Quantified Modal Relevant Logics

by

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Abstract

Applications of formal logic often require the language of the logics to be sufficiently expressive, capturing notions such as necessity, possibility, subject-predicate sentences, quantified sentences, and identity. To this end, logics employ modal operators, first order quantifiers, and an identity relation. A logic with this kind of expressive power is called a *quantified modal logic*. Relevant logics are the logics that ensure the conclusions of an argument are relevant to its premises, and are able capable of making many philosophically motivated inferential distinctions. The quantified modal extensions or relevant logics have not received much attention, partially due to the historical difficulties in developing less expressive relevant logics.

This work constructs a general framework for constructing quantified modal relevant logics with identity, focusing on formal semantics. The proof systems given are all Hilbert style axiom systems for a range of quantified modal logics extending the basic affixing logic **B**. Using insights from work on the regular modal relevant logics, as well as a recent general frame semantics for the quantified relevant logic **RQ**, I construct a general frame relational semantics for a wide range of logics. To this framework, I add an identity relation in a variety of ways, each of which enjoys a different philosophical motivation. Kremer's relevant indiscernibility approach to identity in relevant logics guides both approaches. The first results in a formal semantics that captures Kremer's informal interpretation of identity, the other builds on Kremer's axiom choice by giving it a semantics for which it is sound and complete. In addition, I explore possible applications in modal naïve set theory. Krajíček's open problem of the consistency of his axiomatization of modal set theory in **KT** is solved using a modal version of Curry's

paradox, which can generalize to show the triviality of numerous other axiomatizations.

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Chapter 1

Introduction

In this dissertation, I develop a general framework for *quantified modal relevant logics* with and without identity, and motivate an application for these logics. Recent developments in the semantics of *quantified relevant logics* and *modal relevant logics* are combined to produce semantics for quantified modal relevant logics. Adding identity to these logics is no small task, as many of the typical approaches to identity end up validating formulas anathematic to the philosophical motivations of relevant logics. This dissertation contains two approaches to identity in relevant logics.

Like many projects, this one was an unintended consequence of another project. I was keen on developing default reasoning logics based on relevant logic. I had my eyes set of constructing proof systems and semantics for relevant versions of *default logics* (see Reiter [91]) and *auto-epistemic logics* (see Moore [85, 86]). As I was planning these projects, it became apparent that a possible world semantics for quantified modal relevant logics would be useful, if not necessary. I could find recent developments in either modal or quantified relevant logics separately, but no unifying, general framework for quantified modal relevant logics. Seeing not only that my intended application of relevant logics, but also many potential applications requiring the expressive power of such logics, benefit from such a framework, I set out to create this framework.

1.0.1 Contents of Chapters

This introductory chapter introduces relevant logic and sets up notational conventions that will be used throughout. Having already described the core project of this dissertation, and why I have embarked on this particular adventure, I will now give a brief description of the content of each of the proceeding chapters. Then, relevant logics are defined, and their usual ternary relational semantics are given. We also consider the related general frame semantics for relevant logics, as general frames will play a key role in this project. The chapter culminates with a summary of some key points in the history of relevant logics pertinent to my goals in this dissertation.

In chapter 2, I construct general frame semantics for quantified modal logics based on the relevant logic $\mathbf{R}$. Mares and Goldblatt [74] give a general frame semantics for both $\mathbf{RQ}^{\circ t}$ and $\mathbf{QR}^{\circ t}$.[1] Their alternative semantics for quantified logics models the universal quantifier by a new operator inspired by both an interpretation of the universal quantifier and the Halmos' *functional polyadic algebra* [50]. By modeling the universal quantifier as they do, they are able to prove completeness for a range of quantified modal (classical) logics in [75]. The Mares and Goldblatt style semantics is thus ripe for extending to quantified modal relevant logics. Takahiro Seki [103, 102, 104, 105, 106, 107, 108, 109] develops general frame semantics for modal relevant logics based on the relatively weak logic $\mathbf{B}$ and the usually considered extensions. Chapter 2 will contain an explication of the Mares and Goldblatt style semantics, as well as Seki's semantics. The chapter culminates with the combination of these semantics into a semantics for quantified modal relevant logics extending $\mathbf{QR}^{\circ t}$. These results are generalized to weaker relevant and modal fragments in chapter 4.

Before we are able to extend the semantics developed for modal extensions of $\mathbf{QR}^{\circ t}$, the Mares and Goldblatt style semantics are shown in chapter 3 to be adequate (i.e., give soundness and completeness results) for quantified relevant logics extending $\mathbf{QB}^{\circ t}$. This will involve showing that the semantics of Mares and Goldblatt employs nothing unavailable to us in weak relevant logics. The proofs will be shown for $\mathbf{QB}^{\circ t}$, and a number of points of interest will be explored. For example, the inter-definability of the quantifiers is shown to hold in both the proof system for $\mathbf{QB}^{\circ t}$ and the semantics.

Then, in chapter 4, a general combination of the extended Mares and Goldblatt style semantics with Seki's semantics is completed. This provides a foundation of quantified modal relevant logics for which to pursue applications. However, many applications also call for the expressive power of identity, which is the subject chapters 5

[1] The notation here correctly suggests that these are specific logics. These names, and other naming notations, will be explained later.

and 6. The former uses the philosophical motivation behind the axiom choice of Philip Kremer [62] for identity in non-modal relevant logics, and develops two kinds of semantics for Kremer-like axiomatizations of quantified relevant logics (including $\mathbf{QR}^{\circ t}$ and extensions) with identity. The Kremer-style axiomatizations are based on J. Michael Dunn's relevant predication [29, 31, 32], and only consider substitution and indiscernibility axioms relative to a set of relevant predications. Note that Shawn Standefer [111] has recently and independently developed a semantic approach to identity in relevant logics similar to one of the approaches taken in chapter 5; however, Standefer adds identity to only **QR** and **RQ**, and he considers different substitution/indiscernibility axioms that those considered here.

Finally, Chapter 7 accomplishes a number of goals. First, we explicate modal naïve set theory in order to reiterate and solve an open problem, which is the consistency of Krajíček's modal naïve set theory in **KT** as posed in [58, 59]. This theory will be shown trivia in **KT**. That is, the theory is absolutely inconsistent, as every sentence is provable from it. Next, I offer an analysis of the routes to triviality possible in various modal set theories in order to motivate a mixed approach, in which we employ substructural modal logics in the development of modal naïve set theories. Thus, I motivate an application for the logics of the previous chapters, in order to advance the mixed approach. The general semantic framework can be used to construct modal inconsistency proofs for various modal set theories.

1.1 Notation, Conventions, and Terminology

Here, I will set up various conventions that I will employ in the sequel. That being said, some of the conventions listed here will be purposely broken at times. However, it will be made clear when the conventions are discarded in order to avoid confusion.

Propositional Logic. In the case of propositional logics, including propositional modal logics, I will take there to be a denumerable set of atomic propositions (or propositional variables) *Atoms*$= \{p_1, p_2, \dots\}$. For convenience, I will often use lowercase p's, q's and r's, with or without subscripts, to denote the members of *Atoms*. For a given logic $\mathfrak{L}$, the set of well-formed formula, $wff_{\mathfrak{L}}$, is defined in the usual way, using, when appropriate, $\rightarrow$ (relevant implication), $\wedge$ (extensional conjunction), $\vee$ (extensional disjunction), $\neg$ (negation), $\leftrightarrow$ (biconditional), $\circ$ (fusion or intensional conjunction), t (intensional truth) and, for modal propositional logics, $\Diamond$ (possibility) and $\Box$ (necessity). Intensional truth t is taken to be a 0-ary connective. $\neg$, $\Box$, and $\Diamond$ are unary connectives, and the rest are binary connectives.

Often, I will take the connectives $\vee$, $\leftrightarrow$, $\Diamond$, and $\circ$ to be defined, when appropriate, as follows:

$$\mathcal{A} \vee \mathcal{B} =_{df} \neg(\neg\mathcal{A} \wedge \neg\mathcal{B})$$

$$\mathcal{A} \leftrightarrow \mathcal{B} =_{df} (\mathcal{A} \rightarrow \mathcal{B}) \wedge (\mathcal{B} \rightarrow \mathcal{A})$$

$$\Diamond\mathcal{A} =_{df} \neg\Box\neg\mathcal{A}$$

$$\mathcal{A} \circ \mathcal{B} =_{df} \neg(\mathcal{A} \rightarrow \neg\mathcal{B})$$

In general, I will use capital letters near the beginning of the English alphabet to denote or range over the formulas of a propositional language.

<u>Quantified Logics.</u> We begin by a denumerable set of variables, *Var*, which will be conveniently denoted by lowercase letters near the end of the alphabet (e.g. x, y, z, x_n, w_1).

Following Goldblatt in [47], a possibly denumerable set $\mathcal{L}$ of predicate symbols, function symbols, and individual constants shall be called a *signature*.

Each predicate symbol is of the form P^n, where n is the arity of the predicate. Similarly, each function symbol is of the form f^n, where n is the arity of the function. Often I will omit the superscript if either the arity is obvious or the arity is irrelevant. Each signature is denumerable at most, so the set of predicate (function) symbols can be ordered, if required. Moreover, I shall write P^n_j (f^n_j) to denote the j-th member of an ordering of predicate (function) symbols. For every signature $\mathcal{L}$, its set of individual constants shall be denoted by $con_{\mathcal{L}}$. I shall denote individual constants by c, with or without subscripts.

A *term* will be denoted by τ with or without subscripts, unless otherwise noted. An $\mathcal{L}$*-term*, relative to a signature $\mathcal{L}$, is defined as follows. Every variable v_n is an $\mathcal{L}$*-term*. Every member c of $con_{\mathcal{L}}$ is an $\mathcal{L}$*-term*. If there are function symbols in the signature, then inductively if $\tau_1, \ldots, \tau_m$ are $\mathcal{L}$*-terms* and f^m is in $\mathcal{L}$, then $f^m(\tau_1, \ldots, \tau_m)$ is an $\mathcal{L}$*-term*. No other expression is a term. A term is *closed* when it contains no variables. A term is *open* when it is not closed.

For a given signature $\mathcal{L}$, the atomic formulas (atomic $\mathcal{L}$-formulas) are those of the form $P^n(\tau_1, \ldots, \tau_n)$, where $P^n \in \mathcal{L}$ and each of $\tau_1, \ldots, \tau_n$ is an $\mathcal{L}$-*term.* The set of well-formed formula of a quantified logic with signature $\mathcal{L}$ (denoted by $\mathit{wff}_{\mathcal{L}}$ or, where able, simply *wff*) is defined inductively as above, but with the addition of $\forall x$ and $\exists x$, for each variable $x \in \mathit{Var}$. The existential quantifier, when taken as defined, is done so as follows:

$$\exists x \mathcal{A} =_{df} \neg \forall x \neg \mathcal{A}$$

For the most part I will use uppercase letters near the beginning of the alphabet to denote or range over the formulas of quantified logics. This will usually cause no confusion with the same letters being used to range over formulas of propositional logics. Context will determine which use is employed. However, in the event of possible confusion, I will use Greek letters including ϕ, ψ, ξ with or without subscript to denote or range over the formulas of a quantified logics. Typically, however, I reserve Greek letters for another purpose.

An instance of a variable x is *bound* in the wff $\mathcal{A}$ if either (1) the instance is the x of an expression $\forall x$ or $\exists x$ occurring in $\mathcal{A}$ or (2) the instance of x occurs within the scope of a quantifier, $\forall x$ or $\exists x$. An instance of a variable is *free* when it is not bound. A formula with no free variables is called a *sentence.*

A term τ is *free for* (or freely substitutable for) x in wff $\mathcal{A}$ if, for every variable y in τ, there are no free occurrences of x in $\mathcal{A}$ that are within the scope of a quantifier $\forall y$ or $\exists y$.

A shorthand for substitutions will be convenient for our purposes. Often we will want to replace every free occurrence of a variable x in $\mathcal{A}$ with the term τ. The formula $\mathcal{A}[\tau/x]$ shall be used to denote the formula that results from such a replacement. It will also be convenient to have a notation for the operation of several simultaneous substitutions. We will use $\mathcal{A}[\tau_0/v_0, \ldots, \tau_n/v_n]$ for the result of simultaneously replacing v_0 through v_n with τ_0 through τ_n respectively.

A *variable assignment*, f, assigns to each variable an element of the domain of individuals, U, as follows. There are a denumerable number of variables which can be ordered as $x_1, x_2, \ldots,$ and a variable assignment is an ordered denumerable list of individuals. In other words, a variable assignment is a member of U^ω, where the nth individual in the ordering is the individual assigned to the variable x_n. The set of all variable assignments, relative to a domain U, is the set U^ω.

An *x-variant* of a variable assignment $f \in U^\omega$ for a domain of individuals U is a variable assignment that differs from f only in the individual assigned to the variable x. The set of all x-variants of f will be denoted by xf. Other notations for variants of variable assignments will be used.

We will use propositional functions. Roughly speaking, a propositional function is a function that takes variable assignments and returns propositions. In classical logic these propositions are the truth values True and False. In modal logics propositions are sets of worlds. As we will see, for relevant logics, propositions are particular kinds of sets of worlds. Taking *Prop* to be a set of propositions, a propositional function is of type $U^\omega \longrightarrow Prop$.

Modal Logics For m-ary and n-ary relations M and N, respectively, we shall write

$$Nn_1, \ldots, n_{i-1}, (Mm_1, \ldots, m_{m-1}), n_{i+1}, \ldots, n_n$$

to denote there exists an x such that $Mm_1, \ldots, M_{M-1}x$ and $Nn_1, \ldots, n_{i-1}, x, n_{i+1}, \ldots, n_n$. Of interest is the special case where $M(Mm_1, \ldots, m_{m-1})n_2, \ldots, n_m$. This shall be denoted, in agreement with the usual convention, as $M^2m_1, \ldots, m_{m-1}, n_2, \ldots, n_m$. The following definitions result from this convention.

$$R^2xyzw =_{df} \exists v(Rxyv \wedge Rvzw)$$
$$S^2xy =_{df} \exists z(Sxz \wedge Szy)$$

An additional special case is the n-ary ancestral binary relations. For modal logics, the notation $\Box^n$ is as shorthand for a repetition of the symbol '$\Box$' n times. This superscripting shorthand is used for all modalities. Further, a corresponding binary relation $S^n_\Box$ is a special case of the notational convention just given, where, after some rewriting, $S^n_\Box ab =_{df} \exists y_1, \ldots, y_{n-1}(S_\Box ay_1 \wedge S_\Box y_1y_2 \wedge, \ldots, \wedge S_\Box y_{n-1}b)$. When this ancestral relation is used, it will be explicitly marked.

A convention pertaining to my vocabulary instead of the symbolic notion is that I will, for the most part, use the word 'worlds' or the phrase 'possible worlds' in the sequel to talk about the elements of certain sets in frames and models. I do not intend to make any metaphysical claims by this use. Nevertheless, a discussion on this topic will not be included. Here, I will use the phrases 'worlds', 'possible worlds', 'situations', 'possible situations', 'impossible situations', 'set-ups', etc. without committing myself to any metaphysical connotations of these phrases.

1.2 Relevant Logics

Propositional relevant logics have likely received the most attention of any relevant logics. In this section, I will present propositional relevant logics. First, I present axiom systems for relevant logics. Here, I give a list of axioms and rules capable of defining a range of relevant logics I am interested in. I will also give alternative axiomatizations for some of these logics. Then, I will construct the usual Routley-Meyer ternary relational semantics for these logics.

1.2.1 Axiom systems

A list of axioms capable of axiomatizing a number of relevant logics that extend the basic affixing logics **B** is given.[2] Using the following axioms and rules, we can define a variety of relevant logics. However, I reserve the right to simplify this presentation or completely switch to other axiomatizations when it is convenient.

(A1) $\mathcal{A} \to \mathcal{A}$ — Identity

(A2) $\mathcal{A} \to (\mathcal{A} \vee \mathcal{B})$ — Disjunction Introduction (left)

(A3) $\mathcal{B} \to (\mathcal{A} \vee \mathcal{B})$ — Disjunction Introduction (right)

(A4) $(\mathcal{A} \wedge \mathcal{B}) \to \mathcal{A}$ — Conjunction Elimination (left)

(A5) $(\mathcal{A} \wedge \mathcal{B}) \to \mathcal{B}$ — Conjunction Elimination (right)

(A6) $\mathcal{A} \wedge (\mathcal{B} \vee \mathcal{C}) \to ((\mathcal{A} \wedge \mathcal{B}) \vee (\mathcal{A} \wedge \mathcal{C}))$ — $\wedge\vee$-Distribution

(A7) $((\mathcal{A} \to \mathcal{B}) \wedge (\mathcal{A} \to \mathcal{C})) \to (\mathcal{A} \to (\mathcal{B} \wedge \mathcal{C}))$ — Conjunction Introduction

(A8) $((\mathcal{A} \to \mathcal{C}) \wedge (\mathcal{B} \to \mathcal{C})) \to ((\mathcal{A} \vee \mathcal{B}) \to \mathcal{C})$ — Disjunction Elimination

[2] **B** is a basic relevant logic that lacks the Law of Excluded middle (A15). Some denote by **B** the logic defined below as **BX**. Here, we take **B** to be the "weaker" logic that lacks the Law of Excluded middle.

(A9) $\neg\neg\mathcal{A} \to \mathcal{A}$ Double Negation Elimination

(A10) $(\mathcal{A} \to \neg\mathcal{B}) \to (\mathcal{B} \to \neg\mathcal{A})$ Contraposition

(A11) $(\mathcal{A} \to \mathcal{B}) \to ((\mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{C}))$ Suffixing

(A12) $(\mathcal{A} \to \mathcal{B}) \to ((\mathcal{C} \to \mathcal{A}) \to (\mathcal{C} \to \mathcal{B}))$ Prefixing

(A13) $(\mathcal{A} \to (\mathcal{A} \to \mathcal{B})) \to (\mathcal{A} \to \mathcal{B})$ Contraction

(A14) $\mathcal{A} \to ((\mathcal{A} \to \mathcal{B}) \to \mathcal{B})$ Assertion

(A15) $\mathcal{A} \vee \neg\mathcal{A}$ Excluded Middle

(A16) $(\mathcal{A} \to \neg\mathcal{A}) \to \neg\mathcal{A}$ Reductio

(A17) $\mathcal{A} \to (\mathcal{A} \to \mathcal{A})$ Mingle

(A18) $(\mathcal{A} \to \mathcal{C}) \to (((\mathcal{A} \to \mathcal{C}) \to \mathcal{B}) \to \mathcal{B})$

(A19) $N(\mathcal{A}) \wedge N(\mathcal{B}) \to N(\mathcal{A} \wedge \mathcal{B})$ (where $N(\mathcal{C})$ is defined as $(\mathcal{C} \to \mathcal{C}) \to \mathcal{C}$.)

(R1) From $\mathcal{A}$ and $\mathcal{A} \to \mathcal{B}$ to infer $\mathcal{B}$

(R2) From $\mathcal{A}$ and $\mathcal{B}$ to infer $\mathcal{A} \wedge \mathcal{B}$

(R3) From $\mathcal{A} \to \mathcal{B}$ to infer$((\mathcal{C} \to \mathcal{A}) \to (\mathcal{C} \to \mathcal{B}))$

(R4) From $\mathcal{A} \to \mathcal{B}$ to infer$((\mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{C}))$

(R5) From $\mathcal{A} \to \neg\mathcal{B}$ to infer $\mathcal{B} \to \neg\mathcal{A}$

It is straightforward to see that when (A12), (A11), and (A10) are axioms of a logic, the rules (R3), (R4), and (R5) respectively are derivable using (R1), and may be omitted from the axiomatization. A sentential form of double negation introduction is derivable using (A1) and (R5).

The logic **B** is defined as (A1)–(A9), and (R1)–(R5), and the logic **BX** is **B**+(A15). **DW** is **B**+(A10), and similarly **DWX** is **DW**+(A15). **TW** is **DW**+(A11)+(A12). **TWX** is as expected, namely **TW**+(A15). The logic **T** of ticket entailment results from either **TW** or **TWX** by adding the axioms (A13) and (A16). That is, (A15) is provable in **TW**+(A13)+(A16). The logic of entailment **E** is **T**+(A18)+(A19). On this axiomatization, **R** results from adding (A12) to either **T** or **E**.

I will call a logic **L** stronger than a logic **M** if **L** extends **M**. That is, if all the theorems of **M** are theorems of **L**.

The above axiom system was designed with a larger range of relevant logics in mind. It is possible to axiomatize the logic **E** without using the defined $N(\mathcal{C})$ or the lengthy formula created by replacing $N(\mathcal{C})$ with $(\mathcal{C} \to \mathcal{C}) \to \mathcal{C}$, as will be shown below.

To these logics, it is often useful to add the binary connective $\circ$, which is usually referred to as "fusion" and the intentional truth $\boldsymbol{t}$. Fusion, sometimes called an intensional conjunction, links the logics to their algebraic counterparts, and plays the role of the residual of implication. The intensional truth $\boldsymbol{t}$ seems to capture a sense of logically true, such that $\boldsymbol{t} \to \mathcal{A}$ has been used to capture $\mathcal{A}$ being logically true, as well as being necessarily true. (The latter interpretation is not open to all relevant logics. For example, adding $(\boldsymbol{t} \to \mathcal{A}) \to \mathcal{A}$ to **T** does not produce a conservative extension of **T**.) We can think of $\boldsymbol{t}$ as the conjunction of each of the axioms. This interpretation will be formalized in the semantics.

We can add fusion and intensional truth to the logics using the following rules:

(R6) From $\mathcal{A}$ to infer $\boldsymbol{t} \to \mathcal{A}$

(R7) From $\boldsymbol{t} \to \mathcal{A}$ to infer $\mathcal{A}$

(R8) From $\mathcal{A} \to (\mathcal{B} \to \mathcal{C})$ to infer $(\mathcal{A} \circ \mathcal{B}) \to \mathcal{C}$

(R9) From $(\mathcal{A} \circ \mathcal{B}) \to \mathcal{C}$ to infer $\mathcal{A} \to (\mathcal{B} \to \mathcal{C})$

In some stronger relevant logics, such as **R**, axioms corresponding to these rules are derivable (while treating the connectives as defined).

When a logic contains either $\circ$ or $\boldsymbol{t}$, the name of the logic will usually reflect this fact using a superscript containing which of these symbols are in the logic. The modal relevant logics I will present from Seki will differ in this notation, but the difference will be made clear.

For another presentation of **E** and **R**, we can define **E** as (A1)–(A10), (A12), (A13), (A16), (A22), and (A23) and **R** as **E**+(A24).

(A22) $(\mathcal{A} \to ((\mathcal{B} \to \mathcal{D}) \to \mathcal{C})) \to ((\mathcal{B} \to \mathcal{D}) \to (\mathcal{A} \to \mathcal{C}))$

(A23) $(((\mathcal{A} \to \mathcal{A}) \wedge (\mathcal{B} \to \mathcal{B})) \to \mathcal{C}) \to \mathcal{C}$

(A24) $(\mathcal{A} \to (\mathcal{B} \to \mathcal{C})) \to (\mathcal{B} \to (\mathcal{A} \to \mathcal{C}))$

1.2.2 Semantics

The semantics in this section is more or less the standard Routley-Meyer ternary relational semantics for relevant logic introduced in [99, 97, 98].

First, a definition of frames for the logic **B** is given. From there, a correspondence between axioms and frame conditions is given.

Definition 1.2.1. A **B**-frame is a tuple $\mathfrak{F} = \langle K, 0, R, * \rangle$, where K is a non-empty set, $0 \subseteq K$, $R \subseteq K^3$, and $*$ is a unary function on K such that a list of postulates to follow are satisfied.

A binary relation on K is defined via 0 and R by the definition:

$$a \leq b =_{df} \exists y \in 0(Ryab)$$

Given this definition, where $a, b, c, d \in K$, the following postulates are satisfied by every **B**-frame:

(p1) $\leq$ is reflexive and transitive

(p2) 0 is closed by $\leq$ in the upwards direction

(p3) if $a \leq b$ and $Rbcd$ then $Racd$

(p4) if $a \leq c$ and $Rbcd$ then $Rbad$

(p5) if $d \leq a$ and $Rbcd$ then $Rbca$

(p6) if $a \leq b$ then $b^* \leq a*$

(p7) $a^{**} = a$

Using the ternary relation R, we can define a binary operation $\Rightarrow$ on the powerset $\wp(K)$. For every $X, Y \subseteq K$, let

$$X \Rightarrow Y = \{a \in K : \forall x \forall y (Raxy \text{ and } x \in X \text{ implies } y \in Y)\}$$

It can be shown that if X and Y are up-sets, then so is $X \Rightarrow Y$, where up-sets are sets of worlds closed under upwardly under the defined $\leq$ relation. That is, if $a \in X$ and $a \leq b$ implies $b \in X$, then X is an up-set

We can define a unary operation on subsets of $\wp(K)$ using the $*$ as follows. For every $X \subseteq K$

$$X^* = \{a \in K : a^* \notin X\}$$

Similarly, if X is an up-set, then so is X^*.

Definition 1.2.2. A ***B***-*model* is a tuple $\mathfrak{M} = \langle K, 0, R, *, |-|^{\mathfrak{M}} \rangle$, where $\langle K, 0, R, * \rangle$ is a **B**-frame and $|-|^{\mathfrak{M}}$ is a valuation that maps each atomic proposition to a $\leq$-up-set of K, and is extended to all *wff* inductively as follows:

$|\boldsymbol{t}|^{\mathfrak{M}} = 0$

$|\mathcal{A} \wedge \mathcal{B}|^{\mathfrak{M}} = |\mathcal{A}|^{\mathfrak{M}} \cap |\mathcal{B}|^{\mathfrak{M}}$

$|\neg \mathcal{A}|^{\mathfrak{M}} = (|\mathcal{A}|^{\mathfrak{M}})^*$

$|\mathcal{A} \to \mathcal{B}|^{\mathfrak{M}} = |\mathcal{A}|^{\mathfrak{M}} \Rightarrow |\mathcal{B}|^{\mathfrak{M}}$

Finally, we say that a formula $\mathcal{A}$ is true in the model $\mathfrak{M}$ if $0 \subseteq |\mathcal{A}|^{\mathfrak{M}}$. $\mathcal{A}$ is valid on a frame if it is true in every model based on that frame, and $\mathcal{A}$ is valid in a class of frames if it is valid in every frame in that class.

To extend these semantics for logics that extend **B**, the following table of axioms and their corresponding frame conditions is useful.

Axiom	**Frame Condition**
$\mathcal{A} \vee \neg \mathcal{A}$	if $a \in 0$ then $a^* \leq a$
$(\mathcal{A} \to \neg \mathcal{B}) \to (\mathcal{B} \to \neg \mathcal{A})$	if $Rabc$ then Rac^*b^*
$(\mathcal{A} \to \mathcal{B}) \to ((\mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{C}))$	if R^2abcd then $Rb(Rac)d$
$(\mathcal{A} \to \mathcal{B}) \to ((\mathcal{C} \to \mathcal{A}) \to (\mathcal{C} \to \mathcal{B}))$	if R^2abcd then $Ra(Rbc)d$
$(\mathcal{A} \to (\mathcal{A} \to \mathcal{B})) \to (\mathcal{A} \to \mathcal{B})$	if $Rabc$ then R^2abbc
$\mathcal{A} \to ((\mathcal{A} \to \mathcal{B}) \to \mathcal{B})$	if $Rabc$ then there is an $x \in K$ such that $a \leq x$ and $Rbxc$
$(\mathcal{A} \to \neg \mathcal{A}) \to \neg \mathcal{A}$	Raa^*a

Let's begin to familiarize ourselves with the concepts of general frame semantics.[3]

General frames for modal (classical) logics are a generalization of the usual Kripke-style relational frames, but are closely related to Boolean algebras with operators.[4] General frames are interesting for a number of reasons. Many find relational semantics quite intuitive, and the general frame semantics appear much like the typical relational semantics. However, they are also natural duals of algebras. This fact can be quite useful. For example, we can obtain completeness results for many modal logics using general frames and their dual algebras.

General frame add to the typical frames a restriction on the possible valuations. This set of *admissible propositions*, $Prop$, can be described as "(the carrier of) a complex algebra over [the Kripke-style relation frame]" [16, p. 304]. Models are general frames with (admissible) valuations whose range is the set of admissible propositions. Validity and related notions are defined as expected. The canonical general frame can be constructed from the Kripke-style canonical frame on which it is based, taking the admissible propositions to be the sets of worlds containing each formula. We define the set of worlds containing $\mathcal{A}$ by $||\mathcal{A}||_c = \{a \in K : \mathcal{A} \in a\}$, and then the admissible propositions in the canonical frame by $Prop = \{||\mathcal{A}||_c : \mathcal{A} \text{ is a formula}\}$. Or more simply we take $Prop_c = \{|\mathcal{A}| : \mathcal{A} \text{ is a wff}\}$

Extending the general frame treatment to relevant logics, we thus get the following definition.

Definition 1.2.3. An *general-**B**-frame* is a tuple

$$\mathfrak{F} = \langle K, 0, R, *, Prop \rangle,$$

where $\langle K, 0, R, * \rangle$ is a **B**-frame and $Prop$ is a subset of the up-sets of K called the admissible propositions which is closed under $\cap, \cup, \Rightarrow$ and $*$ and contains 0.

A model based on a general frame is given by a valuation that assigns to each propositional variable a member of $Prop$. This is extended as before. Finally, truth in a model, a frame, and a class of frames is defined as usual.

[3]S.K. Thomason [113] introduced general frames. For an introduction to general frames, see Blackburn, de Rijke, and Venema [16].

[4]Indeed, they are identical in a category-theoretic sense.

The canonical general frame is obtained from the canonical frame as follows. We obtain $Prop_c$ by setting $Prop_c = \{|\mathcal{A}| : \mathcal{A}$ is a wff$\}$.

There are certain general frames for a logic that are of particular interest. These are the *differentiated, tight, compact, refined, descriptive, full, and discrete* general frames. The definition of these properties is sometimes relative to the set of intensional operators (including the relevant conditional), and thus will not be defined here. In the next chapter, we will reproduce the definition of these properties for modal relevant logics as given by Seki [103].

1.3 A Too Brief History of Relevant Logics

Relevant Logics

I have introduced relevant logic, but I have not explained any of the philosophical motivation supporting it. In this brief history, some of the motivation for relevant logic will be touched upon, but will not be the focus. Overviews of the subject of relevant logic include Katalin Bimbó [11] and Dunn and Restall [36]. Recent introductions to the topic of relevant logic, such as Mares in 2004 [73], have started with the topic of the non-sequitur. In the case of relevant logics, an example of a non-sequitur is, "the moon is made of cheese implies that $2 + 2 = 4$". The key word in this sentence is the world *implies*, so far as relevant logic is concerned. Relevant logic is intended to capture implication, which is taken to be a relation which requires a degree of relevance between what is implied and what is doing the implying. The diagnosis of the non-sequitur problem is a lack of relevance between the antecedent and consequent. Thus, relevant logic rejects these kinds of non-sequiturs on the grounds of an irrelevance between the antecedent and the consequent. While this is indeed a valuable feature of relevant logic, this concern does not go back to the beginning of relevant logic.

In 1951, Church [23] introduced the *weak implicational calculus*, which is better known as the logic $\mathbf{R}_{\rightarrow}$. Then, in 1956, Ackermann [1] introduced a number of logics. One of these logics, Π', is equivalent to the relevant logic $\mathbf{E}$ with the rule of disjunctive syllogism (famously known as the rule γ as it was named by Ackerman).

Given their philosophical motivations, Anderson and Belnap went on to explore $\mathbf{R}_{\rightarrow}$, $\mathbf{E}_{\rightarrow}$, their extensions, and various other relevant logics. Anderson and Belnap published a number of papers on the topic during the 1960s. This includes a pair of papers from 1962 [4, 3]. Their work culminated in the first volume of *Entailment: The Logic of Relevance and Necessity* [5], which lead to a second volume with Dunn [6]. What they proposed was *the* logic of both relevance and necessity was the logic **E**. This is not only because it avoids fallacies of relevance and fallacies of modality, but also because it has a definable **S4**-like modality and rejects irrelevance. This first volume also includes motivations for other logics, definitions of other relevant logics, and presentations of other proof systems for these logics. In some sense, the logic **E** was supposed to be a modal relevant logic. The interest in modal relevant logics comes from the beginning of relevant logic, and is also part of the motivation for some relevant logics.

The construction of the ternary relational semantics was a significant event in the history of relevant logics. At the time of its construction, only some relevant logics (typically fragments) had semantics for which they were sound and complete. Of these logics, it is worth mentioning the semi-lattice semantics of Urquhart [115]. Routley has independently constructed equivalent semi-lattice semantics for some fragments in a manuscript sent to Urquhart and Meyer. In a footnote in the aforementioned, Urquhart mentions this manuscript. Routley's ambitious plans for the semi-lattice semantics are seen in the table of contents of this manuscript. He had constructed multiple ways to model negation, many of which are not found in Urquhart's paper; however, Routley's various constructs ultimately fail to provide an adequate semantics. Nevertheless, this manuscript mentions using a ternary relation. A typescript written by Meyer runs with this idea, and eventually the 3 co-authored papers [99, 97, 98] are published, introducing what is known as the Routley-Meyer semantics for relevant logic.

Before turning our focus to developments in modal relevant logic and quantified relevant logic, note that there are many important and interesting developments in relevant logic I have either skimmed over or left out. The exercise of listing every development in the history of relevant logic is a task for a much larger project. Here, I focus on enough history for a reader to understand the significance of constructing adequate semantics for relevant logics and to know where the different semantic systems explicated below come from.

Modal Relevant Logic

The relevant logics described so far have been propositional logics (with modalities in the case of logics like **E** where a modality is definable). The expressive power of both modalities and quantifiers is desirable, as some valid inferences cannot be captured otherwise. Let's begin with the interesting history of modal relevant logics. The logics **E** and **R** were conjectured to be related in a way similar to the relation between classical and intuitionistic logic. That is, the entailment connective of **E** was conjectured to be definable in a modalized **R** by Meyer in [79]. This conjecture has been shown to hold for various fragments. For example, Theorem 2 of [79] shows that the first degree fragment of **E** has this relation to **NR**. This theorem states that, if $\mathcal{A}$ is a first degree formula of **E** (contains no embedded arrows), then $\mathcal{A}$ is a theorem of **E** iff its translation is a theorem of **NR**. This result also applies to the implicational fragment of **E**. That is, if you extend the implicational fragment of **R** with an **S4**-like $\Box$ operator, resulting in the logic $\mathbf{NR}_{\rightarrow}$, then the sentences of the form $\Box(\mathcal{A} \rightarrow \mathcal{B})$ in **NR** correspond to the theorems of the implicational fragment of **E**. In particular, $\vdash_{\mathbf{E}_{\rightarrow}} \mathcal{A} \rightarrow \mathcal{B}$ if and only if $\vdash_{\mathbf{NR}_{\rightarrow}} \Box(\mathcal{A} \rightarrow \mathcal{B})$.

It conjectured by Meyer that this would hold for the "full" logics **NR** and **E** with negation, disjunction, and conjunction added; however, this conjecture was shown to be false by Maksimova in [66]. In particular, adding conjunction and disjunction to the implicational fragments of **R** and **E** "breaks" the neat relationship between these two logics It was demonstrated that

$$\Box(\mathcal{A} \rightarrow \Box(\mathcal{B} \rightarrow \mathcal{C})) \wedge \Box(\mathcal{B} \rightarrow (\mathcal{A} \vee \mathcal{C})) \rightarrow \Box(\mathcal{B} \rightarrow \mathcal{C})$$

is a theorem of **NR** while its translation

$$((\mathcal{A} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C})) \wedge (\mathcal{B} \Rightarrow (\mathcal{A} \vee \mathcal{C})) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C})$$

is not a theorem of **E** (see [5, p. 351]). Thus, extending **NR** would not lead to a remedy for this situation for, in any extension of **NR**, the same formula will be a theorem of the extension, but its translation will still not be a theorem of **E**.

NR was one of the first relevant logics to be extended by a non-definable $\Box$ operator. The logic **NR** does not contain the theorems of the logic **S4** (on a suitable translation).

The axiom

$$\Box(\mathcal{A} \vee \mathcal{B}) \rightarrow (\Diamond\mathcal{A} \vee \Box\mathcal{B})$$

is added to **NR** to create the logic **R4** on the suggestion Nuel Belnap [77]. The purpose of adding this axiom is so that the relevant modal logic **R4** would contain all of the theorems of **S4** on a suitable translation.

André Fuhrmann [41], a student of Richard Routley, developed a semantics for modal relevant logics that do not have Belnap's suggested axiom $\Box(\mathcal{A} \vee \mathcal{B}) \rightarrow (\Diamond\mathcal{A} \vee \Box\mathcal{B})$. Fuhrmann uses the naming convention for modal relevant logics in which the relevant logic base $\mathcal{L}$ is extended to be $\mathcal{N}$-like by the notation $\mathcal{L}.\mathcal{N}$, where $\mathcal{N}$ is the name of a classical modal logic. As we will see, the addition of the Belnap axiom will remove the '**.**' from the notation. Thus, **NR** is written as either **R.KT4** or **R.4** or **R.S4**.

Remark 1.3.1. The axiom $\Box(\mathcal{A} \vee \mathcal{B}) \rightarrow (\Diamond\mathcal{A} \vee \Box\mathcal{B})$ suggested by Belnap is different from the axiom used by Seki, as will be seen. In the background of **R**, these axioms are equivalent. However, when negation is weaker, as it is in **B**, these axioms do not produce equivalent extensions of a logic. The dot notation could lead to confusions when determining which axiom is being added. However, in the sequel it will be made clear what the dot notation means in each circumstance.

Mares and Meyer [77] defined a semantics for **R4** in which "there is an [**S4** model structure] embedded in a natural way into each [**R4** model structure]" [77, p. 109]. In [70], Mares defined semantics for other modal extensions of **R** in which a similar result holds for the corresponding model structures of classical modal logics weaker than **S4**.

The logic **R4**, and the other modal relevant logics with a "weaker" modal fragment considered by Mares in [70], can be given what we will call a *Kripkean semantics*. This notion is best explained with reference to the kind of semantics for which the Kripkean semantics are but a special case. A general frame semantics, also known as admissible semantics, adds a set of admissible propositions to the modal structures. Roughly, not every set of upwardly closed possible worlds counts as a proposition — i.e. not every set of worlds can be the "truth set" of a wff. A Kripkean semantics is a "full" general frame semantics, wherein every set of upwardly closed worlds is an admissible proposition. The details of this will be given in a later section. What is important to note here is that Mares and Meyer [77] and Mares [70] give what we will call a Kripkean semantics to a range of modal relevant logics.

Takahiro Seki gives general frame semantics for a wider range of modal relevant logics [103, 102]. He goes on to prove theorems such as the *admissibility of* γ for these logics [104, 105, 106, 107, 108, 109]. Seki's general frame semantics for modal relevant logics have the Kripkean frames of earlier semantics for modal relevant logics as special cases. A notable features of Seki's general frame semantics is that the modalities $\Box$ and $\Diamond$ are not interdefinable in the usual way in every modal relevant logic considered.

It is the work of Seki, which builds upon previous work in the semantics of modal relevant logic, that I intend to borrow from in constructing a theory of quantified modal relevant logics.

Quantified Relevant Logic

Modal relevant logic has seen many developments that furthered it over the years to the point where a quite large number of modal relevant logics are considered by Seki in his papers. The situation in quantified relevant logics, in contrast, has a number of semantic systems, mostly for stronger logics. Importantly, the semantics for quantified relevant logics that I will combine with semantics for modal relevant logic has only been constructed for quantified relevant logics based on **R**. In fact, the task of the second chapter is the construction of Mares and Goldblatt style semantics for a range of quantified relevant logics.

Kit Fine [37], gave a semantics for the quantified relevant logic **RQ**. Shortly after, Fine [38] showed that, given a constant domain over the Routley-Meyer ternary relational semantics, the logic **RQ** is incomplete. Fine's semantics for **RQ** uses an increasing domain and numerous conditions and operators. While Fine's semantics are adequate for soundness and completeness results (and have been called "powerful and ingenious" by Mares and Goldblatt [74, p. 163]), many relevant logicians have tried to simplify the semantics.

Unable to simplify Fine's semantics, Edwin Mares and Robert Goldblatt [74] give an alternative approach to the semantics for **RQ** and the weaker quantified relevant logic **QR**. This approach takes an idea from the proof theory of these quantified relevant logics and builds a semantics upon this idea. The idea is that a universally quantified formula is a proposition that implies every instantiation of itself, and it is the *weakest* thing that does so. Propositions are thus added to the semantics, generating a general frame semantics for quantified relevant logic. Often, we could think of a universally quantified formula as the potentially infinite conjunction of all its instantiations. However, we are not guaranteed that the infinite conjunction of formulas is itself a proposition. Mares and Goldblatt define an operation that takes a possibly infinite number of instances of a universally quantified formula and always returns a proposition. The exact details of this are critical to the theory of quantified modal relevant logic developed below. Section 2.2 contains an exposition of the semantics Mares and Goldblatt constructed.

Goldblatt, in [47], provides yet another semantics for quantified relevant logic, which differs from his earlier work with Mares in that he uses the notion of a cover from topology in the semantics. Here, Goldblatt defined a frame using a set of points (worlds), but replaces the ternary relation with a binary operation, the unary $*$ with a binary relation of orthogonality or incompatibility, adds a fusion operation (not to be confused with the fusion connective definable in **R**), and adds a distinguished world which is an identity element for fusion. The semantics here uses the notion of a cover in topology, but also requires admissible propositions and admissible propositional functions to work. While adding modalities to this structure might prove interesting, I intend to use the previous work of Mares and Goldblatt while constructing a semantics for quantified modal relevant logic.

Chapter 2

Quantified Modal Relevant Logics Based on R

2.1 Introduction

This chapter constructs semantics for quantified modal relevant logics based on $\mathbf{R}^{\circ t}$ by combining the work on $\mathbf{QR}^{\circ t}$ and $\mathbf{RQ}^{\circ t}$ of Mares and Goldblatt [74] and the work of Seki [103, 102] restricted to a base relevant fragment $\mathbf{R}^{\circ t}$. As such, the following two sections explicate these semantics, beginning with the Mares and Goldblatt semantics. The penultimate section combines these approaches. The final section contains some concluding remarks.

2.2 The Semantics of Mares and Goldblatt

The history of the logics **QR** and **RQ** was described in section 1.3 of the previous chapter. Briefly, the search for a semantics for the logic **RQ** and the related **QR** has lead to a few systems of formal semantics that have introduced or employed interesting ideas. Some of these semantics are more natural than others. The semantics given by Mares and Goldblatt [74] was heavily influenced by certain axiom systems for these logics. Mares and Goldblatt [75], and later Goldblatt [47] have applied this semantic approach to quantified modal classical logics, resulting in a unifying semantics with variable domains for which completeness follows. Additionally, Goldblatt and Kane [49] use this semantic approach to define semantics for propositionally quantified relevant logics, in which quantifiers are added that range over propositional variables.[1]

In this section, I will first present the axiom system from which Mares and Goldblatt drew their inspiration. Then, I will move on to their semantic systems. Finally, I will discuss a number of features of their semantics. These features will be used in the semantics for quantified modal relevant logics in a later section, and a discussion of these features will give the reader a better understanding of the semantics in general.

2.2.1 The logic $\mathbf{QR}^{\circ t}$ and $\mathbf{RQ}^{\circ t}$

Here, I will give an axiomatization of a couple quantified relevant logics, and state some of the important theorems about these systems. There are a number of axiomatizations of $\mathbf{RQ}^{\circ t}$ and $\mathbf{QR}^{\circ t}$. The one presented here requires the addition of only one rule and two axioms to an axiomatization of $\mathbf{R}^{\circ t}$.

The axiomatization of the logic $\mathbf{QR}^{\circ t}$ that inspired Mares and Goldblatt is as follows:

Axioms

(Q1) $\mathcal{A} \to \mathcal{A}$ Identity

(Q2) $(\mathcal{B} \to \mathcal{C}) \to ((\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C}))$ Prefixing

(Q3) $(\mathcal{A} \to (\mathcal{B} \to \mathcal{C})) \to (\mathcal{B} \to (\mathcal{A} \to \mathcal{C}))$ Permutation

(Q4) $(\mathcal{A} \to (\mathcal{A} \to \mathcal{B})) \to (\mathcal{A} \to \mathcal{B})$ Contraction

(Q5) $(\mathcal{A} \wedge \mathcal{B}) \to \mathcal{A}$ Conjunction Elimination

(Q6) $(\mathcal{A} \wedge \mathcal{B}) \to \mathcal{B}$ Conjunction Elimination

(Q7) $((\mathcal{A} \to \mathcal{B}) \wedge (\mathcal{A} \to \mathcal{C})) \to (\mathcal{A} \to (\mathcal{B} \wedge \mathcal{C}))$ Conjunction Introduction

(Q8) $((\mathcal{A} \to \mathcal{C}) \wedge (\mathcal{B} \to \mathcal{C})) \to ((\mathcal{A} \vee \mathcal{B}) \to \mathcal{C})$ Disjunction Elimination

(Q9) $\mathcal{A} \to (\mathcal{A} \vee \mathcal{B})$ Disjunction Elimination

[1]Goldblatt and Kane treat propositionally quantified **B** (and its extensions). Although I extended the Mares and Goldblatt semantics to $\mathbf{QB}^{\circ t}$ and its extensions before discovering the Goldblatt and Kane article, their results nevertheless strongly suggest (some of) the results of the next chapter are provable.

(Q10) $\mathcal{A} \to (\mathcal{B} \vee \mathcal{A})$ Disjunction Elimination

(Q11) $(\mathcal{A} \wedge (\mathcal{B} \vee \mathcal{C})) \to ((\mathcal{A} \wedge \mathcal{B}) \vee (\mathcal{A} \wedge \mathcal{C}))$ $\wedge\vee$-Distribution

(Q12) $(\mathcal{A} \to \neg\mathcal{A}) \to \neg\mathcal{A}$ Reductio

(Q13) $\neg\neg\mathcal{A} \to \mathcal{A}$ Double Negation Elimination

(Q14) $(\mathcal{A} \to \neg\mathcal{B}) \to (\mathcal{B} \to \neg\mathcal{A})$ Contraposition

(Q15) $\boldsymbol{t}$

(Q16) $\mathcal{A} \to (\boldsymbol{t} \to \mathcal{A})$

(Q17) $(\mathcal{A} \to (\mathcal{B} \to \mathcal{C})) \leftrightarrow ((\mathcal{A} \circ \mathcal{B}) \to \mathcal{C})$

(Q18) $\forall x\mathcal{A} \to \mathcal{A}[\tau/x]$, where τ is free for x in $\mathcal{A}$ Universal Instantiation

Rules

(MP) $\dfrac{\vdash \mathcal{A} \to \mathcal{B} \qquad \vdash \mathcal{A}}{\vdash \mathcal{B}}$ (RIC) $\dfrac{\vdash \mathcal{A} \to \mathcal{B}}{\vdash \mathcal{A} \to \forall x\mathcal{B}}$

(ADJ) $\dfrac{\vdash \mathcal{A} \qquad \vdash \mathcal{B}}{\vdash \mathcal{A} \wedge \mathcal{B}}$

The rule (RIC) comes with the condition that x is not free in $\mathcal{A}$. Here, because I will not be proving or reproducing many of the meta-logical theorems concerning this logic, all of the connectives present will be taken as primitive. However, some of these connectives can be taken to be defined in the sense indicated in the section on notational conventions.

The logic $\mathbf{RQ}^{\circ\boldsymbol{t}}$ is results from adding the axiom (Q19).

(Q19) $(\mathcal{A} \wedge \exists x\mathcal{B}) \to \exists x(\mathcal{A} \wedge \mathcal{B})$, where x is not free in $\mathcal{A}$

The reader might make the observation that the rule of universal generalization,

$$\text{(UG)}\ \frac{\vdash \mathcal{A}}{\vdash \forall x\mathcal{A}}$$

is not a primitive rule. However, it is derivable. I will state this as a lemma.

Lemma 2.2.1. *The rule (UG) is derivable in* $\mathbf{RQ}^{\circ\boldsymbol{t}}$ *and* $\mathbf{QR}^{\circ\boldsymbol{t}}$.

The proof of this lemma uses axioms (Q16), (Q15), and the rules (MP) and (RIC) [74, p. 175–6]. This lemma is stated here, for the axiomatizations I will give of quantified modal relevant logics with these axioms will thus also have (UG) as a derivable rule. Mares and Goldblatt also give a proof of the following lemma. I will state the lemma and list the axioms and rules used in Mares and Goldblatt's proof of the lemma. Again, this will be for later use when studying the properties of quantified modal relevant logic.

Lemma 2.2.2. *The following formulas are theorems of both* $\mathbf{RQ}^{\circ\boldsymbol{t}}$ *and* $\mathbf{QR}^{\circ\boldsymbol{t}}$*.*

(i) $\forall x(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \forall x\mathcal{B})$*, where* x *is not free in* $\mathcal{A}$

(ii) $\forall x(\mathcal{A} \to \mathcal{B}) \to (\forall x\mathcal{A} \to \forall x\mathcal{B})$

(iii) $\mathcal{A} \leftrightarrow \forall x\mathcal{A}$*, where* x *is not free in* $\mathcal{A}$

(iv) $\forall x\forall y\mathcal{A} \to \forall y\forall x\mathcal{A}$

(v) $\exists x\forall y\mathcal{A} \to \forall y\exists x\mathcal{A}$

The proof of (i) requires axioms (Q17), (Q18), and rule (RIC). (ii) requires (Q17), (Q18), the rule (RIC) and "transitivity of $\to$" [74, p. 176]. For (iii), axioms (Q17), (Q18) and rules (RIC) and (ADJ) are sufficient. For (iv), (Q18) and (RIC) are enough. Finally, for (v), axioms (Q18), (Q14), and the rule (RIC) suffice.

Having presented the axiom system that provided the inspiration for Mares and Goldblatt's semantics, and having stated a few lemmas that should be applicable to quantified modal relevant logic, I now turn to the semantics of $\mathbf{RQ}^{\circ\boldsymbol{t}}$ and $\mathbf{QR}^{\circ\boldsymbol{t}}$.

2.2.2 Semantics for $\mathbf{QR}^{\circ\boldsymbol{t}}$ and $\mathbf{RQ}^{\circ\boldsymbol{t}}$

I have chosen Mares and Goldblatt's semantics for $\mathbf{RQ}^{\circ\boldsymbol{t}}$ and $\mathbf{QR}^{\circ\boldsymbol{t}}$ for a number of reasons, including the following. First, Mares and Goldblatt's semantics is a general frame semantics. General frame semantics appear to be advantageous for completeness results for a number of logics. The second is that it is more natural than Fine's semantics. This naturalness appears to follow from a motivating idea based on the proof theory given by Mares and Goldblatt:

The idea comes from the proof theory for quantified relevant logic. We start with the schema,

$$\text{(UI)}\ \ \forall x\mathcal{A} \to \mathcal{A}[\tau/x]$$

and the rule,

$$\text{(RIC)}\ \frac{\vdash \mathcal{A} \to \mathcal{B}}{\vdash \mathcal{A} \to \forall x\mathcal{B}}$$

(restriction on the rule: x does not occur free in $\mathcal{A}$). …[Mares and Goldblatt's] truth condition says:

> '$\forall x\mathcal{A}$' is true at a world a if and only if there is some proposition X such that (i) X entails that $\mathcal{A}$ holds of every individual and (ii) X obtains at a.

Thus, in order to incorporate RIC into the semantics, we add propositions to the modal theory. [74, p. 164]

By 'propositions' is meant (certain types of) sets of possible worlds. More specifically, given the relevant logic basis, a proposition is a set of worlds that are upwardly closed under a reflexive and transitive relation on the worlds defined with reference to a special subset of worlds — the worlds that are logically normal — and the ternary relation.

Mares and Goldblatt compare their semantic condition to that found in the interpretation of intuitionistic logic called the Brouwer-Heyting-Komologorov (BHK) interpretation. On the BHK interpretation, "a universally quantified formula $\forall x\mathcal{A}(x)$ is proved if and only if there is a proof that takes any object i and returns a proof of $\mathcal{A}(i)$" [74, p. 164–5]. Mares and Goldblatt make an analogy between the propositions in their semantics and the proofs of the BHK interpretation, in which we can see that a universally quantified formula $\forall x\mathcal{A}(x)$ is a particular proposition that implies all propositions $\mathcal{A}(i)$, for each i.

2.2.3 $\mathbf{QR}^{\circ t}$ and $\mathbf{RQ}^{\circ t}$ Models

Mares and Goldblatt [74] first define frames for $\mathbf{QR}^{\circ t}$, then discuss the modifications required for $\mathbf{RQ}^{\circ t}$-frames. I will follow their order of presentation. This section contains more or less a reproduction of the most pertinent parts of Mares and Goldbatt's work [74] on quantified relevant logic. I will not replicate their soundness and completeness proofs here, for example, but I will fully explicate their semantic structures in order to demonstrate the ideas used in a later section on quantified modal relevant logic. I have also made slight notational changes to facilitate reading this dissertation.

Propositions

We have already seen that Mares and Goldblatt use propositions. The set of propositions in the general frame semantics here will be a subset of the hereditary subsets (up-sets) of possible worlds. We will also call these *admissible propositions.*

As we saw above, in the section for general frame semantics for $\mathbf{R}^{\circ \boldsymbol{t}}$, the operations on $\wp(K)$, the set of possible worlds, are defined as follows.

Induced by the ternary relation, the operator $\Rightarrow$ on $\wp(K)$ is defined as, for every $X, Y \in \wp(K)$,

$$X \Rightarrow Y = \{w : \forall x \forall y (Rwxy \text{ and } x \in X \text{ implies } y \in Y)\}.$$

A unary operator $*$ on $\wp(K)$ is defined by lifting $*$ to sets of worlds. For every $X \in \wp(K)$,

$$X^* = \{w : w^* \notin X\}.$$

We also include the operators $\cup$ and $\cap$ on the powerset $\wp(K)$.

The final operation to discuss before moving on to propositional functions is an operation from sets of sets of worlds to sets of worlds. This operation is determined by *Prop*, as *Prop* is used in the definition of the operation. When *Prop* is a set of hereditary subsets of possible worlds, an operation $\sqcap$ of type $\sqcap : \wp\wp K \to \wp K$ such that, for every $S \subseteq \wp(K)$

$$\sqcap S = \cup\{X \in Prop : X \subseteq \cap S\}$$

This notion is explained by Goldblatt (in his 2011 book) as being "motivated by the intuition that the sentence $\forall x\phi$ expresses the conjunction of all the sentences $\phi[a/x]$" [47, p. 17]. The problem with using the arbitrary conjunction of the sentences $\phi[a/x]$ is that this conjunction is not guaranteed to be admissible. We only guarantee that the binary conjunction of two members of *Prop* is also in *Prop*. Thus, Goldblatt explains that a notion of entailment between propositions (of the sort considered) will work in this case. First, given our notion of proposition, a proposition X entails a proposition Y if $X \subseteq Y$, for whenever you have a world at which X is true, Y is also true. Goldblatt also calls Y *weaker* than X, and X *stronger* than Y [47, p. 17]. Next, the operation similar enough to the arbitrary conjunction of a collection of elements of $Prop$ is a member of $Prop$ that entails every member of the collection. In particular, it is the *weakest* member of $Prop$ that entails every member of the collection. That is, for a collection of members of $Prop$ $\{X_i : i \in I\}$, the proposition sufficiently expressing their conjunction is the X in $Prop$ such that,

(i) $X \subseteq X_i$ for each $i \in I$, and

(ii) if $Z \in Prop$ and $Z \subseteq X_i$ for all $i \in I$, then $Z \subseteq X$. [47, p. 17]

The key point here is that this conjunction-like operation always results in a subset of $\bigcap_{i \in I} X_i$, but not necessarily equal. The latter ($\cap$) is not guaranteed to be a member of $Prop$, while we require that the former ($\sqcap$) is. This conjunction-like operator is denoted by $\sqcap_{i \in I} X_i$.

Propositional Functions

Propositional functions, on the other hand, are functions from value assignments for variables to admissible propositions. In addition to using a set $Prop$ of admissible propositions, Mares and Goldblatt also use a set $PropFun$ of admissible propositional functions.

A propositional function for classical logic is a function from value assignments for the variables into the set $\{True, False\}$, where a proposition is either true or false. For logics with possible world semantics, propositional functions are functions into the set of propositions, where propositions are taken to be a subset of the powerset of worlds. For example, consider the sentence

"Socrates is human."

This sentence is an instance of the function *x is a human* applied to a value assignment to the variables which assigns to x the object *Socrates*. In classical logic, this application of a function would result in either $True$ or $False$. In the case of quantified relevant logic, this function application returns a set of upwardly closed worlds in which "Socrates is a human" is true.

A propositional function in the context of the semantics of Mares and Goldblatt is a function from value assignments of the variables to admissible propositions. That is, a propositional function ϕ is of type ϕ: $U^\omega \longrightarrow Prop$, where U is the domain of individuals in a model. While *admissible* propositional functions are required for the semantics of $\mathbf{RQ}^{\circ \boldsymbol{t}}$, motivations for adopting admissible propositions can be turned into motivations for admissible propositional functions.

Definition 2.2.3. For any two elements ϕ and ψ of $PropFun$, the functions $\phi \cap \psi$, $\phi \Rightarrow \psi$, and ϕ^* of the same type are defined by, for every value assignment to the variables $f \in U^\omega$:

$$(\phi \cap \psi)f = \phi f \cap \psi f$$
$$(\phi \Rightarrow \psi)f = \phi f \Rightarrow \psi f$$
$$(\phi^*)f = (\phi f)^* \quad \text{[74, p. 167]}$$

Often, the set of propositions $\{X_i : i \in I\}$ is constructed by considering all formulas that result from a rewriting of another formula. In particular, we will want to express the conjunction-like operation of every instantiation of a variable. When we have a propositional function ϕ, we will define the functions $\forall_n \phi$ of the same type for each $n \in \omega$ by the following:

$$(\forall_n \phi)f = \bigsqcap_{j \in I} \phi(f[j/n]) \tag{2.1}$$

In this case, '$\bigsqcap_{j \in I} \phi(f[j/n])$' is a convenient shorthand for the $\sqcap$ operation applied to a particular set of propositions.

Frames and Models

Definition 2.2.4. A $\mathbf{QR}^{\circ \boldsymbol{t}}$-frame is a tuple $\mathfrak{F} = \langle K, 0, R, *, U, Prop, PropFun \rangle$, where $\langle K, 0, R, * \rangle$ is an $\mathbf{R}$-frame, U is a non-empty set, $Prop$ is a set of hereditary subsets of K based on $\leq$, $Prop$ contains 0, and $PropFun$ is a subset of the functions from U^ω to $Prop$, such that C1–C6 hold:

C1 If X and Y are in $Prop$, then $X \cap Y, X \Rightarrow Y, X^* \in Prop$.

C2 The constant function ϕ_0 — for every f, $\phi_0 f = 0$ — is in $PropFun$.

C3 If $\phi, \psi \in PropFun$, then $\phi \Rightarrow \psi \in PropFun$.

C4 If $\phi, \psi \in PropFun$, then $\phi \cap \psi \in PropFun$.

C5 If $\phi \in PropFun$, then $\phi^* \in PropFun$.

C6 If $\phi \in PropFun$, then $\forall_n \phi \in PropFun$ for every $n \in \omega$.

A $\mathbf{QR}^{\circ t}$-frame is called *full* if the set $Prop$ contains *every* hereditary subset of K, and $PropFun$ contains *every* function from U^ω to $Prop$.

Remark 2.2.5. All full, or Kripkean, frames are general frames. That is, the frames wherein every up-set of possible worlds is a member of $Prop$ is a $\mathbf{QR}^{\circ t}$-frame. This does appear to lead to an interesting anomaly, if one's motivations for adopting propositions include the rejection that every arbitrary set of worlds, or in this case every hereditary subset of worlds, is a proposition. This kind of motivation might follow from the intuition that there is nothing that guarantees that every hereditary subset of worlds has something in common. Given that all full frames are general frames, there are full frames for which $\mathbf{QR}^{\circ t}$ is sound (given theorems stated below). However, in these full frames, every arbitrary hereditary set of worlds is a proposition, which appears to clash with the motivations given. By expanding the set of frames, we do not get rid of troublesome frames, even though we make distinction between propositions and up-sets of worlds. Is this sufficient to satisfy these motivations for adopting general frames? If the answer is in the negative, then can we show the set of non-full $\mathbf{QR}^{\circ t}$-frames are sufficient for soundness and completeness?

Following the exposition in [47], we will define both pre-models and models based on these frames.

Definition 2.2.6. A *pre-model* $\mathcal{M}$ on a $\mathbf{QR}^{\circ t}$-frame $\mathfrak{F}$ is given by an assignment function $|-|^{\mathcal{M}}$ that assigns

1. an element $|c|^{\mathcal{M}} \in U$ to each constant $c \in \mathcal{L}$;

2. a function $|P|^{\mathcal{M}} : U^n \longrightarrow \wp K$ to each n-ary predicate symbol $P \in \mathcal{L}$;

3. a propositional function $|\mathcal{A}|^{\mathcal{M}} : U^{\omega} \longrightarrow \wp K$ to each $\mathcal{L}$-formula $\mathcal{A}$.

The propositional function assigned to the atomic formula $P\tau_1, \ldots, \tau_n$, is given by

$$|P\tau_1, \ldots, \tau_n|^{\mathcal{M}} f = |P|^{\mathcal{M}}(|\tau_1|^{\mathcal{M}} f, \ldots, |\tau_n|^{\mathcal{M}} f)$$

for each $f \in U^{\omega}$. When $\mathcal{A}$ is not atomic, the propositional function assigned to $\mathcal{A}$ is given by the following:

$$\begin{aligned}
|\boldsymbol{t}|^{\mathcal{M}} &= \phi_0 \\
|\mathcal{A} \wedge \mathcal{B}|^{\mathcal{M}} &= |\mathcal{A}|^{\mathcal{M}} \cap |\mathcal{B}|^{\mathcal{M}} \\
|\neg\mathcal{A}|^{\mathcal{M}} &= (|\mathcal{A}|^{\mathcal{M}})^* \\
|\mathcal{A} \to \mathcal{B}|^{\mathcal{M}} &= |\mathcal{A}|^{\mathcal{M}} \Rightarrow |\mathcal{B}|^{\mathcal{M}} \\
|\forall x\mathcal{A}|^{\mathcal{M}} &= \forall_x |\mathcal{A}|^{\mathcal{M}}
\end{aligned}$$

Corresponding to these propositional function assignments are "truth set" assignments, which assign a set of worlds to each proposition-function pair. Where $|\mathcal{A}|^{\mathfrak{M}}$ is a function, $|\mathcal{A}|^{\mathfrak{M}} f$ is a member of $Prop$. Keeping a variable assignment f constant, the truth sets are defined as follows:

$$\begin{aligned}
|\boldsymbol{t}|^{\mathcal{M}} f &= 0 \\
|\mathcal{A} \wedge \mathcal{B}|^{\mathcal{M}} f &= |\mathcal{A}|^{\mathcal{M}} f \cap |\mathcal{B}|^{\mathcal{M}} f \\
|\neg\mathcal{A}|^{\mathcal{M}} f &= (|\mathcal{A}|^{\mathcal{M}} f)^* \\
|\mathcal{A} \to \mathcal{B}|^{\mathcal{M}} f &= |\mathcal{A}|^{\mathcal{M}} f \Rightarrow |\mathcal{B}|^{\mathcal{M}} f \\
|\forall x\mathcal{A}|^{\mathcal{M}} f &= \sqcap_{g \in xf} |\mathcal{A}|^{\mathcal{M}} g
\end{aligned}$$

Again, where $g \in xf$ is every x-variant of f.

Definition 2.2.7. A *model* based on $\mathbf{QR}^{\circ \boldsymbol{t}}$-frame $\mathfrak{F}$ is a pre-model based on $\mathfrak{F}$ such that $|p|^{\mathcal{M}} \in PropFun$ for every atomic formula p.

Given the closure conditions required of $PropFun$, it follows that in a model $\mathcal{M}$, for every formula $\mathcal{A}$, $|\mathcal{A}|^{\mathcal{M}} \in PropFun$, and for every model $\mathcal{M}$, $|\mathcal{A}|^{\mathcal{M}} f \in Prop$, for every formula $\mathcal{A}$ and every value assignment to the variables $f \in U^{\omega}$ [47, p. 211–2].

Definition 2.2.8. A formula $\mathcal{A}$ is *valid* in the model $\mathcal{M}$ if $0 \subseteq |\mathcal{A}|^{\mathcal{M}} f$ for every $f \in U^{\omega}$. Further, $\mathcal{A}$ is valid on a $\mathbf{QR}^{\circ t}$-frame $\mathfrak{F}$ if $\mathcal{A}$ is valid on every model based on $\mathfrak{F}$. Finally, $\mathcal{A}$ is valid on a class of frames if it is valid for every frame in the class.

As defined, the semantics above is adequate for the logic $\mathbf{QR}^{\circ t}$. I will briefly state the soundness and completeness theorems.

Theorem 2.2.9 (Soundness)**.** *Every theorem of* $\mathbf{QR}^{\circ t}$ *is valid on the class of all* $\mathbf{QR}^{\circ t}$*-frames.*

Theorem 2.2.10 (Completeness)**.** *If* $\mathcal{A}$ *is valid in the class of all* $\mathbf{QR}^{\circ t}$*-frames, then* $\mathcal{A}$ *is a theorem of* $\mathbf{QR}^{\circ t}$*.*

These theorems were proven by Mares and Goldblatt in [74, Theorem 9.7].

The logic $\mathbf{RQ}^{\circ t}$ is obtained from $\mathbf{QR}^{\circ t}$ by the addition of the axiom scheme $\mathcal{A} \wedge \exists x\mathcal{B} \rightarrow \exists x(\mathcal{A} \wedge \mathcal{B})$, where x is not free in $\mathcal{A}$. The models for $\mathbf{RQ}^{\circ t}$ are constructed by adding a further restriction on the $\mathbf{QR}^{\circ t}$-models.

Definition 2.2.11. The set of $\mathbf{RQ}^{\circ t}$*-models* is the subset of $\mathbf{QR}^{\circ t}$-models that satisfy the following condition:

$$X - Y \subseteq \bigcap_{a \in U} \phi(f[a/x]) \text{ implies } X - Y \subseteq (\forall_x \phi) f$$

for all $\phi \in PropFun$, all $x \in Var$, all $X, Y \in Prop$, and all $f \in U^{\omega}$ [47, p. 213].

As defined, $\mathbf{RQ}^{\circ t}$ is sound and complete for the $\mathbf{RQ}^{\circ t}$ -models.

2.2.4 Key Features of Mares and Goldblatt's Semantics

The most important feature of Mares and Goldblatt's semantics for quantified relevant logics that will carry over to the quantified modal relevant logics is the treatment of the quantifier(s) in the semantics. This is a multifaceted feature that involves the use of general frames (admissible propositions), propositional functions, and the naturalness from the motivation of the semantic conditions.

The reasons I am using this treatment of the quantifiers in the semantics over, say, Fine's are as follows. First, the condition is natural. That is, the proof-theoretic origin of this treatment of the quantifiers lets us formalize an interpretation of the syntax into the semantics. This is natural because one role of a semantics for a logic is to interpret the logic. By formalizing an informal interpretation, and finding that the desired results follow, our informal interpretation is given additional support and the semantics is backed by both formal results and informal reasoning.

Second, the semantics is already a general frame semantics, which has been used to better study the Barcan formula and its converse and the semantic conditions that correspond to these formulas. Goldblatt [47], for quantified modal *classical* logic, shows how semantic conditions for the Barcan formulas are quite different from those for the usual Kripkean possible worlds semantics. In particular, the well-known correspondence between the Barcan Formulas and their semantic conditions in full or Kripkean models does not hold in general frame semantics for quantified modal classical logic.

A frame (in the semantics of quantified modal classical logic) is *contracting* if Rxy implies the domain of x is a subset of the domain of y, and it is *expanding* if Rxy implies the domain of y is a subset of the domain of x. Further, a frame has *constant domains* if frame is both contracting and expanding. The Barcan formula $\forall x\Box\mathcal{A} \to \Box\forall x\mathcal{A}$ and the Converse Barcan Formula $\Box\forall x\mathcal{A} \to \forall x\Box\mathcal{A}$ correspond to contracting domains and expanding domains, respectively, in quantified modal classical logic. However, in admissible semantics, this correspondence breaks down. The Converse Barcan formula ends up valid in models with expanding domains, but also in models with constant domains. This fact holds even when the Barcan formula is not valid [47, p. 67].

Quantified modal classical logics with the Converse Barcan Formula are characterized by admissible semantics with expanding domains, as expected. However, these logics are also characterized by contracting domains, even without the Barcan formula being valid [47, p. 67]. Further, the constant domain admissible semantics are sufficient to characterize quantified modal classical logics with the Converse Barcan Formula [47, p. 67]. Again, this holds even when the Barcan formula is not valid.

Importantly, general frame semantics are required, in some sense. This will be the next point. It follows from this requirement that the general frame conditions corresponding to the Barcan Formulas have more force: we ought to think of the Barcan and Converse Barcan formula relative to their role in general frame semantics as opposed to their role in Kripkean models. Further, in section 4.7, I will show the behavior of the Barcan Formula and its converse in quantified modal relevant logics. It is that behavior which ought to guide our interpretations of what the Barcan formulas *really* mean.

Third, Kripkean models are insufficient to characterize quantified modal *classical* logics. Again, Goldblatt shows that some quantified modal classical logics are not complete with respect to their "corresponding" Kripkean models. The Kripkean models do not make fine enough distinctions for some logics. That is, there are truths that cannot be separated, but would need to be for a completeness result. For those that they do, Kripkean models remain a special case of general frame models, so there appears to be no harm in using general frame models.

2.3 Modal Relevant logics

As noted above, the general frame semantics for modal relevant logics were published by Seki [103]. In his 2003 paper, Seki introduces general frame semantics in order to prove Sahlqvist's theorem for modal relevant logics. Seki's other 2003 article [102] proves a duality between general frame semantics and algebras for modal relevant logic.[2] The general frame semantics found in these papers will provide the framework which I will use to base the modal parts of the semantics for quantified modal relevant logic. To familiarize the reader with the features I will use later, I will provide an axiom system for modal relevant logics, then reproduce Seki's general frame semantics.

The naming conventions for modal relevant logics here are like those stated above. Seki's base modal relevant logic is $\mathbf{B.C}_{\Box\Diamond}$, which will be defined below. As a reminder to the reader, the letters before the dot represent the relevant logic fragment, and the letters after the dot represent the modal logic fragment. However, the dot is removed from the notation when an axiom is added to sufficiently strong (w.r.t. negation) modal relevant logics with inter-definable modalities so that the logic contains the corresponding modal logic on suitable translation. Belnap suggests the axiom $\Box(\mathcal{A} \vee \mathcal{B}) \to (\Diamond\mathcal{A} \vee \Box\mathcal{B})$, as mentioned above. However, given a background of $\mathbf{R}$, this axiom is equivalent to the axiom $\Diamond\mathcal{A} \wedge \Box\mathcal{B} \to \Diamond(\mathcal{A} \wedge \mathcal{B})$ which Seki suggests adding to $\mathbf{R.K}$ to produce $\mathbf{RK}$ [103, p. 408].

Thus there are two criteria that have to be met for a modal relevant logic's name

[2] Other relevant works of Seki's include [104, 108, 105, 106, 109, 107].

to lack the dot. It must contain all suitably translated theorems of the corresponding modal classical logic and the modalities must be inter-definable in the usual way.

Drawing on Seki's papers, and the work of Mares and Meyer, I will present modal relevant logics. The next section presents the axioms systems for modal relevant logics, and the section that follows gives the construction of their semantics.

2.3.1 Syntax

Following Seki, we begin with the modal relevant logic $\mathbf{B.C}_{\Box\Diamond}$, in which $\Box$ and $\Diamond$ are not taken to be defined via $\Box\mathcal{A} =_{df} \neg\Diamond\neg\mathcal{A}$ or $\Diamond\mathcal{A} =_{df} \neg\Box\neg\mathcal{A}$. The logic $\mathbf{B.C}_{\Box\Diamond}$ includes a potentially denumerable set of propositional variables, a constant $\boldsymbol{t}$, the logical connectives $\rightarrow, \wedge, \vee, \circ, \neg$ and the modal operators $\Box$ and $\Diamond$. The set of well-formed formulas, atomic propositions, and literals will be defined as usual. Further, Seki defines $\leftrightarrow$ in the usual way via conjunction and implication. Moreover, two additional modal operators are defined by setting

$$\boxdot\mathcal{A} =_{def} \neg\Diamond\neg\mathcal{A}$$

$$\Diamond\!\!\!\cdot\,\mathcal{A} =_{def} \neg\Box\neg\mathcal{A}$$

The axiom system for $\mathbf{B.C}_{\Box\Diamond}$ consists of the axioms and rules of $\mathbf{B}^{\circ\boldsymbol{t}}$ with the following:

BC1 $\Box\mathcal{A} \wedge \Box\mathcal{B} \rightarrow \Box(\mathcal{A} \wedge \mathcal{B})$

BC2 $\Diamond(\mathcal{A} \vee \mathcal{B}) \rightarrow \Diamond\mathcal{A} \vee \Diamond\mathcal{B}$

$$\frac{\mathcal{A} \rightarrow \mathcal{B}}{\Box\mathcal{A} \rightarrow \Box\mathcal{B}}\ (\Box\text{-monotonicity}) \qquad\qquad \frac{\mathcal{A} \rightarrow \mathcal{B}}{\Diamond\mathcal{A} \rightarrow \Diamond\mathcal{B}}\ (\Diamond\text{-monotonicity})$$

The least logic containing the axioms and rules of $\mathbf{B}^{\circ\boldsymbol{t}}$ together with axioms BC1 and BC2 and rules $\Box$-monotnonicity, and $\Diamond$-monotonicity is the logic $\mathbf{B.C}_{\Box\Diamond}$—the base modal relevant logic for Seki.

Let $\mathbb{L}$ be a relevant logic extending $\mathbf{B}^{\circ\boldsymbol{t}}$. Seki defines a *regular relevant modal logic over* $\mathbb{L}$ if it includes axioms BC1 and BC2 and rules $\Box$-monotnonicity, and $\Diamond$-monotonicity [103, p. 386]. The *least regular logic over* $\mathbb{L}$ is called $\mathbb{L}.\mathbf{C}_{\Box\Diamond}$. That is, in this document, all regular relevant modal logics will include both '$\circ$' and '$\boldsymbol{t}$'.

2.3.2 Semantics for Modal Relevant Logics

Here, I will present Seki's general frame semantics for the modal relevant logic $\mathbf{B.C}_{\Box\Diamond}$, then list the correspondences between axioms and frame conditions for the extensions of $\mathbf{B.C}_{\Box\Diamond}$. In particular, the logic **R4** is an important extension.

Definition 2.3.1. A *general* $\boldsymbol{B.C}_{\Box\Diamond}$*-frame* is a tuple $\mathfrak{F} = \langle K, 0, R, S_\Box, S_\Diamond, *, e, Prop\rangle$ where K is a set of elements (worlds, situations, etc.), 0 is a non-empty subset of K, R is a ternary relation on K, $S_\Box$ and $S_\Diamond$ and binary relations on K, $*$ is a unary operation on K, $e \in K$ (the *null* world), and the conditions that follow are satisfied. Where $u = e^*$ and $\leq$ is defined on K as it was for propositional relevant logics via 0,

(c1) $a \leq a$

(c2) $a \leq b$ and $Rbcd \Rightarrow Racd$

(c3) $a \leq c$ and $Rbcd \Rightarrow Rbad$

(c4) $d \leq a$ and $Rbcd \Rightarrow Rbca$

(c5) $Ruab \Rightarrow a = e$ or $b = u$

(c6) $Reue$

(c7) $a \leq b \Rightarrow b^* \leq a^*$

(c8) $a^{**} = a$

(c9) $a \leq b$ and $S_\Box bc \Rightarrow S_\Box ac$

(10) $S_\Box ee$

(c11) $S_\Box ua \Rightarrow a = u$

(c12) $a \leq b$ and $S_\Diamond ac \Rightarrow S_\Diamond bc$

(c13) $S_\Diamond ea \Rightarrow a = e$

(c14) $S_\Diamond uu$

(c15) $a \leq b$ and $a \in 0 \Rightarrow b \in 0$

(c16) $e \neq u$

The definition and notation for the set of up-sets of elements of K is as follows. Given a set S and a relation on the set $\leq$, I will write $\wp(S)^{\uparrow}_{\leq}$ to denote the set of up-sets or cones of elements of S under the relation $\leq$. When the relation is obvious, I will drop the subscript from the notation. For Seki's semantics, we will define the set of cones so that it does not contain the empty set or K itself (without changing the notation). That is,

$$\wp(K)^{\uparrow} = \{X \subseteq K : X \neq \emptyset \text{ and } X \neq K \text{ and } \forall a \forall b (a \in X \ \& \ a \leq b \Rightarrow b \in X)\}$$

Finally, $Prop$ is a non-empty subset of $\wp(K)^{\uparrow}$ which contains 0, is closed under $\cup$ and $\cap$, and the operations $\Rightarrow$, $\cdot$, $*$, $\Box$, and $\Diamond$, which are defined as follows. For every

$X, Y \subseteq K$,

$$X \Rightarrow Y = \{a \in K : \forall b \forall c(Rabc \;\&\; b \in X \Rightarrow c \in Y)\}$$

$$X \cdot Y = \{a \in K : \exists b \exists c(Rbca \;\&\; b \in X \;\&\; c \in Y)\}$$

$$X^* = \{a \in K : a^* \notin X\}$$

$$\Box X = \{a \in K : \forall b(S_\Box ab \Rightarrow b \in X)\}$$

$$\Diamond X = \{a \in K : \exists b(S_\Diamond ab \;\&\; b \in X)\}$$

Seki shows that we can also define the binary relations used to model the dotted modalities as follows. For every $a, b, \in K$,

$$S_{\boxdot} ab \text{ iff } S_\Diamond a^* b^*$$

$$S_{\odot} ab \text{ iff } S_\Box a^* b^*$$

Further, we can define lifted operations for $\odot$ and $\boxdot$ under which $Prop$ is closed, based on these definitions. Such a definition is straightforward.

Definition 2.3.2. A *general* $\boldsymbol{B.C}_{\Box\Diamond}$*-model* is a tuple $\mathfrak{M} = \langle K, 0, R, S_\Box, S_\Diamond, *, e, Prop,$ $|-|^{\mathfrak{M}}\rangle$, where $\langle K, 0, R, S_\Box, S_\Diamond, *, e, Prop\rangle$ is a $\mathbf{B.C}_{\Box\Diamond}$-frame and $|-|^{\mathfrak{M}}$ is a function that assigns to each atomic propositions a member of $Prop$. Thus assignment is extended to wff by the following clauses:

$$|\mathcal{A} \wedge \mathcal{B}|^{\mathfrak{M}} = |\mathcal{A}|^{\mathfrak{M}} \cap |\mathcal{B}|^{\mathfrak{M}}$$

$$|\mathcal{A} \vee \mathcal{B}|^{\mathfrak{M}} = |\mathcal{A}|^{\mathfrak{M}} \cup |\mathcal{B}|^{\mathfrak{M}}$$

$$|\mathcal{A} \to \mathcal{B}|^{\mathfrak{M}} = |\mathcal{A}|^{\mathfrak{M}} \Rightarrow |\mathcal{B}|^{\mathfrak{M}}$$

$$|\mathcal{A} \circ \mathcal{B}|^{\mathfrak{M}} = |\mathcal{A}|^{\mathfrak{M}} \cdot |\mathcal{B}|^{\mathfrak{M}}$$

$$|\neg \mathcal{A}|^{\mathfrak{M}} = (|\mathcal{A}|^{\mathfrak{M}})^*$$

$$|\Box \mathcal{A}|^{\mathfrak{M}} = \Box(|\mathcal{A}|^{\mathfrak{M}})$$

$$|\Diamond \mathcal{A}|^{\mathfrak{M}} = \Diamond(|\mathcal{A}|^{\mathfrak{M}})$$

$$|\boldsymbol{t}|^{\mathfrak{M}} = 0$$

From this, a relation $\vDash_{\mathfrak{M}}$ between K and *wff* is definable for each model. (The subscript will be omitted for convenience.) The relation $\vDash$ is defined inductively as follows:

1. For any atomic proposition p, $a \vDash p$ iff $a \in |p|^{\mathfrak{M}}$
2. $a \vDash \mathcal{A} \wedge \mathcal{B}$ iff $a \vDash \mathcal{A}$ and $a \vDash \mathcal{B}$
3. $a \vDash \mathcal{A} \vee \mathcal{B}$ iff $a \vDash \mathcal{A}$ or $a \vDash \mathcal{B}$
4. $a \vDash \mathcal{A} \rightarrow \mathcal{B}$ iff $\forall b, c((Rabc \;\&\; b \vDash \mathcal{A}) \Rightarrow c \vDash \mathcal{B})$
5. $a \vDash \mathcal{A} \circ \mathcal{B}$ iff $\exists b, c(Rbca \;\&\; b \vDash \mathcal{A} \;\&\; c \vDash \mathcal{B})$
6. $a \vDash \neg\mathcal{A}$ iff $a^* \nvDash \mathcal{A}$
7. $a \vDash \Box\mathcal{A}$ iff $\forall b(S_{\Box}ab \Rightarrow b \vDash \mathcal{A})$
8. $a \vDash \Diamond\mathcal{A}$ iff $\exists b(S_{\Diamond}ab \;\&\; b \vDash \mathcal{A})$
9. $a \vDash \boldsymbol{t}$ iff $a \in 0$

The dotted operators have corresponding relations:

10. $a \vDash \boxdot\mathcal{A}$ iff $\forall b(S_{\boxdot}ab \Rightarrow b \vDash \mathcal{A})$
11. $a \vDash \diamonddot\mathcal{A}$ iff $\exists b(S_{\diamonddot}ab \;\&\; b \vDash \mathcal{A})$

An important result is the hereditary lemma. This can be proven by showing that the defined operations on subsets of K preserve the property of being in $Prop$. In other words, every truth set is a member of $Prop$, all of which are up-sets. Thus the following lemma holds.

Lemma 2.3.3. *Let $\mathfrak{M} = \langle K, 0, R, S_{\Box}, S_{\Diamond}, *, e, Prop, |-|^{\mathfrak{M}}\rangle$ be a $\boldsymbol{B.C}_{\Box\Diamond}$-model. It follows that if $a \leq b$ and $a \vDash \mathcal{A}$ then $b \vDash \mathcal{A}$ for all $\mathcal{A} \in$wff.*

Finally, we say that a wff $\mathcal{A}$ is true in a model iff $0 \subseteq |\mathcal{A}|^{\mathfrak{M}}$. A wff is valid in a frame $\mathfrak{F}$ iff $\mathcal{A}$ is true in every model based on $\mathfrak{F}$. Finally, given a class of frames C, the wff $\mathcal{A}$ is valid in the class of frames C iff it is valid in every frame in the class.

Soundness is easy to show. Completeness is interesting, so I will make note of a few features of the proof in Seki [103].

Let $\mathfrak{M} = \langle K_c, 0_c, R_c, S_{\Box c}, S_{\Diamond c}, *_c, e_c, |{-}|_c^{\mathfrak{M}}\rangle$ be the canonical model for $\mathbf{B.C}_{\Box\Diamond}$. Note that this is for non-general frames. The general frame is constructed by taking

$$Prop_c = \{|\mathcal{A}|^{\mathfrak{M}} : \mathcal{A} \in \mathit{wff}\}$$

The model based on this general frame uses the same valuation.

Using this canonical general frame, and the canonical model based on it, we can prove completeness quite easily, given the completeness proof using the non-general canonical frame.

Extensions of $\mathbf{B.C}_{\Box\Diamond}$

The logic $\mathbf{B.C}_{\Box\Diamond}$ is "weak" in that its relevant logic fragment is the base affixing logic **B** and its modal fragment is strictly weaker than **K**. Extending the relevant logic fragment of this logic is done syntactically by the axioms and semantically by the frame conditions found in the section on relevant logic. Some extensions of $\mathbf{B.C}_{\Box\Diamond}$ that add modal axioms are the focus here.

Here, a list of axioms, a rule, and their corresponding frame conditions is given. These correspondences are found in Seki [103].

	Axiom	**Frame Condition**
K$\Box$	$\Box(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})$	If $Rbcf$ and $S_\Box fd$, there there exist $b', c' \in K$ such that $S_\Box bb'$, $S_\Box cc'$, and $Rb'c'd$.
K$\Diamond$	$\Box(\mathcal{A} \to \mathcal{B}) \to (\Diamond\mathcal{A} \to \Diamond\mathcal{B})$	If $Rbcd$ and $S_\Diamond cf$, then there exists $b', d' \in K$ such that $Rb'fd'$, $S_\Box bb'$, and $S_\Diamond dd'$.
SC	$\Box(\Box\mathcal{A} \to \mathcal{B}) \vee \Box(\Box\mathcal{B} \to \mathcal{A})$	If $S_\Box ab$, $Rbcd$, $S_\Box ab'$, and $Rb'c'd'$, then $S_\Box cd'$ or $S_\Box c'd$.
CON	$\Box((\mathcal{A} \wedge \Box\mathcal{A}) \to \mathcal{B}) \vee$ $\Box((\mathcal{B} \wedge \Box\mathcal{B}) \to \mathcal{A})$	If $S_\Box ab$, $Rbcd$, $S_\Box ab'$, and $Rb'c'd'$, then $c \leq d'$ or $c' \leq d$ or $S_\Box cd'$ or $S_\Box c'd$.
Alt$_n$	$\Box\mathcal{A}_1 \vee \Box(\mathcal{A}_1 \to \mathcal{A}_2) \vee \dots$ $\vee\Box(\mathcal{A}_1 \wedge \dots \wedge \mathcal{A}_n \to \mathcal{A}_{n+1})$	If $S_\Box ad_0$, $\bigwedge_{i=1}^{n}(S_\Box ab'$ and $Rb_ic_id_i)$ and $d_n \neq u$ then $\bigvee_{i=0}^{n-1}\bigvee_{j=i+1}^{n} c_j \leq d_i$.
	Rule	**Frame Condition**
Nec	$\dfrac{\mathcal{A}}{\Box\mathcal{A}}$	If $b \in 0$ and $S_\Box bc$, then $c \in 0$.

Given this list of correspondences, I will now isolate a couple of interesting extensions of $\mathbf{B.C}_{\Box\Diamond}$. As described by Seki [103, p. 408], the logic **R.K** is obtained from

$\mathbf{B.C}_{\Box\Diamond}$ by deleting $\boldsymbol{t}$ and $\Diamond$ (and using $\diamonddot$ as a replacement) from the logic's signature, removing the axioms $\boldsymbol{t}$ and $\Diamond(\mathcal{A}\vee\mathcal{B}) \to (\Diamond\mathcal{A}\vee\Diamond\mathcal{B})$, and the rule $\Diamond$-monotonicity, extending the relevant logic fragment from **B** to **R**, and adding the axiom (K$\Box$) and the rule of necessitation. The logic **RK** results from adding the axiom $(\Diamond\mathcal{A}\wedge\Box\mathcal{B}) \to \Diamond(\mathcal{A}\wedge\mathcal{B})$ to **R**.**K**. Finally, **R4** is obtained by adding to **RK** the axioms $\Box\mathcal{A} \to \mathcal{A}$ and $\Box\Box\mathcal{A} \to \Box\mathcal{A}$.

2.3.3 General Frames and Duality

General frames are useful for many reasons, including that general frames are duals of algebraic semantics. As anticipated, the relevant properties of general frames for modal relevant logics will be listed, and the some of Seki's duality results will be stated. The following definition is from Seki [103].

Definition 2.3.4. Given a general frame $\mathfrak{F}$ (for $\mathbf{B.C}_{\Box\Diamond}$ or its extensions), we say that $\mathfrak{F}$ is

1. *differentiated* if for any $a, b, \in K$, $a = b$ iff $\forall X \in \mathit{Prop}(a \in X \leftrightarrow b \in X)$
2. *r-tight* if for any $a, b, c \in K$, $Rabc$ iff $\forall X, Y \in \mathit{Prop}(a \in X \to Y \;\&\; b \in X \Rightarrow c \in Y)$
3. $\Box$-*tight* if for any $a, b \in K$, $S_{\Box}ab$ iff $\forall X \in \mathit{Prop}(a \in \Box X \Rightarrow b \in X)$
4. $\boxdot$-*tight* if for any $a, b \in K$, $S_{\boxdot}ab$ iff $\forall X \in \mathit{Prop}(a \in \boxdot X \Rightarrow b \in X)$
5. $\Diamond$-*tight* if for any $a, b \in K$, $S_{\Diamond}ab$ iff $\forall X \in \mathit{Prop}(b \in X \Rightarrow a \in \Diamond X)$
6. $\diamonddot$-*tight* if for any $a, b \in K$, $S_{\diamonddot}ab$ iff $\forall X \in \mathit{Prop}(b \in X \Rightarrow a \in \diamonddot X)$
7. *compact* if, for any families $\mathcal{X} \subseteq \mathit{Prop}$ and $\mathcal{Y} \subseteq \overline{\mathit{Prop}}$,[3]
$$\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$$
whenever $\bigcap(\mathcal{X}' \cup \mathcal{Y}') =\neq \emptyset$ for all finite subfamilies $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$
8. *descriptive* if it is differentiated, r-tight, $\Box$-tight, $\boxdot$-tight, $\Diamond$-tight, $\diamonddot$-tight, compact, and satisfies
$$0 = \bigcap\{X \in \mathit{Prop} : 0 \subseteq X\}$$

[3] Here, $\overline{\mathit{Prop}} = \{K - X : X \in \mathit{Prop}\}$

Seki shows that a general frame is isomorphic to its dual algebra's dual frame [103, Theorem 14]. That is, less than descriptive general $\mathbb{L}$-frames have $\mathbb{L}$-algebras as their duals, $\mathbb{L}$-algebras have general $\mathbb{L}$-frames as their duals, and if the frames are descriptive, the bidual of a frame is isomorphic to itself.

The next section and Seki's work suggest a future research project. That is, defining the algebraic semantics for quantified modal relevant logics, and then proving a duality results with the semantics of the next section (and later chapters). The relation of $\sqcap$ to Halmos' functional polyadic algebra means that the project should begin with defining a polyadic algebra for **QR** and **RQ**. This is left as a future project.

2.4 Quantified Modal R

2.4.1 Syntax

Finally, I turn to quantified modal relevant logics. In this section, I will present some axiom systems for QMRLs based on $\mathbf{R}^{\circ t}$ and explore their features including derivable rules. In the section that follows, I will construct semantics for these logics.

The logics $\mathbf{RQ4}^{\circ t}$ and $\mathbf{QR4}^{\circ t}$ — **S4**-ish extensions of $\mathbf{RQ}^{\circ t}$ and $\mathbf{QR}^{\circ t}$ respectively — are axioms systems in a quantified modal language. Axiom systems for these logics are defined as follows. The logic $\mathbf{QR4}^{\circ t}$ is given by

Axioms

- All of the axioms of $\mathbf{QR}^{\circ t}$
- $\Box\mathcal{A} \to \mathcal{A}$ — T
- $\Box\mathcal{A} \to \Box\Box\mathcal{A}$ — 4
- $\Box(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})$ — K
- $\Box(\mathcal{A} \wedge \Box\mathcal{B}) \to \Box(\mathcal{A} \wedge \mathcal{B})$ — $\Box\wedge$-Distribution
- $\Box(\mathcal{A} \vee \mathcal{B}) \to (\Diamond\mathcal{A} \vee \Box\mathcal{B})$

Rules

$$\text{(MP)}\ \frac{\vdash \mathcal{A} \to \mathcal{B} \qquad \vdash \mathcal{A}}{\vdash \mathcal{B}} \qquad\qquad \text{(NEC)}\ \frac{\vdash \mathcal{A}}{\vdash \Box\mathcal{A}}$$

$$\text{(ADJ)}\ \frac{\vdash \mathcal{A} \qquad \vdash \mathcal{B}}{\vdash \mathcal{A} \wedge \mathcal{B}} \qquad\qquad \text{(RIC)}\ \frac{\vdash \mathcal{A} \to \mathcal{B}}{\vdash \mathcal{A} \to \forall x\mathcal{B}}$$

For (RIC) we required that x is not free in $\mathcal{A}$. The logic $\mathbf{RQ4}^{\circ t}$ is the logic $\mathbf{QR4}^{\circ t}$ plus the axiom scheme

- $(\mathcal{A} \wedge \exists x\mathcal{B}) \to \exists x(\mathcal{A} \wedge \mathcal{B})$, where x is not free in $\mathcal{A}$

Note that the theorems of $\mathbf{RQ}^{\circ t}$ and $\mathbf{QR}^{\circ t}$ that I mentioned above and which I claimed were some axioms and rules used by Mares and Goldblatt, are theorems of $\mathbf{RQ4}^{\circ t}$ and $\mathbf{QR4}^{\circ t}$.

The next series of lemmas show that a rule which will be used in the completeness proofs is derivable. I will prove that this rule is derivable using the same arguments as Mares and Goldblatt employ in [74].

Lemma 2.4.1. *For both* $\mathbf{RQ4}^{\circ t}$ *and* $\mathbf{QR4}^{\circ t}$, $\forall y(\mathcal{A}[y/x]) \to \forall x\mathcal{A}$ *is a theorem if* y *is not free in* $\mathcal{A}$.

Proof. The proof is given by Mares and Goldblatt [74, p. 177].

1. $\forall y(\mathcal{A}[y/x]) \to (\mathcal{A}[y/x])[x/y]$ — by axiom (Q18)
2. $(\mathcal{A}[y/x])[x/y]$ is $\mathcal{A}$ — y is not free in $\mathcal{A}$
3. $\forall y(\mathcal{A}[y/x]) \to \mathcal{A}$ — 1,2
4. $\forall y(\mathcal{A}[y/x]) \to \forall x\mathcal{A}$ — 3, (RIC)

□

Lemma 2.4.2. *The rule*

$$\textit{RIC(Con)}\ \frac{\vdash \mathcal{A} \to \mathcal{B}[c/x]}{\vdash \mathcal{A} \to \forall x\mathcal{B}}\ , \textit{ where } c \textit{ is not in } \mathcal{A}$$

is derivable in $\mathbf{RQ4}^{\circ t}$ *and* $\mathbf{QR4}^{\circ t}$.

Proof. Given a proof of $\mathcal{A} \to \mathcal{B}[c/x]$ in which c does not occur in $\mathcal{A}$ or $\mathcal{B}$, we can construct a new proof replacing c with a brand new variable y. The result is a proof of $\mathcal{A} \to \mathcal{B}[y/x]$ with y not in $\mathcal{A}$. By (RIC) we get $\mathcal{A} \to \forall y\mathcal{B}[y/x]$. Using the previous lemma we then get $\mathcal{A} \to \forall x\mathcal{B}$. □

Lemma 2.4.3. *The rule*

$$UG(Con)\frac{\vdash \mathcal{A}[c/x]}{\vdash \forall x\mathcal{A}}\text{, where } c \text{ is not in } \mathcal{A}$$

is derivable in $\mathbf{RQ}\mathit{4}^{\circ \boldsymbol{t}}$ *and* $\mathbf{QR}\mathit{4}^{\circ \boldsymbol{t}}$.

Proof. The proof is straightforward using RIC(Con) and the $\boldsymbol{t}$ rules. □

2.4.2 Semantics

In this section I will follow somewhat the exposition of [74] and [47]. In particular, I will prove similar things in a similar order. As I am adding a modal operator to the semantics found in [74], I will tend to closely follow their exposition adding modal cases to proofs. In other words, I will substitute **R4**-frames for **R**-frames in their semantics and rework the proofs. We will use the following convention from Mares and Meyer [77].

$$Tab =_{df} Sab \wedge Sa^*b^*$$

Definition 2.4.4. A $\mathbf{QR4}^{\circ \boldsymbol{t}}$-frame is a tuple

$$\mathfrak{F} = \langle K, 0, R, S_\Box, *, U, Prop, PropFun\rangle$$

where:

(1) $\langle K, 0, R, S_\Box, *\rangle$ is an **R4**-frame.

(2) $\langle K, 0, R, *, U, Prop, PropFun\rangle$ is a $\mathbf{QR}^{\circ \boldsymbol{t}}$-frame.

That is, where K is a set, 0 is a subset of K, R is a ternary relation on K, $S_\Box$ is a binary relation on K, $*$ is a unary operation on K, U is a set of individuals, $Prop$ is a subset of the hereditary sets of K based on $\leq$, defined as before via 0, and $PropFun$ is a subset of the functions of type $U^\omega \longrightarrow Prop$ such that the following conditions are met.

(c1) if $a \in 0$ and $a \leq b$ then $b \in 0$

(c2) $\leq$ is transitive and reflexive.

(c3) $Raaa$

(c4) $R^2abcd \Rightarrow R^2acbd$

(c5) if $Rabc$ then $Rbac$

(c6) if $Rabc$ then Rac^*b^*

(c7) if $Rbcd$ and $a \leq b$ then $Racd$

(c8) $a^{**} = a$

(c9) if $S_\Box(Rab)c$ then $R(S_\Box a)(S_\Box b)c$

(c10) if $a \leq b$ and $S_\Box bc$ then $S_\Box ac$

(c11) if $S_\Box ab$ then $\exists x \leq b(Tax)$

(c12) if $\exists x \in 0(Txa)$ then $a \in 0$

(c13) $S_\Box aa$

(c14) if $S_\Box ab$ and $S_\Box bc$ then $S_\Box ac$

(c15) If $X, Y \in Prop$, then $X \cap Y$, $X \Rightarrow Y$, $X^* \in Prop$, where these operations are defined as before.

(c16) If $X \in Prop$, then $\Box S \in Prop$, where $\Box S$ is defined as we saw in section 2.3.2

(c17) The constant function ϕ_0 — for every f, $\phi_0 f = 0$ — is in PropFun

(c18) If $\phi, \psi \in PropFun$, then $\phi \Rightarrow \psi$, $\phi \cap \psi$, ϕ^*, $\Box\phi \in PropFun$, where $\Box\phi$ is defined by $(\Box\phi)f = \Box(\phi f)$, and the other functions are defined as before.

(c19) If $\phi \in PropFun$, then $\forall_n \phi \in PropFun$ for every $n \in \omega$.

The conditions above were chosen from the conditions given in the above sections on modal and quantified relevant logics. There may be simpler presentations of the above conditions. I have not included the element e of K which Seki's frames contain. Another notable difference is that in Seki [103], the condition corresponding to our (c12) uses $S_\Box$ instead of T. In the third chapter, I will develop a more general theory of quantified modal relevant logics in which conditions are more carefully chosen.

Lemma 2.4.5. *The condition (c20) is derivable.*

(c20) *if* $b \in 0$ *and* $S_\Box bc$ *then* $c \in 0$.

Proof. Suppose $b \in 0$ and $S_\Box bc$. From the latter and condition (c11) we get that $\exists x(x \leq c \ \& \ Tbx)$. From this and (c12) and (c1) we get that $c \in 0$. $\Box$

Mares and Meyer define $\leq$ by $a \leq b = (R(S0)ab)$, with 0 being a single world. However, with 0 being a set of worlds here, this translates to $\exists x \in 0(R(Sx)ab)$. The following lemma and its proof show why this is equivalent to our definition.

Lemma 2.4.6. $\exists x \in 0(R(Sx)ab)$ *iff* $\exists x \in 0(Rxab)$

Proof. One direction of this proof is straightforward. The other, as we will see, uses a number of conditions on the frames. Suppose $x \in 0$ and $R(Sx)ab$ (that is, $a \leq b$ on the Mares and Meyer-like definition). Let c be such an x. Then $\exists y(Scy \,\&\, Ryab)$. Let d be such a y. Using (c11) we get that $\exists w(w \leq d \,\&\, Tcw)$. Using (c12), Tcw and $c \in 0$ we get $w \in 0$. Using $w \in 0$, $w \leq d$ and (c1) we get that $d \in 0$. But we also have $Rdab$. So, $\exists x \in 0(Rxab)$, as required. $\square$

Definition 2.4.7. A $\mathbf{RQ4}^{\circ \boldsymbol{t}}$-frame is a $\mathbf{QR4}^{\circ \boldsymbol{t}}$-frame that also satisfies the condition that, for every $\alpha \in PropFun$, every $x \in Var$, every $X, Y \in Prop$, and for every $f \in U^{\omega}$,

$$X - Y \subseteq \bigcap_{a \in U} \alpha(f[a/x]) \text{ implies } X - Y \subseteq (\forall_x \alpha) f$$

This condition is the same one that is in **RQ** model structures, but not required for **QR** model structures [47, p. 213].

Pre-models and models are defined in a manner similar to Goldblatt [47], making substitutions with $\mathbf{RQ4}^{\circ \boldsymbol{t}}$ and $\mathbf{QR4}^{\circ \boldsymbol{t}}$ frames and adding conditions for '$\Box$'.

Definition 2.4.8. A *pre-model for* $\mathbf{QR4}^{\circ \boldsymbol{t}}$ ($\mathbf{RQ4}^{\circ \boldsymbol{t}}$),

$$\mathfrak{M} = \langle K, 0, R, S_{\Box}, *, U, Prop, PropFun, |{-}|^{\mathfrak{M}} \rangle$$

is a tuple such that $\langle K, 0, R, S_{\Box}, *, U, Prop, PropFun \rangle$ is a model structure for $\mathbf{QR4}^{\circ \boldsymbol{t}}$ ($\mathbf{RQ4}^{\circ \boldsymbol{t}}$) and $|{-}|^{\mathfrak{M}}$ is a value assignment that assigns,

1. an element $|c|^{\mathfrak{M}} \in U$ to each constant symbol c;

2. a function $|P|^{\mathfrak{M}} : U^n \longrightarrow \wp(K)$ to each n-ary predicate symbol P;

3. a propositional function $|\mathcal{A}|^{\mathfrak{M}} : U^{\omega} \longrightarrow \wp(K)$ to each formula $\mathcal{A}$ such that, when $\mathcal{A}$ is the atomic $P\tau_1, \ldots, \tau_n$, the propositional function assigned to it is given by, for each $f \in U^{\omega}$,

$$|P\tau_1, \ldots, \tau_n|^{\mathfrak{M}} f = |P|^{\mathfrak{M}}(|\tau_1|^{\mathfrak{M}} f, \ldots, |\tau_n|^{\mathfrak{M}} f).$$

Further, when $\mathcal{A}$ is not atomic, the function assigned is given by the following:

$$|\boldsymbol{t}|^{\mathcal{M}} = \phi_0$$
$$|\mathcal{A} \wedge \mathcal{B}|^{\mathcal{M}} = |\mathcal{A}|^{\mathcal{M}} \cap |\mathcal{B}|^{\mathcal{M}}$$
$$|\neg\mathcal{A}|^{\mathcal{M}} = (|\mathcal{A}|^{\mathcal{M}})^*$$
$$|\mathcal{A} \to \mathcal{B}|^{\mathcal{M}} = |\mathcal{A}|^{\mathcal{M}} \Rightarrow |\mathcal{B}|^{\mathcal{M}}$$
$$|\Box\mathcal{A}|^{\mathcal{M}} = \Box|\mathcal{A}|^{\mathcal{M}}$$
$$|\forall x\mathcal{A}|^{\mathcal{M}} = \forall_x|\mathcal{A}|^{\mathcal{M}}$$

Definition 2.4.9. A *model* for $\mathbf{QR4}^{\circ,\boldsymbol{t}}$ ($\mathbf{RQ4}^{\circ,\boldsymbol{t}}$) is a pre-model for $\mathbf{QR4}^{\circ,\boldsymbol{t}}$ ($\mathbf{RQ4}^{\circ,\boldsymbol{t}}$) that assigns a member of $PropFun$ to each atomic formula.

For the following, I will use a notion of an x-variant of a variable assignment f. The notation we will use for the set of x-variants of f is xf The truth sets, the result of applying a propositional function to a variable assignment, is thus given by the following:

$$|\boldsymbol{t}|^{\mathcal{M}}f = 0$$
$$|\mathcal{A} \wedge \mathcal{B}|^{\mathcal{M}}f = |\mathcal{A}|^{\mathcal{M}}f \cap |\mathcal{B}|^{\mathcal{M}}f$$
$$|\neg\mathcal{A}|^{\mathcal{M}}f = (|\mathcal{A}|^{\mathcal{M}}f)^*$$
$$|\mathcal{A} \to \mathcal{B}|^{\mathcal{M}}f = |\mathcal{A}|^{\mathcal{M}}f \Rightarrow |\mathcal{B}|^{\mathcal{M}}f$$
$$|\Box\mathcal{A}|^{\mathcal{M}}f = \Box(|\mathcal{A}|^{\mathcal{M}}f)$$
$$|\forall x\mathcal{A}|^{\mathcal{M}}f = \sqcap_{g \in xf}|\mathcal{A}|^{\mathcal{M}}g$$

We can then define $\vDash_{\mathfrak{M}}$ — or simply $\vDash$ for convenience — inductively as follows:

(i) $a, f \vDash P\tau_1, \ldots, \tau_n$ iff $a \in |P\tau_1, \ldots, \tau_n|^{\mathfrak{M}}f$

(ii) $a, f \vDash \boldsymbol{t}$ iff $a \in 0$

(iii) $a, f \vDash \mathcal{A} \wedge \mathcal{B}$ iff $a, f \vDash \mathcal{A}$ and $a, f \vDash \mathcal{B}$

(iv) $a, f \vDash \neg\mathcal{A}$ iff $a^*, f \nvDash \mathcal{A}$

(v) $a, f \vDash \Box\mathcal{A}$ iff $\forall b(S_{\Box}ab \Rightarrow b, f \vDash \mathcal{A})$

(vi) $a, f \vDash \mathcal{A} \rightarrow \mathcal{B}$ iff $\forall b, c((Rabc$ and $b, f \vDash \mathcal{A}) \Rightarrow c, f \vDash \mathcal{B})$

(vii) $a, f \vDash \forall x\mathcal{A}$ iff there is an $X \in Prop$ such that $X \subseteq \bigcap_{g \in x_n f} |\mathcal{A}|^{\mathfrak{M}} g$ and $a \in X$

Definition 2.4.10. A formula $\mathcal{A}$ is *satisfied* by a variable assignment f in a model $\mathfrak{M}$ if $a, f \vDash \mathcal{A}$, for every $a \in 0$. A formula $\mathcal{A}$ is valid in the model $\mathfrak{M}$, if it is satisfied by every variable assignment in that model. A formula $\mathcal{A}$ is valid in a frame, if it is valid in every model based on the frame. Finally, a formula is valid in a class of frames, if it is valid in every frame in the class.

Importantly, we can show the Semantic Entailment Lemma.

Lemma 2.4.11 (Semantic Entailment)**.** *In a model $\mathfrak{M}$, a formula $\mathcal{A} \rightarrow \mathcal{B}$ is satisfied by the variable assignment f iff for every world $a \in K$, if $a, f \vDash \mathcal{A}$, then $a, f \vDash \mathcal{B}$.*

The original statement of this lemma and its proof can be found as Lemmas 2 and 3 in Routley and Meyer [99].

The next lemma requires the notion of agreeing on all free variables of a formula. For any two variable assignments f and g, they agree on the free variables of the formula $\mathcal{A}$ iff, for every free variable x_n in $\mathcal{A}$, $fn = gn$.

Lemma 2.4.12. *For any formula $\mathcal{A}$, if f and g agree on all of the free variables of $\mathcal{A}$, then $|\mathcal{A}|f = |\mathcal{A}|g$.*

Proof. The proof is by induction on the complexity of $\mathcal{A}$. Mares and Goldblatt arguments may be used for all of the cases except the case of $\mathcal{A} = \Box\mathcal{B}$.

If $\mathcal{A} = \Box\mathcal{B}$, then $|\Box\mathcal{B}|f = \Box|\mathcal{B}|f$. By the induction hypothesis $|\mathcal{B}|f = |\mathcal{B}|g$, so $\Box|\mathcal{B}|f = \Box|\mathcal{B}|g$. □

Lemma 2.4.13. *Let x be free for τ in a formula $\mathcal{A}$. If g is an x-variant of f such that $g_x = f_\tau$, where f_τ is the element of U assigned to the term τ under the variable assignment f, then $|\mathcal{A}[\tau/x]|f = |\mathcal{A}|g$ in a model $\mathfrak{M}$.*

Proof. The proof is by induction on the complexity of $\mathcal{A}$. The arguments of Mares and Goldblatt [74, p. 177] can be used for all cases except for $\mathcal{A} = \Box\mathcal{B}$. Here I show this remaining case.

Let $\mathcal{A} = \Box\mathcal{B}$ and assume the result holds for $\mathcal{B}$. If x is free for τ in $\Box\mathcal{B}$, then x is free for τ in $\mathcal{B}$. By the induction hypothesis, $|\mathcal{B}[\tau/x]|f = |\mathcal{B}|g$. Thus, $\Box|\mathcal{B}[\tau/x]|f = \Box|\mathcal{B}|g$. From this we get that $|\Box\mathcal{B}[\tau/x]|f = |\Box\mathcal{B}|g$, as required. □

With the above lemmas from Mares and Goldblatt, extended as appropriate, we can now prove soundness.

Lemma 2.4.14. *The axioms of* **R** *are valid, and the rules of* **R** *preserve validity, in the class of* $\mathbf{QR}4^{\circ t}$ ($\mathbf{RQ}4^{\circ t}$) *frames.*

The proof of this is standard using the typical methods. The interesting cases for soundness are the axioms and rules with quantifiers and modalities.

Lemma 2.4.15. *The axiom* $\forall x\mathcal{A} \to \mathcal{A}[\tau/x]$ *is valid, and the rule (RIC) preserves validity in all* $\mathbf{QR}4^{\circ t}$ *and* $\mathbf{RQ}4^{\circ t}$ *models.*

Proof. This lemma is given a proof in Mares and Goldblatt [74, Lemmas 7.2 and 7.3]. Their proof uses similar truth conditions and three lemmas [74, Lemmas 4.3, 4.4, and 7.1]. I have proved above that these lemmas hold for $\mathbf{QR4}^{\circ t}$ and $\mathbf{RQ4}^{\circ t}$ above: Lemmas 2.4.11, 2.4.12, and 2.4.13. Using these lemmas we can prove this lemma in the same way. □

Lemma 2.4.16. *The axiom* $\forall x(\mathcal{A} \vee \mathcal{B}) \to (\mathcal{A} \vee \forall x\mathcal{B})$ *where* x *is not free in* $\mathcal{A}$ *is valid in all* $\mathbf{RQ}4^{\circ t}$ *models.*

Proof. Mares and Goldblatt demonstrate the validity of this axiom using one of the lemmas I have shown to hold for $\mathbf{RQ4}^{\circ t}$, Lemma 2.4.11. Thus, their proof that this axiom is valid in $\mathbf{RQ}^{\circ t}$ applies to $\mathbf{RQ4}^{\circ t}$. □

Lemma 2.4.17. *The following axioms are valid.*

1. $\Box\mathcal{A} \to \mathcal{A}$
2. $\Box\mathcal{A} \to \Box\Box\mathcal{A}$
3. $\Box(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})$
4. $(\Box\mathcal{A} \wedge \Box\mathcal{B}) \to \Box(\mathcal{A} \wedge \mathcal{B})$

5. $\Box(\mathcal{A} \vee \mathcal{B}) \to (\Diamond\mathcal{A} \vee \Box\mathcal{B})$

Proof. 1. Suppose $a, f \models \Box\mathcal{A}$. Then, for every b such that $S_\Box ab$, $b, f \models \mathcal{A}$. We have that $S_\Box aa$, so then it follows that $a, f \models \mathcal{A}$. Finally, by Lemma 2.4.11, we have that $\mathcal{A} \to \mathcal{B}$ is satisfied by f. As f was arbitrary, the formula is satisfied by every f and therefore valid.

2. Suppose that $a, f \models \Box\mathcal{A}$. Then, for every b such that $S_\Box ab$, $b, f \models \mathcal{A}$. For reductio, suppose that $a, f \not\models \Box\Box\mathcal{A}$. Then there is a world c such that $S^2_\Box ac$ and $c, f \not\models \mathcal{A}$. However, $S^2_\Box ac$ implies $S_\Box ac$, so we get that $c, f \models \mathcal{A}$, a contradiction. Thus $a, f \models \Box\Box\mathcal{A}$. Again, by Lemma 2.4.11 we get our desired result.

3. Suppose that $a, f \models \Box(\mathcal{A} \to \mathcal{B})$. For reductio, assume that $a, f \not\models \Box\mathcal{A} \to \Box\mathcal{B}$. Then there are worlds b, c such that $Rabc$, $b, f \models \Box\mathcal{A}$ and $c, f \not\models \Box\mathcal{B}$. From $c, f \not\models \Box\mathcal{B}$, we get that there is a world d such that $S_\Box cd$ and $d, f \not\models \mathcal{B}$. We have $S_\Box cd$ and $Rabc$, so by (c9) we get that $\exists a', b'(Ra'b'd \;\&\; S_\Box aa' \;\&\; S_\Box bb')$.

From $a, f \models \Box(\mathcal{A} \to \mathcal{B})$ and $S_\Box aa'$ it follows that $a', f \models (\mathcal{A} \to \mathcal{B})$. From $b, f \models \Box\mathcal{A}$ and $S_\Box bb'$ we get $b', f \models \mathcal{A}$. From the truth condition for implication we get that $d, f \models \mathcal{B}$, our contradiction. This completes the reductio, and using Lemma 2.4.11 we get our result.

4. The proof is straightforward.

5. Suppose that $a, f \models \Box(\mathcal{A} \vee \mathcal{B})$. Then, for reductio, assume that $a, f \not\models \neg\Box\neg\mathcal{A} \vee \Box\mathcal{B}$. It follows that both $a, f \not\models \neg\Box\neg\mathcal{A}$ and $a, f \not\models \Box\mathcal{B}$. From the latter we get that there is a b such that $S_\Box ab$ and $b, f \not\models \mathcal{B}$. From $S_\Box ab$ and condition (c11) we get that there is a c such that $c \leq b$ and $S_\Box ac$ and $S_\Box a^*c^*$. From $S_\Box ac$ and our first assumption it follows that $c, f \models \mathcal{A} \vee \mathcal{B}$. However, we can also show that such a world c cannot exist, giving us our contradiction.

Either $c, f \models \mathcal{A}$ or $c, f \models \mathcal{B}$. If the latter, then by $c \leq b$ we get $b \models \mathcal{B}$, a contradiction. If the former, then $c^*, f \not\models \neg\mathcal{A}$. It follows then, from $S_\Box a^*c^*$, that $a^*, f \not\models \Box\neg\mathcal{A}$. Finally, we get $a, f \models \neg\Box\neg\mathcal{A}$, giving us our contradiction. Thus $a, f \models \neg\Box\neg\mathcal{A} \vee \Box\mathcal{B}$, and by Lemma 2.4.11 we get our result. $\Box$

Lemma 2.4.18. *The rule (NEC) preserves validity.*

Proof. Suppose the formula $\mathcal{A}$ is valid. Then, for every $a \in 0$ and every f, we have $a, f \models \mathcal{A}$. Suppose for reductio that there is a world $a \in 0$ such that $a, f \not\models \Box\mathcal{A}$ for some f. Then there is a world $\mathcal{B}$ such that $S_{\Box}ab$ and $b, f \not\models \mathcal{A}$. However, from condition (c20) we get that $b \in 0$, and it immediately follows that $b, f \models \mathcal{A}$ by our supposition. Thus, we have our contradiction and it follows that there cannot be a world in 0 and a variable assignment f such that $a, f \not\models \Box\mathcal{A}$, as required. $\Box$

Theorem 2.4.19 (Soundness for $\mathbf{QR4}^{\circ t}$ and $\mathbf{RQ4}^{\circ t}$)**.** *If $\mathcal{A}$ is a theorem of $\mathbf{QR4}^{\circ t}$ ($\mathbf{RQ4}^{\circ t}$), then $\mathcal{A}$ is valid in the class of $\mathbf{QR4}^{\circ t}$ ($\mathbf{RQ4}^{\circ t}$) frames.*

Proof. The preceding lemmas suffice to prove this theorem. $\Box$

2.4.3 Theories and Completeness

The proof of completeness (for $\mathbf{QR4}^{\circ t}$ and $\mathbf{RQ4}^{\circ t}$) in this section will again follow Mares and Goldblatt, making adjustments where appropriate for the added necessity operator. I will take the usual detour through theories in what Mares and Goldblatt call a "Henkin-Lemmon-Scott-Routley-Meyer canonical model construction" [74, p. 178].

For notational convenience, let $\Gamma \gg_{\mathbb{L}} \Delta$ mean that there are some $\mathcal{A}_1, \ldots, \mathcal{A}_n \in \Gamma$ and $\mathcal{B}_1, \ldots, \mathcal{B}_m \in \Delta$ such that $\vdash_{\mathbb{L}} (\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_n) \to (\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m)$, where Γ and Δ are sets of formulas and $\mathbb{L}$ is a logic.

Definition 2.4.20. A pair (Γ, Δ) is $\mathbb{L}$-*independent* if and only if $\Gamma \not\gg \Delta$

Definition 2.4.21. An $\mathbb{L}$-theory is a set of formulas Γ such that if $\Gamma \gg \mathcal{A}$, then $\mathcal{A} \in \Gamma$. A theory Γ is *prime* if and only if, if $\mathcal{A} \vee \mathcal{B} \in \Gamma$, then either $\mathcal{A} \in \Gamma$ or $\mathcal{B} \in \Gamma$. A theory Γ is regular if and only if it contains every theorem of $\mathbb{L}$.

The following lemma will be useful.

Lemma 2.4.22. *If $\mathcal{A} \in \Gamma$ and $\vdash_{\mathbb{L}} \mathcal{A} \to \mathcal{B}$, then $\mathcal{B} \in \Gamma$, for $\mathbb{L} = \mathbf{QR4}^{\circ t}$ or $\mathbb{L} = \mathbf{RQ4}^{\circ t}$.*

The proof of this lemma is trivial.

Finally, the extension lemma, is needed to show that, if a formula is not a theorem, then there is a regular prime theory that does not contain the formula.

Lemma 2.4.23 (Extension)**.** *If (Γ, Δ) is $\mathbb{L}$-independent, then there is some prime theory Γ' such that $\Gamma \subseteq \Gamma'$ and (Γ', Δ), for $\mathbb{L} = \mathbf{QR4}^{\circ t}$ or $\mathbb{L} = \mathbf{RQ4}^{\circ t}$.*

The proof of this lemma for the logic **RQ** is due to Belnap (but the result is unpublished) [36, p. 41]. Roughly, one way to prove this lemma is to take the closure of Δ under disjunction, and consider all theories extending Γ that do not include any element of the closure of Δ. We apply Zorn's lemma to get a maximal theory, and then prove that it is prime.

The canonical model is defined as in [74], but with various adjustments for modeling the necessity operator. At this point, either we can assume that the set of constants *Con* is infinite, or we could add a denumerable number of constants to the language by the standard arguments using the finitude of proofs.

Definition 2.4.24. A *canonical frame* for $\mathbf{QR4}^{\circ t}$ is a tuple,

$$\mathfrak{F} = \langle K_c, 0_c, R_c, S_{\Box c}, *_c, U_c, Prop_c, PropFun_c\rangle$$

where

- K_c is the set of all prime theories.
- 0_c is the set of all regular prime theories.
- R_c is defined by $R_c abc$ iff $\{\mathcal{A} \circ \mathcal{B} : \mathcal{A} \in a \;\&\; \mathcal{B} \in b\} \subseteq c$.
- $S_{\Box c}$ is defined by $S_{\Box c}ab$ iff $\{\mathcal{A} : \Box\mathcal{A} \in a\} \subseteq b$.
- $*_c$ is defined by $a^* = \{\mathcal{A} : \neg\mathcal{A} \notin a\}$.
- U_c is the infinite set of constants *Con*.
- For every closed formula $\mathcal{A}$, $||\mathcal{A}||_c$ is defined to be the set $\{a \in K : \mathcal{A} \in a\}$.
- $Prop_c$ is defined as the set $\{||\mathcal{A}||_c : \mathcal{A} \text{ is a closed formula}\}$.
- Given a variable assignment f, the value fn is a constant. Substituting each variable in a formula $\mathcal{A}$ with the constant assigned to it by a variable assignment f results in a closed formula which will be denoted $\mathcal{A}^f$. Therefore $\mathcal{A}^f = \mathcal{A}[f0/x_0, \ldots, fn/x_n, \ldots]$.
- To each formula $\mathcal{A}$, there is a corresponding function $\phi_{\mathcal{A}}$ of type $U^\omega \longrightarrow K$ given by $\phi_{\mathcal{A}}f = ||\mathcal{A}^f||_c$. $PropFun_c$ is the set of all functions $\phi_{\mathcal{A}}$, where $\mathcal{A}$ is a formula.

Definition 2.4.25. A *canonical model* for $\mathbf{QR4}^{\circ t}$ is a tuple,

$$\mathfrak{M} = \langle K_c, 0_c, R_c, S_{\Box c}, *_c, U_c, Prop_c, PropFun_c, |-|_c^{\mathfrak{M}} \rangle, \text{ where}$$

- $\langle K_c, 0_c, R_c, S_{\Box c}, *_c, U_c, Prop_c, PropFun_c \rangle$ is the canonical frame.
- $|c|_c = c$, for every constant symbol c.
- $|P|_c(c_0, \ldots, c_n) = ||P(c_0, \ldots, c_n)||_c$.
- The valuation is extended to all wff as before.

For the remaining lemmas, the following Squeeze Lemmas are useful.

Lemma 2.4.26 (R_c Squeeze Lemma)**.** *If a and b are theories, c is a prime theory, and $R_c abc$, then there are prime theories a' and b' such that $R_c a'b'c$.*

The proof of this lemma is standard.

Lemma 2.4.27 ($S_{\Box c}$ Squeeze Lemma)**.**

- *If b is a theory, a is a prime theory, and $S_{\Box c}ab$, then there is a prime theory b' such that $S_{\Box c}ab'$.*
- *If b is a theory, a is a prime theory, and $S_{\Box c}ab$ and $\mathcal{A} \notin b$, then there is a prime theory b' such that $S_{\Box c}ab'$ and $\mathcal{A} \notin b'$.*

The proof of this lemma is found in Seki [102], Lemma 4.3 (1). A fact about this lemma is worth pointing out. Seki proves this lemma for modal logics with the base non-modal fragment being the logic **B**, so it will be applicable going further in the dissertation.

Lemma 2.4.28. *The canonical frame is a* $\mathbf{QR4}^{\circ t}$*-frame.*

Proof. Conditions (c1)–(c8) can be shown to hold by Routley and Meyer's arguments in [99]. That is, we have an **R**-frame $\langle K_c, 0_c, R_c, S_{\Box c}, *_c \rangle$. By the same arguments, the canonical frame is such that $a \leq b$ iff $a \subseteq b$.

(c9): The proof of this condition uses a similar argument as that for this condition for **NR**'s canonical models in [97]. Suppose that $S_{\Box c}(R_c ab)c$. That is, there is a d such that $S_{\Box c}dc$ and $R_c abd$. That is, $\{\mathcal{A} \circ \mathcal{B} : \mathcal{A} \in a \ \& \ \mathcal{B} \in b\} \subseteq d$ and $\{\mathcal{A} : \Box\mathcal{A} \in d\} \subseteq c$. So the goal is to find prime theories a', b' such that $S_{\Box}aa'$, $S_{\Box}bb'$, and $R_c a'b'c$.

Let a'' be the set of formulas $\mathcal{A}$ such that $\Box\mathcal{A} \in a$. It immediately follows that $S_\Box aa''$. Because of the axiom $(\Box\mathcal{A} \wedge \Box\mathcal{B}) \to \Box(\mathcal{A} \wedge \mathcal{B})$, and the primeness of a, a'' is closed under the the rule (ADJ). Further, suppose $\mathcal{A} \in a''$ and $\mathcal{A} \to \mathcal{B}$ is a theorem. Then $\Box(\mathcal{A} \to \mathcal{B}) \in a$. Using the axiom $(K\Box)$ it follows that $\Box\mathcal{B} \in a$, and so $\mathcal{B} \in a''$. Thus, a'' is a theory. We can apply similar reasoning to get a theory b'' such that $S_\Box bb''$.

Suppose $\mathcal{A} \in a''$ and $\mathcal{B} \in b''$. Then $\Box\mathcal{A} \in a$ and $\Box\mathcal{B} \in b$. It follows that $\Box(\mathcal{A} \circ \mathcal{B}) \in d$. It follows that $(\mathcal{A} \circ \mathcal{B}) \in c$. Thus, $R_c a'' b'' c$. Finally, applying the squeeze lemma gets us prime theories a', b' such that $S_\Box aa'$, $S_\Box bb'$, and $R_c a' b' c$.

(c10) Given that $a \leq b$ iff $a \subseteq b$ and the definition of $S_{\Box c}$, it follows immediately that this condition holds.

(c11) The argument this condition holds is a modified version of that found in Mares and Meyer [77, lemma 4.5]. Assume that $S_{\Box c}ab$. That is, if $\Box\mathcal{A} \in a$, then $\mathcal{A} \in b$. Now, I show that there is a prime theory $x \subseteq b$ such that $T_c ax$. Let c be the set of formulas $\mathcal{A}$ such that $\Diamond\mathcal{A} \notin a$. Let a' be the set of formulas $\mathcal{A}$ such that $\Box\mathcal{A} \in a$. We know that $a' \subseteq b$. Let d be the set of formulas not in b. As was shown above, a' is a theory because a is. Now, the pair $(a', c \cup d)$ is $\mathbb{L}$-independent.

To show this, suppose it is not independent. Then there are formulas $\mathcal{A}_1, \ldots, \mathcal{A}_n \in a'$, $\mathcal{B}_1, \ldots, \mathcal{B}_m \in d$, and $\mathcal{C}_1, \ldots, \mathcal{C}_l \in c$ such that

$$(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_n) \vdash_{\mathbb{L}} (\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m \vee \mathcal{C}_1 \vee \cdots \vee \mathcal{C}_l)$$

It follows that

$$(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_n) \to (\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m \vee \mathcal{C}_1 \vee \cdots \vee \mathcal{C}_l)$$

is a theorem of $\mathbb{L}$. From this we can get that

$$\Box(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_n) \to \Box(\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m \vee \mathcal{C}_1 \vee \cdots \vee \mathcal{C}_l)$$

is a theorem of $\mathbb{L}$. From this, it follows that

$$\Box(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_n) \to (\Box(\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m) \vee \Diamond(\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_l))$$

is a theorem of $\mathbb{L}$. Finally,

$$\Box(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_n) \vdash_{\mathbb{L}} \Box(\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m) \vee \Diamond(\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_l)$$

From this final fact, the primeness of a, and $\Box(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_n) \in a$, we get that either $\Box(\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m) \in a$ or $\Diamond(\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_l) \in a$. If the latter, then $\Diamond\mathcal{C}_i \in a$ for some $i \leq l$. Then $\mathcal{C}_i \notin c$, contradicting our assumption. If the former, then $\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m \in b$. Since b is prime, then $\mathcal{B}_i \in b$, for some $i \leq m$, contradicting our assumption. Thus, the pair $(a', c \cup d)$ is $\mathbb{L}$-independent. By the extension lemma, there is an prime theory x that extends a' such that the pair $(x, c \cup d)$ is $\mathbb{L}$-independent. It follows also that $T_c ax$.

(c12) Assume $x \in 0$ and $T_c xa$. Then, to show that a is regular (contains all theorems), it is sufficient to note that $\Box\mathcal{A} \in x$ for every theorem $\mathcal{A}$.

(c13) Straightforward.

(c14) Straightforward.

(c15) This is proven to hold by Mares and Goldblatt [74, lemma 9.1] for $\mathbf{QR}^{\circ \boldsymbol{t}}$ and $\mathbf{RQ}^{\circ \boldsymbol{t}}$. The same argument suffices here.

(c16) For this, it is sufficient to show that if $||\mathcal{A}||_c \in Prop$, then so is $\Box||\mathcal{A}||_c$. It is enough here to show that $\Box||\mathcal{A}||_c = ||\Box\mathcal{A}||_c$. For every f,

$$\Box||\mathcal{A}^f||_c = \{a : \forall b(S_{\Box c}ab \Rightarrow b \in ||\mathcal{A}^f||_c)\} \tag{1}$$

$$||\Box\mathcal{A}^f||_c = \{a : \Box\mathcal{A}^f \in a\} \tag{2}$$

We show that the left-hand of each equation is equal by showing the right-hand of the equations are equal. The direction from (2) to (1) is fairly trivial, given the definition of $S_{\Box c}$. For the other direction, assume that $c \in \{a : \forall b(S_{\Box c}ab \Rightarrow b \in ||\mathcal{A}^f||_c)\}$. For reductio, further assume that $\Box\mathcal{A}^f \notin c$. Let's construct the theory d by defining it as $\{\mathcal{A} : \Box\mathcal{A} \in c\}$. Clearly $S_{\Box c}cd$. By the squeeze lemma, there is a prime theory d' extending d such that both $S_{\Box c}cd'$ and d' does not contain $\mathcal{A}^f$. However, by our first assumption, d' does contain $\mathcal{A}^f$, giving us a contradiction, so $\Box\mathcal{A}^f \in c$.

(c17) This is proven to hold by Mares and Goldblatt [74, Lemma 9.1] for $\mathbf{QR}^{\circ \boldsymbol{t}}$ and $\mathbf{RQ}^{\circ \boldsymbol{t}}$. The same argument suffices here.

(c18) The arguments given by by Mares and Goldblatt [74, lemma 9.2] can be used for every case except the new $\Box$ case. For $\Box$, we have to show that $\Box\phi_{\mathcal{A}} = \phi_{\Box\mathcal{A}}$. For every f,

$$\begin{aligned}
\phi_{\Box\mathcal{A}} f &= ||\Box\mathcal{A}^f||_c && \text{by definition of } \phi_{\mathcal{B}} \\
&= \Box||\mathcal{A}^f||_c && \text{by equality established in (c16) case} \\
&= \Box\phi_{\mathcal{A}} f && \text{by definition of } \phi_{\mathcal{B}}
\end{aligned}$$

(c19) The argument used in Mares and Goldblatt [74, Lemmas 9.3 and 9.4] can be repeated here to show that (c19) holds. Their proof uses only facts I have shown to hold for the canonical models for $\mathbf{QR4}^{\circ \boldsymbol{t}}$ and $\mathbf{RQ4}^{\circ \boldsymbol{t}}$.

This concluded that the canonical frame for $\mathbf{QR4}^{\circ \boldsymbol{t}}$ is a $\mathbf{QR4}^{\circ \boldsymbol{t}}$-frame. For $\mathbf{RQ4}^{\circ \boldsymbol{t}}$, the the added condition can be shown to hold using the reasoning of Mares and Goldblatt [74, Lemma 10.1]. □

Let the notation τ^f denote the constant assigned to τ by f. The following lemma, Lemma 9.5 in [74] for **QR** and **RQ**, is stated here for $\mathbf{QR4}^{\circ \boldsymbol{t}}$ and $\mathbf{RQ4}^{\circ \boldsymbol{t}}$. The proof remains the same, but will be shown in some detail.

Lemma 2.4.29. *For every n-ary predicate symbol P, every variable assignment, and every set of terms $\tau_0, \ldots, \tau_{n-1}$,*

1. $P(\tau_0, \ldots, \tau_{n-1})^f = P(|\tau_0|_c f, \ldots, |\tau_{n-1}|_c f)$
2. $|P(\tau_0, \ldots, \tau_{n-1})|_c = \phi_{P(\tau_0, \ldots, \tau_{n-1})}$

Proof. For 1, $P(\tau_0, \ldots, \tau_{n-1})^f$ is just $P((\tau_0^f), \ldots, (\tau_{n-1}^f))$. If τ_m, where $m \geq 0$, is variable x_m, then (τ_m^f) is the constant assigned by f to x_m. That is, $(\tau_m^f) = |\tau_m|_c f$. The case where τ_m is a constant is straightforward.

For 2, the following series of identities from Mares and Goldblatt [74, p. 181] is sufficient.

$$\begin{aligned}
|P(\tau_0, \ldots, \tau_{n-1})| f &= |P(|\tau_0|_c f), \ldots, (|\tau_{n-1}|_c f)| && \text{by definition 2.4.8.3} \\
&= ||P(|\tau_0|_c f), \ldots, (|\tau_{n-1}|_c f)||_c && \text{by definition 2.4.25} \\
&= ||P(\tau_0, \ldots, \tau_{n-1})^f||_c && \text{by part 1.} \\
&= \phi_{P(\tau_0, \ldots, \tau_{n-1})} f && \text{by definition 2.4.24}
\end{aligned}$$

□

This demonstrates that the canonical model is in fact a model, by showing that every atomic formula is mapped onto a member of $PropFun$.

Lemma 2.4.30 (Truth Lemma)**.** *For any formula $\mathcal{A}$, $\mathcal{A} = \phi_{\mathcal{A}}$. That is, for all f, $|\mathcal{A}|f = ||\mathcal{A}||_c$. In other words, $a, f \vDash \mathcal{A}$ iff $\mathcal{A}^f \in a$.*

Proof. The proof of this lemma is by induction on the complexity of $\mathcal{A}$ and is shown for every case but $\mathcal{A} = \Box\mathcal{B}$ in Mares and Goldblatt [74].

Let $\mathcal{A} = \Box\mathcal{B}$, and assume the inductive hypothesis. That is, $|\mathcal{B}| = \phi_{\mathcal{B}}$. It follows that $|\mathcal{A}| = \Box|\mathcal{B}| = \Box\phi_{\mathcal{B}} = \phi_{\Box\mathcal{B}}$. □

Theorem 2.4.31 (Completeness for $\mathbf{QR4}^{\circ t}$ and $\mathbf{RQ4}^{\circ t}$)**.** *If $\mathcal{A}$ is valid in every $\mathbf{QR4}^{\circ t}$-model ($\mathbf{RQ4}^{\circ t}$-model), then $\mathcal{A}$ is a theorem of $\mathbf{QR4}^{\circ t}$ ($\mathbf{RQ4}^{\circ t}$).*

Proof. Let $\mathcal{A}$ be valid in every $\mathbf{QR4}^{\circ t}$-model including the canonical model. There is a similar argument for $\mathbf{RQ4}^{\circ t}$. It follows that every regular prime theory includes $\mathcal{A}^f$ for every $f \in U_c$. For every free variable in $\mathcal{A}$, replace the free variable with a different constant not in $\mathcal{A}$. That is, where $x_1, \ldots, x_n$ are the free variables of $\mathcal{A}$, perform the substitution $\mathcal{A}[c_1/x_1, \ldots, c_n/x_n]$ for constants $c_1, \ldots, c_n$ that do not appear in $\mathcal{A}$. This new formula belongs to every regular prime theory, and is therefore a $\mathbf{QR4}^{\circ t}$ theorem. Finally, repeated but finite applications of UG(Con) will produce the proof of $\mathcal{A}$. □

2.5 Remarks

In this chapter I have succeeded in combining the work of Seki with that of Mares and Goldblatt to define general frame semantics for the quantified modal $\mathbf{RQ4}^{\circ t}$ and $\mathbf{QR4}^{\circ t}$. The general idea of this combination will be extended in further chapters. In the following two chapters, I demonstrate that the Mares and Goldblatt semantics can be extended to a wider range of quantified relevant logics, and then construct semantics for quantified modal logics including $\mathbf{QB.C}_{\Box\Diamond}$ (the regular modal extension of $\mathbf{QB}$) and its extensions. The modal fragment of $\mathbf{QB.C}_{\Box\Diamond}$, as shown above, does not guarantee the inter-definability of the modal operators. As the quantifiers and modal operators are similar in some ways, we might expect their inter-definability to break down in these weaker logics as well. However, given the definition of $\sqcup$ and $\sqcap$, and the strength of negation in **B**, the quantifiers end up definable in terms of each other in the usual ways.

After constructing the foundation of semantics for quantified modal relevant logics, I spend two further chapters adding identity to these logics. First, I add identity in two interesting ways to the non-modal logics. Then, I add identity to the modal logics with the goal of covering as many axiomatizations as possible. The relatively weak logic $\mathbf{QB.C}_{\Box\Diamond}$ allows for many possible ways of including identity, given that fewer axioms are (inter)derivable from others. I also pay particular attention to adding identity to the Logic of Entailment **E**, as we must be careful to avoid fallacies of modality. In the final chapter, I motivate an application of quantified modal relevant logics. The application is modal naïve set theory. I will show that most known attempts at modal naïve set theory that have resulting in trivial systems, including a system which I show trivial (solving an open problem), have many routes to their triviality. By analyzing these routes to triviality, I motivate a substructural, modal approach.

Chapter 3

Quantified Relevant Logics

3.1 Introduction

In this chapter, the semantics of Mares and Goldblatt for $\mathbf{RQ}^{\circ t}$ and $\mathbf{QR}^{\circ t}$ will be extended to $\mathbf{BQ}^{\circ t}$ and $\mathbf{QB}^{\circ t}$, which will form a basis upon which the next chapter will add modalities. This chapter is presented for $\mathbf{BQ}^{\circ t}$, but the main results will hold for $\mathbf{QB}^{\circ t}$, as will be recorded. First, I will demonstrate the duality of the quantifiers for $\mathbf{BQ}^{\circ t}$. The duality will initially be shown to hold in the proof system. Mares and Goldblatt style semantics will be constructed in which a duality between operators interpreting the quantifiers will be shown. A general method employed here is to observe that the proofs given by Mares and Goldblatt are applicable to the weaker logics considered, with minor modifications. That is, the arguments used in the demonstration of soundness and completeness are for the most part already given in [74], because the (cases of) proofs dealing with quantifiers typically do not employ anything not available to the weaker logics. Part of my contribution here is some minor modifications to the proofs (mostly due to the number of operations/connective/quantifiers I take to be primitive) and the observation that the natural semantics developed by Mares and Goldblatt is generalizable. Further, I claim to have selected an interesting proof system in which the behavior of the quantifiers is apparent from the start. This is in contrast to a number of axiom systems in which the duality is not as *visible*.

We note that the immediate duality of the quantifiers may not be seen as desirable. For example, as we have seen the modalities in the background of $\mathbf{B}^{\circ t}$ are somewhat constructive/intuitionistic in the way they break down. The modal axiom $\neg\Box\neg\mathcal{A} \to \Diamond\mathcal{A}$ is similar in form to the formula $\neg\forall x\neg\mathcal{A} \to \exists x\mathcal{A}$. The latter, quantified formula is not intuitionistically valid. This, then, will perhaps cause a philosophical tension in the next chapter, where quantified modal logics based on $\mathbf{B}^{\circ \boldsymbol{t}}$ are constructed. As we will show, the duality found in the semantics for the quantifiers is dependent on the typical set theoretic notions of $\cap$ and $\cup$, but also the star operator. The strength of the negation in **B** is then partially responsible for this duality, even though this strength is not sufficient to force the duality of the modal operators in the usual way.

Mares and Goldblatt use the $\sqcap$ to model the universal quantifier. The intuitive idea behind this is that a universally quantified formula is the weakest thing that implies all of its instances. Dually, we can think of an existential quantified sentence as the strongest thing implied by every instance. Syntactically, we can dualize the axioms and rules given for the universal. Mares and Goldblatt define an operation $\sqcup : \wp\wp K \longrightarrow \wp K$ by,

$$\sqcup S = \cap\{X \in Prop : \cup S \subseteq X\}$$

for every $S \subseteq \wp K$ [74, p. 170]. They then show that this operator can be used to model the existential quantifier (which was taken to be defined) in $\mathbf{QR}^{\circ \boldsymbol{t}}$ and $\mathbf{RQ}^{\circ \boldsymbol{t}}$. This is done by showing that $\sqcup S = (\sqcap(S^*))^*$. I will reproduce their proof in greater detail than presented in [74]. The reason for doing so is to determine the nature of these operators, as defined, in the background of the logic **B**. For this purpose, let's use a notation that is easier on the eyes. Let $-S$ be shorthand for S^*. Then, the equation $\sqcup S = (\sqcap(S^*))^*$ is rewritten as $\sqcup S = - \sqcap -S$.

The first step is the proof is to demonstrate that

$$- \cup S = \cap\{-X : X \in S\}$$

Mares and Goldblatt observe that this follows from the fact that the De Morgan Laws hold for $-, \cap$, and $\cup$. Consider the left-hand side. $- \cup S$ is the set $\{x \in K : x^* \notin \cup S\}$ by definition. The right side is the set $\{x \in K : x \in -X$ for every $X \in S\}$. But $x \in -X$ if and only if $x^* \notin X$, given that $a = a^{**}$. So the right-hand side is $\{x \in K : x^* \notin X$ for every $X \in S\}$. We can see that the sides of this equation are equal. So far, we have used only shared properties of frames for **R** and frames for **B**, so this result holds for **B** as well.

The next step in this proof is to show that the following equation holds.

$$- \sqcap S = \sqcup\{-X : X \in S\}$$

The proof involves the equation we have just demonstrated. Following the proof, we begin with the left-hand side of the equation, which, by definition is

$$- \cup \{X \in Prop : X \subseteq \cap S\}$$

Given the first equation, this is equivalent to

$$\cap\{-X : X \in Prop \ \& \ X \subseteq \cap S\}$$

On the other hand, the right side of the equation is by definition

$$\cap\{Y \in Prop : \cup\{-X : X \in S\} \subseteq Y\}$$

The next fact used is that $-X \subseteq Y$ iff $-Y \subseteq X$. Suppose that $-X \subseteq Y$. Let $x \in -Y$. Then $x \notin Y$. It follows that $x \notin -X$. So then, $x \in X$, as required. It follows that

$$\cup\{-X : X \in S\} \subseteq Y \text{ iff } -Y \subseteq \cap S$$

Finally, given the fact that *Prop* is closed under $-$, and that $a = b^*$ iff $a^* = b$, we get that desired equality above.

It appears to follow that $- \sqcup -S = \sqcap S$, even in the background of **B**. The proof above, taken from Mares and Goldblatt and filling in a little more detail, seems to take a huge detour through the interactions between $-$, $\cap$, and $\cup$. This detour, however, is motivated by the definitions of $\sqcap$ and $\sqcup$. Therefore, it appears that these operators are, by definition, going to be duals in this respect. It is also worth emphasizing the role of the star operator in this argument. The regular set theoretic unions and intersections are sufficiently dual given the properties of the star operator.

3.2 Quantified B

Here I will define $\mathbf{BQ}^{\circ \boldsymbol{t}}$ using quantificational axioms and rules equivalent to Ross Brady's in *Universal Logic* [21]. This is due to the discussion above, which shows that the duality of the quantifiers should be derivable, for the duality is provable on the usual definitions of $\mathbf{BQ}^{\circ \boldsymbol{t}}$. Thus, I will define $\mathbf{BQ}^{\circ \boldsymbol{t}}$ in such a way. I will show that the system defined below and Brady's are equivalent. The axiom system is as follows:

Axioms

A1–9 The axioms of **B**

A10 $\forall x\mathcal{A} \to \mathcal{A}[\tau/x]$, where τ is free for x in $\mathcal{A}$ (UUI)

A11 $\forall x(\mathcal{A} \vee \mathcal{B}) \to \mathcal{A} \vee \forall x\mathcal{B}$, where x is not free in $\mathcal{A}$ (EC$_\forall$)

A12 $\mathcal{A}[\tau/x] \to \exists x\mathcal{A}$, where τ is free for x in $\mathcal{A}$ (UEI)

A13 $\mathcal{A} \wedge \exists x\mathcal{B} \to \exists x(\mathcal{A} \wedge \mathcal{B})$, where x is not free in $\mathcal{A}$ (EC$_\exists$)

Rules

$$\frac{\vdash \mathcal{A}}{\vdash \boldsymbol{t} \to \mathcal{A}}\ \boldsymbol{t}\text{-rule} \ (\text{double line}) \qquad \frac{\vdash \mathcal{A} \to (\mathcal{B} \to \mathcal{C})}{\vdash (\mathcal{A} \circ \mathcal{B}) \to \mathcal{C}}\ \circ\text{-rule} \ (\text{double line})$$

$$\frac{\vdash \mathcal{A} \to \mathcal{B}}{\vdash \exists x\mathcal{A} \to \mathcal{B}}\ \exists\text{-Intro} \qquad \frac{\vdash \mathcal{A} \to \mathcal{B}}{\vdash \mathcal{A} \to \forall x\mathcal{B}}\ \text{RIC}$$

$$\frac{\vdash \mathcal{A} \to \mathcal{B} \qquad \vdash \mathcal{A}}{\vdash \mathcal{B}}\ \text{MP} \qquad \frac{\vdash \mathcal{A} \qquad \vdash \mathcal{B}}{\vdash \mathcal{A} \wedge \mathcal{B}}\ \text{ADJ}$$

$$\frac{\vdash \mathcal{A} \to \mathcal{B}}{\vdash (\mathcal{C} \to \mathcal{A}) \to (\mathcal{C} \to \mathcal{B})}\ \text{Prefix} \qquad \frac{\vdash \mathcal{A} \to \mathcal{B}}{\vdash (\mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{C})}\ \text{Suffix}$$

$$\frac{\vdash \mathcal{A} \to \neg\mathcal{B}}{\vdash \mathcal{B} \to \neg\mathcal{A}}\ \text{Contra}$$

The conditions on $\exists$-Intro and *RIC* are that x is not free in $\mathcal{B}$ for $\exists$-Intro and x is not free in $\mathcal{A}$ for RIC. The $\boldsymbol{t}$ and $\circ$ rules are bi-directional, indicated by the double lines. Brady's system is defined using a number of additional axioms, and by substituting UG for RIC and $\exists$-Intro. These additional axioms are as follows:

A14 $\forall x(\mathcal{A} \to \mathcal{B}) \to (\exists x\mathcal{A} \to \mathcal{B})$ where x is not free in $\mathcal{B}$

A15 $\forall x(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \forall x\mathcal{B})$, where x is not free in $\mathcal{A}$.

These axioms are derivable in the system defined above.

Lemma 3.2.1. *Axioms A14 and A15 are derivable.*

Proof. For A15,

1	$\forall x(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{B}[\tau/x])$	A10
2	$\forall x((\mathcal{A} \to \mathcal{B}) \circ \mathcal{A}) \to \mathcal{B}[\tau/x]$	1, $\circ$-rule
3	$\forall x((\mathcal{A} \to \mathcal{B}) \circ \mathcal{A}) \to \forall x\mathcal{B}$	2, RIC
4	$\forall x(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \forall x\mathcal{B})$	3, $\circ$-rule

For A14,

1	$\forall x(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A}[\tau/x] \to \mathcal{B})$	A10
2	$\mathcal{A}[\tau/x] \to \mathcal{A}[\tau/x]$	A1
3	$\exists x\mathcal{A} \to \mathcal{A}[\tau/x]$	2, $\exists$-Intro
4	$(\mathcal{A}[\tau/x] \to \mathcal{B}) \to (\exists x\mathcal{A} \to \mathcal{B})$	3, Suffixing
5	$((\mathcal{A}[\tau/x] \to \mathcal{B}) \to (\exists x\mathcal{A} \to \mathcal{B})) \to$ $\to (\forall x(\mathcal{A} \to \mathcal{B}) \to (\exists x\mathcal{A} \to \mathcal{B}))$	1, Suffixing
6	$\forall x(\mathcal{A} \to \mathcal{B}) \to (\exists x\mathcal{A} \to \mathcal{B})$	4, 5, MP

$\square$

Brady's rule of universal generalization is derivable in the usual way via the $\boldsymbol{t}$-rules, MP and RIC. It follows that all of the axioms and rules of Brady's system are derivable in the system defined above. Next, I will show the other half of the equivalence. It is immediate that RIC is derivable in Brady's system. Further, we can use A14 to derive the rule $\exists$-Intro.

Lemma 3.2.2. *The rule $\exists$-Intro is derivable in* $\mathbf{BQ}^{\circ \boldsymbol{t}}$ *−* $\exists$*-Intro.*

Proof. The proof is a simple derivation.

1	$\mathcal{A} \to \mathcal{B}$	Hyp
2	$\boldsymbol{t} \to (\mathcal{A} \to \mathcal{B})$	1, A11, MP
3	$\boldsymbol{t} \to \forall x(\mathcal{A} \to \mathcal{B})$	2, RIC
4	$\forall x(\mathcal{A} \to \mathcal{B})$	3, $\boldsymbol{t}$-rule
5	$\forall x(\mathcal{A} \to \mathcal{B}) \to (\exists x\mathcal{A} \to \mathcal{B})$	A14
6	$\exists x\mathcal{A} \to \mathcal{B}$	4, 5, MP

$\square$

The axioms of the system defined above are axioms of Brady's system. Therefore the logics are equivalent, justifying the name $\mathbf{BQ}^{\circ \boldsymbol{t}}$. Note that I have implicitly been using an extension of Brady's system with both $\circ$ and $\boldsymbol{t}$ to show the equivalence.

Lemma 3.2.3. *The following are theorems of* $\mathbf{BQ}^{\circ \boldsymbol{t}}$*.*

i. $\forall x\mathcal{A} \to \neg\exists x\neg\mathcal{A}$

ii. $\neg\exists x\neg\mathcal{A} \to \forall x\mathcal{A}$

iii. $\exists x\mathcal{A} \to \neg\forall x\neg\mathcal{A}$

iv. $\neg\forall x\neg\mathcal{A} \to \exists x\mathcal{A}$

Proof. i.

1	$\forall x\mathcal{A} \to \mathcal{A}[\tau/x]$	(A10)
2	$(\mathcal{A}[\tau/x] \to \neg\exists\neg\mathcal{A}) \to (\forall x\mathcal{A} \to \neg\exists x\neg\mathcal{A})$	suffixing
3	$\neg\mathcal{A}[\tau/x] \to \neg\mathcal{A}[\tau/x]$	A1
4	$\exists x\neg\mathcal{A}[\tau/x] \to \neg\mathcal{A}[\tau/x]$	3, $\exists$-Intro
5	$\mathcal{A}[\tau/x] \to \neg\exists\neg\mathcal{A}$	4, Contra
6	$\forall x\mathcal{A} \to \neg\exists x\neg\mathcal{A}$	1, 5, MP

ii.

1	$\neg\mathcal{A}[\tau/x] \to \exists x\neg\mathcal{A}$	A12
2	$\neg\mathcal{A}[\tau/x] \to \neg\neg\exists x\neg\mathcal{A}$	Suffixing, MP, theorem
3	$\neg\exists x\neg\mathcal{A} \to \neg\neg\mathcal{A}[\tau/x]$	2, Contra
4	$\neg\exists x\neg\mathcal{A} \to \mathcal{A}[\tau/x]$	Suffixing, MP, theorem
5	$\neg\exists x\neg\mathcal{A} \to \forall x\mathcal{A}$	4, RIC

iii.

1	$\forall x\neg\mathcal{A} \to \neg\mathcal{A}[\tau/x]$	A10
2	$\mathcal{A}[\tau/x] \to \neg\forall x\neg\mathcal{A}$	1, Contra
3	$\exists x\mathcal{A} \to \neg\forall x\neg\mathcal{A}$	2, $\exists$-Intro

iv.

1	$\neg\exists\neg\neg\mathcal{A} \to \forall x\neg\mathcal{A}$	(ii)
2	$\neg\exists\neg\neg\mathcal{A} \to \neg\neg\forall x\neg\mathcal{A}$	1, Suffixing, MP, Theorem
3	$\neg\forall x\neg\mathcal{A} \to \neg\neg\exists x\neg\neg\mathcal{A}$	2, Contra
4	$\neg\forall x\neg\mathcal{A} \to \exists x\neg\neg\mathcal{A}$	3, Suffixing, MP, Theorem
5	$\exists x\neg\neg\mathcal{A} \to \exists x\mathcal{A}$	Theorem(Straightforward)
6	$\neg\forall x\neg\mathcal{A} \to \exists x\mathcal{A}$	4, 5 Prefixing, MP

□

This lemma demonstrates that the duality, and interdefinability, of $\exists$ and $\forall$ is built into this proof system. A glance at the rules and axioms will, without much squinting, show the duality of these quantifiers.

Next, let's consider a number of theorems and derivable rules that will be useful later.

Lemma 3.2.4. *The rules RIC(con) and UG(con) are derivable. As well, the formulas* $\forall x(\mathcal{A} \to \mathcal{B}) \to (\forall x\mathcal{A} \to \forall x\mathcal{B})$ *and* $\forall x(\mathcal{A} \wedge \mathcal{B}) \to (\forall x\mathcal{A} \wedge \forall x\mathcal{B})$ *are theorem-schemes.*

It is easy to check that the proofs given by Mares and Goldblatt in lemmas 6.1, 6.3, 6.5, and 6.6 work in $\mathbf{BQ}^{\circ t}$ (by expanding some of the condensed steps in the proofs). While some of these facts will be used in later chapters, the derivable UG(con) is important for the completeness proof later in this chapter.

I will consider the extensions of $\mathbf{BQ}^{\circ t}$ that extend the relevant logic fragment. That is, where $\mathbb{L}$ is a propositional relevant logic considered in the last chapter that extends $\mathbf{B}$, I will take $\mathbb{L}\mathbf{Q}^{\circ t}$ to be defined as above, replacing the axioms of $\mathbf{B}$ with the axioms of $\mathbb{L}$.

3.3 Extending the Mares and Goldblatt Style Semantics

Here, we generalize on the semantics of Mares and Goldblatt [74], demonstrating that their style of semantics is adequate for the logic $\mathbf{BQ}^{\circ t}$. Then, we will consider extensions. The semantics is Mares and Goldblatt style, and because of this the proofs and lemmas leading to soundness and completeness will mostly follow their original presentation for $\mathbf{QR}^{\circ t}$ and $\mathbf{RQ}^{\circ t}$. Many of the proofs from Mares and Goldblatt [74] are reproduced with the observation that their proofs of lemmas for $\mathbf{RQ}^{\circ t}$ and $\mathbf{QR}^{\circ t}$ constitute proofs of corresponding lemmas for $\mathbf{BQ}^{\circ t}$ and $\mathbf{QB}^{\circ t}$. As many of the details of their proofs were skimmed over in the first chapter, here the details are focused on to demonstrate the applicability to the weaker logic. Their proofs are modified to work for $\mathbf{BQ}^{\circ t}$ to deal with both the weaker propositional base and the additional primitives

As before, we will use the $\sqcap$ to define functions $\forall_n \phi : U^\omega \longrightarrow Prop$. This is done by setting

$$(\forall_n \phi) f = \sqcap_{j \in U} \phi(f[j/n])$$

for every $n \in \omega$. Similarly, we will define functions $\exists_n \phi : U^\omega \longrightarrow Prop$. This is done by setting, for every $n \in \omega$,

$$(\exists_n \phi) f = \sqcup_{j \in U} \phi(f[j/n]).$$

Definition 3.3.1. A $\mathbf{BQ}^{\circ t}$-frame is a tuple

$$\mathfrak{F} = \langle K, 0, R, *, U, Prop, PropFun \rangle$$

such that $\langle K, 0, R, * \rangle$ is a $\mathbf{B}$-frame, U is a non-empty set, $Prop$ is a subset of the up-sets of K and contains 0, $PropFun$ is a subset of the functions of type $U^\omega \longrightarrow Prop$, and the following conditions hold:

c1 If $X, Y \in Prop$, then $X \cup Y \in Prop$, $X \cap Y \in Prop$, $X^* \in Prop$, and $X \Rightarrow Y \in Prop$, where X^* and $X \Rightarrow Y$ are defined as before.

c2 $PropFun$ contains a constant function ϕ_0 such that $\phi_0 f = 0$ for each $f \in U^\omega$.

c3 If $\phi, \psi \in PropFun$, then $\phi \Rightarrow \psi$, $\phi \cup \psi$, $\phi \cap \psi$, and ϕ^* are in $PropFun$, and are defined, as before, by

$$(\phi \Rightarrow \psi)f = \phi f \Rightarrow \psi f$$
$$(\phi \cup \psi)f = \phi f \cup \psi f$$
$$(\phi \cap \psi)f = \phi f \cap \psi f$$
$$(\phi^*)f = (\phi f)^*$$

c4 If $\phi \in PropFun$, then $\forall_n \phi \in PropFun$, for each $n \in \omega$.

c5 If $\phi \in PropFun$, then $\exists_n \phi \in PropFun$, for each $n \in \omega$.

c6 $X - Y \subseteq \cap_{a \in U} \phi(f[a/n])$ implies $X - Y \subseteq (\forall_n \phi)f$

A frame is *full* when $Prop$ is the set of every up-set of K and $PropFun$ is the set of all functions from U^ω to $Prop$.

Definition 3.3.2. A *pre-model* for $\mathbf{BQ}^{\circ t}$ is a tuple

$$\mathfrak{M} = \langle K, 0, R, *, U, Prop, PropFun, |-|^{\mathfrak{M}} \rangle$$

where $\langle K, 0, R, *, U, Prop, PropFun \rangle$ is a $\mathbf{BQ}^{\circ t}$-frame and $|-|^{\mathfrak{M}}$ is an assignment that assigns

- an element $|c|^{\mathfrak{M}} \in U$ to each constant symbol;
- a function $|P|^{\mathfrak{M}} : U^n \longrightarrow \wp(K)$ to each n-ary predicate symbol P;
- a propositional function $|\mathcal{A}|^{\mathfrak{M}} : U^\omega \longrightarrow \wp(K)$ to each formula $\mathcal{A}$ such that, when $\mathcal{A}$ is the atomic formula $P_{\tau_1,\ldots,\tau_n}$, the propositional function assigned to it is given by

$$|P_{\tau_1,\ldots,\tau_n}|^{\mathfrak{M}} = |P|^{\mathfrak{M}}(|\tau_1|^{\mathfrak{M}}, \ldots, |\tau_n|^{\mathfrak{M}})$$

Further, when $\mathcal{A}$ is not atomic, the function assigned is given by the following:

$$|\boldsymbol{t}|^{\mathfrak{M}} = \phi_0$$
$$|\mathcal{A} \wedge \mathcal{B}|^{\mathfrak{M}} = |\mathcal{A}|^{\mathfrak{M}} \cap |\mathcal{B}|^{\mathfrak{M}}$$
$$|\mathcal{A} \vee \mathcal{B}|^{\mathfrak{M}} = |\mathcal{A}|^{\mathfrak{M}} \cup |\mathcal{B}|^{\mathfrak{M}}$$
$$|\neg\mathcal{A}|^{\mathfrak{M}} = (|\mathcal{A}|^{\mathfrak{M}})^*$$
$$|\mathcal{A} \to \mathcal{B}|^{\mathfrak{M}} = |\mathcal{A}|^{\mathfrak{M}} \Rightarrow |\mathcal{B}|^{\mathfrak{M}}$$
$$|\forall x\mathcal{A}|^{\mathfrak{M}} = \forall_x|\mathcal{A}|^{\mathfrak{M}}$$
$$|\exists x\mathcal{A}|^{\mathfrak{M}} = \exists_x|\mathcal{A}|^{\mathfrak{M}}$$

Definition 3.3.3. A pre-model is a *model* for $\mathbf{BQ}^{\circ\boldsymbol{t}}$ when it assigns an element of $PropFun$ to each atomic formula.

Satisfaction and validity are defined as usual.

The truth sets, given a propositional function and a variable assignment, are as follows:

$$|\boldsymbol{t}|^{\mathfrak{M}}f = 0$$
$$|\mathcal{A} \wedge \mathcal{B}|^{\mathfrak{M}}f = |\mathcal{A}|^{\mathfrak{M}}f \cap |\mathcal{B}|^{\mathfrak{M}}f$$
$$|\mathcal{A} \vee \mathcal{B}|^{\mathfrak{M}}f = |\mathcal{A}|^{\mathfrak{M}}f \cup |\mathcal{B}|^{\mathfrak{M}}f$$
$$|\neg\mathcal{A}|^{\mathfrak{M}}f = (|\mathcal{A}|^{\mathfrak{M}}f)^*$$
$$|\mathcal{A} \to \mathcal{B}|^{\mathfrak{M}}f = |\mathcal{A}|^{\mathfrak{M}}f \Rightarrow |\mathcal{B}|^{\mathfrak{M}}f$$
$$|\forall x\mathcal{A}|^{\mathfrak{M}}f = \sqcap_{g \in xf}|\mathcal{A}|^{\mathfrak{M}}g$$
$$|\exists x\mathcal{A}|^{\mathfrak{M}}f = \sqcup_{g \in xf}|\mathcal{A}|^{\mathfrak{M}}g$$

Finally, we can express the familiar $\vDash$ relation. The cases for the quantifiers are taken from Mares and Goldblatt, and the existential quantifier relies on the duality between the quantifiers. I will discuss this in greater detail below. The $\vDash$ relation is as follows:

$a, f \vDash P(\tau_1, \ldots, \tau_n)$ iff $a \in |P(\tau_1, \ldots, \tau_n)|^{\mathfrak{M}}f$

$a, f \vDash \boldsymbol{t}$ iff $a \in 0$

$a, f \vDash \mathcal{A} \wedge \mathcal{B}$ iff $a, f \vDash \mathcal{A}$ and $a, f \vDash \mathcal{B}$

$a, f \vDash \mathcal{A} \vee \mathcal{B}$ iff $a, f \vDash \mathcal{A}$ or $a, f \vDash \mathcal{B}$

$a, f \vDash \neg\mathcal{A}$ iff $a^*, f \nvDash \mathcal{A}$

$a, f \vDash \mathcal{A} \to \mathcal{B}$ iff $\forall b, c((Rabc \ \&\ b, f \vDash \mathcal{A}) \Rightarrow c, f \vDash \mathcal{B})$

$a, f \vDash \forall x\mathcal{A}$ iff there is a proposition X, $a \in X$ and $X \subseteq \cap_{g \in xf} |\mathcal{A}|^{\mathfrak{M}} g$

$a, f \vDash \exists x\mathcal{A}$ iff, for every $X \in \mathit{Prop}$ such that $a^* \in X$, there is a $b \in X$ and x-variant such that $b^* \in |\mathcal{A}|^{\mathfrak{M}} g$

The existential quantifier case is taken directly from [74, p. 169], and relies on the duality that $\exists x\mathcal{A} \leftrightarrow \neg\forall x\neg\mathcal{A}$. The arguments above shows that the duality in the operators $\sqcap$ and $\sqcup$ holds in the semantics for $\mathbf{BQ}^{\circ \boldsymbol{t}}$. Thus, the same conditions (w.r.t. $\vDash$) hold in this case. Mares and Goldblatt explain this case by stating the meaning of "$\exists x\mathcal{A}$ if true at a" as

> *every proposition compatible with $\mathcal{A}$ is compatible with some x-instantiation of $\mathcal{A}$* [74, p. 170]

The question arises as to how this compares with the informal interpretation of an existentially quantified formula as

> the strongest proposition implied by every instantiation of the formula.

The definition of $\sqcup$ seems to support the informal interpretation. How can we relate this to the $\vDash$-condition?

I state the next lemma, and its proof is by the usual arguments.

Lemma 3.3.4 (Semantic Entailment)**.** *$\mathcal{A} \to \mathcal{B}$ is satisfied by a variable assignment f in a model $\mathfrak{M}$ iff, for every $a \in K$, if $a, f \vDash \mathcal{A}$, then $a, f \vDash \mathcal{B}$.*

Proof. The proof can be found as lemmas 2 and 3 in [99]. □

Lemma 3.3.5. *If f and g agree on all the free variables in the formula $\mathcal{A}$, then $|\mathcal{A}|^{\mathfrak{M}} f = |\mathcal{A}|^{\mathfrak{M}} g$.*

The proof is straightforward.

3.4 Soundness

Lemma 3.4.1. *Axioms A1–9 are valid in every* $\mathbf{BQ}^{\circ t}$*-model. Further, the rules MP, ADJ, Prefix, Suffix, and Contra preserve validity in every* $\mathbf{BQ}^{\circ t}$*-model.*

The proof is as usual.

Lemma 3.4.2. *For any formula* $\mathcal{A}$ *with* x *free for* τ *in* $\mathcal{A}$*, in any* $\mathbf{BQ}^{\circ t}$*-model* $\mathfrak{M}$*, if* $g \in xf$ *and* $|x|g = |\tau|f$*, then* $|\mathcal{A}[\tau/x]|^{\mathfrak{M}}f = |\mathcal{A}|^{\mathfrak{M}}g$*.*

Proof. The proof is modified from Mares and Goldblatt's proof of the same lemma for $\mathbf{QR}^{\circ t}$ [74, Lemma 7.1]. The proof is by induction on the complexity of the formula $\mathcal{A}$. The base cases and the connective cases are fairly straightforward. The cases for the universal and existential quantifiers are shown, with the universal quantifier case proceeding by the arguments for Mares and Goldblatt.

Case $\mathcal{A} = \forall y\mathcal{B}$. Assume the result holds for the formula $\mathcal{B}$ as the induction hypothesis. Either x is free in $\mathcal{A}$ or it isn't. If it isn't then $\mathcal{A}[\tau/x] = \mathcal{A}$. Since g is an x-variant of f, the result is immediate by lemma 3.3.5. On the other hand, if x is free in $\mathcal{A}$, then $x \neq y$. Further, $\mathcal{A}[\tau/x] = \forall y\mathcal{B}[\tau/x]$. Because x is free for τ, it follows that $y \neq \tau$ and x is free for τ in $\mathcal{B}$. Let y be the variable x_n. We have the following identities:

$$|\mathcal{A}[\tau/x]|^{\mathfrak{M}}f = \sqcap_{i\in U}|\mathcal{B}[\tau/x]|^{\mathfrak{M}}f[i/n]$$
$$|\mathcal{A}|^{\mathfrak{M}}g = \sqcap_{i\in U}|\mathcal{B}|^{\mathfrak{M}}g[i/n]$$

To complete the proof we show that the two right sides of these identities and identical to each other.

Note that for any $i \in U$, the assignment $f[i/n]$ is an x-variant of $g[i/n]$, as they are x-variants before the substitution and the substitution is applied to each of them. From $x_n \neq \tau$ and $x \neq x_n$ we get that $|\tau|f[i/n] = |\tau|f = |x|g = |x|g[i/n]$. Using the induction hypothesis we get that

$$|\mathcal{B}[\tau/x]|^{\mathfrak{M}}f[i/n] = |\mathcal{B}|^{\mathfrak{M}}g[i/n]$$

for every $i \in U$. From this we get that $|\mathcal{A}[\tau/x]|^{\mathfrak{M}}f = |\mathcal{A}|^{\mathfrak{M}}f$, as desired.

Case $\mathcal{A} = \exists y\mathcal{B}$. Assume that the result holds for $\mathcal{B}$ as the induction hypothesis. Either x is free in $\mathcal{A}$ or it isn't. If it isn't, then $\mathcal{A}[\tau/x] = \mathcal{A}$. Since g is an x-variant of f, the result is immediate.

On the other hand, if x is free in $\mathcal{A}$, then $x \neq y$. Further, $\mathcal{A}[\tau/x] = \exists y \mathcal{B}[\tau/x]$. Because x is free for τ, it follows that $y \neq \tau$ and x is free for τ in $\mathcal{B}$. Let y be the variable x_n. We have the following identities:

$$|\mathcal{A}[\tau/x]|^{\mathfrak{M}} f = \sqcup_{i \in U} |\mathcal{B}[\tau/x]|^{\mathfrak{M}} f[i/n]$$
$$|\mathcal{A}|^{\mathfrak{M}} g = \sqcup_{i \in U} |\mathcal{B}|^{\mathfrak{M}} g[i/n]$$

To complete the proof we show that the two right sides are identical, as the previous case.

Again, note that for any $i \in U$, the assignment $f[i/n]$ is an x-variant of $g[i/n]$, as they are x-variants before the substitution, and the substitution is applied to each of them. From $x_n \neq \tau$ and $x \neq x_n$ it follows that $|\tau| f[i/n] = |\tau| f = |x| g = |x| g[i/n]$. Using the induction hypothesis we get that

$$|\mathcal{B}[\tau/x]|^{\mathfrak{M}} f[i/n] = |\mathcal{B}|^{\mathfrak{M}} g[i/n]$$

The proof is completed as in the previous case, thus concluding the induction. $\square$

Lemma 3.4.3. *Axiom A10 is valid in every* $\mathbf{BQ}^{\circ \boldsymbol{t}}$*-model.*

Proof. The proof is as in Mares and Goldblatt [74, Lemma 7.2]. Let $a, f \vDash \forall x \mathcal{A}$ and let x be free for τ in $\mathcal{A}$, for some model $\mathfrak{M}$ and some variable assignment f. It follows that $a, g \vDash \mathcal{A}$ for every x-variant g of f. One x-variant of f is a g such that $|x| g = |\tau| f$ and $a \in |\mathcal{A}|^{\mathfrak{M}} g$. By lemma 3.4.2, we have that $|\mathcal{A}|^{\mathfrak{M}} g = |\mathcal{A}[\tau/x]|^{\mathfrak{M}} f$. Therefore, $a, f \vDash \mathcal{A}[\tau/x]$, and by Semantic Entailment axiom A10 is valid in this model. As the model and variable assignment were arbitrary, the axiom is valid in all $\mathbf{BQ}^{\circ \boldsymbol{t}}$-models. $\square$

One modification to Mares and Goldblatt's proofs is the addition of lemmas to demonstrate the soundness of the additional axioms of my proof system.

Lemma 3.4.4. *Axiom A12 is valid in every* $\mathbf{BQ}^{\circ \boldsymbol{t}}$*-model.*

Proof. Let $a, f \vDash \mathcal{A}[\tau/x]$ and let τ be free for x in $\mathcal{A}$. That is, that $a \in |\mathcal{A}[\tau/x]|^{\mathfrak{M}} f$. Let g be an x-variant of f such that $|x| g = |\tau| f$. It follows that $a, g \vDash \mathcal{A}$. That is, $|\mathcal{A}[\tau/x]| f = |\mathcal{A}| g$. Thus, $a \in |\mathcal{A}| g$. It follows that $a \in \cup_{g \in xf} |\mathcal{A}| g$. Thus, a is in every proposition X such that $\cup_{g \in xf} |\mathcal{A}| g \subseteq X$. Therefore $a \in \sqcup_{g \in xf} |\mathcal{A}| g$. By Semantic Entailment the axiom A15 is valid. $\square$

Lemma 3.4.5. *The rule RIC preserves validity in every* $\mathbf{BQ}^{\circ \boldsymbol{t}}$*-model.*

Proof. Suppose that $\mathcal{A} \to \mathcal{B}$ is valid in the model $\mathfrak{M}$ and that x does not occur free in $\mathcal{A}$. It follows that for every variable assignment g, and from Semantic Entailment, that $|\mathcal{A}|^{\mathfrak{M}}g \subseteq |\mathcal{B}|^{\mathfrak{M}}g$. Now take any variable assignment f. If g is an x-variant of f, then they agree on the free variables of $\mathcal{A}$, given that x is not free in $\mathcal{A}$. From this we get that $|\mathcal{A}|^{\mathfrak{M}}f = |\mathcal{A}|^{\mathfrak{M}}g$, and that $|\mathcal{A}|^{\mathfrak{M}}f \subseteq |\mathcal{B}|^{\mathfrak{M}}g$. Since g is an x-variant of f, considering all such x-variants gives us $|\mathcal{A}|^{\mathfrak{M}}f \subseteq \sqcap_{g \in xf}|\mathcal{B}|^{\mathfrak{M}}g$. This is the case for every f, therefore, by Semantic Entailment, we get that $\mathcal{A} \to \forall x\mathcal{B}$ is valid in this model. $\square$

Lemma 3.4.6. *The rule* $\exists$*-Intro preserves validity in every* $\mathbf{BQ}^{\circ \boldsymbol{t}}$*-model.*

Proof. Suppose that $\mathcal{A} \to \mathcal{B}$ is valid in the model $\mathfrak{M}$ and that x does not occur free in $\mathcal{B}$. It follows that for every variable assignment g, and from Semantic Entailment, that $|\mathcal{A}|^{\mathfrak{M}}g \subseteq |\mathcal{B}|^{\mathfrak{M}}g$. Now take any variable assignment f. If g is an x-variant of f, then they agree on the free variables of $\mathcal{B}$, given that x is not free in $\mathcal{B}$. From this we get that $|\mathcal{B}|^{\mathfrak{M}}f = |\mathcal{B}|^{\mathfrak{M}}g$, and that $|\mathcal{A}|^{\mathfrak{M}}g \subseteq |\mathcal{B}|^{\mathfrak{M}}f$.

Since g is an x-variant of f, considering all such x-variants gives us $\sqcup_{g \in xf}|\mathcal{A}|^{\mathfrak{M}}g \subseteq |\mathcal{B}|^{\mathfrak{M}}f$. As $\sqcup_{g \in xf}|\mathcal{A}|^{\mathfrak{M}}g = |\exists x\mathcal{A}|^{\mathfrak{M}}f$, and our choice of f was arbitrary, the result follows by Semantic Entailment. $\square$

Lemma 3.4.7. *The axiom A11 is valid in every* $\mathbf{BQ}^{\circ \boldsymbol{t}}$*-model.*

Proof. The argument here is as in Mares and Goldblatt [74]. Suppose that $a, f \vDash \forall x(\mathcal{A} \vee \mathcal{B})$, where x does not occur free in $\mathcal{A}$. It follows that there is an a and an X such that $a \in X$, $X \in \mathit{Prop}$, and $X - |\mathcal{A}|^{\mathfrak{M}}f \subseteq \cap_{g \in xf}|\mathcal{B}|^{\mathfrak{M}}g$. The latter is established as follows.

Given $a, f \vDash \forall x(\mathcal{A} \vee \mathcal{B})$, then there is an $X \in \mathit{Prop}$ where $a \in X$ and, for every x-variant g of f, $X \subseteq |\mathcal{A} \vee \mathcal{B}|^{\mathfrak{M}}g$. And so, $X \subseteq |\mathcal{A}|^{\mathfrak{M}}g \cup |\mathcal{B}|^{\mathfrak{M}}g$. That is, $X - |\mathcal{A}|^{\mathfrak{M}}g \subseteq |\mathcal{B}|^{\mathfrak{M}}g$. However, given the fact that x is free in $\mathcal{A}$ and g is an x-variant of f, we get that $X - |\mathcal{A}|^{\mathfrak{M}}f \subseteq |\mathcal{B}|^{\mathfrak{M}}g$, for every $g \in xf$, as required.

By condition c6, it follows that $X - |\mathcal{A}|^{\mathfrak{M}}f \subseteq |\forall x\mathcal{B}|^{\mathfrak{M}}f$. Now suppose $a \in |\mathcal{A}|^{\mathfrak{M}}f$. The result follows immediately. On the other hand, suppose $a \notin |\mathcal{A}|^{\mathfrak{M}}f$. Since we have $a \in X$, we can apply what we just proved to show that $a \in |\forall x\mathcal{B}|^{\mathfrak{M}}$. The result follows straightforwardly. Again, by Semantic Entailment the lemma is then completed. $\square$

Lemma 3.4.8. *The axiom A13 valid in every* $\mathbf{BQ}^{\circ t}$*-model.*

Proof. Suppose that $a, f \vDash \mathcal{A} \wedge \exists x\mathcal{B}$, and that x is not free in $\mathcal{A}$. It follows that $a, f \vDash \mathcal{A}$ and $a, f \vDash \exists x\mathcal{B}$.

For reductio, suppose that $a, f \nvDash \exists x(\mathcal{A} \wedge \mathcal{B})$. That is, via convenient dualities (which would have made this lemma redundant if we were to have taken the existential quantifier as defined), $a, f \nvDash \neg\forall x\neg(\mathcal{A} \wedge \mathcal{B})$. Therefore, $a^*, f \vDash \forall x(\neg\mathcal{A} \vee \neg\mathcal{B})$. By the previous lemma, $a^*, f \vDash \neg\mathcal{A} \vee \forall x\neg\mathcal{B}$. However, if $a^*, f \vDash \neg\mathcal{A}$, then $a \nvDash \mathcal{A}$, a contradiction. On the other hand, if $a^*, f \vDash \forall x\neg\mathcal{B}$, then $a, f \nvDash \neg\forall x\neg\mathcal{B}$, which is $a, f \nvDash \exists x\mathcal{B}$, a contradiction. Finally, by Semantic Entailment the result follows. □

Theorem 3.4.9 (Soundness)**.** *All of the theorems of* $\mathbf{BQ}^{\circ t}$ *are valid in every* $\mathbf{BQ}^{\circ t}$*-model.*

The following corollary should be apparent given the only use of c6 was in the proof of the existential confinement axiom and its dual.

Corollary 3.4.10. *All of the theorems of* $\mathbf{QR}^{\circ t}$ *are valid in every* $\mathbf{QB}^{\circ t}$*-model, which is a* $\mathbf{BQ}^{\circ t}$*-model without condition c6.*

3.5 Completeness

The reader is referred back to the previous chapter for the definition of theories, prime theories, and regular theories. The extension lemma is probable in this case as well.

The canonical model is defined as before. That is, the canonical model is defined as it was for $\mathbf{QR}^{\circ t}$ and $\mathbf{RQ}^{\circ t}$ by Mares and Goldblatt, merely changing the underlying logic determining the theories to $\mathbf{BQ}^{\circ t}$.

Definition 3.5.1. A *canonical frame* for $\mathbf{BQ}^{\circ t}$ is a tuple,

$$\mathfrak{F} = \langle K_c, 0_c, R_c, *_c, U_c, Prop_c, PropFun_c \rangle, \text{ where}$$

- K_c is the set of all prime theories.
- 0_c is the set of all regular prime theories.
- R_c is defined by $R_c abc$ iff $\{\mathcal{A} \circ \mathcal{B} : \mathcal{A} \in a \,\&\mathcal{B} \in b\} \subseteq c$.

- $*_c$ is defined by $a^* = \{\mathcal{A} : \neg\mathcal{A} \notin a\}$.

- U_c is the set infinite set of constants *Con*.

- For every closed formula $\mathcal{A}$, $||\mathcal{A}||_c$ is defined to be the set $\{a \in K : \mathcal{A} \in a\}$.

- $Prop_c$ is defined as the set $\{||\mathcal{A}||_c : \mathcal{A}$ is a closed formula$\}$.

- Given a variable assignment f, the value fn is a constant. Substituting each variable in a formula $\mathcal{A}$ with the constant assigned to it by a variable assignment f results in a closed formula which will be denoted $\mathcal{A}^f$. Therefore $\mathcal{A}^f = \mathcal{A}[f0/x_0, \ldots, fn/x_n, \ldots]$.

- To each formula $\mathcal{A}$, there is a corresponding function $\phi_{\mathcal{A}}$ of type $U^\omega \longrightarrow K$ given by $\phi_{\mathcal{A}} f = ||\mathcal{A}^f||_c$. $PropFun_c$ is the set of all function $\phi_{\mathcal{A}}$, where $\mathcal{A}$ is a formula.

Definition 3.5.2. A *canonical model* for $\mathbf{BQ}^{\circ t}$ is a tuple,

$$\mathfrak{M} = \langle K_c, 0_c, R_c, *_c, U_c, Prop_c, PropFun_c, |{-}|_c^{\mathfrak{M}} \rangle, \text{ where}$$

- $\langle K_c, 0_c, R_c, *_c, U_c, Prop_c, PropFun_c \rangle$ is the canonical frame.

- $|c|_c = c$, for every constant symbol c.

- $|P|_c(c_0, \ldots, c_n) = ||P(c_0, \ldots, c_n)||_c$.

- The valuation is extended to all wff as before.

Before demonstrating that the canonical frame is a frame, the squeeze lemma is stated.

Lemma 3.5.3 (Squeeze Lemma)**.** *If a and b are theories, c is a prime theory, and $R_c abc$, then there are prime theories a', b' such that $Ra'b'c$.*

Again, the proof of this lemma is standard.

Lemma 3.5.4. *The canonical frame is a frame.*

Proof. The requirement that $\langle K, 0, R, *\rangle$ is a **B**-frame can be shown using Routley and Meyer's arguments in [99]. What remains to be shown is conditions c1–c6. The remainder of this proof is dedicated to providing the details of Mares and Goldblatt's proofs for $\mathbf{RQ}^{\circ \boldsymbol{t}}$ to show their proofs are applicable to $\mathbf{BQ}^{\circ \boldsymbol{t}}$, by showing that nothing essentially **R**-ish or unavailable in **B** is unavoidable into their proofs.

c1: First, note that $||\boldsymbol{t}||_c = 0$, and $0 \in \mathit{Prop}$. For the remainder of the propositional connectives, and $\cap$, and $\cup$, it is sufficient to note that the following equations hold.

$$||\mathcal{A}||_c \cap ||\mathcal{B}||_c = ||\mathcal{A} \wedge \mathcal{B}||_c$$

$$||\mathcal{A}||_c \cup ||\mathcal{B}||_c = ||\mathcal{A} \vee \mathcal{B}||_c$$

$$||\mathcal{A}||_c^* = ||\neg\mathcal{A}||_c$$

$$||\mathcal{A}||_c \Rightarrow ||\mathcal{B}||_c = ||\mathcal{A} \to \mathcal{B}||_c$$

c2: The constant function ϕ_0 is such that $\phi_0 f = 0$ for every f. Well, $\phi_{\boldsymbol{t}} f = ||\boldsymbol{t}^f||_c$. But since $\boldsymbol{t}$ has no free variables, $||\boldsymbol{t}^f||_c = ||\boldsymbol{t}||_c = 0$.

c3: Let $\oplus$ be one of $\cup$, $\cap$, and $\Rightarrow$, and let the corresponding connective symbol be $\otimes$. It is easy to see that for every f, $\mathcal{A}^f \otimes \mathcal{B}^f = (\mathcal{A} \otimes \mathcal{B})^f$. From this we get that $||\mathcal{A}^f||_c \oplus ||\mathcal{B}^f||_c = ||(\mathcal{A} \otimes \mathcal{B})^f||_c$. Further, this means that $\phi_{\mathcal{A}} f \oplus \phi_{\mathcal{B}} f = (\phi_{\mathcal{A}\otimes\mathcal{B}})f$. This applies to every f, so $\phi_{\mathcal{A}} \oplus \phi_{\mathcal{B}} = (\phi_{\mathcal{A}\otimes\mathcal{B}})$. Similar reasoning applies to the unary operator $*$.

To show that c4 holds, the following sub-lemma proved by Mares and Goldblatt is convenient.

Lemma 3.5.5. *If $\forall x\mathcal{A}$ is a sentence, then, for every prime theory a, $\forall x\mathcal{A} \in a$ iff there is an $X \in \mathit{Prop}$ such that, for ever constant c, $a \in X$ and $X \subseteq ||\mathcal{A}[c/x]||_c$. That is,*

$$||\forall x\mathcal{A}||_c = \sqcap_{c \in con} ||\mathcal{A}[c/x]||_c.$$

Proof. The proof here is an is Mares and Goldblatt [74, Lemma 9.3]. For one direction, assume that $\forall x\mathcal{A} \in a$. Let $X = ||\forall x\mathcal{A}||_c$. By these assumptions we get that $a \in X$ and $X \in \mathit{Prop}$. Given that X is a theory, and given axiom A10, we get that $X \subseteq ||\mathcal{A}[c/x]||_c$, for all constants, as required.

For the other direction, suppose that $a \in X$, $X \in \mathit{Prop}$ and $X \subseteq ||\mathcal{A}[c/x]||_c$ for every constant c. By definition, X must be the set $||\mathcal{B}||_c$, for some $\mathcal{B}$. We get $\mathcal{B} \in a$ as a consequence of this.

Next, we show that $\vdash \mathcal{B} \to \mathcal{A}[c/x]$ for a fresh constant c. The argument is by reductio. Suppose $\nvdash \mathcal{B} \to \mathcal{A}[c/x]$. Then the pair $(\{\mathcal{B}\}, \{\mathcal{A}[c/x]\})$ is $\mathbf{BQ}^{\circ \boldsymbol{t}}$-independent. By the extension lemma, we get a prime theory Γ extending $\{\mathcal{B}\}$ such that $(\Gamma, \{\mathcal{A}[c/x]\})$ is an independent pair and $\mathcal{A}[c/x] \notin \Gamma$. However, this implies that $\gamma \in ||\mathcal{B}||_c - ||\mathcal{A}[c/x]||_c$, giving our contradiction. Thus, we have established $\vdash \mathcal{B} \to \mathcal{A}[c/x]$. By the derivable rule RIC(con), $\vdash \mathcal{B} \to \forall x \mathcal{A}$. Given that a is a theory and $\mathcal{B} \in a$, we get that $\forall x \mathcal{A} \in a$. $\square$

c4: Here we will show that, for every $n \in \omega$, for any $\mathcal{A}$, $\forall_n \phi_{\mathcal{A}} = \phi_{\forall_{x_n} \mathcal{A}}$. Mares and Goldblatt introduce another notation for this proof. We will write $\mathcal{A}^{f \setminus n}$ for the formula

$$\mathcal{A}[f0/x_0, \ldots, f(n-1)/x_{n-1}, x_n/x_n, f(n+1)/x_{n+1}, \ldots]$$

This formula is the result of applying the substitution determined by f with the exception of x_n. Given this, it follows that $\mathcal{A}^{f \setminus n}[c/x_n] = \mathcal{A}^f c/n$. It can also be seen that $\forall x_n(\mathcal{A}^{f \setminus n}) = (\forall x_n \mathcal{A})^f$. Given these two facts, the following derivation from Mares and Goldblatt is possible.

$$\begin{aligned}
(\forall_n \phi_{\mathcal{A}}) f &= \phi_{\mathcal{A}}(f[c/n]) && \text{by the definition of } \forall_n \\
&= \sqcap_{c \in U} ||\mathcal{A}^{f[c/n]}||_c && \text{by definition of } \phi_A \\
&= \sqcap_{c \in U} ||\mathcal{A}^{f \setminus n}[c/x_n]||_c && \text{by an equality just established} \\
&= ||\forall x_n(\mathcal{A}^{f \setminus n})||_c && \text{by our sub-lemma} \\
&= ||(\forall x_n \mathcal{A})^f||_c && \text{by an equality just established} \\
&= \phi_{\forall x_n \mathcal{A}} f && \text{by definition of } \phi_{\forall x_n \mathcal{A}}
\end{aligned}$$

Thus $\mathit{PropFun}$ is closed under $\forall_n$ for every $n \in \omega$.

c5: The proof can be reduced the previous proof by the duality of the quantifiers.

c6: Suppose the antecedent of the condition holds in the canonical model. That is, suppose $X - Y \subseteq \cap_{c \in U} \phi(f[c/n])$. The definition of the canonical model means that $Y = ||\mathcal{A}||_c$ for some sentence $\mathcal{A}$. Also, $\phi = \phi_{\mathcal{B}}$ for some possibly open $\mathcal{B}$. Thus, our supposition is that, for every constant c, $X - ||\mathcal{A}||_c \subseteq \phi_{\mathcal{B}}(f[c/n])$. Thus we get that $X \subseteq ||\mathcal{A} \vee (\mathcal{B}^{f[c/n]})||_c$. By an equality proved for condition c4 above, and the sentencehood of $\mathcal{A}$ we get the following equality.

$$\mathcal{A} \vee (\mathcal{B}^{f[c/n]}) = \mathcal{A} \vee (\mathcal{B}^{f\setminus n}[c/x_n]) = (\mathcal{A} \vee \mathcal{B})^{f\setminus n}[c/x_n]$$

Now, assume that $a \in X - Y$. So $a \in X$, but also $a \in ||(\mathcal{A} \vee \mathcal{B})^{f\setminus n}[c/x_n]||_c$ for every constant c. Given that $(\mathcal{A} \vee \mathcal{B})^{f\setminus n}[c/x_n]$ is closed, by 3.5.5 we get that $\forall x_n((\mathcal{A} \vee \mathcal{B})^{f\setminus n}) \in a$. Using the axiom of extensional confinement and the fact that a is a theory, it follows that $\mathcal{A} \vee \forall x_n(\mathcal{B}^{f\setminus n}) \in a$.

We also have that $a \notin ||\mathcal{A}||_c$, meaning $\mathcal{A} \notin a$. Thus, $\forall x_n(\mathcal{B}^{f\setminus n}) \in a$. Using another equality established in the c4 case, we get that

$$a \in ||(\forall x_n \mathcal{B})^f||_c = (\phi_{\forall x_n \mathcal{B}})f = (\forall_n \phi_{\mathcal{B}})f$$

Thus, we have shown that $X - Y \subseteq (\forall_n \phi)f$. $\square$

The following lemma, Lemma 9.5 in [74] for **QR** and **RQ**, is stated here for $\mathbf{BQ}^{\circ \boldsymbol{t}}$. The proof remains the same, and the reader is directed to the previous chapter, or the Mares and Goldblatt article, for the details.

Lemma 3.5.6. *For every n-ary predicate symbol P, every variable assignment, and every set of terms $\tau_0, \ldots, \tau_{n-1}$,*

1. $P(\tau_0, \ldots, \tau_{n-1})^f = P(|\tau_0|_c f, \ldots, |\tau_{n-1}|_c f)$
2. $|P(\tau_0, \ldots, \tau_{n-1})|_c = \phi_{P(\tau_0, \ldots, \tau_{n-1})}$

This lemma ensures that every atomic proposition is assigned a member of $PropFun$. Thus the corollary follows.

Corollary 3.5.7. *The canonical model is a model.*

Lemma 3.5.8 (Truth lemma for $\mathbf{BQ}^{\circ \boldsymbol{t}}$)**.** *For any formula $\mathcal{A}$, $\mathcal{A} = \phi_{\mathcal{A}}$. That is, for all f, $|\mathcal{A}|f = ||\mathcal{A}||_c$. In other words, $a, f \vDash \mathcal{A}$ iff $\mathcal{A}^f \in a$.*

The proof, again, is the same as shown in Mares and Goldblatt [74], and is a straightforward induction on the complexity of a formula.

Theorem 3.5.9 (Completeness for $\mathbf{BQ}^{\circ t}$). *If $\mathcal{A}$ is valid in every $\mathbf{BQ}^{\circ t}$-model, then $\mathcal{A}$ is a theorem of $\mathbf{BQ}^{\circ t}$.*

The proof is as before.

Corollary 3.5.10. *Let the set of $\mathbf{QB}^{\circ t}$-models being constructed as $\mathbf{BQ}^{\circ t}$-models with the deletion of condition c6. It follows that the logic $\mathbf{QB}^{\circ t} = \mathbf{BQ}^{\circ t} - A14 - A16$ is sound and complete for the set of $\mathbf{QB}^{\circ t}$-models.*

The reader may be assured that c6 was only used to ensure the valid of A14 and A16. Further, the proof that the canonical model is a model only used axioms A14 and A16 to demonstrate that condition c6 held. Thus, it is easy to see that the weaker logic $\mathbf{QB}^{\circ t}$ is characterized by the models that result from the deletion of frame condition c6.

Let $\mathbb{L}$ be a relevant logic considered in chapter 1. We may define $\mathbb{L}\mathbf{Q}$ and $\mathbf{Q}\mathbb{L}$ by adding frame conditions for the relevant logic fragment.

Corollary 3.5.11. *The logic $\mathbb{L}\boldsymbol{Q}$ ($\boldsymbol{Q}\mathbb{L}$) is sound and complete for the $\mathbb{L}\boldsymbol{Q}$-models ($\boldsymbol{Q}\mathbb{L}$-models).*

Again, this corollary is rather straightforward to see.

I have now shown that the Mares and Goldblatt style semantics for quantified relevant logics generalizes to $\mathbf{QB}^{\circ t}$, $\mathbf{BQ}^{\circ t}$, and extensions given by strengthening the relevant logic fragment of these logics. The next chapter adds modalities to these logics.

3.6 Lines of Future Research

The semantics of Mares and Goldblatt for **QR** and **RQ** improves on Fine's more complicated semantics, and provides a natural way of interpreting the quantifiers. Further, Mares and Goldblatt's semantics for quantified modal (classical) logics [75], in which the universal quantifier is treated similarly, provides completeness results for a wide range of quantified modal logics. The canonical models don't require ω-complete theories, and the usual incompleteness (which results from conflicting requirements in the truth lemma for $\Box$ and $\forall x$) is bypassed. This kind of semantics for quantified modal logics is powerful, as seen in Mares and Goldblatt [75], Goldblatt [47], and the next chapter. However, despite how natural and useful the semantics are, it is an open question how it can be used to model quantified relevant logics with non-dual quantifiers.

Chapter 4

Quantified Modal Relevant Logics

4.1 Introduction

In the first chapter, we saw the Mares and Goldblatt semantics [74] extended with a binary relation in order to model the modal operator $\Box$. The logics $\mathbf{QR4}^{\circ t}$ and $\mathbf{RQ4}^{\circ t}$, treated in detail, were found to be sound and complete with respect to these semantics. To extend this treatment of quantified modal relevant logics by varying both the relevant and the modal fragments of these logics, the previous chapter demonstrated that the Mares and Goldblatt style semantics can indeed be used to treat quantified **B** and its standard relevant extensions. The aim of this chapter is to combine the extended Mares and Goldblatt style semantics with Seki's semantics [102, 103] for modal relevant logics.

Note well that the duality of the modalities is seen to break down in weak relevant logics, but the duality of the quantifiers remains. This may seem counterintuitive. Belnap's suggested axiom and its dual, which are used to recover the theorems of **S4** (under translation), are of the same form as a formula containing the quantifiers. This formula with quantifiers is not intuitionistically valid, given a lack of duality between the quantifiers. As the Mares and Goldblatt semantics' treatment of the quantifiers ensures their duality, we must ask what modifications must be made in order to break this duality (both to the semantics and to the syntax). The chapter will end with a discussion of the Barcan formula. I will begin by defining axiomatic systems for these logics, and we will consider what is provable in these systems and their extensions.

4.2 Syntax

4.2.1 Regular Quantified Modal Relevant Logics

Let $\mathbb{L}$ be a quantified relevant logic extending **QB** considered in the previous chapter. In particular, let this logic have both $\boldsymbol{t}$ and $\circ$. We now define quantified modal relevant logics based on $\mathbb{L}$.

Definition 4.2.1. A *(regular) quantified modal logic over* $\mathbb{L}$ is any logic which contains the axioms and rules of $\mathbb{L}$ and the following axioms and rules.

Axioms

- $(\Box\mathcal{A} \wedge \Box\mathcal{B}) \to \Box(\mathcal{A} \wedge \mathcal{B})$
- $\Diamond(\mathcal{A} \vee \mathcal{B}) \to (\Diamond\mathcal{A} \vee \Diamond\mathcal{B})$

Rules

$$\frac{\vdash \mathcal{A} \to \mathcal{B}}{\vdash \Box\mathcal{A} \to \Box\mathcal{B}}\ (\Box\text{-Monotonicity}) \qquad\qquad \frac{\vdash \mathcal{A} \to \mathcal{B}}{\vdash \Diamond\mathcal{A} \to \Diamond\mathcal{B}}\ (\Diamond\text{-Monotonicity})$$

Furthermore, the least (regular) quantified modal logic over $\mathbb{L}$ will be denoted by $\mathbb{L}.\mathbf{C}_{\Box\Diamond}$. In proofs, for convenience, '$\Box$-Monotonicity' will be shortened to simply '$\Box$-M', and similarly for '$\Diamond$-Monotonicity'.

Seki uses the phrase "regular modal logic over..." to describe the modal relevant logics he considers. However, I will generally omit the adjective 'regular'. I may use the phrase "quantified modal logic over $\mathbb{L}$".

A note on the history of the term 'regular modal logic'. Chellas's *Introduction to Modal Logic* calls a logic regular if and only if it contains the theorem scheme $\Diamond\mathcal{A} \leftrightarrow \neg\Box\neg\mathcal{A}$ and the rule RR [22, p. 234]:

$$\frac{\vdash (\mathcal{A} \wedge \mathcal{B}) \to \mathcal{C}}{\vdash (\Box\mathcal{A} \wedge \Box\mathcal{B}) \to \Box\mathcal{C}}\ \text{RR}$$

Any of what are called regular quantified modal logics here contain the similar theorem scheme $\dot{\Diamond}\mathcal{A} \leftrightarrow \neg\Box\neg\mathcal{A}$, for the defined dotted modality. Further, using the distribution axiom for $\Box$ and $\Box$-Monotonicity, the rule RR is derivable. Thus, regular quantified modal relevant logics are regular in this sense.

Naming conventions are interesting, as **R**.**K** and **RK** technically lack the $\Diamond$ as defined, although the dotted diamond may have similar behavior. Thus, **R**.**K** will be rewritten as **R**.$\mathbf{K}_\Box$ for the usual logic. For any two regular quantified modal logics $\mathbb{L}.\mathbb{M}_{\Box\Diamond}$ and $\mathbb{L}.\mathbb{M}_\Box$, they are the same logic when the $\Diamond$ of the latter is definable in the former as simple $\diamonddot$. That is, by the definition $\Diamond\mathcal{A} =_{df} \diamonddot\mathcal{A}$.

Lemma 4.2.2. *The formulas of Lemma 3.2.3, which show the duality of the quantifiers, are provable.*

The proof is the same.

While we have the full set of theorems of either **QB** or **BQ**, depending on which quantified modal relevant logic we are dealing with, the more interesting formulas are those with both quantifiers and modalities. Some of these formulas are only theorems of quantified modal relevant logics with stronger modal fragments. However, note that the following lemma demonstrates the provability of an important formula in even the weakest quantified modal relevant logic considered.

Lemma 4.2.3. *The Converse Barcan Formula is provable in* **QB**.$\boldsymbol{C}_{\Box\Diamond}$.

Proof.

1	$\forall x\mathcal{A} \to \mathcal{A}$	Axiom UI
2	$\Box\forall x\mathcal{A} \to \Box\mathcal{A}$	$\Box$-M
3	$\Box\forall x\mathcal{A} \to \forall x\Box\mathcal{A}$	RIC

$\Box$

Here we see that the $\Box$-Monotonicity rule actually shortens the proof of this formula from the proof given for the axiom system in chapter 1. This monotonicity rule encodes all we need from the rule of necessitation and the K axiom.

Lemma 4.2.4. *The dual of the Converse Barcan Formula,* $\exists x\Diamond\mathcal{A} \to \Diamond\exists x\mathcal{A}$ *, is also provable in* **QB**.$\boldsymbol{C}_{\Box\Diamond}$.

Proof.

1	$\mathcal{A} \to \exists x\mathcal{A}$	Axiom
2	$\Diamond\mathcal{A} \to \Diamond\exists x\mathcal{A}$	$\Diamond$-M
3	$\exists x\Diamond\mathcal{A} \to \Diamond\exists x\mathcal{A}$	$\exists$ -Intro

$\Box$

Lemma 4.2.5. *The rule NEC is equivalent to axiom* $\Box \boldsymbol{t}$.

Proof. One direction of the equivalence is trivial. The other direction, proving NEC is derivable given the axiom $\Box \boldsymbol{t}$, is given by the following:

1	$\mathcal{B}$	Hypothesis
2	$\boldsymbol{t} \to \mathcal{B}$	$\boldsymbol{t}$-rule
3	$\Box \boldsymbol{t} \to \Box \mathcal{B}$	2, $\Box$-M
4	$\Box \boldsymbol{t}$	Axiom
5	$\Box \mathcal{B}$	3, 4, MP

$\Box$

Lemma 4.2.6. *Any regular quantified modal relevant logic contains the rules RIC(Con) and UG(Con).*

The proof of this lemma is as in chapter 1.

4.2.2 Extensions

There are numerous extensions of $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ and $\mathbf{BQ}.\mathbf{C}_{\Box\Diamond}$ that we are interested in. First, extending the relevant logic fragment of the language is done as in the first chapter, with range of propositional relevant logics from **B** to **R** as the focus. Further, extending and cutting the modal fragment of these logics allows for an even larger set of logics. First, we may consider cutting out one of the modalities by eliminating every primitive axiom and rule in which it occurs. Second, we can extend the logic, which may cut down our need to take every modality as primitive, as some may become definable. Further, some modalities may become equivalent to others, which is somewhat like cutting out one of the equivalent modalities. For instance, an extension that equates $\Diamond$ with $\diamonddot$ essentially cuts out $\Diamond$, making its axioms and rules redundant.

Third, we can add some axiom schemes or rules that do not strictly fall into a purely modal and purely propositional fragment. As example of an axiom scheme of this kind is the Barcan formula, which is not a theorem of these logics.

The following list of axioms and rules are worth considering. We will later discuss how many logics are possible, as some different axiomatizations may form equivalent logics.[1]

[1]Many of the axioms considered here are paired, one for $\Box$, the other a "dual" for $\Diamond$. Or, in the

K$\Box$ $\Box(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})$

K$\Diamond$ $\Box(\mathcal{A} \to \mathcal{B}) \to (\Diamond\mathcal{A} \to \Diamond\mathcal{B})$

SC $\Box(\Box\mathcal{A} \to \mathcal{B}) \vee \Box(\Box\mathcal{B} \to \Box\mathcal{A})$

CON $\Box((\mathcal{A} \wedge \Box\mathcal{A}) \to \mathcal{B}) \vee \Box((\mathcal{B} \wedge \Box\mathcal{B}) \to \mathcal{A})$

Alt$_n$ $\Box\mathcal{A}_1 \vee \Box(\mathcal{A}_1 \to \mathcal{A}_2) \vee \cdots \vee \Box(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_n \to \mathcal{A}_{n+1})$

T $\Box\mathcal{A} \to \mathcal{A}$

T$_\Diamond$ $\mathcal{A} \to \Diamond\mathcal{A}$

D $\Box\mathcal{A} \to \Diamond\mathcal{A}$

B $\mathcal{A} \to \Box\Diamond\mathcal{A}$

B$_\Diamond$ $\Diamond\Box\mathcal{A} \to \mathcal{A}$

4 $\Box\mathcal{A} \to \Box\Box\mathcal{A}$

4$_\Diamond$ $\Diamond\Diamond\mathcal{A} \to \Diamond\mathcal{A}$

5 $\Diamond\mathcal{A} \to \Box\Diamond\mathcal{A}$

5$_\Diamond$ $\Diamond\Box\mathcal{A} \to \Box\mathcal{A}$

N$\boldsymbol{t}$ $\Box\boldsymbol{t}$

BF $\forall x\Box\mathcal{A} \to \Box\forall x\mathcal{A}$

BF$_\Diamond$ $\Diamond\exists x\mathcal{A} \to \exists x\Diamond\mathcal{A}$

NEC $\dfrac{\vdash \mathcal{A}}{\vdash \Box\mathcal{A}}$

Many additional axiom schemes result from substituting a modality for its corresponded dotted version. These will be explored in some detail later, and it will be made explicit when such axioms are being used.

case of both modalities in the axiom or rule, the unaltered name is the common formulation, while the subscripted diamond represents the dual. Here, for notational convenience, I will only add a subscript for the axiom or rule corresponding to $\Diamond$ in order to save myself some typing. In addition, this is in keeping with a naming convention found in Chellas [22].

4.3 Semantics

Here we will combine the extended Mares and Goldblatt style semantics with Seki's semantics, as we have done earlier. Given these two general frame semantics, we combine the defining tuples of their frames (and models) together, identifying the overlapping elements, and stipulating additional conditions where necessary.

The frames which are defined here require an additional operation to be defined. The set *Prop* will be closed under a number of operations already defined, but here we add an operation for $\Diamond$. For every $X \in \wp(K)$, let

$$\Diamond X = \{a \in K : \exists b(S_\Diamond ab \;\&\; b \in X)\}.$$

Then, just as we did for the other operations, let

$$\Diamond(\phi f) = (\Diamond\phi)f$$

for every propositional function ϕ.

Definition 4.3.1. A $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ *frame* is a tuple

$$\mathfrak{F} = \langle K, 0, R, *, S_\Box, S_\Diamond, U, Prop, PropFun\rangle$$

where $\langle K, 0, R, *\rangle$ is an $\mathbf{B}$-frame, $S_\Box, S_\Diamond \subseteq K^2$, U is a non-empty set, *Prop* is a subset of the up-sets of K which contains 0, *PropFun* is a subset of the functions of type $U^\omega \longrightarrow Prop$, and the following conditions are satisfied:

c1 if $a \in 0$ and $a \leq b$ then $b \in 0$;

c2 $\leq$ is reflexive and transitive;

c3 if $a \leq b$ and $Rbcd$, then $Racd$;

c4 if $a \leq c$ and $Rbcd$, then $Rbad$;

c5 if $d \leq a$ and $Rbcd$, then $Rbca$;

c6 if $a \leq b$ then $b^* \leq a^*$;

c7 $a^{**} = a$;

c8 if $a \le b$ and $S_\Box bc$ then $S_\Box ac$;

c9 if $a \le b$ and $S_\Diamond ac$ then $S_\Diamond bc$;

c10 if $X, Y \in Prop$, then $X \cup Y$, $X \cap Y$, $X \Rightarrow Y$, $X^* \in Prop$, where these operations are defined as before;

c11 if $X \in Prop$, then $\Box X, \Diamond X \in Prop$;

c12 the constant function ϕ_0 is in $PropFun$;

c13 if $\phi, \psi \in PropFun$, then $\phi \cup \psi$, $\phi \cap \psi$, $\phi \Rightarrow \psi$, ϕ^*, $\Box\phi$, $\Diamond\phi \in PropFun$;

c14 if $\phi \in PropFun$, then $\exists_n \phi$, $\forall_n \phi \in PropFun$, for every $n \in \omega$.

Using condition c4, and the fact that 0 is an up-set, we can derive the transitivity requirements of $\le$.

Definition 4.3.2. A *pre-model* for $\mathbf{QB.C}_{\Box\Diamond}$ is a tuple

$$\mathfrak{M} = \langle K, 0, R, *, S_\Box, S_\Diamond, U, Prop, PropFun, |-|^{\mathfrak{M}} \rangle$$

where $\langle K, 0, R, *, S_\Box, S_\Diamond, U, Prop, PropFun \rangle$ is a $\mathbf{QB.C}_{\Box\Diamond}$-frame and $|-|^{\mathfrak{M}}$ is a value assignment that assigns

1. an element $|c|^{\mathfrak{M}} \in U$ to each constant symbol c;
2. a function $|P|^{\mathfrak{M}} : U^n \longrightarrow \wp(K)$ to each n-ary predicate symbol P;
3. a propositional function $|\mathcal{A}|^{\mathfrak{M}} : U^\omega \longrightarrow \wp(K)$ to each formula $\mathcal{A}$ such that, when $\mathcal{A}$ is the atomic $P\tau_1, \ldots, \tau_n$, the propositional function assigned to it is given by, for each $f \in U^\omega$,

$$|P\tau_1, \ldots, \tau_n|^{\mathfrak{M}} f = |P|^{\mathfrak{M}}(|\tau_1|^{\mathfrak{M}} f, \ldots, |\tau_n|^{\mathfrak{M}} f).$$

Further, when $\mathcal{A}$ is not atomic, the function assigned to the formula is given by the following:

$$|\boldsymbol{t}|^{\mathfrak{M}} = \phi_0$$
$$|\mathcal{A} \wedge \mathcal{B}|^{\mathcal{M}} = |\mathcal{A}|^{\mathcal{M}} \cap |\mathcal{B}|^{\mathcal{M}}$$
$$|\mathcal{A} \vee \mathcal{B}|^{\mathcal{M}} = |\mathcal{A}|^{\mathcal{M}} \cup |\mathcal{B}|^{\mathcal{M}}$$
$$|\neg\mathcal{A}|^{\mathcal{M}} = (|\mathcal{A}|^{\mathcal{M}})^*$$
$$|\mathcal{A} \to \mathcal{B}|^{\mathcal{M}} = |\mathcal{A}|^{\mathcal{M}} \Rightarrow |\mathcal{B}|^{\mathcal{M}}$$
$$|\Box\mathcal{A}|^{\mathcal{M}} = \Box|\mathcal{A}|^{\mathcal{M}}$$
$$|\Diamond\mathcal{A}|^{\mathcal{M}} = \Diamond|\mathcal{A}|^{\mathcal{M}}$$
$$|\forall x\mathcal{A}|^{\mathcal{M}} = \forall_x|\mathcal{A}|^{\mathcal{M}}$$
$$|\exists x\mathcal{A}|^{\mathcal{M}} = \exists_x|\mathcal{A}|^{\mathcal{M}}$$

Definition 4.3.3. A *model* for $\mathbf{QB.C}_{\Box\Diamond}$ is a pre-model for $\mathbf{QB.C}_{\Box\Diamond}$ that assigns a member of $PropFun$ to each atomic formula.

Given the functions determined by a (pre-)model, we can then consider the truth sets of each function as it is applied to a variable assignment. These truth sets are given, for each $f \in U^{\omega}$, as follows:

$$|\boldsymbol{t}|^{\mathcal{M}} f = 0$$
$$|\mathcal{A} \wedge \mathcal{B}|^{\mathcal{M}} f = |\mathcal{A}|^{\mathcal{M}} f \cap |\mathcal{B}|^{\mathcal{M}} f$$
$$|\mathcal{A} \vee \mathcal{B}|^{\mathcal{M}} f = |\mathcal{A}|^{\mathcal{M}} f \cup |\mathcal{B}|^{\mathcal{M}} f$$
$$|\neg\mathcal{A}|^{\mathcal{M}} f = (|\mathcal{A}|^{\mathcal{M}} f)^*$$
$$|\mathcal{A} \to \mathcal{B}|^{\mathcal{M}} f = |\mathcal{A}|^{\mathcal{M}} f \Rightarrow |\mathcal{B}|^{\mathcal{M}} f$$
$$|\Box\mathcal{A}|^{\mathcal{M}} f = \Box(|\mathcal{A}|^{\mathcal{M}} f)$$
$$|\Diamond\mathcal{A}|^{\mathcal{M}} f = \Diamond(|\mathcal{A}|^{\mathcal{M}} f)$$
$$|\forall x\mathcal{A}|^{\mathcal{M}} f = \sqcap_{g \in xf}|\mathcal{A}|^{\mathcal{M}} g$$
$$|\exists x\mathcal{A}|^{\mathcal{M}} f = \sqcup_{g \in xf}|\mathcal{A}|^{\mathcal{M}} g$$

Finally, a familiar looking $\vDash_{\mathfrak{M}}$ relation — or simply $\vDash$ for convenience — is determined as follows:

(i) $a, f \vDash P\tau_1, \ldots, \tau_n$ iff $a \in |P\tau_1, \ldots, \tau_n|^{\mathfrak{M}} f$

(ii) $a, f \vDash \boldsymbol{t}$ iff $a \in 0$

(iii) $a, f \vDash \mathcal{A} \wedge \mathcal{B}$ iff $a, f \vDash \mathcal{A}$ and $a, f \vDash \mathcal{B}$

(iii) $a, f \vDash \mathcal{A} \vee \mathcal{B}$ iff $a, f \vDash \mathcal{A}$ or $a, f \vDash \mathcal{B}$

(iv) $a, f \vDash \neg\mathcal{A}$ iff $a^*, f \nvDash \mathcal{A}$

(v) $a, f \vDash \mathcal{A} \to \mathcal{B}$ iff $\forall b, c((Rabc$ and $b, f \vDash \mathcal{A}) \Rightarrow c, f \vDash \mathcal{B})$

(vi) $a, f \vDash \Box\mathcal{A}$ iff $\forall b(S_{\Box}ab \Rightarrow b, f \vDash \mathcal{A})$

(vii) $a, f \vDash \Diamond\mathcal{A}$ iff $\exists b(S_{\Diamond}ab \;\&\; b, f \vDash \mathcal{A})$

(viii) $a, f \vDash \forall x\mathcal{A}$ iff there is an $X \in Prop$ such that $X \subseteq \bigcap_{g \in x_n f} |\mathcal{A}|^{\mathfrak{M}} g$ and $a \in X$

(ix) $a, f \vDash \exists x\mathcal{A}$ iff, for every $X \in Prop$ such that $a^* \in X$, there is a $b \in X$ and x-variant such that $b^* \in |\mathcal{A}|^{\mathfrak{M}} g$

Definition 4.3.4. A formula $\mathcal{A}$ is *satisfied* by a variable assignment f in a model $\mathfrak{M}$ if $a, f \vDash \mathcal{A}$, for every $a \in 0$. A formula $\mathcal{A}$ is *valid* in the model $\mathfrak{M}$, if it is satisfied by every variable assignment in that model. A formula $\mathcal{A}$ is valid in a frame, if it is valid in every model based on the frame. Further, a formula is valid is a class of frames, if it is valid in every frame in the class.

Lemma 4.3.5 (Semantic Entailment)**.** *In a model $\mathfrak{M}$, a formula $\mathcal{A} \to \mathcal{B}$ is satisfied by the variable assignment f iff for every $a \in K$, if $a, f \vDash \mathcal{A}$, then $a, f \vDash \mathcal{B}$.*

The proof of this lemma is as before.

Lemma 4.3.6. *For any formula $\mathcal{A}$, if f and g agree on all free variable of $\mathcal{A}$, then $|\mathcal{A}|f = |\mathcal{A}|g$.*

Proof. The proof is by induction on the complexity of $\mathcal{A}$. The arguments of Mares and Goldblatt may be used for most cases. The cases for $\mathcal{A} = \Box\mathcal{B}$ and $\mathcal{A} = \exists x\mathcal{B}$ are covered by arguments from previous chapters. Finally, the case of $\mathcal{A} = \Diamond\mathcal{B}$ is similar to the case for $\Box$. $\Box$

Lemma 4.3.7. *Let τ be free for x in a formula $\mathcal{A}$. If $g \in xf$ and $|x|g = |\tau|f$, then $|\mathcal{A}[\tau/x]|f = |\mathcal{A}|g$ in a model $\mathfrak{M}$.*

Proof. The proof is by induction on the complexity of $\mathcal{A}$. Previous arguments again apply here for every case except $\mathcal{A} = \Diamond\mathcal{B}$, which is straightforward. $\square$

4.4 Soundness

Lemma 4.4.1. *The axioms of* $\mathbf{B}^{\circ t}$ *are valid, and the rules of* $\mathbf{B}^{\circ t}$ *preserve validity, in the class of* $\mathbf{QB.C}_{\square\Diamond}$*-frames.*

The proof of this is standard.

Lemma 4.4.2. *The axioms* $\forall x\mathcal{A} \to \mathcal{A}[\tau/x]$*, where* τ *is free for* x *in* $\mathcal{A}$ *and* $\mathcal{A}[\tau/x] \to \exists x\mathcal{A}$*, where* τ *is free for* x *in* $\mathcal{A}$ *are valid .*

The arguments of the previous chapter are again sufficient here.

Lemma 4.4.3. *The rules RIC and* $\exists$*-Intro preserve validity in every* $\mathbf{QB.C}_{\square\Diamond}$*-model.*

Again, the arguments of the previous chapter are sufficient for a proof here.

Finally, we are left with the axioms and rules governing $\Diamond$ and $\square$.

Lemma 4.4.4. *The axioms* $(\square\mathcal{A} \wedge \square\mathcal{B}) \to \square(\mathcal{A} \wedge \mathcal{B})$ *and* $\Diamond(\mathcal{A} \vee \mathcal{B}) \to (\Diamond\mathcal{A} \vee \Diamond\mathcal{B})$ *are valid in the class of* $\mathbf{QB.C}_{\square\Diamond}$*-frames.*

Proof. For the former, begin by supposing that $a, f \vDash \square\mathcal{A} \wedge \square\mathcal{B}$. It follows that $a, f \vDash \square\mathcal{A}$ and $a, f \vDash \square\mathcal{B}$. For reductio, let $a, f \nvDash \square(\mathcal{A} \wedge \mathcal{B})$. Then there exists a b such that $S_\square ab$ and $b, f \nvDash \mathcal{A} \wedge \mathcal{B}$. Thus, either $b, f \nvDash \mathcal{A}$ or $b, f \nvDash \mathcal{B}$. However, given $a, f \vDash \square\mathcal{A}$ and $a, f \vDash \square\mathcal{B}$ and $S_\square ab$, we get that both $b, f \vDash \mathcal{A}$ and $b, f \vDash \mathcal{B}$, producing our contradiction. Thus $a, f \vDash \square(\mathcal{A} \wedge \mathcal{B})$. Finally, by Semantic Entailment we get our result.

For the latter, begin by supposing that $a, f \vDash \Diamond(\mathcal{A} \vee \mathcal{B})$. Then there is a b such that $S_\Diamond ab$ and $b, f \vDash \mathcal{A} \vee \mathcal{B}$. That is, either $b, f \vDash \mathcal{A}$ or $b, f \vDash \mathcal{B}$. If the former, then $a, f, \vDash \Diamond\mathcal{A}$ and also $a, f, \vDash \Diamond\mathcal{A} \vee \Diamond\mathcal{B}$. If the latter, then $a, f, \vDash \Diamond\mathcal{B}$ and also $a, f, \vDash \Diamond\mathcal{A} \vee \Diamond\mathcal{B}$. Either way we have $a, f, \vDash \Diamond\mathcal{A} \vee \Diamond\mathcal{B}$, and by Semantic Entailment we have our result. $\square$

Lemma 4.4.5. *The rules of* $\square$*-Monotonicity and* $\Diamond$*-Monotonicity preserve validity in every* $\mathbf{QB.C}_{\square\Diamond}$*-model.*

Proof. For the former, suppose that $\mathcal{A} \to \mathcal{B}$ is valid in a model. For some f and a, let $a, f, \vDash \Box\mathcal{A}$. For reductio, let $a, f \nvDash \Box\mathcal{B}$. Then there is a b such that $S_\Box ab$ and $b, f \nvDash \mathcal{B}$. Also, we have that $b, f \vDash \mathcal{A}$. Given the validity of $\mathcal{A} \to \mathcal{B}$, $b, f \vDash \mathcal{A}$, and Semantic Entailment, we have that $b, f \vDash \mathcal{B}$, which gives us our contradiction. Therefore $a, f \vDash \Box\mathcal{B}$. The result follows by Semantic Entailment.

For the latter, the proof is just as straightforward. $\Box$

Theorem 4.4.6 (Soundness for $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ and $\mathbf{BQ}.\mathbf{C}_{\Box\Diamond}$)**.** *The logic* $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ *is sound for the defined semantics, and the logic* $\mathbf{BQ}.\mathbf{C}_{\Box\Diamond}$ *is sound for the defined semantics with the additional condition for extensional confinement as in chapter 1.*

Proof. For $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$, the previous lemmas suffice. For $\mathbf{BQ}.\mathbf{C}_{\Box\Diamond}$, the arguments of the previous chapter give us a straightforward proof of the validity of the additional axioms, which will give us our result. $\Box$

Given soundness for the basic logics $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ and $\mathbf{BQ}.\mathbf{C}_{\Box\Diamond}$, we can now turn to their extensions. Of course, adding the semantic conditions corresponding to axioms of propositional relevant logics extending $\mathbf{B}^{\circ \boldsymbol{t}}$ will produce models for the logics resulting by the addition of the axioms. More interesting is the isolation of piecemeal conditions which build up to (something equivalent to) the conditions Mares and Meyer state with the binary relation T.

4.4.1 Extending $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ and $\mathbf{BQ}.\mathbf{C}_{\Box\Diamond}$

There are three kinds of formulas which may be taken as axiom schemes to extend $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ and $\mathbf{BQ}.\mathbf{C}_{\Box\Diamond}$. The first extend the quantified relevant logic fragment, which affects the name of the logic to the left of the dot. The second is axioms which extend the modal fragment of these logics. These affect the name to the right of the dot. Finally, there are formulas like the Barcan Formula which involve both modalities and quantifiers. The naming convention for this last kind of axiom scheme, since these schemes don't fit nicely into one of the first two categories, will be to append the logic's name with "+$\mathfrak{A}$" for each axiom $\mathfrak{A}$ of this category.

The first kind of extension — i.e. extension via strengthening the quantified relevant fragment — is straightforward. The list of axioms and corresponding semantic conditions is given earlier in the first chapter.

Lemma 4.4.7. *The following table gives axioms and rules, and corresponding conditions which, when a model satisfied the condition, the axiom is valid or the rule preserves validity.*

T	$\Box\mathcal{A} \to \mathcal{A}$	$\forall a(S_\Box aa)$
$\mathrm{T}_\Diamond$	$\mathcal{A} \to \Diamond\mathcal{A}$	$\forall a(S_\Diamond aa)$
D	$\Box\mathcal{A} \to \Diamond\mathcal{A}$	$\forall a \exists b(S_\Box ab \ \& \ S_\Diamond ab)$
B	$\mathcal{A} \to \Box\Diamond\mathcal{A}$	$\forall a, b(S_\Box ab \Rightarrow S_\Diamond ba)$
$\mathrm{B}_\Diamond$	$\Diamond\Box\mathcal{A} \to \mathcal{A}$	$\forall a, b(S_\Diamond ab \Rightarrow S_\Box ba)$
4	$\Box\mathcal{A} \to \Box\Box\mathcal{A}$	$S^2_\Box ab \Rightarrow S_\Box ab$
$4_\Diamond$	$\Diamond\Diamond\mathcal{A} \to \Diamond\mathcal{A}$	$S^2_\Diamond ab \Rightarrow S_\Diamond ab$
5	$\Diamond\mathcal{A} \to \Box\Diamond\mathcal{A}$	$\forall a, b, c(S_\Diamond ab \ \& \ S_\Box ac \Rightarrow S_\Diamond cb)$
$5_\Diamond$	$\Diamond\Box\mathcal{A} \to \Box\mathcal{A}$	$\forall a, b, c(S_\Diamond ab \ \& \ S_\Box ac \Rightarrow S_\Box bc)$

Proof. Case T: Suppose that $a, f \vDash \Box\mathcal{A}$. Given $S_\Box aa$, by the truth condition for $\Box\mathcal{A}$, we have that $a, f \vDash \mathcal{A}$. The result follows by Semantic Entailment.

Case $\mathrm{T}_\Diamond$: Suppose that $a, f \vDash \mathcal{A}$. Given that $S_\Diamond aa$ and the truth condition for $\Diamond\mathcal{A}$, we have that $a, f \vDash \Diamond\mathcal{A}$. The result follows by Semantic Entailment.

Case D: Suppose that $a, f \vDash \Box\mathcal{A}$. It follows by the condition assumed that there is a b such that $S_\Box ab$ & $S_\Diamond ab$. By the truth condition for $\Box\mathcal{A}$, it follows then that $b, f \vDash \mathcal{A}$, which then gives us $a, f \vDash \Diamond\mathcal{A}$. The result follows by Semantic Entailment.

Case B: Suppose that $a, f \vDash \mathcal{A}$. Either $S_\Box ab$ for some b, or there is no such b. If the latter, then the result follows immediately. If the former, then by the condition assumed, we have that $S_\Diamond ba$. Thus, $b, f \vDash \Diamond\mathcal{A}$. Further, since this is provable for every such b, we have that $a, f \vDash \Box\Diamond\mathcal{A}$. The result follows by Semantic Entailment.

Case $\mathrm{B}_\Diamond$: Suppose that $a, f \vDash \Diamond\Box\mathcal{A}$. It follows that there is a b such that $S_\Diamond ab$. It follows, by our assumption, that we have $S_\Box ba$. By the truth conditions, we have first $b, f \vDash \Box\mathcal{A}$, and then $a, f \vDash \mathcal{A}$. The result follows by Semantic Entailment.

Case 4: Suppose $a, f \vDash \Box\mathcal{A}$. For reductio, suppose that $a, f \nvDash \Box\Box\mathcal{A}$. Then there is a b such that $S_\Box ab$ and $b, f \nvDash \Box\mathcal{A}$. Thus, there is a c such that $S_\Box bc$ and $c, f \nvDash \mathcal{A}$. By our assumption, we get that $S_\Box ac$, and so $c, f \vDash \mathcal{A}$, which gives us our contradiction, so $a, f \vDash \Box\Box\mathcal{A}$. The result follows by Semantic Entailment.

Case $4_\Diamond$: Suppose that $a, f \vDash \Diamond\Diamond\mathcal{A}$. Then there is a b such that $S_\Diamond ab$ and $b, f \vDash \Diamond\mathcal{A}$. Thus there is a c such that $S_\Diamond bc$ and $cf \vDash \mathcal{A}$. By assumption, we get that $S_\Diamond ac$ and thus $a, f \vDash \Diamond\mathcal{A}$. The result follows by Semantic Entailment.

Case 5: Suppose that $a, f \vDash \Diamond\mathcal{A}$. Then there is a b such that $S_\Diamond ab$ and $b, f \vDash \mathcal{A}$. For reductio, suppose that $a, f \nvDash \Box\Diamond\mathcal{A}$. Then there is a c such that $S_\Box ac$ and $c, f \nvDash \Diamond\mathcal{A}$. However, by our assumption we get that $S_\Diamond cb$, and so $c, f \vDash \Diamond\mathcal{A}$, a contradiction. So, we have $a, f \vDash \Box\Diamond\mathcal{A}$. The result follows by Semantic Entailment.

Case $5_\Diamond$: Suppose that $a, f \vDash \Diamond\Box\mathcal{A}$. Then there is a b such that $S_\Diamond ab$ and $b, f \vDash \Box\mathcal{A}$. For reductio, suppose that $a, f \nvDash \Box\mathcal{A}$. Then there is a c such that $S_\Box ac$ and $c, f \nvDash \mathcal{A}$. By assumption we get $S_\Box bc$. Thus, $c, f \vDash \mathcal{A}$, which gives a contradiction. Thus, $a, f \vDash \Box\mathcal{A}$. The result follows by Semantic Entailment. $\Box$

Lemma 4.4.8. *The following table gives axioms and rules, and corresponding conditions given by Seki which, when a model satisfies the condition, the axiom is valid or the rule preserves validity.*

$K\Box$	$\Box(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})$	*if $Rbcf$ and $S_\Box fd$, then there exist $b', c' \in K$ such that $S_\Box bb'$, $S_\Box cc'$, and $Rb'c'd$*
$K\Diamond$	$\Box(\mathcal{A} \to \mathcal{B}) \to (\Diamond\mathcal{A} \to \Diamond\mathcal{B})$	*if $Rbcd$ and $S_\Diamond cf$, then there exists $b', d' \in K$ such that $Rb'fd'$, $S_\Box bb'$, and $S_\Diamond dd'$*
SC	$\Box(\Box\mathcal{A} \to \mathcal{B}) \vee \Box(\Box\mathcal{B} \to \mathcal{A})$	*If $S_\Box ab$, $Rbcd$, $S_\Box ab'$, and $Rb'c'd'$, then $S_\Box cd'$ or $S_\Box c'd$*
CON	$\Box(\mathcal{A} \wedge \Box\mathcal{A} \to \mathcal{B}) \vee$ $\Box(\mathcal{B} \wedge \Box\mathcal{B} \to \mathcal{A})$	*If $S_\Box ab$, $Rbcd$, $S_\Box ab'$, and $Rb'c'd'$, then $c \leq d'$ or $c' \leq d$ or $S_\Box cd'$ or $S_\Box c'd$*
Alt_n	$\Box\mathcal{A}_1 \vee \Box(\mathcal{A}_1 \to \mathcal{A}_2) \vee \ldots$ $\vee\Box(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_n \to \mathcal{A}_{n+1})$	*If $S_\Box ad_0$, $\bigwedge_{i=1}^{n}(S_\Box ab'$ and $Rb_i, c_i, d_i)$ and $d_n \neq u$ then $\bigvee_{i=0}^{n-1} \bigvee_{j=i+1}^{n} c_j \leq d_i$*
	Rule	***Frame Condition***
NEC	$\dfrac{\mathcal{A}}{\Box\mathcal{A}}$	*if $b \in 0$ and $S_\Box bc$, then $c \in 0$*

The frame condition for the rule NEC is equivalent to the frame condition for the axiom $\Box\boldsymbol{t}$ in regular modal logics.

The Barcan formula will be discussed in section 4.7.

4.5 Completeness

Theories are defined as before. Further, the extension lemma is still provable. For the construction of a canonical model, we either assume that the set of constants Con is denumerable, or we add a denumerable number of constants to the set Con in the usual way.

Definition 4.5.1. A *canonical frame for* $\mathbf{QB.C}_{\Box\Diamond}$ is a tuple,

$$\mathfrak{F}_c = \langle K_c, 0_c, R_c, *_c, S_{\Box c}, S_{\Diamond c}, U_c, Prop_c, PropFun_c \rangle$$

where

- K_c is the set of all prime theories.
- 0_c is the set of all regular prime theories.
- R_c is defined by $R_c abc$ iff $\{\mathcal{A} \circ \mathcal{B} : \mathcal{A} \in a \ \& \ \mathcal{B} \in b\} \subseteq c$.
- $S_{\Box c}$ is defined by $S_{\Box c} ab$ iff $\{\mathcal{A} : \Box\mathcal{A} \in a\} \subseteq b$.
- $S_{\Diamond c}$ is defined by $S_{\Diamond c} ab$ iff $\{\Diamond\mathcal{A} : \mathcal{A} \in b\} \subseteq a$.
- $*_c$ is defined by $a^* = \{\mathcal{A} : \neg\mathcal{A} \notin a\}$.
- U_c is the infinite set of constants Con.
- For every closed formula $\mathcal{A}$, $||\mathcal{A}||_c$ is defined to be the set $\{a \in K : \mathcal{A} \in a\}$.
- $Prop_c$ is defined as the set $\{||\mathcal{A}||_c : \mathcal{A}$ is a closed formula$\}$.
- Given a variable assignment f, the value fn is a constant. Substituting each variable in a formula $\mathcal{A}$ with the constant assigned to it by a variable assignment f results in a closed formula which will be denoted $\mathcal{A}^f$. Therefore $\mathcal{A}^f = \mathcal{A}[f0/x_0, \ldots, fn/x_n, \ldots]$.
- To each formula $\mathcal{A}$, there is a corresponding function $\phi_{\mathcal{A}}$ of type $U^\omega \longrightarrow K$ given by $\phi_{\mathcal{A}} f = ||\mathcal{A}^f||_c$. $PropFun_c$ is the set of all functions $\phi_{\mathcal{A}}$, where $\mathcal{A}$ is a formula.

Definition 4.5.2. A *canonical model* for $\mathbf{QB.C}_{\Box\Diamond}$ is a tuple,

$$\mathfrak{M}_c = \langle K_c, 0_c, R_c, *_c, S_{\Box c}, S_{\Diamond c}, U_c, Prop_c, PropFun_c, |-|_c^{\mathfrak{M}} \rangle, \text{ where}$$

- $\langle K_c, 0_c, R_c, S_{\Box c}, *_c, U_c, Prop_c, PropFun_c \rangle$ is the canonical frame.
- $|c|_c = c$, for every constant symbol c.
- $|P|_c(c_0, \ldots, c_n) = ||P(c_0, \ldots, c_n)||_c$.

- The valuation is extended to all wff as before.

Lemma 4.5.3. *If a and b are theories, c is a prime theory, and $R_c abc$, then there are prime theories a' and b' extending a and b respectively such that $R_c a'b'c$.*

The proof of this lemma is standard.

Lemma 4.5.4. *If a is a prime theory, b is a theory, and $S_{\Box c}ab$, and $\mathcal{A} \notin b$, then there is a prime theory b' extending b such that $S_{\Box c}ab'$ and $\mathcal{A} \notin b'$.*

The proof is straightforward using the arguments of the first chapter.

Lemma 4.5.5. *If a is a prime theory, b is a theory, and $S_{\Diamond}ab$, then there is a prime theory b' extending b such that $S_{\Diamond}ab'$.*

Lemma 4.5.6. *Conditions c1–c7 are satisfied by the canonical model.*

The conditions c1–c7 can be shown to hold by the usual arguments, which will also demonstrate that $\langle K_c, 0_c, R_c, *_c\rangle$ is a **B**-frame and that $a \leq b$ iff $a \subseteq b$.

Lemma 4.5.7. *Conditions c8 and c9 are satisfied by the canonical model. That is, if $a \leq b$, $S_{\Box c}bc$, and $S_{\Diamond c}ad$, then $S_{\Box c}ac$ and $S_{\Diamond c}bd$.*

Proof. Assume that $a \leq b$, which is $a \subseteq b$. Further, let $S_{\Box c}bc$. From the latter we have that $\{\mathcal{A} : \Box\mathcal{A} \in b\} \subseteq c$, but also that if $\Box\mathcal{A} \in a$, then $\Box\mathcal{A} \in b$. Thus, $S_{\Box c}ac$. For the other condition, suppose in addition that $S_{\Diamond c}ad$. Then $\{\Diamond\mathcal{A} : \mathcal{A} \in d\} \subseteq a \subseteq b$. Thus $S_{\Diamond c}bd$. □

Lemma 4.5.8. *Prop is closed under $\cup, \cap, \Rightarrow, *$.*

Proof. The proof is as before. □

Lemma 4.5.9. *Prop is closed under $\Box$ and $\Diamond$.*

Proof. The proof for $\Box$ is as before, using the appropriate squeeze lemma. For $\Diamond$, I am required to show that $\Diamond||\mathcal{A}^f||_c = ||\Diamond(\mathcal{A})^f||_c$. We first note the following equalities given by the definition of the canonical frame and the $\Diamond$ operator.

$$\Diamond||\mathcal{A}^f||_c = \{a : \exists b(S_{\Diamond c}ab \;\&\; b \in ||\mathcal{A}^f||_c)\}$$
$$||\Diamond(\mathcal{A})^f||_c = \{a : \Diamond\mathcal{A}^f \in a\}$$

For one direction, let $c \in \{a : \exists b(S_{\Diamond c}ab \ \& \ b \in ||\mathcal{A}^f||_c)\}$. Then $\exists b(S_{\Diamond c}cb \ \& \ b \in ||\mathcal{A}^f||_c)$. It follows that $\{\Diamond\mathcal{A} : \mathcal{A} \in b\} \subseteq c$. Thus, $\Diamond\mathcal{A}^f \in c$, as required.

For the other direction, let $c \in \{a : \Diamond\mathcal{A}^f \in a\}$. Thus, $\Diamond\mathcal{A}^f \in c$. We are required to show that there is a b such that $S_{\Diamond c}cb$ and $\mathcal{A}^f \in b$.

Consider the set $b' = \{\mathcal{A} : \Diamond\mathcal{A} \in c\}$. Clearly the $S_{\Diamond c}cb'$. Further, b' is a theory. Let $\mathcal{B} \in b'$ and $\vdash \mathcal{B} \to \mathcal{C}$. Then $\Diamond\mathcal{B} \in c$. But $\vdash \Diamond\mathcal{B} \to \Diamond\mathcal{C}$ and the theoryhood of c imply that $\mathcal{C} \in b'$. Applying the squeeze lemma, we get a b extending b' such that $S_{\Diamond c}cb$. Further, $\mathcal{A}^f \in b$, as required.

□

Lemma 4.5.10. *The constant function ϕ_0 is in PropFun, and PropFun is closed under $\cup, \cap, \Rightarrow, *, \Box$, and $\Diamond$.*

Proof. The proof is much as before. The only new case is for $\Diamond$, which I will now consider. The argument is similar, however, and establishes the more general equality $\Diamond\phi_{\mathcal{A}} = \phi_{\Diamond\mathcal{A}}$. For every f,

$$\begin{aligned} \phi_{\Diamond\mathcal{A}}f &= ||\Diamond\mathcal{A}^f||_c && \text{By definition of } \phi_{\mathcal{B}} \\ &= \Diamond||\mathcal{A}^f||_c && \text{lemma 4.5.9} \\ &= \Diamond\phi_{\mathcal{A}}f && \text{by definition of } \phi_{\mathcal{B}} \end{aligned}$$

□

Lemma 4.5.11. *PropFun is closed under $\exists_n$ and $\forall_n$.*

Proof. The proof is as in the previous chapter. □

To show that this canonical model is in fact a model for $\mathbf{QB.C}_{\Box\Diamond}$, the next lemma suffices.

Lemma 4.5.12. *For every n-ary predicate symbol P, every variable assignment, and every set of terms $\tau_1, \ldots, \tau_n$,*

1. $P(\tau_1, \ldots, \tau_n)^f = P(|\tau_1|f, \ldots, |\tau_n|f)$

2. $|P(\tau_1, \ldots, \tau_n)|_c = \phi_{P(\tau_1, \ldots, \tau_n)}$

Proof. Mares and Goldblatt's arguments may be used here, as they were in earlier chapters. $\square$

Corollary 4.5.13. *The canonical model is a* $\mathbf{QB.C}_{\square\Diamond}$*-model.*

Our final step before completeness is again a truth lemma.

Lemma 4.5.14 (Truth Lemma for $\mathbf{QB.C}_{\square\Diamond}$)**.** *For any formula* $\mathcal{A}$, $\mathcal{A} = \phi_{\mathcal{A}}$. *That is, for all* f, $|\mathcal{A}|cf = ||\mathcal{A}||_c$. *In other words,* $a, f \vDash \mathcal{A}$ *iff* $\mathcal{A}^f \in a$.

Proof. Again, our only new case is that for $\Diamond$, as the $\square$ can be handled by the arguments of chapter 1, and the rest by those of chapter 2. The following argument is enough for the $\Diamond$ case.

$$\begin{aligned}
|\mathcal{A}| &= |\Diamond\mathcal{B}| && \text{Case Hyp}\\
&= \Diamond|\mathcal{B}| && \text{Definition of } |-|\\
&= \Diamond\phi_{\mathcal{B}} && \text{Inductive Hypothesis}\\
&= \phi_{\Diamond\mathcal{B}} && \text{Lemma 4.5.10}\\
&= \phi_{\mathcal{A}} && \text{Case Hyp}
\end{aligned}$$

$\square$

Theorem 4.5.15 (Completeness for $\mathbf{QB.C}_{\square\Diamond}$)**.** *If* $\mathcal{A}$ *is valid in every* $\mathbf{QB.C}_{\square\Diamond}$*-model, then* $\mathcal{A}$ *is a theorem of* $\mathbf{QB.C}_{\square\Diamond}$.

Proof. Let $\mathcal{A}$ be valid in every $\mathbf{QB.C}_{\square\Diamond}$-model including the canonical model. (The same argument may be used for $\mathbf{BQ.C}_{\square\Diamond}$.) It follows that every regular prime theory includes $\mathcal{A}^f$ for every f. For every free variable in $\mathcal{A}$, replace it with a different constant not in $\mathcal{A}$. This new formula belongs to every regular prime theory, and is therefore a $\mathbf{QB.C}_{\square\Diamond}$ theorem. Repeated but finite applications of UG(Con) followed by repeated by finite applications of the axiom $\forall x\mathcal{A} \to \mathcal{A}[t/x]$, with τ free for x in $\mathcal{A}$, will produce a proof of $\mathcal{A}$. $\square$

4.5.1 Extensions of $\mathbf{QB.C}_{\square\Diamond}$ and $\mathbf{BQ.C}_{\square\Diamond}$

Finally, we will show that the canonical model corresponding to various extensions of $\mathbf{QB.C}_{\square\Diamond}$ and $\mathbf{BQ.C}_{\square\Diamond}$ satisfies the conditions corresponding to the additional axioms or rules. For the axioms of the relevant fragment, the proof is standard. Thus, we turn to the modal axioms. In section 4.7, the Barcan formula is briefly given some attention.

Lemma 4.5.16. *The canonical model for a regular quantified modal relevant logic containing an axiom from the table in Lemma 4.4.7 satisfies the corresponding condition.*

Proof. Case T: Suppose the logic in consideration has T as an axiom (theorem) scheme. Let $\Box\mathcal{A} \in a$, for some a. It follows by the theoryhood of a and the theorem T that $\mathcal{A} \in a$. Thus, by the definition of $S_{\Box c}$, we have that $S_{\Box c}aa$, for every a, since our choice of a was arbitrary.

Case $\mathrm{T}_{\Diamond}$: This case is similar to the previous one.

Case D: Take any $a \in K_c$. Create the theory $b' = \{\mathcal{A} : \Diamond\mathcal{A} \in a\}$. Clearly $S_{\Diamond c}ab'$. By the squeeze lemma, we have b which extends b' such that $S_{\Diamond c}ab$. Further, if $\Box\mathcal{A} \in a$, then $\Diamond\mathcal{A} \in a$ by the axiom scheme D, and so $\mathcal{A} \in b$, which means that $S_{\Box c}ab$, as required.

Case B: Suppose $a, b \in K_c$ and $S_{\Box c}ab$. Further, suppose that $\mathcal{A} \in a$. It follows that $\Box\Diamond\mathcal{A} \in a$, by the axiom B. Thus, $\Diamond\mathcal{A} \in b$. This is the case for every $\mathcal{A} \in a$, so $S_{\Diamond c}ba$, as required.

Case $\mathrm{B}_{\Diamond}$: This case is similar to the previous one.

Case 4: This case is straightforward.

Case $4_{\Diamond}$: This case is straightforward.

Case 5: Suppose $a, b, c \in K_c$, $S_{\Box c}ab$ and $S_{\Diamond c}ac$. Further, suppose that $\mathcal{A} \in b$. It follows that $\Diamond\mathcal{A} \in a$. From this, using axiom 5, we get that $\Box\Diamond\mathcal{A} \in a$, and this $\Diamond\mathcal{A} \in c$, as required, so $S_{\Diamond c}cb$.

Case $5_{\Diamond}$: This case is similar to the previous one.

$\Box$

4.6 Dropping the Dot

As mentioned in chapter 1, for the diamond and box to behave sufficiently similar to how they do in the classical case, and to warrant the removal of the dot in a logic's name, we must add $\Box(\mathcal{A} \vee \mathcal{B}) \to (\Box\mathcal{A} \vee \Diamond\mathcal{B})$ or the dual $(\Diamond\mathcal{A} \wedge \Box\mathcal{B}) \to \Diamond(\mathcal{A} \wedge \mathcal{B})$. In an earlier chapter, we had the duality of the modalities $\Box$ and $\Diamond$. Consequently, would were able to give a frame condition for the axiom $\Box(\mathcal{A} \vee \mathcal{B}) \to (\Box\mathcal{A} \vee \Diamond\mathcal{B})$ using only $S_\Box$. Here, we will consider this axiom, and its dual, without assuming duality. We will take $\Box$ and $\Diamond$ as primitive (and the dotted duals as defined).

Lemma 4.6.1. *The axiom scheme $\Box(\mathcal{A} \vee \mathcal{B}) \to (\Box\mathcal{A} \vee \Diamond\mathcal{B})$ is valid in all* $\mathbf{QB.C}_{\Box\Diamond}$*-models that satisfy the following condition:*

(c11b) if $S_\Box ab$, then $\exists x \leq b(S_\Box ax\ \&\ S_\Diamond ax)$

Proof. Suppose that $a, f \vDash \Box(\mathcal{A} \vee \mathcal{B})$. For reductio, let $a, f \nvDash \Box\mathcal{A}$ and $a, f \nvDash \Diamond\mathcal{B}$. From the former we get that $S_\Box ab$ and $b, f \nvDash \mathcal{A}$. From the condition above, $S_\Box ac$ and $S_\Diamond ac$ for $c \leq b$.

So in particular, it follows that $c, f \vDash \mathcal{A} \vee \mathcal{B}$. If $c, f \vDash \mathcal{B}$, then we get a contradiction, as it entails $a, f \vDash \Diamond\mathcal{B}$. On the other hand, if $c, f \vDash \mathcal{A}$, then $b, f \vDash \mathcal{A}$, which gives us another contradiction. Thus the reductio assumption is wrong and $a, f \vDash \Box\mathcal{A} \vee \Diamond\mathcal{B}$. The result follows by Semantic Entailment. $\Box$

In chapter 2, the condition from Mares and Meyer [77] we labeled c11 — if $S_\Box ab$, then $\exists x \leq b(Tax)$ — is used to validate $\Box(\mathcal{A} \vee \mathcal{B}) \to (\Box\mathcal{A} \vee \Diamond\mathcal{B})$ in a modal relevant logic in which the box is primitive, and the diamond is a defined dual. The T relation in c11 is used in particular because of the essential role of negation in defining the diamond. The proof requires looking at star worlds. However, the condition I give above can be used in both positive fragments and in logics without the duality of the modalities. The condition above is essentially c11 restricted to the relations for two possibly non-dual modalities. If the modalities are in fact dual, then $S_\Diamond ab$ iff $S_\Box a^*b^*$, and the condition above becomes c11.

Lemma 4.6.2. *The axiom scheme $(\Diamond\mathcal{A} \wedge \Box\mathcal{B}) \to \Diamond(\mathcal{A} \wedge \mathcal{B})$ is valid in all* $\mathbf{QB.C}_{\Box\Diamond}$*-models that satisfy the following condition:*

c11d if $S_\Diamond ab$, then $\exists x \leq b(S_\Box ax\ \&\ S_\Diamond ax)$

Proof. Let $a, f \vDash \Diamond\mathcal{A} \wedge \Box\mathcal{B}$. From this we get $S_\Diamond ab$ and $b, f \vDash \mathcal{A}$. By our condition, there is a $c \leq b$ such that $S_\Box ac$ and $S_\Diamond ac$. Thus, $c, f \vDash \mathcal{B}$, which entails $b, f \vDash \mathcal{B}$. From here, we know that $b, f \vDash \mathcal{A} \wedge \mathcal{B}$, and so $a, f \vDash \Diamond(\mathcal{A} \wedge \mathcal{B})$, as required. The result follows by Semantic Entailment. $\Box$

Lemma 4.6.3. *Any logic extending* $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}+(\Diamond\mathcal{A}\wedge\Box\mathcal{B})\rightarrow\Diamond(\mathcal{A}\wedge\mathcal{B})$ *is complete w.r.t the class of all* $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$*-models satisfying (c11b). Further, any logic extending* $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}+\Box(\mathcal{A}\vee\mathcal{B})\rightarrow(\Box\mathcal{A}\vee\Diamond\mathcal{B})$ *is complete w.r.t the class of all* $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$*-models satisfying (c11d).*

Proof. The completeness proofs, of which we will only show that for the axiom $\Box(\mathcal{A}\vee\mathcal{B})\rightarrow(\Box\mathcal{A}\vee\Diamond\mathcal{B})$, are essentially the proof given by Mares and Meyer [77, Lemma 4.5], with obvious modifications to handle the star-less condition given above.

First, assume that $S_\Box ab$. That is, if $\Box\mathcal{A}\in a$, then $\mathcal{A}\in b$. We show that there is a prime theory $x\subseteq b$ such that $S_\Box ax$ and $S_\Diamond ax$. Let c be the set of formulas $\mathcal{A}$ such that $\Diamond\mathcal{A}\notin a$, and let a' be the set of formulas $\mathcal{A}$ such that $\Box\mathcal{A}\in a$. Thus, $a'\subseteq b$, and is a theory. Further suppose that d is the set of formulas not in b. We have here that the pair $(a', c\cup d)$ is independent by the following:

Suppose for reductio that the pair is not independent. Then there are $\mathcal{A}_1,\ldots,\mathcal{A}_n\in a'$, $\mathcal{B}_1,\ldots,\mathcal{B}_m\in d$ and $\mathcal{D}_1,\ldots,\mathcal{D}_l\in c$ such that

$$\vdash(\mathcal{A}_1\wedge,\ldots,\wedge\mathcal{A}_n)\rightarrow(\mathcal{B}_1\vee,\ldots,\vee\mathcal{B}_m\vee\mathcal{D}_1\vee,\ldots,\vee\mathcal{D}_l)$$

from which we can infer

$$\vdash\Box(\mathcal{A}_1\wedge,\ldots,\wedge\mathcal{A}_n)\rightarrow\Box(\mathcal{B}_1\vee,\ldots,\vee\mathcal{B}_m\vee\mathcal{D}_1\vee,\ldots,\vee\mathcal{D}_l).$$

Then, given the axiom in question, we can derive

$$\vdash\Box(\mathcal{A}_1\wedge,\ldots,\wedge\mathcal{A}_n)\rightarrow\Box(\mathcal{B}_1\vee,\ldots,\vee\mathcal{B}_m)\vee\Diamond(\mathcal{D}_1\vee,\ldots,\vee\mathcal{D}_l).$$

The primeness of the element a further implies either the $\Box(\mathcal{B}_1\vee,\ldots,\vee\mathcal{B}_m)\in a$ or that $\Diamond(\mathcal{D}_1\vee,\ldots,\vee\mathcal{D}_l)\in a$. If the latter, then $\Diamond\mathcal{D}_i\in a$ for some $i\leq l$, which entails $\mathcal{D}_i\in c$, a contradiction. On the other hand, $\Box(\mathcal{B}_1\vee,\ldots,\vee\mathcal{B}_m)\in a$, then $\mathcal{B}_1\vee,\ldots,\vee\mathcal{B}_m\in b$, which also leads to contradiction. Thus, $(a', c\cup d)$ is independent. Using the extension lemma, There is a prime theory x extending a' such that the pair $(x, c\cup d)$ is independent. It is obvious that $S_\Box ax$ by our definition of a'. To show that $S_\Diamond ax$, suppose that it did not hold. Then there is a formula $\mathcal{A}$ such that $\mathcal{A}\in x$ but $\Diamond\mathcal{A}\notin a$. Such formulas make up the set c, as defined above. All such formulas were excluded from x in the extension lemma. So $S_\Diamond ax$. □

This brings us to the title of the current section. In the naming scheme of the logics of this chapter, as given in an earlier chapter, my convention is to drop the dot in the name of the logic when (1) the logic contains both the diamond and the box (with possibly one being defined), (2) the diamond and box are duals, and (3) both the formula schemes discussed in this section are theorems.

4.7 Barcan Formula

Here we briefly discuss the role of the Barcan formula in modal relevant logics. In the logics defined above, using UI, the Barcan formula is not derivable. However, the Converse Barcan Formula is derivable.

Lemma 4.7.1. *The Barcan formula is not a theorem of* $\mathbf{RQ4}^{\circ t}$.

Proof. All of the theorems of $\mathbf{RQ4}^{\circ t}$ are theorems of quantified S4 with UI. One cannot prove the Barcan Formula in quantified S4 with UI [47, p. 82]. Therefore, the Barcan Formula cannot be a theorem of $\mathbf{RQ4}^{\circ t}$. □

While the Barcan Formula is not a theorem of any regular quantified modal logic weaker or equivalent to $\mathbf{RQ4}^{\circ t}$, a model demonstrating the invalidity of this Formula is desirable.

Corollary 4.7.2. *The Barcan Formula is not a theorem of any logic of the form* $\mathbb{L}.\mathbf{M}^{\circ t}$*, as defined earlier in the chapter.*

This follows because $\mathbf{RQ4}^{\circ t}$ is the strongest logic of the form $\mathbb{L}.\mathbf{M}^{\circ t}$ defined in this chapter.

In classical logics, the Barcan formula is related to Kripke/full models, in which *Prop* and *PropFun* are full and the truth condition for quantified formula is Tarkian. Mares and Goldblatt [75] show that, for a class of frames $\mathbb{C}$ of classical modal logics, a formula $\mathcal{A}$ is valid in all standard constant domain models based on members of $\mathbb{C}$ iff $\mathcal{A}$ is valid in all Tarskian general frames based on members of $\mathbb{C}$ iff $\mathcal{A}$ is valid in all full general frames based on members of $\mathbb{C}$ [75, Theorem 7]. They go on to show that models that are not Tarskian can validate the Barcan Formula, showing that "the

Tasrskian condition is sufficient to ensure validity of BF, but it is not necessary" [75, p. 14].

The question then remains as to what the role of the Barcan formula is in modal relevant logics.

Lemma 4.7.3. *In any of the models defined in this chapter, if the model is Kripkean, then the truth condition for* $\forall x\mathcal{A}$ *is Tarskian.*

If a model is full, then the infinite intersection is admissible — $\bigcap S = \prod S$. The truth condition therefore becomes Tarskian.

Lemma 4.7.4. *If* $\mathfrak{M}$ *is a Kripkean/full model, then* $\mathfrak{M}$ *validates BF.*

Proof. The proof is much like [47, Lemma 2.2.7]. Let $a, f \vDash \forall x\Box\mathcal{A}$. Then suppose that $S_\Box ab$. The domains are constant, or at least universal, in the sense that there is a single domain for the entire model. So, if $a \in U$, then $a, f[a/x] \vDash \Box\mathcal{A}$, which implies that $b, f[a/x] \vDash \mathcal{A}$.

So we have that, for every $a \in U$, $b, f[a/x] \vDash \mathcal{A}$. From this, and the assumption that the model is Kripkean and therefore Tarskian, we get that $b, f \vDash \forall x\mathcal{A}$. As this holds for every b such that $S_\Box ab$, it follows that $a, f \vDash \Box\forall x\mathcal{A}$. The result follows by Semantic Entailment. □

Note that Goldblatt and Ian Hodkinson [48] shows that Kripkean models also validate commuting quantifiers (CQ — $\forall x\forall y\mathcal{A} \to \forall y\forall x\mathcal{A}$) in classical modal logics. In the relevant logics we consider, we can prove that $\vdash$ CC using UI (and RIC) without the Barcan Formula [74, Lemma 6.3 (d)].

What is left to do is prove, of certain classes of models, that the relevant logics considered plus the Barcan Formula are characterized by these models. That is, give soundness and completeness results for whichever classes of models we are able. Here, we must be careful that the class of models we choose does not imply incompleteness for the quantified relevant fragment of the logic by, for example, becoming vulnerable to Fine's incompleteness proof [38].

Although Goldblatt [47] and Mares and Goldblatt [74, 75] prove many relations between logics with the Barcan Formula (and CQ), the tightest characterization so far is not the most illuminating. That characterization, for relevant logics, is given in the following theorem. While related to Kripkean models with Tarskian conditions, the following fact about relevant logics with the Barcan formula is the furthest we will venture into the Barcan formula in this work.

Theorem 4.7.5. *Where $\mathbb{L}$ is a quantified modal relevant logic without BF, the logic $\mathbb{L}$+BF is characterized by the class of all admissible function models that satisfy*

$$\forall_x \Box \phi \subseteq \Box \forall_x \phi,$$

and are based on a general $\mathbb{L}$-frame. (Where $\Box$ in the condition above is the type-lifted operation on propositional functions.)

Proof. The proof is essentially unaltered from Goldblatt [47]. For soundness, we have the following:

$$\begin{aligned} |\forall x \Box \mathcal{A}| f &= \forall_x \Box |\mathcal{A}| f \\ &\subseteq \Box \forall_x |\mathcal{A}| f \\ &= |\Box \forall x \mathcal{A}| f \end{aligned}$$

For completeness, the goal is to show that $\phi_{\forall x \Box \mathcal{A}} \subseteq \phi_{\Box \forall x \mathcal{A}}$ using the fact that $\vdash \forall x \Box \mathcal{A} \to \Box \forall x \mathcal{A}$. For this, it is sufficient to show that, if $\vdash \mathcal{A} \to \mathcal{B}$, then $\phi_{\mathcal{A}} \subseteq \phi_{\mathcal{B}}$. Take an arbitrary $a \in K$ and $f \in U^{\omega}$, and suppose that $a \in \phi_{\mathcal{A}} f$. By definition, this is $a \in ||\mathcal{A}^f||_c$, and so $\mathcal{A}^f \in a$. But a is a theory, and so $\mathcal{B}^f \in a$. This gives us $a \in \phi_{\mathcal{B}}$, as required. ∎

This theorem might not seem very insightful for the role the Barcan formula plays. Goldblatt proves a similar theorem [47, Theorem 4.5.4]. Goldblatt explains the importance of this and related theorems.

> It might be thought that characterizations using the condition [defined above] do little to advance our understanding of the Barcan Formula, since this is essentially a translation of BF into structural/algebraic form. On the other hand, for [some logics considered], the results given here would appear to be the first characterizations of any kind that are based on possible-worlds style relational semantics. [47, p. 157]

Thus, proving this theorem will at least be a good starting point in exploring characterizations of relevant logics with the Barcan Formula.

Chapter 5

Identity in Relevant Logics

5.1 Introduction

Philip Kremer [62] pointed out in 1999 that *identity* in relevant logics has received little attention. To this day, it is still not settled how to interpret identity in relevant logics, let alone which axiomatizations are satisfactory. Nevertheless, we now have additional criteria for which to judge such axiomatizations. Kremer [62] proposed that an axiomatization of identity ought to be *stable* (in a sense explicated later in this chapter) and this results in the rejection of particular axiomatizations.

While Kremer has pointed out that identity has not been given much attention in relevant logic, he also notes that Mares [69] deals with multiple axiomatizations of identity in relevant logic. Mares provides a way of extending Fine's semantics [37] in order to model identity, and does so for various axiomatizations of identity. Given Kremer's proposed interpretation of identity, novel semantics extending the models of Mares and Goldblatt with identity will be constructed here.

My original intent was to provide a rough-and-ready recipe for creating adequate semantics for a variety of axiomatizations of identity in modal relevant logic, and to largely leave the task of finding the right interpretation to those seeking to apply these logics. This was to begin in this chapter with adding identity to relevant logics, and then extend this base to modal logics in the next chapter. However, I was swayed to reject some axiomatizations by the philosophical arguments of Philip Kremer (with regards to *stability*) combined with the introduction of opaque contexts given by the modal fragments of the logics considered. Thus, this chapter will first examine possible axiomatizations for identity, and will offer some argument as to how the list should be narrowed down. Different relevant logics as a base seem to entail the *prima facie* feasibility of different axiomatizations of identity, and most arguments on identity in relevant logics have focused on only a couple of the relevant logics at most. As a result, it may be that stronger axiomatizations for identity are feasible in weaker relevant logics, as some of the results for stronger logics may require the strength of those logics. That is, because weaker logics in terms of theoremhood are stronger in terms of making finer distinctions and blocking more inferences.

Modalities will be added, and various axioms for identity in the presence of modalities will be considered, in the next chapter. We will make our way to Kremer's take on indiscernibility, which he gives as both an intuitive semantic idea and a set of axiom schemes. However, we will point out an essential mismatch between his semantic idea and axiom choice (in first order logic) by developing a semantics on Kremer's suggested idea and showing it validates more than his axiom system, and by developing a semantics for his axiom system which does not quite capture his semantic suggestion. The latter semantics is not only motivated by the relevant indiscernibility interpretation of identity offered by Kremer, which uses the notion of a relevant predication as found in Dunn [29, 31, 32], but also by the semantics of Mares [69].

5.2 Relevant Indiscernibility

The interpretation of identity on which we will base the logic defined here is Kremer's relevant indiscernibility interpretation. This interpretation relies on Dunn's *relevant predication* [29, 31, 32]. However, I will argue for two ways to refine Kremer's project that are motivated by a closer look at the nature of relevant predications and their relations to relevant properties. As such, we will look at Dunn's project before explicating Kremer's proposal. However, we will first take a detour by examining why the usual approaches to identity do not work. Later we will also briefly look at Mares' approach to identity, which builds on Fine's semantics for quantified relevant logics.

5.2.1 The Problems with Traditional Approaches to Identity

As identified by both Kremer and Mares, traditional approaches to identity in relevant logic lead to the validity of some irrelevant implications. The *traditional interpretation* *of identity*, as labeled by Kremer, states that the identity relation holds between two terms if and only if they refer to the same individual. As made clear in Kremer [62], relevant logicians using this interpretation of identity would require that a semantics in which is it possible for a variable to refer to different individuals (1) at different worlds, and (2) at one and the same world. Without the former, the scheme $\mathcal{A} \to (x = y \to y = x)$ becomes valid. Without the latter, $\mathcal{A} \to x = x$ is valid.

Another classical approach is what Kremer labels as the *indiscernibility interpretation of identity*. On this view, the identity relation holds between two terms if and only if they are "interchangeable, *salva veritate*, in all contexts" [62, p. 202]. That is, $t_1 = t_2$ is true if and only if $\mathcal{A}[t_1/x] \leftrightarrow \mathcal{A}[t_2/x]$ is the case for *every* $\mathcal{A}$. This includes when $\mathcal{A}$ does not contain x, and thus we validate $x = y \to (\mathcal{A} \leftrightarrow \mathcal{A})$, for every $\mathcal{A}$.

Thus, these interpretations of identity are not appealing to the relevant logician.

5.2.2 Dunn's Relevant Predication

One of the most significant, unique applications of relevant logic with identity is Dunn's *relevant predication* [29, 31, 32]. The relevant predication project has been developed further by Kremer [60], wherein one of Dunn's conjectures is refined and proved/disproved in numerous forms. The relevant predication project should restrict our choice of indiscernibility axioms, as will be shown, lest the project become "irrelevant" by validating inferences such as $x = y \to (\mathcal{A} \to \mathcal{A})$ where $\mathcal{A}$ contains neither x nor y. That is, our first philosophical constraint is on the choice of if and how we add indiscernibility to the logic. Kremer's proposes a kind of indiscernibility axiom that is available to us, given the restrictions of relevant predication, which Kremer takes to add to the case for Dunn's project [62, p. 218]. We shall discuss this in a further section. Given the importance of Dunn's project, and its impact on the selection of indiscernibility axioms, it must be explicated here.

Dunn's relevant predication is proposed as a "way to sort out those properties that have an intimate life with an object from those that do not" [29, p. 347]. The thrust of Dunn's proposal to someone without a grasp of relevant logic is perhaps better acquired by examining Dunn's example about Socrates and Reagan.[1] Suppose that Socrates is

[1] Dunn also suggest we could call these properties "real" or "natural" for polemical reasons [29, p. 348].

wise. It follows, in natural language made to resemble logical syntax, that "if anyone is Socrates, then he is wise". Classically, irrelevantly, it also follows that "if anyone is Reagan, then Socrates is wise". Here, there are two properties or predications; being wise and being such that Socrates is wise. The first appears a relevant or real property of Socrates, while the second is not relevant to Reagan. In fact, Dunn suggests that the Reagan example relies on some form of thinning, while the Socrates example is some form of indiscernibility [29, p. 350]. Thus, the task of relevant predication is to separate the relevant from the irrelevant predications.

Now suppose that the predicate S was a relevant/real/natural/essential/etc. property of the object (named by constant symbol) r. The naming of the following equation, Shakespeare's Law, intended by Dunn to capture the Shakespearean quote that "A rose by any other name would smell as sweet," is used to give half of what is the sentence expressing our presumption. From Shakespeare's Law,

$$Sr \to \forall x(x = r \to Sx),$$

which is the rule RIC applied to an instance of indiscernibility, we may obtain the relevant predication of smelling sweet to a rose,

$$Sr \leftrightarrow \forall x(x = r \to Sx),$$

by fairly standard means. Given that Shakespeare's Law is expected to hold for relevant predications, and given that the other direction of the biconditional is always a theorem, we may obtain a definition of relevant predication. To avoid a lambda-like abstraction notation used by Dunn, I will use the formulation of the definition given by Kremer:

> "c relevantly has the property of being (an x) such that $\mathcal{A}x =_{df} \forall x(x = c \to \mathcal{A}x)$ [62, p. 203]

To bring back Dunn's earlier example, the Reagan case is thus translated as $\forall x(x =$ Reagan $\to$ Socrates is wise), which is an irrelevant implication. According to Dunn, there is thinning involved in the inference this Reagan formula: Socrates is wise $\to x =$ Reagan $\to$ Socrates is wise.

From here, Dunn gives a definition of when a formula $\mathcal{A}x$ "is a formula of a *kind that determines relevant properties (with respect to x)*" [29, p. 361]. This formula labeled as Uniform Indiscernibility, $\forall x(\mathcal{A}x \to \forall y(x = y \to \mathcal{A}y))$, says that $\mathcal{A}x$ determines a relevant property. For any predication $\mathcal{A}x$ which satisfies Uniform Indiscernibility, if $\mathcal{A}a$, then $\mathcal{A}$ is a relevant predication on a [29, Fact 1]. Now, full indiscernibility, $\mathcal{A}(a) \to (x = a \to \mathcal{A}(x))$, and its permuted alternative are equivalent in **R**. Unfortunately for full Indiscernibility, the inclusion of full indiscernibility would make it so that every predicate determines a relevant property. This likely lead to Kremer developing the relevant indiscernibility interpretation of identity.

Here I will note my first concern with the relevant predication project and my attempt to give an adequate semantics for quantified modal relevant logics with identity. Part of my aim is to produce a general semantic treatment capable of modeling a number of axiomatizations of identity. Mares in [69] did something similar for quantified relevant logics with identity based on Fine's semantics. However, my aim is to consider quantified modal relevant logics with identity that are based in **B** and its extensions. The problem here is that the relevant predication uses the base logic **R**, and thus restrictions given by the project will seemingly only apply to logics based on **R**, and not weaker logics. Now Dunn states that even the strong system **E** is too much for the project [29, p. 349]. It might be the case that other relevant logics need not be concerned with the restrictions given by relevant predication. That is, we may not want to add certain axioms because of their consequences in **R**; however, they may not have these consequences in weaker relevant logics. This, I leave as mere conjecture for the time being.

5.2.3 Kremer's Relevant Indiscernibility

As far as axiom choice goes for identity in quantified relevant logics, the most developed account I know of is Kremer's, which I take to be accumulated in [62] and [61]. Kremer argues for a relevant indiscernibility interpretation of identity, and considers numerous axioms for reflexivity, symmetry, and transitivity. These axioms, keeping his labels, are as follows:

Reflexivity	$x = x$
Relevant Symmetry	$x = y \to y = x$
Truth-functional Symmetry	$\neg x = y \vee y = x$
Nested Transitivity	$x = y \to (y = z \to x = z)$
Conjoined Transitivity	$(x = y \wedge y = z) \to x = z$
Truth-functional Transitivity	$\neg x = y \vee \neg y = z \vee x = z$

Kremer is seemingly the first to consider truth-functional versions of these axioms, which illuminated the possibilities of axiomatizing identity in relevant logics. However, he ultimately opts for the Relevant Symmetry and Nested Transitivity, with Reflexivity and a special indiscernibility axiom due to his interpretation of identity.

The indiscernibility Kremer calls "relevant indiscernibility" states that "$s = t$ can be interpreted, intuitively, as an infinite conjunction of biconditionals $(\mathcal{B}[s/x] \leftrightarrow \mathcal{B}[t/x]) \wedge (\mathcal{C}[s/x] \leftrightarrow \mathcal{C}[t/x] \wedge \ldots$, where $\mathcal{B}x, \mathcal{C}x$, etc., run through *only those formulas that express relevant properties*" [62, p. 204]. That is, two terms are equal when the same set of relevant predications are true of each of them. The axiom of indiscernibility chosen is "$x = y \to (Gx \to Gy)$, where G is a relevant predicate constant" [62, p. 218]. Here, Kremer uses a typed language where the predicate symbols are divided into relevant predicate constants $G_1, G_2, \ldots$, and non-relevant predicate constants $F_1, F_2, \ldots$.

This interpretation of identity thus presupposes that we have a list of relevant predications. Dunn's theory of relevant predications depends on the axiomatization of identity. Kremer points out this problem [62, p. 210], and proposes a way to test whether an axiomatization is *stable*, given a set of relevant properties. He then uses this test to show that certain axiomatizations should be rejected, but the circular connection to the problem of determining which properties are relevant seem to remain largely unsolved. We will discuss some suggestions about which predicates should be taken as relevant below. First, let us explicate the notion of stability.

The following definitions, taken from Kremer [62, Definitions 1, 2], are displayed here with irrelevant notational modifications.

Definition 5.2.1. Given a logic $\mathfrak{L}$ whose language contains $\forall$, a formula $\mathcal{A}$ is $\mathfrak{L}$-*relevant in the variable* x if and only if $\vdash_{\mathfrak{L}} \forall x(\mathcal{A} \to \forall y(x = y \to \mathcal{A}[y/x]))$, where y does not occur free in $\mathcal{A}$. Further, when $\mathfrak{L}$'s language does not contain $\forall$, a formula $\mathcal{A}$ is $\mathfrak{L}$-relevant in x iff $\vdash_{\mathfrak{L}} \mathcal{A}[u/x] \to (u = v \to \mathcal{A}[v/x])$, where u and v are not free in $\mathcal{A}$.

Definition 5.2.2. The logic $\mathfrak{L}$ is *stable* (with respect to the R.I. interpretation of identity) iff it can be reaxiomatized by $\mathfrak{L}$ + Reflexivity + Relevant Symmetry + Nested Transitivity together with the axiom scheme $x = y \to (\mathcal{A}[x/u] \to \mathcal{A}[y/u])$, where $\mathcal{A}u$ is $\mathfrak{L}$-relevant in u, together with the rules for $\mathfrak{L}$.

The first definition can be seen as a definition of relevant *properties*, which include the relevant *predications*. That is, from the G-atomic formulas we can derive (in some systems) that other properties are relevant (i.e. the relevant indiscernibility formula for the property is a theorem). This marks the difference between a relevant predication and a relevant property: a relevant predication is a predicate that is explicitly given as relevant by the axiomatization, while the relevant properties are all those properties whose relevance is derivable in the system. It might be expected that in some systems the set of relevant properties remains the same when the set of relevant predicates are changed. To repeat, the set of relevant predicates of an axiom system is given by the set of atomic predicates $G_1, G_2, \ldots$, while the set of relevant properties (in x) are all those formulas $\mathcal{A}$ for which we can derive $\vdash_{\mathfrak{L}} \forall x(\mathcal{A} \to \forall y(x = y \to \mathcal{A}[y/x]))$.

Briefly, a logic is stable iff it makes no difference whether you use the RI axiom for relevant predicates or an RI axiom scheme for all relevant properties of the logic. Stability, as defined, uses a logic's own relevant properties to judge itself. Using this notion, Kremer demonstrates that a full substitution axiom leads to instability. Thus, some logics with substitution are not internally coherent; they are unstable.

Because it leads to instability, Kremer does not include an axiom of substitution. It is unlikely that a quantified *modal* logic will have an axiom of substitution, as we shall see, but Kremer offers additional reasons to reject substitution axioms for the quantified relevant logic fragment. First, Kremer notes there is no consecution calculi solution, elegant or not, that corresponds to the substitution axiom. Further, semantics based on Fine's semantics would require additional semantic primitives to be added to an already complex system. However, the more philosophical reason Kremer offers that the logics with substitution are *philosophically unstable*. That is, he proves that the addition of the substitution axiom leads to logics which are not stable as in definition 5.2.2.

Kremer's axiomatization for the logic he labels $\mathbf{R}^{\forall\exists x=}$ begins with a language wherein we have separate n-place predicate constants for relevant and non-relevant predicates. Given how Kremer talks of an infinite conjunction of biconditionals, it seems that Kremer assumes the set of relevant predicate constants $G_1, \ldots, G_n, \ldots$ to be infinite. Further, we can suppose that the set of non-relevant predicate constants $F_1, \ldots, F_n, \ldots$ is also infinite. Given this, the axiomatization of Kremer is as follows **RQ** + Reflexivity (REF) + Relevant Symmetry (SYM) + Nested Transitivity (NT) + the axiom

$$RI: \quad x = y \to (Gx \to Gy), \text{ where } G \text{ is a relevant predicate constant}$$

N.B. the difference between a relevant predicate and a relevant property, which is important for later discussion.

> So far, we have postulated indiscernibility to atomic formulas. The r.i. interpretation for the non-atomic formula $\mathcal{A}x$ whenever $\mathcal{A}x$ expresses a relevant property with respect to x. Unfortunately, we do not have a clear idea which non-atomic formulas express relevant properties with respect to x. At this point we hold off on extending indiscernibility. Depending on other axioms, the postulation of indiscernibility for atomic formulas produces indiscernibility for non-atomic formulas. We can think of ourselves as taking up a suggestion made by Urquhart and presented in Dunn (1987): postulate indiscernibility for atomic formulas, 'letting induction on formulas take us where it may with respect to Indiscernibility for compound formulas' (p. 452). [62, p. 206]

In effect, the suggestion in the latter part of the quotation is exactly the method we will take here, building upon Kremer. The relation between relevant predicates and relevant properties will be explored later, both for finite and infinite sets of relevant predicates.

5.2.4 A Closer Look at Relevant Predications

For a relevant indiscernibility interpretation to get off the ground, we must be able to say something about the set of relevant predicates. Here, I will argue for two main points. The first, is that there are arguments for both finite and infinite versions of the R.I. interpretation. That is, it is plausible to assume either that the set is finite or that it is infinite. We will explore both options in this chapter. The second point is that we ought to (or, rather, we are able to) take the set of relevant predications to be a (proper) subset of the atomic formulas.

Finite

To argue for the plausibility of a finite set of relevant predicates, we must at some point address a potential objection. It might be objected that, in certain domains, a finite set of relevant predicates is insufficient, and that there are domains wherein we must, if guided by intuitions, have an infinite set of relevant predicates. For example, if our domain is the set of natural numbers, the set of infinite predicates "is the number n" for all n might be seen as relevant. It is my hope to address this potential objection by showing that what is desired by our intuitions about these cases is merely that the set of relevant *properties* be infinite. It will be shown that an infinite set of relevant properties can be obtained from a finite set of relevant predicates. Given this fact, I hope to dissolve the potential objection. If the reader is able to forgive the suspense, we will first show the plausibility of the finite interpretation before returning to this potential objection in more detail.

There are of course formal niceties that motivate the finite interpretation. As will be shown below, given only finitely many relevant predicates, the identity of two terms is relevantly equivalent to a formula expressible in the language — and without the use of second order quantification. A finite conjunction of biconditionals is a formula one could write down or use in a proof (given enough time, ink, paper, etc.) Moreover, only a single axiom for identity is required to axiomatize the logic. However, it is the semantic simplicity of the models that really demonstrates the benefits of this interpretation. As the reader can verify below, allowing only a finite number of relevant predicates allows us to essentially re-use the models that lacked identity, without requiring that the sets of admissible proportions and admissible propositional function be closed under infinite intersection or some new operation. This gives the advantage of parsimony to the finite interpretation. In addition, the finite approach given here has been constructed in the spirit of Kremer's semantic idea for identity in relevant logic.

A finite set of relevant predicates can give rise to an infinite set of relevant properties, as will be shown. The main thrust of plausibility for the finite approach is that it can do everything we want the infinite approach to do — albeit sometimes in a roundabout way — with simpler semantics. The next section will demonstrate that we can build up infinite number of relevant properties from finite number of relevant predicates.

Infinite from Finite

The objection hinted at in the last section, when explicated in full, relies on equivocating relevant predicates with relevant properties. In some domains of reasoning, an infinite number of relevant properties is required. However, this does not imply that an infinite number of relevant predicates is also required.

Let us first add to our notation for relevant predicates, so that we can facilitate talking of n-ary predicates relevant in one or several of their argument places. For convenience and readability, I will often employ a special variable symbol z which is restricted to only appear in relevant predicates when collected as a set external to the axiom system (e.g., $RPred$ and RP below). On this convenient "z-notation", we must replace z to obtain a wff in the language we are using, so $\mathcal{A}\tau_1, \ldots, z, \ldots, \tau_n$ is not in the language, but $(\mathcal{A}\tau_1, \ldots, z, \ldots, \tau_n)[x/z]$ is a wff. Further, we interpret $\mathcal{A}\tau_1, \ldots, z, \ldots, \tau_n$ as saying that the predicate symbol with τ_1 through τ_n in their respective place is relevant in the place in which z occurs. If multiple occurrences of z occur, then the predicate will be said to be relevant in all places z occurs simultaneously.[2] However, we will often make the simplifying assumption of only a single occurrence of z and monadic predicates.

Let us now show how to construct an infinite set of relevant properties from a finite set of relevant predicates. Let us for the moment consider only monadic predicates for simplicity. Let $RPred$ be a finite set of relevant monadic predicates (relevant in z). That is, on our axiomatization, let the RI axiom-scheme hold for these predicates. We

[2]To illustrate this, consider the following.

$$(1)\mathcal{A}(z,z) \in RPred \qquad\qquad (2)\mathcal{A}(z,\tau), \mathcal{A}(\tau,z) \in RPred$$

In (1), the same variable must be substituted in for both occurrence of z to produce a relevant predication. On the other hand, according to (2), $\mathcal{A}(c,\tau)$ is a relevant predication on the constant c.

show that there is an infinite set of properties $\mathcal{A}_i$ (with $i \in I$ for an infinite index I) such that

$$x = y \to (\mathcal{A}_i x \leftrightarrow \mathcal{A}_i y)$$

To see how to construct such a set, let us consider the sublanguage of the logic consisting of the set $RPred$ and the implicational connective '$\to$'. It can be shown from a single relevant predicate G that from the RI axiom

$$x = y \to (G[x/z] \leftrightarrow G[y/z])$$

that we can find infinitely many formulas $\mathcal{A}$ for which the RI scheme holds. For example, let $\mathcal{B}$ be any of infinitely many well formed formulas. Using our one instance of the RI axiom for our single relevant predicate we can derive

$$x = y \to ((\mathcal{B} \to G[x/z]) \leftrightarrow (\mathcal{B} \to G[y/z])).$$

This shows that there are an infinite number of relevant properties derivable in this system.

The proof is as follows.

1.	$x = y \to (G[x/z] \leftrightarrow G[y/z])$	Axiom
2.	$(G[x/z] \leftrightarrow G[y/z]) \to (G[x/z] \to G[y/z])$	Theorem
3.	$x = y \to (G[x/z] \to G[y/z])$	1, 2, MP
4.	$(G[x/z] \to G[y/z]) \to ((\mathcal{B} \to G[x/z]) \to (\mathcal{B} \to G[y/z]))$	Theorem
5.	$x = y \to ((\mathcal{B} \to G[x/z]) \to (\mathcal{B} \to G[y/z]))$	3, 4
6.	$x = y \to ((\mathcal{B} \to G[x/z]) \leftarrow (\mathcal{B} \to G[y/z]))$	Similar
7.	$x = y \to ((\mathcal{B} \to G[x/z]) \leftrightarrow (\mathcal{B} \to G[y/z]))$	5, 6

This may seem odd, in that from the predicate x *smells sweet* being relevant to a rose we may derive that *being such that the moon is made of cheese implies that* x *smells sweet* is relevant to a rose. That is, $\mathcal{B} \to Gz$ does not have the intuitive appearance of a relevant property. In other words, it might not seem as if it bears an intimate relation with whatever it holds of. We thus have the sentence

$$(\mathcal{B} \to G[rose/z]) \to (x = rose \to ((\mathcal{B} \to G[rose/z])))$$

which is not as harmful as it seems. This states that *if* a rose has the property that if $\mathcal{B}$ then it smells sweet — and that's a big if! — then it has this property relevantly.

This example is illustrative of the kinds of relevant properties there can be, as well as the infinitude of relevant properties from finitely many relevant predicates. To this first point we note the difference between a relevant predicate and a relevant property can be significant. We will contrast the relevant properties/predicates to sparse properties later in this chapter. To the second point, we have shown how to construct some fairly complicated relevant properties that might never be particularly useful. It is left as an open question whether or not the properties constructed like those the example are useful in any particular application. However, there is another way to construct an infinite number of relevant properties from a finite base of predications, which does not require increasingly complex formulas. Instead, the use of quantification over an infinite domain is relied upon.

Let us construct an example in which we will provide a useful, infinite set of relevant properties. One possible application of relevant logics with identity is the construction of relevant mathematics. In fact, relevant mathematics is a common theme in the history of relevant logic.[3] In arithmetic, limited to the set of the natural numbers, one may have the intuition that there are an infinite number of relevant properties of the form "is the number n". Given an infinite abundance of relevant predicates, we would construct these properties as predicates of the form $x =$ 'n', where 'n' is a constant (name) for the number n. However, given only a finite set of relevant predicates, we have to take another route.

We will show that an infinite number of properties of the form "is the successor of 'n' "are relevant, *given only a single relevant predicate*. To do so, for the remainder of this section let the predicate Sxy mean that x is the successor of y. Then, let our set of relevant predicates include the predicate (Szw) (relevant in z). Thus, we have as an axiom the universal closure of

$$\tau_1 = \tau_2 \to ((Szw)[\tau_1/z] \leftrightarrow (Szw)[\tau_2/z]).$$

Now, let our language contain a constant symbol that names every natural number. From the axiom resulting from the universal closure of the above formula, we get the following proof. For every constant c_n naming a natural number, we have the following:

1.	$\forall w[\tau_1 = \tau_2 \to ((Szw)[\tau_1/z] \leftrightarrow (Szw)[\tau_2/z])]$	Axiom
2.	$\tau_1 = \tau_2 \to ((Szc_n)[\tau_1/z] \leftrightarrow (Szc_n)[\tau_2/z])$	1

[3]For example, see Meyer [81], Meyer and Mortensen [83] or §73 in [6].

The second line of this proof is derivable for every constant naming a natural number, but is also a formula which gives a relevant property. That property is *being the successor of* c_n, and is derivable for infinite c_n where $n \in \mathbb{N}$.

Infinite

It may appear to some, namely those with more finely tuned intuitions about metaphysics and the nature of things, that there is an important distinction between relevant predicates and relevant properties (which is not merely the formal distinction in the logic given here). To someone with these intuitions, it might appear as though there are in fact infinitely many relevant predicates. To satisfy the intuitions of this kind of metaphysician, we will consider what happens when we assume an infinite number of relevant predicates.

One particular metaphysical consideration that could motivate infinite relevant predicates is the difference between sparse and abundant properties. This distinction, found for example in Lewis in [64], is an attempt to differentiate properties in general (i.e. collection of objects) from the special properties which have some sort of metaphysical prestige for their supposed "carving of nature at the joints". While there are some accounts of what these properties are, such as Schaffer [101], it is more important to our current discussion to focus on what the distinction does. That is, we want to analyze what a sparse property is supposed to do. Lewis lists the features of sparse properties as follows:

> The Sparse properties are another story. Sharing of them makes for qualitative similarity, they carve at the joints, they are intrinsic, they are highly specific, the set of their instances are *ipso facto* not entirely miscellaneous, there are only just enough of them to characterize things completely and without redundancy. [64, p. 60][4]

It is clear that relevant properties are not always sparse properties. The first example of infinitely many properties derived from finitely many predicates in the previous

[4]Note, however, that Lewis' Laundry list of the properties of sparse properties does not guarantee existence. Schaffer [101] gives an account, or rather two competing accounts, of existing properties that might satisfy Lewis' Laundry list.

section *appears* to have demonstrated this; there are relevant properties whose formal structure appear to preclude their membership into the set of sparse properties.

A more likely candidate for being a sparse property is the relevant predications. However, given the arbitrary selection of relevant predications, it appears that not every selection will align with some metaphysically motivated distinction between sparse and abundant properties. But, beginning with an account of sparse/abundant properties, the metaphysician may make a selection of relevant predicates such that the relevant predicates are the sparse properties. It is unclear what role the abundant, but relevant, properties would play in such an account. Nevertheless, if the metaphysician requires an infinite number of sparse properties, the corresponding set of relevant predicates must also be infinite. While this may not be the most convincing argument for an infinite number of relevant predicates, it need not offer certainty, but merely a possible motivation. However, the next argument may be more convincing.

Of course, there is an additional concern that can be raised by even one who is not a practiced metaphysician. *Logic is not in the business of telling us how many relevant predicates there are.* That job is left, presumably, to the empirical sciences, the metaphysicians, and others. Logic should not limit the number of *predicates* we take (or discover) to be intimately connected to an object, especially if one is to make a distinction between relevant predicates and relevant properties.

In conclusion, I hope to have shown that there are at least plausible arguments for both the finite and infinite R.I. interpretation of identity. Each of these interpretations gives rise to its own proof system and semantics. Further, as was hinted at, we will show that one preserves the semantic idea of Kremer, the other preserves the axiom choice of Kremer.

5.3 The *Finite* R.I. Interpretation: $\mathbf{QR}^{\circ t}_{=}$ and $\mathbf{RQ}^{\circ t}_{=}$

The finite R.I. interpretation of identity gives rise to a simple semantics, requiring no more than was required for $\mathbf{QR}^{\circ t}$ or $\mathbf{RQ}^{\circ t}$. Further, the axiom system of Kremer can be refined to better express the informal interpretation of identity.

5.3.1 Axiom systems

Given the reasoning above, we now wish to axiomatize the finite R.I. interpretation. Given that the R.I. axiom itself depends on our choice of relevant predications, there are (as the title of this section suggests) many ways to extend $\mathbf{QR}^{\circ \boldsymbol{t}}$ with identity using the R.I axiom. Here we will concern ourselves with a subset of these possibilities.

The language of the logic we will explore first will contain the full language of $\mathbf{QR}^{\circ \boldsymbol{t}}$, and its signature will include a special 2-place relation for identity, for which will we use the familiar '=' symbol.

Here, the logics we are interested in will be determined by a set of relevant predications, which are a *finite* subset of the (non-identity) atomic formulas. The set of relevant predications will be denoted by RP, and will consist of formulas $G_i(\dots z \dots)$, where the formula is taken to be a relevant predication in z. For $G_i(\dots z \dots)$, we will often use the shorthand Gz. Further, when needed we will take the set RP to be ordered.

Fixing a finite set RP constant, the logic $\mathbf{QR}^{\circ \boldsymbol{t}}_{=}$ determined by this set is given by the following axiomatization (via the universal closure of the following formula schemes):

$\mathbf{QR}^{\circ \boldsymbol{t}}$ The axioms and rules of $\mathbf{QR}^{\circ \boldsymbol{t}}$, in the language of $\mathbf{QR}^{\circ \boldsymbol{t}}_{=}$.

Ref $\tau_1 = \tau_1$

Sym $\tau_1 = \tau_2 \to \tau_2 = \tau_1$

Trans $\tau_1 = \tau_2 \to (\tau_2 = \tau_3 \to \tau_1 = \tau_3)$

RI $\tau_1 = \tau_2 \to (G[\tau_1/z] \leftrightarrow G[\tau_2/z])$, where $Gz \in RP$

=-I $((G_1[\tau_1/z] \leftrightarrow G_1[\tau_2/z]) \wedge, \dots, \wedge (G_n[\tau_1/z] \leftrightarrow G_n[\tau_2/z])) \to \tau_1 = \tau_2$, where the antecedent is the finite conjunction of $(G[\tau_1/z] \leftrightarrow G[\tau_2/z])$, for every $Gz \in RP$.

This axiomatization extends Kremer's by the addition of the =-I (and the inclusion of $\boldsymbol{t}$ and $\circ$). This last axiom, however, is motivated by the relevant indiscernibility interpretation of identity, where the set of relevant predications are taken to be finite.

Indeed, given the assumption of the finiteness of RP, the additional acts as a first order *identity of indiscernibles* axiom.

Moreover, we can do some simplification to this axiom systems. Indeed, motivated by the interpretation of identity we may introduce identity with the following biconditional:

$$=_{df} \quad ((G_1[\tau_1/z] \leftrightarrow G_1[\tau_2/z]) \wedge, \ldots, \wedge (G_n[\tau_1/z] \leftrightarrow G_n[\tau_2/z])) \leftrightarrow \tau_1 = \tau_2$$

where the finite conjunction on the left is the finite conjunction of $(G[\tau_1/z] \leftrightarrow G[\tau_2/z])$, for every $Gz \in RP$. It is trivial to show that, with this new axiom, we can derive =-I and RI. Indeed, with little thought we can also show that Ref, Sym, and Trans are derivable in $\mathbf{QR}^{\circ \boldsymbol{t}}$ using the axiom $= df$. Thus, in this finite R.I. interpretation, identity can be taken as defined, in much the same way as we can take $\circ$ to be a defined connective.

It is worth noting that, given a finite but incredibly large set of relevant predicate symbols, some proofs in the system are so large that they require to write down more symbols than the number of atoms in the universe. Using a defined rule from a biconditional to one of its conjuncts, deriving =-I from $=_{df}$ requires over 4 times as many symbols as there are relevant predicates. If there are 2^{500} relevant predicates, then this two-line proof could never be written down. Nevertheless, the impossibly large but finite proofs of other systems are not generally taken to be problematic; and furthermore we can prove things about what can be proven in this system.

Lemma 5.3.1. *Relevant/restricted SUB is a theorem. That is, the formula*

$$(G[x/z] \wedge x = y) \to G[y/z]$$

is for every relevant predicate G in z.

Proof. The transitivity of the arrow is suppressed in the justification column, as it is derivable.

1	$x = y \to (G[x/z] \to G[y/z])$	RI, $\wedge$-elim
2	$G[x/z] \to (x = y \to G[y/z])$	Permutation
3	$(G[x/z] \circ x = y) \to G[y/z]$	$\circ$-rule
4	$(G[x/z] \wedge x = y) \to (G[x/z] \circ x = y)$	Theorem
5	$(G[x/z] \wedge x = y) \to G[y/z]$	3, 4

$\square$

The logic given here is not *stable* in the sense that Kremer's preferred axiomatization is stable. However, this is not to detract from the support of the logic above. Stability was defined by Kremer as stable with respect to the R.I. interpretation, which means roughly that it is reaxiomatizable using Kremer's axiomatization. As our added axiom is not a theorem of Kremer's logic, the defined logics are obviously not stable under Kremer's definition of stability. But if we define a new type of stability with respect to the finite R.I. interpretation of identity, then this new version of stability can be used as a test of internal coherence in much the same way. (And it is easy to see that a full substitution axiom is still unstable on this new definition.)

5.3.2 Models

Kremer's relevant indiscernibility takes the meaning of $s = t$ to be an infinite conjunction of biconditionals, $G_i[s/x] \leftrightarrow G_i[t/x]$, for each relevant predication G_ix. The finite R.I. interpretation only requires a finite conjunction of biconditionals. We will construct a semantics for quantified relevant logics with identity, where identity is modeled as in the finite R.I. interpretation. Given that we will use $\cap$ to help model identity, we have to determine the exact type of propositional function onto which identity statements will be mapped. In the case of the universal quantifier, which takes a propositional function as an argument, our propositional function was $\forall_n \phi$, a function that was built using the propositional function ϕ. Switching to identity, the propositional function that models $x = y$ will not be built using other propositional function — it is an atomic formula. However, it will *depend* on other propositional functions; i.e. the propositional functions to which non-identity atomic formulas in RP are mapped.

Thus, let us introduce a function to model identity. Let $\approx^x_y$: $U^\omega \longrightarrow \wp(K)$ be a function defined by

$$(\approx^x_y)f = \cap_{Gz \in RP}(|G[x/z] \leftrightarrow G[y/z]|f)$$

This propositional function depends the propositional functions assigned to the members of the set RP.

Definition 5.3.2. A $\mathbf{QR}^{\circ\boldsymbol{t}}_{=}$-*frame* is a tuple

$$\mathfrak{F} = \langle K, 0, R, *, U, Prop, PropFun\rangle,$$

where $\langle K, 0, R, *, U, Prop, PropFun\rangle$ is a $\mathbf{QR}^{\circ\boldsymbol{t}}$-frame.

These model structures are exactly the model structures of Mares and Goldblatt [74]. If the set RP were infinite, we would require $Prop$ to be closed under an infinite intersection or a $\sqcap$ of the correct type. The condition

$$\approx\text{-Closure:} \qquad \text{If } \tau_1, \tau_2 \text{ are terms, then} (\approx^{\tau_1}_{\tau_2}) \in PropFun.$$

would be required if RP were infinite, as there is no guarantee that $(\approx^{\tau_1}_{\tau_2})$ — infinite conjunction of biconditionals — is in $PropFun$. Given the finiteness of RP, this condition is fairly trivially satisfied.

Definition 5.3.3. A *pre-model for* $\mathbf{QR}^{\circ\boldsymbol{t}}_{=}$ is a tuple

$$\mathfrak{M} = \langle K, 0, R, *, U, Prop, PropFun, |{-}|^{\mathfrak{M}}\rangle,$$

such that $\langle K, 0, R, *, U, Prop, PropFun\rangle$ is a $\mathbf{QR}^{\circ\boldsymbol{t}}_{=}$-frame and $|{-}|^{\mathfrak{M}}$ is a value assignment that assigns,

1. an element $|c|^{\mathfrak{M}} \in U$ to each constant symbol c;
2. a function $|P|^{\mathfrak{M}} : U^n \longrightarrow \wp(K)$ to each n-ary predicate symbol P;
3. a propositional function $|\mathcal{A}|^{\mathfrak{M}} : U^\omega \longrightarrow \wp(K)$ to each formula $\mathcal{A}$ such that, when $\mathcal{A}$ is the atomic $P\tau_1, \ldots, \tau_n$, the propositional function assigned to it is given by, for each $f \in U^\omega$,

$$|P\tau_1, \ldots, \tau_n|^{\mathfrak{M}} f = |P|^{\mathfrak{M}}(|\tau_1|^{\mathfrak{M}} f, \ldots, |\tau_n|^{\mathfrak{M}} f).$$

Further, when $\mathcal{A}$ is not atomic or not an identity, the function assigned is given by the following:

$$|\boldsymbol{t}|^{\mathfrak{M}} = \phi_0$$
$$|\mathcal{A} \wedge \mathcal{B}|^{\mathcal{M}} = |\mathcal{A}|^{\mathcal{M}} \cap |\mathcal{B}|^{\mathcal{M}}$$
$$|\mathcal{A} \vee \mathcal{B}|^{\mathcal{M}} = |\mathcal{A}|^{\mathcal{M}} \cup |\mathcal{B}|^{\mathcal{M}}$$
$$|\neg\mathcal{A}|^{\mathcal{M}} = (|\mathcal{A}|^{\mathcal{M}})^*$$
$$|\mathcal{A} \to \mathcal{B}|^{\mathcal{M}} = |\mathcal{A}|^{\mathcal{M}} \Rightarrow |\mathcal{B}|^{\mathcal{M}}$$
$$|\forall x\mathcal{A}|^{\mathcal{M}} = \forall_x |\mathcal{A}|^{\mathcal{M}}$$

Finally, the pre-models must satisfy the following equation:

$$|\tau_1 = \tau_2|^{\mathfrak{M}} = \ (\approx^{\tau_1}_{\tau_2})$$

This last equation lets us focus on the role of identity statements. We could have treated them as atomic formulas, but as we will see this is not required when we make the move from pre-models to models. Identity is treated as a shorthand for a finite intersection of biconditionals, and is therefore not treated as atomic here. This is partially due to the following lemma.

Lemma 5.3.4. *If every $G \in RP$ is mapped onto a member of $PropFun$, then every identity statement is mapped onto a member of $PropFun$.*

Proof. Here I provide a sketch of the proof. From the assumption, we can by the usual means derive that the biconditionals and conjunctions of biconditionals of atomic formulas with predicate symbols in RP are all mapped onto members of $PropFun$. Identity, by the definition of $\approx^{\tau_1}_{\tau_2}$, is mapped onto one of these conjunctions of biconditionals. □

Given this lemma, our models require that non-identity atomic formulas are assigned members of $PropFun$.

Definition 5.3.5. Finally, for pre-model $\mathfrak{M}$ to be a *model,* the value assignment must assign a propositional function in $PropFun$ to each non-identity atomic formula. The identity atomic formulas are then assigned a member of $PropFun$ by means of the definition of a pre-model.

From this, a relation $\models$ can be defined as usual by the following:

(ia) $a, f \models P\tau_1, \ldots, \tau_n$ iff $a \in |P\tau_1, \ldots, \tau_n|^{\mathfrak{M}} f$

(ib) $a, f \models t_n = t_m$ iff $a \in \cap_{G \in RP}(|G[t_n/z] \leftrightarrow G[t_m/z]| f)$

(ii) $a, f \models \boldsymbol{t}$ iff $a \in 0$

(iii) $a, f \models \mathcal{A} \wedge \mathcal{B}$ iff $a, f \models \mathcal{A}$ and $a, f \models \mathcal{B}$

(iii) $a, f \models \mathcal{A} \vee \mathcal{B}$ iff $a, f \models \mathcal{A}$ or $a, f \models \mathcal{B}$

(iv) $a, f \models \neg\mathcal{A}$ iff $a^*, f \not\models \mathcal{A}$

(v) $a, f \models \mathcal{A} \to \mathcal{B}$ iff $\forall b, c((Rabc$ and $b, f \models \mathcal{A}) \Rightarrow c, f \models \mathcal{B})$

(vii) $a, f \models \forall x\mathcal{A}$ iff there is an $X \in Prop$ such that $X \subseteq \bigcap_{g \in x_n f} |\mathcal{A}|^{\mathfrak{M}} g$ and $a \in X$

Definition 5.3.6. A formula $\mathcal{A}$ is *satisfied* by assignment f in model $\mathfrak{M}$ if $a, f \models \mathcal{A}$, for every $a \in 0$. A formula $\mathcal{A}$ is *valid in a model* if it is satisfied by every $f \in U_\omega$ in that model. A formula is *valid in a frame*, if it is valid in every model based on that frame. A formula is *valid in a class of frames*, if it is valid in every frame in the class.

5.3.3 Metatheorems

Soundness and completeness for this logic are fairly trivial. None of the new cases for the defined identity symbol are given. However, we will quickly note why Kremer's semantic idea cannot produce a semantics (for a first order logic) with respect to which his axiom system is complete.

Remark 5.3.7. There are prime theories constructed using Kremer's axiom system (not the one introduced for the semantics here) whose presence in a canonical model invalidate theorems of the logic defined in this section.

Proof. Consider a prime theory generated by the principal filter $[\mathbb{G})$, where $\mathbb{G}$ is the conjunction of biconditionals on the right-hand side of the definition $=_{df}$. On Kremer's axiomatization, there is no proof from $\mathbb{G}$ to the corresponding left-hand side identity statement. Let the identity statement be $x = y$. Then $(\{\mathbb{G}\},\{x=y\})$ is $\mathbf{QR}^{\circ t}_{=}$-independent. $\square$

This is why Kremer's semantic idea is at odds with his axiom choice in first order logic. The axioms are not enough to interpret identity *as* the conjunction of biconditionals.

5.3.4 Extending to the Infinitely Many Relevant Predicates

Without a second order axiom of the identity of relevant indiscernibles, in the infinite case the semantics cannot be adequately constructed using the method employed in the finite case. That is, the pair $(\Gamma, x = y)$, where $\Gamma = \{G[x/z] \leftrightarrow G[y/z] : G(z) \in RP\}$, is an independent pair. There is no proof from Γ to $x = y$. Thus, there must be situations in the model at which Γ is satisfied, yet $x = y$ is not. It follows, therefore, that we cannot model identity statements by the infinite intersection of biconditionals as we did before. A quick review of remark 5.3.7 should also convince the reader that using the $\sqcap$ operation as defined before will also lead to problems.

The reader is advised to note the notational difference between $\mathbf{QR}^{\overset{\circ \boldsymbol{t}}{=}}$ (of the next section) and $\mathbf{QR}^{\circ \boldsymbol{t}}_{=}$, and to take this difference as indication that a new logic is being defined. The later notation was used for the finite interpretation of Kremer's axiom system for relevant indiscernibility. The former is introduced below, in §5.4.6, for a different axiomatization. We begin by discussing the role of the RI axiom scheme.

The RI axiom scheme depends on our choice of relevant predications, and should be treated as a set of proper axioms. This will be defended here, justifying the new notation and choice of axioms. There are two reasons given for treating instances of RI as proper axioms. The first is that logic should not determine which predicates are relevant.

The logics based on Kremer's axiomatization can be criticized on the grounds that the set of relevant properties was determined by the axiom system. In $\mathbf{QR}^{\circ \boldsymbol{t}}_{=}$, this criticism was addressed by choosing the set RP *prior to* constructing the logic. That is, the relevant predicates were determined before constructing the logic. However, each axiom system therefore contains extra-logical content. This extra-logic content is to be avoided, because out aim is a pure logic in which we can construct and reason about theories with non-logical content. Having extra-logical content built in would be an unwanted limitation. A logic without any instances of RI takes no stand on the (presumably) metaphysical question of which properties are relevant. However, taking instances of RI as proper axioms allows us to consider *theories* of relevant predication. Canonical[5] models based on any theory given by RI proper axioms may be constructed. Thus, we separate the purely logical content of identity (reflexivity, symmetry, and transitivity) from extra-logical, essentially metaphysical claims (instances of RI).

5.4 Relevant Logics with Identity

This section will provide a semantics for relevant logics with identity. The main technique is similar to that used by Mares in [69], in which Mares builds upon Fine's semantics for quantified relevant logics. Here 5.4.1, we first explicate Mares' semantics and the logics they interpret, ending with a guide to the rest of the section.

5.4.1 Mares and Identity

The axiom system Mares begins with is the logic **BQ** with reflexivity, symmetry, and

(wSub) $u = v \wedge \mathcal{A} \to \mathcal{A}[u/v]$, where $\mathcal{A}$ does not contain $\to$.

to axiomatize identity. Using this weak substitutivity axiom, weak transitivity (i.e. $x = y \wedge y = z \to x = z$) is provable as a theorem (-scheme) [69, p. 12]. Mares considers adding the axiom of nested transitivity and a full substitution axiom to this logic, but I will instead use a restricted substitution axiom.

Mares [69] defined semantics for quantified relevant logics with identity using Fine's semantics as a starting point. In these semantics, identity is not treated as a predicate. A primitive relation is added to the models, and a list of conditions is given to constrain its behavior to model an identity predicate. Thus, Mares introduced a relation of *semantic equivalence* at a world, which is a subset of the pairs of elements in the domain of a world. This semantics equivalence relation is denoted by $\approx_a$, and is made to pick out an equivalence class of objects at each world by means of a set of conditions.

The conditions placed on the semantic equivalence relation can be divided into a couple of classes. The first is the set of conditions that give us reflexivity, symmetry; (i), (vii).[6] We will treat transitivity as a separate issue, as the weak variety is derivable using the axiom of substitution. There are conditions that are used to ensure the axiom

[5] That is, in the sense of Chellas [22].

[6] The Roman numeral labels are Mares', taken from [69].

of weak substitutivity is valid (here we can exclude the condition used explicitly for weak transitivity); (viii), (ix), (x). There are also conditions ensuring up, down, and across lemmas are provable;[7] (iv), (v), (vi). Finally, there is a condition given to aid the heredity proof; (iii). This last condition which appears similar to the assumption for atomic cases — here, it is needed because the semantic equivalence relation is not given as an atomic predicate. Finally, so that the semantic equivalence relation is an equivalence at each world, a condition for weak transitivity is given; (ii).

For nested transitivity, the additional condition (xi) is given. This condition relies on notation for worlds sharing the same domains, as well as the fusion operation on worlds. As the models we will construct here use a single domain, we will utilize a simpler condition.

The semantics given by Mares are for a system in which (1) an axiom of substitution is assumed, and (2) weak transitivity is derivable using the aforementioned axiom. The logic defined below does not have an axiom of substitution (of even the relevant variety), but takes nested transitivity as an axiom — though a recipe will be given for systems with weaker forms of transitivity. In the next few sections, we will begin by defining a logic with identity and its extensions (and some weakenings). Then, we will define models for these logics that something similar to Mares' *semantic equivalence* relation, that for other reasons we will call a *left-to-right* identity relation. We will then show that these models can be simplified by replacing the left-to-right identity relation and its constraints with constraints on a function assigned to the identity symbol. We will then show soundness for the simplified models (and thus for the non-simplified models), and completeness for the non-simplified models. Finally, Kremer's relevant indiscernibility axioms will be added as proper axioms, and the conditions required for these axioms will be given. In this chapter, model conditions are given for the RI axiom. In the next chapter, frame conditions are given instead, with slight modifications to the models.

[7]The up, down, and across lemmas are needed in Fine's semantics, but will be neither reproduced nor explained here, because they are not required.

Standefer

I have discovered that I am not the first to extend the Mares and Goldblatt semantics with identity building on the semantics of Mares [69]. Shawn Standefer [111] has independently constructed such a semantics, which I came to discover after completing this chapter.[8] Despite our similar starting points, my work on relevant logics with identity is considerably different from Standefer's. In particular, the only substitutions and indiscernibility axioms I consider are dependent on a set of relevant predications, and highly influenced by Kremer's axiomatizations. Standefer does not consider relevant indiscernibility or relevant substitution axioms, as will be shown. Here I briefly explicate the key points of Standefer's work, noting the similarities and key differences between his work and the material of this chapter.

The first key difference is the base relevant fragments. In an earlier chapter, we generalized the Mares and Goldblatt style semantics to **B** and its extensions, and so this chapter adds identity to logics based on **B** and its typical relevant extensions. Standefer adds identity only to the relevant logics $\mathbf{QR}^{\circ \boldsymbol{t}}$ and $\mathbf{RQ}^{\circ \boldsymbol{t}}$.

Standefer's axiom choice includes reflexivity, symmetry, and the substitution axiom

$$\mathcal{A} \wedge x = y \to \mathcal{A}',$$

where $\mathcal{A}$ does not contain $\to$ and $\mathcal{A}'$ is the result of substituting at least one instance of x with y in $\mathcal{A}$. Weak transitivity is a consequence of this axiom choice. He also constructs semantics for this axiom without the $\to$-free restriction. The substitution axiom assumed by Standefer guarantees the validity of formulas such as $x = y \wedge x \neq z \to y \neq z$. The semantics given in this chapter do not validate this axiom, except when the relevant substitution axiom is given and $x \neq z$ is a relevant predicate in the position of x. In this dissertation, frame and model conditions for relevant indiscernibility and relevant substitution are given, and differ from Standefer's conditions required for the full substitution axiom. The example of substitution in the negation of identity statements can be given a corresponding frame condition — $a \in |\Rrightarrow|(x, y) \& a^* \in |\Rrightarrow|(yz) \Rightarrow a^* \in |\Rrightarrow|(xz)$ — which is Standefer's ID5. For full substitution within the scope of negation, Standefer also requires another condition. With another condition my models lack, the full substitution axiom is made valid.

[8] I would like to thank Shawn for his sending me a copy of his paper.

On the other hand, the semantics given here does not have a full substitution axiom. The conditions required for it in Standefer are not needed. This makes the logics here noticeably weaker, even when relevant indiscernibility/substitution axioms are added. Further, when the relevant indiscernibility/substitution are added, the frame and model conditions given in this document differ from ID5–7 of Standefer. That being said, we will also note that the models given here ought to be easily modified using Standefer's conditions to model logics with the substitution axioms he considers.

I have chosen not to adopt stronger substitution and indiscernibility axioms than those dependent on a set of relevant predicates. Of course, when the set of relevant predicates is large enough, the axioms are equivalent to those which Standefer models. In such situations, I direct the reader to Standefer [111] for frame/model conditions. However, my choice to exclude such stronger axioms was made before I became aware of Standefer's results. I am sympathetic to the relevant predication project and Kremer's arguments for relevant indiscernibility (and thus, at least in $\mathbf{R}^{\circ \boldsymbol{t}}$, relevant substitution).

5.4.2 The Logics $\mathbf{QB}^{\circ \boldsymbol{t}}_{=}$ and $\mathbf{BQ}^{\circ \boldsymbol{t}}_{=}$

The logics defined here have no axioms of substitution or indiscernibility. Thus, to retain as much of the identity-ness as possible, a full axiom a transitivity is given. However, we will describe how to weaken the system in various ways including weaker/no axiom of transitivity. The axiom systems are as follows.

Definition 5.4.1. The logics $\mathbf{QB}^{\circ \boldsymbol{t}}_{=}$ and $\mathbf{BQ}^{\circ \boldsymbol{t}}_{=}$ as defined by adding the following axiom schemes to the logics $\mathbf{QB}^{\circ \boldsymbol{t}}$ and $\mathbf{BQ}^{\circ \boldsymbol{t}}$, respectively.

REF $\tau_1 = \tau_1$

SYM $\tau_1 = \tau_2 \to \tau_2 = \tau_1$

NT $\tau_1 = \tau_2 \to (\tau_2 = \tau_3 \to \tau_1 = \tau_3)$

We will take this as a minimal logic of fully transitive, symmetric identity.[9] This logic will form the basis for the Kremer axiomatizations which include axioms of relevant indiscernibility. All of the RI axioms, one for each of our relevant predicates

[9]By this is meant the inclusion of the symmetry axiom above, and nested transitivity (NT).

given by extra-logical considerations, are added to the minimal logic of fully transitive identity (and perhaps strengthened to a basis of $\mathbf{RQ}^{\circ t}$, if desired). Thus, Kremer's axiomatization is an extension of this minimal, fully transitive logic.

We can obtain a minimal logic of weakly transitive identity by replacing NT with weak transitivity. Mares' base logic extends the minimal logic of weakly transitive identity by the addition of a weak substitution axiom (at which point the weak transitivity axiom becomes derivable from the rest of the system).

5.4.3 Models with Left-to-Right Identity Sets

Inspired by the semantic equivalence relation from Mares, we define a function that determines a left-to-right identity relation at each world for each object in the domain. The function is of type $\mathrel{=\!\!\!\rightharpoonup} : K \times U \longrightarrow \wp(U)$. We write $y \in \mathrel{=\!\!\!\rightharpoonup}(a, x)$ to mean that at the world a, x is left-to-right identical to y. Note the reader is not to confuse this use of variables referring to objects in the domain in the metalanguage with the variables in the language which require a valuation function to refer to objections in the domain.

The function introduced determines a 2-place relation between objects in the domain at each world. Using this, we will define models and demonstrate that they satisfy certain conditions. Then, we will simplify these models to only satisfy those conditions, and demonstrate soundness with respect to these simpler models, and thus *a fortiori* demonstrate the soundness of the models using this left-to-right identity relation.

Unsimplified Models

Using the function $\mathrel{=\!\!\!\rightharpoonup}$, we may define, via a method known as 'Currying', a function $|{\approx}| : U^2 \longrightarrow \wp(K)$ such that

$$y \in \mathrel{=\!\!\!\rightharpoonup}(a, x) \text{ iff } a \in |{\approx}|(x, y).$$

As a function from an n-tuple of elements of U to a set of worlds, this new function can serve as the basis of an atomic predicate. In fact, the simplified models take this defined function as primitive rather than the left-to-right identity relation. The seeming benefit of the unsimplified models is that the conditions required to capture identity using the left-to-right identity relation are all frame conditions rather than model conditions.

Definition 5.4.2. A *frame* for $\mathbf{QB}^{\circ t}_{=}$ ($\mathbf{BQ}^{\circ t}_{=}$) is a tuple

$$\langle K, 0, R, *, U, Prop, PropFun, \Rrightarrow \rangle,$$

where $\langle K, 0, R, *, U, Prop, PropFun \rangle$ is a frame for $\mathbf{QB}^{\circ t}$ ($\mathbf{BQ}^{\circ t}$), and the following conditions are satisfied:

1. $a \in 0 \Rightarrow x \in \Rrightarrow(a, x)$
2. $a \leq b \ \&\ y \in \Rrightarrow(a, x) \Rightarrow x \in \Rrightarrow(b, y)$
3. $Rabc \ \&\ y \in \Rrightarrow(a, x) \ \&\ z \in \Rrightarrow(b, y) \Rightarrow z \in \Rrightarrow(c, x)$

Again, we say a frame is *full* when $Prop$ is the set of every up-set of K and $PropFun$ is the set of all functions from variable assigns to $Prop$.

Definition 5.4.3. A *pre-model for* $\mathbf{QB}^{\circ t}_{=}$ ($\mathbf{BQ}^{\circ t}_{=}$) is a tuple

$$\langle K, 0, R, *, U, Prop, PropFun, \Rrightarrow, |{-}|^{\mathfrak{M}} \rangle,$$

where $\langle K, 0, R, *, U, Prop, PropFun, \Rrightarrow \rangle$ is a $\mathbf{QB}^{\circ t}_{=}$- ($\mathbf{BQ}^{\circ t}_{=}$-) frame and $|{-}|^{\mathfrak{M}}$ is an assignment as in models for $\mathbf{QB}^{\circ t}$ ($\mathbf{BQ}^{\circ t}$) with the exception/addition that:

- where $\mathcal{A}$ is the atomic $\tau_1 = \tau_2$, the propositional function assigned to $\mathcal{A}$ is given by
$$|\tau_1 = \tau_2|^{\mathfrak{M}} f = |{\approx}|^{\mathfrak{M}}(|\tau_1|^{\mathfrak{M}} f, |\tau_2|^{\mathfrak{M}} f),$$
where $|{\approx}|^{\mathfrak{M}}$ is the function $|{\approx}|$ obtained by the $\Rrightarrow$ function of the model.

Definition 5.4.4. A pre-model is a *model* for $\mathbf{QB}^{\circ t}_{=}$ ($\mathbf{BQ}^{\circ t}_{=}$) when it assigns an element of $PropFun$ to each atomic formula, including all identity statements.

The $\models$ relation is much like it was, but for the case of identity, which is

$a, f \models \tau_1 = \tau_2$ iff $a \in |\tau_1 = \tau_2|^{\mathfrak{M}} f$

Satisfaction and validity are defined as usual.

Lemma 5.4.5 (Heredity)**.** *If $a \leq b$ and $a, f \models \mathcal{A}$, then $b, f \models \mathcal{A}$.*

The new case for identity statements is trivially satisfied by the definition of models.

Lemma 5.4.6 (Semantic Entailment)**.** *$\mathcal{A} \to \mathcal{B}$ is satisfied by a variable assignment f in a model $\mathfrak{M}$ iff, for every $a \in K$, if $a, f \vDash \mathcal{A}$, then $a, f \vDash \mathcal{B}$.*

The proof follows the usual arguments based on the heredity lemma.

I now turn to the lemma that allows us to simplify the models. The lemma proves that certain conditions hold for the current models. These conditions will be critical in showing soundness for the simplified models. It will be shown that the relation between and simplified and unsimplified models gives us an *a fortiori* argument for the soundness of the unsimplified models, given a proof of the soundness for the simplified models.

Lemma 5.4.7. *The following conditions are satisfied in every model $\mathfrak{M}$*

(i) $0 \subseteq |{\approx}|^{\mathfrak{M}}(x, x)$

(ii) $a \leq b \;\&\; a \in |{\approx}|^{\mathfrak{M}}(x, y) \Rightarrow b \in |{\approx}|^{\mathfrak{M}}(y, x)$

(iii) $Rabc \;\&\; a \in |{\approx}|^{\mathfrak{M}}(x, y) \;\&\; b \in |{\approx}|^{\mathfrak{M}}(y, z) \Rightarrow c \in |{\approx}|^{\mathfrak{M}}(x, z)$

Proof. (i) Suppose that $a \in 0$. By condition 1. in the definition of frames, $x \in \rightharpoonup(a, x)$. It follows that $a \in |{\approx}|^{\mathfrak{M}}(x, x)$, as required.

(ii) Suppose that $a \leq b$ and $a \in |{\approx}|^{\mathfrak{M}}(x, y)$. By the definition of $|{\approx}|^{\mathfrak{M}}$, we have that $y \in \rightharpoonup(a, x)$. By condition 2. of the definition of frames, $x \in \rightharpoonup(b, y)$, and thus $b \in |{\approx}|^{\mathfrak{M}}(y, x)$, as required.

(iii)Suppose that $Rabc$, $a \in |{\approx}|^{\mathfrak{M}}(x, y)$, and $b \in |{\approx}|^{\mathfrak{M}}(y, z)$. By reasoning similar to the previous case, using condition 3. in the definition of frames, we get that $c \in |{\approx}|^{\mathfrak{M}}(x, z)$, as required. ☐

Simplified Models

Lemma 5.4.7 demonstrates that the function introduced in the previous section to represent the left-to-right identity relation in the models (in a way that parallels the treatment of non-identity atomic formulas) is sufficient to prove a list of conditions. In this section, we will show that such conditions are sufficient to prove soundness, and that a semantics simplified in this way is therefore sufficient.

Definition 5.4.8. A simplified *pre-model for* $\mathbf{QB}^{\circ t}_{=}$ ($\mathbf{BQ}^{\circ t}_{=}$) is a pre-model for $\mathbf{QB}^{\circ t}$ ($\mathbf{BQ}^{\circ t}$) with the following exceptions:

- a function $|\approx|^{\mathfrak{M}} : U^2 \longrightarrow \wp(K)$ is assigned to $=$ such that

  (i) $0 \subseteq |\approx|^{\mathfrak{M}}(x, x)$

  (ii) $a \leq b \ \&\ a \in |\approx|^{\mathfrak{M}}(x, y) \Rightarrow b \in |\approx|^{\mathfrak{M}}(y, x)$

  (iii) $Rabc \ \&\ a \in |\approx|^{\mathfrak{M}}(x, y) \ \&\ b \in |\approx|^{\mathfrak{M}}(y, z) \Rightarrow c \in |\approx|^{\mathfrak{M}}(x, z)$

- where $\mathcal{A}$ is the atomic $\tau_1 = \tau_2$, the propositional function assigned to $\mathcal{A}$ is given by

$$|\tau_1 = \tau_2|^{\mathfrak{M}} f = |\approx|^{\mathfrak{M}}(|\tau_1|f, |\tau_2|f)$$

Definition 5.4.9. A pre-model is a *model* for $\mathbf{QB}^{\circ t}_{=}$ ($\mathbf{BQ}^{\circ t}_{=}$) when it assigns an element of *PropFun* to each atomic formula, including all identity statements.

The $\vDash$ relation is much like it was, except for the case of identity, which is

$$a, f \vDash \tau_1 = \tau_2 \text{ iff } a \in |\tau_1 = \tau_2|^{\mathfrak{M}} f$$

This fact is in both simplified and unsimplified models a special case: the way the models are defined allows us to cover this case by the usual treatment of atomic formulas. This is due to the fact that we have defined $|\approx|^{\mathfrak{M}}$ to have all the properties we require. Otherwise, identity statements are just like any other atomic proposition.

Satisfaction and validity are defined as usual, as usual.

Lemma 5.4.10 (Heredity). *If $a \leq b$ and $a, f \vDash \mathcal{A}$, then $b, f \vDash \mathcal{A}$.*

The proof of this lemma remains the same, with the exception of atomic, identity formulas. However, the definition of a model ensures that every atomic sentence, including the identities, are assigned to admissible propositional functions, which are all upwardly closed sets. Thus, the case for identity statements in the proof of this lemma is no different from the case for the non-identity atomic formulas.

5.4.4 Soundness

For the soundness proof, we will show that all of the lemmas required for the axioms of $\mathbf{QB}^{\circ t}$ (and $\mathbf{BQ}^{\circ t}$) are provable. Then, we will show that the axioms for identity are also valid.

Lemma 5.4.11 (Semantic Entailment)**.** *$\mathcal{A} \to \mathcal{B}$ is satisfied by a variable assignment f in a model $\mathfrak{M}$ iff, for every $a \in K$, if $a, f \vDash \mathcal{A}$, then $a, f \vDash \mathcal{B}$.*

As in Lemma 5.4.6, the proof of this lemma is as per the usual arguments.

Lemma 5.4.12. *The relativity, symmetry, and nested transitivity axioms are valid in every $\mathbf{QB}^{\circ t}_{=}$ (and $\mathbf{BQ}^{\circ t}_{=}$) model.*

Proof. Reflexivity: We are required to show that $a, f \vDash \tau_1 = \tau_1$, for every $a \in 0$, and every $f \in U^\omega$. Take an arbitrary f and $a \in 0$. By the definition of models we have that $a \in |\tau_1 = \tau_1|^{\mathfrak{M}} f$, which gives us $a, f \vDash \tau_1 = \tau_1$.

Symmetry: Suppose that $a, f \vDash \tau_1 = \tau_2$ for some a and f. Thus, $a \in |\tau_1 = \tau_2|^{\mathfrak{M}} f$. That is, $a \in |{\approx}|^{\mathfrak{M}}(\tau_1, \tau_2)$. Given that $a \leq a$, by the definition of our models we get that $a \in |{\approx}|^{\mathfrak{M}}(\tau_2, \tau_1)$, and so $a \in |\tau_2 = \tau_1|^{\mathfrak{M}} f$. The result follows by Semantic Entailment.

Nested Transitivity: Suppose that $a, f \vDash \tau_1 = \tau_2$. Then $a \in |{\approx}|^{\mathfrak{M}}(\tau_1, \tau_2)$. Suppose further that $Rabc$ and $b, f \vDash \tau_2 = \tau_3$. Thus $b \in |{\approx}|^{\mathfrak{M}}(\tau_2, \tau_3)$. By a condition imposed on the models, we have that $c \in |{\approx}|^{\mathfrak{M}}(\tau_1, \tau_3)$, in other words $c, f \vDash \tau_1 = \tau_3$. The result follows by Semantic Entailment. □

To demonstrate the soundness of the rest of the system, the lemmas used in previous chapters for $\mathbf{QB}^{\circ t}$ and $\mathbf{BQ}^{\circ t}$ must be shown to hold for the $\mathbf{QB}^{\circ t}_{=}$ and $\mathbf{BQ}^{\circ t}_{=}$. This will involve demonstrating that the various proofs by induction on the complexity of a formula in the relevant lemmas survive the addition of identity formulas as a base case. The proof of the next two lemmas is straightforward.

Lemma 5.4.13. *If f and g agree on all the free variables in a formula $\mathcal{A}$, then $|\mathcal{A}|f = |\mathcal{A}|g$.*

Lemma 5.4.14. *For any formula $\mathcal{A}$ with x free for τ in $\mathcal{A}$, in any $\mathbf{QB}^{\circ t}_{=}$-model ($\mathbf{BQ}^{\circ t}_{=}$-models) $\mathfrak{M}$, if $g \in xf$ and $|x|g = |\tau|f$, then $|\mathcal{A}[\tau/x]|f = |\mathcal{A}|g$.*

Lemma 5.4.15. *The axioms and rules of* $\mathbf{QB}^{\circ t}$ *(and* $\mathbf{BQ}^{\circ t}$*) and validity and validity preserving, respectively, in the* $\mathbf{QB}^{\overset{\circ t}{=}}$*- (and* $\mathbf{BQ}^{\overset{\circ t}{=}}$*-)models.*

This can be shown given the material of the previous chapters and the previous two lemmas.

Theorem 5.4.16 (Soundness for $\mathbf{QB}^{\overset{\circ t}{=}}$ (and $\mathbf{BQ}^{\overset{\circ t}{=}}$))**.** *All of the orems of* $\mathbf{QB}^{\overset{\circ t}{=}}$ *(*$\mathbf{BQ}^{\overset{\circ t}{=}}$*) are valid in every* $\mathbf{QB}^{\overset{\circ t}{=}}$*-model (*$\mathbf{BQ}^{\overset{\circ t}{=}}$*-model).*

This theorem follows from 5.4.15 and 5.4.12.

This demonstrates the soundness of the simplified models, and a fortiori the unsimplified models. We now move to completeness, constructing a canonical, unsimplified model.

5.4.5 Completeness

Theories, prime theories, regular theories, and independent pairs are defined as usual. An extension lemma is provable.

Definition 5.4.17 (Canonical Frames)**.** A *canonical frame* for $\mathbf{QB}^{\overset{\circ t}{=}}$ ($\mathbf{BQ}^{\overset{\circ t}{=}}$) is a tuple

$$\langle K_c, 0_c, R_c, *_c, U_c, Prop_c, PropFun_c, \Rightarrow_c\rangle, \text{ where}$$

- K_c is the set of all prime theories.
- 0_c is the set of all regular prime theories.
- R_c is defined by $R_c abc$ iff $\{\mathcal{A} \circ \mathcal{B} : \mathcal{A} \in a \ \& \mathcal{B} \in b\} \subseteq c$.
- $*_c$ is defined by $a^* = \{\mathcal{A} : \neg\mathcal{A} \notin a\}$.
- U_c is the set infinite set of constants Con.
- $\Rightarrow_c$ is defined by $\Rightarrow_c(a, \tau_1) = \{\tau_i : \tau_1 = \tau_i \in a\}$.
- For every closed formula $\mathcal{A}$, $||\mathcal{A}||_c$ is defined to be the set $\{a \in K : \mathcal{A} \in a\}$.
- $Prop_c$ is defined as the set $\{||\mathcal{A}||_c : \mathcal{A}$ is a closed formula$\}$.

- Given a variable assignment f, the value fn is a constant. Substituting each variable in a formula $\mathcal{A}$ with the constant assigned to it by a variable assignment f results in a closed formula which will be denoted $\mathcal{A}^f$. Therefore $\mathcal{A}^f = \mathcal{A}[f0/x_0, \ldots, fn/x_n, \ldots]$.

- To each formula $\mathcal{A}$, there is a corresponding function $\phi_{\mathcal{A}}$ of type $U^\omega \longrightarrow K$ given by $\phi_{\mathcal{A}} f = ||\mathcal{A}^f||_c$. $PropFun_c$ is the set of all function $\phi_{\mathcal{A}}$, where $\mathcal{A}$ is a formula.

Definition 5.4.18. A *canonical model* for $\mathbf{BQ}^{\circ \boldsymbol{t}}$ is a tuple,

$$\mathfrak{M} = \langle K_c, 0_c, R_c, *_c, U_c, Prop_c, PropFun_c, \rightleftharpoons_c, |-|_c^{\mathfrak{M}} \rangle, \text{ where}$$

- $\langle K_c, 0_c, R_c, *_c, U_c, Prop_c, PropFun_c, \rightleftharpoons_c \rangle$ is the canonical frame.

- $|c|_c = c$, for every constant symbol c.

- $|P|_c(c_0, \ldots, c_n) = ||P(c_0, \ldots, c_n)||_c$, for every n-ary predicate symbol P .

- The valuation is then given for all wff as before.

Before demonstrating that the canonical frame is a frame, the squeeze lemma is stated.

Lemma 5.4.19 (Squeeze Lemma)**.** *If a and b are theories, c is a prime theory, and $R_c abc$, then there are prime theories a', b' such that $Ra'b'c$.*

Again, the proof of this lemma is standard.

Lemma 5.4.20. *The canonical frame is a frame.*

Proof. By the arguments of lemma 3.5.4 we can demonstrate everything except the conditions for $\rightleftharpoons$. Here, we demonstrate that the conditions hold.

Suppose that $a \in 0$, and this $a \in ||\boldsymbol{t}||_c$. As $\boldsymbol{t}$ is a closed formula, $\boldsymbol{t} \in a$. The formula $\boldsymbol{t} \to x = x$ is a theorem, so $x = x \in a$. Thus, $x \in \rightleftharpoons(a, x)$, as required.

Suppose that $a \leq b$ and $y \in \rightleftharpoons(a, x)$. It follows that $a \subseteq b$ and $x = y \in a$. By the axiom $x = y \to y = x$ we get that $y = x \in a$, and thus $y = x \in b$. By definition, $x \in \rightleftharpoons(b, y)$, as required.

Suppose that $Rabc$, $y \in \rightleftharpoons(a, x)$ and $z \in \rightleftharpoons(b, y)$. By definition, $x = y \in a$ and $y = z \in b$, and so $x = y \circ y = z \in c$. It is also a theorem that $(x = y \circ y = z) \to x = z$, so $x = z \in c$, which by definition means $z \in \rightleftharpoons(c, x)$, as required. $\square$

Lemma 5.4.21. *For every n-ary predicate symbol P (including $=$), every variable assignment, and every set of terms $\tau_1, \ldots, \tau_n$*

1. $P(\tau_1, \ldots, \tau_n)^f = P(|\tau_1|_c f, \ldots, |\tau_n|_c f)$

2. $|P(\tau_1, \ldots, \tau_n)|_c = \phi_{P((\tau_1, \ldots, \tau_n))}$

This lemma shows that every atomic proposition is assigned a member of $PropFun$ by the canonical valuation, and thus we have the following corollary.

Corollary 5.4.22. *The canonical model is a model.*

Lemma 5.4.23 (Truth Lemma)**.** *For any formula, $\mathcal{A}$, $\mathcal{A} = \phi_{\mathcal{A}}$. In other words, for every f, $|\mathcal{A}|_c f = ||\mathcal{A}^f||_c$. That is, $a, f \vDash \mathcal{A}$ iff $\mathcal{A}^f \in a$.*

Again, the proof is as in Mares and Goldblatt [74].

Theorem 5.4.24 (Completeness for $\mathbf{QB}^{\circ t}_{=}$ ($\mathbf{BQ}^{\circ t}_{=}$))**.** *If $\mathcal{A}$ is valid in every $\mathbf{QB}^{\circ t}_{=}$-model ($\mathbf{BQ}^{\circ t}_{=}$-model), then $\mathcal{A}$ is a theorem of $\mathbf{QB}^{\circ t}_{=}$ ($\mathbf{BQ}^{\circ t}_{=}$).*

The proof is as before.

Let $\mathbb{L}$ be a relevant logic considered in chapter 1. We define the quantified relevant logics with identity based on $\mathbb{L}$ by adding the corresponding axioms to $\mathbf{QB}^{\circ t}_{=}$ (or $\mathbf{BQ}^{\circ t}_{=}$). Semantically, we define their models by treating identity as in we have here, and the remainder of the model as in chapter 2.

Corollary 5.4.25. *The logics $\mathbf{LQ}^{\circ t}_{=}$ and $\mathbf{QL}^{\circ t}_{=}$ are sound and complete for their models as defined by the proceeding paragraph.*

The models, informally defined by the paragraph proceeding the statement of the corollary can be given a rigorous definition, and the corollary will follow.

5.4.6 Weakened Transitivity, Relevant Indiscernibility, and Substitution

As was mentioned earlier, we may instead use weak transitivity in place of nested transitivity. Such a logic requires a semantics based on frames that lack the condition used to ensure nested transitivity is valid. Instead, we replace that condition in the models with the condition that

$$y \in \mathrel{=\!\!\rightharpoonup}(a, x) \;\&\; z \in \mathrel{=\!\!\rightharpoonup}(a, y) \Rightarrow z \in \mathrel{=\!\!\rightharpoonup}(a, x)$$

Lemma 5.4.26. *Weak transitivity is valid in* $\mathbf{QB}^{\circ t}_{=}$*-models that satisfy the above condition.*

Proof. Suppose that $a, f \vDash x = y \wedge y = z$. Then $a, f \vDash x = y$ and $a, f \vDash y = z$. That is, $y \in \mathrel{=\!\!\rightharpoonup}(a, x)$ & $z \in \mathrel{=\!\!\rightharpoonup}(a, y)$. By the condition assumed, $z \in \mathrel{=\!\!\rightharpoonup}(a, x)$, which by definition means $a, f \vDash x = z$. The result follows by Semantic Entailment. $\square$

Further, if we define the canonical models as in the case of $\mathbf{QB}^{\circ t}_{=}$, the canonical model is a model.

Lemma 5.4.27. *The canonical model satisfies the condition* $y \in \mathrel{=\!\!\rightharpoonup}(a, x)$ & $z \in \mathrel{=\!\!\rightharpoonup}(a, y) \Rightarrow z \in \mathrel{=\!\!\rightharpoonup}(a, x)$.

Proof. Suppose that $y \in \mathrel{=\!\!\rightharpoonup}(a, x)$ and $z \in \mathrel{=\!\!\rightharpoonup}(a, y)$. By definition, $x = y, y = z \in a$ By weak transitivity, $x = z \in a$, which is $z \in \mathrel{=\!\!\rightharpoonup}(a, x)$. $\square$

Thus, for every logic with nested transitivity discussed in this section (including those defined below), the logics resulting by replacing nested transitivity with weak transitivity are given a semantics for which they are sound and complete.

Let us now consider adding proper axioms to these logics to constrain the behavior of identity with respect to relevant predications. There are two kinds of axioms to consider. The first are the RI axioms argued for by Kremer. The second is the relevant substitution axioms, which follows from RI given permutation and fusions implied by conjunctions.

For any extension of $\mathbf{QB}^{\circ t}_{=}$ ($\mathbf{BQ}^{\circ t}_{=}$) by proper axioms of the form of a relevant indiscernibility axiom, a possibly infinite set of relevant predications must be given. Let this set be the set $\mathbb{RP}$, and let it contain *monadic* atomic relevant predicate symbols. (The case for polyadic predicate symbols should be a fairly straightforward generalization.) Let us write $\mathfrak{A}_{\mathbb{RP}}$ to denote a set of relevant indiscernibility axioms for the set of relevant predications $\mathbb{RP}$.

N.B. that the base logic dealt with here can be extended with all of the results maintained. Thus, such a logic could serve as the base logic for Kremer, by extending the base to $\mathbf{R}^{\circ t}$.

Lemma 5.4.28. *Any logic* $\mathbf{QB}^{\circ t}_{=} + \mathfrak{A}_{\mathbb{RP}}$ *is sound with respect the class of* $\mathbf{QB}^{\circ t}_{=}$*-models that satisfy the following condition:*

$$|\tau_1 = \tau_2|f \subseteq \cap_{G\in\mathbb{RP}}|G[\tau_1/z] \leftrightarrow G[\tau_2/z]|f$$

Proof. Suppose that $a, f \vDash \tau_1 = \tau_2$. By the above condition, $a \in \cap_{G\in\mathbb{RP}}|G[\tau_1/z] \leftrightarrow G[\tau_2/z]|f$, and thus $a, f \vDash G[\tau_1/z] \leftrightarrow G[\tau_2/z]$. The result follows for every RI axiom by Semantic Entailment. □

The condition given here is a model condition rather than a frame condition. This can be defended by reference to the reason we are assuming RI axioms to be proper axioms. That is, the content of an RI axiom is extra-logical: the relevant predicates are not so because of logic. This need not be the case. In the next chapter, frame conditions are given for RI and related axioms.

If the canonical model for $\mathbf{QB}^{\circ t}_{=} + \mathfrak{A}_{\mathbb{RP}}$ is defined as for $\mathbf{QB}^{\circ t}_{=}$, the following lemma and corollary hold.

Lemma 5.4.29. *The canonical model for* $\mathbf{QB}^{\circ t}_{=} + \mathfrak{A}_{\mathbb{RP}}$ *is a model that satisfies the condition in lemma 5.4.28*

Proof. Suppose that $\tau_1 = \tau_2 \in a$. It follows by the relevant RI axioms that $a \in \cap_{G\in\mathbb{RP}}|G[\tau_1/z] \leftrightarrow G[\tau_2/z]|f$. This fact is sufficient to establish that the condition holds. □

Corollary 5.4.30. *Any logic* $\mathbf{QB}^{\circ t}_{=} + \mathfrak{A}_{\mathbb{RP}}$ *is sound and complete with respect the class of* $\mathbf{QB}^{\circ t}_{=}$*-models that satisfy the following condition:*

$$|\tau_1 = \tau_2|f \subseteq \cap_{G\in\mathbb{RP}}|G[\tau_1/z] \leftrightarrow G[\tau_2/z]|f$$

Kremer's axiomatization is thus given a semantics here; however, the semantics are not quite in line with Kremer's suggested interpretation of identity.

The axioms of relevant substitution are very much similar to those or RI. Let $\mathfrak{A}_{\mathbb{S}}$ be a set of relevant substitution axioms.

Lemma 5.4.31. *Any logic* $\mathbf{QB}^{\circ t}_{=} + \mathfrak{A}_{\mathbb{S}}$ *is sound with respect the class of* $\mathbf{QB}^{\circ t}_{=}$*-models that satisfy the following condition:*

$$|\tau_1 = \tau_2 \wedge G[\tau_1/z]|f \subseteq |G[\tau_2/z]|f$$

Proof. Suppose that $a, f \vDash \tau_1 = \tau_2 \wedge G[\tau_1/z]$. Then $a \in |\tau_1 = \tau_2 \wedge G[\tau_1/z]|f$ and thus $a \in |G[\tau_2/z]|f$ by the condition above, which by definition means $a, f \vDash G[\tau_2/z]$. □

Let us define the canonical model for $\mathbf{QB}^{\circ t}_{=} + \mathfrak{A}_{\mathbb{S}}$ as in $\mathbf{QB}^{\circ t}_{=}$.

Lemma 5.4.32. *The canonical model for* $\mathbf{QB}^{\circ t}_{=} + \mathfrak{A}_{\mathbb{S}}$ *is a model that satisfies the condition in lemma 5.4.31*

Corollary 5.4.33. *Any logic* $\mathbf{QB}^{\circ t}_{=} + \mathfrak{A}_{\mathbb{S}}$ *is sound and complete with respect the class of* $\mathbf{QB}^{\circ t}_{=}$*-models that satisfy the following condition:*

$$|\tau_1 = \tau_2 \wedge G[\tau_1/z]|f \subseteq |G[\tau_2/z]|f$$

To summarize, identity in relevant logics is less messy and overall easier when based on the semantics of Mares and Goldblatt [74] than on Fine's semantics. At least compared to semantics based on Fine's for first order relevant logics. We have defended certain axiomatizations of relevant logics with identity, and shown different possible interpretations of Kremer's proposed axiom system. Further, we have constructed semantics for not only each interpretation of Kremer's proposed system, but for a wide range of relevant logics with identity. Further, we have argued for the extra-logical nature of Kremer's RI axioms, and the related relevant substitution axioms.

In the next chapter, we will see what happens when we add modalities to the logics defined in this section.

5.4.7 Context-Dependent Relevance

Yaroslav Shramko's article [110] on relevant properties suggests that relevant predications are context dependent. For example, tasting acidic might be relevant to a certain kind of coffee in the context of selecting a coffee to drink or sorting roasts of coffee. But one can imagine contexts in which the taste of a coffee is not relevant to it: selecting a liquid to fill a water bed; a contest between different beverages based solely on visual properties; someone without a sense of taste selecting beverages from a cafe.

In the above models a single set of relevant predicates RP is set a fixed, and two objects are identical when they agree on all of the relevant predicates. We may now ask whether identity is context dependent. How should the models be changed if it is?

Finally, how compatible is the above axiomatization and semantics with relevant predication? Shramko's? Dunn's? Kremer's? We have already seen that Kremer's treatment of identity and thus relevant predication has an essential conflict between syntax and semantics in the first order.

We may also consider some properties relevant to the identity of individuals are modal properties. Although Gibbard [45] argues for contingent identity, the case of Lumpl and Goliath could be used to motivate modal properties. The same piece of clay can be thought of as two objects. One, a lump of clay, which is not destroyed by clumping up the clay. The second, a statue, which is destroyed by the clumping. All non-modal properties of these two objects are the same. We would thus benefit from having a list of relevant properties that includes non-atomic, even modal properties. This is quite important if we are to use this logic in such applications.

Thus, while this chapter has provided multiple axiomatizations and semantics for identity in relevant logic, there are still many questions left in the project of relevant predication the solutions to which may motivate treatments of identity not covered in this chapter.

Chapter 6

Quantified Modal Logics with Identity

6.1 Introduction

In the previous chapter, I gave a semantic framework for identity extending quantified relevant logics as weak as $\mathbf{QB}^{\circ t}$ and its extensions, proving a correspondence between various axioms with identity and semantic conditions. In this chapter, I extend this result to modal logics, in which a plenitude of (seemingly) independent axioms involving identity and modal operators arises.

The chapter is divided into sections as follows. First, we explore the syntax of the modal logics with identity, providing just enough detail to introduce the axiom systems and a convention for naming them. A discussion of particular axiom systems and their philosophical coherence or motivation will be delayed until after semantic systems are developed in the next section, which comprises correspondence results for a number of axioms for identity. In particular, we first focus on the relationship between identity and the modal operators. Then, we consider indiscernibility and substitution axioms. The key insight here is how we turn the model conditions for RI and the related relevant substitution axioms into frame conditions, which is achieved by first giving a sort of name to some elements in PropFun, then restricting the model in ways using these names. (The final section of the chapter will briefly discuss other potential frame conditions.)

Having the axiom systems and semantics in place for a range of logics, we then turn to discussing particular logics.[1] The latter part of the chapter will concern axiomatizations of a specific model relevant logic with identity. Our example will be based on the logic **E**, for which RI, relevant substitution, and nested transitivity are all seen as sentences which commit fallacies of modality. Fallacies of modality will be explicated, and some coherent axiomatizations will be suggested, one of which will answer the question of what it takes to ensure that RI does not commit such a fallacy. Finally, I close the chapter by suggesting further areas of research.

6.2 Logics with Identity

Let $\mathbb{L}$ be a logic extending $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ as defined in an earlier chapter. That is, the extensions of $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ that add either the extensional confinement axiom, the Barcan Formula, axioms to strengthen to relevant fragment, or axioms that strengthen the modal fragment. To the languages of these logics, we of course add the 2-place predicate '=' for identity. Further, we again split the atomic predicates in two, where RP is a subset of the predicates. We also make the simplifying assumption that all of the predicates in RP are monadic.

For a complete list of axioms we will consider, see the tables in the next section. Here, we will discuss only a few. Where $\mathbb{A}_=$ is a set of identity axioms (from our complete list of axioms considered), we write $\mathbb{L} + \mathbb{A}_=$ to be the logic given by the axioms and rules of $\mathbb{L}$ together with the set of axiom $\mathbb{A}_=$.

The base logics with identity that will we consider are as follows. Let $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$) be the logic defined as $\mathbf{QB}.\mathbf{C}_{\Box\Diamond} + \mathbb{A}_=$ ($\mathbf{BQ}.\mathbf{C}_{\Box\Diamond} + \mathbb{A}_=$), where $\mathbb{A}_=$ is empty. That is, it is just the logic $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ in a language including the identity symbol. Here it may seem a bit odd that the "basic" quantified modal logic *with identity* has no axioms for identity, and is therefore just the logic $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ or $\mathbf{BQ}.\mathbf{C}_{\Box\Diamond}$ with an extended language. The reason I have made this choice is to list a set of frame conditions for each axiom governing identity, not relying on any other properties identity has in the logic — i.e. a relative frame definability.

[1]The range of logics chosen to axiomatize in this (and the last) chapter deserves a remark in regards to the potential applications and the kinds of terms assumed. For example, we will not separate the terms into rigid designators and non-rigid designators, which might appear to differ in terms of the necessity of identity statements. I intend to extend the work of this and the previous chapter in future work to address this concern.

6.3 Semantics for Logics with Identity

The previous chapter has undoubtedly anticipated the treatment of identity in the current chapter, with the exception of the modified models seemingly needed to have frame conditions for RI and related axioms.

Again, we model identity using a relation between objects relative to each world. Let's continue to call this relation a left-to-right identity relation. It is given by a function $\mathrel{=\!\!\!\rightharpoonup} : K \times U \longrightarrow \wp(U)$. We define the function $|{\approx}| : U^2 \longrightarrow \wp(K)$ as before by setting

$$y \in \mathrel{=\!\!\!\rightharpoonup}(a, x) \text{ iff } a \in |{\approx}|(x, y).$$

The base model for $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$) lacks frame conditions for identity, as was anticipated by the non-inclusion of any identity axioms. Besides having extra machinery for modeling the identity predicate, these models treat identity "just like" any other predicate, where they are mapped onto propositional functions without restriction by the valuation.

6.3.1 Models for $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$)

Definition 6.3.1. A *frame* for $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$) is a tuple

$$\langle K, 0, R, *, S_\Box, S_\Diamond, UProp, PropFun, \mathrel{=\!\!\!\rightharpoonup}\rangle$$

where $\langle K, 0, R, *, S_\Box, S_\Diamond, UProp, PropFun\rangle$ is a $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}_{\Box\Diamond}$)

Definition 6.3.2. A *pre-model* for $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$) is a tuple

$$\langle K, 0, R, *, S_\Box, S_\Diamond, UProp, PropFun, \mathrel{=\!\!\!\rightharpoonup}, |{-}|\rangle$$

where $\langle K, 0, R, *, S_\Box, S_\Diamond, UProp, PropFun, \mathrel{=\!\!\!\rightharpoonup}\rangle$ is a $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$- ($\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$-)frame and $|{-}|$ is a value assignment like that for $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}_{\Box\Diamond}$) except that

- where $\mathcal{A}$ is atomic $\tau_1 = \tau_2$, the propositional function assigned to $\mathcal{A}$ is given by

$$|\tau_1 = \tau_2|^{\mathfrak{M}} f = |{\approx}|^{\mathfrak{M}}(|\tau_1|^{\mathfrak{M}} f, |\tau_2|^{\mathfrak{M}} f)$$

Definition 6.3.3. I define *models* to be pre-models that assign every atomic formula (including identity statements) to elements of $PropFun$, and *satisfaction*, *validity* (in frames, models, and classes of models) as usual.

Given the arguments of previous chapters (particularly for the corresponding lemmas for $\mathbf{QB.C}_{\Box\Diamond}$ ($\mathbf{BQ.C}_{\Box\Diamond}$)), the following lemmas are provable, and will be stated in succession. (I will continue to omit $\mathfrak{M}$ in the notation when convenient.)

Lemma 6.3.4 (Heredity). *If $a \leq b$ and $a, f \vDash \mathcal{A}$, then $b, f \vDash \mathcal{A}$.*

Lemma 6.3.5 (Semantic Entailment). *$\mathcal{A} \to \mathcal{B}$ is satisfied by a variable assignment f in a model $\mathfrak{M}$ iff, for every $a \in K$, if $a, f \vDash \mathcal{A}$, then $a, f \vDash \mathcal{B}$.*

Lemma 6.3.6. *If f and g agree on all the free variables in a formula $\mathcal{A}$, then $|\mathcal{A}|f = |\mathcal{A}|g$.*

Lemma 6.3.7. *For any formula $\mathcal{A}$ with x free for τ in $\mathcal{A}$, in any $\mathbf{QB.C}^{=}_{\Box\Diamond}$-model ($\mathbf{BQ.C}^{=}_{\Box\Diamond}$-model) $\mathfrak{M}$, if $g \in xf$ and $|x|g = |\tau|f$, then $|\mathcal{A}[\tau/x]|f = |\mathcal{A}|g$.*

As we have merely extended the language of $\mathbf{QB.C}_{\Box\Diamond}$ ($\mathbf{BQ.C}_{\Box\Diamond}$) with identity, and treated it effectively as any other 2-place predicate, the soundness results for $\mathbf{QB.C}_{\Box\Diamond}$ ($\mathbf{BQ.C}_{\Box\Diamond}$) will naturally apply for the new logics $\mathbf{QB.C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ.C}^{=}_{\Box\Diamond}$). I record this result as follows.

Theorem 6.3.8 (Soundness for $\mathbf{QB.C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ.C}^{=}_{\Box\Diamond}$)). *All of the theorems of $\mathbf{QB.C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ.C}^{=}_{\Box\Diamond}$) are valid in every $\mathbf{QB.C}^{=}_{\Box\Diamond}$-model ($\mathbf{BQ.C}^{=}_{\Box\Diamond}$-model).*

Let's now consider the completeness of $\mathbf{QB.C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ.C}^{=}_{\Box\Diamond}$) for the defined semantics. As with the soundness results above, completeness results are trivial given the arguments for completeness for $\mathbf{QB.C}_{\Box\Diamond}$ ($\mathbf{BQ.C}_{\Box\Diamond}$), and thus we will merely state some definitions and then record some lemmas without proof, and their proof is by earlier arguments. The only difference here is the definition of the canonical model, in which our clause for identity will be shown to cause no harm to earlier arguments.

Definition 6.3.9. We define *theories, prime theories*, and *regular theories* as usual, and an extension lemma is provable.

Definition 6.3.10 (Canonical Frames). A *canonical frame* for $\mathbf{QB.C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ.C}^{=}_{\Box\Diamond}$) is a tuple

$$\langle K_c, 0_c, R_c, *_c, U_c, S_{\Box c}, S_{\Diamond c}, Prop_c, PropFun_c, \Rrightarrow_c \rangle$$

where

- K_c is the set of all prime theories.
- 0_c is the set of all regular prime theories.
- R_c is defined by $R_c abc$ iff $\{\mathcal{A} \circ \mathcal{B} : \mathcal{A} \in a \,\& \mathcal{B} \in b\} \subseteq c$.
- $S_{\Box c}$ is defined by $S_{\Box c}ab$ iff $\{\mathcal{A} : \Box\mathcal{A} \in a\} \subseteq b$.
- $S_{\Diamond c}$ is defined by $S_{\Diamond c}ab$ iff $\{\Diamond\mathcal{A} : \mathcal{A} \in b\} \subseteq a$
- $*_c$ is defined by $a^* = \{\mathcal{A} : \neg\mathcal{A} \notin a\}$.
- U_c is the set infinite set of constants Con.
- $\Rrightarrow_c$ is defined by $\Rrightarrow_c(a, \tau_1) = \{\tau_i : \tau_1 = \tau_i \in a\}$.
- For every closed formula $\mathcal{A}$, $||\mathcal{A}||_c$ is defined to be the set $\{a \in K : \mathcal{A} \in a\}$.
- $Prop_c$ is defined as the set $\{||\mathcal{A}||_c : \mathcal{A}$ is a closed formula$\}$.
- Given a variable assignment f, the value fn is a constant. Substituting each variable in a formula $\mathcal{A}$ with the constant assigned to it by a variable assignment f results in a closed formula which will be denoted $\mathcal{A}^f$. Therefore $\mathcal{A}^f = \mathcal{A}[f0/x_0, \ldots, fn/x_n, \ldots]$.
- To each formula $\mathcal{A}$, there is a corresponding function $\phi_{\mathcal{A}}$ of type $U^\omega \longrightarrow K$ given by $\phi_{\mathcal{A}} f = ||\mathcal{A}^f||_c$. $PropFun_c$ is the set of all function $\phi_{\mathcal{A}}$, where $\mathcal{A}$ is a formula.

Definition 6.3.11. A *canonical model* for $\mathbf{QB.C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ.C}^{=}_{\Box\Diamond}$) is a tuple

$$\langle K_c, 0_c, R_c, *_c, U_c, S_{\Box c}, S_{\Diamond c}, Prop_c, PropFun_c, \Rrightarrow_c\rangle$$

where

- $\langle K_c, 0_c, R_c, *_c, U_c, S_{\Box c}, S_{\Diamond c}, Prop_c, PropFun_c, \Rrightarrow_c\rangle$ is the canonical frame
- $|c|_c = c$, for every constant symbol c
- $|P|_c(c_1, \ldots, c_n) = ||P(c_1, \ldots, c_n)||_c$
- The valuation is extended to all wff as before

The completeness results for $\mathbf{QB}.\mathbf{C}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}_{\Box\Diamond}$) will carry over straightforwardly to $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$) because the addition of the $\mathrel{=\!\!\!\rightharpoonup}$ does not affect the proofs. This is because we have that

$$|\tau_2|f \in \mathrel{=\!\!\!\rightharpoonup}(a, |\tau_1|f) \text{ iff } (\tau_1 = \tau_2)^f \in a \text{ iff } a \in ||(\tau_1 = \tau_2)^f||_c$$

That is, identity statements are still treated like any other atomic proposition in the canonical model, and the definition of $\mathrel{=\!\!\!\rightharpoonup}$ can be seen as merely epiphenomenal to the "core" of the model. For ease, let us sum up in a single theorem.

Theorem 6.3.12 (Completeness for $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$))**.** *If $\mathcal{A}$ is valid in every* $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$) *model, then $\mathcal{A}$ is a theorem of* $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$).

6.4 Extending the Semantics

Let us now consider adding axioms governing the behavior of identity in the system. This section will consist of tables of axioms and frame/model conditions and the proof of their correspondence when extending the systems of the previous section.

In the following table, the axioms are given using terms in general, which is equivalent in this setting to specifying them with just variables. The variables in the conditions are ranging over the objects in the domain U in the model, and the reader should not confuse them with the variables in the logics.

The proof of the correctness of the table below — that each axiom is sound and complete for the corresponding frame condition in $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$), and *a fortiori* any extension — is to be recorded in the following lemmas. In many of the proofs below, the steps using the equivalence of $a, f \vDash x = y$ with $|y| \in \mathrel{=\!\!\!\rightharpoonup}(a, |x|f)$ will often remain implicit in the proof.

Name	Axiom	Frame Condition
Ref	$\tau = \tau$	$a \in 0 \Rightarrow x \in \Rrightarrow(a, x)$
Sym	$\tau_1 = \tau_2 \to \tau_2 = \tau_1$	$a \leq b \;\&\; y \in \Rrightarrow(a, x) \Rightarrow x \in \Rrightarrow(b, y)$
E.Sym	$\tau_1 \neq \tau_2 \vee \tau_2 = \tau_1$	$a \in 0 \;\&\; y \in \Rrightarrow(a^*, x) \Rightarrow x \in \Rrightarrow(a, y)$
N. Tr	$\tau_1 = \tau_2 \to (\tau_2 = \tau_3 \to \tau_1 = \tau_3)$	$Rabc \;\&\; y \in \Rrightarrow(a, x) \;\&\; z \in \Rrightarrow(b, y) \Rightarrow z \in \Rrightarrow(c, x)$
W. Tr	$(\tau_1 = \tau_2 \wedge \tau_2 = \tau_3) \to \tau_1 = \tau_3$	$y \in \Rrightarrow(a, x) \;\&\; z \in \Rrightarrow(a, y) \Rightarrow z \in \Rrightarrow(a, x)$
E. Tr	$\tau_1 \neq \tau_2 \vee \tau_2 \neq \tau_3 \vee \tau_1 = \tau_3$	$a \in 0 \;\&\; y \in \Rrightarrow(a^*, x) \;\&\; z \in \Rrightarrow(a^*, y) \Rightarrow z \in \Rrightarrow(a, x)$
Ref$^{\Box}$	$\Box\tau = \tau$	$a \in 0 \;\&\; S_{\Box}ab \Rightarrow x \in \Rrightarrow(b, x)$
Sym$^{\Box}$	$\Box(\tau_1 = \tau_2 \to \tau_2 = \tau_1)$	$a \in 0 \;\&\; S_{\Box}ab \;\&\; Rbcd \;\&\; y \in \Rrightarrow(c, x) \Rightarrow x \in \Rrightarrow(d, y)$
E.Sym$^{\Box}$	$\Box(\tau_1 \neq \tau_2 \vee \tau_2 = \tau_1)$	$a \in 0 \;\&\; S_{\Box}ab \;\&\; y \in \Rrightarrow(b^*, x) \Rightarrow x \in \Rrightarrow(b, y)$
N. Tr$^{\Box}$	$\Box(\tau_1 = \tau_2 \to (\tau_2 = \tau_3 \to \tau_1 = \tau_3))$	$a \in 0 \;\&\; S_{\Box}ab \;\&\; Rbcd \;\&\; Rdef \;\&\; y \in \Rrightarrow(c, x) \;\&$ $\&\; z \in \Rrightarrow(x, y) \Rightarrow z \in \Rrightarrow(f, x)$
W. Tr$^{\Box}$	$\Box((\tau_1 = \tau_2 \wedge \tau_2 = \tau_3) \to \tau_1 = \tau_3)$	$a \in 0 \;\&\; S_{\Box}ab \;\&\; Rbcd \;\&\; y \in \Rrightarrow(c, x) \;\&$ $\&\; z \in \Rrightarrow(c, y) \Rightarrow z \in \Rrightarrow(d, x)$
E. Tr$^{\Box}$	$\Box(\tau_1 \neq \tau_2 \vee \tau_2 \neq \tau_3 \vee \tau_1 = \tau_3)$	$a \in 0 \;\&\; S_{\Box}ab \;\&\; y \in \Rrightarrow(b^*, x) \;\&$ $\&\; z \in \Rrightarrow(b^*, y) \Rightarrow z \in \Rrightarrow(b, x)$
NSID	$\tau = \tau \to \Box(\tau = \tau)$	$S_{\Box}ab \;\&\; x \in \Rrightarrow(a, x) \Rightarrow x \in \Rrightarrow(b, x)$
NID	$\tau_1 = \tau_2 \to \Box(\tau_1 = \tau_2)$	$S_{\Box}ab \;\&\; y \in \Rrightarrow(a, x) \Rightarrow x \in \Rrightarrow(b, y)$
NSID$^{\Box}$	$\Box(\tau = \tau \to \Box(\tau = \tau))$	$a \in 0 \;\&\; S_{\Box}ab \;\&\; Rbcd \;\&\; S_{\Box}de \;\&$ $\&\; x \in \Rrightarrow(c, x) \Rightarrow x \in \Rrightarrow(e, x)$
NID$^{\Box}$	$\Box(\tau_1 = \tau_2 \to \Box(\tau_1 = \tau_2))$	$a \in 0 \;\&\; S_{\Box}ab \;\&\; Rbcd \;\&\; S_{\Box}de \;\&$ $\&\; y \in \Rrightarrow(c, x) \Rightarrow y \in \Rrightarrow(e, x)$

Lemma 6.4.1. *Logics with Reflexivity (Ref), Symmetry (Sym), Nested Transitivity (N. Tr), Weak Transitivity (W. Tr), and Relevant Indiscernibility (RI) are sound and complete for the class of* $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$) *models satisfying the corresponding conditions.*

Arguments in the previous chapter suffice to demonstrate this.

Lemma 6.4.2. *Logics with Necessary Reflexivity (Ref$^{\Box}$), Necessary Symmetry (Sym$^{\Box}$), Extensional Symmetry (E. Sym), Extensional Transitivity (E. Tr), Necessary Extensional Symmetry (E. Sym$^{\Box}$), Necessary Nested Transitivity (N. Tr$^{\Box}$), Necessary Weak Transitivity (W. Tr$^{\Box}$), and Necessary Extensional Transitivity (E. Tr$^{\Box}$) are sound and complete for the class of* $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$ ($\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$) *models satisfying the corresponding conditions.*

Proof. Here we simplify the cases to just variables as terms, but the results can obviously be extended to all terms.

Case Ref$^{\Box}$: Validity — Suppose that $a \in 0$ and that $a, f \not\models \Box x = x$. Then there is a b such that $S_{\Box}ab$ and $b, f \not\models x = x$. But our condition can be applied giving us $b, f \models x = x$, and thus a contradiction. Completeness — Suppose $a \in 0$. Then a is regular and $(\Box x = x)^f \in a$. Further suppose that $S_{\Box c}ab$. From this it follows by the definition of the canonical model that $(x = x)^f \in b$, as required.

Case Sym$^\Box$: Validity — For reductio, suppose that $a \in 0$ and $a, f \nvDash \Box(x = y \to y = x)$. Then we get a b such that $S_\Box ab$ and $b, f \nvDash (x = y \to y = x)$. Which implies that $Rbcd$ for some c and d such that $c, f \vDash x = y$ but $d, f \nvDash y = x$. Applying the condition, we get that $d, f \vDash y = x$, giving us the required contradiction. Completeness — Suppose $y \in \Rightarrow(c, x), S_{\Box c}ab, a \in 0$, and $R_c bcd$. Then we have that $x = y^f \in c$. As a is regular and thus contains $(\text{Sym}^\Box)^f$, we get that $(x = y \to y = x)^f \in b$. Given that $\vdash ((x = y \to y = x) \circ x = y) \to y = x$, the definition of R_c gives us $y = x^f \in d$ which is the required $x \in \Rightarrow(d, y)$.

Case E.Sym: Validity — Suppose $a \in 0$ and $a, f \nvDash x \neq y \vee y = x$. Then both $a, f \nvDash x \neq y$ and $a, f \nvDash y = x$. The former is if and only if $a^*, f \vDash x = y$ which is $|y|f \in \Rightarrow(a^*, |x|f)$. Applying the condition we get that $|x|f \in \Rightarrow(a, |y|f)$, which gives us $a, f \vDash y = x$, and our contradiction. Thus $a \vDash x \neq y \vee y = x$. Completeness — Suppose $a \in 0$ and $y \in \Rightarrow(a^*, x)$. Then $x \neq y \vee y = x \in a$. As a is prime, one of the two disjuncts must be in a too. It cannot be the first disjunct, as the valuation gives that $a^*, f \nvDash x = y$, which is in contradiction with our assumptions, thus it must be the second disjunct, which gives us $|x|f \in \Rightarrow(a, |y|f)$, as required.

Case E. Tr: Validity — Suppose that $a \in 0$, and for reductio that $a, f \nvDash x \neq y \vee y \neq z \vee x = z$. So $a, f \nvDash$ each of the disjuncts, the first two give us both $y \in \Rightarrow(a^*, x)$ and $z \in \Rightarrow(a^*, y)$, and our condition thus gives us $a, f \vDash x = z$, contradicting our assumption, so $a, f \vDash x \neq y \vee y \neq z \vee x = z$, for all f and all $a \in 0$. Completeness — Suppose that $a \in 0$ and $y \in \Rightarrow(a^*, x)$ and $z \in \Rightarrow(a^*, y)$. Thus, $(x \neq y \vee y \neq z \vee x = z)^f \in a$. Given the primeness of a, at least one of the disjuncts is also in a. Our assumptions rule out the first two, so it follows that $x = z^f \in a$ and thus $z \in \Rightarrow(a, x)$, as required.

Case E. Sym$^\Box$: Validity — Assume $a \in 0$ and for reductio that $a, f \nvDash \Box(x \neq y \vee y = x)$. It is easy to check that with the condition in hand there is a contradiction as required along the lines of reasoning similar to that for E. Sym and any boxed axiom. Completeness — Suppose that $a \in 0$, so $\Box(x \neq y \vee y = x) \in a$. Further let $S_{\Box c}ab$ and $y \in \Rightarrow(b^*, x)$. We can infer that $x \neq y \vee y = x \in b$. Then, given the primeness of b the only disjunct possible without contradiction is the second, so $x \in \Rightarrow(b, y)$.

Case N. Tr$^{\Box}$: Validity — Suppose that $a \in 0$, and for reductio that $a, f \not\models \Box(x = y \to (y = z \to x = z))$. Then there is a b such that $S_{\Box}ab$ and $b, f \not\models x = y \to (y = z \to x = z)$. This implies that $Rbcd$ for some c, d such that $c, f \models x = y$ and $d, f \not\models x = z$. The latter giving us $Rdeg$ for some e, g such that $e, f \models y = z$ but $g, f \not\models x = z$. However, we have all conjuncts of the antecedent of the corresponding condition, giving us $g, f \models x = z$, which contradicts our last conclusion. Thus $a, f \not\models \Box(x = y \to (y = z \to x = z))$, for every $a \in 0$ as required.

Completeness — Suppose that $a \in 0, S_{\Box c}ab$, $R_c bcd$, $R_c deg$, $x = y^f \in c$ and $y = z^f \in e$. From a being regular and the definition of $S_{\Box c}ab$ we can infer that $x = y \to (y = z \to x = z)^f \in b$. It is also a theorem that $((x = y \to (y = z \to x = z) \in b) \circ x = y) \to (y = z \to x = z)$, so from the definition of R_c we get that $(y = z \to x = z)^f \in d$. Using a similar argument we get that $x = z^f \in g$, as required.

Case W. Tr$^{\Box}$: Validity — Suppose that $a \in 0$ and then for reductio that $a, f \not\models \Box((x = y \wedge y = z) \to x = z)$. It follows that $S_{\Box}ab$ for some b such that $b, f \not\models ((x = y \wedge y = z) \to x = z)$. But then we get a c and d such that $Rbcd$, $c, f \models x = y \wedge y = z$, but $d, f \not\models x = z$. We can now apply the condition to get that $d, f \models x = z$, giving us our contradiction. So $a, f \models \Box((x = y \wedge y = z) \to x = z)$ for all $a \in 0$.

Completeness — Suppose that $a \in 0$, $S_{\Box c}ab$, $R_c bcd$ and that $x = y^f, y = z^f \in c$. By the usual arguments, $((x = y \wedge y = z) \to x = z)^f \in b$. Then, given the definition of R_c and the fact that theories are closed under adjunction, we can see that $x = z^f \in d$, as required.

Case E. Tr$^{\Box}$: Validity — Suppose that $a \in 0$ and for reductio that $a, f \not\models \Box(x \neq y \vee y \neq z \vee x = z)$. Then $S_{\Box}ab$ for some b such that $b, f \not\models x \neq y$, $b, f \not\models y \neq z$, and $b, f \not\models x = z$. From the former we get that $b^*, f \models x \neq y$ and $b^*, f \models y \neq z$. By our condition this gives us $b, f \models x = z$, and thus our contradiction.

Completeness — Suppose that $a \in 0$, $S_{\Box c}ab$, $x = y^f \in b^*$, and $y = z^f \in b^*$. As a is regular, it contains the disjunction $(x \neq y \vee y \neq z \vee x = z)^f$. As a is prime, it contains at least one of the three disjuncts. It is easy to check that, on pain of contradiction, the only disjunct it can contain is $x = z^f$, as required. $\Box$

Lemma 6.4.3. *Logics with Necessity of Self Identity (NSID), Necessity of Identity (NID), and their boxed cousins (NSID$^{\Box}$) and ((NID$^{\Box}$) are sound and complete for the* *class of* $\mathbf{QB.C}^{=}_{\Box\Diamond}$ *(*$\mathbf{BQ.C}^{=}_{\Box\Diamond}$*) models satisfying the corresponding conditions.*

Proof. The proofs are straightforward. Let us only consider a couple cases.

NID: Validity — Suppose that $a, f \vDash x = y$. For reductio, suppose that $a, f \nvDash \Box x = y$. Then $S_\Box ab$ and $b, f \nvDash x = y$. But applying our condition we get that $b, f \vDash x = y$, giving us a contradiction. So $a, f \vDash \Box x = y$. The result follows by Semantic Entailment.

Case NID$^\Box$: Validity — Suppose that $a \in 0$ and for reductio that $a, f \nvDash \Box(x = y \to \Box(x = y))$. So $S_\Box ab$ and $b, f \nvDash x = y \to \Box(x = y)$. Thus, $Rbcd$ for some c, d such that $c, f \vDash x = y$ and $d, f \nvDash \Box x = y$. So $S_\Box de$ for an e such that $e, f \nvDash x = y$. But applying our condition we get that $e, f \vDash x = y$, and thus our contradiction.

Completeness — Suppose that $a \in 0$, $S_{\Box c}ab$, $R_c bcd$, $S_{\Box c}de$, and $x = y^f \in c$. From a being regular we have that $(\Box(x = y \to \Box(x = y)))^f \in a$, so $(x = y \to \Box(x = y))^f \in b$. Using the R_c definition and similar arguments are before we arrive at $\Box(x = y)^f \in d$. But then, as required, $x = y^f \in e$. $\Box$

Remark 6.4.4. It is worth remarking on the increased complexity of the boxed axioms when compared to their non-boxed counterparts. Indeed, without the box, we often have implicative formulas with which we could employ Semantic Entailment, taking any arbitrary world at which the antecedent was true. In particular, the advantage the definition of $\leq$ enables the worlds we look at to be related via the $\leq$ relation. However, the boxed formulas require us to look just at the worlds in 0 and what they can access or "see" along the $S_\Box$ relation, and these worlds don't have the same advantage. We may have an implication to evaluate at one of these worlds, and must in that case look "down" the R relation. If we are at world b, we must look at worlds c, d such that $Rbcd$. But we are not guaranteed that $c \leq d$, so we must employ extra machinery, giving rise to the more complex conditions.

However, we run into an additional complication when constructing logics: the boxed axioms are not equivalent to a system with necessitation. According to the condition for $Ref^\Box$, all worlds one step away have $x = x$, but this condition will not guarantee that the worlds two steps away make $x = x$ true. The first part of these conditions $a \in 0$ & $S_\Box ab$ is required so that any logically normal world (i.e., any world in 0) can only "see" worlds that are normal with respect to identity in certain ways (at a distance of 1). Without an axiom $\Box\mathcal{A} \to \Box\Box\mathcal{A}$ or the rule of necessitation, we can see need to generalize these axioms and their conditions. We can generalize to any finite number of boxes in from of an axiom. When where are n boxes before an axiom, for *finite* n, we subscript the axiom with that number and add a box in the superscript. This $\text{Ref}^\Box$ is really $\text{Ref}^\Box_1$, and $\text{Ref}^\Box_5$ is $\Box\Box\Box\Box\Box x = x$. For the axiom conditions, it is sufficient to just change the initial $a \in 0 \ \& \ S_\Box ab$ to $a \in 0 \ \& \ S^n_\Box ab$.

6.4.1 Indiscernibility

To give the conditions for the following axioms, we will need to make slight modifications to our frames. When constructing models, we set as fixed a set of relevant predicates, *in this case with the simplification that they are monadic.* To state the conditions for the indiscernibility axioms, we require the ability to single out both the set of admissible propositional functions to which the relevant predications are mapped and the corresponding admissible propositions. Thus, let us write $PropFun^G$ to denote a subset of the admissible propositional functions to which it is acceptable to map relevant predicates. Similarly, we write $Prop^G$ to denote the admissible propositions to which $PropFun^G$ can map onto. Thus, a relevant predication Gx will be mapped only by a member of $PropFun^G$ to a proposition in $Prop^G$. Using this notation, we are able to construct conditions for RI and its variants.

The previous chapter contained a model condition for RI, and here we are constructing a frame condition for it. Unfortunately, it appears we may either make a seemingly arbitrary stipulation[2], or in some way pack more of the model into the frame. The way I propose to do this lets us identify what will be the free variable in a formula which

[2] We could stipulate that there is no function in $PropFun^G$ that returns the same member of $Prop^G$ for each $f \in U^\omega$. This stipulation can be seen as formalizing a somewhat intuitive notion that no relevant predication holds of everything — relevant predications make significant divisions of the domain. However, this requirement is motivated by an apparent formal necessity. We could have a condition for RI of the form

$$y \in \Rightarrow a, x \ \& \ Rabc \ \& \ b \in \Phi f \ \& \ b \notin \Phi g (\text{for some } g \in xf) \Rightarrow c \in \Phi h \text{ for some } h \in xf \text{ such that } |x|h = |y|f$$

In the condition for RI, the antecedent contains $b \in \Phi f \ \& \ b \notin \Phi g(\text{for some } g \in xf)$, where Φ is a member of $PropFun^G$. This is to ensure that the condition does not accidentally validate $x = y \to Gz \to Gz$, *by requiring that the relevant predication mapped onto* Φ *has* x *free.* Now, to prove soundness, we will have to demonstrate something similar to what we show below, that the propositional function to which Gy is mapped is similar enough to the function to which Gx is mapped applied to the assignment h in the condition.

will be mapped into $PropFun^G$. The first step is to name the elements of $PropFun^G$. The second step is to use the names of propositional function when assigning formulas to them in the model.

Step 1: For each of our monadic predicates $G \in RP$, for each term τ, let $: G\tau :$ be a name. We then consider a surjective function that assigns names to each member of $PropFun^G$.

Step 2: We additionally require for models that, for every $G \in RP$ and every term τ, the valuation function assigns $G\tau$ the element of $PropFun^G$ named by $: G\tau :$.

This lets us identify propositional functions that *will* be assigned to Gx, which enables us to use the fact that x will be free.

Let us now expand upon these two steps, describing the exact changes to make to our models.

To complete Step 1, we could name every propositional function, but we will find that it is sufficient just to name those in $PropFun^G \subseteq PropFun$. In fact, this set will be closed under replacement of variables, as will be shown after we introduce some additional notation.

For Step 2, we must consider both changing the definition of models, and the definition of the canonical model. Let us start with the definition of models. The valuation function $|-|$ will be modified so that, when $G \in RP$, $|G\tau| =: G\tau :$. For this, it is sufficient to define a model as that which assigns a propositional function to every atomic proposition, and further only assigns members in $PropFun^G$ to relevant predication atomic formulas.

More importantly, we have to describe the canonical model. We define $PropFun^G$ to be the set of all functions $\Phi_{\mathcal{A}}$ where $\mathcal{A}$ is a relevant predication — that is, is of the form $G\tau$ for some term τ and some $G \in RP$. Further, $\Phi_{\mathcal{A}} \in PropFun^G$ will also have the name $: \mathcal{A} :$. Thus, by the definition of $PropFun$, $\Phi_{\mathcal{A}} f = ||\mathcal{A}^f||_c$ for all formulas $\mathcal{A}$. Thus, substituting names, we have that for all relevant predications, $: G\tau : f = ||G\tau^f||_c$. Using this, we get that $b \in: Gx : f$ iff $Gx^f \in b$.

Remark 6.4.5. The frame conditions for RI and related axioms are achieved through squeezing more of the models into the frames. More accurately, we squeeze more of the canonical model into the frame. $\Phi_{\mathcal{A}}$ is a unique function for the formula $\mathcal{A}$ in the canonical model. In the models constructed here the frames build in this correspondence between propositional functions (named) and relevant predications. In other words, instead of having relevant predications mapped onto propositional functions, we have them map onto specific propositional functions in such a way that mimics/anticipates the canonical frame.

It will be helpful to have notation for propositional functions that differ from each other only by the propositions assigned to certain x-variants.

Definition 6.4.6. Where Φ is a propositional function, let $\Phi[y/x]$, the propositional function that differs at most from Φ by what propositions are assigned to x-variants, be defined by

$$\Phi[y/x]f = \Phi f[|y|f/x].$$

That is, $\Phi[y/x]f$ behaves exactly as Φ, except that the function f assigns $|y|f$ to the variable x. In other words, $\Phi[y/x]$ is as if all x's were substituted for y's. In fact, that is the point. The formulas assigned to $\Phi[y/x]$ will differ from that assigned to Φ by just this substitution. We prove this below.

Lemma 6.4.7. *$PropFun^G$ is closed under term substitution. That is, if Φ is in $PropFun^G$, then so is $\Phi[y/x]$, for all variables x and y.*

The proof is obvious from the definition.

Lemma 6.4.8. *The relevant predication Gx is mapped to $: Gx :$ iff Gy is mapped to $: Gx : [|y|f/x]$.*

Proof. We know that Gx is mapped to $: Gx :$ and that Gy is mapped to $: Gy :$. Thus, it suffices to show that $: Gx : [|y|f/x] =: Gy :$. Both Gx and Gy have one free variable. For any assignment g, $: Gx : [|y|g/x]$ and $: Gy :$ can only differ if g assigns a different value to x than it does y. It is easy to see that this is in fact not the case. With $: Gx : [|y|g/x]g$ iff $: Gx : g[|y|g/x]$ by definition, we have that g must assign $|y|g$ to x. But by definition, this is also the object that g assigns to y. □

Lemma 6.4.9. *For all $a \in K$ and $f \in U^\omega$ and $G \in RP$, $a \in: Gx : [|y|f/x]$ iff $a, f \vDash Gy$.*

Name	Axiom	Frame Condition
RI	$\tau_1 = \tau_2 \to (G\tau_1 \to G\tau_2)$	For every $G \in RP$ and $f \in U^\omega$, $y \in\Rightarrow(a,x)$ & $Rabc$ & $b \in: Gx : f \Rightarrow c \in: Gx : [\lvert y\rvert f/x]f$
NRI	$\tau_1 = \tau_2 \to \Box(G\tau_1 \to G\tau_2)$	For every $G \in RP$ and $f \in U^\omega$, $y \in\Rightarrow(a,x)$ & Sab & $Rbcd$ & $c \in: Gx : f \Rightarrow d \in: Gx : [\lvert y\rvert f/x]$
RNI	$\Box\tau_1 = \tau_2 \to (G\tau_1 \to G\tau_2)$	For every $G \in RP$ and $f \in U^\omega$, if $a \in \{e : \forall d \in K(S_{\Box c}ed \Rightarrow \lvert y\rvert f \in\Rightarrow(d, \lvert x\rvert f))\}$ & $Rabc$ & $b \in: Gx : f \Rightarrow c \in: Gx : [\lvert y\rvert f/x]f$
DRI	$\Box\tau_1 = \tau_2 \to \Box(G\tau_1 \to G\tau_2)$	For every $G \in RP$ and $f \in U^\omega$, $a \in \{e : \forall d \in K(S_{\Box c}ed \Rightarrow \lvert y\rvert f \in\Rightarrow(d, \lvert x\rvert f))\}$, and $c \in: Gx : f$ & $S_\Box ab$ & $Rbcd$ & $y \in\Rightarrow(b,x)$ & $c \in: Gx : f \Rightarrow d \in: Gy : f$

Table 6.1: Indiscernibility Axioms

Proof. The proof is straightforward from lemma 6.4.8 and the fact that $a, f \vDash Gx$ iff $a \in: Gx : f$. $\Box$

Lemma 6.4.10. *The logics with RI and its variants listed in Table 6.1 are sound and complete for the class of* $\mathbf{QB}.\mathbf{C}^{=}_{\Box\Diamond}$ *(*$\mathbf{BQ}.\mathbf{C}^{=}_{\Box\Diamond}$*) models satisfying the corresponding conditions.*

Proof. **Case RI**: Validity — Suppose that $a, f \vDash x = y$. For reductio, let $a, f \nvDash Gx \to Gy$. Then $Rabc$ for some b, c such that $b, f \vDash Gx$ but $c, f \nvDash Gy$. From $b, f \vDash Gx$ we get that $b \in |Gx|f$. As Gx is relevant predication, it is mapped onto the element of $PropFun^G$ named $: Gx :$, and so we have that $b \in: Gx : f$. We can now apply the frame condition for RI and infer that $c \in: Gx : [y/x]f$. By lemma 6.4.8, we get that $c \in: Gy : f$, and so $c, f \nvDash Gy$, giving us our contradiction. Thus $a, f \vDash Gx \to Gy$, and the result follows from Semantic Entailment.

Completeness — Suppose that $x = y^f \in a$, $R_c abc$, and $b \in: Gx : f$ for $G \in RP$. It is a theorem that $(x = y \circ Gx) \to Gy$. By the definition of R_c and the fact that we can further infer that $Gx^f \in b$, we have that $Gy^f \in c$. From this we can infer that $b \in: Gy : f$, which by lemma 6.4.9 is equivalent to $b \in: Gx : [|y|f/x]$, as required.

Case NRI: Validity — Suppose that $a, f \vDash x = y$. For reductio, let $a, f \nvDash \Box(Gx \to Gy)$. Then $Rbcd$, $c \in: Gx : f$, and $d \notin: Gy : f$. However, with similar reasoning to the previous case, and applying the condition, we get that $d \in: Gy : f$, and so $a, f \vDash \Box(Gx \to Gy)$. The result follows from Semantic Entailment.

Completeness — Suppose that $y \in\Rightarrow(a, x)$, $S_{\Box c}ab$, $R_c bcd$ and $c \in: Gx : f$. It is a theorem that $x = y \to \Box(Gx \to Gy)$, so we can infer that $(Gx \to Gy)^f \in b$. Thus, it follows that $d \in: Gy : f$, as required.

Case RNI: Validity — Suppose that $a, f \vDash \Box x = y$. This entails that $a \in \{e : \forall d \in K(S_{\Box c}ed \Rightarrow |y|f \in \rightharpoondown\!\!\!= (d, |x|f))\}$. Then, for reductio, let $a, f \nvDash Gx \to Gy$. Then we have $b, c \in K$ such that $Rabc$ and $b \in: Gx : f$ and $c \notin: gy : f$. Using the condition above and Lemma 6.4.8 we can derive that $c \in: gy : f$, giving us our contradiction. So $a, f \vDash Gx \to Gy$, and the result follows from Semantic Entailment.

Completeness — Suppose that $a \in \{e : \forall d \in K(S_{\Box c}ed \Rightarrow |y|f \in \rightharpoondown\!\!\!= (d, |x|f))\}$, $R_c abc$, and $b \in: Gx : f$. It follows that $(\Box x = y)^f \in a$, because if a bears the $S_\Box$ relation to any situation d, then that situation contain $x = y^f$. This would lead to contradiction if $a, f \nvDash \Box x = y$. The rest of the proof is similar to that of the case for RI. Note that the readability of the condition here is greatly improved by making it a model condition, replacing $a \in \{e : \forall d \in K(S_{\Box c}ed \Rightarrow |y|f \in \rightharpoondown\!\!\!= (d, |x|f))\}$ with $a, f \vDash \Box x = y$.

Case DRI: Validity — Suppose that $a, f \vDash \Box x = y$, and so $a \in \{e : \forall d \in K(S_{\Box c}ed \Rightarrow |y|f \in \rightharpoondown\!\!\!= (d, |x|f))\}$, and $c \in: Gx : f$. For reduction, let $a, f \nvDash \Box(Gx \to Gy)$. Then we have $S_\Box ab$ and $b, f \nvDash (Gx \to Gy)$, and also $y \in \rightharpoondown\!\!\!= (b, x)$. But then $Rbcd$, $c \in: Gx : f$, and $d \notin: Gy : f$. But this last fact together with the antecedent of the relevant condition give us a contradiction, so $a, f \vDash \Box(Gx \to Gy)$. As usual, the result then follows by Semantic Entailment.

Completeness — Suppose that $S_{\Box c}ab$, $R_c bcd$, $a \in \{e : \forall d \in K(S_{\Box c}ed \Rightarrow |y|f \in \rightharpoondown\!\!\!= (d, |x|f))\}$, and $c \in: Gx : f$. Thus, $x = y^f \in b$ and $Gx^f \in c$. We then have that $(\Box x = y)^f \in a$. It is a theorem that $\Box x = y \to \Box(Gx \to Gy)$, and so $(Gx \to Gy)^f \in b$. It follows that $d \in: Gy : f$, as required.

$\Box$

The conditions for boxed versions of the axioms can be derived from the general format of the conditions for boxed axioms in the table, as needed.

In the above conditions, we used $a \in \{e : \forall d \in K(S_{\Box c}ed \Rightarrow |y|f \in \rightharpoondown\!\!\!= (d, |x|f))\}$, and $c \in: Gx : f$ when $\Box x = y$ appeared in the antecedent of the axioms. This is because it is equivalent to the model condition that $a, f \vDash \Box x = y$, but expressed as a frame condition. However, the presence of $\rightharpoondown\!\!\!=$ and the names $: Gx :$ allow us to give frame conditions for the axioms. That is, for axioms that otherwise look like $\mathcal{A} \to (\mathcal{B} \to \mathcal{C})$. The naming of propositions and propositional functions and the $\rightharpoondown$ let us capture in the frames the exact relations between $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ in the case of the RI axioms. It is then not at all surprising that we require the frames to look more like models, as what looks like contingent information about the implicational relationship between $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ in RI is really "logical" and invariant over mappings of meaningful declarative sentences into the logic (so long as we keep a fixed set of relevant predications).

Corollary 6.4.11. *The frame conditions for relevant substitution axioms can be read off of the conditions for RI axioms above, making the required permutation.*

Remark 6.4.12. We end this section with a remark. We have two approaches to RI and similar axioms. The first is to use model conditions, which gives us relatively simple systems at the cost of using model conditions. (Again, all the frame conditions given in this section can be turned into model conditions.) The second approach, the one demonstrated here, was to use the kind of correspondence between propositional functions and sentences found in the canonical model in the frame, by including names for some propositional functions. While this method works, it is somewhat aesthetically unpleasing. As either model or frame conditions will work, for all practical purposes it seems we can use the systems defined by model conditions. Further, this approach using frame conditions does not seem to reveal some deep fact about the correspondence between identity and relevant predications.

Near the beginning of 6.4.1, I mentioned a third approach, which was to specify that no relevant predication holds of everything. I conjecture this will work.

To conclude the remark, there is to my knowledge no approach to modeling logics with RI that is completely satisfactory. Either this is to reveal something about the nature of identity in these logics, or there is a satisfactory approach yet to be taken.

6.5 A Note on Necessitated Axioms

Normal modal logics are often defined using the rule of necessitation. But in the classical setting this rule is equivalent to the requirement that, if $\mathcal{A}$ is an axiom, then so too is $\Box\mathcal{A}$. The reasoning is as follows. Let us have a proof in which necessitation is used. We want to replace the proof with one that does not have necessitation, but has the extra, necessitated axioms. Consider for example the following proof:

Proof $\mathbb{P}$

(1)	$\vdash \mathcal{A}$	Axiom
(2)	$\vdash \mathcal{A} \to \mathcal{B}$	Derived wff
(3)	$\vdash \mathcal{B}$	(1),(2), MP
(4)	$\vdash \Box\mathcal{B}$	(3), NEC

The general idea of the equivalence is to use the regularity of the system to derive $\Box\mathcal{A} \to \Box\mathcal{B}$ from line (2), and to use the extra condition to confirm that $\Box\mathcal{A}$ is also an axiom. So the transformed proof will look like this:

Proof $\mathbb{P}'$

(1)	$\vdash \Box\mathcal{A}$	Axiom
(2)	$\vdash \mathcal{A} \to \mathcal{B}$	Derived wff
(3)	$\vdash \Box\mathcal{A} \to \Box\mathcal{B}$	(2), $\Box$-Monotonicity
(4)	$\vdash \Box\mathcal{B}$	(1),(3), MP

The example just given does not constitute a proof of the equivalence, but does facilitate our discussion of necessitated axioms of identity.

Here, the condition was that if $\mathcal{A}$ is an axiom, then so is $\Box\mathcal{A}$. However, we just discussed limiting necessitation to particular axioms, namely some set of identity axioms.

The equivalence in the classical settings entails that the same frame condition may be used for both Necessitation and the equivalent enlargements of the set of axioms. The same will not be true when only a subset of the axioms can be necessitated. In fact, for each axiom of identity, for each natural number of boxed prefixing the axiom, there is a distinct frame condition. Thus, to mimic necessitation for identity axioms, an infinite number of distinct frame conditions must be met. It is left for a future project to define a finite (set of) frame condition(s) to model the logics with the condition 'if

$\mathbb{A}_=$ is a set of identity axioms, then so is $\Box\mathbb{A}_=$' (defined by the set of all necessitated axioms in $\mathbb{A}_=$). I conjecture that such a condition may be easy to find, and will express what is informally given as "everywhere that logically normal worlds can 'see' (via $S_\Box$) is a place that is normal with respect to identity".

6.6 Entailment, Identity, and Fallacies of Modality

Axiomatizing identity in the logic **E** turns out to be a bit tricky. This is because several of the axioms we have thus far considered for identity have the potential to commit fallacies of modality. In particular, the RI axiom, Nested Transitivity (NT), and Relevant Substitution (RS) must be examined. In this section I will examine these axioms, demonstrate their potential for committing fallacies of modality, motivate adopting RI and Nested Transitivity, and then show what extra axiom can be adopted which ought to ensure that they do not result in the logic committing any modal fallacy.

6.6.1 Fallacies of Modality

Extending **E** with identity is an interesting special case of adding identity to a relevant logic. This is due the fact that **E**, Anderson and Belnap's favorite logic, was the logic they identified as *the* logic of entailment. Among many things, this meant for them that the logic avoids both fallacies of relevance and fallacies of modality. Here, we are not only adding identity but also necessity,[3] and I will show that identity and modality must be added in coherent ways to **E** in order to preserve the philosophical motivation of **E**. That is, in order that the resulting logic does not commit the fallacies a logic of Entailment should not commit. Here, I will ignore fallacies of relevance and focus solely on fallacies of modality.

As a rough summary of Anderson and Belnap's (A&B) exposition of modal fallacies, one could say that *you don't go looking at Matters of Fact out there in the world in order to figure out Relations of Ideas.* For A&B, an entailment ($\mathcal{A} \Rightarrow \mathcal{B}$) falls under the umbrella Relation of Ideas and therefore when it is true it is necessary. That is, true logical entailments are necessary. And so you don't go looking out there in the

[3]Strictly speaking, **E** is a modal logic with a definable modal operator approximating the box of **S4**. However, we will consider both using this defined modality and adding primitive modalities.

world to determine which entailments are true, just as any observation of the world cannot reveal anything to be necessary.[4]

An example A&B use to illustrate their point is whether or not the observation that 'Crater Lake is blue' entails that 'it is necessarily possible that Crater Lake is blue' [5, p. 39]. In the logic **S5**, this in-question entailment is a theorem: $\vdash_{S4} \mathcal{A} \to \Box\Diamond\mathcal{A}$. This formula's theoremhood (with respect to $\to$ being entailment) is a mistake according to A&B because it commits a fallacy of modality. It commits this fallacy *even if it is necessarily possible that Crater Lake is blue.* The intuitive idea here is that the contingent fact that Crater Lake is blue does not logically entail that it is necessarily possible that it is so. However, in formulating what is meant, we must be careful. Routley and Routley [100] point out that the condition[5]

$$(*) \qquad \text{If } \mathcal{A} \text{ entails } \mathcal{B} \text{ and } \mathcal{A} \text{ is contingent, then so is } \mathcal{B}$$

is (1) not satisfied by **E** (cf. $p \to p \vee \neg p$) and (2) just plain false . For this second point, Routley and Routley's give an analogy with the fallacy of affirming the antecedent. If $\mathcal{A}$ entails $\mathcal{B}$ is a true entailment, then if $\mathcal{B}$ is contingent, then so is $\mathcal{A}$ (given $\Box$-monotonicity.) But the (*) above reverses the inference, just as in the fallacy of affirming the antecedent.

A&B respond to the Routleys by refining their account, which requires considering more than just what is necessary and contingent. Let's go on to reproduce some of the definitions given by A&B to explain the fallacies of modality. Key to their explanation is the notation of a *necessitive.* This notion is in turn explained by analogy with the more familiar notions of *negative* and *conjunctive.* A proposition $\mathcal{A}$ is negative iff "there is some proposition $\mathcal{B}$ such that $\mathcal{A}$ is equivalent (in the sense of co-entailment) to the denial of $\mathcal{B}$" [5, p. 35]. Similarly, a proposition $\mathcal{A}$ is conjunctive iff there are propositions $\mathcal{B}$ and $\mathcal{C}$ such that $\mathcal{A}$ is equivalent to $\mathcal{B} \wedge \mathcal{C}$. The latter is intuitively limited by A&B to cases where $\mathcal{B}$ and $\mathcal{C}$ are distinct, so that not every proposition is conjunctive.

Following suit, a proposition $\mathcal{A}$ is *necessitive* iff there is a proposition $\mathcal{B}$ such that $\mathcal{A}$

[4]Of course, in science observations are used to support claims of necessity, but one would not say that the observations entailed the truth of the supporting theory.

[5]Conditions (*) and (**) are reproduces in both name and content from J. Alberto Coffa's "Fallacies of Modality" section in Anderson and Belnap's *Entailment* vol. 1 [24] (§22.1.2).

is equivalent to $\Box\mathcal{B}$. That is, such that $\mathcal{A}$ co-entails $\Box\mathcal{B}$. This is different from necessary propositions in that there can be necessary, non-necessitive propositions; non-necessary necessitive propositions; necessary necessitive propositions; and non-necessary, non-necessitive propositions. For example, $\Box p$ for some contingent propositional variable p is false, and thus not necessary; however, it is necessitive as it is equivalent to $\Box\Box p$ in **S4**. One caveat to the definition of necessitives is that true necessitive propositions are also necessary. Further, a *pure non-necessitive* is a non-necessitive that cannot be expressed as a conjunction with a necessitive part. This is to distinguish between a propositional variable p, a pure non-necessitive, from formulas such as $p \wedge (\mathcal{A} \rightarrow \mathcal{A})$, which is a non-necessitive, but not a pure non-necessitive.

A fallacy of modality occurs when it is claimed that a pure non-necessitive entails a necessitive. Thus, the logic of entailment should never contain a formula expressing that an entailment is entailed by a pure non-necessitive. This is the idea behind A&B's analysis of fallacies of modality, and the key issue we will address in axiomatizing identity in **E**. What has been said about these fallacies thus far is sufficient to identify the problematic axioms for identity, but further analysis is required to really solve the problem.

Now, the idea of a pure non-necessitive is developed by J. Alberto Coffa later in volume 1 of *Entailment* (244–252), wherein Coffa introduces the idea of *weak formulas*. Coffa defines a tree structure in order to first identity the *strong formulas* of a system, which are the formulas at least as strong as a necessitive. The exact details of this process are omitted here, but we end up with a division of formulas which captures the right ideas.

> Given a class N of necessitives, we say that S(N) is the class of formulas at least as strong as formulas in N, and that W(N) is its complement. [24, p. 247]

With this distinction in place, we can reformulate (*) as

$$(**) \qquad \text{If } \mathcal{A} \rightarrow \mathcal{B} \text{ and } \mathcal{A} \in \mathrm{W(N)}, \text{ then } \mathcal{B} \in \mathrm{W(N)},$$

which Coffa claims serves as the basis for A&B's identification of modal fallacies.

Now the explanation of Coffa's developments so far have been relevant, but ultimately a bit tangential, to the current topic. However, there is a further distinction made by Coffa that is pertinent to later discussion. Coffa points out that, on his construction, necessitives have the interesting property of being sensitive to the modal logic governing the behavior of $\Box$. Our class of necessitives can be determined within the logic itself, or by external philosophical reasoning. The latter is hopeless, but the former and internal determination of necessitives is mathematically rigorous. We can determine this class of internal necessitives and thus identity which formulas commit fallacies of modality by violating (**).

We can now see why we should worry about RI, RS, and NT; they are of the form of an entailment with an atomic antecedent and an entailment for the consequent. Now, RI and NT have identity statements as antecedent, while RS has a simple predication as an antecedent. This will matter in deciding which to add and under what circumstances.

I am not the first to have these kinds of worries. Mares [69] worries about NT. Kremer is also aware of the potential to create modally fallacious **E**:

> For example, [Mares] recognizes that expressing transitivity as $(x = y \rightarrow (y = z \rightarrow x = z))$ might be a problem in the logic **E**, since in **E** this implies that all identities are necessary. [62, p. 200]

Note that, given our above explication of the fallacies of modality, adding NT only implies that identity statements are necessitive, not necessary. Only true necessitives are necessary, and NT is bound to have false antecedents in some instances, so NT merely implies that identity statements are necessitives. NT is in question then because, according to Mares, "it is not clear that I should take identities to be necessary" [69, p. 2]. Rephrasing in terms of necessitives, we get that it is not obvious that identity statements are necessitive either. This seems just as true, leading to just as strong of worries about NT. Similar worries are then directed towards RI.

To anticipate some answers below, note that it appears that the antecedent of RS is at least sometimes clearly a Matter of Fact and also a member of W(N) and so RS will not be a likely addition to **E**. On the other hand, RI (the permuted version of RS) has an identity statement as an antecedent, which is not clearly (sometimes) merely a Matter of Fact. (Further discussion below of Coffa's internal and external classification of necessitives will bear on this issue.)

We are able to see that the answer to whether or not we can adopt RI or NT without committing modal fallacies depends on whether or not identity statements are necessitives. That is, whether for all $x = y$ there is a $\mathcal{B}$ such that $x = y$ is equivalent to $\Box\mathcal{B}$. This, as we will see, is linked to the question of whether or not identity statements are necessary.

6.6.2 Reasons to Adopt RI and NT

As it turns out, the informal interpretation of identity in relevant logics given by Kremer provides some motivation for thinking about identity as a necessitive and motivates both RI (and NT). I will show how Kremer's informal interpretation of identity makes identity statements necessitive without using the properties of $\Box$. This motivation leads to the adoption of both RI and NT, as will be shown.

Note that Shramko suggests that **E** is the logic in which we should pursue Dunn's Relevant Predication [110, p. 109]. However, Shramko's suggestion is brief and, for example, does not give any details as to which of RS and RI should be adopted in the background of **E** in which they are not equivalent. Much of Dunn's exposition on the project uses RS instead of RI. We will see that in **E** we ought to adopt RI and reject RS, if we choose to adopt either.

Here, however, we will use Kremer's suggested interpretation of identity in relevant logics to motivate both NT and RI independently. That is, the interpretation of identity will motivate these axioms, and produce a coherent system wherein identity is a necessitive. We have already seen that Kremer's interpretation motivates RI and NT, but not RS. What we now need to show is just that identity is not a pure non-necessitive.

Kremer's RI interpretation of identity, explicated in the previous chapter, is that $x = y$ is interpreted as the (infinite) conjunction of biconditions[6] $\bigcap_{G \in RP} Gx \leftrightarrow Gy$, where RP is the set of relevant predicates. For simplicity, we assume that the relevant predicates are monadic. Thus, an identity statement in **E** on this interpretation is a conjunction of entailments. This conjunction of entailments is clearly not purely non-necessitive.

[6] Or an infinite conjunction of conditionals, as the biconditional is just a conjunction of conditionals.

To clarify a bit further, let us add a necessity operator to the language. All entailments are necessary, so we can replace each entailment in $\bigcap_{G \in RP} Gx \leftrightarrow Gy$ with a necessary entailment. Further assume that our box satisfies $(\Box\mathcal{A} \wedge \Box\mathcal{B}) \leftrightarrow \Box(\mathcal{A} \wedge \mathcal{B})$. It follows that our conjunction of entailments is equivalent (entails and is entailed by) $\Box(\bigcap_{G \in RP} Gx \leftrightarrow Gy)$. While this interpretation of identity may involve an infinite conjunction, the interpretation is informal a system formalizing this interpretation would avoid external fallacies of modality according to Coffa, where the external fallacies of modality, explained below, are determined by a set of necessitives given outside the logic plus (**) above. Nevertheless, turning to Kremer's interpretation gives support to adopting both RI and NT and at worst an external judgement of necessitives.[7] Nevertheless, RI and NT have external justification which at least avoids external fallacies of modality.

Thus, the next goal is to show how to extend these external considerations into a system which avoids internal fallacies of modality. We will claim that at least 2 systems are coherent (and begin a formal proof of this claim), and that certain sets of axioms are incoherent, when extending **E**. Further, one coherent system will lack RI and NT, and the other will contain them and a modal operator.

6.6.3 What it takes to coherently adopt RI and NT

To answer is question of what it takes to coherently axiomatize identity in **E** in order to avoid fallacies of modality, we first explicate Coffa's distinction between internal and external fallacies of modality. A system has fallacies of modality if it violates (**) for some set of necessitives. For some logics, the system itself can be used to identify necessitives, in that it either has a modality primitive, a definable modality, or in some cases we could consider extensions with modalities. Let us ignore the latter for now. These necessitives could be used to identity sets W(N), and thus (**) becomes defined using a set of internally accessible necessitives. In fact, this gives us the Coffa's definition of *internal fallacies of modality*, which are committed by systems which violate (**) for the internal necessitives.

On the other hand, an *external fallacy of modality* is an entailment that violates

[7] Given a finite number of relevant predicates and a modal language, we can turn it into an internal judgement.

(**) defined by external considerations.

> In this case one does not go to [the logic in question] in order to find out what things are to be counted as members of N.. . . Whether a system has [external fallacies of modality] or not is a philosophical matter, and therefore one never to be solved to everybody's satisfaction. [24, p. 248]

However, Coffa goes on to say that

> But whether it has [internal fallacies of modality] is a mathematical question that in principle allows an unambiguous answer. [24, p. 248]

Here we show a couple systems which appear to have no internal fallacies of modality.

Before we move away from external fallacies of modality and focus on the internal fallacies, note that whether identity belongs to Relations of Ideas or Matters of Fact appears to be a philosophical question in the philosophy of identity which will not be covered here. However, if identity is found to be necessitive as on Kremer's interpretation, then identity is a Relation of Ideas, and not something to be discovered out there in the world. This at least superficially appears to clash with the seemingly new information that was found when we learned that Hesperus=Phosphorus. So there is interesting philosophical work to be done in selecting an external set of necessitives, and I hope to give here a couple systems with no internal fallacies of modality that roughly correspond to different answers to the external questions.

Now, adding RI and NT to the system **E** (plus the reflexivity and symmetry of identity) appears to avoid external fallacies of modality on Kremer's interpretation of identity, but it is an open question as to whether the system of **E**, the necessity of identity, RI, and NT does in fact avoid fallacies of modality. The axioms avoid these fallacies, but we require a proof that one cannot prove a theorem that commits one of these modal fallacies such as Coffa's proof for propositional **E** [24].

An important fact which will be useful for us is that **E** has a definable modality using the definition $\Box\mathcal{A} =_{df} ((\mathcal{A} \to \mathcal{A}) \to \mathcal{A})$. Using this fact, we will also conveniently ignore extensions of **E** with modal axioms for now. Thus, we focus currently on adding axioms of identity to **E**.

Thus, in order to avoid violating (**) with the addition of RI and NT, we must find or construct a formula $\mathcal{B}$ for every identity statement $x = y$ such that $x = y$ entails and is entailed by $((\mathcal{B} \to \mathcal{B}) \to \mathcal{B})$, which is $\Box\mathcal{B}$ in **E**. As the defined box acts much like the box of **S4** (with some exceptions), an obvious candidate is to choose $x = y$ for $\mathcal{B}$, because this gives us the entailment $\Box(x = y) \to x = y$. Thus, we only need to ensure that we also have $x = y \to \Box(x = y)$. To do this, we will add this formula, **E**'s version of the necessity of identity, as an additional axiom, and then conjecture that the resulting system does not violate (**). A similar move with a non-definable modality can be done, and the T axiom will give the direction of the desired co-entailment that we got for free with the defined necessity operator.

Thus, consider the system **E** with definable $\Box$ extended by reflexivity, symmetry, NT, and RI, together with the necessity of identity.[8] Call this system $\mathbf{E}^{NEQ}$.

Conjecture: $\mathbf{E}^{NEQ}$ does not violate (**) with respect to internal fallacies of modality.

At the time of writing this conjecture remains unproven. Modifying the tree structured defined by Coffa in his proof for **E** to include quantifiers in the obvious way (a universally quantified statement is necessitive iff the statement quantified is necessitive) fails to produce a tree structure that such that the conjunction of its leaves entails the root formula. However, the proof may be possible for an extension of $\mathbf{E}^{NEQ}$ such that if the extension is free of modal fallacies, then so is $\mathbf{E}^{NEQ}$.

Conjecture: **EQ**+REF+SYM does not commit fallacies of modality.

Informally, REF and SYM appear to be innocuous with respect to (**). This noting of the appearance of these formulas clearly does not constitute a rigorous proof. However, note two details. First, this conjecture is a corollary of the previous conjecture. Second, this logic forms the base of many logics with identity. For example, this logic may be extended by Weak or Extensional Transitivity, which also seem innocuous.

We follow these conjectures with some facts.

Fact: Extending **EQ** with RI or NT but without NID results in a system that commits fallacies of modality.

Fact: Extending **EQ** with RS results in a system that commits fallacies of modality.

The proof of the last two facts is obvious.

[8]This is of course with Anderson and Belnap's 14 axiom system for **E**, which includes axiom E7: $(\Box\mathcal{A} \wedge \Box\mathcal{B}) \to \Box(\mathcal{A} \wedge \mathcal{B})$, with the definable $\Box$.

6.7 $\mathbf{E}^{NEQ}$ is Fallacy Free, if EQ is Fallacy Free

Here we make progress in proving that $\mathbf{E}^{NEQ}$ does not violate (**) with respect to any internal fallacies of modality. The proof idea is as follows. First, we make the assumption that **EQ** does not violate (**) with respect to any internal fallacies of modality.[9] Then, to give our conditional results, the general proof idea is to assume for reductio that a fallacious formula is provable in $\mathbf{E}^{NEQ}$, and then transform the proof into a proof in **EQ**. The transformed proof is a proof of a fallacious formula in **EQ**, so the assumption that **EQ** is fallacy free is sufficient to show that $\mathbf{E}^{NEQ}$ is fallacy free.

This section will only deal with internal fallacies of modality, thus violations of (**) are only considered with respect to internal fallacies of modality. Each use of (**) is then with respect to a particular logic, which should be apparent from context.

We will begin with the easy case of $\mathbf{E}^{NEQ} - RI$. This is the logic $\mathbf{E}^{NEQ}$ with RP being an empty set, so we will call it $\mathbf{E}^{NEQ}_{\emptyset}$.

Definition 6.7.1. For every formula $\mathcal{A}$ of $\mathbf{E}^{NEQ}_{\emptyset}$, we define the transformation $\mathcal{A}'$ of $\mathcal{A}$ as the replacements of every identity formula in $\mathcal{A}$ with a biconditional such that:

- $\tau_1 = \tau_2$ is replaces uniformly with $Q^{\tau_1} \leftrightarrow Q^{\tau_2}$.
- Each Q^{τ} is a new zero-ary predicate extending the signature of the logic.

Lemma 6.7.2. *The formulas obtained from transforming Ref, Sym, and NT, and necessity of identity are all theorems of* **EQ**.

Proof. Here we show just the case for NT. One instance of the transformed NT′ is $(Q^x \leftrightarrow Q^y) \rightarrow ((Q^y \leftrightarrow Q^z) \rightarrow (Q^x \leftrightarrow Q^z))$. In this form is it easy to see that the NT′ is a theorem. □

Theorem 6.7.3. *The logic* $\mathbf{E}^{NEQ}_{\emptyset}$ *does not violate (**), if* **EQ** *does not violate (**).*

Proof. For reductio, let us assume that $\mathbf{E}^{NEQ}_{\emptyset}$ violates (**). That means there is a theorem of the form $p \rightarrow \mathcal{A}$, where $p \in W(N)$ and $\mathcal{A} \notin W(N)$. From our assumption, the proof must contain a use of NT, Ref, or Sym.

[9] Coffa [24] proved that **E** was free of modal fallacies, but his proof does not extend easily to the **EQ**. So far my own attempts at proving **EQ** fallacy free have tried to prove a stronger result to the effect that a consequence relation for **EQ** with a strengthened ∀-introduction rule is fallacy free. The general idea of Coffa's proof could then be applied. However, I have no significant results currently.

Consider the transformed formula $(p \to \mathcal{A})'$. Clearly the use of Ref, Sym, and NT (after transformation) are acceptable in a proof in **EQ** (with the extended signature). Thus, we can transform the proof of the formula in $\mathbf{E}_{\emptyset}^{NEQ}$ into a proof in **EQ**. Thus, **EQ** violates (**). But given our assumption, this gives us a contradiction. Therefore, completing the reductio, $\mathbf{E}_{\emptyset}^{NEQ}$ does not violate (**). □

Thus we have completed a conditional proof, making progress towards the conjectures above. Again, with the addition of RI we will find that the finite and infinite cases (for the size of the set of relevant predicates) leads us to two treatments. The finite case here is more satisfying, and the infinite case involves a much more elaborate assumption on our part. Additionally, Kremer's interpretation of identity formulas is helpful.

First, we assume that the set RP is finite. Let us define a new transformation with Kremer's interpretation in mind.

Definition 6.7.4. For every formula $\mathcal{A}$ of $\mathbf{E}^{NEQ}$, we define the transformation $\mathcal{A}'$ of $\mathcal{A}$ as the replacements of every identity formula in $\mathcal{A}$ with a biconditional such that:

- $\tau_1 = \tau_2$ is replaced uniformly with $\bigwedge(G\tau_1 \leftrightarrow G\tau_2)$, for all $G_z \in RP$.

Lemma 6.7.5. *The formulas obtained from transforming Ref, Sym, and NT are all theorems of* **EQ**.

This lemma was shown in the previous chapter for the finite case.

Theorem 6.7.6. *The logic* $\mathbf{E}^{NEQ}$ *does not violate (**), if* **EQ** *does not violate (**).*

Proof. The proof is by transformation of proofs as in the previous case. However, we have not yet shown that the resulting proofs are proofs in **EQ**. In particular, we have not considered the transformation of RI. However, RI is transformed into a large case of the $\wedge$-elimination formula, and is therefore a theorem of **EQ**. Thus, the transformed proof is a proof **EQ**, and our proof is complete. □

So far our progress towards the conjectures above have only required us to assume that **EQ** is fallacy free. However, the case for denumerable RP is not as straightforward. In the finite case we transformed identity statements (1) into necessitives and

(2) into formulas that entailed $Gx \leftrightarrow Gy$ for every $Gz \in RP$. There appears to be no such formula in **EQ**. (Otherwise we could define identity using it.) Thus, consider the extension of **EQ** in which formulas consisting of at most denumerable many symbols are allowed, and proofs are defined accordingly.[10] We assume that this logic, which I have left somewhat vague, does not violate (**). For convenience, let us name this logic $\mathbf{EQ}_{\mathbb{N}}$

We define the transformation as we did in the finite case, which now implies that identity statements are replaced with infinite conjunctions of biconditionals. As can be confirmed by inspection, the transformed identity axioms are all theorems of $\mathbf{EQ}_{\mathbb{N}}$. Thus, assuming that this vaguely defined logic is fallacy free entails that $\mathbf{E}^{NEQ}$ does not violate (**).

The next step is proving the conjectures is to prove that **EQ** does not violate (**), which I suspect is provable. Then, either a different route to the conjecture for infinite RP can be found, or $\mathbf{EQ}_{\mathbb{N}}$ (or something close) can be nailed down and also shown to be free of modal fallacies. Currently, I believe Coffa's proof (which relies on a result from Maksimova which needs to be generalized to **EQ**) can be extended to show a stronger result for **EQ**, namely that if $\mathcal{A} \vdash \mathcal{B} \to \mathcal{C}$, then $\mathcal{A}$ is at least as strong as a necessitive. This would imply the result for **EQ**. Two remarks must be made. The first is that the move to a consequence relation is related to Coffa's trees. A node's children (conjoined) must entail the node.[11] For the case for $\forall \mathcal{A}x$, the most straightforward tree rules do not ensure that the children of a $\forall \mathcal{A}x$ node entail $\forall \mathcal{A}x$. This leads us to the second remark, which concerns the universal generalization rule. By moving to a consequence relation, we can consider $\vdash$ as our notion of entailment that children nodes bear to their parents. That is, for node ψ, $\phi \vdash \psi$, where the conjunction of the ψ's children nodes is ϕ. Thus, by the rule of universal generalization we get that $\mathcal{A}x \vdash \forall x \mathcal{A}$. Thus the resulting tree structures should have this new property, and a corresponding modification of Coffa's proof is nearer to completion.[12]

[10] That is, proofs are finite ordered lists of formulas, each of which follows by the application of a rule from previous formulas. Only the formulas are allowed to be of denumerable size.

[11] For exact details, the reader is referred to Coffa [24]. For now, the details given suffice for the discussion.

[12] The proofs for the consequence relation will also require tracking which premises are used. ADJ poses a problem, as we can take ϕ and conjoin it with a theorem, and then use conjunction elimination to end up with just the theorem. Cases like this should not count ϕ as used in the derivation of the

Thus, we have made progress towards the conjectures, which will demonstrate that there are ways to add NT and RI to **E** without violating (**). This gives us more freedom to define entailment logics with identity.

To conclude the chapter, I have developed a general framework for constructing models for a variety of modal relevant logics with identity. Further, one particular philosophical issue for identity in relevant logics was identified. Exploring this issue, a number of conjectures were made (after our analysis made them seem plausible). Finally, some progress towards proving the conjectures paved the way for future work.

theorem. Thus, there is work to do defining these proofs.

Chapter 7

The Future of Modal Naïve Set Theory and select Applications of Modal Relevant Logics

7.1 Introduction

In this chapter, I aim to accomplish a number of related goals. The first is to solve an open problem in modal naïve set theory, which was posed by Krajíček [58]. This problem is introduced below. The solution of this problem will bring us to a more thorough examination of the routes to triviality found in various modal set theories. This analysis points to an overlap in the types of routes to triviality for both naïve set theory and modal naïve set theory. Substructural logics are capable of treating these routes separately, adopting different solutions to block different routes to triviality. The substructural solutions are motivated separately and not merely *ad hoc* solutions. The modal aspect of modal naïve set theory is also motivated separately. Given these motivations, and given the analysis of the routes to triviality for modal naïve set theories, I set out a number of conjectures about the power and non-triviality of mixed strategies, which employ both substructural logics and a modal comprehension axiom. This leads us to the second goal of the chapter.

The logics defined in the previous chapters have the expressive power to serve as the base for such a mixed approach, being modal extensions of substructural first order logics. The second goal of this chapter is to outline the application of these logics in the mixed approach to naïve set theory. In the previous chapters, semantics were constructed for these logics. It is the hope that these models can be used in the search for consistency proofs for various set theories in the mixed approach.

The final goal of this chapter is to discuss a couple not-necessarily-non-substructural approaches to set theory for which the preceding chapters may prove useful. There are many ways to explore modal set theory, even after the results of Fritz et al. [40], and the results of this chapter. One of which is the mixed approach. Another which I find appealing is the adoption of a regular, but non-normal, modal logic as base. In particular, my interest lies in the result of giving up the rule of necessitation. Regular model logics lack the rule of necessitation, and so we will look at the possibliity of modal naïve set theories based on regular modal logics. We will call these regular model naïve set theories, and trust that the reader will not confuse them with the regular theories used in the construction of canonical models. These regular naïve set theories will be suggested in some detail in this chapter. There are other routes which may lead to non-trivial systems such as dropping the T axiom or finding other formulas that capture the gist of the intuition behind naïve comprehension. These other theories will be briefly only mentioned, if at all.

Thus the chapter is split into two main sections following this introduction. The first provides the bulk of the chapter. Here, I introduce modal naïve set theory, motivate the modal approach in general, set up and solve Krajíček's open problem, show that other modal set theories have similar routes to triviality, and then conjecture that mixed approaches constitute an interesting line of future research. This section is intended to be able to stand alone, and it shares its title with this chapter. The next section contains some details of future research projects in non-standard set theory. My main hope is to show the plausibility of theories which reject necessitation.

7.2 The Future of Modal Naïve Set Theory

7.2.1 Introduction

Modal naïve set theory, a type of set theory axiomatized in quantified modal logic, adds one or more modal operators to the naïve comprehension scheme. The resulting comprehension scheme preserves something of our intuitions found in naïve set theory; however, the triviality results for many modal set theories are discouraging. Fritz et al. [40] demonstrate the triviality of many modal set theories, and the relative weakness of some consistent modal set theories.[1] Fritz et al. leave open the question whether or not the modal set theory set out by Krajíček in [58] is consistent in **KT** (also known simply as the normal modal logic **T**). Krajíček [59] had shown his theory trivial in **S5**. Here, I prove that Krajíček's set theory is also trivial in **KT**. Finally, given all the discouraging results for modal set theory, and given extant consistent non-modal naïve set theories, I believe the future of modal set theory should employ substructural logics. I will suggest some non-*ad hoc* reasons to explore these theories.

First, modal naïve set theories will be explicated in section 7.2.2. The naïvety of these theories will be assessed, motivating the adoption of some modal comprehension axiom. Then, in section 7.2.3, the open problem of the consistency of Krajíček's set theory will be posed, and in the following section the theory will be proven trivial by a modal version of Curry's paradox.[2] In the final section, I discuss the possibility of some consistent theories, and show how an analysis of the problems facing naïve set theories gives rise to the motivation for modal comprehension axioms in substructural logics.

7.2.2 Modal Set Theory

Modal set theories are perhaps best understood as retaining some of the intuition of naïve set theory, as making the comprehension axiom essentially modal is often done as an attempt to avoid restricting the kinds of conditions or predications which can determine sets. The naïve comprehension axiom

$$\exists y \forall x (x \in y \leftrightarrow \mathcal{A}(x)) \tag{Comp}$$

is known to lead to triviality by a variety of routes. Here, and in what follows, the notation $\mathcal{A}(x)$ is used to indicate a formula in which x may, but need not, occur free. Consequently, x may or may not occur free in the right-hand side of the formula in naïve set theory. Often a naïve abstraction axiom

$$x \in \{z : \mathcal{A}(z)\} \leftrightarrow \mathcal{A}(x) \tag{Abs}$$

[1] Generally, when a set theory is weak, this means that it is not useful in a practical sense for many of the applications for which set theory has found itself playing a star role. E.g., in mathematics.

[2] Here I refer to Curry's original, syntactic paradox set out for combinatory logic [25, 26], and not the related semantic paradox. In this, I follow Meyer, Routley, and Dunn's presentation of the paradox in [84].

is used in the presence of a set forming operator, when the uniqueness of the set formed is provable from the extensionality axiom. Note, however, that some (e.g. [95, 94, 92]) have used the two halves of Abs for naïve comprehension in systems which don't contain conjunction, and so cannot define the biconditional.

Quine's New Foundations (NF) and the familiar Zermelo-Fraenkel set theory with the axiom of choice (ZFC) restrict what can be substituted for $\mathcal{A}$ in this axiom scheme. Modal set theory, on the other hand, preserves our intuitions that any and every condition determines a set, but the emphasis is placed on *how* an object satisfies the condition, or satisfies the 'is an element of' relation. Consider the modal comprehension axiom

$$\exists y \forall x (x \in y \leftrightarrow \Box \mathcal{A}(x)).^{3} \tag{Comp$\Box$}$$

Fritz et al. analyze modal comprehension axioms like this as expressing that "for *every* condition, [this] comprehension axiom will assert the existence of a set containing all and only the sets satisfying that condition *in a special way*" [40, p. 22]. The modal operator in (Comp$\Box$) can be interpreted in numerous ways, as suggested by Fritz et al. Nevertheless, (Comp$\Box$) preserves something of our naïve intuition.

Here I am perhaps fighting an uphill battle, arguing that most are mistaken as to what *the* naïve comprehension axiom/intuition is. However, there are reasons to doubt the non-modal axiom correctly formalizes our intuition. Furthermore, there is another intuitive aspect of a set that requires modality to fully express.

Beginning with the reasons to doubt the non-modal axiom, we first note that Fitch, in [39], explains the modal aspect of his modal set theory as follows:

> It was not originally intended that the desired system should be a modal logic, but the modal character of the system appears to be a natural outgrowth of the way it is constructed. [39, p. 93]

As this seems to be the only explanation Fitch offers, it leaves open the possibility that Fitch's original intuition was essentially modal; however, going that far would be mere speculation. Accordingly, note that the mere fact that modality emerged in

[3] Note that with the multiple places you could add a modality to the naïve axiom, there are at least as many modal comprehension axioms to choose from as there are modalities in the background modal logic.

Fitch's theory leaves open the possibility that the theory's construction was guided by a modal intuition.

Next, it is obvious that expressing an intuition or piece of natural language in formal logic requires sufficient machinery to fully capture the meaning. If one was working in classical logic and did not have modalities, then the typical naïve comprehension axiom would then be a good approximation of several modal naïve comprehension axioms. Thus, if our intuitions were essentially modal, we would expect the classical approximation. Now this does not demonstrate the modal nature of our intuitions of set theory, and neither does Fitch's theory, but it seems enough to cast doubt on the non-modal naïve comprehension axiom.

To defend modal set theory, the issue of how a set of elements is bound together as one object can be used to show that the *how* is essential to the intuition.

To demonstrate this, let's consider what the comprehension axiom is. It seemingly links a set with some condition in some way. Weber argues that the way it links a set to a condition is by an intension.

> The full naïve set concept has an intensional aspect. A set is any collection of objects, itself an object. Intension is what binds the set and makes it an object in its own right. . . . Intensional comprehension explains 'how a many becomes a one,' which extensionality alone leaves mysterious. [118, p. 89–90]

Here, Weber's 'intension' is an intensional conditional, but it seems this point can be extended to the modal operator in, for instance, Comp$\Box$. That is, the binding intension of a set is that all the elements in the set satisfy a condition *in some special way*, and without this special way, the set is not sufficiently determined.

Weber uses Cantor to support this claim for naïve set theory. On this concept of sets, a set is a collection of every object that satisfies some intension, which we have interpreted as including both the satisfying of a condition/predication as well as doing so in a particular way. This is what a comprehension axiom should be, what it should express.

Among many of the axioms considered, Fritz et al. [40] explore the modal comprehension axioms $\exists y \forall x(x \in y \leftrightarrow \Box \mathcal{A}(x))$ (Comp$\Box$) and the stronger

$$\exists y \forall x \Box(x \in y \leftrightarrow \Box \mathcal{A}(x)), \tag{$\Box$Comp$\Box$}$$

and find that the former produces a relatively weak, consistent theory, while the latter trivializes in the logic **KT**. Fritz et al. suggest that "a reasonable response to the results of [their] article might be to investigate [Krajíček's modal comprehension axiom] with renewed energy" [40, p. 27]. Being such a response to their article, this paper contains discouraging results, but ultimately motivates entirely novel investigations.

7.2.3 Setting up the Problem

There are inconsistency results for some modal set theories in **KT**, which is a relatively weak normal modal logic. The open question is whether or not Krajíček's modal comprehension axiom is consistent in **KT**. To answer this question, first let's define two quantified modal logics based on **KT**. The quantified modal logic used by Fritz et al. has a first-order language with a denumerable set of variables, two binary predicates $=$ and $\in$, the connectives $\neg$ and $\wedge$, and the universal quantifier. We will also assume a denumerable set of constants is available to us. Other connectives, quantifiers, and the shorthands $\neq$ and $\notin$ are taken to be defined as usual. They include a definable existence predicate by $\mathcal{E}x =_{df} \exists y(y = x)$. Where Λ is a normal modal propositional logic, the (actualist) quantified modal logic based on Λ, $\mathcal{Q}\Lambda$, is given by the following axiomatization:[4]

[4]Here I have taken the liberty of renaming some of the axiom schemes and rules.

(Λ)	any substitution instance of a theorem of Λ
$(\forall 1\mathcal{E})$	$(\forall x\mathcal{A}x \wedge \mathcal{E}y) \to \mathcal{A}[y/x]$
$(\forall \to \text{dist})$	$\forall x(\mathcal{A} \to \mathcal{B}) \to (\forall x\mathcal{A} \to \forall x\mathcal{B})$
(VQ)	$\mathcal{A} \to \forall x\mathcal{A}$, where x is not free in $\mathcal{A}$
$(\mathrm{U}\mathcal{E})$	$\forall x\mathcal{E}x$
(Refl)	$x = x$
(Sub)	$x = y \to (\mathcal{A} \to \mathcal{B})$, where $\mathcal{B}$ is $\mathcal{A}$ potentially with some free y in place of free x
(LN1)	$x \neq y \to \Box(x \neq y)$
(MP)	$\dfrac{\mathcal{A} \qquad \mathcal{A} \to \mathcal{B}}{\mathcal{B}}$
(Nec)	$\dfrac{\mathcal{A}}{\Box\mathcal{A}}$
(UG)	$\dfrac{\mathcal{A}}{\forall x\mathcal{A}}$
$(\mathrm{UGL}\forall^n)$	$\dfrac{\mathcal{A}_1 \to (\Box(\mathcal{A}_2 \to \cdots \to \Box(\mathcal{A}_n \to \Box\mathcal{B})\ldots))}{\mathcal{A}_1 \to (\Box(\mathcal{A}_2 \to \cdots \to \Box(\mathcal{A}_n \to \Box\forall x\mathcal{B})\ldots))}$, where x is not free in $\mathcal{A}_i$, for any $1 \leq i \leq n$.

The usual restrictions to these rules are included. Note that this axiomatization typically corresponds to a variable domain, actualist interpretation of quantified modal logic. This is done on purpose by Fritz et al. to ensure that neither the Barcan Formula nor its converse are valid.

On the other hand, the problem originally posed by Krajíček used a quantified modal logic with the full universal instantiation axiom (UI) $\forall x\mathcal{A} \to \mathcal{A}[y/x]$ where y is free for x in $\mathcal{A}$. In addition, Krajíček has the Barcan formula, and from UI we can derive the converse Barcan formula. Where Λ is a normal modal propositional logic, the (possiblist) quantified modal logic based on Λ, $\mathbb{Q}\Lambda$, extends $\mathcal{Q}\Lambda$ with UI and the Barcan Formula.

7.2.4 Krajíček in KT

To solve the open problem of the consistency of the Krajíček comprehension axiom, let us first construct a corresponding abstraction axiom. Krajíček's comprehension axiom

$$\exists y\forall x((\Box x \in y \leftrightarrow \Box\mathcal{B}(x)) \wedge (\Box\neg x \in y \leftrightarrow \Box\neg\mathcal{B}(x))), \qquad \text{(KCA)}$$

was introduced in [58]. Semantically, an existentially quantified formula is true when there is an object in the domain of which the formula is true. Given the content of the axiom, this object is a set. Let us use a set forming operator to name the set satisfying the existential. The notation for the set forming operator we will use is new, because the non-modal or usual notation does not bear the modal weight of the Krajíček comprehension axiom. Therefore, we will use the set forming operator notation $\{z :_{\Box}: \mathcal{B}(z)\}$ to denote the set satisfying KCA. Here, we have $\forall x((\Box x \in \{z :_{\Box}: \mathcal{B}(z)\} \leftrightarrow \Box\mathcal{B}(x)) \wedge (\Box\neg x \in \{z :_{\Box}: \mathcal{B}(z)\} \leftrightarrow \Box\neg\mathcal{B}(x)))$. Given a full Universal Instantiation (UI) axiom, we then are left with an abstraction axiom of the form

$$(\Box x \in \{z :_{\Box}: \mathcal{B}(z)\} \leftrightarrow \Box\mathcal{B}(x)) \wedge (\Box\neg x \in \{z :_{\Box}: \mathcal{B}(z)\} \leftrightarrow \Box\neg\mathcal{B}(x)). \tag{KAA}$$

Note, however, that Fritz et al. do not use a logic with UI, but rather an actualist universal elimination axiom. On the other hand, Krajíček uses UI.

The Krajíček abstraction formula just shown can be translated, taking the box to stand for "determinately", as

> something is determinately a member of the set determined by a condition if and only if it determinately satisfies the condition and something is determinately not a member of the set determined by a condition if and only if it determinately does not satisfy the condition.

Note that "determinately" is merely a placeholder for whichever interpretation of $\Box$ is most apt. Thus, the Krajíček abstraction formula is to the non-modal abstraction formula as the Krajíček comprehension axiom is to the non-modal comprehension axiom. As a corollary of the above reasoning, given the new set forming operator and the rule UI, the Krajíček abstraction formula is equivalent to the Krajíček comprehension axiom.

Krajíček demonstrates that the usual Curry's paradox is not derivable for the set $a = \{y :_{\Box}: y \in y \to \mathcal{A}\}$, and in fact shows that the system is consistent for all instances of the modal comprehension axiom with non-modal conditions [58, Corollary 8.8]. However, the version of Curry's paradox that we give here has a modal condition. This version of Curry's paradox is derivable from the formula $(\Box a \in a \leftrightarrow \Box(\Box a \in a \to \mathcal{A})) \wedge (\Box\neg a \in a \leftrightarrow \Box\neg(\Box a \in a \to \mathcal{A}))$, where $a = \{z :_{\Box}: \Box a \in a \to \mathcal{A}\}$, which is an instance of the Krajíček abstraction axiom. This is recorded in the following theorem.

Theorem 7.2.1. *Krajíček abstraction formula trivializes in* $\boldsymbol{KT}$ *(with UI).*

Proof

(1)	$\vdash (\Box a \in a \leftrightarrow \Box(\Box a \in a \rightarrow \mathcal{A})) \wedge (\Box\neg a \in a \leftrightarrow \Box\neg(\Box a \in a \rightarrow \mathcal{A}))$	Krajíček Abstraction
(2)	$\vdash \Box a \in a \leftrightarrow \Box(\Box a \in a \rightarrow \mathcal{A})$	(1), $\wedge$-Elim
(3)	$\vdash \Box a \in a \rightarrow \Box(\Box a \in a \rightarrow \mathcal{A})$	(2), $\wedge$-Elim
(4)	$\vdash \Box a \in a \rightarrow (\Box a \in a \rightarrow \mathcal{A})$	(3), T, Trans
(5)	$\vdash \Box a \in a \rightarrow \mathcal{A}$	(4), Contraction
(6)	$\vdash \Box(\Box a \in a \rightarrow \mathcal{A})$	(5), Necessitation
(7)	$\vdash \Box a \in a$	(2), (6)
(8)	$\vdash \mathcal{A}$	(5), (7), MP

The premise of the proof is the Abstraction axiom which we derived using UI, and only the first conjunct is needed for the proof. Note that the set abstraction operator is not required, as the constant a only needs to satisfy the existential quantifier in the chosen instance of KCA. We take the left-to-right direction of the biconditional, and using the transitivity of the arrow we get a formula we can apply contraction to. The necessitation of this formula and the other direction of the biconditional gives the formula on line 7, which we can use with modus ponens to derive $\mathcal{A}$, for arbitrary $\mathcal{A}$. Thus the proof is just like the non-modal Curry's paradox, but requires one application of the T axiom, and one application of the necessitation rule.

Krajíček has proposed KCA as a route to modal set theory, and proved it trivial in **S5**. It was left as an open problem whether or not it was consistent in weaker logics, and in particular in **KT**. Here, I have shown that the KCA is trivial in **KT**.

However, this is not the end for Krajíček's axiom. The alternative formulation of the question where the background logic is actualist, as in Fritz et al., remains open. The route to Curry is possible under the actualist interpretation, for example, if one adopts a particular kind of serious actualism wherein, for every formula $\mathcal{A}$, every set a is such that $\mathcal{A}a \rightarrow \mathcal{E}a$. Serious actualism is the view that only objects that exist at a world may have properties in that world, and some have argued that serious actualism follows from actualism [10]. Thus, if the Krajíček comprehension axiom happens to be consistent in $\mathcal{Q}$**KT** (without UI), a response to the actualism/serious actualism debate is warranted. On the other hand, looking to substructural logics might produce consistent theories, with UI and with the modal fragment of **S5**. The next section looks forward to a substructural approach to modal set theory.

7.2.5 Outline of the Future of Modal Set Theory

It appears as if many identified modal set theories are either trivial or are not very useful. This need not be discouraging. To show this, let's begin with another triviality proof for a system already proven trivial. That is, I will show that some modal set theories—in particular ones with the rule of necessitation and the axiom T—suffer from the same problems afflicting non-modal naïve set theories in order to motivate a mixed approach to naïve set theory.

Fritz et al. show that ($\Box$Comp$\Box$) is inconsistent with **KT** [40, p. 34–5]. The proof of this fact, taking $*$ to denote any modality in **KT**, involves inferences that are irrelevant according to some substructural logics. For example, take the inference from $\neg\Box(y \in y \leftrightarrow *y \notin y)$ and $(\forall x\Box(x \in y \leftrightarrow *x \notin x) \wedge \mathcal{E}y) \rightarrow \Box(y \in y \leftrightarrow *y \notin y)$ to $\mathcal{E}y \rightarrow \neg\forall x\Box(x \in y \leftrightarrow *x \notin x)$. This leads to triviality *because the logic is not paraconsistent* and the fact that $\exists y\forall x\Box(x \in y \leftrightarrow *x \notin x)$, labeled as ($\Box$ Russell$*$) by Fritz et al., is derivable. That is, the problem we run into here is that a formula and its negation are derivable. Those who defend naïve set theory are wont to adopt paraconsistent logics based on the Russell paradox, and a similar move here would take away the significance of the derivability of such a contradiction by blocking the move from there to triviality.

Not only do we have a modal paradox with similarities to Russell's paradox, but we can also derive a modal Curry's paradox to demonstrate another route to triviality for this modal set theory. For the following proof, we use the abstraction axiom corresponding to ($\Box$Comp$\Box$). More will be said about this below. By choosing the usual Curry set, which we will call a, we derive the abstraction axiom and then get the following:

Proof

(1)	$\vdash \Box(a \in a \leftrightarrow \Box(a \in a \rightarrow \mathcal{A}))$	Abstraction
(2)	$\vdash (a \in a \leftrightarrow \Box(a \in a \rightarrow \mathcal{A}))$	(1), T, MP
(3)	$\vdash (a \in a \rightarrow \Box(a \in a \rightarrow \mathcal{A}))$	(2), $\wedge$-Elim
(4)	$\vdash (a \in a \rightarrow (a \in a \rightarrow \mathcal{A}))$	(3), T, Trans
(5)	$\vdash (a \in a \rightarrow \mathcal{A})$	(4), Contraction
(6)	$\vdash \Box(a \in a \rightarrow \mathcal{A})$	(5), Necessitation
(7)	$\vdash a \in a$	(2), (6)
(8)	$\vdash \mathcal{A}$	(5), (7), MP

Fritz et al. may have stopped at the Russell's paradox, as it demonstrates triviality, given their choice of logic, but knowing that the Curry's paradox is derivable in this set theory informs us that possible solutions here must drop contraction and go paraconsistent. That is, some naïve modal set theories suffer from the same kinds of problems as their non-modal counterparts.

Using necessitation and the T axiom, many of the problems for naïve set theory seem to present themselves.[5] I suggest the adoption of a mixed strategy for modal naïve set theory. That is, we should adopt a substructural modal logic base with a modal comprehension axiom. We have seen that the modal approach is motivated by the essential modal nature of our intuitions. I will not argue here that the substructural approach is also motivated. Nevertheless, the substructural approach produces consistent set theories.

There are consistent modal set theories based on classical logic, and there are

[5]One may be temped to say that something is a member of a set *in a special way* if and only if it satisfies a condition. However, we can verify that a corresponding Comprehension scheme

$$\exists y \forall x(\Box x \in y \leftrightarrow \mathcal{A}(x)) \qquad (\Box(Comp))$$

is easily trivialized by Curry's paradox. Here, we get $\Box a \in a \leftrightarrow (\Box a \in a \rightarrow \mathcal{B})$, which is trivializable without using Nec or the T axiom.

non-trivial non-modal set theories using substructural logics (e.g. Brady's [17].) The problem of finding the right logic and the problem of expressing the naïve intuition are related, because the logic must be able to express the intuition. However, the problems are distinct in that they only overlap when it comes to expressive power, but not necessarily the deductive strength of a logic. Thus, arguments motivating the deductive strength of weaker implicational fragments can be combined with arguments for modal comprehension axioms. Doing so should create non-*ad hoc* support for the mixed approach.

The *optimistic conjecture* is that the mixed strategy will produce consistent modal set theories whose non-modal counterparts are trivial. If theories are constructed that confirm this conjecture, then these theories will gain the benefit of being stronger than currently known consistent substructural set theories while enjoying the motivation of modal naïve comprehension axioms.

On the other hand, if the *pessimistic conjecture* is true, that the substructural logics used in constructing consistent modal naïve set theories are just the logics consistent with the naïve comprehension axiom, then we can hope to gain a better understanding of comprehension axioms. Moreover, if our intuitions correctly justify a modal comprehension axiom, logics employing such axioms will be better motivated than their counterparts.

7.2.6 Concluding Remarks

The mixed strategy is of philosophical interest due to the variety of problems of naïve set theory and the essentially modal intuitions we are trying to express with a naïve theory. I believe that switching to a modal comprehension axiom will not trivialize certain consistent substructural theories. If that is right, we can take substructural solutions to the problems of naïve set theory and modalize the comprehension axiom to better express our intuitions.

It remains an open question whether or not we can produce models to prove the above conjectures. There are philosophical motivations for attempting to answer these questions beyond mere curiosity and the seeming lack of any work on such a mixed strategy. Thus, the triviality of KCA in **KT** and many of the triviality results of Fritz et al. [40] need not be discouraging. These trivial theories suffer from many of the same problems of their non-modal counterparts, which have found solutions in substructural logics. There is hope that a mixed strategy will produce stronger, better motivated theories than either single strategy alone.

7.3 Other Applications

The following sections briefly discuss other attempts at set theory for which the material of the previous chapters will be useful.

7.3.1 Non-Naïve Relevant Predicate Set Theory

Dunn [29] contains a general line of proposals for set theory that offer different restrictions to the comprehension axiom. In particular, these proposals would use a restriction based on relevant predication that would aim to fix the treatment of identity, presumably as defined in second order logic.[6] The problems identified for identity included the fact that many treatments seemed to unavoidably validate $x = y \to (\mathcal{A} \to \mathcal{A})$. The proposals to solve this problem come (mostly) from conversations with Belnap, Meyer, and Urquhart, and Dunn summarizes them as follows:

> Thus it was proposed that atomic formulas should always be counted as determining (relevant) properties, and that compound formulas should be counted as determining properties only when they met certain restrictions about "dependence" on their free variables. The actual detailed restrictions varies, the action centering around conjunction (and disjunction). [29, p. 370]

Thus, the proposals differed in what ought to be taken as a relevant property. Dunn offers a relevant comprehension axiom wherein the right-hand side condition is restricted to relevant properties.

While the work of the previous chapters does not decide between the proposals concerning relevant properties, and while I have not argued for a particular base set of relevant predicates or even the inclusion of particular predicates in the set, the semantics of the previous chapters can be used to better analyze the proposals and resulting set theory.

[6] Dunn gives the comprehension axiom in second order logic: $\exists F \forall x (Fx \leftrightarrow \phi x)$.

The restriction of the conditions to relevant properties in the comprehension axiom is argued against earlier this chapter. In particular, by the motivation for adopting modal naïve set theory. However, the modal part of the theory need not exclude a restriction of the conditions to relevant properties. In fact, if the modal character of the comprehension axiom is justified by our intuition, we may still expect any restriction on the conditions. The treatment of identity (perhaps with modification) above may then be useful in modeling these set theories, depending on the underlying logic. These possibilities are all marked as possible future research, building on what has been done in this document.

7.3.2 Regular Modal Set Theory

When I presented section 7.2 at the Society for Exact Philosophy (SEP) in Toronto on May 17, 2019, I emphasized that "the" future of modal naïve set theory I was describing way really the research that I had plans to work on. These plans were the mixed approaches described above. Several other directions of research were mentioned, and I paid particular, but brief, attention to regular modal logics because (1) all the Curry results that I have shown in detail involve necessitation, and (2) necessitation is seemingly invalid on at least one motivated interpretation of the box operator for modal set theory. Discussion with Philip Kremer at the SEP lead to the addition of this section to the chapter. This is because the reasons listed above motivate the regular modal set theories as an interesting line of research in the future of modal set theory, and therefore also justifies adding it to my research plans. This section in included in this chapter now because it does appear to be an interesting line of research in the future of modal naïve set theory, and I have been convinced to include it in my own research plans. This section as written, however, is only aimed at convincing the reader that this line of research is in fact interesting and plausible.

Let's first examine what the rule of necessitation will generally say, keeping the interpretation of the box operator largely unspecified. Then the rule of necessitation says that if something is provable in the set theory axiomatization, then it is "true in a special way". It is obvious that being provable does not entail being true in every special way. So there are ways for things to be true such that a sentence can be provable but not true in that way. (A toy example might be the negation operator.)

The goal is to motivate an interpretation of the box which is one of these ways, and such that it gives a decent interpretation of the comprehension axiom. Finding such an interpretation will philosophically motivate the project, but is left for future research. However, we will consider a sample interpretation, the problems of which will be noted.

Let's use an interpretation of the box which is described in Fritz et al. as an interpretation on which the T axiom appears to be false. On this interpretation, the set theory being constructed is a sort of fiction, and the box operator means "true in the fiction". Let's examine what happens when we look at the rule of necessitation under this interpretation.

On the fictionalist interpretation, sets are objects of a fiction, and can only be said to exist in the fiction.[7] If a set has a property in a fiction, then it cannot also have that property out of the fiction, because it does not exist outside the fiction [40, p. 24].[8] We will motivate regular modal set theories on this interpretation.

Intuitively, the necessitation rule appears invalid on this interpretation. Just because something exists and has a property does not entail either that it exists in some fiction or that it has that property in the fiction. For one thing, fictional works are sometimes incomplete theories, in which not every expressible sentence is given a truth value. So intuitively the fictionalist approach does not seem to motivate the rule of necessitation.[9] However, an example of an object satisfying a property, but either not existing or not satisfying the property in the fiction of set theory is desirable. Further, not satisfying a property in actuality and not in a fiction is insufficient, as we only want to consider the properties that it is provable that an object satisfies. This offers a harder challenge. We will thus consider the possibility of something existing in actuality, but not in the fiction.

Consider a formula $\neg\exists y(a \in y)$, for some actual object a. A fictionalist should endorse this formula as actually true, because to them no set exists in actuality.[10] All

[7] For fictionalist in the philosophy of mathematics, Fritz et al [40] provide a short list of references, among which is an overview by Balaguer [7].

[8] This may also motivate an actualist interpretation of quantified modal logic.

[9] Technically, while $\Box\mathcal{A} \vee \Box\neg\mathcal{A}$ is not provable in general from $\mathcal{A} \vee \neg\mathcal{A}$, the formula $\Box(\mathcal{A} \vee \neg\mathcal{A})$ is (by necessitation). Thus one could hold that a fiction is not complete in the usual sense, but rather pseudo-complete, in the sense that every situation in the model describing it (accessible via $S_\Box$) is complete.

[10] A fictionalist may in fact claim that the formula is nonsense or a category mistake, for the set-theoretic inclusion relation can only be true or false of fictional objects. This is a serious objection

models should make it actually true, and so it ought to be provable in a fictionalist set theory.

Clearly, this sentence is false in the set theoretic fiction, and a counterexample to the rule of necessitation.

Whether or not the rejection of the rule of necessitation on grounds of some informal interpretation of the modal operator is motivated, such motivation is not the only applicable kind of judgement of set theories. We are also interested in the practical usage of set theories. In particular, we are interested in how much of mathematics can be recovered in non-standard set theories. This is a question of what formal results are provable in these systems. The standard set theories can be accused of only having, at some foundational level, *ad hoc* justifications stemming from the triviality results of naïve set theory.[11] Thus, the regular modal set theories are interesting from a purely formal perspective (in addition to the philosophical interest of finding further non-trivial systems with a naïve comprehension axiom).

I take it to be at least plausible that regular modal logics should serve as the basis of modal set theory, whether the sentential part of the logic is relevant, classical, or otherwise. One possible route to explore is regular substructural set theories. Thus, we extend the conjectures of the main section in this chapter to include logics whose modal fragments are regular. Such work is left as future research projects.

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to the motivation of regular logics considered, but will not be fully addressed here. The aim is to provide a plausible motivation for a line of formal research. Either an adequate response can be found, another motivation can be given (perhaps using a different interpretation of the modal operator), or the results of the future formal research won't be altogether that interesting.

[11] If the reader is disposed to find, say, the iterative conception of set theory a non-*ad hoc* justification of a standard set theory, I challenge the reader to provide the necessary justification that such a foundation defines "set" and not some other notion.

Chapter 8

Conclusion

To conclude, I briefly summarize the previous chapters, situate the dissertation more broadly in the literature, and say a bit more about potential future projects.

In the preceding chapters, I have accomplished my goal of constructing a general framework for quantified modal relevant logics. For a wide range of relevant logics, their quantified modal extensions with and without identity have been given semantics. Furthermore, the semantics constructed is one unified framework, with the exception of identity, for which a number of approaches were considered.

The main results of the dissertation and soundness and completeness results for the general semantic framework for a wide range of relevant logics, the extension of the Mares and Goldblatt style semantics to weaker quantified relevant logics, the adequacy of defining an identity predicate in first order relevant logics when given a finite set of relevant predicates, the soundness and completeness (using frame conditions) for logics with identity and infinite relevant predicates, the conditional proof that **E** with NT and RI is fallacy free, and solving the open problem of the consistency of Krajíček's axiom in **KT**. The general semantic framework I've developed here should be useful for many applications, some of which I have already provided some detail for, and should be extendable to other logics. Let's situate this dissertation while discussing potential extensions/applications.

Algebras

The semantics given here is a general frame semantics. General frames, while having many of the advantages of frames, also benefit from being closely related to algebras. There are many well known results about the relation between propositional logics and algebras. Stone [112], with his representation theorem, showed that Boolean algebras are isomorphic to a field of sets (or set algebra). Jónsson and Tarski [56, 57] extend Stone's results to modal logics — i.e., Boolean algebras with operators. Similar results can be found, for instance, in Priestly [89] and in Dunn's representation theorem for distributoids found in his gaggle theory [33] (also found in Dunn [34] and Bimbó and Dunn [15], and Bimbó [12]). Rasiowa and Sikorski [90], Dunn and Hardegree [35] are examples of algebraic approaches to logic.

One of the benefits of the algebraic approach is the construction of frames from the logic's algebra. The link between logics and algebras has also given rise to links between logic and topology, for example see Johnstone [55]. For general frames, the relation is even more interesting, for it is often the case that the dual of a general frame is an algebra, and the dual of an algebra is a general frame. (For certain classes of general frames, a frame is isomorphic to its bidual.) Thomason [113] introduced the idea of general frames, and he [114] proved some duality results with some modal algebras. Goldblatt [46] proved a full duality between descriptive general frames and modal algebras. Moving on to relevant logics, duality results for modal relevant logic were given by Seki [102, 103].

When we add quantifiers into the mixture, the corresponding algebras are rather complex. In quantified classical logic, for example, there are cylindrical algebras [52, 53], and *polyadic algebras* [50]. Meyer, Dunn and Leblanc [78] have an algebraic semantics for **RQ**. For a more detailed history of algebras and general frames, the reader is directed to Blackburn, de Rijke, and Venema [16], where many of details just given are expanded upon.

Hence, the general frame semantics of this dissertation should lead to a dual algebra (which, as I noted earlier with reference to Mares and Goldblatt's mention of Halmos, ought to be closely related to polyadic algebras). This future project extends the work done in this dissertation. Further, it will open up more possibilities for quantified modal relevant logics. For example, algebraic methods and results will become available.

Gamma

In the history of relevant logics, γ admissibility has been an interesting issue. A detailed history of the rule in relevant logics can be found in Urquhart [116]. Anderson and Belnap defined the logic **E**, in part, by deleting the rule γ, from $\vdash \neg\mathcal{A} \vee \mathcal{B}$ and $\vdash \mathcal{A}$ to infer $\vdash \mathcal{B}$, from Ackermann's "Strenge Implikation" [1]. It was posed by Anderson [2] as an open question. As Urquhart [116] makes clear, from the proof of the antecedents of the rule, it is not obvious how to prove the consequent. Using algebraic methods, Meyer and Dunn [82] were able to solve the open problem in 1968. A second solution using the ternary relational semantics if given by Routley and Meyer in 1973 [98]. Then, a solution using metavaluations is given by Meyer in 1976 [80]. Seki [105, 106, 107] gives proofs of γ-admissibility using a variety of methods. For **RQ**, the admissibility proof is given in Meyer, Dunn, and Leblanc [78]. Some other results include Mares and Meyer [76].

The open questions, and lines of future research, suggested here are to prove γ-admissibility, if possible, for each of the quantified modal relevant logics considered in this dissertation. The methods for proving γ-admissibility as well-established, and finding a dual algebra for the general frames opens up the algebraic method.

Quantified Modal Logic

The difficulties surrounding completeness for quantified modal logic are excellently summarized by James Garson [42] (or in [43]). There are many ways of defining quantified modal logics, and this dissertation has dealt with only one, which could be given a constant domain semantics. For some approaches to quantified classical modal logics, one can see Hughes and Cresswell [54], Goldblatt [47], Garson [42]. There is much philosophy surrounding the differences in semantics and syntax for various quantified modal relevant logics. For example, see Routley [96], Lewis [64, 63], Williamson [119], Hazen [51], Linsky and Zalta [65] and countless others.

Modifying the semantics of the dissertation with varying domains and an existence predicate, in order to capture other axiomatizations of quantified modal relevant logics, ought to be feasible. The semantics of Goldblatt [47] and Goldblatt and Mares [75] for the classical case can serve as a basis. However, this task is not straightforward. The set theoretic conditional in $\sqcap(Ea \Rightarrow |\phi|)$ corresponds to a material conditional in the syntax. It is not clear philosophically whether the conditional in the relevant logic in this case should be relevant or classical. If relevant, then it appears that the models will require a slight overhauling.

Proof Theory

In the dissertation I used Hilbert-style axiom systems as proof systems for the logics considered. Other proof systems are desirable. An important area in proof theory is that of sequent calculi, which were introduced by Gentzen [44]. Unfortunately, proof theory for both relevant logics and for modal logics is an area of research with many disappointing results. It turns out to be difficult to give sequent calculi for many relevant and modal logics. Display logics, introduced by Belnap [8] (see also [9]), and hypersequents, can solve some of the problems encountered. Examples of sequent calculi, display calculi, and hypersequents for some modal and relevant logics can be found in Wansing [117], Bimbó [13], Restall [93], and Paoli [87], for example.

It would be beneficial to have additional proof systems for the logics considered in this document. Many of the extant systems for fragments of these logics ought to be useful in defining such proof systems. Again, this is left as a future project.

As mentioned in the introduction, this project began as merely an aim at constructing a foundation for creating relevant default and auto-epistemic logics. In particular, this was to tackle the frame problem. (Note that relevant logic, and in particular relevant predication, has already been used to approach the problem by Dunn [30].) However, the work of the dissertation opens many doors for potential applications, as detailed throughout the dissertation. Moreover, the preceding paragraphs show that there is much work to be done extending what I have developed here. I intend to continue to develop this project in the ways described.

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