# University of Alberta

Bezout Orders and Krull Rings

by

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# Abstract


Arithmetic rings in commutative algebra are Krull and Prüfer rings and therefore include principal ideal rings, Dedekind, valuation and unique factorization rings. The main tool in the study of arithmetic rings is an appropriate valuation theory. For commutative domains, it is the theory of valuations on fields initiated by Krull in 1932. The most successful valuation theory for commutative rings with zero divisors is the theory introduced by M. E. Manis in 1967. In the non-commutative case, Schilling valuations were used to determine the Brauer group over local fields and Dubrovin valuation rings, introduced in 1984, do not only have a rich extension theory, but are very useful in the investigation of Bezout orders in simple artinian rings. This thesis concentrates on the study of the ideal structure of non-commutative valuation rings and commutative Krull rings with zero divisors.

We first consider the structure of prime ideals of a Dubrovin valuation ring. A prime ideal $P$ of a Dubrovin valuation ring $R$ in a simple artinian ring $Q$ is called Goldie prime if $R/P$ is a prime Goldie ring. We show that in the special case when $R$ is a total valuation ring, Goldie prime ideals are exactly completely prime ideals. Then we proceed to show that any intersection and union of Goldie prime ideals is again Goldie prime, any idempotent proper ideal is Goldie prime and the intersection of all powers of any proper ideal is Goldie prime. Using these results we show that in the case of a rank one Dubrovin valuation ring $R$, the set $D(R)$ of all divisorial ideals of $R$ is a group, order isomorphic to a subgroup of $(\mathbb{R}, +)$.

Next, we show that there exists a complete analogy between the structure of ideals of a cone in a right-ordered group and the structure of ideals of a Dubrovin valuation ring. In the rank one case, this structure is described completely. Also, we show that for a discrete Dubrovin valuation ring $R$, the Jacobson radical $\mathcal{J}(R)$ is principal as a right $R$-ideal, $D(R) = \langle \mathcal{J}(R) \rangle$ and $\bigcap \mathcal{J}(R)^n = (0)$.

Finally, we consider an application of the valuation theory in the study of commutative and non-commutative arithmetic rings. If $R$ is a commutative Krull domain with the quotient field $K$, $R \neq K$, then for any finite set $\{v_{\mathfrak{P}_1}, v_{\mathfrak{P}_2}, \dots, v_{\mathfrak{P}_n}\}$ of essential valuations of the ring $R$ and any set $\{m_1, m_2, \dots, m_n\}$ of integers, there exists an element $x \in K$ such that $v_{\mathfrak{P}_i}(x) = m_i$ for all $i \in \{1, 2, \dots, n\}$ and $v_{\mathfrak{P}}(x) \geq 0$ for all other essential valuations $v_{\mathfrak{P}}$ of $R$. We prove an analogous approximation theorem for Krull rings with zero divisors. This result allows us to characterize divisorial fractional ideals of an additively regular Krull ring.

**University of Alberta**

**Faculty of Graduate Studies and Research**

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Bezout Orders and Krull Rings** submitted by Enver Osmanagic in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

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To my parents

To my wife Slavinka and my daughters

Tanja and Maja for their love, patience and understanding

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# Contents

# Introduction

A successful partial proof of the famous Fermat Conjecture (before A.Wiles proved the Conjecture), i.e., the impossibility of the equation $x^p + y^p = z^p$ for p an odd prime number and $x, y, z$ integers $\neq 0$, did depend on the *uniqueness* of the factorization in the ring $\mathbb{Z}[\zeta]$ ($\zeta \neq 1$ is a $p$-th root of unity). A mistake in his first attempt to prove the Conjecture led Kummer to his study of the arithmetic of cyclotomic fields which occupied him for almost 25 years. Using the *theory of ideals* and *ideal classes* in algebraic number fields, Kummer proved the Fermat Conjecture for every regular odd prime number $p$ , i.e., $p \nmid h$, where $h$ is the class number in $\mathbb{Q}(\zeta)$. His "ideal prime numbers", that is, "the exponent" with which a factor appears in the decomposition of a number $x \in \mathbb{Z}[\zeta]$, is in the modern language the value of a *valuation* on the field $\mathbb{Q}(\zeta)$ at $x$. This development of the theory of algebraic numbers between 1830 and 1860 was one of the two central motivations for the creation of modern commutative algebra and the study of commutative integral domains, principal ideal domains (PID), Prüfer, Krull, Dedekind and unique factorization domains (UFD). The second source was Algebraic Geometry.

Even though a strict definition of the notion of an *arithmetic ring* does not exist, many arguments suggest that by arithmetic rings in commutative algebra we mean Krull and Prüfer rings which include principal ideal rings (PIR), Dedekind, valuation and unique factorization rings (UFR).

The main object of study in this thesis are commutative and non-commutative arithmetic rings. We contribute some results concerning the ideal structure of non-commutative valuation rings and some results about essential valuations of commutative Krull rings with zero divisors.

In Chapter 1, we present background material necessary for the subject. We discuss the basic results and present the classical theorems from the general theory of non-commutative rings, the general theory of orders with an emphasis on Bezout orders in an artinian ring and the theory of commutative and non-commutative valuation rings. For some of these results modified proofs are given. To a certain extent, this chapter makes the thesis independent of the other sources.

In 1984, N. Dubrovin introduced a new class of non-commutative valuation rings in a simple artinian ring, now called Dubrovin valuation rings. In Chapter 2, we examine the ideal theory of Dubrovin valuation rings. We start with presenting recent results by J. Gräter about Bezout orders in an artinian ring and proceed to give a modified proof of a characterization theorem for Dubrovin valuation rings which shows that this class of rings consists exactly of local Bezout orders in simple artinian rings. In particular, we consider prime ideals $P$ of a Dubrovin valuation ring $R$ with the property that $R/P$ is a prime Goldie ring. Such ideals are called Goldie prime ideals. We first show that in the special case when $R$ is a total valuation ring, then a prime ideal $P$ of $R$ is a Goldie prime if and only if $P$ is completely prime, see Proposition 2.3.16. Then we show that Goldie prime ideals have many "nice" properties.

If $P_i$, $i \in \Lambda$, are Goldie primes in $R$, then $\bigcap P_i$ and $\bigcup P_i$ are Goldie prime ideals, see Proposition 2.3.19 and Corollary 2.3.22. Also, if $I^2 = I \neq R$ is an idempotent ideal of $R$, then $I$ is a Goldie prime, Lemma 2.3.20. The essential result for further exploration of the ideal structure of Dubrovin valuation rings is the result in Theorem 2.3.24. This theorem shows that the intersection of all powers of any proper ideal of $R$ is a Goldie prime.

The end of Chapter 2 looks at divisorial ideals of a Dubrovin valuation ring $R$. A right $R$-ideal $A$ is called divisorial if $A = A^* = \bigcap cS$, where $c$ runs over all elements in $Q$ with $cS \supseteq A$ and $S = O_r(A)$. On the set $D(R)$ of all divisorial ideals we define the partial operation "$\circ$" by $A \circ B = (AB)^*$ if $A, B \in D(R)$ with $O_r(A) = O_l(B)$. With respect to the operation "$\circ$", $D(R)$ becomes an algebraic structure known as Brandt groupoid . We show that in the case when $R$ is a Dubrovin valuation ring of rank one, $(D(R), \circ)$ is a group, order isomorphic to a subgroup of $(\mathbb{R}, +)$, the additive group of real numbers, Theorem 2.3.15 and Lemma 2.3.25.

Chapter 3 explores the ideal structure in a Dubrovin valuation ring. In particular, our interest lies in pairs $P_1 \supset P_2$ of two distinct Goldie prime ideals $P_1$ and $P_2$ such that no further Goldie prime ideal exists between $P_1$ and $P_2$. Such a pair is called a prime segment. We show that in a Dubrovin valuation ring there are exactly three types of prime segments: archimedean, simple and exceptional. Applying this result to a Dubrovin valuation ring $R$ of rank one, we describe all possibilities for the group $D(R)$ and the subgroup $H(R)$ of $D(R)$ of all non-zero ideals which are principal as right $R$-ideals. Since, in this case, for any non-zero $R$-ideal $A$ in $Q$ either $A = A^* \in D(R)$ or $A \subset A^*$ and then $A^* = cR$, $A = c\mathcal{J}(R)$ for some $c \in U(Q)$, knowing the groups $D(R)$ and $H(R)$ we are able to describe completely the structure of all ideals in a Dubrovin valuation ring of rank one. This result shows that rank one Dubrovin valuation rings have an ideal structure completely analogous to that of cones of right ordered groups and, equivalently, of chain domains, which has been described earlier in papers by H.H. Brungs and G. Törner, [BT76], H.H. Brungs and N. Dubrovin, [BD], T.V. Dubrovina and N. Dubrovin, [DD96] and H.H. Brungs and M. Schröder, [BS95]. Also, we show that a rank one Dubrovin valuation ring $R$ in a simple artinian ring $Q$ with finite dimension over its center has only an archimedean prime segment. We conclude Chapter 3 by applying these results to a discrete Dubrovin valuation ring. We show that every discrete Dubrovin valuation ring $R$ has an archimedean prime segment $\mathcal{J}(R) \supset (0)$, with $\mathcal{J}(R) \neq \mathcal{J}(R)^2$, the Jacobson radical $\mathcal{J}(R)$ is principal as a right $R$-ideal and $\bigcap \mathcal{J}(R)^n = (0)$.

The final Chapter 4 is devoted to an application of valuation theory to commutative and non-commutative arithmetic rings. In particular, we consider commutative and non-commutative Krull rings. The class of commutative Krull rings with zero divisors was introduced by R. Kennedy, [Ken73] and later explored by R. Matsuda, [Mat82] and [Mat85], and J. Alajbegovic and E. Osmanagic, [AO90b] and [AO90a]. We prove an approximation theorem for essential valuations of Krull rings with zero divisors: Let $R$ be a Krull ring with total quotient ring $K$, such that $R \neq K$, and $\{v_{\mathfrak{P}} \mid \mathfrak{P} \in P(R)\}$ is the family of essential valuations of $R$. If $\{v_{\mathfrak{P}_1}, v_{\mathfrak{P}_2}, \ldots, v_{\mathfrak{P}_n}\}$ is a finite set of essential valuations of the ring $R$ and $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, then there exists an element $x \in K$ such that $v_{\mathfrak{P}_i}(x) = m_i$ for all $i \in \{1, 2, \ldots, n\}$ and $v_{\mathfrak{P}}(x) \geq 0$ for all other essential valuations $v_{\mathfrak{P}}$ of $R$. This results mirrors the analogous approximation theorem for essential valuations of Krull domains. The element $x$ constructed in the proof of the above result is not necessarily regular. We show that for the class of additively regular rings the element $x$ can be chosen to be regular. In this case, we give a characterization of divisorial fractional ideals in terms of principal divisorial ideals. Also, we briefly discuss the notions of non-commutative Prüfer rings and non-commutative Krull rings.

# Chapter 1

# Preliminaries

In this chapter we give the definitions and results from the general theory of associative rings which are basic for this subject. The main results are taken from [RM87], [MMU97], [LM71], [Sch45] and [Dub84]. Throughout this chapter, a ring is always an associative ring with identity 1 . A subring of a ring $R$ always contains the identity element of $R$. For a ring $R$, we denote by $U(R)$ the set of all units of $R$ and by $\mathcal{C}_R(0)$, or sometimes by $\text{Reg}(R)$, the set of all regular elements of $R$. A right (left) ideal $A$ of $R$ is regular if it contains a regular element. The set of all zero divisors of $R$ is denoted by $Z(R)$.

## 1.1 Elementary properties of orders

A ring $R$ is called *simple* if $R$ does not contain nonzero proper two sided ideals. If $M$ is any ideal of $R$, $R/M$ is a simple ring if and only if $M$ is a maximal ideal. Clearly, every division ring is simple. By Wedderburn's Theorem, if $R$ is a simple artinian ring then $R \cong M_n(D)$ for some division ring $D$.

Let $\mathcal{C}$ be a multiplicatively closed subset of a ring $R$. Then we say that $R$ satisfies *the right Ore condition* with respect to $\mathcal{C}$ or that $\mathcal{C}$ is a *right Ore set* of $R$ if for all

$a \in R$ and $c \in \mathcal{C}$, there exist $b \in R$ and $d \in \mathcal{C}$ such that $ad = cb$. If $\mathcal{C} \subseteq \mathcal{C}_R(0)$, then it is called a *regular right Ore* set of $R$. A (regular) left Ore set of $R$ is defined similarly. If $\mathcal{C}$ is a (regular) right and left Ore set of $R$, then it is simply called a (regular) *Ore set of R*.

**Theorem 1.1.1 ( [Her68])** *If $\mathcal{C}$ is a regular right Ore set of $R$, then there exists an overring $T = R\mathcal{C}^{-1} = R_{\mathcal{C}}$ of $R$, called the right quotient ring of $R$ with respect to $\mathcal{C}$ such that*

**i)** *any $c \in \mathcal{C}$ is a unit of $T$;*

**ii)** *for any $q \in T$, there exist $a \in R$ and $c \in \mathcal{C}$ such that $q = ac^{-1}$.*

*For arbitrary elements $q_1, q_2, \ldots, q_n \in T$ there exists a common denominator, i.e., a regular element $c \in \mathcal{C}$ such that $q_i = r_i c^{-1}$ , $i = 1, 2, \ldots, n$ for some $r_i \in R$.*

Let $R$ be a subring of a ring $Q$. If $Q = R\mathcal{C}_R(0)^{-1}$, then $R$ is called a *right order* in $Q$. Sometimes, the ring $Q$ is denoted by $Q = Q(R)$. Hence, a subring $R$ of a ring $Q$ is a right order in $Q$ if and only if $R$ satisfies the right Ore condition with respect to $\mathcal{C}_R(0)$. A *left order* in $Q$ is defined similarly, and a ring $R$ which is both a right and a left order in $Q$ is called *an order* in $Q$.

For a subset $A$ of a ring $R$ we define $r_R(A) = \{x \in R \mid Ax = 0\}$ and call it the *right annihilator* of $A$. The *left annihilator* $l_R(A)$ is defined similarly. A right annihilator of $R$ is a right ideal of $R$ and a left annihilator is a left ideal of $R$.

In an artinian ring all one-sided regular elements are units. More precisely the following result holds:

**Lemma 1.1.2 ( [Rob67a])** *Let $Q$ be a right artinian ring and $c \in Q$. Then the following conditions are equivalent:*

**i)** *$r_Q(c) = 0$;*

ii) *c is regular;*

iii) *c is a unit.*

**Proof.** The only implication we need to prove is $(i) \Rightarrow (iii)$. Assume that $r_Q(c) = 0$ and consider the chain

$$cQ \supseteq c^2Q \supseteq c^3Q \supseteq \cdots$$

of right ideals of Q. Then there exits $n$ such that $c^nQ = c^{n+1}Q$. Hence, $c^n(1-cx) = 0$ for some $x \in Q$. By $(i)$, $1 = cx$. But, $cx - 1 = 0$, implies $cxc - c = 0$, i.e., $c(xc-1) = 0$, and again by $(i)$, $xc = 1$, i.e., $c$ is a unit.

□

The next two results are often used in this work; they describe very useful properties of orders in an artinian ring.

**Lemma 1.1.3 ( [Rob67a])** *Let $R$ be a right order in a right artinian ring $Q$ and $c \in R$. Then, the following are equivalent:*

i) $r_R(c) = 0$;

ii) $r_Q(c) = 0$;

iii) *c is regular in R;*

iv) *c is regular in Q;*

v) *c is a unit in Q.*

**Proof.** By Lemma 1.1.2, it is enough to prove $(i) \Rightarrow (ii)$. Assume that $r_R(c) = 0$ and let $cq = 0$ for some $q \in Q$. Since $q = ab^{-1}$, where $a, b \in R$ and $b$ is regular, then $ca = 0$. By the assumption, $a = 0$. Hence, $q = 0$.

□

**Corollary 1.1.4 ( [Rob67a])** *Let $R$ be a right order in a right artinian ring $Q$ and $a, b \in R$. Then $ab$ is regular if and only if $a$ and $b$ are regular.*

**Proof.** If $ab$ is regular, then $r_R(b) = 0$. By Lemma 1.1.3, $b$ is unit in $Q$. Then $a = (ab)b^{-1}$ is regular in $Q$, and again by Lemma 1.1.3, $a$ is regular in $R$.

□

Note that if a ring $R$ has a right quotient ring $Q$ and $Q$ satisfies $ACC$ for right or left annihilators, then the property in Corollary 1.1.4 holds. Rings do exits which do not satisfy this property.

Let $R$ be a ring and $M$ be a right $R$-module. An $R$-submodule $L$ of $M$ is said to be an *essential submodule* if $L \bigcap N \neq 0$ for any nonzero $R$-submodule $N$ of $M$. If $L$ is an $R$-submodule of $M$, then there exists an $R$-submodule $L'$ such that $L \bigcap L' = 0$ and $L \oplus L'$ is an essential submodule in $M$. If a right ideal $I$ of $R$ is an essential $R$-submodule of $R$, then $I$ is called an *essential right ideal.*

A right $R$-module $U$ is said to be a *uniform module* if every nonzero $R$-submodule of $U$ is essential. A right $R$-module $M$ is said to have *finite Goldie dimension* if it contains no infinite direct sum of nonzero $R$ submodules.

A ring $R$ is called a *right Goldie ring* if $R$ satisfies the ascending chain condition (ACC) for right annihilators and $R$ does not contain an infinite direct sum of nonzero right ideals. A *left Goldie* ring is defined similarly, and if $R$ is a right and left Goldie ring, then $R$ is called *Goldie.* By Goldie's Theorem, a ring $R$ is a (semiprime) prime right Goldie ring if and only if $R$ is a right order in a (semisimple) simple artinian ring $Q$. Some basic properties of Goldie's rings are summarized in the following two results:

**Theorem 1.1.5 ( [RM87])** *Suppose that $R$ is a semiprime right Goldie ring and let $I$ be a right ideal of $R$.Then:*

i) *$I$ is essential if and only if $I$ contains a regular element;*

**ii)** *If $I$ is essential, then $I$ is generated by regular elements.*

**Theorem 1.1.6 ( [RM87])** *If a right $R$-module $M$ has finite Goldie dimension, then there exist uniform $R$-submodules $U_1, U_2, \dots, U_n$ of $M$ such that $U_1 \oplus U_2 \oplus \dots \oplus U_n$ is an essential $R$-submodule of $M$. In this case $n$ is independent of the choice of the $U_i$. We call $n$ the Goldie dimension of $M$ and denote it by $d_R(M)$ or $d(M)$.*

**Definition 1.1.7** *Let $R$ be an order in a ring $Q$. Then a right $R$-submodule $I$ of $Q$ is called a* right $R$-ideal *of $Q$ if*

**i)** $U(Q) \bigcap I \neq \emptyset$;

**ii)** *there exists an element $c \in U(Q)$ such that $cI \subseteq R$.*

*A right $R$-ideal $I$ of $Q$ is called* integral *if $I \subseteq R$. A* left $R$-ideal *is defined similarly. A right and left $R$-ideal of $Q$ is called an* $R$-ideal.

Let $R$ be a semiprime Goldie ring and $Q$ be the semisimple artinian ring of quotients of $R$. If $I$ is a right ideal of $R$, then $I$ is a right $R$-ideal if and only if $I$ is essential. In particular, any nonzero two sided ideal $I$ of a prime Goldie ring $R$ is an $R$-ideal since for any nonzero right ideal $X$ of $R$ we have $0 \neq XI \subseteq X \bigcap I$, i.e., $I$ is essential and by Theorem 1.1.5, $I$ contains a regular element.

Let $R$ be an order in a ring $Q$. For any two subsets $A$ and $B$ of $Q$, we define:

$$(A : B)_r = \{q \in Q \mid \; Bq \subseteq A\}$$

$$(A : B)_l = \{q \in Q \mid \; qB \subseteq A\}$$

$$A^{-1} = \{q \in Q \mid \; AqA \subseteq A\}.$$

For a right $R$-ideal $I$ of $Q$ we set

$$O_r(I) = (I : I)_r = \{q \in Q \mid \; Iq \subseteq I\}$$

$$O_l(I) = (I : I)_l = \{q \in Q \mid qI \subseteq I\}.$$

Then the following holds:

**Lemma 1.1.8** ( [MMU97]) *Let $R$ be an order in a ring $Q$ and $I$ be a right $R$-ideal of $Q$. Then:*

**i)** *$O_r(I)$ and $O_l(I)$ are orders in $Q$;*

**ii)** *$I$ is a left $O_l(I)$-ideal and a right $O_r(I)$-ideal;*

**iii)** *$(R : I)_l$ is a left $R$-ideal and a right $O_l(I)$-ideal.*

**Definition 1.1.9** *A ring $R$ is called a right (left) chain ring if right (left) ideals of $R$ are linearly ordered by inclusion. A right and left chain ring is called a chain ring.*

We can extend this concept in the following way:

**Definition 1.1.10** *Let $R$ be a subring of a ring $Q$. Then $R$ is called a right $n$-chain ring in $Q$ if for any elements $a_0, a_1, \ldots, a_n$ in $Q$ there exists an element $a_i$, $i \in \{0, 1, \ldots, n\}$, such that $a_i$ belongs to the right $R$-submodule generated by the remaining elements $a_j$, $j \neq i$, that is, $a_i \in \sum_{i \neq j} a_j R$. A right $n$-chain ring in itself is called a right* $n$-chain ring. *A* left $n$-chain ring *is defined similarly. A ring which is both right and left $n$-chain ring is called* $n$-chain ring.

**Remark 1.1.11 a)** *Let $S$ be an overring of a ring $R$ in a ring $Q$. If $R$ is a right $n$-chain ring in $Q$, then $R$ is a right $n$-chain ring in $S$ and $S$ is a right $n$-chain ring in $Q$. Furthermore, if $I \subseteq R$ is an ideal of $Q$ and $R$ is a right $n$-chain ring in $Q$, then $R/I$ is a right $n$-chain ring in $Q/I$.*

**b)** *The class of 1-chain rings coincides with the class of chain rings.*

c) *In every chain ring $R$, the Jacobson radical $\mathcal{J}(R)$ is a maximal right and left ideal.*

By the Remark 1.1.11 c), if $R$ is a semi-simple chain ring, then $R$ is a division ring. The analogous result for $n$-chain rings is:

**Lemma 1.1.12** ( **[MMU97]**) *Let $R$ be a semi-simple ring, i.e, $\mathcal{J}(R) = 0$. Then $R$ is artinian if and only if $R$ is a right $n$-chain ring for some $n$.*

Let $R$ be a commutative ring with identity and $K$ its total quotient ring, and $S$ a multiplicative closed system of $R$. Then the *large quotient ring of $R$* with respect to $S$ is defined to be the set $R_{[S]} = \{x \in K \mid (\exists s \in S)\ xs \in R\}$. If $A$ is an ideal of $R$, then the *extension of $A$* is $[A]R_{[S]} = \{x \in K \mid (\exists s \in S)\ xs \in A\}$. A ring $R$ is called *r-noetherian* if it satisfies the ascending chain condition ($ACC$) for regular ideals.

**Proposition 1.1.13** ( **[Gri69]**) *For a commutative ring $R$ the following conditions are equivalent:*

i) *$R$ is r-noetherian;*

ii) *Every nonempty set of regular ideals of $R$ has a maximal element;*

iii) *Every regular ideal of $R$ is finitely generated;*

iv) *Every regular prime ideal of $R$ is finitely generated.*

An r-noetherian ring need not be noetherian ( [Mat81]).

Let $R$ be a subring of a ring $R'$ and let $a \in R'$. If $a$ is a root of a monic polynomial with coefficients from $R$, then $a$ is called *integral over $R$*. If there exists a finitely generated $R-$submodule $M$ of $R'$, such that $a^n \in M$ for all $n$, then $a$ is called *almost integral over $R$*. The subset $R_0$ of all elements in $R'$ which are almost integral over $R$ is called a *complete integral closure* of $R$ in $R'$. If $R_0 = R$, we say that $R$ is *completely integrally closed* in $R'$. In the case that $R'$ is total quotient ring of $R$, $R_0$

is called a *complete integral closure* of $R$ and, in the case $R_0 = R$, we say that $R$ is *comletely integrally closed* . Notions of *integrally closed* and *integral closure* are defined similarly. If $a$ is an integral element over $R$ then $a$ is almost integral over $R$. The converse does not hold. For example, $T = \mathbb{Z}_{(2)} + X\mathbb{Q}[[X]]$ is a valuation domain with value group $\mathbb{Z} \times \mathbb{Z}$, and hence, $T$ is integrally closed but $T$ is not completely integrally closed.

For the class of $r$-noetherian rings the notions of completely integrally closed and integrally closed coincide.

**Proposition 1.1.14 ( [AO90a])** *If a ring $R$ is r-noetherian and integrally closed, then $R$ is completely integrally closed.*

A ring $R$ has *few zero divisors* if $Z(R)$ is a finite union of prime ideals; and $R$ is *additively regular* if for each $z$ in its total quotient ring $K$, there exists $u \in R$ such that $z + u \in C_K(0)$. A ring $R$ is called a *Marot ring* or a ring with the *property (P)* if every regular ideal is generated by regular elements. The relationships between these classes of rings are summarized in the next theorem:

**Theorem 1.1.15 ( [Huc88])** *Consider the following four conditions on a commutative ring $R$*

**1)** *$R$ is a noetherian ring;*

**2)** *$R$ has few zero divisors;*

**3)** *$R$ is an additively regular ring;*

**4)** *$R$ is a Marot ring.*

*Then* 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4).

None of the implications of the Theorem are reversible ( [Huc88], Examples).

## 1.2 Valuation rings

Valuation Theory is an important and powerful tool in many areas of mathematics: Number Theory, Theory of Local and Global Fields, Algebraic Geometry and Model Theory. The first axiomatic definition of a valuation has been stated by Josef Kürschák in 1913 and the further development, at the beginning of the 20−th century, is associated with works of Kurt Hensel, Alexander Ostrowski and Helmut Hasse. While this stage of the development of the valuation theory is mainly concerned with valuations of rank one, the notion of a general valuation was introduced by Wolfgang Krull in 1932, [Kru32]:

**Definition 1.2.1** *Let $K$ be a field and let $V$ be a map of $K$ onto a totally ordered abelian group $G_v$ with an added element $\infty$, such that $\infty + \infty = g + \infty = \infty$, $\infty > g$ for all $g \in G$. The map $v : K \to G_v \bigcup \{\infty\}$ is called a* (Krull) valuation *on $K$ and $G_v$ is called the* value group *of $v$ if the following conditions are satisfied*

**V1)** $(\forall x \in K)\ \ v(x) = \infty$ *if and only if* $x = 0$

**V2)** $(\forall x, y \in K)\ \ v(xy) = v(x) + v(y)$

**V3)** $(\forall x, y \in K)\ \ v(x + y) \geq min\{v(x), v(y)\}$

If $G_v = 0$ then $v$ is called a *trivial valuation*. If the group $G_v$ is the group of integers, then $v$ is called the *discrete valuation of rank one*. For example, for every $x \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ and a prime number $p$ there exist integers $\alpha$, $r$, $s$ such that $x = p^{\alpha}\frac{r}{s}$, where $r$ and $s$ are not divisible by $p$; then the map $v_p : \mathbb{Q}^* \to \mathbb{Z}$ defined by $v_p(x) = \alpha$ and $v_p(0) = \infty$ is a discrete valuation of rank one of $\mathbb{Q}$. The ring $R_v = \{x \in K \mid v(x) \geq 0\}$ is called the *ring of the valuation $v$* and the prime ideal $P_v = \{x \in K \mid v(x) > 0\}$ is called the *positive ideal of $v$*. The ring $R_v$ satisfies the following condition: for each $x \in K$, $x \in R_v$ or $x^{-1} \in R_v$. Conversely, if $R$ is a subring of $K$ such that $x \in K \setminus R$ implies $x^{-1} \in R$, then there exists a valuation $v$ of $K$ such that $R = R_v$. Many authors contributed to this theory, among them O.F.G. Schilling, E. Artin, O. Zariski, P.J. Jaffard, P. Ribenboim, O. Endler and M. Fukava. Krull valuations are greatly used by Zariski and his school of Algebraic Geometry. For example, power series rings are used in the study of the algebraic varieties in projective spaces; abstract Riemann surfaces are developed.

Among competing attempts to develop a theory of valuations for commutative rings which may have nontrivial zero divisors, the most successful was the definition of the valuation ring by M.E. Manis in [Man67].

**Definition 1.2.2 ( [Man67])** *A map $v$ from a commutative ring $T$ onto a totally ordered abelian group $G_v$ with $\infty$ adjoined, is called a* valuation *of $T$ if the following conditions hold:*

(i) $v(xy) = v(x) + v(y)$;

(ii) $v(x + y) \geq min\{v(x), v(y)\}$.

*The ring $R_v = \{x \in T \mid v(x) \geq 0\}$ is called the ring of the valuation $v$. The set $P_v = \{x \in T \mid v(x) > 0\}$ is a prime ideal of $R_v$ and is called the positive ideal of $v$.*

The ring $R_v$ and the ideal $P_v$ satisfy the following condition:

**(iii)** For each $x \in T \setminus R_v$ there exists $y \in P_v$ such that $xy \in R_v \setminus P_v$.

Conversely, if $R$ is a subring of a ring $T$ and $P$ is a prime ideal of $R$ such that for each $x \in T \setminus R$ there exists $y \in P$ with $xy \in R \setminus P$, then there exists a Manis valuation $v$ on $T$ such that $R_v = R$ and $P_v = P$. A pair $(R, P)$ satisfying *(iii)* is called a *Manis valuation pair* of $T$. For example, the pair $(R, P) = (\mathbb{Z}[X], X\mathbb{Z}[X])$ is a Manis valuation pair of the ring $T = \mathbb{Z}[X, X^{-1}]$. If the group $G_v$ is equal to (0), then $v$ is called a *trivial* valuation. If the group $G_v$ is the group of integers, then $v$ is called the *discrete valuation of rank one.*

For the theory of Manis valuations refer to [LM71], [Huc88] and [AM92]. In the last three decades an extension of the theory of commutative integral domains to commutative rings with zero divisors was developed, see R. Gilmer [Gil72], M. Larsen, P. McCarthy [LM71], J. Huckaba [Huc88], R. Matsuda [Mat85] and J. Alajbegovic, J. Močkoř [AM92].

In the non-commutative case, a natural question is: Which class of rings plays the same role for the study of non-commutative arithmetic rings as the classes of valuation domains and Manis valuation rings do for commutative arithmetic rings? First, note that a successful "candidate" should share many properties with commutative valuation rings, in particular, the properties of extension of valuations. For example, we would like to know a non-commutative analogue of the following result due to Chevalley: If $V$ is a valuation ring of a field $F$ and $K$ is an extension of $F$ , then there exists a valuation ring $W$ of $K$ with $W \bigcap F = V$. In this case we say that $W$ is an *extension* of $V$ in $K$. In order to consider these questions one must decide what it means to have a *valuation on a division ring.* It seemed natural to suggest that this role is played by Schilling's valuations on a skew field, see [Sch45]:

**Definition 1.2.3** *A valuation $v$ on a division ring $D$ is a function $v$ from $D^* = D \setminus \{0\}$ onto an ordered group $G_v$, such that for all $a, b \in D^*$, $v(ab) = v(a) + v(b)$ and $v(a+b) \geq min\{v(a), v(b)\}$.*

For convenience we extend $v$ to $D$ by $v(0) = \infty$ where $\infty > \gamma$ and $\infty + \gamma = \gamma + \infty = \infty$ for all $\gamma \in G_v$. Such valuations $v$ on $D$ correspond to subrings $R$ of $D$ (called total invariant valuation rings of $D$) with the following two properties:

**(T)** If $x \in D^*$ then $x \in R$ or $x^{-1} \in R$; we say that $R$ is a *total valuation ring* of $D$.

**(I)** $dRd^{-1} = R$, for all $d \neq 0$ in $D$; we say that $R$ is *invariant.*

Note that valuations on the finite dimensional central simple division algebras are used to show that the Brauer groups $B_r(F)$ over a local field $F$ are isomorphic to $\mathbb{Q}/\mathbb{Z}$. Using valuations, the central simple algebras over *global fields*, i.e., algebraic number fields or algebraic function fields, are classified. This work is associated with the famous names of H. Hasse, R. Brauer, E. Noether and A. Albert, see [Pie82]. However, the valuation of the center $F$ of a division ring $D$ such that $[D : F] < \infty$, in general, cannot be extended to the whole algebra. For example, for $p$ an odd prime number, the valuation ring $\mathbb{Z}_{(p)}$ in the center $\mathbb{Q} = Z(\mathbb{H}(\mathbb{Q}))$ of the 4-dimensional algebra of quaternions over the rationals cannot be extended by a total invariant valuation ring in $\mathbb{H}(\mathbb{Q})$, see Example 1.4.2. This difficulty has been overcome by the introduction of Dubrovin valuation rings in a simple artinian ring, see [Dub84]. A very well developed ideal theory and extension theory in the finite dimensional case justify the name *valuation rings.*

**Definition 1.2.4** *Let $R$ be a subring in a simple artinian ring $Q$ and assume that $M$ is an ideal of $R$ such that $R/M$ is a simple artinian ring, and for each $q \in Q \setminus R$ there are elements $r, r' \in R$ with $qr, r'q \in R \setminus M$. Then $R$ is called a Dubrovin valuation ring of $Q$.*

Examples of Dubrovin valuation rings are given in Section 1.4.

Note, that in the case $Q = K$ is a commutative field the classes of Schilling valuation rings, i.e., total invariant rings in $Q$, total valuation rings in $Q$ and Dubrovin valuation rings of $Q$ all agree and are in fact equal to the class of the classical commutative valuation domains in $Q = K$. More precisely:

**Theorem 1.2.5** *Let $R$ be a Bezout order in a simple artinian ring $Q$. Then:*

1. *$R$ is a Dubrovin valuation ring of $Q$ if and only if $R/\mathcal{J}(R)$ is a simple artinian ring.*

*2. $R$ is a total valuation ring of $Q$ if and only if $R/\mathcal{J}(R)$ and $Q$ are skew fields.*

*3. $R$ is a Krull valuation ring in $Q$ if and only if $R/\mathcal{J}(R)$ and $Q$ are fields.*

For the general theory of Dubrovin valuation rings see [Dub84], [Dub85], [Dub91a], [Grä92b] and [MMU97].

We mention here the following extension theorem for Dubrovin valuation rings.

**Theorem 1.2.6 ( [Dub85] and [BG90])** *For every valuation ring $V$ in the center $F$ of a finite dimensional central simple algebra $D$, there exists a Dubrovin valuation ring $R$ of $D$ such that $R \bigcap F = V$.*

A very rich extension theory using Dubrovin valuation rings has been developed in the last decade, see for example papers by N. Dubrovin [Dub85], H.H. Brungs, J. Gräter [BG90], J. Gräter [Grä92a], P. Morandi [Mor89], P. Morandi, A. Wadsworth [MW89] and A. Wadsworth [Wad89]. This suggests that among three competing concepts of noncommutative valuation rings, i.e., Schilling's total invariant valuation rings, total valuation rings in a skew field (equivalently, chain domains) and Dubrovin valuation rings in a simple artinian ring, the last one is the most effective for the study of *noncommutative arithmetic rings*.

## 1.3 Cones in groups and chain domains

Let $G$ be a group. A *cone* of a group $G$ is a subset $P$ of $G$ such that $PP \subseteq P$ and $P \bigcup P^{-1} = G$. It follows that the identity element $e$ of $G$ belongs to $P$. If in addition $P \bigcap P^{-1} = \{e\}$, then $P$ is called a *basic cone* of $G$. In this case, we say that $(G, P)$ is a *right ordered group* with $a \leq_r b$ if and only if $ba^{-1} \in P$ for $a$, $b \in G$. The right order $\leq_r$ will also be a left order if and only if $aPa^{-1} = P$ for all $a \in G$. The group $(G, P)$ is then a *linearly ordered* group.

Let $P$ be a cone of $G$. A non-empty subset $I$ of $G$ is called a *right $P$-ideal* if $IP \subseteq I$ and $I \subseteq aP$ for some $a \in G$. Left $P$-ideals and $P$-ideals are defined similarly. If $I \subseteq P$ then we omit the prefix "$P-$" and call $I$ a right (left) ideal of $P$. Moreover, if $I \neq P$, then we say that (right or left) ideal $I$ is *proper*. A proper ideal $I$ of the cone $P$ is said to be *prime (completely prime)* if for $a,\ b \in P$ we have $a \in I$ or $b \in I$ whenever $aPb \subseteq I$ (respectively, $ab \in I$).

The proofs of the following results are similar to those of the corresponding assertions about chain rings:

**C1)** The set of right (left) $P$-ideals is totally ordered with respect to inclusion.

**C2)** The set of elements invertible in the cone $P$ is the intersection $P \bigcap P^{-1}$ and it is a subgroup of $G$.

**C3)** The subset $J(P)$ of elements non-invertible in $P$ is a maximal right (left) ideal. This ideal is a completely prime ideal and $P = J(P) \bigcup U(P)$ and $U(P) \bigcap J(P) = \emptyset$, where $U(P)$ is the set of all units of $P$.

We note that *a subring $R$ of a skew field $D$ is a total valuation ring if and only if $(R^*, \cdot)$ is a cone in $(D^*, \cdot)$. Any total valuation subring $R$ of a skew field $D$ is a Bezout order in $D$ with $R/J(R)$ a skew field.*

Let $G$ be a group with a cone $P$. We say that a total valuation ring $R$ of a skew field $D$ is *associated* with $P$ if the following conditions hold:

**a)** $G \subseteq (D^*, \cdot)$;

**b)** Every element $d \in D^*$ can be written as $d = g_1u_1 = u_2g_2$ with $g_1,\ g_2 \in G$ and $u_1,\ u_2 \in U(R)$ and $Pg_1P = Pg_2P$;

**c)** $R \bigcap G = P$.

The existence of total valuation rings with prescribed associated cones has been studied recently, see papers by H.H. Brungs and G. Törner, [BT98], T.V. Dubrovina and N.I. Dubrovin, [DD96], and H.H. Brungs and N.I. Dubrovin, [BD]. Many examples are constructed by the authors mentioned above.

The following result describes the correspondence between the chain of right $P$-ideals in a group $G$ with a cone $P$ and the chain of right $R$-ideals in the total valuation ring $R$ of a skew field $D$ associated with a cone $P$.

**Theorem 1.3.1** *Let a total valuation ring $R$ of a skew field $D$ be associated with a cone $P$ of a group $G$. Then $\varphi(I) = IR$, for a right $P$-ideal $I$, defines an isomorphism between the chain of right $P$-ideals in $G$ and the chain of right $R$-ideals in $D$. The inverse mapping $\psi$ assigns to each right $R$-ideal $A$ the right $P$-ideal $\psi(A) = A \bigcap G$. This correspondence preserves the properties of being an ideal, a completely prime ideal and a prime ideal, and $R$-ideals correspond to $P$-ideals.*

Since the cone $R^*$ of $D^*$ is associated with the total valuation ring $R$ in the skew field $D$, an immediate consequence of Theorem 1.3.1 is the following result:

**Corollary 1.3.2** *Let $R$ be a total valuation subring of a skew field $D$ with right (left) $R$-ideal $A$. Then $A \setminus \{0\}$ is a right (left) $R^*$-ideal in $D^*$. Conversely, if $I$ is a right (left) $R^*$-ideal in $D^*$, then $I \bigcup \{0\}$ is a right (left) $R$-ideal in $D$. This correspondence preserves the properties of being an ideal, a prime ideal, a completely prime ideal and of being proper.*

We conclude this section with the result by N.Dubrovin, which will allow the construction of total valuation rings with prescribed associated cones.

**Theorem 1.3.3 ( [Dub93])** *Let $P$ be a cone in a group $G$ and $F$ a skew field such that the following conditions hold:*

**a)** *$P$ does not contain a minimal ideal;*

b) *For every right P-ideal I, the group ring $FU(O_l(I))$ is an Ore domain.*

*Then there exists a skew field D and a total valuation ring S of D associated with P.*

## 1.4 Examples

We have seen in Theorem 1.2.5 that if $R$ is a Dubrovin valuation ring in a simple artinian ring $Q$ with $R/\mathcal{J}(R)$ a skew field, then $Q$ is a skew field and $R$ is a chain domain. Hence, chain domains are special Dubrovin valuation rings. Also, matrix rings over chain domains are Dubrovin valuation rings.

A particular example of a Dubrovin valuation ring is the following:

**Example 1.4.1**

*The ring $R = \mathbb{H}(\mathbb{Z}_{(p)})$ of quaternions over the valuation ring $\mathbb{Z}_{(p)}$, where $p$ is an odd prime number, is a Dubrovin valuation ring in the ring $Q = \mathbb{H}(\mathbb{Q})$ of quaternions over the rationals.*

**Proof.** $M = pR$ is a two sided ideal of $R$, $R/M = \mathbb{F}_p \bigoplus \mathbb{F}_p i \bigoplus \mathbb{F}_p j \bigoplus \mathbb{F}_p k$, where $\mathbb{F}_p$ is a field of $p$ elements. Since $R/M$ contains nontrivial zero divisors, we have $R/M \cong M_2(\mathbb{F}_p)$. To check the second condition of Definition 1.2.4 we take an arbitrary element $q = q_0 + q_1 i + q_2 j + q_3 k \in Q \setminus R$. Then at least one of $q_l$ does not belong to $\mathbb{Z}_{(p)}$. Among these, we choose one, say $q_s$, with smallest $p$-adic valuation. Then $q_s = \frac{r_s}{p^{\alpha_s} t_s}$, where $\alpha_s > 0$ and $p \nmid r_s$, $p \nmid t_s$. Let $r = p^{\alpha_s} t_s \in R$. Then $qr \in R \setminus M$.

The next example was announced in Section 1.2.

**Example 1.4.2** *A valuation domain V of the center F of a central simple division algebra D with $[D : F] < \infty$, cannot be extended to a total valuation ring in D.*

**Proof.** Let $p$ be an odd prime number, $D = \mathbb{H}(\mathbb{Q})$ be the 4-dimensional algebra of quaternions over the rationals, $F = \mathbb{Q}$ is the center of $D$ and $V = \mathbb{Z}_{(p)}$ the valuation ring of the $p$-adic valuation on $F = \mathbb{Q}$. Assume that there exists a total valuation ring $R$ in $D$ which extends $V$, i.e., $R \bigcap \mathbb{Q} = V$. Since $R$ is a chain domain, the Jacobson radical $M = J(R)$ is a maximal right (left) ideal of $R$. Hence, $R/M$ is a division ring. Also $[R/M : \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}] \leq [D : \mathbb{Q}] = 4 < \infty$. Therefore, $R/M$ is a finite division ring and by Wedderburn Theorem, $R/M$ is commutative. So $\overline{ij} = \overline{ji}$ ,i.e., $2ji \in M$. But, $i$,$j$ are units in $D$. Hence, $2 \in M$, a contradiction which shows that $V = \mathbb{Z}_{(p)}$ cannot be extended to a total valuation ring in $D$.

□

**Example 1.4.3**

*The valuation ring $V = \mathbb{Z}_{(2)}$ of $\mathbb{Q}$ has a total invariant extension in $D = \mathbb{H}(\mathbb{Q})$.*

**Proof.** For an element $\alpha = a_0 + a_1 i + a_2 j + a_3 k \in D$, where $a_i \in \mathbb{Q}$, we define $\alpha^* = a_0 - a_1 i - a_2 j - a_3 k$ and $N(\alpha) = \alpha\alpha^* = a_0^2 + a_1^2 + a_2^2 + a_3^2$. It can be shown that:

1. $(\alpha + \beta)^* = \alpha^* + \beta^*$, $(\alpha\beta)^* = \beta^*\alpha^*$, for all $\alpha$, $\beta \in D$;
2. $N(\alpha\beta) = N(\alpha)N(\beta)$, for all $\alpha$, $\beta \in D$;
3. $N(\alpha^{-1}) = N(\alpha)^{-1}$, for all $\alpha \in D$;
4. $B = \{\alpha \in D \mid N(\alpha) \in \mathbb{Z}_{(2)}\}$ is a subring of $D$;
5. $B \bigcap \mathbb{Q} = \mathbb{Z}_{(2)}$.

Now, for $\alpha \in D \setminus B$, $N(\alpha) \in \mathbb{Q} \setminus \mathbb{Z}_{(2)}$. Hence, $N(\alpha^{-1}) = N(\alpha)^{-1} \in \mathbb{Z}_{(2)}$. So, $\alpha^{-1} \in B$, i.e., $B$ is a total valuation ring in $D$. Also, $\alpha B = B\alpha$ for all $\alpha \in D$, i.e., $B$ is invariant. Hence, $B$ is a total invariant extension of $V$ in $D$.

□

# Chapter 2

# Bezout Orders in a Simple Artinian Ring

A ring $R$ is said to be *local* if $\overline{R} = R/\mathcal{J}(R)$ is a simple artinian ring. In this chapter we consider Bezout rings, and, in particular, local Bezout orders in a simple artinian ring. These orders are exactly Dubrovin valuation rings (see Theorem 2.2.8). In Section 2.3., the prime spectrum of a Dubrovin valuation ring is investigated.

## 2.1 Bezout rings

A commutative integral domain is called a *Bezout domain* if each finitely generated ideal of R is principal. Hence, every principal ideal domain (PID) is a Bezout domain and every noetherian Bezout domain is a PID.

**Example 2.1.1** *There exists a Bezout domain which is not a PID.*

**Proof.** Let $(G, P)$ be a commutative ordered group with the positive cone $P$, i.e., $PP \subseteq P$, $P \bigcup P^{-1} = G$ and $P \bigcap P^{-1} = \{e\}$. Let $F$ be any field and consider the group ring $FG$ and the subring $T = FP = \{\sum a_p p \mid a_p \in F,\ p \in P\}$. Then

$S = \{\sum a_p p \mid a_e \neq 0\} \subset T$ is a multiplicatively closed system in $T$. Since $F$ is a field and $G$ is a commutative ordered group, $FG$ does not have nontrivial zero divisors and it is a commutative domain. Consider the localization $V = TS^{-1}$ of $T$ at $S$ in the quotient field $K = Q(FG)$ of the domain $FG$.

The set of all nonzero principal ideals of $V$ is exactly the set $\{gV \mid g \in P\}$ and $g_1V = g_2V$ if and only if $g_1 = g_2$.

For, let $0 \neq t = a_{g_0}g_0 + a_{g_1}g_1 + \cdots + a_{g_n}g_n \in T = FP$ where $g_i \in P$, $a_{g_i} \in F \setminus \{0\}$, be any element and assume that $e \leq g_0 < g_1 < g_2 < \cdots < g_n$. Then $t = g_0 s_1$, where $s_1 = a_{g_0} + a_{g_1}g_0^{-1}g_1 + \cdots + a_{g_n}g_0^{-1}g_n \in S$. Since $V = TS^{-1}$, each element $0 \neq v \in V$ has the form $v = ts^{-1}$ for some $t \in T$ and $s \in S$.Hence, $vV = g_0 s_1 s^{-1} V = g_0 V$.Assume $g_1V = g_2V$ but $g_1 < g_2$. Then $V = g_1^{-1}g_2V = pV$ for some $p \in P$.This implies $1 = pts^{-1}$, i.e., $s = pt \in S$, a contradiction.

Hence, the lattice of principal ideals of $V$ is a chain, and if $I = (v_1, v_2, \cdots, v_n)V$ is a finitely generated ideal of $V$, then $I$ is principal, i.e., $V$ is a Bezout domain. Note that $V$ is a valuation domain and the valuation associated to $V$ on $K$ has $G$ as the value group. Choosing the group $G$ to be $(\mathbb{R}, +)$, the valuation domain $V$ is not discrete and hence, $V$ is not a PID.

□

Every Bezout domain is a Prüfer domain, i.e., every finitely generated ideal of $R$ is invertible. Therefore, for each commutative Bezout domain $R$ and a prime ideal $P$ of $R$, the localization $R_P$ is a valuation ring and $R = \bigcap R_P$, where $P$ runs over all maximal ideals of $R$.

Noncommutative Bezout rings have been studied by many authors, see P. Cohn [Coh63], [Coh85], J. Robson [Rob67b], H.H. Brungs [Bru86], R. Beauregard [Bea73]. Following [Rob67b], a ring $R$ is called a *right Bezout* ring if the following conditions are satisfied:

**i)** $ab$ is regular in $R$ if and only if $a$ and $b$ are regular;

ii) $aR \bigcap bR$ contains a regular element for every pair of regular elements $a$, $b$ of $R$;

iii) For every pair of regular elements $a, b \in R$ the right ideals $aR \bigcap bR$ and $aR + bR$ are principal.

A *left Bezout* ring is defined similarly and a *Bezout* ring is a ring which is right and left Bezout. A right Bezout ring is not necessarily a left Bezout ring but if a ring $R$ has a left and right quotient ring $Q$ which is artinian, then $R$ is right Bezout if and only if $R$ is left Bezout. Also, *if $R$ is semiprime Goldie, then $R$ is a Bezout ring if and only if every finitely generated right ideal of $R$ is principal*, see [Rob67b], Th.2.4. and Th.3.5.

**Example 2.1.2** *There exists a ring $R$ that is a right Bezout ring but it is not a left Bezout ring.*

**Proof.** Consider the field $K = \mathbb{Q}(t)$ and define the map $\sigma : K \to K$ with $\sigma(q) = q$, for all $q \in \mathbb{Q}$ and $\sigma(t) = t^2$. Then $\sigma$ is a monomorphism from $K$ into $K$. Define $R := \{\sum_{i=0}^{\infty} x^i a_i \mid a_i \in K\}$ where $ax = x\sigma(a)$, for all $a \in K$. Then $R$ is a ring. An element $0 \neq r = a_0 + xa_1 + x^2a_2 + \cdots \in R$ is a unit if and only if $a_0 \neq 0$. Let $f = x^n a_n + x^{n+1}a_{n+1} + \cdots \in R$ where $n \geq 1$ and $a_n \neq 0$. Then $fR = x^n uR$, where $u = a_n + xa_{n+1} + \cdots \in U(R)$, i.e., $fR = x^n R$. Hence, all principal right ideals of $R$ are of the form $x^n R$, $n = 1, 2, \ldots$, i.e., $R$ is a right Bezout ring.

On the other hand, for the left ideals $Rx$ and $Rxt$, the intersection $Rx \bigcap Rxt = 0$, i.e., $R$ is not a left Bezout ring.

□

In [Bea73] it is shown that any ring $T$ between a Bezout domain $R$ and its quotient field $K$ is a quotient ring of $R$ and $T$ is also a Bezout domain. The following theorem generalizes this result.

**Theorem 2.1.3** ( [Grä92b]) *Let $R$ be a (right) Bezout ring having a right and left quotient ring $Q$ which is right artinian and let $T$ be an overring of $R$ in $Q$. Furthermore, let $S = U(T) \bigcap R$. Then the following hold:*

**i)** *$S$ is a left Ore set of $R$;*

**ii)** *$T =_S R$;*

**iii)** *$T$ is a Bezout ring*

## 2.2 The basic properties of Dubrovin valuation rings

Throughout this section, $Q$ is a simple artinian ring, $R$ is a Dubrovin valuation ring of $Q$ and $M$ is the ideal satisfying conditions of Definition 1.2.4. The following results are the main steps to prove that $M$ is the Jacobson radical of $R$ and that it is a unique maximal ideal of $R$. The results are important by themselves and very often used in this work.

**Lemma 2.2.1** ( [Dub84]) *For any right (left) $R$-submodule $I$ of $Q$ the following holds:*

$$I \bigcap R \subseteq M \Longrightarrow I \subseteq M$$

**Proof.** Let $I$ be a right $R$-submodule of $Q$ so that $I \bigcap R \subseteq M$. Let $x \in I$. If $x \in Q \setminus R$, then there exists an element $r \in R$ such that $xr \in R \setminus M$. But, $xr \in I$, i.e., $xr \in I \bigcap R \subseteq M$, a contradiction. So, $x \in R$, i.e., $I \subseteq R$. Hence, $I = I \bigcap R \subseteq M$.

□

**Lemma 2.2.2** ( [Dub84]) *$M$ does not contain nonzero right (left) ideals of the ring $Q$.*

**Proof.** Let $a \in Q$, $a \neq 0$ and $aQ \subseteq M$. If $1 + aQ \subseteq U(Q)$, then $aQ \subseteq \mathcal{J}(Q) = 0$, i.e., $a = 0$. So, $1 + aQ \nsubseteq U(Q)$. Therefore, there exists an element $q \in Q$ such that $1 + aq \notin U(Q)$. But, $Q$ is an artinian ring and by Lemma 1.1.2, every one sided regular element of $Q$ is invertible. So, $L := l_Q(1+aq) \neq 0$. Also, $L \bigcap R \subseteq M$. Hence, by Lemma 2.2.1, $L \subseteq M$. Since $Laq \neq 0$, $LaQ$ is a nonzero 2-sided ideal of the ring $Q$. Hence, $LaQ = Q$. Then $1 \in LaQ \subseteq M$, a contradiction.

□

**Lemma 2.2.3 ( [Dub84])** $1 + M \subseteq U(R)$

**Proof.** Let $m \in M$ and $L := l_Q(1 + m)$. Then $L \bigcap R \subseteq M$. By Lemma 2.2.1 and Lemma 2.2.2, $L = 0$. Since $Q$ is an artinian ring, $1 + m \in U(Q)$. So, for any $m \in M$, $(1 + m)^{-1}$ exists. If $(1 + m)^{-1} \in Q \setminus R$, then $(1 + m)^{-1} - 1 \notin R$ and there exists an element $s \in R$ such that $r := [(1 + m)^{-1} - 1]s \in R \setminus M$. Then $(1 + m)r = -ms$, so that $r = -m(r + s) \in M$, a contradiction.

□

**Theorem 2.2.4 ( [Dub84])** *Let $R$ be a Dubrovin valuation ring of $Q$. Then $M$ is the Jacobson radical of $R$. In particular, $M$ is the unique maximal ideal of $R$.*

**Proof.** Since $R/M$ is a simple artinian ring , $\mathcal{J}(R) \subseteq M$. By Lemma 2.2.3, $M \subseteq \mathcal{J}(R)$.

□

**Corollary 2.2.5** *Let $S$ be an overring of $R$ in $Q$ and let $I$ be a proper ideal of $S$. Then $I \subseteq \mathcal{J}(R)$. In particular, $\mathcal{J}(S) \subseteq \mathcal{J}(R)$*

**Proof.** Let $I \neq S$ be any two-sided ideal of $S$. Then $I \bigcap R \neq R$, i.e., $I \bigcap R$ is a proper ideal of $R$. Hence, by Theorem 2.2.4, $I \bigcap R \subseteq \mathcal{J}(R)$ and by Lemma 2.2.1, $I \subseteq \mathcal{J}(R)$.

□

To characterize Dubrovin valuation rings in terms of Bezout orders and $n$-chain rings, we need the next two results.

**Lemma 2.2.6 ( [Dub84])** *Let $R$ be a Dubrovin valuation ring of $Q$ and $I$ be a right $R$-submodule of $Q$. Then*

$$d_{\overline{R}}(I/I\mathcal{J}(R)) \leq d_{\overline{R}}(\overline{R})$$

*where* $\overline{R} = R/\mathcal{J}(R)$.

**Proof.** First note that $I/I\mathcal{J}(R)$ is a right $\overline{R}$-module. Let $n = d_{\overline{R}}(\overline{R})$ be the Goldie dimension of the $\overline{R}$-module $\overline{R}$ and assume that the opposite holds, i.e., $d_{\overline{R}}(I/I\mathcal{J}(R)) > n$. Then, since $\overline{R}$ is a simple artinian ring there exist $a,\ b \in I,\ b \notin I\mathcal{J}(R)$ such that

$$(aR + bR + I\mathcal{J}(R))/I\mathcal{J}(R) = (aR + I\mathcal{J}(R))/I\mathcal{J}(R) \oplus (bR + I\mathcal{J}(R))/I\mathcal{J}(R) \qquad (*)$$

and

$$(aR + I\mathcal{J}(R))/I\mathcal{J}(R) \cong \overline{R} = R/\mathcal{J}(R). \qquad (**)$$

Since $(aR+I\mathcal{J}(R))/I\mathcal{J}(R) \cong aR/aR\bigcap I\mathcal{J}(R)$ and $a\mathcal{J}(R) \subseteq aR\bigcap I\mathcal{J}(R)$, it follows from $aR/aR\bigcap I\mathcal{J}(R) \cong R/\mathcal{J}(R)$ that $aR\bigcap I\mathcal{J}(R) = a\mathcal{J}(R)$ and hence, $aR/a\mathcal{J}(R) \cong R/\mathcal{J}(R)$. So, $r_R(a) \subseteq \mathcal{J}(R)$. On the other hand, $r_R(a) = r_Q(a)\bigcap R$. Hence, by Lemma 2.2.1, $r_Q(a) \subseteq \mathcal{J}(R)$. Then, by Lemma 2.2.2 and Lemma 1.1.2, $a$ is a unit in $Q$. So, we can rewrite $aR\bigcap I\mathcal{J}(R) = a\mathcal{J}(R)$ in the form $R\bigcap a^{-1}I\mathcal{J}(R) = \mathcal{J}(R)$. Again, by Lemma 2.2.1, $a^{-1}I\mathcal{J}(R) \subseteq \mathcal{J}(R)$, i.e., $I\mathcal{J}(R) = a\mathcal{J}(R)$. Now,

$$a\mathcal{J}(R) = I\mathcal{J}(R) \overset{*}{=} (aR + I\mathcal{J}(R))\bigcap(bR + I\mathcal{J}(R))$$

$$= (aR + a\mathcal{J}(R))\bigcap(bR + a\mathcal{J}(R)) = aR\bigcap(bR + a\mathcal{J}(R)) = (aR\bigcap bR) + a\mathcal{J}(R).$$

Hence, $aR\bigcap bR \subseteq a\mathcal{J}(R)$, i.e., $R\bigcap a^{-1}bR \subseteq \mathcal{J}(R)$. By Lemma 2.2.1, $a^{-1}bR \subseteq \mathcal{J}(R)$. This implies $b \in I\mathcal{J}(R)$, a contradiction.

□

**Lemma 2.2.7** ( **[Dub84]**) *Let $R$ be a Dubrovin valuation ring of $Q$. Then any finitely generated right (left) $R$-submodule of $Q$ is principal.*

**Proof.** Let $I$ be a right finitely generated $R$-submodule of $Q$. By Lemma 2.2.6, $d_{\overline{R}}(I/I\mathcal{J}(R)) \leq d_{\overline{R}}(\overline{R})$, where $\overline{R} = R/\mathcal{J}(R)$ is a simple artinian ring. Hence, $\overline{R} = I/I\mathcal{J}(R) \oplus N$ for some right $\overline{R}$-submodule $N$ of $\overline{R}$. Since $\overline{R}$ is a cyclic $\overline{R}$-module, $I/I\mathcal{J}(R)$ is a cyclic $\overline{R}$-module and hence $I/I\mathcal{J}(R)$ is a cyclic $R$-module, i.e., $aR + I\mathcal{J}(R) = I$, for some $a \in I$. But, $I$ is a finitely generated $R$-module. By Nakayama's lemma, this implies that $I = aR$, i.e., $I$ is a cyclic $R$-module. Similarly for left $R$-submodules.

□

The next result is the main characterization theorem for Dubrovin valuation rings.

**Theorem 2.2.8** ( **[Dub84]** and **[MMU97]**) *Let $R$ be a subring of a simple artinian ring $Q$. Then the following conditions are equivalent:*

**(1)** *$R$ is a Dubrovin valuation ring of $Q$.*

**(2)** *$R$ is a local Bezout order in $Q$.*

**(3)** *$R$ is a local $n$-chain ring in $Q$ for some $n$ with $d(\overline{R}) \geq n$, where $\overline{R} = R/\mathcal{J}(R)$.*

**Proof.**

$$(1) \Longrightarrow (2)$$

If we assume that (1) holds, then by Lemma 2.2.7, $R$ is a local Bezout ring.

Let $q \in Q$ be any element. Consider the finitely generated right $R$-submodule $R + qR$ of $Q$. Then $R + qR$ is principal, i.e., $R + qR = sR$ for some $s \in Q$. Hence, $1 = sr$ for $r \in R$. This implies that the right annihilator $r_Q(r)$ of $r$ in $Q$ is equal to 0, $r_Q(r) = 0$. Hence, by Lemma 1.1.2, $r$ is a unit in $Q$. On the other hand, $q = sx$ for some $x \in R$. So, $q = r^{-1}x$.

We are left to prove that every regular element in $R$ is invertible in $Q$. Let $c \in R$ be a regular element and consider $q \in r_Q(c)$. Then, $cq = 0$. If $q \in Q \setminus R$, then by (1), there exists an element $r \in R$ so that $qr \in R \setminus \mathcal{J}(R)$. Then $cqr = 0$ and $qr \in R$ imply that $qr = 0$, since $c$ is regular in $R$. Hence, $qr \in \mathcal{J}(R)$, a contradiction which proves that $q \in R$, and hence, $q = 0$. So, $r_Q(c) = 0$, and by Lemma 1.1.2, $c$ is a unit in $Q$.

This proves that $R$ is a left order in $Q$. Similarly, $R$ is a right order in $Q$.

$$(2) \Longrightarrow (3)$$

Assume that (2) holds and let $n = d(\overline{R})$. Let $q_0, q_1, \cdots, q_n \in Q$. Then $q_0R + q_1R + \cdots + q_nR$ is a finitely generated right $R$-submodule of $Q$. By (2), we can write $q_i = s_i^{-1}r_i$, where $r_i,\ s_i \in R$ with $s_i$ regular elements. For elements $s_i^{-1} \in Q$, there exist a common denominator, i.e., a regular element $t \in R$ such that $s_i^{-1} = t^{-1}\overline{r_i}$, for some $\overline{r_i} \in R$. Then $q_0R + q_1R + \cdots + q_nR = t^{-1}(\overline{r_0}r_0R + \overline{r_1}r_1R + \cdots + \overline{r_n}r_nR)$. Since $R$ is a Bezout order in a simple artinian ring $Q$, the finitely generated right ideal $\overline{r_0}r_0R + \overline{r_1}r_1R + \cdots + \overline{r_n}r_nR$ of $R$ is principal. So, there exists an element $s \in Q$ so that $q_0R + q_1R + \cdots + q_nR = sR$. Since $d(sR/s\mathcal{J}(R)) \leq n = d(\overline{R})$, there exists $i$, say $i = 0$, such that $q_1R + \cdots + q_nR + s\mathcal{J}(R) = sR$. By Nakayama's lemma, $q_1R + \cdots + q_nR = sR$. So, $q_0 \in sR = q_1R + \cdots + q_nR$, i.e., $R$ is a right $n$-chain ring in $Q$. Similarly, $R$ is a left $n$-chain ring in $Q$.

$$(3) \Longrightarrow (1)$$

Suppose that (3) holds and let $m = d(\overline{R})$, i.e., $m$ is the length of "the longest" direct sum of uniform right (left) ideals of $\overline{R}$ which is essential. We only need to prove that for all $q \in Q \setminus R$ there exist $r,\ r' \in R$ such that $qr,\ r'q \in R \setminus \mathcal{J}(R)$.

Let $q \in Q \setminus R$. Since $\overline{R}$ is a simple artinian ring, right ideals of $\overline{R}$ are generated by idempotents. Hence, there exist a set $\{\overline{e_1}, \overline{e_2}, \cdots, \overline{e_m}\}$ of primitive orthogonal idempotents of $\overline{R}$. Let $e_1, e_2, \cdots, e_m$ be the inverse images of $\overline{e_1}, \overline{e_2}, \cdots, \overline{e_m}$ under the map $R \longrightarrow \overline{R} = R/\mathcal{J}(R)$ respectively. By Remark 1.1.11 $a$), $\overline{R}$ is an $n$-chain ring. Since $m \geq n$, $\overline{R}$ is an $m$-chain ring. Consider the elements $\overline{q}, \overline{e_1}, \overline{e_2}, \cdots, \overline{e_m}$. There are two possibilities, $qR + \mathcal{J}(R) \subseteq \sum_{i=1}^{m} e_iR + \mathcal{J}(R)$ or there exists $i$ such that $e_iR \subseteq qR + \sum\limits_{j \neq i} e_jR + \mathcal{J}(R)$. The first case is impossible, since then $q \in R$. Hence, $e_i = qr + \sum\limits_{j \neq i} e_jr_j + x$, where $x \in \mathcal{J}(R)$ and $r_j$, $r \in R$. Then $qr \in R \backslash \mathcal{J}(R)$. Otherwise, $e_i - \sum\limits_{j \neq i} e_jr_j \in \mathcal{J}(R)$, i.e., $\overline{e_i} = \sum\limits_{j \neq i} \overline{e_jr_j}$,, i.e., the idempotents $\{\overline{e_1}, \overline{e_2}, \cdots, \overline{e_m}\}$ are not primitive.

Similarly, there exists $r' \in R$ such that $r'q \in R \setminus \mathcal{J}(R)$.

□

Note that every condition in Theorem 2.2.8 is equivalent to the condition that $R$ is a local semi-hereditary order in $Q$. Since the notion of semi-hereditary order is not essential for this work we omit the details of this characterization. For details see [MMU97].

## 2.3 The Ideal Theory of Dubrovin valuation rings

Throughout this section, $R$ is a Dubrovin valuation ring in a simple artinian ring $Q$. In this section we study the properties of $R$-ideals, divisorial ideals and overrings of $R$. In particular, we consider prime ideals of $R$. The following result is crucial for the study of the ideal theory of Dubrovin valuation rings. The detailed proof can be found in [Dub84] and [MMU97].

**Lemma 2.3.1 ( [Dub84] and [MMU97])** *Let $R$ be Dubrovin valuation ring of $Q$ and let $T_2 \subseteq T_1$ be right $R$-submodules of $Q$ such that*

(1) *$T_1$ is regular and*

**(2)** *there exists a subring $S$ of $O_l(T_2)$ such that for any regular elements $t_1, t_2 \in T_1$ there is a regular element $t \in T_1$ with $St_1 + St_2 \subseteq St$.*

*Then either $T_1 = T_2$ or there is a regular element $t_0 \in T_1$ such that $T_2 \subseteq t_0\mathcal{J}(R)$.*

The next result shows that Dubrovin valuation rings share a very important property with other types of valuation rings.

**Proposition 2.3.2 ( [Dub84])** *Let $R$ be a Dubrovin valuation ring of $Q$ and let $S$ be a Bezout order in $Q$. Then the set of regular $S-R$-sub-bimodules of $Q$ is linearly ordered by inclusion. In particular, the set of all $R$-ideals of $Q$ and hence, the set of all two-sided ideals of $R$, are linearly ordered by inclusion.*

**Proof.** Let $T_1, T_2$ be regular $S-R$-sub-bimodules of $Q$. Set $T_0 = T_1 \bigcap T_2$. If $T_0 = T_1$, then $T_1 \subseteq T_2$. Let $T_0 \subset T_1$. Then, by Lemma 2.3.1 there exists a regular element $t_0 \in T_1$ such that $T_0 \subseteq t_0\mathcal{J}(R) \subset t_0R \subseteq T_1$. Hence, $T_2 \bigcap t_0R \subseteq T_2 \bigcap T_1 = T_0 \subseteq t_0\mathcal{J}(R)$, that is, $t_0^{-1}T_2 \bigcap R \subseteq \mathcal{J}(R)$. By Lemma 2.2.1, $t_0^{-1}T_2 \subseteq \mathcal{J}(R)$, i.e. $T_2 \subseteq t_0\mathcal{J}(R) \subseteq T_1$.

□

Let $P$ be a prime ideal of a ring $R$. If $\mathcal{C}_R(P) := \{r \in R \mid r + P \text{ is regular in } R/P\}$ is a regular Ore set of $R$, then the quotient ring $R\mathcal{C}_R(P)^{-1} = \mathcal{C}_R(P)^{-1}R$ of $R$ with respect to $\mathcal{C}_R(P)$ is denoted by $R_P = {}_PR$ and is called the *localization* of $R$. The overrings $T$ of a Dubrovin valuation ring $R$ in $Q$ are again Dubrovin valuation rings of $Q$ that are in one-to-one correspondence with the prime ideals $P$ of $R$ for which $R/P$ is prime Goldie. More generally, the following results hold:

**Theorem 2.3.3 ( [Dub84], [Grä92b])** *Let $R$ be a Bezout order in a simple artinian ring $Q$, let $B$ be a Dubrovin valuation ring of $Q$ containing $R$ and let $P = \mathcal{J}(B) \bigcap R$. Then $P$ is a prime ideal of $R$ such that $R/P$ is Goldie, $\mathcal{C}_R(P) = \{r \in R \mid r + P \text{ is regular in } R/P\}$ is a regular left and right Ore set and $B = R_P = {}_PR$.*

*Conversely, if $R$ is a Bezout order in a simple artinian ring $Q$ and $P$ is a prime ideal of $R$ such that $R/P$ is Goldie, then $\mathcal{C}_R(P)$ is a regular left and right Ore set and $R_P = {}_PR$ is a Dubrovin valuation ring of $Q$ such that $\mathcal{J}(R_P) \bigcap R = P$.*

**Theorem 2.3.4 ( [Dub84])** *Let $R$ be a Dubrovin valuation ring of a simple artinian ring $Q$ and let $S$ be an overring of $R$ in $Q$. Then $\widetilde{R} = R/\mathcal{J}(S)$ is a Dubrovin valuation ring of $\overline{S} = S/\mathcal{J}(S)$, $S$ is a Dubrovin valuation ring of $Q$ and $S = R_{\mathcal{J}(S)}$.*

**Proof.** By Corollary 2.2.5, $\mathcal{J}(S) \subseteq R$ and by Theorem 2.2.8, $R$ is an $n$-chain ring in $Q$ with $d(\overline{R}) = d(R/\mathcal{J}(R)) \geq n$. Hence, by Remark 1.1.11, $S$ is an $n$-chain ring in $Q$ and $\overline{S}$ is an $n$-chain ring in $\overline{S}$. Since $\overline{S}$ is a semi-simple ring, by Lemma 1.1.12, $\overline{S}$ is an artinian ring. If $I \subset S$ is any ideal of $S$, then by Corollary 2.2.5, $I \subseteq \mathcal{J}(R)$ and hence, $1 + I \subseteq 1 + \mathcal{J}(R) \subseteq U(R) \subseteq U(S)$, i.e., $I \subseteq \mathcal{J}(S)$. So, $\overline{S} = S/\mathcal{J}(S)$ is a simple artinian ring . By Theorem 2.1.3, $S$ is a Bezout order in $Q$. Hence, $S$ is a Dubrovin valuation ring of $Q$. Since $\widetilde{R}/\mathcal{J}(\widetilde{R}) \cong R/\mathcal{J}(R)$ is a simple artinian ring and $d(\widetilde{R}/\mathcal{J}(\widetilde{R})) \geq n$, by Theorem 2.2.8, $\widetilde{R}$ is a Dubrovin valuation ring in $\bar{S}$. Hence, $R/\mathcal{J}(S)$ is a prime Goldie ring and by Theorem 2.3.3, $S = R_{\mathcal{J}(S)}$.

□

It follows, that prime ideals $P$ of a Dubrovin valuation ring $R$ such that $R/P$ is a prime Goldie ring, are very important. Therefore, the following definition is natural:

**Definition 2.3.5** *The prime ideal $P$ of a Dubrovin valuation ring $R$ is called a Goldie prime ideal if $R/P$ is a prime Goldie ring.*

Note that $J(R)$ and $(0)$ are Goldie primes of a Dubrovin valuation ring $R$.

**Remark 2.3.6** *In the special cases, when $R$ is a total valuation ring in a skew field $Q$, i.e., $R$ is a Bezout order in $Q$ with $R/\mathcal{J}(R)$ being a skew field, or when $R$ is a Bezout order in a central simple algebra $Q$, the bijection given in Theorem 2.3.3, is described by the following results:*

a) *If $R$ is a Bezout order in a skew field $Q$ such that $R/\mathcal{J}(R)$ is a skew field, i.e., $R$ is a total valuation ring in $Q$, then there exists a one-to-one correspondence between overrings $B$ of $R$ in $Q$ and the set of completely prime ideals of $R$ given by $B \to \mathcal{J}(B)\bigcap R$ and $P \to RS^{-1} = R_P$, where $S = R \setminus P$.*

b) ( [Grä92b]) *If $R$ is a Bezout order in a central simple algebra $Q$, $\mathcal{B}$ is the set of all Dubrovin valuation rings of $Q$ containing $R$, and $\mathcal{P}$ is the set of all prime ideals of $R$ , then the map $f : \mathcal{B} \to \mathcal{P}$ given by $B \mapsto \mathcal{J}(B)\bigcap R$ is a well defined anti-order isomorphism where $f^{-1}(P) = R_P$. Moreover, $R = \bigcap R_P$ where the intersection runs over all maximal ideals of $R$, i.e., $R$ is an intersection of Dubrovin valuation rings.*

**Remark 2.3.7** *From the result in Remark 2.3.6 b), it follows that if $R$ is a Dubrovin valuation ring in a simple artinian ring $Q$ with finite dimension over its center $K$, then every prime ideal $P$ of $R$ is Goldie prime and there is a bijection between spec$R$ and the set of all overrings of $R$ in $Q$ given by $P \longrightarrow R_P$ ($R_P$ is the localization with respect to the set $\mathcal{C}_R(P)$ of elements $r \in R$ regular modulo $P$). This result was obtained by Dubrovin,* [Dub85], *Theorem 1.*

In the following part of this section we discus a theory of divisors of a Dubrovin valuation ring. The following results are needed.

**Lemma 2.3.8 ( [MMU97])** *Let $R$ be a Dubrovin valuation ring of $Q$. Then $O_r(\mathcal{J}(R)) = O_l(\mathcal{J}(R)) = R$.*

**Proof.** Assume $R \subset O_r(\mathcal{J}(R))$. Then by Lemma 2.3.1, there exists a regular element $t_0 \in O_r(\mathcal{J}(R))$ such that $R \subseteq t_0\mathcal{J}(R)$ and hence, $t_0^{-1} \in \mathcal{J}(R)$. But then, $1 \in \mathcal{J}(R)$, a contradiction. Thus, $R = O_r(\mathcal{J}(R))$. Similarly, $R = O_l(\mathcal{J}(R))$.

□

**Lemma 2.3.9** *Let $R$ be a Dubrovin valuation ring of $Q$ and $P$ be a prime ideal of $R$. Then $O_r(P) = O_l(P)$.*

**Proof.** Since the set of all overrings of $R$ in $Q$ is linearly ordered by inclusion, we may assume $O_l(P) \subseteq O_r(P)$.Then, $\mathcal{J}(O_r(P)) \subseteq \mathcal{J}(O_l(P))$. Hence,$\mathcal{J}(O_r(P))(O_r(P)P) = (\mathcal{J}(O_r(P))O_r(P))P \subseteq \mathcal{J}(O_r(P))P \subseteq \mathcal{J}(O_l(P))P \subseteq O_l(P)P \subseteq P$. But $\mathcal{J}(O_r(P))$ and $O_r(P)P$ are 2-sided ideals of $R$. Thus, since $P$ is a prime ideal of $R$, we have $\mathcal{J}(O_r(P)) \subseteq P$ or $O_r(P)P \subseteq P$, i.e, $P = \mathcal{J}(O_r(P))$ or $O_r(P) = O_l(P)$. Thus, $O_l(P) = O_l(\mathcal{J}(O_r(P)) = O_r(\mathcal{J}(O_r(P)) = O_r(P)$ by Lemma 2.3.8, or $O_r(P) = O_l(P)$. So, in both cases $O_r(P) = O_l(P)$.

□

**Lemma 2.3.10** ( [MMU97]) *Let $R$ be a Dubrovin valuation ring of $Q$, $A$ be an $R$-ideal of $Q$ and let $S = O_r(A)$. Then the following are equivalent:*

**(1)** *$A$ is principal as a right $S$-ideal.*

**(2)** $A^{-1}A = S$.

**(3)** $A \supset A\mathcal{J}(S)$

**Proof.**
(1) $\Rightarrow$ (2): Let $A = aS$ for some $a \in A$. Then $A^{-1} = Sa^{-1}$ and (2) follows.
(2) $\Rightarrow$ (3): Let $A^{-1}A = S$. If $A = A\mathcal{J}(S)$, then $A^{-1}A = A^{-1}A\mathcal{J}(S)$, i.e., $S = \mathcal{J}(S)$, a contradiction.
(3) $\Rightarrow$ (1): Let $A \supset A\mathcal{J}(S)$. Then by Lemma 2.3.1, there exists a regular element $t \in A$ such that $A\mathcal{J}(S) \subseteq t\mathcal{J}(S) \subseteq A\mathcal{J}(S)$, that is, $t\mathcal{J}(S) = A\mathcal{J}(S)$. Hence, $t^{-1}A\mathcal{J}(S) = \mathcal{J}(S)$. Therefore, $t^{-1}A \subseteq O_l(\mathcal{J}(S)) = S$. Thus, $A \subseteq tS$, that is, $A = tS$.

□

On the set of right (left) $R$-ideals we introduce the following operation:

**Definition 2.3.11** *Let $I$ be a right $R$-ideal and $S = O_r(I)$. We define $I^* = \bigcap cS$, where $c$ runs over all elements in $Q$ with $cS \supseteq I$. Similarly, for any left $R$-ideal $L$ with $T = O_l(L)$ we define ${}^*L = \bigcap Tc$, where $c$ runs over all elements in $Q$ with $Tc \supseteq L$.*

That $I^* = {}^*I$ for an $R$-ideal $I$ follows from Proposition 2.3.13. In the next result, the basic properties of the "$*$"- operation are given. Note that in Chapter 4, we briefly discuss an analogous operation for commutative rings.

**Proposition 2.3.12 ( [MMU97])** *Let $R$ be a Dubrovin valuation ring of $Q$ and let $I$ be a right $R$-ideal of $Q$. Then:*

**(1)** $I \subseteq I^*$

**(2)** $(I^*)^* = I^*$

**(3)** $(cI)^* = cI^*$, *for any $c \in U(Q)$*

**(4)** $(cI)^{-1} = I^{-1}c^{-1}$, *for any $c \in U(Q)$*

**Proof.** <u>**(1)**</u> : This is trivial.

<u>**(2)**</u> : By (1), $I^* \subseteq (I^*)^*$. Assume $I^* \subset (I^*)^*$. Then, there exists an element $x \in (I^*)^*$ such that $x \notin I^*$. Therefore, there exists an element $b_0 \in Q$ with $b_0 S \supseteq I$ but $x \notin b_0 S$. Also, $x \in cS$, for all $c \in Q$ with $cS \supseteq I^*$. But, $b_0 S \supseteq I$ implies $I^* \subseteq b_0 S$ and thus, $x \in b_0 S$, a contradiction.

<u>**(3)**</u> : Let $c \in U(Q)$ and $x \in I^*$ be an arbitrary element. Consider any $b \in Q$ such that $bS \supseteq cI$. Then $I \subseteq c^{-1}bS$ and since $x \in I^*$, by the Definition 2.3.11, $x \in c^{-1}bS$ so that $cx \in bS$. Thus, $cx \in (cI)^*$. Since $x$ was an arbitrary element, $cI^* \subseteq (cI)^*$. Conversely, let $x \in (cI)^*$. Let $b \in Q$ be any element with $bS \supseteq I$. Then $cI \subseteq cbS$.

Hence, $(cI)^* \subseteq cbS$, that is, $x \in cbS$. But, $b$ was an arbitrary element. Therefore, $x \in \bigcap cbS$, with $bS \supseteq I$, that is, $x \in c(\bigcap bS) = cI^*$ and $(cI)^* \subseteq cI^*$ follows.

(4) : Let $c \in U(Q)$ and $x \in I^{-1}$. Then $cIxc^{-1}cI = cIxI \subseteq cI$ implies $xc^{-1} \in (cI)^{-1}$. Thus, $I^{-1}c^{-1} \subseteq (cI)^{-1}$. Conversely, $x \in (cI)^{-1}$, implies $cIxcI \subseteq cI$. Hence, $IxcI \subseteq I$, that is, $xc \in I^{-1}$. Thus, $x \in I^{-1}c^{-1}$ and $(cI)^{-1} \subseteq I^{-1}c^{-1}$ follows.

□

The proof of the following result can be found in [MMU97].

**Proposition 2.3.13** *Let $R$ be a Dubrovin valuation ring of $Q$ and let $A$ be an $R$-ideal of $Q$. Set $S = O_r(A)$ and $T = O_l(A)$. Then:*

**(1)** $A_v := (S : (S : A)_l)_r = A^* = {}^*A = (T : (T : A)_r)_l =: {}_vA$ *and* $A^* = (A^{-1})^{-1}$.

**(2)** $A^{**} = A^*$ *and* $(A^{-1})^* = A^{-1}$

**(3)** *If $A$ is not principal as a right $S$-ideal then $A^{-1}A = \mathcal{J}(S)$ and $\mathcal{J}(S)$ is not a principal right $S$-ideal.*

**(4)** *If $A \subset A^*$, then $A^* = cS$ and $A = c\mathcal{J}(S)$ for some regular element $c \in A^*$. In particular, $A = A^*\mathcal{J}(S)$.*

An $R$-ideal $A$ of $Q$ is called *divisorial* if $A = A^*$. With $D(R)$ we denote the set of all divisorial $R$-ideals of $Q$. On the set $D(R)$ we define the partial operation "$\circ$" by $A \circ B = (AB)^*$ if $A, B \in D(R)$ with $O_r(A) = O_l(B)$. The next result shows that $(D(R), \circ)$ is an algebraic structure known as Brandt groupoid without connectivity property.

**Theorem 2.3.14 ( [Dub84])** *Let $R$ be a Dubrovin valuation ring of $Q$. Then $(D(R), \circ)$ has the following properties:*

**(1)** *For every $A \in D(R)$, there exist unique elements $E_r(A)$ and $E_l(A)$ in $D(R)$ such that*

$$A \circ E_r(A) = A = E_l(A) \circ A.$$

**(2)** *For $A,\ B \in D(R)$, the product $A \circ B$ is defined if and only if $E_r(A) = E_l(B)$.*

**(3)** *If for $A,\ B,\ C \in D(R)$ the products $A \circ B$ and $B \circ C$ are defined, then $(A \circ B) \circ C$ and $A \circ (B \circ C)$ are defined and they are equal.*

**(4)** *For every element $A \in D(R)$ there exists a unique element $\widetilde{A} \in D(R)$ such that*

$$A \circ \widetilde{A} = E_l(A), \quad \widetilde{A} \circ A = E_r(A).$$

**Proof.** <u>(1) :</u> Let $A \in D(R)$ and set $E_r(A) := O_r(A),\ E_l(A) := O_l(A)$. Then:

- <u>$\mathbf{E_r(A) \in D(R)}$ :</u> First, we prove that $S := O_r(E_r(A)) = O_r(A)$. For $q \in S$, we have $q \in O_r(A)q \subseteq O_r(A)$. Conversely, $q \in O_r(A)$ implies $O_r(A)q \subseteq O_r(A)$, i.e., $q \in S$. Hence,

  $$E_r(A) \subseteq E_r(A)^* \overset{def}{=} \bigcap_{cS \supseteq E_r(A)} cS \subseteq 1 \cdot S = O_r(A) = E_r(A),$$

  that is, $E_r(A) = E_r(A)^* \in D(R)$.

- <u>$\mathbf{A \circ E_r(A)}$ **is defined**</u> since $O_r(A) = O_l(O_r(A)) = O_l(E_r(A))$.

- <u>$\mathbf{A \circ E_r(A) = A}$ :</u> $A \circ E_r(A) = (AE_r(A))^* = (AO_r(A))^* = A^* = A$.

Similarly, $E_l(A) \in D(R)$, $E_l(A) \circ A$ is defined and $E_l(A) \circ A = A$. The uniqueness of the right and the left units follows by a familiar argument.

<u>(2) :</u> This follows immediately from the definition.

<u>(3)</u> : Let $A,\ B,\ C \in D(R)$ and let $A \circ B$ and $B \circ C$ are defined, i.e., $O_r(A) = O_l(B)$ and $O_r(B) = O_l(C)$. It is enough to prove that

$$((AB)^*C)^* = (ABC)^* = (A(BC)^*)^*$$

If $AB = (AB)^*$, then the first equality holds. Let $AB \subset (AB)^*$ and let $S = O_r(AB)$. Then $S \supseteq O_r(B)$ and since $O_l(B) = O_r(A) \supseteq A^{-1}A$, we have $A^{-1}AB \subseteq B$. If $A^{-1}AB = B$, then $O_r(B) \supseteq O_r(AB) = S$, i.e., $S = O_r(B)$. If $A^{-1}AB \subset B$, then $A^{-1}A \subset O_l(B) = O_r(A)$. By Lemma 2.3.10, this implies that $A$ is not principal as a right $O_r(A)$-ideal. Hence, by Proposition 2.3.13 (3), $A^{-1}A = \mathcal{J}(O_r(A)) = \mathcal{J}(O_l(B))$. Hence, $B \supset \mathcal{J}(O_l(B))B$ and by the left version of Lemma 2.3.10, $B$ is principal as a left $O_l(B)$-ideal. So, $B = O_l(B)b$ , for some $b \in B$. Therefore, $S = O_r(AB) = O_r(AO_l(B)b) = O_r(AO_r(A)b) = O_r(Ab) = b^{-1}O_r(A)b = b^{-1}O_l(B)b = O_r(O_l(B)b) = O_r(B)$. Hence, $S = O_r(AB) = O_r(B)$. Now, since $AB$ is not divisorial, by Proposition 2.3.13 (3) and (4), $(AB)^* = cS$ and $AB = c\mathcal{J}(S)$, for some $c \in U(Q)$, and $\mathcal{J}(S)$ is not principal right $S$-ideal. Then $\mathcal{J}(S)^{-1}\mathcal{J}(S) \subset O_r(\mathcal{J}(S)) = S$ by Lemma 2.3.10. Hence, $S \subseteq \mathcal{J}(S)^{-1}$ implies $\mathcal{J}(S) = S\mathcal{J}(S) \subseteq \mathcal{J}(S)^{-1}\mathcal{J}(S) \subseteq \mathcal{J}(S)$, i.e., $\mathcal{J}(S)^{-1}\mathcal{J}(S) = \mathcal{J}(S)$. This implies $\mathcal{J}(S)^{-1} \subseteq O_l(\mathcal{J}(S)) = S$, i.e., $\mathcal{J}(S)^{-1} = S$. Finally, by Propositions 2.3.12 and 2.3.13, since $O_l(C) = O_r(B) = S$ and $\mathcal{J}(S) = \mathcal{J}(O_l(C)) = C$ we have $(ABC)^* = (c\mathcal{J}(S)C)^* = c(\mathcal{J}(S)C)^* = c(C)^* = (cC)^* = (cO_l(C)C)^* = (cSC)^* = ((AB)^*C)^*$. Similarly, $(ABC)^* = (A(BC)^*)^*$.

<u>(4)</u> : Let $A \in D(R)$ and denote $S = O_r(A)$, $T = O_l(A^{-1})$. Then $S = T$. To prove this, first note that $R$ and $S$ are overrings of $R$. Since by Theorem 2.3.3 and Proposition 2.3.2, overrings of $R$ are linearly ordered by inclusion, we can assume that $R \subseteq S \subseteq T$.

If $A^{-1}A = S$, then by Lemma 2.3.10, $A = aS$ for some regular element $a \in A$ and also $A^{-1} = Sa^{-1}$. Hence, $q \in T$ implies $qSa^{-1} \subseteq Sa^{-1}$, i.e., $q \in S$. So, $T \subseteq S$.

If $A^{-1}A \subset S$, then $A^{-1}A \subseteq \mathcal{J}(S)$. Hence, $1 + A^{-1}A \subseteq 1 + \mathcal{J}(S) \subseteq U(S) \subseteq U(T)$, i.e., $A^{-1}A \subseteq \mathcal{J}(T)$. Assume that $S \subset T$. Then $A \subset AT$, since otherwise $A = AT$ and $T \subseteq O_r(A) = S$. By Lemma 2.3.1, there exists a regular element $c \in AT$ such that $A \subseteq c\mathcal{J}(S) \subseteq cS \subseteq ATS \subseteq ATT = AT$. Multiplying by $T$ on the right, we have $AT = cT$. Furthermore, $A \subseteq cS$ implies $c^{-1} \in A^{-1}$. Thus, $T = c^{-1}AT \subseteq A^{-1}AT \subseteq \mathcal{J}(T)T = \mathcal{J}(T)$, a contradiction that shows that $S = T$, i.e., $O_r(A) = O_l(A^{-1})$.

Similarly, $O_r(A^{-1}) = O_l(A)$.

This implies that the products $A^{-1} \circ A$ and $A \circ A^{-1}$ are defined and by the definition $A^{-1} \circ A = (A^{-1}A)^* \subseteq S^* = O_r(A)^* \overset{(1)}{=} O_r(A) = S$. Hence, $A^{-1} \circ A \subseteq O_r(A)$. Let $c \in U(Q)$ be any element such that $A^{-1}A \subseteq cS$. Then $c^{-1}A^{-1}A \subseteq S$, i.e., $c^{-1}A^{-1} \subseteq A^{-1}$. Hence, $c^{-1} \in O_l(A^{-1}) = O_r(A)$, i.e., $1 \in cS$. This implies $S \subseteq \bigcap_{cS \supseteq A^{-1}A} cS = (A^{-1}A)^* = A^{-1} \circ A$, i.e., $A^{-1} \circ A = O_r(A)$. Similarly, $A \circ A^{-1} = O_l(A)$.

□

Now, we consider a Dubrovin valuation ring $R$ of *rank one*, that is, $\mathcal{J}(R)$ and $(0)$ are the only Goldie prime ideals. In this case, the only proper overring of $R$ is $Q$, and the operation "$\circ$" is defined on the whole of $D(R)$. The following result shows that for a rank one Dubrovin valuation ring $R$, the structure $(D(R), \circ)$ from Theorem 2.3.14 becomes a group.

**Theorem 2.3.15** *Let $R$ be a Dubrovin valuation ring of $Q$ of rank one. Then $(D(R), \circ)$ is a group.*

**Proof.** By Theorem 2.3.3 there are no proper overrings of $R$ in $Q$. Hence, for every $A \in D(R)$, $O_r(A) = O_l(A) = R$. Thus, for any two elements $A$, $B \in D(R)$, $O_r(A) = O_l(B) = R$, that is, $A \circ B$ is defined. So, "$\circ$" is defined on the whole of $D(R)$.

Furthermore, there exists a unique element $E = R \in D(R)$ such that for every $A \in D(R)$, $A \circ E = E \circ A = A$, that is, there is an identity element in $D(R)$.

Finally, for every $A \in D(R)$, there is a unique element $A^{-1} \in D(R)$ (see the proof of Theorem 2.3.14) such that $A \circ A^{-1} = O_l(A) = O_r(A) = A^{-1} \circ A = R = E$. Since "$\circ$" is associative, $(D(R), \circ)$ is a group.

□

Now, we consider Goldie prime ideals of a Dubrovin valuation ring. We first show that in the special case of the total valuation ring, the classes of Goldie prime ideals and completely prime ideals coincide.

**Proposition 2.3.16** *Let $R$ be a total valuation ring in a skew field $Q$, and let $P$ be a prime ideal of $R$. Then, $P$ is a Goldie prime if and only if $P$ is completely prime.*

**Proof.** Assume that $P$ is a Goldie prime. Then, by Theorem 2.3.3, $S = R_P = {}_PR$ is a Dubrovin valuation ring of $Q$ and $\mathcal{J}(S) \bigcap R = P$. But, by Corollary 2.2.5, $\mathcal{J}(S) \subseteq \mathcal{J}(R) \subset R$, so that $P = \mathcal{J}(S)$. On the other hand, $S$ as an overring of $R$ in $Q$, is a total valuation ring in $Q$. Hence, $\overline{S} = S/\mathcal{J}(S)$ is a skew field. Thus, $\widetilde{R} = R/P = R/\mathcal{J}(S)$ is a subring of a skew field $\overline{S}$, that is, $R/P$ is an integral domain. So, $P$ is a completely prime ideal.

Conversely, if $P$ is completely prime ideal, then $R/P$ is an integral domain. Also, $R$ is a Dubrovin valuation ring in $Q$ with $\overline{R} = R/\mathcal{J}(R)$ is a skew field. Hence, $d(\overline{R}) = d(R/\mathcal{J}(R)) = 1$. By Theorem 2.2.8, $R$ is a 1-chain domain; so by Remark 1.1.11, $\widetilde{R} = R/P$ is also 1-chain domain such that $\widetilde{R}/\mathcal{J}(\widetilde{R}) = R/P/\mathcal{J}(R)/P \cong R/\mathcal{J}(R)$ is a skew field. Again, by Theorem 2.2.8, $R/P$ is a Dubrovin valuation ring in $\overline{R}$, i.e., $R/P$ is a prime Goldie ring. Thus, $P$ is a Goldie prime.

□

The natural question is: Does there exist a Dubrovin valuation ring with a prime ideal that is not Goldie prime? This question was raised by H.H.Brungs and G.Törner in [BT76], in the form: Does there exist a prime chain ring with zero divisors, i.e., a chain ring with a prime ideal that is not completely prime? N.Dubrovin, [Dub93], gave the positive answer to this question.

**Example 2.3.17** ( **[Dub93]**) *There exists a Dubrovin valuation ring with a prime ideal that is not Goldie prime.*

**Proof.** Consider the group $SL(2,\mathbb{R})$ of $2 \times 2$ matrices with real entries and determinant 1. Then

$$\mathbb{U} = \{u = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid b,\ 0 < a \in \mathbb{R}\},$$

$$\mathbb{S} = \{r(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \mid t \in \mathbb{R}\}$$

are subgroups of $SL(2,\mathbb{R})$ and every element $s \in SL(2,\mathbb{R})$ can be written in a unique way as $s = r(\varphi)u$ for $r(\varphi) \in \mathbb{S}, 0 \leq \varphi < 2\pi$ , $u \in \mathbb{U}$.

Consider now first the universal covering group $\Gamma$ of $\mathbb{S}$ where $\Gamma = \{x^t \mid t \in \mathbb{R}\}$ with $x^{t_1}x^{t_2} = x^{t_1+t_2}$ as operation, and $x^{t_1} \leq x^{t_2}$ if and only if $t_1 \leq t_2$. As a linearly ordered group, $\Gamma$ is isomorphic to $(\mathbb{R}, +, \leq)$. The covering map $\tau$ from $\mathbb{R}$ to $\mathbb{S}$ with $\tau(x^t) = r(t)$ is a homomorphism with $Ker\ \tau = \langle x^{2\pi}\rangle$, the cyclic subgroup of $\mathbb{R}$ generated by $x^{2\pi}$.

Next we define the covering group $G$ of $SL(2,\mathbb{R})$ as the set $G = \{x^t u \mid x^t \in \Gamma,\ u \in \mathbb{U}\}$ with the operation given by:

$$x^{t_1}u_1x^{t_2}u_2 = x^{t_1}u_1x^{2\pi k+\varphi}u_2 = x^{t_1+2\pi k+\psi}u_1^{'}u_2,$$

where $\varphi \in [0, 2\pi]$ and $u_1 r(\varphi) = r(\psi)u_1^{'}$ for $\psi \in [0, 2\pi)$. The mapping $\tau$ above is extended to a map from $G$ to $SL(2,\mathbb{R})$ by defining $\tau(x^t u) = r(t)u$.

Finally, we consider the subsemigroup $P = \{x^r u \mid 0 \leq r \in \mathbb{R},\ u \in \mathbb{U}\}$ of the group $G$. Then $P \bigcup P^{-1} = G$ and $P \bigcap P^{-1} = \mathbb{U}$. The element $x^\pi$ is contained in $P$ and is central in $P$. So, $Q = x^\pi P$ is an ideal of $P$, properly contained in the maximal ideal $J(P)$ and, in addition, $\bigcap Q^m = \bigcap x^{\pi m} P = \emptyset$. Since $x^{\pi/2} x^{\pi/2} \in Q$ but $x^{\pi/2} \notin Q$, $Q$ is a prime ideal in $P$ that is not completely prime. It can be shown that $P$ satisfies the conditions of Theorem 1.3.3. hence, there exists a skew field $D$ and a total valuation ring $S$ in $D$ associated to the cone $P$. By Theorem 1.3.1, the corresponding ideal $\varphi(Q) = QS = x^\pi PS$ is a prime ideal of the ring $S$ that is not completely prime. Now, applying Theorem 1.2.5 and Proposition 2.3.16, $S$ is a Dubrovin valuation ring with a prime ideal $\varphi(Q)$ that is not Goldie prime.

□

The following results show that Goldie prime ideals of a Dubrovin valuation ring $R$ in $Q$ have nice properties. These results are important steps for describing the structure of ideals in Dubrovin valuation rings. But first, we prove the following basic fact which we will use frequently.

**Lemma 2.3.18** *Let $R$ be a Dubrovin valuation ring of a simple artinian ring $Q$ and let $S$ be an overring of $R$ in $Q$. If $I \neq R$ is a non-zero ideal of $R$ such that $I \supset \mathcal{J}(S)$, then $IS = S$.*

**Proof.** The ideal $I/\mathcal{J}(S)$ is a nonzero ideal in $\widetilde{R} = R/\mathcal{J}(S)$ which is a Dubrovin valuation ring in $\overline{S} = S/\mathcal{J}(S)$ by Theorem 2.3.4. Since $\widetilde{R}$ is a prime Goldie ring, the ideal $I/\mathcal{J}(S)$ is essential and by Theorem 1.1.5, it is a regular ideal. So, there exists an element $r \in I$ so that $r + \mathcal{J}(S)$ is a regular element in $\widetilde{R} = R/\mathcal{J}(S)$. Hence, $r + \mathcal{J}(S)$ is a unit in $\overline{S} = S/\mathcal{J}(S)$, that is, $r \in U(S)$. Thus $IS = S$.

□

**Proposition 2.3.19** *Let $P_i$ be Goldie primes in $R$, $i \in \Lambda$. Then $P = \bigcap P_i$ is a Goldie prime.*

**Proof.** Since $P_i$ is Goldie, the localization $R_{P_i}$ exists for every $i$ and we set $S = \bigcup R_{P_i}$. The overrings $R_{P_i}$ of $R$ in $Q$ are totally ordered by inclusion. Hence, $S$ is a ring in $Q$ containing $R$. By Theorem 2.3.4, $S$ is a Dubrovin valuation ring of $Q$ and $\mathcal{J}(S) \subseteq P_i$ since $R_{P_i} \subseteq S$. It follows that $\mathcal{J}(S)$ is a Goldie prime contained in $P$ (Theorem 2.3.3).

Assume that $P \supset \mathcal{J}(S)$. Then by Lemma 2.3.18 we have $PS = S$. Hence, $1 = \sum p_i s_i$ for elements $p_i$ in $P$ and $s_i$ in $S = \bigcup R_{P_i}$. But, the rings $R_{P_i}$ are totally ordered by inclusion. So, there exists an index $j_0 \in \Lambda$ with $s_i \in R_{P_{j_0}}$ for all $i$ and $p_i \in P \subseteq P_{j_0}$.

Therefore, $1 = \sum p_i s_i \in P_{j_0} R_{P_{j_0}} = P_{j_0}$, a contradiction that shows $P = \mathcal{J}(S)$, which is a Goldie prime.

□

**Lemma 2.3.20** *Let $I \neq R$ be a non-zero ideal in $R$ with $I = I^2$. Then $I$ is neither a principal right $O_r(I)$-ideal nor a principal left $O_l(I)$-ideal.*

**Proof.** Both $S = O_r(I)$ and $T = O_l(I)$ are overrings of $R$ and hence either $T \subseteq S$ or $S \subseteq T$, since the overrings of $R$ are linearly ordered by inclusion. It is enough to consider the case $T \subseteq S$. We show first that $I \subseteq \mathcal{J}(S)$. Otherwise, $I \supset \mathcal{J}(S)$ and by Lemma 2.3.18, it follows that $S = IS = I$, a contradiction that proves $I \subseteq \mathcal{J}(S)$. Assume that $I$ is principal as a right $S$-ideal, i.e., $I = aS$. Then, by Corollary 1.1.4, $a$ is regular element. Then, $aS = I = I^2 = aSaS$ implies $S = SaS$ and $1 = \sum_{i=1}^{n} s_i a t_i$, $s_i$, $t_i \in S$, follows. Since $S = R_{\mathcal{J}(S)} = {}_{\mathcal{J}(S)}R$, there exist elements $c$, $d$ in $\mathcal{C}_R(\mathcal{J}(S))$ with $cs_i \in R$, $t_i d \in R$ for all $i$. Hence $cd = \sum cs_i a t_i d \in I \subseteq \mathcal{J}(S)$, a contradiction which proves that $I$ is not a principal right $S$-ideal. Since $T \subseteq S$ implies $I \subseteq \mathcal{J}(S) \subseteq \mathcal{J}(T)$, a similar argument shows that $I$ is not a principal left $T$-ideal.

□

The next result shows that idempotent ideals $\neq R$ are Goldie primes.

**Proposition 2.3.21** *Let $I^2 = I \neq R$ be an idempotent ideal. Then the following hold:*

a) $O_r(I) = S = O_l(I)$;

b) $I = \mathcal{J}(S)$ *is a Goldie prime with* $S = R_{\mathcal{J}(S)}$.

**Proof.** Let $S = O_r(I)$ and $T = O_l(I)$. It is enough to consider the case $S \subseteq T$. From Lemma 2.3.20, it follows that $I$ is neither a principal right $S$-ideal nor a principal left $T$-ideal. Hence, $I^{-1}I = \mathcal{J}(S)$ and $II^{-1} = \mathcal{J}(T)$ by Proposition 2.3.13(3). Further, $I^{-1} = \{x \in Q \mid xI \subseteq S\} = (S : I)_l \supseteq T$. Conversely, if $x \in (S : I)_l$, then $xI \subseteq S$ and $xI = xI^2 \subseteq SI \subseteq I$, and $x \in T$ follows; we have proved that $I^{-1} = T$. However, $II^{-1} = \mathcal{J}(T) \subseteq \mathcal{J}(S) \subseteq \mathcal{J}(R) \subset R$ implies $I^{-1} \subseteq (R : I)_r$. Further, if $q \in (R : I)_r$, then $Iq \subseteq R \subseteq O_r(I)$, i.e., $Iq = IIq \subseteq I$. Hence, $q \in O_r(I) = S$, i.e., $(R : I)_r \subseteq S$. Thus $T = I^{-1} \subseteq S$ and $T = S$ follows which proves a).

Now, $\mathcal{J}(S) = I^{-1}I = TI = I$ which proves that $I$ is Goldie prime, since $\mathcal{J}(S)$ is a Goldie prime. In addition, $S = R_{\mathcal{J}(S)}$ follows and both parts of b) are proven.

□

The next result shows that the union of Goldie primes is again a Goldie prime.

**Corollary 2.3.22** *Let $R$ be a Dubrovin valuation ring and let $R \supset P_i$, $i \in \Lambda$, be Goldie primes in $R$. Then:*

a) $P = \bigcup P_i$ *is Goldie prime;*

b) $R_P = \bigcap R_{P_i}$;

c) $O_l(P) = R_P = O_r(P)$.

**Proof.** If there exists a $P_j$ with $P_j \supseteq P_i$ for all $i$, then $P = P_j$ is a Goldie prime, $R_P = R_{P_j} = \bigcap R_{P_i}$, and $P = \mathcal{J}(R_P)$ implies $O_l(P) = O_r(P) = R_P$, by Lemma 2.3.8.

We can therefore assume that for every $P_i$ there exists a $P_j$ with $P_j \supset P_i$. Hence, $P \supset P_i$ for all $i$, and $P \supseteq P^2 \supset P_i$ for all $i$, since otherwise $P^2 \subseteq P_i$, which would imply $P \subseteq P_i$, since $P_i$ is prime. It follows from Proposition 2.3.21 that $P = P^2$ is a Goldie prime with $R_P = O_l(P) = O_r(P)$. It remains to prove that $S = O_l(P)$ where $S := \bigcap R_{P_i}$. Let $x \in O_l(P)$, hence $xP \subseteq P$. Since $P \supset P_i$ is an ideal in $R$ and $R/P_i$ is Goldie, $P$ contains an element in $\mathcal{C}_R(P_i)$ and $PR_{P_i} = R_{P_i}$. Therefore, $xR_{P_i} = xPR_{P_i} \subseteq PR_{P_i} = R_{P_i}$; so $x \in R_{P_i}$ for all $i$; hence $x \in S$ follows. Conversely, if $x \in S$ and $a \in P$, then there exists $P_j$ with $a \in P_j$, and $xa \in SP_j \subseteq R_{P_j}P_j = P_j \subset P$ proves $x \in O_l(P)$; so $S = O_l(P)$ follows.

□

Now, let $I \neq R$ be an ideal of $R$ that is not Goldie prime. Then, the families:

$$\mathcal{R} = \{P \mid P \text{ is a Goldie prime, } I \subset P\}$$

$$\mathcal{S} = \{P \mid P \text{ is a Goldie prime, } P \subset I\}$$

are not empty since $\mathcal{J}(R) \in \mathcal{R}$ and $(0) \in \mathcal{S}$. By Proposition 2.3.19 and Corollary 2.3.22, $P_1 = \bigcap\{P \mid P \in \mathcal{R}\}$ and $P_2 = \bigcup\{P \mid P \in \mathcal{S}\}$ are Goldie primes, $P_1 \supset I \supset P_2$ and and no further Goldie prime ideal exists between $P_1$ and $P_2$.

**Definition 2.3.23** *Let $R$ be a Dubrovin valuation ring of $Q$. A prime segment of $R$ (and of $Q$) is a pair of two distinct Goldie primes $P_1 \supset P_2 \supseteq (0)$ in $R$ so that no further Goldie prime exists between $P_1$ and $P_2$.*

The next result is the most important for the study of prime segments of a Dubrovin valuation ring.

**Theorem 2.3.24** *Let $I \neq R$ be an ideal in the Dubrovin valuation ring $R$. Then $\bigcap I^n = P$ is Goldie prime.*

**Proof.** The result follows if $\bigcap I^n = I^m$ for a certain $m$, since then $(I^m)^2 = I^m$ is idempotent and we can apply Proposition 2.3.21. We can therefore assume that

$$I \supset I^2 \supset \cdots \supset I^n \supset I^{n+1} \supset \cdots \supset P = \bigcap I^n$$

and show that the assuming $P$ not to be Goldie prime leads to a contradiction.

First, let $A$ and $B$ be ideals of $R$ such that $P \subset A$ and $P \subset B$. Then there exists an $n$ such that $I^n \subseteq A$ and $I^n \subseteq B$. Otherwise, for all $n$, $A \subsetneq I^n \subset I$ or $B \subsetneq I^n \subset I$, i.e., $A \subseteq \bigcap I^n = P$ or $B \subseteq \bigcap I^n = P$, a contradiction. Thus, $AB \supseteq I^n I^n = I^{2n} \supset P$; so, $P$ is a prime ideal.

If $I$ itself is not Goldie prime, then by the remark above, there exists a prime segment $P_1 \supset P_2$ in $R$ with $P_1 \supset I \supset P_2$.

If $I$ itself is a Goldie prime that does not have a lower neighbor among Goldie primes, then $I = \bigcup P_i$ for Goldie primes $I \supset P_i$. In this case, $I \supseteq I^2 \supset P_i$ for all $i$ and hence $I = I^2$. So, $P = \bigcap I^n = I$ is a Goldie prime, which is a contradiction, since $P$ is assumed not to be Goldie prime.

Therefore, we can assume that there exists a prime segment $P_1 \supset P_2$ in $R$ with $P_1 \supseteq I \supset P_2$. We define $N := P_1 I P_1 \subseteq I$. Then $I^3 = III \subsetneq P_1 I P_1 = N$ and therefore $\bigcap I^n = \bigcap N^n = P$ follows; in addition, $N$ and $P$ are $R_{P_1}$-ideals. After localizing at $P_1$ we obtain $R_{P_1} \supset P_1 \supseteq N \supset \bigcap N^n = P \supset P_2$ and $P$ is not Goldie prime in $R_{P_1}$. We therefore can consider $R_{P_1}/P_2$ and can assume from now on that $R$ has rank one with $R \supset \mathcal{J}(R) = P_1 \supseteq N \supset \bigcap N^n = P \supset (0)$.

We consider the following set $W$ of ideals in $R$ :

$$W = \{L \mid P_1 \supset L \supset P,\ L \text{ an ideal of } R\};$$

note that $W$ contains $N^n$ for $n \geq 2$. Two cases may happen.

**Case 1.** : *W contains an ideal L which is not divisorial*

Then $L \subset L^*$ where $L^* = \bigcap cR$ with $cR \supseteq L$ (by Definition 2.3.11 and the fact that $R$ is of rank one and hence $O_r(L) = R$).

It follows from Proposition 2.3.13 (1) and (4), that $L^* = (L^{-1})^{-1}$, $L^* = aR$ and $L = a\mathcal{J}(R) = aP_1$ for some unit $a$ in $Q$. Note that $a$ is regular but not a unit in $R$ since otherwise $L = a\mathcal{J}(R) = aR\mathcal{J}(R) = R\mathcal{J}(R) = \mathcal{J}(R) = P_1$. Also, $L^* = aR \subsetneq P_1$. Since $O_l(L^*) = aRa^{-1} = R$ we have $L^* = aR = Ra$ and $(L^*)^n = a^nR = Ra^n$ for $n \geq 1$.

It follows that the set $C = \{a^n \mid n = 1, 2, \dots\}$ is an Ore system in $R$, and since $a^{-1} \notin R$, we have $R \subset RC^{-1} \subseteq Q$, i.e., $RC^{-1}$ is a proper overring of $R$. Hence, $RC^{-1} = Q$. Since $P$ is a non-zero ideal in $R$, it contains a regular element $c$ and $c^{-1} = ra^{-n}$ for some $r$ in $R$ and some $n \geq 1$. Hence, $a^n = cr \in P$, which implies $(L^*)^n = a^nR \subseteq P$; so $L^* \subsetneq P \subset L \subset L^*$, which is a contradiction.

**Case 2.** : *$L = L^*$ for all $L \in W$*

Let $L \in W$ and consider the $R$-ideal $L^{-1}$; it is divisorial since $(L^{-1})^* = ((L^{-1})^{-1})^{-1} = (L^*)^{-1} = L^{-1}$. We claim that $L^{-1} \supset R$. Otherwise, $L^{-1} = R$ and $(L^{-1}L)^* = L^* = L$. On the other hand, by Theorem 2.3.15, $(L^{-1}L)^* = L^{-1} \circ L = O_r(L) = R$, a contradiction.

We consider $A_0 = \bigcup L^{-1}$, $L \in W$, and want to prove that $A_0$ is an overring of $R$, hence equal to $Q$. Let $x, y$ be elements in $A_0$ and $x \in L_1^{-1}$, $y \in L_2^{-1}$ for $L_1, L_2 \in W$ follows. Either $L_1 \subseteq L_2$ or $L_2 \subseteq L_1$ and we can assume $L_1 \subseteq L_2$.

Since

$$L^{-1} = \{x \in Q \mid LxL \subseteq L\} = \{x \in Q \mid xL \subseteq O_r(L) = R\} = (R : L)_l$$

for any non-zero ideal $L$ of $R$, it follows that $L_2^{-1} \subseteq L_1^{-1}$ and $x \pm y \in L_1^{-1}$.

Further, $L_1^{-1}L_2^{-1}L_2L_1 \subseteq L_1^{-1}RL_1 \subseteq L_1^{-1}L_1 \subseteq R$ shows that $L_1^{-1}L_2^{-1} \subseteq (R : L_2L_1)_l = (L_2L_1)^{-1}$, and $xy \in (L_2L_1)^{-1}$. In order to see that $xy \in A_0$, note that $P_1 \supset L_1 \supset P$ and $P_1 \supset L_2 \supset P$ imply $P_1 \supset L_2L_1 \supset P$ since $P = \bigcap N^n$ and $N^{n+1} \subset N^n$ for all $n$.

To reach the final contradiction we choose a regular element $c$ in $P \neq (0)$. Since $Q = A_0 = \bigcup L^{-1}, \quad L \in W$, there exists $L$ in $W$ with $c^{-1} \in L^{-1}$. Therefore, $c^{-1}L \subseteq L^{-1}L \subseteq R$ and $L \subseteq cR \subseteq P$, but $P \subset L$. It follows that $P = \bigcap I^n$ is Goldie prime.

□

**Lemma 2.3.25** *Let $R$ be a rank one Dubrovin valuation ring. Then the group $(D(R), \circ)$ is order isomorphic to a subgroup of $(\mathbb{R}, +)$, the additive group of real numbers.*

**Proof.** We define a binary relation " $\succeq$ " on $D(R)$ by $A \succeq B$ if and only if $A \subseteq B$, for all $A,\ B \in D(R)$. Clearly, the relation " $\succeq$ " is reflexive, antisymmetric and transitive, i.e., it is a partial order on $D(R)$. Since the elements in $D(R)$ are totally ordered by inclusion, the relation " $\succeq$ " is a total order. Furthermore, $A \subseteq B$ implies $AC \subseteq BC$ and $CA \subseteq CB$ for all $A,\ B,\ C \in D(R)$. Hence, $A \subseteq B$ implies $A \circ C = (AC)^* \subseteq (BC)^* = B \circ C$ and $C \circ A = (CA)^* \subseteq (CB)^* = C \circ B$ for all $A,\ B,\ C \in D(R)$. Therefore, " $\succeq$ " is compatible with " $\circ$ " and $(D(R), \circ, \succeq)$ is a totally ordered group.

We prove that $D(R)$ is an archimedean group, i.e., for all $I,\ B \in D(R)$ with $I \subset R$, there exists an integer $n$ such that $(I^n)^* \subset B$.

Clearly, it is enough to prove that for every $I \in D(R)$, $I \subset R$ , the intersection $K = \bigcap (I^n)^* = (0)$.

Let $I \in D(R)$, $I \subset R$. Then $I \subseteq J = \mathcal{J}(R)$. If $I \subset J$, then by Theorem 2.3.24, $\bigcap I^n = (0)$, since $J \supset (0)$ is a prime segment. If $I = J \supset J^2$, then by the same argument, again $\bigcap I^n = (0)$. We are left with the case $I = J = J^2$. In this case, by Lemma 2.3.10, $J$ is not principal as a right $R$-ideal, since $O_r(J) = R$. This implies that $J^* = R$. In fact, if $J \subseteq cR$ for some $c \in U(Q)$, then $c^{-1}J \subseteq R$ and, hence, $Rc^{-1}RJ \subseteq R$. If $Rc^{-1}RJ = R$, then $(R : J)_l J = R$, i.e. $J^{-1} = (R : J)_l$. This implies

$J^{-1}J = R = O_r(J)$ and by Lemma 2.3.10, $J$ is principal as a right $R$-ideal, a contradiction. So, $Rc^{-1}RJ \subset R$, i.e. $Rc^{-1}RJ \subseteq J$ which implies $Rc^{-1}R \subseteq O_l(J) = R$, i.e. $R \subseteq cR$. This means that $I^* = J^* = R \supset J = I$, that is, $I$ is not divisorial, a contradiction. Therefore, the case $I = J = J^2$ cannot arise; so $\bigcap I^n = (0)$ holds for any $I \in D(R)$, $I \subset R$.

Now, assume that $K = \bigcap (I^n)^* \neq (0)$. First note that $K$ is divisorial, since $K \subseteq (I^n)^*$ for all $n$ implies $K^* \subseteq ((I^n)^*)^* = (I^n)^*$ for all $n$, i.e., $K^* \subseteq \bigcap (I^n)^* = K$ and $K = K^*$ follows. Hence, there exists an integer $k$, such that $K \supset I^k$. Otherwise, $K \subseteq \bigcap I^k \subseteq \bigcap (I^k)^* = K$, i.e. $(0) = K$. Now, $K^* = K \supseteq (I^k)^* \supset (I^{k+1})^*$, since if $(I^k)^* = (I^{k+1})^*$, then $I = R$, because $(D(R), \circ)$ is a group. Hence, $K \supset (I^{k+1})^* \supseteq \bigcap (I^k)^* = K$, a contradiction which shows that $K = \bigcap (I^n)^* = (0)$, i.e. $(D(R), \circ)$ is an archimedean group. By Hölders Theorem, (see for example, [Sch50], Theorem 1, Chapter 1), $D(R)$ is order isomorphic to a subgroup of $(\mathbb{R}, +)$.

□

In the case of a Dubrovin valuation ring $R$ of rank one, the group $(D(R), \circ, \succeq)$ contains a subgroup we denote by $H(R) := \{I \in D(R) \mid I = aR,\ 0 \neq a \in Q\}$, i.e. the set of all those non-zero ideals $I$ which are principal as right $R$-ideals. For, if $I = aR \in H(R)$, then $a$ is a regular element, i.e. $a$ is a unit in $Q$. Hence, $O_l(aR) = aRa^{-1}$ and since $R$ is of rank one, $aRa^{-1} = R$, i.e. $aR = Ra$. Therefore, for $aR$, $bR \in H(R)$ we have $aR \circ bR = (aRbR)^* = (abR)^* = abR \in H(R)$ and $I^{-1} = Ra^{-1} = a^{-1}R \in H(R)$.

Let $I$ be any non-zero $R$-ideal in $Q$. Then $I$ is either divisorial and then $I = I^* \in D(R)$ or $I \subset I^*$ and then by Proposition 2.3.13 (4), $I^* = aR = Ra \in D(R)$ and $I = a\mathcal{J}(R)(R)$ for some $a \in U(Q)$. Therefore, the lattice of all ideals of rank one Dubrovin valuation ring $R$ is known completely if $D(R)$ and $H(R)$ are known.

The next result describes two possibilities for an arbitrary prime ideal in a Dubrovin valuation ring $R$.

**Proposition 2.3.26** *Let $R$ be a Dubrovin valuation ring of $Q$, let $P$ be a prime ideal of $R$, and $S = O_r(P) = O_l(P)$. Then $P$ is either a Goldie prime and $P = \mathcal{J}(S)$, or $P$ is not Goldie prime and $\mathcal{J}(S)$ is the minimal Goldie prime containing $P$. Moreover, in the second case, there is no ideal of $R$ properly between $P$ and $\mathcal{J}(S)$.*

**Proof.** Since $1 + P \subseteq U(R) \subseteq U(S)$, clearly $P \subseteq \mathcal{J}(S)$.

If $P = \mathcal{J}(S)$, then $P$ is a Goldie prime, by Theorem 2.3.3.

If $P \subset \mathcal{J}(S)$, then $P$ is not a Goldie prime, since otherwise $R_P P = P$, i.e., $R_P \subseteq O_l(P) = S$, which contradicts $S \subset R_P$.

Let $I$ be an ideal of $R$ such that $P \subset I \subseteq \mathcal{J}(S)$. Then by Theorem 2.3.24, $P' := \bigcap I^n$ is a Goldie prime. Also, $P \subset P'$, since $P \subset I^n$, for all $n$. Otherwise, there exists an $n$ such that $I^n \subseteq P$ and then $I \subseteq P$, a contradiction. Furthermore, $PO_r(P') \cdot P' = PO_l(P')P' \subseteq PP' \subseteq P$. Since $PO_r(P') \subseteq P' \subset R$, clearly $PO_r(P')$ is a two sided ideal of $R$. Hence, $PO_r(P') \subseteq P$ since $P$ is a prime ideal and $P' \nsubseteq P$. So, $O_r(P') \subseteq O_r(P) = S$.

Hence, by Lemma 2.3.8,

$$P \subset P' \subseteq \mathcal{J}(S) \subset S = O_r(\mathcal{J}(S)) = O_r(P) \subseteq O_r(P') \subseteq O_r(P) = S$$

Thus, $S = O_r(P') = O_r(P) = O_r(\mathcal{J}(S))$, i.e., $P' = \mathcal{J}(S)$. Therefore, $P \subset P' = \bigcap I^n \subseteq I \subseteq \mathcal{J}(S)$. So, $I = \mathcal{J}(S)$.

□

**Corollary 2.3.27** *If $P$ is a prime ideal of a Dubrovin valuation ring $R$ that is not Goldie prime and $S = O_r(P) = O_l(P)$, then $\mathcal{J}(S) \supset P \supset \bigcap P^n$ is the "highest" prime segment in ring $S$.*

# Chapter 3

# Prime Segments

The existence of a Dubrovin valuation ring $R$ in a simple artinian ring $Q$ with a prime ideal $I$ that is not Goldie prime implies the existence of a prime segment $P_1 \supset I \supset P_2$, consisting of a pair of two distinct Goldie primes $P_1$ and $P_2$ in $R$ such that no further Goldie prime exists between $P_1$ and $P_2$ (see Example 2.3.17 and the remark after Corollary 2.3.22).

In the last decade, prime segments were defined and studied in different situations. In [BD], H.H. Brungs and N.I. Dubrovin considered prime segments in total valuation rings . They classified prime segments and in the rank one case described the structure of all ideals. Also, examples for all cases were constructed. In [BS95], H.H. Brungs and M. Schröder defined prime segments in valued skew fields and gave methods for constructing valued skew fields with prescribed types of prime segments. In [DD96], the structure of ideals and prime ideals in a cone $P$ of a right ordered group $G$ is studied. The rank one case is described completely and corresponding examples are constructed. In [BT98], H.H. Brungs and G. Törner defined right cones and classified their prime segments.

Note that prime segments correspond to jumps in ordered and right ordered groups. Let $(G, P)$ be a right ordered group with the positive cone $P$, i.e., $PP \subseteq P$,

$P\bigcap P^{-1} = \{e\}$ and $P\bigcup P^{-1} = G$. A subset $H \subset G$ is called *convex* if $x,\ y \in H$ $g \in G$, $x \le g \le y$ implies $g \in H$. The set $\Sigma$ of all convex subgroups of $G$ is a chain containing $\{e\}$ and $G$. If $C_\lambda$ ($\lambda \in \Lambda$ ) is in $\Sigma$, then $\bigcap C_\lambda$ and $\bigcup C_\lambda$ are again in $\Sigma$. Hence, every element $g \neq e$ in $G$ defines two subgroups $D,\ C \in \Sigma$ so that $C \supset D$, $g \in C \setminus D$, and $\Sigma$ contains no convex subgroups strictly between $C$ and $D$. Such a pair $C \supset D$ is called a *jump* in $\Sigma$ (see [Kop98] and [Fuc63]). In this situation there exists a $1-1$ correspondence between $\Sigma$ and the set of all completely prime ideals of the cone $P$. More precisely:

**Theorem 3.0.28** *Let $(G, P)$ be a right ordered group.*

*If $A \in \Sigma$, $A \neq G$, is a convex subgroup of $G$ then $I = P\setminus(P\bigcap A)$ is a completely prime ideal of the cone $P$.*

*Conversely, if $I \neq P$ is a completely prime ideal of the cone $P$, then $A = (P \setminus I)\bigcup(P \setminus I)^{-1}$ is a convex subgroup of $G$.*

By this correspondence, jumps in $\Sigma$ correspond to the prime segments in the cone $P$.

In this chapter, prime segments in a Dubrovin valuation ring $R$ are classified and the structure of the lattice of all ideals in the rank one case is completely described. The results obtained show that there exists a complete analogy between the ideal structure of a cone in a right-ordered group and the structure of ideals of a Dubrovin valuation ring in a simple artinian ring.

## 3.1 Prime segments in Dubrovin valuation rings

In this section, $R$ denotes a Dubrovin valuation ring in a simple artinian ring $Q$.

**Definition 3.1.1** *Let $P_1 \supset P_2$ be a prime segment of $R$.*

*The prime segment* $P_1 \supset P_2$ *is called* archimedean *if for all* $a \in P_1 \setminus P_2$ *there exists an ideal* $I \subseteq P_1$ *such that* $a \in I$ *and* $\bigcap I^n = P_2$.

*The prime segment* $P_1 \supset P_2$ *is called* simple *if there are no ideals between* $P_1$ *and* $P_2$.

*The is called* exceptional *if there exists a prime ideal* $I$ *in* $R$ *that is not Goldie prime such that* $P_1 \supset I \supset P_2$.

**Lemma 3.1.2** *Let* $P_1 \supset P_2$ *be a prime segment in* $R$.

**i)** *If* $P_1 \supset P_2$ *is exceptional and* $I$ *is a prime ideal that is not Goldie prime such that* $P_1 \supset I \supset P_2$, *then there are no ideals between* $P_1$ *and* $I$, $P_1 = P_1^2$ *and* $\bigcap I^n = P_2$;

**ii)** *If* $P_1 \supset P_1^2$ *or* $P_1 = \bigcup I$, $I \subset P_1$, *i.e.,* $P_1$ *is the union of ideals* $I$ *properly contained in* $P_1$, *then the prime segment* $P_1 \supset P_2$ *is archimedean.*

**Proof.** i) Let $L$ be an ideal of $R$ such that $P_1 \supset L \supset I$. Then by Theorem 2.3.24, $P = \bigcap L^n$ is a Goldie prime ideal of $R$ and $P_1 \supset L \supseteq P \supseteq I \supset P_2$, a contradiction. Furthermore, $P_1^2 \supset I$ since otherwise $P_1^2 \subseteq I$, which implies $P_1 \subseteq I$. Hence, $P_1 = P_1^2$. Also, $\bigcap I^n = P_2$.

ii) First, assume that $P_1 \supset P_1^2$ and let $a \in P_1 \setminus P_2$. Set $I := P_1$. Then $a \in I$ and $\bigcap I^n = \bigcap P_1^n \subseteq P_1^2 \subset P_1$ Hence, $\bigcap I^n = P_2$ since $\bigcap I^n$ is a Goldie prime.

Next, assume that $P_1 = \bigcup I$, the union of all ideals $I$ properly contained in $P_1$ and let $a \in P_1 \setminus P_2$. Then there exists an ideal $I \subset P_1$ such that $a \in I$. Then $P_1 \supset I \supseteq \bigcap I^n \supseteq P_2$, i.e $\bigcap I^n = P_2$, by the same argument as in the first case.

□

The next result shows that there are exactly three types of prime segments in a Dubrovin valuation ring $R$.

**Theorem 3.1.3** *For a prime segment $P_1 \supset P_2$ of a Dubrovin valuation ring $R$ exactly one of the following possibilities occurs:*

a) *$P_1 \supset P_2$ is archimedean;*

b) *$P_1 \supset P_2$ is simple;*

c) *$P_1 \supset P_2$ is exceptional.*

**Proof.** Let $L := \bigcup I$ be the union of all ideals $I$ of $R$ properly contained in $P_1$. If $L = P_2$, then there are no ideals between $P_1$ and $P_2$, i.e., the prime segment $P_1 \supset P_2$ is simple.

Next, we prove that the prime segment $P_1 \supset P_2$ is exceptional if and only if $P_1 \supset L \supset P_2$ and $P_1 = P_1^2$.

Let these conditions be satisfied. Then $L$ is a prime ideal of $R$. For, if $B$ and $C$ are ideals of $R$ such that $B \supset L$ and $C \supset L$ then $B \supseteq P_1$ and $C \supseteq P_1$ since otherwise $B \subset P_1$ or $C \subset P_1$, i.e., $B \subseteq L$ or $C \subseteq L$. Hence, $BC \supseteq P_1^2 = P_1 \supset L$. So, $L$ is a prime ideal that is not Goldie, i.e., the prime segment $P_1 \supset P_2$ is exceptional. The converse was proved in Lemma 3.1.2.

We are left with the case that $P_1 \supset P_1^2$ or $P_1 = \bigcup I$ for ideals $I$ of $R$ with $P_1 \supset I \supset P_2$. In both of these cases the prime segment $P_1 \supset P_2$ is archimedean, as we have shown in Lemma 3.1.2.

□

Consider the set $K(P_1) := \{a \in P_1 \mid P_1 a P_1 \subset P_1\}$. Then, for any $r \in R$ and $a \in K(P_1)$, $P_1 r a P_1 \subseteq P_1 a P_1 \subset P_1$ and $P_1 a r P_1 \subseteq P_1 a P_1 \subset P_1$, i.e., $ra,\ ar \in K(P_1)$. Also, for $a,\ b \in K(P_1)$ we have $P_1 a P_1 \subseteq P_1 b P_1$ or $P_1 b P_1 \subseteq P_1 a P_1$. In the first case, $P_1(a+b)P_1 \subseteq P_1 b P_1 \subset P_1$. Hence, $a + b \in K(P_1)$. Similarly, in the second case, $a + b \in K(P_1)$. Therefore, $K(P_1)$ is an ideal of $R$. Using the ideal $K(P_1)$, we can prove the following result:

**Corollary 3.1.4** *The prime segment $P_1 \supset P_2$ of $R$ is archimedean if and only if $K(P_1) = P_1$, it is simple if and only if $K(P_1) = P_2$ and it is exceptional if and only if $P_1 \supset K(P_1) \supset P_2$. In the last case, $K(P_1)$ is a prime ideal that is not Goldie.*

**Proof.** First, let $P_1 \supset P_2$ be archimedean. For any $a \in P_2$, $P_1aP_1 \subseteq P_2 \subset P_1$, i.e., $a \in K(P_1)$. If $a \in P_1 \setminus P_2$ then there exists an ideal $I \subset P_1$, such that $a \in I$. Hence, $P_1aP_1 \subseteq I \subset P_1$, i.e., $a \in K(P_1)$. So, $P_1 \subseteq K(P_1) \subseteq P_1$, i.e., $K(P_1) = P_1$.

Conversely, let $K(P_1) = P_1$ and assume that $P_1 \supset P_2$ is not archimedean. If $P_1 \supset P_2$ is simple and $a \in P_1 \setminus P_2$ then $P_1 \supset P_1aP_1 \supset P_2$, a contradiction. If $P_1 \supset P_2$ is exceptional, then for a prime ideal $I$ that is not Goldie such that $P_1 \supset I \supset P_2$ and $a \in P_1 \setminus I$, we have $P_1aP_1 = P_1RaRP_1 \subseteq I \subset P_1$, since there are no ideals between $P_1$ and $I$ by Lemma 3.1.2 i). But $I$ is a prime ideal of $R$. Hence, $P_1 \subseteq I$ or $RaR \subseteq I$, a contradiction which shows that $P_1 \supset P_2$ is archimedean.

Next, let $P_1 \supset P_2$ be simple and assume that there exists an element $a \in K(P_1) \setminus P_2$. Then $P_1 \supset P_1aP_1 \supset P_2$, since $P_1aP_1 = P_1RaRP_1 \subseteq P_2$ implies $P_1 \subseteq P_2$ or $a \in P_2$. But this is impossible. So, $K(P_1) = P_2$.

Conversely, let $K(P_1) = P_2$ and assume that $P_1 \supset P_2$ is not simple. If $P_1 \supset P_2$ is archimedean, then for $a \in P_1 \setminus P_2$ there exists an ideal $I \subset P_1$ with $a \in I$. Then $P_1aP_1 \subseteq I \subset P_1$, i.e., $a \in K(P_1) = P_2$, a contradiction. If $P_1 \supset P_2$ is exceptional than for a prime ideal $I$ of $R$ that is not Goldie with $P_1 \supset I \supset P_2$ and $a \in I \setminus P_2$, we have $P_1aP_1 \subseteq I \subset P_1$, i.e., $a \in K(P_1) = P_2$, a contradiction.

Finally, let $P_1 \supset P_2$ be exceptional and let $I$ be a prime ideal of $R$ that is not Goldie such that $P_1 \supset I \supset P_2$. If $a \in K(P_1) \setminus I$ then $P_1aP_1 \subseteq I \subset P_1$ . Hence, $P_1 \subseteq I$ or $a \in I$, a contradiction which shows that $I = K(P_1)$.

Conversely, let $P_1 \supset K(P_1) \supset P_2$. First, there are no ideals of $R$ between $P_1$ and $K(P_1)$. For, if $I$ was an ideal of $R$ such that $P_1 \supset I \supset K(P_1)$, then for $a \in I \setminus K(P_1)$ we would have $P_1aP_1 \subseteq I \subset P_1$, i.e., $a \in K(P_1)$. Secondly, $K(P_1)$ is a prime ideal of

$R$. For, if $B$ and $C$ are ideals of $R$ such that $B \supset K(P_1)$ and $C \supset K(P_1)$, then, as noted first, $B \supseteq P_1$ and $C \supseteq P_1$. But $P_1 = P_1^2$, since $P_1^2 \subset P_1$ implies $P_1^3 \subset P_1$, i.e., $P_1 \subseteq K(P_1)$, a contradiction. Hence, $BC \supseteq P_1^2 = P_1 \supset K(P_1)$. So, $K(P_1)$ is a prime ideal of $R$ that is not Goldie, and the prime segment $P_1 \supset P_2$ is exceptional.

□

It follows from this result that the type of the prime segment $P_1 \supset P_2$ is the same for any Dubrovin valuation ring $R$ of $Q$ that contains this prime segment.

## 3.2 Rank one and discrete Dubrovin valuation rings

Let $R$ be a rank one Dubrovin valuation ring of $Q$. Then $Q$ is the only proper overring of $R$ and we have shown, see Theorem 2.3.15 and Lemma 2.3.25, that $(D(R), \circ, \succeq)$ is a group, order isomorphic to a subgroup of $(\mathbb{R}, +)$. Also $D(R)$ contains the subgroup $H(R)$ of all nonzero ideals which are principal as right ideals. Using these facts and Theorem 3.1.3, we now proceed to give a complete description of the lattice of two sided ideals for rank one Dubrovin valuation rings.

**Theorem 3.2.1** *Let $R$ be a rank one Dubrovin valuation ring of the simple artinian ring $Q$ with maximal ideal $J = \mathcal{J}(R)$.*

*Then exactly one of the following possibilities occurs:*

**a)** *The segment $J \supset (0)$ is archimedean and either*

- **i)** *$J \supset J^2$ and then $D(R) \cong \langle J \rangle \cong H(R)$ is an infinite cyclic group; or*
- **ii)** *$J = J^2$ and then $D(R) \cong (\mathbb{R}, +)$ and $H(R)$ is a dense subgroup of $D(R)$.*

**b)** *The segment $J \supset (0)$ is simple and then $D(R) = H(R) = \{R\}$ is the trivial group.*

**c)** *The segment $J \supset (0)$ is exceptional. In this case if $I$ is a non Goldie prime of $R$ with $J \supset I \supset (0)$, then $D(R) = \langle I \rangle$ is the infinite cyclic group generated by*

*$I = I^*$ and an integer $k \geq 0$ exists with $H(R) = \langle (I^k)^* \rangle$ (the segment is said to be exceptional of type $k$).*

**Proof.** We saw in Lemma 2.3.25 that $D(R)$ is an archimedean group.

<u>CLAIM:</u> *Assume that $R$ contains a maximal divisorial ideal $I \subset R$, and let $C \subset R$ be any divisorial ideal. Then there exists an integer $n \geq 1$ such that $C = (I^n)^*$.*

Since $(D(R), \circ)$ is archimedean with the identity element $R$, there exists a minimal $n$ with $n \geq 1$ and $C \supseteq I^n$. Hence $I^{n-1} \supset C \supseteq (I^n)^*$ and $(I^{n-1})^* \supset C$. Therefore, since " $\circ$ " is compatible with "$\succeq$" we have

$$R = (I^{n-1})^* \circ (I^{-(n-1)})^* \supset C \circ (I^{-(n-1)})^* \supseteq (I^n)^* \circ (I^{-(n-1)})^* = I$$

and, by the maximality of $I$, since $C \circ (I^{-(n-1)})^* = (C(I^{-(n-1)})^*)^*$ is divisorial, we have $I = C \circ (I^{-(n-1)})^*$. Therefore, $C = (I^n)^*$ and the claim is proved.

a) Let $J \supset (0)$ be archimedean. Two cases occur, $J \supset J^2$ and $J = J^2$.

By Lemma 2.3.10, we have $J \supset J^2$ if and only if $J$ is principal as a right $R$-ideal, since $S = O_r(J) = R$, i.e., $J = aR$. Then, as noted after the proof of Lemma 2.3.25, $O_l(J) = aRa^{-1} = R$. Hence, $J = aR = Ra \subset R$ is a maximal divisorial ideal of $R$. By the claim, $J$ is a generator of the group $D(R)$; so this proves the case a) i).

Next we consider the case a) ii) where $J = J^2$ and $J \supset (0)$ is an archimedean segment. For every non-zero element $a$ in $J$ there exists therefore an ideal $I_1 \subsetneq J$ such that $a \in I_1$ and $\bigcap I_1^n = 0$. Then $RaR \subseteq I_1 \subset J$, since $I_1 = J$ would imply $J = \bigcap J^n = \bigcap I_1^n = 0$, which is a contradiction. We want to show that the ideal $I = RaR$ is a principal right $R$-ideal for any $0 \neq a$ in $J$ and hence that $I \in H(R)$.

If $I$ is not right principal, then $IJ = I$ (Lemma 2.3.10) and $a = \sum_{i=1}^{n} r_i a s_i$ with $r_i \in R$, $s_i \in J$, for all $i$. Since $R$ is a left Bezout order there exists $s$ in $R$ with $Rs_1 + \cdots + Rs_n = Rs$; hence $0 \neq s \in J$, $T_2 = RsR \subset J$ and $I = IT_2$ follows.

Since $T_1 = J$ and $O_r(T_2) = R$ is right Bezout, we can apply the left-right symmetric version of Lemma 2.3.1 to obtain a regular element $t_0$ in $T_1 = J$ such that

$T_2 = RsR \subseteq Jt_0$. Hence, $I = IT_2 \subseteq IJt_0 \subseteq IJ = I$ and $I = IJt_0 = It_0$ follows. Since $t_0$ is regular, $t_0^{-1}$ exists in $Q$. Now, $It_0^{-1} = I$; so $t_0^{-1} \in O_r(I) = R$, a contradiction since $t_0 \in J$.

This proves that $I = RaR$ is in $H(R)$ for any $0 \neq a \in J$. Finally, for every $RaR \subset J$ there exist $b \in J \setminus RaR$ and $RaR \subset RbR \subset J$ follows; $H(R)$ and $D(R)$ are therefore isomorphic to dense subgroups of $(\mathbb{R}, +)$. Since the intersection $K = \bigcap I_i$ of divisorial ideals of $R$ is divisorial if $K \neq (0)$, $D(R)$ is also complete and $D(R) \cong (\mathbb{R}, +)$ follows.

The assertion in case b) is trivial.

It remains to consider the case c) where $J \supset I \supset (0)$ is an exceptional prime segment. In this case, $J = J^2$ is not principal as a right $R$-ideal, and as in the proof of Lemma 2.3.25, $J^* = R$, i.e $J$ is not divisorial. But $I^* = I$, since otherwise $I^* \supset I$, $J \supset I$, yet $I = I^*J$ by Proposition 2.3.13 (4), which is a contradiction for the prime ideal $I$. Hence, $I$ is a maximal divisorial ideal (there are no ideals of $R$ between $J$ and $I$). Therefore, $D(R) = \langle I \rangle$ by the claim and $H(R)$ is then equal to $\langle (I^k)^* \rangle$ for some $k \geq 0$.

□

The exceptional case is the most interesting. It splits into countably many subcases, depending on the integer $k \geq 0$. We give chains of all ideals in every case.

If $k = 0$, then $H(R) = \{R\}$, i.e., there are no principal right ideals. If $L$ is any ideal of $R$, then either $L = L^* \in D(R) = \langle I \rangle$ or $L \subset L^*$ and then $L^*$ is a principal right ideal of $R$, i.e., $L^* \in H(R)$. Hence, $L^* = R$ and $L = J$. Therefore, the proper ideals of $R$ are $J$, $(0)$ and powers of $I$, i.e,

$$R \supset J \supset I \supset I^2 \supset \cdots \supset (0)$$

is the chain of all ideals of $R$.

If $k = 1$, then $H(R) = \langle I \rangle = D(R)$. Hence, $I^* = I = aR = Ra$ is principal, $(I^n)^* = aR \circ aR \circ \cdots \circ aR = (aRaR \cdots aR)^* = a^n R = Ra^n$ and

$$R \supset J \supset aR \supset aJ \supset a^2 R \supset a^2 J \supset \cdots \supset (0)$$

is the chain of all ideals of $R$.

If $k > 1$, then $H(R) = \langle (I^k)^* \rangle$. Hence, $(I^k)^* = aR = Ra$ and $I^k = aJ$. This follows, since $I^k \subset (I^k)^*$. Otherwise, $I^k = (I^k)^* = aR$ and then $I^k J = aRJ = aJ \subset aR = I^k$. Hence, $IJ \subset I$. By Lemma 2.3.10, $I$ is a principal right $R$-ideal and the contradiction $k = 1$ follows. In the case $k > 1$, the chain of all ideals of $R$ is therefore:

$$R \supset J \supset I \supset I^2 \supset \cdots \supset I^{k-1} \supset aR \supset aJ = I^k$$

$$\supset I^{k+1} \supset \cdots \supset I^{2k-1} \supset a^2 R \supset a^2 J = I^{2k} \supset \cdots \supset (0).$$

To analyze the prime segment of a rank one Dubrovin valuation ring $R$ in a simple artinian ring $Q$ with finite dimension over its center $K$, the following result is needed:

**Lemma 3.2.2 ( [MMU97], Lemma 7.9 Chapter II)** *Let $R$ be a Dubrovin valuation ring in a simple artinian ring $Q$ with finite dimension over its center $K$. If $\mathcal{J}(R)$ is finitely generated as an ideal, then $\mathcal{J}(R) \neq \mathcal{J}(R)^2$.*

The next result shows that in the finite dimensional case the prime segment of a rank one Dubrovin valuation ring $R$ is archimedean.

**Lemma 3.2.3** *If $R$ is a rank one Dubrovin valuation ring in a simple artinian ring $Q$ with finite dimension over its center $K$, then the prime segment $\mathcal{J}(R) \supset (0)$ must be archimedean.*

**Proof.** By Remark 2.3.7, every prime ideal of $R$ is Goldie. Hence, $\mathcal{J}(R) \supset (0)$ is not exceptional.

Assume that the prime segment $\mathcal{J}(R) \supset (0)$ is simple.

<u>**Case 1.** : *$\mathcal{J}(R)$ is finitely generated as an ideal*</u>

Then, by Lemma 3.2.2, $\mathcal{J}(R) \supset \mathcal{J}(R)^2 \supseteq (0)$. But, by Lemma 2.3.10, $\mathcal{J}(R) = aR = Ra$, since $O_l(aR) = aRa^{-1} = R$. Hence, if $\mathcal{J}(R)^2 = (0)$, then $a^2R = (0)$, i.e., $R = (0)$. This shows that $\mathcal{J}(R) \supset \mathcal{J}(R)^2 \supset (0)$, a contradiction.

<u>**Case 2.** : *$\mathcal{J}(R)$ is not finitely generated as an ideal*</u>

Let $0 \neq x \in \mathcal{J}(R)$. Then, $RxR \neq (0)$ and $\mathcal{J}(R) \supset RxR \supset (0)$, since otherwise $RxR = \mathcal{J}(R)$ is finitely generated as an ideal.

□

To conclude this section we briefly discuss discrete Dubrovin valuation rings.

**Definition 3.2.4** *A Dubrovin valuation ring $R$ in a simple artinian ring $Q$ is called discrete if $R$ is of rank one and $J \neq J^2$.*

Let $R$ be a discrete Dubrovin valuation ring of $Q$. By Theorem 3.2.1, the prime segment $J \supset (0)$ is archimedean with $J \neq J^2$, $D(R) \cong \langle J \rangle \cong H(R)$, and $\bigcap J^n = (0)$.

For, by Lemma 2.3.10, $J$ is a principal as a right $R$-ideal, i.e., $J = aR = Ra$ for some regular element $a \in J$. The prime segment $J \supset (0)$ is not simple, since otherwise $J^2 = a^2R = (0)$, i.e., $R = (0)$. Also, $J \supset (0)$ is not exceptional; for otherwise $J = J^2$ by Lemma 3.1.2.

Hence, when $Q$ is a commutative field, then Definition 3.2.4 coincides with the definition of discrete valuation domains in a field, see for example, [Bou72], Chapter VI. Note that a discrete Dubrovin valuation ring $R$ is a non-artinian ring, since otherwise, by Lemma 1.1.2, every regular element $r \in R$ is a unit in $R$, i.e., $R = Q$, and the contradiction $J = J^2 = (0)$ follows.

## 3.3 Examples

In this section, we discus some examples which show that all cases in Theorem 3.2.1 can be realized.

**Example 3.3.1** *There exists a Dubrovin valuation ring of rank one with an archimedean prime segment $J \supset (0)$ of the type described in Theorem 3.2.1, case a) i).*

**Proof.** It follows from the remark after Definition 3.2.4, that any discrete Dubrovin valuation ring $R$ provides an example to illustrate the case $a)$ $i)$. In particular, the ring $R = M_2(\mathbb{Z}_{(p)})$, where $p$ is a prime, is an example of a discrete Dubrovin valuation ring in the simple artinian ring $Q = M_2(\mathbb{Q})$. Also, any discrete rank one commutative valuation domain $R$, like $\mathbb{Z}_{(p)}$ or the power series ring $K[[X]]$ over a field $K$, is an example for the case $a)$ $i)$.

□

**Example 3.3.2 ( [BS95])** *There exists a valuation domain $V$ of rank one with quotient field $F$ whose associated valuation has the value group $H$, a dense subgroup in $(\mathbb{R}, +)$, illustrating the case a) ii) in Theorem 3.2.1.*

**Proof.** Krull's construction from 1932 of a commutative valuation domain with given commutative ordered group as the value group of the associated valuation, can be used to illustrate the case $a)$ $ii)$. We take $(H, H^+)$ to be any dense subgroup of $(\mathbb{R}, +)$, where $H^+$, is the positive cone of $H$ and as in Example 2.1.1, we construct a commutative valuation domain $V$ as the localization of the subring $KH^+$ of the group ring $KH$ over a field $K$ at the multiplicatively closed set $S = \{\sum a_h h \in KH^+ \mid a_0 \neq 0\}$, i.e., $V = (KH^+)S^{-1}$. The set of all principal ideals of $V$ is exactly the set $\{hV \mid h \in H^+\}$, and $h_1 V = h_2 V$ if and only if $h_1 = h_2$. Then $V$ is a valuation domain of rank one in the quotient field $F = Q(KH)$, and the associated valuation on $F$ has

$H$ as its value group. Since $H$ is dense in $\mathbb{R}$, $V$ is an example that illustrates the case $a)$ $ii)$ in Theorem 3.2.1.

□

**Example 3.3.3** ( [BS95]) *There exists a Dubrovin valuation ring $R$ of rank one with a simple prime segment.*

**Proof.** In order to construct a total valuation ring $R$ of rank one with a simple prime segment, we consider the dense subgroup $H = \mathbb{Q}$ of $(\mathbb{R}, +)$ and the commutative valuation domain $V = (FH^+)S^{-1}$ constructed in Example 3.3.2, where $F$ is a field. We denote by $K = Q(FH)$ the field of quotients of the group ring $FH$. Then $\sigma : K \longrightarrow K$, defined by $\sigma(h) = 2h$ for $h \in H$, is an automorphism of $K$ compatible with the valued field $(K, V)$. Hence, $\sigma$ induces an automorphism $\sigma : V \longrightarrow V$ with $\sigma(h) = 2h$, $h \in H$.

Now, consider the skew-polynomial rings $T := V[x, \sigma] \subset K[x, \sigma]$. Then, $K[x, \sigma]$ is a domain with a left and right division algorithm. This implies that $K[x, \sigma]$ is a left and right PID and hence a noetherian domain. Therefore, $K[x, \sigma]$ is a left and right Ore domain. So, there exist the classical ring of quotients $D := Q(K[x, \sigma])$.

Consider the set $S = \{\sum x^i a_i \mid a_i \in V, \text{ at least one } a_i \text{ is a unit in } V\}$. Then, $S$ is multiplicatively closed and $S$ satisfies the Ore conditions. Therefore, $R = TS^{-1} = \{ts^{-1} \mid t \in T, s \in S\}$ is a subring of $D = Q(K[x, \sigma])$. Furthermore, $R = V[x, \sigma]S^{-1}$ is a rank one total valuation ring in a division ring $D$. Its nonzero principal right ideals have the form $hR$, $h \in H^+ = \mathbb{Q}^+$. However, $xh = x\sigma(\frac{h}{2}) = \frac{h}{2}x$, i.e., $xhR = \frac{h}{2}R \supset hR$. This shows that $J(R) \supset (0)$ is a simple prime segment.

□

**Example 3.3.4** ( [Dub93], [BD]) *For every nonnegative integer $k$ there exits a rank one total valuation ring whose prime segment is exceptional of type $k$.*

**Proof.** The skew field of quotients of the group ring $FG$ of the covering group $G$ over a skew field $F$ of the group SL(2,$\mathbb{R}$) contains total valuation rings with exceptional prime segments of type $k$, for each integer $k \geq 0$. For $k = 1$, the construction is described, with more details, in Example 2.3.17. In fact, for all $k = 0, 1, 2, \ldots$, the group $G$ contains certain subgroups $H_k$ with the exceptional positive cone $P_k$ of type corresponding to $k$. The cone $P_k$ satisfies the conditions in Theorem 1.3.3 and hence, there exists a total valuation ring $S_k$ associated with $P_k$. Now, applying Theorem 1.3.1 to the ring $S_k$, it follows that $S_k$ is a rank one total valuation ring with the exceptional prime segment of type $k$. Note, that for this construction, Dubrovin's results from [Dub93] are needed. No easy construction is known.

□

Some results of Chapters 2 and 3 will appear in [BMO].

# Chapter 4

# Krull Rings

For a commutative Krull ring, the role that the group of divisorial ideals of a rank one Dubrovin valuation ring plays in the previous chapters, is played by the group of divisors of regular fractional ideals. In this chapter, we discuss commutative and non-commutative Krull rings. We use the group of divisors of a commutative Krull ring $R$ with the total quotient ring $K$, $R \neq K$, to prove an approximation theorem. A version of this chapter has been published, [Osm99].

## 4.1 Krull rings

Commutative Prüfer domains play a central role in the study of classical commutative integral domains. A commutative domain $R$ is called *Prüfer domain* if every non-zero finitely generated ideal of $R$ is invertible. Clearly, commutative PIDs and commutative Bezout domains are Prüfer domains. The classical ideal theory of commutative domains has been extended to commutative rings with zero divisors. M. Griffin [Gri69] introduced the notion of commutative Prüfer rings with zero divisors and gave many characterizations of that class of rings.

**Definition 4.1.1** *A commutative ring $R$ is called Prüfer ring if every regular finitely generated ideal of $R$ is invertible.*

**Theorem 4.1.2 ( [Gri69], [LM71])** *Let $R$ be a commutative ring, and let $K$ be the total quotient ring of $R$. Then the following statements are equivalent:*

1. *$R$ is a Prüfer ring.*
2. *For every regular prime ideal $P$ of $R$, the pair $(R_{[P]}, [P]R_{[P]})$ is a Manis valuation pair of $K$.*
3. *For every regular maximal ideal $M$ of $R$, the pair $(R_{[M]}, [M]R_{[M]})$ is a Manis valuation pair of $K$.*
4. *$R$ is integrally closed and and for all $a,\ b \in R$, where at least one of $a$ and $b$ is regular, $(a,b)^n \subseteq (a^n, b^n)$.*

In the non-commutative case, Prüfer rings are defined and studied in [AD90] and [MMU97].

**Definition 4.1.3 ( [AD90])** *A prime Goldie ring $R$ is called a right Prüfer ring if every finitely generated right $R-$ideal $I$ of $R$ satisfies the equalities*

$$I^{-1}I = R, \quad II^{-1} = O_l(I).$$

*A prime Goldie ring $R$ is called a left Prüfer ring if every finitely generated left $R-$ideal $J$ of $R$ satisfies the equalities*

$$J^{-1}J = O_r(J), \quad JJ^{-1} = R.$$

It has been shown in [AD90] that a prime Goldie ring $R$ is left Prüfer if and only if it is right Prüfer. Also, every Dubrovin valuation ring $R$ in a simple artinian ring

$Q$ is Prüfer ring. For the theory of non-commutative Prüfer rings we refer to [AD90] and [MMU97].

The notion of commutative Dedekind domains also has been extended to commutative rings with zero divisors.

**Definition 4.1.4** *A commutative ring $R$ is called Dedekind ring if it is Prüfer and r-noetherian.*

The complete proof of the following characterization can be found in [AO90a].

**Theorem 4.1.5 ( [AO90a])** *For a commutative ring $R$ the following conditions are equivalent:*

1. *All regular prime ideals of $R$ are invertible.*
2. *$R$ is a Dedekind ring.*
3. *The semigroup of regular fractional ideals of $R$ is a group.*
4. *Every regular ideal of $R$ is a product of prime ideals.*
5. *Every regular element of $R$ is contained only in a finite number of prime ideals of $R$, and the semigroup of regular ideals can be embedded in a direct product of ordered cyclic groups*

Recall that an $R$-ideal $I$ of a Dubrovin valuation ring $R$ in a simple artinian ring $Q$ is called divisorial if $I = I^*$, where $I^* = \bigcap cS$, $S = O_r(I)$ and $c$ runs over all elements in $Q$ with $cS \supseteq I$ (Definition 2.3.11). In the case of a rank one Dubrovin valuation ring $R$ we have the following result:

**Proposition 4.1.6** *Let $R$ be a rank one Dubrovin valuation ring of $Q$ and let $I$ be an $R$-ideal. Then*

$$I^* = (R : (R : I)_l)_r.$$

**Proof.** Since the only proper overring of $R$ is $Q$, we have $S = O_r(I) = R$. So, $I^* = \bigcap cR$, where $cR \supseteq I$.

Let $s \in (R : (R : I)_l)_r$ and $c \in U(Q)$ be such that $cR \supseteq I$. Then, $c^{-1}I \subseteq R$, i.e., $c^{-1} \in (R : I)_l$. Hence, $c^{-1}s \in R$, i.e., $s \in cR$. Therefore, $s \in I^*$ and $(R : (R : I)_l)_r \subseteq I^*$ follows.

Assume that $s \in I^*$ and $s \notin (R : (R : I)_l)_r$. Then $(R : I)_l s \nsubseteq R$, and hence there exists a regular element $c \in (R : I)_l$ such that $cs \notin R$.

Otherwise, for all regular elements $c \in (R : I)_l$, we have $cs \in R$. But, by Lemma 1.1.8, $(R : I)_l$ is a left $R$–ideal. Then, by Definition 1.1.7, there exists $c \in U(Q)$ with $(R : I)_l c \subseteq R$. Since $(R : I)_l \bigcap U(Q) \neq \emptyset$ (also by Definition 1.1.7), $(R : I)_l c$ contains a regular element of $R$ and is therefore an essential left ideal of $R$. Thus, by Theorem 1.1.5, $(R : I)_l c$ is generated by a set $\{c_i\}$ of regular elements; so, as a left $R$-submodule of $Q$, $(R : I)_l$ is generated by the set $\{c_i c^{-1}\}$ of regular elements. So, $(R : I)_l s \subseteq R$, a contradiction.

Then, $s \notin c^{-1}R$. On the other hand, $c \in (R : I)_l$ implies $cI \subseteq R$ and $I \subseteq c^{-1}R$. Since $s \in I^*$, this implies $s \in c^{-1}R$.

□

**Remark 4.1.7** *Proposition 4.1.6 holds in the more general situation when $R$ is an order in a simple artinian ring $Q$ and for an $R$-ideal $I$, the $*$-operation is defined by $I^* = \bigcap cR$, where $cR \supseteq I$.*

In the commutative case, an analogous operation is defined in the following way. Let $R$ be a commutative ring; we denote by $K$ the total quotient ring of $R$ and by $F(R)$ the set of all regular fractional ideals of $R$. If $A, B \in F(R)$, then $(A : B) = \{x \in K \mid xB \subseteq A\}$ is also in $F(R)$. On the set $F(R)$ we define an equivalence relation $\sim$ by: $A \sim B \Leftrightarrow (R : A) = (R : B)$. The set of all equivalence classes is denoted by $D(R)$, and the class containing $A \in F(R)$ is denoted by $\operatorname{div}(A)$, the *divisor* of $A$. If

$A \in F(R)$ we define $\bar{A}$ to be the set $(R : (R : A)) \in F(R)$. A regular fractional ideal $A$ of $R$ is *divisorial* if $A = \bar{A}$.

**Remark 4.1.8** *If $R$ is a commutative integral domain, then $\bar{A}$ is equal to the intersection of all nonzero principal fractional ideals containing $A$. This is no longer true if $R$ is a commutative ring with zero divisors, see* [AM85].

However, we still have a satisfactory theory of divisors which is similar to the theory of divisors for integral domains. For $A, B \in F(R)$ we define $\mathrm{div}(A) + \mathrm{div}(B) = \mathrm{div}(AB)$. Under this operation, $D(R)$ is a commutative semigroup with identity element $\mathrm{div}(R)$. We also define a relation $\leq$ on $D(R)$ by setting $\mathrm{div}(A) \leq \mathrm{div}(B)$ if and only if $\overline{B} \subseteq \overline{A}$. Then $D(R)$ is a partially ordered semigroup. $D(R)$ is a group if and only if $R$ is completely integrally closed, see [Huc88], Th.8.4.

R.Kennedy [Ken73] introduced the class of commutative Krull rings with zero divisors and developed their theory of divisors.

**Definition 4.1.9 ( [Ken73])** *Let $R$ be a commutative ring with the total quotient ring $K$ such that $R \neq K$. Then $R$ is called a Krull ring if there exists a family $\{(V_\alpha, P_\alpha) \mid \alpha \in I\}$ of discrete rank one valuation pairs of $K$ with associated valuations $\{v_\alpha \mid \alpha \in I\}$ such that*

**(K1)** $R = \bigcap V_\alpha$;

**(K2)** $v_\alpha(a) = 0$ *almost everywhere on $I$ for each regular element $a \in K$ and each $P_\alpha$ is a regular ideal of $V_\alpha$.*

The following theorem provides the main characterization for commutative Krull rings with zero divisors.

**Theorem 4.1.10 ( [Ken73], Conjec. 3.26 and [Mat82], Th 5.)** *Let $R$ be a commutative ring which is not equal to its total quotient ring. Then $R$ is a Krull* *ring if and only if R is completely integrally closed and each nonempty collection of divisorial ideals of R has a maximal element.*

Note that if $R$ is an integral domain different from its quotient field, then $R$ is a Krull domain if and only if $R$ is a Krull ring.

**Corollary 4.1.11** *Every commutative Dedekind ring is a Krull ring.*

**Proof.** If $R$ is a Dedekind ring, then $R$ is $r$-noetherian and Prüfer. By Theorem 4.1.2, $R$ is integrally closed, and hence, by Proposition 1.1.14, $R$ is completely integrally closed. So, by Theorem 4.1.10, $R$ is a Krull ring.

□

We mention now some results which show that commutative Krull rings with zero divisors not only share many common properties with Krull integral domains but also show that there are differences. The following result is an immediate consequence of Proposition 1.1.14 and Theorem 4.1.10.

**Proposition 4.1.12 ( [AO90b], Prop.2.3.)** *Let $R$ be an $r$-noetherian ring that is distinct from its total quotient ring. Then $R$ is a Krull ring if and only if $R$ is integrally closed.*

**Proposition 4.1.13 ( [Bou72], Prop13, p. 488)** *Let $R$ be a Krull domain and $x_1, x_2, \ldots, x_n$ be indeterminates. Then the ring $R[x_1, x_2, \ldots, x_n]$ is a Krull ring.*

Note, that in the case of rings with zero divisors this is not true. For example, the ring $R = \mathbb{Z} \bigoplus \mathbb{Z}/\mathbb{Z}p^n$, where $p$ is a prime and $n \geq 2$, is a Krull ring, but $R[X]$ is not a Krull ring since it is not completely integrally closed.

Recall that a commutative ring $R$ is called a *Marot ring* or a ring with the *property (P)* if every regular ideal is generated by regular elements.

**Example 4.1.14** *A Krull ring R which is not Marot.*

**Proof.** Let $D = \mathbb{Z}[\sqrt{-5}]$. Then $D$ is a Dedekind domain with a maximal ideal $M = (1+\sqrt{-5})\mathbb{Z} + 2\mathbb{Z}$ which is not principal but $M^2 = 2D$ is principal. Let $A$ be the $D$-module $\bigoplus\{D/Q \mid Q$ a maximal ideal of $D$ , $Q \neq M\}$, and let $R$ be the ring $D(+)A$, the idealization of the $D$-module $A$, i.e., $(d,a)+(d',a') = (d+d', a+a')$ and $(d,a)(d',a') = (dd', ad' + a'd)$ for all $a$, $a' \in A$, $d, d' \in D$. For more details see, for example, [Huc88]. Then:

1. The ideal $P = M(+)A$ of $R$ is not principal.

2. $P$ is the unique regular prime ideal of $R$.

3. The set of all regular ideals of $R$ is the set of the form $\{P^n\}_{n=0}^{\infty}$.

4. The ideal $P$ is invertible.

5. The set of all regularly generated ideals of $R$ is $\{t^n R\}_{n=0}^{\infty}$, for some $t \in R$.

Hence, by Theorem 4.1.5 and Corollary 4.1.11, $R$ is a Krull ring. But, the ideal $P$ is a regular ideal which is not generated by regular elements since $P$ is not a principal ideal and the set of all regularly generated ideals of $R$ contains only principal ideals. Hence, $R$ is not a Marot ring.

Finally, at the end of this section we briefly discuss how discrete Dubrovin valuation rings are used in studying non-commutative Krull rings.

Non-commutative Krull rings have been defined and studied by H. Marubayashi, [Mar75], [Mar76], [Mar78], M. Chamarie, [Cha81] and N. Dubrovin, [Dub91b]. In [Dub91b], Dubrovin uses discrete Dubrovin valuation rings in a simple artinian ring to define non-commutative Krull rings.

**Definition 4.1.15 ( [Dub91b])** *A subring $R$ in a simple artinian ring $Q$ is called a non-commutative Krull ring if there exists a family $\{R_i \mid i \in \Lambda\}$ of discrete Dubrovin valuation rings of $Q$ such that*

**(K1)** $R = \bigcap R_i$;

**(K2)** *for each regular element $q \in Q$, the equation $qR_i = R_i$ holds for almost all $i$ in $\Lambda$;*

**(K3)** *for any finite set $\{i_0, i_1, \cdots, i_n\}$ in $\Lambda$, there exits an element $q \in Q$ such that $q \equiv 1 \pmod{\mathcal{J}(R_{i_0})}$, $q \equiv 0 \pmod{\mathcal{J}(R_{i_t})}$ for $1 \leq t \leq n$, and $q \in R_j$ for all other $j \in \Lambda$.*

It follows from the results in [Dub91b] that the class of non-commutative Krull rings defined in Definition 4.1.15 coincides with the class of non-commutative Krull rings defined in [Mar75], where valuation rings are not used.

## 4.2 Essential valuations of Krull rings

We denote by $R$ a commutative Krull ring with total quotient ring $K$. Then $D(R)$ is a group. Let $M(R)$ be the set of all maximal divisorial integral ideals of $R$. If $P \in M(R)$ then $\text{div}(P)$ is a minimal positive element of the group $D(R)$. We denote the set $\{\text{div}(P) \mid P \in M(R)\}$ by $P(R)$. Each divisor $\text{div}(A) \in D(R)$ can be written uniquely as:

$$\sum\{n_{\mathfrak{P}}\mathfrak{P} \mid \mathfrak{P} \in P(R)\},\ n_{\mathfrak{P}} \in \mathbb{Z},\ n_{\mathfrak{P}} = 0 \text{ for almost all } \mathfrak{P} \text{ in } P(R).$$

If $\text{div}(A) = \sum n_{\mathfrak{P}}\mathfrak{P}$ and $\text{div}(B) = \sum m_{\mathfrak{P}}\mathfrak{P}$, then $\text{div}(A) \leq \text{div}(B)$ if and only if $n_{\mathfrak{P}} \leq m_{\mathfrak{P}}$ for all $\mathfrak{P} \in P(R)$.

For each divisor $\mathfrak{P} \in P(R)$ we define a map from $F(R)$ to the set of integers $\mathbb{Z}$ by $v_{\mathfrak{P}}(A) = n_{\mathfrak{P}}$, where $\text{div}(A) = \sum n_{\mathfrak{Q}} \cdot \mathfrak{Q}$. The map $v_{\mathfrak{P}}$ has the following properties:

a) $v_{\mathfrak{P}}(AB) = v_{\mathfrak{P}}(A) + v_{\mathfrak{P}}(B)$ for all $A$, $B \in F(R)$;

b) $v_{\mathfrak{P}}(A + B) = \min\{v_{\mathfrak{P}}(A), v_{\mathfrak{P}}(B)\}$ for all $A$, $B \in F(R)$;

c) $A \subseteq B$ is equivalent to $v_{\mathfrak{P}}(A) \geq v_{\mathfrak{P}}(B)$ for all $\mathfrak{P}$ in $P(R)$, where $A$, $B \in F(R)$ and $B = \overline{B}$.

The family of the valuations $\{v_{\mathfrak{P}} : \mathfrak{P} \in P(R)\}$ defined by:

$$(\forall\, x \in K)\ v_{\mathfrak{P}}(x) = \sup\ \{v_{\mathfrak{P}}(Rx + A) \mid A \in F(R)\}$$

is said to be the family of *essential valuations* of a Krull ring $R$. The ring of the valuation $v_{\mathfrak{P}}$ is denoted by $V_{\mathfrak{P}}$ and the positive ideal by $P_{\mathfrak{P}}$. The essential valuations of a Krull ring $R$ are discrete of rank one and they define the ring $R$, i.e., $R = \bigcap\{V_{\mathfrak{P}} \mid \mathfrak{P} \in P(R)\}$ and $v_{\mathfrak{P}}(a) = 0$ for almost all $\mathfrak{P}$ in $P(R)$, for all $a \in \mathcal{C}_K(0)$, see [Mat82].

Essential valuations of a Krull ring have the following properties:

**Proposition 4.2.1 ( [AO90b], Prop.2.5)** *If $x \in \mathcal{C}_K(0)$ and $\mathfrak{P} \in P(R)$, then $v_{\mathfrak{P}}(x) = v_{\mathfrak{P}}(Rx)$.*

**Proposition 4.2.2 ( [AO90b], Prop.2.6)** *Let $A$ be a divisorial ideal of $R$, and $\operatorname{div}(A) = \sum\{n_{\mathfrak{P}}\mathfrak{P} \mid \mathfrak{P} \in P(R)\}$. Then $a \in A$ if and only if $v_{\mathfrak{P}}(a) \geq n_{\mathfrak{P}}$.*

**Proposition 4.2.3 ( [AO90b], Prop.2.7)** *Let $n_{\mathfrak{P}} \in \mathbb{Z}$ with almost all $n_{\mathfrak{P}} = 0$ for $\mathfrak{P} \in P(R)$, and set $A = \{x \in K \mid v_{\mathfrak{P}}(x) \geq n_{\mathfrak{P}}\ (\forall \mathfrak{P} \in P(R))\}$. Then $A$ is a regular divisorial ideal, and $\operatorname{div}(A) = \sum\{n_{\mathfrak{P}}\mathfrak{P} \mid \mathfrak{P} \in P(R)\}$.*

**Proposition 4.2.4 ( [AO90b], Prop.2.8)** *Let $\mathfrak{P} = \operatorname{div}(P) \in P(R)$, where $P$ is a maximal divisorial ideal of $R$. Then $P$ is a minimal regular prime (r-prime) ideal of $R$. The valuation ring of $v_{\mathfrak{P}}$ is equal to $R_{[P]}$, and the positive ideal of $v_{\mathfrak{P}}$ is $[P]R_{[P]}$.*

**Corollary 4.2.5 ( [AO90b], Cor.2.9)** *The family $\{P \mid \operatorname{div}(P) \in P(R)\}$ of all maximal divisorial ideals of $R$ is precisely the family of all minimal regular prime ideals of $R$.*

## 4.3 An approximation theorem for Krull rings

Approximation theorems are very common in many areas of mathematics, for example, in the valuation theory of fields and commutative rings, in multiplicative ideal theory and in the theory of partially ordered abelian groups. They are used as a tool in studying topological structures and play an important role in the description of the division class group. The Chinese Remainder Theorem is the oldest form of an approximation theorem for congruences. In this section we prove an approximation theorem for essential valuations of commutative Krull rings with zero divisors. Throughout, rings are assumed to be commutative with identity 1.

**Theorem 4.3.1** *Let $R$ be a commutative Krull ring with total quotient ring $K$ such that $R \neq K$ and $\{v_{\mathfrak{P}} \mid \mathfrak{P} \in P(R)\}$ is the family of essential valuations of $R$. If $\{v_{\mathfrak{P}_1}, v_{\mathfrak{P}_2}, \ldots, v_{\mathfrak{P}_n}\}$ is a finite set of essential valuations of the ring $R$ and $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, then there exists an element $x \in K$ such that $v_{\mathfrak{P}_i}(x) = m_i$ for all $i \in \{1, 2, \ldots, n\}$ and $v_{\mathfrak{P}}(x) \geq 0$ for all other $\mathfrak{P} \in P(R)$.*

**Proof.** It suffices to prove the theorem in the case where at most one of the $m_i$ is not equal to 0.

Let $m_2 = m_3 = \cdots = m_n = 0$. If $m_1 = 0$ then $x = 1$.

**Case 1.** Let $m_1 > 0$ and $\mathfrak{P}_i = \operatorname{div} P_i$, $i = 1, 2, \ldots, n$ be minimal positive elements of the group $D(R)$, where the $P_i$ are maximal divisorial integral ideals of $R$. By Proposition 4.2.4, $P_i$ is a minimal regular prime ideal of $R$, $i \in \{1, 2, \ldots, n\}$. Then $\overline{P_1^{m_1}}$ and $\overline{P_1^{m_1+1}}$ are divisorial integral ideals of $R$. Also:

$$\overline{P_1^{m_1}} \not\subseteq \overline{P_1^{m_1+1}} \bigcup P_2 \bigcup P_3 \bigcup \cdots \bigcup P_n.$$

Otherwise, by [Bou72, Proposition 2, Ch.II, §1], $\overline{P_1^{m_1}} \subseteq \overline{P_1^{m_1+1}}$ or $\overline{P_1^{m_1}} \subseteq P_i$, for some $i \in \{2, 3, \ldots, n\}$. But then $\operatorname{div}(P_1) \leq 0$ or $P_1 = P_i$, which is impossible.

Let $x \in \overline{P_1^{m_1}} \setminus (\overline{P_1^{m_1+1}} \bigcup P_2 \bigcup \cdots \bigcup P_n)$. Then $Rx + \overline{P_1^{m_1+1}} \subseteq \overline{P_1^{m_1}}$. Hence, by the property c) of the map $v_{\mathfrak{P}}$ :

$$(\forall\, \mathfrak{P} \in P(R))\ v_{\mathfrak{P}}(Rx + \overline{P_1^{m_1+1}}) \geq v_{\mathfrak{P}}(\overline{P_1^{m_1}}).$$

In particular:

$$v_{\mathfrak{P}_1}(x) = \sup\{v_{\mathfrak{P}_1}(Rx + L) \mid L \in F(R)\} \geq v_{\mathfrak{P}_1}(Rx + \overline{P_1^{m_1+1}}) \geq v_{\mathfrak{P}_1}(P_1^{m_1}) = m_1.$$

Assume that $v_{\mathfrak{P}_1}(x) \geq m_1 + 1$. Then, there exists an element $L \in F(R)$ such that $v_{\mathfrak{P}_1}(Rx + L) \geq m_1 + 1$. We can assume that $L \subseteq R$. For, there exists a regular element $d \in R$ such that $dL \subseteq R$. Then $v_{\mathfrak{P}}(Rx + L) \geq v_{\mathfrak{P}}(\overline{P_1^{m_1+1}})$ for all $\mathfrak{P} \in P(R)$, since $Rx + L \subseteq R$ implies $v_{\mathfrak{P}}(Rx + L) \geq 0 = v_{\mathfrak{P}}(\overline{P_1^{m_1+1}})$ for all $\mathfrak{P} \neq \mathfrak{P}_1$. By the property c) of the map $v_{\mathfrak{P}}$, we have $Rx + L \subseteq \overline{P_1^{m_1+1}}$, i.e., $x \in \overline{P_1^{m_1+1}}$, a contradiction.

Therefore, $v_{\mathfrak{P}_1}(x) < m_1 + 1$, i.e., $v_{\mathfrak{P}_1}(x) = m_1$.

It remains to prove that $v_{\mathfrak{P}_i}(x) = 0 = m_i$ for all $i \in \{2, 3, \ldots, n\}$. Since $x \in \overline{P_1^{m_1}} \subset R = \bigcap V_{\mathfrak{P}} \subseteq V_{\mathfrak{P}}$ for all $\mathfrak{P} \in P(R)$, $v_{\mathfrak{P}}(x) \geq 0$ for all $\mathfrak{P} \in P(R)$. Assume that $v_{\mathfrak{P}_i}(x) > 0$ for some $i \in \{2, 3, \ldots, n\}$. Then $x$ belongs to $[P_i]R_{[P_i]}$, by Proposition 4.2.4. Hence, $x \in P_i$, which is a contradiction.

**Case 2.** Let $m_1 < 0$ and let $r \in \operatorname{Reg}(P_1^{-m_1})$. By Proposition 4.2.1, $v_{\mathfrak{P}_1}(r) = v_{\mathfrak{P}_1}(Rr) \geq -m_1 > 0$. Then $v_{\mathfrak{P}_1}(r^{-1}) < 0$. We prove that the set $\{\mathfrak{P} \in P(R) \mid v_{\mathfrak{P}}(r^{-1}) < 0\}$ is finite. If not, then the divisor of $Rr^{-1} \in F(R)$ in its decomposition $\operatorname{div}(Rr^{-1}) = \sum_{\mathfrak{P} \in P(R)} (v_{\mathfrak{P}}(Rr^{-1}))\mathfrak{P}$ has infinitely many coefficients different from 0. This is impossible since $D(R)$ is a free $\mathbb{Z}$-module with a free bases $P(R)$.

Let $\{\mathfrak{P}_1, \mathfrak{P}_{\alpha_2}, \ldots, \mathfrak{P}_{\alpha_t}\}$ be the set of all elements $\mathfrak{P}$ in $P(R)$ such that $v_{\mathfrak{P}}(r^{-1}) < 0$. Then $v_{\mathfrak{P}}(r^{-1}) = 0$ for all $\mathfrak{P} \in P(R) \setminus \{\mathfrak{P}_1, \mathfrak{P}_{\alpha_2}, \ldots, \mathfrak{P}_{\alpha_t}\}$, since $r \in P_1^{-m_1} \subset R$ implies $v_{\mathfrak{P}}(r^{-1}) \leq 0$ for all $\mathfrak{P} \in P(R)$.

Consider the family $\{v_{\mathfrak{P}_1}, v_{\mathfrak{P}_{\alpha_2}}, \ldots, v_{\mathfrak{P}_{\alpha_t}}\}$ of essential valuations of the ring $R$ and $(v_{\mathfrak{P}_1}(r) + m_1, v_{\mathfrak{P}_{\alpha_2}}(r), \ldots, v_{\mathfrak{P}_{\alpha_t}}(r)) \in \mathbb{Z}^t$. Since $v_{\mathfrak{P}_1}(r) + m_1 \geq 0$ and $v_{\mathfrak{P}_{\alpha_i}}(r) > 0$ for all $i \in \{2, \ldots, t\}$ we can apply Case 1. Hence, there exists an element $x_1 \in R$ such that $v_{\mathfrak{P}_1}(x_1) = v_{\mathfrak{P}_1}(r) + m_1, v_{\mathfrak{P}_{\alpha_i}}(x_1) = v_{\mathfrak{P}_{\alpha_i}}(r)$ for all $i \in \{2, \ldots, t\}$ and $v_{\mathfrak{P}}(x_1) \geq 0$ for all other $\mathfrak{P} \in P(R)$. Then $x = r^{-1} \cdot x_1 \in K$ is an element satisfying the required condition.

□

The element $x$ constructed in the proof of the Theorem 4.3.1 is not necessarily regular. For the class of additively regular rings it follows from the next result that the element $x$ can be chosen to be regular. This result is also an easy consequence of Theorem 4.3.1. But first, we recall the definition of an additively regular ring. A ring $R$ is additively regular if for each $z$ in the total quotient ring $K$ of $R$, there exists $u \in R$ such that $z + u \in \mathcal{C}_K(0)$.

**Theorem 4.3.2** *Let $R$ be an additively regular Krull ring such that $R \neq K$, and let $\{v_{\mathfrak{P}} : \mathfrak{P} \in P(R)\}$ be the family of essential valuations of $R$. If $\{v_{\mathfrak{P}_1}, v_{\mathfrak{P}_2}, \ldots, v_{\mathfrak{P}_n}\}$ is a finite set of essential valuations and if $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, then there exists a regular element $x \in K$ such that $v_{\mathfrak{P}_i}(x) = m_i$ for all $i \in \{1, 2, \ldots, n\}$ and $v_{\mathfrak{P}}(x) \geq 0$ for all other $\mathfrak{P} \in P(R)$.*

**Proof.** In the proof of Theorem 4.3.1, Case 1, if $x \in \overline{P_1^{m_1}} \setminus (\overline{P_1^{m_1+1}} \bigcup P_2 \bigcup P_3 \bigcup \cdots \bigcup P_n)$, then, by the additive regularity of $R$, for $b \in \operatorname{Reg}(\overline{P_1^{m_1+1}} \bigcap P_2 \bigcap P_3 \bigcap \cdots \bigcap P_n)$, there exists an element $u \in R$ such that $t = x + bu$ is regular and $t \in \overline{P_1^{m_1}} \setminus (\overline{P_1^{m_1+1}} \bigcup P_2 \bigcup \cdots \bigcup P_n)$. The element $t$ satisfies the required conditions.

□

This allows us to characterize divisorial fractional ideals of an additively regular Krull ring in terms of principal fractional ideals in the same way as in the case of domains, see [Bou72], Ch. VII, §5, Corollary 2.

**Corollary 4.3.3** *Let $R$ be an additively regular Krull ring and let $A, B, C$ be divisorial fractional ideals of $R$ such that $A \subseteq B$. Then there exists an element $x \in K$ such that*

$$A = B \bigcap RxC.$$

**Proof.** Let $\{v_{\mathfrak{P}} \mid \mathfrak{P} \in P(R)\}$ be the family of essential valuations of a Krull ring $R$ and let

$$\text{div}(A) = \sum m_{\mathfrak{P}} \cdot \mathfrak{P};\ \text{div}(B) = \sum n_{\mathfrak{P}} \cdot \mathfrak{P};\ \text{div}(C) = \sum p_{\mathfrak{P}} \cdot \mathfrak{P}.$$

Then $A \subseteq B$ implies $\text{div}(A) \geq \text{div}(B)$ and hence $m_{\mathfrak{P}} \geq n_{\mathfrak{P}}$ for all $\mathfrak{P} \in P(R)$. Let $J_1 = \{\mathfrak{P} \in P(R) \mid m_{\mathfrak{P}} > n_{\mathfrak{P}}\}$. Also, there exists a finite subset $J_2$ of $P(R)$ such that $p_{\mathfrak{P}} = m_{\mathfrak{P}} = 0$ for all $\mathfrak{P} \in P(R) \setminus J_2$. Let $J = J_1 \bigcup J_2$. Then $\{v_{\mathfrak{P}} \mid \mathfrak{P} \in J\}$ is a finite family of essential valuations of the ring $R$ and $\{p_{\mathfrak{P}} - m_{\mathfrak{P}} \mid \mathfrak{P} \in J\}$ is a finite family of integers. By Theorem 4.3.2, there exists a regular element $x \in K$ such that $v_{\mathfrak{P}}(x^{-1}) = p_{\mathfrak{P}} - m_{\mathfrak{P}}$ for all $\mathfrak{P} \in J$ and $v_{\mathfrak{P}}(x^{-1}) \geq 0 = p_{\mathfrak{P}} - m_{\mathfrak{P}}$ for all $\mathfrak{P} \in P(R) \setminus J$. Hence:

$$(\forall \mathfrak{P} \in P(R))\ \sup\{n_{\mathfrak{P}}, v_{\mathfrak{P}}(Rx) + p_{\mathfrak{P}}\} = m_{\mathfrak{P}}$$

since $v_{\mathfrak{P}}(x) = v_{\mathfrak{P}}(Rx)$.

Suppose that $a \in A$. By Proposition 4.2.2, $v_{\mathfrak{P}}(a) \geq m_{\mathfrak{P}} = \sup\{n_{\mathfrak{P}}, v_{\mathfrak{P}}(Rx) + p_{\mathfrak{P}}\}$ for all $\mathfrak{P} \in P(R)$. Therefore, $v_{\mathfrak{P}}(a) \geq n_{\mathfrak{P}}$ and $v_{\mathfrak{P}}(a) \geq v_{\mathfrak{P}}(Rx) + p_{\mathfrak{P}}$ for all $\mathfrak{P} \in P(R)$. On the other hand, $Rx \in F(R)$ and

$$\text{div}(RxC) = \text{div}(Rx) + \text{div}(C) = \sum (v_{\mathfrak{P}}(Rx) + p_{\mathfrak{P}})\mathfrak{P}.$$

Furthermore, since the fractional ideal $Rx$ is invertible, $Rx \cdot C$ is a divisorial ideal of the ring $R$. Applying Proposition 4.2.2 again, we get $a \in B \bigcap RxC$, i.e., $A \subseteq B \bigcap RxC$.

Conversely, for $a \in B \bigcap RxC$, $v_{\mathfrak{P}}(a) \geq n_{\mathfrak{P}}$ and $v_{\mathfrak{P}}(a) \geq v_{\mathfrak{P}}(Rx) + p_{\mathfrak{P}}$ for all $\mathfrak{P} \in P(R)$. Hence, $v_{\mathfrak{P}}(a) \geq \sup\{n_{\mathfrak{P}}, v_{\mathfrak{P}}(Rx) + p_{\mathfrak{P}}\} = m_{\mathfrak{P}}$ for all $\mathfrak{P} \in P(R)$ i.e., $a \in A$ by the same argument.

□

**Corollary 4.3.4** *Let $R$ be an additively regular Krull ring such that $R \neq K$ and let $A$ be a fractional regular ideal of $R$. Then $A$ is a divisorial ideal of $R$ if and only if $A$ is the intersection of two fractional principal ideals of $R$.*

**Proof.** If $A$ is a divisorial ideal, then $A = \overline{A} \subseteq \bigcap\{Ra \mid a \in \mathcal{C}_K(0),\ \ A \subseteq Ra\}$. Let $B = Ra$, $a \in \mathcal{C}_K(0)$ be any regular principal fractional ideal of $R$ such that $A \subseteq B$ and let $C = Rb$, $b \in \mathcal{C}_K(0)$. Then, by Corollary 4.3.3, there exists a regular element $x \in K$ such that $A = B \bigcap RxC = Ra \bigcap Rxb$.

Conversely, if $A = Ra \bigcap Rb$ where $a, b \in \mathcal{C}_K(0)$, then $Ra \bigcap Rb = A \subseteq \overline{A} \subseteq \bigcap\{Ry \mid y \in \mathcal{C}_K(0),\ A \subseteq Ry\} \subseteq Ra \bigcap Rb = A$, i.e., $A = \overline{A}$.

□

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