mwp(S, X)$. The action of $(\mathfrak{M}(S), weak^*)$ on $(\mathfrak{M}(A), weak^*)$ by convolution we defined previously is separately continuous.*

Proof. Considering the argument we give in the beginning of this section. It suffices to prove that for each $N \in \mathfrak{M}(S)$, the map $(\mathfrak{M}(A), weak^*) \to (\mathfrak{M}(A), weak^*)$ by $M \mapsto N \odot M$ is continuous.

Let $M_\alpha \xrightarrow{weak^*} M$, $F \in A$, then $M_{\alpha_L} F \xrightarrow{weak^*} M_L F$. Since $\{M_L F;\ M \in \mathfrak{M}(A)\}$ is weakly compact, passing through a subnet if necessary, $M_{\alpha_L} F \xrightarrow{weak} M_L F$. Therefore

$$N \odot M_\alpha(F) = N(M_{\alpha_L} F) \to N(M_L F) = N \odot M(F)$$

for any $N \in \mathfrak{M}(S)$. Thus $N \odot M_\alpha \xrightarrow{weak^*} N \odot M$. $\square$

3.3 Generalised Functions and Topological S-invariant Means

3.3.1 Properties and Connections

Let X be a locally compact space. Let $\mu \in M^+(X)$, let h be a Borel measurable function on X. The essential supremum norm corresponding to μ on X is

$$\|h\|_{\mu,\infty} = \inf_{\mu(N)=0} \sup_{x \notin N} |h(x)| = \inf\{M \in \mathbb{R};\ \ |h(x)| \leq M \ \ \mu - \text{a.e.}\}$$

For each $\mu \in M(X)$, we denote by $\mathcal{L}_\infty(|\mu|)$ the space of Borel measurable functions h such that $\|h\|_{|\mu|,\infty} < \infty$. The space $\mathcal{L}_\infty(|\mu|)$ is Banach space with the essential supremum $\|\cdot\|_{|\mu|,\infty}$ (see [28]). As there will be no conflicts in the future in this thesis, we will write $\|\cdot\|_{\mu,\infty}$ for $\|\cdot\|_{|\mu|,\infty}$.

Definition 3.3.1. We say $f = (f_\mu)_{\mu \in M(X)} \in \prod\limits_{\mu \in M(X)} \mathcal{L}_\infty(|\mu|)$ is a generalised function on X, if it satisfies:

i) $\|f\| = \sup\limits_{\mu \in M(X)} \|f_\mu\|_{\mu,\infty} < \infty$

ii) Whenever $\mu \ll \sigma$ $(\mu, \sigma \in M(X))$, $f_\mu = f_\sigma$ $|\mu|$-a.e..

Here we say $\mu \ll \sigma$ if $\mu(F) = 0$ for each compact $F \in X$ with $|\sigma|(F) = 0$ (see [20, 14.19]). We denote by $GL(X)$ the set of generalized functions on X.

Remark 3.3.2.

1. Note that for any $\mu \in M(X)$, $a \in \mathbb{R}$, we have $f_\mu = f_{|\mu|}$ $|\mu|$-a.e., $f_{a\mu} = f_\mu$ $|\mu|$-a.e.. Thus $\|\cdot\|$ in Definition 3.3.1 can be equivalently defined as $\|f\| = \sup_{\mu \in P(X)} \|f\|_{\mu,\infty}$.

2. It is clear that $\|\cdot\|$ satisfies positivity and triangle inequality. Let $f \in GL(X)$, $\|f\| = 0$ implies $f_\mu = 0$ $|\mu| -$ a.e. for all $\mu \in M(X)$. Thus $\|\cdot\|$ defined in Definition 3.3.1 is a norm on $GL(X)$.

Theorem 3.3.3. *The space $(GL(X), \|\cdot\|)$ is a Banach space.*

Proof. Let $\{f^\alpha\}_{\alpha\in I}$ be a Cauchy net in $GL(S)$. Then $\{f^\alpha_\mu\}$ is a Cauchy net in $\mathcal{L}_\infty(|\mu|)$ for any $\mu \in M(X)$. Since $(\mathcal{L}_\infty(|\mu|), \|\cdot\|_{\mu,\infty})$ is Banach, there exists $f_\mu \in \mathcal{L}_\infty(|\mu|)$, such that $f^\alpha_\mu \xrightarrow{\|\cdot\|_{\mu,\infty}} f_\mu$. Let $f = (f_\mu)_{\mu\in M(X)}$. Since $\{f^\alpha\}_{\alpha\in I}$ is Cauchy in $GL(X)$, for any $\epsilon > 0$, there exists $\beta \in I$, such that

$$\left\|f^\alpha - f^\beta\right\| = \sup_{\mu\in M(X)} \left\|f^\alpha_\mu - f^\beta_\mu\right\|_{\mu,\infty} < \frac{\epsilon}{2} \qquad (\alpha > \beta)$$

Moreover, since $f^\alpha_\mu \xrightarrow{\|\cdot\|_{\mu,\infty}} f_\mu$, there exists $\alpha_\mu > \beta$, such that $\left\|f^{\alpha_\mu}_\mu - f_\mu\right\|_{\mu,\infty} < \frac{\epsilon}{2}$ for each $\mu \in M(X)$. Hence

$$\left\|f_\mu - f^\beta_\mu\right\|_{\mu,\infty} \leq \left\|f^{\alpha_\mu}_\mu - f_\mu\right\|_{\mu,\infty} + \left\|f^{\alpha_\mu}_\mu - f^\beta_\mu\right\|_{\mu,\infty} < \epsilon$$

This implies $f^\alpha \xrightarrow{\|\cdot\|} f$. In addition, $\|f\| \leq \left\|f^\beta\right\| + \left\|f - f^\beta\right\| < \left\|f^\beta\right\| + \epsilon < \infty$.

Let $\mu, \nu \in M(X)$ such that $\mu \ll \nu$. Then $f_\nu - f^\alpha_\nu \to 0$ $|\nu|$-a.e.and $f^\alpha_\nu = f^\alpha_\mu \to f_\mu$ $|\mu|$-a.e.. Hence $f_\mu = f_\nu$ $|\mu|$-a.e.. Therefore $f \in GL(X)$, $(GL(X), \|\cdot\|)$ is a Banach space. □

Further, we order $GL(X)$ by saying $f \geq 0$ if $f_\mu \geq 0$ μ-a.e. for any $\mu \in M(S)$. We denote by $\mathbb{1}$ the function $f \in GL(X)$ such that $f_\mu = 1$ for any $\mu \in M(X)$. We also denote by $\mathbb{1}$ a functional in $M(X)^*$ such that $\mathbb{1}(\mu) = \mu(X)$ for any $\mu \in M(X)$. There will be no confusions in future. The norm $\|\cdot\|$ on $M(X)^*$ is the usual dual norm, i.e. for each $F \in M(X)^*$, $\|F\| = \sup\{|F(\mu)|\,;\ \mu \in M(X), \|\mu\| = 1\}$.

Theorem 3.3.4. *The linear map $T : GL(X) \to M(X)^*$ by $Tf(\mu) = \int f_\mu d\mu$, where $f = (f_\mu)_{\mu\in M(X)} \in GL(X)$, is an isometric order preserving isomorphism. In particular, $T(\mathbb{1}) = \mathbb{1}$.*

Proof. First we prove that T maps $GL(X)$ into the continuous dual of $M(X)$.

Let $\mu, \nu \in M(X)$, $f = (f_\sigma)_{\sigma \in M(X)} \in GL(X)$, we have

$$\begin{aligned} Tf(\mu+\nu) &= \int f_{\mu+\nu} d(\mu+\nu) = \int f_{|\mu|+|\nu|} d(\mu+\nu) \quad \mu+\nu \ll |\mu|+|\nu| \\ &= \int f_{|\mu|+|\nu|} d\mu + \int f_{|\mu|+|\nu|} d\nu \\ &= \int f_\mu d\mu + \int f_\nu d\nu \\ &= Tf(\mu) + Tf(\nu) \end{aligned}$$

Let $a \in \mathbb{R}$, $\mu \in M(X)$, $f \in GL(X)$,

$$Tf(a\mu) = \int f_{a\mu} da\mu = a \int f_{a\mu} d\mu = a \int f_\mu d\mu = aT(f) \quad |\mu| \ll |a\mu|$$

Moreover,

$$\begin{aligned} \|Tf\| &= \sup\{|Tf(\nu)|\,;\ \mu \in M(X), \|\mu\| = 1\} \\ &= \sup\{\left|\int f_\mu d\mu\right|;\ \mu \in M(X), \|\mu\| = 1\} \\ &\leq \sup\{\|f_\mu\|_{\nu,\infty}\,;\ \mu \in M(X), \|\mu\| = 1\} = \|f\| \qquad \text{By Remark 3} \end{aligned}$$

Thus T maps $GL(X)$ into $M(X)^*$.

Let $F \in M(X)^*$, then for each $\mu \in M(X)$, F introduces a linear functional F_μ on $\mathcal{L}_1(|\mu|) = \{\sigma \in M(X);\ \sigma \ll |\mu|\}$ by setting $F_\mu(\sigma) = F(\sigma)$ for all $\sigma \in \mathcal{L}_1(|\mu|)$. Thus there exists some $f_\mu \in \mathcal{L}_\infty(|\mu|)$, such that $F_\mu(\sigma) = \int f_\mu d\sigma$, for all $\sigma \in \mathcal{L}_1(|\mu|)$. In particular, $F_\mu(\mu) = \int f_\mu d\mu$.

Let $f = (f_\mu)_{\mu \in M(X)}$, then for each $\sigma, \tau \in M(X)$ such that $\sigma \ll \tau$, $\int f_\sigma d\sigma = F(\sigma) = F_\tau(\sigma) = \int f_\sigma d\tau$. Thus $f \in GL(X)$. Moreover,

$$\|f_\mu\|_{\nu,\infty} = \|F_\mu\| = \sup\{f_\mu(\sigma);\ \sigma \in \mathcal{L}_1(|\nu|), \|\sigma\| = 1\} \leq \|F\|$$

for all $\mu \in M(X)$. Thus $\|f\| \leq \|F\|$. Conversely

$$\begin{aligned}\|F\| &= \sup\{|F(\mu)|\,;\ \mu \in M(X), \|\mu\| = 1\}\\ &= \sup\{\left|\int f_\mu d\mu\right|;\ \mu \in M(X), \|\mu\| = 1\}\\ &\leq \|f\|\end{aligned}$$

Thus $\|f\| = \|F\|$, T is an isometric isomorphism.

For each $f \in GL(X)$, such that $f \geq 0$. We have $Tf(\mu) = \int f_\mu d\mu \geq 0$ for all $\mu \in M^+(X)$. Thus the order is preserved by T. □

3.3.2 Convolutions and Topological Invariant Means

Let S be a locally compact semitopological semigroup, let X be a locally compact space. Assume the left action of S on X is separately continuous. We denote by $BM(X)$ the space of bounded Borel measurable functions on X.

Let $\nu \in M(S)$, $h \in BM(X)$, $\mu \in M(X)$, we define

$$\begin{aligned}\nu \odot h(x) &= \int h(sx)d\nu(s) = \int r_x h d\nu\\ h \odot \mu(s) &= \int h(sx)d\mu(x) = \int l_s h d\mu\end{aligned}$$

Since $h \in BM(X) \subset \mathcal{L}_1(|\nu| * |\sigma|)$ for all $\sigma \in M(X)$, by Theorem 3.1.15, $\nu \odot h$ is defined σ-almost everywhere and $\nu \odot h \in \mathcal{L}_1(|\sigma|)$. Moreover, $\|\nu \odot h\|_{\sigma,\infty} \leq \|h\|_{|\nu|*|\sigma|,\infty} \|\nu\|$. Hence $\nu \odot h \in \mathcal{L}_\infty(|\sigma|)$ for all $\sigma \in M(X)$. Similarly, $h \odot \mu$ is defined τ-almost everywhere and $h \odot \mu \in \mathcal{L}_\infty(|\tau|)$ for all $\tau \in M(S)$.

Let $f \in GL(X)$, $\nu \in M(S)$, $\mu \in M(X)$, we define $\nu \odot f \in \prod_{\mu \in M(X)} \mathcal{L}_\infty(|\mu|)$ by setting $(\nu \odot f)_\mu = \nu \odot f_{|\nu|*|\mu|}$. This definition is proper since $f_{|\nu|*|\mu|} \in \mathcal{L}_\infty(|\nu| * |\mu|) \subset \mathcal{L}_1(|\nu| * |\mu|)$. Hence by Theorem 3.1.15, $(\nu \odot f)_\mu \in \mathcal{L}_\infty(|\mu|)$.

Let $\sigma \in M(X)$, such that $\mu \ll \sigma$. By Lemma 3.1.13, we have $|\nu| * |\mu| \ll$

$|\nu| * |\sigma|$. Thus by Theorem 3.1.15, we have

$$\begin{aligned}\int (\nu \odot f)_\mu d\,|\mu| &= \int \nu \odot f_{|\nu|*|\mu|} d\,|\mu| \\ &= \iint f_{|\nu|*|\mu|}(sx) d\nu(s) d\,|\mu| = \int f_{|\nu|*|\mu|} d\nu * |\mu| \\ &= \int f_{|\nu|*|\sigma|} d\nu * |\mu| = \int (\nu \odot f)_\sigma d\,|\mu|\end{aligned}$$

This implies $(\nu \odot f)_\mu = (\nu \odot f)_\sigma$ $|\mu|$-a.e.. It is clear that $\|\nu \odot f\| = \|f\|$. Thus $\{\nu \odot f;\ \nu \in M(S), f \in GL(X)\} \subset GL(X)$.

Assume $f = f' \in GL(X)$, then $f_{|\nu|*|\mu|} = f'_{|\nu|*|\mu|}$ $|\nu| * |\mu|$-a.e. for any $\mu \in M(X)$. Thus by Theorem 3.1.15

$$\begin{aligned}\int \nu \odot f_{|\nu|*|\mu|} d\,|\mu| &= \iint f_{|\nu|*|\mu|}(sx) d\nu(s) d\,|\mu| = \int f_{|\nu|*|\mu|} d\nu * |\mu| \\ &= \int f'_{|\nu|*|\mu|} d\nu * |\mu| = \int \nu \odot f'_{|\nu|*|\mu|} d\,|\mu|\end{aligned}$$

Hence $(\nu \odot f)_\mu = \nu \odot f_{|\nu|*|\mu|} = \nu \odot f'_{|\nu|*|\mu|} = (\nu \odot f')_\mu$ $|\mu|$-a.e.. Therefore the convolution "$\odot$" is well-defined.

Similarly, for each $f \in GL(X)$, $\mu \in M(X)$, we may define $f \odot \mu \in GL(S)$ by setting $(f \odot \mu)_\nu = f_{|\nu|*|\mu|} \odot \mu$ and we have $\|f \odot \mu\| \leq \|f\|$.

Theorem 3.3.5. *Let $T : GL(X) \to M(X)^*$ be the isometric isomorphism as we defined in Theorem 3.3.4. Then $\nu \odot Tf = T(\nu \odot f)$, $Tf \odot \mu = T(f \odot \mu)$, for all $\nu \in M(S)$, $\mu \in M(X)$, $f \in GL(X)$.*

Proof. Let $\nu \in M(S)$, $f \in GL(X)$. Then by Theorem 3.1.15, we have

$$\begin{aligned}\nu \odot Tf(\mu) = Tf(\nu * \mu) &= \int f_{\nu*\mu} d\nu * \mu \\ &= \int f_{|\nu|*|\mu|} d\nu * \mu = \iint f_{|\nu|*|\mu|}(sx) d\nu(s) d\mu(x) \\ &= \int (\nu \odot f)_\mu d\mu = T(\nu \odot f)(\mu)\end{aligned}$$

The proof of $Tf \odot \mu = T(f \odot \mu)$ is similar. $\square$

Definition 3.3.6. A functional m on $GL(X)$ is a mean on $GL(X)$ if $\|m\| = 1 = m(\mathbb{1})$. Further, we say m is a topological S-invariant mean if $m(\nu \odot f) = m(f)$ holds for all $\nu \in P(S)$, $f \in GL(X)$.

The commutativity of T with the convolutions allows us to relate the topological S-invariant mean on $M(X)^*$ with topological S-invariant mean on $GL(X)$. This gives another characterization of the existence of topological S-invariant mean on $M(X)^*$.

Theorem 3.3.7. *$GL(X)$ has a topological S-invariant mean if and only if $M(X)^*$ has one.*

Proof. Assume that $M(X)^*$ has a topological S-invariant mean M. We denote by T^* the adjoint operator of $T : GL(X) \to M(X)^*$ as we introduced in Theorem 3.3.4. We claim that T^*M is a topological S-invariant mean on $GL(X)$.

It is clear that T^*M is in the algebraic dual of $GL(X)$. Besides

$$\begin{aligned}\|T^*M\| &= \sup\{T^*M(f) = M(Tf);\ f \in GL(X), \|f\| \leq 1\} \\ &\leq \|M\| \, \|Tf\| = \|M\| \, \|f\| \leq 1\end{aligned}$$

In addition, $T^*M(\mathbb{1}) = M(T(\mathbb{1})) = M(\mathbb{1}) = 1$. Thus T^*M is a mean on $GL(X)$. Moreover, $T^*M(\nu \odot f) = M(T(\nu \odot f)) = M(\nu \odot Tf) = M(T(f)) = T^*M(f)$. Therefore T^*M is a topological S-invariant mean on $GL(X)$.

Conversely, for each $F \in M(X)^*$, by Theorem 3.3.4, there exists a unique $f \in GL(X)$, such that $Tf = F$. Then

$$T^{-1}(\nu \odot F) = T^{-1}(\nu \odot Tf) = T^{-1}(T(\nu \odot f)) = \nu \odot f = \nu \odot T^{-1}F$$

Thus T^{-1} commutes with convolution of $M(S)$. Similar as we argued above, we complete the proof. □

As we have proved in Theorem 3.2.7, the existence of topological S-invariant means on $M(X)$ is equivalent to the existence of a net in $P(X)$

that is weakly convergent to its topological S-invariance. Since it is more concrete and easier to work with $M(X)$ compared to working on the dual of $GL(X)$. We want a similar characterization for topological S-invariant mean on $GL(X)$.

Theorem 3.3.8. *Let $\mathcal{L}_\infty(|\mu|) = \mathcal{L}_1(|\mu|)^*$ be equipped with weak* topology and $GL(X) \subset \prod_{\mu \in M(X)} \mathcal{L}_\infty(|\mu|)$ be equipped with relative product topology τ. Then*

$$T : (GL(X), \tau) \to (M(X)^*, weak^*)$$

is a linear homeomorphism.

Proof. It suffices to prove both T and T^{-1} are continuous. Let $f^\alpha \xrightarrow{\tau} f$ be a net in $GL(X)$. For each $\mu \in M(X)$, since $f^\alpha_\mu \xrightarrow{weak^*} f_\mu$ and $\mu \in \mathcal{L}_1(|\mu|)$, we have

$$Tf^\alpha(\mu) = \int f^\alpha_\mu d\mu \to \int f_\mu d\mu = Tf(\mu)$$

Conversely, assume $F_\alpha \xrightarrow{weak^*} F$. Let $f^\alpha = T^{-1}F^\alpha, f = T^{-1}F$. Let $\mu \in M(X)$. Then

$$\int f^\alpha_\mu d\sigma = F_\alpha(\sigma) \to F(\sigma) = \int f_\mu d\sigma$$

holds for any $\sigma \in M(X)$, $\sigma \in \mathcal{L}_1(|\mu|)$. This implies $f^\alpha_\mu \xrightarrow{weak^*} f_\mu$. Hence $f^\alpha \xrightarrow{\tau} f$. □

Corollary 3.3.9. *Assume $M(X)^*$ is topological X-stationary, then for each $f \in GL(X)$, there exists a net μ_α in $P(X)$, $c \in \mathbb{R}$,such that $f \odot \mu_\alpha \xrightarrow{\tau} c\mathbb{1}$.*

Proof. Directly apply Theorem 3.3.5 and Theorem 3.3.8. □

Theorem 3.3.10. *The following are equivalent*

i) $M(X)^*$ has topological S-invariant mean.

ii) *There exist a net* $\{\mu_\alpha\}$ *in* $P(X)$, *such that* $\nu * \mu_\alpha - \mu_\alpha \xrightarrow{weak} 0$, *for any* $\nu \in P(S)$.

iii) $GL(X)$ *has topological S-invariant mean.*

iv) *There exist a net* $\{\mu_\alpha\}$ *in* $P(X)$, *such that* $(\nu \odot f)_{\mu_\alpha} - f_{\mu_\alpha} \to 0$, $|\mu_\alpha| - a.e.$, *for any* $\nu \in P(S)$, $f \in GL(X)$.

Proof. We have proved i)$\Leftrightarrow$ ii) in Theorem 3.2.7, i) $\Leftrightarrow$ iii) in Theorem 3.3.7.

ii) $\Rightarrow$ iv) Let $f \in GL(X)$. Since $\nu * \mu_\alpha - \mu_\alpha \xrightarrow{weak} 0$, we have $Tf(\nu * \mu_\alpha) - Tf(\mu_\alpha) \to 0$. This implies $T(\nu \odot f)(\mu_\alpha) - Tf(\mu_\alpha) \to 0$. Hence $\int((\nu \odot f)_{\mu_\alpha} - f_{\mu_\alpha})d\mu_\alpha \to 0$, which implies $(\nu \odot f)_{\mu_\alpha} - f_{\mu_\alpha} \to 0$ $|\mu_\alpha|$- a.e..

iv) $\Rightarrow$ i) Define $m(f) = \lim_\alpha \int f_{\mu_\alpha} d\mu_\alpha$ for any $f = (f_\mu)_{\mu \in M(X)}$. It is clear that m is linear, $|m(f)| \leq \|f\|$ and $m(\mathbb{1}) = 1$. Thus m is a mean on $GL(X)$. In addition,

$$m(f) = \lim_\alpha \int f_{\mu_\alpha} d\mu_\alpha = \lim_\alpha \int (\nu \odot f)_{\mu_\alpha} d\mu_\alpha = m(\nu \odot f)$$

Therefore m is a topological S-invariant mean on $GL(X)$. $\square$

By the definition of generalised function, we may embed $BM(X)$ into $GL(X)$ by $f \mapsto (f)_{\mu \in M(X)}$ where $f \in BM(X)$. In Theorem 3.2.4, we showed that if $M(S)^*$ has a topological left invariant mean, then $M(X)^*$ has a topological S-invariant mean. We shall use generalised function to show in the following that if a subset of $M(S)^*$ has a topological left invariant mean, then $M(X)^*$ has a topological S-invariant mean when X has S-absolute continuous probability measure.

Definition 3.3.11. Let $M(X)$ be equipped with uniform topology. A measure $\mu \in M(X)$ is said to be S-absolutely continuous if the map $s \to \delta_s * \mu$ is continuous, where δ_s represents the Dirac measure of s. We denote by $M_a(S, X)$ the set of all S-absolutely continuous measure on X.

Lemma 3.3.12. *Let* $\mu \in M_a(S, X) \cap P(X)$, $\nu \in M(S)$, $s \in$ *supp* (ν). *Then* $\delta_s * \mu \ll |\nu| * \mu$.

Proof. Let K be any compact set, such that $|\nu| * \mu(K) = 0$. Suppose $\delta_s * \mu(K) > 0$, by the continuity of $s \to \delta_s * \mu$, there exists $\epsilon > 0$ and an open neighborhood U of s in S, such that $\delta_t * \mu(K) \geq \epsilon$ for any $t \in U$. Since $s \in \text{supp }(\nu)$, we have $|\nu|(U) > 0$.

Therefore, by Theorem 3.1.15

$$\begin{aligned} |\nu| * \mu(K) &= \iint \chi_K(sx) d|\nu|(s) d\mu(x) = \int \mu(s^{-1}K) d|\nu|(s) \\ &= \int \delta_s * \mu(K) d|\nu|(s) \geq \epsilon |\nu|(U) > 0 \end{aligned}$$

This contradicts with the fact that $|\nu| * \mu(K) = 0$. Thus $\delta_s * \mu \ll |\nu| * \mu$. □

Lemma 3.3.13. *Let $\mu \in M_a(S, X) \cap P(X)$. Then $F(\nu * \mu) = \int F(\delta_s * \mu) d\nu(s)$ for all $\nu \in M(S)$, $F \in M(X)^*$.*

Proof. Since T is bijective, for each F in $M(X)^*$, there exists $f \in GL(X)$, such that $Tf = F$. Then by Theorem 3.3.5 and Lemma 3.3.12,

$$\begin{aligned} F(\nu * \mu) &= \nu \odot F(\mu) = T(\nu \odot f)(\mu) = \int (v \odot f)_\mu d\mu \\ &= \iint f_{|\nu| * \mu}(sx) d\mu(s) d\nu(s) = \iint \delta_s \odot f_{|\nu| * \mu}(x) d\mu(x) d\nu(s) \\ &= \iint f_{|\nu| * \mu} d\delta_s * \mu d\nu(s) = \iint f_{\delta_s * \mu} d\delta_s * \mu d\nu(s) \\ &= \int F(\delta_s * \mu) d\nu(s) \end{aligned}$$

□

Definition 3.3.14. Let $CB(S)$ be equipped with the supremum norm topology. We say $f \in CB(S)$ is right uniformly continuous on X if the map $s \to r_t f$ is continuous, where $r_t f(s) = f(st)$ for any $s \in S$. We denote by $RUC(S)$ the set of right uniformly continuous functions on S.

Lemma 3.3.15. *Let $f \in RUC(S)$. Then $\nu \odot f \in RUC(S)$ for any $\nu \in M(S)$.*

Proof. Let $t_1, t_2 \in X$, $s \in S$,

$$\begin{aligned}\|r_{t_1}(\nu \odot f) - r_{t_2}(\nu \odot f)\| &= \sup_{s\in S} |r_{t_1}(\nu \odot f)(s) - r_{t_2}(\nu \odot f)(s)| \\ &= \sup_{s\in S} \left| \int (f(wst_1) - f(wst_2)) d\nu(w) \right| \\ &\leq \|r_{t_1}f - r_{t_2}f\| \, \|\nu\|\end{aligned}$$

□

We denote by $\mathbb{1}$ the constant 1 function on S, it is obviously in $RUC(S)$. We say $RUC(S)$ has topological left invariant mean m in $RUC(S)^*$, if m satisfies,

1) $m(\mathbb{1}) = \|m\| = 1$
2) $m(\nu \odot f) = m(f)$ for any $\nu \in P(S)$, where $P(S)$ is the set of probability measures.

Theorem 3.3.16. *Assume $M_a(S, X) \cap P(X) \neq \emptyset$, then whenever $RUC(S)$ has topological left invariant mean, $M(X)^*$ has topological S-invariant mean.*

Proof. Let $\mu \in M_a(S, X) \cap P(X)$. For each $F \in M(X)^*$, define $f(s) := F(\delta_s * \mu)$ for any $s \in S$. It is easy to check $\|f\| \leq \|F\|$. Since f is the composition of continuous function F and $s \to \delta_s * \mu$, it is also continuous. Moreover,

$$\|r_{t_1}f - r_{t_2}f\| = \|F(\delta_{st_1} * \mu) - F(\delta_{st_2} * \mu)\| \leq \|F\| \, \|\delta_{t_1} * \mu - \delta_{t_2} * \mu\|$$

Thus f is in $RUC(S)$.

Let $\nu \in P(S)$. Note that if $t \in \text{supp}\,(\nu)$, then $ts \in \text{supp}\,(\nu * \delta_s)$ by Corollary 3.1.17. By Lemma 3.3.13

$$\begin{aligned}\nu \odot f(s) &= \int f(ts) d\nu(t) = \int F(\delta_{ts} * \mu) d\nu(t) \\ &= \int F(\delta_a * \mu) d\nu * \delta_s(a) = F(\nu * \delta_s * \mu) = \nu \odot F(\delta_s * \mu)\end{aligned}$$

Let m be a topological S-invariant mean on $RUC(S)$, we define $M(F) = m(f)$ for any $F \in M(X)$ and correspondingly $f = F(\delta_s * \mu)$. Consequently, we have $m(\mathbb{1}) = M(\mathbb{1}) = 1 \leq \|M\| \leq \|m\| \, \|f\| \, / \, \|F\| \leq 1$. Thus M is a mean on $M(X)$. Moreover, $M(\nu \odot F) = m(\nu \odot F) = m(F) = M(f)$. Therefore, M is a topological S-invariant mean on $M(X)^*$. □

3.4 Support of Topological S-invariant Mean

Throughout this section, we let S be a locally compact semitopological semigroup, X be a locally compact space that is closed under separately continuous left action of S. Let A be a topological S-invariant subspace of $M(X)^*$ containing $\mathbb{1}$. Let T be a Borel subset of X, we denote by χ_T the characterization functional of T on $M(X)^*$, i.e., $\chi_T(\mu) = \mu(T)$ for all $\mu \in M(X)$. We let $A_T := \{F \in A;\ \chi_T \leq F \leq 1\}$.

Definition 3.4.1. A Borel subset $T \subset X$ is said to be topological (S,A)-lumpy, if for any triple (ν, ϵ, F), where $\nu \in P_c(S)$, $\epsilon > 0$, $F \in A_T$, there exists $\mu \in P(X)$, such that $F(\nu * \mu) > 1 - \epsilon$. When A contains χ_T, we usually write topological S-lumpy for topological (S, A)-lumpy. In the case where $X = S$, we write topological left A-lumpy for topological (S, A)-lumpy, topological left lumpy for topological S-lumpy.

Remark 3.4.2. It is clear that if a Borel subset T is topological S-lumpy, then T is topological (S, A)-lumpy for all topological S-invariant subspace A of $M(X)^*$ containing $\mathbb{1}$.

Let A_1, A_2 be topological S-invariant subspaces of $M(X)^*$ containing $\mathbb{1}$, if $A_1 \subset A_2$, then $T \subset X$ being topological (S, A_2)-lumpy implies T being topological (S, A_1)-lumpy.

In the following whenever we mention triple (ν, ϵ, F), we mean that $\nu \in P_c(S)$, $\epsilon > 0$, $F \in A_T$ unless specify otherwise.

Theorem 3.4.3. *Let T be a Borel subset in X. Then the following are equivalent:*

i) T is topological (S, A)-lumpy.

ii) For each triple (ν, ϵ, F), where $\nu \in P(S)$, there exists $\mu \in P_c(X)$, such that $F(\nu * \mu) > 1 - \epsilon$, $F(\mu) > 1 - \epsilon$.

iii) For each triple (ν, ϵ, F), where $\nu \in P(S)$, there exists $x \in X$, such that $F(\nu * \delta_x) > 1 - \epsilon$, $F(\delta_x) > 1 - \epsilon$.

Proof. It is clear that iii) implies ii) implies i).

i) $\Rightarrow$ ii) Let $\gamma \in P(S)$. Since $P_c(S)$ is norm dense in $P(S)$, there exists $\tau \in P_c(S)$ such that $\left\|\tau - \frac{\nu*\gamma+\gamma}{2}\right\| \leq \frac{\epsilon}{8}$. Since T is topological (S,A)-lumpy, there exists $\sigma \in P(X)$ such that $F(\tau * \sigma) > 1 - \frac{\epsilon}{8}$. Hence

$$\begin{aligned} F(\frac{\nu * \gamma + \gamma}{2} * \sigma) &= F(\tau * \sigma) - F((\tau - \frac{\nu * \gamma + \gamma}{2}) * \sigma) \\ &\geq F(\tau * \sigma) - \frac{\epsilon}{8} \|F\| > 1 - \frac{\epsilon}{4} \end{aligned}$$

As we have $F \leq \mathbb{1}$, it implies that $F(\nu * \gamma * \sigma) > 1 - \frac{\epsilon}{2}$, $F(\gamma * \sigma) > 1 - \frac{\epsilon}{2}$. Since $P_c(X)$ is norm dense in $P(X)$, there exists $\mu \in P_c(X)$, such that $\|\mu - \gamma * \sigma\| \leq \frac{\epsilon}{2}$. Therefore, $F(\nu * \mu) > 1 - \epsilon$, $F(\mu) > 1 - \epsilon$.

ii) $\Rightarrow$ iii) By Theorem 3.3.4, there exists $f \in GL(X)$, such that $Tf = F$. Let $\theta = \frac{\nu*\delta_s+\delta_s}{2}$ there exists $\mu \in P_c(X)$, such that,

$$\begin{aligned} 1 - \epsilon/2 < F(\theta * \mu) &= \int f_{\theta*\mu} d\theta * \mu = \int f_{\theta*\mu}(sx) d\theta(s) d\mu(x) \\ &= \int_{\text{supp }(\mu)} f_{\theta*\mu} d\theta * \delta_x d\mu(x) \\ &= \int_{\text{supp }(\mu)} f_{\theta*\delta_x} d\theta * \delta_x d\mu(x) \qquad \text{By Lemma 3.1.13} \\ &= \int_{\text{supp }(\mu)} F(\theta * \delta_x) d\mu(x) \end{aligned}$$

Since $\mu \in P(X)$, there must exist some $x \in \text{supp }(\mu)$, such that $F(\theta * \delta_x) = F(\frac{\nu*\delta_s+\delta_s}{2} * \delta_x) > 1 - \epsilon/2$. Hence it implies that $F(\nu * \delta_{sx}) > 1 - \epsilon$ and $F(\delta_{sx}) > 1 - \epsilon$ since $F \leq \mathbb{1}$.

□

Remark 3.4.4. Assume A contains χ_T. By Theorem 3.4.3 iii), T is topological S-lumpy if and only if for each triple (ν, ϵ, F), there exists $x \in T$, such that $\nu * \delta_x(T) > 1 - \epsilon$. Assume further that T is compact, then there exists $y \in T$, such that $\nu \odot \delta_y(T) = 1$ for any $\nu \in P(S)$.

It is interesting to note that even for the case when T is dense in X, T may not be topological S-lumpy. An example of this is given in the following.

Example 3.4.5. Let $S = X = \mathbb{R}$ be equipped with the usual topology and $\mathbb{R}$ act on itself by addition. Let $T = \mathbb{Q}$ be equipped with the subspace topology of $\mathbb{R}$. We denote by m the Lebesgue measure on $\mathbb{R}$. For each Borel subset E of $\mathbb{R}$, define $\mu(E) = m(E \cap [0, 1])$. Thus $\mu \in P(\mathbb{R})$. For any $x \in X$, we have $\mu * \delta_x(\mathbb{Q}) = \mu(\mathbb{Q} - x) \leq m(\mathbb{Q} - x) = m(\mathbb{Q}) = 0$. Thus by the remark above, $\mathbb{Q}$ is not topological S-lumpy in $\mathbb{R}$ despite the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Example 3.4.6. Let T be a Borel subset of X. If there exists $x \in X$, such that $Sx = \{sx;\ s \in S\} \subset T$. Then for each $\nu \in P(S)$, $\nu * \delta_x(T) = \int \chi_T(sx) d\nu(s) = 1$. Thus T is topological S-lumpy.

However the converse statement of Example 3.4.6 is not true.

Example 3.4.7. Let $S = X = \mathbb{R}$ be equipped with the usual topology and $\mathbb{R}$ act on itself by addition. Let T be the set of irrational numbers in $\mathbb{R}$. Then $Sx \not\subset T$ for all $x \in \mathbb{R}$. However, let μ be defined as Example 3.4.5. We have

$$\nu * \mu(T) = \int \mu(s^{-1}T) d\nu(s) = 1$$

holds for all $\nu \in P(S)$. Thus T is topological S-lumpy and hence is topological (S, A)-lumpy for all topological invariant subspace A of $M(X)^*$.

Lemma 3.4.8. *Let T be a Borel subset in X. Then the following are equivalent:*

a) T is topological (S,A)-lumpy.

b) There exists $M \in \mathfrak{M}(A)$, such that $M(\nu \odot F) = M(F) = 1$, for any $\nu \in P(S)$, $F \in A_T$.

c) There exists $M \in \mathfrak{M}(A)$, such that $N \odot M(F) = M(F) = 1$, for any $F \in A_T$, $N \in \mathfrak{M}(S)$.

Proof. a) $\Rightarrow$ b) Let $C = \{\nu_1, \cdots, \nu_n\}$ be a finite subset in $P_c(S)$, $D = \{F_1, \cdots, F_m\}$ be a finite subset in A_T. Then $\frac{\sum_{j=1}^m F_j}{m} \in A_T$ and $\frac{\sum_{i=1}^n \nu_i}{n} \in P_c(S)$.

Let $\alpha = (\epsilon, C, D)$, we say $\alpha \geq \alpha'$ if $\epsilon < \epsilon'$, $C \supset C'$, $D \supset D'$. Since T is topological (S,A)-lumpy, thus by Theorem 3.4.3 iii), for each α, there exists $x_\alpha \in X$, such that

$$\frac{\sum_{j=1}^m F_j}{m}(\delta_{x_\alpha}) > 1 - \frac{\epsilon}{nm} \qquad \frac{\sum_{j=1}^m F_j}{m}(\frac{\sum_{i=1}^n \nu_i * \delta_{x_\alpha}}{n}) > 1 - \frac{\epsilon}{nm}$$

This implies $F_j(\delta_{x_\alpha}) > 1-\epsilon/n \geq 1-\epsilon$ and $F_j(\nu_i * \delta_{x_\alpha}) > 1-\epsilon$. Equivalently, we have $1 \geq Q(\delta_{x_\alpha})(\nu_i \odot F_j) > 1 - \epsilon$ and $1 \geq Q(\delta_{x_\alpha})(F_j) > 1 - \epsilon$, for all $1 \leq i \leq n$, $1 \leq j \leq m$.

Since $\mathfrak{M}(A)$ is weak* compact, passing through its subnet if necessary, there exists $M \in \mathfrak{M}(A)$, such that $Q(\delta_{x_\alpha}) \xrightarrow{weak^*} M$. Hence, $M(\nu \odot F) = M(F) = 1$ for any $F \in A_T$, $\nu \in P(S)$.

b) $\Rightarrow$ c) Let $M \in \mathfrak{M}(A)$, such that $M(\nu \odot F) = M(F) = 1$ for any $\nu \in P(S)$, $F \in A_T$. Let $N \in \mathfrak{M}(S)$, by Proposition 3.2.3, there exists a net ν_β in P(S), such that $Q\nu_\beta \xrightarrow{weak^*} N$. Therefore,

$$N \odot M(F) = \lim_\beta Q\nu_\beta \odot M(F) = \lim_\beta M(\nu_\beta \odot F) = 1$$

c) $\Rightarrow$ a) If T is not topological (S,A)-lumpy, there exists a triple (ν, ϵ, F) such that $F(\nu * \mu) \leq 1 - \epsilon$ for all $\mu \in P(X)$. Equivalently, $Q\mu(\nu \odot F) \leq 1 - \epsilon$, for all $\mu \in P(X)$. Since $QP(X)$ is weak* dense in $\mathfrak{M}(A)$, we have $M(\nu \odot F) \leq 1 - \epsilon$ for all $M \in \mathfrak{M}(A)$. Therefore $Q\nu \odot M(F) \leq 1 - \epsilon$ which contradicts c). $\square$

Remark 3.4.9. Let $\mathfrak{M}_T(A) := \{M \in \mathfrak{M}(A);\ M_L F = \mathbb{1}, F \in A_T\}$. The subset $T \subset X$ is topological (S,A)-lumpy if and only if $\mathfrak{M}_T(A)$ is not empty by Lemma 3.4.8 b). Actually, $\mathfrak{M}_T(A)$ is an $\mathfrak{M}(S)$-invariant subset of $\mathfrak{M}(A)$. In particular, if $X = S$, $\mathfrak{M}_T(A)$ is an ideal in $\mathfrak{M}(A)$.

Corollary 3.4.10. *Let T be a Borel subset of X. Assume A has a topological S-invariant mean M such that $M(F) = 1$ for all $F \in A_T$. Then T is topological (S, A)-lumpy.*

Proof. By Lemma 3.4.8 and the fact that M is positive. □

Denote $ZA := \{M_L F;\ M \in \mathfrak{M}(A), F \in A\} \subset M(S)^*$, where $\mathfrak{M}(A)$ is the set of all the means on A as we mentioned in Section 3.2.

Let B be a topological S-invariant subspace of $M(S)^*$. Conventionally, we say that B is topological left invariant. A topological left invariant subspace B of $M(S)^*$ is said to be topological left introverted if $\{M_L F;\ M \in \mathfrak{M}(B), F \in B\} \subset B$. It makes convolution of means on B defined by Arens product from convolution of measures on S well-defined.

Lemma 3.4.11. *For each S-invariant subspace A of $M(X)^*$, ZA is topological left introverted.*

Proof. Let $\nu \in M(S)$, $M_L F \in ZA$. Then

$$\begin{aligned}\nu \odot M_L F(\theta) &= M_L F(\nu * \theta) = M((\nu * \theta) \odot F)\\ &= M(\theta \odot (\nu \odot F)) = M_L(\nu \odot F)(\theta)\end{aligned}$$

holds for all $\theta \in M(S)$. Since A is topological left invariant, $\nu \odot M_L F = M_L(\nu \odot F) \in ZA$. Thus ZA is topological S-invariant.

Let $N \in \mathfrak{M}(ZA)$, $M_L F \in ZA$. We have

$$\begin{aligned}N_L(M_L F)(\nu) &= N(\nu \odot M_L F) = N(M_L(\nu \odot F))\\ &= N \odot M(\nu \odot F) = (N \odot M)_L F(\nu)\end{aligned}$$

holds for all $\nu \in M(S)$. Therefore ZA is topological S-introverted. □

Corollary 3.4.12. *Assume ZA has a topological left invariant mean and T is a (S,A)-lumpy subset of X. Then A has a topological S-invariant mean $M \in M_T(A)$.*

Proof. Let N be a topological left invariant mean on ZA. Since T is (S, A)-lumpy, $\mathfrak{M}_T(A) \neq \emptyset$. Let $M' \in \mathfrak{M}_T(A)$. Then $N \odot M' \in \mathfrak{M}_T(A)$ by Remark 3.4.9. Moreover,

$$\begin{aligned} N \odot M'(\nu \odot F) &= N(M'_L(\nu \odot F)) = N(\nu \odot M'_L F) \\ &= N(M'_L F) = N \odot M'(F) \end{aligned}$$

holds for all $\nu \in P(S)$, $F \in A$. Therefore, $M = N \odot M'$ is a topological S-invariant mean in $\mathfrak{M}_T(A)$. □

Let $X = S$. Let T be a Borel subset of X. Day [9] shows that if $M(S)^*$ has a topological invariant mean, then the existence of topological left invariant mean M on $M(S)^*$, such that $M(\chi_T) = 1$ is equivalent with T being topological left lumpy. We generalize this result to the case where T is (S, A)-lumpy for some left introverted subspace A of $M(S)^*$ that has a topological left invariant mean.

Theorem 3.4.13. *Let S be locally compact semitopological semigroup and A be a topological left introverted subspace of $M(S)^*$ with topological left invariant means. Then the following are equivalent:*

a) T is topological left A-lumpy.

b) There exist a topological left invariant mean M on A, such that $M(F) = 1$ for all $F \in A_T$.

Proof. b) ⇒ a) Followed from Lemma 3.4.8.

a) ⇒ b) Let N be a topological left invariant mean on A. By Lemma 3.4.8, there exists $M_1 \in \mathfrak{M}(A)$, such that $M_1(\nu \odot F) = M_1(F) = 1$ holds for all $\nu \in P(S)$, $F \in A_T$. Let $M = N \odot M_1$. Then M is a mean on A and for all $F \in A$, $\nu \in P(S)$, $M(\nu \odot F) = N(M_{1_L}(\nu \odot F)) = N(\nu \odot M_{1_L} F) = N \odot$

$M_1(F) = M(F)$. Moreover, we have $M(\nu \odot F) = M(F) = N(M_{1_L}(F)) = 1$ for all $F \in A_T$. $\square$

In the case when X is a general locally compact space, the author do not know if the above equivalence holds for general topological S-invariant subspace of $M(X)^*$. However, we can construct a topological S-invariant subspace A of $M(X)^*$, such that T being topological S-lumpy implies the existence of a topological S-invariant mean M on A with $M(\chi_T) = 1$.

Theorem 3.4.14. *Let T be topological S-lumpy subset of X. Then there exists a topological S-invariant norm closed subspace A of $M(X)^*$ containing $\mathbb{1}$ and χ_T, such that it has a topological S-invariant mean M on A with $M(\chi_T) = 1$.*

Proof. Since T is topological S-lumpy, there exists $M \in \mathfrak{M}(X)$, such that $M(\nu \odot \chi_T) = M(\chi_T) = 1$ for all $\nu \in P(S)$. Thus the set $H = \{F \in M(X)^*;\ M(\nu \odot F) = M(F) \text{ for all } \nu \in P(S)\}$ is not trivial. It is clear that H is a S-invariant linear subspace of $M(X)^*$ containing $\mathbb{1}$. Let A be the norm closure of H. Let $F \in A$, then there exists a net $F_\alpha \xrightarrow{\|\cdot\|} F$ in H. Let $\nu \in P(S)$, then $\|\nu \odot F - \nu \odot F_\alpha\| \leq \|F - F_\alpha\| \to 0$. Then $\nu \odot F \in A$ since H is topological S-invariant. Thus A is topological S-invariant norm closed subspace of $M(X)^*$ containing $\mathbb{1}$. Let $N(F) = M(F)$ for all $F \in A$. Then $1 = M(\mathbb{1}) = N(\mathbb{1}) \leq \|N\| \leq \|M\| = 1$ and $N(\nu \odot G) = M(\nu \odot G) = M(G) = N(G)$ for all $\nu \in P(S)$, $G \in H$. Hence, let $F \in A$, $F_\alpha \xrightarrow{\|\cdot\|} F$ be a net in H, then $\lim_\alpha N(\nu \odot F) - N(\nu \odot F_\alpha) \leq \lim_\alpha \|\nu \odot F - \nu \odot F_\alpha\| = 0$ for all $\nu \in P(S)$, $\lim_\alpha N(F) - N(F_\alpha) \leq \lim_\alpha \|F - F_\alpha\| = 0$. Thus $N(\nu \odot F) = N(F)$ for all $\nu \in P(S)$, $F \in A$. Therefore N is a topological S-invariant mean on A.

$\square$

3.5 Topological S-invariant Means on Locally Compact Subspace

In this section, we first let X be a locally compact space, T be a locally compact Borel subspace of X.

Remark 3.5.1. A Borel subspace of a locally compact space may not be locally compact. For example. Let $\mathbb{R}$ be equipped with usual topology, $\mathbb{Q} \subset \mathbb{R}$ is Borel since it is a countable union of rational points in $\mathbb{R}$. However, $\mathbb{Q}$ is not locally compact since all of its compact subsets have empty interior.

On the other hand, open or closed subspace of locally compact space is again locally compact (see [27, 2.3.29]).

Lemma 3.5.2. *For each $\mu \in M(X)$, we define a set function on T by setting $\mu_T(E) = \mu(E)$ for all Borel subset $E \subset T$. Then μ_T is a bounded Borel measure on T and $\|\mu_T\| \leq \|\mu\|$.*

In particular, let $f \in BM(T)$, we have $\int \bar{f} d\mu = \int f d\mu_T$, where $\bar{f}(x) = \begin{cases} f(x) & x \in T \\ 0 & otherwise \end{cases}$.

Proof. Since μ is a regular bounded Borel measure, it is clear that μ_T is a Borel and finite measure on T and $\|\mu_T\| \leq \|\mu\|$ by definition. So it suffices to prove regularity of μ_T.

Let $K \subset T$ be compact, it is then compact in T, since for any net V_α of open sets in T that covers K, it corresponds with a net U_α of open sets, such that $V_\alpha = U_\alpha \cap T$. Conversely, if K is compact in T, K is then compact, since for any open net U_β that covers K, $U_\beta \cap T$ is a net of sets open in T

covers K. Therefore

$$\begin{aligned}
\mu_T(E) = \mu(E) &= \sup\{\mu(K);\ K \subset E, K \text{ compact}\} \\
&= \sup\{\mu_T(K);\ K \subset E, K \text{ compact in } T\} \\
&= \mu(E) = \inf\{\mu(U);\ U \supset E, U \text{ is open }\} \\
&\geq \inf\{\mu(V);\ V = U \cap T \supset E \text{ for some open set } U\} \\
&= \inf\{\mu_T(V);\ V \supset E \text{ is open in } T\}
\end{aligned}$$

Thus μ_T is regular Borel measure on T.

Let $f \in BM(T)$, $\int \bar{f} d\mu = \int f d\mu_T$ follows directly from the fact that if E is Borel subset of T, $\int \xi_E d\mu_T = \mu_T(E) = \mu(E) = \int \xi_E d\mu$. □

Lemma 3.5.3. *For each $\mu \in M(T)$, there exists a unique measure $\bar{\mu}$ in $M(X)$, such that $\bar{\mu}(E) = \mu(E \cap T)$ whenever $E \cap T$ is μ-measurable. Moreover, $\|\mu\| = \|\bar{\mu}\|$.*

Proof. Without loss of generality, we assume $\mu \in M^+(T)$. Let $f \in C_0(X)$, we denote by $f|_T$ the restriction of f on T. Then $I(f) = \int f|_T d\mu$ is a linear functional on $C_0(X)$. By Riesz representation theorem, let $\bar{\mu}$ be the unique corresponding bounded Borel measure of I on X.

Let U be an open subset in X, then ξ_U is lower semicontinuous as it is shown in Lemma 3.1.5. Thus

$$\begin{aligned}
\bar{\mu}(U) = \int \xi_U d\bar{\mu} &= \sup\{\int f d\bar{\mu};\ f \in C_c(X), 0 \leq f \leq \xi_U\} \qquad [20, 11.13] \\
&= \sup\{\int f|_T d\mu;\ f \in C_c(X), 0 \leq f \leq \xi_U\} \\
&\leq \sup\{\int g d\mu;\ g \in C_c(T), 0 \leq g \leq \xi_{U \cap T}\} = \mu(U \cap T)
\end{aligned}$$

The last inequivalence holds since $f|_T \in C_c(T)$ whenever $f \in C_c(X)$ as we have shown in the proof of Lemma 3.5.2.

On the other hand, let $\epsilon > 0$, by the regularity of μ, there exists a compact subset K in T such that $K \subset U \cap T$, $\mu(U \cap T) \leq \mu(K) + \epsilon$. In addition, since X is locally compact, there exists $h \in C_c(X)$, such that $h(K) = 1$, $h(X - U) = 0$. Thus

$$\bar{\mu}(U) \leq \mu(U \cap T) \leq \mu(K) + \epsilon \leq \int h|_T d\mu + \epsilon = \int h d\bar{\mu} \leq \bar{\mu}(U) + \epsilon$$

Therefore, $\bar{\mu}(U) = \mu(U \cap T)$ for any open subset U in X.

Let C be a closed subset of X. Then $\bar{\mu}(C) = \bar{\mu}(X) - \bar{\mu}(X - C) = \mu(T) - \mu((X - C) \cap T) = \mu(C \cap T)$.

Let B be any Borel subset of X. Let $\epsilon > 0$, by the regularity of $\bar{\mu}$, there exists an open subset $U \supset B$ such that

$$\bar{\mu}(B) \geq \bar{\mu}(U) - \epsilon = \mu(U \cap T) - \epsilon \geq \mu(B \cap T) - \epsilon$$

This implies $\bar{\mu}(B) \geq \mu(B \cap T)$. On the other hand, there exists a compact subset $K \subset B$, such that

$$\bar{\mu}(B) \leq \bar{\mu}(K) + \epsilon = \mu(K \cap T) + \epsilon \leq \mu(B \cap T) + \epsilon$$

Thus $\bar{\mu}(B) = \mu(B \cap T)$. In particular, $\|\bar{\mu}\| = \bar{\mu}(X) = \mu(T) = \|\mu\|$. $\square$

Remark 3.5.4. Lemma 3.5.3 implies that the map from $M(X)$ to $M(T)$ defined in Lemma 3.5.2 is surjective. Thus from now on, we let μ_T denote measure in $M(T)$ while $\mu \in M(X)$.

Note that for each $\mu \in M(X)$, $\mu = \overline{\mu_T} + \overline{\mu_{(X-T)}}$, since

$$\begin{aligned}
\mu(E) &= \mu(E \cap T) + \mu(E \cap (X - T)) \\
&= \mu_T(E \cap T) + \mu_{(X-T)}(E \cap (X - T)) \\
&= \overline{\mu_T}(E) + \overline{\mu_{(X-T)}}(E)
\end{aligned}$$

for all Borel subset E of X.

Now let S be a locally compact semitopological semigroup, X be a locally compact space that is closed under separately continuous left action of S.

Let R be a locally compact semitopological subsemigroup of S, T be locally compact subspace of X that is closed under separately continuous left action of R. We say $M(X)^*$ has a topological R-invariant mean M, if M is a mean that satisfies $M(\nu \odot F) = M(F)$, for all $\nu \in P(S)$ with $\nu(R) = 1$.

Lemma 3.5.5. *Let $\nu \in M(S)$, $\mu \in M(X)$. Let ν_R, μ_T, $(\nu * \mu)_T$ defined respectively as in Lemma 3.5.2. Then*

$$\|(\nu * \mu)_T - \nu_R * \mu_T\| \leq \int \mu(E_s) d\nu(s) + \int_T \nu(E_x) d\mu(x)$$

where $E_s = \{x \in X - T;\ sx \in T\}$, $E_x = \{s \in S - R;\ sx \in T\}$.

Proof. Let $f \in BM(T)$, define $\bar{f}$ as in Lemma 3.5.2, we have,

$$\begin{aligned}
\int f d\nu_R * \mu_T &= \iint f(sx) d\nu_R(s) d\mu_T(x) \\
&= \int_T \int_R \bar{f}(sx) d\nu(s) d\mu(x) \qquad\qquad \text{By Lemma 3.5.2} \\
\int f d(\nu * \mu)_T &= \int \bar{f} d\nu * \mu = \iint \bar{f}(sx) d\nu(s) d\mu(x)
\end{aligned}$$

Thus, by Theorem 3.1.15,

$$\begin{aligned}
\left| \int f d\nu_R * \mu_T - \int f d(\nu * \mu)_T \right| &\leq \left| \iint_{X-T} f(sx) d\mu(x) d\nu(s) \right| \\
&\quad + \left| \int_T \int_{S-R} f(sx) d\nu(s) d\nu(x) \right| \\
&\leq \|f\| \left(\int \mu(E_s) d\nu(s) + \int_T \nu(E_x) d\mu(x) \right)
\end{aligned}$$

Therefore

$$\|(\nu * \mu)_T - \nu_R * \mu_T\| \leq \int \mu(E_s) d\nu(s) + \int_T \nu(E_x) d\mu(x)$$

□

Theorem 3.5.6. *There is a topological R-invariant mean on $M(T)^*$ if and only if $M(X)^*$ has a topological R-invariant mean M such that $M(\chi_T) = 1$.*

Proof. Assume $M(T)^*$ has a topological R-invariant mean M_T. Let $F \in M(X)^*$, define $F_T(\mu_T) = F(\overline{\mu_T})$ for any $\mu_T \in M(T)$, where $\overline{\mu_T}$ is defined as in Lemma 3.5.3. In particular, $\mathbb{1}_T(\mu_T) = \mu_T(T)$. It is clear that F_T is linear and $\|F_T\| \leq \|F\|$. Further we define $M(F) = M_T(F_T)$. The function M is well-defined since $F_T \neq G_T$ implies $F \neq G$. It is clear that M is linear, $M(\mathbb{1}) = M_T(\mathbb{1}_T) = 1 = \|M\|$. Thus M is a mean on $M(T)^*$.

Moreover, let $\nu \in P(S)$ with $\nu(R) = 1$, $\mu \in M(X)^*$ with $\mu(X - T) = 0$. By Lemma 3.5.5, we have

$$\nu \odot F(\mu) = F(\nu * \mu) = F_T((\nu * \mu)_T) = F_T(\nu_R * \mu_T) = V_R \odot F_T(\mu_T)$$

Hence $M(\nu \odot F) = M_T(V_R \odot F_T) = M_T(F_T) = M(F)$.

Conversely, assume that $M(X)^*$ has topological R-invariant mean M with $M(\chi_T) = 1$. Hence$M(\chi_{(X-T)}) = 0$. Let $F \in M(T)^*$, define $\overline{F}(\mu) = F(\mu_T)$ for all $\mu \in M(X)$ while μ_T defined as in Lemma 3.5.2. Since $\mu_T \neq \sigma_T$ implies $\mu \neq \sigma$ for any $\mu, \sigma \in M(X)$, $\overline{F}$ is well defined. It is easy to check that F is linear and $\left\|\overline{F}\right\| \leq \|F\|$. Thus $\overline{F} \in M(X)^*$.

Now define $M_T(F) = M(\overline{F})$. It is easy to check that M_T is linear and $M_T(\mathbb{1}_T) = 1 \leq \|M_T\| \leq \|M\| = 1$. Moreover, let $\nu \in P(S)$ with $\nu(R) = 1$

$$\begin{aligned}
\left|\overline{\nu_R \odot F}(\mu) - \nu \odot \overline{F}(\mu)\right| &= |F(\nu_R * \mu_T) - F((\nu * \mu)_T)| \\
&\leq \|F\| \, \|(\nu * \mu)_T - \nu_R * \mu_T\| \\
&\leq \|F\| \int_R \mu(E_s) d\nu(s) && \text{Lemma 3.5.5} \\
&\leq \|F\| \, \mu(X - T) = |F| \, \chi_{(X-T)}(\mu)
\end{aligned}$$

holds for any $\mu \in P(X)$. Thus

$$\left|M(\overline{\nu_R \odot F}) - M(\nu \odot \overline{F})\right| \leq |F| \, M(\chi_{X-T}) = 0$$

Therefore $M_T(\nu_R \odot F) = M(\overline{\nu_R \odot F}) = M(\nu \odot \overline{F}) = M(\overline{F}) = M_T(F)$. Then M_T is a topological R-invariant mean on $M(T)^*$ since ν_R runs out of

$M(R)$ by Remark 3.5.4. $\square$

Theorem 3.5.7. *Assume further that R is a topological S-lumpy subsemigroup of S. Then $M(T)^*$ has a topological R-invariant mean if and only if $M(X)^*$ has a topological S-invariant mean M and $M(\chi_T) = 1$.*

Proof. By Theorem 3.5.6, we have the later implies the former. Now we assume $M(T)^*$ has a topological R-invariant mean M_T. Define $M(F) = M_T(\mu_T)$ for any $F \in M(X)^*$. Then M is a topological R-invariant mean on $M(X)^*$ by Theorem 3.5.6.

Since R is topological S-lumpy, by Lemma 3.4.8, there exists $P \in \mathfrak{M}(S)$, such that $P(\nu \odot \chi_R) = P(\chi_R) = 1$. Since $\mathfrak{M}(S)$ is weak* compact. Let θ_α be a net in $P(S)$, such that $Q(\theta_\alpha) \xrightarrow{weak^*} P$. Then $\lim\limits_\alpha \nu * \theta_\alpha(R) = \lim\limits_\alpha \theta_\alpha(R) = 1$.

Let $N = P \odot M$. It is clear that N is a mean on $M(X)^*$. Let $\nu \in P(S)$, then by Remark 3.5.4

$$
\begin{aligned}
N(\nu \odot F) &= P(M_L(\nu \odot F)) = P(\nu \odot M_L(F)) = \lim_\alpha \nu \odot M_L(F)(\theta_\alpha) \\
&= \lim_\alpha M((\nu * \theta_\alpha) \odot F) \\
&= \lim M(\overline{(\nu * \theta_\alpha)_R} \odot F) + \lim M(\overline{(\nu * \theta_\alpha)_{S-R}} \odot F) \\
&= M(F) \\
N(F) &= P(M_L F) = \lim_\alpha M(\theta_\alpha \odot F) \\
&= \lim_\alpha M(\overline{\theta_{\alpha R}} \odot F) + \lim_\alpha M(\overline{\theta_{\alpha S-R}} \odot F) = M(F)
\end{aligned}
$$

Therefore N is a topological S-invariant mean on $M(X)^*$. $\square$

Chapter 4

Related Results and Open Problems

4.1 Definitions and Notations

Let S be a locally compact semitopological semigroup, X be a locally compact space. We assume the left action of S on X is separately continuous.

We denote by $CB(X)$ the Banach algebra of bounded continuous functions on X with supremum norm and pointwise product. Let $f \in CB(X)$, $s \in S$, $x \in X$, we define $l_s f(x) := f(sx)$, $r_x f(s) := f(sx)$.

Let A be a subspace of $CB(X)$, we say A is S-invariant if $l_s f \in A$ for any $s \in S$, $f \in A$.

Let $f \in CB(X)$, we say f is S-uniformly continuous if $s \to l_s f$ is continuous. We denote by $LUC(S, X)$ the space of all S-uniformly continuous functions on X. It is clear that $LUC(S, X)$ is a norm closed S-invariant subspace of $CB(X)$.

Let A be a subspace of $CB(X)$ containing constants. A functional $\phi \in A^*$ is a mean on A if it satisfies $\phi(\mathbb{1}) = \|\phi\| = 1$, where $\mathbb{1}$ is the constant 1 function on X. We denoted by $\mathfrak{M}(A)$ the set of all means on A. By Banach-Alaoglu theorem, $\mathfrak{M}(A)$ is weak* compact and convex. In this chapter, we usually write $\mathfrak{M}(X)$ for $\mathfrak{M}(LUC(S, X))$.

Suppose A is a subalgebra of $CB(X)$ containing constants, a mean $\phi \in \mathfrak{M}(A)$ is multiplicative if $\phi(fg) = \phi(f)\phi(g)$ for all $f, g \in A$.

Let A be a subspace of $CB(X)$ containing constants. Let $Q : X \to A^*$ by $Q_x(f) = f(x)$ for all $x \in X$, $f \in A$. It is easy to check that $Q_x \in \mathfrak{M}(A)$. Actually if A is in further an algebra, Q_x is a multiplicative mean on A, for all $x \in X$. Since $\mathfrak{M}(A)$ is convex, $Co\ \{Q_x\} \subset \mathfrak{M}(A)$, where $Co\ \{Q_x\}$ stands for the convex hull of $\{Q_x\}$. Means in $Co\ \{Q_x\}$ are called finite means.

Proposition 4.1.1. *1. The set of finite means is weak* dense in the set of all means on A.*

2. $QX := \{Q_x;\ x \in X\}$ is weak dense in the set of all multiplicative means on A.*

Proof. 1. Similar as the proof of Proposition 3.2.3

2. Paterson [26, 2.27] □

Throughout this section, we shall let A be a S-invariant subspace of $CB(X)$ containing constants. Let $s \in S$, $\phi \in \mathfrak{M}(A)$, we define $L_s\phi(f) = \phi(l_s f)$ for all $f \in A$. It is clear that $L_s\phi \in \mathfrak{M}(A)$. We say a mean ϕ is S-invariant if $L_s\phi = \phi$ for all $s \in S$. Let $\phi \in \mathfrak{M}(A)$, $f \in A$, we define $\phi_l f(s) = \phi(l_s f)$ for all $s \in S$.

Remark 4.1.2. Let $\phi \in \mathfrak{M}(X)$ and $\mathfrak{M}(X)$ be equipped with weak* topology. Then the map $T_\phi : S \to \mathfrak{M}(X)$ by $s \mapsto L_s\phi$ is continuous. This is not true for an arbitrary S-invariant subspace of $CB(X)$.

Corollary 4.1.3. *1. Let ϕ in $\mathfrak{M}(A)$. Then there exists a net of finite means $\{\phi_\alpha\}$ such that for each $f \in A$, we have $(\phi_\alpha)_l f \xrightarrow{pointwise} \phi_l f$*

2. Assume A is in further a subalgebra of $CB(X)$. Let ϕ be a multiplicative mean on A. Then there exists a net $\{x_\alpha\}$ in X, such that for each $f \in A$, we have $(Q_{x_\alpha})_l f \xrightarrow{pointwise} \phi_l f$.

Proof. 1. Since the set of finite means is weak* dense in $\mathfrak{M}(A)$ as we have shown in Proposition 4.1.1, let $\{\phi_\alpha\}$ be a net of finite means such that $\phi_\alpha \xrightarrow{weak^*} \phi$. Then $(\phi_\alpha)_l f(s) = \phi_\alpha(l_s f) \to \phi(l_s f) = \phi_l f(s)$.

2. Similar to the above argument, we finish the proof.

□

Proposition 4.1.4. *For each $f \in LUC(S,X)$, let $Z(f) := \{\phi_l f;\ \phi \in \mathfrak{M}(X)\} \subset LUC(S)$. Then $\mathcal{T}_p$ coincide with $\mathcal{T}_c$ on $Z(f)$, where $\mathcal{T}_p$ is the topology of pointwise convergence and $\mathcal{T}_c$ is the topology of uniform convergence on compacta.*

Proof. Let $\phi \in \mathfrak{M}(X)$, $f \in LUC(S,X)$,

$$|\phi_l f(s) - \phi_l f(t)| = |\phi(l_s f - l_t f)| \leq \|l_s f - l_t f\|$$

Thus $Z(f)$ is equicontinuous. Hence by Kelley [22, 232], $\mathcal{T}_p$ coincide with $\mathcal{T}_c$ on $Z(f)$. Moreover,

$$\begin{aligned} \|l_s(\phi_l f) - l_t(\phi_l f)\| &= \sup_{a \in S} |\phi_l f(sa) - \phi_l f(ta)| = \sup_{a \in S} |L_a \phi(l_s f) - L_a \phi(l_t f)| \\ &\leq \|l_s f - l_t f\| \end{aligned}$$

Therefore $Z(f) \subset LUC(S)$. □

Corollary 4.1.5. *Let $\phi \in \mathfrak{M}(X)$, then there exists a net of finite means $\{\phi_\alpha\}$ such that $(\phi_\alpha)_l f \to \phi_l f$ uniformly on compacta for all $f \in LUC(S,X)$. In particular, if ϕ is a multiplicative mean on $LUC(S,X)$, there exists a net $\{x_\alpha\}$ in X, such that $r_{x_\alpha} f \to \phi_l f$ uniformly on compacta.*

Proof. Directly apply Corollary 4.1.3 and Proposition 4.1.4. □

4.2 In the Convex Hull of Multiplicative Means

Throughout this section, we let X be a topological subspace under the separately continuous left action of a semitopological semigroup S. We shall focus our attention on the subalgebra $LUC(S,X)$ of $CB(X)$. We shall show that the existence of S-invariant means in the convex hull of multiplicative mean reflects the structure of S. We denote by $\Delta(X)$ the set of all multiplicative means on $LUC(S,X)$.

Lemma 4.2.1. *1. $\Delta(X)$ is weak* compact.*

2. Elements in $\Delta(X)$ are linear independent.

Proof. 1. Let ϕ_α be a net in $\Delta(X)$, since $\mathfrak{M}(X)$ is weak* compact, there exists $\phi \in \mathfrak{M}(X)$ such that $\phi_\alpha \xrightarrow{weak^*} \phi$. Let $f, g \in LUC(S,X)$, we have

$$\phi(fg) = \lim_\alpha \phi_\alpha(fg) = \lim_\alpha \phi_\alpha(f)\phi_\alpha(g) = \phi(f)\phi(g)$$

Thus $\phi \in \Delta(X)$ and $\Delta(X)$ is weak* closed in $\mathfrak{M}(X)$. Hence $\Delta(X)$ is weak* compact.

2. Let $\Delta(X)$ equipped with weak* topology. Consider the natural embedding $Q : LUC(S,X) \to C(\Delta(X))$ by $Qf(\phi) = \phi(f)$ for any $f \in LUC(S,X)$, $\phi \in \Delta(X)$. It is clear that Q is an isometric isomorphism when $C(\Delta(X))$ equipped with pointwise product and supremum norm. Then the adjoint operator Q^* of Q maps point measure on $C(\Delta(X))$ onto elements in $\Delta(X)$. Therefore elements in $\Delta(X)$ are linear independent by the linearity of Q^*. □

Theorem 4.2.2. *If $LUC(S,X)$ admits a S-invariant mean in the convex hull of $\Delta(X)$, then S can be decomposed into a finite union of disjoint open and closed cosets of a finite quotient group.*

Proof. Let $\psi \in \mathrm{Co}(\Delta(X))$, then $\psi = \sum_{i=1}^n a_i\phi_i$, where $a_i > 0$, $\sum_{i=1}^n a_i = 1$, $\{\phi_1, \ldots, \phi_m\}$ are distinct in $\Delta(X)$. Let $H = \{\phi_i\}_{i=1}^n$. Since elements in $\Delta(X)$ are independent, we have $L_sH = H$ for all $s \in S$.

Since H is finite, the restriction of L_s on H is bijective for all $s \in S$. We define an equivalence "$\sim$" on S by setting $s \sim t$ if $L_s\phi = L_t\phi$ for any $\phi \in H$. It is easy to check that "$\sim$" is a two sided equivalence by the bijectivity of L_s. Thus $S/(\sim)$ is finite and cancellative, hence it is a group. We let e denote the identity of $S/(\sim)$.

Let $\pi : S \to S/(\sim)$ be the canonical quotient map. Let $E := \pi^{-1}(e) = \{s \in S;\ L_s\phi = \phi,$ for all $\phi \in H\}$. Then E is a closed subsemigroup of S. Let $\{t_i\}_{i=1}^m$ be a set of representatives of $S/(\sim)$, where $t_1 \in E$. Since $S/(\sim)$ is a group, for each $s \in S$, there exist a unique i ($i \in \mathbb{N}, 1 \leq i \leq m$), such that $\pi(t_i s) = \pi(t_i)\pi(s) = e$. The first equivalence holds since "$\sim$" is a two-sided equivalence. Thus we have $S = \bigcup_{i=1}^m t_i^{-1}E$ and the family $\{t_i^{-1}E\}_{i=1}^m$ is pairwise disjoint. By the separate continuity of the action of S on X, we have $t_i^{-1}E$ is closed for all $1 \leq i \leq m$. Hence $E = S - \bigcup_{i=2}^m t_i^{-1}E$ is open. Further, $t_i^{-1}E$ is open for all $1 \leq i \leq m$, $i \in \mathbb{N}$. $\square$

4.3 Open Problems

The following problems are open.

1. Let S be a locally compact semitopological semigroup. When does $M(S)^*$ has a multiplicative S-invariant mean? We know that when S is discrete, this is equivalent with, for any $a, b \in S$, there exists $c \in S$, such that $ac = bc = c$ (see [16]). Do we have similar characterization for the existence of multiplicative S-invariant mean on $M(S)^*$ when S is a locally compact semitopological semigroup?

2. Let S be a locally compact semitopological semigroup, X be a locally compact space. The left action of S on X is separately continuous. Does topological X-stationary of a S-invariant subspace A implies the existence of topological S-invariant mean on A? Does the sublinear functional p on A always exists as we mentioned in Theorem 3.2.7?

3. Let S be a locally compact semitopological semigroup, X be a locally compact space. The left action of S on X is separately continuous. From Theorem 3.3.4, we know that we can identify $M(X)^*$ with $GL(X)$. Can we identify $mwp(S, X)$ with a subspace of $GL(X)$? We denote by $mwp(S, X)$ the set of function F in $M(X)^*$ such that $\{F \odot \mu;\ \mu \in P(X)\}$ is weakly relatively compact.

4. As we have shown in Theorem 3.4.14, for each topological S-invariant subset T of X, there exists a topological S-invariant subspace A of $M(X)^*$ containing χ_T, such that A has a topological S-invariant mean M and $M(\chi_T) = 1$. A natural question is for each [topological] S-invariant subspace A of $M(X)^*$ that has [topological] S-invariant means, does it guarantee the existence of [topological] (S, A)-lumpy subset T of X such that χ_T is included in A?

5. Let S be a locally compact semitopological semigroup, X is a locally compact space under separately continuous left action of S. It is known that $M(S)^*$ has a topological left invariant mean if and only if there exists a net $\mu_\alpha \in P(S)$, such that $\|\nu * \mu_\alpha - \mu_\alpha\| \to 0$ for all $\nu \in P(S)$. Does there exist a net $\mu_\alpha \in P(X)$ such that $\|\nu * \mu_\alpha - \mu_\alpha\| \to 0$ for all $\nu \in P(S)$ when $M(X)^*$ has a topological left invariant mean?

6. Suppose that S is a semitopological semigroup and for each $f \in LUC(S)$ there exists a mean $m_f = \frac{1}{n} \sum\limits_{i=1} n\phi_i$, where $\phi_i \in \Delta(S)$, such that $m_f(l_s f) = m_f(f)$ for all $s \in S$. Does $LUC(S)$ has a left invariant mean $m = \sum\limits_{i=1} n a_i \phi_i$, where $a_i \in \mathbb{R}$, $\phi_i \in \Delta(X)$?

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