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$\widehat{T}$-Surfaces in the Affine Grassmannian

by

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Abstract

In this thesis we examine singularities of surfaces and affine Schubert varieties in the affine Grassmannian $\mathcal{G}/\mathcal{P}$ of type $A^{(1)}$, by considering the action of a particular torus $\widehat{T}$ on $\mathcal{G}/\mathcal{P}$. Let Σ be an irreducible $\widehat{T}$-stable surface in $\mathcal{G}/\mathcal{P}$ and let u be an attractive $\widehat{T}$-fixed point with $\widehat{T}$-stable affine neighborhood Σ_u. We give a description of the $\widehat{T}$-weights of the tangent space $T_u(\Sigma)$ of Σ at u, give some conditions under which Σ is nonsingular at u, and provide some explicit criteria for Σ_u to be normal in terms of the weights of $T_u(\Sigma)$. We will also prove a conjecture regarding the singular locus of an affine Schubert variety in $\mathcal{G}/\mathcal{P}$.

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Chapter 1

Introduction

In the theory of algebraic groups, the flag variety plays an important role. Given an algebraic group G (eg. $\mathrm{GL}_n(\mathbb{C})$) and a Borel subgroup B (eg. upper triangular matrices) of G, we form the flag variety G/B. Now G (and hence any of its subgroups) acts as a group of transformations on G/B. One particularly important subgroup action is the action of a maximal torus T (eg. for $G = \mathrm{GL}_n(\mathbb{C})$, the subset of diagonal matrices) contained in B on G/B. The homogeneous space G/B has been studied by many authors and a key object in some of these investigations is the closure of a T-orbit. In particular, irreducible T-stable curves and surfaces contained in G/B are T-orbit closures which are well understood: the case for T-stable curves has been addressed by Carrell and Peterson in [4] and the case for T-surfaces has been covered by Carrell and Kurth in [5]. The theory of curves and surfaces has applications to the theory of Schubert varieties, combinatorics, and representation theory as outlined in [6] and [1].

Now let us consider an infinite dimensional analogue to the above situation in the context known as the Kac-Moody setting. The role of the flag variety is played by the affine Grassmannian $\mathcal{G}/\mathcal{P}$, where $\mathcal{G} := \mathrm{SL}_n(\mathbb{C}((x)))$ and $\mathcal{P} := \mathrm{SL}_n(\mathbb{C}[[x]])$ for some $n \in \mathbb{N}$. In this case, $\mathcal{G}/\mathcal{P}$ is an ind-variety, ie. a direct limit of finite dimensional projective varieties. The torus under consideration is $\widehat{T} = T \times S$, where T is the subset of $\mathcal{G}$ consisting of the diagonal matrices with constant entries and $S = \mathbb{C}^*$. The irreducible $\widehat{T}$-stable curves are well understood (cf. Proposition 12.1.7 in [10]) with a description similar to those in the classical case. As the next logical progression in complexity, we are interested in studying irreducible T-stable surfaces. In the classical setting, irreducible T-stable curves and surfaces are useful tools in understanding the singular loci of Schubert varieties. In the Kac-Moody setting, there is a natural generalization of the concept of a Schubert variety called affine Schubert variety. As was hoped would be the case, a firm understanding of T-surfaces is not only as instrumental in studying the singular loci of affine Schubert varieties as in the classical backdrop, but in fact, similar techniques can be employed.

Before we provide any specific details about the contents of this thesis we will establish some notation and state our universal assumptions. We will always work over $\mathbb{C}$. We will view varieties as sets of closed points. We assume that the set $\mathbb{N}$ of natural numbers does not include 0 and will write $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. Despite the fact that we work over $\mathbb{C}$, we will denote the coordinate ring of an affine variety X by $k[X]$ and its field of regular functions by $K(X)$. The tangent space of an affine variety X at a point $x \in X$, will be denoted by $T_x(X)$. We assume all algebraic groups are affine varieties. Whenever we discuss an action of an algebraic group on a variety we assume that it acts morphically. Finally, the Lie algebras of G, B, and T will be denoted by $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{h}$, respectively.

Returning to the classical case, in [5], Carrell and Kurth determine the singularities of T-stable surfaces in G/P, where P is a parabolic subgroup of G. In Chapter 3, we will examine in great detail a restatement of these findings in G/B as given in Proposition 5.2 in [6]. The technique applied to this problem is as follows: Given any T-stable surface Σ in the projective variety G/B and a T-fixed point u in Σ (which we identify with an element of the Weyl group $N_G(T)/T$), there exists an open affine T-stable neighborhood of u, denoted Σ_u. Now, Σ_u is a T-orbit closure and has the property that there is a T-equivariant embedding of Σ_u into $T_u(\Sigma)$. Since we may assume $u = e$, we have the following situation:

$$\Sigma_u \hookrightarrow T_u(\Sigma) \subset T_u(G/B) = \mathfrak{g}/\mathfrak{b} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha},$$

where Φ are the roots of G with respect to T. Thus, determining the singularities of Σ and the form of Σ_u reduces to root system considerations.

We will use the same technique as in the classical setting to examine singularities of $\widehat{T}$-stable surfaces in $\mathcal{G}/\mathcal{P}$, which is possible, in part, because any $\widehat{T}$-stable surface Σ in $\mathcal{G}/\mathcal{P}$ sits in a finite dimensional projective variety. Again, for any $\widehat{T}$-fixed point u in Σ we have that there is an open affine $\widehat{T}$-stable neighborhood of u such that

$$\Sigma_u \hookrightarrow T_u(\Sigma) \subset T_u(\mathcal{G}/\mathcal{P}),$$

only in this case the problem reduces to considering roots which result from the action of $\widehat{T}$ on $\mathfrak{g} \otimes \mathbb{C}[x, x^{-1}]$, where $\mathfrak{g}$ is the Lie algebra of $\mathrm{SL}_n(\mathbb{C})$. Although initial considerations deal with $\mathcal{G} = \mathrm{SL}_n(\mathbb{C}((x)))$, our results in Chapter 4 hold for $G(\mathbb{C}((x)))$, where G is any simply laced connected semi-simple algebraic group. We provide a description of the possible weights of $T_u(\Sigma)$ (via the weights of the dual space $T_u(\Sigma)^*$) in terms of the two weights that we know occur and place bounds on the number of weights that may occur. We state some conditions under which Σ is nonsingular at u and give a description of Σ_u in certain cases. In addition, we give explicit conditions on the weights of $T_u(\Sigma)^*$ under which Σ_u is normal.

As mentioned above, T-stable surfaces are used in studying singularities of Schubert varieties in G/B and this is also the case for G/P. In deed, in [6], Carrell and Kuttler use T-stable surfaces in conjunction with an object called a Peterson translate of a Schubert variety to prove a generalized version of Peterson's ADE-Theorem (describes the singular locus of a Schubert variety in G/P). In the affine case, Kuttler and Lakshmibai conjectured as to the singular locus of a Schubert variety X in $\mathcal{G}/\mathcal{P}$ (Remark 4.19 in [11]). In Chapter 5, using our partial description of the regular locus of a $\widehat{T}$-surface contained X, we provide a proof of this remark, which utilizes the same technique as the proof of the ADE-Theorem.

In Chapter 2, we will present some well-known results about torus actions, T-orbit closures, and T-fixed points. In particular, we will discuss the notion of an attractive fixed point, whose usefulness alone makes it deserving of its name. In addition to presenting the proof of Proposition 5.2 from [6], we present some facts about G/B and $\mathfrak{g}/\mathfrak{b}$ in Chapter 3. In Chapter 4, we will comment on how the techniques in Chapter 3 translate into the new setting and present our findings on the $\widehat{T}$-stable surfaces in $\mathcal{G}/\mathcal{P}$ and the open affine $\widehat{T}$-stable neighborhood of a $\widehat{T}$-fixed point. In Chapter 5, we discuss the notion of a Peterson translate of a Schubert variety along a $\widehat{T}$-stable curve and provide a proof of Remark 4.19 in [11].

Chapter 2

Preliminaries

We will begin this chapter with the definition of a torus, some important properties of objects associated with a torus, and some of the basic properties of torus actions on vector spaces and on affine and projective varieties. Throughout Chapters 2 and 3 we will use facts about linear algebraic groups and varieties which can be found in the books by Borel (see [2]), Humphreys (see [8]), and Hartshorne (see [7]). All of the results presented in these two chapters are well-known and the proofs have been provided for the pleasure of the reader.

2.1 Tori

Definition 2.1. A *torus* is an algebraic group which is isomorphic to $(\mathbb{C}^*)^n$ for some $n \in \mathbb{N}$. An algebraic group is called *diagonalizable* if it is isomorphic to a closed subgroup of a torus.

An example of a torus is $\mathrm{D}_n(\mathbb{C})$, the subgroup of the general linear group $\mathrm{GL}_n(\mathbb{C})$ consisting of $n \times n$ diagonal matrices. Every finite abelian group is a diagonalizable group. Any connected diagonalizable group over $\mathbb{C}$ is a torus and since any homomorphic image of a connected diagonalizable group is again connected and diagonalizable, any homomorphic image of a torus is again a torus. In particular, connected subgroups and quotients of a torus are tori.

Of particular interest in this thesis is the case in which $T \subset B \subset G$, where G is a connected semi-simple algebraic group, B a Borel subgroup, and T a maximal torus. We will discuss this case further in Chapter 3.

2.2 Characters and One-Parameter Subgroups

A *character* of an algebraic group G is a homomorphism $\chi : G \to \mathbb{C}^*$. The set of all characters of G, denoted $X(G)$, forms an abelian group under the rule $(\chi_1 + \chi_2)(g) = \chi_1(g)\chi_2(g)$, for all $g \in G$, and can be viewed as a subset of the coordinate ring $k[G]$. Note that $(-\chi)(g) = \chi(g)^{-1} = \chi(g^{-1})$. The characters of a torus $(\mathbb{C}^*)^n$ have the form

$$\chi(c_1, c_2, \ldots, c_n) = c_1^{a_1} c_2^{a_2} \cdots c_n^{a_n}$$

for some $a_1, a_2, \ldots, a_n \in \mathbb{Z}$. It is clear that $X(T)$ is a finitely generated group isomorphic to $\mathbb{Z}^n$ (so $X(T)$ is a torsion-free $\mathbb{Z}$-module) and the generators are the projections

$$\pi_i : (c_1, c_2, \ldots, c_n) \mapsto c_i.$$

In particular, $X(\mathbb{C}^*) \simeq \mathbb{Z}$. The character group gives us another method of identifying a torus, ie. a torus is an abelian group whose character group is a torsion-free $\mathbb{Z}$-module with rank equal to the dimension of the group as a variety. Likewise we can define the concept of a diagonalizable group in terms of its characters. Indeed, a diagonalizable group is an algebraic group G such that $X(G)$ spans the coordinate ring $k[G]$. Note that there is an anti-equivalence between the category of diagonalizable groups and the category of finitely generated abelian groups where A corresponds to $X(A)$.

A *one-parameter subgroup* of a torus T is a homomorphism $\lambda : \mathbb{C}^* \to T$. Every one-parameter subgroup λ is given by

$$\lambda(c) = (c^{b_1}, c^{b_2}, \ldots, c^{b_n}),$$

for some $b_1, b_2, \ldots, b_n \in \mathbb{Z}$. As with characters, the set of all one-parameter subgroups of T, denoted $Y(T)$, forms a finitely generated abelian group, with generators

$$\gamma_i : c \mapsto (1, 1, \ldots, 1, c, 1, \ldots, 1),$$

where c is in the i^{th} entry.

It is well-known that $X(T)$ and $Y(T)$ are dual $\mathbb{Z}$-modules, ie. $X(T) \simeq Y(T)^* = \mathrm{Hom}_{\mathbb{Z}}(Y(T), \mathbb{Z})$, or equivalently there is a non-degenerate pairing $X(T) \times Y(T) \to X(\mathbb{C}^*)$ given by $(\chi, \lambda) \mapsto \langle \chi, \lambda \rangle$, where $\langle \chi, \lambda \rangle$ is the integer which corresponds to $\chi \circ \lambda \in X(\mathbb{C}^*)$, ie.

$$\begin{aligned} \chi \circ \lambda : \mathbb{C}^* &\to \mathbb{C}^*. \\ c &\mapsto c^{\langle \chi, \lambda \rangle} \end{aligned}$$

2.3 Actions on Vector Spaces

We will frequently make use of finite dimensional complex vector spaces with torus actions. Indeed, they possess a number of useful properties and will play a key role in subsequent discussions as we will often be able to reduce problems concerning varieties with T-actions to problems concerning vector spaces. Given a finite dimensional representation $\rho : T \to \mathrm{GL}(V)$, we define a linear action of T on V by the rule $t \cdot v = \rho(t)(v)$. A vector space with such a T-action is called a T-module. One particularly important T-action occurs when T is a subgroup of an algebraic group G. If we restrict the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ to T, we obtain an action of T on $\mathfrak{g}$.

We begin our examination of T-modules with a well-known result, which will be applied ad nauseam.

Lemma 2.2. *Let T be a torus and V a T-module, then*

$$V = \bigoplus_{\alpha \in X(T)} V_\alpha,$$

where

$$V_\alpha = \{v \in V \mid t \cdot v = \alpha(t)v, \text{ for all } t \in T\}.$$

This decomposition is referred to as the *weight space decomposition* of V. The α for which $V_\alpha \neq 0$ are called the *weights* of T in V (or simply the weights of V, if the torus involved is clear) and V_α is called a *weight space*. We denote the set of weights by $\Omega(V)$. If $V = \mathfrak{g}$, then the nonzero elements of $\Omega(\mathfrak{g})$ are called the *roots* of G relative to T and the set of roots will be denoted Φ.

Remark 2.3. If $T = \mathbb{C}^*$, then we can specify a $\mathbb{Z}$-grading on a T-module V by defining $V_d := V_\alpha$, where $\alpha(c) = c^d$. For an arbitrary T, given a $\lambda \in Y(T)$, we can make any T-module into a $\mathbb{C}^*$-module by defining $c \cdot v := \lambda(c) \cdot v$. The induced $\mathbb{Z}$-grading on V in terms of the weight space decomposition of V with respect to T is then

$$V_d = \bigoplus_{\substack{\alpha \in X(T) \\ \langle \alpha, \lambda \rangle = d}} V_\alpha.$$

In either case, define

$$V^+ := \bigoplus_{d>0} V_d.$$

Of course, if we want to apply Lemma 2.2 to a subspace of a T-module, then the subspace itself must be a T-module, ie. it must be T-stable. If this is the case, then the structure of the subspaces is known. Given a T-stable subspace W of a T-module V, by definition we have that $\Omega(W) \subseteq \Omega(V)$ and that

$W_\alpha \subseteq V_\alpha$, for every $\alpha \in \Omega(V)$. Thus $W_\alpha = V_\alpha \cap W$ and hence

$$W = \bigoplus_{\alpha \in X(T)} (W \cap V_\alpha).$$

A T-module V is called *multiplicity free* if $\dim V_\alpha = 1$, for all $\alpha \in \Omega(V)$. In this event, the T-stable subspace $W \cap V_\alpha$ of W is either $\{0\}$ or V_α. This gives us the following lemma:

Lemma 2.4. *Suppose* $V = \bigoplus_{\alpha \in X(T)} V_\alpha$ *is a multiplicity free* T*-module. If* W *is a* T*-stable subspace of* V*, then* $W = \bigoplus_{\alpha \in \Gamma} V_\alpha$*, where* $\Gamma \subseteq \Omega(V)$*.*

When dealing with a T-module V, it is natural to consider the action of T on the dual space V^* of V and on the quotient V/W of V by a T-stable subspace W. The action of T on V^* is defined by $(t \cdot f)(v) = f(t^{-1} \cdot v)$ for all $t \in T$, $f \in V^*$, $v \in V$ and the action of T on V/W is defined in the obvious way. In particular, we want to be able to relate the weights of V^* and V/W to the weights of V. We will begin with the weight space decomposition of V^*.

Lemma 2.5. *Let* V *be a* T*-module, then* $V^* = \bigoplus_{\alpha \in \Omega(V)} (V_\alpha)^*$*.*

Note that $\Omega((V_\alpha)^*) = \{-\alpha\}$. We now move on to consider the weight space decomposition of a quotient space.

Lemma 2.6. *Let* V *be a* T*-module and* W *a* T*-stable subspace. Then* $(V/W)_\alpha = V_\alpha / W_\alpha$*.*

Remark 2.7. As a consequence of this, $\Omega(V/W) \subseteq \Omega(V)$.

2.4 T-Orbit Closures

In this section we will examine some of the properties of T-orbits and their closures and consider torus actions on projective and affine varieties. We will first fix some notation and define some basic terms. Let X be a variety with a T-action and let $x \in X$. We will use X^T, $T \cdot x$, and T_x to denote the set of fixed points of T, the orbit of x, and the stabilizer of x, respectively. A map $f : X \to Y$ of sets with T-actions for which $t \cdot (f(x)) = f(t \cdot x)$ for all $x \in X$, $t \in T$, is called T*-equivariant*. An irreducible T-stable curve (resp. surface) is called a T*-curve* (resp. T*-surface*). Although our focus is on T-actions, we begin with some useful and well-known properties which apply to G-actions for any algebraic group G.

Lemma 2.8. *Let G be any algebraic group and let X be variety with a G-action.*

1) *G-orbits are open in their closures.*
2) *G-orbit closures are G-stable.*
3) *Every G-orbit closure contains a closed orbit.*
4) *G-orbits are irreducible, if G is connected.*
5) *X^G is closed in X.*
6) *G_x is a closed subgroup of G, for all $x \in X$.*

Proof. See chapters 7 and 8 of [8]. □

As with all group actions, we have the Orbit-Stabilizer Theorem:

$$G/G_x \simeq G \cdot x$$

(as varieties with a G-action).

So in the case that G is a torus, it follows immediately from the Orbit-Stabilizer Theorem that a T-orbit, $T \cdot x$, is isomorphic to the torus T/T_x as a variety. This fact proves helpful when determining the dimension of a T-orbit. Let V be a T-module and let $v \in V$. Then $v = \sum v_\alpha$, where $v_\alpha \in V_\alpha$, and the set $s(v) := \{\alpha \mid v_\alpha \neq 0\}$ in $X(T)$ is called the *support* of v. Now let M be the $\mathbb{Z}$-module generated by the support $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ of v. We will use the notation $M = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$. The following lemma provides us with a nice relationship between $T \cdot v$ and M.

Lemma 2.9. *Suppose v is an element of a T-module V with support $s(v) = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. Then $\dim T \cdot v = \operatorname{rank} M$, where $M = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$.*

Proof. Let v be an element of a T-module V with support $s(v) = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and let $M = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$. As a consequence of the Orbit-Stabilizer Theorem we have that

$$\dim T \cdot v = \dim T - \dim T_v.$$

Recall that there is an anti-equivalence of categories between the categories of finitely generated abelian groups and diagonalizable groups. Thus, given the inclusion map $\iota : T_v \hookrightarrow T$, we obtain the map $X(T) \twoheadrightarrow X(T_v)$, given by $\alpha \mapsto \alpha|_{T_v}$ whose kernel will be denoted as K.

The claim is that $K = M$, from which it follows that

$$\dim T_v = \operatorname{rank} X(T_v) = \operatorname{rank} X(T) - \operatorname{rank} M = \dim T - \operatorname{rank} M$$

and subsequently $\dim T \cdot v = \operatorname{rank} M$.

To see that the claim is true we first note that

$$T_v = \bigcap_{\alpha \in M} \ker \alpha$$

and so $M \subseteq K$.

For the other direction, since M is a finitely generated abelian group, $M = X(S)$, for some diagonalizable group S. So we have the inclusion map

$$\varphi^* : X(S) \hookrightarrow X(T)$$

with corresponding map

$$\varphi : T \twoheadrightarrow S.$$

Let $H = \ker \varphi$. From the exact sequence

$$1 \to H \hookrightarrow T \twoheadrightarrow S \to 1,$$

we obtain the exact sequence

$$0 \to X(S) \hookrightarrow X(T) \twoheadrightarrow X(H) \to 0.$$

Therefore, $X(S) = \{\alpha \in X(T) \mid \alpha|_H = 0\}$.

Now for all $\alpha \in X(S), h \in H$, we have that $\varphi^*(\alpha)(h) = \alpha(\varphi(h)) = 1$ and hence $H \subseteq T_v$. Accordingly, if $\alpha \in K$, ie. $\alpha|_{T_v} = 0$, then $\alpha|_H = 0$ and so $\alpha \in X(S)$. Consequently, $K \subseteq X(S) = M$. $\square$

Remark 2.10. It follows immediately that the dimension of a T-orbit $T \cdot v$ is 1 if and only if the elements of the support of V are proportional over $\mathbb{Q}$ and at least one is nonzero.

In addition to considering T-actions on varieties, we want to consider torus actions on objects associated with these varieties. The coordinate ring $k[X]$ of an affine variety X with T-action and the tangent space $T_x(X)$ of X at the point $x \in X$ will be instrumental in subsequent sections.

The action of T on $k[X]$ is defined to be $(t \cdot f)(x) = f(t^{-1} \cdot x)$, for all $t \in T$, $f \in k[X]$, and $x \in X$. It is well-known that T acts locally finite (ie. for each $f \in k[X]$, there exists a finite dimensional subspace of $k[X]$ that contains $T \cdot f$). In particular, this gives us that $k[X]$ is the union of T-stable finite dimensional subspaces. Also, the action is rational, so when T acts on these subspaces, we obtain a weight space decomposition for each and hence

$$k[X] = \bigoplus_{\alpha \in X(T)} k[X]_\alpha.$$

The action of T on the quotient field $K(X)$ is defined by $t\cdot(f/g) = (t\cdot f)/(t\cdot g)$. If f and g are elements of $k[X]$ of weight of α and β, respectively, then f/g has weight $\alpha - \beta$.

Let $t \in T$ and use t to denote the map $t : X \to X$ given by $y \mapsto t \cdot y$. Then for $x \in X^T$, the action of T on $T_x(X)$ is defined by $t \cdot \delta = d_x t(\delta)$, where $d_x t$ is the differential of the map t at x. If Y is a T-stable subvariety of X and $y \in Y^T$, then $T_y(Y)$ is a T-stable subspace of $T_y(X)$.

Now let $x \in X^T$ and let $\mathfrak{m}_x$ be the maximal ideal in $k[X]$ consisting of elements that vanish at x. Thus, $\mathfrak{m}_x$ is T-stable since if $f \in \mathfrak{m}_x$, then

$$(t \cdot f)(x) = f(t^{-1} \cdot x) = f(x) = 0.$$

It follows that $\mathfrak{m}_x^2$ is also T-stable and so $\mathfrak{m}_x$ and $\mathfrak{m}_x/\mathfrak{m}_x^2$ are T-modules with weight space decompositions such that $\Omega(\mathfrak{m}_x/\mathfrak{m}_x^2) \subseteq \Omega(\mathfrak{m}_x)$. Recall that $\mathfrak{m}_x/\mathfrak{m}_x^2 \simeq T_x(X)^*$ by the identification $\bar{f} \mapsto d_x f$, where $\bar{f}$ is the coset $f + \mathfrak{m}_x^2$. Thus if $d_x f \neq 0$ is an element of $(T_x(X)^*)_\omega$, for some $\omega \in \Omega(T_x(X)^*)$, then we obtain a nonzero $f \in (\mathfrak{m}_x)_\omega$ which is then also an element of $k[X]_\omega$.

Remark 2.11. Any weight of $T_x(X)^*$ is a weight of $k[X]$.

This is significant as it allows us to use results about the weights of $k[X]$ to understand the weights of $T_x(X)^*$.

We now turn our attention to torus actions on varieties. We begin with an important definition.

Definition 2.12. A *T-variety* X is an irreducible variety with a T-action such that X^T is finite and for all $x \in X^T$ there is an open T-stable affine neighborhood of x, which we will denote by X_x.

Our work in subsequent chapters relies heavily on T-varieties and so we will now present many of their properties that we will require.

Given a vector space V of dimension $n+1$, projective space $\mathbb{P}^n$ can be identified with the set $\mathbb{P}(V)$ of all lines $[v]$ through the origin in V. If V is a T-module, then the action of T on $\mathbb{P}(V)$ is given by $t \cdot [v] = [t \cdot v]$.

Lemma 2.13. *Let V be a multiplicity free T-module, then any irreducible T-stable subvariety of $\mathbb{P}(V)$ is a T-variety.*

Proof. Once we have proven this for $\mathbb{P}(V)$, then the result follows immediately for any irreducible T-stable subvariety X since $X^T \subseteq \mathbb{P}(V)^T$ is finite and for each $x \in X^T$, a T-stable open affine neighborhood in X is $X \cap U$, where U is a neighborhood in $\mathbb{P}(V)$.

Let $x \in \mathbb{P}(V)^T$. The T-fixed points of $\mathbb{P}(V)$ are the T-stable lines of V. It follows from Lemma 2.4, that the only T-stable lines in a multiplicity free T-module V are the weight spaces V_α, of which there are finitely many. For this reason, $\mathbb{P}(V)^T$ is finite. We fix a T-homogeneous coordinate system on V and denote the dual basis by $\{x_0, x_1, \ldots, x_n\}$. Every point in projective space has an open affine neighborhood U which is the complement of a hyperplane $Z(x_i)$. To see that this is T-stable let $[u] \in U$, then $x_i(t \cdot u) = \alpha_i(t)x_i(u)$, giving that $t \cdot [u] \in U$. $\square$

Lemma 2.15 below is a crucial element of the proof of Proposition 5.2 in [6] and of our analogous work in the affine setting. In order to prove Lemma 2.15, we will use the next lemma, which follows from Theorem 2.2 and Corollary 2.3 in [12].

Lemma 2.14. *If an irreducible affine variety X has a G-action such that the maximal dimension of an orbit is d, then the transcendence degree of $K(X)^G$ over $\mathbb{C}$ is equal to $\dim X - d$.*

Lemma 2.15. *Let X be an affine T-variety. The following are equivalent:*

1) *X has finitely many orbits.*

2) *X has an open dense orbit.*

3) *$k[X]$ is multiplicity free.*

Proof.
For 1) implies 2):
Since $X = \bigcup(T \cdot x)$, if X has finitely many orbits, then $X = \overline{\bigcup(T \cdot x)} = \bigcup(\overline{T \cdot x})$. Consequently, $X = \overline{T \cdot x}$, for some $x \in X$, due to the fact that X is irreducible and thus by Lemma 2.8 1), $T \cdot x$ is an open dense orbit of X.

For 2) implies 3):
Assume that X has an open dense orbit $T \cdot x$. Let $f, g \neq 0 \in k[X]_\alpha$ and let $y \in T \cdot x$ such that $g(y) \neq 0$. Thus $f(t^{-1} \cdot y) = \alpha(t)f(y)$ and $g(t^{-1} \cdot y) = \alpha(t)g(y)$ and so we have

$$f(t^{-1} \cdot y) = \frac{f(y)}{g(y)} g(t^{-1} \cdot y).$$

Consequently, since the action of T is transitive on $T \cdot x$,

$$f = \frac{f(y)}{g(y)} g$$

on the open dense orbit and hence on its closure X. Thus $\dim k[X]_\alpha = 1$.

For 3) implies 1):
Suppose that $k[X]$ is multiplicity free and let $T \cdot x$ be an orbit in X of maximal dimension. Set $d := \dim(T \cdot x)$. Now let $f \in K(X)^T$, so $f = f_1/f_2$ for some $f_1, f_2 \in k[X]$ such that $f_2 \neq 0$. Then the set $V := \{g \in k[X] \mid gf \in k[X]\}$ is a nonzero T-stable subspace of $k[X]$ and so there exists a nonzero $g \in V_\alpha$ for some α. Thus $fg \in (k[X])_\alpha$, but since $k[X]$ is multiplicity free, $fg = ag$ for some $a \in \mathbb{C}$ and hence $f \in \mathbb{C}$. Consequently, $K(X)^T = \mathbb{C}$, and so has transcendence degree 0. Therefore, by Lemma 2.14, $\dim X = d$ which implies that $T \cdot x$ is dense in X. If $d = 0$, then X is finite and hence has finitely many orbits. Now proceeding by induction on the dimension of X, we consider the closed subvariety $Y = X \backslash (T \cdot x)$, whose coordinate ring is multiplicity free since it is a quotient of $k[X]$. By induction, each of the finitely many irreducible components of Y have finitely many orbits. Hence Y has finitely many orbits and subsequently the same is true for X. $\square$

2.5 T-Fixed Points

We have and will focus a great deal of attention on T-fixed points as they are indicators of whether or not a variety possesses certain properties. Before we discuss their use, we will establish that they actually occur.

Lemma 2.16. *Every T-stable closed subset of a projective T-variety X contains a T-fixed point. In particular, $X^T \neq \emptyset$.*

Proof. Let Y be a T-stable subvariety of a projective variety X and let $y \in Y$. If $y \in Y^T$, we are done. Otherwise, since Y is T-stable, $T \cdot y \subseteq Y$ and hence $\overline{T \cdot y} \subseteq Y$, since Y is closed. Now $\overline{T \cdot y}$ contains a closed orbit $T \cdot z$ (Lemma 2.8, 3)) which is simultaneously an irreducible projective and affine variety. Therefore, $k[T \cdot z] = \mathbb{C}$, which implies that $T \cdot z$ is a single point, a T-fixed point. $\square$

Note that this is a special case of Borel's fixed point theorem. Now that we know that they occur, we shall examine their usefulness.

Lemma 2.17. *If X is a T-variety, then the set of singular points* $\operatorname{Sing} X$ *of X is a proper closed T-stable subset.*

Proof. For a proof that $\operatorname{Sing} X$ is a proper closed subset of X, see Theorem 5.3 of [7]. To see that it is T-stable note that for any $t \in T$ the map $\psi : X \to X$ given by $x \mapsto t \cdot x$ is an automorphism and hence its differential at any $x \in \operatorname{Sing} X$ is an isomorphism. Consequently, $\dim X \neq \dim T_x(X) = \dim T_{t \cdot x}(X)$. $\square$

Since every T-stable closed subvariety of a projective T-variety contains a T-fixed point, it follows from this lemma that if $\operatorname{Sing} X$ is nonempty then it has a T-fixed point. Thus, proving a variety is nonsingular reduces to showing that none of the T-fixed points are singular. Of course, the importance of T-fixed points is not limited to identifying the presence of singular points. Whenever it is the case that the set of all points with a particular property forms a closed T-stable subset, if there is a point with that property, then there is a T-fixed point with that property.

2.6 T-Curves

Our focus in the next two chapters will be T-surfaces and in our explorations we will exploit the properties of the T-curves contained in them. As we are interested in T-actions, it is only fitting that we begin with a Lemma that relates T-curves to T-orbits.

Lemma 2.18. *Let C be a T-curve such that $C \neq C^T$, then C is a T-orbit closure.*

Proof. Clear. □

Remark 2.19. Since, by definition, a T-variety has only finitely many T-fixed points, any curve in a T-variety is the closure of a T-orbit.

We have focused a great deal of attention on the properties of T-modules and have mentioned that we want to relate our objects, such as T-curves, to vector spaces. One reason for this is given the following lemma.

Lemma 2.20. *Suppose that V is a multiplicity free T-module such that no two weights are proportional, then the only T-curves contained in V are the weight spaces V_α.*

Proof. Suppose C is a T-curve in V. If $C = C^T$, then $C \subseteq V_0$. Hence $C = V_0$ for dimensional reasons and so $V = V_0$ as 0 is proportional to all other weights. Otherwise, by Lemma 2.18, $C = \overline{T \cdot v}$, for some $v = \sum v_\alpha \neq 0 \in V$, where $v_\alpha \in V_\alpha$. Suppose that the support of v, $s(v) = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, contains at least two elements. Since the weights of V are nonproportional, $M = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$ has rank at least 2, but then

$$\dim \overline{T \cdot v} = \dim T \cdot v = \operatorname{rank} M \geq 2,$$

by Lemma 2.9, which contradicts the fact that $\dim \overline{T \cdot v} = 1$. Thus the support of v consists of one element, ie. $v \in V_\alpha$, for some $\alpha \in \Omega(V)$, and since V_α is T-stable, closed, and has dimension 1, $\overline{T \cdot v} = V_\alpha$. □

Denote by $E(X, x)$ the set of T-curves in a T-variety X containing a point $x \in X^T$. It follows that for any multiplicity free T-module V such that no two weights are proportional, $|E(V, 0)|$ is finite.

Of course, we can have many beautiful results regarding T-curves, but these results are useless unless we know that T-curves actually occur. The final lemma of this section guarantees that T-varieties of dimension at least 1 with a T-fixed point contain a T-curve through that point.

Lemma 2.21. *Let $x \in X^T$ where X is a T-variety, then $|E(X, x)| \geq \dim X$.*

Proof. See Lemma 2 in [4]. □

2.7 Attractive Points

Definition 2.22. Let X be a T-variety. A T-fixed point x and X_x are called *attractive* if there is a $\lambda \in Y(T)$ such that $\langle \alpha, \lambda \rangle > 0$, for all $\alpha \in \Omega(T_x(X))$.

Lemma 2.23. *If X is an affine variety with T-action, then there is a T-equivariant embedding $X \hookrightarrow V$, for some T-module V. In the case that x is an attractive fixed point of some T-variety X, then X_x embeds into $T_x(X)$.*

Proof. Since X is an affine variety, its coordinate ring $k[X]$ is a finitely generated $\mathbb{C}$-algebra and each generator is contained in a finite dimensional T-stable subspace. Taking V to be the sum of these subspaces, we have obtained a finite dimensional T-stable subspace which generates $k[X]$ as a $\mathbb{C}$-algebra. Since $V \simeq V^{**}$ generates $k[V^*]$ as a $\mathbb{C}$-algebra, we obtain a surjective T-equivariant $\mathbb{C}$-algebra homomorphism $k[V^*] \twoheadrightarrow k[X]$ which yields the T-equivariant embedding $X \hookrightarrow V^*$.

Assume x is attractive and so there is a $\lambda \in Y(T)$ such that $\langle \alpha, \lambda \rangle > 0$, for all $\alpha \in \Omega(T_x(X))$. Let $\mathfrak{m}_x$ be the ideal of vanishing of x. Using $-\lambda$ to obtain a grading on $T_x(X)$, we have that $T_x(X)$ is negatively graded, ie. $(T_x(X))_d = 0$, for all $d \geq 0$, and hence $(\mathfrak{m}_x/\mathfrak{m}_x^2)_d = (T_x(X)^*)_d = 0$, for all $d \leq 0$. Let $S := \{\bar{f}_i\}_{i=1}^m$ be generators of $\mathfrak{m}_x/\mathfrak{m}_x^2$ (chosen so that $f_i \in (k[X_x])_{d_i}$, where $d_i > 0$). The claim is that for any choice of homogeneous representatives, the set $\{f_i\}_{i=1}^m$ generates $k[X_x]$ as a $\mathbb{C}$-algebra. Given that the claim holds, taking $V := \text{Span}(S)$, we get a surjective T-equivariant $\mathbb{C}$-algebra homomorphism $k[V^*] \twoheadrightarrow k[X_x]$ and hence we have a T-equivariant embedding $X_x \hookrightarrow V^* \simeq T_x(X)$.

To see that the claim holds, we first note that for any $\ell \in \mathbb{N}$, $(\mathfrak{m}_x)^\ell/(\mathfrak{m}_x)^{\ell+1}$ is generated by the images of homogeneous polynomials of degree ℓ in the functions f_i. Indeed, any element of $\mathfrak{m}_x$ has the form $f = \sum a_i f_i + \tilde{f}$ where $a_i \in \mathbb{C}$

and $\tilde{f} \in \mathfrak{m}_x^2$ and so any element g of $\mathfrak{m}_x^\ell$ has the form $g = \sum g_1 g_2 \cdots g_\ell$, where $g_j = \sum a_{ji} f_i + \tilde{g}_j$. Multiplying this out one observes that $g = p(f_1, f_2, \ldots, f_m) + \tilde{g}$ where $p(f_1, f_2, \ldots, f_m)$ is a homogeneous polynomial of degree ℓ in the functions f_i and $\tilde{g} \in (\mathfrak{m}_x)^{\ell+1}$. Consequently, $(\mathfrak{m}_x)^\ell/(\mathfrak{m}_x)^{\ell+1}$ is generated by the images of monomials of degree ℓ in the functions f_i. As a consequence of this, if $\bar{f} \neq 0 \in \mathfrak{m}_x^\ell/\mathfrak{m}_x^{\ell+1}$ is homogeneous of degree d (with respect to the grading on $k[X_x]$ obtained from $-\lambda$), then

$$\bar{f} = \sum_{j=1}^{k} a_j \prod_{i=1}^{m} \bar{f}_i^{\ell_{ij}},$$

where $\sum \ell_{ij} = \ell$, for each j, and $d = \sum d_i \ell_{ij}$. Thus, since $d_i > 0$, $d \geq \ell$.

We will prove by induction that for each $\ell \in \mathbb{Z}_{>0}$, $k[X_x] = \mathfrak{m}_x^\ell \oplus U_\ell$, where

$$U_\ell \subseteq \mathrm{Span}(\text{polynomials in the functions } f_i \text{ of degree strictly less than } \ell)$$

is T-stable. For $\ell = 1$, $k[X_x]/\mathfrak{m}_x \simeq \mathbb{C}$ and the base case holds. For $\ell > 1$, we have $k[X_x] = \mathfrak{m}_x^{\ell-1} \oplus U_{\ell-1}$ and from the above discussion we have that $\mathfrak{m}_x^{\ell-1} = \mathfrak{m}_x^\ell \oplus V_{\ell-1}$, where

$$V_{\ell-1} \subseteq \mathrm{Span}(\text{monomials of degree } \ell-1 \text{ in the functions } f_i).$$

It follows that $k[X_x] = \mathfrak{m}_x^\ell \oplus V_{\ell-1} \oplus U_{\ell-1}$, so taking $U_\ell = V_{\ell-1} \oplus U_{\ell-1}$, we are done.

Finally, let $g \in (k[X_x])_d$. Assume $g \in \mathfrak{m}_x^\ell$, for some $\ell > d$. But then there exists a k such that $g \in \mathfrak{m}_x^k \setminus \mathfrak{m}_x^{k+1}$ (embed $k[X_x]$ into the Noetherian local ring $k[X_x]_{\mathfrak{m}_x}$), so that $d \geq k \geq \ell$ by the above argument, which gives us a contradiction. Thus, there exists an ℓ such that $\mathfrak{m}_x^\ell$ contains no element of degree d. Consequently, $g \in U_\ell$ and hence $k[X_x]$ is generated as a $\mathbb{C}$-algebra by $\{f_i\}_{i=1}^m$, as required. $\square$

Remark 2.24. If x is a smooth attractive T-fixed point of a T-variety X, then $X_x \simeq T_x(X)$ (T-equivariantly).

We will state two equivalent notions to Definition 2.22, but first we provide some clarifying remarks. Given a morphism $f : \mathbb{C}^* \to X$ of varieties, we use the expression

$$\lim_{c \to 0} f(c) = y$$

to mean that we extend f to a morphism $\tilde{f} : \mathbb{C} \to X$, by defining $\tilde{f}(0) = y$. Working in a vector space, we have that this gives us the usual notion of a limit.

Also, to say that a subset S of $\Omega(V)$ lies on one side of a hyperplane in

$X(T) \otimes \mathbb{Q} \simeq \mathbb{Q}^n$ means that there is a linear function on $\mathbb{Q}^n$ that is strictly positive on S.

Lemma 2.25. *Let X be a T-variety and let $x \in X^T$. The following are equivalent:*

1) *x is attractive.*
2) *$\Omega(T_x(X))$ lies on one side of a hyperplane in $X(T) \otimes \mathbb{Q}$.*
3) *There exists a $\lambda \in Y(T)$ such that $\lim\limits_{c \to 0} \lambda(c) \cdot y = x$, for all $y \in X_x$.*

Proof.
For 1) implies 2):
Use $\langle \cdot, \lambda \rangle$ extended $\mathbb{Q}$-linearly.

For 2) implies 1):
Let $T \simeq (\mathbb{C}^*)^n$, so that $X(T) \simeq \mathbb{Z}^n$. Assume $\Omega(T_x(X))$ lies on one side of a hyperplane f in $\mathbb{Q}^n$. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{Q}^n$. Suppose $f(e_i) = \frac{a_i}{b_i}$, for some $a_i \in \mathbb{Z}, b_i \in \mathbb{N}$ and let $m = b_1 b_2 \cdots b_n$. Thus $\tilde{f}$, defined by $\tilde{f}(e_i) = mf(e_i)$, when restricted to $\mathbb{Z}^n$ is an element of $Y(T)$ which is strictly positive on $\Omega(T_x(X))$.

For 1) implies 3):
By Lemma 2.23, there is a T-equivariant embedding of X_x into $T_x(X)$. Let $y = \sum v_{\alpha_i} \in X_x$, for some $\alpha_i \in \Omega(T_x(X))$. Therefore,

$$\lambda(c) \cdot y = \lambda(c) \cdot \sum v_{\alpha_i} = \sum \alpha_i(\lambda(c)) v_{\alpha_i} = \sum c^{d_i} v_{\alpha_i},$$

for some $d_i \in \mathbb{N}$, since $\langle \alpha_i, \lambda \rangle > 0$. Thus,

$$\lim_{c \to 0} \lambda(c) \cdot y = \lim_{c \to 0} \sum c^{d_i} v_{\alpha_i} = 0 = \lim_{c \to 0} \lambda(c) \cdot x = x,$$

since $x \in X^T$.

For 3) implies 1):
Assume there exists such a λ. Using this λ, we obtain a $\mathbb{Z}$-grading on $T_x(X)$. We can embed X_x into a T-module V, which also has a λ induced $\mathbb{Z}$-grading. We may assume that $x = 0$. Then

$$\lim_{c \to 0} c \cdot y = \lim_{c \to 0} \lambda(c) \cdot y = 0,$$

for all $y \in X_x$ implies that $X_x \subseteq V^+$. It follows from the definition of the action of T on $k[V^+]$ that $k[V^+]_d = 0$ for $d > 0$ and $k[V^+]_0 = \mathbb{C}$ and since $k[X_x]$ is a quotient of $k[V^+]$, the same holds for $k[X_x]$. The ideal $\mathfrak{m}_x$ is homogeneous and so

$$\mathfrak{m}_x = \bigoplus_{d<0} \mathfrak{m}_x \cap k[X_x]_d = \bigoplus_{d<0} k[X_x]_d.$$

Consequently, since $\mathfrak{m}_x/\mathfrak{m}_x^2 \simeq T_x(X)^*$, $T_x(X)_d^* = 0$ for $d \geq 0$ and by duality $T_x(X)_d = 0$, for all $d \leq 0$. Therefore, $\langle \alpha, \lambda \rangle > 0$, for all $\alpha \in \Omega(T_x(X))$. □

Property 3 gives justification for the name attractive point and two immediate consequences. The first is that X_x is unique. Indeed, assume that X_x and X_x' are both open T-stable affine neighborhoods of the attractive point x, let $y \in X_x$ and consider the continuous map $f : \mathbb{C} \to X_x$ given by $c \mapsto \lambda(c) \cdot y$. Since $f^{-1}(X_x \cap X_x')$ is open (ie. infinite), there is a nonzero $c \in \mathbb{C}$ such that $f(c) = \lambda(c) \cdot y \in X_x'$, which implies that $y \in X_x'$. By symmetry, we have $X_x = X_x'$.

The second consequence is that X_x contains only one T-fixed point, namely x. For if $y \in (X_x)^T$, then

$$x = \lim_{c \to 0} \lambda(c) \cdot y = \lim_{c \to 0} y = y.$$

Thus attractive points of T-varieties are isolated. Also, note that an attractive point x is equal to 0, when viewed as an element of $T_x(X)$, as obtained in the proof 1) implies 3).

We mentioned above that given a T-variety X and a T-fixed point x, $|E(X, x)| \geq \dim X$, but $|E(X, x)|$ is actually determined in the following case.

Lemma 2.26. *Suppose X is a T-surface with attractive point x such that $|E(X, x)|$ is finite, then $|E(X, x)| = 2$.*

Proof. See Corollary 1 and Corollary 2 in [3]. □

Finally, we will require the following lemma concerning attractive points in Chapter 3. It is Lemma 2.1 in [6].

Lemma 2.27. *Given a map $f : X \to Y$ of affine T-varieties, where X contains an attractive T-fixed point x, then f is a finite morphism if and only if the fibre $f^{-1}(f(x))$ is a finite set.*

Chapter 3

G/B

Let G be a connected semi-simple algebraic group, B a Borel subgroup, and $T \subset B$ a maximal torus. In this chapter we will examine Proposition 5.2 by Carrell and Kuttler given in [6] regarding singularities of T-surfaces in G/B and the open affine neighborhood of a T-fixed point. The same technique will be applied in the next chapter.

3.1 The structure of G/B

The homogeneous space G/B is called the *flag variety* of G, a name which stems from the concept of the *flag variety* of a vector space. A *full flag* of an n-dimensional vector space V is a chain $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ of subspaces of V in which $\dim V_i = i$. The collection of all full flags of a vector space is called the flag variety of V, denoted $\mathcal{F}(V)$. For $G = \mathrm{GL}_n(\mathbb{C})$ or $SL_n(\mathbb{C})$ and B the corresponding subgroup of upper triangular matrices G/B is the flag variety of $\mathbb{C}^n$. The flag variety of a vector space can be given the structure of a projective variety. Indeed, G/B is an irreducible smooth projective variety.

Finally, G/B is a T-variety. We define the action of T on G/B as usual by $t \cdot gB = (tg)B$. Since G/B is a projective variety, by Lemma 2.16 $(G/B)^T \neq \emptyset$. In fact, $(G/B)^T$ is a finite set, since there is a one-to-one correspondence between the Weyl group $W = N_G(T)/T$ (which is finite) and $(G/B)^T$ in which w corresponds to $\dot{w}B$, where $\dot{w}$ is a representative of w in $N_G(T)$. Henceforth, we will identify elements of $(G/B)^T$ with the corresponding element of W.

Now G/B is a T-variety, a fact which follows easily from Lemma 2.13. However, we will provide a concrete proof, so that we have an explicit description of the objects involved. To that end, let $u \in (G/B)^T$. We will show that

u has an open affine T-stable neighborhood. Suppose that T acts on G by conjugation and let U^- be the unipotent radical of B^-, the Borel subgroup opposite B. As a normal subgroup of $B^- = TU^-$, U^- is T-stable. Now consider the usual projection map $\pi : G \twoheadrightarrow G/B$. The restriction of π to U^- is a T-equivariant open immersion and hence $U^- \simeq \pi|_{U^-}(U^-) = U^- \cdot e$, where U^- acts on G by left multiplication. Thus, $U^- \cdot e$ is a T-stable open smooth affine variety. Clearly, $e \in (G/B)^T$ and since W acts transitively on $(G/B)^T$, there is a $w \in W$ such that $u = w \cdot e$. Now, we choose a representative $w_0 \in N_G(T)$ of the coset w, so that $u = w_0 \cdot e$. Let $U := (w_0U^-) \cdot e = (w_0U^-w_0^{-1}w_0) \cdot e$. Then U is the open T-stable affine neighborhood of u, ie. $(G/B)_u = U$.

Consequently, any closed irreducible T-stable subvariety X of G/B is also a T-variety since $X^T \subseteq (G/B)^T$ and for any $x \in X^T$, $(G/B)_x \cap X$ is the open affine T-stable neighborhood of x.

3.2 T-actions on G/B and $\mathfrak{g}/\mathfrak{b}$

As previously mentioned, the action of T on $\mathfrak{g}$ is given by the adjoint representation Ad. Since T and B are T-stable, $\mathfrak{h}$ and $\mathfrak{b}$ are T-stable. We define the action of T on $\mathfrak{g}/\mathfrak{b}$ in the canonical way. Let $\Phi = \Omega(\mathfrak{g}) \setminus \{0\}$ be the set of roots of G relative to T. Since G is a semi-simple algebraic group and $\mathbb{C}$ has characteristic 0, $\mathfrak{g}$ is a semi-simple Lie algebra.

We established in Chapter 2 that

$$\mathfrak{g} = (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha) \oplus \mathfrak{h},$$

where $\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid t \cdot g = \alpha(t)g, \text{ for all } t \in T\}$. It is well-known that $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$ and thus from Lemma 2.4,

$$\mathfrak{b} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

for some $\Delta \subset \Omega(\mathfrak{g})$.

We will introduce a notion of positivity on Φ, in which case we denote the set of positive roots by Φ^+ and the set of negative roots by Φ^-. We let

$$\Phi^+ := \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \subset \mathfrak{b}\}$$

which is equivalent to defining

$$\Phi^+ := \{\alpha \in \Phi \mid \langle \alpha, \lambda \rangle > 0\},$$

where λ is an element of $Y(T)$ that depends on the choice of B. We write $\alpha > 0$ to indicate that $\alpha \in \Phi^+$ and $\alpha < 0$ to indicate that $\alpha \in \Phi^-$. Note that $\Phi^- = -\Phi^+$. Also, since Φ is a reduced root system, the roots in Φ^- are nonproportional.

Thus
$$\mathfrak{g} = (\bigoplus_{\alpha\in\Phi^-} \mathfrak{g}_\alpha) \oplus \mathfrak{b}$$
and hence
$$\mathfrak{g}/\mathfrak{b} = \bigoplus_{\alpha\in\Phi^-} \mathfrak{g}_\alpha.$$

We have one additional description of $\mathfrak{g}/\mathfrak{b}$ and that is
$$\mathfrak{g}/\mathfrak{b} \simeq T_e(G/B).$$

Related to this, we have that the tangent space at an arbitrary $u \in (G/B)^T$ is
$$T_u(G/B) \simeq \mathfrak{g}/u\mathfrak{b}u^{-1} \simeq \bigoplus_{w^{-1}(\alpha)\in\Phi^-} \mathfrak{g}_\alpha,$$
where $u = w \cdot e$, for some $w \in W$.

From these descriptions, we obtain the following important fact about the T-fixed points of G/B.

Lemma 3.1. *Every element of $(G/B)^T$ is attractive.*

Proof. We will first consider the T-fixed point e. Since
$$T_e(G/B) = \bigoplus_{\alpha\in\Phi^-} \mathfrak{g}_\alpha,$$
we know from our notion of positivity on Φ that there is a $\lambda \in Y(T)$ such that $\langle\alpha, \lambda\rangle < 0$, for all $\alpha \in \Phi^-$. Hence $\langle\alpha, -\lambda\rangle > 0$ and it follows that e is attractive. For arbitrary $u \in (G/B)^T$, we know that $\Omega(T_u(G/B)) = \{\alpha \in X(T) \mid w^{-1}(\alpha) \in \Phi^-\}$ where $u = w \cdot e$, for some $w \in W$. Then $\langle w^{-1}(\alpha), -\lambda\rangle = \langle\alpha, w(-\lambda)\rangle > 0$, for all $\alpha \in \Omega(T_u(G/B))$ and hence u is attractive. $\square$

3.3 T-Curves in G/B

The study of singularities of T-surfaces in G/B heavily involves the use of T-curves and a great deal is known about these objects. We know from the previous chapter that every T-curve C in G/B is a T-orbit closure, since C is not equal to the finite set C^T. Denote by $E(X)$ the set of all T-curves in X, where X is any T-stable subvariety of G/B.

Lemma 3.2. *Let X be a nonempty T-stable subvariety of G/B. Then the following hold:*

1) *$E(X)$ is finite.*
2) *Every element of $E(X)$ is smooth.*
3) *Every element of $E(X)$ contains exactly two T-fixed points.*
4) *If $C \neq D$ are elements of $E(X)$ with nonempty intersection, then $C \cap D = \{u\}$, for some $u \in X^T$.*

Proof. See Theorems D and F in [4]. $\square$

Recall that $E(G/B, u)$ is the set of T-curves in G/B containing u and let U_α be the subgroup of G with Lie algebra $\mathfrak{g}_\alpha$. The following description of the elements of $E(G/B, u)$ can be found in Theorem F of [4].

Lemma 3.3. *Let $u \in (G/B)^T$ and suppose that $C \in E(G/B, u)$, then $C = \overline{U_\alpha \cdot x}$, for some $\alpha \in \Phi$ and $C^T = \{u, r_\alpha u\}$, where r_α is the reflection in the Weyl group W corresponding to α.*

So now we have a characterization of the elements of $E(u, G/B)$ and we know that $|E(u, G/B)|$ is finite, but it is also the case that $|E(u, G/B)|$ is known.

Lemma 3.4. *There are precisely d T-curves through $u \in (G/B)^T$, where $d = \dim G/B$.*

Proof. Since $(G/B)_u \simeq T_u(G/B)$, the result follows from Lemma 2.20. $\square$

3.4 T-Surfaces in G/B

Given that we have focused a great deal of attention on torus actions, it should come as no surprised that every T-surface in G/B is, in fact, a T-orbit closure.

Lemma 3.5. *A T-surface in G/B is a T-orbit closure.*

Proof. Let Σ be any T-surface in G/B and assume that Σ does not contain an orbit of dimension 2. Since any orbit of dimension 0 is a fixed point, there are only finitely many 0-dimensional orbits. Also, the closure of any 1-dimensional orbit is a curve C which contains exactly two fixed points and any two distinct curves contain at most one fixed point in common. Therefore, there are only finitely many curves and subsequently 1-dimensional orbits, but Σ cannot be the union of finitely many curves and fixed points. Thus, Σ contains a 2-dimensional orbit and is equal to its closure. $\square$

For the remainder of this section we will present the proof of Proposition 5.2 in [6], which will be stated after the proof. We assume that G has no G_2 factors, ie. Φ does not contain a copy of G_2. Let $\Sigma \subseteq G/B$ be any T-surface. Thus $\Sigma = \overline{T \cdot y}$, for some $y \in G/B$. Let $u \in \Sigma^T$ which we know to be attractive. We may assume that $u = e$, since $\Sigma \simeq u^{-1}\Sigma$ as varieties, where $u^{-1}\Sigma$ is also a T-surface. The goal of this section is to determine when Σ is smooth and to describe Σ_u. We will first provide a general picture of how all of our main objects relate to each other.

As above, $\Sigma_u = \Sigma \cap (G/B)_u = \overline{T \cdot y} \cap U$, where $U := U^- \cdot e$. We have

$$\Sigma_u \subseteq U \simeq T_u(U) \simeq T_u(G/B) \simeq \mathfrak{g}/\mathfrak{b}.$$

But since $\Sigma_u \cap T \cdot y$ is nonempty and T-stable, $T \cdot y \subseteq \Sigma_u$ and hence $\Sigma_u = \overline{T \cdot y}$ in $\mathfrak{g}/\mathfrak{b}$.

By Lemma 2.23, there is a T-equivariant embedding of Σ_u into $T_u(\Sigma)$ and so

$$\Sigma_u \hookrightarrow T_u(\Sigma) \subset T_u(G/B) = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}.$$

Now since $T_u(\Sigma)$ is a T-stable subspace of $T_u(G/B)$,

$$T_u(\Sigma) = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha,$$

for $\Gamma \subseteq \Phi^-$, where each $\mathfrak{g}_\alpha$ has dimension 1. In particular, this means that $\dim T_u(\Sigma) = |\Gamma|$. Thus determining whether or not a T-fixed point $u \in \Sigma$ is singular reduces to determining Γ. To do this we use the fact that

$$-\Gamma = \Omega(T_u(\Sigma)^*) \subseteq \Omega(k[\Sigma_u])$$

and employ Lemma 3.8, a result concerning the weights of $k[\Sigma_u]$. Note that since Γ is a subset of Φ^-, the weights of $T_u(\Sigma)^*$ are positive roots.

Let $h \in T_u(\Sigma)^*$. Since Σ_u embeds in $T_u(\Sigma)$, we can restrict h to Σ_u. Now taking the differential of $h|_{\Sigma_u}$ at u gives $d_u(h|_{\Sigma_u}) = h$. As in the previous chapter we let $\mathfrak{m}_u$ be the ideal in $k[\Sigma_u]$ of all functions vanishing at u and we have that $T_u(\Sigma)^* \simeq \mathfrak{m}_u/\mathfrak{m}_u^2$ by $d_u f \mapsto \bar{f}$. Thus $h|_{\Sigma_u} \in \mathfrak{m}_u$. Note also that if $h|_{\Sigma_u} = 0$, then $d_u(h|_{\Sigma_u}) = h = 0$ or equivalently if $h \neq 0$, then $h|_{\Sigma_u} \neq 0$. There is a nice relationship between weights of $T_u(\Sigma)^*$ and the restrictions to Σ_u of variables on the weight spaces $\mathfrak{g}_\alpha$ contained in $T_u(\Sigma)$.

Lemma 3.6. *Let $\{\alpha_1, \alpha_2, \ldots, \alpha_m\} \subseteq \Omega(T_u(\Sigma)^*)$ and let $x_{\alpha_i} \in k[\Sigma_u]$ be the restriction to Σ_u of the variable on $\mathfrak{g}_{-\alpha_i}$, for each i. If*

$$\sum_{i=1}^m a_i\alpha_i = \sum_{i=1}^m b_i\alpha_i,$$

where $a_i, b_i \in \mathbb{N}_0$, then

$$\prod_{i=1}^m x_{\alpha_i}^{a_i} = \prod_{i=1}^m x_{\alpha_i}^{b_i},$$

up to scalars.

Proof. Let

$$a = \sum_{i=1}^m a_i\alpha_i.$$

Then,

$$\prod_{i=1}^m x_{\alpha_i}^{a_i}, \prod_{i=1}^m x_{\alpha_i}^{b_i} \in k[\Sigma_u]_a.$$

Since Σ_u is a T-orbit closure, by Lemma 2.15, the integral domain $k[\Sigma_u]$ is a multiplicity free T-module. Therefore,

$$\prod_{i=1}^m x_{\alpha_i}^{a_i} = c\prod_{i=1}^m x_{\alpha_i}^{b_i},$$

for some $c \in \mathbb{C}$ (which we can assume is 1 by an appropriate choice of variables). $\square$

From the above discussion and the previous Lemma we obtain the following:

Lemma 3.7. *Any linear combination $\sum_{i=1}^m a_i\alpha_i$ of weights of $T_u(\Sigma)^*$, with $m, a_i \in \mathbb{N}$ and $\sum_{i=1}^m a_i > 1$ is not a weight of $T_u(\Sigma)^*$.*

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be weights of $T_u(\Sigma)^*$ and let $x_{\alpha_i} \in k[\Sigma_u]$ be the restriction to Σ_u of the variable on $\mathfrak{g}_{-\alpha_i}$, for each i. Assume that $\omega = \sum a_i\alpha_i$ is a weight of $T_u(\Sigma)^*$, such that $\sum a_i > 1$ and let $x_\omega \in k[\Sigma_u]$ be the restriction to Σ_u of the variable on $\mathfrak{g}_{-\omega}$. Thus, Lemma 3.6 implies that $x_\omega = c\prod x_{\alpha_i}^{a_i}$, for some $c \in \mathbb{C}$. From the above discussion, $x_{\alpha_i} \in \mathfrak{m}_u$, for each i, and since $\sum a_i > 1$, $\prod x_{\alpha_i}^{a_i} \in \mathfrak{m}_u^2$. Therefore, $d_u x_\omega = 0$ (in $\mathfrak{m}_u/\mathfrak{m}_u^2$), which is a contradiction. $\square$

As previously mentioned, we will use the T-curves in Σ to analyze Σ. By Lemma 2.26, $|E(\Sigma_u, u)| = 2$, so let C_u and D_u be the two curves in $E(\Sigma_u, u)$. Then by Lemma 2.20, $C_u = \mathfrak{g}_{-\alpha}$ and $D_u = \mathfrak{g}_{-\beta}$, for some $\alpha, \beta \in -\Gamma \subseteq \Phi^+$. So $\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\beta} \subseteq T_u(\Sigma)$, which gives us that α and β occur as weights of $T_u(\Sigma)^*$. Now we need to determine whether there are any others.

Let $x_\alpha, x_\beta \in k[\Sigma_u]$ be the restrictions to Σ_u of variables on $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-\beta}$ in $T_u(\Sigma)$, respectively (so both are nonzero). Note that x_α and x_β have weight α and β, respectively, in the T-representation on $k[\Sigma_u]$ and that $x_\alpha, x_\beta \in \mathfrak{m}_u$.

Lemma 3.8. *Let $\omega \in \Omega(k[\Sigma_u])$. Then there exists an $N \in \mathbb{N}$ such that $N\omega = a\alpha + b\beta$, for some $a, b \in \mathbb{N}_0$.*

Proof. Let $\rho : \Sigma_u \to \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\beta}$ be the restriction to Σ_u of the unique T-equivariant projection $\tilde{\rho} : T_u(\Sigma) \twoheadrightarrow \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\beta}$, so then $d_u\rho = \tilde{\rho}$. Now $\rho(u) = 0$ since u is attractive, ie. $u = 0$. If the fibre over 0 is infinite, then its dimension is at least 1 and furthermore, by attractiveness, every irreducible component contains u. Hence, by Lemma 2.21, it contains at least one T-curve through u, which we may assume is C_u. But then $\rho(C_u) = 0$ implies that $d_u\rho(T_u(C_u)) = 0$. Consequently,

$$\tilde{\rho}(\mathfrak{g}_{-\alpha}) = \tilde{\rho}(T_u(C_u)) = 0,$$

which is a contradiction. Thus the fibre over $\rho(u)$ is finite and hence by Lemma 2.27, ρ is a finite morphism. Therefore, by definition, since $\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\beta}$ is affine, the $\mathbb{C}[x_\alpha, x_\beta]$-algebra $k[\Sigma_u]$ is a finitely generated $\mathbb{C}[x_\alpha, x_\beta]$-module, ie. $\rho^* : \mathbb{C}[x_\alpha, x_\beta] \to k[\Sigma_u]$ is a finite ring homomorphism. In particular, ρ^* is integral and so $k[\Sigma_u]$ is integral over $\mathbb{C}[x_\alpha, x_\beta]$.

Let $f \neq 0 \in k[\Sigma_u]_\omega$. There is an $N \in \mathbb{N}$ such that

$$f^N = h_{N-1}f^{N-1} + \cdots + h_1 f + h_0,$$

where $h_{N-i} \in \mathbb{C}[x_\alpha, x_\beta]$, for $1 \leq i \leq N$, and $h_0 \neq 0$. Since $\mathbb{C}[x_\alpha, x_\beta]$ has a weight space decomposition,

$$h_{N-i} = \sum_{\mu \in X(T)} h_{N-i,\mu},$$

where $h_{N-i,\mu} \in \mathbb{C}[x_\alpha, x_\beta]_\mu$. Since $f \in k[\Sigma_u]_\omega$, f^N has weight $N\omega$, so in fact

$$f^N = \sum_{i=1}^{N} h_{N-i,i\omega} f^{N-i},$$

where $h_{0,N\omega} \neq 0$ and each summand has weight $N\omega$. Therefore, $N\omega$ is a weight of $\mathbb{C}[x_\alpha, x_\beta]$. $\square$

Assume $\omega \in \Omega(T_u(\Sigma)^*) \subseteq k[\Sigma_u]$ and let $x_\omega \in k[\Sigma_u]$ be the restriction to Σ_u of a variable on $\mathfrak{g}_{-\omega}$. By Lemma 3.8, there exists an $N \in \mathbb{N}$ such that $N\omega = a\alpha + b\beta$, for some $a, b \in \mathbb{N}_0$. Since the positive roots α, β, and ω generate a root system of rank 2, we can analyze the equation $N\omega = a\alpha + b\beta$ for α, β and ω in a copy of A_2, $A_1 \oplus A_1$, or B_2 (we have excluded G_2 by assumption). If α, β and ω sit in a copy of A_2 or $A_1 \oplus A_1$, then we immediately obtain that $\omega = a'\alpha + b'\beta$, for some $a', b' \in \mathbb{N}_0$. But by Lemma 3.7, $a' + b' = 1$, since ω is a weight of $T_u(\Sigma)^*$. Therefore, $\omega = \alpha$ or β. If ω, α, and β are contained in a copy of B_2, then either $\omega = a'\alpha + b'\beta$ or $2\omega = a'\alpha + b'\beta$, for some $a', b' \in \mathbb{N}_0$, in which case $\omega = \alpha$ or β, as above, or $\omega = \frac{1}{2}(\alpha + \beta)$.

If it is always the case that $\omega = \alpha$ or β, ie. α and β are the only two weights of $T_u(\Sigma)^*$, then $T_u(\Sigma) = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\beta}$ and hence $\dim T_u(\Sigma) = 2 = \dim \Sigma$, giving us that Σ is nonsingular at u. In particular, if G is simply laced, ie. all roots have the same length, then α, β and ω sit in a copy of A_2 or $A_1 \oplus A_1$ and hence Σ is nonsingular at u.

In the case that there is an $\omega = \frac{1}{2}(\alpha + \beta)$, $T_u(\Sigma) = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\beta} \oplus \mathfrak{g}_{-\frac{1}{2}(\alpha+\beta)}$. Consequently, u is a singularity of Σ and $x_\omega^2 = x_\alpha x_\beta$ (Lemma 3.6). Let $\mathfrak{a} = \langle x_\omega^2 - x_\alpha x_\beta \rangle$, then the prime ideal $\mathfrak{a}$ is T-stable since $x_\omega^2 - x_\alpha x_\beta$ is an eigenvector of weight $\alpha + \beta$. Thus, $Z(\mathfrak{a})$ is a T-surface in $T_u(\Sigma)$. Consequently, $\Sigma_u \subseteq Z(\mathfrak{a})$ and is equal for dimensional reasons.

Thus we have proven the following Proposition.

Proposition 3.9 (Proposition 5.2 of [6])**.** *Let Σ be a T-surface in G/B, where G is assumed to have no G_2 factors. Then for $u \in \Sigma^T$, Σ is either nonsingular at u or the two elements of $E(\Sigma, u)$ have weights α and β which are orthogonal long roots sitting in a copy of B_2 in Φ. In that event, the open neighborhood Σ_u is isomorphic to a surface given by the equation $z^2 = xy$ for the variables $x, y, z \in k[\Sigma_u]$ of weights α, β, and $\frac{1}{2}(\alpha + \beta)$, respectively.*

Chapter 4

$\mathcal{G}/\mathcal{P}$

In this chapter we consider an infinite-dimensional analogue to the proposition examined in the previous chapter. For general results presented in this chapter on the affine Grassmannian see [10].

4.1 The Structure of $\mathcal{G}/\mathcal{P}$

Let $\mathcal{G} := \mathrm{SL}_n(\mathbb{C}((x)))$ and $\mathcal{P} := \mathrm{SL}_n(\mathbb{C}[[x]])$, for some $n \in \mathbb{N}$, then the quotient $\mathcal{G}/\mathcal{P}$ is a projective ind-variety, known as the *affine Grassmannian*. So,

$$\mathcal{G}/\mathcal{P} = \lim_{\rightarrow} X_i,$$

where each X_i is an irreducible normal finite dimensional projective variety. Now let B be the lower triangular Borel subgroup of $\mathrm{SL}_n(\mathbb{C})$ and let $\mathcal{B} = ev^{-1}(B)$, where $ev : \mathcal{P} \to \mathrm{SL}_n(\mathbb{C})$ is entry-wise evaluation at $x = 0$. Then $\mathcal{B}$ is the set of matrices in $\mathcal{P}$ whose entries above the diagonal have no constant term. Let $T \subset \mathcal{B}$ be the maximal torus consisting of diagonal matrices in $\mathrm{SL}_n(\mathbb{C})$, ie. the subset of diagonal matrices in $\mathcal{G}$ with constant entries (so $\widehat{T} \simeq (\mathbb{C}^*)^{n-1}$) and let T act on $\mathcal{G}$ by conjugation. Let $S := \mathbb{C}^*$, where S acts on each $g \in \mathcal{G}$ by acting on each entry of g by the rule

$$s \cdot (\sum_{i=\ell}^{\infty} x^i) = \sum_{i=\ell}^{\infty} s^i x^i,$$

for all $s \in S$. Since these actions commute we can set $\widehat{T} := T \times S \simeq (\mathbb{C}^*)^n$. Denote the *affine Weyl group* of $\mathcal{G}$ by $\widehat{W} := N_{\mathcal{G}}(T)/T$. Then for each $w \in \widehat{W}$ the set

$$X(w) = \overline{\mathcal{B}w\mathcal{P}}$$

is called a *Schubert variety*. This object is extremely useful as it is an irreducible finite dimensional normal projective variety such that $E(X(w), y)$ is finite, for all $y \in X(w)^{\widehat{T}}$. Returning to $\mathcal{G}/\mathcal{P}$, for each i, one may choose $X_i = X(w_i)$, for some $w_i \in \widehat{W}$. We have that the set of $\widehat{T}$-fixed points of $\mathcal{G}/\mathcal{P}$ is in a one-to-one correspondence with the points of the set $\widehat{W}^{\mathcal{P}}$, the set of minimal length representatives of $\widehat{W}/W$, where W is the Weyl group of $\mathrm{SL}_n(\mathbb{C})$. Henceforth, we will identify the fixed points of $\mathcal{G}/\mathcal{P}$ with the corresponding element of $\widehat{W}^{\mathcal{P}}$.

The set of $\widehat{T}$-fixed points of $\mathcal{G}/\mathcal{P}$ is discrete and hence

$$X(w_i)^{\widehat{T}} = (\mathcal{G}/\mathcal{P})^{\widehat{T}} \cap X(w_i)$$

is finite. Thus, since $X(w_i)$ is normal and irreducible, $X(w_i)$ and any of its $\widehat{T}$-stable irreducible subvarieties are $\widehat{T}$-varieties (See Sumihiro's Theorem in [13], [14]). Also, the $\widehat{T}$-fixed points of $\mathcal{G}/\mathcal{P}$ form a single orbit under the action of the affine Weyl group.

4.2 $\widehat{T}$-actions on $\hat{\mathfrak{g}}/\hat{\mathfrak{p}}$

We will denote the Lie algebra of $\mathrm{SL}_n(\mathbb{C})$ by $\mathfrak{g}$, ie. $\mathfrak{g}$ is the set of $n \times n$ matrices with trace 0. The set Φ of roots of $\mathrm{SL}_n(\mathbb{C})$ with respect to T is a root system of type A_{n-1}. Now $\mathfrak{g}$ has root space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha \oplus \mathfrak{b},$$

where $\mathfrak{b}$ is the subspace of $\mathfrak{g}$ consisting of upper triangular matrices.

Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[x, x^{-1}]$ and let $\hat{\mathfrak{p}} = \mathfrak{g} \otimes \mathbb{C}[x]$. The action of $\widehat{T}$ on $\hat{\mathfrak{g}}$ is given by

$$(t, s) \cdot (g \otimes x^i) = tgt^{-1} \otimes s^i x^i.$$

So if $g \in \mathfrak{g}_\alpha$, then we have

$$(t, s) \cdot (g \otimes x^i) = \alpha(t) g \otimes s^i x^i = \alpha(t) s^i (g \otimes x^i).$$

Defining $(\alpha + h\delta)(t, s) := \alpha(t) s^h$ we see that the roots of $\hat{\mathfrak{g}}$ are

$$\hat{\Phi} = \{\alpha + h\delta \mid \alpha \in \Phi \text{ and } h \in \mathbb{Z}\} \cup \{h\delta \mid h \in \mathbb{Z} \setminus \{0\}\}$$

and

$$\hat{\Phi}^+ = \{\alpha + h\delta \mid h > 0 \text{ or } h = 0 \text{ and } \alpha > 0\}.$$

A root is said to be *imaginary* if it is an element of $\{h\delta \mid h \in \mathbb{Z} \setminus \{0\}\}$, otherwise it is called *real*. Set

$$\hat{\Phi}^+_{h>0} = \{\hat{\alpha} \in \hat{\Phi}^+ \mid h > 0\}$$

and

$$\hat{\Phi}^-_{h<0} = \{\hat{\alpha} \in \hat{\Phi}^- \mid h < 0\}.$$

As in the classical case, there is an alternate way of expressing $\hat{\Phi}^+$. First write $\lambda + d\delta$ for the element $(\lambda, d) \in Y(\widehat{T}) \simeq Y(T) \oplus \mathbb{Z}$. Now we have that $\Phi^+ = \{\alpha \in \Phi \mid \langle \alpha, \lambda \rangle \geq 0\}$, for some $\lambda \in Y(T)$ and since Φ is finite there is a $d \in \mathbb{N}$ such that $|\langle \alpha, \lambda \rangle| < d$ for all $\alpha \in \Phi$. Thus $\langle \alpha + h\delta, \lambda + d\delta \rangle = b + dh$, where $|b| < d$ and from this we have that

$$\hat{\Phi}^+ = \{\alpha + h\delta \mid \langle \alpha + h\delta, \lambda + d\delta \rangle \geq 0\}.$$

If $\hat{\alpha} = \alpha + h_\alpha \delta$, where $\alpha \neq 0$, then $\hat{\mathfrak{g}}_{\hat{\alpha}} = \mathfrak{g}_\alpha \otimes (\mathbb{C}[x, x^{-1}])_{h_\alpha}$, where $\dim \mathfrak{g}_\alpha$ and $\dim(\mathbb{C}[x, x^{-1}])_{h_\alpha}$ are both 1 and hence $\dim \hat{\mathfrak{g}}_{\hat{\alpha}} = 1$. Since a $\widehat{T}$-eigenvector appears in $\hat{\mathfrak{p}}$ if and only if $h \geq 0$, we have

$$\hat{\mathfrak{g}}/\hat{\mathfrak{p}} = \bigoplus_{\hat{\Phi}^-_{h<0}} \hat{\mathfrak{g}}_{\hat{\alpha}}$$

As in the classical case,

$$T_e(\mathcal{G}/\mathcal{P}) = \hat{\mathfrak{g}}/\hat{\mathfrak{p}}$$

and for arbitrary $u = w \cdot e \in (\mathcal{G}/\mathcal{P})^{\widehat{T}}$, for some $w \in \widehat{W}$, we have

$$T_u(\mathcal{G}/\mathcal{P}) = \bigoplus_{\hat{\alpha} \in w^{-1}(\hat{\Phi}^-_{h<0})} \hat{\mathfrak{g}}_{\hat{\alpha}}$$

From this and the definition of $\widehat{\Phi}^+$ it follows that the elements of $(\mathcal{G}/\mathcal{P})^{\widehat{T}}$ are attractive (in every Schubert variety), using a proof similar to that of Lemma 3.1.

4.3 $\widehat{T}$-Surfaces in $\mathcal{G}/\mathcal{P}$

In this section we will apply the techniques of the classical case to our new objects, but first we need to verify that this is possible. Let $\Sigma \subset \mathcal{G}/\mathcal{P}$ be a $\widehat{T}$-surface. So Σ is a closed irreducible subvariety of some Schubert variety $X(w)$ and hence is a $\widehat{T}$-variety. Now, let $u \in \Sigma^{\widehat{T}}$, with open affine neighborhood Σ_u. As before we may assume that $u = e$. By Lemma 2.12,

$$\Sigma_u \hookrightarrow T_u(\Sigma) \subset \bigoplus_{\hat{\alpha} \in \hat{\Phi}^-_{h>0}} \hat{\mathfrak{g}}_{\hat{\alpha}}.$$

Since Σ_u contains a dense orbit, by Lemma 2.15 $k[\Sigma_u]$ is multiplicity free and hence $T_u(\Sigma)$ is as well. Thus

$$T_u(\Sigma) = (\bigoplus_{\hat{\alpha} \in \Gamma} \hat{\mathfrak{g}}_{\hat{\alpha}}) \oplus (\bigoplus_{h \in \Delta \subseteq \mathbb{Z}_{<0}} L_{h\delta}),$$

where Γ is some subset of $\hat{\Phi}^-_{h<0}$ consisting of real roots and where $L_{h\delta}$ is a line in $\mathfrak{h} \otimes (\mathbb{C}[x, x^{-1}])_h$. Let C_u and D_u be the two $\widehat{T}$-curves in $E(\Sigma_u, u)$, with corresponding $\widehat{T}$-curves C and D, respectively, in $E(X(w), u)$. Then $T_u(C) = \hat{\mathfrak{g}}_{\hat{\alpha}}$ for some $\hat{\alpha} \in \hat{\Phi}^-_{h<0}$ with $\alpha \neq 0$ (see Proposition 12.1.7 in [10]). Let $C_u = \overline{\widehat{T} \cdot v}$, for some $v \in T_u(\Sigma)$. Now by Lemma 2.9, since $\dim C_u = 1$, the rank of the $\mathbb{Z}$-module M generated by the support $s(v)$ of v is 1 and hence $M \simeq \mathbb{Z}$ and the elements of $s(v)$ are proportional. Recall that

$$\widehat{T}_v = \bigcap_{\hat{\chi} \in M} \ker \hat{\chi}.$$

So the connected component $(\widehat{T}_v)^\circ$ of $\widehat{T}_v$ is a codimension 1 torus which acts trivially on $\widehat{T} \cdot v$, hence on C_u, and thus on $T_u(C) = \hat{\mathfrak{g}}_{\hat{\alpha}}$. Therefore, $(\widehat{T}_v)^\circ \subseteq \ker \hat{\alpha}$. Let $N = \ker(X(\widehat{T}) \twoheadrightarrow X((\widehat{T}_v)^\circ))$ (restriction), so $N \simeq \mathbb{Z}$ and contains M. Therefore, $M = nN$ for some $n \in \mathbb{Z}$, which gives us that $n\hat{\alpha} \in M$. Thus the elements of $s(v)$ are proportional to $\hat{\alpha}$ and since no other element of $\hat{\Phi}^-_{h<0}$ is proportional to $\hat{\alpha}$, $v \in \hat{\mathfrak{g}}_{\hat{\alpha}}$. Consequently, $C_u = \mathfrak{g}_{\hat{\alpha}}$, which we will write instead as $\hat{\mathfrak{g}}_{-\hat{\alpha}}$, for $\hat{\alpha} = \alpha + h_\alpha \delta \in \widehat{\Phi}^+_{h>0}$. Similarly, we have $D_u = \mathfrak{g}_{-\hat{\beta}}$, where $\hat{\beta} = \beta + h_\beta \delta \in \widehat{\Phi}^+_{h>0}$ and $\beta \neq 0$.

As before, $\hat{\alpha}$ and $\hat{\beta}$ are weights of $T_u(\Sigma)^*$ and we want to determine if there are any others. To that end, let $\hat{\omega}$ be any weight of $T_u(\Sigma)^*$. Since $\hat{\omega} \in \Omega(k[\Sigma_u])$, by Lemma 3.8, $N\hat{\omega} = a\hat{\alpha} + b\hat{\beta}$, for some $N \in \mathbb{N}$ and some $a, b \in \mathbb{N}_0$, where at least one of a or b is nonzero, since $N \neq 0$.

Remark 4.1. Since we can consider the roots α and β as being contained in a root system of rank 2, ie. a copy of A_2 or $A_1 \oplus A_1$ in Φ and since these are the only rank 2 roots systems contained in the roots systems of type A, D or E, it now becomes clear as to why all results to follow hold if we replace $\mathrm{SL}_n(\mathbb{C}((x)))$ by $G(\mathbb{C}((x)))$, where G is a connected semi-simple algebraic group of type A, D or E.

Lemma 4.2 below gives us the possible forms of $\hat{\omega}$ based on the above equation.

Lemma 4.2. *Let $\hat{\omega} = \omega + h_\omega \delta$ such that $N\hat{\omega} = a\hat{\alpha} + b\hat{\beta}$, for some $N \in \mathbb{N}$ and* *some* $a, b \in \mathbb{N}_0$*, then*

$$\hat{\omega} = \hat{\alpha}, \hat{\beta}, \hat{\alpha} + \hat{\beta}$$

except when $\beta = \pm\alpha$*.*
If $\beta = \alpha$*, then*

$$\hat{\omega} = \alpha + h_\omega \delta, \text{ where } h_\alpha \leq h_\omega \leq h_\beta.$$

If $\beta = -\alpha$*, then* $\hat{\omega} =$

$$((h_\alpha + h_\beta)c_r + r)\delta,$$

$$\hat{\alpha} + ((h_\alpha + h_\beta)c_r + r)\delta, \text{ or}$$

$$\hat{\beta} + ((h_\alpha + h_\beta)c_r + r)\delta,$$

where $c_r \in \mathbb{N}_0$ *and* $0 \leq r < (h_\alpha + h_\beta)$*.*

Proof. From the equation $N(\omega + h_\omega \delta) = a(\alpha + h_\alpha \delta) + b(\beta + h_\beta \delta)$, we obtain the two equations $N\omega = a\alpha + b\beta$ and $Nh_\omega = ah_\alpha + bh_\beta$.

We will first consider the case where $\beta \neq \pm\alpha$ and $\omega \neq 0$. Although α, $\beta \neq 0$, according to our notion of positivity on $\hat{\Phi}$, α could be either positive or negative and the same holds for β. These roots are elements of A_{n-1}, but since $\beta \neq \pm\alpha$, α and β form a root system of rank 2 and hence we only consider that they sit in a copy of A_2 or $A_1 \oplus A_1$. From the relation $N\hat{\omega} = a\hat{\alpha} + b\hat{\beta}$, for some $N \in \mathbb{N}$ and some $a, b \in \mathbb{N}_0$, we obtain $\omega = \alpha$, β, or $\alpha + \beta$. If $\omega = \alpha$, then $N\alpha = a\alpha + b\beta$ implies that $N = a$ and $b = 0$, since α and β are linearly independent. Then from the equation $Nh_\omega = ah_\alpha + bh_\beta$, we obtain $h_\omega = h_\alpha$ and thus $\hat{\omega} = \hat{\alpha}$. Similarly, if $\omega = \beta$, then $\hat{\omega} = \hat{\beta}$. If $\omega = \alpha + \beta$, then from $N(\alpha + \beta) = a\alpha + b\beta$ we obtain that $N = a = b$ and hence $h_\omega = h_\alpha + h_\beta$. Therefore $\hat{\omega} = \hat{\alpha} + \hat{\beta}$. Now if $\omega = 0$, then $0 = a\alpha + b\beta$ implies that $\beta = -\alpha$, which we have assumed is not the case.

In the case where $\beta = \alpha$, we have $N\omega = (a+b)\alpha$, so $\omega = \frac{a+b}{N}\alpha$. Now since Φ is reduced, $a + b \geq 0$, and $N > 0$, we obtain that $\omega = \alpha$ and $\frac{a+b}{N} = 1$. Therefore, $N = a+b$ and $(a+b)h_\omega = ah_\alpha + bh_\beta$. It follows that $a(h_\omega - h_\alpha) + b(h_\omega - h_\beta) = 0$ and so exactly one of $h_\omega - h_\alpha$ and $h_\omega - h_\beta$ is 0 (not both since $\hat{\alpha} \neq \hat{\beta}$ implies $h_\alpha \neq h_\beta$) or exactly one is strictly greater than 0. Without loss of generality, assume $h_\alpha \leq h_\omega \leq h_\beta$. Therefore, $\hat{\omega} = \alpha + h_\omega \delta$, where $h_\alpha \leq h_\omega \leq h_\beta$, as required.

In the case where $\beta = -\alpha$, we have $N\omega = (a-b)\alpha$ which implies that $\omega = \frac{a-b}{N}\alpha$. Therefore, either $\omega = \alpha$ and $\frac{a-b}{N} = 1$, $\omega = -\alpha$ and $\frac{a-b}{N} = -1$ or $\omega = 0$ and $a = b$. For the first possibility, $N = a - b$ and so $(a-b)h_\omega = ah_\alpha + bh_\beta$. Thus

$$a(h_\omega - h_\alpha) = b(h_\omega + h_\beta) \geq 0$$

and hence $h_\alpha \leq h_\omega$. By the Division algorithm applied to $h_\omega - h_\alpha$, there exist an r where $0 \leq r < (h_\alpha + h_\beta)$ and a $c_r \in \mathbb{N}_0$ such that,

$$\hat{\omega} = \alpha + h_\omega \delta = \alpha + h_\alpha \delta + ((h_\alpha + h_\beta)c_r + r)\delta = \hat{\alpha} + ((h_\alpha + h_\beta)c_r + r)\delta.$$

For the second possibility, $N = b - a$. From this we obtain $h_\beta \leq h_\omega$ and $\hat{\omega} = \hat{\beta} + ((h_\alpha + h_\beta)c_r + r)\delta$. For the remaining possibility, it follows immediately that $\hat{\omega} = ((h_\alpha + h_\beta)c_r + r)\delta$. □

This lemma gives rise to the following theorem.

Theorem 4.3. *If $\beta \neq \pm\alpha$, then Σ is nonsingular at u and Σ_u is smooth.*

Proof. By Lemma 4.2, since $\beta \neq \pm\alpha$, $\hat{\omega} = \hat{\alpha}, \hat{\beta}, \hat{\alpha} + \hat{\beta}$. However, if $\hat{\omega} = \hat{\alpha} + \hat{\beta}$, then by Lemma 3.7, $\hat{\omega}$ is not a weight of $T_u(\Sigma)^*$. Hence $\hat{\alpha}$ and $\hat{\beta}$ are the only weights of $T_u(\Sigma)^*$ and the result follows. □

Unfortunately, if $\beta = \pm\alpha$, then we can not eliminate the possibility that $T_u(\Sigma)^*$ has weights other than $\hat{\alpha}$ and $\hat{\beta}$ since not all $\widehat{T}$-surfaces are smooth. Although we do not know exactly how many weights occur, we can place bounds on the number of possible weights. From these bounds we obtain some conditions under which Σ is nonsingular at u. We begin by examining the case where $\beta = \alpha$ (so $h_\alpha \neq h_\beta$).

Lemma 4.4. *If $\beta = \alpha$, then there are at most $h_\beta - h_\alpha + 1$ weights of $T_u(\Sigma)^*$.*

Proof. This follows immediately from 4.2, since there are $h_\beta - h_\alpha + 1$ natural numbers between h_α and h_β, inclusively. □

We will now consider what happens when $h_\beta - h_\alpha = 1$, 2, or 3.

Theorem 4.5. *Suppose $\beta = \alpha$. If $h_\beta - h_\alpha = 1$, then Σ is nonsingular at u and Σ_u is smooth.*

Proof. By Lemma 4.4, there are at most $h_\beta - h_\alpha + 1 = 2$ weights of $T_u(\Sigma)^*$ and hence α and β are the only weights. □

Theorem 4.6. *Suppose $\beta = \alpha$. If $h_\beta - h_\alpha = 2$, then either Σ is nonsingular at u or Σ_u is isomorphic to a surface given by $x_{\hat{\alpha}} x_{\hat{\beta}} = x_{\hat{\omega}}^2$, where $x_{\hat{\alpha}}, x_{\hat{\beta}}, x_{\hat{\omega}} \in k[\Sigma_u]$ of weights $\hat{\alpha}$, $\hat{\beta}$, $\frac{1}{2}(\hat{\alpha} + \hat{\beta})$, respectively.*

Proof. If $h_\beta - h_\alpha = 2$, then there are at most 3 weights of $T_u(\Sigma)^*$. If there are only two weights then Σ is nonsingular at u, otherwise there is a third weight $\hat{\omega}$ which, by Lemma 4.2, has the form $\alpha + h_\omega \delta$, where $h_\alpha < h_\omega < h_\alpha + 2$, ie. $h_\omega = h_\alpha + 1$. Consequently, $\hat{\alpha} + \hat{\beta} = 2\hat{\omega}$ and hence $\hat{\omega} = \frac{1}{2}(\hat{\alpha} + \hat{\beta})$. The result follows as in Chapter 3. □

Remark 4.7. If $h_\beta - h_\alpha = 3$, then there are at most 4 weights of $T_u(\Sigma)^*$ where $\hat{\alpha} = \alpha + h_\alpha\delta$, $\hat{\beta} = \alpha + (h_\alpha + 3)\delta$, and the two possible weights have the form $\hat{\omega_1} = \alpha + (h_\alpha + 1)\delta$ and $\hat{\omega_2} = \alpha + (h_\alpha + 2)$. Again, either 2, 3, or 4 weights occur. In the first case, Σ is nonsingular at u. In the second case, if $\hat{\omega_1}$ appears, then $2\hat{\alpha} + \hat{\beta} = 3\hat{\omega_1}$ implies that Σ_u is isomorphic to a surface given by $x_{\hat{\alpha}}^2 x_{\hat{\beta}} = x_{\hat{\omega}_1}^3$, where $x_{\hat{\alpha}}, x_{\hat{\beta}}, x_{\hat{\omega}_1} \in k[\Sigma_u]$ of weights $\hat{\alpha}$, $\hat{\beta}$, $\hat{\omega_1}$. Likewise, if $\hat{\omega_2}$ appears, then Σ_u is isomorphic to a surface given by $x_{\hat{\alpha}} x_{\hat{\beta}}^2 = x_{\hat{\omega}_2}^3$. Now in the case that all 4 appear we have that

$$\hat{\alpha} + \hat{\beta} = \hat{\omega_1} + \hat{\omega_2},\ \ \hat{\alpha} + \hat{\omega_2} = 2\hat{\omega_1},\ \text{ and }\ \hat{\beta} + \hat{\omega_1} = 2\hat{\omega_2}.$$

Thus

$$x_{\hat{\alpha}} x_{\hat{\beta}} = x_{\hat{\omega}_1} x_{\hat{\omega}_2},\ \ x_{\hat{\alpha}} x_{\hat{\omega}_2} = x_{\hat{\omega}_1}^2,\ \text{ and }\ x_{\hat{\beta}} x_{\hat{\omega}_1} = x_{\hat{\omega}_2}^2.$$

The ideal

$$\mathfrak{a} := \langle x_{\hat{\alpha}} x_{\hat{\beta}} - x_{\hat{\omega}_1} x_{\hat{\omega}_2}, x_{\hat{\alpha}} x_{\hat{\omega}_2} - x_{\hat{\omega}_1}^2, x_{\hat{\beta}} x_{\hat{\omega}_1} - x_{\hat{\omega}_2}^2 \rangle$$

is prime (verified by Maple) and so defines an irreducible surface which contains Σ_u. Thus $\Sigma_u \simeq Z(\mathfrak{a})$.

We will now consider the situation where $\beta = -\alpha$. This case is more complicated than the previous one as any additional weight can assume one of three forms. We shall refer to these three forms so frequently that the following notation may be useful.

Definition 4.8. Let $\hat{\alpha} = \alpha + h_\alpha\delta$ and $\hat{\beta} = -\alpha + h_\beta\delta$ and let $r \in \mathbb{Z}$ such that $0 \leq r < d = h_\alpha + h_\beta$. We say a weight $\hat{\omega}$ of $T_u(\Sigma)^*$ is of type ra, rb, or rc if $\hat{\omega} = ((h_\alpha + h_\beta)c_r + r)\delta$, $\hat{\alpha} + ((h_\alpha + h_\beta)c_r + r)\delta$, or $\hat{\beta} + ((h_\alpha + h_\beta)c_r + r)\delta$, respectively.

For simplicity, let $d = h_\alpha + h_\beta$. Thus $d\delta = \hat{\alpha} + \hat{\beta}$. We will work towards producing an upper limit for the number of weights that could occur. At the moment, we have that there are d possible values for the remainder r. So now, for a fixed r, we want to determine how many weights of each type can possibly occur and whether or not weights of different types can simultaneously occur. We start with a Lemma which places bounds on the quotient c_r of any weight that occurs.

Lemma 4.9. *If $(dc_r + r)\delta$ is a weight of $T_u(\Sigma)^*$ and $(dc'_r + r)\delta$ is a different weight of $k[\Sigma_u]$, ie. $(dc'_r + r)\delta$ can be expressed as a positive linear combination of weights of $T_u(\Sigma)^*$, then $c_r < c'_r$. If $\hat{\alpha} + (dc_r + r)\delta$ is a weight of $T_u(\Sigma)^*$ and $\hat{\alpha} + (dc'_r + r)\delta$ is a different weight of $k[\Sigma_u]$, then $c_r < c'_r$. If $\hat{\beta} + (dc_r + r)\delta$ is a weight of $T_u(\Sigma)^*$ and $\hat{\beta} + (dc'_r + r)\delta$ is a different weight of $k[\Sigma_u]$, then $c_r < c'_r$.*

Proof. Assume $c'_r < c_r$. For type ra we have

$$(dc_r + r)\delta = d(c_r - c'_r)\delta + (dc'_r + r)\delta = (c_r - c'_r)\hat{\alpha} + (c_r - c'_r)\hat{\beta} + (dc'_r + r)\delta$$

and hence is a positive linear combination of weights of $T_u(\Sigma)^*$ in which the sum of the coefficients is at least 1. Thus by Lemma 3.7, $(dc_r + r)\delta$ is not a weight of $T_u(\Sigma)^*$, a contradiction.

For type rb, we obtain

$$\hat{\alpha} + (dc_r + r)\delta = (c_r - c_r')\hat{\alpha} + (c_r - c_r')\hat{\beta} + (\hat{\alpha} + (dc_r' + r)\delta),$$

which as above leads to a contradiction and for type rc,

$$\hat{\beta} + (dc_r + r)\delta = (c_r - c_r')\hat{\alpha} + (c_r - c_r')\hat{\beta} + (\hat{\beta} + (dc_r' + r)\delta),$$

which also leads to a contradiction. $\square$

With this condition in place, we can now restrict the number of weights that appear of a given type.

Lemma 4.10. *If $\beta = -\alpha$, then for each $0 \le r < d$, there can occur at most one weight of $T_u(\Sigma)^*$ of each type ra, rb, and rc.*

Proof. Fix $0 \le r < d$. Assume $\hat{\omega} = (dc_r + r)\delta$ and $\hat{\nu} = (dc_r' + r)\delta$ are distinct weights of type ra such that $c_r < c_r'$. By viewing $\hat{\omega}$ as linear combination of itself, from Lemma 4.9 we obtain that $c_r' < c_r$. Thus we have a contradiction. For type rb (resp. rc) , we add the term $\hat{\alpha}$ (resp. $\hat{\beta}$) to $\hat{\omega}$ and $\hat{\nu}$ and proceed as above. $\square$

As was hoped would be the case, we can limit the number of possible weights further by comparing weights of different types.

Lemma 4.11. *For a fixed remainder r, weights of types ra and rb cannot both occur as weights of $T_u(\Sigma)^*$ and weights of type ra and rc cannot both occur as weights of $T_u(\Sigma)^*$. If $\hat{\alpha} + (dc_r' + r)\delta$ and $\hat{\beta} + (dc_r'' + r)\delta$ both occur as weights as weights of $T_u(\Sigma)^*$ for a fixed r, then $c_r' = c_r''$.*

Proof. Fix $0 \le r < d$. Let $\hat{\omega} = (dc_r + r)\delta$, $\hat{\nu} = \hat{\alpha} + (dc_r' + r)\delta$, and $\hat{\mu} = \hat{\beta} + (dc_r'' + r)\delta$.

Assume both $\hat{\omega}$ and $\hat{\nu}$ occur. Then since $\hat{\alpha} + \hat{\omega} = \hat{\alpha} + (dc_r + r)\delta$, by Lemma 4.9 we have $c_r' < c_r$. Since

$$\hat{\beta} + \hat{\nu} = (-\alpha) + h_\beta\delta + \alpha + h_\alpha\delta + (dc_r' + r)\delta = (d(c_r' + 1) + r)\delta,$$

we also have $c_r < c_r' + 1$, by Lemma 4.9. Thus, $c_r - 1 < c_r' < c_r$, which is a contradiction. The case for type ra and type rc is proved similarly.

Now if $\hat{\nu}$ and $\hat{\mu}$ both occur as weights, then $\hat{\alpha}+2\hat{\beta}+\hat{\nu} = \hat{\beta}+(d(c'_r+1)+r)\delta$ gives us that $c''_r \leq c'_r$, by Lemma 4.9. Similarly, since $2\hat{\alpha}+\hat{\alpha}+\hat{\mu} = \hat{\alpha}+(d(c''_r+1)+r)\delta$, we have that $c'_r \leq c''_r$. Hence, $c'_r = c''_r$. □

Taking into account these restriction, we have the following upper bound for the number of weights that may appear:

Lemma 4.12. *If $\beta = -\alpha$, then there are at most $2(h_\alpha+h_\beta)$ weights of $T_u(\Sigma)^*$.*

Proof. For each r such that $0 \leq r < d$, there is at most one weight of each type ra, rb, and rc, by Lemma 4.10. By Lemma 4.11 at most two types can occur for a given r and hence there are at most $2d$ weights. □

Remark 4.13. It was noted above that for the fixed point $u = e$, the weights $-(\alpha + h_\alpha\delta)$ and $-(\beta + h_\beta\delta)$ of the $\widehat{T}$-curves contained in Σ_u have h_α, $h_\beta > 0$. For an arbitrary fixed point, this may not be the case (ie. h_α or h_β could be 0) since our roots are elements of $w(\hat{\Phi}^-_{h<0})$, for some $w \in \widehat{W}$. In addition, although we are working with a $\widehat{T}$-surface in $\mathcal{G}/\mathcal{P}$, all of the results presented are true for $\widehat{T}$-surfaces in $\mathcal{G}/\mathcal{B}$. Indeed, for the $\widehat{T}$-fixed point $u = e\mathcal{B}$, we have

$$\Sigma_u \hookrightarrow \hat{\mathfrak{g}}/\hat{\mathfrak{b}} = \bigoplus_{\hat{\omega}\in\hat{\Phi}^-} \hat{\mathfrak{g}}_{\hat{\omega}},$$

which differs from our original case only by the addition of roots $\hat{\omega} = \omega + h_\omega\delta$, where $h_\omega = 0$ and $\omega < 0$. However, we did not use the fact that $h_\alpha, h_\beta \neq 0$ in our root considerations above. Consequently, in either of these contexts, the following theorem holds.

Theorem 4.14. *Let $\beta = -\alpha$. If $h_\alpha + h_\beta = 1$, then Σ is nonsingular at u and Σ_u is smooth.*

Proof. Since $h_\alpha+h_\beta = 1$, Lemma 4.12 gives us that there are exactly 2 weights of $T_u(\Sigma)^*$ and hence of $T_u(\Sigma)$. □

Remark 4.15. If $\beta = \pm\alpha$, then α and β are contained in a copy of A_1. So we may assume that T is a subset of $\mathrm{SL}_2(\mathbb{C})$ and so the weights of the action of T on $\mathfrak{sl}_2(\mathbb{C})$ are ± 2 and 0. For $\mathcal{G} = \mathrm{SL}_2(\mathbb{C}((x)))$ and $\mathcal{P} = \mathrm{SL}_2(\mathbb{C}[[x]])$ and where $Z(\mathcal{G})$ is the centre of $\mathcal{G}$, we have

$$\mathcal{G}/\mathcal{P} \simeq \big(\mathcal{G}/Z(\mathcal{G})\big)/\big(\mathcal{P}/Z(\mathcal{G})\big) = \mathrm{PGL}_2(\mathbb{C}((x)))/\big(\mathcal{P}/Z(\mathcal{G})\big).$$

Hence we can consider instead the action of the 1-dimensional torus $T/H \subseteq \mathrm{PGL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/H$, where H is the normal subgroup of $\mathrm{SL}_2(\mathbb{C})$ containing the identity matrix I_2 and $-I_2$, on the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of $\mathrm{PGL}_2(\mathbb{C})$. In this case, the weights are ± 1 and 0. So we may assume that $\alpha = 1$ and $\beta = \pm 1$. We will write $a + b\delta$ for the weight (a, b).

A subvariety of a vector space is said to be a *cone* if it is closed under scalar multiplication.

Theorem 4.16. *If $\beta \neq \pm\alpha$ or $\beta = \alpha$, the affine variety $\Sigma_u \subseteq T_u(\Sigma)$ is a cone.*

Proof. Viewing Σ_u as a subvariety of $V := T_u(\Sigma)$, we have that if $\beta \neq \pm\alpha$, then $\Sigma_u = T_u(\Sigma)$ and hence is a cone.

If $\beta = \alpha = 1$, then

$$V = \bigoplus_{h \in \mathbb{Z}} V_{-1+h\delta}.$$

Let $v \in \Sigma_u$ and $t \in T \simeq \mathbb{C}^*$, then $(t^{-1}, 1) \cdot v = tv$. which is again in Σ_u since Σ_u is $\widehat{T}$-stable. Also, $0 = u \in \Sigma_u$. $\square$

After having examined the subject of singularities, one is naturally lead to consider the weaker property of normality. An affine variety X is said to be *normal* if $k[X]$ is integrally closed in $K(X)$. We have been able to determine conditions under which Σ_u is normal in terms of the weights of $T_u(\Sigma)^*$.

If Σ_u is smooth, then it is normal. Thus the only cases left to consider are those where Σ_u is singular and either $\beta = -\alpha$ or $\beta = \alpha$. To address these situations, we will use the following criterion (cf. [9] page 5):

Theorem 4.17. *Let T be a torus and let X be an affine T-variety. X is normal if and only if the following holds: if $\chi \in \mathrm{Span}_{\mathbb{Z}}(\Omega(k[X]))$ such that $N\chi \in \Omega(k[X])$, for some $N \in \mathbb{N}$, then $\chi \in \Omega(k[X])$.*

Although the possible forms of weights of $T_u(\Sigma)^*$ are more complicated in the case that $\beta = -\alpha$ than in the case where $\alpha = \beta$, the normality condition for Σ_u is simpler for $\beta = -\alpha$.

So assume $\alpha = 1$ and $\beta = -1$, then any weights of $T_u(\Sigma)^*$ has the form

$$(dc'_r + r')\delta,\ \hat{\alpha} + (dc'_r + r')\delta, \text{ or } \hat{\beta} + (dc'_r + r')\delta,$$

where $d = h_\alpha + h_\beta$, $c'_r \in \mathbb{Z}_{\geq 0}$ and $0 \leq r' < (h_\alpha + h_\beta)$. Recall that $d\delta = \hat{\alpha} + \hat{\beta} \in \Omega(k[\Sigma_u])$.

Theorem 4.18. *Suppose Σ_u is singular and that $\beta = -\alpha$. Then Σ_u is normal if and only if $\Omega(T_u(\Sigma)^*) = \{\hat{\alpha}, \hat{\beta}, r\delta\}$, where r is a divisor of $d = h_\alpha + h_\beta$ other than d.*

Proof. Assume Σ_u is normal and that $\hat{\alpha} = 1 + h_\alpha\delta$ and $\hat{\beta} = -1 + h_\beta\delta$. Let r be the gcd of d and the remainders r' of the elements of $\Omega(T_u(\Sigma)^*)$. If $(dc'_r + r')\delta$ is a weight of $T_u(\Sigma)^*$ (and hence of $k[\Sigma_u]$), then since

$$r'\delta = (dc'_r + r')\delta - c'_r\hat{\alpha} - c'_r\hat{\beta} \in \mathrm{Span}_{\mathbb{Z}}(\Omega(k[\Sigma_u]))$$

and

$$d(r'\delta) = r'(d\delta) \in \Omega(k[\Sigma_u])$$

by Theorem 4.17, $r'\delta \in \Omega(k[\Sigma_u])$. Similarly, if either $\hat{\alpha} + (dc'_r + r')\delta$ or $\hat{\beta} + (dc'_r + r')\delta$ is a weight of $T_u(\Sigma)^*$, then $r'\delta \in \Omega(k[\Sigma_u])$. Thus $r'\delta \in \Omega(k[\Sigma_u])$ for every remainder r' of the elements of $\Omega(T_u(\Sigma)^*)$ and since there is a linear combination of $d\delta$ and the weights $r'\delta$ which is equal to $r\delta$ and $d(r\delta) = r(d\delta)$, $r\delta \in \Omega(k[\Sigma_u]) = \mathrm{Span}_{\mathbb{N}_0}(\Omega(T_u(\Sigma)^*))$. But since $d > r$, $r' \geq r$ and $dc'_r \geq 0$, the only way for $r\delta$ to be a linear combination of weights of $T_u(\Sigma)^*$ with positive coefficients is for $r\delta$ to be an element of $\Omega(T_u(\Sigma)^*)$. But then,

$$(dc'_r + r')\delta = c'_r\hat{\alpha} + c'_r\hat{\beta} + cr\delta$$

$$\hat{\alpha} + (dc'_r + r')\delta = (c'_r + 1)\hat{\alpha} + c'_r\hat{\beta} + cr\delta$$
$$\hat{\beta} + (dc'_r + r')\delta = c'_r\hat{\alpha} + (c'_r + 1)\hat{\beta} + cr\delta,$$

where $r' = cr$ and so by Lemma 3.7, the only weights of $T_u(\Sigma)^*$ are h_α, h_β and $r\delta$.

For the other direction, assume that $\Omega(T_u(\Sigma)^*) = \{\hat{\alpha}, \hat{\beta}, r\delta\}$ and set $d = \tilde{r}r$. Let $a + b\delta \in \mathrm{Span}_{\mathbb{Z}}(\hat{\alpha}, \hat{\beta}, r\delta) = \mathrm{Span}_{\mathbb{Z}}(\Omega(k[\Sigma_u]))$ be an nonzero element such that $N(a + b\delta) \in \Omega(k[\Sigma_u])$ for some $N \in \mathbb{N}$. Thus,

$$a + b\delta = a_1(1 + h_\alpha\delta) + a_2(-1 + h_\beta\delta) + a_3r\delta,$$

so that

$$a = a_1 - a_2 \ \text{ and } \ b = a_1h_\alpha + a_2h_\beta + a_3r, \tag{4.1}$$

where $a_i \in \mathbb{Z}$, for $i = 1, 2, 3$. Also,

$$N(a + b\delta) = b_1(1 + h_\alpha\delta) + b_2(-1 + h_\beta\delta) + b_3r\delta,$$

which gives

$$Na = b_1 - b_2 \ \text{ and } \ Nb = b_1h_\alpha + b_2h_\beta + b_3r, \tag{4.2}$$

where $b_i \in \mathbb{N}$, for $i = 1, 2, 3$. From 4.2 we have that $b \geq 0$.

If $a = 0$, then from 4.1 we have that $a_1 = a_2$ and hence

$$b = a_1d + a_3r = (a_1\tilde{r} + a_3)r,$$

where $(a_1\tilde{r} + a_3) \geq 0$. Consequently,

$$a + b\delta = (a_1\tilde{r} + a_3)r\delta \in \Omega(k[\Sigma_u]).$$

So now assume that $a \neq 0$, then from 4.2 we obtain:

$$b(b_1 - b_2) = ab_1h_\alpha + ab_2h_\beta + ab_3r$$

and hence

$$(ah_\alpha - b)b_1 + (ah_\beta + b)b_2 + arb_3 = 0.$$

If $a > 0$, then $arb_3, (ah_\beta + b)b_2 \geq 0$ and so $ah_\alpha - b \leq 0$, ie. $b = ah_\alpha + \tilde{b}$ for some $\tilde{b} \in \mathbb{N}_0$. From 4.1 we have that

$$\tilde{b} = -(a_1 - a_2)h_\alpha + a_1 h_\alpha + a_2 h_\beta + a_3 r = a_2(h_\alpha + h_\beta) + a_3 r = (a_2\tilde{r} + a_3)r$$

where $a_2\tilde{r} + a_3 \geq 0$. Thus,

$$a + b\delta = a(1 + h_\alpha\delta) + (a_2\tilde{r} + a_3)r\delta \in \Omega(k[\Sigma_u]).$$

If $a < 0$, then $(ah_\alpha - b)b_1, arb_3 \leq 0$ and so $b \geq -ah_\beta$. As above $b = -ah_\beta + \tilde{b}$, where $\tilde{b} = (a_1\tilde{r} + a_3)r \geq 0$, which implies that

$$a + b\delta = -a(1 + h_\beta\delta) + (a_1\tilde{r} + a_3)r\delta \in \Omega(k[\Sigma_u]).$$

Therefore, by Theorem 4.17 Σ_u is normal. $\square$

Remark 4.19. If $\Omega(T_u(\Sigma)^*) = \{\hat{\alpha}, \hat{\beta}, r\delta\}$, where $r|d$, then $\Sigma_u = Z(x^k_{r\delta} - x_{\hat{\alpha}}x_{\hat{\beta}})$, where $kr = d$ and hence Σ_u is normal. Surfaces of this form are the only singular normal surfaces in this context.

Now for the case where $\beta = \alpha$, we again assume that $\alpha = \beta = 1$, so that any weights of $T_u(\Sigma)^*$ has the form $1 + (h_\alpha + j)\delta$, for some $0 \leq j \leq h_\beta - h_\alpha$.

Theorem 4.20. *Suppose Σ_u is nonsingular and that $\beta = \alpha$. Then Σ_u is normal if and only if $\Omega(T_u(\Sigma)^*) = \{1 + (h_\alpha + ic)\delta \mid 0 \leq i \leq k$ and $c \in \mathbb{N}\}$, where $h_\beta = h_\alpha + kc$.*

Proof. Suppose that Σ_u is normal. Let

$$J = \{j \in \mathbb{N} \mid 1 + (h_\alpha + j)\delta \in \Omega(T_u(\Sigma)^*)\},$$

let c be the gcd of all elements of J and set $h_\beta = h_\alpha + kc$, then

$$j\delta = 1 + (h_\alpha + j)\delta - (1 + h_\alpha\delta) \in \mathrm{Span}_{\mathbb{Z}}(\Omega(k[\Sigma_u])),$$

for all $j \in J$, and hence $c\delta \in \mathrm{Span}_{\mathbb{Z}}(\Omega(k[\Sigma_u]))$. It follows that $1 + (h_\alpha + ci)\delta \in \mathrm{Span}_{\mathbb{Z}}(\Omega(k[\Sigma_u]))$, for all $1 \leq i \leq k - 1$. Consequently,

$$k(1 + (h_\alpha + ci)\delta) = (k - i)(1 + h_\alpha\delta) + i(1 + (h_\alpha + kc)\delta)$$

$$= (k - 1)\hat{\alpha} + i\hat{\beta} \in \Omega(k[\Sigma_u]),$$

by Theorem 4.17, gives that $1 + (h_\alpha + ci)\delta$ is in $\Omega(k[\Sigma_u]) = \mathrm{Span}_{\mathbb{N}_0}(\Omega(T_u(\Sigma)^*))$, which implies that $1 + (h_\alpha + ci)\delta \in \Omega(T_u(\Sigma)^*)$. By our choice of c, every element of $\Omega(T_u(\Sigma)^*)$ is of the form $1 + (h_\alpha + ci)\delta$, for some $0 \leq i \leq k$.

Now assume that $\Omega(T_u(\Sigma)^*) = \{1 + (h_\alpha + ic)\delta \mid 0 \le i \le k \text{ and } c \in \mathbb{N}\}$ and let $a+b\delta$ be a nonzero element of $\mathrm{Span}_{\mathbb{Z}}(\Omega(k[\Sigma_u]))$ such that $N(a+b\delta) \in \Omega(k[\Sigma_u])$. Thus

$$a + b\delta = \sum_{i=0}^{k} a_i(1 + (h_\alpha + ic)\delta),$$

where the $a_i \in \mathbb{Z}$. Consequently,

$$a = \sum_{i=0}^{k} a_i \;\text{ and }\; b = \sum_{i=0}^{k} a_i h_\alpha + \sum_{i=0}^{k} a_i ic = ah_\alpha + \sum_{i=0}^{k} a_i ic, \tag{4.3}$$

Also,

$$N(a + b\delta) = \sum_{i=0}^{k} b_i(1 + (h_\alpha + ic)\delta),$$

from which we obtain

$$Na = \sum_{i=0}^{k} b_i \;\text{ and }\; Nb = \sum_{i=0}^{k} b_i(h_\alpha + ic) = Nah_\alpha + \sum_{i=0}^{k} b_i ic. \tag{4.4}$$

where the $b_i \in \mathbb{N}_0$. From equation 4.4 we have that both a and b are in $\mathbb{N}$, $Nah_\alpha \le Nb$, and

$$Nb \le Nah_\alpha + \sum_{i=0}^{k} b_i kc = Na(h_\alpha + kc).$$

Thus

$$ah_\alpha \le b \;\text{ and }\; b \le a(h_\alpha + kc) = ah_\beta.$$

If $b = a(h_\alpha + ci)$, for some $0 \le i \le k$, then $a + b\delta = a(1 + (h_\alpha + ci)\delta)$ is in $\Omega(k[\Sigma_u])$.

Now assume $b \ne a(h_\alpha + ci)$, for any $0 \le i \le k$, so, in particular, $ah_\alpha < b < ah_\beta$. There exists an m where $1 \le m \le a$ such that

$$(m-1)kc < b - ah_\alpha \le mkc,$$

which is equivalent to

$$(m-1)k < \sum_{i=0}^{k} a_i i \le mk$$

(from equation 4.3). Thus,

$$\sum_{i=0}^{k} a_i i = (m-1)k + j$$

for some $1 \leq j \leq k$ and hence

$$b - (a-m)h_\alpha = ((m-1)kc + jc) + mh_\alpha$$

$$= (m-1)(h_\alpha + kc) + (h_\alpha + jc) = (m-1)h_\beta + (h_\alpha + jc).$$

Accordingly,

$$a + b\delta = (a-m)(1 + h_\alpha\delta) + (m + (b-(a-m))h_\alpha\delta)$$

$$= (a-m)\hat{\alpha} + (m-1)\hat{\beta} + (1 + (h_\alpha + jc)\delta) \in \Omega(k[\Sigma_u]).$$

Therefore, by Theorem 4.17 Σ_u is normal. □

Remark 4.21. There are two additional proofs for the "if" direction of Theorem 4.20. As the affine cone over the k-uple embedding of $\mathbb{P}^1$ into $\mathbb{P}^k$, Σ_u is normal. Alternately, Σ_u is the closure of the minimal orbit of $\mathrm{SL}_2(\mathbb{C})$ on the irreducible representation of dimension $k+1$ and, as such, is normal. We decided to include the above proof as it demonstrates that the same criterion (Theorem 4.17) can be used to prove both directions of the theorems for each of the exceptional cases (ie. $\beta = \alpha$ and $\beta = -\alpha$) and that this can be done using elementary techniques.

Remark 4.22. The results presented in this chapter hold for an arbitrary $\widehat{T}$-surface in $\mathcal{G}/\mathcal{P}$. For any such $\widehat{T}$-surface, we have considered all possible descriptions of the weights of the tangent space at e, but given a "possible" set S of weights, we do not know if there is a $\widehat{T}$-surface Σ in $\mathcal{G}/\mathcal{P}$ such that $\Omega(T_u(\Sigma)) = S$. Indeed, the issue remains as to which $\widehat{T}$-surfaces actually appear in the affine Grassmannian. It is, however, clear that not all $\widehat{T}$-surfaces can be smooth, since there are rationally smooth singular Schubert varieties.

Chapter 5

Smooth Points of Schubert Varieties in $\mathcal{G}/\mathcal{P}$

In this section we will provide a proof of Remark 4.19 in [11] (appearing here as Theorem 5.9), which addresses the topic of smooth points of Schubert varieties in $\mathcal{G}/\mathcal{P}$, where $\mathcal{G} := \mathrm{SL}_n(\mathbb{C}((x)))$ and $\mathcal{P} := \mathrm{SL}_n(\mathbb{C}[[x]])$, for some $n \in \mathbb{N}$. A key object in this proof is the Peterson translate of a Schubert variety along a $\widehat{T}$-curve. A result (see Theorem 5.4) given by Carrell and Kuttler in [6] reduces the problem to considering the Peterson translate of $\widehat{T}$-surfaces along the $\widehat{T}$-curve, enabling us to use our results from the previous chapter. This chapter is based on material presented in [6], [10] and [11].

5.1 Peterson Translates

Let T be a torus, X an affine T-variety, and $u \in X^T$ an attractive fixed point (so $X = X_u$). Let $C \in E(X, u)$ and define d to be the common dimension of $T_c(X)$, for all c in the open orbit $C \setminus C^T$. Using $V := T_u(X)$, set $G(d, V)$ to be the Grassmannian of d-planes in V.

Now consider the map:

$$\begin{aligned} \varphi\colon C \setminus C^T &\to G(d, V), \\ c &\mapsto T_c(X) \end{aligned}$$

This map extends uniquely to C_u, where the object defined below is the point $\varphi(u)$ of $G(d, V)$.

Definition 5.1. The *Peterson translate* of X along C is the limit

$$\tau_C(X,u) := \lim_{c \to u} T_c(X),$$

where $c \in C \setminus C^T$.

Thus

$$\dim X \leq \dim \tau_C(X,u) \leq \dim T_u(X).$$

Let $\Sigma(X,C)$ be the set of all T-surfaces in X which contain C. A T-curve $C = \overline{T \cdot z}$ in X is said to be *good* if X is nonsingular at z. If $C \in E(X,u)$ is good, then $\tau_C(X,u)$ depends on $\tau_C(\Sigma,u)$, for $\Sigma \in \Sigma(X,C)$ as follows:

Lemma 5.2. *Let X be a T-variety, let $u \in X^T$ be attractive, and let C be a good T-curve in $E(X,u)$, then*

$$\tau_C(X,u) = \sum_{\Sigma \in \Sigma(X,C)} \tau_C(\Sigma,u).$$

Proof. See Lemma 5.1 in [6]. □

In addition to Peterson translates, we will also use the following object.

Definition 5.3. Let X be any T-variety with T-fixed point u.

$$TE(X,u) := \sum_{C \in E(X,u)} T_u(C)$$

is called the *tangent space* to $E(X,u)$ at u.

Now $TE(X,u) \subseteq T_u(X)$ and since $\tau_C(X,u)$ is also a subspaces of $T_u(X)$, it is natural to ask if there is a relationship between $TE(X,u)$ and $\tau_C(X,u)$. If X is nonsingular at u, then $\tau_C(X,u) = T_u(X)$, for all $C \in E(X,u)$ and since every T-stable line in $T_u(X)$ is tangent to some T-curve $C \in E(X,u)$, $TE(X,u) = T_u(X)$. Therefore, $\tau_C(X,u) = TE(X,u)$. So then the question arises: are there conditions under which $\tau_C(X,u) = TE(X,u)$ implies that X is nonsingular at u? One such set of conditions is given in the theorem below. This is Theorem 1.4 in [6].

Theorem 5.4. *Let X be a T-variety with an attractive T-fixed point u. Assume that $E(X,u)$ contains only smooth curves and that for $C \neq D \in E(X,u)$, the T-weights of $T_u(C)$ and $T_u(D)$ as T-modules are not equal. If either*

1) *$TE(X,u) = \tau_C(X,u)$ for at least two good T-curves $C \in E(X,u)$, or*

2) *X is Cohen-Macaulay at u and $TE(X,u) = \tau_C(X,u)$ for at least one good $C \in E(X,u)$,*

holds, then X is nonsingular at u.

5.2 Smooth Points of Schubert Varieties

We return to our initial situation in the previous chapter, ie. $\mathcal{G} := \mathrm{SL}_n(\mathbb{C}((x)))$, $\mathcal{P} := \mathrm{SL}_n(\mathbb{C}[[x]])$, for some $n \in \mathbb{N}$, $\widehat{W} = N_{\mathcal{G}}(T)/T$ is the affine Weyl group, $W = S_n$ is the Weyl group of $\mathrm{SL}_n(\mathbb{C})$, and $\widehat{W}^{\mathcal{P}}$, the set of minimal length representatives of $\widehat{W}/W$.

Remark 5.5. The affine Weyl group can be realized as a subgroup of the group of permutations on $\mathbb{Z}$ as a subset of permutations which commute with shifting by n.

Let $X(w) = \overline{\mathcal{B}w}$, for $w \in \widehat{W}^{\mathcal{P}}$ be any Schubert variety in $\mathcal{G}/\mathcal{P}$. Recall that we identify the $\widehat{T}$-fixed points of $X(w)$ with elements of $\widehat{W}^{\mathcal{P}}$.

Define a partial order on $\widehat{W}^{\mathcal{P}}$, called the *Bruhat-Chevalley order*, by setting $u < y$ if and only if $u \neq y$ and $u \in X(y)$. Let

$$[u,w] = \{y \in \widehat{W}^{\mathcal{P}} | u \leq y \leq w\}.$$

Thus $[e,w] = X(w)^{\widehat{T}}$ and we have that

$$X(w) = \bigcup_{y \in [e,w]} \mathcal{B}y.$$

Consequently, for $u, y \in \widehat{W}^{\mathcal{P}}$, $u \leq y$ implies $X(u) \subseteq X(y)$.

Let $\mathcal{U}_{\hat{\alpha}}$ be the unique subgroup of $\mathcal{G}$ normalized by $\widehat{T}$ with Lie algebra $\hat{\mathfrak{g}}_{\hat{\alpha}}$ and let $\mathcal{G}_{\hat{\alpha}} := \langle \mathcal{U}_{\hat{\alpha}}, \mathcal{U}_{-\hat{\alpha}} \rangle$, ie. the copy of $\mathrm{SL}_2(\mathbb{C})$ in $\mathcal{G}$ which is the group generated by $\mathcal{U}_{\hat{\alpha}}$ and $\mathcal{U}_{-\hat{\alpha}}$ in $\mathcal{G}$. The $\widehat{T}$-curves in $X(w)$ have the following description:

$$\begin{aligned}
&E(X(w),u) \\
&\qquad = \{\mathcal{G}_{\hat{\alpha}} u \mid \hat{\alpha} \in \hat{\Phi}^+ \text{ is real and } u \neq r_{\hat{\alpha}} u \leq w\} \\
&\qquad = \{\overline{\mathcal{U}_{\hat{\alpha}} u} \mid \hat{\alpha} \in \hat{\Phi}^+ \text{ is real and } r_{\hat{\alpha}} u < u\} \\
&\qquad\quad \cup \{\overline{\mathcal{U}_{-\hat{\alpha}} u} \mid \hat{\alpha} \in \hat{\Phi}^+ \text{ is real and } u < r_{\hat{\alpha}} u \leq w\}
\end{aligned}$$

All of these curves are distinct and smooth. The curve $C = \overline{\mathcal{U}_{\hat{\alpha}} u}$ (respectively, $\overline{\mathcal{U}_{-\hat{\alpha}} u}$) has $C^{\widehat{T}} = \{u, r_{\hat{\alpha}} u\}$ and $T_u(C) = \hat{\mathfrak{g}}_{\hat{\alpha}}$ (respectively, $\hat{\mathfrak{g}}_{-\hat{\alpha}}$) If $C, D \in E(X(w), u)$ are distinct, then $C \cap D = \{u\}$ and $T_u(C) \cap T_u(D) = \{0\}$. (See [4] or [10]).

In this context,

$$TE(X(w), u) = \bigoplus_{C \in E(X(w),u)} T_u(C)$$

and has dimension $|E(X(w), u)|$. If $X(w)$ is nonsingular at $u \leq w$, then $\dim TE(X(w), u) = \dim X(w)$. Note also that the requirements of Theorem 5.4 are fulfilled by any $X(w)$ in $\mathcal{G}/\mathcal{P}$.

We are missing only one key ingredient for Theorem 5.9 below and it is this ingredient which distinguishes the affine case from the classical case and enables us to use our results from the previous chapter.

Definition 5.6. Let $\hat{\alpha} = \alpha + h_\alpha \delta$. A reflection $r_{\hat{\alpha}} \in \widehat{W}$ is said to be *small* if $|r_{\hat{\alpha}}(z) - z| < n$ (see Remark 5.5), for all $z \in \mathbb{Z}$, or equivalently if $\alpha > 0$ and $h_\alpha = 0$, or $\alpha < 0$ and $h_\alpha = 0$ or 1. Otherwise, $r_{\hat{\alpha}}$ is called *large*.

Lemma 5.7. *Let Σ be any $\widehat{T}$-surface in $\mathcal{G}/\mathcal{P}$ with $\widehat{T}$-fixed point u and let C and D be the two elements of $E(\Sigma, u)$. Let $C^{\widehat{T}} = \{u, r_{\hat{\alpha}} u\}$ and $D^{\widehat{T}} = \{u, r_{\hat{\beta}} u\}$, where $r_{\hat{\alpha}}, r_{\hat{\beta}} \in \widehat{W}$. If $r_{\hat{\alpha}}$ and $r_{\hat{\beta}}$ are both small, then Σ is nonsingular at u.*

Proof. Let $\hat{\alpha} = \alpha + h_\alpha \delta$ and $\hat{\beta} = \beta + h_\beta \delta$. By Theorem 4.3, if $\beta \neq \alpha$, then Σ is nonsingular at u.

If $\beta = \alpha$, since $r_{\hat{\alpha}}$ and $r_{\hat{\beta}}$ are both small and $\hat{\alpha} \neq \hat{\beta}$, then $h_\beta - h_\alpha = 1$ (assuming $h_\alpha < h_\beta$) and hence by Theorem 4.5, Σ is nonsingular at u.

In the case that $\beta = -\alpha$, $r_{\hat{\alpha}}$ and $r_{\hat{\beta}}$ are both small, exactly one of α and β is positive, and $\hat{\alpha}$ and $\hat{\beta}$ are nonproportional imply that exactly one of h_α and h_β is 0 and the other is equal to 1, ie. $h_\alpha + h_\beta = 1$. Hence, by Theorem 4.14, Σ is smooth at u. $\square$

We are now in a position to prove Theorem 5.9 (Remark 4.19), but first we will provide the inspiration from the classical setting for this remark, namely a result by Dale Peterson called the ADE-Theorem. Let P be a parabolic subgroup of G.

Theorem 5.8 (ADE-Theorem)**.** *Let X be a Schubert variety in G/P, where G is of type A, D, or E and let $u \in X^T$, then u is a smooth point of X if and only if $|E(X, y)| = \dim X$, for all T-fixed points y in X such that $y \geq u$.*

Proof. See [6]. $\square$

Although $|E(X(w), y)| = \dim X(w)$, for all $y \in [x, w]$ is still necessary in the affine case, it no longer guarantees that $X(w)$ is nonsingular at u. Indeed, Kuttler and Lakshmibai prove in Lemma 4.18 in [11], that $X(w)$ is singular at a point $u \in X(w)^{\widehat{T}}$, if $s_{\hat{\alpha}}$ is a large reflection such that $u < s_{\hat{\alpha}} u \le w$.

The proof for the following theorem is a modified version of the proof of the ADE-Theorem given in [6].

Theorem 5.9 (Remark 4.19 of [11]). *Let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{P}$, where $\mathcal{G} := \mathrm{SL}_n(\mathbb{C}((x)))$ and $\mathcal{P} := \mathrm{SL}_n(\mathbb{C}[[x]])$, for some $n \in \mathbb{N}$ and let $u \in X(w)^{\widehat{T}}$, then $X(w)$ is nonsingular at u if and only if $|E(X(w), y)| = \dim X(w)$, for all $y \in [u, w]$ and for every $y \in [u, w]$, any $s_{\hat{\alpha}}$ such that $y < s_{\hat{\alpha}} y \le w$ is small.*

Proof. If $X(w)$ is nonsingular at u, then $|E(X(w), u)| = \dim X(w)$ and if $u < y \le w$, then $X(w)$ must also be nonsingular at y since otherwise $X(y)$ would be contained in the singular locus of $X(w)$ and thus $X(u) \subset X(y)$ would be as well. Consequently, $|E(X(w), y)| = \dim X(w)$, for all $y \in [u, w]$. Also, for $y \in [u, w]$, if there exists a large reflection $s_{\hat{\alpha}}$ such that $y < s_{\hat{\alpha}} y \le w$, then by Lemma 4.18 in [11], y is a singular point of $X(w)$. Thus, X is nonsingular at u implies that for every $y \in [u, w]$, any $s_{\hat{\alpha}}$ such that $y < s_{\hat{\alpha}} y \le w$ is small.

Now, for any $y \in \widehat{W}^{\mathcal{P}}$, let $\ell(y) := \dim X(y)$. We will give a proof by induction on $\ell(w) - \ell(u)$. The claim holds for $\ell(w) - \ell(u) = 0$ or 1, since $X(w)$ is nonsingular at w and Schubert varieties are nonsingular in codimension 1. Thus we may assume that $\ell(w) - \ell(u) \ge 2$, $|E(X(w), u)| = \dim X(w)$, and that all $y \in (u, w]$ are smooth points of $X(w)$. Now $|E(X(w), u)| = \dim X(w)$ is equivalent to

$$\{s_{\hat{\alpha}} | u < s_{\hat{\alpha}} u \le w\} = \ell(w) - \ell(u).$$

In other words, there are at least two $\widehat{T}$-curves in $X(w)$ whose $\widehat{T}$-fixed points are in $[u, w]$. Let C be any such $\widehat{T}$-curve, ie. $C = \overline{\mathcal{U}_{\hat{\alpha}} u}$, where $\hat{\alpha} \in \hat{\Phi}^+$ is real and $C^{\widehat{T}} = \{u, r_{\hat{\alpha}} u\}$ such that $u < r_{\hat{\alpha}} u$ and $r_{\hat{\alpha}}$ is small. Thus $X(w)$ is smooth at $r_{\hat{\alpha}} u$ and hence $\dim T_z(X(w)) = \dim T_{r_{\hat{\alpha}} u}(X(w)) = \dim X(w)$, for all $z \in C \setminus C^{\widehat{T}}$. Therefore, C is good.

Let $D \in E(X(w), u)$ other than C, and let $D^{\widehat{T}} = \{u, r_{\hat{\beta}} u\}$, for some $r_{\hat{\beta}} \in \widehat{W}$.

If $u < r_{\hat{\beta}} u$, then $D = \overline{\mathcal{U}_{-\hat{\beta}} u}$, where $\hat{\beta} \in \hat{\Phi}^+$ is real. As in the proof of Lemma 3.8, there is a finite surjective morphism $\pi : X(w)_u \to TE(X(w), u)$. Let $\Sigma' = T_u(C) \oplus T_u(D) = \hat{\mathfrak{g}}_{\hat{\alpha}} \oplus \hat{\mathfrak{g}}_{-\hat{\beta}}$. Then, for dimensional reasons, some irreducible component Σ of $\pi^{-1}(\Sigma')$ is a surface containing C_u and D_u. Since both $r_{\hat{\alpha}}$ and $r_{\hat{\beta}}$ are small, by Lemma 5.7, Σ is nonsingular at u. Thus

$$T_u(D) \subseteq T_u(\Sigma) = \tau_C(\Sigma, u) \subseteq \tau_C(X(w), u),$$

with the last inclusion coming from Lemma 5.2.

In the case where $r_{\hat{\beta}}u < u$, we have that $D = \overline{\mathcal{U}_{\hat{\beta}}u}$, where $\hat{\beta} \in \hat{\Phi}^+$ is real. Recall that using $V := T_u(X(w))$ and $d := \dim \tau_C(X(w), u)$, by definition $\tau_C(X(w), u)$ is the point $\varphi(u)$ in $G(d, V)$. Let $C' = \varphi(C_u)$.

Now consider the map

$$\begin{aligned} \psi\colon C_u &\to G(d, V) \times V \\ c &\mapsto (c', d_{(1,c)}\mu(1, 0)) \end{aligned}$$

where $\mu\colon \mathcal{U}_{\hat{\beta}} \times X(w) \to X(w)$ is the action map and $c' = \varphi(c)$. Then $\psi(C_u) \subseteq E$, where

$$E = \{(Z, v) \mid v \in Z \text{ and } Z \in C'\} \subseteq G(d, V) \times V$$

is the tautological bundle over C'. Consequently, $\psi(u) = (u', d_{(1,u)}\mu(1, 0))$ lies in the fibre of E over u', which is $\tau_C(X(w), u)$. But since $T_u(\mathcal{U}_{\hat{\beta}}u) = T_u(D)$ is spanned by $d_{(1,u)}\mu(1, 0)$,

$$T_u(D) \subseteq \tau_C(X(w), u).$$

Consequently, $TE(X(w), u) \subseteq \tau_C(X(w), u)$. From the induction assumption and the fact that C is good we have

$$\dim \tau_C(X(w), u), \dim TE(X(w), u) = \dim X(w),$$

and thus $\tau_C(X(w), u) = TE(X(w), u)$. Therefore, since there are at least two good curves for which this holds, by Lemma 5.4, $X(w)$ is nonsingular at u. $\square$

Remark 5.10. For a Schubert variety in $\mathcal{G}/\mathcal{B}$, the "if" direction of Theorem 5.9 still holds. The question remains as to whether or not the opposite direction holds or if not what condition should replace the hypothesis that all $r_{\hat{\alpha}}$ are small.

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